

Chapter 11

Diffusion Problems and Fourier Series

In previous chapters we have studied ordinary differential equations that govern the way in which a finite number of quantities evolve with respect to an independent variable t , usually thought of as time. We began by studying single equations governing the evolution of a single quantity and then studied coupled systems of n equations governing the simultaneous evolution of n quantities.

In this chapter and the next we undertake an introductory study of differential equations that govern the evolution in time of *functions* defined on some domain in space. Such equations involve derivatives with respect to one or more *spatial* variables (as well as time derivatives); thus they are *partial differential equations*.

Partial differential equations comprise a subject that is far too vast and far too complex for us to do more than scratch its surface here. Our purpose is only to introduce the most basic ideas. We will therefore restrict ourselves in this chapter to *one-dimensional* problems (i.e., problems involving only a single spatial variable x). In Chapter 12 we will look at a few problems involving two spatial variables.

11.1 The Basic Diffusion Problem

One of the most common processes in nature is the random movement of matter from regions of high concentration to regions of lower concentration. This “averaging-out” process is known as *diffusion*. Examples include a drop of ink diffusing in a glass of water and water vapor diffusing in air. Another quantity that diffuses is heat energy. For instance, when heat is applied to one end of a metal rod, it diffuses toward the other end of the rod. Diffusion is also a common part of population models that account for the tendency of species to disperse in order to avoid crowding.

Suppose that U is a quantity that diffuses in a one-dimensional, uniform medium—such as heat energy in a thin, laterally insulated iron rod or a drop of ink in a thin tube of water. Let $u(t, x)$ be the concentration of that quantity at time t and position x . Our fundamental postulate is that the spatial flow rate of U (per unit area) is proportional to its concentration gradient u_x *; that is,

$$\text{flow rate} = -k u_x. \quad (1)$$

* Subscripts signify partial derivatives: $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $u_t = \frac{\partial u}{\partial t}$, and so on.

This is known as *Fick's law*. In the context of heat conduction, it is *Fourier's law*. The coefficient k , generically called the *diffusivity*, is positive and in certain applications may depend upon x , t , or even the concentration u . However, for the sake of simplicity we will assume (for now) that k is constant. Note that the positivity of k implies that U flows in the direction of decreasing concentration.

Let us now derive a differential equation satisfied by u . Suppose that the rod has constant cross-sectional area α . Let $a < x < b$, and consider a small, interior section of the medium corresponding to the interval $[x, x + h]$. (See Figure 1.) Within this section at time t , the amount of U is

$$\alpha \int_x^{x+h} u(t, y) dy,$$

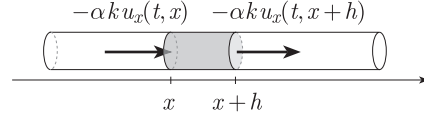


Figure 1

and the rate of “accumulation” of U is the time derivative of the amount:

$$\begin{aligned} \text{rate of} \\ \text{accumulation of } U &= \alpha \frac{d}{dt} \int_x^{x+h} u(t, y) dy = \alpha \int_x^{x+h} u_t(t, y) dy. \end{aligned}$$

Furthermore, because of (1) we also have

$$\begin{aligned} \text{rate of} \\ \text{accumulation of } U &= (\text{flow rate at } x) - (\text{flow rate at } x+h) \\ &= -\alpha k u_x(t, x) + \alpha k u_x(t, x+h). \end{aligned}$$

Subtracting the second of these two rate expressions from the first, and dividing by h , we arrive at

$$\frac{\alpha}{h} \int_x^{x+h} u_t(t, y) dy - \alpha k \frac{u_x(t, x+h) - u_x(t, x)}{h} = 0.$$

Dividing by α and taking the limit as $h \rightarrow 0$, we obtain the one-dimensional **diffusion equation** with constant diffusivity:

$$u_t - k u_{xx} = 0. \quad (2)$$

Note that this is a *partial differential equation*, since it involves partial derivatives with respect to each of the two independent variables. It is a *homogeneous, linear* partial differential equation, because the operator \mathcal{H} associated with the left side of the equation, that is, defined by

$$\mathcal{H}u = u_t - k u_{xx},$$

is a linear operator and any linear combination of solutions is also a solution. (See Problem 10.)

- **Example 1** Each of the expressions

$$e^{-k\omega^2 t} \cos \omega x \quad \text{and} \quad e^{-k\omega^2 t} \sin \omega x,$$

where ω is any constant, is easily shown to satisfy (2) for x in any interval. Thus a family of solutions of (2) is given by

$$u(t, x) = e^{-k\omega^2 t} (A \cos \omega x + B \sin \omega x),$$

where A, B , and ω are constants. (See Problem 1.) It is interesting to note that any such solution, with $\omega \neq 0$, decays exponentially (in time) toward the zero solution. Also, the larger ω is (i.e., the higher the spatial frequency), the greater will be the rate of the exponential decay. With $k = 1$, a particular solution is

$$u(t, x) = e^{-t} \cos x,$$

which is plotted in Figures 2ab. Figure 2a shows the graph of u as a function of x (for $-2\pi \leq x \leq 2\pi$) at each of the discrete times $t = 0, 1, 2$, and 3. Figure 2b shows the graph of $u(x, t)$ for $-2\pi \leq x \leq 2\pi$ and $0 \leq t \leq 3$. Notice the averaging out, or “smoothing,” of the solution as t increases.

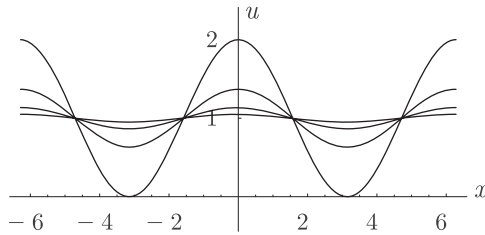


Figure 2a

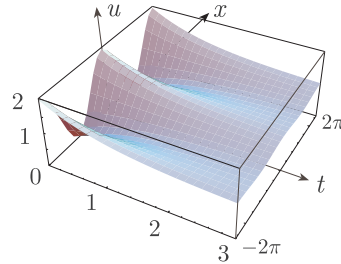


Figure 2b

- **Example 2** Routine calculation shows that

$$u(t, x) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

is a solution of (2) for all $x \in \mathbb{R}$ and $t > 0$. (See Problem 2.) Figure 2a shows the graph of u (with $k = 1$) as a function of x , for $-1 \leq x \leq 1$, at each of the discrete times $t = 0.005, 0.055, \dots, 0.255$. Figure 3b shows the graph of $u(x, t)$ for $-1 \leq x \leq 1$ and $0 \leq t \leq 0.25$. This solution is particularly interesting because, as $t \rightarrow 0^+$, $u(x, t)$ approaches the Dirac distribution (see Section 7.5), indicating that $u(x, t)$ describes the diffusion of a unit mass concentrated at the point $x = 0$ at time $t = 0$.

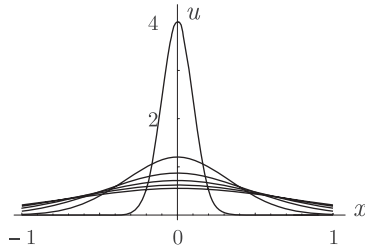


Figure 3a

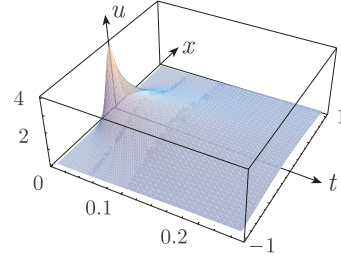


Figure 3b

■ x -Dependence and Sources

It is often natural for the rate at which diffusion occurs to depend upon location. For instance, if a metal rod is made of a heterogeneous mixture of copper and tin, its ability to conduct heat will vary along its length. In such a case, the diffusivity coefficient k in postulate (1) becomes a positive function of x (and possibly t), and as a result the correct form of the diffusion equation is

$$u_t - (k u_x)_x = 0. \quad (3)$$

If the diffusing quantity U is generated or otherwise enters the medium at a rate per unit length given by $f(t, x)$, then (3) becomes

$$u_t - (k u_x)_x = f(t, x). \quad (4)$$

The term $f(t, x)$ in (4) generically represents a *source* of the quantity U . When $f = 0$, the equation is *homogeneous*; otherwise it is nonhomogeneous. When k and f are independent of t , the equation is *autonomous*. Each of these notions is thoroughly consistent with what we have seen previously in the context of ordinary differential equations.

■ Initial Data and Boundary Conditions

Suppose that we are interested in solutions of the diffusion equation (4) for $t \geq 0$ and on some bounded (one-dimensional) spatial domain $a \leq x \leq b$. As one would expect, the diffusion equation itself will have infinitely many solutions. Our experience with ordinary differential equations suggests that it is appropriate to specify *initial data*, which in the present case will be of the form

$$u(0, x) = \phi(x), \quad \text{for } a \leq x \leq b, \quad (5)$$

where ϕ is a given function on $[a, b]$, providing an initial value for the solution at each x in $[a, b]$.

For each fixed t the diffusion equation is a second-order differential equation in x . Thus two independent conditions relative to the solution's x -dependence are appropriate. Conditions that are physically meaningful in a wide range of

applications are *boundary conditions* of the general form

$$\begin{aligned}\alpha u(t, a) - (1 - \alpha)u_x(t, a) &= \gamma_a, \\ \beta u(t, b) + (1 - \beta)u_x(t, b) &= \gamma_b,\end{aligned}\quad \text{for } t > 0, \quad (6)$$

where α , β , γ_a , and γ_b are functions of t (perhaps constants) with $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. When $\gamma_a = \gamma_b = 0$, the boundary conditions are said to be *homogeneous*. In that case, any linear combination of functions that satisfy (6) will satisfy (6) as well.

Note that when $\alpha = \beta = 1$, (6) specifies endpoint values of the solution:

$$\begin{aligned}u(t, a) &= \gamma_a, \\ u(t, b) &= \gamma_b,\end{aligned}\quad \text{for } t > 0,$$

as would be appropriate for a heated rod whose ends were held at fixed temperatures, for example. Such boundary conditions are known as **Dirichlet boundary conditions**.

When $\alpha = \beta = 0$, (6) specifies the *outward** flow rate (cf. (1)), or outward *flux*, of the solution at each endpoint:

$$\begin{aligned}-u_x(t, a) &= \gamma_a, \\ u_x(t, b) &= \gamma_b,\end{aligned}\quad \text{for } t > 0.$$

Such conditions are *flux boundary conditions* and are often called **Neumann boundary conditions**. The more specific case with $\gamma_a = \gamma_b = 0$ is appropriate for a heated rod with (perfectly) insulated ends or any other model in which the diffusing quantity is prohibited from flowing either into or out of the medium through its boundary.

For α and β with values in $(0, 1)$, the equations in (6) may be thought of as specifying the outward flux at each endpoint in terms of “exterior” concentrations *outside* of (a, b) , which equate to the endpoint values $u(t, a)$ and $u(t, b)$ by continuity. An exterior concentration less than γ_a or γ_b results in a positive outward flux, and an exterior concentration greater than γ_a or γ_b results in a negative outward (or positive inward) flux. Boundary conditions of this type are called **Robin boundary conditions**.

It is not necessary that the boundary conditions at the two endpoints be of the same type. For instance, if $\alpha = \gamma_a = 1/2$ and $\beta = \gamma_b = 0$, then (6) becomes

$$\begin{aligned}-u_x(t, a) &= 1 - u(t, a), \\ u_x(t, b) &= 0,\end{aligned}\quad \text{for } t > 0.$$

* This is the reason for the minus sign before $u_x(t, a)$. Without it, $u_x(t, a)$ represents the rate of flow *into* the interval.

In summary, our *initial-boundary value problem* for the diffusion equation in one spatial dimension is

$$\left. \begin{aligned} u_t - (k u_x)_x &= f && \text{for } a < x < b, \ t > 0, \\ u(0, x) &= \phi(x) && \text{for } a \leq x \leq b, \\ \alpha u(t, a) - (1 - \alpha)u_x(t, a) &= \gamma_a && \text{for } t > 0, \\ \beta u(t, b) + (1 - \beta)u_x(t, b) &= \gamma_b && \text{for } t > 0. \end{aligned} \right\} \quad (7)$$

In order to keep our present discussion relatively simple and to avoid some technical details that are more appropriately left for a course in partial differential equations, we make the following assumptions throughout the remainder of this section as well as the next section.

- The diffusivity k is a positive, continuously differentiable function of x for $a \leq x \leq b$ and is independent of t .
- The source term f is continuous for $t \geq 0$ and $a \leq x \leq b$.
- The function ϕ that specifies the initial data is continuous on $[a, b]$.
- The coefficients α , β , γ_a , and γ_b in the boundary conditions are constants.

We now state (without proof) the following existence and uniqueness theorem. We emphasize that the stated assumptions are merely *sufficient* for the conclusion of the theorem; they are not *necessary*.

THEOREM 1 *Under the preceding assumptions, (7) has a unique solution u , which is continuously differentiable in t and twice continuously differentiable in x for all $t > 0$ and $a \leq x \leq b$.*

The following are simple examples of diffusion problems and their solutions. (Bear in mind that we have yet to discuss any method for *finding* solutions.)

- **Example 3** Consider the problem

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{for } 0 < x < \pi, \ t > 0, \\ u(0, x) &= 1 - \cos x && \text{for } 0 \leq x \leq \pi, \\ u_x(t, 0) &= u_x(t, \pi) = 0 && \text{for } t > 0, \end{aligned}$$

in which homogeneous Neumann boundary conditions require zero outward flux at each endpoint for all $t > 0$. Routine calculation verifies that the solution is

$$u(t, x) = 1 - e^{-t} \cos x.$$

Figure 4a shows the graph of u as a function of x at each of $t = 0, 1, 2$, and 3 , while Figure 4b shows the graph of u for $0 \leq x \leq \pi$ and $0 \leq t \leq 3$.

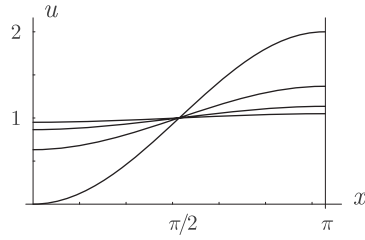


Figure 4a

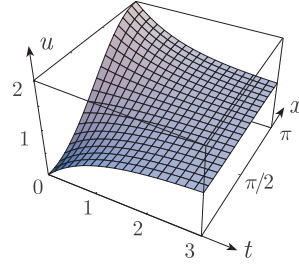


Figure 4b

- **Example 4** Consider the problem

$$\begin{aligned} u_t - u_{xx} &= -\frac{1}{8}\pi^2 \cos \frac{\pi x}{2} \quad \text{for } 0 < x < 2, \ t > 0, \\ u(0, x) &= \frac{1}{2} \left(1 - \cos \frac{\pi x}{2} + \sin \pi x \right) \quad \text{for } 0 \leq x \leq 2, \\ u(t, 0) &= 0, \ u(t, 2) = 1 \quad \text{for } t > 0, \end{aligned}$$

in which Dirichlet boundary conditions require that the solution's endpoint values be fixed for all $t > 0$. Routine calculation shows that the solution is

$$u(t, x) = \frac{1}{2} \left(1 - \cos \frac{\pi x}{2} + e^{-\pi^2 t} \sin \pi x \right),$$

which is illustrated in Figures 5a and b.

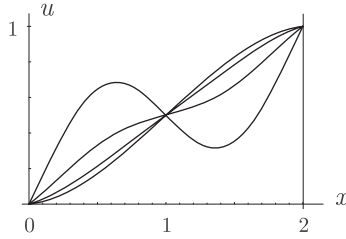


Figure 5a

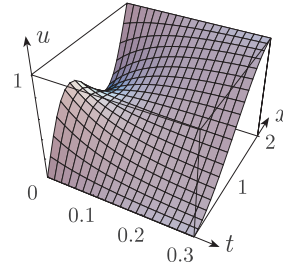


Figure 5b

- **Example 5** Consider the problem

$$\begin{aligned} u_t - \frac{4}{81\pi^2} u_{xx} &= 8 \quad \text{for } 0 < x < 1, \ t > 0, \\ u(0, x) &= \pi^2 (1 - x^2) + 5 \cos \left(\frac{9\pi x}{2} \right) \quad \text{for } 0 \leq x \leq 1, \\ u_x(t, 0) &= 0, \ u(t, 1) = 0 \quad \text{for } t > 0, \end{aligned}$$

in which a homogeneous Neumann condition and a homogeneous Dirichlet condition require, respectively, that the solution's outward flux at $x = 0$ and value

at $x = 1$ be zero for all $t > 0$. The solution,

$$u(t, x) = \pi^2 (1 - x^2) + 5 e^{-t} \cos\left(\frac{9\pi x}{2}\right),$$

is easily verified and is illustrated in Figures 6a and b.

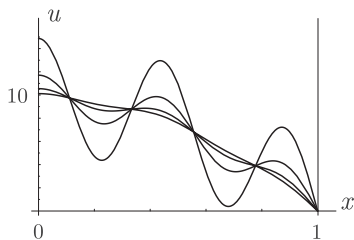


Figure 6a

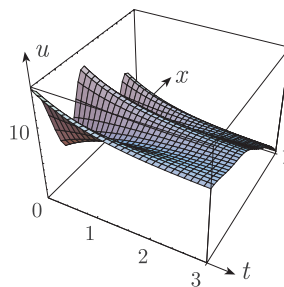


Figure 6b

Problems

- Let k , ω , A , and B be constants. Verify that

$$u(t, x) = e^{-k\omega^2 t} (A \sin \omega x + B \cos \omega x)$$

satisfies $u_t - k u_{xx} = 0$ for $-\infty < x < \infty$ and $-\infty < t < \infty$.

- Let k be a positive constant. Verify that

$$u(t, x) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

satisfies $u_t - k u_{xx} = 0$ for $-\infty < x < \infty$ and $t > 0$.

- Let k be a constant. Verify that, for any integer n , the function

$$u(t, x) = e^{-kn^2\pi^2 t} \cos(n\pi x)$$

satisfies the diffusion equation

$$u_t - k u_{xx} = 0 \quad \text{for } 0 < x < 1, \quad t > 0,$$

and the boundary conditions

$$u_x(t, 0) = u_x(t, 1) = 0 \quad \text{for } t > 0.$$

- Let k and ω be constants, and let u be defined by

$$u(t, x) = e^{-k\omega^2 t} (\omega \cos(\omega x) + \sin(\omega x)).$$

- Verify that u satisfies the diffusion equation

$$u_t - k u_{xx} = 0 \quad \text{for } 0 < x < 1, \quad t > 0.$$

- (b) Verify that, if ω satisfies $\omega \tan \omega = 1$, then u satisfies the boundary conditions

$$u(t, 0) - u_x(t, 0) = u_x(t, 1) = 0 \quad \text{for } t > 0.$$

- (c) Give a graphical argument that the equation $\omega \tan \omega = 1$ has infinitely many positive solutions. (Suggestion: Rewrite the equation as $\omega = \cot \omega$.)

5. (a) Verify that the function $u(t, x) = e^{-6t}(1 - x^2)$ satisfies the diffusion problem

$$\begin{aligned} u_t - ((3 - x^2)u_x)_x &= 0 \quad \text{for } 0 < x < 1, \quad t > 0, \\ u(0, x) &= 1 - x^2 \quad \text{for } 0 \leq x \leq 1, \\ u_x(t, 0) &= 0, \quad u(t, 1) = 0 \quad \text{for } t > 0. \end{aligned}$$

- (b) Show that this solution is unique by verifying the hypotheses of Theorem 1.

6. Let

$$k(x) = \begin{cases} 1 & \text{if } x = 0; \\ \frac{(1 + x^2)^2 \arctan x}{x} & \text{if } 0 < x \leq 1. \end{cases}$$

- (a) Verify that the function

$$u(t, x) = \frac{e^{-2t}}{1 + x^2}$$

satisfies the diffusion problem

$$\begin{aligned} u_t - (k u_x)_x &= 0 \quad \text{for } 0 < x < 1, \quad t > 0, \\ u(0, x) &= \frac{1}{1 + x^2} \quad \text{for } 0 \leq x \leq 1, \\ u_x(t, 0) &= 0, \quad u(t, 1) + u_x(t, 1) = 0 \quad \text{for } t > 0. \end{aligned}$$

- (b) Show that this solution is unique by verifying the hypotheses of Theorem 1.

7. Verify the solution in Example 3.

8. Verify the solution in Example 4.

9. Verify the solution in Example 5.

10. Let k be a differentiable function defined on $[a, b]$, and let \mathcal{H} be the operator defined by

$$\mathcal{H}u = u_t - (k u_x)_x$$

for all functions u that are differentiable in t for $t > 0$ and twice differentiable in x for x in $[a, b]$. Show that \mathcal{H} is a linear operator.

11. Let λ be a constant, and let X be a twice differentiable function of x (i.e., independent of t). Show that $u(t, x) = e^{\lambda t}X(x)$ satisfies the diffusion equation

$$u_t - (k u_x)_x = 0 \quad \text{for } a < x < b, \quad t > 0,$$

if and only if X satisfies

$$(kX')' = \lambda X \quad \text{for } a < x < b.$$

12. Suppose that \tilde{u} and \tilde{v} each satisfy the following diffusion equation and boundary conditions:

$$\begin{aligned} u_t - (ku_x)_x &= f & \text{for } a < x < b, \ t > 0, \\ \alpha u(t, a) - (1 - \alpha)u_x(t, a) &= \gamma_a & \text{for } t > 0, \\ \beta u(t, b) + (1 - \beta)u_x(t, b) &= \gamma_b & \text{for } t > 0. \end{aligned}$$

Show that $\tilde{u} - \tilde{v}$ must satisfy the following *homogeneous* diffusion equation and boundary conditions:

$$\begin{aligned} u_t - (ku_x)_x &= 0 & \text{for } a < x < b, \ t > 0, \\ \alpha u(t, a) - (1 - \alpha)u_x(t, a) &= 0 & \text{for } t > 0, \\ \beta u(t, b) + (1 - \beta)u_x(t, b) &= 0 & \text{for } t > 0. \end{aligned}$$

A *steady-state* (i.e., *time-independent*) solution of (7) is a solution of the boundary-value problem

$$\begin{aligned} -(kz')' &= f & \text{for } a < x < b, \\ \alpha z(a) - (1 - \alpha)z'(a) &= \gamma_a, \\ \beta z(b) + (1 - \beta)z'(b) &= \gamma_b. \end{aligned}$$

Find the solution(s) of each the boundary-value problems in 13 through 16.

