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E6VRS **CRYPTOLOGY** E

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QISV LECTURE NOTES 8

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Contents

1. Introduction

1.1 Cryptography vs. Cryptoanalysis

Cryptography: The constructive part (e.g. building systems, “constructing the encryption”).
Cryptanalysis: The destructive part (e.g. breaking systems, “cracking the code”).

This course covers both parts of Cryptology. Both parts are needed in deployed systems, since we want to know how efficient the system is **and** how hard it is to break it:

■ **Example 1.1**

- 2^{60} is inconvenient to compute, but totally doable;
- 2^{70} is doable for academic clusters;
- 2^{80} is doable for the NSA;
- 2^{128} nobody can do that (including a security margin);

■

R These numbers depend on the model of computation. Algorithms for example have different runtimes on quantum computers.

1.2 Caesar Cipher

A standard example in cryptology is a CAESAR CIPHER. In this example ALICE (denoted A in figure 1.1) wants to transmit a secret message ($E(m)$) to BOB (B), since the eavesdropper EVE (E) can see or hear everything on the channel.

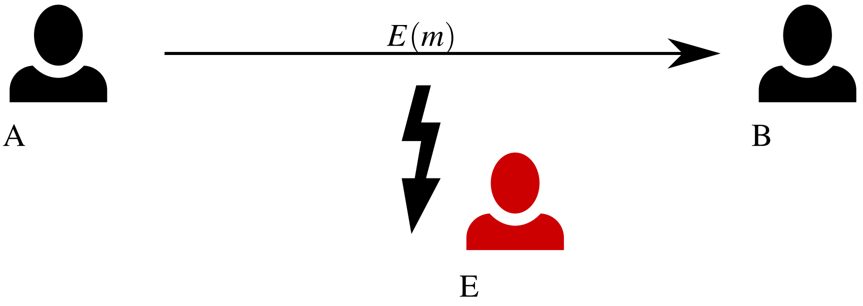


Figure 1.1: Encrypted communication

So ALICE and BOB decide to use a simple form of encryption - the CAESARS CIPHER. They replace the letters of the message by a letter some fixed number (in this case it is 3) of positions down the alphabet (see figure 1.2).

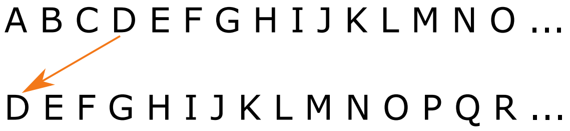

A B C D E F G H I J K L M N O ...
D E F G H I J K L M N O P Q R ...

Figure 1.2: Idea of CAESARS CIPHER

■ Example 1.2
The (original) message
HELLO (Plaintext), would be encrypted into
KHOOR (Ciphertext) ■

This method of encryption is easy to cryptanalyze.

It is possible to turn this into a keyed encryption: $E_k(m)$, where the key k gives the shifting distance. The decryption therefore is: $D_k(c)$, which should satisfy

$$D_k(E_k(m)) = m.$$

1.3 Two types of crypto systems

There are two types of crypto systems: *symmetric* and *asymmetric* (= public key) systems. In the *symmetric* crypto A and B share the same key (e.g. the shifting distance of the CAESARS CIPHER, 128 bits in AES).

In *public key* crypto each user has 2 keys:

- A *public* one which can be used by anybody to encrypt messages to this user and
- A *private* (= secret key) one which the user uses to decrypt messages.

Therefore in *asymmetric* systems this means $c = E_{P_k}(m), D_{S_k}(c) = m$, with P_k the *public key* and S_k the *secret key*. P_k and S_k can be of very different nature.

In a good crypto system one should not be able to recover S_k from P_k (= *complete break*) or to decrypt c without knowing S_k.

R This means that there are two possiblities of cryptanalyzing a system. One can either find the *secret key* or find the message. Obviously the first case means a *complete break*, meaning that every other message can be decrypted as well.

We will see different crypto systems such as RSA, Elliptic-Curve Cryptography (ECC) and McEliece encryption including attacks on these.

Crypto is also about authentication, e.g. digital signatures.

2. Mathematic Repetition

2.1 Modulus Calculation

Notation 2.1.
$\mathbb{Z}$: *integers* $\{\dots,-3,-2,-1,0,1,2,3,\dots\}$
$\mathbb{Z}/n$: *residue classes of* $\mathbb{Z}$ *modulo* n
$\mathbb{Z}/6 = \{0,1,2,3,4,5\} = \{-2,-1,0,1,2,3\}$.

We use integers to represent classes, so 0 stands for the set of integers which are congruent to 0 modulo 6, i.e. $\{0,6,12,18,\dots,-6,-12,\dots\}$. You might have learned this as $\bar{0}$ to denote the class.

$7 \equiv 1 \mod 6$ ($\widehat{=}$7 is congruent to 1 modulo 6). Usually we want the small represantative on the right side; depends on set chosen, e.g. $17 \equiv 5 \equiv -1 \mod 6$.

2.1.1 Multiplication

If we have a multiplication with modulus n, e.g. $a \cdot b \mod n$. We can compute the result and then reduce - or we could reduce at any intermediate step. $17 \cdot 35 \equiv -1 \cdot (-1) \equiv 1 \mod 6$.

2.1.2 Exponentiation

If we want to compute an exponentiation with modulus n: $a^b \mod n$, we write b in binary:

$$b = \sum_{i=0}^{l-1} b_i 2^i;\ b_i \in \{0,1\} \text{ with } b_{l-1} = 1$$

■ **Example 2.2**
$17 = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$;
$35 = 1 \cdot 2^5 + 1 \cdot 2^1 + 1 \cdot 2^0$. ■

$$a^b = a^{\sum b_i \cdot 2^i} = \left(\dots\left(\left(a^{b_{l-1}2+b_{l-2}}\right)^{2+b_{l-3}}\right)^{2+\dots+2+b_0}$$

Square and Multiply

```
$c \leftarrow a$ ;
for $i = l-2$ down to $0$ do
    $c \leftarrow c^2 \bmod n$ ;
    if $b_i = 1$ then
        $c \leftarrow c \cdot a \bmod n$ ;
    end
    output $c$;
end
```

Algorithm 1: Square and Multiply - pseudocode

This algorithm reduces at every intermediate step, making the exponentiation a lot simpler and faster.

2.1.3 Inverses

Sometimes we need to calculate inverses, e.g. divide equation $\bmod 7$ by 4, this means undo a multiplication by 4. What we are looking for is to find an integer $a \bmod 7$, such that $4 \cdot a \equiv 1 \bmod 7$, other notation $a^{-1} \equiv 4 \bmod 7$.

Here $a \equiv 2 \bmod 7$ because $4 \cdot 2 \equiv 8 \equiv 1 \bmod 7$.

In general, use EUCLIDEAN ALGORITHM to compute a.

Look up XGCD-EXTENDED EUCLIDEAN ALGORITHM.

Inverses do not exist if the numbers are not coprime, e.g. 4 is not inversible $\bmod 6$.

Notation 2.3. *Set of invertible integers* $\bmod n$ *is denoted by* $(\mathbb{Z}/n)^*$

■ Example 2.4

- $(\mathbb{Z}/7)^* = \{1,2,3,4,5,6\}$ with inverses $1^{-1} \equiv 1 \bmod 7$, $2^{-1} \equiv 4 \bmod 7$, $3^{-1} \equiv 5 \bmod 7$, $6^{-1} \equiv 6 \bmod 7$; these are exactly the integers coprime with 7.
- $(\mathbb{Z}/6)^* = \{1,5\}$
- $(\mathbb{Z}/6)^* = \{1,2,3,\dots,p-1\}$ for p prime.

■

EULER'S PHI FUNCTION gives the size of $|(\mathbb{Z}/6)^*| = \varphi(n)$.

■ Example 2.5

- $\varphi(7) = 6$
- $\varphi(6) = 2$
- $\varphi(p) = p-1$

■

For the multiplication of two primes p,q $(p \neq q)$ you can calculate the PHI FUNCTION as follows:

$$\begin{array}{c|cccccc}
0 & 1 & 2 & 3 & 4 & \dots & q-1 \\
q & q+1 & q+2 & q+3 & \dots & & 2q-1 \\
2q & \dots & & & & & \\
\vdots & & & & & & \vdots \\
(p-1)q & & & & & & p\cdot q-1
\end{array}$$

The numbers in the first column are all divisible by q (there are p of them). They can be removed (since p and q are primes, this means that only numbers which are multiples of p or q are not coprime).

Do the same with p:

$$0 \quad 1 \quad 2 \quad \dots \quad p-1$$

this removes q values,

$$\varphi(p\cdot q) = p\cdot q - p - q + 1$$

Since the 0 is removed twice, we have to add $+1$.

■ **Example 2.6**

- $\varphi(2\cdot 3) = 2\cdot 3 - 2 - 3 + 1 = 2$ (same)
- $\varphi(p^2) = p^2 - p$, in general:
- $\varphi(p^b) = p^b - p^{b-1}$
- if you have: $n = \prod_{i=0}^{l-1} p_i^{e_i}$, $p_i \neq p_j$ then $(e_i \in \mathbb{N} \leq 1)$

$$\varphi(n) = \prod (p_i^{e_i} - p_i^{e_i-1})$$

Test: $\varphi(p\cdot q) = (p-1)(q-1) = p\cdot q - p - q + 1$ as above.

■

2.2 Lagrange's Theorem

Theorem 2.7 — Lagrange's Theorem. Let G be a finite group of size $|G| = l$ then for any $a \in G$ we have

$$a^l = 1,$$

where 1 is the neutral element and G is written multiplicatively

■ **Example 2.8** $a^6 \equiv 1 \bmod 7$ for $a \in (\mathbb{Z}/7)^*$ ■

3. RSA

The name RSA comes from the initials of RIVEST, SHAMIR, ADEMAN. In 1977 it was the first encryption and signature system with a public key.

3.1 Generating the Keys

We pick two large primes $p \neq q$, put $n = p \cdot q$ and compute $\varphi(n) = (p-1)(q-1)$.

Now pick an integer e, which is coprime with $\varphi(n)$.

Compute $e^{-1} \equiv d \bmod \varphi(n)$ using XGCD.

RSA public key: (e,n) private key: d

■ Example 3.1

$p = 5$, $q = 7$,
$n = 5 \cdot 7 = 35$
$\varphi(n) = (5-1)(7-1) = 24$

- Pick $e = 5$, $gcd(5,24) = 1$,
  $5^{-1} \equiv 5 \bmod 24$, so $d = 5$
- Other options, e.g. $e = 7$. Compute gcd:

24	1	0	
7	0	1	$q = 3$
3	1	-3	$q = 2$
1	-2	7	
$r\uparrow$	$a\uparrow$	$b\uparrow$	

For every row, q is the number of times the first item in the row “fits into” the item above it. So for 24 and $e = 7$ this means: $q = 34$ (since $7 \cdot 3 = 21$ with rest 3). You

then subtract 3 times this row from the previous row, or more general: subtract q times row from previous.
Every row now satisfies that $r = a \cdot 24 + b\ \cdot 7$, e.g. $1 = -2 \cdot 24 + 7 \cdot 7 \Rightarrow 7^{-1} \equiv 7 \bmod 24$

Then $(5, 35)$ public key and $d = 5$ secret key or $(7, 35)$ public key and $d = 7$ secret key.

R It is pure coincidence (and due to the fact that we use small numbers) that d and e are equivalent)

■

3.2 Encryption and Decryption of message

3.2.1 Encryption

We want to encrypt the message $m < n, m \in \mathbb{N}$.

$$c \equiv m^e \bmod n$$

R At this point in an implementation we would use the Square and Multiply method.

3.2.2 Decryption

To decrypt of c, we calculate:

$$m' \equiv c^d \bmod n$$

R We now want to proof that the decryption undos the encryption, or as we stated earlier $D(E(m)) = m$, or in this case $m' = m$

This decryption works because

$$m' \equiv c^d \equiv (m^e)^d \equiv m^{e \cdot d} \equiv m^{1+k\varphi(n)} \equiv m \bmod n$$

where $e \cdot d = 1 \bmod \varphi(n)$, i.e. $e \cdot d = 1 + k \cdot \varphi(n)$ for some k.

For the last step, we used LAGRANGE with group $(\mathbb{Z}/n)^*$ because there $m^{\varphi(n)} \equiv 1 \bmod n$. You can check that $m^{1+k\varphi(n)} \equiv m \bmod n$ even when $gcd(m, n) \neq 1$.

To show this, use the CHINESE REMAINDER THEOREM (CRT)

$$m \equiv 0 \bmod p \qquad m \equiv a \bmod q,\ a \neq 0, \text{ then} \qquad m^{e \cdot d} \equiv 0^{e \cdot d} \equiv 0 \bmod p$$

In $(\mathbb{Z}/q)^*$ we have $a^{q-1} \equiv 1 \bmod q$.
Thus

$$m^{e \cdot d} \equiv a^{e \cdot d} \equiv a^{1+k\varphi(n)} \equiv a^{1+(p-1)(q-1)} \equiv a \cdot (\underbrace{a^{q-1}}_{=1})^{p-1} \equiv a \cdot 1^{p-1} \equiv a \bmod q$$

$$m^{e \cdot d} \equiv 0 \equiv m \bmod p$$
$$m^{e \cdot d} \equiv a \equiv m \bmod q,$$

thus CRT gives $m^{e \cdot d} \equiv m \bmod n$.
Last case $0^{e \cdot d} \equiv 0 \bmod d \Rightarrow$ RSA gives $m' = m$.

