Algebra Through Problem Solving

by

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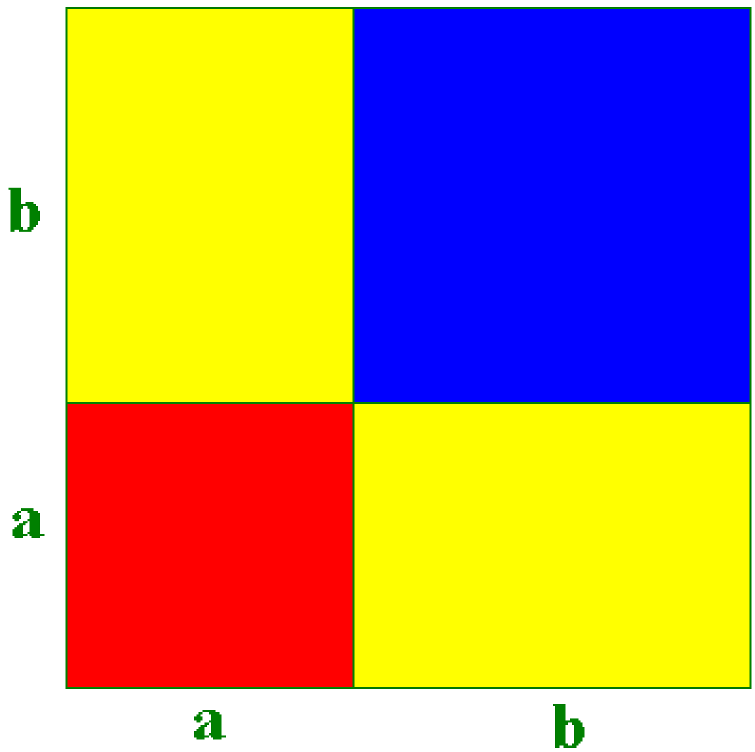

$$(a + b)^2 = ???$$

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Chapter 1

THE PASCAL TRIANGLE

In modern mathematics, more and more stress is placed on the context in which statements are true. In elementary mathematics this generally means an emphasis on a clear understanding of which number systems possess certain properties. We begin, then, by describing briefly the number systems with which we will be concerned. These number systems have developed through successive enlargements of previous systems.

At one time a "number" meant one of the **natural numbers**: 1,2,3,4,5,6,... . The next numbers to be introduced were the fractions:1/2, 1/3, 2/3, 1/4, 3/4, 1/5,..., and later the set of numbers was expanded to include zero and the negative integers and fractions. The number system consisting of zero and the positive and negative integers and fractions is called the system of **rational numbers**, the word "rational" being used to indicate that the numbers are ratios of integers. The integers themselves can be thought of as ratios of integers since 1 = 1/1, -1 = -1/1, 2 = 2/1, -2 = -2/1, 3 = 3/1, etc.

The need to enlarge the rational number system became evident when mathematicians proved that certain constructible lengths, such as the length $\sqrt{2}$ of a diagonal of a unit square, *cannot* be expressed as rational numbers. The system of **real numbers** then came into use. The real numbers include all the natural numbers; all the fractions; numbers such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{(2+\sqrt{6})/3}$ which represent constructible lengths; numbers such as $\sqrt[3]{2}$ and π which do not represent lengths constructible from a given unit with compass and straightedge; and the negatives of all these numbers. Modern technology and science make great use of a still larger number system, called the **complex numbers**, consisting of the numbers of the form $a + bi$ with a and b real numbers and $i^2 = -1$.

Our first topic is the Pascal Triangle, an infinite array of natural numbers. We begin by considering expansions of the powers $(a + b)^n$ of a sum of two terms. Clearly, $(a + b)^2 = (a + b)(a + b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$. Then $(a + b)^3 = (a + b)^2(a + b) = (a^2 + 2ab + b^2)(a + b)$. We expand this last expression as the sum of all products of a term of $a^2 + 2ab + b^2$ by a term of $a + b$ in the following manner:

$$
\begin{array}{llll}
a^2 & +\,2ab & +\,b^2 & \\
a & +\,b & & \\
\hline
a^3 & +\,2a^2b & +\,ab^2 & \\
 & \quad a^2b & +\,2ab^2 & +\,b^3 \\
\hline
a^3 & +\,3a^2b & +\,3ab^2 & +\,b^3
\end{array}
$$

Hence $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. If $a \neq 0$ and $b \neq 0$ (a is not equal to zero and b is not equal to zero), this may be written

$$(a + b)^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3$$

The terms of the expanded form are such that the exponent for a starts as 3 and decreases by one each time, while the exponent of b starts as 0 and increases by one each time. Thus the sum of the exponents is 3 in each term.

One might guess that by analogy the expansion of $(a + b)^4$ involves a^4, a^3b, a^2b^2, ab^3, and b^4. This is verified by expanding $(a + b)^4 = (a + b)^3(a + b) = (a^3 + 3a^2b + 3ab^2 + b^3)(a + b)$ as follows:

(1)	a^3	$+ 3a^2b$	$+ 3ab^2$	$+ b^3$	
(2)	a	$+ b$			
(3)	a^4	$+ 3a^3b$	$+ 3a^2b^2$	$+ ab^3$	
(4)		a^3b	$+ 3a^2b^2$	$+ 3ab^3$	$+ b^4$
(5)	a^4	$+ 4a^3b$	$+ 6a^2b^2$	$+ 4ab^3$	$+ b^4$

Thus we see that a^4, a^3b, a^2b^2, ab^3, and b^4 are multiplied by 1, 4, 6, 4, 1 to form the terms of the expansion. The numbers 1, 4, 6, 4, 1 are the coefficients of the expansion. Examination of expressions (1) to (5), above, shows that these coefficients are obtainable from the coefficients 1, 3, 3, 1 of $(a + b)^3$ by means of the following condensed versions of (3), (4), and (5):

(3*)	1	3	3	1	
(4*)		1	3	3	1
(5*)	1	4	6	4	1

We now tabulate the coefficients of $(a + b)^n$ for $n = 0, 1, 2, 3, 4, \ldots$ in a triangular array:

n	Coefficients of $(a + b)^n$
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
...	. . . . . .

One may observe that the array is bordered with 1's and that each number inside the border is the sum of the two closest numbers on the previous line. This observaion simplifies the generation of additional lines of the array. For example, the coefficients for $n = 5$ are 1, 1 + 4 = 5, 4 + 6 = 10, 6 + 4 = 10, 4 + 1 = 5, and 1.

The above triangular array is called the **<u>Pascal Triangle</u>** in honor of the mathematician Blaise Pascal (1623-1662). A notation for the coefficients of $(a + b)^n$ is

$$\binom{n}{0}, \binom{n}{1}, \ldots , \binom{n}{n}. \tag{6}$$

For example, one writes $(a + b)^4$ in this notation as

$$\binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4$$

where $\binom{4}{0} = 1 = \binom{4}{4}$, $\binom{4}{1} = 4 = \binom{4}{3}$, and $\binom{4}{2} = 6$.

A two-term expression is called a **<u>binomial</u>**, and an expansion for an expression such as $(a + b)^n$ is called a **<u>binomial</u>** **<u>expansion</u>**. The coefficients listed in (6) above are called **<u>binomial coefficients</u>**.

Note that the symbol $\binom{n}{k}$ denotes the coefficient of $a^{n-k}b^k$, or of a^kb^{n-k}, in the expansion of $(a + b)^n$. Thus $\binom{3}{1}$ is the coefficient 3 of a^2b or of ab^2 in the expansion of $(a + b)^3$,and $\binom{4}{2}$ is the coefficient 6 of x^2y^2 in $(x + y)^4$. One reads $\binom{n}{k}$ as "binomial coefficient n choose k" or simply as "n choose <u>k</u>." The reason for this terminology is given in Chapter 7.

In Figure 1, (on page 4) we see how n and k give us the location of $\binom{n}{k}$ in the Pascal Triangle. The number n in $\binom{n}{k}$ is the row number and k is the diagonal number if one adopts the convention of labeling the rows or diagonals as 0, 1, 2, ...

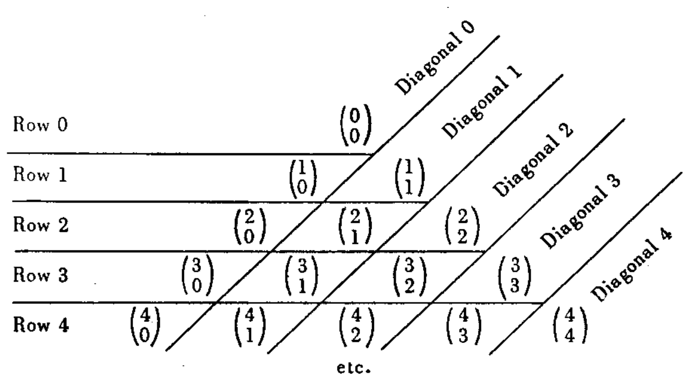

FIGURE 1

The formulas $\binom{n}{0} = 1 = \binom{n}{n}$ recall the fact that the Pascal Triangle is bordered with 1's. The rule that each number inside the border of 1's in the Pascal Triangle is the sum of the two closest entries on the previous line may be written as

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Example 1. Expand $(2x + 3y^2)^3$.

Solution: The expansion $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ is an identity which remains true when one substitutes $a = 2x$ and $b = 3y^2$ and thus obtains

$$\begin{aligned}(2x + 3y^2)^3 &= (2x)^3 + 3(2x)^2(3y^2) + 3(2x)(3y^2)^2 + (3y^2)^3 \\ &= 8x^3 + 3(4x^2)(3y^2) + 3(2x)(9y^4) + 27\,y^6 \\ &= 8x^3 + 36x^2y^2 + 54xy^4 + 27y^6.\end{aligned}$$

Example 2. Show that $\binom{4}{0} + \binom{5}{1} + \binom{6}{2} + \binom{7}{3} + \binom{8}{4} = \binom{9}{4}$.

Solution: Using the fact that $\binom{4}{0} = 1 = \binom{5}{0}$ and the formula

$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ we see that

$$\binom{8}{4}+\binom{7}{3}+\binom{6}{2}+\binom{5}{1}+\binom{4}{0} = \binom{8}{4}+\binom{7}{3}+\binom{6}{2}+\binom{5}{1}+\binom{5}{0}$$

$$= \binom{8}{4}+\binom{7}{3}+\binom{6}{2}+\binom{6}{1}$$

$$= \binom{8}{4}+\binom{7}{3}+\binom{7}{2}$$

$$= \binom{8}{4}+\binom{8}{3}$$

$$= \binom{9}{4}.$$

Problems for Chapter 1

1. Give the value of $\binom{5}{2}$, that is, of the coefficient of a^3b^2 in $(a + b)^5$.

2. Give the value of $\binom{5}{4}$.

3. Find s if $\binom{5}{4} = \binom{5}{s}$ and s is not 4.

4. Find t if $\binom{5}{t} = \binom{5}{0}$ and t is not 0.

5. Obtain the binomial coefficients for $(a + b)^3$ from those for $(a + b)^2$ in the style of lines (3*), (4*), (5*) on page 2.

6. Obtain the binomial coefficients for $(a + b)^6$ from those for $(a + b)^5$ in the style of lines (3*), (4*), (5*) on page 2.

7. Generate the lines of the Pascal Triangle for $n = 6$ and $n = 7$, using the technique described at the top of page 3.

8. Find $\binom{8}{0}$, $\binom{8}{1}$, $\binom{8}{2}$, $\binom{8}{3}$, and $\binom{8}{4}$.

9. Use $\binom{9}{1} = 9$ and $\binom{9}{2} = 36$ to find $\binom{9}{7}$ and $\binom{9}{8}$.

10. Use $\binom{9}{1} = 9$ and $\binom{9}{2} = 36$ to find $\binom{10}{2}$ and $\binom{10}{8}$.

11. Expand $(5x + 2y)^3$.

12. Expand $(x^2 - 4y^2)^3$ by letting $a = x^2$ and $b = -4y^2$ in the expansion of $(a + b)^3$.

13. Show that:

(a) $(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.