13.
$$\begin{cases} -z'' = 6x & \text{for } 0 < x < 2, \\ z(0) - z'(0) = 1, \ z'(2) = 0. \end{cases}$$
14.
$$\begin{cases} -((x+1)z')' = 1 & \text{for } 0 < x < 1, \\ z(0) = 1, \ z(1) = 0. \end{cases}$$
15.
$$\begin{cases} -z'' = \sin x & \text{for } 0 < x < \pi, \\ z'(0) = z'(\pi) = 0. \end{cases}$$
16.
$$\begin{cases} -(e^x z')' = e^{2x} & \text{for } 0 < x < 1, \\ z(0) - z'(0) = z(\ln 2) + z'(\ln 2) = 0. \end{cases}$$

17. Suppose that z is a solution of the “steady-state” diffusion problem

$$\begin{aligned} -(kz')' &= f & \text{for } a < x < b, \\ \alpha z(a) - (1 - \alpha)z'(a) &= \gamma_a, \\ \beta z(b) + (1 - \beta)z'(b) &= \gamma_b, \end{aligned}$$

under the following assumptions:

- i) $k(x) > 0$ for $a < x < b$,
- ii) $f(x) > 0$ for $a < x < b$,
- iii) γ_a and γ_b are each nonnegative, and
- iv) α and β are each in $[0, 1]$.

Argue as follows to show that z is nonnegative on $[a, b]$.

- a) Rewrite the differential equation as $kz'' + k'z' = -f$, and argue that the minimum value of z for $a \leq x \leq b$ cannot be attained in (a, b) and therefore must be attained either at a or at b .
- b) For each of the three cases: $\alpha = 0$, $0 < \alpha < 1$, and $\alpha = 1$, argue that a *negative* minimum cannot be attained at a . (The case where $\alpha = \gamma_a = 0$ is the tricky one.)
- c) For each of the three cases: $\beta = 0$, $0 < \beta < 1$, and $\beta = 1$, argue that a *negative* minimum cannot be attained at b .
- d) Conclude that the minimum value of z for $a \leq x \leq b$ must be nonnegative.

11.2 Solutions by Separation of Variables

We now turn our attention to the following diffusion problem on the spatial interval $[0, \ell]$ and with constant diffusivity and fixed endpoint values:

$$\left. \begin{aligned} u_t - k u_{xx} &= f && \text{for } 0 < x < \ell, \ t > 0, \\ u(0, x) &= \phi(x) && \text{for } 0 \leq x \leq \ell, \\ u(t, 0) &= \gamma_a && \text{for } t > 0, \\ u(t, \ell) &= \gamma_b && \text{for } t > 0. \end{aligned} \right\} \quad (1)$$

We assume that (1) is fully *autonomous*; that is, we assume that f (as well as k , γ_a , and γ_b as usual) does not depend upon t .

An important *boundary-value problem* related to (1) is satisfied by any *time-independent* solution z of (1):

$$\left. \begin{aligned} -k z_{xx} &= f && \text{for } 0 < x < \ell, \\ z(0) &= \gamma_0, \ z(\ell) = \gamma_\ell. \end{aligned} \right\} \quad (2)$$

A solution of (2) is called a **steady-state solution** of (1). This problem is one of class of boundary-value problems that we will study in more detail later in this chapter. It serves our present needs simply to observe that one can solve (2) for z by straightforward integration. (See Problem 13.) So suppose that the solution z of (2) is known, and set

$$\varphi = \phi - z,$$

where ϕ is the initial data function in (1). Because of the linearity of the differential equation, the solution of (1) may be obtained by adding z to the solution of

the following *homogeneous* problem with zero source and zero endpoint values:

$$\left. \begin{aligned} w_t - k w_{xx} &= 0 && \text{for } 0 < x < \ell, \ t > 0, \\ w(0, x) &= \varphi(x) && \text{for } 0 \leq x \leq \ell, \\ w(t, 0) &= w(t, \ell) = 0 && \text{for } t > 0. \end{aligned} \right\} \quad (3)$$

We now focus upon solving (3). A classical technique, dating back to the mid-eighteenth century, is commonly known as *separation of variables*. This technique begins with the supposition that certain simple solutions of the differential equation and boundary conditions can be expressed as the product of a function of t and a function of x —that is, in the form

$$w(t, x) = T(t)X(x).$$

Note that for such a product we have $w_t = T'(t)X(x)$ and $w_{xx} = T(t)X''(x)$. Thus, were such a product to satisfy the differential equation in (3), it would follow that

$$T'(t)X(x) - kT(t)X''(x) = 0,$$

and, consequently,

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}. \quad (4)$$

We now make the key observation that in order for (4) to be true it is necessary that *both sides of the equation be constant*. To see this, think of varying t while holding x fixed. It follows that the left side of (4) is constant; thus the right side is as well. So let us denote this common constant value by λ . Then (4) gives rise to a pair of ordinary differential equations:

$$T'(t) = \lambda k T(t) \quad (5)$$

and

$$X''(x) = \lambda X(x). \quad (6)$$

The upshot of the development so far is that if T and X satisfy (5) and (6), respectively, *for some constant* λ , then their product will satisfy $w_t - k w_{xx} = 0$. The next step is to enforce the boundary conditions in (3) upon $T(t)X(x)$. So we require that

$$T(t)X(0) = T(t)X(\ell) = 0 \quad \text{for all } t > 0,$$

which becomes

$$X(0) = X(\ell) = 0, \quad (7)$$

since we are not interested in the alternative that $T(t) = 0$ for all $t > 0$. Equations (6) and (7) together comprise a boundary-value problem for X . Our immediate goal now is to determine the value(s) of λ for which that problem has solutions other than the obvious trivial (i.e., identically zero) solution.

We begin by eliminating the case $\lambda = 0$. If $\lambda = 0$, then (6) becomes $X''(x) = 0$, which implies that $X(x) = ax + b$ for some constants a and b . From $X(0) = 0$ we conclude that $b = 0$, and then from $X(\ell) = 0$ we conclude that $a = 0$. Thus $X(x) = 0$ for all x .

In the case where $\lambda > 0$, every solution of $X'' = \lambda X$ is of the form

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x},$$

where A and B are constants. The boundary conditions thus result in the system of equations

$$\begin{aligned} A + B &= 0, \\ Ae^{\sqrt{\lambda}\ell} + Be^{-\sqrt{\lambda}\ell} &= 0, \end{aligned}$$

of which $A = B = 0$ is the unique solution. This shows that a positive value of λ produces only the trivial solution of (6) and (7).

So we now consider (6) and (7) with λ negative. For convenience, we set $\lambda = -\sigma^2$, with $\sigma > 0$, and restate (6) and (7) as

$$\begin{aligned} X''(x) &= -\sigma^2 X(x) \\ X(0) &= X(\ell) = 0. \end{aligned} \tag{8}$$

We know that every solution of the differential equation can be written as

$$X(x) = A \cos \sigma x + B \sin \sigma x$$

for some choice of constants A and B . The boundary condition $X(0) = 0$ forces $A = 0$. The other boundary condition, $X(\ell) = 0$, then forces

$$B \sin \sigma \ell = 0;$$

that is, either $B = 0$ or $\sin \sigma \ell = 0$. With $B = 0$ (since A is also 0), we simply obtain the trivial solution $X = 0$. So it is the equation $\sin \sigma \ell = 0$ that reveals the interesting values of σ , which comprise the sequence

$$\sigma_n = \frac{n\pi}{\ell}, \quad n = 1, 2, 3, \dots \tag{9}$$

and allow (8) to possess nontrivial solutions.

We can now state that, for any positive integer n and $\sigma_n = \frac{n\pi}{\ell}$, the function

$$w(t, x) = T(t) \sin \sigma_n x$$

satisfies both the differential equation and the boundary conditions in (3), *provided that* T satisfies (5) with $\lambda = -\sigma_n^2$, that is,

$$T'(t) = -\sigma_n^2 k T(t).$$

The solutions of this equation are all constant multiples of

$$T_n(t) = e^{-\sigma_n^2 k t}.$$

Thus we conclude that each of functions

$$w_n(t, x) = e^{-\sigma_n^2 k t} \sin \sigma_n x, \quad n = 1, 2, 3, \dots, \quad (10)$$

where $\sigma_n = \frac{n\pi}{\ell}$, satisfies the differential equation and the boundary conditions in (3). Moreover, so does any linear combination of them:

$$w(t, x) = \sum_{n=1}^N a_n e^{-\sigma_n^2 k t} \sin \sigma_n x, \quad (11)$$

as is easily verified. Indeed, if the initial data in (3) can be expressed as

$$\varphi(x) = \sum_{n=1}^N a_n \sin \sigma_n x,$$

then the solution of (3) is given precisely by (11), as is also easily verified. Thus we have solved (3) for a large class of initial functions φ —functions that are expressible as linear combinations of trigonometric functions that satisfy the boundary conditions.

• **Example 1** Consider the problem

$$w_t - k w_{xx} = 0 \quad \text{for } 0 < x < 1, \quad t > 0,$$

$$w(0, x) = 2 \sin \pi x + 3 \sin 3\pi x + \sin 5\pi x \quad \text{for } 0 \leq x \leq 1,$$

$$w(t, 0) = w(t, 1) = 0 \quad \text{for } t > 0.$$

Since $\ell = 1$ here, we have $\lambda_n = n\pi$, and so we look for the solution in the form

$$w(t, x) = \sum_{n=1}^N a_n e^{-n^2 \pi^2 k t} \sin n\pi x.$$

This function already satisfies the differential equation and the boundary conditions; so we need only choose the coefficients $a_1, a_2, a_3, \dots, a_N$ so that the initial condition is met. At $t = 0$ we have

$$w(0, x) = \sum_{n=1}^N a_n \sin n\pi x,$$

which corresponds to the desired initial data if we let $N = 5$ and choose

$$a_1 = 2, \quad a_3 = 3, \quad a_5 = 1, \quad \text{and} \quad a_2 = a_4 = 0.$$

Therefore, the solution is

$$w(t, x) = 2e^{-\pi^2 k t} \sin \pi x + 3e^{-9\pi^2 k t} \sin 3\pi x + e^{-25\pi^2 k t} \sin 5\pi x.$$

Figures 1a and b indicate the behavior of the solution in the case where

$$k = \frac{1}{\pi^2},$$

with Figure 1a showing the solution at each of $t = 0, 0.1$, and 0.3 . ◇

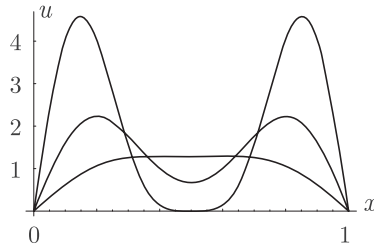


Figure 1a

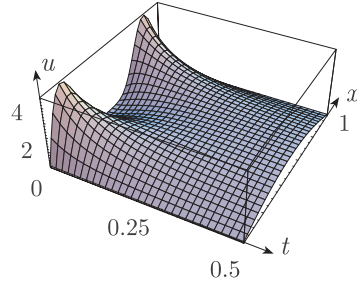


Figure 1b

The crux of our investigation so far is that the differential equation and boundary conditions in (3) are satisfied by any finite sum of the form

$$w(t, x) = \sum_{n=1}^N a_n e^{-\sigma_n^2 k t} \sin \sigma_n x,$$

where $\sigma_n = n\pi/\ell$; thus we are able to solve (3) if the initial data are of the form

$$\varphi(x) = \sum_{n=1}^N a_n \sin \sigma_n x.$$

Let us now consider u given instead by an *infinite series* of the form

$$w(t, x) = \sum_{n=1}^{\infty} a_n e^{-\sigma_n^2 k t} \sin \sigma_n x,$$

while assuming that the series converges for all $t \geq 0$ and $0 \leq x \leq \ell$. Assuming also that term-by-term differentiation is valid, it is easy to see that such a function satisfies (3) with the initial data

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \sin \sigma_n x.$$

The following example provides a simple illustration of this idea.

- **Example 2** Consider the trigonometric series

$$\varphi(x) = \frac{8}{\pi^2} \left(\sin \pi x - \frac{1}{9} \sin 3\pi x + \frac{1}{25} \sin 5\pi x - \frac{1}{49} \sin 7\pi x + \cdots \right),$$

noting first that

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x),$$

where

$$a_n = \begin{cases} \frac{8(-1)^{\frac{n+1}{2}}}{\pi^2 n^2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

At each x , the value of $\varphi(x)$ is, *by definition*, the limit of the sequence of partial sums:

$$\begin{aligned} & \frac{8}{\pi^2} \sin \pi x, \\ & \frac{8}{\pi^2} \left(\sin \pi x - \frac{1}{9} \sin 3\pi x \right), \\ & \frac{8}{\pi^2} \left(\sin \pi x - \frac{1}{9} \sin 3\pi x + \frac{1}{25} \sin 5\pi x \right), \\ & \frac{8}{\pi^2} \left(\sin \pi x - \frac{1}{9} \sin 3\pi x + \frac{1}{25} \sin 5\pi x - \frac{1}{49} \sin 7\pi x \right), \dots, \end{aligned}$$

provided that the limit exists. For instance, at $x = 1/2$, the first five terms in this sequence are

$$0.810569, 0.900633, 0.933056, 0.949598, 0.959605,$$

and the 100th, 200th, ..., 500th terms are

$$0.997974, 0.998987, 0.999325, 0.999493, 0.999595,$$

suggesting that the value of $\varphi(1/2)$ may be 1. (This is indeed true.) Figure 2 shows the graph of each the first, second, third, tenth, and twentieth of the partial sums on the interval $[0, 1]$.

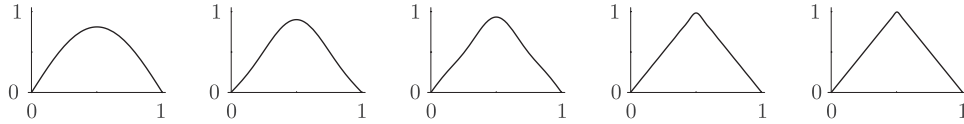


Figure 2

The last of the graphs in Figure 2 is a reasonably good approximation of the graph of the series $\varphi(x)$, which has, in fact, the “closed form”

$$\varphi(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2(1-x) & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

(We will learn in the next section how to find trigonometric series representations such as the one examined here.) The solution of (3) with initial data given by $\varphi(x)$ is given by

$$w(t, x) = \frac{8}{\pi^2} \left(e^{-\pi^2 k t} \sin \pi x - \frac{1}{9} e^{-9\pi^2 k t} \sin 3\pi x + \frac{1}{25} e^{-25\pi^2 k t} \sin 5\pi x + \cdots \right).$$

Figures 3a and b indicate the behavior of the solution in the case where $k = 1$, with Figure 3a showing the solution at each of $t = 0, 0.02, 0.04, \dots, 0.20$. \diamond

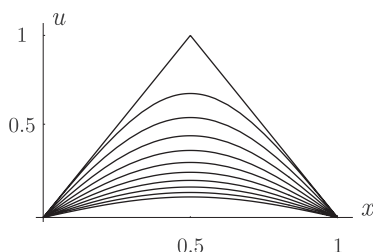


Figure 3a

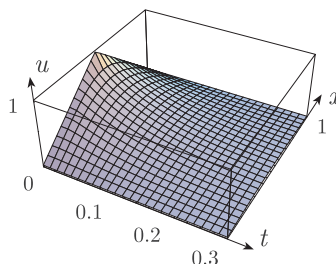


Figure 3b

The essential result of this section is that the solution of (3) is easily obtained (indeed may be written down by inspection) provided that we can express the initial data as a trigonometric series of the form

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{\ell} x.$$

Similar results are true for problems with other types of boundary conditions, as you will be asked to show in the problems that follow. Thus we are faced with these questions:

What functions φ , defined on $[0, \ell]$, can be expressed as convergent trigonometric series of the appropriate form?

Given such a function φ , how can the coefficients of the series be calculated?

The next two sections will be devoted to answering these questions.

Problems

Find the solution of (3) in each of the instances described in Problems 1 through 6.

1. $\ell = 1$, $w(t, 0) = \sin 3\pi x$
2. $\ell = 2$, $w(t, 0) = \sin 3\pi x$
3. $\ell = 1$, $w(t, 0) = 2 \sin \pi x + \sin 2\pi x$
4. $\ell = 2$, $w(t, 0) = 2 \sin \pi x + \sin 2\pi x$
5. $\ell = \pi$, $w(t, 0) = 3 \sin x + \sin 2x - \frac{1}{2} \sin 3x$
6. $\ell = \frac{\pi}{2}$, $w(t, 0) = 5 \sin 2x - \sin 4x$
7. Consider the diffusion problem with zero-flux boundary conditions:

$$\begin{aligned} w_t - k w_{xx} &= 0 & \text{for } 0 < x < \ell, \ t > 0, \\ w(0, x) &= \varphi(x) & \text{for } 0 < x < \ell, \\ w_x(t, 0) &= w_x(t, \ell) = 0 & \text{for } t > 0. \end{aligned}$$

- (a) Use the separation of variables technique to derive the following family of functions that satisfy the differential equation and boundary conditions:

$$w_n(t, x) = e^{-\sigma_n^2 k t} \cos \sigma_n x, \quad \sigma_n = \frac{n\pi}{\ell}, \quad n = 0, 1, 2, \dots$$

- (b) Let $\ell = 1$, and write down the solution $w(t, x)$ satisfying

$$w(0, x) = 2 + 5 \cos \pi x - \cos 3\pi x.$$

- (c) Let $\ell = \pi$, and write down the solution $w(t, x)$ satisfying

$$w(0, x) = 1 + \cos 2x + \cos 3x.$$

- (d) Write down the solution $w(t, x)$ satisfying

$$w(0, x) = \sum_{n=0}^{\infty} a_n \cos \sigma_n x, \quad \text{where } \sigma_n = n\pi/\ell.$$

8. Consider the diffusion problem with a zero-value condition at $x = 0$ and a zero-flux condition at $x = \ell$:

$$\begin{aligned} w_t - k w_{xx} &= 0 & \text{for } 0 < x < \ell, \ t > 0, \\ w(0, x) &= \varphi(x) & \text{for } 0 < x < \ell, \\ w(t, 0) &= 0, \ w_x(t, \ell) = 0 & \text{for } t > 0. \end{aligned}$$

- a) Use the separation of variables technique to derive the following family of functions that satisfy the differential equation and boundary conditions:

$$w_n(t, x) = e^{-\sigma_n^2 k t} \sin \sigma_n x, \quad \sigma_n = \frac{(2n-1)\pi}{2\ell} \quad n = 1, 2, 3, \dots$$

- b) Let $\ell = 1$, and write down the solution $w(t, x)$ satisfying

$$w(0, x) = 5 \sin \frac{\pi x}{2} - \sin \frac{3\pi x}{2}.$$

- c) Let $\ell = \frac{\pi}{2}$, and write down the solution $w(t, x)$ satisfying

$$w(0, x) = \sin x + \frac{1}{2} \sin 3x - \frac{1}{10} \sin 7x.$$

- d) Write down the solution $w(t, x)$ satisfying

$$w(0, x) = \sum_{n=1}^{\infty} a_n \sin \sigma_n x, \quad \text{where } \sigma_n = \frac{(2n-1)\pi}{2\ell}.$$

9. Consider the diffusion problem with a zero-flux condition at $x = 0$ and a value-dependent flux condition at $x = \ell$:

$$\begin{aligned} w_t - k w_{xx} &= 0 & \text{for } 0 < x < \ell, \ t > 0, \\ w(0, x) &= \varphi(x) & \text{for } 0 < x < \ell, \\ w_x(t, 0) &= 0, \ w(t, \ell) + w_x(t, \ell) = 0 & \text{for } t > 0. \end{aligned}$$

- (a) Show that the differential equation and boundary conditions are satisfied by

$$w(t, x) = e^{-\sigma^2 k t} \cos \sigma x,$$

provided that σ is a solution of the equation

$$\sigma = \cot \sigma \ell.$$

- (b) Give a graphical argument that the positive solutions of $\sigma = \cot \sigma \ell$ define an increasing, divergent sequence $\sigma_1, \sigma_2, \sigma_3, \dots$. Conclude that there is an infinite family of functions, given by

$$w_n(t, x) = e^{-\sigma_n^2 k t} \cos \sigma_n x, \quad n = 1, 2, 3, \dots,$$

that satisfy the differential equation and boundary conditions.

- (c) Assuming that $\sigma_1, \sigma_2, \sigma_3, \dots$ are known, write down the solution $w(t, x)$ satisfying

$$w(0, x) = \sum_{n=1}^{\infty} a_n \cos \sigma_n x.$$

10. Diffusion problems involving heat energy (measured as temperature) are often called heat conduction problems. In such problems, the diffusivity coefficient k is the *thermal diffusivity* and is related to material properties through $k = \frac{\kappa}{\rho c}$, where κ is thermal conductivity, ρ is density, and c is specific heat. For copper, the value of k is approximately 1.14.

