MATH3024 Project

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Introduction

This is the project for Fraser Paterson. All dynamics were generated via from scipy.integrate.odeint and each call to odeint performed an integral over 10000 time steps.

Questions

Question 1

The system for question 1 is as follows:

$$\dot{x} = \alpha(y - x - f(x))$$
$$\dot{y} = x - y + z$$
$$\dot{x} = -\beta y$$

where

$$f(x) = bx + \frac{1}{2}(a - b)[|x + 1| - |x - 1|]$$

and $\alpha = 10.0, \beta = 14.87, a = -1.27, b = -0.68$

1.a This is as described in my interem report.

1.b For an x to x coupling we take a copy of the entire system and add the difference in the x coordinates, multiplied by the coupling strength: σ, to both x coordinates in each respective system. Let each coordinate dynamics be given by a primed value, hence we have:

$$\dot{x} = \alpha(y - x - f(x)) + \sigma(x' - x)$$
$$\dot{y} = x - y + z$$
$$\dot{x} = -\beta y$$
$$\dot{x'} = \alpha(y' - x' - f(x')) + \sigma(x - x')$$
$$\dot{y'} = x' - y' + z'$$
$$\dot{x'} = -\beta y'$$

where

$$f(x') = bx + \frac{1}{2}(a - b)[|x' + 1| - |x' - 1|]$$

and the parameters remain as above.

See figures 1, 2 and 3 for plots of each respective coordinate dynamics (uncoupled dynamics).

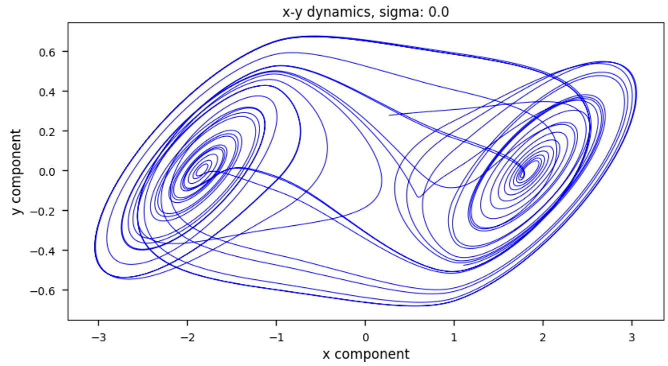

Figure 1: x - y dynamics

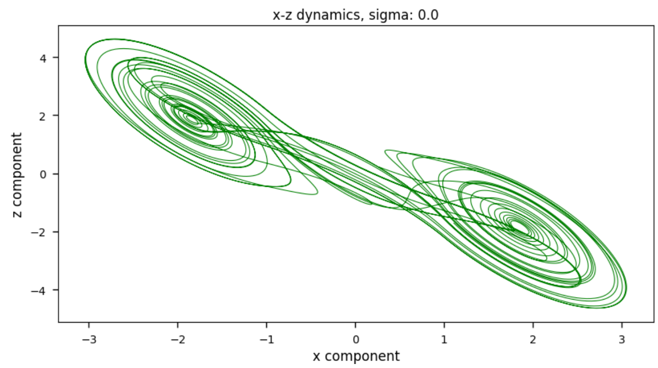

Figure 2: x - z dynamics

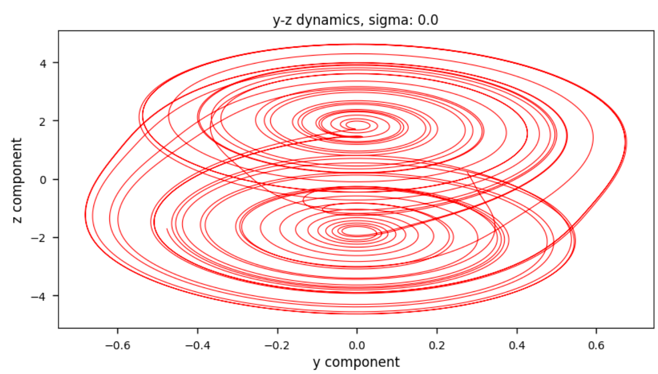

Figure 3: y - z dynamics

The uncoupled dynamics in time are as follows:

