Localized interpretation to compute all the coefficients g_n^m in the generalized Lorenz–Mie theory

G. Gouesbet, G. Gréhan, and B. Maheu

Laboratoire d'Energétique des Systèmes et Procédés, Unité de Recherche Associée au Centre National de la Recherche Scientifique, No. 230, Institut National des Sciences Appliquées de Rouen, B.P. 08, 76131 Mont-Saint-Aignan Cedex, France

Received May 5, 1989; accepted January 6, 1990

Numerical computations in the framework of the generalized Lorenz–Mie theory require the evaluation of a new double set of coefficients $g_{n,\mathrm{TM}}^m$ and $g_{n,\mathrm{TE}}^m$ ($n = 1, \ldots, \infty$; $m = -n, \ldots, +n$). A localized interpretation of these coefficients is designed to permit fast and accurate computations, even on microcomputers. When the scatter center is located on the axis of the beam, a previously published localized approximation for a simpler set of coefficients g_n is recovered as a special case. The subscript n in coefficients g_n and g_n^m is associated with ray localization and discretization of space in directions perpendicular to the beam axis, while superscript m in coefficients g_n^m is associated with azimuthal wave modes.

1. INTRODUCTION

Our original motivation in the development of a generalized Lorenz–Mie theory (GLMT) was to achieve an understanding and a better design of optical techniques for simultaneous measurements of velocities and sizes of particles transported by flows. Reference 1 gives a good account of the present state of the art in optical sizing techniques. It appears that most data processing, design of instruments, and theory principles rely on the classical Lorenz–Mie theory, i.e., the theory of interaction between an ideal spherical particle (with an isotropic, homogeneous, nonmagnetic material) and a plane wave,[2] although the light source in experiments is often a laser source. Clearly, when the laser beam is focused and particle diameters are large in comparison with the laser beam diameter, the use of the Lorenz–Mie theory is misleading, thus requiring the use of a GLMT. Other applications of the GLMT might be the understanding and design of optical levitation experiments[3–6] and the computation of radiation pressure forces in the development of laser thermonuclear fusion experiments, plus a host of other expected applications that might exist although we are not clever enough to list them.

We developed a GLMT step by step, from Ref. 7 to Refs. 8 and 9, in which other intermediary references may be found. Reference 8 concerns the case of light scattering from a sphere that is arbitrarily located in a Gaussian beam, while Ref. 9 concerns another generalization for arbitrary beams and/or different descriptions of the laser beam. Notations in this paper are those of Ref. 8, to which the reader should refer before reading this paper. References 10 and 11 provide short and long reviews, respectively, of the GLMT from which all our references in this field are quoted.

For applications of the theory, numerical calculations by a computer are required. Beside the more-or-less classical mathematical functions that are not discussed here, the GLMT involves new sets of coefficients that contain all the information on the incident beam.

Let us focus our attention on the case when the incident beam is a Gaussian beam. Then, in the general case of the arbitrary location of the particle, the new sets are named $g_{n,\mathrm{TM}}^m$ and $g_{n,\mathrm{TE}}^m$ ($n = 1, \ldots, \infty$; $m = -n, \ldots, +n$; TM, transverse magnetic; TE, transverse electric). In the special case of the location of the particle on the beam axis, the GLMT simplifies dramatically, and the double set reduces to a single set of coefficients g_n ($n = 1, \ldots, +\infty$). Computation of these coefficients is not a trivial matter.

For the g_n's, we know three methods. In the first one, g_n's are evaluated as quadratures. The method is rigorous and arises from the original development of the theory. However, functions to integrate are oscillating, leading to a loss of accuracy and a costly and time-consuming task requiring the use of mainframe computers. In a second method developed later, the g_n's are evaluated by using finite series. This method also is rigorous, and, formally, it is strictly equivalent to the previous one. It is advantageous in terms of cost and rapidity. However, there is a faster method, namely, the use of a localized interpretation of the coefficients to provide simple formulas that can in some cases be handled with pocket calculators. Although it is not rigorous, the approximation is good and results in an insignificant loss of accuracy in most relevant situations. Using localized approximation formulas, we can handle the GLMT for particle axis location with fast-running codes on microcomputers.[12] A complete discussion of these methods is available in Ref. 13. Several discussions of the localized interpretation appeared in the literature. References for these discussions are available in Ref. 13 and date back to the original, simplest (but special) form in Ref. 14.

For the g_n^m's, we also know two rigorous methods (quadratures and finite series) from which we recover the two rigorous methods for the g_n's, when the center of the particle approaches the beam axis. Quadrature expressions are available from Ref. 8, while finite-series expressions are developed in Ref. 15. Owing to their expected cost and time-consuming character, quadrature computations (which are necessarily more complex than for the g_n's) have never been attempted for evaluating the g_n^m's. Finite-series expressions for the g_n^m's are lengthy and awkward. Furthermore a single expression is not available but rather a list of expressions is,

0740-3232/90/060998-10$02.00

depending for instance on the parity of n and m. They will not be recalled in this paper. However, although they are lengthy and awkward, these expressions can be readily evaluated by using microcomputers. Being rigorous, these finite series computations will serve to provide reference data in the present paper.

Our aim has then been to generalize the localized approximation to the g_n's to provide a localized approximation for computing the coefficients g_n^m, because we expect that this third method to the g_n^m's will be faster and cheaper than the two previous, rigorous ones. In a first attempt we have been successful in developing only a localized approximation to the g_n^m's, with $|m| = 1$.[16,17] This approximation was rather simple because the g_n^m's, $|m| \neq 1$, tend to zero when the particle location approaches the beam axis. In other words, the g_n^m's, $|m| = 1$, which remain finite for axis location, play a special role. Furthermore, for axis location, these coefficients g_n^{+1} and g_n^{-1} are simply expressed in terms of the coefficients g_n.[8] However, more generally, the set $\{g_n\}$ and the set $\{g_{n,\mathrm{TM}}^m, g_{n,\mathrm{TE}}^m\}$ essentially contain the same information, namely, the description of the incident beam. Consequently, a fourth method for the g_n^m's is in principle feasible; we must have, namely, $g_{n,\mathrm{TM}}^m = f_{n,\mathrm{TM}}^m(g_i)$ and $g_{n,\mathrm{TE}}^m = f_{n,\mathrm{TE}}^m(g_i)$. Finding the functions f_n^m is an open subject of research. But knowing that they exist and that a successful localized approximation to the g_n's has been developed, we conclude that a successful localized approximation to the g_n^m's (whatever n and m) is possible. This paper is devoted to this generalization of the localized approximation, leading to a third method to compute all the coefficients g_n^m quickly and accurately.

The paper is organized as follows. In Section 2, the localized approximation to the g_n's is briefly recalled. Although the expressions provided are not new, a new procedure to derive them is given and supplies us with significant clues to the final generalization. Section 3 is devoted to the derivation of the localized approximation expressions to the g_n^m's. Section 4 provides numerical results and discussions, mainly comparisons between values of g_n^m's that are computed by using localized approximation and rigorous finite-series computations, providing evidence for the success of the generalization. Section 5 is a conclusion.

2. LOCALIZED APPROXIMATION TO THE COEFFICIENTS g_n

A. Generalities

We consider a Gaussian beam; the middle of the waist is located at point O_G (Fig. 1), and the axis O_Gw is the beam axis. Another Cartesian coordinate system (O_Pxyz) is used with O_Gu parallel to O_Px and similarly for the other axes. In solving the scattering problem of the GLMT, the center of the scatterer is located at point O_P, and scattering properties are evaluated at point $P(r, \theta, \varphi)$. Actually, in this paper, we need not refer to any scatter center but only have to think of describing the incident beam in the system (O_Pxyz) instead of describing it in the original Gaussian beam system (O_Guvw). However, for the sake of convenience, we discuss features in terms of scatter center location.

