1

(a).

Tangent vector:

$$\tau(t) = (\frac{dx(t)}{dt}, \frac{dy(t)}{dt}) = (a, -gt + b)$$

Since $\tau(t) \times n(t) = 0$
It's easy to find out that normal vector is:

$$n(t) = (gt - b, a)$$

(b).
Set

$$-\frac{1}{2}gt^2 + bt + h = 0$$

We get two solutions

$$t = \frac{b - \sqrt{b^2 + 4gh}}{g}$$

And

$$t = \frac{b + \sqrt{b^2 + 4gh}}{g}$$

Since $t_i > 0$, we know that second solution is the time of impact

$$t_i = \frac{b + \sqrt{b^2 + 4gh}}{g}$$

Location is

$$(x(t_i), y(t_i)) = (\frac{ab + a\sqrt{b^2 + 4gh}}{g}, 0)$$

Velocity is

$$(\frac{dx(t_i)}{dt}, \frac{dy(t_i)}{dt}) = (a, -\sqrt{b^2 + 4gh})$$

2

(a) A translation and a shear in x
Consider a translation T and a shear S

$$TS = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & x & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

$$ST = \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & x & a+bx \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

From the above computation we can see that

$$TS \neq ST$$

Hence a translation and a shear in x do not commute

(b) Two different rotations
Consider two rotations R_θ and R_ϕ

$$R_\theta R_\phi = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\phi) & -sin(\phi) & 0 \\ sin(\phi) & cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} cos(\theta)cos(\phi) - sin(\theta)sin(\phi) & -cos(\theta)sin(\phi) - sin(\theta)cos(\phi) & 0 \\ sin(\theta)cos(\phi) + cos(\theta)sin(\phi) & -sin(\theta)sin(\phi) + cos(\theta)cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_\phi R_\theta = \begin{bmatrix} cos(\phi) & -sin(\phi) & 0 \\ sin(\phi) & cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} cos(\theta)cos(\phi) - sin(\theta)sin(\phi) & -cos(\theta)sin(\phi) - sin(\theta)cos(\phi) & 0 \\ sin(\theta)cos(\phi) + cos(\theta)sin(\phi) & -sin(\theta)sin(\phi) + cos(\theta)cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From the above computation we know that

$$R_\theta R_\phi = R_\phi R_\theta$$

Hence two different rotations commute

(c) A rotation and a uniform scaling
Consider a rotation R and a uniform scaling S

$$RS = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} scos(\theta) & -ssin(\theta) & 0 \\ ssin(\theta) & scos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$SR = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} scos(\theta) & -ssin(\theta) & 0 \\ ssin(\theta) & scos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From the above computation we know that

$$RS = SR$$

Hence a rotation and a uniform scaling commute

(d) A rotation and a non-uniform scaling
Consider a rotation R and a non-uniform scaling S

$$RS = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_a & 0 & 0 \\ 0 & s_b & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} s_a cos(\theta) & -s_b sin(\theta) & 0 \\ s_a sin(\theta) & s_b cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$SR = \begin{bmatrix} s_a & 0 & 0 \\ 0 & s_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} s_a cos(\theta) & -s_a sin(\theta) & 0 \\ s_b sin(\theta) & s_b cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From the above computation we can see that

$$RS \neq SR$$

Hence a rotation and a non-uniform scaling do not commute

3

(a).
Let $\mathbf{v} = \mathbf{p}_{i+1} - \mathbf{p}_i$
Then normal vector $\mathbf{n}_i = (-v_y, v_x) = (y_i - y_{i+1}, x_{i+1} - x_i)$

(b).
Let $\mathbf{t} = \mathbf{q} - \mathbf{p}_i$
If $\mathbf{t} \cdot \mathbf{n} > 0$, then $\mathbf{q}$ on the same side
If $\mathbf{t} \cdot \mathbf{n} = 0$, then $\mathbf{q}$ on line $\mathbf{l}_i$
If $\mathbf{t} \cdot \mathbf{n} < 0$, then $\mathbf{q}$ on the opposite side
(c).

