Functions of several variables

1. Partial derivatives

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ABSTRACT: The notion of partial derivative is presented with an emphasis on two and three dimensional applications. The geometric point of view is first approached through the notions of surfaces and level sets. Then the concepts of limit and continuity for functions of several variables are discussed, avoiding an overly formal presentation, then the idea of partial derivative (first and second order) is presented. This tool is subsequently applied to the study of tangent planes and to the calculation of linear approximations. The notions of directional derivatives, gradient and chain rules are also presented in detail with many examples. Finally, the problem of minimizing and maximizing a function of several variables is approached with, once again, many application examples. Exercises accompanying each theme are provided in the appendix.

Contents

1 Introduction

Not surprisingly, the notion of partial derivative is central in the study of functions of several variables. To convince ourselves of this it suffices to consider the importance, both from an historical and mathematical point of view, of the notion of derivative for a function of a single variable, with which we are already familiar. Consider for instance a function f depending on a single real variable x. A natural question is how does f vary when its variable x also varies, and we know the answer to this question. Now we want to consider functions of the type $f(x, y)$ or $F(x, y, z)$ with two or three or more variables varying independently. Analogously to the one-variable case, we now want to connect the small (infinitesimal) variations of the parameters x and y (and z), which we denote by Δx and Δy (and Δz), to a small variation of f (or f), denoted Δf (or ΔF).

This problem is solved in a totally analogous way to the construction of the derivative for a function with only one variable, that is to say by passing to the limit. Indeed, we know that df/dx is the limit of the rate of increase $\Delta f/\Delta x$ when Δx becomes infinitely small. In the multidimensional framework, this rate of increase is always associated with the slope of a tangent but it is rather a tangent to a *surface*. Our goal is therefore to make sense of the derived notion in order to study the variations of functions such as

$$f(x, y) = x^2 - y^2 \qquad \text{or} \qquad f(x, y) = \; sqrtx^2 + y^2 \qquad \text{or} \qquad f(x, y, z) = 2x + 3y + 4z$$

Heuristically, we can be convinced that these functions have graphs such that they therefore admit tangents with their associated derivatives. It is these that we want to evaluate.

To summarize, our subject is the differential calculus for the functions of several variables, that is to say the study of the variations of functions with several parameters according to the (small) variations of these. Our presentation can be summarized in six essential points presented in the table below. In this one we put in correspondence the usual operations concerning the differential calculus for functions of a variable with their counterparts for a function with two variables.

2 Surfaces and Level Curves

The graph of $y = f(x)$ is a curve in the xy plane. There are two variables, x is independent and free, y is dependent on x. Above x on the base line is the point (x, y) on the curve. The curve can be displayed on a two-dimensional printed page.

The graph of $z = f(x, y)$ is a **surface in** xyz **space**. There are *three variables* - x and y are independent, z is dependent. Above (x, y) in the base plane is the point (x, y, z) on the surface (**Figure 1 (left)**). Since the printed page remains two-dimensional, we shade or color or project the surface. The eyes are extremely good at converting two-dimensional images into three-dimensional understanding, they get a lot of practice. The mathematical part of our brain also has something new to work on, **two partial derivatives**.

This section uses examples and figures to illustrate surfaces and their level curves. The next section is also short. Then the work begins.

Exemple 1. Describe the surface and the level curves for $z = f(x, y) = \sqrt{x^2 + y^2}$.

Solution 1. *The surface is a cone.* Reason : $\sqrt{x^2 + y^2}$ is the distance in the base plane from $(0, 0)$ to (x, y). When we go out a distance 5 in the base plane, we go up the same distance 5 to the

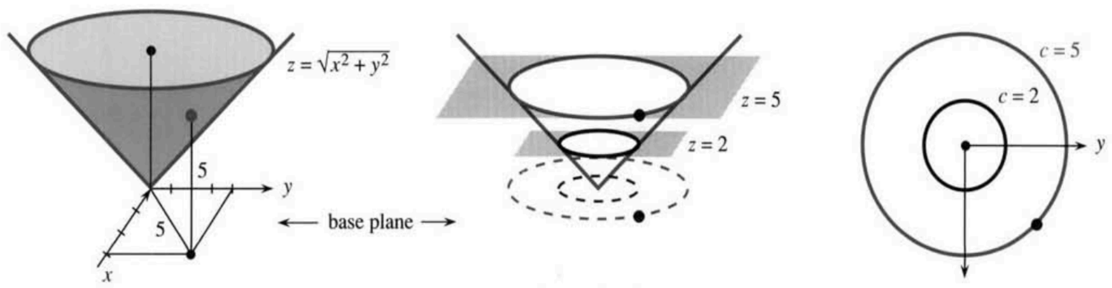

Figure 1: The surface for $z = f(x, y) = \sqrt{x^2 + y^2}$ is a cone. The level curves are circles.

surface. The cone climbs with slope 1. The distance out to (x, y) equals the distance up to z (this is a 45° cone).

The level curves are circles. At height 5, the cone contains a circle of points - all at the same « level » on the surface. The plane $z = 5$ meets the surface $z = \sqrt{x^2 + y^2}$ at those points (**Figure 1 (middle and right)**). The circle below them (in the base plane) is the level curve.

Définition 1. *A **level curve** or **contour line** of $z = f(x, y)$ contains all points (x, y) that share the same value$f(x, y) = c$. Above those points, the surface is at the height $z = c$.*

There are different level curves for different c. To see the curve for $c = 2$, cut through the surface with the horizontal plane $z = 2$. The plane meets the surface above the points where $f(x, y) = 2$. **The level curve in the base plane has the equation** $f(x, y) = 2$. Above it are all the points at « level 2 » or « level c » on the surface.

Every curve $f(x, y) = c$ is labeled by its constant c. This produces a **contour map** (the base plane is full of curves). For the cone, the level curves are given by $\sqrt{x^2 + y^2} = c$, and the contour map consists of circles of radius c.

Question 1. *What are the level curves of $z = f(x, y) = x^2 + y^2$?*

Answer 1. *Still circles.* But the surface is not a cone (it bends up like a parabola). The circle of radius 3 is the level curve $x^2 + y^2 = 9$. On the surface above, the height is 9.

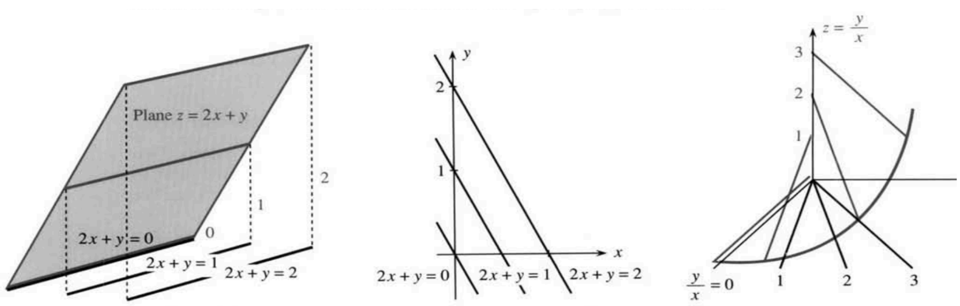

Figure 2: A plane has parallel level lines. The spiral slide $z = y/x$ has lines $y/x = c$.