The Gaussian beam is described by following Davis's formulation[8,18,19] and neglecting all terms with powers of s higher than 1, s being a small parameter defined by

$$s = w_0/l = 1/(kw_0), \tag{1}$$

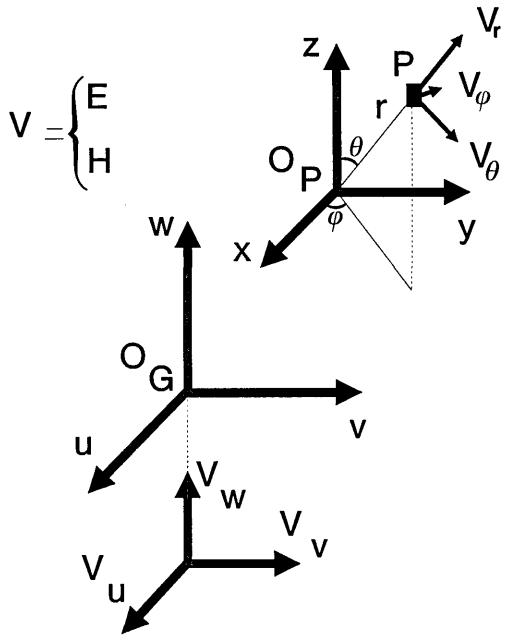

Fig. 1. Geometry of the problem.

in which w_0 is the beam-waist radius, l its spreading length, and k the wave number of the incident light. We call this description the order L (lowest order) of approximation to the description of the Gaussian beam. Then, to that order, we choose axes u and v so that the nonzero Cartesian field components are E_u, H_v, E_w, H_w. (E is the electric field; H is the magnetic field.) We also use a simpler approximation, called L^-, in which the field components E_w and H_w are taken as equal to zero. We have

$$O_PO_G = (x_0, y_0, z_0). \tag{2}$$

In that section, we assume that $x_0 = y_0 = 0$, $z_0 \neq 0$, which is the case of the location of a particle along the axis.

B. Localized Approximation to the Coefficients g_n

The localized approximation to the coefficients g_n is extensively discussed in Ref. 13. Results are briefly recalled in this section, however, by using a slightly different procedure that is more helpful to the final generalization in Section 3.

We show (e.g., Ref. 8) that the radial field components E_r and H_r read as

$$\begin{pmatrix} E_r \\ H_r \end{pmatrix} = \left\{ \begin{pmatrix} E_0 \cos\varphi \\ H_0 \sin\varphi \end{pmatrix} \sin\theta \exp(-ikz) \right\} \times \left[1 - \epsilon_L \frac{2Q}{l} z \right] iQ \exp\left[-iQ \frac{r^2 \sin^2\theta}{w_0^2} \right] \exp(ikz_0), \tag{3}$$

in whch $\epsilon_L = 1$ at the order L of approximation and 0 at the order L^-, and Q is given by

$$Q = \frac{1}{i + 2\dfrac{z - z_0}{l}}. \tag{4}$$

In Eq. (3), the term in braces is the only one that would appear if the incident beam were a plane wave. The multiplication factor in Eq. (3) (which we call $\mathcal{A}$) is then specific to the Gaussian beam that is under examination.

The radial components E_r and H_r play a special role in the theory because we need to know them only to derive two Bromwich scalar potentials, U^i_{TM} and U^i_{TE} (respectively), from which all the incident field components can be rederived in terms of spherical Bessel functions $\Psi^1_n(kr)$ (or equivalently in terms of Ricatti–Bessel functions) and of associated Legendre polynomials $p^1_n(\cos\theta)$. These Bromwich scalar potentials also contain coefficients g_n. The superscript i refers to an incident wave.

From this discussion we conclude that coefficients g_n must be expressed in terms of the Gaussian term $\mathcal{A}$ in Eq. (3). Then we must have

$$g_n = f(\mathcal{A}), \tag{5}$$

in which f is an operator acting on $\mathcal{A}$ to produce the g_n's. The coefficients g_n being constants, the operator f must comply with three simple requirements: f must (i) eliminate variable z in $\mathcal{A}$, (ii) eliminate variable $r\sin\theta$ in $\mathcal{A}$, and (iii) introduce a subscript n. We are now going to guess the operator of f in view of these requirements. For this guess, two guides are invoked: (i) use physics when convincing arguments can be found and (ii) be as simple as possible because it is assumed that nature likes simplicity. The quality of the guess will be then checked by direct comparisons between guess results and rigorous computations. It is remarkable that our first trial was immediately successful.

Our first guess concerns the way to eliminate variable z. The simplest way to proceed is to set it to a special value, namely, $z = 0$, since this value is the only one playing a special role (it defines the center of the coordinate system O_Pxyz).

To eliminate $r\sin\theta$, we rely on the van de Hulst principle of localization[20] according to which a term of order n corresponds to a ray passing the origin at a distance $(n + 1/2)(\lambda/2\pi)$. Here, λ is the wavelength of the light, and the term of order n is the term of a Lorenz–Mie series that is associated with a coefficient g_n in GLMT. Then we replace the distance $r\sin\theta$ to the axis z by $(n + 1/2)(\lambda/2\pi)$, which simultaneously eliminates $r\sin\theta$ and introduces a subscript n as required.

To sum up, from such simple considerations, we tentatively assume that the effect of operator f is (i) to replace z with 0 and (ii) to replace $r\sin\theta$ with $(n + 1/2)(\lambda/2\pi)$. Consequently, we obtain a localized approximation to the coefficients g_n according to

$$g_n = i\bar{Q}\exp\left[-i\bar{Q}\left(\frac{\rho_n}{w_0}\right)^2\right]\exp(ikz_0), \tag{6}$$

in which

$$\bar{Q} = Q(z=0) = \frac{1}{i - \dfrac{2z_0}{l}}, \tag{7}$$

$$\rho_n = \frac{(n+1/2)}{2\pi}\lambda. \tag{8}$$

When O_P is located at the beam waist center $O_G(z_0 = 0)$, Eq. (6) is reduced to a simple expression that can be computed by using pocket calculators:

$$g_n = \exp\left[-\left(\frac{\rho_n}{w_0}\right)^2\right], \tag{9}$$

which is actually the first result that we obtained in developing the localized approximation.[14] Equation (9) results from a mere generalization of the van de Hulst principle of localization and has been originally obtained without having to use the more complicated procedure explained above. This procedure (actually a similar one to that in Ref. 13) has been developed to permit us to generalize from Eq. (9) to Eq. (6), with the condition that Eq. (9) should be recovered as a special case.

From Eq. (9) we notice that, in the limit $w_0 \to \infty$, all g_n's become equal to 1, in agreement with a result that we know to be rigorously true for the plane-wave case.[7,8] The same limit leads to $g_n = \exp(ikz_0)$ from Eq. (6), i.e., g_n's are all equal to a phase term, having no physical significance for the plane-wave-scattering problem.

We also notice that the double-valued quantity ϵ_L does not appear in Eq. (6), i.e., the localized approximation results are identical, irrespective of the original order of approximation to the description of the Gaussian beam (L or L^-). This statement will remain true in the general case of arbitrary location of the scatter center (actually here, arbitrary location of O_P with respect to O_G).