3.3 Making the Algorithm Faster

To get a faster encryption, we can use a small e when generating the key.

Choosing small d gives EVE your key; but we can decrypt faster using RSA-CRT: Compute

$$c^d \equiv m_p \bmod p \qquad\qquad c^d \equiv m_q \bmod q$$

This is faster because the operands are smaller (naivley this save a factor of 4 in each computation, do this twice, so save factor of 2).

Combine m_p and m_q using CRT to get m.

We can save more effort by noticing that $c^{p-1} \equiv \bmod p$, so compute $d_p \equiv d \bmod p-1$ and $d_q \equiv d \bmod q-1$; d_p and d_q have half the bit length of d, so the SQUARE AND MULTIPLY loop is half as long in computing

$$c^{d_p} \equiv m_p \bmod p \qquad\qquad c^{d_q} \equiv m_q \bmod q$$

Combine m_p and m_q to m using CRT. This needs just one multiplication given $u \equiv p^{-1} \bmod q$ as precomputed value.

Then $m = m_p q \cdot v + p \cdot u \cdot m_q$ with $v \equiv q^{-1} \bmod p$. “Naively” because with KARATSUBA, doublelength costs only 3 times as much and for very long integers the FAST FOURIER TRANSFORMATION (FFT) takes only twice as long.

4. Finite Fields

Definition 4.1 — Fields. A set K is a *field* with respect to $\circ$ and $\diamond$, denoted $(K,\circ,\diamond)$, if

i $(K,\circ)$ is an abelian group:
- **closure** for all $a,b \in K \Rightarrow a\circ b \in K$
- **associativity** $(a\circ b)\circ c = a\circ(b\circ c)\ \ a,b,c \in K$
- **identity** there is a $e_0 \in K$ such that for all $a \in K$, $a\circ e_0 = e_0\circ a = a$
- **inverse** for all $a \in K$, there is a $b \in K$, such that $a\circ b = e_0$
- **commutive** $a\circ b = b\circ a$

ii $(K^* = K\ \{e_0\},\diamond)$ is an abelian group

iii the distributive law holds in K:

$$a\diamond(b\circ c) = a\diamond b\circ a\diamond c$$

■ Example 4.2

- $(\mathbb{N},+,\cdot)$ is **NOT** a finite field (e.g. there is no inverse for $+$)
- $(\mathbb{Z},+,\cdot)$ is **NOT** a finite field (e.g. there is no inverse for $\cdot$)
- $(\mathbb{Q},+,\cdot)$ is a finite field
- $(\mathbb{R},+,\cdot)$ is a finite field
- $(\mathbb{C},+,\cdot)$ is a finite field
- $K = \{0,1\}$ is the smallest set we can get:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

relating to:

$\circ$	$e_\circ$	$e_\diamond$
$e_\circ$	$e_\circ$	$e_\diamond$
$e_\circ$	$e_\diamond$	$e_\circ$

$\diamond$	$e_\circ$	$e_\diamond$
$e_\circ$	$e_\circ$	$e_\circ$
$e_\diamond$	$e_\circ$	$e_\diamond$

The $+$ corresponds to XOR, the $\cdot$ to AND

■

Definition 4.3 — Subfield. If $(K,\circ,\diamond)$ and $(L,\circ,\diamond)$ are field and $K \subseteq L$ then K is a *subfield* of L.

$\Rightarrow$ We can add elements of L to and multiply them with elements of $K \Rightarrow L$ is a vectorspace over K (other properties work because of the distributive laws.

Definition 4.4 — Extension Degree. Let L be a field and let K be a subfield of L. The *extension degree* $[L:K]$ is defined as $\dim_k L$, the dimension of L as a K vectorspace.

Definition 4.5 — Characteristic. Let K be a field. The *characteristic* of K, denoted $\text{char}(K)$, is the smallest positive integer m such that $\underbrace{e_\diamond \circ e_\diamond \circ \cdots \circ e_\diamond}_{m \text{ copies of } e_\diamond \text{ denoted as } [m]e_\diamond} = e_\circ$; if no such integer exists, $\text{char}(K) = 0$.

Lemma 4.6 The characteristic of a field is 0 or a prime

Proof. Let $\text{char}(K) = n = a \cdot b \;\; 1 < a,b < n$. Then

$$e_\circ = [m]e_\diamond = [a\cdot b]e_\diamond = [a]e_\diamond \diamond [b]e_\diamond$$

Since a field has no zero divisors it must be that $[a]e_\diamond = e_\circ$ or $[b]e_\diamond = e_\circ$ ↯ to minimality of $\text{char}(K) = n$. ■

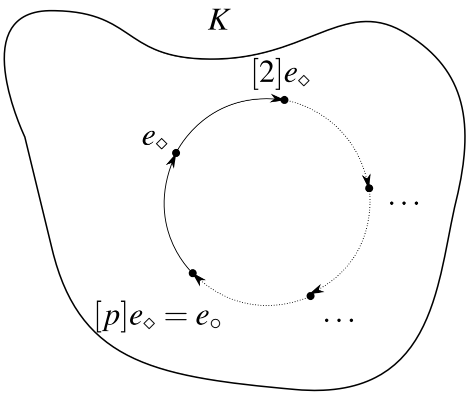
In the proof of Lemma 4.6 we repeatedly used the distributive law:

$$\begin{aligned}
&[a]e_\diamond \diamond [b]e_\diamond \\
\iff\; &(e_\diamond \circ [a-1]e_\diamond) \diamond [b]e_\diamond \iff \underbrace{e_\diamond \diamond [b]e_\diamond}_{\substack{=[b]e_\diamond,\\ \text{since } e_\diamond \text{ is the neutral}\\ \text{element for } \diamond}} \circ [a-1]e_\diamond \diamond [b]e_\diamond \\
\iff\; &[b]e_\diamond \circ (e_\diamond \circ [a-2]e_\diamond) \diamond [b]e_\diamond \iff \ldots \iff \underbrace{[b]e_\diamond \circ [b]e_\diamond \cdots \circ [b]e_\diamond}_{a \text{ times } \to [a\cdot b]e_\diamond}
\end{aligned}$$

Lemma 4.7 A finite field K has characteristic p for some prime p

Proof. Since K is finite, there must be $i,j \in \mathbb{N}$ with $[i]e_\diamond = [j]e_\diamond$. Let $i > j$ then $[i-j]e_\diamond = e_\circ$ and so $\text{char}(K) | (i-j)$. ■

Let K be a finite field. We will now explore its structure. We know already: $\text{char}(K) = p$ for a prime p, and there exists $e_\circ, e_\diamond \in K$ with $e_\circ \neq e_\diamond$. Since K is closed under $\circ$ we do also find $[2]e_\diamond, [3]e_\diamond, \ldots, [p-1]e_\diamond, [p]e_\diamond = e_\circ, [p+1]e_\diamond = e_\diamond, \ldots$ a cyclic subgroup of order p of $(K,\circ)$. Multiplying two such elements $[i]e_\diamond \diamond [j]e_\diamond = [ij]e_\diamond$ again gives us an element of the set$\{[i]e_\diamond | 0 \leq i < p\}$. The scalars are considered modulo p because $[p]e_\diamond = e_\circ$. Since p is a prime, $i \cdot j \not\equiv 0 \bmod p$ for $0 < i,j < p$. This means that $\{[i]e_\diamond | 0 < i < p\}$ forms a subgroup of K^* (the multiplicative group in K; $K^* = K \backslash \{e_\circ\}$).



If two structures (groups, rings, fields,...) behave exactly the same way so that one can give a one-to-one map between them, mathematicians call these twostructures *isomorphic*. Out considerations have found a subfield of K which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ with map $[i]e_\diamond \longmapsto i + p\mathbb{Z}$.

Definition 4.8 — Prime Field. Let K be a field. The smallest subfield contained in K is called the *prime field* of K.

Lemma 4.9 Let K be a finite field of $\mathrm{char}(K) = p$.
The prime field of K is isomorphic to $\mathbb{Z}/p\mathbb{Z}$

$$\begin{aligned} e_\circ &\longmapsto 0 && \text{in } \mathbb{Z}/p\mathbb{Z} \\ e_\diamond &\longmapsto 1 && \text{in } \mathbb{Z}/p\mathbb{Z} \end{aligned}$$

Above we found that an extension field can be considered as a vectorspace over its subfield. From now on we identify the prime field of a finite field with $\mathbb{Z}/p\mathbb{Z}$ and write 0 for $e_\circ$ and 1 for $e_\diamond$. Let $[K : \mathbb{Z}/p\mathbb{Z}] = n$, i.e., the dimension of K as a vectorspace over $\mathbb{Z}/p\mathbb{Z}$ is n. This means that there exists a basis of n linearly independent "vectors" $\alpha_1, \alpha_2, \dots, \alpha_n$. This being a basis means that every element in K can be written in a unique way as $\sum_{i=1}^{n} c_i \alpha_i$ with $c_i \in \mathbb{Z}/p\mathbb{Z}$. The p^n different choices for $(c_1, c_2, \dots, c_n) \in \mathbb{Z}/p\mathbb{Z}^n$ mean that K has p^n elements.

Lemma 4.10 Let K be a finite field. There exists a prime p and an integer $n \in \mathbb{N}_{>0}$ such that $|K| = p^n$ and $\mathrm{char}(K) = p$.

Notation 4.11. *A field of characteristic p and dimension n is $\mathbb{F}_{p^n}$ or $GF(p^n)$ (for "*GALOIS *field")*

This implies that every finite field has a prime power as its cardinality, so in particular there are no fields of size $6, 10, 14, 15$ etc.

In this representation it is very easy to add elements:

$$\left(\sum_{i=1}^{n} c_i \alpha_i\right) + \left(\sum_{i=1}^{n} d_i \alpha_i\right) = \sum_{i=1}^{n} (c_i + d_i)\, \alpha_i$$

but for multiplying them we need to know $\alpha_i \cdot \alpha_j$ for $1 \le i, j \le n$.

R From now on we write $+$ for the first operation $\circ$ and $\cdot$ for the second operation $\diamond$, since we see K as an extension of $\mathbb{Z}/p\mathbb{Z}$.

Let's see whether we can find out more about the multiplicative strucure. Remember that for a group G we have $[|G|]a = e$ for any $a \in G$ by the properties of the order of a group. Since K is a field, K^* is a group and it has one element, namely 0, less than K; thus $|K^*| = p^n - 1$.

Lemma 4.12 Let K be a finite field. The multiplicative group K^* is cyclic.

Thus, for every $a \in K^*$ we have $a^{p^n - 1} = 1$.

Let's look at another field beyond $\mathbb{Z}/p\mathbb{Z}$. We know that they must have p^n elements for some p and n - so what about a field with $2^2 = 4$ elements? This should have a basis of size 2, let's use $\alpha_i = 1$ and $\alpha_2 = a$ then $\mathbb{F}_4 = 0, 1, a, a+1$ and we can simply write out the addition table using the vectorspace structure:

$+$	0	1	a	$a+1$
0	0	1	a	$a+1$
1	1	0	$a+1$	a
a	a	$a+1$	0	1
$a+1$	$a+1$	a	1	0

To write the multiplication table - if possible - we need to know what a^2 is in terms of $1, a$ and $a+1$. A table of a group has each element exactly once per row and column. So defining $a^2 = a$ conflict with having already entry a in the first entry of this row (see left table below). Using $a^2 = 1$ means that $a \cdot (a+1) = a^2 + a = 1 + a$ - but then the third column (in the middle table) has already $a+1$ in the first entry. Try $a^2 = a+1$ then $a \cdot (a+1) = a^2 + a = (a+1) + a = 1$ and $(a+1) \cdot (a+1) = a^2 + a + a + 1 = a^2 + 1 = (a+1) + 1 = a$.

$\cdot$	1	a	$a+1$
1	1	a	$a+1$
a	a	a	
$a+1$	$a+1$		

$\cdot$	1	a	$a+1$
1	1	a	$a+1$
a	a	1	$a+1$
$a+1$	$a+1$		

$\cdot$	1	a	$a+1$
1	1	a	$a+1$
a	a	$a+1$	1
$a+1$	$a+1$	1	a

The tables show all group properties except for associativity. We could probe this by checking all combinations but that is very cumbersome.

We could do the same that we just did with $\mathbb{F}_4$ now with $\mathbb{F}_8$, but this would take a while (we would now use $1, a$ and b as a basis). The multiplication table is now 8 × 8. So the question arises:

Problem 4.13 How can we get this "automatically"? How do we compute $\alpha_i \cdot \alpha_j$ without a lookup table?

The idea is to use a polynomial ring to represent the field elements. A polynomial ring also spans a vector space - but contrast to the vector space, the multiplication of polynomials is well defined.