Algorithm 1 Determine if $\mathbf{q}$ in shaded region

```
inside = True
# Make sure point is inside outer polygon
for i = 1, 2, ..., n - 1 do
    v = p_{i+1} − p_i
    n_1 = (−v_y, v_x)
    if n_1 · (q − p_i) < 0 then
        inside = False
    end if
end for
# Make sure point is outside inner polygon
for j = 1, 2, ..., m - 1 do
    w = r_{i+1} − r_i
    n_2 = (w_y, −w_x)
    if n_2 · (q − r_i) < 0 then
        inside = False
    end if
end for
return inside
```

4

(a). Define homography as

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix}$$

From mapping

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

Obtained c = -4, f = 1
From mapping

$$\begin{bmatrix} a & b & -4 \\ d & e & 1 \\ g & h & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -0.5 \\ 1 \end{bmatrix}$$

Obtained

$$\frac{a-4}{g+1} = -1.5$$

$$\frac{d+1}{g+1} = -0.5$$

From mapping

$$\begin{bmatrix} a & b & -4 \\ d & e & 1 \\ g & h & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \\ 1 \end{bmatrix}$$

Obtained

$$\frac{b-4}{h+1} = 0.5$$

$$\frac{e+1}{h+1} = -0.5$$

From mapping

$$\begin{bmatrix} a & b & -4 \\ d & e & 1 \\ g & h & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Obtained

$$\frac{a+b-4}{g+h+1} = 0$$

$$\frac{d+e+1}{g+h+1}=1$$

From the above six formulas we get:

$$a+1.5g=2.5$$

$$d+0.5g=-1.5$$

$$b-0.5h=4.5$$

$$e+0.5h=-1.5$$

$$a+b=4$$

$$d+e-g-h=0$$

Solve the following coordinate system

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1.5 & 0 \\ 0 & 0 & 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 1 & 0 & 0.5 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ d \\ e \\ g \\ h \end{bmatrix} = \begin{bmatrix} 2.5 \\ -1.5 \\ 4.5 \\ -1.5 \\ 4 \\ 0 \end{bmatrix}$$

Got

$$\begin{bmatrix} a \\ b \\ d \\ e \\ g \\ h \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

The homography is

$$\begin{bmatrix} 1 & 3 & -4 \\ -2 & 0 & 1 \\ 1 & -3 & 1 \end{bmatrix}$$

(b).

$$\begin{bmatrix} 1 & 3 & -4 \\ -2 & 0 & 1 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -0.75 \\ 0.25 \\ 1 \end{bmatrix}$$

In Euclidean coordinates, the point gets mapped to (-0.75. 0.25)
(c). No, because the homography is not in the general affine transformation matrix form

$$\begin{bmatrix} A & & t \\ 0 & 0 & 1 \end{bmatrix}$$

5

Affine transformation is closed under composition.

$$M = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 23 \\ \frac{3}{5} & \frac{4}{5} & -7 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 23 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} sin(53°) & -cos(53°) & 0 \\ cos(53°) & sin(53°) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 23 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}$$

Now prove a rotation and a translation commute:
Consider a rotation R and a translation T:

$$RT = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} cos(\theta) & -sin(\theta) & a \\ sin(\theta) & cos(\theta) & b \\ 0 & 0 & 1 \end{bmatrix}$$

$$TR = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} cos(\theta) & -sin(\theta) & a \\ sin(\theta) & cos(\theta) & b \\ 0 & 0 & 1 \end{bmatrix}$$

From the above computation we can see that

$$RT = TR$$

Thus a rotation and a translation commute
Hence

$$M = \begin{bmatrix} 1 & 0 & 23 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sin(53^\circ) & -cos(53^\circ) & 0 \\ cos(53^\circ) & sin(53^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}$$

Which is decomposed into a translation followed by a rotation followed by a translation.