Suppose that an 11-cm length of copper wire is perfectly insulated along its length, with its ends held at a fixed temperature of 0°C . Suppose also that the initial temperature distribution along the wire is given by

$$w(0, x) = 50 \sin \frac{\pi x}{11} + 20 \sin \pi x, \quad 0 \leq x \leq 11.$$

- (a) Write down the solution $w(t, x)$ of the relevant heat conduction problem.
 (b) Graph the temperature distribution at each of the times $t = 0, 1$, and 10 seconds.

11. Consider the special case of (1):

$$\begin{aligned} u_t - \frac{1}{2} u_{xx} &= 3x \quad \text{for } 0 < x < 1, \ t > 0, \\ u(0, x) &= x \quad \text{for } 0 \leq x \leq 1, \\ u(t, 0) &= 1, \ u(t, 1) = 0 \quad \text{for } t > 0. \end{aligned}$$

- (a) Find the steady-state solution by solving the boundary-value problem

$$\begin{aligned} -\frac{1}{2} z_{xx} &= 3x \quad \text{for } 0 < x < 1, \\ z(0) &= 1, \ z(1) = 0. \end{aligned}$$

- (b) Write the homogeneous diffusion problem (as in (3)) satisfied by $w = u - z$.

12. Consider the special case of (1):

$$\begin{aligned} u_t - u_{xx} &= \sin x \quad \text{for } 0 < x < \pi, \ t > 0, \\ u(0, x) &= 1 + \sin 2x \quad \text{for } 0 \leq x \leq \pi, \\ u(t, 0) &= u(t, \pi) = 1 \quad \text{for } t > 0, \end{aligned}$$

- (a) Find the steady-state solution by solving the boundary-value problem

$$\begin{aligned} -z_{xx} &= \sin x \quad \text{for } 0 < x < \pi, \\ z(0) &= z(\pi) = 1. \end{aligned}$$

- (b) Write *and solve* the homogeneous diffusion problem satisfied by $w = u - z$.

- (c) Write the solution $u = w + z$.

13. Verify that the solution of (2) is given by

$$z(x) = \gamma_0 + \left(\gamma_\ell - \gamma_0 + \frac{1}{k} \int_0^\ell \int_0^\xi f(\eta) d\eta d\xi \right) \frac{x}{\ell} - \frac{1}{k} \int_0^x \int_0^\xi f(\eta) d\eta d\xi.$$

14. Consider the boundary-value problem satisfied by any steady-state solution of the diffusion problem in Problem 8:

$$\begin{aligned} -k z_{xx} &= f \quad \text{for } 0 < x < \ell, \\ z(0) &= z_x(\ell) = 0. \end{aligned}$$

Find the solution z in a form similar to that in Problem 13.

15. Consider the boundary-value problem satisfied by any steady-state solution of the diffusion problem in Problem 7:

$$\begin{aligned} -k z_{xx} &= f \quad \text{for } 0 < x < \ell, \\ z_x(0) &= z_x(\ell) = 0. \end{aligned}$$

- (a) Show that, if $\int_0^\ell f(x) dx \neq 0$, then this boundary-value problem has no solution.
 (b) Show that, if $\int_0^\ell f(x) dx = 0$, then this boundary-value problem has infinitely many solutions, and any two solutions differ by a constant.

16. (a) Suppose that the solution w of (3) is known and c is a constant. Show that v , defined by $v(t, x) = e^{-ct}w(t, x)$, satisfies the differential equation

$$v_t - k v_{xx} + c v = 0 \quad \text{for } 0 < x < \ell, \ t > 0,$$

and the same initial and boundary conditions as w .

- (b) Find the solution of

$$\begin{aligned} v_t - v_{xx} + v &= 0 & \text{for } 0 < x < \pi, \ t > 0, \\ v(0, x) &= 3 \sin x + \sin 3x & \text{for } 0 \leq x \leq \pi, \\ v(t, 0) = v(t, \pi) &= 0 & \text{for } t > 0. \end{aligned}$$

17. Let $\sigma_1, \sigma_2, \dots$ be any sequence of real numbers. Use the comparison test (consult your calculus book) to show that if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then each of the series

$$\sum_{n=1}^{\infty} a_n \sin \sigma_n x \quad \text{and} \quad \sum_{n=1}^{\infty} a_n \cos \sigma_n x$$

converges absolutely for all x .

18. Let $\sigma_n = n\pi/\ell$, $n = 1, 2, 3, \dots$, and define the function

$$g(x) = \sum_{n=1}^{\infty} a_n \sin \sigma_n x.$$

Show that if the series defining g converges for all x , then g is periodic with period 2ℓ ; that is,

$$g(x + 2\ell) = g(x) \quad \text{for all } x.$$

11.3 Fourier Series

We found in the preceding section that certain simple diffusion problems are easily solved, provided that the initial data are expressed as an appropriate series of trigonometric terms. So it is our goal in this section to explore the issue of expressing functions in that form.

First recall that a function f defined on $(-\infty, \infty)$ is **periodic** with period $p > 0$, if $f(x+p) = f(x)$ for all x . For brevity we will refer to a periodic function with period p as *p-periodic*.

Now let φ be a function defined on an interval $[-\ell, \ell]$. There corresponds to φ a unique 2ℓ -periodic function $\bar{\varphi}$ for which

$$\bar{\varphi}(x) = \varphi(x) \quad \text{for all } x \text{ in } (-\ell, \ell].$$

This function $\bar{\varphi}$ is called the **periodic extension** of φ . The idea is illustrated in Figures 1ab. A generic function on $[-\ell, \ell]$ is shown in Figure 1a, and its periodic

extension is shown in Figure 1b. (Note that $\bar{\varphi}$ and φ will not agree at $x = -\ell$ unless $\varphi(-\ell) = \varphi(\ell)$.)

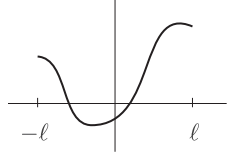


Figure 1a

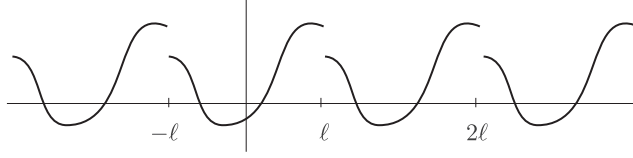


Figure 1b

Let us now suppose that φ is a function defined on $[-\ell, \ell]$ with periodic extension $\bar{\varphi}$. Since each of the functions

$$1, \cos \frac{\pi x}{\ell}, \sin \frac{\pi x}{\ell}, \cos \frac{2\pi x}{\ell}, \sin \frac{2\pi x}{\ell}, \cos \frac{3\pi x}{\ell}, \sin \frac{3\pi x}{\ell}, \dots \quad (1)$$

is periodic with common period $p = 2\ell$, we will investigate the possibility of expanding $\bar{\varphi}(x)$ —and thus $\varphi(x)$ for x in $(-\ell, \ell]$ —in an infinite series of the form*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right). \quad (2)$$

Our first goal is to find formulae for the coefficients a_0, a_1, a_2, \dots and b_1, b_2, b_3, \dots in terms of computable properties of φ . For this purpose we will *assume* that the series in (2) does indeed converge to $\varphi(x)$ for all x in $(-\ell, \ell]$ and that φ has whatever properties are necessary to justify the steps in our calculations.

The key to this endeavor is found in a collection of integral properties possessed by the family of simple cosine and sine functions listed in (1).

THEOREM 1 *Let m and n be positive integers. Then*

$$\begin{aligned} \text{(i)} \quad & \int_{-\ell}^{\ell} \cos \frac{n\pi x}{\ell} dx = \int_{-\ell}^{\ell} \sin \frac{n\pi x}{\ell} dx = 0, \\ \text{(ii)} \quad & \int_{-\ell}^{\ell} \cos \frac{m\pi x}{\ell} \sin \frac{n\pi x}{\ell} dx = 0, \\ \text{(iii)} \quad & \int_{-\ell}^{\ell} \cos \frac{m\pi x}{\ell} \cos \frac{n\pi x}{\ell} dx = \begin{cases} 0 & \text{if } m \neq n, \\ \ell & \text{if } m = n, \end{cases} \\ \text{(iv)} \quad & \int_{-\ell}^{\ell} \sin \frac{m\pi x}{\ell} \sin \frac{n\pi x}{\ell} dx = \begin{cases} 0 & \text{if } m \neq n, \\ \ell & \text{if } m = n. \end{cases} \end{aligned}$$

Each of the formulae asserted in Theorem 1 can be derived by means of elementary integration techniques. This is the subject of Problem 13 at the end

* The reason for the factor of $\frac{1}{2}$ in the constant term is so that later a single formula can be given for all of a_0, a_1, a_2, \dots .

of this section. An alternative derivation of (ii)–(iv) with $m \neq n$ is the subject of Problem 14.

Our derivation of formulae for the coefficients a_0, a_1, a_2, \dots and b_1, b_2, b_3, \dots will require term-by-term integration over $[-\ell, \ell]$ of certain infinite series related to the one in (2). Such operations are not automatically justified; however they are justified for a large class of functions defined by series of the type in (2). So we proceed under the assumptions that

$$\varphi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \quad \text{for all } x \text{ in } (-\ell, \ell], \quad (3)$$

and that the necessary term-by-term integrations over $[-\ell, \ell]$ are valid.

We begin by integrating φ over $[-\ell, \ell]$:

$$\int_{-\ell}^{\ell} \varphi(x) dx = \int_{-\ell}^{\ell} \frac{a_0}{2} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\ell}^{\ell} \cos \frac{n\pi x}{\ell} dx + b_n \int_{-\ell}^{\ell} \sin \frac{n\pi x}{\ell} dx \right).$$

By part (i) of Theorem 1, the only nonzero term on the right side is the one involving a_0 . That term is $\int_{-\ell}^{\ell} \frac{a_0}{2} dx = a_0 \ell$; thus it follows that

$$a_0 = \frac{1}{\ell} \int_{-\ell}^{\ell} \varphi(x) dx.$$

Next we let m be an arbitrary positive integer and multiply each side of (3) by $\cos \frac{m\pi x}{\ell}$ before integrating over $[-\ell, \ell]$:

$$\begin{aligned} \int_{-\ell}^{\ell} \varphi(x) \cos \frac{m\pi x}{\ell} dx &= \int_{-\ell}^{\ell} \frac{a_0}{2} \cos \frac{m\pi x}{\ell} dx \\ &+ \sum_{n=1}^{\infty} \left(a_n \int_{-\ell}^{\ell} \cos \frac{n\pi x}{\ell} \cos \frac{m\pi x}{\ell} dx + b_n \int_{-\ell}^{\ell} \sin \frac{n\pi x}{\ell} \cos \frac{m\pi x}{\ell} dx \right). \end{aligned}$$

By parts (i), (ii), and (iii) of Theorem 1, the only nonzero term on the right side is the one involving a_m . Thus

$$\begin{aligned} \int_{-\ell}^{\ell} \varphi(x) \cos \frac{m\pi x}{\ell} dx &= a_m \int_{-\ell}^{\ell} \cos^2 \frac{m\pi x}{\ell} dx \\ &= \frac{a_m}{2} \int_{-\ell}^{\ell} \left(1 + \cos \frac{2m\pi x}{\ell} \right) dx = a_m \ell, \end{aligned}$$

and so we have

$$a_m = \frac{1}{\ell} \int_{-\ell}^{\ell} \varphi(x) \cos \frac{m\pi x}{\ell} dx.$$

Finally, we let m be an arbitrary positive integer and multiply each side of (3) by $\sin \frac{m\pi x}{\ell}$ before integrating over $[-\ell, \ell]$:

$$\begin{aligned} \int_{-\ell}^{\ell} \varphi(x) \sin \frac{m\pi x}{\ell} dx &= \int_{-\ell}^{\ell} \frac{a_0}{2} \sin \frac{m\pi x}{\ell} dx \\ &+ \sum_{n=1}^{\infty} \left(a_n \int_{-\ell}^{\ell} \cos \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} dx + b_n \int_{-\ell}^{\ell} \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} dx \right). \end{aligned}$$

By parts (i), (ii), and (iv) of Theorem 1, the only nonzero term on the right side is the one involving b_m . Thus

$$\begin{aligned} \int_{-\ell}^{\ell} \varphi(x) \sin \frac{m\pi x}{\ell} dx &= b_m \int_{-\ell}^{\ell} \sin^2 \frac{m\pi x}{\ell} dx \\ &= \frac{b_m}{2} \int_{-\ell}^{\ell} \left(1 - \cos \frac{2m\pi x}{\ell} \right) dx = b_m \ell, \end{aligned}$$

and so we have

$$b_m = \frac{1}{\ell} \int_{-\ell}^{\ell} \varphi(x) \sin \frac{m\pi x}{\ell} dx.$$

To summarize, we have found that if φ is given by (3) and if the term-by-term integrations in the preceding development are valid, then the coefficients in (3) are given by

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} \varphi(x) \cos \frac{n\pi x}{\ell} dx, \quad n = 0, 1, 2, \dots \quad (4)$$

and

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} \varphi(x) \sin \frac{n\pi x}{\ell} dx, \quad n = 1, 2, 3, \dots \quad (5)$$

These formulae for a_n and b_n —obtained by formal calculations—motivate the following definition.

Definition. Given a function φ defined on $[-\ell, \ell]$, the numbers a_0, a_1, a_2, \dots and b_1, b_2, b_3, \dots given by (4) and (5) are called the **Fourier coefficients** of φ (with respect to $[-\ell, \ell]$). The function φ_{\sim} defined by

$$\varphi_{\sim}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \quad (6)$$

is called the **Fourier series** associated with φ on $[-\ell, \ell]$. We also call φ_{\sim} the *Fourier series representation* of φ on the interval $[-\ell, \ell]$.

We point out that the definition of φ_{\sim} is not dependent upon convergence properties and requires only that the coefficients a_0, a_1, a_2, \dots and b_1, b_2, b_3, \dots

exist, which is the case if φ is integrable on $[-\ell, \ell]$. (Why?) It may be the case that, at certain x , the series defining $\varphi_{\sim}(x)$ either does not converge or converges to something other than $\varphi(x)$. The fundamental question regarding φ_{\sim} is this: *For what values of x , if any, does $\varphi_{\sim}(x) = \varphi(x)$?* Before we state a theorem that answers this question, let's examine a couple of examples.

- **Example 1** Let φ be defined on $[-1, 1]$ by

$$\varphi(x) = \begin{cases} 0, & \text{if } -1 \leq x \leq 0; \\ 4x(1-x), & \text{if } 0 < x \leq 1. \end{cases}$$

The graph of the periodic extension $\bar{\varphi}$ of φ is indicated in Figure 2. Note that $\bar{\varphi}$ is continuous.

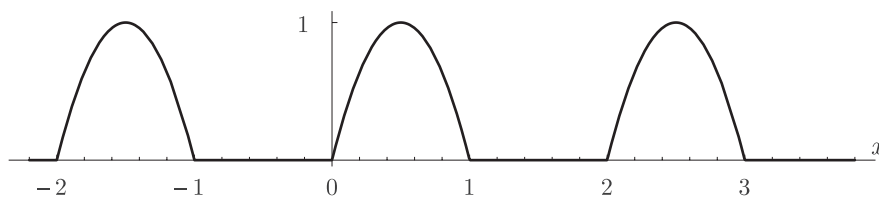


Figure 2

Since $\ell = 1$ and $\varphi(x) = 0$ for $-1 \leq x \leq 0$, the Fourier coefficients of φ are

$$\begin{aligned} a_n &= \int_0^1 \cos(n\pi x) 4x(1-x) dx, \quad n = 0, 1, 2, \dots, \\ b_n &= \int_0^1 \sin(n\pi x) 4x(1-x) dx, \quad n = 1, 2, 3, \dots \end{aligned}$$

Further computation—either by hand using integration by parts or with the help of a computer—reveals that

$$\begin{aligned} a_0 &= \frac{2}{3}, \quad a_n = -(1 + (-1)^n) \frac{4}{n^2\pi^2}, \quad n = 1, 2, 3, \dots, \\ b_n &= (1 - (-1)^n) \frac{8}{n^3\pi^3}, \quad n = 1, 2, 3, \dots \end{aligned}$$

Thus the Fourier series associated with $\bar{\varphi}$ —and the Fourier series representation of φ on $[-1, 1]$ —is

$$\varphi_{\sim}(x) = \frac{1}{3} + \frac{8}{\pi^2} \left(\frac{2}{\pi} \sin \pi x - \frac{1}{2^2} \cos 2\pi x + \frac{2}{3^3\pi} \sin 3\pi x - \frac{1}{4^2} \cos 4\pi x + \dots \right).$$

Figure 3 shows graphs of partial sums containing two, three, four, and five terms, each along with a dashed version of the graph of $\bar{\varphi}$.

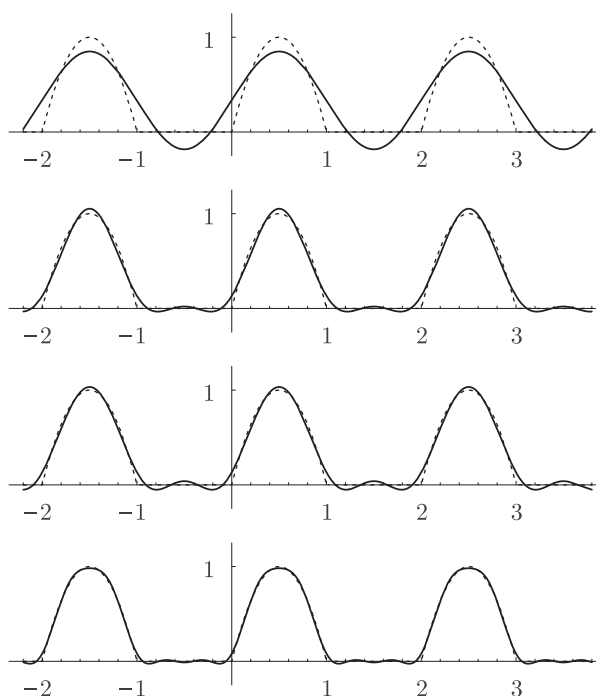


Figure 3

Graphs of subsequent partial sums show even more accurate approximations of $\bar{\varphi}$. So it is evident that the value of Fourier series associated with φ on $[-1, 1]$ indeed coincides with $\bar{\varphi}(x)$ for all x , and, in particular,

$$\varphi_{\sim}(x) = \varphi(x) \quad \text{for all } x \text{ in } [-1, 1].$$

◇

- **Example 2** Let φ be defined on $[-\pi, \pi]$ by

$$\varphi(x) = \begin{cases} 0, & \text{if } -\pi \leq x \leq 0; \\ \pi, & \text{if } 0 < x \leq \pi. \end{cases}$$

The graph of the periodic extension $\bar{\varphi}$ of φ is indicated in Figure 4. Note that $\bar{\varphi}$ is continuous except at each integer multiple of π .

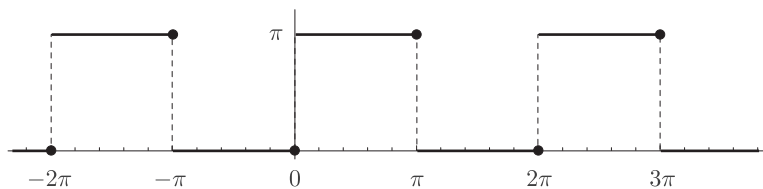


Figure 4

The Fourier coefficients of φ are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) \varphi(x) dx = \frac{1}{\pi} \int_0^{\pi} \cos(nx) \pi dx, \quad n = 0, 1, 2, \dots,$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) \varphi(x) dx = \frac{1}{\pi} \int_0^{\pi} \sin(nx) \pi dx, \quad n = 1, 2, 3, \dots,$$

from which we find that $a_0 = \pi$ and

$$a_n = 0, \quad b_n = \frac{1 - (-1)^n}{n}, \quad n = 1, 2, 3, \dots$$

Thus the Fourier series associated with φ is

$$\varphi_{\sim}(x) = \frac{\pi}{2} + 2 \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right).$$

Figure 5a shows graphs of partial sums containing two, three, and four terms, and Figure 5b shows graphs of partial sums containing ten, twenty, and thirty terms. For comparison each of the graphs is superimposed over a dashed version of the graph of $\bar{\varphi}$.

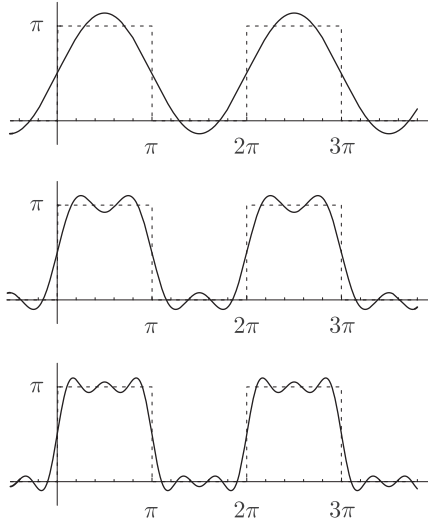


Figure 5a

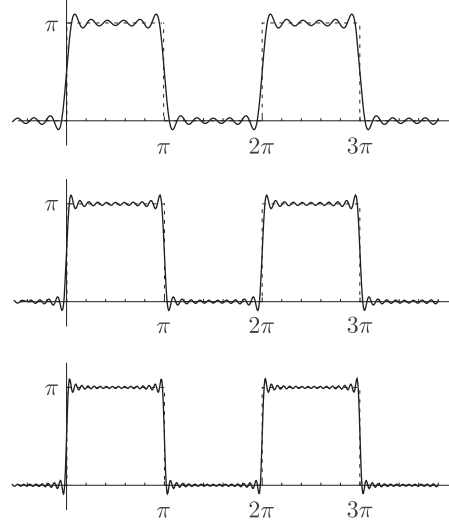


Figure 5b

These plots suggest that the partial sums defining $\varphi_{\sim}(x)$ converge to $\bar{\varphi}(x)$ at each x except those where $\bar{\varphi}$ is discontinuous. In fact, for any integer j , it is easy to see that

$$\varphi_{\sim}(j\pi) = \frac{1}{2\pi}, \quad \text{while } \bar{\varphi}(j\pi) = 0.$$

◇

■ A Convergence Theorem

The preceding examples are illustrative of general convergence properties of Fourier series associated with *piecewise-continuous* periodic functions. Before stating these convergence properties as a theorem, we first need to give a precise definition of piecewise continuity.