Finally, although the derivation of Eq. (6) cannot be considered to be firmly demonstrated, its quality and validity have been shown several times. See Ref. 13, in which earlier references can be found. We can then think of a further generalization to the coefficients g^m_n. However, a new difficulty arises because we have now to cope with a subscript n and with a superscript m. As far as subscript n is concerned, we can assume that the reduction procedure explained above remains valid. Through the localization principle, this procedure induces a discretization of space in the directions perpendicular to z (ρ directions) conveyed by the introduction of discrete radial locations ρ_n. Because of this discretization, the wave character of the beam in ρ directions is lost, permitting us to discuss the problem in terms of geometry, that is to say, in terms of rays, or more exactly, in terms of an annular bundle of rays, each bundle of rays being associated with subscript n and passing at approximately a distance ρ_n from axis z. Subsection 2.C will show that, conversely, m is associated not with a new discretization of space but with m-wave modes.

C. Localized Approximation to the Coefficients g^{+1}_n, g^{-1}_n for Axis Location

For axis location, the formulation can be expressed in terms of g_n's but also in terms of g^{+1}_n's and g^{-1}_n's, all the other coefficients g^m_n, $|m| \neq 1$, being equal to 0. Furthermore, in this case, we have[8]

$$g^1_{n,TM} = g^{-1}_{n,TM} = g_n/2 = g_{n,TM}/2, \tag{10}$$

$$g^1_{n,TE} = -g^{-1}_{n,TE} = -ig_n/2 = -ig_{n,TE}/2. \tag{11}$$

Let us first discuss the TM coefficients that are associated with the field component E_r.[8] After we express $\cos\varphi$ in terms of exponentials, relation (3) for E_r can be rewritten as a two-term expansion:

$$E_r = E^{+1}_r + E^{-1}_r = \sum_{m=1,-1} E^m_r, \tag{12}$$

$$E_r^{+1} = \frac{E_0}{2} e^{i\varphi} \sin\theta \exp(-ikz)\mathcal{A}, \tag{13}$$

$$E_r^{-1} = \frac{E_0}{2} e^{-i\varphi} \sin\theta \exp(-ikz)\mathcal{A}, \tag{14}$$

in which $m = +1$ and $m = -1$ are associated with $e^{i\varphi}$ and $e^{-i\varphi}$ modes, respectively. From results for g_n in Subsection 2.B and relation (10), we have

$$g_{n,\mathrm{TM}}^{\pm 1} = \frac{1}{2} f(\mathcal{A}). \tag{15}$$

In other words, $g_{n,\mathrm{TM}}^{+1}$ is obtained from E_r^{+1} by removing a plane-wave contribution $E_0 \sin\theta \exp(-ikz)$ and the exponential term $e^{i\varphi}$ and by applying the operator f to the remaining term $\mathcal{A}/2$. Similarly, $g_{n,\mathrm{TM}}^{-1}$ is obtained from E_r^{-1} by removing a plane-wave contribution $E_0 \sin\theta \exp(-ikz)$ and the exponential term $e^{-i\varphi}$ and, again, by applying the operator f to the remaining term $\mathcal{A}/2$.

We now similarly consider the TE coefficients that are associated with H_r. Expressing $\sin\varphi$ in terms of exponentials, we can rewrite relation (3) for H_r as a two-term expansion,

$$H_r = H_r^{+1} + H_r^{-1} = \sum_{m=1,-1} H_r^m, \tag{16}$$

$$H_r^{+1} = \frac{H_0}{2i} e^{i\varphi} \sin\theta \exp(-ikz)\mathcal{A}, \tag{17}$$

$$H_r^{-1} = \frac{H_0}{2i} (-e^{-i\varphi}) \sin\theta \exp(-ikz)\mathcal{A}, \tag{18}$$

and we readily find that

$$g_{n,\mathrm{TE}}^{1} = \frac{1}{2i} f(\mathcal{A}), \tag{19}$$

$$g_{n,\mathrm{TE}}^{-1} = \frac{-1}{2i} f(\mathcal{A}). \tag{20}$$

In other words, $g_{n,\mathrm{TE}}^{+1}$ is obtained from H_r^{+1} by removing a plane-wave contribution $H_0 \sin\theta \exp(-ikz)$ and the exponential term $e^{i\varphi}$ and by applying the operator f to the remaining term $\mathcal{A}/(2i)$. Similarly, $g_{n,\mathrm{TE}}^{-1}$ is obtained from H_r^{-1} by removing a plane-wave contribution $H_0 \sin\theta \exp(-ikz)$ and the exponential term $e^{-i\varphi}$ and by applying the operator f to the remaining term $\mathcal{A}/(-2i)$.

As we stated above, this new procedure shows that m must be associated not with any discretization but with m waves $e^{i\varphi}$ or $e^{-i\varphi}$ (generally $e^{im\varphi}$), associated with angle φ. Besides, as additional support for this conclusion, we remark that the circumference at distance ρ_n from the axis is $2\pi\rho_n = (2n+1)\lambda/2$, i.e., an integer times the half wavelength. The above procedure also suggests a way for generalizing the localized approximation to all the coefficients g_n^m in the general case, namely, to (i) expand E_r, H_r in m modes E_r^m, H_r^m containing $e^{im\varphi}$, (ii) isolate in E_r^m, H_r^m the plane-wave contributions $E_0 \sin\theta \exp(-ikz)$, $H_0 \sin\theta \exp(-ikz)$, and also the exponential term $e^{im\varphi}$, and (iii) apply the operator f to the remaining parts of the expressions. We now examine whether such a procedure is successful.

3. LOCALIZED APPROXIMATION TO THE COEFFICIENTS g_n^m

A. First Attempt

In the general case (arbitrary location in a Gaussian beam at orders L and L^-) the radial field components E_r and H_r are given by[8]

$$E_r = E_0 \frac{F}{2}\left[\sum^{jp} \Psi_{jp} \exp(ij_+\varphi) + \sum^{jp} \Psi_{jp} \exp(ij_-\varphi)\right] + E_0 x_0 G \epsilon_L \sum^{jp} \Psi_{jp} \exp(ij_0\varphi), \tag{21}$$

$$H_r = H_0 \frac{F}{2i}\left[\sum^{jp} \Psi_{jp} \exp(ij_+\varphi) - \sum^{jp} \Psi_{jp} \exp(ij_-\varphi)\right] + H_0 y_0 G \epsilon_L \sum^{jp} \Psi_{jp} \exp(ij_0\varphi), \tag{22}$$

in which ϵ_L has been defined in Subsection 2.B and

$$F = \Psi_0^0 \sin\theta\left(1 - \frac{2Q}{l}\epsilon_L z\right)\exp(-ikz)\exp(ikz_0), \tag{23}$$

$$G = \Psi_0^0 \frac{2Q}{l} \cos\theta \exp(-ikz)\exp(ikz_0), \tag{24}$$

$$\Psi_0^0 = iQ \exp\left(-iQ\frac{r^2\sin^2\theta}{w_0^2}\right)\exp\left(-iQ\frac{x_0^2+y_0^2}{w_0^2}\right), \tag{25}$$

$$\Psi_{jp} = \left(\frac{iQr\sin\theta}{w_0^2}\right)^j \frac{(x_0 - iy_0)^{j-p}(x_0+iy_0)^p}{(j-p)!p!}, \tag{26}$$

$$\sum^{jp} = \sum_{j=0}^{\infty}\sum_{p=0}^{j}, \tag{27}$$

$$j_+ = j + 1 - 2p = j_0 + 1, \tag{28}$$

$$j_- = j - 1 - 2p = j_0 - 1. \tag{29}$$

According to the procedure proposed at the end of Section 2, we now expand E_r and H_r into m wave modes:

$$\begin{pmatrix} E_r \\ H_r \end{pmatrix} = \sum_{m=-\infty}^{+\infty} \begin{pmatrix} E_r^m \\ H_r^m \end{pmatrix}. \tag{30}$$