Exemple 2. For the linear function $f(x, y) = 2x + y$, the surface is a plane. Its level curves are straight lines. **The surface** $z = 2x + y$ **meets the plane** $z = c$ **in the line** $2x + y = c$. That line is above the base plane when c is positive, and below when c is negative. The contour lines are

in the base plane. **Figure 2 (center)** labels these parallel lines according to their height in the surface.

Question 2. *If the level curves are all straight lines, must they be parallel ?*

Answer 2. No. The surface $z = y/x$ has level curves $y/x = c$. Those lines $y = cx$ swing around the origin, as the surface climbs like a spiral playground slide.

Exemple 3. The weather map shows contour lines of the temperature function. Each level curve connects points at a constant temperature. One line runs from Seattle to Omaha to Cincinnati to Washington. In winter it is painful even to think about the line through L.A. and Texas and Florida. USA Today separates the contours by color, which is better. We had never seen a map of universities.

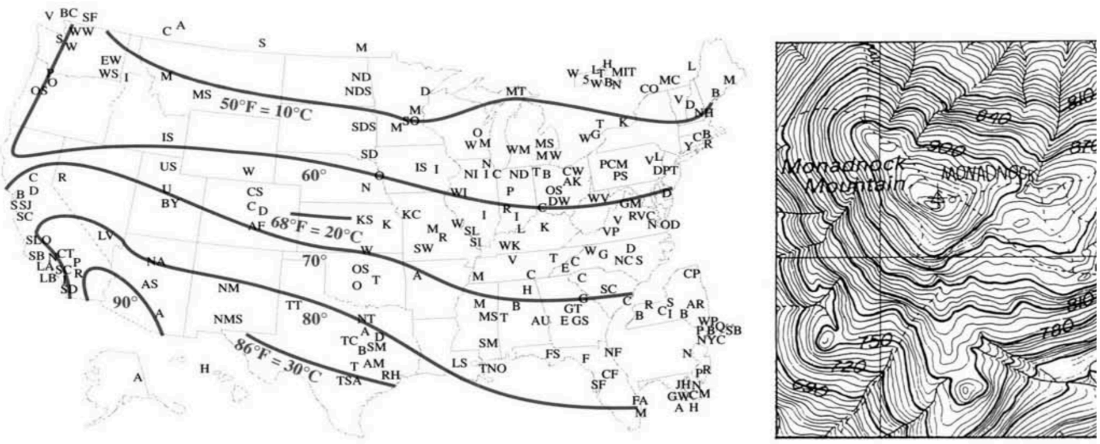

Figure 3: The temperature at many U.S. and Canadian universities. Mt. Monadnock in New Hampshire is said to be the most climbed mountain (except Fuji ?) at 125,000/year. Contour lines every 6 meters.

Question 3. *From a contour map, how do you find the highest point ?*

Answer 3. The level curves form *loops* around the maximum point. As c increases the loops become tighter. Similarly the curves squeeze to the lowest point as c decreases.

Exemple 4. A contour map of a mountain may be the best example of all. Normally the level curves are separated by 100 feet in height. On a steep trail those curves are bunched together - the trail climbs quickly. In a flat region the contour lines are far apart. Water runs perpendicular to the level curves. On my map of New Hampshire that is true of creeks but looks doubtful for rivers.

Question 4. *Which direction in the base plane is uphill on the surface ?*

Answer 4. The steepest direction is perpendicular to the level curves. This is important. Proof to come.

Exemple 5. In economics x^2y is a utility function and $x^2y = c$ is an indifference curve. The utility function x^2y gives the value of x hours awake and y hours asleep. Two hours awake and fifteen minutes asleep have the value $f = (2^2)(1/4)$. This is the same as one hour of each : $f = (1^2)(1)$.

Those lie on the same level curve in **Figure 4 (left)**. We are indifferent, and willing to exchange any two points on a level curve.

The indifference curve is « convex » . **We prefer the average of any two points**. The line between two points is up on higher level curves.

Figure 4 (center) shows an extreme case. The level curves are straight lines $4x + y = c$. Four quarters are freely substituted for one dollar. The value is $f = 4x + y$ dollars.

Figure 4 (right) shows the other extreme. Extra left shoes or extra right shoes are useless. The value (or utility) is the smaller of x and y. That counts pairs of shoes.

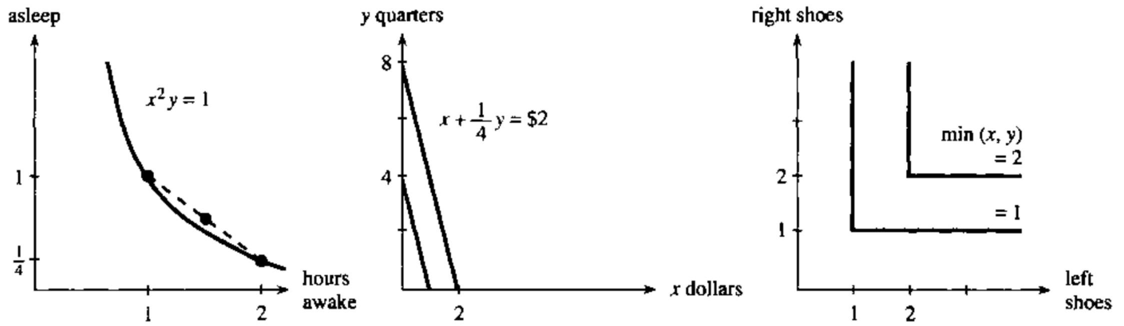

Figure 4: Utility functions x^y, $4x + y$, $\min(x, y)$. Convex, straight substitution, complements.

3 Limits and Continuity

We have now examined functions of more than one variable and seen how to graph them. In this section, we see how to take the limit of a function of more than one variable, and what it means for a function of more than one variable to be continuous at a point in its domain. It turns out these concepts have aspects that just don't occur with functions of one variable.

3.1 Limit of a Function of Two Variables

We recall the definition of a limit of a function of one variable : Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Let L be a real number. Then

$$\lim_{a \to a} f(x) = L \tag{3.1}$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$ for all x in the domain of f, then

$$|f(x) - L| > \varepsilon \tag{3.2}$$

Before we can adapt this definition to define a limit of a function of two variables, we first need to see how to extend the idea of an open interval in one variable to an open interval in two variables.

Définition 2. *Consider a point $(a, b) \in \mathbb{R}^2$. A δ **disk** centered at point (a, b) is defined to be an open disk of radius δ centered at point (a, b) that is,*

$$\{(x, y) \in \mathbb{R}^2 | (x - a)^2 + (y - b)^2 < \delta^2\} \tag{3.3}$$

*as shown in **Figure 5**.*

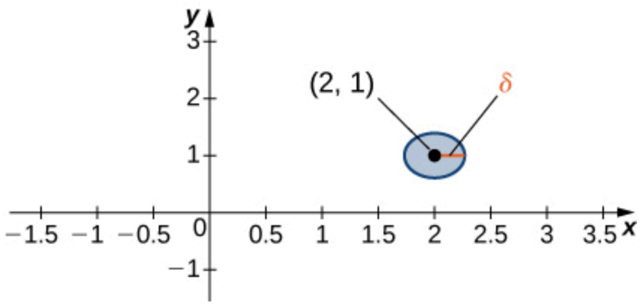

Figure 5: A δ disk centered around the point $(2, 1)$.

