Logical Formalism

Set Theory

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What Is a Set?

Collecting objects

- We need to be able to **gather** and manipulate objects.
- A **set** is merely a well-defined collection of objects.

E $\mathbb{N}$, $\mathbb{R}$, the prime numbers $\mathbb{P}$, the points of a line, etc.

What Is a Set?

The $\in$ predicate

- Formally, a set E is matched to a **predicate** $\in$ such that x **belongs** to E if and only the statement $x \in E$ is true.
- We also introduce the predicate $x \notin E \iff \neg(x \in E)$.
- We define the **empty set** $\emptyset$ such that no element belongs to $\emptyset$: the proposition $x \in \emptyset$ is always false.

What Is a Set?

Representing sets

- Each object in a set can appear **only once**: there are no duplicates.
- A finite set containing n elements can be written $\{x_1, \ldots, x_n\}$. This indexing is a purely arbitrary choice and may be **unordered**.
- The set $\{x\}$ containing only the element x is called a **singleton**. It should not be mistaken with x itself.

E $\{0, 1, 2\}$ and $\{1, 0, 2\}$ represent the same set, whereas $\{1, 1, 2\}$ is not a set.

What Is a Set?

Shaky foundations

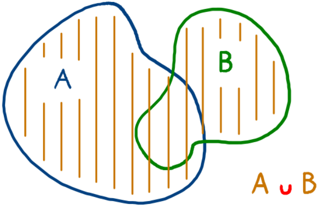
Set theory is somewhat flawed because of the **catalogue paradox**.

Consider a library that follows **three rules**:

1. Each library section features a catalogue listing its contents.
2. Some catalogues may list themselves, others do not.
3. A special catalogue lists all the catalogues that do not list themselves.

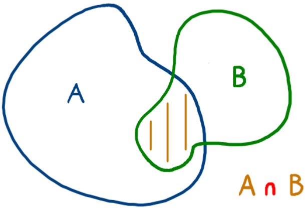
? The paradox consists in the following question: **does the special catalogue list itself**?

Given two sets A and B, we introduce their **union** $A \cup B$ such that $x \in A \cup B$ if and only if $(x \in A) \vee (x \in B)$.



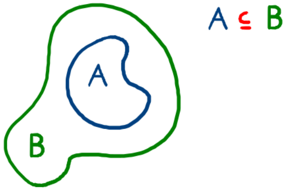
E Consider $\mathbb{Z} = \mathbb{Z}_+ \cup \mathbb{Z}_-$. Note that 0 belongs to **both** $\mathbb{Z}_+$ and $\mathbb{Z}_-$.

Given two sets A and B, we introduce their **intersection** $A \cap B$ such that $x \in A \cap B$ if and only if $(x \in A) \wedge (x \in B)$.



E The intersection of two distinct lines in $\mathbb{R}^2$ is either $\emptyset$ or a singleton.

Given two sets A and B, we introduce the **subset relation**: $A \subseteq B$ if and only if $(x \in A) \implies (x \in B)$.



E You should already know that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

The logical implication $\implies$ yields the following **proof pattern**:

Goal. Prove that $A \subseteq B$.

> Let $x \in A$ ...
> ... then $x \in B$.

Practical Application

Exercise 1. Let $A = \{a, b, c\}$ and $B = \{a, d\}$. Explicit the sets:

$$A \cup A$$

$$A \cap A$$

$$A \cup B$$

$$A \cap B$$

$$A \cap B \cup \{e\}$$

Answer

Operations on Sets

Equality of sets

- Given two sets A and B, we introduce the **equality relation**: $A = B$ if and only if $(x \in A) \iff (x \in B)$.
- Remember that $(P \iff Q) \iff (P \implies Q) \land (Q \implies P)$.
- As a consequence, $(A = B) \iff (A \subseteq B) \land (B \subseteq A)$.

Goal. Prove that $A = B$.

Subgoal 1. Prove that $A \subseteq B$.

Let $x \in A$ ...
... then $x \in B$.

Subgoal 2. Prove that $B \subseteq A$.

Let $x \in B$ ...
... then $x \in A$.

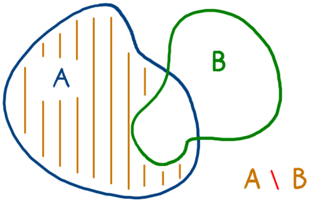
- Given a set E, we introduce $\mathcal{P}(E)$, the **power set** of E such that $X \in \mathcal{P}(E) \iff X \subseteq E$.
- $\mathcal{P}(E)$ is a **set of sets**.
- Note that $\emptyset \in \mathcal{P}(E)$ and $E \in \mathcal{P}(E)$ for any set E.

Practical Application

Exercise 2. Explicit the set $\mathcal{P}(\{a, b, c\})$.