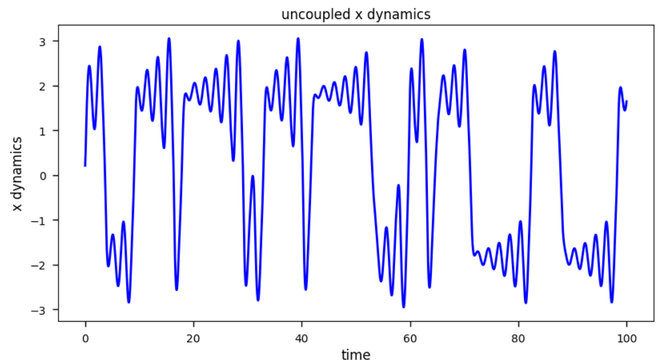

Figure 4: uncoupled x component dynamics

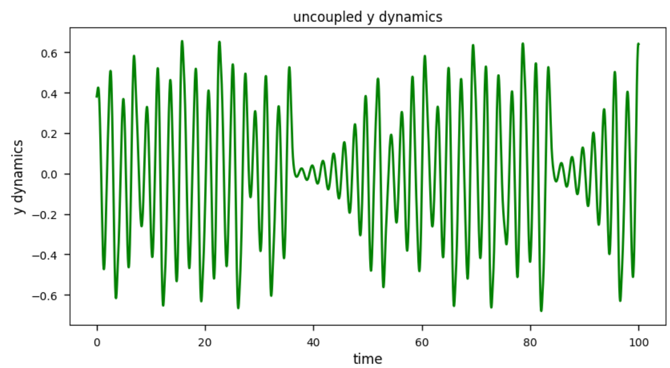

Figure 5: uncoupled y component dynamics

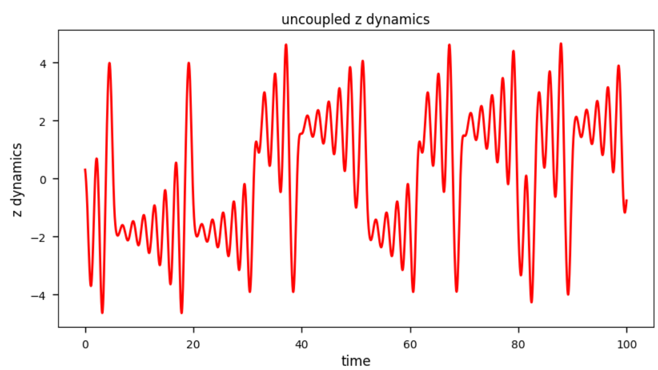

Figure 6: uncoupled z component dynamics

To analyse the stability of the coupling for any given $\sigma \in [0, 10.0]$ we define the following errors in each respective component of the coupled dynamics:

$$e_x = x - x'$$

$$e_y = y - y'$$

$$e_z = z - z'$$

Substituting in each respective expression for x and x' into these error equations, we have:

$$e_x = \alpha(e_y - e_x - (f(x) - f(x'))) - 2\sigma e_x$$

Note, that by the intermidiate value theorem $f(x) - f(x') = f'(\eta)(x - x')$ we may express the above as:

$$e_x = \alpha(e_y - e_x - f'(\eta)e_x) - 2\sigma e_x$$

Now, $f'(\eta)$ = a or b. Hence let $f'(\eta) = \xi$ and so we have: $e_x = (-\alpha - \alpha\xi - 2\sigma)e_x + \alpha e_y$. Performing a similar set of substitutions for the y and z errors yields the full error dynamics:

$$e_x = (-\alpha - \alpha\xi - 2\sigma)e_x + \alpha e_y$$

$$e_y = e_x - e_y + e_z$$

$$e_z = -\beta e_y$$

Expressing the error dynamics as a matrix equation:

$$\begin{pmatrix} \dot{e_x} \\ \dot{e_y} \\ \dot{e_z} \end{pmatrix} = \begin{pmatrix} -\alpha - \alpha\xi - 2\sigma & \alpha & 0 \\ 1 & -1 & 0 \\ 0 & -\beta & 0 \end{pmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}$$

Now if we numerically calculate each respective component error and plot this against the coupling strength, for $\sigma \in [0, 10.0]$, where we take step sizes of 0.01 for σ we yield the plots shown in figures 4, 5 and 6. Note, there were no further changes to the x component error for $\sigma \in (5, 10]$