We readily obtain

$$E_r^m = \{E_0 \exp(-ikz)\exp(im\varphi)\sin\theta\}\exp(ikz_0)iQ \times \exp\left(-iQ\frac{r^2\sin^2\theta}{w_0^2}\right)\exp\left(-iQ\frac{x_0^2+y_0^2}{w_0^2}\right)\left[\frac{1}{2}\left(1 - \frac{2Q}{l}\epsilon_L z\right) \times \left(\sum_{j_+=m}^{jp} \Psi_{jp} + \sum_{j_-=m}^{jp} \Psi_{jp}\right) + x_0\epsilon_L \frac{2Q}{l}\frac{z}{r\sin\theta}\sum_{j_0=m}^{jp} \Psi_{jp}\right], \tag{31}$$

$$H_r^m = \{H_0 \exp(-ikz)\exp(im\varphi)\sin\theta\}\exp(ikz_0)iQ \times \exp\left(-iQ\frac{r^2\sin^2\theta}{w_0^2}\right)\exp\left(-iQ\frac{x_0^2+y_0^2}{w_0^2}\right)\left[\frac{1}{2i}\left(1-\frac{2Q}{l}\epsilon_L z\right)\times\left(\sum_{j_+=m}^{jp}\Psi_{jp}-\sum_{j_-=m}^{jp}\Psi_{jp}\right)+y_0\epsilon_L\frac{2Q}{l}\frac{z}{r\sin\theta}\sum_{j_0=m}^{jp}\Psi_{jp}\right], \quad (32)$$

in which $\sum_c^{jp}$ designates summation $\sum^{jp}$ restricted to condition c.

The terms in braces in Eqs. (31) and (32) are the ones to remove according to our proposal. Then, applying the operator f to the other parts, we readily obtain

$$g_{n,\mathrm{TM}}^{m,\mathrm{old}} = \exp(ikz_0)i\bar{Q}\exp\left[-i\bar{Q}\left(\frac{\rho_n}{w_0}\right)^2\right] \times \exp\left(-i\bar{Q}\frac{x_0^2+y_0^2}{w_0^2}\right)\frac{1}{2}\left(\sum_{j_+=m}^{jp}\bar{\Psi}_{jp}+\sum_{j_-=m}^{jp}\bar{\Psi}_{jp}\right), \quad (33)$$

$$g_{n,\mathrm{TE}}^{m,\mathrm{old}} = \exp(ikz_0)i\bar{Q}\exp\left[-i\bar{Q}\left(\frac{\rho_n}{w_0}\right)^2\right]\exp\left(-i\bar{Q}\frac{x_0^2+y_0^2}{w_0^2}\right) \times\frac{1}{2i}\left(\sum_{j_+=m}^{jp}\bar{\Psi}_{jp}-\sum_{j_-=m}^{jp}\bar{\Psi}_{jp}\right), \quad (34)$$

in which $\bar{\Psi}_{jp}$ stands for $f(\Psi_{jp})$, that is to say, relation (26) with Q replaced by $\bar{Q}$ and $r\sin\theta$ by ρ_n. The superscript "old" has been added for further convenience. Again, we observe that the doubled-valued symbol ϵ_L has disappeared; that is to say, our procedure does not distinguish between orders L and L^-. For the purpose of numerical computations, it is advantageous to replace the double summation $\sum_c^{jp}$ by a single summation. This leads to different expressions depending on the parity of m ($m = 2l$ or $2l+1$) and on the sign of l. These expressions are available from Ref. 17.

As a way of checking the coherence of these results with the exact theory, we can consider the special case of axis location ($x_0 = y_0 = 0$). Then we find that $\Psi_{jp} = 0$, but for $\Psi_{00} = 1$ and from Eqs. (33) and (34), we observe that

$$g_{n,\mathrm{TM}}^{m,\mathrm{old}} = g_{n,\mathrm{TE}}^{m,\mathrm{old}} = 0, \qquad |m| \neq 1,$$

$$g_{n,\mathrm{TM}}^{1,\mathrm{old}} = g_{n,\mathrm{TE}}^{-1,\mathrm{old}} = \frac{1}{2}g_n,$$

$$g_{n,\mathrm{TE}}^{1,\mathrm{old}} = -g_{n,\mathrm{TE}}^{-1,\mathrm{old}} = -\frac{i}{2}g_n, \quad (35)$$

which link the coefficients $g_n^{m,\mathrm{old}}$ in the present state of the localized approximation and the successful localized approximation to the g_n's (for axis location) as it should because relation (35) is valid for the rigorous coefficients g_n^m and g_n.[8]

For an arbitrary location, extensive numerical computations have been carried out to compare g_n^m values computed either from the rigorous finite-series method[15] or from Eqs. (33) and (34). Results are reported in Refs. 16 and 17. They are summarized by the following features.

For $|m| = 1$, the agreement between rigorous values and Eqs. (33) and (34) is satisfactory. However, the discrepancy between the two sets of results is wide for $|m| \neq 1$. Consequently, another step is required to design a successful localized approximation.

B. Final Generalization

Although the discrepancy between rigorous and approximated values is strong when $|m| \neq 1$, we must note two satisfactory features: (i) relations (35) are satisfied, i.e., the coefficients $g_{n,\mathrm{TM}}^{m,\mathrm{old}}$ and $g_{n,\mathrm{TE}}^{m,\mathrm{old}}$ tend toward zero when we approach the case of axis location as they should, and (ii) each time we numerically observe symmetries for the rigorous g_n^m's, these symmetries are also observed for the approximated values (extensive examples of this statement are given in Ref. 17). These facts strongly suggest that a normalization factor is lacking in Eqs. (33) and (34). Consequently, we tentatively conjecture that

$$\begin{pmatrix} g_{n,\mathrm{TM}}^m \\ g_{n,\mathrm{TE}}^m \end{pmatrix} = \begin{pmatrix} Z_{n,\mathrm{TM}}^m & \\ & Z_{n,\mathrm{TE}}^m \end{pmatrix}\begin{pmatrix} g_{n,\mathrm{TM}}^{m,\mathrm{old}} \\ g_{n,\mathrm{TE}}^{m,\mathrm{old}} \end{pmatrix}. \quad (36)$$

The normalization factors are expected to be complex; then we have

$$Z_n^m = R_n^m\phi_n^m, \quad (37)$$

in which we omit the subscripts TM and TE. The R_n^m's are real numbers, while the ϕ_n^m are phase terms that are expressed as

$$\phi_n^m = \exp(i\phi_n^m). \quad (38)$$

We also conjecture that the Z_n^m's depend only on n and m; i.e., they do not depend on the other input parameters defining the problem. Examining the rigorous theoretical expressions of the GLMT,[8] we find that these input parameters are four dimensionless groups that can be chosen as x_0/w_0, y_0/w_0, z_0/l, and s. The first three groups define the location of O_G with respect to O_P in a dimensionless way [relation (2)], while the last one is the small parameter defined by relation (1), characterizing the laser beam and accounting simultaneously for w_0/l (the ratio of the two beam-length scales) and for λ/w_0 (the light wavelength made dimensionless with the beam length scale w_0).

We have not yet found analytical evidence of the conjecture that $Z_n^m = Z(n, m)$. It will be numerically checked (Section 4). Also, we do not know at the present time how to determine analytically the expressions for the normalization factors. Consequently, we also rely on numerical works. For instance, let us consider the real factors R_n^0. From Eqs. (36) and (37),

$$R_n^0 = |Z_n^0| = \frac{|g_n^0|}{|g_n^{0,\mathrm{old}}|}. \quad (39)$$

We consider TM coefficients. For $x_0 = y_0 = 2\ \mu\mathrm{m}$, $w_0 = 5\ \mu\mathrm{m}$, $\lambda = 0.5\ \mu\mathrm{m}$, and $z_0 = 10\ \mu\mathrm{m}$ ($x_0/w_0 = 0.4$, $y_0/w_0 = 0.4$, $z_0/l = 0.031830989$, $s = 0.015915494$), Table 1 shows computed values of R_n^0 (for $n = 1, 2, 3, 4$) from Eq. (39), in which $|g_n^0|$ is obtained from rigorous finite series computations and $|g_n^{0,\mathrm{old}}|$ from the localized approximation in Subsection 3.A. Examining the GLMT expressions,[8] we note that the quantity $(2n+1)$ appears in a permanent way, leading us to the conclusion that it might play a role in the expressions for R_n^0.