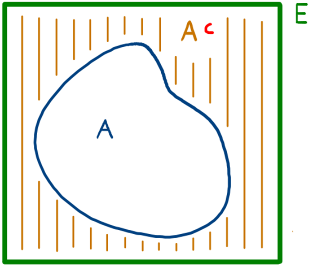
Answer

Given two sets A and B, we introduce the **subtraction** $A \setminus B$ of A by B such that $x \in A \setminus B$ if and only if $(x \in A) \wedge (x \notin B)$.



E B isn't necessarily a subset of A: $\{0, 1, 2, 3\} \setminus \{1, 3, 4\} = \{0, 2\}$.

Given an **implicit** set E and a subset $A \subseteq E$, the **complement** of A with regard to E is the set $A^{\complement} = E \setminus A$.



E Given $A = \{0\}$, $A^{\complement} = \mathbb{N}^*$ **with regard to** $\mathbb{N}$ and $A^{\complement} = \mathbb{R}^*$ **with regard to** $\mathbb{R}$.

Properties of Sets

Sets and propositional logic

Sets	Logical Operators
$A \cup B$	$P \vee Q$
$A \cap B$	$P \wedge Q$
$A \subseteq B$	$P \implies Q$
$A = B$	$P \iff Q$
$A^{\complement}$	$\neg P$

Properties of Sets

Common equalities

The following properties on sets hold, assuming $A, B, C \in \mathcal{P}(E)$:

$$\begin{array}{rcl}
A^{\complement^\complement} & = & A \\
A \cup A^\complement & = & E \\
A \setminus B & = & A \cap B^\complement \\
(A \cup B)^\complement & = & A^\complement \cap B^\complement \\
(A \cap B)^\complement & = & A^\complement \cup B^\complement \\
\emptyset^\complement & = & E \\
A \subseteq B & \iff & A \cap B^\complement = \emptyset \\
A \cap (B \cup C) & = & (A \cap B) \cup (A \cap C) \\
A \cup (B \cap C) & = & (A \cup B) \cap (A \cup C)
\end{array}$$

Practical Application

Exercise 3. Are the following properties true in the general case, assuming $A, B, C \in \mathcal{P}(E)$? If they're not, use **counter-examples** to falsify them.

$$\begin{aligned}
A = B &\iff A \cup C = B \cup C \\
A = B &\iff A \cap B = A \\
A \cup B = C &\iff A \cup C = B \\
A \subseteq B \cup C &\iff (A \subseteq B) \vee (A \subseteq C) \\
A \subseteq B \cap C &\iff (A \subseteq B) \wedge (A \subseteq C) \\
A = B &\iff A \setminus C = B \setminus C \\
(A \subseteq C) \wedge (B \subseteq C) &\implies A \subseteq B \\
A \cup B = C &\iff A = C \setminus B \\
A \cup B + A \cap B = A + B &\iff A \cup B = A + B - A \cup B
\end{aligned}$$

Answer I

Answer II

Practical Application

Exercise 4. Are the following equalities true in the general case, assuming $A, B, C \in \mathcal{P}(E)$? If they're not, use **counter-examples** to falsify them.

$$
\begin{aligned}
(A \setminus B) \cup B &= A \\
(A \cup B) \setminus B &= A \\
(A \cup B \cup B) \setminus B &= A \cup B \\
(A \setminus B) \cup (A \cap B) &= A \\
\emptyset \cup A &= E \cap A \\
\emptyset \cap A &= E \cup A \\
A + B - B &= A \\
A \cup B \cup B &= A + 2B \\
A \cup B &= (A \cup B) \setminus (A \cap B)
\end{aligned}
$$

Answer I

Answer II

A Last Few Definitions

The Cartesian product $\times$

- We consider **ordered pairs** of objects. We use the notation (x, y).
- Do not mistake the pair (x, y) for the set (also called an unordered pair) $\{x, y\}$.
- Given two sets U and V, we introduce their **Cartesian product** $U \times V$ such that $(x, y) \in U \times V$ if and only if $(x \in U) \land (y \in V)$.

E Consider $\mathbb{R}^2 = (\mathbb{R}_+ \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times \mathbb{R}_-) \cup (\mathbb{R}_- \times \mathbb{R}_-) \cup (\mathbb{R}_- \times \mathbb{R}_+)$.

A Last Few Definitions

Qualifying sets

- A common way to define a set is to select all the elements of a set **verifying a given proposition**.
- Formally, given a proposition $P(x)$ and a set E, we can introduce a subset S of E such that $x \in S \iff (x \in E) \wedge P(x)$.
- We often write $S = \{x \in E \mid P(x)\}$ instead.

E For $a, b \in \mathbb{R}$, the set $S = \{(x, y) \in \mathbb{R}^2 \mid y = a \cdot x + b\}$ is a line of the real plane $\mathbb{R}^2$.