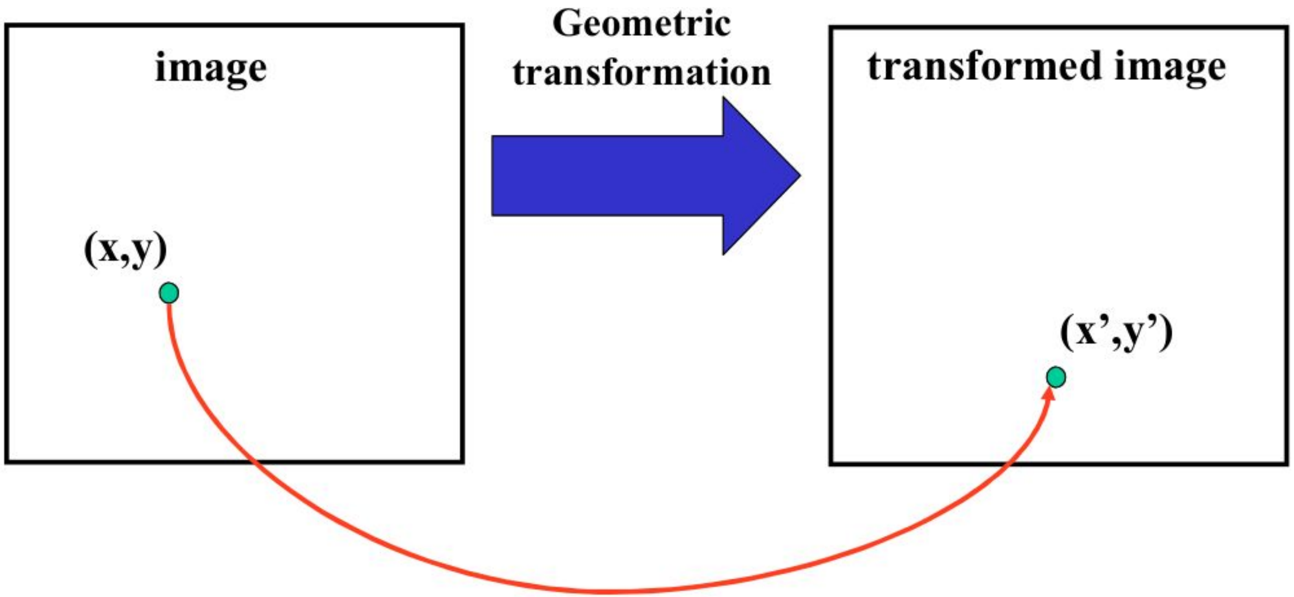
MLRF Lecture 03

J. Chazalon, LRE/EPITA, 2025

Projective transformations

Lecture 03 part 03

A linear transformation of pixel coordinates



$$x' = f(x, y, \{parameters\})$$
$$y' = g(x, y, \{parameters\})$$

Illustration: Robert Collins

Image Mappings Overview

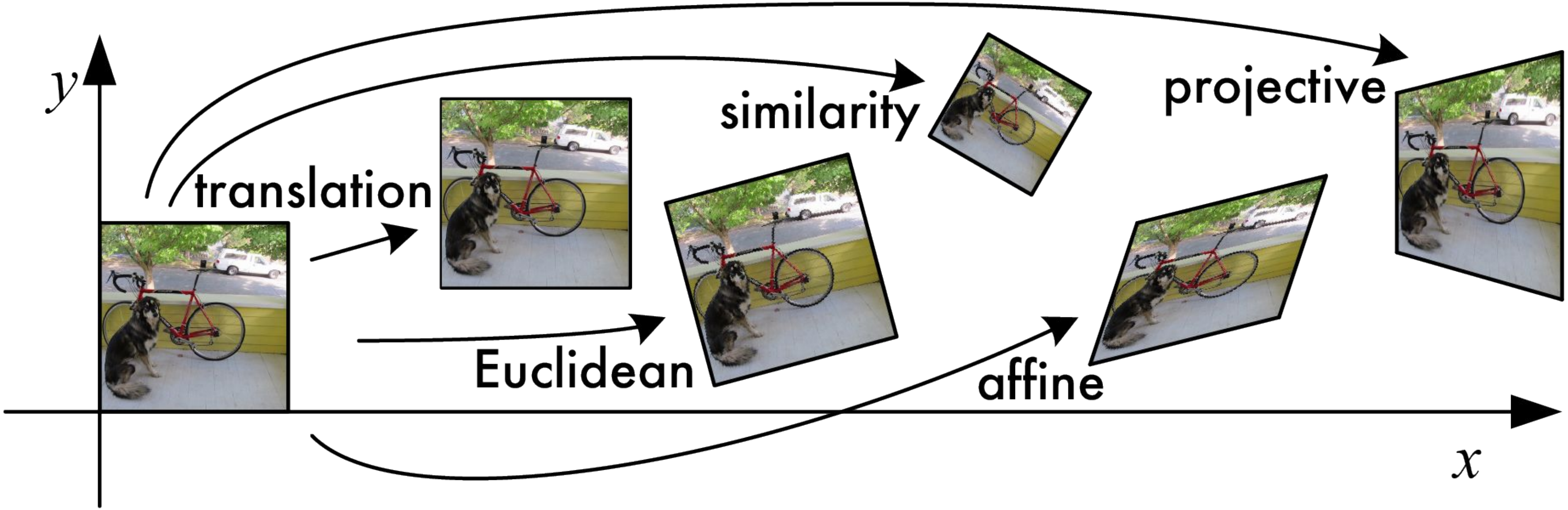


Image adapted from R. Szeliski by JP. Reddie

Math. foundations & assumptions

Homography H
(planar projective
transformation)