The idea of a δ disk appears in the definition of the limit of a function of two variables. If δ is small, then all the points (x, y) in the δ disk are close to (a, b). This is completely analogous to x being close to a in the definition of a limit of a function of one variable. In one dimension, we express this restriction as $a - \delta < x < a + \delta$. In more than one dimension, we use a δ disk.

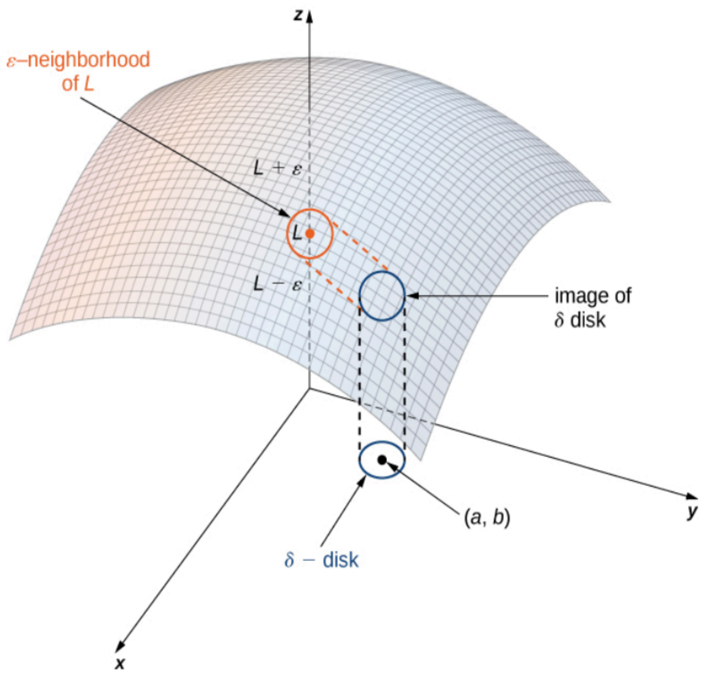

Figure 6: The limit of a function involving two variables requires that $f(x, y)$ be within ε of L whenever (x, y) is within δ of (a, b). The smaller the value of ε, the smaller the value of δ.

Définition 3. *Let f be a function of two variables, x and y. The limit of $f(x, y)$ as (x, y) approaches (a, b) is L, written*

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \tag{3.4}$$

*if for each $\delta > 0$ there exists a small enough $\delta > 0$ such that for all points (x, y) in a δ disk around (a, b), except possibly for (a, b) itself, the value of $f(x, y)$ is no more than ε away from L (**Figure***

6). Using symbols, we write the following : For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \textit{whenever} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \tag{3.5}$$

Proving that a limit exists using the definition of a limit of a function of two variables can be challenging. Instead, we use the following theorem, which gives us shortcuts to finding limits.

Théorème 1. *Let $f(x, y)$ and $g(x, y)$ be defined for all $(x, y) \neq (a, b)$ in a neighborhood around (a, b), and assume the neighborhood is contained completely inside the domain of f. Assume that L and M are real numbers such that $\lim_{(x,y)\to(a,b)} f(x, y) = L$ and $\lim_{(x,y)\to(a,b)} g(x, y) = M$, and let c be a constant. Then each of the following statements holds :*

- ***Constant Law :*** $\displaystyle\lim_{(x,y)\to(a,b)} c = c$
- ***Identity Laws :*** $\displaystyle\lim_{(x,y)\to(a,b)} x = a$ *et* $\displaystyle\lim_{(x,y)\to(a,b)} y = b$
- ***Sum Law :*** $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) + g(x, y) = L + M$
- ***Difference Law :*** $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) - g(x, y) = L - M$
- ***Constant Multiple Law :*** $\displaystyle\lim_{(x,y)\to(a,b)} cf(x, y) = cL$
- ***Product Law :*** $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y)\, g(x, y) = L\, M$
- ***Quotient Law :*** $\displaystyle\lim_{(x,y)\to(a,b)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}$ *for* $M \neq 0$
- ***Power Law :*** $\displaystyle\lim_{(x,y)\to(a,b)} (f(x, y))^n = L^n$ *for any positive integer n.*
- ***Root Law :*** $\displaystyle\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x, y)} = \sqrt[n]{L}$ *for all L if n is odd and positive, and for $L \geq 0$ if n is even and positive.*

Exemple 6. Suppose that we want to find the following limits :

(1) $\displaystyle\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6)$ **(2)** $\displaystyle\lim_{(x,y)\to(2,-1)} \frac{2x + 3y}{4x - 3y}$

For the case (1), first use the sum and difference laws to separate the terms :

$$\begin{aligned}\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6) = & \lim_{(x,y)\to(2,-1)} x^2 - \lim_{(x,y)\to(2,-1)} 2xy + \lim_{(x,y)\to(2,-1)} 3y^2 \\ & - \lim_{(x,y)\to(2,-1)} 4x + \lim_{(x,y)\to(2,-1)} 3y - \lim_{(x,y)\to(2,-1)} 6\end{aligned}$$

Next, use the constant multiple law on the second, third, fourth, and fifth limits :

$$\begin{aligned}= & \left(\lim_{(x,y)\to(2,-1)} x^2\right) - 2\left(\lim_{(x,y)\to(2,-1)} xy\right) + 3\left(\lim_{(x,y)\to(2,-1)} y^2\right) \\ & -4\left(\lim_{(x,y)\to(2,-1)} x\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right) - \left(\lim_{(x,y)\to(2,-1)} 6\right)\end{aligned}$$

Now, use the power law on the first and third limits, and the product law on the second limit :

$$\begin{aligned}= & \left(\lim_{(x,y)\to(2,-1)} x\right)^2 - 2\left(\lim_{(x,y)\to(2,-1)} x\right)\left(\lim_{(x,y)\to(2,-1)} x\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right)^2 \\ & -4\left(\lim_{(x,y)\to(2,-1)} x\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right) - \left(\lim_{(x,y)\to(2,-1)} 6\right)\end{aligned}$$

Last, use the identity laws on the first six limits and the constant law on the last limit :

$$\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6) = (2^2) - 2(2)(-1) = 3(-1)^2 - 4(2) + 3(-1) - 6 = 6$$

For the case **(2)**, before applying the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, constant multiple law, and identity law,

$$\begin{aligned}\lim_{(x,y)\to(2,-1)} (4x - 3y) &= \lim_{(x,y)\to(2,-1)} 4x - \lim_{(x,y)\to(2,-1)} -3y \\ &= 4\left(\lim_{(x,y)\to(2,-1)} x\right) - 3\left(\lim_{(x,y)\to(2,-1)} y\right) = 4(2) - 3(-1) = 11\end{aligned}$$

Since the limit of the denominator is nonzero, the quotient law applies. We now calculate the limit of the numerator using the difference law, constant multiple law, and identity law:

$$\begin{aligned}\lim_{(x,y)\to(2,-1)} (2x + 3y) &= \lim_{(x,y)\to(2,-1)} 2x + \lim_{(x,y)\to(2,-1)} -3y \\ &= 2\left(\lim_{(x,y)\to(2,-1)} x\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right) = 2(2) + 3(-1) = 1\end{aligned}$$

Therefore, according to the quotient law we have

$$\lim_{(x,y)\to(2,-1)} \frac{2x+3y}{4x-3y} = \frac{\lim_{(x,y)\to(2,-1)}(2x+3y)}{\lim_{(x,y)\to(2,-1)}(4x-3y)} = \frac{1}{11}$$

Since we are taking the limit of a function of two variables, the point (a, b) is in $\mathbb{R}^2$, and it is possible to approach this point from an infinite number of directions. Sometimes when calculating a limit, the answer varies depending on the path taken toward (a, b). If this is the case, then the limit fails to exist. In other words, the limit must be unique, regardless of path taken.