Table 1. Example of Evaluation of the Normalization, Real Factors R_n^0

n	R_n^0	R'^0_n	R''^0_n
1	1.3339	$4.002 \simeq 4$	1
2	2.4031	$12.01 \simeq 12$	3
3	3.4332	$24.03 \simeq 24$	6
4	4.4497	$40.05 \simeq 40$	10

Multiplying R_n^0 by $(2n + 1)$, we obtain R'^0_n, which is given in the third column. They can be well approximated by integers that can clearly be divided by 4 to yield R''^0_n, which is given in the last column. Clearly,

$$\left.\begin{aligned} R''^0_1 &= 1 \\ R''^0_n &= R''^0_{n-1} + n \end{aligned}\right\}, \tag{40}$$

which leads to

$$R''^0_n = \sum_{p=1}^{n} p = \frac{n(n+1)}{2}. \tag{41}$$

Consequently, our proposal for R_n^0 is

$$R_n^0 = \frac{2n(n+1)}{(2n+1)}. \tag{42}$$

With a similar procedure for all Z_n^m's, our complete proposal is given by the following set of expressions:

$$R^0_{n,\mathrm{TM}} = R^0_{n,\mathrm{TE}} = \frac{2n(n+1)}{(2n+1)}, \tag{43}$$

$$R^m_{n,\mathrm{TM}} = R^m_{n,\mathrm{TE}} = \left(\frac{2}{2n+1}\right)^{|m|-1} \qquad |m| \geq 1, \tag{44}$$

$$\phi^m_{n,\mathrm{TM}} = \phi^m_{n,\mathrm{TE}} = i(-i)^{|m|} \; \forall m. \tag{45}$$

For $|m| = 1$, these expressions yield

$$Z^1_{n,\mathrm{TM}} = Z^{-1}_{n,\mathrm{TM}} = Z^1_{n,\mathrm{TE}} = Z^{-1}_{n,\mathrm{TE}} = 1 \tag{46}$$

as they should since the old localized approximation performed perfectly well for the coefficients g_n^m, $|m| = 1$.

Proposals (43)–(45) have been determined by relying on special values of input parameters. Section 4 is devoted to evidencing the quality of these expressions by testing them with a large number of different input parameters. The verification being correct, we conclude that our generalized approximation is likely to be fully completed.

4. VALIDATION OF THE NEW LOCALIZED APPROXIMATION

A. Checking the Validity of the Conjecture $Z_n^m = Z(n, m)$

To check the validity of the conjecture $Z_n^m = Z(n, m)$, we rewrite the expressions of $g^{m,\mathrm{old}}_{n,\mathrm{TM}}$ and $g^{m,\mathrm{old}}_{n,\mathrm{TE}}$ in a dimensionless way. This method, for example, transforms relation (34):

$$g^{m,\mathrm{old}}_{n,\mathrm{TM}} = \exp(iz_{0ad}/s/s)i\bar{Q} \exp\{-i\bar{Q}[(n+1/2)s]^2\}$$

$$\times \exp[-i\bar{Q}(x_{0\mathrm{ad}}^2 + y_{0\mathrm{ad}}^2)] \frac{1}{2}\left(\sum_{j_+=m}^{jp} \bar{\Psi}_{jp} + \sum_{j_-=m}^{jp} \bar{\Psi}_{jp}\right), \tag{47}$$

with

$$x_{0ad} = x_0/w_0, \qquad y_{0\mathrm{ad}} = y_0/w_0, \qquad z_{0\mathrm{ad}} = z_0/l, \tag{48}$$

$$\bar{Q} = \frac{1}{(i - 2z_{0\mathrm{ad}})}, \tag{49}$$

$$\bar{\Psi}_{jp} = [i\bar{Q}(n+1/2)s]^j \frac{(x_{0\mathrm{ad}} - iy_{0\mathrm{ad}})^{j-p}}{j-p!} \frac{(x_{0\mathrm{ad}} + iy_{0\mathrm{ad}})^p}{p!}. \tag{50}$$

$g^m_{n,\mathrm{TM}}$ have been computed for the following values of each dimensionless variable: $x_{0\mathrm{ad}} = 0.4, 0.8, 1.2$; $y_{0\mathrm{ad}} = 0.4, 0.8, 1.2$; $z_{0\mathrm{ad}} \simeq 0.0318, 0.06367, 10^{-9}$; $s \simeq 0.0159, 0.00159$. These computations have been carried out with an Alliant FX-80 computer. For some values, the same computations have been carried out with a personal computer (Apricot XEN-i XD45). No difference has been noted in the results.

We computed the 54 files corresponding to these values by using finite series and localized approximation. Then, for each couple of linked files (the same dimensionless variables) we computed the $R_n^m = |g^{m,\mathrm{exact}}_{n,\mathrm{TM}}|/|g^{m,\mathrm{old}}_{n,\mathrm{TM}}|$ and phase ϕ_n^m.

Then, for each (n, m) couple, we computed the average $\bar{R}$ and the standard deviation σ (both for real and imaginary parts when needed) over the 54 dimensionless values.

Figure 2 shows $(\bar{R} \pm \sigma)$ versus n, for $m = 0$. Although relation (42) is nonlinear, the relation between R_n^0 and n in Fig. 2 looks linear. This is not a discrepancy but simply results from the fact that $R_n^0 \simeq (n + 1)$ when n is large enough. The curves $(\bar{R} + \sigma)$ and $(\bar{R} - \sigma)$ cannot actually be distinguished owing to the small value of σ, confirming our conjecture that R_n^0 should depend only on n (with a good accuracy). Similarly, Fig. 3 shows $\bar{R}_n^m \pm \sigma$ versus n, for several m, leading again to the same conclusion. For $m = 1$, we just display the value R_n^1 that we know to be equal to 1, from previous research [see relation (46) and the comments following this relation].

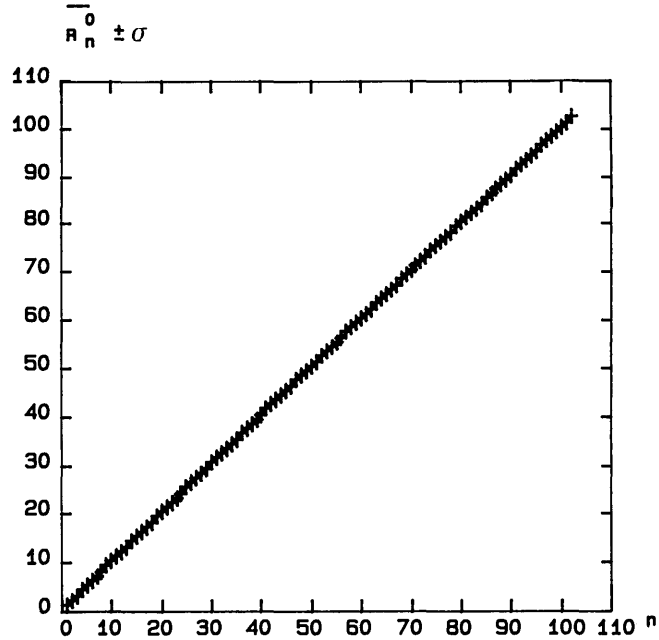

Fig. 2. $\bar{R}_n^0 \pm \sigma$ versus n in which $\bar{R}$ is the average of R_n^0 obtained for 54 dimensionless values and σ is the corresponding standard deviation.

