Logical Formalism

Simple Proof Patterns

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Manipulating Variables

- Learning how to manipulate mathematical objects, be they simple or complex.
- We will name them unambiguously using **variables**: two different objects must be named differently.
- A **rule of thumb**: two similar objects should have similar names.

E The statement "$n, m \in \mathbb{N}$ and $x, y \in \mathbb{R}$" is easier to parse than "$n, x \in \mathbb{N}$ and $y, m \in \mathbb{R}$".

Manipulating Propositions

Propositions are merely statements to which we match a truth value.

The following principle applies:

Law of the excluded middle

A proposition P is either **true** or **false**.

This axiom states that there are only two possible and mutually exclusive truth values.

- Mathematical truths are often expressed as **theorems**: if a hypothesis P is true, then a conclusion Q must be true as well.
- Given two propositions P and Q, we use the connector $\implies$ in order to define a new proposition $P \implies Q$ expressing that **if** P is true **then** Q must be true as well.

? Q being true and P being false does not contradict $P \implies Q$.

This **proof pattern** can be used to prove theorems of the form $P \implies Q$.

Goal. Prove that $P \implies Q$ is true.

If P is true ...

Remember your definitions. Express the hypothesis P in a detailed manner by making the definitions explicit.

Write common properties. Can you think of some obvious, immediate consequences of P?

...then Q is true.

Practical Application

Exercise 1. Prove that if $n \in \mathbb{N}$ is even, then n^2 is even as well.

Answer

Logical Connectors

The logical and $\wedge$

- Given two propositions P and Q, we use the connector $\wedge$ in order to define a new proposition $P \wedge Q$ that is true if and only if **both** P and Q are true.
- Thus, $P \wedge Q$ is **false** in each of the following three cases:
  - Only P is false.
  - Only Q is false.
  - Both P and Q are false.
- Note that you may have to **rewrite** a proposition to make the $\wedge$ obvious.

A proof pattern for $\wedge$ as a conclusion

Goal. Prove that $P \implies (Q \wedge R)$ is true.

Suppose that P is true ...

Split a complex goal into subgoals. Split the proof into more manageable subproofs by detailing the original goal.

Subgoal 1. Prove that Q is true.

Subgoal 2. Prove that R is true.

Practical Application

Exercise 2. Prove that $\forall x, y \in \mathbb{R}$, $||x| - |y|| \leq |x - y|$. Note that:

- Given $u \in \mathbb{R}$ and $v \in \mathbb{R}^+$, $|u| \leq v$ is equivalent to $-v \leq u \leq v$.
- The **triangle inequality** states that, $\forall u, v \in \mathbb{R}$, $|u + v| \leq |u| + |v|$.
- Try applying it to x and $(y - x)$ as well as y and $(x - y)$.

Answer I

Answer II

Goal. Prove that $(P \wedge Q) \implies R$ is true.

If P is true ...
... and Q is true ...
... then R is true.

Logical Connectors

The logical or $\vee$

- Given two propositions P and Q, we use the connector $\vee$ in order to define a new proposition $P \vee Q$ that is true if and only if **at least one** of the two propositions P and Q is true.
- Thus, $P \vee Q$ is false if and only if **both** P and Q are false.
- Note that the mathematical $\vee$ should not be mistaken for the common, everyday or.

E The sentence *'Pay a fine or go to jail.'* features an **exclusive** or.

Goal. Prove that $P \implies (Q \vee R)$ is true.

If P is true . . .

Use a **case disjunction.** Q can either be true or false; in both cases, we want $Q \vee R$ to be true.

Assume that Q is true.

Then obviously $Q \vee R$ is true. No further proof is needed.

Assume that Q is false.

If P is true and Q is false . . .
. . . then R must be true.

A proof pattern for $\vee$ as a hypothesis

Goal. Prove that $(P \vee Q) \implies R$ is true.

Subgoal 1. Prove that $P \implies R$ is true.

If P is true ...
... then R is true.

Subgoal 2. Prove that $Q \implies R$ is true.

If Q is true ...
... then R is true.

Given two propositions P and Q, we use the connector $\iff$ in order to define a new proposition $P \iff Q$ that is true **if and only if** P and Q have the **same truth value**: they're either both true or both false.

The following property holds:

Double implication

$P \iff Q$ is true if and only if $P \implies Q$ and $Q \implies P$ are true.

Goal. Prove that $P \iff Q$ is true.

Subgoal 1. Prove that $P \implies Q$ is true.

If P is true . . .
. . . then Q is true.

Subgoal 2. Prove that $Q \implies P$ is true.

If Q is true . . .
. . . then P is true.

Practical Application

Exercise 3. Prove that $\forall n \in \mathbb{N}$, n is a multiple of 9 if and only if the sum of its digits is a multiple of 9 as well. To do so:

- Remember that $\forall k \in \mathbb{N}$, $(10^k - 1)$ is a multiple of 9.
- Use the **modulo** notation.

Answer I

Answer II