Exemple 7. Suppose that we want to find the following limits :

(1) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{3x^2+3y^2}$ **(2)** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2+3y^4}$

In the case **(1)**, The domain of the function $f(x, y) = 2xy/(3x^2 + y^2)$ consists of all points in the xy-plane except for the point $(0, 0)$ (**Figure 7**). To show that the limit does not exist as (x, y) approaches $(0, 0)$, we note that it is impossible to satisfy the definition of a limit of a function of two variables because of the fact that the function takes different values along different lines passing through point $(0, 0)$. First, consider the line $y = 0$ in the xy-plane. Substituting $y = 0$ into $f(x, y)$ gives

$$f(x,0) = \frac{2x(0)}{3x^2+0^2} = 0 \tag{3.6}$$

for any value of x. Therefore the value of f remains constant for any point on the x-axis, and as y approaches zero, the function remains fixed at zero.

Next, consider the line $y = x$. Substituting $y = x$ into $f(x, y)$ gives

$$f(x,x) = \frac{2x(x)}{3x^2+x^2} = \frac{2x^2}{4x^2} = \frac{1}{2} \tag{3.7}$$

This is true for any point on the line $y = x$. If we let x approach zero while staying on this line, the value of the function remains fixed at 1/2 regardless of how small x is.

Choose a value for ε that is less than $1/2$, say $1/4$. Then, no matter how small a δ disk we draw around $(0,0)$, the values of $f(x,y)$ for points inside that δ disk will include both 0 and 2. Therefore, the definition of limit at a point is never satisfied and the limit fails to exist.

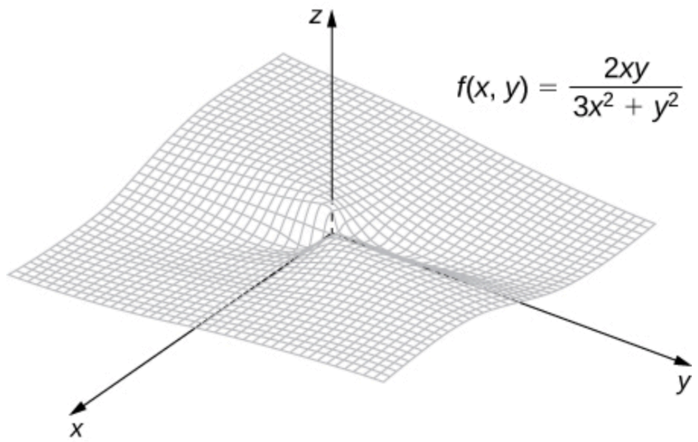

Figure 7: Graph of the function $f(x,y) = (2xy)/(3x^2 + y^2)$. Along the line $y = 0$, the function is equal to zero; along the line $y = x$, the function is equal to $1/2$.

For the case **(2)**, in a similar fashion to **(1)**, we can approach the origin along any straight line passing through the origin. If we try the x-axis (i.e., $y = 0$), then the function remains fixed at zero. The same is true for the y-axis. Suppose we approach the origin along a straight line of slope k. The equation of this line is $y = kx$. Then the limit becomes

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2+3y^4} &= \lim_{(x,y)\to(0,0)} \frac{4x(kx)}{x^2+3(kx)^4} = \lim_{(x,y)\to(0,0)} \frac{4k^2x^3)}{x^2+3k^4x^4} \\ &= \lim_{(x,y)\to(0,0)} \frac{4k^2x)}{1+3k^4x^2} = \frac{\lim_{(x,y)\to(0,0)}(4k^2x)}{\lim_{(x,y)\to(0,0)}(1+3k^4x^2)} = 0\end{aligned}$$

regardless of the value of k. It would seem that the limit is equal to zero. What if we chose a curve passing through the origin instead ? For example, we can consider the parabola given by the equation $x = y^2$. Substituting in y^2 in place of x in $f(x,y)$ gives

$$\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2+3y^4} = \lim_{(x,y)\to(0,0)} \frac{4(y^2)y^2}{(y^2)^2+3y^4} = \lim_{(x,y)\to(0,0)} \frac{4y^4}{y^4+3y^4} = \lim_{(x,y)\to(0,0)} = 1$$

By the same logic in **(1)**, it is impossible to find a δ disk around the origin that satisfies the definition of the limit for any value of $\varepsilon < 1$. Therefore, $\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2+3y^4}$ does not exist.

3.2 Interior Points and Boundary Points

To study continuity and differentiability of a function of two or more variables, we first need to learn some new terminology.

Définition 4. *Let S be a subset of $\mathbb{R}^2$ (see **Figure 8**).*

(a) *A point P_0 is called an **interior point** of S if there is a δ disk centered around P_0 contained completely in S.*

(b) *A point P_0 is called a **boundary point** of S if every δ disk centered around P_0 contains points both inside and outside S.*

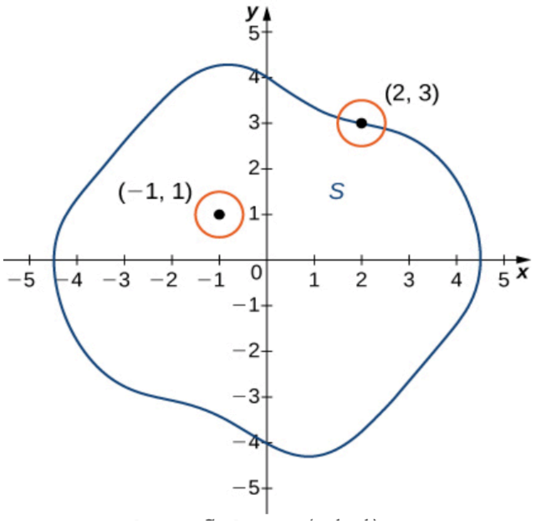

Figure 8: In the set S shown, $(-1, 1)$ is an interior point and $(2, 3)$ is a boundary point.

Définition 5. *Let S be a subset of $\mathbb{R}^2$ (see **Figure 8**).*

(a) *S is called an **open set** if every point of S is an interior point.*

(b) *S is called a **closed set** if it contains all its boundary points.*

An example of an open set is a δ disk. If we include the boundary of the disk, then it becomes a closed set. A set that contains some, but not all, of its boundary points is neither open nor closed. For example if we include half the boundary of a δ disk but not the other half, then the set is neither open nor closed.

Définition 6. *Let S be a subset of $\mathbb{R}^2$ (see **Figure 8**).*

(a) *An open set S is a **connected set** if it cannot be represented as the union of two or more disjoint, nonempty open subsets.*

(b) *A set S is a **region** if it is open, connected, and nonempty.*

The definition of a limit of a function of two variables requires the δ disk to be contained inside the domain of the function. However, if we wish to find the limit of a function at a boundary point of the domain, the δ disk is not contained inside the domain. By definition, some of the points of the δ disk are inside the domain and some are outside. Therefore, we need only consider points that are inside both the δ disk and the domain of the function. This leads to the definition of the limit of a function at a boundary point.

