SEQUENCES OF FUNCTIONS

Theorems proofs

1 Some useful properties and definitions

The properties and definitions that are given in this section are not proofs that you must learn. They are just notions that are frequently used, in the theorems proofs of next section as well as in exercises done in the class.

Proposition (Triangle inequality). *For all $(x, y) \in \mathbb{R}^2$, the following holds:* $|x + y| \leqslant |x| + |y|$

Definition (Supremum of a function).

Let $I \subset \mathbb{R}$ be an interval and f a function defined and upper bounded on I. Then the «supremum of f» is its lowest upper bound. It is denoted by:

$$\operatorname*{Sup}_{x \in I} f(x) \qquad \text{or, more concisely,} \qquad \operatorname*{Sup}_{I} f$$

Remarks:

1. If the function f is not upper bounded on I, then it admits no supremum since it admits no upper bound.
2. If the function f is upper bounded on I, the supremum of f is the unique $M_0 \in \mathbb{R}$ such that:
$$\underbrace{\forall x \in I,\ f(x) \leqslant M_0}_{M_0 \text{ is an upper bound of } f} \qquad \text{and} \qquad \forall M < M_0,\ \underbrace{\exists x \in I,\ f(x) > M}_{M \text{ is not an upper bound of } f}$$
3. All function which is upper bounded admits a supremum.
4. Do not confuse the supremum of f with its maximum. The maximum of f, when it exists, is a value that f can take: there must exist $x \in I$ such that $f(x) = \max f$. An upper bounded function may admit no maximum. However, it always admits a supremum.
Example: assume that f is an increasing function on $\mathbb{R}$, which tends to 1 as x approaches $+\infty$. Then $\operatorname*{Sup}_{\mathbb{R}} f = 1$ but f admits no maximum because no $x \in \mathbb{R}$ satisfies to $f(x) = 1$.
5. The following property is widely used in theorems proofs or in exercises: for all $M \in \mathbb{R}$,
$$\Big(\forall x \in I,\ f(x) \leqslant M\Big) \Longrightarrow \begin{cases} f \text{ is upper bounded on } I\text{, it hence admits a supremum} \\ \operatorname*{Sup}_{I} f \leqslant M \end{cases}$$

Definition (Uniform convergence).

The sequence (f_n) converges uniformly to the function f if one of the properties below is satisfied (these properties are equivalent).

1. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \forall x \in I,\ n > n_0 \Longrightarrow \big|f_n(x) - f(x)\big| < \varepsilon$

2. *There exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, the function $f_n - f$ is bounded on I, and*

$$\operatorname*{Sup}_I |f_n - f| \xrightarrow[n\to+\infty]{} 0$$

We accept without proof that properties 1 and 2 are equivalent. About property 2, note that the sequence $\left(\operatorname*{Sup}_I |f_n - f|\right)$ is well-defined above the rank n_0, because for all $n \geqslant n_0$,

$$\begin{aligned} f_n - f \text{ bounded on } I &\iff \exists M \in \mathbb{R} \text{ such that } \forall x \in I, \left|f_n(x) - f(x)\right| \leqslant M \\ &\iff |f_n - f| \text{ is upper bounded on } I \\ &\iff \operatorname*{Sup}_I |f_n - f| \text{ exists} \end{aligned}$$

2 Theorems proofs

In this section, we give the theorems proofs that you have to know. At the exams, you will be asked to do some of them.

In the whole section, I denotes an interval of $\mathbb{R}$, (f_n) and (g_n) two sequences of functions defined on I, f and g two functions defined on I.

Proposition (A necessary condition).

If the sequence (f_n) converges uniformly to f on I, then for all sequence (u_n) of I-elements,

$$f_n(u_n) - f(u_n) \xrightarrow[n\to+\infty]{} 0$$

Remark: this proposition is useful for its contrapositive. If there exists a sequence (u_n) of I-elements such that $\left(f_n(u_n) - f(u_n)\right)$ does not converge to 0, then (f_n) does not converge uniformly to f.

Proof: assume that (f_n) converges uniformly to f. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, the function $f_n - f$ is bounded on I. Thus, for all $n \geqslant n_0$, $\operatorname*{Sup}_I |f_n - f|$ exists and

$$u_n \in I \Longrightarrow 0 \leqslant \left|f_n(u_n) - f(u_n)\right| \leqslant \operatorname*{Sup}_I |f_n - f|$$

Furthermore, if (f_n) converges uniformly to f, then $\left(\operatorname*{Sup}_I |f_n - f|\right)$ converges vers 0. Using squeeze theorem, we get:

$$\left|f_n(u_n) - f(u_n)\right| \xrightarrow[n\to+\infty]{} 0, \text{ that is, } f_n(u_n) - f(u_n) \xrightarrow[n\to+\infty]{} 0$$

Theorem (Uniform convergence and linear combinations).