(b) $(x - y)^5 = \binom{5}{0}x^5 - \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 - \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 - \binom{5}{5}y^5$.

14. Show that:

(a) $(x + 1)^6 + (x - 1)^6 = 2\left[\binom{6}{0}x^6 + \binom{6}{2}x^4 + \binom{6}{4}x^2 + \binom{6}{6}\right]$.

(b) $(x + y)^6 - (x - y)^6 = 2\left[\binom{6}{1}x^5y + \binom{6}{3}x^3y^3 + \binom{6}{5}xy^5\right]$.

15. Show that $(x + h)^3 - x^3 = h(3x^2 + 3xh + h^2)$.

16. Show that $(x + h)^{100} - x^{100} = h\left[\binom{100}{1}x^{99} + \binom{100}{2}x^{98}h + \binom{100}{3}x^{97}h^2 + \ldots + \binom{100}{100}h^{99}\right]$.

17. Find numerical values of c and m such that cx^3y^m is a term of the expansion of $(x + y)^8$.

18. Find d and n such that dx^5y^4 is a term of $(x + y)^n$.

19. Find each of the following:

(a) $\binom{2}{0} + \binom{2}{1} + \binom{2}{2}$.

(b) $\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$.

(c) $2\binom{4}{0} + 2\binom{4}{1} + \binom{4}{2}$.

(d) $2\left[\binom{5}{0} + \binom{5}{1} + \binom{5}{2}\right]$.

(e) $2\left[\binom{6}{0} + \binom{6}{1} + \binom{6}{2}\right] + \binom{6}{3}$.

(f) $2\left[\binom{7}{7} + \binom{7}{6} + \binom{7}{5} + \binom{7}{4}\right]$.

20. Find the sum of the 101 binomial coefficients for $n = 100$ by assigning specific values to a and b in the identity

$$(a+b)^{100} = \binom{100}{0}a^{100} + \binom{100}{1}a^{99}b + \binom{100}{2}a^{98}b^2 + \dots$$
$$+ \binom{100}{98}a^2b^{98} + \binom{100}{99}ab^{99} + \binom{100}{100}b^{100}.$$

21. Find each of the following:

(a) $\binom{2}{0} - \binom{2}{1} + \binom{2}{2}$.

(b) $\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4}$.

22. Find each of the following:

(a) $\binom{100}{0} - \binom{100}{1} + \binom{100}{2} - \binom{100}{3} + \ldots - \binom{100}{99} + \binom{100}{100}$.

(b) $\binom{101}{0} - \binom{101}{1} + \binom{101}{2} - \binom{101}{3} + \ldots + \binom{101}{100} - \binom{101}{101}$.

23. Find each of the following:

(a) $\binom{4}{0} + \binom{4}{2} + \binom{4}{4}$.

(b) $\binom{5}{0} + \binom{5}{2} + \binom{5}{4}$.

(c) $\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6}$.

(d) $\binom{7}{1} + \binom{7}{3} + \binom{7}{5} + \binom{7}{7}$.

24. Find each of the following:

(a) $\binom{1000}{0} + \binom{1000}{2} + \binom{1000}{4} + \ldots + \binom{1000}{1000}$.

(b) $\binom{1000}{1} + \binom{1000}{3} + \binom{1000}{5} + \ldots + \binom{1000}{999}$.

25. Find r, s, t, and u, given the following:

(a) $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = \binom{r}{3}$.

(b) $\binom{2}{0} + \binom{3}{1} + \binom{4}{2} + \binom{5}{3} + \binom{6}{4} = \binom{s}{4}$.

(c) $\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \binom{4}{1} + \binom{5}{1} = \binom{t}{2}$.

(d) $\binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3} = \binom{u}{3}$.

26. Express $\binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4} + \cdots + \binom{100}{4}$ as a binomial coefficient.

27. Express $\binom{5}{0} + \binom{6}{1} + \binom{7}{2} + \cdots + \binom{995}{990}$ as a binomial coefficient.

28. Show that

(a) $n = \binom{0}{0} + \binom{1}{0} + \binom{2}{0} + \binom{3}{0} + \cdots + \binom{n-1}{0} = \binom{n}{1}$.

(b) $\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1} = \binom{n+1}{2}$.

(c) $\binom{n}{k} + 2\binom{n}{k+1} + \binom{n}{k+2} = \binom{n+2}{k+2}$ for $0 \le k \le n - 2$.

29. Show that $\binom{9}{4}\binom{5}{3} = \binom{9}{3}\binom{6}{4} = \binom{9}{2}\binom{7}{3}$.

30. Show that $\binom{10}{1}\binom{9}{2}\binom{7}{3} = \binom{10}{4}\binom{6}{3}\binom{3}{2} = \binom{10}{2}\binom{8}{4}\binom{4}{1}$.

31. Expand $(x + y + z)^4$ by expanding $(w + z)^4$, then replacing w by $x + y$, and expanding further.

32. Expand $(x + y - z)^4$.

33. The sum of squares $\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2$ is expressible in the form $\binom{2m}{m}$.

Find m.

34. Express each of the following in the form $\binom{2m}{m}$:

(a) $\binom{4}{0}\binom{4}{4} + \binom{4}{1}\binom{4}{3} + \binom{4}{2}\binom{4}{2} + \binom{4}{3}\binom{4}{1} + \binom{4}{4}\binom{4}{0}$.

(b) $2\left[\binom{5}{0}^2 + \binom{5}{1}^2 + \binom{5}{2}^2\right]$.

(c) $\binom{6}{3}^2 + 2\left[\binom{6}{2}^2 + \binom{6}{1}^2 + \binom{6}{0}^2\right]$.

35. Use $(x^3 + 3x^2 + 3x + 1)^2 = [(x + 1)^3]^2 = (x + 1)^6$ to show the following:

(a) $\binom{3}{0}^2 = \binom{6}{0}$.

(b) $\binom{3}{0}\binom{3}{1} + \binom{3}{1}\binom{3}{0} = \binom{6}{1}$.

(c) $\binom{3}{0}\binom{3}{2} + \binom{3}{1}\binom{3}{1} + \binom{3}{2}\binom{3}{0} = \binom{6}{2}$.

(d) $\binom{3}{0}\binom{3}{3} + \binom{3}{1}\binom{3}{2} + \binom{3}{2}\binom{3}{1} + \binom{3}{3}\binom{3}{0} = \binom{6}{3}$.

36. Express $\binom{100}{0}^2 + \binom{100}{1}^2 + \binom{100}{2}^2 + \cdots + \binom{100}{100}^2$ in the form $\binom{2m}{m}$.

37. How many of the 3 binomial coefficients $\binom{n}{k}$ with n = 0 or 1 are odd?

38. How many of the 10 binomial coefficients $\binom{n}{k}$ with n = 0, 1, 2, or 3 are odd?

39. How many binomial coefficients $\binom{n}{k}$ are there with n = 0, 1, 2, 3, 4, 5, 6, or 7, and how many of these are odd?

40. Show that all eight of the coefficients $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, \binom{7}{3}, \ldots, \binom{7}{7}$ are odd.

41. For what values of *n* among 0, 1, 2,..., 20 are all the coefficients $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ on row *n* of the Pascal Triangle odd?

42. If *n* is an answer to the previous problem, how many of the binomial coefficients on row *n* + 1 of the Pascal Triangle are odd?

*43. How many binomial coefficients $\binom{n}{k}$ are there with n = 0, 1, 2,..., 1022, or 1023, and how many of these are even?

Chapter 2

THE FIBONACCI AND LUCAS NUMBERS

The great Italian mathematician, Leonardo of Pisa (c. 1170-1250), who is known today as Fibonacci (an abbreviation of filius Bonacci), expanded on the Arabic algebra of North Africa and introduced algebra into Europe. The solution of a problem in his book *Liber Abacci* uses the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots \tag{F}$$

One of the many applications of this ***Fibonacci sequence*** is a theorem about the number of steps in an algorithm for finding the greatest common divisor of a pair of large integers. We study the sequence here because it provides a wonderful opportunity for discovering mathematical patterns.

The numbers shown in (F) are just the beginning of the unending Fibonacci sequence. The rule for obtaining more terms is as follows:

RECURSIVE PROPERTY. The sum of two consecutive terms in (F) is the term immediately after them.

For example, the term after 55 in (F) is 34 + 55 = 89 and the term after that is 55 + 89 = 144.

To aid in stating properties of the Fibonacci sequence, we use the customary notation F_0, F_1, F_2, F_3, ÿ for the integers of the Fibonacci sequence. That is, $F_0 = 0$, $F_1 = 1$, $F_2 = F_0 + F_1 = 1$, $F_3 = F_1 + F_2 = 2$, $F_4 = F_2 + F_3 = 3$, $F_5 = F_3 + F_4 = 5$, $F_6 = F_4 + F_5 = 8$, and so on. When F_n stands for some term of the sequence, the term just after F_n is represented by F_{n+1}, the term after F_{n+1} is F_{n+2}, and so on. Also, the term just before F_n is F_{n-1}, the term just before F_{n-1} is F_{n-2} and so on.

We now can define the Fibonacci numbers formally as the sequence F_0, F_1, ... having the two following properties.

INITIAL CONDITIONS $F_0 = 0$ and $F_1 = 1$.

RECURSION RULE $F_n + F_{n+1} = F_{n+2}$ for $n = 0, 1, 2, \ldots$.

Next let S_n stand for the sum of the Fibonacci numbers from F_0 through F_n. That is,

$$\begin{array}{lll}
S_0 = F_0 = 0 & & \\
S_1 = F_0 + F_1 & = 0 + 1 & = 1 \\
S_2 = F_0 + F_1 + F_2 & = 0 + 1 + 1 = 2 & \\
S_3 = F_0 + F_1 + F_2 + F_3 & = S_2 + F_3 & = 2 + 2 = 4
\end{array}$$

and in general, $S_n = F_0 + F_1 + F_2 + \ldots + F_n = S_{n-1} + F_n$.

We tabulate some of the values and look for a pattern.

n	0	1	2	3	4	5	6	7	...
F_n	0	1	1	2	3	5	8	13	...
S_n	0	1	2	4	7	12	20	33	...

Is there a relationship between the numbers on the third line of this table and the Fibonacci numbers? One pattern is that each of the terms of the sequence $S_0, S_1, S_2, \ldots$ is 1 less than a Fibonacci number. Specifically, we have

$$\begin{aligned} S_0 &= F_2 - 1 = 0 \\ S_1 &= F_3 - 1 = 1 \\ S_2 &= F_4 - 1 = 2 \\ S_3 &= F_5 - 1 = 4 \\ S_4 &= F_6 - 1 = 7 \\ S_5 &= F_7 - 1 = 12 \end{aligned}$$

and might conjecture that $S_n = F_{n+2} - 1$ for $n = 0, 1, 2, \ldots$. Does this formula hold for $n = 6$? Yes, since

$$\begin{aligned} S_6 &= F_0 + F_1 + F_2 + F_3 + F_4 + F_5 + F_6 \\ &= S_5 + F_6 \\ &= (F_7 - 1) + F_6 \\ &= (F_6 + F_7) - 1 \\ &= F_8 - 1. \end{aligned}$$

The first steps in proving our conjecture correct for all the terms in the unending sequence $S_0, S_1, S_2, \ldots$ are rewriting the recursion formula $F_n + F_{n+1} = F_{n+2}$ as $F_n = F_{n+2} - F_{n+1}$ and then using this to replace each Fibonacci number in the sum S_n by a difference as follows:

$$\begin{aligned} S_n &= F_0 + F_1 + F_2 + \ldots + F_{n-1} + F_n \\ S_n &= (F_2 - F_1) + (F_3 - F_2) + (F_4 - F_3) + \ldots + (F_{n+1} - F_n) + (F_{n+2} + F_{n+1}). \end{aligned}$$

Next we rearrange the terms and get

$$\begin{aligned} S_n &= -F_1 + (F_2 - F_2) + (F_3 - F_3) + \ldots + (F_{n-1} + F_{n-1}) + F_{n+2} \\ S_n &= -F_1 + 0 + 0 + \ldots + 0 + F_{n+2} \\ S_n &= F_{n+2} - F_1 = F_{n+2} - 1. \end{aligned}$$

Thus we have made our conjecture (that is, educated guess) into a theorem.