Définition 7. *Let f be a function of two variables, x and y, and suppose (a, b) is on the boundary of the domain of f. Then, the limit of $f(x, y)$ as (x, y) approaches (a, b) is L, written*

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \tag{3.8}$$

if for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that for any point (x, y) inside the domain of f and within a suitably small distance positive δ of (a, b), the value of $f(x, y)$ is no more than δ away from L (Figure 4.15). Using symbols, we write the following : For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \textit{whenever} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \tag{3.9}$$

Exemple 8. (Limit of a Function at a Boundary Point)

Suppose we want to prove the following :

$$\lim_{(x,y)\to(4,3)} \sqrt{25 - x^2 - y^2} = 0 \tag{3.10}$$

The domain of the function $f(x, y) = \sqrt{25 - x^2 - y^2}$ is $\{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 25\}$, which is a circle of radius 5 centered at the origin, along with its interior as shown in **Figure 9** (left).

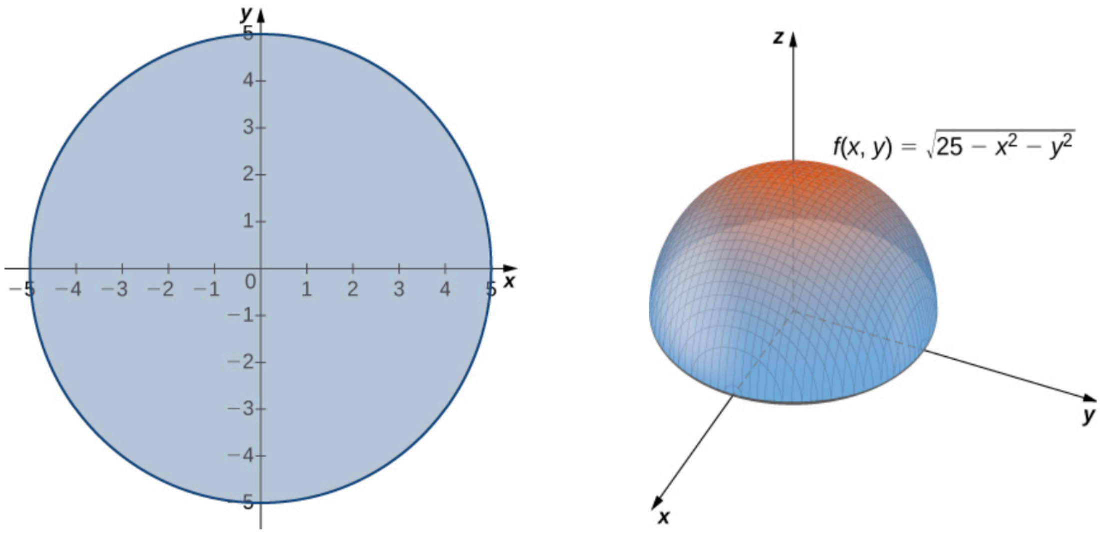

Figure 9: Domain (left) and graph (right) of the function $f(x, y) = \sqrt{25 - x^2 - y^2}$.

We can use the limit laws, which apply to limits at the boundary of domains as well as interior points (see **Figure 9** (right)) :

$$\begin{aligned}
\lim_{(x,y)\to(4,3)} \sqrt{25 - x^2 - y^2} &= \sqrt{\lim_{(x,y)\to(4,3)} (25 - x^2 - y^2)} \\
&= \sqrt{\lim_{(x,y)\to(4,3)} 25 - \lim_{(x,y)\to(4,3)} x^2 - \lim_{(x,y)\to(4,3)} y^2} = \sqrt{25 - 4^2 - 3^2} = 0
\end{aligned}$$

3.3 Continuity of Functions of Two Variables

We know that the continuity of a function of one variable rely on the limit of a function of one variable. In particular, three conditions are necessary for $f(x)$ to be continuous at point $x = a$:

(a) $f(a)$ exists.

(b) $\lim_{x\to a} f(x)$ exists

(c) $\lim_{x\to a} f(x) = f(a)$

These three conditions are necessary for continuity of a function of two variables as well.

Définition 8. *A function $f(x,y)$ is continuous at a point (a,b) in its domain if the following conditions are satisfied :*

(a) *$f(a,b)$ exists.*

(b) *$\lim_{(x,y)\to(a,b)} f(x)$ exists.*

(c) *$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$*

Exemple 9. (Demonstrating Continuity for a Function of Two Variables)

We want to show that the function $f(x,y) = 3x+2yx+y+1$ is continuous at point $(5,-3)$. To show this, there are three conditions to be satisfied, per the definition of continuity. In this example, $a = 5$ and $b = -3$.

(a) $\underline{f(a,b) \text{ exists}}$. This is true because the domain of the function f consists of those ordered pairs for which the denominator is nonzero (i.e., $x+y+1 \neq 0$). Point $(5,-3)$ satisfies this condition. Furthermore,

$$f(a,b) = f(5,-3) = \frac{3(5)+2(-3)}{5+(-3)+1} = \frac{15-6}{2+1} = 3$$

(b) $\underline{\lim_{(x,y)\to(a,b)} f(x) \text{ exists}}$. This is also true :

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(5,-3)} f(x,y) \frac{3x+2y}{x+y+1} = \frac{\lim_{(x,y)\to(5,-3)}(3x+2y)}{\lim_{(x,y)\to(5,-3)}(x+y+1)} = \frac{15-6}{5-3+1} = 3$$

(c) $\underline{\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)}$. This is true because we have just shown that both sides of this equation equal three.

Continuity of a function of any number of variables can also be defined in terms of delta and epsilon. A function of two variables is continuous at a point (x_0,y_0) in its domain if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, whenever $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$ it is true, $|f(x,y)-f(a,b)| < \varepsilon$. This definition can be combined with the formal definition (that is, the epsilon-delta definition) of continuity of a function of one variable to prove the following theorems :

Théorème 2. *(The Sum of continuous functions is continuous) If $f(x,y)$ is continuous at (x_0,y_0), and $g(x,y)$ is continuous at (x_0,y_0), then $f(x,y)+g(x,y)$ is continuous at (x_0,y_0).*

Théorème 3. *(The product of continuous functions is continuous) If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x,y) = g(x)h(y)$ is continuous at (x_0,y_0).*

Théorème 4. *(The Composition of continuous functions is continuous) Let g be a function of two variables from a domain $D \subseteq \mathbb{R}^2$ to a range $R \subseteq \mathbb{R}$. Suppose g is continuous at some point $(x_0,y_0) \in D$ and define $z_0 = g(x_0,y_0)$. Let f be a function that maps $\mathbb{R}$ to $\mathbb{R}$ such that z_0 is in the domain of f. Last, assume f is continuous at z_0. Then $f \circ g$ is continuous at (x_0,y_0) as shown in **Figure 10**.*

Let's now use the previous theorems to show continuity of functions in the following examples.