Definition. A function φ is **piecewise continuous** on a closed and bounded interval $[a, b]$ if there are numbers $x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ such that

- i) φ is continuous on each open subinterval (x_{i-1}, x_i) , $i = 1, 2, \dots, n$,
- ii) the left-sided limit $\lim_{x \rightarrow x_i^-} \varphi(x)$ exists for all $i = 1, 2, \dots, n$, and
- iii) the right-sided limit $\lim_{x \rightarrow x_i^+} \varphi(x)$ exists for all $i = 0, 1, \dots, n-1$.

A function is piecewise continuous on $(-\infty, \infty)$ if it is piecewise continuous on every closed and bounded interval $[a, b]$.

One may view piecewise continuous functions on $[a, b]$ as functions that either are continuous on $[a, b]$ or have finitely many discontinuities in $[a, b]$, each of which is either a removable or “jump” discontinuity. Excluded are vertical asymptotes and oscillatory discontinuities such as that of $\sin(1/x)$ at $x = 0$.

Consequences of the preceding definition include the observations that

- a piecewise-continuous function on $[a, b]$ is also piecewise continuous on every closed subinterval of $[a, b]$, and
- a function that is piecewise continuous on both $[a, b]$ and $[b, c]$ is also piecewise continuous on $[a, c]$,

from which it follows that:

- the periodic extension $\bar{\varphi}$ of a piecewise-continuous function φ on $[-\ell, \ell]$ is piecewise continuous on $(-\infty, \infty)$.

We now state the precise relationship between a piecewise-continuous function φ and its Fourier series representation on $[-\ell, \ell]$. The proof is omitted, since it requires methods from advanced calculus that are beyond the scope of this text.

THEOREM 2 Suppose that φ and φ' are each piecewise continuous on $[-\ell, \ell]$ and that $\bar{\varphi}$ is the periodic extension of φ . Then the Fourier series φ_{\sim} defined by (6) converges for all x , and

$$\varphi_{\sim}(x) = \begin{cases} \bar{\varphi}(x), & \text{wherever } \bar{\varphi} \text{ is continuous;} \\ \frac{1}{2} \left(\lim_{\xi \rightarrow x^-} \bar{\varphi}(\xi) + \lim_{\xi \rightarrow x^+} \bar{\varphi}(\xi) \right), & \text{wherever } \bar{\varphi} \text{ is discontinuous.} \end{cases}$$

In particular, if $\bar{\varphi}$ is continuous, then $\varphi_{\sim} = \bar{\varphi}$.

Note that according to Theorem 2 the Fourier series φ_{\sim} associated with a piecewise-continuous function φ on $[-\ell, \ell]$ is a piecewise-continuous, 2ℓ -periodic function whose discontinuities in $[-\ell, \ell]$ coincide with those of φ and whose value at each of its discontinuities is the average of its two one-sided limits. (Note that this is also true at points where φ_{\sim} is continuous.) This fact is illustrated by Figure 6, which shows the graph of the Fourier series computed in Example 2 and associated with the function φ whose 2π -periodic extension is plotted in Figure 4. Note that changing the value of φ at any or all of its discontinuities—or at any finite collection of points in $[-\ell, \ell]$ —would have no effect on the resulting Fourier series.

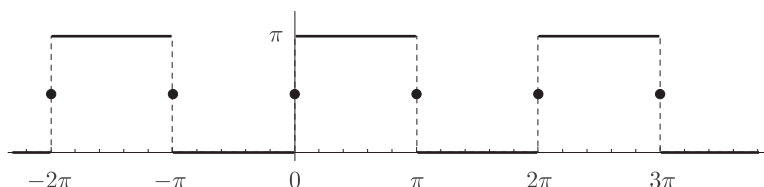


Figure 6

■ Remarks

We remark that, for any two distinct functions u, v from those listed in (1), Theorem 1 tells us that

$$\int_{-\ell}^{\ell} u(x)v(x) dx = 0.$$

Two functions with this property are said to be **orthogonal** on the interval $[-\ell, \ell]$. Thus Theorem 1 states that the functions listed in (1) comprise an *orthogonal family* of functions on $[-\ell, \ell]$. Indeed, orthogonality was the crucial property that enabled us to calculate the coefficients in (2) in a simple way.

The analogy with the notion of orthogonal vectors is worth noting. Recall that two vectors $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ in \mathbb{R}^n are orthogonal if their *inner product* (or “dot product”) is zero:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^n u_i v_i = 0.$$

Furthermore, if $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are an orthogonal family of vectors in \mathbb{R}^n , then any vector \mathbf{u} in \mathbb{R}^n can be expressed as the sum of its *orthogonal projections* onto $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$:

$$\mathbf{u} = \sum_{i=1}^n \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\langle \mathbf{v}_i, \mathbf{v}_i \rangle} \mathbf{v}_i.$$

By defining the *inner product of two functions* u and v on $[-\ell, \ell]$ to be

$$\langle u, v \rangle = \int_{-\ell}^{\ell} u(x)v(x) dx,$$

we can analogously express the Fourier series of a function φ on $[-\ell, \ell]$ as

$$\varphi_{\sim} = \sum_{i=0}^{\infty} \frac{\langle \varphi, q_i \rangle}{\langle q_i, q_i \rangle} q_i,$$

where q_1, q_2, q_3, \dots are the orthogonal family of functions

$$1, \cos \frac{\pi x}{\ell}, \sin \frac{\pi x}{\ell}, \cos \frac{2\pi x}{\ell}, \sin \frac{2\pi x}{\ell}, \cos \frac{3\pi x}{\ell}, \sin \frac{3\pi x}{\ell}, \dots$$

Problems

For each function φ in Problems 1 through 8, (a) find φ_{\sim} , the Fourier series representation on $[-1, 1]$; (b) use a computer or graphing calculator to plot on $[-1, 1]$ the graph of the partial sum $\frac{a_0}{2} + \sum_{n=1}^3 (a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell})$; and (c) sketch the graph of φ_{\sim} on $[-3, 3]$ (using Theorem 2).

$$1. \varphi(x) = \begin{cases} 0, & \text{if } -1 \leq x \leq 1/2 \\ 1, & \text{if } 1/2 < x \leq 1 \end{cases} \quad 2. \varphi(x) = \begin{cases} 0, & \text{if } -1 \leq x \leq 0 \\ 1, & \text{if } 0 < x \leq 1/2 \\ 0, & \text{if } 1/2 < x \leq 1 \end{cases}$$

$$3. \varphi(x) = \begin{cases} 0, & \text{if } -1 \leq x \leq -1/2 \\ 1, & \text{if } -1/2 < x \leq 1/2 \\ 0, & \text{if } 1/2 < x \leq 1 \end{cases} \quad 4. \varphi(x) = \begin{cases} 0, & \text{if } -1 \leq x \leq 0 \\ x, & \text{if } 0 < x \leq 1 \end{cases}$$

$$5. \varphi(x) = x \quad 6. \varphi(x) = |x| \quad 7. \varphi(x) = \frac{x}{|x|} \quad 8. \varphi(x) = e^{-|x|}$$

For each function φ in Problems 9 through 12, let $\bar{\varphi}$ be the 2-periodic function that agrees with φ on $(-1, 1]$, and let φ_{\sim} be the Fourier series representation of φ on $[-1, 1]$. Sketch the graphs of $\bar{\varphi}$ and φ_{\sim} for $-3 \leq x \leq 3$, taking care to indicate the behavior at each discontinuity. (Do not compute the Fourier coefficients; use Theorem 2 to sketch the graph of φ_{\sim} .)

$$9. \varphi(x) = \frac{x}{|x|} - x \quad 10. \varphi(x) = \sin \pi x - \frac{x}{|x|}$$

$$11. \varphi(x) = x^3 \quad 12. \varphi(x) = \frac{x}{2} (x + |x|)$$

13. Prove each of the properties listed in Theorem 1 directly by means of elementary integration techniques.
14. This problem concerns an alternative proof of the orthogonality properties stated in Theorem 1.

- a) Suppose that u and v are nonzero solutions on $[-\ell, \ell]$ of $u'' = \lambda u$ $v'' = \mu v$, respectively, where $\lambda \neq \mu$. Integrate by parts to show that

$$\int_{-\ell}^{\ell} (v u'' - u v'') dx = (u'v - u v') \Big|_{-\ell}^{\ell}.$$

Then conclude that

$$\int_{-\ell}^{\ell} u v dx = \frac{1}{\lambda - \mu} (u'v - u v') \Big|_{-\ell}^{\ell}.$$

- b) Let u and v be as in (a). Use the result of (a) to show that, if u and v are 2ℓ -periodic, then they are orthogonal on $[-\ell, \ell]$. (See Problems 15a and b.)
- c) Use the result of (b) to show that each of parts (ii), (iii), and (iv) of Theorem 1 is true when $m \neq n$. Finally, use the double-angle formula for sine to prove part (ii) in the case where $m = n$.
15. (a) Use the definition of the derivative to prove that, if a differentiable function f is p -periodic, then so is f' .
- (b) Show that, if f and g are each p -periodic functions, then so is fg , and so is any linear combination $\alpha f + \beta g$.
- (c) Suppose that f is p -periodic and g is q -periodic. Show that, if there are integers m and n such that $mp = nq$, then $f + g$ is mp -periodic.
- (d) Find the smallest period of $f(x) = \sin 7x + \cos \frac{3x}{11}$.
- (e) Prove that $f(x) = \cos x + \cos \pi x$ is not a periodic function. (*Hint*: Consider the equation $f(p) = f(0)$, which would be true if f were p -periodic.)

16. Use the Fourier series derived in Example 2 to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

17. Find the Fourier series representation of $|x|$ on $[-1, 1]$ and use it to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

18. The *Legendre polynomials*, P_0, P_1, P_2, \dots , comprise an orthogonal family of functions on $[-1, 1]$. (See Problems 1 and 2 in Section 5.7.) The first four Legendre polynomials (scaled so that $P_n(1) = 1$) are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

Let f be a given function defined on $[-1, 1]$, and suppose that we wish to approximate f with a polynomial p of degree N or less, in such a way that

$$\int_{-1}^1 f(x)q(x) dx = \int_{-1}^1 p(x)q(x) dx$$

for *all* polynomials q of degree N or less. Since any polynomial of degree N or less can be written as a linear combination of the first $N + 1$ Legendre polynomials, we will look for p the form

$$p(x) = \sum_{n=0}^N c_n P_n(x)$$

and choose the coefficients c_0, c_1, c_2, \dots so that

$$\int_{-1}^1 f(x) P_m(x) dx = \int_{-1}^1 p(x) P_m(x) dx, \quad m = 0, 1, 2, \dots, N.$$

(a) Show that this plan produces coefficients

$$c_n = \frac{\int_{-1}^1 f(x) P_n(x) dx}{\int_{-1}^1 P_n(x)^2 dx}, \quad n = 0, 1, 2, \dots, N.$$

(b) Find a cubic polynomial approximation on $[-1, 1]$ for the function

$$f(x) = \begin{cases} 0, & \text{if } -1 \leq x < 0; \\ 1, & \text{if } 0 \leq x \leq 1. \end{cases}$$

Sketch its graph along with the graph of f .

11.4 Fourier Sine and Cosine Series

Recall that in Section 11.2 we solved the diffusion problem

$$\left. \begin{aligned} w_t - k w_{xx} &= 0 & \text{for } 0 < x < \ell, \ t > 0 \\ w(0, x) &= \varphi(x) & \text{for } 0 \leq x \leq \ell \\ w(t, 0) &= w(t, \ell) = 0 & \text{for } t > 0 \end{aligned} \right\} \quad (1)$$

in the case where φ is expressed as a Fourier series of the form

$$\varphi_{\sim}(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}. \quad (2)$$

The solution of (1) then is

$$w(t, x) = \sum_{n=1}^{\infty} b_n e^{-n^2 \pi^2 k t / \ell^2} \sin \frac{n\pi x}{\ell}.$$

For the analogous problem with homogeneous Neumann (i.e., zero-flux) boundary conditions,

$$\left. \begin{aligned} w_t - k w_{xx} &= 0 & \text{for } 0 < x < \ell, \ t > 0 \\ w(0, x) &= \varphi(x) & \text{for } 0 \leq x \leq \ell \\ w_x(t, 0) &= w_x(t, \ell) = 0 & \text{for } t > 0, \end{aligned} \right\} \quad (3)$$

the desired Fourier series representation of φ is

$$\varphi_{\sim}(x) = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{\ell}, \quad (4)$$

and the consequent solution of (3) is

$$w(t, x) = \sum_{n=1}^{\infty} a_n e^{-n^2 \pi^2 k t / \ell^2} \cos \frac{n\pi x}{\ell},$$

since each of $\cos \frac{n\pi x}{\ell}$, $n = 0, 1, 2, \dots$, satisfies the differential equation and the boundary conditions. (See Problem 7 in Section 11.2.)

Our goal in the present section is learn how to find Fourier series of the forms in (2) and (4) for a function φ defined on $[0, \ell]$. We refer to these as Fourier *sine* and *cosine* series, respectively. Since the functions φ we are considering are defined only on $[0, \ell]$, we are free to extend the definition of φ to the interval $[-\ell, \ell]$ by defining φ on $[-\ell, 0]$ however we wish. Indeed, the key to obtaining a Fourier sine or cosine series is the manner in which we extend the definition of φ to $[-\ell, \ell]$.

■ Odd Functions and Even Functions

Recall that a function g defined on an interval of the form $[-\ell, \ell]$ is

- **odd** if $g(-x) = -g(x)$ for all x in $[-\ell, \ell]$;
- **even** if $g(-x) = g(x)$ for all x in $[-\ell, \ell]$.

For example, basic properties of sine and cosine tell us that $\sin \omega x$ is an odd function, and $\cos \omega x$ is an even function (each for any ω and on any interval $[-\ell, \ell]$). Recall also that the graph of an even function is symmetric about the y -axis, and the graph of an odd function is symmetric about the origin. These symmetries are illustrated in Figures 1a and b, where the graph of an odd function is shown in Figure 1a, and the graph of an even function is shown in Figure 1b.

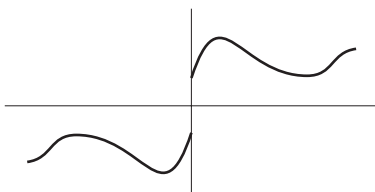


Figure 1a

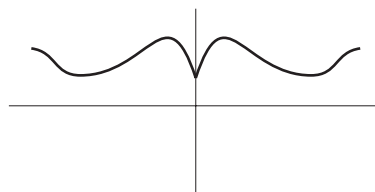


Figure 1b

Further important properties of odd functions and even functions are as follows. Verification of these properties is the subject of Problem 7.

THEOREM 1

i) If f is odd and integrable on $[-\ell, \ell]$, then

$$\int_{-\ell}^{\ell} f(x) dx = 0.$$

ii) If f is even and integrable on $[-\ell, \ell]$, then

$$\int_{-\ell}^{\ell} f(x) dx = 2 \int_0^{\ell} f(x) dx.$$

iii) If f and g are two functions that are either even or odd, then the product fg is also even or odd according to the following table.

f	g	fg
odd	odd	even
odd	even	odd
even	even	even



Let us now consider the Fourier series representation of a function φ on $[-\ell, \ell]$, where φ is either even or odd. Recall that the Fourier series representation of any function φ on $[-\ell, \ell]$ is defined by

$$\varphi_{\sim}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right),$$

where

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} \varphi(x) \cos \frac{n\pi x}{\ell} dx, \quad n = 0, 1, 2, \dots$$

and

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} \varphi(x) \sin \frac{n\pi x}{\ell} dx, \quad n = 1, 2, 3, \dots$$

If φ is even, then $\varphi(x) \cos \frac{n\pi x}{\ell}$ is even and $\varphi(x) \sin \frac{n\pi x}{\ell}$ is odd for all n . Thus

$$a_n = \frac{2}{\ell} \int_0^{\ell} \varphi(x) \cos \frac{n\pi x}{\ell} dx, \quad n = 0, 1, 2, \dots$$

and

$$b_n = 0, \quad n = 1, 2, 3, \dots$$

If φ is odd, then $\varphi(x) \cos \frac{n\pi x}{\ell}$ is odd and $\varphi(x) \sin \frac{n\pi x}{\ell}$ is even for all n . Thus

$$a_n = 0, \quad n = 1, 2, 3, \dots$$

and

$$b_n = \frac{2}{\ell} \int_0^\ell \varphi(x) \sin \frac{n\pi x}{\ell} dx, \quad n = 1, 2, 3, \dots$$

The upshot of all this is that the Fourier series of an even function contains no nonzero sine terms, the Fourier series of an odd function contains no nonzero cosine terms, and in either of those cases the nonzero coefficients can be expressed as integrals over $[0, \ell]$. For reference we state the details of these conclusions as follows:

THEOREM 2 *If φ is an even function on $[-\ell, \ell]$, then the Fourier series representation of φ on $[-\ell, \ell]$ is given by*

$$\varphi_\sim(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell},$$

where

$$a_n = \frac{2}{\ell} \int_0^\ell \varphi(x) \cos \frac{n\pi x}{\ell} dx, \quad n = 0, 1, 2, \dots$$

If φ is an odd function on $[-\ell, \ell]$, then the Fourier series representation of φ on $[-\ell, \ell]$ is given by

$$\varphi_\sim(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell},$$

where

$$b_n = \frac{2}{\ell} \int_0^\ell \varphi(x) \sin \frac{n\pi x}{\ell} dx, \quad n = 1, 2, 3, \dots \quad \blacktriangleleft$$

■ Functions on $[0, \ell]$

Suppose that φ is a given function defined on $[0, \ell]$. Then φ determines an *even extension* φ_e on $[-\ell, \ell]$ defined by

$$\varphi_e(x) = \begin{cases} \varphi(-x), & \text{if } -\ell \leq x < 0; \\ \varphi(x), & \text{if } 0 \leq x \leq \ell. \end{cases}$$

We similarly define an *odd extension* φ_o of φ by

$$\varphi_o(x) = \begin{cases} -\varphi(-x), & \text{if } -\ell \leq x < 0; \\ 0, & \text{if } x = 0; \\ \varphi(x), & \text{if } 0 < x \leq \ell. \end{cases}$$

- **Example 1** Consider the function φ defined on $[0, 1]$ by

$$\varphi(x) = x \quad \text{for } 0 \leq x \leq 1.$$

The graphs of the even extension φ_e and the odd extension φ_o on $[-1, 1]$ are shown in Figures 2a and b, respectively. \diamond

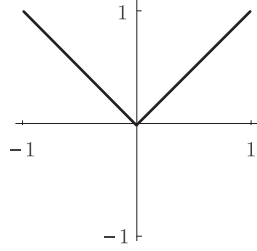


Figure 2a

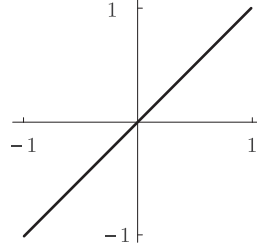


Figure 2b

Given a function φ defined on $[0, \ell]$, the even periodic extension φ_e and the odd periodic extension φ_o provide us with two distinct Fourier series representations of φ on $[0, \ell]$, which we define as follows.

Definition. Let φ be defined on $[0, \ell]$, and let φ_e and φ_o be the even and odd extensions, respectively, of φ on $[-\ell, \ell]$. The **Fourier cosine series** representation of φ on $[0, \ell]$ is the Fourier series representation of φ_e on $[-\ell, \ell]$:

$$\varphi_{\cos}(x) = \varphi_{e\sim}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell},$$

where

$$a_n = \frac{2}{\ell} \int_0^{\ell} \varphi(x) \cos \frac{n\pi x}{\ell} dx, \quad n = 0, 1, 2, \dots$$

The **Fourier sine series** representation of φ on $[0, \ell]$ is the Fourier series representation of φ_o on $[-\ell, \ell]$:

$$\varphi_{\sin}(x) = \varphi_{o\sim}(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell},$$

where

$$b_n = \frac{2}{\ell} \int_0^{\ell} \varphi(x) \sin \frac{n\pi x}{\ell} dx, \quad n = 1, 2, 3, \dots$$

In the following example, we derive the Fourier cosine and sine series for a specific function on the interval $[0, 1]$.

