Partial Determinant

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Abstract. The partial determinant of a matrix has the geometrical interpretation of the volume of the parallelotope of an $n \times m$ matrix, also the partial determinant is equal to the determinant when the vectors x_1 to x_n are a basis. The motivation of creating a partial determinant was to find a more intuitive and easy method for calculating the determinant because the determinant which has a very intuitive geometric interpretation has rather a very obscure and complicated algebraic formula except in the case of 2 and 3 dimension, that is in function of the magnitude and the angle between the vectors which I extend to n dimensions . . .

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First of all, I lack mathematic experience so if I happen to have something bogus, sorry, I believe I don't, in any case, if is there a mistake the theorem is a postulate that I believe seem easy to prove. I wasn't able to find such a thing like the one I expose here, maybe there is, in any case, this way of thinking about the determinant I found it more intuitive than others and should be teased.

1 Definition of the Partial Determinant

The partial determinant will be denoted by $pdet(x_1, x_2, ...x_n)$ where x_1 trough x_n are the columns of a matrix A. The partial determinan is define by the next properties:

1. $pdet(x_1, x_2, x_3, ...x_n) = |x_1|\ sin(\theta)\ pdet(x_2, x_3, ...x_n)$
2. $pdet(x) = |x|$
3. $pdet(x_1, x_2) = |x_1||x_2|sin(\theta)$
4. $\forall A\ if\ (A\,A^T)^{-1}\ \exists \Rightarrow pdet(x_1, x_2, x_3, ...x_n)\ \exists$
5. $if\ pdet(x_1, x_2, x_3, ...x_n) = det(x_1, x_2, x_3, ...x_n) \iff A$ is a basis

Here all the $x's$ are vectors and $|x|$ means the magnitude of the vector.

The angle θ is the internal angle between the vector x_1 and the subspace span by $(x_2, x_3, ...x_n)$. I show later a way to compute de angle.

By this definition, we can use the properties 1 to calculate the determinant of A when A is invertible (is the same as the columns of A are a basis).

Provided the property 2 the property 3 is a consequence of 1 and 2, anyway I show it because is well know that $det(x_1, x_2)$ is indeed $|x_1||x_2|sin(\theta)$ and in fact there is also a similar definition for $n = 3$ from where my inspiration of this simplification arise from. This definition simplifies the calculation for matrices of size $n = 2$.

2 Calculate the angle between a vector and a subspace

In order to calculate the partial determinant we first need to calculate the angle between a vector and the subspace of the remaining vectors, to that we will make use of the projection of the vector x_1 onto the subspace $x_2, x_3, ...x_n$:

$$p = (A^T A)^{-1} A x_1$$

A: is the matrix containing $x_2, x_3, ...x_n$
p: is the projection of x_1 in A

Now we can calculate the Normal vector n that is equal to the error vector e times some constant c:

$$e = x_1 - p$$
$$n = ce$$

Knowing this the angle θ is the inner angle of x_1 and n:

$$\theta = arccos(\frac{x_1 * n}{|x_1||n|})$$

3 Proofs

Corollary 1. *The pdet($x_1, x_2, x_3, ...x_n$) is equal to det($x_1, x_2, x_3, ...x_n$) iff $x_1, x_2, x_3, ...x_n$ are a basis.*

Theorem 1. *The pdet($x_1, x_2, x_3, ...x_n$) is equal to the n-volume of the parallelotope given by the independent vectors $x_1, x_2, x_3, ...x_n$.*

$$P(n) : pdet(x_1, x_2, ...x_n) = Vol_n(x_1, x_2, ...x_n) \tag{1}$$

Vol_n is the volume of the parallelotope given by the vectors $x_1, x_2, ...x_n$

Lemma 1. *The n-volume of a n-orthotope (n dimension rectangle) is given by the (n-1)-volume of the n-orthotope times the longitude of the n'th side.* $Vol_n(l_1, l_2, ...l_n) = l_1 \times Vol_{n-1}(l_2, l_3, ...l_n)$

Proof. Lema 1

$$Vol_n = \int_0^{l_1} Vol_{n-1} dx \tag{2}$$

Where x is in the direction of l_1, Vol_{n-1} is the volume of the n-orthotope formed by the sides l_2 up to l_n. Since Vol_{n-1} do not deppend on x.

$$Vol_n = Vol_{n-1} x|_0^{l_1}$$
$$Vol_n = Vol_{n-1}\, l_1$$

Q.E.D.

Proof. Theorem 1 The volume of the n-Parallelotope (n dimension parallepiped) is given by:

$$Vol_n = \int_0^{|x_1|sin(\theta)} Vol_{n-1}dx$$

$|x_1|sin(\theta)$ is the longitude of component of x_1 in the direction of the normal vector to $x_2, x_3, ...x_n$ so that we can use the $lema1$, so this becomes the n-volume of an n-orthotope

$$Vol_n = |x_1|sin(\theta)Vol_{n-1}$$

Using matemtical induction on (5) ($P(n) : pdet(x_1, x_2, ...x_n) = Vol_n(x_1, x_2, ...x_n)$) we have that:

$$\begin{aligned} P(2) : pdet(x_1, x_2) &= |x_1||x_2|sin(\theta) \\ P(n) : pdet(x_1, x_2, ...x_n) &= Vol_n(x_1, x_2, ...x_n) \\ P(n+1) : pdet(x_1, x_2, ...x_n, x_{n+1}) &= Vol_n(x_1, x_2, ...x_n, x_{n+1}) \Rightarrow \\ &= |x_1|sin(\theta)Vol_n(x_1, x_2, ...x_n) \\ By\, P(n) \qquad &= |x_1|sin(\theta)pdet(x_1, x_2, ...x_n) \end{aligned}$$

Q.E.D.

Note here that the vectors for the n-Parallelotope do not need to be a basis for the 'pdet' to work with them since if we have for example two vectors a = (a_1, a_2) and b = (b_1, b_2), and calculate the det(a, b) that would be equal to de area, but in other situation that would not be possible, if a' = (a_1', a_2', a_3') and b' = (b_1', b_2', b_3') given that $||a'|| = ||a||$ and $||b'|| = ||b||$ where the angle between a and b, and the angle between a' and b' are the same they will have the same area even so the det(a,b) would not be defined.

Proof. Corollary 1 Using the Theorem 1 is very clear why the det = pdet when the vectors form a basis. The determinant is the signed volume of a n-Parallelotope formed by the vectors $x_2, x_3, ...x_n$ only when the matrix A is n×n and the partial determinant is the volume of any matrix A by the Theorem 1.

References

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2. https://ocw.mit.edu/
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