The fundamental relation $F_n = F_{n+2} - F_{n+1}$ can also be used to define F_n when n is a negative integer. Letting $n = -1$ in this formula gives us $F_{-1} = F_1 - F_0 = 1 - 0 = 1$. Similarly, one finds that $F_{-2} = F_0 - F_{-1} = 0 - 1 = -1$ and $F_{-3} = F_{-1} - F_{-2} = 1 - (-1) = 2$. In this way one can obtain F_n for any negative integer n.

Some of the values of F_n for negative integers n are shown in the following table:

n	...	-6	-5	-4	-3	-2	-1
F_n	...	-8	5	-3	2	-1	1

Perhaps the greatest investigator of properties of the Fibonacci and related sequences was François Edouard Anatole Lucas (1842-1891). A sequence related to the F_n bears his name. The ***Lucas sequence***, 2, 1, 3, 4, 7, 11, 18, 29, 47, ..., is defined by

$$L_0 = 2,\ L_1 = 1,\ L_2 = L_1 + L_0,\ L_3 = L_2 + L_1,\ ...,\ L_{n+2} = L_{n+1} + L_n, ...$$

Some of the many relations involving the F_n and the L_n are suggested in the problems below. These are only a very small fraction of the large number of known properties of the Fibonacci and Lucas numbers. In fact, there is a mathematical journal, *The Fibonacci Quarterly*, devoted to them and to related material.

Problems for Chapter 2

1. For the Fibonacci numbers F_n show that:

   (a) $F_3 = 2F_1 + F_0$. (b) $F_4 = 2F_2 + F_1$. (c) $F_5 = 2F_3 + F_2$.

2. The relation $F_{n+2} = F_{n+1} + F_n$ holds for all integers n and hence so does $F_{n+3} = F_{n+2} + F_{n+1}$. Combine these two formulas to find an expression for F_{n+3} in terms of F_{n+1} and F_n.

3. Find r, given that $F_r = 2F_{101} + F_{100}$.

4. Express $F_{157} + 2F_{158}$ in the form F_s.

5. Show the following:

   (a) $F_4 = 3F_1 + 2F_0$. (b) $F_5 = 3F_2 + 2F_1$.

6. Add corresponding sides of the formulas of the previous problem and use this to show that $F_6 = 3F_3 + 2F_2$.

7. Express F_{n+4} in terms of F_{n+1} and F_n.

8. Find s, given that $F_s = 3F_{200} + 2F_{199}$.

9. Find t, given that $F_t = 5F_{317} + 3F_{316}$.

10. Find numbers a and b such that $F_{n+6} = aF_{n+1} + bF_n$ for all integers n.

11. Show the following:

(a) $F_0 + F_2 + F_4 + F_6 = F_7 - 1$.

(b) $F_0 + F_2 + F_4 + F_6 + F_8 = F_9 - 1$.

(c) $F_1 + F_3 + F_5 + F_7 = F_8$.

(d) $F_1 + F_3 + F_5 + F_7 + F_9 = F_{10}$.

12. The relation $F_{n+2} = F_{n+1} + F_n$ can be rewritten as $F_{n+1} = F_{n+2} - F_n$. Use this form to find a compact expression for $F_a + F_{a+2} + F_{a+4} + F_{a+6} + \ldots + F_{a+2m}$.

13. Find p, given that $F_p = F_1 + F_3 + F_5 + F_7 + \ldots + F_{701}$.

14. Find u and v, given that $F_u - F_v = F_{200} + F_{202} + F_{204} + \ldots + F_{800}$.

15. Show the following:

(a) $F_4 = 3F_2 - F_0$.

(b) $F_5 = 3F_3 - F_1$.

(c) $F_6 = 3F_4 - F_2$.

16. Use the formulas $F_{n+4} = 3F_{n+1} + 2F_n$ and $F_{n+1} = F_{n+2} - F_n$ to express F_{n+4} in terms of F_{n+2} and F_n.

17. Show the following:

(a) $2(F_0 + F_3 + F_6 + F_9 + F_{12}) = F_{14} - 1$.

(b) $2(F_0 + F_3 + F_6 + F_9 + F_{12} + F_{15}) = F_{17} - 1$.

18. Show the following:

(a) $2(F_1 + F_4 + F_7 + F_{10} + F_{13}) = F_{15}$.

(b) $2(F_1 + F_4 + F_7 + F_{10} + F_{13} + F_{16}) = F_{18}$.

19. By addition of corresponding sides of formulas of the two previous problems, find expressions for:

(a) $2(F_2 + F_5 + F_8 + F_{11} + F_{14})$.

(b) $2(F_2 + F_5 + F_8 + F_{11} + F_{14} + F_{17})$.

20. (i) Prove that $F_{n+3} - 4F_n - F_{n-3} = 0$ for $n \geq 3$.
(ii) Prove that $F_{n+4} - 7F_n + F_{n-4} = 0$ for $n \geq 4$.
(iii) For $m \geq 4$, find a compact expression for $F_a + F_{a+4} + F_{a+8} + \cdots + F_{a+4m}$.

21. Evaluate each of the following sums:

a) $\binom{2}{0} + \binom{1}{1}$

b) $\binom{3}{0} + \binom{2}{1}$

c) $\binom{4}{0} + \binom{3}{1} + \binom{2}{2}$

d) $\binom{5}{0} + \binom{4}{1} + \binom{3}{2}$

e) $\binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3}$

f) $\binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3}$

22. Find m, given that $\binom{9}{0} + \binom{8}{1} + \binom{7}{2} + \binom{6}{3} + \binom{5}{4} = F_m$

22. Find m, given that $\binom{9}{0} + \binom{8}{1} + \binom{7}{2} + \binom{6}{3} + \binom{5}{4} = F_m$.

23. Find r, s, and t given that:

(a) $\binom{2}{0}F_0 + \binom{2}{1}F_1 + \binom{2}{2}F_2 = F_r$. (b) $\binom{2}{0}F_1 + \binom{2}{1}F_2 + \binom{2}{2}F_3 = F_s$.

(c) $\binom{2}{0}F_7 + \binom{2}{1}F_8 + \binom{2}{2}F_9 = F_t$.

24. Find u, v, and w, given that:

(a) $\binom{3}{0}F_0 + \binom{3}{1}F_1 + \binom{3}{2}F_2 + \binom{3}{3}F_3 = F_u$.

(b) $\binom{3}{0}F_1 + \binom{3}{1}F_2 + \binom{3}{2}F_3 + \binom{3}{3}F_4 = F_v$.

(c) $\binom{3}{0}F_7 + \binom{3}{1}F_8 + \binom{3}{2}F_9 + \binom{3}{3}F_{10} = F_w$.

25. Find r,s, and t, given that:

(a) $(F_7)^2 + (F_8)^2 = F_r$. (b) $(F_8)^2 + (F_9)^2 = F_s$. (c) $(F_9)^2 + (F_{10})^2 = F_t$.

26. Find u, v, *and* w, given that:

(a) $(F_3)^2 - (F_2)^2 = F_u F_{u+3}$. (b) $(F_4)^2 - (F_3)^2 = F_v F_{v+3}$. (c) $(F_9)^2 - (F_8)^2 = F_w F_{w+3}$.

27. Let L_0, L_1, L_2 ,... be the Lucas sequence. Prove that

$$L_0 + L_1 + L_2 + L_3 + \ldots + L_n = L_{n+2} - 1.$$

28. Find r, given that $L_r = 2L_{100} + L_{99}$.

29. Find s, given that $L_s = 3L_{201} + 2L_{200}$.

30. Find t, given that $L_t = 8L_{999} + 5L_{998}$.

31. Show that $L_0 + L_2 + L_4 + L_6 + L_8 + L_{10} = L_{11} + 1$.

32. Find m, given that $L_0 + L_2 + L_4 + L_6 + \ldots + L_{400} = L_m + 1$.

33. Derive a formula for $L_1 + L_3 + L_5 + L_7 + \ldots + L_{2m+1}$.

34. Conjecture, and test in several cases, formulas for:

(a) $L_0 + L_3 + L_6 + L_9 + \ldots + L_{3m}$.

(b) $L_1 + L_4 + L_7 + L_{10} + \ldots + L_{3m+1}$.

(c) $L_2 + L_5 + L_8 + L_{11} + \ldots + L_{3m+2}$.

(d) $\binom{n}{0} L_k + \binom{n}{1} L_{k+1} + \binom{n}{2} L_{k+2} + \ldots + \binom{n}{n} L_{k+n}$.

35. Find *r*, *s*, and *t*, given that $F_2 L_2 = F_r$, $F_3 L_3 = F_s$, and $F_4 L_4 = F_t$.

36. Find *u*, *v*, and *w*, given that $F_{10}/F_5 = L_u$, $F_{12}/F_6 = L_v$, and $F_{14}/F_7 = L_w$.

37. Evaluate the following:

(a) $(F_1)^2 - F_0 F_2$.

(b) $(F_2)^2 - F_1 F_3$.

(c) $(F_3)^2 - F_2 F_4$.

(d) $(F_4)^2 - F_3 F_5$.

38. Evaluate the expressions of the previous problem with each Fibonacci number replaced by the corresponding Lucas number.

39. Which of the Fibonacci numbers F_{800}, F_{801}, F_{802}, F_{803}, F_{804}, and F_{805} are even?

40. Which of the Fibonacci numbers of the previous problem are exactly divisible by 3?

Chapter 3

FACTORIALS

A sequence which occurs frequently in mathematics is

$$1,\ 1,\ 1\cdot 2,\ 1\cdot 2\cdot 3,\ 1\cdot 2\cdot 3\cdot 4,\ \ldots.$$

We tabulate this in the form

n	0	1	2	3	4	5	6	...
$n!$	1	1	2	6	24	120	720	...

where the notation $n!$ (read as **n factorial**) is used for the number on the second line that is thus associated with n. Clearly $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, etc. The definition of $n!$ can be given as follows:

$$0! = 1,\ 1! = 1(0!),\ 2! = 2(1!),$$
$$3! = 3(2!),\ \ldots,\ (n+1)! = (n+1)(n!),\ \ldots\ .$$

The expression $n!$ is not defined for negative integers n. One reason is that the relation $(n + 1)! = (n + 1)(n!)$ becomes $1 = 0\cdot(-1)!$ when $n = -1$, and hence there is no way to define $(-1)!$ so that this relation is preserved.

Problems for Chapter 3

1. Find the following:

   (a) $7!$.
   (b) $(3!)^2$.
   (c) $(3^2)!$.
   (d) $(3!)!$.

2. Find the following:

   (a) $8!$.
   (b) $(2!)(3!)$.
   (c) $(2\cdot 3)!$.

3. Show that $\binom{5}{2}(2!)(3!) = 5!$ and $\binom{7}{3}(3!)(4!) = 7!$.

4. Find c and d, given that $\binom{6}{2}(2!)(4!) = c!$ and $\binom{8}{3}(3!)(5!) = d!$.

5. Write as a single factorial:

   (a) $3!\cdot 4\cdot 5$.
   (b) $4!\cdot 210$.
   (c) $n!(n + 1)$.

6. Express $a!(a^2 + 3a + 2)$ as a single factorial.

7. Find a and b such that $11\cdot 12\cdot 13\cdot 14 = a!/b!$.

8. Find e, given that $(n + e)!/n! = n^3 + 6n^2 + 11n + 6$.

9. Express $(n + 4)!/n!$ as a polynomial in n.

10. Find numbers $a, b, c, d,$ and e such that $(n + 5)!/n! = n^5 + an^4 + bn^3 + cn^2 + dn + e$.

11. Calculate the following sums:

   (a) $1!\cdot 1 + 2!\cdot 2 + 3!\cdot 3$.
   (b) $1!\cdot 1 + 2!\cdot 2 + 3!\cdot 3 + 4!\cdot 4$.
   (c) $1!\cdot 1 + 2!\cdot 2 + 3!\cdot 3 + 4!\cdot 4 + 5!\cdot 5$.