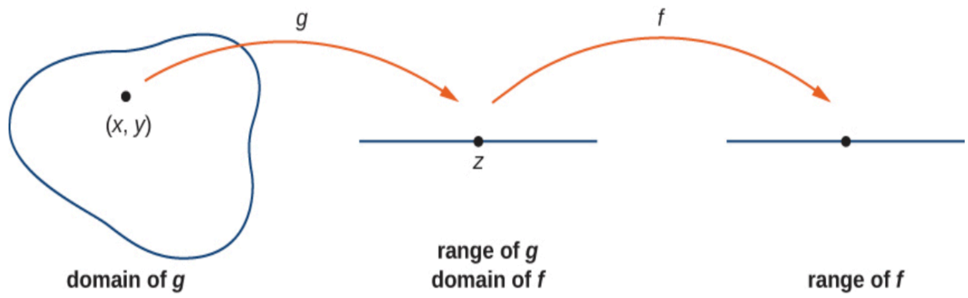

Figure 10: The composition of two continuous functions is continuous.

Exemple 10. We want to show that the functions $f(x, y) = 4x^3y^2$ and $g(x, y) = \cos(4x^3y^2)$ are continuous everywhere. To do that, we first notice that he polynomials $g(x) = 4x^3$ and $h(y) = y^2$ are continuous at every real number, and therefore by the product of continuous functions theorem, $f(x, y) = 4x^3y^2$ is continuous at every point (x, y) in the xy-plane. Since $f(x, y) = 4x^3y^2$ is continuous at every point (x, y) in the xy-plane and $g(x) = \cos x$ is continuous at every real number x, the continuity of the composition of functions tells us that $g(x, y) = \cos(4x^3y^2)$ is continuous at every point (x, y) in the xy-plane.

3.4 Functions of Three or More Variables

The limit of a function of three or more variables occurs readily in applications. For example, suppose we have a function $f(x, y, z)$ that gives the temperature at a physical location (x, y, z) in three dimensions. Or perhaps a function $g(x, y, z, t)$ can indicate air pressure at a location (x, y, z) at time t. How can we take a limit at a point in $\mathbb{R}^3$? What does it mean to be continuous at a point in four dimensions ? The answers to these questions rely on extending the concept of a δ disk into more than two dimensions. Then, the ideas of the limit of a function of three or more variables and the continuity of a function of three or more variables are very similar to the definitions given earlier for a function of two variables.

Définition 9. *Let (x,y_0, z_0) be a point in $\mathbb{R}^3$. Then, a δ ball in three dimensions consists of all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x,y_0, z_0), that is*

$$\left\{(x, y, z) \in \mathbb{R}^3 \, , \, \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta\right\} \tag{3.11}$$

To define a δ ball in higher dimensions, add additional terms under the radical to correspond to each additional dimension. For example, given a point $P = (w_0, x,y_0, z_0)$ in $\mathbb{R}^4$, a δ ball around P can be described by

$$\left\{(w, x, y, z) \in \mathbb{R}^3 \, , \, \sqrt{(w - w_0)^2 + (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta\right\} \tag{3.12}$$

To show that a limit of a function of three variables exists at a point (x,y_0, z_0), it suffices to show that for any point in a ball centered at (x,y_0, z_0), the value of the function at that point is arbitrarily close to a fixed value (the limit value). All the limit laws for functions of two variables hold for functions of more than two variables as well.

Exemple 11. (Finding the Limit of a Function of Three Variables) We want to find the following limit :

$$\lim_{(x,y,z)\to(4,1,-3)} \frac{x^2y - 3z}{2x + 5y - z} \tag{3.13}$$

Before we can apply the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, the identity law, and the constant law,

$$\begin{aligned}\lim_{(x,y,z)\to(4,1,-3)} 2x+5y-z \;&=\; 2\left(\lim_{(x,y,z)\to(4,1,-3)} x\right) + 5\left(\lim_{(x,y,z)\to(4,1,-3)} y\right) - 2\left(\lim_{(x,y,z)\to(4,1,-3)} z\right)\\ &=\; 2(4)+5(1)-(-3) \;=\; 16\end{aligned}$$

Since this is nonzero, we next find the limit of the numerator. Using the product law, difference law, constant multiple law, and identity law,

$$\begin{aligned}\lim_{(x,y,z)\to(4,1,-3)} x^2y-3z \;&=\; \left(\lim_{(x,y,z)\to(4,1,-3)} x\right)^2\left(\lim_{(x,y,z)\to(4,1,-3)} y\right) - 3\left(\lim_{(x,y,z)\to(4,1,-3)} z\right)\\ &=\; (4^2)(1)-3(-3) \;=\; 16+9 \;=\; 25\end{aligned}$$

Last, applying the quotient law :

$$\lim_{(x,y,z)\to(4,1,-3)} \frac{x^2y-3z}{2x+5y-z} = \frac{\lim_{(x,y,z)\to(4,1,-3)}(x^2y-3z)}{\lim_{(x,y,z)\to(4,1,-3)}(x^2y-3z)} = \frac{25}{16}$$

4 Partial Derivatives

The central idea of differential calculus is the derivative. A change in x produces a change in f. The ratio $\Delta f/\Delta x$ approaches the derivative, or slope, or rate of change. What to do if f depends on both x and y ?

The new idea is to vary x and y one at a time. First, only x moves. If the function is $x+xy$, then Δf is $\Delta x + y\,\Delta x$. The ratio $\Delta f/\Delta x$ is $1+y$. The « x derivative » of $x+xy$ is $1+y$. For all functions the method is the same : Keep y constant, change x, take the limit of $\Delta f/\Delta x$.

Définition 10. *We have*

$$\frac{\partial f}{\partial x}(x,y) = \lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x} \tag{4.1}$$

*On the left is a new symbol $\frac{\partial f}{\partial x}$. It signals that only x is allowed to vary - $\frac{\partial f}{\partial x}$ is a **partial derivative**. The different form « ∂ »of the same letter (still say « d ») is a reminder that x is not the only variable. Another variable y is present but not moving.*

Exemple 12. We consider

$$f(x,y) = x^2y^2+xy+y \qquad \text{and} \qquad y = 2xy^2+y+0 \tag{4.2}$$

Do not treat y as zero ! Treat it as a constant, like 6. Its x derivative is zero. If $f(x)=\sin 6x$ then $\partial f/\partial x = 6\cos 6x$. If $f(x,y)=\sin xy$ then $\partial f/\partial x = y\cos xy$.

Spoken aloud, $\partial f/\partial x$ is still « *df dx* » . It is a function of x and y. When more is needed, call it « the partial of f with respect to x » . The symbol f' is no longer available, since it gives no special indication about x. its replacement f_x, is pronounced « $f\,x$ » or « f sub x » which is shorter than $\partial f/\partial x$ and means the same thing.

We may also want to indicate the point (x_0,y_0) where the derivative is computed :

$$\frac{\partial f}{\partial x}(x_0,y_0) \quad \text{or} \quad f_x(x_0,y_0) \quad \text{or} \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \quad \text{or just} \quad \left(\frac{\partial f}{\partial x}\right)_0 \tag{4.3}$$

Exemple 13. We consider

$$f(x,y) = \sin 2x \cos y \qquad \text{therefore} \qquad f_x = 2\cos 2x \cos y \tag{4.4}$$

because $\cos y$ is constant for $\partial/\partial x$. The particular point (x_0, y_0) is $(0,0)$. The height of the surface is $f(0,0) = 0$. The slope in the x direction is $f_x = 2$. At a different point $x_0 = \pi$, $y_0 = \pi$ we find $f_x(\pi,\pi) = -2$.