The polynomial ring over field K:

$$K[x] = \left\{ \sum_{i=1}^{n} a_i x^i | n \in \mathbb{N}, a_i \in K \right\},\ f \in K[x],\ f = \sum f_i x_i.$$

Let n be the largest integer with $f_n \neq 0$ then $\deg(f) = n$, *leading coefficient* $\mathrm{LC}(f) = f_n$, *leading term* $\mathrm{LT} = f_x x^n$.

Definition 4.14 — Irreducible. A polynomial $f \in K[x]$ is called *irreducible* if $\deg(f) \geq 1$ and it cannot be written as a product of polynomials of lower degree over the same field, i.e. if $u|f$ then $u \in K$ or $u = f$ for all $u \in K[x]$ and $\deg(u) \leq \deg(f)$ Otherwise f is *reducible*. Note that this depends on the field K.

■ Example 4.15

- $x^2 - 1 = (x+1)(x-1)$ is reducible in $\mathbb{R}[x]$.
- $x^4 + 2x + 1 = (x^2+1)^2$ in $\mathbb{R}[x]$ has no roots but is reducible.
- $x^2 + 1$ is irreducible in $\mathbb{R}[x]$ but reducible in $\mathbb{C}[x]$ by $(x-i)(x+i)$.
- $x^3 + 6x^2 + 4$ is irreducible in $\mathbb{Z}/7\mathbb{Z}$

■

We will now look again at $GF(8) = GF(2^3)$. The question arises, what is $a \cdot b$ or $a \cdot a^2 = a^3$ since $b = a \cdot a = a^2$? We chose $a^3 = a + 1$ and then all operations followed by using this equality. This polynomial - $a^3 + a + 1$ - does not factor over $GF(2)$. Other choices we considered, e.g. $a^3 + 1$ do in fact factor and it was exactly by considering these factors, e.g. $(a+1)$ and (a^2+a+1) that we derived contradictions, e.g. $(a+1)\cdot(a^2+a+1) = a^3 + 1 = 0$ (using $a^3 = 1$). In the end we worked in $GF(2)[a]/_{(a^3+a+1)GF(2)[a]}$ - the polynomial ring over $GF(2)$ modulo the irreducible polynomial $a^3 + a + 1$.

■ Example 4.16

We want to compute $a \cdot (a^2 + a) = a^3 + a^2$ and $(a+1)\cdot(a^2+a) = a^3 + \cancel{a^2} + \cancel{a^2} + a = a^3 + a$. For this we use the irreducible polynomial $a^3 + a + 1$ and do a polynomial long division:

$$\begin{array}{r|ll} a^3+a+1 & a^3+a^2 & = 1 \\ & a^3+a+1 & \\ \cline{2-2} & a^2+a+1 & \end{array} \qquad\qquad \begin{array}{r|ll} a^3+a+1 & a^3+a & = 1 \\ & a^3+a+1 & \\ \cline{2-2} & \hfill 1 & \end{array}$$

■

In general, this construction gives a finite field: Let f be an irreducible polynomial of degree n over $GF(p)$. We define addition and multiplication on

$$GF(p)[x]/_{f(x)GF(p)[x]} = \left\{ \sum_{i=0}^{n-1} a_i x^i | a_i \in GF(p) \right\}$$

as addition and multiplication in $GF(p)[x]$ followed by reduction modulo $f(x)$. $\rightsquigarrow GF(p^n)$

Let $g \in GF(p^n)$, f irreducible in $GF(p)$. $\gcd(f,g) = 1$ (The gcd of f and g has to be 1 since f is irreducible) and XGCD computes polynomials h and l with

$$\begin{aligned} 1 &= g \cdot h + f \cdot l \\ 1 &\equiv g \cdot h \bmod f \\ h &\equiv g^{-1} \bmod f \end{aligned}$$

■ **Example 4.17**

The polynomial $f = x^3 + x^2 + 1$ is irreducible over $GF(2)$. What is the inverse of $g = x^2 + 1$ over $GF(2)$ mod f?

$$\begin{array}{r|l} x^2+1 & \overline{x^3+x^2+1} \quad = x+1 \\ & \quad -(x^3+x) \\ \hline & \quad x^2+x+1 \\ & \quad -(x^2+1) \\ \hline & \qquad\quad x \end{array}$$

$$\Rightarrow x^3 + x^2 + 1 = (x^2+1)(x+1) + x \qquad (*)$$

$$\begin{array}{r|l} x & \overline{x^2+1} \quad = x \\ & -x^2 \\ \hline & \quad 1 \end{array}$$

$$\Rightarrow x^2 + 1 = x \cdot x + 1 \qquad (**)$$

$$1 = f \cdot ? + g \cdot ?$$

$$\begin{aligned} 1 &= (x^2+1) + x \cdot x && \text{from } (**) \\ &= (x^2+1) + [(x^3+x^2+1) + (x^2+1)(x+1)]x && \text{from inserting } (*) \\ &= (x^2+1) + (x^3+x^2+1)x + (x^2+1)(x+1)x \\ &= (x^3+x^2+1)x + (x^2+1) + (x^{+}1)(x+1)x \\ &= (x^3+x^2+1)x + (x^2+1)[1 + (x+1)x] \\ &= (x^3+x^2+1)x + (x^2+1)[x^2+x+1] \end{aligned}$$

■

Alternative approach:

We know that $a^{p^n} = a$ and $a^{p^2-1} = 1$ for $a \in GF(p^n)$ (LAGRANGE'S Theorem). Thus $a \cdot a^{p^n-2} = a^{p^n-1} = 1$.

So we can compute the inverse of (x^2+1) as $x(^2+1)^6$ in $GF(8)$:

$$(x^2+1)^{8-2} = (x^2+1)^6 = (x^2+1)^4 \cdot (x^2+1)^2 = ((x^2+1)^2)^2 \cdot (x^2+1)^2$$

How do we find irreducible polynomials? We can pick a random polynomial and check if it is irreducible using “RABIN’S test of irreducibility”.

Notation 4.18. *We denote $\mathbb{F}_q^*$ as the multiplicative group of $\mathbb{F}_q$. $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$ is a cyclic group and if $\mathbb{F}_q^* = \langle g \rangle$ then the generator g is called a primitive element.*

■ Example 4.19

$$\mathbb{F}_7^* : \ \langle 2 \rangle = \{2^0 = 1, 2^1 = 2, 2^2 = 4\}$$

$2^3 = 1$, etc. so $\langle 2 \rangle$ contains only 3 elements, so 2 is not primitive.

$$\langle 3 \rangle = \{1,3,2,6,4,5\} = \mathbb{F}_7^*$$

so 3 is a primitive element. ■

Lemma 4.20 Let G be a cyclic group of order n, then the order of any $a \in G$ satisfies:

$$\mathrm{ord}(a) | n$$

$\mathrm{ord}(a)|n$ states how many times we need to multiply a to receive 1, i. e. $a^n = 1$.

■ Example 4.21

- $\mathrm{ord}(4) = 6$ in $\mathbb{F}_7^*$
- $\mathrm{ord}(2) = 3$ in $\mathbb{F}_7^*$
- $\mathrm{ord}(1) = 1$ in $\mathbb{F}_7^*$
- $\mathrm{ord}(6) = 2$ in $\mathbb{F}_7^*$

■

Lemma 4.22 Let G be a cyclic group of order n, let $\langle g \rangle = G$, and let $l|n$.

$$\text{Then } \mathrm{ord}(g^{n/l}) = l$$

Proof.

$$1, g^{n/l}, g^{2n/l}, g^{3n/l}, \ldots, g^{ln/l} = 1$$

so $\mathrm{ord}(g^{n/l})$ is no larger than l and $i \cdot {}^{n}/_{l}$ for $0 \le i < n$ is less than n. Because n is minimum for g, l is minimal for ${}^{n}/_{l}$. ■

■ Example 4.23

$$\langle 3 \rangle = \mathbb{F}_7^*; \ \mathrm{ord}(3^{6/2} = 3^3) = \mathrm{ord}(6) = 2$$

■

There are multiple elements of order l: all the $g^{i \cdot n/l}$ have order dividing l and if $gcd(i,l) = 1$ then $\mathrm{ord}(g^{i \cdot n/l}) = l$.

■ **Example 4.24**

$$\mathbb{F}_{19}^{*} = \langle 2 \rangle \text{ , } 19 - 1 = 18 = 2 \cdot 3^2$$

$$\{1,2,4,8,-3,-6,7,-5,9,-1,-2,-4,-8,3,6,-7,5,9\} = \langle 2 \rangle$$

$$2 \text{ has } \mathrm{ord}(6) \text{ , but } \mathrm{ord}(2^{2 \cdot 18/6}) = 3 \text{ , } \mathrm{ord}(2^{3 \cdot 18/6}) = 2$$

This means there are $\varphi(l)$ elements of order l. ■

5. Diffie-Hellman Key Exchange

5.1 Introduction

The Diffie-Hellman Key Exchange is used to secretly exchange keys for an encryption. The key exchange can be explained easily with an example:

A(lice) and (B)ob want to exchange a key, without letting the eavesdropper E(ve) know. Everybody (including) Eve knows a cyclic group G, a generator g and its order n:

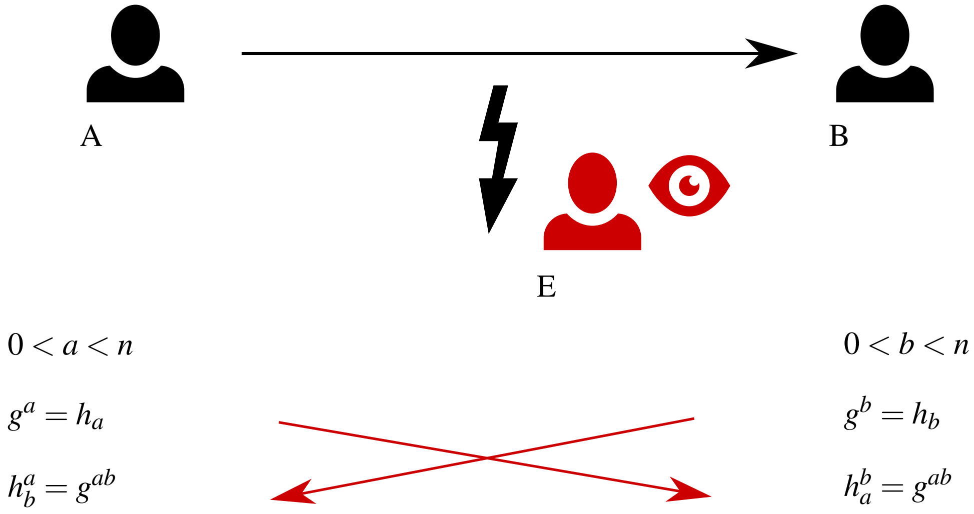


Figure 5.1: Illustration of theDiffie-Hellman key exchange
A and B both think of a number between 0 and n (this is a and b respectively). Now the compute g^a and g^b and exchange this information (so Eve knows g^a and g^b. Then A computes g^{b^a} and B computes g^{a^b}. So they both now know g^{ab}.

In summary, this means that A computes $h_b^a = (g^b)^a = g^{ba} = g^{ab}$ and B computes

$h_a^b = (g^a)^b = g^{ab}$. So A and B both share g^{ab}, while Eve only knows g, $h_a = g^a$, $h_b = g^b$.

5.1.1 Computational DH Problem (CDHP)

Given g, g^a, g^b compute g^{ab}.

In groups that are good for crypto there are no efficient attacks on the CDHP.

■ **Example 5.1** Examples for these "good groups" are:

- elliptic curves over finite fields
- multipl. groups of finite fields

■

5.1.2 Decisional DH Problem (DDHP)

Given g, g^a, g^b and g^c.

Decide whether $g^c = g^{ab}$.

Proofs for protocols often use DDHP rathen than CDHP.

Bad groups would be: $\langle 2 \rangle \subseteq \mathbb{Q}^*$:

$$1, 2, 4, 18, 16 \cdots, 2^i, \ldots \qquad \text{no reduction}$$

$h_a = 2^a$, E takes log and gets a.

5.1.3 Discrete Logarithm Problem (DLP)

Given g and g^a, compute a.

Solving DLP implies solving CDHP implies solving DDHP. Usually DLP is the best attack we know on DHP, but there is no equivalence.

In browser DH or DHE indicates that DH in finite fields is used.

DHE: ephemeral DH, i. e. use a new key for every connection or for each time interval. Perfectly forward secrecy. Somebody taking over your system at time t should not be able to decrypt any message prior to time t (or $t - t_0$, where t_0 is the time for the ephemeral key).

DH: DIFFIE-HELLMAN with longterm keys.

Crypto parameters choose p to have ≥ 2048 bits and prime, work in $\mathbb{F}_p^*$, are okay with long-term use; ephemeral is to deal with stuff outside crypto.