12. Conjecture a compact expression for the sum $1!\cdot 1 + 2!\cdot 2 + 3!\cdot 3 + \cdots + n!\cdot n$ and test it for several values of n.

13. Show that $(n + 1)! - n! = n!\cdot n$.

14. Show that $(n + 2)! - n! = n!(n^2 + 3n + 1)$.

15. Find numbers $a, b,$ and c such that $(n + 3)! - n! = n!(n^3 + an^2 + bn + c)$ holds for $n = 0, 1, 2, \ldots$.

16. Use the formula in Problem 13 to derive a compact expression for the sum in Problem 12.

17. Use the formula in Problem 14 to derive a compact expression for

$$0! + 11(2!) + 29(4!) + ... + (4m^2 + 6m + 1)[(2m)!].$$

18. Derive a compact expression for

$$5(1!) + 19(3!) + 41(5!) + ... + (4m^2 + 2m - 1)[(2m - 1)!].$$

19. Derive compact expressions for:

(a) $0! + 5(1!) + 11(2!) + ... + (n^2 + 3n + 1)(n!).$

*(b) $0! + 2(1!) + 5(2!) + ... + (n^2 + 1)(n!).$

20. Derive a compact expression for $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + ... + \dfrac{n}{(n + 1)!}$

21. Show that:

(a) $6! = 3! \cdot 2^3 \cdot 3 \cdot 5.$

(b) $8! = 4! \cdot 2^4 \cdot 3 \cdot 5 \cdot 7.$

(c) $10! = 5! \cdot 2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 9.$

22. Express r, s, and t in terms of m so that

$$1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2m - 1) = r!/(s! \cdot 2^t).$$

Chapter 4

ARITHMETIC AND GEOMETRIC PROGRESSIONS

A finite sequence such as

$$2, 5, 8, 11, 14, \ldots , 101$$

in which each succeeding term is obtained by adding a fixed number to the preceding term is called an **<u>arithmetic progression</u>**. The general form of an arithmetic progression with n terms is therefore

$$a,\ a + d,\ a + 2d,\ a + 3d,\ \ldots\ ,\ a + (n - 1)d$$

where a is the first term and d is the fixed **<u>difference</u>** between successive terms.

In the arithmetic progression above, the first term is 2 and the common difference is 3. The second term is $2 + 3 \cdot 1$. the third term is $2 + 3 \cdot 2$. the fourth is $2 + 3 \cdot 3$. and the nth is $2 + 3(n - 1)$. Since $101 - 2 = 99 = 3 \cdot 33$ or $101 = 2 + 3(34 - 1)$, one has to add 3 thirty-three times to obtain the nth term. This shows that there are thirty-four terms here. The sum S of these thirty-four terms may be found by the following technique. We write the sum with the terms in the above order and also in reverse order, and add:

$$\begin{array}{rcccccccccc}
S & = & 2 & + & 5 & + & 8 & + \cdots + & 98 & + & 101 \\
S & = & 101 & + & 98 & + & 95 & + \cdots + & 5 & + & 2 \\
\hline
2S & = & (2 + 101) & + & (5 + 98) & + & (8 + 95) & + \cdots + & (98 + 5) & + & (101 + 2) \\
 & = & 103 & + & 103 & + & 103 & + \cdots + & 103 & + & 103 \\
2S & = & 34 \cdot 103 & = & 3502. & & & & & &
\end{array}$$

Hence $S = 3502/2 = 1751$.

Using this method, one can show that the sum

$$T_n = 1 + 2 + 3 + 4 + \ldots + n$$

of the first n positive integers is $n(n + 1)/2$. Some values of T_n are given in the table which follows.

n	1	2	3	4	5	6	...
T_n	1	3	6	10	15	21	...

The sequence T_n may be defined for all positive integers n by

$$T_1 = 1,\ T_2 = T_1 + 2,\ T_3 = T_2 + 3,$$

$$T_4 = T_3 + 4, \ldots, T_{n+1} = T_n + (n + 1), \ldots .$$

The values 1, 3, 6, 10, 15, ... of T_n are called **triangular numbers** because they give the number of objects in triangular arrays of the type shown in Figure 2.

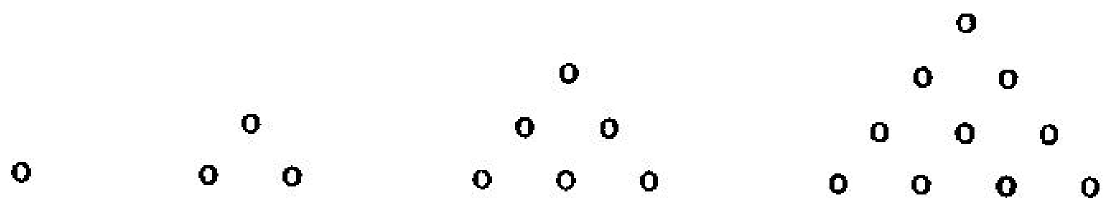

Figure 2

An arithmetic progression may have a negative common difference d. One with $a = 7/3$, $d = -5/3$, and $n = 8$ is:

$$7/3,\ 2/3,\ -1,\ -8/3,\ -13/3,\ -6,\ -23/3,\ -28/3.$$

The **average** (or **arithmetic mean**) of n numbers is their sum divided by n. For example, the average of 1, 3, and 7 is 11/3. If each of the terms of a sum is replaced by the average of the terms, the sum is not altered. We note that the average of all the terms of an arithmetic progression is the average of the first and last terms, and that the average is the middle term when the number of terms is odd, that is, whenever there is a middle term.

If a is the average of r and s, then it can easily be seen that r, a, s are consecutive terms of an arithmetic progression. (The proof is left to the reader.) This is why the average is also called the arithmetic mean.

A finite sequence such as

$$3, 6, 12, 24, 48, 96, 192, 384$$

in which each term after the first is obtained by multiplying the preceding term by a fixed number, is called a **geometric progression**. The general form of a geometric progression with n terms is therefore

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}.$$

Here a is the first term and r is the fixed multiplier. The number r is called the **ratio** of the progression, since it is the ratio (i.e., quotient) of a term to the preceding term.

We now illustrate a useful technique for summing the terms of a geometric progression.

Example. Sum $5 + 5 \cdot 2^2 + 5 \cdot 2^4 + 5 \cdot 2^6 + 5 \cdot 2^8 + \ldots + 5 \cdot 2^{100}$.

Solution: Here the ratio r is $2^2 = 4$. We let S designate the desired sum and write S and rS as follows:

$$\begin{aligned} S &= 5 + 5 \cdot 2^2 + 5 \cdot 2^4 + 5 \cdot 2^6 + \ldots + 5 \cdot 2^{100} \\ 4S &= \phantom{5 + {}} 5 \cdot 2^2 + 5 \cdot 2^4 + 5 \cdot 2^6 + \ldots + 5 \cdot 2^{100} + 5 \cdot 2^{102}. \end{aligned}$$

Subtracting, we note that all but two terms on the right cancel out and we obtain

$$3S = 5 \cdot 2^{102} - 5$$

or

$$3S = 5(2^{102} - 1).$$

Hence we have the compact expression for the sum:

$$S = \frac{5(2^{102} - 1)}{3}.$$

If the ratio r is negative, the terms of the geometric progression alternate in signs. Such a progression with $a = 125$, $r = -1/5$, and $n = 8$ is

$$125, -25, 5, -1, 1/5, -1/25, 1/125, -1/625.$$

The geometric mean of two positive real numbers a and b is $\sqrt{ab}$, the positive square root of their product; the geometric mean of three positive numbers a, b, and c is $\sqrt[3]{abc}$. In general, the **geometric mean** of n positive numbers is the nth root of their product. For example, the geometric mean of 2, 3, and 4 is $\sqrt[3]{2 \cdot 3 \cdot 4} = \sqrt[3]{8 \cdot 3} = 2\sqrt[3]{3}$.

Problems for Chapter 4

1. (a) Find the second, third, and fourth terms of the arithmetic progression with the first term -11 and difference 7.
   (b) Find the next three terms of the arithmetic progression -3, -7, -11, -15, ...

2. (a) Find the second, third, and fourth terms of the arithmetic progression with the first term 8 and difference -3.
   (b) Find the next three terms of the arithmetic progression 7/4, 1, 1/4, -1/2, -5/4, -2, ...

3. Find the 90th term of each of the following arithmetic progressions:

   (a) 11, 22, 33, 44, ...
   (b) 14, 25, 36, 47, ...
   (c) 9, 20, 31, 42, ...

4. For each of the following geometric progressions, find e so that 3^e is the 80th term.

   (a) 3, 9, 27, ...
   (b) 1, 3, 9, ...
   (c) 81, 243, 729, ...

5. Find x, given that 15, x, 18 are consecutive terms of an arithmetic progression.

6. Find x and y so that 14, x, y, 9 are consecutive terms of an arithmetic progression.

7. Sum the following:

   (a) 7/3 + 2/3 + (-1) + (-8/3) + (-13/3) + ... + (-1003/3).
   (b) (-6) + (-2) + 2 + 6 + 10 + ... + 2002.
   (c) The first ninety terms of 7/4, 1, 1/4, -1/2, -5/4, -2, ...
   (d) The first n odd positive integers, that is, $1 + 3 + 5 + ... + (2n - 1)$.

8. Sum the following:

   (a) 12 + 5 + (-2) + (-9) + ... + (-1073).
   (b) (-9/5) + (-1) + (-1/5) + 3/5 + 7/5 + 11/5 + ... + 2407.
   (c) The first eighty terms of -3, -7, -11, -15, ...
   (d) The first n terms of the arithmetic progression a, $a + d$, $a + 2d$, ...

9. Find the fourth, seventh, and ninth terms of the geometric progression with first term 2 and ratio 3.

10. Find the fourth and sixth terms of the geometric progression with first term 2 and ratio -3.

11. Find the next three terms of the geometric progression 2, 14, 98, ...

12. Find the next three terms of the geometric progression 6, -2, 2/3, -2/9, ...

13. Find both possible values of x if 7, x, 252 are three consecutive terms of a geometric progression.

14. Find all possible values of y if 400, y, 16 are three consecutive terms of a geometric progression.

15. Find a compact expression for each of the following:

(a) $1 + 7 + 7^2 + 7^3 + ... + 7^{999}$.
(b) $1 - 7 + 7^2 - 7^3 + ... - 7^{999}$.
(c) $1 + 7 + 7^2 + 7^3 + ... + 7^{n-1}$.

16. Find a compact expression for each of the following:

(a) $1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + ... + \frac{1}{3^{88}}$.

(b) $8 + \frac{8}{3^2} + \frac{8}{3^4} + \frac{8}{3^6} + ... + \frac{8}{3^{188}}$.

(c) $8 + 8 \cdot 3^{-2} + 8 \cdot 3^{-4} + 8 \cdot 3^{-6} + ... + 8 \cdot 3^{-2m}$.

17. Find n, given that $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \binom{n}{2}$.

*18. Find m, given that $1 + 2 + 3 + 4 + ... + 1000 = \binom{m}{m-2}$.

19. Given that a is the average of the numbers r and s, show that r, a, s are three consecutive terms of an arithmetic progression and that their sum is $3a$.

20. Show that r^3, r^2s, rs^2, s^3 are four consecutive terms of a geometric progression and that their sum is $(r^4 - s^4)/(r - s)$.

21. Find the geometric mean of each of the following sets of positive numbers:

(a) 6, 18.
(b) 2, 6, 18, 54.
(c) 2, 4, 8.
(d) 1, 2, 4, 8, 16.