- **Example 2** Consider the function φ defined on $[0, 1]$ by

$$\varphi(x) = x^2, \quad \text{for } 0 \leq x \leq 1.$$

The coefficients in the Fourier cosine series φ_{\cos} are

$$a_n = 2 \int_0^1 x^2 \cos(n\pi x) dx, \quad n = 0, 1, 2, \dots$$

Integration reveals that

$$a_0 = \frac{2}{3} \quad \text{and} \quad a_n = \frac{4(-1)^n}{n^2\pi^2}, \quad n = 1, 2, 3, \dots$$

Therefore

$$\varphi_{\cos}(x) = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x).$$

The coefficients in the Fourier sine series φ_{\sin} are

$$b_n = 2 \int_0^1 x^2 \sin(n\pi x) dx, \quad n = 1, 2, 3, \dots$$

Integration reveals that

$$b_n = 4 \frac{(-1)^n - 1}{n^3\pi^3} + \frac{(-1)^n}{n\pi}, \quad n = 1, 2, 3, \dots$$

Therefore,

$$\varphi_{\sin}(x) = \sum_{n=1}^{\infty} \left(4 \frac{(-1)^n - 1}{n^3\pi^3} + \frac{(-1)^n}{n\pi} \right) \sin(n\pi x).$$

Figure 3a shows the graph of the tenth cosine-series partial sum

$$\frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{10} \frac{(-1)^n}{n^2} \cos(n\pi x),$$

and Figure 3b shows the graph of the tenth sine-series partial sum

$$\sum_{n=1}^{10} \left(4 \frac{(-1)^n - 1}{n^3\pi^3} + \frac{(-1)^n}{n\pi} \right) \sin(n\pi x).$$

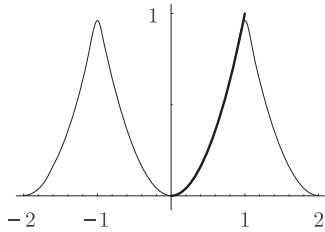


Figure 3a

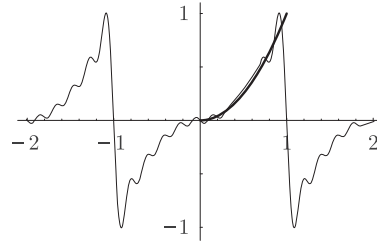


Figure 3b

It is apparent from Figures 3a and b that the tenth cosine-series partial sum gives a much better approximation of $\varphi(x)$ on $[0, 1]$ than does the tenth partial

sum of the sine series. Note that this is to be expected, since the coefficients in the sine series are approximately proportional to $1/n$ for large n and thus decay more slowly than those in the cosine series, which are proportional to $1/n^2$. \diamond

- **Example 3** Consider the diffusion problem with zero-flux boundary conditions:

$$\begin{aligned} w_t - w_{xx} &= 0 & \text{for } 0 < x < 1, \ t > 0, \\ w(0, x) &= \sin(\pi x) & \text{for } 0 \leq x \leq 1, \\ w_x(t, 0) &= w_x(t, 1) = 0 & \text{for } t > 0. \end{aligned}$$

Because of the nature of the boundary conditions, we need to express the initial data as a Fourier cosine series on $[0, 1]$. The coefficients for this purpose are

$$a_n = 2 \int_0^1 \cos(n\pi x) \sin \pi x \, dx, \quad n = 0, 1, 2, \dots$$

Computer-aided computation reveals that

$$a_{2i} = \frac{-4}{(4i^2 - 1)\pi} \quad \text{and} \quad a_{2i+1} = 0, \quad i = 0, 1, 2, \dots$$

Thus the Fourier cosine series for $\sin(\pi x)$ on $[0, 1]$ is

$$\varphi_{\cos}(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n\pi x)}{4n^2 - 1}.$$

Therefore, the solution of our diffusion problem can be expressed as

$$w(x, t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{e^{-4n^2\pi^2 t} \cos(2n\pi x)}{4n^2 - 1}.$$

Figures 4a and b* illustrate this solution. Figure 4a shows the graphs of the solution (versus x) at each of $t = 0, 0.01, 0.02$, and 0.05 , while Figure 4b is a surface plot of $w(x, t)$ for $0 \leq x \leq 1$ and $0 \leq t \leq 0.75$. A couple of things are worth

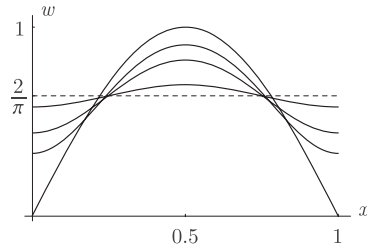


Figure 4a

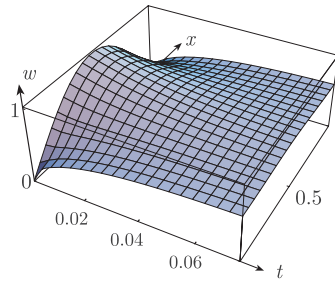


Figure 4b

* These plots were generated with the 21-term partial sum:

$$\frac{2}{\pi} - \frac{4}{\pi} \left(\frac{1}{3} e^{-4\pi^2 t} \cos(2\pi x) + \frac{1}{15} e^{-16\pi^2 t} \cos(4\pi x) + \dots + \frac{1}{399} e^{-400\pi^2 t} \cos(20\pi x) \right).$$

noting here. It is easy to see—both from the series representation of the solution and from Figure 4a—that the solution approaches the constant $2/\pi$ as $t \rightarrow \infty$. (This constant is the average value of the initial data. See Problem 16.) Also, we note that the initial data do not satisfy the boundary conditions, yet the solution does satisfy the (zero-flux) boundary conditions for all $t > 0$. This behavior is typical of diffusion problems: A solution will satisfy the boundary conditions for all $t > 0$, but it need not satisfy the boundary conditions at $t = 0$.

Problems

For each function φ in Problems 1 through 4, (a) sketch the graph of the odd extension φ_o on $[-1, 1]$; (b) find φ_{\sin} , the Fourier sine series representation on $[0, 1]$; (c) with the help of a computer or graphing calculator, sketch the graph of the partial sum with three nonzero terms on $[-1, 1]$; and (d) sketch the graph of φ_{\sin} on $[-1, 1]$.

- | | |
|---|--|
| 1. $\varphi(x) = x$ | 2. $\varphi(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1/2 \\ 1, & \text{if } 1/2 \leq x \leq 1 \end{cases}$ |
| 3. $\varphi(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1/2 \\ 0, & \text{if } 1/2 \leq x \leq 1 \end{cases}$ | 4. $\varphi(x) = \begin{cases} -1, & \text{if } 0 \leq x < 1/2 \\ 1, & \text{if } 1/2 \leq x \leq 1 \end{cases}$ |

For each function φ in Problems 5 through 8, (a) sketch the graph of the even extension φ_e on $[-1, 1]$; (b) find φ_{\cos} , the Fourier cosine series representation on $[0, 1]$; (c) with the help of a computer or graphing calculator, sketch the graph of the partial sum with three nonzero terms on $[-1, 1]$; and (d) sketch the graph of φ_{\cos} on $[-1, 1]$.

- | | |
|-----------------------------|-----------------------------|
| 5. φ from Problem 1 | 6. φ from Problem 2 |
| 7. φ from Problem 3 | 8. φ from Problem 4 |
9. Find the Fourier sine series representation of $\varphi(x) = \cos x$ on $[0, \pi]$.
10. Find the Fourier cosine series representation of $\varphi(x) = \sin x$ on $[0, \pi]$.
11. Find the Fourier sine series representation of $\varphi(x) = \sin x$ on $[0, \pi]$.
12. Find the Fourier sine series representation of $\varphi(x) = \sin x$ on $[0, \frac{\pi}{2}]$.
13. a) Find the solution of the diffusion problem

$$\begin{aligned} w_t - w_{xx} &= 0 && \text{for } 0 < x < 1, \ t > 0, \\ w(0, x) &= 1 && \text{for } 0 \leq x \leq 1, \\ w(t, 0) = w(t, 1) &= 0 && \text{for } t > 0. \end{aligned}$$

- b) Obtain an approximation to $w(0.01, x)$ on $[0, 1]$ by discarding from its Fourier sine series all terms with coefficients less than 0.005 in absolute value. Graph this approximation to $w(0.01, x)$. Repeat for $w(0.1, x)$.

14. a) Find the solution of the diffusion problem

$$\begin{aligned} w_t - w_{xx} &= 0 & \text{for } 0 < x < 1, \ t > 0, \\ w(0, x) &= \begin{cases} 0, & \text{if } 0 \leq x < 1/2 \\ 1, & \text{if } 1/2 \leq x \leq 1 \end{cases}, \\ w_x(t, 0) &= w_x(t, 1) = 0 & \text{for } t > 0. \end{aligned}$$

- b) Obtain an approximation to $w(0.01, x)$ on $[0, 1]$ by discarding from its Fourier cosine series all terms with coefficients less than 0.005 in absolute value. Graph this approximation to $w(0.01, x)$. Repeat for $w(0.1, x)$.

15. Prove Theorem 1.

16. Consider the diffusion problem with homogeneous Neumann boundary conditions:

$$\begin{aligned} w_t - w_{xx} &= 0 & \text{for } 0 < x < \ell, \ t > 0, \\ w(0, x) &= \varphi(x) & \text{for } 0 \leq x \leq \ell, \\ w_x(t, 0) &= w_x(t, \ell) = 0 & \text{for } t > 0. \end{aligned}$$

- a) Let $\bar{w}(t)$ be the average value of w on $[0, \ell]$ at time t :

$$\bar{w}(t) = \frac{1}{\ell} \int_0^\ell w(t, x) dx \quad \text{for all } t \geq 0.$$

Integrate each side of the differential equation over $[0, \ell]$ and apply the boundary conditions to show that $\bar{w}(t)$ is constant; that is, show that

$$\bar{w}'(t) = 0 \quad \text{for all } t > 0.$$

(Assume that differentiation through the integral sign is valid, that is, $\frac{d}{dt} \int_0^\ell w dx = \int_0^\ell w_t dx$.) Thus conclude that

$$\bar{w}(t) = \frac{1}{\ell} \int_0^\ell \varphi(x) dx \quad \text{for all } t \geq 0.$$

- b) Use the Fourier cosine series representation of φ on $[0, \ell]$ to write the solution w . Then show that, as $t \rightarrow \infty$, w converges *pointwise* to the average value of φ ; that is, show that

$$\lim_{t \rightarrow \infty} w(t, x) = \frac{1}{\ell} \int_0^\ell \varphi(x) dx \quad \text{for all } x \text{ in } [0, \ell].$$

17. Consider the diffusion problem with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} w_t - w_{xx} &= 0 & \text{for } 0 < x < \ell, \ t > 0, \\ w(0, x) &= \varphi(x) & \text{for } 0 \leq x \leq \ell, \\ w(t, 0) &= w(t, \ell) = 0 & \text{for } t > 0. \end{aligned}$$

Use the Fourier sine series representation of φ on $[0, \ell]$ to write the solution w . Then show that, as $t \rightarrow \infty$, w converges *pointwise* to zero; that is, show that

$$\lim_{t \rightarrow \infty} w(t, x) = 0 \quad \text{for all } x \text{ in } [0, \ell].$$

18. Use the Fourier cosine series for x^2 from Example 2 to show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

19. Use the Fourier cosine series for $\sin \pi x$ from Example 3 to show that

$$\frac{\pi}{4} = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = \frac{1}{2} + \frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \frac{1}{63} + \cdots.$$

11.5 Sturm-Liouville Eigenvalue Problems

We begin this section by considering a general, linear, homogeneous, diffusion problem (cf. Section 11.1) of the form

$$\left. \begin{aligned} w_t - (k w_x)_x &= 0 && \text{for } a < x < b, \quad t > 0, \\ w(0, x) &= \varphi(x) && \text{for } a \leq x \leq b, \\ \alpha w(t, a) - (1 - \alpha)w_x(t, a) &= 0 && \text{for } t > 0, \\ \beta w(t, b) + (1 - \beta)w_x(t, b) &= 0 && \text{for } t > 0, \end{aligned} \right\} \quad (1)$$

under the “usual” assumptions:

- k is positive and continuously differentiable on $[a, b]$;
- φ is piecewise continuous on $[a, b]$;
- α and β are constants in $[0, 1]$.

Our ultimate goal is to derive the solution of (1) based upon a *generalized* Fourier series representation for the initial data $\varphi(x)$.

It will serve the overall theme of this section for us to introduce an operator \mathcal{S} defined by

$$\mathcal{S}\psi = (k\psi')'$$

for all twice-continuously differentiable functions ψ on $[a, b]$. (For brevity, we will henceforth refer to these functions as the \mathcal{C}^2 *functions* on $[a, b]$.) Since for each $t > 0$ the solution w of (1) is such a function on $[a, b]$, it makes sense to write the differential equation in (1) as

$$w_t - \mathcal{S}w = 0,$$

with the understanding that in this context $\mathcal{S}w$ means $(k w_x)_x$; that is, derivatives are partial derivatives with respect to x .

Either separation of variables (as discussed in Section 11.2) or direct substitution reveals that the differential equation in (1) is satisfied by

$$w(t, x) = e^{\lambda t} X(x),$$

provided that λ is a constant and X satisfies

$$\mathcal{S}X = \lambda X.$$

If, in addition, X satisfies the boundary conditions in (1); that is, if

$$\begin{aligned}\alpha X(a) - (1 - \alpha)X'(a) &= 0, \\ \beta X(b) + (1 - \beta)X'(b) &= 0,\end{aligned}$$

then $w = e^{\lambda t} X$ will satisfy both the differential equation and the boundary conditions in (1). Thus we seek nontrivial solutions of

$$\left. \begin{aligned}\mathcal{S}X &= \lambda X && \text{for } a < x < b, \\ \alpha X(a) - (1 - \alpha)X'(a) &= 0, \\ \beta X(b) + (1 - \beta)X'(b) &= 0.\end{aligned}\right\} \quad (2)$$

A problem such as (2) is called an *eigenvalue problem** for the operator \mathcal{S} . Any constant λ for which (2) has a nontrivial solution is called an **eigenvalue** of the operator \mathcal{S} (with respect to the stated boundary conditions). A corresponding nontrivial solution X is called an **eigenfunction** of \mathcal{S} (also with respect to the stated boundary conditions).

- **Example 1** Recall that in Section 11.2 we encountered the eigenvalue problem

$$kX'' = \lambda X \quad \text{for } 0 < x < \ell,$$

$$X(0) = X(\ell) = 0,$$

which was found to possess a nontrivial solution if and only if

$$\lambda = \lambda_n = -\frac{n^2\pi^2 k}{\ell^2} \quad \text{for some } n = 1, 2, 3, \dots$$

A nontrivial solution corresponding to each λ_n was determined to be

$$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right).$$

Thus, for the operator \mathcal{S} defined by

$$\mathcal{S}\psi = k\psi'',$$

* Note the similarity to the matrix eigenvalue problem: $A\mathbf{x} = \lambda\mathbf{x}$, where A is an $n \times n$ matrix and \mathbf{x} is a column vector of length n .

the described λ_n , $n = 1, 2, 3, \dots$, are the eigenvalues of \mathcal{S} with respect to the boundary conditions $X(0) = X(\ell) = 0$. Corresponding eigenfunctions are X_n , $n = 1, 2, 3, \dots$, as described previously.

- **Example 2** Consider (1) with constant diffusivity k , $\alpha = 1/2$, and $\beta = 0$. Then (2) becomes

$$kX'' = \lambda X \quad \text{for } 0 < x < \ell,$$

$$X(0) - X'(0) = X'(\ell) = 0.$$

We first observe that nontrivial solutions exist only if $\lambda < 0$. (See Problem 12.) Setting $\omega = \sqrt{-\lambda/k}$ for economy of notation, we conclude that all nontrivial solutions will be of the form

$$X(x) = A \cos \omega x + B \sin \omega x$$

where A and B are constants. The boundary conditions then require that A and B satisfy

$$A - \omega B = 0 \quad \text{and} \quad -\omega A \sin \omega \ell + \omega B \cos \omega \ell = 0.$$

If we set $A = \omega B$, then the first of these equations is satisfied, and the second becomes

$$-\omega^2 B \sin \omega \ell + \omega B \cos \omega \ell = 0.$$

We dismiss the case of $B = 0$, since that leads to the trivial solution $X = 0$. Consequently, the desired values of ω satisfy the transcendental equation

$$\omega = \cot \omega \ell.$$

Figure 1 shows the graphs of ω and $\cot \omega \ell$ versus ω and reveals that there is a sequence of solutions $\omega_0, \omega_1, \omega_2, \dots$, for which

$$n\pi < \omega_n < (n+1)\pi \quad \text{for all } n = 0, 1, 2, \dots$$

and

$$\frac{\omega_n}{n\pi} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Rounded to three decimal places, the first four of these are

$$\omega_0 = 0.860, \quad \omega_1 = 3.426, \quad \omega_2 = 6.437, \quad \omega_3 = 9.529.$$

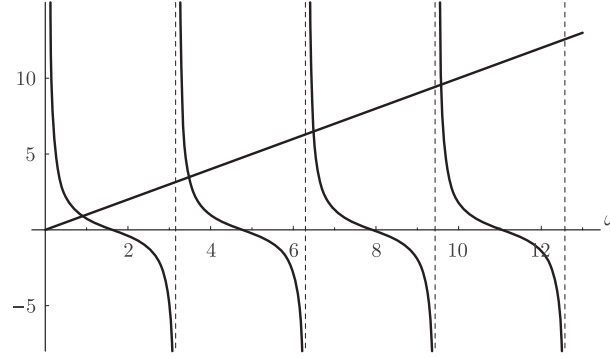


Figure 1

The result is that, with respect to the stated boundary conditions, the operator \mathcal{S} defined by $\mathcal{S}\psi = k\psi''$ has eigenvalues $\lambda_n = -k\omega_n^2$, $n = 0, 1, 2, 3, \dots$, where the ω_n are as previously indicated. Moreover, corresponding eigenfunctions are given by

$$X_n(x) = B_n (\omega_n \cos \omega_n x + \sin \omega_n x),$$

where B_n is any nonzero constant. The constants B_n may be chosen to *scale* the eigenfunctions as desired. An important type of scaling, done to achieve

$$\int_0^\ell (X_n(x))^2 dx = 1, \quad n = 0, 1, 2, 3, \dots,$$

uses

$$B_n = \left(\int_0^\ell (\omega_n \cos \omega_n x + \sin \omega_n x)^2 dx \right)^{-1/2}.$$

The resulting eigenfunctions are

$$X_n(x) = \frac{\omega_n \cos \omega_n x + \sin \omega_n x}{\sqrt{\int_0^\ell (\omega_n \cos \omega_n x + \sin \omega_n x)^2 dx}},$$

The first four of these are plotted in Figure 2. With coefficients rounded to four significant digits, they are

$$\begin{aligned} X_0(x) &= 0.8543 (0.8603 \cos(0.8603 x) - \sin(0.8603 x)), \\ X_1(x) &= 0.3816 (3.426 \cos(3.426 x) - \sin(3.426 x)), \\ X_2(x) &= 0.2146 (6.437 \cos(6.437 x) - \sin(6.437 x)), \\ X_3(x) &= 0.1468 (9.529 \cos(9.529 x) - \sin(9.529 x)). \end{aligned}$$

◇

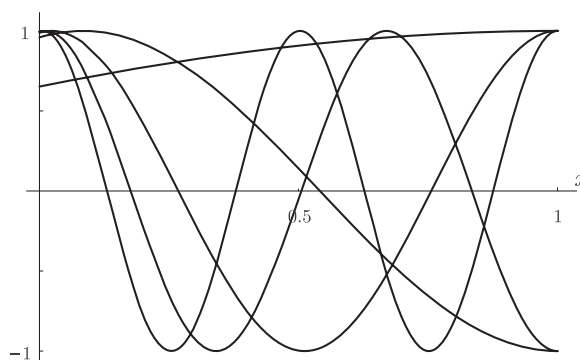


Figure 2

Eigenvalue problems such as (2) comprise a subset of a class of problems known as *Sturm-Liouville* eigenvalue problems*. In order to give a general description of these problems, let us first define what we mean by “Sturm-Liouville operator.” Let I be an interval, and suppose that the functions k , p , and q are such that

- k is positive and continuously differentiable on I ;
- p is positive and continuous on I ;
- q is continuous on I .

An operator \mathcal{S} , acting on \mathcal{C}^2 functions ψ on I , for which $\mathcal{S}\psi$ may be expressed as

$$\mathcal{S}\psi = \frac{1}{p} \left((k\psi')' + q\psi \right), \quad (3)$$

is a **Sturm-Liouville operator**. We note that Theorem 1 of Section 5.4 guarantees that, for *any* constant λ , the differential equation

$$\mathcal{S}\psi = \lambda\psi \quad (4a)$$

possesses a pair of linearly independent solutions on I , linear combinations of which provide the general solution on I .

Equation (4a) together with some suitable set of boundary conditions is a **Sturm-Liouville eigenvalue problem**. Any λ for which the problem possesses a nontrivial solution ψ is said to be an eigenvalue of \mathcal{S} with respect to the stated boundary conditions, the nontrivial solution ψ being a corresponding eigenfunction. The set of all eigenvalues is called the **spectrum** of \mathcal{S} (with respect to the stated boundary conditions).

* The work of the French mathematicians Charles Sturm and Joseph Liouville was published in the 1830s. (Note carefully the order of the vowels in “Liouville.”)

If I is a closed and bounded interval $[a, b]$, then \mathcal{S} is a *regular* Sturm-Liouville operator, and equation (4a) together with boundary conditions of the form

$$\begin{aligned}\alpha \psi(a) - (1 - \alpha)\psi'(a) &= 0, \\ \beta \psi(b) + (1 - \beta)\psi'(b) &= 0,\end{aligned}\tag{4b}$$

is a *regular* Sturm-Liouville eigenvalue problem.

Clearly the operator \mathcal{S} in Examples 1 and 2, defined by $\mathcal{S}\psi = k\psi''$ for \mathcal{C}^2 functions ψ on $[0, \ell]$, is a regular Sturm-Liouville operator (with $p = 1$ and $q = 0$), and the eigenvalue problems solved in those examples are regular Sturm-Liouville eigenvalue problems. The following example suggests that Sturm-Liouville operators are much wider in scope. (See also Problem 8.)