A(LICE) and B(OB) use share g^{ab} after running it through a hash function to get a fixed-length string (128 or 256 bits) with good distribution. Cryptographic hash functions are also:

- *pre-image resistant*,i. e. cannot find g^{ab} from $h(g^{ab})$
- *second-pre-image resistant*, i. e. cannot find another pre-image given the first one
- *collision resistant*, i. e. cannot find two strings m_1 and m_2 with $h(m_1) = h(m_2)$.

R We know that m_1 and m_2 exist, but we don't want to be able to compute them.

In DH we use $h(g^{ab})$ as shared key in symmetric crypto, e. g. AES.

5.2 ElGamal Encryption

General parameters: g, n

ALICE'S public key: $h_a = g^a$

ALICE'S secret key: a

Encryption: Pick random $0 < k \leq n$, compute $r = g^k$, $c = h_a^k \cdot m$ (assume $m \in G$), with m the message. We then send (r, c).

Decryption: $c/r^a = m'$. This works, i. e. $m = m'$ because

$$\frac{c}{r^a} = \frac{h_a^k \cdot m}{(g^k)^a} = \frac{\cancel{(g^a)^k} \cdot m}{\cancel{(g^a)^k}} = m$$

In practice, ELGAMAL is not used like this because $m \notin G$. Instead $c = AES_{h(h_a^k)}(m)$ and m' is computed by first computing $K = h(r^a)$ and then $AES_K(c) = m'$. This corresponds to asymmetric DH.

A has longterm key, sender has K, r as one time keys, but DH uses h_a^K directly as key instead of transmitting m.

5.2.1 ElGamal Signature Scheme

Parameters as above; signature proves that the signer has access to a. Signature is on $h(m)$ not m - fixed length; no algebraic relations.

Sign: Pick **one-time** $0 < k < n$ nonce (= number used only once), compute $r = g^k$, compute $s = k^{-1}(h(m) - a \cdot r) \bmod n$. The signature is (r, s).

Verify: Does $g^{h(m)}$ equal $h_a^r \cdot r^s$? A valid signature passes verification because:

$$h_a^r \cdot r^s = g^{ar} \cdot g^{k(k^{-1}(h(m)-ar))} = g^{h(m)} \qquad \surd$$

Anybody knowing a can sign. This becomes a problem if the a being used, becomes known:

We see (r, s) on m, so we can compute

$$s \cdot k = h(m) - a \cdot r \overset{\substack{\text{since} \\ s,k,r,h(m) \\ \text{are known}}}{\Longrightarrow} a = \frac{h(m) - s \cdot k}{r} \bmod n$$

So we can recover A's longterm secret a with just one leaked nonce k. We can also recover a if k is reused (see the homework), we can detect this from seeing repeated r.

R The problem of re-using k, made the first PLAYSTATION open for attacks

This means that ELGAMAL signatures are somewhat fragile. We can work around this by choosing k pseudorandomly, i. e. A has two secrets: a and k_a, compute k as $k = h(k_a, m)$. This gives the same k for repeated m, but the attack on repeated k needs different m's. If you want a shorter secret have a master secret key s_a and derive $a = h(s_a, \alpha)$ - with α a string "nonce key".

Note that biases in k also lead to breaks via the hidden number problem (we may take about this later), so would need really good randomness for each k - or this construction.

Properties of hash functions are important here because a second pre-image of $h(m)$, i. e. a message m' with $h(m) = h(m')$ can just use the same (r, s).

If E has colliding messages m_1 and m_2, where m_1 is innocent, she can ask A to sign m_1 and use that (r, s) as signature on m_2.

5.3 Pohlig-Hellman Attack

For the DDH we know g, g^a, g^b, g^c, and the problem is, is $g^c = g^{ab}$?

Sometimes we can solve this without solving CDH:

$$\mathbb{F}_{19}^* = \langle 2 \rangle \qquad h_a = 7, \qquad h_b = 11$$
$$g^c = 13 \overset{?}{=} DH(7,11)$$

We can figure out if a was even or odd. $\text{ord}(2) = 18$, so 18 is smallest power of 2 giving 1.

If $a = 2a'$, then $h_a^9 = 2^{a9} = 2^{2a'9} = 2^{18a'} = 1$
else $a = 2a'+1$, then $h_a^9 = 2^{9(2a'+1)} = 2^9 = 18$
$7^9 = 1$ same for h_b: $11^9 = 1$, so a and b are both even, so $a \cdot b$ is even.
Try 13^9 to see whether it can be 2^{ab} : $13^9 = 18 \implies$ this is not $g^{ab} \implies$ we solved DDH.
This way we can solve the DDH whenever parity of c and ab does not match.

To make statements about probability of solving DDH we work with sequences of triples and try to distinguish (g^a, g^b, g^{ab}) from (g^a, g^b, g^c) with random c.

Avoid this DDH weakness by picking g to have odd order, so in $\mathbb{F}_19^*$ use the subgroup generated by $2^2 = 4$ of order 9. This doesn't solve all our problems, we can check whether g^{ab} and g^c match as third powers:

$7^{18/3} = 7^6 = 1$ shows that a is a multiple of 3, thus $a \cdot b \equiv 0 \bmod 3$. Try whether 13 matches this, here $13^6 = 11$, so another proof that $(7, 11, 13)$ is not a valid DDH triple.

Can we find out more about a and c? We know:

$$\left.\begin{array}{l} a \equiv 0 \bmod 2 \\ a \equiv 0 \bmod 3 \end{array}\right\} a \equiv 0 \bmod 6$$

we also know $0 < a < 18$, i. e. $a = 6$ or $a = 12$. We can check the two possibilities and get $2^6 = 7$, so $a = 6$.

R This is sort of a “brute-force” approach.

We know $c \equiv 1 \bmod 2$ and $c \not\equiv 0 \bmod 3$. So $c = c_0 + 3 \cdot c_1$, so $2^c = 2^{c_0+3c_1}$, thus $2^{c \cdot 6} = 2^{6(c_0+3c_1)} = 2^{6c_0}$ this gave $13^6 = 11$, so determine c_0, compare 11 with 2^6 (i. e. $c_0 = 1$) and 2^{12} (i. e. $c_0 = 2$).

$$\text{Here: } 2^6 = 7 \qquad 2^{12} = 11 \qquad \implies \text{ so } c_0 = 2, c = 2 + 3 \cdot c_1$$

With CRT we know $c \equiv 1 \bmod 2$ $c \equiv 2 \bmod 3 \implies c \equiv 5 \bmod 6$, so it could be 5, 11 or 17.

We could just try these 3 but we want a systematic way of computing DL in groups of composite orders.

$$\begin{aligned} &2^c = 2^{2+3c_1} && \text{,so} \\ &2^{3c_1} = {}^{13}\!/_{4} = 8 && \text{, get } c_1 \text{ by computing} \\ &2^{3c_1} = 2^{6c_1} = 8^2 = 7 && \text{and then comparing to} \\ &2^0 = 1, 2^6 = 7 \text{ and } 2^{12} = 11 \implies c_1 = 1 \end{aligned}$$

$\implies$ complete set:

$$c \equiv 1 \bmod 2 \qquad\qquad c \equiv 5 \bmod 9 \qquad\qquad \rightarrow c = 5$$

This POHLIG-HELLMAN *attack* solves the DLP in all subgroups of prime order and then combines the results using CRT.

■ Example 5.2

$\mathbb{F}_{37}^* = \langle 2 \rangle$, find $a = \log_2 17$,

the group has subgroups of order $2, 3, 4, 9$ (and others of non-prime order)

$$17^{18} = 36 \implies a \equiv 1 \bmod 2$$

$$17^{12} = 26, \text{ so } a \not\equiv 0 \bmod 3$$

compare with $2^{12} = 26$ and with $2^{2 \cdot 12} = 10 \rightarrow a \equiv 1 \bmod 3$.
To get a mod 4, compute $({}^{17}\!/_{2})^9 = 27^9 = 36 \Rightarrow a \equiv 1 + 1 \cdot 2 \equiv 3 \bmod 4$.
Same for mod 9: $({}^{17}\!/_{2})^4 = 27^4 = 10 \implies a = 1 + 2 \cdot 3 \equiv 7 \bmod 9 \implies a \equiv 7 \bmod 36$. ■

5.3.1 Pohlig-Hellman & DDH

Computation no harder than in largest prime order group and factors weaken DDH $\implies$ work with g of prime order l $(n = l)$. New problem: solve DLP in $\langle g \rangle$ of prime order l.

Approaches:

- **Brute Force Search:** Takes at most l steps, on average done after ${}^{l}\!/_{2}$.
- **Randomized Brute Force:** (if E suspects that A chooses a in upper part of interval). Attack h_a^b for some known b, i.e. compute DL of $h_a^b = g^{ab}$, get ab, get a by dividing by b modulo l. E can turn this into multi-target attack by searching for $h_a, h_a^{b_1}, h_a^{b_2}, h_a^{b_3}, \ldots$ for known b_i; each would give a after division.
  But each costs an exponensiation while a step in brute force costs a multiplication. Make the multiple targets cheaper by going after $1, h_a, h_a^2, h_a^3, h_a^4, \ldots, h_a^m$. Compare to $1, g, g^2, g^3, g^4, \ldots, g^k$ until we find a match.
  Birthday paradox says that after rougly $\sqrt{l}$ tries we get a collision, so choose $m \approx k \approx \sqrt{l}$

R The birthday paradox states that in a group of only 23 people the probability for a collision of birthdays (two people were born on the same date) is more than 50%.

- **Baby-Step-Giant-Step Algorithm:** Do this systematically, get deterministic algorithm, using $a = a_0 + ma_1$ for $m = \lfloor\sqrt{l}\rfloor$ with $0 \le a_0 < m$ and $0 \le a_1 \le m \underbrace{+1}_{\text{no need}}$.

  1. Compute Baby Steps:
  $$g^0, g^1, g^2, \dots, g^{m-1}$$
  2. Put pairs (g^i, i) in sorted table
  3. Compute $g^{-m} = G$
  4. Compute $h_a, h_aG, h_aG^2, \dots$ the Giant-Steps at every result, compare with table
  5. Once a match is found, i. e. $g^i = h_a \cdot G^j = h_a \cdot g^{-mj} \iff g^{i+mj} = h_a$, so output $i + mj$.

  Running time: Step 2 takes m mults, Step 4 takes at most m mults $\implies$ Total $\le 2\sqrt{l}$ and on average $1.5\sqrt{l}$ (can randomize to avoid corner cases)
  Downside: algorithm needs $\sqrt{l}$ storage. If one trades space for time, e. g. does only $l^{1/3}$ BS, then one needs $\approx l^{2/3}$ GS, so overall time goes up.
  Summary: DLP in any group can be solved in $\mathcal{O}(\sqrt{l})$.

Pollard's Rho Algorithm

Probabilistic algorithm, runs in $\sqrt{\frac{\pi}{2}l}$ (average case) steps. Perform a random walk in $\langle g\rangle$, picking powers of g and h_a at random. If $g^ih_a^j = g^kh_a^m$ then $g^{i-k} = h_a^{m-j}$ so $a \equiv {}^{(i-k)}/_{(m-j)} \bmod l$ (if invertible).

Define random walk $f(W_i) = W_{i+1}$, depending only on current position $\implies$ no memory of how we got there.

Expensive but very random

$$f(W_i) = \underbrace{g}_{F(W_i)}\underbrace{h_a}_{G(W_i)} \qquad \text{two exponents}$$

Cheaper and fairly random: Make a small table of random steps $g_ih_a^j$ and select next step from this table $\implies$ just one mult per step.

Schoolbook version for POLLARD'S RHO in $\mathbb{F}_p^*\langle g\rangle$. Take $W_a = h_a^r$, r random, update as

$$W_{i+1} = \begin{cases} W_i \cdot g & , W_i \equiv 0 \mod 3 \\ W_i \cdot h & , W_i \equiv 1 \mod 3 \\ W_i^2 & , W_i \equiv 2 \mod 3 \end{cases}$$

That means that if $W_i = g^jh_a^k$ then

$$W_{i+1} = \begin{cases} g^{i+1}h_a^k & , W_i \equiv 0 \mod 3 \\ g^jh_a^{k+1} & , W_i \equiv 1 \mod 3 \\ g^{2j}h_a^{2k} & , W_i \equiv 2 \mod 3 \end{cases}$$

Avoid using memory by using FLOYD'S CYCLIC FINDING METHOD Detecht whether your walk has entered a cycle by comparing $W_i \overset{?}{=} W_{2i}; W_{i+1} \overset{?}{=} W_{2(i+1)}$ etc. Do this as one fast walk taking 2 Steps at once and one slow walk. At every step compare $\Longrightarrow$ twice the work, but only 2 group elements to store. Needs more computation and we won't find the first point that's visited twice - but only a small constant worse. Better than schoolbook: make table with random powers, e. g. 2048 steps precomputed, choose these as powers of g only: h_a only for entrance points.

Parallel Pollard Rho

About factor n speed up for n computers, some storage, some communication.

Mark some group elements as "distinguished points", e. g. elements with top 30 bits equal to 0.

Whenever a walk reaches such a point, it reports the point and the powers of g and h_a to a server, the server sorts and stores these; eventually finds a collision between walks $\Longrightarrow$ DLP.