22. Find the geometric mean of each of the following sets of numbers:

(a) 3, 4, 5.
(b) 3, 4, 5, 6.
(c) $1, 7, 7^2, 7^3$.
(d) a, ar, ar^2, ar^3, ar^4.

23. Find the geometric mean of 8, 27, and 125.

24. Find the geometric mean of a^4, b^4, c^4, and d^4.

25. Let b be the middle term of a geometric progression with $2m + 1$ positive terms and let r be the common ratio. Show that:

(a) The terms are $br^{-m}, br^{-m+1}, \dots , br^{-1}, b, br, \dots , br^m$.
(b) The geometric mean of the $2m + 1$ numbers is the middle term.

26. Show that the geometric mean of the terms in a geometric progression of positive numbers is equal to the geometric mean of any two terms equally spaced from the two ends of the progression.

27. Find a compact expression for the sum $x^n + x^{n-1}y + x^{n-2}y^2 + \dots + xy^{n-1} + y^n$.

28. Find a compact expression for the arithmetic mean of $x^n, x^{n-1}y, x^{n-2}y^2, \dots, xy^{n-1}, y^n$.

29. A 60-mile trip was made at 30 miles per hour and the return at 20 miles per hour.

(a) How many hours did it take to travel the 120 mile round trip?
(b) What was the average speed for the round trip?

30. Find x, given that 1/30, $1/x$, and 1/20 are in arithmetic progression. What is the relation between x and the answer to Part (b) of problem 29?

31. Verify the factorization $1 - x^7 = (1 - x)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6)$ and use it with $x = 1/2$ to find a compact expression for

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6.$$

32. Use the factorization $1 + x^{99} = (1 + x)(1 - x + x^2 - x^3 + x^4 - \dots + x^{98})$ to find compact expressions for the following sums:

(a) $1 - 5^{-1} + 5^{-2} - 5^{-3} + \dots - 5^{-97} + 5^{-98}$.
(b) $a - ar + ar^2 - ar^3 + \dots - ar^{97} + ar^{98}$.

33. Let $a_1, a_2, a_3, \dots, a_{3m}$ be an arithmetic progression, and for $n = 1, 2, \dots, 3m$ let A_n be the arithmetic mean of its first n terms. Show that A_{2m} is the arithmetic mean of the two numbers A_m and A_{3m}.

34. Let $g_1, g_2, \dots, g_{3m}$ be a geometric progression of positive terms. Let A, B, and C be the geometric means of the first m terms, the first $2m$ terms, and all $3m$ terms, respectively. Show that $B^2 = AC$.

35. Let S be the set consisting of those of the integers 0, 1, 2, ..., 30 which are divisible exactly by 3 or 5 (or both), and let T consist of those divisible by neither 3 nor 5.

(a) Write out the sequence of numbers in S in their natural order.
(b) In the sequence of Part (a), what is the arithmetic mean of terms equally spaced from the two ends of the sequence?
(c) What is the arithmetic mean of all the numbers in T?
(d) Find the sum of the numbers in T.

36. Find the sum 4 + 5 + 6 + 8 + 10 + 12 + 15 + ... + 60,000 of all the positive integers not exceeding 60,000 which are integral multiples of at least one of 4, 5, and 6.

37. Let $u_1, u_2, ..., u_t$ satisfy $u_{n+2} = 2u_{n+1} - u_n$ for $n = 1, 2, ..., t - 2$. Show that the t terms are in arithmetic progression.

38. Find a compact expression for the sum $v_1 + v_2 + ... + v_t$ in terms of v_1 and v_2, given that $v_{n+2} = (v_{n+1})^2/v_n$ for $n = 1, 2, ..., t - 2$.

39. Let $a_n = 2^n$ be the nth term of the geometric progression $2, 2^2, 2^3, ..., 2^t$. Show that $a_{n+2} - 5a_{n+1} + 6a_n = 0$ for $n = 1, 2, ..., t - 2$.

40. For what values of r does the sequence $b_n = r^n$ satisfy $b_{n+2} - 5b_{n+1} + 6b_n = 0$ for all n?

41. Let a be one of the roots of $x^2 - x - 1 = 0$. Let the sequence $c_0, c_1, c_2, ...$ be the geometric progression $1, a, a^2, ...$. Show that:

(a) $c_{n+2} = c_{n+1} + c_n$.
(b) $c_2 = a^2 = a + 1$.
(c) $c_3 = a^3 = 2a + 1$.
(d) $c_4 = 3a + 2$.
(e) $c_5 = 5a + 3$.
(f) $c_6 = 8a + 5$.

42. For the sequence $c_0, c_1, ...$ of the previous problem, express c_{12} in the form $aF_u + F_v$, where F_u and F_v are Fibonacci numbers, and conjecture a similar expression for c_m.

*43. In the sequence 1/5, 3/5, 4/5, 9/10, 19/20, 39/40, ... each succeeding term is the average of the previous term and 1. Thus:

$$\frac{3}{5} = \frac{1}{2}\left(\frac{1}{5} + 1\right), \ \frac{4}{5} = \frac{1}{2}\left(\frac{3}{5} + 1\right), \ \frac{9}{10} = \frac{1}{2}\left(\frac{4}{5} + 1\right), \ \ldots .$$

(a) Show that the twenty-first term is $1 - \dfrac{1}{5 \cdot 2^{18}}$.

(b) Express the *n*th term similarly.

(c) Sum the first five hundred terms.

*44. In the sequence 1, 2, 3, 6, 7, 14, 15, 30, 31, ... a term in an even numbered position is double the previous term, and a term in an odd numbered position (after the first term) is one more than the previous term.

(a) What is the millionth term of this sequence?

(b) Express the sum of the first million terms compactly.

Chapter 5

MATHEMATICAL INDUCTION

In mathematics, as in science, there are two general methods by which we can arrive at new results. One, deduction, involves the assumption of a set of axioms from which we deduce other statements, called theorems, according to prescribed rules of logic. This method is essentially that used in standard courses in Euclidean geometry.

The second method, induction, involves the guessing or discovery of general patterns from observed data. While in most branches of science and mathematics the guesses based on induction may remain merely conjectures, with varying degrees of probability of correctness, certain conjectures in mathematics which involve the integers frequently can be proved by a technique of Pascal called mathematical induction. Actually, this technique in not induction, but is rather an aid in proving conjectures arrived at by induction.

THE PRINCIPLE OF MATHEMATICAL INDUCTION: *A statement concerning positive integers is true for all positive integers if (a) it is true for 1, and (b) its being true for any integer k implies that it is true for the next integer k + 1.*

If one replaces (a) by (a'),"it is true for some integer *s*," then (a') and (b) prove the statement true for all integers greater than or equal to *s*. Part (a) gives only a starting point; this starting point may be any integer - positive, negative, or zero.

Let us see if mathematical induction is a reasonable method of proof of a statement involving integers n. Part (a) tells us that the statement is true for $n = 1$. Using (b) and the fact that the statement is true for 1, we obtain the fact that it is true for the next integer 2. Then (b) implies that it is true for $2 + 1 = 3$. Continuing in this way, we would ultimately reach any fixed positive integer.

Let us use this approach on the problem of determining a formula which will give us the number of diagonals of a convex polygon in terms of the number of sides. The three-sided polygon, the triangle, has no diagonals; the four-sided polygon has two. An examination of other cases yields the data included in the following table:

n = number of sides	3	4	5	6	7	8	9	...	n	...
D_n = number of diagonals	0	2	5	9	14	20	27	...	D_n	...

The task of guessing the formula, if a formula exists, is not necessarily an easy one, and there is no sure approach to this part of the over-all problem. However, if one is perspicacious, one observes the following pattern:

$$\begin{aligned} 2D_3 &= 0 = 3\cdot 0 \\ 2D_4 &= 4 = 4\cdot 1 \\ 2D_5 &= 10 = 5\cdot 2 \\ 2D_6 &= 18 = 6\cdot 3 \\ 2D_7 &= 28 = 7\cdot 4. \end{aligned}$$

This leads us to conjecture that

$$2D_n = n(n - 3)$$

or

$$D_n = \frac{n(n-3)}{2}$$

Now we shall use mathematical induction to prove this formula. We shall use as a starting point $n = 3$, since for n less than 3 no polygon exists. It is clear from the data that the formula holds for the case $n = 3$. Now we assume that a k-sided polygon has $k(k - 3)/2$ diagonals. If we can conclude from this that a $(k + 1)$-sided polygon has $(k + 1)[(k + 1) - 3]/2 = (k + 1)(k - 2)/2$ diagonals, we will have proved that the formula holds for all positive integers greater than or equal to 3.

Consider a k-sided polygon. By assumption it has $k(k - 3)/2$ diagonals. If we place a triangle on a side AB of the polygon, we make it into a $(k + 1)$-sided polygon. It has all the diagonals of the k-sided polygon plus the diagonals drawn from the new vertex N to all the

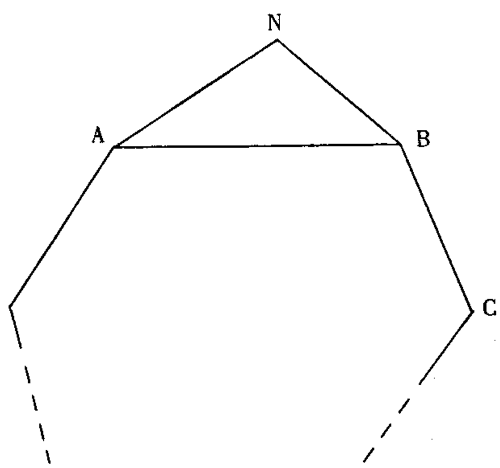

Figure 3

vertices of the previous k-sided polygon except 2, namely A and B. In addition, the former side AB has become a diagonal of the new $(k + 1)$-sided polygon. Thus a $(k + 1)$-sided polygon has a total of $\frac{k(k-3)}{2} + (k - 2) + 1$ diagonals. But:

$$\frac{k(k-3)}{2} + (k - 2) + 1$$
$$= \frac{k^2 - 3k + 2k - 2}{2}$$

$$= \frac{k^2 - k - 2}{2}$$
$$= \frac{(k + 1)(k - 2)}{2}$$
$$= \frac{(k + 1)[(k + 1) - 3]}{2}$$

This is the desired formula for $n = k + 1$.

So, by assuming that the formula $D_n = n(n - 3)/2$ is true for $n = k$, we have been able to show it true for $n = k + 1$. This, in addition to the fact that it is true for $n = 3$, proves that it is true for all integers greater than or equal to 3. (The reader may have discovered a more direct method of obtaining the above formula.)

The method of mathematical induction is based on something that may be considered one of the axioms for the positive integers: If a set S contains 1, and if, whenever S contains an integer k, S contains the next integer $k + 1$, then S contains all the positive integers. It can be shown that this is equivalent to the principle that in every non-empty set of positive integers there is a least positive integer.

Example 1. Find and prove by mathematical induction a formula for the sum of the first n cubes, that is, $1^3 + 2^3 + 3^3 + ... + n^3$.

Solution: We consider the first few cases:

$$\begin{aligned} 1^3 &= 1 \\ 1^3 + 2^3 &= 9 \\ 1^3 + 2^3 + 3^3 &= 36 \\ 1^3 + 2^3 + 3^3 + 4^3 &= 100. \end{aligned}$$

We observe that $1 = 1^2$, $9 = 3^2$, $36 = 6^2$, and $100 = 10^2$. Thus it appears that the sums are the squares of triangular numbers 1, 3, 6, 10, ... In Chapter 4 we saw that the triangular numbers are of the form $n(n + 1)/2$. This suggests that

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n + 1)}{2}\right]^2.$$

It is clearly true for $n = 1$. Now we assume that it is true for $n = k$:

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k + 1)}{2}\right]^2.$$

Can we conclude from this that

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \left[\frac{(k+1)[(k+1)+1]}{2}\right]^2?$$

We can add $(k + 1)^3$ to both sides of the known expression, obtaining:

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \\
&= (k+1)^2 \frac{k^2}{4} + (k+1)^3 \\
&= \frac{(k+1)^2 (k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2 (k+2)^2}{4} \\
&= \left[\frac{(k+1)(k+2)}{2}\right]^2 \\
&= \left[\frac{(k+1)[(k+1)+1]}{2}\right]^2.
\end{aligned}$$

Hence the sum when $n = k + 1$ is $[n(n + 1)/2]^2$, with n replaced by $k + 1$, and the formula is proved for all positive integers n.