Now keep x constant and vary y. The ratio $\Delta f/\Delta y$ approaches $\partial f/\partial y$: We have

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{\Delta f}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \tag{4.5}$$

This is the slope in the y direction. Please realize that a surface can go up in the x direction and down in the y direction. The plane $f(x,y) = 3x - 4y$ has $f_x = 3$ (up) and $f_y = -4$ (down). We will soon ask what happens in the 45° direction.

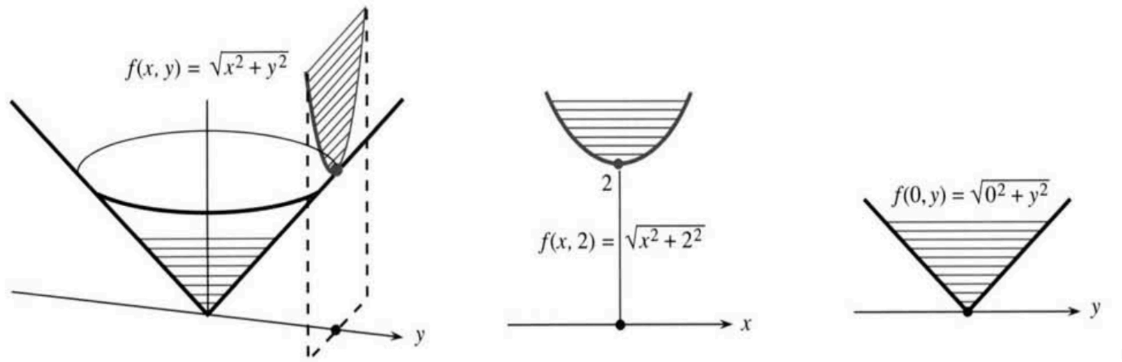

Figure 11: Partial functions $\sqrt{x^2+y^2}$ and $\sqrt{0^2+2^2}$ of the distance function $f = \sqrt{x^2+y^2}$.

Exemple 14. We consider

$$f(x,y) = \sqrt{x^2+y^2} \quad , \quad \frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{f} \quad , \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{f} \tag{4.6}$$

The x derivative of $\sqrt{x^2+y^2}$ is really one-variable calculus, because y is constant. The exponent drops from $1/2$ to -4, and there is $2x$ from the chain rule. This distance function has the curious derivative $\frac{\partial f}{\partial x} = \frac{x}{f}$.

The graph is a cone. Above the point $(0,2)$ the height is $\sqrt{0^2+2^2} = 2$. The partial derivatives are $f_x = 0/2$ and $f_y = 2/2$. At that point, **Figure 11** climbs in the y direction. It is level in the x direction. An actual step Δx will increase $0^2 + 2^2$ to $(\Delta x)^2 + 2^2$. But this change is of order $(\Delta x)^2$ and the x derivative is zero.

Figure 11 is rather important. It shows how $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are the ordinary derivatives of $f(x, y_0)$ and $f(x_0, y)$. It is natural to call these **partial functions**. The first has y fixed at y_0 while x varies. The second has x fixed at x_0 while y varies. Their graphs are **cross sections down the surface** - cut out by the vertical planes $y = y_0$ and $x = x_0$. Remember that the level curve is cut out by the horizontal plane $z = c$.

The limits of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are computed as always. With partial functions we are back to a single variable. **The partial derivative is the ordinary derivative of a partial function** (constant y or constant x). For the cone, $\frac{\partial f}{\partial y}$ exists at all points except $(0, 0)$. The figure shows how the cross section down the middle of the cone produces the absolute value function: $f(0, y) = |y|$. It has one-sided derivatives but not a two-sided derivative.

Similarly $\frac{\partial f}{\partial x}$ will not exist at the sharp point of the cone. We develop the idea of a *continuous function* $f(x, y)$ as needed (the definition is in **Exercise 12**). Each partial derivative involves one direction, but limits and continuity involve all directions. The distance function is continuous at $(0, 0)$, where it is not differentiable.

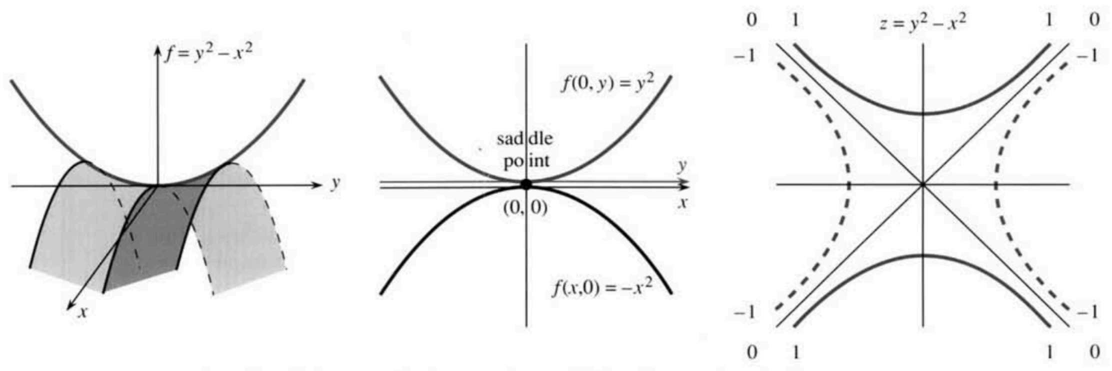

Figure 12: A saddle function, its partial functions, and its level curves.

Exemple 15. We consider

$$f(x, y) = x^2 - y^2 \quad , \quad \frac{\partial f}{\partial x} = -2x \quad , \quad \frac{\partial f}{\partial y} = 2y \tag{4.7}$$

Move in the x direction from $(1, 3)$. Then $x^2 - y^2$ has the partial function $9 - x^2$. With y fixed at 3, a parabola opens downward. In the y direction (along $x = 1$) the partial function $y^2 - 1$ opens upward. The surface in **Figure 12** is called a hyperbolic paraboloid, because the level curves $x^2 - y^2 = c$ are hyperbolas. Most people call it a saddle, and the special point at the origin is a **saddle point**.

The origin is special for $x^2 - y^2$ because both derivatives are zero. **The bottom of the** y **parabola at** $(0, 0)$ **is the top of the** x **parabola**. The surface is momentarily flat in all directions. It is the top of a hill and the bottom of a mountain range at the same time. A saddle point is neither a maximum nor a minimum, although both derivatives are zero.

Note Do not think that $f(x, y)$ must contain y^2 and x^2 to have a saddle point. The function $2xy$ does just as well. The level curves $2xy = c$ are still hyperbolas. The partial functions $2xy_0$ and $2x_0y$ now give straight lines - which is remarkable. Along the $45°$ line $x = y$, the function is $2x^2$ and climbing. Along the $-45°$ line $x = -y$, the function is $-2x^2$ and falling. The graph of $2xy$ is **Figure 12** rotated by $45°$.