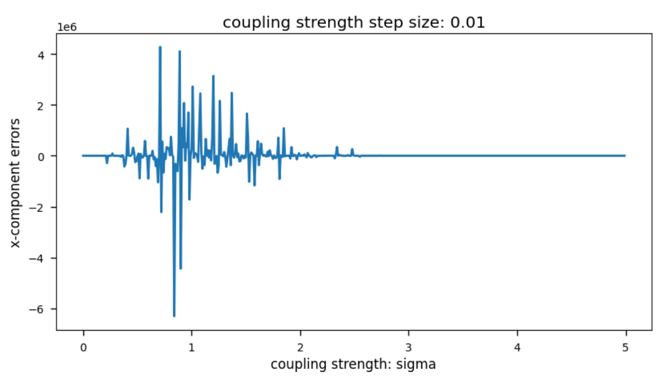

Figure 7: x component error dynamics

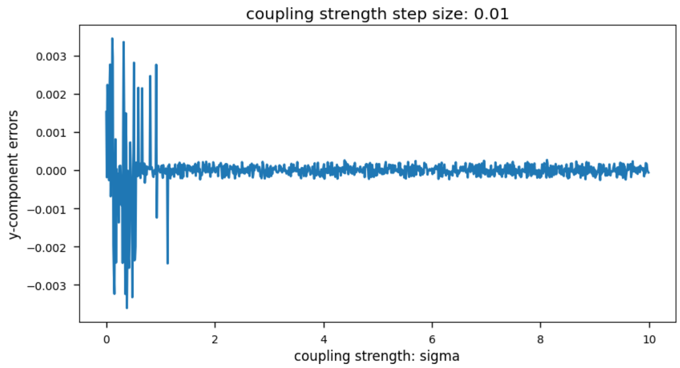

Figure 8: y component error dynamics

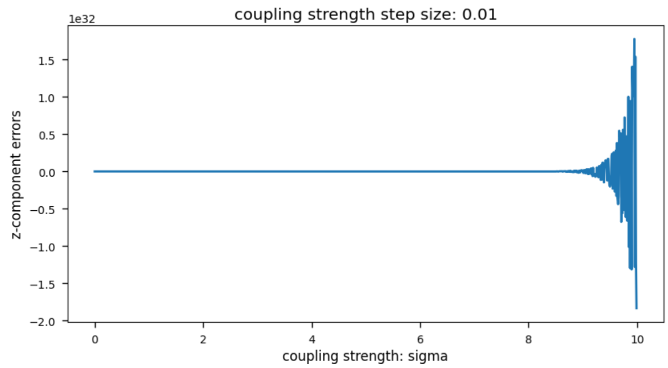

Figure 9: z component error dynamics

Numerically integrating the system of x-x coupled dynamics yields the plots shown in figure 7 to 12. The plots in navy blue correspond to sigma = 2.6 while those in red correspond to sigma = 2.8.