Our guessed expression for the sum was a fortunate one!

Example2. Prove that $a - b$ is a factor of $a^n - b^n$ for all positive integers n.

Proof: Clearly, $a - b$ is a factor of $a^1 - b^1$; hence the first part of the induction is verified, that is, the statement is true for $n = 1$. Now we assume that $a^k - b^k$ has $a - b$ as a factor:

$$a^k - b^k = (a - b)M.$$

Next we must show that $a - b$ is a factor of $a^{k+1} - b^{k+1}$. But

$$\begin{aligned}
a^{k+1} - b^{k+1} &= a{\cdot}a^k - b{\cdot}b^k \\
&= a{\cdot}a^k - b{\cdot}a^k + b{\cdot}a^k - b{\cdot}b^k \\
&= (a - b)a^k + b(a^k - b^k).
\end{aligned}$$

Now, using the assumption that $a^k - b^k = (a - b)M$ and substituting, we obtain:

$$\begin{aligned} a^{k+1} - b^{k+1} &= (a - b)a^k + b(a - b)M \\ &= (a - b)\ [a^k + bM]. \end{aligned}$$

We see from this that $a - b$ is a factor of $a^{k+1} - b^{k+1}$ and hence $a - b$ is a factor of $a^n - b^n$ for n equal to any positive integer. It is easily seen that the explicit factorization is

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1}).$$

Example 3. Prove that $n(n^2 + 5)$ is an integral multiple of 6 for all integers n, that is, there is an integer u such that $n(n^2 + 5) = 6u$.

Proof: We begin by proving the desired result for all the integers greater than or equal to 0 by mathematical induction.

When $n = 0$, $n(n^2 + 5)$ is 0. Since $0 = 6 \cdot 0$ is a multiple of 6, the result holds for $n = 0$.

We now assume it true for $n = k$, and seek to derive from this its truth for $n = k + 1$. Hence we assume that

$$(1) \qquad k(k^2 + 5) = 6r$$

with r an integer. We then wish to show that

$$(2) \qquad (k + 1)[(k + 1)^2 + 5] = 6s$$

with s an integer. Simplifying the difference between the left-hand sides of (2) and (1), we obtain

$$(3) \qquad (k + 1)[(k + 1)^2 + 5] - k(k^2 + 5) = 3k(k + 1) + 6.$$

Since k and $k + 1$ are consecutive integers, one of them is even. Then their product $k(k + 1)$ is even, and may be written as $2t$, with t an integer. Now (3) becomes

$$(4) \qquad (k + 1)[(k + 1)^2 + 5] - k(k^2 + 5) = 6t + 6 = 6(t + 1).$$

Transposing, we have

$$(k + 1)[(k + 1)^2 + 5] = k(k^2 + 5) + 6(t + 1).$$

Using (1), we can substitute $6r$ for $k(k^2 + 5)$. Hence

$$(k + 1)[(k + 1)^2 + 5] = 6r + 6(t + 1) = 6(r + t + 1).$$

Letting s be the integer $r + t + l$, we establish (2), which is the desired result when $n = k + 1$. This completes the induction and proves the statement for $n \geq 0$.

Now let n be a negative integer, that is, let $n = -m$, with m a positive integer. The previous part of the proof shows that $m(m^2 + 5)$ is of the form $6q$, with q an integer. Then

$$n(n^2 + 5) = (-m)[(-m)^2 + 5] = -m(m^2 + 5) = -6q = 6(-q),$$

a multiple of 6. The proof is now complete.

We have seen that binomial coefficients, Fibonacci and Lucas numbers, and factorials may be defined inductively, that is, by giving their initial values and describing how to get new values from previous values. Similarly, one may define an arithmetic progression $a_1, a_2, \dots, a_t$ as one for which there is a fixed number d such that $a_{n+1} = a_n + d$ for $n = 1, 2, \dots, t - 1$. Then the values of a_1 and d would determine the values of all the terms. A geometric progression $b_1, \dots, b_t$ is one for which there is a fixed number r such that $b_{n+1} = b_n r$ for $n = 1, 2, \dots, t - 1$; its terms are determined by b_1 and r.

It is not surprising that mathematical induction is very useful in proving results concerning quantities that are defined inductively, however, it is sometimes necessary or convenient to use an alternate principle, called **strong mathematical induction**.

STRONG MATHEMATICAL INDUCTION: *A statement concerning positive integers is true for all the positive integers if there is an integer q such that (a) the statement is true for 1, 2, ... , q, and (b) when $k \geq q$, the statement being true for 1, 2, ... , k implies that it is true for k + 1.*

As in the case of the previous principle, this can be modified to apply to statements in which the starting value is an integer different from 1.

We illustrate strong induction in the following:

Example 4. Let $a, b, c, r, s,$ and t be fixed integers. Let $L_0, L_1, \dots$ be the Lucas sequence. Prove that

$$rL_{n+a} = sL_{n+b} + tL_{n+c} \qquad \text{(A)}$$

is true for $n = 0, 1, 2, \dots$ if it is true for $n = 0$ and $n = 1$.

Proof: We use strong induction. It is given that (A) is true for $n = 0$ and $n = 1$. Hence, it remains to assume that $k \geq 1$ and that (A) is true for $n = 0, 1, 2, \dots, k$, and to use these assumptions to prove that (A) holds for $n = k + 1$.

We therefore assume that

$$\begin{aligned}
rL_a &= sL_b + tL_c \\
rL_{1+a} &= sL_{1+b} + tL_{1+c} \\
rL_{2+a} &= sL_{2+b} + tL_{2+c} \\
&\dots \\
rL_{k-1+a} &= sL_{k-1+b} + tL_{k-1+c} \\
rL_{k+a} &= sL_{k+b} + L_{k+c}
\end{aligned}$$

and that there are at least two equations in this list. Adding corresponding sides of the last two of these equations and combining like terms, we obtain

$$r(L_{k+a} + L_{k-1+a}) = s(L_{k+b} + L_{k-1+b}) + t(L_{k+c} + L_{k-1+c}).$$

Using the relation $L_{n+1} + L_n = L_{n+2}$ for the Lucas numbers, this becomes

$$rL_{k+1+a} = sL_{k+1+b} + tL_{k+1+c}$$

which is (A) when $n = k + 1$. This completes the proof.

Problems for Chapter 5

In Problems 1 to 10 below, use mathematical induction to prove each statement true for all positive integers n.

1. The sum of the interior angles of a convex $(n + 2)$-sided polygon is $180n$ degrees.

2. $1^3 + 3^3 + 5^3 + \ldots + (2n - 1)^3 = n^2(2n^2 - 1)$.

3. (a) $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = n(4n^2 - 1)/3$.

   (b) $1{\cdot}3 + 3{\cdot}5 + 5{\cdot}7 + \ldots + (2n - 1)(2n + 1) = n(4n^2 + 6n - 1)/3$.

   (c) $\dfrac{1}{1{\cdot}3} + \dfrac{1}{2{\cdot}4} + \dfrac{1}{3{\cdot}5} + \ldots + \dfrac{1}{n(n + 2)} = \dfrac{n(3n + 5)}{4(n + 1)(n + 2)}$.

   (d) $1 + 2a + 3a^2 + \ldots + na^{n-1} = [1 - (n + 1)a^n + na^{n+1}]/(1 - a)^2$.

4. (a) $1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n + 1)(2n + 1)/6$.

   (b) $1{\cdot}3 + 2{\cdot}4 + 3{\cdot}5 + \ldots + n(n + 2) = n(n + 1)(2n + 7)/6$.

   (c) $\dfrac{5}{1{\cdot}2}{\cdot}\dfrac{1}{3} + \dfrac{7}{2{\cdot}3}{\cdot}\dfrac{1}{3^2} + \dfrac{9}{3{\cdot}4}{\cdot}\dfrac{1}{3^3} + \ldots + \dfrac{2n + 3}{n(n + 1)}{\cdot}\dfrac{1}{3^n} = 1 - \dfrac{1}{3^n(n + 1)}$.

5. $(1^3 + 2^3 + 3^3 + \ldots + n^3) + 3(1^5 + 2^5 + 3^5 + \ldots + n^5) = 4(1 + 2 + 3 + \ldots + n)^3$.

6. $(1^5 + 2^5 + 3^5 + \ldots + n^5) + (1^7 + 2^7 + 3^7 + \ldots + n^7) = 2(1 + 2 + 3 + \ldots + n)^4$

*7. $3^n + 7^n - 2$ is an integral multiple of 8.

*8. $2 \cdot 7^n + 3 \cdot 5^n - 5$ is an integral multiple of 24.

9. $x^{2n} - y^{2n}$ has $x + y$ as a factor.

10. $x^{2n+1} + y^{2n+1}$ has $x + y$ as a factor.

11. For all integers n, prove the following:

 (a) $2n^3 + 3n^2 + n$ is an integral multiple of 6.
 (b) $n^5 - 5n^3 + 4n$ is an integral multiple of 120.

12. Prove that $n(n^2 - 1)(3n + 2)$ is an integral multiple of 24 for all integers n.

13. Guess a formula for each of the following and prove it by mathematical induction:

 (a) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \ldots + \dfrac{1}{n(n+1)}$.

 (b) $(x + y)(x^2 + y^2)(x^4 + y^4)(x^8 + y^8) \ldots (x^{2^n} + y^{2^n})$.

14. Guess a formula for each of the following and prove it by mathematical induction:

 (a) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n + 1)$.

 (b) $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \ldots + \dfrac{1}{(2n - 1)(2n + 1)}$.

15. Guess a simple expression for the following and prove it by mathematical induction:

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{n^2}\right).$$

16. Find a simple expression for the product in Problem 15, using the factorization

$$x^2 - y^2 = (x - y)(x + y).$$

17. Prove the following properties of the Fibonacci numbers F_n for all integers n greater than or equal to 0:

(a) $2(F_s + F_{s+3} + F_{s+6} + \ldots + F_{s+3n}) = F_{s+3n+2} - F_{s-1}$.

(b) $F_{-n} = (-1)^{n+1}F_n$.

(c) $\binom{n}{0}F_s + \binom{n}{1}F_{s+1} + \binom{n}{2}F_{s+2} + \ldots + \binom{n}{n}F_{s+n} = F_{s+2n}$.

18. Discover and prove formulas similar to those of Problem 17 for the Lucas numbers L_n.

19. Use Example 4, in the text above, to prove the following properties of the Lucas numbers for $n = 0, 1, 2, \ldots$, and then prove them for all negative integers n.

(a) $L_{n+4} = 3L_{n+2} - L_n$.

(b) $L_{n+6} = 4L_{n+3} + L_n$.

(c) $L_{n+8} = 7L_{n+4} - L_n$.

(d) $L_{n+10} = 11L_{n+5} + L_n$.

20. State an analogue of Example 4 for the Fibonacci numbers instead of the Lucas numbers and use it to prove analogues of the formulas of Problem 19.

21. In each of the following parts, evaluate the expression for some small values of n, use this data to make a conjecture, and then prove the conjecture true for all integers n.

(a) $F_{n+1}^2 - F_n F_{n+2}$.

(b) $\dfrac{F_{n+2}^2 - F_{n+1}^2}{F_n}$.

(c) $F_{n-1} + F_{n+1}$.

22. Discover and prove formulas similar to the first two parts of the previous problem for the Lucas numbers.

23. Prove the following for all integers m and n:

(a) $L_{m+n+1} = F_{m+1}L_{n+1} + F_mL_n$.

(b) $F_{m+n+1} = F_{m+1}F_{n+1} - F_mF_n$.