Exemple 16. We consider the two functions

$$f(x, y, z) = x^2 + y^2 + z^2 \qquad \text{and} \qquad P(T, V) = aRT/V \tag{4.8}$$

The function f shows more variables. The function P shows that the variables may not be named x and y. Also, the function may not be named f! Pressure and temperature and volume are P and T and V. The letters change but nothing else:

$$\frac{\partial P}{\partial T} = \frac{nR}{V} \qquad \text{and} \qquad \frac{\partial P}{\partial V} = \frac{nRT}{V^2} \qquad \text{(note the derivative of } 1/V\text{)} \tag{4.9}$$

There is no $\frac{\partial P}{\partial R}$ because R is a constant from chemistry - not a variable.

Physics produces six variables for a moving body - the coordinates x, y, z and the momenta p_x, p_y, p_y. Economics and the social sciences do better than that. If there are 26 products there are 26 variables - sometimes 52, to show prices as well as amounts. The profit can be a complicated function of these variables. **The partial derivatives are the marginal profits**, as one of the 52 variables is changed. A spreadsheet shows the 52 values and the effect of a change. An infinitesimal spreadsheet shows the derivative.

4.1 Second derivative

Genius is not essential, to move to second derivatives. The only difficulty is that two first derivatives f_x and f_y lead to four second derivatives f_{xx} and f_{yy} and f_{xy} and f_{yx}. (Two subscripts: f_{xx} is the x derivative of the x derivative. Other notations are $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x^2}$.) Fortunately f_{xx} equals f_{yy}, as we see first by example.

Exemple 17. The function $f = x/y$ has $f_x = 1/y$, which has $f_{xx} = 0$ and $f_{xy} = -1/y^2$.
The function x/y is linear in x (which explains $f_{xx} = 0$). Its y derivative is $f_y = -x/y^2$. This has the x derivative $f_{xx} = -1/y^2$. **The mixed derivatives f_{xy} and f_{yx} are equal**. In the pure y direction, the second derivative is $f_{yy} = 2x/y^3$. One-variable calculus is sufficient for all these derivatives, because only one variable is moving.

Exemple 18. The function $f = 4x^2 + 3xy + y^2$ has $f_x = 8x + 3y$ and $f_y = 3x + 2y$.
Both « cross derivatives» f_{xy} and f_{yx} equal 3. The second derivative in the x direction is $\frac{\partial^2 f}{\partial x^2} = 8$ or $f_{xx} = 8$. Thus « $f\ x\ x$» is « d second f dx squared » . Similarly $\frac{\partial^2 f}{\partial y^2} = 8$. The only change is from d to ∂.

If $f(x, y)$ has continuous second derivatives then $f_{xy} = f_{yx}$. Problem 43 sketches a proof based on the Mean Value Theorem. For third derivatives almost any example shows that $f_{xxy} = f_{xyx} = f_{yxx}$ from $f_{yyx} = f_{yxy} = f_{xyy}$.

Question : How do you plot a space curve $x(t), y(t), z(t)$ in a plane?
One way is to look parallel to the direction $(1, 1, 1)$. On your XY screen, plot $X = (y - x)\sqrt{2}$ and $Y = (2z - x - y)/\sqrt{6}$. The line $x = y = z$ goes to the point $(0, 0)$!

How do you graph a surface $z = f(x, y)$?
Use the same X and Y. Fix x and let y vary, for curves one way in the surface. Then fix y and vary x, for the other partial function. For a parametric surface like $x = (2 + v \sin u/2) \cos u$, $y = (2 + v \sin y/2) \sin u$, $z = v \cos u/2$, vary u and then v. Dick Williamson showed how this draws a one-sided « Möbius strip » .

5 Tangent Planes and Linear Approximations

Over a short range, a smooth curve $y = f(x)$ is almost straight. The curve changes direction, but the tangent line $y - y_0 = f'(x_0)(x - x_0)$ keeps the same slope forever. The tangent line immediately gives the linear approximation to $y = f(x) : y \approx y_0 + f'(x_0)(x - x_0)$.

What happens with two variables? The function is $z = f(x, y)$, and its graph is a **surface**. We are at a point on that surface, and we are near-sighted. We don't see far away. The surface may curve out of sight at the horizon, or it my be a bowl or a saddle. To our myopic vision, the surface looks flat. We believe we are on a plane (not necessarily horizontal), and we want the equation of this **tangent plane**.

Notation The basepoint has coordinates x_0 and y_0. The height on the surface is $z_0 = f(x_0, y_0)$. Other letters are possible: the point can be (a, b) with height w. The subscript $_0$ indicates the value of x or y or z or $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$ at the point.

With one variable the tangent line has slope $\frac{df}{dx}$. With two variables there are two derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. At the particular point, they are $(\frac{\partial f}{\partial x})_0$ and $(\frac{\partial f}{\partial y})_0$. **Those are the slopes of the tangent plane**. Its equation is an important result :

The tangent plane at (x_0, y_0, z_0) has the same slopes as the surface $z = f(x, y)$. The equation of the tangent plane (a linear equation) is

$$z - z_0 = \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0) \tag{5.1}$$

The normal vector $\boldsymbol{N}$ to that plane has components $(\partial f/\partial x)_0$, $(\partial f/\partial y)_0$, -1.

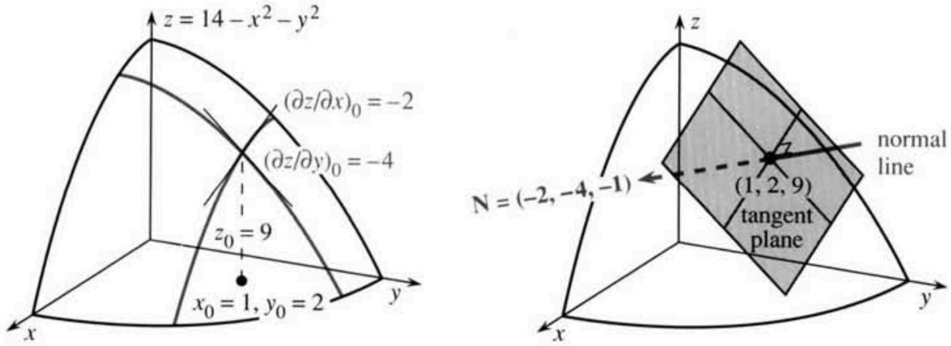

Figure 13: The tangent plane contains the x and y tangent lines, perpendicular to $\boldsymbol{N}$.

Exemple 19. Find the tangent plane to $z = 14 - x^2 - y^2$ at $(x_0, y_0, z_0) = (1, 2, 9)$.

Answer 19 The derivatives are $(\partial f/\partial x)_0 = -2x$ and $(\partial f/\partial y)_0 = -2y$. When $x = 1$ and $y = 2$ those are $(\partial f/\partial x)_0 = -2$ and $(\partial f/\partial y)_0 = -4$. The equation of the tangent plane is

$$z - 9 = -2(x - 1) - 4(4y - 2) \qquad \text{or} \qquad z + 2x + 4y = 19 \tag{5.2}$$

This $z(x, y)$ has derivatives -2 and -4, just like the surface. So the plane is tangent. The normal vector $\boldsymbol{N}$ has components -2, -4, -1. **The equation of the normal line is** $(x, y, z) = (1, 2, 9) + t(-2, -4, -1)$**.** Starting from $(1, 2, 9)$ the line goes out along $\boldsymbol{N}$, perpendicular to the