SAPIENZA
UNIVERSITÀ DI ROMA

MASTER'S DEGREE IN CYBERSECURITY

An intuitive explanation of Itô calculus with practical code examples

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Contents

Chapter 1

Brief introduction to Itô calculus

Itô calculus is a method, named after the Japanese mathematician Kiyosi Itô in the 1940s, and has since become a fundamental tool in the study of stochastic differential equations. It has an important relevance in multiple mathematical applications, most famously in mathematical finance.

In particular, Itô calculus defines a particular mathematical operator, named Itô integral, which is the stochastic version of the traditional integral. Despite the name, it is not used to evaluate the area under the curve of a particular function, but rather to integrate functions with respect to a specific stochastic process, being the Wiener process.

1.1 Geometric Brownian Motion - GBM

A Geometric Brownian Motion is defined as follows

$$\Delta S = \mu S \Delta t + \sigma S \sqrt{\Delta t} \mathcal{Z}$$

If we interpret S the stock price of any stock, then for the previous definition we could define

- ΔS: change in stock price
- μ: expected rate of return
- σ: volatility of the stock
- $\mathcal{Z}$: normally distributed random variable, that is exactly defined as

$$\mathcal{Z} \sim \mathcal{N}(\mu = 0, \sigma = 1)$$

Note that this definition is discrete, and that

$$\sigma S\sqrt{\Delta t}$$

is the stochastic component of the above equation, for which we cannot use standard calculus to solve. We'd need a new type of calculus for this type of mathematics, and thanks to the discoveries of Kiyosi Itô, most noticeably thanks to Itô's lemma, that we know how to operate when we encounter these types of processes.

Chapter 2

Itô Process

An Itô process is defined as a generalized Wiener process in which in general the parameters μ and σ are functions of X_t and time t, where, if we consider X_t to be the stock price of any stock at any given time t, we could define the above functions as:

- μ represents the rate of growth
- σ represents the volatility of the stock price

The Itô process is then a process that can be represented by the following relation:

$$X_t = X_0 + \int_0^t \mu(X_\tau, \tau)d\tau + \int_0^t \sigma(X_\tau, \tau)dW_\tau$$

Where W is a standard Brownian motion, also known as Wiener process.

We could also express the above Itô process as a stochastic differential equation, simply by derivating both sides of the equation

$$X_t' = dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

If we wanted to solve the above relation, we'd notice that we could solve the first component of the equation

$$\int_0^t \mu(X_\tau, \tau)d\tau$$

using standard calculus rules, but the second part

$$\int_0^t \sigma(X_\tau, \tau)dW_\tau$$

we are not able to, because we cannot integrate a Wiener process with what we know from standard calculus. This is one of the trivial reasons of why Itô calculus is so important.

Chapter 3

Itô's Lemma

Itô's lemma is a very powerful statement that allows us to proceed in stochastic calculus. More specifically, suppose to have obtained the following Itô process

$$dx = \mu(x,t)dt + \sigma(x,t)dW_t$$

Where W_t is a Wiener process. If this is true, if we have any function $f(X_t, t)$ then the following statement is always true

$$df = \left(\frac{\partial f}{\partial x}\mu(x,t) + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2(x,t)\right)dt + \left(\frac{\partial f}{\partial x}\sigma(x,t)\right)dW_t$$

Where we have substituted X_t for x.

We can kind of make the argument that stocks are modelled like Itô processes, so such that it's got a randomness and an expected rate of return, it's not exactly perfect, since in the Itô process it could be possible to go in negative due to possibly large variations due to the volatility of the Wiener process.

Thanks to Itô's lemma, if we have any Itô process X_t, then we can make up any function $f(X_t, t)$ that takes it as input, and Itô's lemma relation will always be true

A consequence of this is that the made-up function f itself is another Itô process, because it is also composed by the growth and volatility components

$$df = \underbrace{\left(\frac{\partial f}{\partial x}\mu_f(x,t) + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2(x,t)\right)}_{\mu(x,t)}dt + \underbrace{\left(\frac{\partial f}{\partial x}\sigma_t\right)}_{\sigma_f(x,t)}dW_t = \mu_f(x,t)dt + \sigma_f(x,t)dW_t$$

Since the Itô process has a drift and volatility component, it also implies that:

$$\begin{aligned} \text{drift rate: } & \left(\frac{\partial f}{\partial x}\mu(x,t) + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2(x,t)\right) \\ \text{variance: } & \left(\frac{\partial f}{\partial x}\sigma(x,t)\right)^2 \end{aligned}$$

This implies, consequently, that we can repeat the same process over and over again.