24. Prove that $(F_{n+1})^2 + (F_n)^2 = F_{2n+1}$ for all integers n.

25. Let a and b be the roots of the quadratic equation $x^2 - x - 1 = 0$. Prove that:

(a) $F_n = \dfrac{a^n - b^n}{a - b}$.

(b) $L_n = a^n + b^n$.

(c) $F_n L_n = F_{2n}$.

(d) $a^n = aF_n + F_{n-1}$ and $b^n = bF_n + F_{n-1}$.

26. The sequence 0, 1, ½, 3/4, 5/8, 11/16, ... is defined by

$$u_0 = 0,\ u_1 = 1,\ u_2 = \frac{u_1 + u_0}{2},\ \ldots,\ u_{n+2} = \frac{u_{n+1} + u_n}{2},\ \ldots .$$

Discover and prove a compact formula for u_n as a function of n.

27. The Pell sequence 0, 1, 2, 5, 12, 29, ... is defined by

$$P_0 = 0,\ P_1 = 1,\ P_2 = 2P_1 + P_0,\ \ldots,\ P_{n+2} = 2P_{n+1} + P_n,\ \ldots .$$

Let $x_n = P_{n+1}^2 - P_n^2$, $y_n = 2P_{n+1}P_n$, and $z_n = P_{n+1}^2 + P_n^2$. Prove that for every positive integer n the numbers x_n, y_n, and z_n are the lengths of the sides of a right triangle and that x_n and y_n are consecutive integers.

28. Discover and prove properties of the Pell sequence that are analogous to those of the Fibonacci sequence.

29. Let the sequence 1, 5, 85, 21845, ... be defined by

$$c_1 = 1,\ c_2 = c_1(3c_1 + 2),\ \ldots,\ c_{n+1} = c_n(3c_n + 2),\ \ldots .$$

Prove that $c_n = \dfrac{4^{2^{n-1}} - 1}{3}$ for all positive integers n.

30. Let a sequence be defined by $d_1 = 4$, $d_2 = (d_1)^2$, ..., $d_{n+1} = (d_n)^2$, ...

Show that $d_n = 3c_n + 1$, where c_n is as defined in the previous problem.

31. Prove that $1{\cdot}2{\cdot}3 + 2{\cdot}3{\cdot}4 + \ldots + n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$.

*32. Certain of the above formulas suggest the following:

$$1 \cdot 2 \cdots m + 2 \cdot 3 \cdots (m + 1) + \cdots + n(n + 1) \cdots (n + m - 1) = \frac{n(n + 1) \ldots (n + m)}{m + 1}.$$

Prove it for general m.

*33. Prove that $n^5 - n$ is an integral multiple of 30 for all integers n.

*34. Prove that $n^7 - n$ is an integral multiple of 42 for all integers n.

*35. Show that every integer from 1 to $2^{n+1} - 1$ is expressible uniquely as a sum of distinct powers of 2 chosen from $1, 2, 2^2, \ldots, 2^n$.

*36. Show that every integer s from $-\frac{3^{n+1} - 1}{2}$ to $\frac{3^{n+1} - 1}{2}$ has a unique expression of the form

$$s = c_0 + 3c_1 + 3^2 c_2 + \ldots + 3^n c_n$$

where each of $c_0, c_1, \ldots, c_n$ is 0, 1, or -1.

Chapter 6

THE BINOMIAL THEOREM

In Chapter 1 we defined $\binom{n}{r}$ as the coefficient of $a^{n-r}b^r$ in the expansion of $(a + b)^n$, and tabulated these coefficients in the arrangement of the Pascal Triangle:

n	Coefficients of $(a + b)^n$
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
...	. . . .

We then observed that this array is bordered with 1's; that is, $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ for $n =$ 0, 1, 2, ... We also noted that each number inside the border of 1's is the sum of the two closest numbers on the previous line. This property may be expressed in the form

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}. \tag{1}$$

This formula provides an efficient method of generating successive lines of the Pascal Triangle, but the method is not the best one if we want only the value of a single binomial coefficient for a large n, such as $\binom{100}{3}$. We therefore seek a more direct approach.

It is clear that the binomial coefficients in a diagonal adjacent to a diagonal of 1's are the numbers 1, 2, 3, ... ; that is, $\binom{n}{1} = n$. Now let us consider the ratios of binomial coefficients to the previous ones on the same row. For $n = 4$, these ratios are:

(2) $$4/1,\ 6/4 = 3/2,\ 4/6 = 2/3,\ 1/4.$$

For $n = 5$, they are

(3) $$5/1,\ 10/5 = 2,\ 10/10 = 1,\ 5/10 = 1/2,\ 1/5.$$

The ratios in (3) have the same pattern as those in (2) if they are rewritten as

$$5/1,\ 4/2,\ 3/3,\ 2/4,\ 1/5.$$

It is easily seen that this pattern also holds on the line for $n = 8$, and that the coefficients on that line are therefore:

(4) $$1,\ \frac{8}{1},\ \frac{8}{1}\cdot\frac{7}{2},\ \frac{8}{1}\cdot\frac{7}{2}\cdot\frac{6}{3},\ \frac{8}{1}\cdot\frac{7}{2}\cdot\frac{6}{3}\cdot\frac{5}{4},\ \ldots .$$

The binomial coefficient $\binom{8}{3} = \frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3}$ can be rewritten as

$$\frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{8!}{3!5!}.$$

Similarly, $\binom{8}{2} = \frac{8!}{2!6!}$ and $\binom{8}{4} = \frac{8!}{4!4!}$.

This leads us to conjecture that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ holds in all cases. We prove this by mathematical induction in the following theorem.

THEOREM: If n and r are integers with $0 \le r \le n$, then

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proof: If $n = 0$, the only allowable value of r is 0 and $\binom{0}{0} = 1$. Since

$$\frac{n!}{r!(n-r)!} = \frac{0!}{0!0!} = 1$$

the formula holds for $n = 0$.

Now let us assume that it holds for $n = k$. Then

$$\binom{k}{r-1} = \frac{k!}{(r-1)!(k-r+1)!}, \binom{k}{r} = \frac{k!}{r!(k-r)!}.$$

Using (1), above, we now have

$$\begin{aligned}
\binom{k+1}{r} = \binom{k}{r-1} + \binom{k}{r} &= \frac{k!}{(r-1)!(k-r+1)!} + \frac{k!}{r!(k-r)!} \\
&= \frac{k!r}{(r-1)!r(k-r+1)!} + \frac{k!(k-r+1)}{r!(k-r)!(k-r+1)} \\
&= \frac{k!(r+k-r+1)}{r!(k-r+1)!} \\
&= \frac{k!(k+1)}{r!(k-r+1)!} \\
&= \frac{(k+1)!}{r!(k-r+1)!}.
\end{aligned}$$

Since the formula

$$\binom{k+1}{r} = \frac{(k+1)!}{r!(k-r+1)!}$$

is the theorem for $n = k + 1$, the formula is proved for all integers $n \geq 0$, with the exception that our proof tacitly assumes that r is neither 0 nor $k + 1$; that is, it deals only with the coefficients inside the border of 1's. But the formula

$$\binom{k+1}{r} = \frac{(k+1)!}{r!(k-r+1)!}$$

shows that each of $\begin{pmatrix} k+1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} k+1 \\ k+1 \end{pmatrix}$ is $\dfrac{(k+1)!}{0!(k+1)!} = 1.$

Hence the theorem holds in all cases.

The above theorem tells us that the coefficient of $x^r y^s$ in $(x+y)^n$ is

$$\frac{n!}{r!s!}.$$

Since this expression has the same value when r and s are interchanged, we again see that the binomial coefficients have the symmetry relation

$$\begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n \\ n-r \end{pmatrix}.$$

By writing out the factorials more explicitly, we see that

$$\begin{aligned} \begin{pmatrix} n \\ r \end{pmatrix} &= \frac{n!}{r!(n-r)\,!} \\ &= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots 2\cdot 1}{1\cdot 2\cdot 3\ldots r(n-r)(n-r-1)\ldots 2\cdot 1}. \end{aligned}$$

Cancelling common factors, we now have

$$\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n(n-1)(n-2)\ \ldots\ (n-r+1)}{1\cdot 2\cdot 3\ \ldots\ r}.$$

This is the alternate form of the theorem illustrated for $n = 8$ in (4), above.

We can now rewrite the expansion of $(a+b)^n$ in the form

$$\begin{aligned} (a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \ldots \\ + \frac{n(n-1)\ldots(n-r+1)}{1\cdot 2\ \ldots\ r}a^{n-r}b^r + \cdots + b^n. \end{aligned}$$

This last formula it generally called the **<u>Binomial</u>** **<u>Theorem.</u>**

The formulas

$$\binom{n}{0} = 1, \quad \binom{n}{r} = \frac{n(n-1)\ldots(n-r+1)}{1\cdot 2\ \ldots\ r} \quad \text{for } r > 0$$

enable us to extend the definition of $\binom{n}{r}$, previously defined only for integers *n* and *r* with $0 \le r \le n$, to allow *n* to be any integer. We then have, for example, $\binom{2}{5} = 0$, $\binom{-2}{7} = -8$, and $\binom{-3}{8} = 45$.

It can easily be shown that the formula

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

holds with the extended definition as it did with the original definition.

Now the identity

$$2\binom{m}{2} + \binom{m}{1} = 2\frac{m(m-1)}{2} + m = m^2 - m + m = m^2$$

holds for all integers *m*, and we can use the formulas

$$\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \ldots + \binom{n}{1} = \binom{n+1}{2}$$

$$\binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \ldots + \binom{n}{2} = \binom{n+1}{3}$$

to show that

$$
\begin{aligned}
1^2 + 2^2 + \ldots + n^2 \\
&= \left[2\binom{1}{2} + \binom{1}{1}\right] + \left[2\binom{2}{2} + \binom{2}{1}\right] + \ldots + \left[2\binom{n}{2} + \binom{n}{1}\right] \\
&= 2\left[\binom{1}{2} + \binom{2}{2} + \ldots + \binom{n}{2}\right] + \left[\binom{1}{1} + \binom{2}{1} + \ldots + \binom{n}{1}\right] \\
&= 2\binom{n+1}{3} + \binom{n+1}{2} \\
&= 2\frac{(n+1)n(n-1)}{6} + \frac{(n+1)n}{2} \\
&= \frac{2n^3 - 2n}{6} + \frac{3n^2 + 3n}{6} \\
&= \frac{2n^3 + 3n^2 + n}{6} \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}
$$

Frequently in mathematical literature a short notation for sums is used which involves the Greek capital letter sigma, written $\sum$. In this notation,

$$a_1 + a_2 + \ldots + a_n$$

is written as

$$\sum_{i=1}^{n} a_i$$

and the auxiliary variable i is called the **index of summation**. Thus, for example,

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{i=1}^{6} 1 = 1 + 1 + 1 + 1 + 1 + 1 = 6$$

$$\sum_{j=1}^{n-1} j^2 = 1^2 + 2^2 + 3^2 + \ldots + (n-1)^2.$$

Under the capital sigma, one indicates the symbol that is used as the index of summation and the initial value of this index. Above the sigma, one indicates the final value. The general polynomial $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ of degree n can be written as

$$\sum_{k=0}^{n} a_k x^{n-k}.$$

One easily sees that

$$\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (a_i + b_i)$$

since

$$\begin{aligned} \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i &= (a_1 + a_2 + \ldots + a_n) + (b_1 + b_2 + \ldots + b_n) \\ &= (a_1 + b_1) + (a_2 + b_2) + \ldots + (a_n + b_n) \\ &= \sum_{i=1}^{n} (a_i + b_i). \end{aligned}$$

Also, $\sum_{i=1}^{n} (ca_i) = c\sum_{i=1}^{n} a_i$, the proof of which is left to the reader. However,

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \neq \sum_{i=1}^{n} (a_ib_i)$$

as can easily be shown by counterexample. (See Problem 19 of this chapter.)