- **Example 3** Let the operator \mathcal{S} be defined for \mathcal{C}^2 functions ψ on $[0, 1]$ by

$$\mathcal{S}\psi = \psi'' + 4\psi' + \psi.$$

A simple computation shows that

$$\mathcal{S}\psi = e^{-4x} \left((e^{4x}\psi')' + e^{4x}\psi \right).$$

Thus \mathcal{S} is a regular Sturm-Liouville operator as described by (3) with

$$p(x) = k(x) = q(x) = e^{4x},$$

and problem (4a,b) is a regular Sturm-Liouville eigenvalue problem. Let us proceed now to solve this problem in the case where $\alpha = \beta = 1$:

$$\begin{aligned}\psi'' + 4\psi' + \psi &= \lambda\psi \quad \text{for } 0 < x < 1, \\ \psi(0) &= \psi(1) = 0.\end{aligned}$$

First we write the differential equation as

$$\psi'' + 4\psi' + (1 - \lambda)\psi = 0.$$

The associated characteristic equation is $z^2 + 4z + 1 - \lambda = 0$, whose solutions are $-2 \pm \sqrt{3 + \lambda}$. So, by the methods of Chapter 5, we find that ψ must be of the form

$$\psi(x) = e^{-2x} (A e^{\eta x} + B e^{-\eta x}),$$

where $\eta = \sqrt{3 + \lambda}$, and A and B are arbitrary (complex) constants. The boundary condition $\psi(0) = 0$ requires that $B = -A$; so

$$\psi(x) = A e^{-2x} (e^{\eta x} - e^{-\eta x}).$$

The boundary condition $\psi(1) = 0$ now requires that

$$e^{\eta} = e^{-\eta},$$

whose only real solution, $\eta = 0$, leads to $\psi = 0$. Consequently, η must be imaginary. Thus we write $\eta = i\omega$, where $\omega = \sqrt{-3 - \lambda}$ is real, and note that

$$\begin{aligned}\psi(x) &= A e^{-2x} (e^{i\omega x} - e^{-i\omega x}) \\ &= 2iA e^{-2x} \sin \omega x \\ &= C e^{-2x} \sin \omega x,\end{aligned}$$

where C is an arbitrary constant that we may assume is real. The boundary condition $\psi(1) = 0$ now requires also that $\sin \omega = 0$; that is,

$$\omega = n\pi \quad \text{for some } n = 1, 2, 3, \dots$$

Therefore, since $\lambda = -3 - \omega^2$, the eigenvalues of \mathcal{S} are

$$\lambda_n = -3 - n^2\pi^2, \quad n = 1, 2, 3, \dots$$

with corresponding eigenfunctions

$$\psi_n(x) = C_n e^{-2x} \sin(n\pi x), \quad n = 1, 2, 3, \dots$$

The nonzero constants C_n may be chosen to scale the eigenfunctions as desired. Figure 3 is a plot of the first four of these eigenfunctions with the simple choice of $C_n = 1$ for all n . ◇

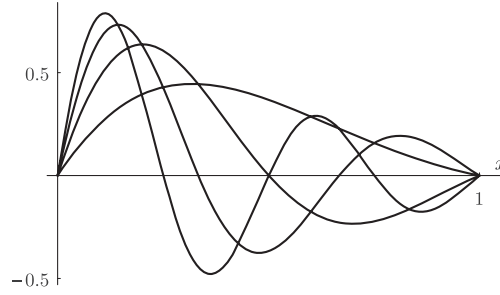


Figure 3

Each of Examples 1 through 3 illustrates several fundamental properties of the eigenvalues and eigenfunctions of any regular Sturm-Liouville operator. The spectrum is a decreasing, infinite sequence of real numbers that diverges to $-\infty$. Moreover, for each eigenvalue, the corresponding set of eigenfunctions is “one dimensional;” that is, it consists of constant multiples of a single eigenfunction. These properties, as well as the crucial orthogonality property of the eigenfunctions, are stated for the record in the following theorem.

THEOREM 1 *Let \mathcal{S} be a regular Sturm-Liouville operator defined for \mathcal{C}^2 functions on $[a, b]$. The following are true for the eigenvalue problem (4a,b).*

- i) *The eigenvalues are real and form an infinite, decreasing sequence*

$$\dots < \lambda_3 < \lambda_2 < \lambda_1 < \lambda_0$$

in which $\lambda_n \rightarrow -\infty$ as $n \rightarrow \infty$.

- ii) Each eigenvalue is simple; that is, for each λ_i the family of corresponding eigenfunctions consists of scalar multiples of a single eigenfunction ψ_i .
- iii) Eigenfunctions corresponding to distinct eigenvalues are orthogonal on $[a, b]$ with respect to the “weight function” p ; that is, if λ_i and λ_j are eigenvalues with corresponding eigenfunctions ψ_i and ψ_j , and if $i \neq j$, then

$$\int_a^b \psi_i \psi_j p dx = 0. \quad \blacktriangleleft$$

The proofs of some parts of Theorem 1 are beyond the scope of this book—in particular, the existence of eigenvalues and the fact that they form an infinite sequence that diverges to $-\infty$. However, two assertions of the theorem,

(1) that there are no nonreal complex eigenvalues, and

(2) that eigenfunctions corresponding to distinct eigenvalues are orthogonal, are elementary consequences of the fact that \mathcal{S} is a *symmetric* operator. The remainder of this section is devoted to the discussion of this concept and to the proof of the aforementioned parts of Theorem 1. We begin with the following theorem.

THEOREM 2 A regular Sturm-Liouville operator \mathcal{S} is **symmetric*** (with respect to the weight function p and the boundary conditions (4b)); that is,

$$\int_a^b \gamma \mathcal{S}\psi p dx = \int_a^b \psi \mathcal{S}\gamma p dx$$

for all \mathcal{C}^2 functions ψ and γ on $[a, b]$ satisfying the boundary conditions (4b). \blacktriangleleft

The proof of Theorem 2 is mainly an exercise in integration by parts and proceeds as follows. Suppose that ψ and γ are \mathcal{C}^2 functions on $[a, b]$ and satisfy the boundary conditions in (4b). Then

$$\begin{aligned} \int_a^b \gamma \mathcal{S}\psi p dx &= \int_a^b \gamma \cdot \left((k\psi')' + q\psi \right) dx \\ &= \int_a^b \gamma \cdot (k\psi')' dx + \int_a^b \gamma q \psi dx. \end{aligned}$$

Next we use integration by parts on the first integral on the right side, obtaining

$$\int_a^b \gamma \mathcal{S}\psi p dx = \gamma k \psi' \Big|_{x=a}^{x=b} - \int_a^b \gamma' k \psi' dx + \int_a^b \gamma q \psi dx. \quad (5)$$

* A real $n \times n$ matrix A is symmetric if $A^T = A$. An important consequence of symmetry in A (or an equivalent definition) is that $\mathbf{y}^T A \mathbf{x} = \mathbf{x}^T A \mathbf{y}$ for all real vectors of length n . Note the similarity between this and the conclusion of Theorem 2.

Similarly, it also follows that

$$\int_a^b \psi \mathcal{S} \gamma p dx = \psi k \gamma' \Big|_{x=a}^{x=b} - \int_a^b \psi' k \gamma' dx + \int_a^b \psi q \gamma dx. \quad (6)$$

Therefore, by subtracting (6) from (5), we find that

$$\int_a^b \gamma \mathcal{S} \psi p dx - \int_a^b \psi \mathcal{S} \gamma p dx = \left(\gamma k \psi' - \psi k \gamma' \right) \Big|_{x=a}^{x=b}. \quad (7)$$

Now we examine the right side of (7), first noting that

$$\begin{aligned} \left(\gamma k \psi' - \psi k \gamma' \right) \Big|_{x=a}^{x=b} &= k(b) \left(\gamma(b) \psi'(b) - \psi(b) \gamma'(b) \right) \\ &\quad - k(a) \left(\gamma(a) \psi'(a) - \psi(a) \gamma'(a) \right). \end{aligned} \quad (8)$$

If $\beta = 0$, then the boundary condition at $x = b$ in (4b) demands that $\psi'(b) = \gamma'(b) = 0$, and so $\gamma(b) \psi'(b) - \psi(b) \gamma'(b) = 0$. If $\beta \neq 0$, then the boundary condition at $x = b$ in (4b) demands that

$$\psi(b) = \frac{\beta - 1}{\beta} \psi'(b) \quad \text{and} \quad \gamma(b) = \frac{\beta - 1}{\beta} \gamma'(b).$$

These facts in turn imply that

$$\gamma(b) \psi'(b) - \psi(b) \gamma'(b) = \frac{\beta - 1}{\beta} (\gamma'(b) \psi'(b) - \psi'(b) \gamma'(b)) = 0.$$

Thus, for any β in $[0, 1]$, we have that

$$\gamma(b) \psi'(b) - \psi(b) \gamma'(b) = 0.$$

Similarly, for any α in $[0, 1]$, it also follows that

$$\gamma(a) \psi'(a) - \psi(a) \gamma'(a) = 0.$$

So we finally conclude from (7) and (8) that \mathcal{S} is symmetric:

$$\int_a^b \gamma \mathcal{S} \psi p dx = \int_a^b \psi \mathcal{S} \gamma p dx. \quad \blacksquare$$

In order to prove that \mathcal{S} can have only real eigenvalues, we first need a complex version of Theorem 2, which we will state as Corollary 1. Its proof, which we omit, follows easily from Theorem 2 and the fact that \mathcal{S} maps real functions to real functions. (A proof that closely parallels that of Theorem 3 is also straightforward.) For the statement of Corollary 1, we must recall that the *conjugate* of a complex number $z = x + iy$ is $\bar{z} = x - iy$; thus we define the conjugate of a complex-valued function $\zeta = u + iv$ to be

$$\bar{\zeta} = u - iv.$$

COROLLARY 1 Let u, v, ρ , and μ be C^2 functions defined on $[a, b]$, each satisfying the boundary conditions in (4b). Also let $\psi = u + i v$ and $\gamma = \rho + i \mu$, and let \mathcal{S} be a Sturm-Liouville operator as defined in (3). Then ψ and γ each satisfy the boundary conditions in (4b), and

$$\int_a^b (\rho - i \mu)(\mathcal{S}u + i \mathcal{S}v) p dx = \int_a^b (u + i v)(\mathcal{S}\rho - i \mathcal{S}\mu) p dx.$$

Since $\mathcal{S}\psi = \mathcal{S}u + i \mathcal{S}v$ and $\mathcal{S}\gamma = \mathcal{S}\rho + i \mathcal{S}\mu$, it follows that

$$\int_a^b \bar{\gamma} \mathcal{S}\psi p dx = \int_a^b \psi \overline{\mathcal{S}\gamma} p dx. \quad \blacktriangleleft$$

To demonstrate that \mathcal{S} has only real eigenvalues, we begin by supposing that $\lambda = \eta + i \omega$ is a complex eigenvalue[†] with corresponding (complex-valued) eigenfunction $\psi = u + i v$. Then, according to Corollary 1 with $Y = \psi$, we have

$$\int_a^b \bar{\psi} \mathcal{S}\psi p dx = \int_a^b \psi \overline{\mathcal{S}\psi} p dx.$$

But since $\mathcal{S}\psi = \lambda \psi$ and $\overline{\mathcal{S}\psi} = \bar{\lambda} \bar{\psi}$, this becomes

$$\lambda \int_a^b \bar{\psi} \psi p dx = \bar{\lambda} \int_a^b \psi \bar{\psi} p dx.$$

Since $\psi \bar{\psi} = \bar{\psi} \psi = u^2 + v^2$, it follows that the two integrals are equal and neither is zero. Therefore, $\lambda = \bar{\lambda}$; that is, $\eta + i \omega = \eta - i \omega$. Consequently, $\omega = 0$, and so it follows that λ is a real number.

To show that eigenfunctions corresponding to distinct eigenvalues of \mathcal{S} are orthogonal on $[a, b]$ with respect to the weight function p , we begin by supposing that λ_i and λ_j are distinct eigenvalues of \mathcal{S} with corresponding eigenfunctions ψ_i and ψ_j , respectively.[‡] Then, by Theorem 2,

$$\int_a^b \psi_i \mathcal{S}\psi_j p dx = \int_a^b \psi_j \mathcal{S}\psi_i p dx.$$

Now since $\mathcal{S}\psi_i = \lambda_i \psi_i$ and $\mathcal{S}\psi_j = \lambda_j \psi_j$, we have

$$\lambda_j \int_a^b \psi_i \psi_j p dx = \lambda_i \int_a^b \psi_j \psi_i p dx.$$

Since $\lambda_i \neq \lambda_j$, the desired orthogonality property follows:

$$\int_a^b \psi_i \psi_j p dx = 0. \quad \blacksquare$$

[†] Recall that the complex numbers include the real numbers.

[‡] Note that, since \mathcal{S} can have only real eigenvalues, we may assume that all eigenfunctions are real-valued. (Why?)

Problems

1. Define the operator \mathcal{S} by $\mathcal{S}\psi = \psi'' + \psi'$ for \mathcal{C}^2 functions on $[0, 1]$. Show that \mathcal{S} is a regular Sturm-Liouville operator.

For the operator \mathcal{S} in Problem 1, find the eigenvalues and corresponding eigenfunctions of \mathcal{S} with respect to the boundary conditions in Problems 2 through 4.

2. $\psi(0) = \psi(1) = 0$ 3. $\psi'(0) = \psi'(1) = 0$ 4. $\psi'(0) = \psi(1) = 0$

5. Let \mathcal{S} defined by $\mathcal{S}\psi = k\psi''$ (with k a constant) for \mathcal{C}^2 functions on $[0, \pi]$.

- (a) Find the eigenvalues and corresponding eigenfunctions of \mathcal{S} with respect to the boundary conditions

$$\psi(0) - \psi'(0) = \psi(\pi) + \psi'(\pi) = 0.$$

- (b) Normalize the eigenfunctions (i.e., scaled them so that $\int_0^\pi \psi_n^2 dx = 1$).

6. Let β be a constant, and define \mathcal{S} by $\mathcal{S}\psi = \psi'' + \beta\psi'$ for \mathcal{C}^2 functions on $[0, 1]$.

- (a) Show that \mathcal{S} is a regular Sturm-Liouville operator.
 (b) Find, in terms of β , the eigenvalues and corresponding eigenfunctions of \mathcal{S} with respect to the boundary conditions $\psi(0) = \psi(1) = 0$.

7. Let γ be a constant, and define \mathcal{S} by $\mathcal{S}\psi = \psi'' + \gamma\psi$ for \mathcal{C}^2 functions on $[0, 1]$.

- (a) Show that \mathcal{S} is a regular Sturm-Liouville operator.
 (b) Find, in terms of γ , the eigenvalues and corresponding normalized eigenfunctions of \mathcal{S} with respect to the boundary conditions $\psi(0) = \psi(1) = 0$.

8. Define the operator \mathcal{S} by $\mathcal{S}\psi = \alpha\psi'' + \beta\psi' + \gamma\psi$ for \mathcal{C}^2 functions ψ on $[a, b]$, where α , β , and γ are continuous functions on $[a, b]$. Show that

$$\mathcal{S}\psi = \alpha e^{-\int \beta/\alpha} \left((e^{\int \beta/\alpha} \psi')' + \gamma e^{\int \beta/\alpha} \psi \right),$$

and conclude, therefore, that \mathcal{S} is a regular Sturm-Liouville operator, if α is positive on $[a, b]$.

In Problems 9 and 10, use the result of Problem 8 to write $\mathcal{S}\psi$ in the form of (3).

9. $\mathcal{S}\psi = (1 + x^2)\psi'' + 4x\psi'$ 10. $\mathcal{S}\psi = \psi'' + x\psi'$

11. Let \mathcal{S} be a regular Sturm-Liouville operator defined by

$$\mathcal{S}\psi = \frac{1}{p}((k\psi')' + q\psi)$$

for \mathcal{C}^2 functions ψ on $[a, b]$, and consider the eigenvalue problem (4a,b). In this problem we will prove that, if $q(x) \leq 0$ for all x in $[a, b]$, then

$$\int_a^b \psi \mathcal{S}\psi p dx \leq 0$$

for all \mathcal{C}^2 functions ψ on $[a, b]$ that satisfy (4b). Because of this property \mathcal{S} is said to be **negative semidefinite**.^{*} Furthermore, as a result, \mathcal{S} has no positive eigenvalues.

- (a) Let ψ be a \mathcal{C}^2 function on $[a, b]$. Use integration by parts (as in the proof of Theorem 2) to obtain

$$\int_a^b \psi \mathcal{S}\psi p dx = k\psi\psi' \Big|_a^b - \int_a^b (k(\psi')^2 - q\psi^2) dx.$$

- (b) Show that, if ψ satisfies the boundary conditions (4b), then $k\psi\psi' \Big|_a^b \leq 0$ for any α and β in $[0, 1]$. Conclude that

$$\int_a^b \psi \mathcal{S}\psi p dx \leq - \int_a^b (k(\psi')^2 - q\psi^2) dx$$

and that, if $q(x) \leq 0$ for all x in $[a, b]$, then

$$\int_a^b \psi \mathcal{S}\psi p dx \leq 0.$$

- (c) Now suppose that λ is an eigenvalue of \mathcal{S} with corresponding eigenfunction ψ . Show that

$$\lambda \int_a^b \psi^2 p dx \leq - \int_a^b (k(\psi')^2 - q\psi^2) dx,$$

and conclude that, if $q(x) \leq 0$ for all x in $[a, b]$, then $\lambda \leq 0$.

12. Consider again the eigenvalue problem in Problem 11. Show that, if either of the following is true,

- (a) $q(x) \leq 0$ for all x in $[a, b]$, and either α or β is not zero in (4a,b);
 (b) $q(x) \leq 0$ for all x in $[a, b]$, $q(x) < 0$ for some x in $[a, b]$;

then, for all \mathcal{C}^2 functions ψ on $[a, b]$,

$$\int_a^b \psi \mathcal{S}\psi p dx < 0$$

and all eigenvalues of \mathcal{S} are negative. In other words, \mathcal{S} is **negative definite**.

13. Let \mathcal{S} be defined for \mathcal{C}^2 functions ψ on $[0, e - 1]$ by

$$\mathcal{S}\psi = ((1+x)^2\psi')'.$$

- (a) Verify that, if $\lambda \leq -1/4$, then the general solution of $\mathcal{S}\psi = \lambda\psi$ is given by

$$\psi = \frac{1}{\sqrt{1+x}} \left(c_1 \cos(\sqrt{-\lambda - 1/4} \ln(1+x)) + c_2 \sin(\sqrt{-\lambda - 1/4} \ln(1+x)) \right).$$

- (b) Verify that, if $\lambda > -1/4$, then the general solution of $\mathcal{S}\psi = \lambda\psi$ is given by

$$\psi = \frac{1}{\sqrt{1+x}} \left(c_1 \cosh(\sqrt{\lambda + 1/4} \ln(1+x)) + c_2 \sinh(\sqrt{\lambda + 1/4} \ln(1+x)) \right).$$

^{*} This is analogous to negative semidefiniteness for matrices. A symmetric matrix A is negative semidefinite, if $\mathbf{x}^T A \mathbf{x} \leq 0$ for all \mathbf{x} , which implies that A has no positive eigenvalues.

- (c) Find the eigenvalues and corresponding eigenfunctions of \mathcal{S} with respect to the boundary conditions $\psi(0) = \psi(e-1) = 0$.

14. Let the operator \mathcal{S} be defined by

$$\mathcal{S}\psi = \psi'' - 2x\psi'$$

for \mathcal{C}^2 functions ψ on an interval $[a, b]$. This operator is associated with Hermite's equation. (See Problems 3 and 4 in Section 5.7.)

- (a) Show that \mathcal{S} is a regular Sturm-Liouville operator. (*Hint*: See Problem 10.)
 (b) Consider the eigenvalue problem

$$\mathcal{S}\psi = \lambda\psi, \quad \psi(0) = \psi(1) = 0,$$

in which the differential equation is Hermite's equation

$$\psi'' - 2x\psi' + 2m\psi = 0, \quad \text{where } 2m = -\lambda.$$

Show that fundamental power-series solutions are

$$u_m = 1 - \frac{2mx^2}{2!} + \frac{2^2m(m-2)x^4}{4!} - \frac{2^3m(m-2)(m-4)x^6}{6!} + \dots$$

$$\text{and } v_m = x - \frac{2(m-1)x^3}{3!} + \frac{2^2(m-1)(m-3)x^5}{5!} - \dots,$$

and conclude that every solution of Hermite's equation is given by $c_1u_m + c_2v_m$ for some choice of c_1 and c_2 .

- (c) Show that $\lambda = -2m$ is an eigenvalue of \mathcal{S} , with corresponding eigenfunction $\psi = v_m$, if and only if m is a solution of

$$1 - \frac{2(m-1)}{3!} + \frac{2^2(m-1)(m-3)}{5!} - \frac{2^3(m-1)(m-3)(m-5)}{7!} + \dots = 0. \quad (*)$$

Conclude from Theorem 1 that this equation has an increasing, divergent, infinite sequence of solutions. Also, show that this equation has only positive solutions, and conclude that \mathcal{S} has only negative eigenvalues.

- (d) Truncate the series in (*) after five terms and use the resulting polynomial to approximate the least solution of (*). Compute the resulting (approximate) greatest eigenvalue λ_0 , and graph the corresponding (approximate) eigenfunction.

15. Let the operator \mathcal{S} be defined by $\mathcal{S}\psi = \psi'' + x\psi$ for \mathcal{C}^2 functions ψ on $[0, 1]$.