3.1 Proof

Itô's original proof is quite complicated and requires the knowledge of many details, so we will abstrain from showing it. Instead, we will prove it using Taylor's expansion series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \longrightarrow \text{assuming } f(x) \text{ is differentiable infinitely}$$

Suppose x is an Itô process that satisfies the following stochastic differential equation

$$dx = \mu_t dt + \sigma_t dW_t$$

Where W_t is a Wiener process, then if $f(x,t)$ is differentiable at least twice, its Taylor expansion series is

$$df = \frac{\partial f}{\partial t}\, dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\, dt^2 + \cdots + \frac{\partial f}{\partial x}\, dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\, dx^2 + \cdots$$

Substituting $\mu_t + \sigma_t dW_t$ for dx gives us the following

$$df = \frac{\partial f}{\partial t}\, dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\, dt^2 + \cdots + \frac{\partial f}{\partial x}(\mu_t\, dt + \sigma_t\, dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(\mu_t^2\, dt^2 + 2\mu_t\sigma_t\, dt\, dW_t + \sigma_t^2\, dW_t^2\right) + \cdots$$

If now we try to evaluate

$$\lim_{dt\to 0} df \longrightarrow \begin{cases} dt^2 \to 0 \\ dt\, dW_t \to 0 \end{cases} \quad \text{faster than } dW_t^2$$

If we set them to zero and we substitute dt for dW_t^2, we obtain Itô's lemma

$$df = \left(\frac{\partial f}{\partial x}\mu_t + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2\right) dt + \left(\frac{\partial f}{\partial x}\sigma_t\right) dW_t.$$

3.2 Example

Let's try to apply Itô's lemma to a financial example:

$$F = Se^{r(T-t)}$$

This is also known as the *forward price* of a stock, where

- F: new stock price
- S: current stock price
- r: interest rate

- T: total time period

Suppose that the current stock price S is modelled after an Itô process like so

$$dS = \mu S dt + \sigma S dW_t$$

This also means that F is an Itô process because it uses S, this implies that we can apply Itô's lemma to it

$$\frac{\partial F}{\partial S} = e^{r(T-t)} \quad \frac{\partial^2 F}{\partial S^2} = 0 \quad \frac{\partial F}{\partial t} = -rSe^{r(T-t)}$$

Which means that we get the following Itô process

$$\begin{aligned} dF = (e^{r(T-t)}\mu S - rSe^{r(T-t)})dt + \sigma Se^{r(T-t)}dW_t &= Se^{r(T-t)}(\mu - r)dt + \sigma Se^{r(T-t)}dW_t \\ &= (\mu - r)Fdt + \sigma F dW_t \\ &= \mu_F F dt + \sigma_F F dW_t \end{aligned}$$

We can then deduct that the forward price F has the same variation of the stock price S but an expected rate of return that is shifted backwards by r

$$F = \begin{cases} \mu_F = (\mu - r) \\ \sigma_F = \sigma \end{cases}$$

As we have seen, Itô's lemma is particularly useful to model any Itô process that is function of any other Itô process.

Chapter 4

Itô Integral

One result of Itô calculus and a fundamental tool of it is the Itô integral.

Formally, given a stochastic process X_t and a partition of the time axis such that

$$0 = t_0 < t_1 < ... < t_n = t$$

then the Itô integral of a function $f(t, X_t)$ is defined as follows

$$\int_0^t f(\tau, X_\tau) dW_\tau = \int_0^t dX_\tau = \int_0^t \mu(X_\tau, \tau) d\tau + \int_0^t \sigma(X_\tau, \tau) dW_\tau$$

And evaluated as a limit of Riemann sums, as follows:

$$\int_0^t dX_t = \lim_{|\Delta t_i| \to 0} \sum_{i=1}^{n} f(t_i, X_{t_i})(W_{t_i} - W_{t_{i-1}})$$

Where W_t is a Wiener process whose increments are normally distributed

$$(W_{t_i} - W_{t_{i-1}}) = \Delta W_i \sim \mathcal{N}(0, \Delta t_i)$$

The essence of the Itô integral is to express the accumulated effect of the integrated function as the stochastic process X_t evolves over time, with the added touch of randomness introduced by the Wiener process W_t

Note that the result of the Itô integral is a stochastic process, which assumes rather interesting and useful properties

4.1 Properties

These are the properties respected by Itô's integral

1. **Linearity**: consider a, b to be constants and X_t, Y_t be integrable processes, then

$$\int_0^t (aX_\tau + bY_\tau) dW_\tau = a \int_0^t X_\tau dW_\tau + b \int_0^t Y_\tau dW_\tau$$

2. **Isometry**: if X_t is a simple process, that is a finite linear combination of indicator functions of intervals, then the integral is isometric, which implies that
$$\mathbb{E}\left[\left(\int_0^t X_\tau dW_\tau\right)^2\right] = \mathbb{E}\left[\int_0^t X_\tau^2 d\tau\right]$$
Where $\mathbb{E}$ is the expected value.

Chapter 5

Itô integral practical simulations

I have written the following brief Python program to show what are the effects of the Itô integral.