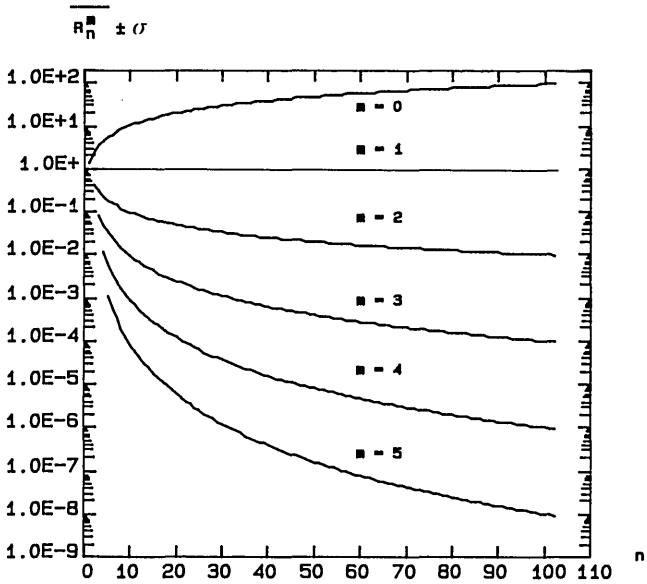

Fig. 3. $\overline{R_n^m} \pm \sigma$ versus n, for $m = 0, 2, 3, 4$, and 5. Curves $(\bar{R} + \sigma)$ and $(\bar{R} - \sigma)$ cannot be distinguished. The value for $m = 1$ has been added as a constant line.

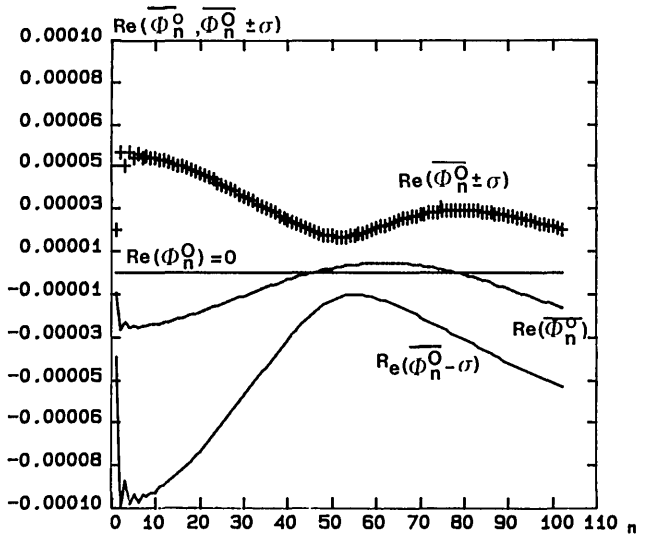

Fig. 4. Real part of $\overline{\phi_n^0}$, $\overline{\phi_n^0} \pm \sigma$ versus n. The value $\mathrm{Re}(\phi_n^0) = 0$ from Eq. (45) is also indicated.

From relation (45), ϕ_n^0 should be i, i.e., $\mathrm{Re}(\phi_n^0) = 0$, $\mathrm{Im}(\phi_n^0) = 1$. Figure 4 shows the expected values $\mathrm{Re}(\phi_n^0) = 0$, $\mathrm{Re}(\overline{\phi_n^0})$, and $\mathrm{Re}(\overline{\phi_n^0} \pm \sigma)$. Note the strongly expanded ordinate scale. Variances being small, our conjecture $\mathrm{Re}(\phi_n^0) = \mathrm{Re}[\phi_n^0(n)]$ is satisfied. Figure 5 displays similar data for $\mathrm{Im}(\phi_n^0)$ and also confirms our conjecture. Finally, the same conclusion holds for all the ϕ_n^m's (whatever m) and also for the $Z_{n,\mathrm{TE}}^m$ coefficients (figures are not provided to avoid overwhelming the reader).

B. Validation of the Localized Approximation

After checking the conjecture $Z_n^m = Z(n, m)$, we must now validate the localized approximation. Figure 6 shows the relative difference ϵ (in percent) between $\overline{R_n^0}$ [average over dimensionless variables of R_n^0 expressed by relation (39), taken as rigorous reference values] and our localized approximation proposal [relation (42)], versus n:

$$\epsilon = 100 \times \frac{\overline{R_n^0} - R_n^0}{\overline{R_n^0}}. \tag{51}$$

The absolute value of the relative difference is always less than approximately 0.02%. Figures 4 and 5 enable us to judge the quality of our proposal [relation (45)] for the phase term ϕ_n^0.

More generally, having previously shown (Subsection 4.A) that coefficients Z_n^m do not depend on the location in the dimensionless parameter space $[Z_n^m = Z(n, m)]$, we can carry out a validation of our proposal by comparing coefficients g_n^m computed by using the rigorous finite-series method[15] and our localized approximation [relations (33), (34), (36), (37),

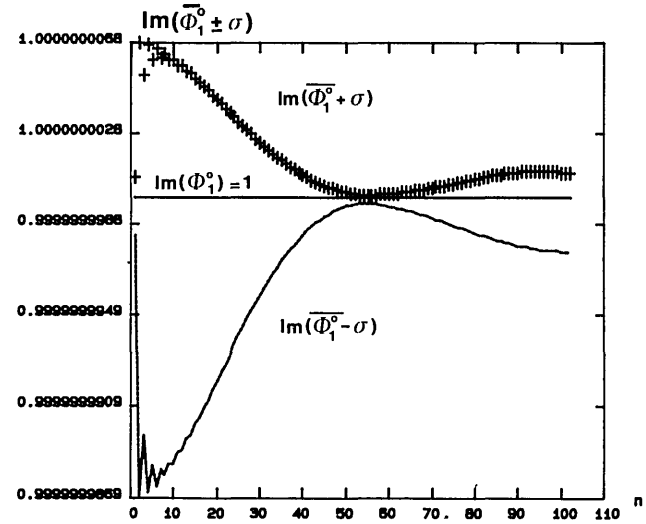

Fig. 5. Imaginary part of $\overline{\phi_n^0} \pm \sigma$ versus n. The value $\mathrm{Im}(\phi_n^0) = 1$ from Eq. (45) is also indicated.

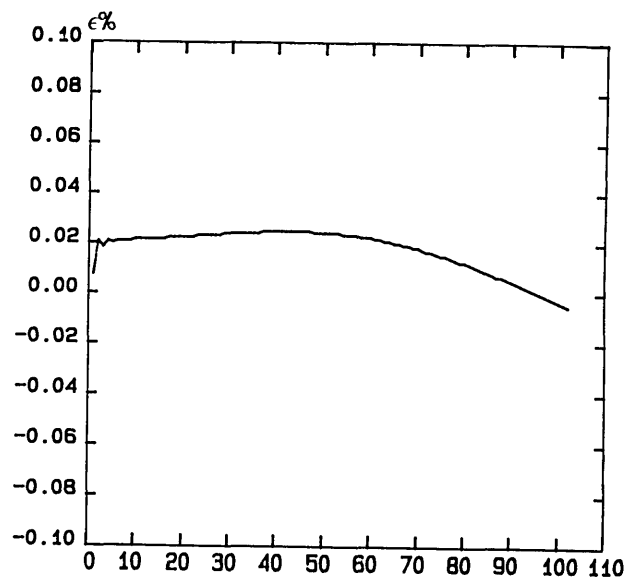

Fig. 6. Percentage of error ϵ for R_n^0.