In practice: Communicate and store as little information as possible, so stop walk at distinguished point and start a new walk at some h_a^r. Report only (DP,r). Do not compute the power of g, this happens only on the server and only once a collision is found. If DP-property is 30 bits equal 0, then each walk takes about 2^{30} steps.

Total number of DPs reported to the server is roughly $l^{1/2}/2^{30}$ (or divide by 2^n if n bits are 0 for DPs).

$\rightarrow$ works in every group not just $\mathbb{F}_q^*$.

The next attack is:

5.4 Index Calculus

Does use the structure of $\mathbb{F}_q^*$, works in some groups but not in all.

In $\mathbb{F}_p^* 10 = 2 \cdot 5 (p > 10)$ is meaning full, so idea is to lift elements from $\mathbb{F}_p$ to $\mathbb{Z}$ and factor them there; or to lift from $\mathbb{F}_{2^n}$ to $\mathbb{F}_2[x]$ using $\mathbb{F}_{2^n} \cong \mathbb{F}_2[x]/f$ with f irreducible of degree n.

R The "lift" operation can be seen as the opposite of the "reduce" (e. g. taking a large integer $\bmod 17$, would result in a number between 0 and 16) operation.

In $\mathbb{F}_{p^n}$ can lift to $\mathbb{F}_p$ or $\mathbb{F}_p[x]$; depending on size of p and n pick one or the other.

Take DLP: g, h find $\log_g h = a$

■ Example 5.3

$h = 10$, so $h = 2 \cdot 5$ and thus $a = \log_g 2 + \log_g 5$ because $h = g^a = 2 \cdot 5 = g^{\log_g 2} \cdot g^{\log_g 5}$ assuming g generates the entire group. Else we can use factorizations involving primes that are powers of g. ■

Why does turning one DLP into two DLPs help?

Idea: find DLs of all small primes by getting lots of relations of the form:

$$g^{b_i} = \prod p_j^{e_ij}$$

where the p_i are in $\mathscr{F} = \{\text{primes smaller than } B\}, B$ chosen appropriately. $\mathscr{F}$ is called the *factorbase*

$$\begin{aligned} b_i &= e_{11}\log_g p_{11} + e_{12}\log_g p_{12} + \cdots + e_{1m}\log_g p_{1m} \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ b_k &= e_{k1}\log_g p_{k1} + e_{k2}\log_g p_{k2} + \cdots + e_{km}\log_g p_{km} \end{aligned}$$

This is a linear system of equations in the unknowns $\log_g p_{ij}$, use all primes in $\mathscr{F}$ for all expressions, i. e. $g^{b_i} = \prod_{j=1}^{m} p_j^{e_{ij}}$ using $e_{ij} = 0$ if a prime does not appear. We have m unknowns $\log_g p_j$, so we need $\geq m$ equations.

■ **Example 5.4**
$p = 107, \mathbb{F}_{107}^{*} = \langle 2 \rangle, h = 57$, find $\log_2 57$. Choosing $\mathscr{F} = \{2,3,5,7\}$ know already $\log_2 2 = 1$

$$\begin{aligned} 2^7 &= 21 = 3 \cdot 7 \\ 2^{11} &= 2 \cdot 3 \cdot 5 \\ 2^{77} &= 3^2 \cdot 7 \end{aligned}$$

$$\begin{array}{c} \underline{\log 3}\ \underline{\log 5}\ \underline{\log 7} \\ \left(\begin{array}{ccc|c} 1 & 0 & 1 & 7 \\ 1 & 1 & 0 & 11 \\ 2 & 0 & 1 & 77 \end{array}\right) \end{array} \Longrightarrow \begin{array}{c} \underline{\log 3}\ \underline{\log 5}\ \underline{\log 7} \\ \left(\begin{array}{ccc|c} 1 & 0 & 1 & 7 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -1 & 63 \end{array}\right) \end{array} \Longrightarrow \begin{array}{c} \underline{\log 3}\ \underline{\log 5}\ \underline{\log 7} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 70 \\ 0 & 1 & 0 & 47 \\ 0 & 0 & 1 & 43 \end{array}\right) \end{array}$$

In the last step we used: entries in this matrix are exponents of g, so we compute $\bmod \operatorname{ord}(g) = 106$.

Now we know

$$3 = 2^{70}; \qquad 5 = 2^{47}; \qquad 7 = 2^{43}$$

Note: We haven't used h here, so this computation can be used to quickly attack any DLP in $\mathbb{F}_{107}^{*}$, at the cost of finding one relation involving h with factors over $\mathscr{F}$. Here:

$$\begin{aligned} h = 57 &= 3 \cdot 19 \ \lightning \\ h \cdot g = 57 \cdot 2 &= 7 \ \in \mathscr{F} \\ \Longrightarrow \ \log_g h = \log_g 7 - 1 &= 43 - 1 = 42 \end{aligned}$$

■

A number is *smooth* if it factors into small $\cong \in \mathbb{F}$ primes. Work on index calculation to choose g^{b_i} more likely to be smooth.

Definition 5.5 — L-Notation. In size N set:

$$L_N(\alpha, c) = \exp(c(\log N)^{\alpha} (\log\log N)^{1-\alpha})$$

■ **Example 5.6**

- $\alpha = 1, L_N(1,c) = \exp(c(\log N)^{\alpha}) = N^c$

- $\alpha = 0$, $L_N(0,c) = \exp(c \log\log N) = \log N^c$

α between 0 and 1 interpolates between polynomial and exponential time. ■

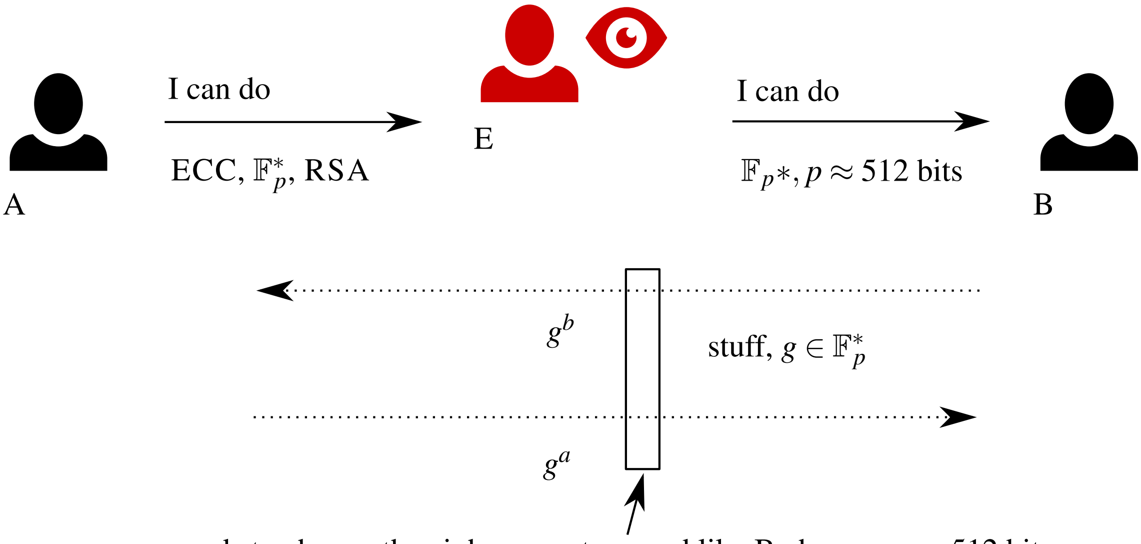
Note: These are relative to size of N, i. e. relative to $\log N$.

Algorithm shown so far runs in time $L_q({}^1\!/_2, c)$, c depending on q. Current research $L_q({}^1\!/_3, c)$ in general and $L_q({}^1\!/_4, c)$ and even less for small p_i, i. e. $p \in \{2,3\}$ and $q = p^n$.

5.4.1 Logjam

Can downgrade TLS to using DH in $\mathbb{F}_p^*$ with $\log_2 p \approx 512$.

There are very few different primes in use, so one can just precompute the relations for each of those primes; the attack per target is so fast that one can mount a man-in-the-middle attack.



I can do
ECC, $\mathbb{F}_p^*$, RSA
A
E
I can do
$\mathbb{F}_p*, p \approx 512$ bits
B
g^b
stuff, $g \in \mathbb{F}_p^*$
g^a

needs to change the cipher spec to sound like B chooses $p \approx 512$ bits

Figure 5.2: Man-in-the-Middle Attack

The change of the cipher spec is secured by a cryptographic checksum In order to fake this, E needs to have the DL of g^b. WEAKDH.ORG has done this for 512; 768 is within reach. Maybe big agencies can do 1024 bits. Interesting about larger sizes that are still used without downgrade.

Note: $L_q({}^1\!/_3, c)$ does not depend on group size l, so balance sizes in $l | q-1$ so that $\sqrt{l} \approx L_q({}^1\!/_3, c)$.

KEYLENGTH.ORG:

$\log_2 l$	$\log_2 q$	security level
160	1024	80 bits, weak
192	2048	96 bits, weak
256	3072	128 bits, OK

Interesting for signatures where computation is done mod l as well. DSA digital signature algorithm (NIST/NSA 1992) uses g as generator of small subgroup; arranges signature equation so that $R = g^k$ is lifted to $\mathbb{Z}$ and $r \equiv R \bmod l$, so (r,s) has length $2\log_2 l$.

Doesn't help much in DH, so many suggestions use $\mathbb{F}_p$, where $p = 2p' + 1$, p' is prime, so use $l = p'$ and s. t. 2 has order p'.

Common groups, e. g. Oakley primes are fixed in protocols - no need for this from mathematical stand point but safer for implementation.

Logjam-Paper

Check out part on DSA parameters: Send (p, g, q) with $q = \operatorname{ord}_p(g)$, usually q is prime. Computer understands (p, q, g), so use q as generator for DH; ignore g. This q has same order dividing $p - 1$, likely to have small factors → Pohlig-Hellman to get secret mod some small factors, remember that a is chosen $< q$, so these small factors might determine a using CRT.

To find $\operatorname{ord}_p(g)$ factor $p - 1$ (later), for each prime factor p_i of $(p - 1)$ check whether $q^{(p-1)/p_i} = 1$, then $q^{(p-1)/p_i^2} \overset{?}{=} 1$ etc. to get the order. We don't even need the complete factorization, as long as the known factors are small enough to compute DLs & their product is larger than q.

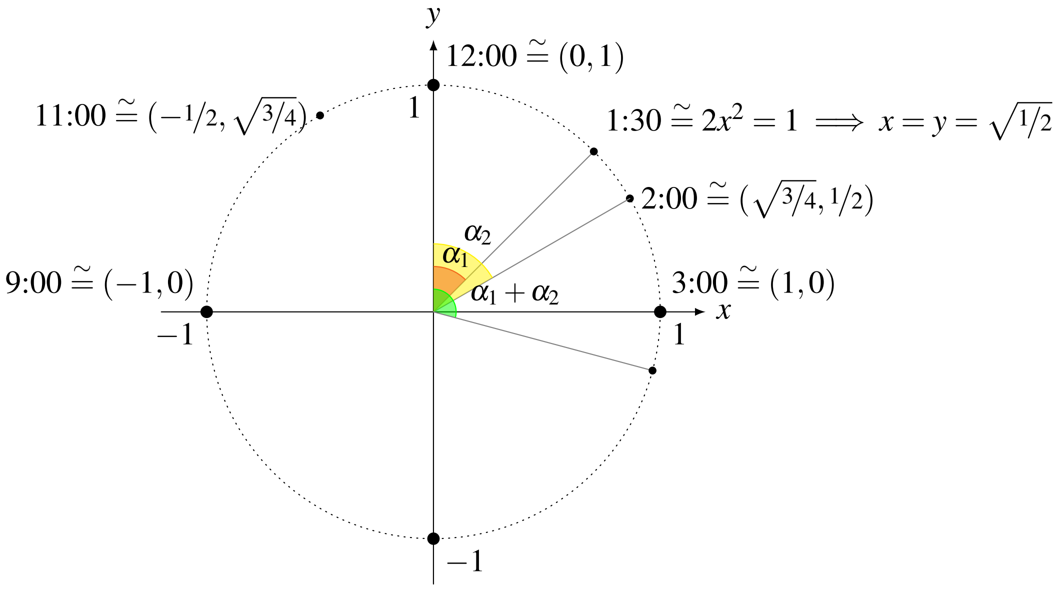
R Check out Nadia Heninger's slides from ECC 2015 (Bordeaux); add keyword elliptic to search.

6. Hash Functions

7. Elliptic Curves

Warm Up

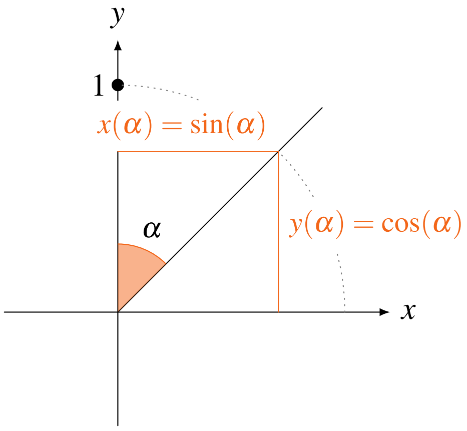
The clock: $x^2 + y^2 = 1$ (not elliptic)



We also have “strange” (= non clock) points, such as $(3/5, 4/5)$.