A corresponding notation for products uses the Greek letter pi:

$$\prod_{i=1}^{n} a_i = a_1a_2 \ldots a_n.$$

In this notation, $n!$ for $n \geq 1$ can be expressed as $\prod_{k=1}^{n} k.$

In solving problems stated in terms of the sigma or pi notation, it is sometimes helpful to rewrite the expression in the original notation.

Problems for Chapter 6

1. Find each of the following:

   (a) The coefficient of x^4y^{16} in $(x + y)^{20}$.
   (b) The coefficient of x^5 in $(1 + x)^{15}$.
   (c) The coefficient of x^3y^{11} in $(2x - y)^{14}$.

2. Find each of the following:

   (a) The coefficient of $a^{13}b^4$ in $(a + b)^{17}$.
   (b) The coefficient of a^{11} in $(a - 1)^{16}$.
   (c) The coefficient of a^6b^6 in $(a - 3b)^{12}$.

3. Find integers a, b, and c such that $6\binom{n}{3} = n^3 + an^2 + bn + c$ for all integers n.

4. Find integers p, q, r, and s such that $4!\binom{n}{4} = n^4 + pn^3 + qn^2 + rn + s$ for all integers n.

5. Prove that $\binom{n}{3} = 0$ for $n = 0, 1, 2$.

6. Given that k is a positive integer, prove that $\binom{n}{k} = 0$ for $n = 0, 1, \ldots, k - 1$.

7. Find $\binom{-1}{r}$ for $r = 0, 1, 2, 3, 4,$ and $5.$.

8. Find $\binom{-2}{r}$ for $r = 0, 1, 2, 3, 4,$ and $5.$.

9. Prove that $\binom{-3}{r} = (-1)^r\binom{r+2}{2}$ for $r = 0, 1, 2, \ldots$.

10. Prove that $\binom{-4}{r} = (-1)^r \binom{r+3}{3}$ for $r = 0, 1, 2, \ldots$.

11. Let m be a positive integer and r a non-negative integer. Express $\binom{-m}{r}$ in terms of a binomial coefficient $\binom{n}{k}$ with $0 \le k \le n$.

12. In the original definition of $\binom{n}{r}$ as a binomial coefficient, it was clear that it was always an integer. Explain why this is still true in the extended definition.

13. Show that $\binom{n}{a}\binom{n-a}{b} = \dfrac{n!}{a!b!(n-a-b)!}$ for integers a, b, and n, with $a \ge 0$, $b \ge 0$, and $n \ge a + b$.

14. Given that $n = a + b + c + d$ and that a, b, c, and d are non-negative integers, show that

$$\binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}\binom{n-a-b-c}{d} = \frac{n!}{a!b!c!d!}.$$

15. Express $\sum_{k=1}^{n} [a + (k-1)d]$ as a polynomial in n.

16. Express $\prod_{k=1}^{n} (2k)$ compactly without using the Π notation.

17. Show that $\prod_{k=1}^{n} a_k = \prod_{j=0}^{n-1} a_{j+1}$.

18. Show that $\sum_{k=1}^{n-2} b_k = \sum_{i=3}^{n} b_{i-2}$.

19. Evaluate $\left(\sum_{i=1}^{2} a_i\right)\left(\sum_{i=1}^{2} b_i\right)$ and $\sum_{i=1}^{2} (a_i b_i)$ and show that they are not always equal.

20. Show that $\left(\prod_{i=1}^{n} a_i\right)\left(\prod_{i=1}^{n} b_i\right) = \prod_{i=1}^{n} (a_i b_i)$.

21. Prove by mathematical induction that $\sum_{i=0}^{n} \binom{s+i}{s} = \binom{s+1+n}{s+1}$.

22. Prove that $\sum_{j=0}^{n} \binom{s+j}{j} = \binom{s+1+n}{n}$.

23. Express $\sum_{k=1}^{n-2} \frac{k(k+1)}{2}$ as a polynomial in n.

24. Express $\sum_{k=1}^{n-2} \binom{k+1}{k-1}$ as a polynomial in n.

25. Write $6\left[\binom{n}{3} + \binom{n}{2} + \binom{n}{1}\right]$ as a polynomial in n, and then use the fact that $\binom{n}{r}$ is always an integer to give a new proof that $n(n^2 + 5)$ is an integral multiple of 6 for all integers n.

26. (a) Write $4!\left[\binom{n}{4} + \binom{n}{3} + \binom{n}{2} + \binom{n}{1}\right]$ as a polynomial in n.

(b) Show that $n^4 - 2n^3 + 11n^2 + 14n$ is an integral multiple of 24 for all integers n.

27. Find numbers s and t such that $n^3 = n(n - 1)(n - 2) + sn(n - 1) + tn$ holds for $n = 1$ and $n = 2$.

28. Find numbers a and b such that $n^3 = 6\binom{n}{3} + a\binom{n}{2} + b\binom{n}{1}$ for all integers n.

29. Find numbers r, s, and t such that $n^4 = n(n - 1)(n - 2)(n - 3) + rn(n - 1)(n - 2) + sn(n - 1) + tn$ for $n = 1, 2,$ and 3. Using these values of r, s, and t, show that

$$n^4 = 24\binom{n}{4} + 6r\binom{n}{3} + 2s\binom{n}{2} + t\binom{n}{1}$$

for all integers n.

30. Find numbers a, b, c, and d such that

$$n^5 = 5!\binom{n}{5} + a\binom{n}{4} + b\binom{n}{3} + c\binom{n}{2} + d\binom{n}{1}.$$

31. Express $\sum_{k=1}^{n} k^4$ as a polynomial in n.

32. Express $\sum_{k=1}^{n} k^5$ as a polynomial in n.

33. We define a sequence $S_0, S_1, S_2, \ldots$ as follows: When n is an even integer $2t$, let

$S_n = S_{2t} = \sum_{j=0}^{t}\binom{t+j}{t-j}$. When n is an odd integer $2t + 1$, let $S_n = S_{2t+1} = \sum_{j=0}^{t}\binom{t+1+j}{t-j}$.

Prove that $S_{2t} + S_{2t+1} = S_{2t+2}$ and $S_{2t+1} + S_{2t+2} = S_{2t+3}$ for $t = 0, 1, 2, \ldots$.

34. For the sequence defined in Problem 33, prove that S_n is the Fibonacci number F_{n+1}.

*35. Prove the following property of the Fibonacci numbers:

$$\sum_{j=0}^{n}\binom{n}{j}(-1)^j F_{s+2n-2j} = F_{s+n}.$$

*36 Prove an analogue of the formula of Problem 35 for the Lucas numbers.

37 Find a compact expression, without using the sigma notation, for

$$1 \cdot n + 2(n-1) + 3(n-2) + \cdots + (n-1) \cdot 2 + n \cdot 1,$$

that is, for $\sum_{k=0}^{n-1}(k+1)(n-k)$.

ANSWERS TO THE
ODD-NUMBERED PROBLEMS

Chapter1, page 5

1. 10.

3. 1.

5.
$$\begin{array}{cccc} 1 & 2 & 1 & \\ \underline{\ } & \underline{1} & \underline{2} & \underline{1} \\ 1 & 3 & 3 & 1. \end{array}$$

7. 1, 1 + 5 = 6, 5 + 10 = 15, 10 + 10 = 20, 10 + 5 = 15, 5 + 1 = 6, and 1;
1, 1 + 6 = 7, 6 + 15 = 21, 15 + 20 = 35, 20 + 15 = 35, 15 + 6 = 21, 6 + 1 = 7, and 1.

9. $\binom{9}{7} = \binom{9}{2} = 36, \binom{9}{8} = \binom{9}{1} = 9.$

11. $125x^3 + 150x^2y + 60xy^2 + 8y^3$.

17. $c = 56$, $m = 5$.

19. (a) 4. (b) 8. (c) 16. (d) 32. (e) 64. (f) 128.

21. (a) 0. (b) 0.

23. (a) 8. (b) 16. (c) 32. (d) 64.

25. $r = 6$, $s = 7$, $t = 6$, $u = 7$.

27. $\binom{996}{990}$

31. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 + 4x^3z + 12x^2yz + 12xy^2z +$
$4y^3z + 6x^2z^2 + 12xyz^2 + 6y^2z^2 + 4xz^3 + 4yz^3 + z^4.$

33. 3.

37. 3.

39. 36, of which 27 are odd.

41. 0, 1, 3, 7, and 15.

Chapter 2, page 14

3. 103.

7. $F_{n+4} = 3F_{n+1} + 2F_n$.

9. 321.

13. 702.

19. (a) $F_{16} - 1$. (b) $F_{19} - 1$.

21. (a) 2. (b) 3. (c) 5. (d) 8. (e) 13. (f) 21.

23. $r = 4$, $s = 5$, and $t = 11$.

25. $r = 15$, $s = 17$, and $t = 19$.

29. 204

33. $L_{2m+2} - 2$.

35. $r = 4$, $s = 6$, and $t = 8$.

37. (a) 1. (b) -1. (c) 1. (d) -1.

Chapter 3, page 18

1. (a) 5040. (b) 36. (c) 362,880. (d) 720.

5. (a) 5!. (b) 7!. (c) $(n + 1)!$.

7. $a = 14$ and $b = 10$ or $a = 24024$ and $b = 24023$.

9. $n^4 + 10n^3 + 35n^2 + 50n + 24$.

11. (a) 23. (b) 119. (c) 719.

15. $a = 6$, $b = 11$, and $c = 5$.

17. $(2m + 2)! - 1$.

19. (a) $(n + 2)! + (n + 1)! - 2$.

Chapter 4, page 23

1. (a) -4, 3, 10. (b) -19, -23, -27.

3. (a) 990. (b) 993. (c) 988.

5. 33/2.

7. (a) -33,698. (b) 501,994. (c) -11,385/4. (d) n^2.

9. 54, 1458, 13122.

11. 686, 4802, 33614.

13. ± 42.

15. (a) $(7^{1000} - 1)/6$. (b) $(1 - 7^{1000})/8$. (c) $(7^n - 1)/6$.

17. 9.

21. (a) $6\sqrt{3}$. (b) $6\sqrt{3}$. (c) 4. (d) 4.

23. 30.

27. $(x^{n+1} - y^{n+1})/(x - y)$.

29. (a) 5. (b) 24 miles per hour.

31. $2[1 - (1/2)^7]$.

35. (a) 0, 3, 5, 6, 9, 10, 12, 15, 18, 20, 21, 24, 25, 27, 30.
(b) 15. (c) 15. (d) $16 \bullet 15 = 240$.

Chapter 5, page 35

13. (a) $1 - \dfrac{1}{n+1}$. (b) $\dfrac{x^{2^{n+1}} - y^{2^{n+1}}}{x - y}$.

15. $(n + 1)/2n$ for $n > 1$.

21. Using mathematical induction, one can show that:

(a) $F_{n+1}^2 - F_n F_{n+2} = (-1)^n$.

(b) $\dfrac{F_{n+2}^2 - F_{n+1}^2}{F_n} = F_{n+3}$.

(c) $F_{n-1} + F_{n+1} = L_n$.

Chapter 6, page 47

1. (a) 4,845. (b) 3,003. (c) -2,912.

3. $a = -3$, $b = 2$, $c = 0$.

7. 1, -1, 1, -1, 1, -1.

11. $\dbinom{-m}{r} = (-1)^r \dbinom{r+m-1}{m-1}$.

15. $(d/2)n^2 + [a - (d/2)]n$.

19. $\left(\sum_{i=1}^{2} a_i\right)\left(\sum_{i=1}^{2} b_i\right) = a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2$ and $\sum_{i=1}^{2} (a_ib_i) = a_1b_1 + a_2b_2$. These are not always equal, since, for example, they are unequal for $a_1 = a_2 = b_1 = b_2 = 1$.

23. $(1/6)n^3 - (1/2)n^2 + (1/3)n$.

25. $n^3 + 5n$.

27. $s = 3$, $t = 1$.

29. $r = 6$, $s = 7$, $t = 1$.

31. $(1/5)n^5 + (1/2)n^4 + (1/3)n^3 - (1/30)n$.

37. $(n^3 + 3n^2 + 2n)/6$.