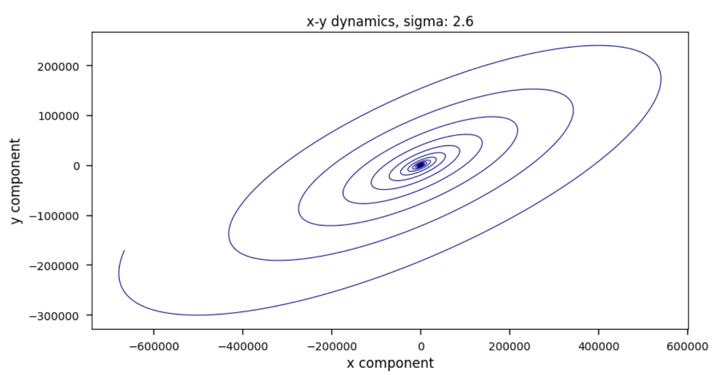

Figure 10: x - y dynamics for x-x coupling

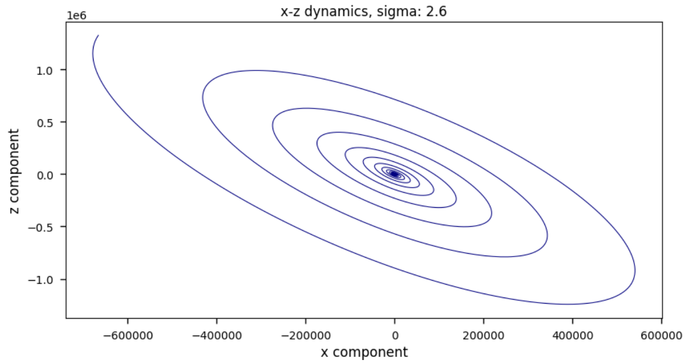

Figure 11: x - z dynamics for x-x coupling

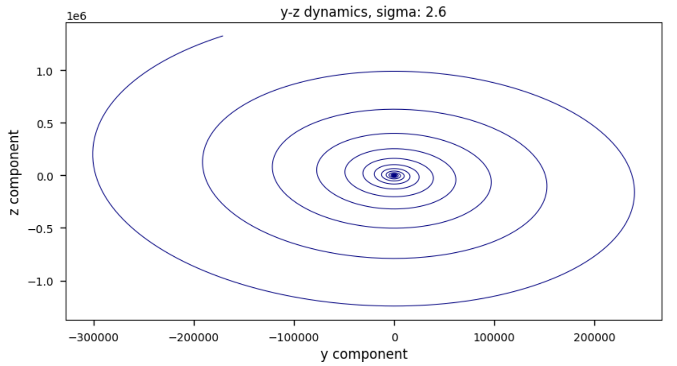

Figure 12: x - z dynamics for x-x coupling

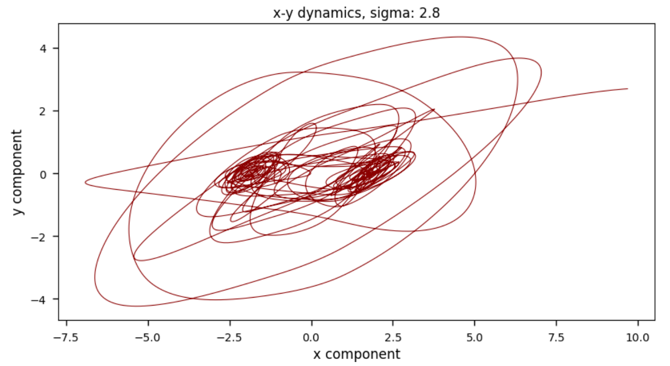

Figure 13: x - y dynamics for x-x coupling

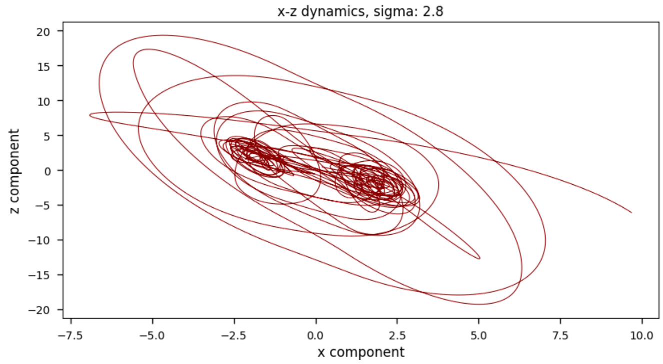

Figure 14: x - z dynamics for x-x coupling

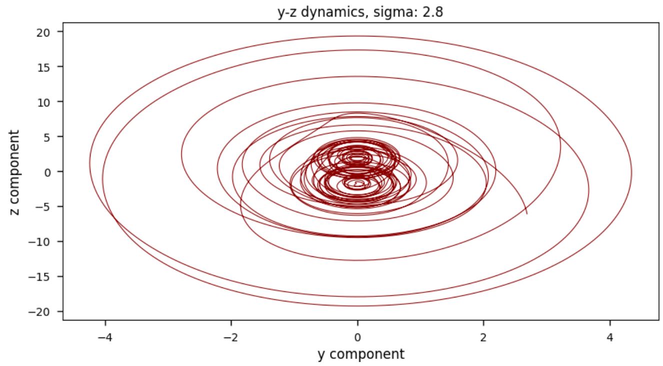

Figure 15: x - z dynamics for x-x coupling

The dynamics in navy correspond to an unstable node; as can be seen if we plot the component dynamics in time for $\sigma = 2.6$