Define addition of clock points: Add points by adding angles; this matches addition of areas, addition of arch length and addition of times (e. g. $1:30 + 2:00 = 3:30$).

7.1 Trigonometric Formulas



$$x(\alpha_1+\alpha_2) = \sin(\alpha_1+\alpha_2) = \sin\alpha_1\cos\alpha_2 + \sin\alpha_2\cos\alpha_1 = x(\alpha_1)y(\alpha_2)+x(\alpha_2)y(\alpha_1)$$
$$y(\alpha_1)+\alpha_2) = \cos(\alpha_1+\alpha_2) = \cos\alpha_1\cos\alpha_2 - \sin\alpha_1\sin\alpha_2 = y(\alpha_1)y(\alpha_2)-x(\alpha_1)x(\alpha_2)$$

Addition law:

$$(x_1,y_1)+(x_2,y_2) = (x_1y_1+x_2y_1, y_1y_2-x_1x_2)$$

No more angles, can use any (x,y) on circle.

■ Example 7.1

$$({}^3/_5,{}^4/_5)+({}^3/_5,{}^4/_5) = \left(2\,{}^3/_5\,{}^4/_5,({}^4/_5)^2-({}^3/_5)^2\right) = ({}^{24}/_{25},{}^7/_{25}) = 2\,({}^3/_5,{}^4/_5)$$
$$({}^3/_5,{}^4/_5)+({}^{24}/_5,{}^7/_5) = ({}^{117}/_{125},{}^{-44}/_{125}) = 3\cdot({}^3/_5,{}^4/_5)$$
$$4\cdot({}^3/_5,{}^4/_5) = ({}^{336}/_{625},{}^{-527}/_{625})\ldots$$

■

These all lie on the curve (= circle), lots of points.
Does this addition match the clock?

$$({}^3/_5,{}^4/_5)+\underbrace{(0,1)}_{\text{12 o'clock, no change to first part?}} = ({}^3/_5\cdot 1+0\cdot{}^4/_5,{}^4/_5\cdot 1-{}^3/_5\cdot 0) = ({}^3/_5,{}^4/_5)$$

In general

$$(x,y)+(0,1) = (x\cdot 1+0\cdot y, y\cdot 1-x\cdot 0) = (x,y) \implies (0,1) \text{ is the neutral element}$$

Show that $(x_1,y_1)+(x_2,y_2)$ is on circle.

Associativity holds because $\alpha_1+(\alpha_2+\alpha_3) = (\alpha_1+\alpha_2)+\alpha_3$ and the link $x=\sin\alpha$, $y=\cos\alpha$.

$-(x,y) = (-x,y)$ (matching $-\alpha$, or just test $(x,y)+(-x,y) = \left(xy-xy.y^2-(-x^2)\right) = (0,\underbrace{x^2+y^2}_{=1\text{ because we're on the circle}}) = (0,1)$

$\implies$ group; also commutative: $\alpha_1+\alpha_2 = \alpha_2+\alpha_1$

7.2 Using Finite Fields

Can use the circle addition law for $(x,y) \in k^2$ for any field k with $x^2+y^2=1$ (no divisions in the formulas, could even use a ring here). For cryptography don't use $k=\mathbb{Q}$, else $a\cdot(x,y)$ gives away easy information on a, e. g. $a\cdot({}^3\!/_5,{}^4\!/_5)=({}^{\cdots}\!/_{5^a},{}^{\cdots}\!/_{5^a})$, instead use finite field $\mathbb{F}_q$; this "wraps around" so we don't get size information. "Bad" points for crypto:

- $(0,1)$ (neutral element, so $a\cdot(0,1)=(0,1)$)
- $(0,-1)$ point of order 2: $2a\cdot(0,-1)=(0,1),(2a+1)(0,-1)=(0,-1)$
- $(\pm1,0)$ have order 4
- (x,x), i. e. 1 : 30 and the other sign combinations have order 8
- other points of small finite order
- Modulo p there are no points of infinite order - there are at most p^2 candidate points and of these only $p-1$ or $p+1$ are on the circle. (Plug in x, check whether $1-x^2$ is a square, if so we've found two points: $(x,y),(x,-y)$).

Example 7.2

$p=17$

$$x=0: \qquad 1-0^2=1=(\pm1)^2 \implies (0,\pm1)$$
$$x=1: \qquad 1-1^2=0=0^2 \implies (1,0)$$

symmetry gives also $(-1,0)$

$$x=\pm2: \qquad 1-4=-3=14 \bmod 17=\square\ ??$$

z	0	±1	±2	±3	±4	±5	±6	±7	±8
z^2	0	1	4	9	16	8	2	15	13

$\implies$ "half" of all numbers appear, clear becaus $\mathbb{F}_q^*=\langle g\rangle$, so g^{2a} are squares, g^{2a+1} are non squares, also 0 is a square $\implies$ half means ${}^{p+1}\!/_2$; in $\mathbb{F}_{2^n}$ every element is a square. 14 is not in the table $\implies$ no point with $x=\pm2$

$$x=\pm3: \qquad 1-9=9=(\pm3)^2 \implies (\pm3,\pm3)$$
$$x=\pm4: \qquad 1-16=2=(\pm6)^2 \implies (\pm4,\pm6)$$

$\implies$ 4 more points using symmetries $(\pm6,\pm4)$

$$x=\pm5: \qquad 1-25=10\neq\square \text{ no points}$$
$$x=\pm7: \qquad 1-49=1-15=3\neq\square \text{ no points}$$
$$x=\pm8: \qquad \text{by symmetry, only candidates are } (\pm8,\pm8):$$
$$\text{test } 8^2+8^2=-4-4=-8\neq1 \implies \text{no points}$$

Set of all points: $\{(0,\pm1),(\pm1,0),(\pm3,\pm3),(\pm4,\pm6),(\pm6,\pm4)\} \implies 16=17-1$ points. Try $p=19$ yourself.

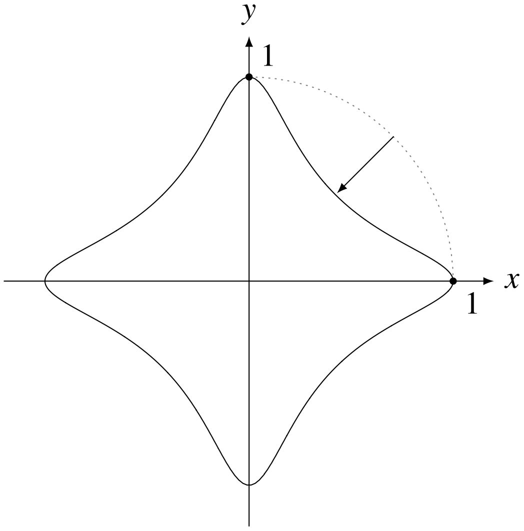
R For the case $x = \pm 8$, we used symmetry. If 8 would pair up with another point, say $(\pm 8, \pm 3)$ then by symmetry the point $(\pm 3, \pm 8)$ would also work. Since this was not the case in any of the numbers before 8, this means that 8 can only "pair" with itself.

■

For crypto choose p large enough and so that a large prime order subgroup appears. Sadly enough index calculus attacks apply to the circle; this is just a complicated description of a subgroup of $(\mathbb{F}_p^2)^*$ (or $\mathbb{F}_p \times \mathbb{F}_p$?).

7.3 Edwards Curves

Tweak circle equation to $x^2 + y^2 = 1 + dx^2y^2$ (avoid $d = 0$ and $d = 1$). This keeps all symmetries, so $(x,y) \implies (\pm x, \pm y)$ and $(\pm y, \pm x)$.



This looks like a 4-legged starfish.
We now want do define the addition $(x_1,y_1)+(x_2,y_2) = (x_3,y_3)$ with

$$x_3 = \frac{(x_1y_2 + x_2y_1)}{(1 + dx_1x_2y_1y_2)} \qquad y_3 = \frac{(y_1y_2 - x_1x_2)}{(1 - dx_1x_2y_1y_2)}$$

if x or $y = 0$ then same as on circle; $\implies$ $(0,1)$ is still the neutral element.
We now can check the inverse element $-(x,y) = (-x,y)$ because:

$$x_3 = \frac{(xy - xy)}{(1 - dx^2y^2)} = 0 \checkmark \qquad y_3 = \frac{(y^2 + x^2)}{(1 + dx^2y^2)}$$

on EDWARDS CURVE $x^2 + y^2 = 1 + dx^2y^2$, the numerator and denominator are equal, so $y_3 = 1$ (giving us $(0,1)$ as a result, which is in fact the neutral element).

Use computer to verify that (x_3,y_3) is on curve and that the addition is associative.

Attention: We have divisions here!!

Over $\mathbb{R}$ we can argue with sizes to see that $0 \le |dx_1x_2y_1y_2| < 1 \implies$ Ok. In $\mathbb{F}_q$ we need a different proof, shows that the denominators are $\neq 0$ if d is not a square in $\mathbb{F}_q \implies$ always pick a non square.

EDWARDS CURVES for crypto: Pick $\mathbb{F}_q$ with q odd, pick $d \in \mathbb{F}_q \backslash \{0,1\}$ which is not a square. The points on $E_d : x^2 + y^2 = 1 + dx^2y^2$ form a group under the addition law above.

7.3.1 Twisted Edwards Curves

Generalization:

$$ax^2 + y^2 = 1 + dx^2y^2 \qquad\qquad , a, d \in \mathbb{F}_q^*, a \neq d;$$

complete addition (= no exceptions) if $a = \square; d \neq \square$. Here

$$(x_1, y_1) + (x_2, y_2) = \left(\frac{x_1y_2 + x_2y_1}{1 + dx_1x_2y_1y_2}, \frac{y_1y_2 - ax_1x_2}{1 - dx_1x_2y_1y_2} \right)$$

We want a to be small, so no cost for multiplication. Save costs in general, i. e. compute both numerators: $x_1y_1 + x_2y_1$, and $y_1y_2 - ax_1x_2$ in 3 mults. instead of 4 by computing: $A = y_1y_2$, $B = x_1x_2$ and then:

$$(x_1 + y_1) \cdot (x_2 + y_2) = \underbrace{x_1 \cdot x_2}_{B} + \underbrace{x_1x_2 + y_1x_2}_{\text{what we want}} + \underbrace{y_1y_2}_{A}$$

so $x_1y_2 + x_2y_1 = (x_1 + y_1) \cdot (x_2 + y_2) - A - B$.

Denominator needs 1 Mults extra for $A \cdot B$, then one by d (want this small) and 2 divisions these are really expensive!

Bring this down to 1 division using MONTGOMERY'S TRICK. Compute $1/C$ and $1/D$ by computing: $E = C \cdot D$, the $F = 1/E\,(= 1/C\cdot D)$ and $1/C = F \cdot D$, $1/D = F \cdot D = F \cdot C$, using 2 M(ultiplication) , 1 I(nversion).

Inversions cost much more than Ms, even using XGCD, but if we need to have constant-time implementation, we need to use LITTLE FERMAT'S THEOREM.

$$1/F = F^{q-2}$$

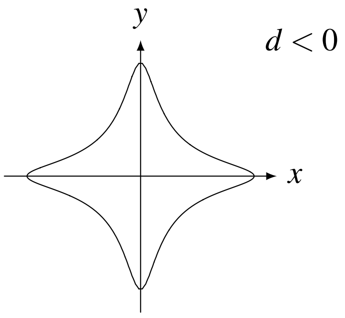
(this holds because $F^{q-1} = 1$, and we divide by F on both sides.)

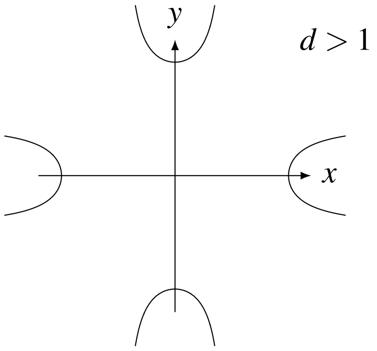
$\Longrightarrow$ use "projective" coordinates, i. e. represent $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$ as $(X : Y : Z)$ with $Z \neq 0$, and get new formulas using

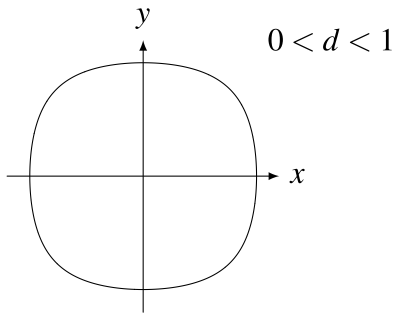
$$(x_1 + y_1) \cdot (x_2 + y_2) = \left(\frac{X_1}{Z_1} + \frac{Y_1}{Z_1} \right) \cdot \left(\frac{X_2}{Z_2} + \frac{Y_2}{Z_2} \right) = \frac{X_1 + Y_1}{Z_1} \cdot \frac{X_2 + Y_2}{Z_2} = \frac{(X_1 + Y_1)(X_2 + Y_2)}{\underbrace{Z_1Z_2}_{\text{new } Z \text{ coordinate}}}$$

trace through the Z coords, simplify etc. $\Longrightarrow$ explicit formulas.