If the sequences (f_n) and (g_n) both converge uniformly on I to the functions f and g, then for all $(\alpha, \beta) \in \mathbb{R}^2$, $(\alpha f_n + \beta g_n)$ converges uniformly on I to $(\alpha f + \beta g)$.

Proof: we will show that:

1. The functions $(\alpha f_n + \beta g_n) - (\alpha f - \beta g)$ are bounded above a certain rank: there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$n \geqslant n_0 \Longrightarrow (\alpha f_n + \beta g_n) - (\alpha f - \beta g) \text{ bounded}$$

2. The numerical sequence $\left(\operatorname*{Sup}_I \left|(\alpha f_n + \beta g_n) - (\alpha f + \beta g)\right|\right)$ converges to 0.

According to the hypotheses, we know that:

1. The functions $f_n - f$ are bounded above a certain rank: there exists $n_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,
$$n \geqslant n_1 \Longrightarrow f_n - f \text{ bounded}$$
2. The functions $g_n - g$ are bounded above a certain rank: there exists $n_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,
$$n \geqslant n_2 \Longrightarrow g_n - g \text{ bounded}$$

Consider the number $n_0 = \max(n_1, n_2)$. Then for all $n \geqslant n_0$,

$$n \geqslant n_1 \Longrightarrow (f_n - f) \text{ bounded} \qquad \text{and} \qquad n \geqslant n_2 \Longrightarrow (g_n - g) \text{ bounded}$$

Let $n \geqslant n_0$. Then for all $x \in I$:

$$\begin{aligned}
\big|(\alpha f_n(x) + \beta g_n(x)) - (\alpha f(x) + \beta g(x))\big| &= \big|\alpha\big(f_n(x) - f(x)\big) + \beta\big(g_n(x) - g(x)\big)\big| \\
&\leqslant \big|\alpha\big(f_n(x) - f(x)\big)\big| + \big|\beta\big(g_n(x) - g(x)\big)\big| \\
&\leqslant |\alpha|.\big|f_n(x) - f(x)\big| + |\beta|.\big|g_n(x) - g(x)\big| \\
&\leqslant |\alpha| \operatorname*{Sup}_I |f_n - f| + |\beta| \operatorname*{Sup}_I |g_n - g|
\end{aligned}$$

The latter inequality shows that:

1. The function $|(\alpha f_n + \beta g_n) - (\alpha f + \beta g)|$ is upper bounded by $|\alpha| \operatorname*{Sup}_I |f_n - f| + |\beta| \operatorname*{Sup}_I |g_n - g|$, it hence admits a supremum.
2. Furthermore,
$$0 \leqslant \left(\operatorname*{Sup}_I \big|(\alpha f_n + \beta g_n) - (\alpha f + \beta g)\big|\right) \leqslant \left(|\alpha| \operatorname*{Sup}_I |f_n - f| + |\beta| \operatorname*{Sup}_I |g_n - g|\right)$$

   Using the hypotheses, we also know that
$$\operatorname*{Sup}_I |f_n - f| \xrightarrow[n \to +\infty]{} 0 \qquad \text{and} \qquad \operatorname*{Sup}_I |g_n - g| \xrightarrow[n \to +\infty]{} 0$$

Thus,
$$\left(|\alpha| \operatorname*{Sup}_I |f_n - f| + |\beta| \operatorname*{Sup}_I |g_n - g|\right) \xrightarrow[n \to +\infty]{} 0$$

Using squeeze theorem, we get
$$\left(\operatorname*{Sup}_I \big|(\alpha f_n + \beta g_n) - (\alpha f + \beta g)\big|\right) \xrightarrow[n \to +\infty]{} 0$$

Theorem (Continuity of the uniform limit).

If the functions f_n are continuous on I above a certain rank and if (f_n) converges uniformly to f, then f is continuous on I too.

Proof: we will show that, for all $a \in I$, the function f is continuous in a. Let $a \in I$. We have to show that $\lim_{x \to a} f(x)$ exists and is $f(a)$, that is,

$$\forall \varepsilon > 0,\ \exists \alpha \in \mathbb{R} \text{ such that } \forall x \in I,\ |x - a| < \alpha \Longrightarrow \big|f(x) - f(a)\big| < \varepsilon \qquad (*)$$

Using the hypotheses, we know that:

— The functions f_n are continuous above a certain rank. Hence, there exists $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$, f_n is continuous.

— The functions $f_n - f$ are bounded above a certain rank. Hence, there exists $n_2 \in \mathbb{N}$ such that for all $n \geqslant n_2$, $f_n - f$ is bounded, so $\operatorname*{Sup}_I |f_n - f|$ exists.

Consider the number $n_0 = \max(n_1, n_2)$. Then for all $n \geqslant n_0$, both properties are satisfied: f_n is continuous on I **and** $f_n - f$ is bounded.