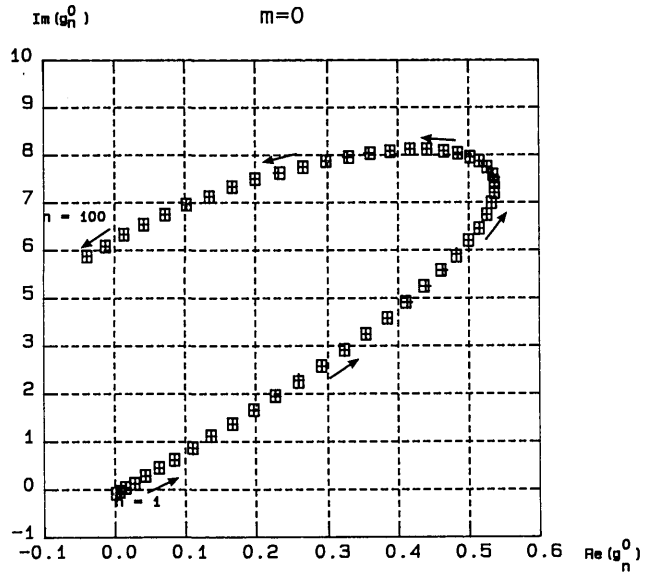

Fig. 7. Imaginary part of $g^0_{n,\mathrm{TM}}$ versus real part of $g^0_{n,\mathrm{TM}}$. The comparisons are between the finite series (+) and the localized approximation (□).

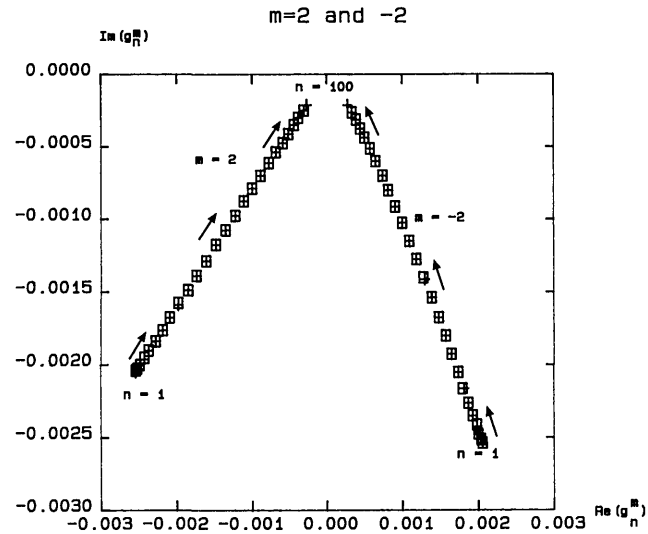

Fig. 8. Imaginary part of $g^m_{n,\mathrm{TM}}$ versus real part of $g^m_{n,\mathrm{TM}}$ for $m = 2$ and -2. The comparisons are between the finite series (+) and the localized approximation (□).

(43)–(45)], at a single location in the parameter space ($x_{0\mathrm{ad}} = y_{0\mathrm{ad}} = 0.4$, $z_{0\mathrm{ad}} \simeq 0.032$, $s \simeq 0.0159$). Results are given only for the TM coefficients. Computations for the TE coefficients lead to the same conclusions.

Figure 7 displays $\mathrm{Im}(g^m_{n,\mathrm{TM}})$ versus $\mathrm{Re}(g^m_{n,\mathrm{TM}})$, for n ranging from 1 to 100; $m = 0$. The crosses represent the finite-series computations, which coincide with the localized approximations represented by squares. The agreement between the methods is perfect. The same representation is used in Fig. 8 for $m = 2, -2$, Fig. 9 for $m = 3, -3$, and Fig. 10 for $m = 4, -4$. Again the agreement between the two methods is perfect, providing a firm validation of our localized approximation.

By the way, note the differences in the symmetry of the figures for $|m|$ odd or even. Also, $|g^m_n|$ decreases quickly when $|m|$ increases. Then we expect that higher-order m terms should contribute little to scattering phenomena.

C. Computation Times

Tables 2 and 3 show a compilation of computation times obtained with finite series and localized approximation in two cases. Time in Table 2 corresponds to the time required to compute all the g^m_n's, for n between 1 and $n_{\max}$ ($m = -n, \ldots, +n$). We note that, for n higher than 12, the localized approximation is more efficient than the finite series. As n grows, the difference between the two computation times becomes impressive, by a factor larger than 10 for $n_{\max}$ larger than approximately 40.

Most of the extensive computations presented in this paper have been carried out with an Alliant FX-80 computer. Nevertheless, some computations have been carried out with

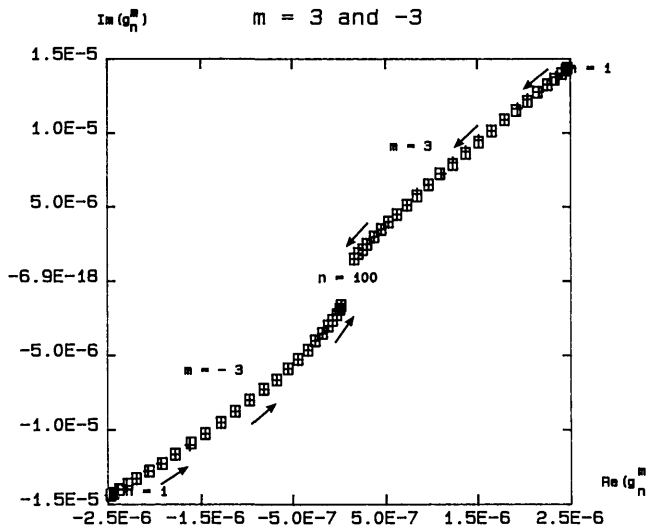

Fig. 9. Imaginary part of $g^m_{n,\mathrm{TM}}$ versus real part of $g^m_{n,\mathrm{TM}}$ for $m = 3$ and -3. The comparisons are between the finite series (+) and the localized approximation (□).

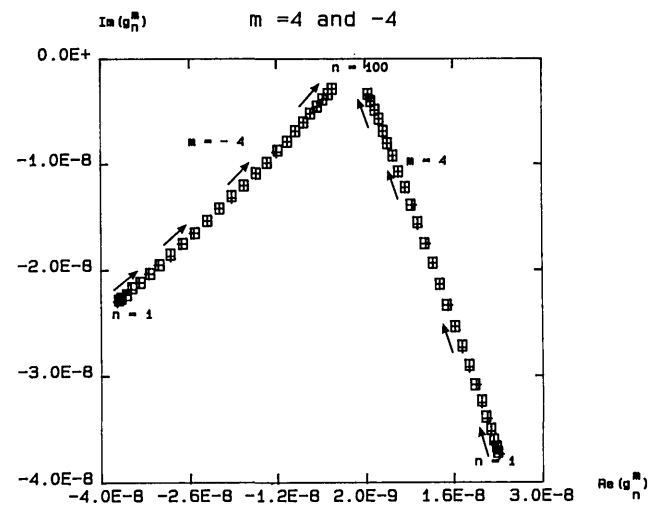

Fig. 10. Imaginary part of $g^m_{n,\mathrm{TM}}$ versus real part of $g^m_{n,\mathrm{TM}}$, for $m = 4$ and -4. The comparisons are between the finite series (+) and the localized approximation (□).

Table 2. Computation Times in Seconds for the Computation of All the g_n^m's[a]

n_{max}	4	12	22	32	42
Alliant series	0.15121	0.6310	6.5748	34.2079	119.044
Alliant localized	0.21486	0.6946	2.5095	6.2000	12.3384
Apricot series	0.77	20.	266.	1500	5460.
Apricot localized	3.0	26.0	114	288	628

[a] $m = -n, \ldots, n; n = 1, \ldots, n_{max}$.