So far we have seen the EDWARDS FORM:





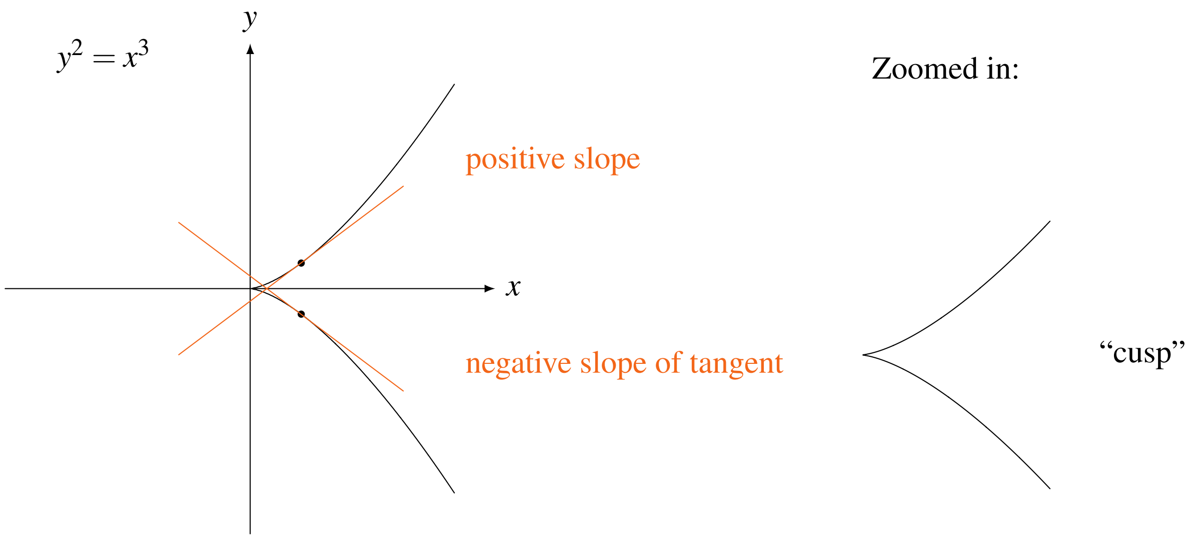


7.4 Weierstrass Curve

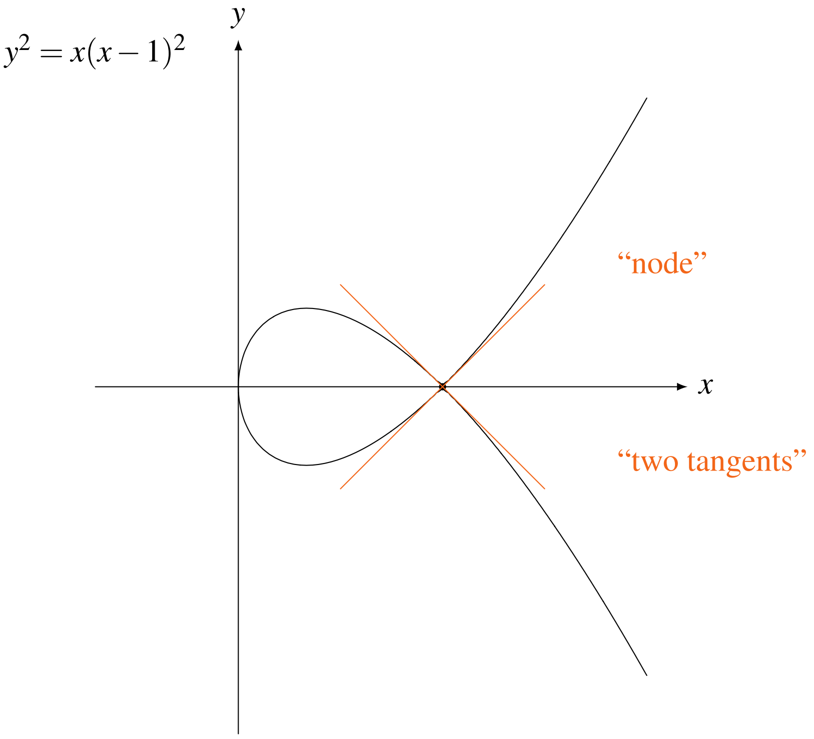
$$E:\ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

where $a_i \in k$, E is an elliptic curve if it is nonsingular.

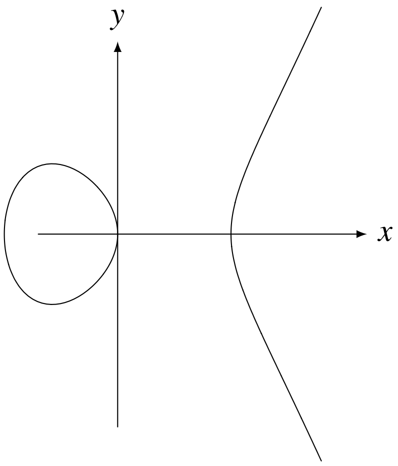
Definition 7.3 — Singular.
A curve is *singular* if any of its points over the field of definition k or any extension field of k is singular.
A point is *singular* if the tangent to the curve in that point is not uniquely determined.



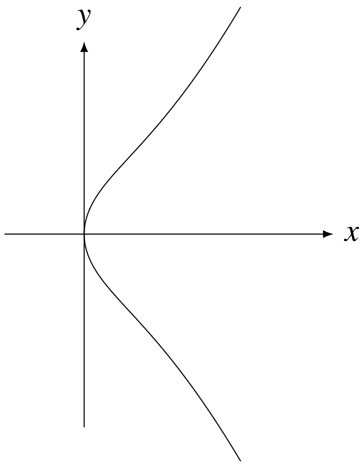
Other examples:



$y^2 = (x+1)x(x-1)$



$y^2 = x(x^2+1)$



> R The last two examples will go to ∞ for $x \to \infty$. This point is in fact on every vertical line and on E.

Computational test for singularity in $(x,y) \in E(k')$: P is singular if the partial derivatives of the curves equation

$$\begin{aligned} E_x: &\qquad a_1 y = 3x^2 + 2a_2 x + a_4 \\ E_y: &\qquad 2y + a_1 x + a_3 = 0 \end{aligned}$$

are both satisfied by P. E. g.:

$$y^2 = x^3: \qquad 0 = 3x^2, 2y = 0$$

both hold in curve point $(0,0) \implies$ this curve is singular.

■ Example 7.4
$y^2 = x^3 - x$ we get the derivatives:

$$0 = 3x^2 - 1, \qquad 2y = 0 \implies y = 0$$

which will give us $0 = x^3 - x \implies x \in \{-1,0,1\}$(from $y = 0$ inserted into the curves equation). So now we test for $(-1,0)$, $(0,0)$ and $(1,0)$ whether they satisfy the derivative with respect to x:

$$3(\pm 1)^2 - 1 = 2 \neq 0 \text{ unless the characteristic of } k \text{ is } 2.$$

$$3 \cdot (0)^2 - 1 = -1 \neq 0, \text{ same}$$

So this curve is nonsingular if $\mathrm{char}(k) \neq 2$. ■

7.4.1 Simplify Curve Equation

Substitute $y - \frac{a_1}{2}x - \frac{a_3}{2}$ for y (if $\text{char}(k) \neq 2$)

$$\left(y - \frac{a_1}{2}x - \frac{a_3}{2}\right)^2 + a_1 x \cdot \left(y - \frac{a_1}{2}x - \frac{a_3}{2}\right) + a_3\left(y - \frac{a_1}{2}x - \frac{a_3}{2}\right)$$

$$= y^2 \underbrace{-2\frac{a_1}{2}xy}_{}\underbrace{-2\frac{a_3}{2}y}_{}+2\frac{a_1}{2}x\frac{a_3}{2} + \frac{a_1^2}{4}x^2 + \frac{a_3^3}{2} + \underbrace{a_1xy}_{} - \frac{a_1^2}{2}x^2 - \frac{a_1a_3}{2}x + \underbrace{a_3y}_{} - \frac{a_1a_3}{2}x - \frac{a_3^2}{2}$$

$$= y^2 + \text{ Terms in } x \text{ of degree } \leq 2 = x^3 + a_2x^2 +_4 x + a_6$$

$$\implies y^2 = x^3 + b_2x^2 + b_4x + b_6 \implies y^2 = f(x).$$

If $\text{char}(k) \neq 3$ subst. $x - \frac{b_2}{3}$ for x to get $y^2 = x^3 + ax + b$. Nonsingular if $4a^3 + 27b^2 \neq 0$. So we have LONG WEIERSTRASS EQUATION with the constants:

$$a_1 \ldots a_6$$

and the SHORT WEIERSTRASS EQUATION in the form (if $\text{char}(k) \neq 2$:

$$y^2 = f(x)$$

and in the case of $\text{char}(k) \neq 2, 3$:

$$y^2 = x^3 + ax + b$$

if $\text{char}(k) = 2$ than either:

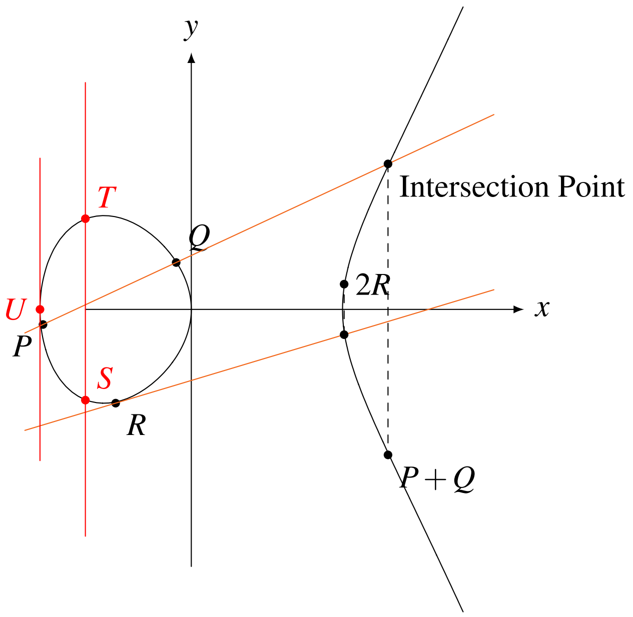
$$y^2 + xy = x^3 + a_2x^2 + a_6$$

or:

$$y^2 + a_3y = x^3 + a_4x + a_6$$

in both cases you can restrict a_2, a_3, a_4 to smaller sets.

7.4.2 Addition Law



Addition law on $(x,y) \in E \cup \{\infty\}$:

Draw line through P and Q, find third intersection point with E, result is the other point with the same x coordinate.
To double a point take the tangent.
If I add $T+S$, they intersect at ∞ and we define the result as ∞, so $T+S=\infty$.
Same goes for $2U$, where we get $2U = \infty$.
Since we want a group (and that means we can add any point in the group), we can also add ∞. If we use a point V' and calculate $V' + \infty$ we get V' again! Since ∞ lies on a vertical line through V', that means that the intersection point V has the same x coordinate as V' and the result is again V'. This is equally true for $U + \infty = U$.

- If one of the inputs is ∞, output the other point ($\infty + \infty = \infty$, $V + \infty = V$, $\infty + U = U$)
- Else (bot inputs are of the form $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$)
  - If $x_P = x_Q$ and $y_P = -y_Q$, output is ∞
  - Else
    - if $P = Q$, double
    - else add using geometric formulas.

Do NOT implement with IF\ELSE!

In formulas, line $y = \lambda x + \mu$

$$\lambda = \frac{3x_P^2 + a}{2y_P} \qquad \text{for DBL; on } y^2 = x^3 + ax + b \qquad \text{(calculate the derivative)}$$

$$\lambda = \frac{y_P - y_Q}{x_P - x_Q} \qquad \text{for ADD, NEVER use for DBL!}$$

Intersection point: $y^2 = x^3 + ax + b = (\lambda x + \mu)^2 = \lambda^2 x^2 + 2\lambda\mu x + \mu^2$. We know P, Q as intersection points and we want to find the third point.

All three points are roots of:

$$\begin{aligned}
0 &= x^3 - \underline{\lambda^2 x^2} + (a - 2\lambda\mu)x + b - \mu^2 \\
&= (x - x_P)(x - x_Q)(x - \underbrace{x_3}_{\text{want this}}) \\
&= x^3 - \underline{(x_P + x_Q + x_3)}x^2 + \text{ stuff}
\end{aligned}$$

$$\implies \lambda^2 = x_P + x_Q + x_3 \iff x_3 = \lambda^2 - x_P - x_Q$$

To find y_3 note that (x_3, y_3) is on $\lambda x + \mu = y$, so $y_3 = \lambda x_3 + \mu$ and $\lambda x_P + \mu = y_P \implies \mu = y_P - \lambda x_P$.

$$\begin{aligned}
x_{P+Q} &= \lambda^2 - x_P - x_Q \\
y_{P+Q} &= -\lambda x_{P+Q} - (y_P - \lambda x_P) \\
&= \lambda(x_P - x_{P+Q}) - y_P
\end{aligned}$$

7.5 Montgomery Curves

$$By^2 = x^3 + Ax^2 + x$$

Very close to WEIERSTRASS CURVES, but extra B (often $B = 1$).