In particular, I have utilized the numpy and matplotlib Python libraries to visualize my results in a simple fashion.

```python
import numpy as np
import matplotlib.pyplot as plt

# parameters
N = 10000                    # number of steps
dt = 1 / N                   # time differential
t = np.linspace(0, 1, N)     # time array

# generate a Wiener process using Donsker's theorem (sum of i.i.d.)
dW = np.random.randn(N) * np.sqrt(dt)
W = np.cumsum(dW)

# parameters for the SDE
mu = 3      # drift
sigma = 5   # volatility
S = np.zeros(N)
S[0] = 1    # first value

# simulate a geometric brownian motion SDE using Euler-Maruyama method
for i in range(1, N):
    S[i] = S[i-1] * np.exp((mu - 0.5 * sigma**2) * dt + sigma * dW[i-1])

# evaluate the ito integral as a sum of the products
ito_integral = np.cumsum(S * dW)

# plot
plt.plot(t, W, label='Brownian Motion')
plt.plot(t, ito_integral, label='Ito Integral')
plt.xlabel('Time')
plt.ylabel('Value')
plt.title('Ito Calculus Simulation')
plt.legend()
plt.tight_layout()
plt.show()
```

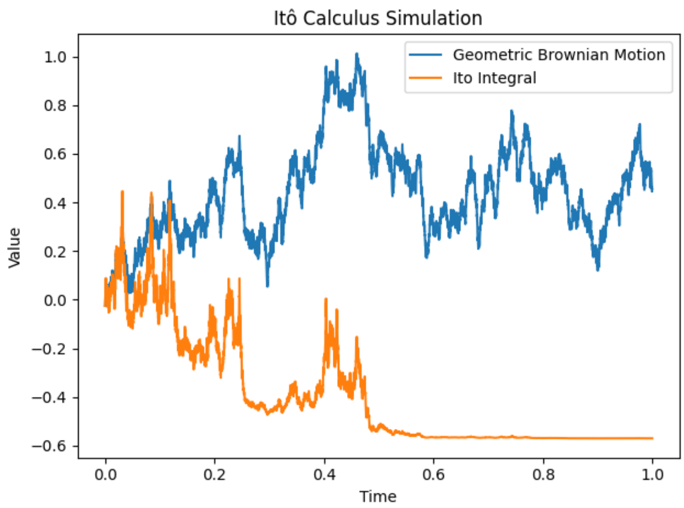
The Itô integral is in fact evaluated as we have explained. A random Geometric Brownian Motion SDE was generated using *Euler-Maruyama method*, that is

$$dS = \mu S dt + \sigma S dW \longrightarrow S_i = S_{i-1} e^{\mu - 0.5\sigma^2} \Delta t + \sigma \sqrt{\Delta t}\, dW_i, \quad \Delta t = \frac{1}{N}$$

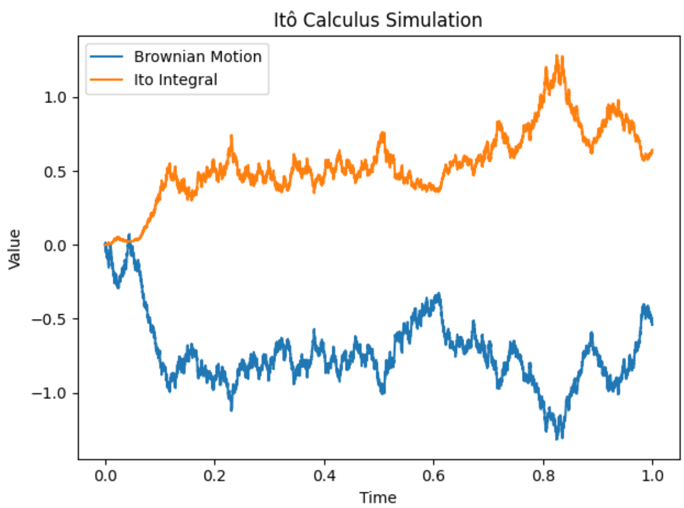
And the Brownian Motion W_t was generated as a consequence of Donsker's theorem, that is by plotting the sum of i.i.d. random variables divided by $\frac{1}{\sqrt{N}}$.

$$W_t = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i, \quad X_i = \text{i.i.d. r.v.}$$

If we execute the program, one of its outputs can be the following



A particularly interesting consequence of the Itô integral is that the integral of the Brownian motion, that is with respect to itself, yields to the following result



In fact, the Itô integral Y_t of the Brownian motion W_t with respect equals to the following

$$Y_t = \int_0^t W_\tau dW_\tau = \frac{1}{2}(W_t^2 - t)$$

Bibliography

[1] Bryne, B. Ito's lemma.

[2] Goodman, J. Stochastic calculus - lesson 4, 2018.

[3] Itô, K. Stochastic integral. *Proc. Imperial Acad. Tokyo 20* (1944), 519–524.

[4] Itô, K. On stochastic differential equations. *Memoirs, American Mathematical Society 4* (1951), 1–51.