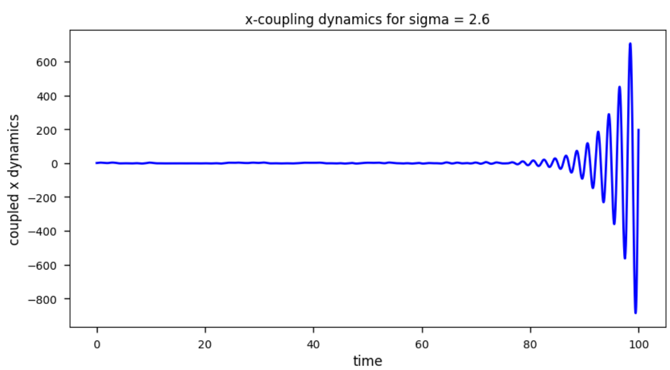

Figure 16: x component dynamics through time for x-x coupling

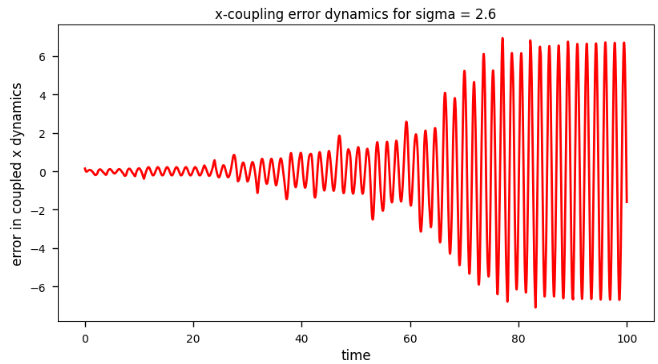

Figure 17: x component error dynamics through time for x-x coupling

Although the x component error dynamics for $\sigma = 2.8$ is not yet 0, we can see that the error in the x component reaches a maximum and then begins to decrease with time:

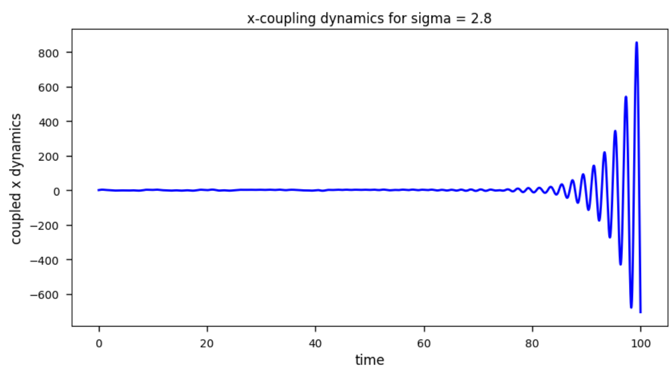

Figure 18: x component dynamics through time for x-x coupling

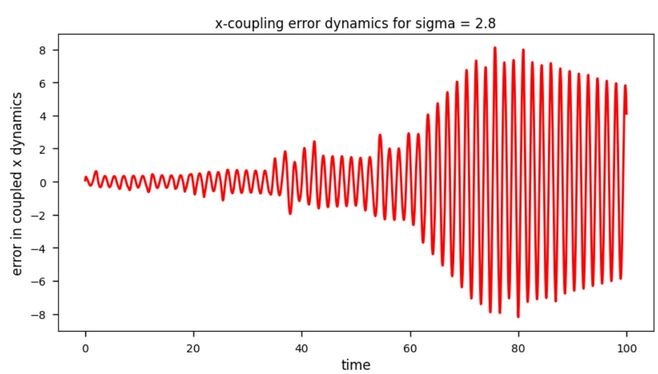

Figure 19: x component error dynamics through time for x-x coupling

Indeed by $\sigma = 3.9$ at the latest, the error in the x component becomes negligible over the domain displayed:

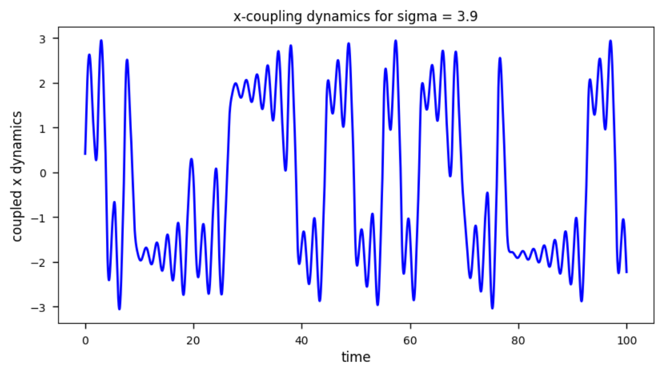

Figure 20: x component dynamics through time

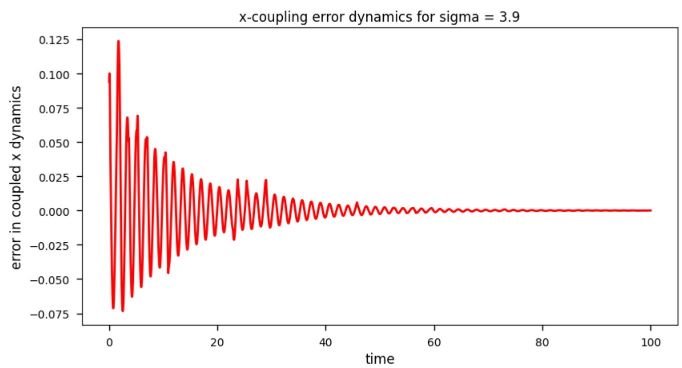

Figure 21: x component error dynamics through time

1.c In the case of a y to y and z to z coupling, we perform the same series of operations as in the above x to x coupling, making sure to add the coupling term to the correct dynamical component