Famous example CURVE 25519 used for crypto, has $A = 486662$, $B = 1$ and is defined over $\mathbb{F}_p$ with $p = 2^{255} - 19$ (prime). Addition works same (up to B):

$$\begin{aligned}
\lambda_{DBL} &= \frac{3x_P^2 + 2Ax_P + 1}{2By_P} \\
x_{P+Q} &= B\lambda^2 - A - x_P - x_Q \\
y_{P+Q} &= \lambda(x_P - x_{P+Q}) - y_P
\end{aligned}$$

MONTGOMERY CURVES are a bit more special than general WEIERSTRASS CURVES, e. g. $(0,0)$ is on the curve and has order 2, over a finite field the order is divisible by 4; same for TWISTED EDWARDS CURVES.

Actually these describe the same curves, for $au^2+v^2 = 1+du^2v^2$ and $A = {}^{2(a+d)}/_{(a-d)}, B = {}^{4}/_{(a-d)}$:

$$\varphi : \mathrm{Ed} \to \mathrm{Mont} : x = \frac{1+v}{1-v}, y = \frac{1+v}{u(1-v)} = \frac{x}{u}$$

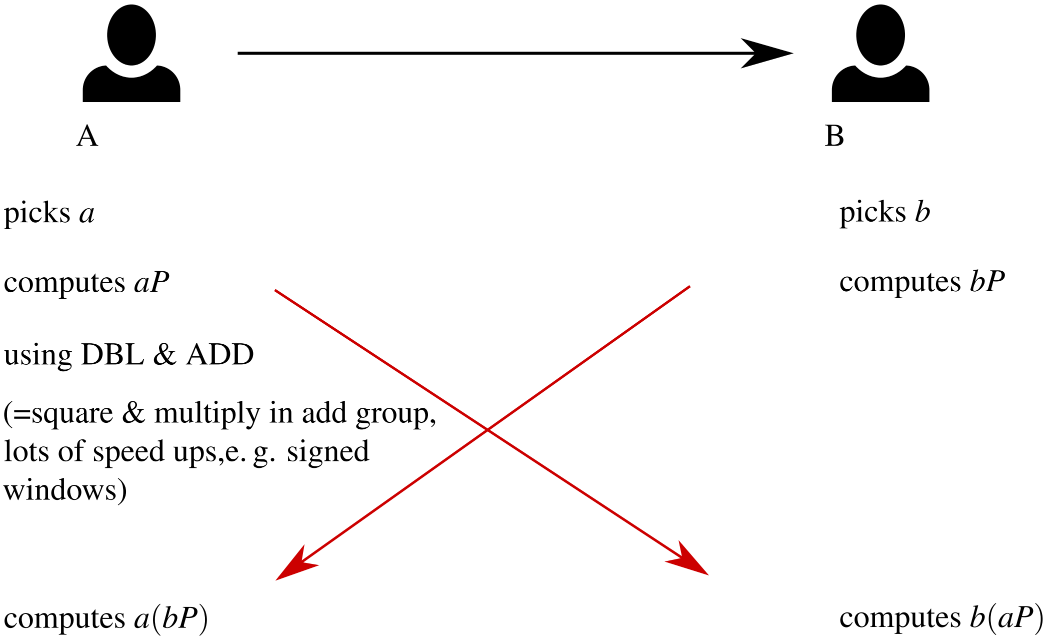
$$\varphi^{-1} : \mathrm{Mont} \to \mathrm{Ed} : u = \frac{x}{y}, v = \frac{x-1}{x+1}$$

are defined almost everywhere and $\varphi(P+Q) = \varphi(P)+\varphi(Q)$, $\varphi^{-1}(\varphi(P)) = P$ etc. MONTGOMERY and EDWARDS CURVES are *birationally equivalent*.

7.6 ECDH

= DH on ELLIPTIC CURVE.

Everybody knows $P \in E(\mathbb{F}_p)$, $\mathrm{ord}(P) = l, l$ prime.



picks a

picks b

computes aP

computes bP

using DBL & ADD

(=square & multiply in add group, lots of speed ups,e. g. signed windows)

computes $a(bP)$

computes $b(aP)$

Figure 7.1: ECDH Key Exchange

Both share $(ab)P$; use hash function on this point (or a coordinate) to derive a key for symmetric crypto (block or stream cipher; also need MAC).

R In exams there are sometimes questions about DH in $(\mathbb{F}_p,+)$ - the additive group. Don't use this in practice; this is totally broken. $h_a = a \cdot g$ (instead of g^a), divide by g (modulo p) to get a; done). On ECs we do not have a multiplication structure, just addition, so no division by D.

Attacks on ECC: generic attacks on the DLP in a group (POHLIG-HELLMAN, BABY-STEP-GIANT-STEP, POLLARD RHO), but no generalization of index calculus for curves over $\mathbb{F}_p$.

Avoid implementation errors:

Use EDWARDS CURVES with $a = \square$, $d \neq \square \implies$ no exceptions.

Use MONTGOMERY CURVES with MONTGOMERY LADDER: P_0, P_1, $P_1 - P_0 = P_i$ update both points for every bit in $a = \sum_{t=0} a_i 2^i$

■ **Example 7.5**

$a = 13 = 8 + 4 + 1 = (1101)_2$.

$P_0 = \infty$	$P_i = P$	initialize
$P_0 = P$	$P_i = 2P$	bit is set, add into first component, double second
$P_0 = 3P$	$P_i = 4P$	bit is set, add into first component, double second
$P_0 = 6P$	$P_i = 7P$	bit is zero; double first
$P_0 = 13P$	$P_i = 14P$	

■

Nice fact on MONTGOMERY CURVES we can share some arithmetic between the DBL & ADD steps; use that we add points at known difference; and we can do all of this using only the x coordinate (using proj. coords this means X and Z).

We use $a_{24} = {}^{(A-2)}/_{4}$ (that's why CURVE 25519's A is $\equiv 2 \bmod 4$; this value is as small as possible to have $\#E(\mathbb{F}_p) = 8 \cdot l$, & $\tilde{E} = 4 \cdot l'$, where $\tilde{E}$ is the quadratic twist of E, that is the curve with the same A and "the other B" (there are only **2** classes for B).

We now use P_2 and P_3 as points in ladder. We initialize:

$$x_1 = x \text{ (of input)} \qquad x_2 = 0 \qquad z_2 = 1$$

$$x_3 = x_1 \qquad z_3 = 1$$

For $t = 254$ down to 0:

$(P_2, P_3) = cswap(a_t, P_2, P_3)$ if $a_t = 1$, swap; else not

$A = x_2 + z_2$; $AA = A^2$; $B = x_2 - z_2$; $BB = B^2$; $E = AA - BB$; $C = x_3 + z_3$; $D = x_3 - z_3$, $DA = D \cdot A$; $CB = C \cdot B$

$$x_3 = (DA + CB)^2 \qquad z_3 = x_1 \cdot (DA - CB)^2 \qquad x_2 = AA \cdot BB \qquad z_2 = E \cdot (AA + a_{24} \cdot E)$$

$(P_2, P_3) = cswap(a_t, P_2, P_3)$. $\implies$ this needs 4S(quarings), 5M(ultiplications), 1 mult. by a_{24} per bit. Once the loop finishes, output ${}^{x_2}/_{z_2} = x_2 \cdot z_x^{p-2}$.

What is *cswap*?

dummy $= a_t \cdot (x_2 - x_3)$ the multiplication is by 0 or 1

$x_2 = x_2 -$ dummy

$x_3 = x_3 +$ dummy

same for z-coordinate. If $a_t = 0$, we get dummy $= 0$, so $x_2 = x_2$, $x_3 = x_3$, else dummy $= x_2 - x_3$, so $x_2 = x_2 - (x_2 - x_3) = x_3$, $x_3 = x_3 + (x_2 - x_3) = x_2$.

In implementation, merge the swap on a_t after the loop with that on a_{t-1} before next as

$$cswap(a_t + a_{t-1}, P_2, P_3)$$

Alternative: dummy $=$ mask (a_t) AND $(x_2$ XOR $x_3)$

$$x_2 = x_2 \text{ XOR dummy}$$
$$x_3 = x_3 \text{ XOR dummy}$$

where mask is $00\ldots0, 11\ldots1$ of the same length as p.

Instead of ECDSA use EDDSA

7.7 Factorizing RSA

Factorizing RSA numbers: $n = p \cdot q$. If we are lucky and know

$$a^2 \equiv b^2 \bmod n$$

for some $a \not\equiv \pm b \bmod n$. Then

$$a^2 - b^2 = (a+b)(a-b) \equiv 0 \bmod n$$

and $gcd(a-b,n)$ is a factor of n, i. e. p or q.

7.7.1 Dixon’s Method

It is not the fastest but easy to understand.

Build ourselves some a and b: Pick a_i, compute $a_i^2 \bmod n$, consider this an integer in $[0, n-1]$ and factor it $\prod p_j^{e_{ij}}$. Similar to index calculus we have a factor base $\mathscr{F}$ consisting of primes. Keep those a_i that factor completely.

■ Example 7.6

$$a_1^2 = 2^2 \cdot 3 \cdot 7$$
$$a_2^2 = 2 \cdot 5 \cdot 7$$
$$a_3^2 = 2 \cdot 3 \cdot 5$$

multiply and get $(a_1 \cdot a_2 \cdot a_3)^2 = (2^2 \cdot 3 \cdot 5 \cdot 7)^2$ This means one factorization per relation but these are “easy” factorizations, we only want relations that factor over $\mathscr{F}$. ■

For easy examples we do all of this by trial division; in real there is still some portion of trial division, e. g. primes up to 2^{13}; but $\mathscr{F}$ contains larger primes; so we’ll do some “easy” factorizations.

Once we have enough relations over $\mathscr{F}$, solve the matrix, Note: we only care about odd and even exponents, so solve this over $\mathbb{F}_2$.

■ **Example 7.7**

$$\begin{array}{cccc} \underline{\log 2} & \underline{\log 3} & \underline{\log 5} & \underline{\log 7} \\ \end{array}$$
$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

■

Here: solving shows that we want product of all 3 relations; usually a selection of relations. Left side is a square, as a product of squares, right side is build to be a square. Note: will need more than $|\mathscr{F}|$ many relations, might hit $a \equiv \pm b \bmod n$. If so, find more relations.

Improvements

choose $a_i = \lceil\sqrt{n}\rceil + s$ (small) to get $a_i^2 = n + 2\lceil\sqrt{n}\rceil s + s^2 + error(\approx \sqrt{n})$. $\bmod n$ this gives $\approx 2\lceil\sqrt{n}\rceil \cdot s + s^2 + \lceil\sqrt{n}\rceil = \mathscr{O}(\sqrt{n})$

$\implies$ easier to factor, more likely to factor over $\mathscr{F}$.

More improvements and you end up at the number field sieve, allow one or two larger primes per relation & combine those.

7.7.2 Pollard's Rho for Factoring

Start with random x_0, pick random c and iterate $x_{i+1} = x_i^2 + c \bmod m$ (with m, easy to factor. If we hit $x_{i+1} = x_{j+1}$, i. e. $x_i^2 + c \equiv x_j^2 + c \bmod m \implies x_i^2 \equiv x_j^2 \bmod m$. Maybe get a factor from $gcd(x_i - x_j, m)$, e. g. $m \equiv a_i^2 \bmod n$.

Actually, we only need $x_i^2 \equiv x_j^2 \bmod p$, so we can think of this as doing a random walk on $\mathbb{F}_p$, find collision after $\mathscr{O}(\sqrt{p})$ steps.

Avoid storage using FLOYD'S, so compute $gcd(x_{2i} - x_i, m)$ at every step, so each step costs 3 squarings and a gcd. Reduce the number of gcds by batching, e. g.

$$gcd\left((x_{2i} - x_i)(x_{2i+2} - x_{i+1}) \dots (x_{2i+2k} - x_{i+k}), m\right)$$

7.7.3 Pollard's $p-1$ Method

Know $b^{p-1} \equiv 1 \bmod p$, so if $p|m$ we have $p|gcd(a^{p-1} - 1, m)$. I don't have p so I cannot compute b^{p-1} - but I can pick some s, compute $gcd(b^s - 1, m)$. This finds p, if $\mathrm{ord}_p(b)|s$ (order of b modulo p). So, hope that $p-1$ has many small divisors and take $s = lcm(1, 2, 3, 4, \dots, B)$ for some bound B, or some other way of choosing s.

Pick b, compute $b^s - 1 \bmod m$, then compute gcd; repeat with different b or larger s.

R Stage 2: more big primes.

Generalization: ECM = ELLIPTIC CURVE METHOD of factorization.

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