Let $\varepsilon > 0$. Since $\left(\operatorname*{Sup}_I |f_n - f|\right)$ converges to 0, there exists $n_3 \geqslant n_0$ such that for all $n \geqslant n_3$, $\operatorname*{Sup}_I |f_n - f| \leqslant \varepsilon$.

Let $n \geqslant n_3$. Then, for all $x \in I$, $\big|f_n(x) - f(x)\big| \leqslant \operatorname*{Sup}_I |f_n - f| \leqslant \varepsilon$.

Furthermore, since the function f_n is continuous on I, it is continuous in a. Thus, there exists $\alpha > 0$ such that for all $x \in I$,

$$|x-a| < \alpha \Longrightarrow \big|f_n(x) - f_n(a)\big| < \varepsilon$$

Then we can claim that, for all $x \in I$,

$$\begin{aligned} |x-a| < \alpha \Longrightarrow \ \big|f(x) - f(a)\big| &= \big|(f(x) - f_n(x)) + (f_n(x) - f_n(a)) + (f_n(a) - f(a))\big| \\ &\leqslant \underbrace{\big|f(x) - f_n(x)\big|}_{\leqslant \varepsilon} + \underbrace{\big|f_n(x) - f_n(a)\big|}_{< \varepsilon} + \underbrace{\big|f_n(a) - f(a)\big|}_{\leqslant \varepsilon} \\ &< 3\varepsilon \end{aligned}$$

We have hence proven the property:

$$\forall \varepsilon > 0, \exists \alpha \in \mathbb{R} \text{ such that } \forall x \in I, |x-a| < \alpha \Longrightarrow \big|f(x) - f(a)\big| < 3 \times \varepsilon \qquad (**)$$

To prove property $(*)$, we can inject $\frac{\varepsilon}{3}$ in property $(**)$: let $\varepsilon > 0$. Since $\frac{\varepsilon}{3} > 0$, we know according to $(**)$ that

$$\begin{aligned} \exists \alpha \in \mathbb{R} \text{ such that } \forall x \in I, \ \ |x-a| < \alpha \ &\Longrightarrow \ \big|f(x) - f(a)\big| < 3 \times \frac{\varepsilon}{3} \\ &\Longrightarrow \ \big|f(x) - f(a)\big| < \varepsilon \end{aligned}$$

Theorem (Convergence of the integrals).

If I is a finite closed interval $[a, b] \subset \mathbb{R}$, if the functions f_n are continuous above a certain rank and if the sequence of functions (f_n) converges uniformly to f on $[a, b]$, then

$$\int_a^b f_n(x)\,dx \xrightarrow[n \to +\infty]{} \int_a^b f(x)\,dx$$

Proof: to start with, using the hypotheses, we know that there exists $n_1 \in \mathbb{N}$ such that, for all $n \geqslant n_1$, f_n is continuous. Hence:

1. The integrals $\displaystyle\int_a^b f_n(x)\,\mathrm{d}x$ are defined for all $n \geqslant n_1$.
2. Since the convergence of (f_n) to f is uniform, we know that the function f is continuous too. The integral $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ is hence defined too.

Thus, the statement of the theorem has a sense.

Furthermore, using the definition of the uniform convergence, we know that there exists a rank $n_2 \in \mathbb{N}$ such that for all $n \geqslant n_2$, the function $f_n - f$ is bounded, that is, $\operatorname*{Sup}_{[a,b]} |f_n - f|$ exist.

Let $n_0 = \max(n_1, n_2)$. Then for all $n \geqslant n_0$,

$$n \geqslant n_1 \Longrightarrow f_n \text{ continuous} \qquad \text{and} \qquad n \geqslant n_2 \Longrightarrow f_n - f \text{ bounded}$$

Let us show the convergence of the integrals' sequence: for all $n \geqslant n_0$,

$$\begin{aligned}
0 \leqslant \left| \int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x \right| &= \left| \int_a^b (f_n(x) - f(x))\,\mathrm{d}x \right| \\
&\leqslant \int_a^b |f_n(x) - f(x)|\,\mathrm{d}x \\
&\leqslant \int_a^b \operatorname*{Sup}_{[a,b]} |f_n - f|\,\mathrm{d}x \qquad \text{because } |f_n(x) - f(x)| \leqslant \operatorname*{Sup}_{[a,b]} |f_n - f| \\
&\leqslant \operatorname*{Sup}_{[a,b]} |f_n - f| \times \int_a^b 1\,\mathrm{d}x \\
&\leqslant \operatorname*{Sup}_{[a,b]} |f_n - f| \times (b - a)
\end{aligned}$$

But we know that $\operatorname*{Sup}_{[a,b]} |f_n - f| \xrightarrow[n\to+\infty]{} 0$. Thus, using squeeze theorem,

$$\left| \int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x \right| \xrightarrow[n\to+\infty]{} 0, \qquad \text{that is,} \qquad \int_a^b f_n(x)\,\mathrm{d}x \xrightarrow[n\to+\infty]{} \int_a^b f(x)\,\mathrm{d}x$$