For the y to y coupling we have:

$$\dot{x} = \alpha(y - x - f(x))$$
$$\dot{y} = x - y + z + \sigma(y - y')$$
$$\dot{x} = -\beta y$$
$$\dot{x'} = \alpha(y' - x' - f(x'))$$
$$\dot{y'} = x' - y' + z' + \sigma(y' - y)$$
$$\dot{x'} = -\beta y'$$

and $f(x')$ is as before.

The error dynamics in the y-y coupled system appeared to converge to 0 for $\sigma \approx 1$:

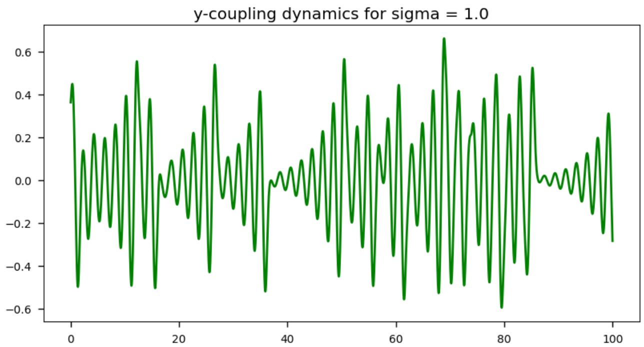

Figure 22: y component dynamics through time y-y coupling

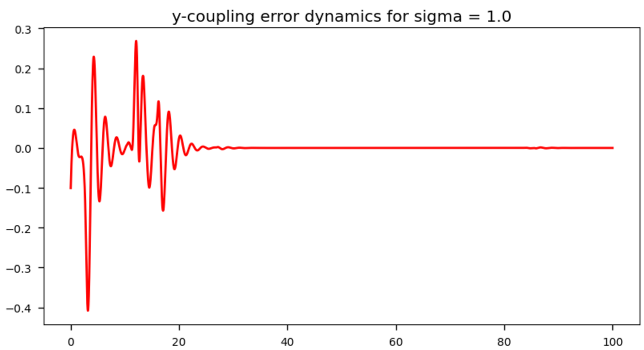

Figure 23: y component error dynamics through time y-y coupling

Hence the system exhibits stable synchronisation for $\sigma >\approx 1$

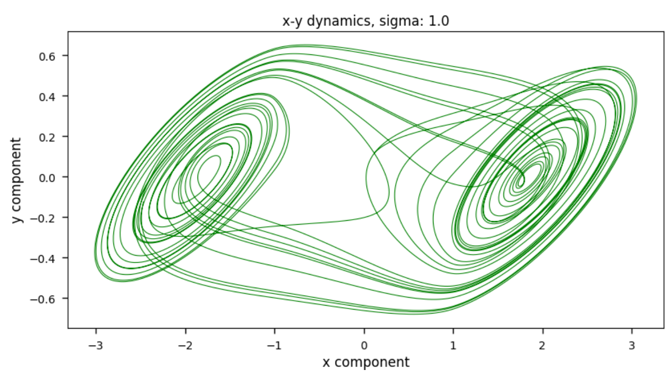

Figure 24: x - y dynamics for the y - y coupling

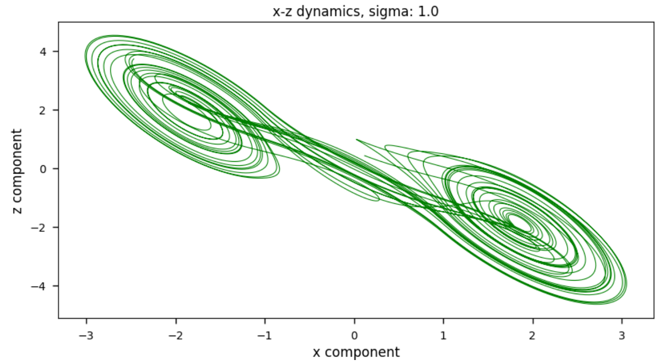

Figure 25: x - z dynamics for the y - y coupling

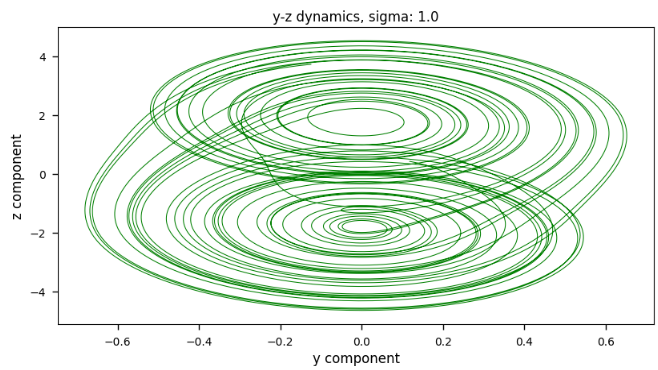