Table 3. Computation Times in Seconds[a]

n_{max}	4	12	22	32	42
Alliant series	0.13891	0.5373	4.2831	17.1096	47.4687
Alliant localized	0.18457	0.5504	1.0920	1.7520	2.5076
Apricot series	0.77	17.1	172.8	739.	2135.
Apricot localized	2.9	18.8	45	76	112

[a] Computations are stopped when $|g_n^m| < 10^{-15}$.

a personal computer (Apricot XEN-i XD45, 80286 + 80287) without any loss of accuracy. The third and fourth rows in Table 2 show the computation times obtained with this kind of computer.

In finite series, computations are stopped when all the terms of the finite series have been summed up. In the localized approximation, the series contain an infinite number of terms and a truncation is required. Data in Table 2 for the localized approximation would depend a bit on the severity of the truncation but not enough to modify our conclusion concerning the increase in computational efficiency.

For practical purposes, we have not actually computed all g_n^m's but only those that remain great enough to contribute significantly to scattering phenomena. Hence Table 3 displays the same data as Table 2 but for $|g_n^m| > 10^{-15}$. For $n_{max} = 42$, the gain of efficiency is now by a factor of 20.

When n increases, we note from Tables 2 and 3 that the computational efficiency of the localized approximation with respect to finite-series computations improves quickly. This feature is a general one. When looking at the finite-series formulas,[15] we observe that the number of terms to sum up increases sharply with n in a nonlinear way. Conversely, in the localized approximation, the computational time does not significantly depend on n.[17] Consequently, in Table 2 (for instance), the accumulated time to compute all the g_n^m's, with $|g_n^m| > 10^{-15}$, increases fairly linearly with n. When n_{max} increases from 4 to 42, the computation time with the localized approximation increases by a factor of ~14 against more than 300 with an Alliant computer (and by 40 against 2800 for an Apricot).

We recall that, for the g_n's (axis location case) the gain may be more impressive in some cases when the localized formulas can be instantaneously evaluated by using pocket calculators.[13]

5. CONCLUSION

The GLMT is a new tool to study the interaction between an ideal spherical particle and an incident beam, for instance, a laser beam in its fundamental Gaussian mode. Besides more-or-less classical mathematical functions, the theory involves a double set of coefficients $g_{n,TM}^m$ and $g_{n,TE}^m$, which can be reduced to a simpler single set g_n when the scatter center is located on the beam axis. For practical purposes, it is necessary to provide efficient numerical means to compute these coefficients, in order to permit extensive computations at a reasonable cost. In this paper we successfully developed a localized approximation leading to an efficient way to evaluate all the coefficients g_n^m, with an insignificant loss of accuracy. This approximation should permit better availability of the GLMT among potential users.

In this conclusion we would like also to emphasize that the generalization of the localized approximation has been obtained with nonstandard methods; this step is unusual, at least in optics. Efficient and accurate expressions have been established with a drastic economy of mathematical and physical tools. It is to some extent fascinating to observe the nearly perfect agreement between rigorous and localized approximation computations. Owing to numerical inaccuracies that are always involved in complex computations, we can even wonder whether the most precise results would not be those from the localized approximation, at least in some cases. These results are also challenging and exciting from the point of view of understanding. We would like to know which physical reality is hidden behind the obtained agreement and the simple procedure that led to it. We are hoping that a rather deep meaning of this research will be revealed. We hope that other workers more clever than we are or with a different background will be able to produce this revelation. Anyway, while waiting for this enlightenment, we can now proceed to the next step, which is producing a complete and full code for extensive computations in the framework of the GLMT.

ACKNOWLEDGMENT

We are grateful to A. Souillard for correcting the English of this paper.

REFERENCES

1. G. Gouesbet and G. Gréhan, eds., *Proceedings of the International Symposium on Optical Particle Sizing: Theory and Practice* (Plenum, New York, 1988).
2. G. Mie, "Beiträge zur Optik Trüber Medien, Speziell Kolloidaler Metallösungen," Ann. Phys. **25**, 377–452 (1908).
3. A. Ashkin, "Acceleration and trapping of particles by radiation pressure," Phys. Rev. Lett. **24**, 156–159 (1970).
4. G. Roosen, "A theoretical and experimental study of the stable equilibrium positions of spheres levitated by two horizontal laser beams," Opt. Commun. **21**, 99–194 (1977).
5. G. Gréhan and G. Gouesbet, "Optical levitation of a single particle to study the theory of the quasi-elastic scattering of light," Appl. Opt. **19**, 2485–2487 (1980).
6. B. Maheu, G. Gréhan, and G. Gouesbet, "Laser beam scattering by individual spherical particles: numerical results and application to optical sizing," Part. Part. Syst. Character. **4**, 141–147 (1987).
7. G. Gouesbet and G. Gréhan, "Sur la généralisation de la théorie de Lorenz-Mie," J. Opt. (Paris) **13**, 97–103 (1982).
8. G. Gouesbet, B. Maheu, and G. Gréhan, "Light scattering from a sphere arbitrarily located in a Gaussian beam, using a Bromwich formulation," J. Opt. Soc. Am. A **5**, 1427–1443 (1988).
9. B. Maheu, G. Gouesbet, and G. Gréhan, "A concise presentation of the generalized Lorenz-Mie theory for arbitrary location of the scatterer in an arbitrary incident profile," J. Opt. (Paris) **19**, 59–67 (1988).

10. G. Gouesbet, "Advanced theory of light scattering. Application to optical sizing," in *6th Symposium of Japan Association of Aerosol Science and Technology* (JAAST, Osaka, 1988), pp. 106–108.
11. G. Gouesbet, G. Gréhan, and B. Maheu, "Generalized Lorenz-Mie theory and application to optical sizing," in *Combustion Measurements*, N. Chigier, ed. (Hemisphere, New York, to be published).
12. F. Corbin, G. Gréhan, G. Gouesbet, and B. Maheu, "Interaction between a sphere and a Gaussian beam: computation on a microcomputer," Part. Part. Syst. Character. **5,** 103–108 (1988).
13. G. Gouesbet, G. Gréhan, and B. Maheu, "Computations of the g_n coefficients in the generalized Lorenz-Mie theory using three different methods," Appl. Opt. **27,** 4874–4833 (1988).
14. G. Gréhan, B. Maheu, and G. Gouesbet, "Scattering of laser beams by Mie scatter centers: numerical results using a localized approximation," Appl. Opt. **25,** 3539–3548 (1986).
15. G. Gouesbet, G. Gréhan, and B. Maheu, "Expressions to compute the coefficients g_n^m in the generalized Lorenz-Mie theory using finite series," J. Opt. (Paris) **19,** 35–48 (1988).
16. G. Gouesbet, G. Gréhan, and B. Maheu, "The localized interpretation to compute the coefficients g_n, g_n^1, and g_n^{-1} in the framework of the generalized Lorenz–Mie theory," in *Optical Methods in Flows and Particle Diagnostics* (International Congress on Applications of Lasers and Electro-Optics, Santa Clara, Calif., 1988), Vol. 67, pp. 263–273.
17. G. Gouesbet, G. Gréhan, and B. Maheu, "On the generalized Lorenz-Mie theory: first attempt to design a localized approximation to the computation of the coefficients g_n^m," J. Opt. (Paris) **20,** 31–43 (1989).
18. L. W. Davis, "Theory of electromagnetic beams," Phys. Rev. A **19,** 1177–1179 (1979).
19. G. Gouesbet, G. Gréhan, and B. Maheu, "Scattering of a Gaussian beam by a Mie scatter center using a Bromwich formalism," J. Opt. (Paris) **16,** 2, 83–93 (1985).
20. H. C. van de Hulst, *Light Scattering by Small Particles* (Wiley, New York, 1957).