- (a) Show that \mathcal{S} is a regular Sturm-Liouville operator.
 (b) Treating λ as a parameter, find fundamental solutions u and v of $\mathcal{S}\psi = \lambda\psi$ in the form of power-series expansions (about $t = 0$).
 (c) Let $\lambda_0, \lambda_1, \lambda_2, \dots$ be the eigenvalues of \mathcal{S} with respect to the boundary conditions $\psi(0) = \psi(1) = 0$. Find the corresponding eigenfunctions, and describe the eigenvalues as the zeros of a certain power series in λ . (cf. Problem 14c)
 (d) Compute a numerical approximation to the greatest eigenvalue λ_0 , and plot the graph of the corresponding (approximate) eigenfunction. (cf. Problem 14d)

16. Let \mathcal{S} be a regular Sturm-Liouville operator defined by

$$\mathcal{S}\psi = \frac{1}{p} ((k\psi')' + q\psi)$$

for \mathcal{C}^2 functions ψ on $[a, b]$, and consider the eigenvalue problem

$$\mathcal{S}\psi = \lambda\psi, \quad \psi(a) = \psi(b) = 0.$$

In this problem we will show that the greatest eigenvalue λ_0 satisfies the inequality

$$\lambda_0 \leq \max_{a \leq x \leq b} \frac{q(x)}{p(x)}. \quad (**)$$

Begin by supposing that λ is a number for which

$$\lambda p(x) - q(x) > 0 \quad \text{for all } x \text{ in } [a, b]$$

and writing the equation $\mathcal{S}\psi = \lambda\psi$ in the form

$$k\psi'' + k'\psi' = (\lambda p - q)\psi.$$

- Show that, if a solution ψ attains its maximum value on $[a, b]$ at x^* , then $\psi(x^*) \leq 0$. Conclude that $\psi(x) \leq 0$ for all x in $[a, b]$.
 - Show that, if a solution ψ attains its minimum value on $[a, b]$ at x^* , then $\psi(x^*) \geq 0$. Conclude that $\psi(x) \geq 0$ for all x in $[a, b]$.
 - Conclude that $\psi = 0$ is the only solution and that therefore λ cannot be an eigenvalue.
 - Conclude that, if λ is an eigenvalue, then $\lambda p(x) \leq q(x)$ for *some* x in $[a, b]$; consequently, $(**)$ is true.
17. Suppose that \mathcal{S} is any linear operator with eigenvalues λ_n , $n = 0, 1, 2, \dots$, and corresponding eigenfunctions ψ_n , $n = 0, 1, 2, \dots$; that is,

$$\mathcal{S}\psi_n = \lambda_n\psi_n, \quad n = 0, 1, 2, \dots,$$

where each $\psi_n \neq 0$. Let c be a constant, and let \mathcal{S}_c be defined by $\mathcal{S}_c\psi = \mathcal{S}\psi + c\psi$. Show that the eigenvalues of \mathcal{S}_c are $\lambda_n + c$, $n = 0, 1, 2, \dots$, and that ψ_n , $n = 0, 1, 2, \dots$, are corresponding eigenfunctions.

18. Let \mathcal{S} be a regular Sturm-Liouville operator defined by

$$\mathcal{S}\psi = \frac{1}{p} ((k\psi')' + q\psi)$$

for \mathcal{C}^2 functions ψ on $[a, b]$. Show that \mathcal{S} is symmetric and negative semidefinite (see Problem 11) with respect to the weight function p and “periodic” boundary conditions

$$\psi(a) = \psi(b) \quad \text{and} \quad k(a)\psi'(a) = k(b)\psi'(b).$$

19. Define \mathcal{S} by $\mathcal{S}\psi = \psi''$ for \mathcal{C}^2 functions ψ on $[0, \pi]$.

- a) Find the eigenvalues of \mathcal{S} with respect to the boundary conditions

$$\psi(0) = \psi(\pi) \quad \text{and} \quad \psi'(0) = \psi'(\pi).$$

- b) Show that each eigenvalue has two linearly independent eigenfunctions. (Thus part (ii) of Theorem 1 is not true for problems with periodic boundary conditions.)

11.6 Singular Sturm-Liouville Problems

Recall from the preceding section that an operator \mathcal{S} , acting on \mathcal{C}^2 functions ψ on I , is a Sturm-Liouville operator if $\mathcal{S}\psi$ can be written in the form

$$\mathcal{S}\psi = \frac{1}{p} \left((k\psi')' + q\psi \right),$$

where

- k is positive and continuously differentiable on I ;
- p is positive and continuous on I ;
- q is continuous on I .

We have seen that, if I is a closed, bounded interval, then \mathcal{S} is called a regular Sturm-Liouville operator. If, on the other hand, either

- (i) I is not bounded, or
- (ii) I is an open or half-open, bounded interval with endpoints a and b , and one of the above conditions on k , p , and q fails on $[a, b]$,

then \mathcal{S} is called a **singular Sturm-Liouville operator**. A few simple examples are:

$$\mathcal{S}\psi = (x\psi')', \quad I = (0, 1];$$

$$\mathcal{S}\psi = ((1-x^2)\psi')', \quad I = (-1, 1);$$

$$\mathcal{S}\psi = x(x^{-1}\psi')', \quad I = (0, 1];$$

$$\mathcal{S}\psi = e^{x^2}(e^{-x^2}\psi')', \quad I = (-\infty, \infty).$$

The important thing to notice about the first three of these operators is that if I were “closed up” by including the missing endpoint(s), then the coefficient k would no longer be positive and continuously differentiable on I .

A singular Sturm-Liouville eigenvalue problem consists of the differential equation

$$\mathcal{S}\psi = \lambda\psi \quad \text{on } I,$$

where \mathcal{S} is a singular Sturm-Liouville operator, together with the “usual” boundary condition at any endpoint contained in I and some suitable side condition

replacing the usual boundary condition at any “missing endpoint” of I (which may include ∞ or $-\infty$). In some cases, with appropriately chosen side conditions replacing the usual boundary conditions as necessary, the eigenvalues and eigenfunctions of a singular Sturm-Liouville eigenvalue problem will share crucial properties with the eigenvalues and eigenfunctions of regular Sturm-Liouville eigenvalue problems; namely, the spectrum is a sequence of real numbers (and therefore said to be *discrete*), and eigenfunctions comprise an orthogonal family. In other cases, however, the spectrum may fail to be discrete and, instead, forms an interval. When this happens we say the operator has a *continuous spectrum*. (There are also operators with “mixed” spectra, which contain both intervals and discrete points.)

Since our primary aim here is to generalize the notion of Fourier series, our criteria for the suitability of side conditions will be that (1) the spectrum of \mathcal{S} must be discrete, and (2) eigenfunctions corresponding to distinct eigenvalues must be orthogonal on I . Thus we require \mathcal{S} to have the same symmetry property as in the regular case; that is,

$$\int_a^b \gamma \mathcal{S}\psi p dx = \int_a^b \psi \mathcal{S}\gamma p dx$$

for all \mathcal{C}^2 functions ψ and γ on I that satisfy the boundary conditions. (Note that the integrals may be improper.) Examination of the proof of Theorem 2 in Section 11.6 reveals that what is crucial for this purpose is that side conditions be such that

$$\lim_{x \rightarrow a^+} k(x) \left(\gamma(x) \psi'(x) - \psi(x) \gamma'(x) \right) = \lim_{x \rightarrow b^-} k(x) \left(\gamma(x) \psi'(x) - \psi(x) \gamma'(x) \right) \quad (1)$$

for all \mathcal{C}^2 functions ψ and γ that satisfy the boundary conditions.

A common and important type of singular problem occurs when I is bounded and k vanishes at one or both endpoints of I . If, for example, $I = (a, b)$ and $k(x) \rightarrow 0$ as $x \rightarrow a^+$ or $x \rightarrow b^-$, then (1) is true if the quantity $\gamma \psi' - \psi \gamma'$ is bounded on (a, b) . Thus, symmetry of \mathcal{S} is guaranteed by considering only bounded functions on (a, b) with bounded derivatives, and the eigenvalue problem for \mathcal{S} becomes

$$\mathcal{S}\psi = \lambda\psi, \quad \psi \text{ and } \psi' \text{ bounded on } (a, b).$$

If $I = (a, b]$ and $k(x) \rightarrow 0$ as $x \rightarrow a^+$, then the appropriate condition at $x = b$ is the usual

$$\beta\psi(b) + (1 - \beta)\psi'(b) = 0,$$

which we augment with the condition that ψ and ψ' be bounded on $(a, b]$. Similarly, if $I = [a, b)$ and $k(x) \rightarrow 0$ as $x \rightarrow b^-$, then the appropriate condition

at $x = a$ is the usual

$$\alpha\psi(a) + (1 - \alpha)\psi'(a) = 0,$$

which we also augment with the condition that ψ and ψ' be bounded on $[a, b)$.

- **Example 1** Let \mathcal{S} be the singular Sturm-Liouville operator defined by

$$\mathcal{S}\psi = \sqrt{1-x^2} \left(\sqrt{1-x^2} \psi' \right)' = (1-x^2)\psi'' - x\psi'$$

for C^2 functions ψ on $(-1, 1)$. This operator is associated with Chebyshev's equation. (See Section 5.6.) Since

$$k(x) = \sqrt{1-x^2} = 0 \quad \text{at } x = \pm 1,$$

the role of boundary conditions is played by the requirement that ψ and ψ' be bounded on $(-1, 1)$. Thus the eigenvalue problem associated with \mathcal{S} is

$$\mathcal{S}\psi = \lambda\psi \quad \text{with } \psi \text{ and } \psi' \text{ bounded on } (-1, 1).$$

Let us now find the eigenvalues and corresponding eigenfunctions. The differential equation $\mathcal{S}\psi = \lambda\psi$ is equivalent to Chebyshev's equation

$$(1-x^2)\psi'' - x\psi' + m^2\psi = 0, \quad \text{where } m^2 = -\lambda. \quad (2)$$

When $m = 0$, any constant is a solution. So we note that $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $\psi = 1$, and henceforth we may assume that $m \neq 0$. Section 5.7, Problem 7, tells us that, for any m , Chebyshev's equation has the general solution

$$\psi = c_1 \cos(m \cos^{-1} x) + c_2 \sin(m \cos^{-1} x)$$

on $[-1, 1]$, which is easily verified. All of these solutions are bounded; so, to determine the eigenvalues, we must rely on the boundedness requirement on ψ' . So we compute the derivative,

$$\psi'(x) = \frac{m(c_1 \sin(m \cos^{-1} x) - c_2 \cos(m \cos^{-1} x))}{\sqrt{1-x^2}},$$

and note that the denominator approaches 0 as $x \rightarrow \pm 1$ from within $(-1, 1)$. As $x \rightarrow 1^-$, the quantity in the numerator approaches

$$m(c_1 \sin(0) - c_2 \cos(0)) = -m c_2;$$

therefore, ψ' can be bounded near $x = 1$ only if $c_2 = 0$. So we set $c_2 = 0$ and $c_1 = 1$ and note then that

$$\psi' = \frac{m \sin(m \cos^{-1} x)}{\sqrt{1-x^2}}.$$

In order for ψ' to be bounded, it is necessary that

$$\sin(m \cos^{-1}(\pm 1)) = 0.$$

Observing that $\sin(m \cos^{-1}(1)) = \sin(m \cdot 0) = 0$ for any m , we then look to

$$\sin(m \cos^{-1}(-1)) = \sin(m\pi) = 0,$$

which requires that m be an integer (which we may as well assume is positive). Now, an application of l'Hôpital's rule reveals that, for any positive integer m ,

$$\begin{aligned} \lim_{x \rightarrow 1^-} \psi'(x) &= \lim_{x \rightarrow 1^-} \frac{m \sin(m \cos^{-1} x)}{\sqrt{1-x^2}} \\ &= \lim_{x \rightarrow 1^-} \frac{m^2 \cos(m \cos^{-1} x) / \sqrt{1-x^2}}{x / \sqrt{1-x^2}} \\ &= \lim_{x \rightarrow 1^-} \frac{m^2 \cos(m \cos^{-1} x)}{x} \\ &= m^2 \cos(m \cdot 0) = m^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \lim_{x \rightarrow -1^+} \psi'(x) &= \lim_{x \rightarrow -1^+} \frac{m^2 \cos(m \cos^{-1} x)}{x} \\ &= -m^2 \cos(m\pi) = -m^2(-1)^m. \end{aligned}$$

So we have found that (2) has a nontrivial solution ψ on $(-1, 1)$, with ψ and ψ' bounded, precisely when m is a nonnegative integer. A nontrivial solution corresponding to any nonnegative integer m is $\psi_m = \cos(m \cos^{-1} x)$, which is, in fact, the m^{th} Chebyshev polynomial. (See Section 5.7, Problems 8 and 9.) Therefore, we conclude that the eigenvalues of \mathcal{S} are

$$\lambda_m = -m^2, \quad m = 0, 1, 2, 3, \dots,$$

and corresponding eigenfunctions are the Chebyshev polynomials

$$T_m(x) = \cos(m \cos^{-1} x), \quad m = 0, 1, 2, 3, \dots$$

In particular, we note that the eigenvalues are real, and eigenfunctions corresponding to distinct eigenvalues are orthogonal on $(-1, 1)$. (See Section 5.7.)

Problems

- Let \mathcal{S} be defined (as in Example 1) by

$$\mathcal{S}\psi = \sqrt{1-x^2} \left(\sqrt{1-x^2} \psi' \right)'$$

for \mathcal{C}^2 functions ψ on $[0, 1)$. Find the eigenvalues and corresponding eigenfunctions of \mathcal{S} with respect to the conditions $\psi(0) = 0$ and ψ and ψ' bounded.

2. Let \mathcal{S} be defined as in Problem 1. Find the eigenvalues and corresponding eigenfunctions of \mathcal{S} with respect to the conditions $\psi'(0) = 0$ and ψ and ψ' bounded.
3. Let the operator \mathcal{S} be defined by

$$\mathcal{S}\psi = (1 - x^2)\psi'' - 2x\psi',$$

for \mathcal{C}^2 functions ψ on $(-1, 1)$, and consider the eigenvalue problem

$$\mathcal{S}\psi = \lambda\psi \text{ with } \psi \text{ and } \psi' \text{ bounded on } (-1, 1).$$

The operator \mathcal{S} is associated with Legendre's equation

$$(1 - x^2)\psi'' - 2x\psi' + m(m + 1)\psi = 0.$$

(See Section 5.7, Problems 1 and 2.)

- (a) Show that \mathcal{S} is a singular Sturm-Liouville operator. (*Hint*: Let $k = 1 - x^2$.)
 - (b) Show that, for each nonnegative integer m , $\lambda_m = -m(m + 1)$ is an eigenvalue of \mathcal{S} with corresponding eigenfunction P_m , where P_0, P_1, P_2, \dots are the Legendre polynomials.
 - (c) Give an informal argument that the derivative of every nonterminating power-series solution of $\mathcal{S}\psi = \lambda\psi$ approaches $\pm\infty$ as $x \rightarrow -1^+$ or $x \rightarrow 1^-$. Conclude that \mathcal{S} has no eigenvalues other than those described in part (b).
4. Define \mathcal{S} by $\mathcal{S}\psi = \psi''$ for \mathcal{C}^2 functions ψ on $[0, \infty)$. Find the eigenvalues and corresponding eigenfunctions of \mathcal{S} with respect to the conditions $\psi(0) = 0$ and ψ, ψ' bounded on $[0, \infty)$.
 5. Let \mathcal{S} be defined by $\mathcal{S}\psi = x^2(x^2\psi')'$ for \mathcal{C}^2 functions on $(0, 1/\pi]$.
 - (a) Show that, if $y(t) = \psi(x)$, where $t = 1/x$, then $y''(t) = \mathcal{S}\psi(x)$. Conclude that ψ is a solution of

$$\mathcal{S}\psi = \lambda\psi \quad \text{on } (0, 1/\pi], \quad \psi(1/\pi) = 0, \tag{*}$$

if and only if $y(x) = \psi(1/x)$ is a solution of

$$y'' = \lambda y \quad \text{on } [\pi, \infty), \quad y(\pi) = 0. \tag{**}$$

- (b) Show that (**)—and therefore (*)—has a nontrivial solution for any real λ . State a corresponding nontrivial solution ψ of (*) for each λ .
- (c) Show that if $\lambda \geq 0$, then the corresponding nontrivial solution of (*) is unbounded on $(0, 1/\pi]$. Then show that if $\lambda < 0$, then the corresponding nontrivial solution of (*) is bounded but has an unbounded derivative on $(0, 1/\pi]$.

11.7 Eigenfunction Expansions

The essence of Theorem 1 in Section 11.5 is that the eigenvalues and eigenfunctions associated with any regular Sturm-Liouville operator (with respect to

suitable boundary conditions) share certain crucial properties with the familiar eigenvalues and eigenfunctions of the operator \mathcal{S} defined by $\mathcal{S}X = X''$, which are (cf. Example 1 in Section 11.5)

$$\lambda_n = -\frac{n^2\pi^2}{\ell^2}, \quad X_n = \sin\left(\frac{n\pi x}{\ell}\right), \quad n = 1, 2, 3, \dots$$

Those properties—namely, that the spectrum is a sequence of real numbers and that eigenfunctions comprise a sequence of orthogonal functions on $[0, \ell]$ —were the crucial tools in the development of (trigonometric) Fourier Series representations of the form

$$\varphi_{\sim}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right),$$

in which the coefficients are given by

$$b_n = \frac{\int_0^{\ell} \varphi(x) \sin\left(\frac{n\pi x}{\ell}\right) dx}{\int_0^{\ell} \sin^2\left(\frac{n\pi x}{\ell}\right) dx}, \quad n = 1, 2, 3, \dots$$

These observations motivate the following definition.

Definition. Let $\psi_0, \psi_1, \psi_2, \dots$ be the orthogonal sequence of eigenfunctions on $[a, b]$ of a regular Sturm-Liouville operator \mathcal{S} with weight function p . Given a function φ defined on $[a, b]$, the **eigenfunction expansion** of φ relative to \mathcal{S} is the series defined by

$$\Phi(x) = \sum_{n=0}^{\infty} c_n \psi_n(x), \quad \text{where } c_n = \frac{\int_a^b \varphi \psi_n p dx}{\int_a^b \psi_n^2 p dx}, \quad n = 0, 1, 2, \dots \quad (1)$$

Note that, if the eigenfunctions $\psi_0, \psi_1, \psi_2, \dots$ are scaled so that

$$\int_a^b \psi_n^2 p dx = 1$$

(cf. Example 2), then the formula for c_n in (1) becomes simply

$$c_n = \int_a^b \varphi \psi_n p dx, \quad n = 0, 1, 2, \dots$$

When scaled in this way, each ψ_n is said to be *normalized*, and the sequence $\psi_0, \psi_1, \psi_2, \dots$ is called an *orthonormal* sequence.

The following theorem, a generalization of Theorem 1 in Section 11.3, describes convergence properties of eigenfunction expansions. Note that these are merely analogues of convergence properties of trigonometric Fourier series.

THEOREM 1 Let \mathcal{S} be a regular Sturm-Liouville operator defined for \mathcal{C}^2 functions on $[a, b]$. Let $\lambda_0, \lambda_1, \lambda_2, \dots$ be the eigenvalues of \mathcal{S} with corresponding

eigenfunctions $\psi_0, \psi_1, \psi_2, \dots$. Suppose that φ and φ' are piecewise continuous functions on $[a, b]$, and let Φ be the eigenfunction expansion of φ relative to \mathcal{S} given by (1). Then for each x in (a, b) ,

$$\Phi(x) = \frac{1}{2} \left(\lim_{\xi \rightarrow x^-} \varphi(\xi) + \lim_{\xi \rightarrow x^+} \varphi(\xi) \right).$$

In particular, $\Phi(x) = \varphi(x)$ at each x in (a, b) where φ is continuous.

- **Example 1** Consider again the operator \mathcal{S} in Example 3 of Section 11.5, defined for \mathcal{C}^2 functions ψ on $[0, 1]$ by

$$\mathcal{S}\psi = \psi'' + 4\psi' + \psi.$$

Recall that \mathcal{S} is a regular Sturm-Liouville operator, since

$$\mathcal{S}\psi = e^{-4x} \left((e^{4x} \psi')' + e^{4x} \psi \right).$$

Recall also that, with respect to the boundary conditions $\psi(0) = \psi(1) = 0$, the eigenvalues of \mathcal{S} are

$$\lambda_n = 3 - n^2 \pi^2, \quad n = 1, 2, 3, \dots,$$

with corresponding eigenfunctions given by

$$\psi_n(x) = A_n e^{-2x} \sin(n\pi x), \quad n = 1, 2, 3, \dots$$

Since the relevant weight function here is $p(x) = e^{4x}$, normalized eigenfunctions are obtained with

$$\begin{aligned} A_n &= \left(\int_0^1 (e^{-2x} \sin(n\pi x))^2 e^{4x} dx \right)^{-1/2} \\ &= \left(\int_0^1 \sin^2(n\pi x) dx \right)^{-1/2} = \sqrt{2}, \end{aligned}$$

resulting in the orthonormal sequence of eigenfunctions

$$\psi_n(x) = \sqrt{2} e^{-2x} \sin(n\pi x), \quad n = 1, 2, 3, \dots$$

Thus the coefficients in (1) for the eigenfunction expansion of φ take the form

$$\begin{aligned} c_n &= \int_0^1 \varphi(x) \sqrt{2} e^{-2x} \sin(n\pi x) e^{4x} dx \\ &= \sqrt{2} \int_0^1 \varphi(x) \sin(n\pi x) e^{2x} dx, \quad n = 1, 2, 3, \dots \end{aligned}$$

For the sake of illustration, let's consider the eigenfunction expansion of the function φ defined by

$$\varphi(x) = \begin{cases} 0, & \text{if } 0 \leq x < \frac{1}{2}, \\ -\sin(2\pi x), & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$

for which the appropriate coefficients are

$$c_n = -\sqrt{2} \int_{\frac{1}{2}}^1 \sin(2\pi x) \sin(n\pi x) e^{2x} dx, \quad n = 1, 2, 3, \dots$$

With computer assistance, we find that a “closed form” for c_n is

$$c_n = -2\sqrt{2} e \pi \frac{4n\pi ((-1)^n e + \cos(\frac{n\pi}{2})) + ((n^2 - 4)\pi^2 - 4) \sin(\frac{n\pi}{2})}{16 + 8(n^2 + 4)\pi^2 + (n^2 - 4)^2 \pi^4}. \quad (2)$$

With these coefficients, and noting that $c_0 = 0$, we have the eigenfunction expansion

$$\Phi(x) = \sum_{n=1}^{\infty} c_n \sqrt{2} e^{-2x} \sin(n\pi x).$$

Since φ is continuous on $[0, 1]$ and φ' is piecewise continuous on $[0, 1]$, Theorem 1 tells us that

$$\Phi(x) = \varphi(x) \quad \text{for all } x \text{ in } (0, 1).$$

Inspection of the series itself reveals that $\Phi(x) = \varphi(x)$ at $x = 0$ and 1 as well. The convergence of the series is illustrated in Figure 1, where the four plots shown are graphs of partial sums

$$\sum_{n=1}^N c_n \sqrt{2} e^{-2x} \sin(n\pi x)$$

with $N = 3, 6, 12$, and 24, respectively. ◇

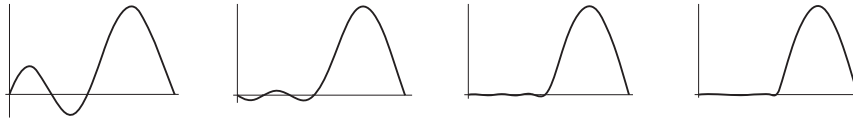


Figure 1

■ The Singular Case

If \mathcal{S} is a singular Sturm-Liouville operator defined for \mathcal{C}^2 functions on $(a, b]$, $[a, b)$, or (a, b) , then Theorem 1 remains true, *provided* that the spectrum of \mathcal{S} is a sequence and the corresponding sequence of eigenfunctions is an orthogonal sequence.