Figure 26: y - z dynamics for the y - y coupling

The error dynamics in the z-z coupled system converged for all values of $\sigma < 9$ (approx). any small increase in the coupling strength past approximately 9 lead to a dramatic change in the error dynamics. For example, the z-z dynamics for $\sigma = 10.0$ blew up to $-\infty$ while the error for $\sigma = 10.1$ blew up to ∞

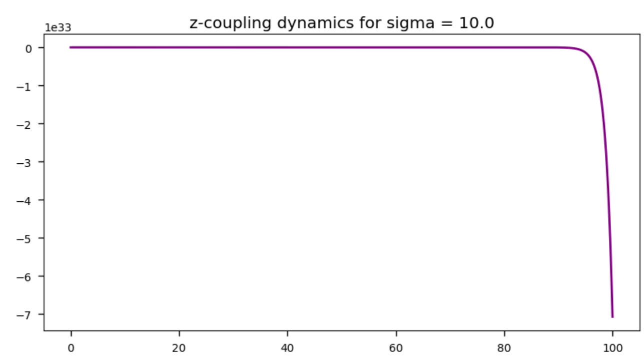

Figure 27: z component dynamics through time, z-z coupling

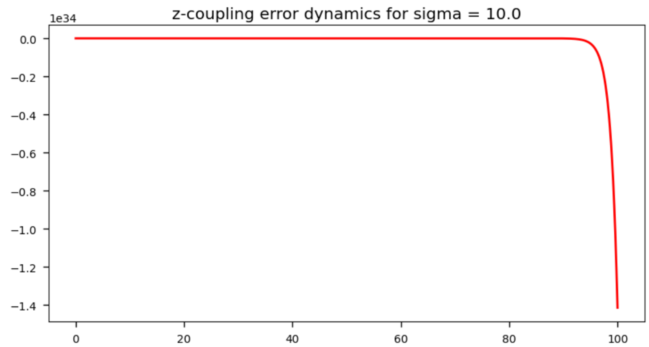

Figure 28: z component error dynamics through time, z-z coupling

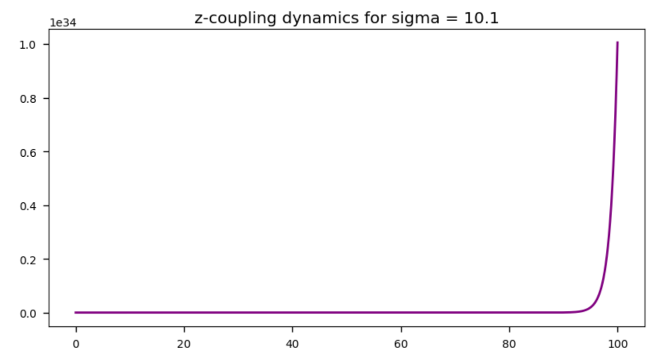

Figure 29: z component dynamics through time, z-z coupling

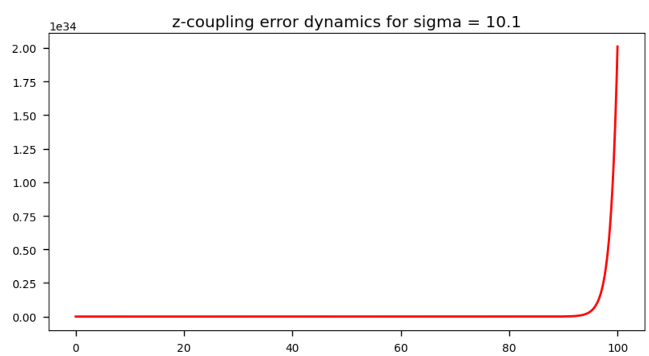

Figure 30: z component error dynamics through time, z-z coupling

Hence the system exhibits stable synchronisation for $\sigma \approx 1$. Indeed we have:

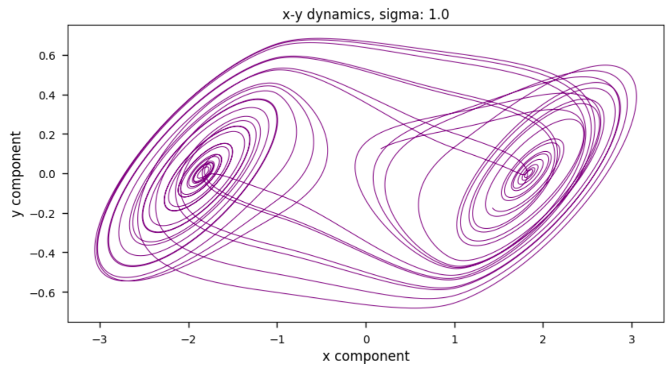

Figure 31: x-y dynamics for the z-z coupling

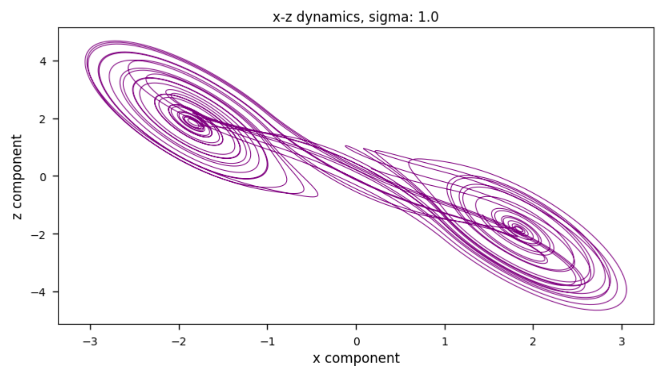

Figure 32: x-z dynamics for the z-z coupling

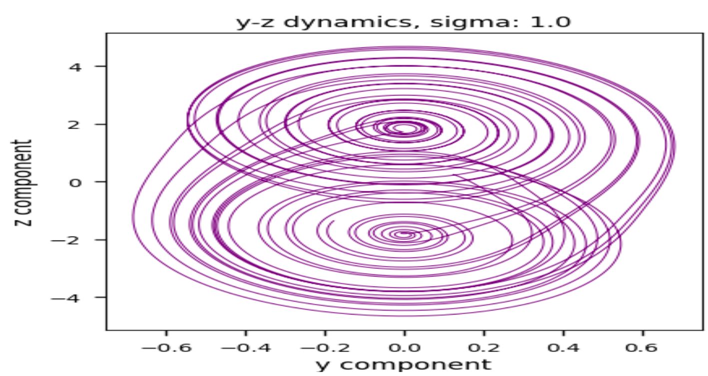

Figure 33: y-z dynamics for the z-z coupling

For which the corresponding error dynamics in the z component is:

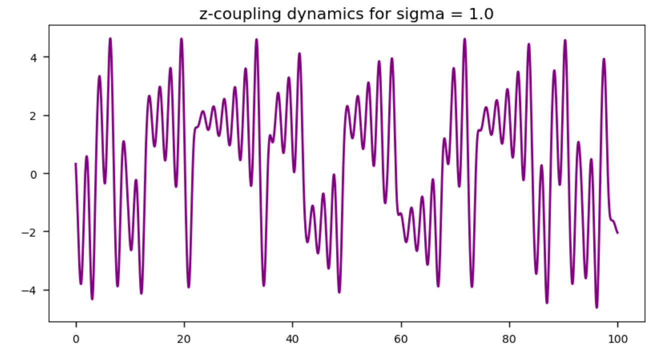

Figure 34: z component dynamics through time z-z coupling

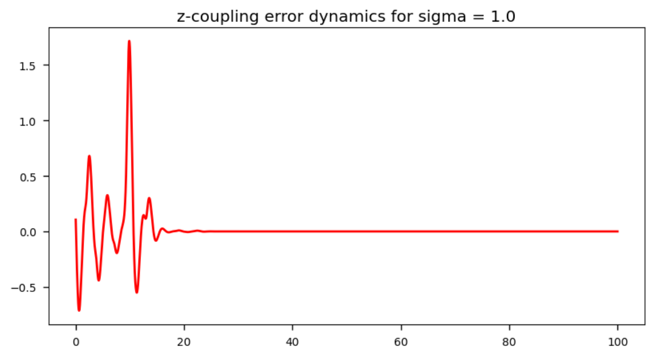

Figure 35: z component error dynamics through time z-z coupling

Question 2

The maths for question 2 The following is a graph:

Question 3

Maths for question 3