- **Example 2** Consider again the operator from Example 1 of Section 11.6:

$$\mathcal{S}\psi = \sqrt{1-x^2} \left(\sqrt{1-x^2} \psi' \right)'$$

for \mathcal{C}^2 functions ψ on $(-1, 1)$. Recall that the eigenvalues of \mathcal{S} are

$$\lambda_n = -n^2, \quad n = 0, 1, 2, 3, \dots,$$

and corresponding eigenfunctions are the Chebyshev polynomials T_n , $n = 0, 1, 2, 3, \dots$, which are

$$1, \quad x, \quad 2x^2 - 1, \quad 4x^3 - 3x, \quad 8x^4 - 8x^2 + 1, \quad 16x^5 - 20x^3 + 5x, \quad \dots$$

Computer-aided computations reveal that

$$\int_{-1}^1 T_0(x)^2 \frac{dx}{\sqrt{1-x^2}} = \pi$$

and

$$\int_{-1}^1 T_n(x)^2 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}, \quad n = 1, 2, 3, \dots;$$

thus the coefficients in (1) are

$$c_0 = \frac{1}{\pi} \int_{-1}^1 \varphi(x) \frac{dx}{\sqrt{1-x^2}}$$

and

$$c_n = \frac{2}{\pi} \int_{-1}^1 \varphi(x) T_n(x) \frac{dx}{\sqrt{1-x^2}}, \quad n = 1, 2, 3, \dots$$

For the sake of illustration, let's find the eigenfunction expansion of $\varphi(x) = \sin \pi x$ on $[-1, 1]$. The coefficients in (1) are

$$c_0 = \frac{1}{\pi} \int_{-1}^1 \sin \pi x \frac{dx}{\sqrt{1-x^2}}$$

and

$$c_n = \frac{2}{\pi} \int_{-1}^1 \sin \pi x T_n(x) \frac{dx}{\sqrt{1-x^2}}, \quad n = 1, 2, 3, \dots$$

When n is even, the integrand is an odd function; hence $c_n = 0$ when n is even. Therefore, by Theorem 1,

$$\sin \pi x = \sum_{n=0}^{\infty} c_{2n+1} T_{2n+1}(x) \quad \text{for } -1 < x < 1.$$

Numerical evidence shows that the series agrees with $\sin \pi x$ at $x = \pm 1$ as well. The first four nonzero coefficients, computed numerically and rounded to six

significant digits, are

$$c_1 = 0.569231, \quad c_3 = -0.666917, \quad c_5 = 0.104282, \quad c_7 = -0.00684063, \quad \dots$$

The degree-five partial sum (for example) of the resulting generalized Fourier series is

$$\begin{aligned} 0.569231 T_1(x) - 0.666917 T_3(x) + 0.104282 T_5(x) \\ = 3.09139 x - 4.75331 x^3 + 1.66852 x^5. \end{aligned}$$

(The reader is invited to plot this polynomial on $[-1, 1]$ along with $\sin \pi x$ for comparison.) \diamond

■ Initial-Value Problems

We complete our current discussion by returning to initial-value problems of the form

$$\left. \begin{aligned} w_t - \mathcal{S}w &= 0 \quad \text{for } x \text{ in } I \text{ and } t > 0, \\ w(0, x) &= \varphi(x) \quad \text{for } x \text{ in } I, \\ w &\text{ satisfies suitable side conditions on } I \text{ for } t > 0, \end{aligned} \right\} \quad (3)$$

where \mathcal{S} is a Sturm-Liouville operator defined for \mathcal{C}^2 functions on the interval I . We assume that the side conditions guarantee that the spectrum of \mathcal{S} is a divergent sequence of real numbers $\lambda_0, \lambda_1, \lambda_2, \dots$, and corresponding eigenfunctions $\psi_0, \psi_1, \psi_2, \dots$, form an orthogonal sequence. We also assume that φ and φ' are piecewise continuous on the closed interval consisting of all points in I together with any finite endpoints of I . (For instance, if $I = (0, 1)$, we assume that φ and φ' are piecewise continuous on $[0, 1]$.) Then φ has the eigenfunction expansion

$$\Phi(x) = \sum_{n=0}^{\infty} c_n \psi_n(x),$$

where the coefficients are as given in (1), and the solution of the initial-value problem (3) for $t > 0$ is given by

$$w(t, x) = \sum_{n=0}^{\infty} c_n e^{\lambda_n t} \psi_n(x).$$

- **Example 3** Consider the problem

$$\begin{aligned} w_t - (w_{xx} + 4w_x + w) &= 0 \quad \text{for } -1 \leq x \leq 1 \text{ and } t > 0, \\ w(0, x) &= \begin{cases} 0, & \text{if } 0 \leq x < \frac{1}{2}, \\ -\sin(2\pi x), & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases} \\ w(t, 0) = w(t, 1) &= 0 \quad \text{for } t > 0, \end{aligned}$$

which involves the regular Sturm-Liouville operator \mathcal{S} from Example 1 in this section and Example 3 in Section 11.5. In Example 1 it was shown that the

initial data have the eigenfunction expansion

$$\Phi(x) = \sum_{n=1}^{\infty} c_n \sqrt{2} e^{-2x} \sin(n\pi x)$$

where c_n is given by formula (2). (See Example 1.) Since the eigenvalues of \mathcal{S} are

$$\lambda_n = 3 - n^2\pi^2, \quad n = 1, 2, 3, \dots,$$

it follows that the solution is

$$w(t, x) = \sum_{n=1}^{\infty} c_n \sqrt{2} e^{(3-n^2\pi^2)t} e^{-2x} \sin(n\pi x),$$

Snapshots of the solution at $t = 0.001, 0.1, 0.025, 0.05, 0.1$, and 0.25 are shown in Figure 2. (Can you explain why the “wave” moves to the left?)

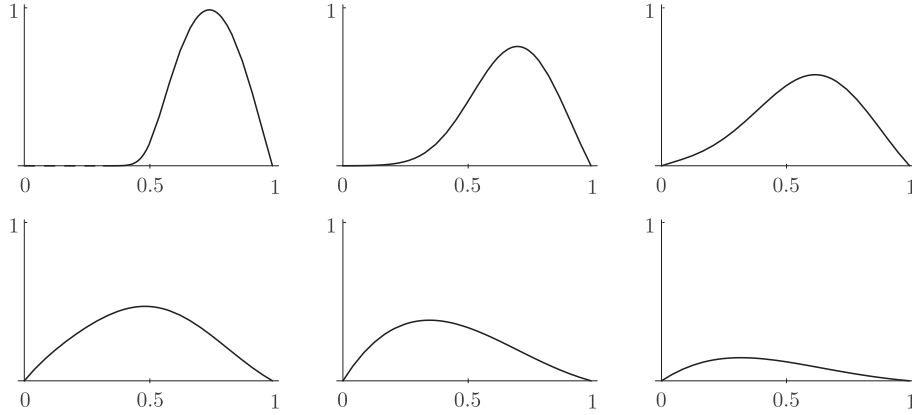


Figure 2

• **Example 4** Consider the problem

$$\begin{aligned} w_t - ((1-x^2)w_{xx} - xw_x) &= 0 \quad \text{for } -1 < x < 1 \text{ and } t > 0, \\ w(0, x) &= -16x + 185x^3 - 394x^5 + 225x^7 \quad \text{for } -1 < x < 1, \\ w(t, x), w_x(t, x) &\text{ bounded on } (-1, 1) \text{ for each } t > 0, \end{aligned}$$

which involves the singular Sturm-Liouville operator \mathcal{S} from Example 2. Recall that the eigenvalues of \mathcal{S} are $\lambda_n = -n^2$, $n = 0, 1, 2, \dots$ and corresponding eigenfunctions are the Chebyshev polynomials $T_n(x)$, $n = 0, 1, 2, \dots$. The initial data have the terminating eigenfunction expansion

$$w(0, x) = -\frac{1}{64} (29T_1(x) + 195T_3(x) + T_4(x) - 225T_7(x)).$$

Therefore, the solution for $t \geq 0$ is

$$w(t, x) = -\frac{1}{64} (29 e^{-t} T_1(x) + 195 e^{-9t} T_3(x) + e^{-16t} T_4(x) - 225 e^{-49t} T_7(x)).$$

Figure 3 shows snapshots of the solution at $t = 0, 0.02, 0.03, 0.5, 0.2$, and 0.5 .

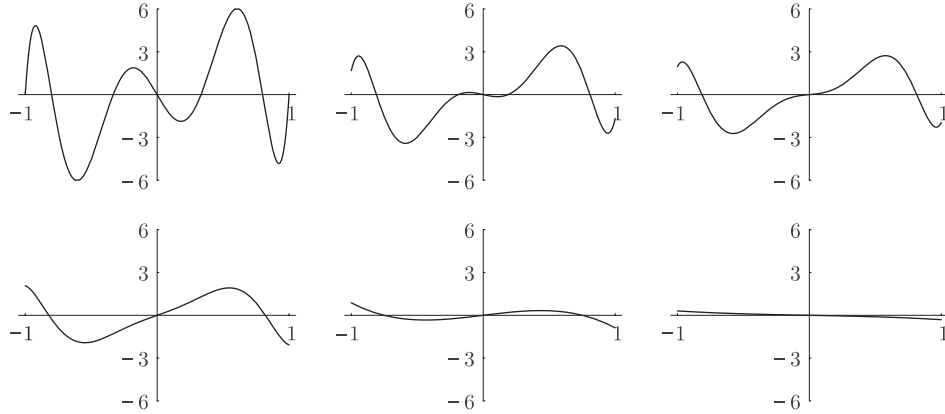


Figure 3

Problems

- Let \mathcal{S} be the operator defined for \mathcal{C}^2 functions ψ on $[0, \pi]$ by

$$\mathcal{S}\psi = e^{2x}((e^{-2x}\psi')' + e^{-2x}\psi) = \psi'' - 2\psi' + \psi.$$

- Find the eigenvalues and corresponding *normalized* eigenfunctions of \mathcal{S} with respect to the boundary conditions $\psi(0) = \psi(\pi) = 0$.
 - Find the eigenfunction expansion of $\varphi(x) = e^x$.
- Rework Problem 1 with the boundary conditions $\psi'(0) = \psi'(\pi) = 0$.
 - Rework Problem 1 with the boundary conditions $\psi'(0) = \psi(\pi) = 0$.
 - Let \mathcal{S} be the operator defined for \mathcal{C}^2 functions ψ on $[0, \pi]$ by

$$\mathcal{S}\psi = e^{-4x}(e^{4x}\psi')' = \psi'' + 4\psi'.$$

- Find the eigenvalues and corresponding *normalized* eigenfunctions of \mathcal{S} with respect to the boundary conditions $\psi(0) = \psi(\pi) = 0$.
 - Find the eigenfunction expansion of $\varphi(x) = e^{-2x}$.
- Rework Problem 4 with the boundary conditions $\psi'(0) = \psi'(\pi) = 0$.
 - Rework Problem 4 with the boundary conditions $\psi'(0) = \psi(\pi) = 0$.
 - Let \mathcal{S} be the operator defined for \mathcal{C}^2 functions ψ on $[0, 1]$ by

$$\mathcal{S}\psi = e^{-2x}(e^{2x}\psi')' = \psi'' + 2\psi'.$$

- (a) Find the eigenvalues and corresponding *normalized* eigenfunctions of \mathcal{S} with respect to the boundary conditions $\psi(0) = \psi(1) = 0$.
- (b) Find (numerically) the first four terms in the eigenfunction expansion of $\varphi(x) = 4x(1-x)$. Plot the resulting approximation of φ along with φ on $[0, 1]$.

In each of Problems 8 through 10 find the eigenfunction expansion of the polynomial relative to the operator \mathcal{S} in Example 2. (In other words, express the polynomial as a sum of Chebyshev polynomials. *Hint:* This requires no integration!)

8. $\varphi(x) = x^3$ 9. $\varphi(x) = x^3 - x^2 + 1$ 10. $\varphi(x) = x^5 + x$

11. Let \mathcal{S} be the operator defined by $\mathcal{S}\psi = \psi''$ for \mathcal{C}^2 functions ψ on $[0, \pi]$, and consider the initial-value problem

$$\begin{aligned} w_t - \mathcal{S}w &= 0 & \text{for } 0 < x < \pi, \ t > 0, \\ w(0, x) &= 1 & \text{for } 0 \leq x \leq \pi, \\ w(t, 0) &= w(t, \pi) + w_x(t, \pi) = 0 & \text{for } t > 0. \end{aligned}$$

- (a) Find the first five eigenvalues $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ (rounded to three significant digits) and corresponding eigenfunctions $\psi_0, \psi_1, \psi_2, \psi_3, \psi_4$ of \mathcal{S} with respect to the boundary conditions $\psi(0) = \psi(\pi) + \psi'(\pi) = 0$. Normalize each ψ_n so that $\int_0^\pi \psi_n^2 dx = 1$.
 - (b) Find the first five terms of the eigenfunction expansion of $\varphi(x) = 1$, and write the resulting approximation of the solution w of the initial-value problem. Then plot the graphs of $w(0.1, x)$, $w(1, x)$, $w(2, x)$, and $w(3, x)$.
12. Rework Problem 11 with the boundary conditions

$$w_x(t, 0) = w(t, \pi) + w_x(t, \pi) = 0$$

(and $\psi'(0) = \psi(\pi) + \psi'(\pi) = 0$). How and why does w behave differently?

13. Let \mathcal{S} be the operator defined by $\mathcal{S}\psi = e^{2x}(e^{-2x}\psi')'$ for \mathcal{C}^2 functions ψ on $[0, \pi]$, and consider the initial-value problem

$$\begin{aligned} w_t - \mathcal{S}w &= 0 & \text{for } 0 < x < \pi, \ t > 0, \\ w(0, x) &= 1 & \text{for } 0 \leq x \leq \pi, \\ w(t, 0) &= w_x(t, \pi) = 0 & \text{for } t > 0. \end{aligned}$$

- (a) Find the first five eigenvalues $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ (rounded to three significant digits) and the corresponding eigenfunctions $\psi_0, \psi_1, \psi_2, \psi_3, \psi_4$ of \mathcal{S} with respect to the boundary conditions $\psi(0) = \psi(\pi) + \psi'(\pi) = 0$. Normalize each ψ_n so that $\int_0^\pi \psi_n^2 e^{-2x} dx = 1$.
 - (b) Find the first five terms of the eigenfunction expansion of $\varphi(x) = 1$, and write the resulting approximation of the solution w of the initial-value problem. Then plot the graphs of $w(0.2, x)$, $w(1, x)$, $w(2, x)$, and $w(3, x)$.
14. Rework Problem 13 with the boundary conditions

$$w_x(t, 0) = w(t, \pi) = 0$$

(and $\psi'(0) = \psi(\pi) = 0$). How and why does w behave differently?

15. Let \mathcal{S} be the operator defined by $\mathcal{S}\psi = \psi'' + \psi$ for \mathcal{C}^2 functions ψ on $[0, \pi]$, and consider the initial-value problem

$$\begin{aligned} w_t - \mathcal{S}w &= 0 & \text{for } 0 < x < \pi, \ t > 0, \\ w(0, x) &= x & \text{for } 0 \leq x \leq \pi, \\ w(t, 0) = w(t, \pi) &= 0 & \text{for } t > 0. \end{aligned}$$

- (a) Find the eigenvalues $\lambda_0, \lambda_1, \lambda_2, \dots$ and their corresponding eigenfunctions $\psi_0, \psi_1, \psi_2, \dots$ of \mathcal{S} with respect to the boundary conditions $\psi(0) = \psi(\pi) = 0$. Normalize each ψ_n so that $\int_0^\pi \psi_n^2 dx = 1$.
- (b) Find the eigenfunction expansion of $\varphi(x) = x$, and use it to write the solution w of the initial-value problem. Then use the first five terms to plot graphs of $w(0.1, x)$, $w(0.25, x)$, $w(0.5, x)$, and $w(1, x)$. What is $\lim_{t \rightarrow \infty} w(t, x)$?
16. Rework Problem 15 with the operator \mathcal{S} defined by $\mathcal{S}\psi = \psi'' + 2\psi$. Explain the difference in the behavior of w .
17. Let \mathcal{S} be the operator defined by $\mathcal{S}\psi = ((1 - x^2)\psi')'$ for \mathcal{C}^2 functions ψ on $(-1, 1)$. With respect to the side conditions that ψ and ψ' be bounded on $(-1, 1)$, the eigenvalues of \mathcal{S} are

$$\lambda_n = -n(n+1), \quad n = 0, 1, 2, 3, \dots,$$

and corresponding eigenfunctions $\psi_0, \psi_1, \psi_2, \dots$ are the Legendre polynomials

$$1, \ x, \ \frac{1}{2}(3x^2 - 1), \ \frac{1}{2}(5x^3 - 3x), \ \frac{1}{8}(3 - 30x^2 + 35x^4), \dots$$

- (a) Find the eigenfunction expansion of $\varphi(x) = x^4 - x^3 + x$ relative to \mathcal{S} .
- (b) Write down the solution w of the initial-value problem
- $$\begin{aligned} w_t - \mathcal{S}w &= 0 & \text{for } -1 < x < 1, \ t > 0, \\ w(0, x) &= x^4 - x^3 + x & \text{for } -1 < x < 1, \\ w(t, x), \ w_x(t, x) &\text{ bounded on } (-1, 1) & \text{for } t > 0. \end{aligned}$$
- (c) Plot the graphs of $w(0, x)$, $w(0.1, x)$, $w(0.5, x)$, and $w(1, x)$.

18. Let \mathcal{S} be the operator from Example 2, and consider the initial-value problem

$$\begin{aligned} w_t - \mathcal{S}w &= 0 & \text{for } -1 < x < 1, \ t > 0, \\ w(0, x) &= \sin \pi x & \text{for } -1 < x < 1, \\ w(t, x), \ w_x(t, x) &\text{ bounded on } (-1, 1) & \text{for } t > 0. \end{aligned}$$

Find, and plot on $[-1, 1]$, the degree-five partial sum of the eigenfunction expansion of $w(x, t)$ at each of $t = 0, 0.1$, and 0.5 .

19. Let $\ell = e^2 - 1$, and let \mathcal{S} be defined for \mathcal{C}^2 functions ψ on $[0, \ell]$ by

$$\mathcal{S}\psi = ((1+x)^2\psi')'.$$

- (a) Verify that the general solution of $\mathcal{S}\psi = \lambda\psi$ on $[0, \ell]$ is given by

$$\psi = \frac{1}{\sqrt{1+x}} \left(c_1 \cos \left(\sqrt{-\lambda - 1/4} \ln(1+x) \right) + c_2 \sin \left(\sqrt{-\lambda - 1/4} \ln(1+x) \right) \right),$$

if $\lambda \leq -1/4$; otherwise (if $\lambda > -1/4$) it is given by

$$\psi = \frac{1}{\sqrt{1+x}} \left(c_1 \cosh \left(\sqrt{\lambda + 1/4} \ln(1+x) \right) + c_2 \sinh \left(\sqrt{\lambda + 1/4} \ln(1+x) \right) \right).$$

- (b) Find the eigenvalues and corresponding eigenfunctions of \mathcal{S} with respect to the boundary conditions

$$\psi(0) = 0, \quad \psi(\ell) = 0.$$

Normalize each ψ_n so that $\int_0^\ell \psi_n^2 dx = 1$.

- (c) Write down the solution w of the initial-value problem

$$\begin{aligned} w_t - \mathcal{S}w &= 0 && \text{for } 0 \leq x \leq \ell, \ t > 0, \\ w(0, x) &= \psi_0(x) - \frac{1}{3} \psi_2(x) + \frac{1}{2} \psi_4(x) && \text{for } 0 \leq x \leq \ell, \\ w(t, 0) &= w(t, \ell) = 0 && \text{for } t > 0. \end{aligned}$$

- d) Plot the graph of $w(t, x)$ on $[0, \ell]$ for each of $t = 0, 0.02, 0.06, 0.2$, and 0.5 .

20. Rework Problem 19 with the boundary conditions $\psi'(0) = \psi'(\ell) = 0$ (and $w_x(t, 0) = w_x(t, \ell) = 0$). How and why does w behave differently?