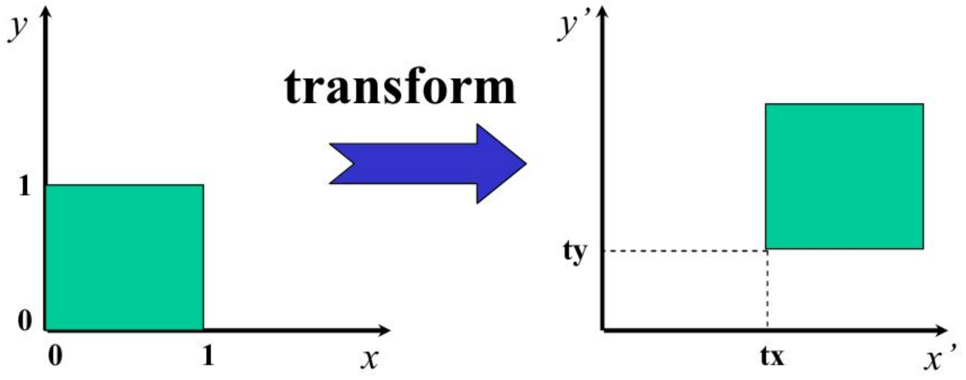
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

For **planar surfaces**, 3D to 2D perspective projection reduces to a 2D to 2D transformation.

This is just a change of coordinate system.

This transformation is **INVERTIBLE**!

Translation



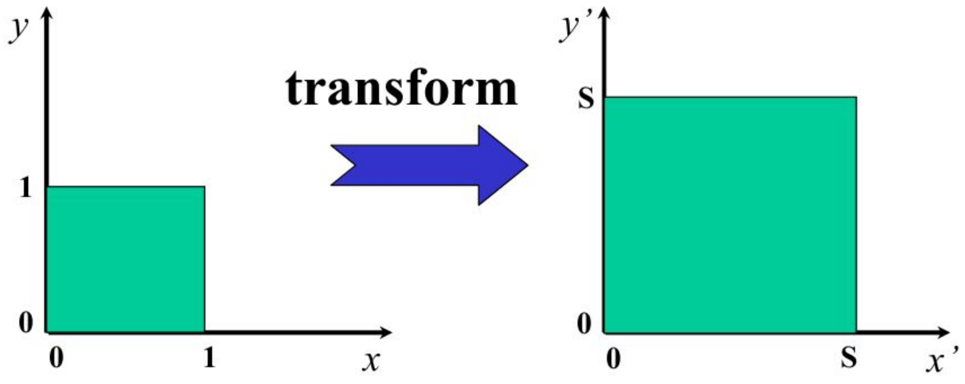
$$
\begin{aligned}
x' &= x + t_x \\
y' &= y + t_y
\end{aligned}
$$

$$
\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
$$

equations **matrix form**

Illustration: Robert Collins

Scale



$$x' = s\,x_i$$
$$y' = s\,y_i$$

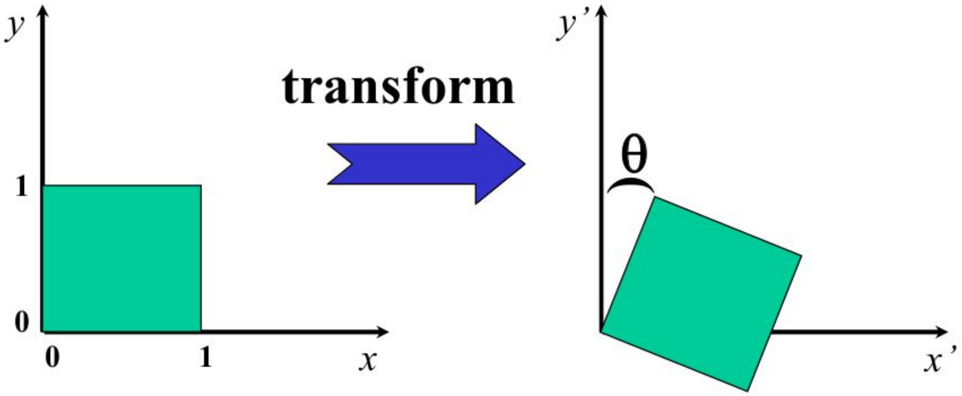
equations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

matrix form

Illustration: Robert Collins

Rotation



$$x' = x_i \cos\theta - y_i \sin\theta$$
$$y' = x_i \sin\theta + y_i \cos\theta$$

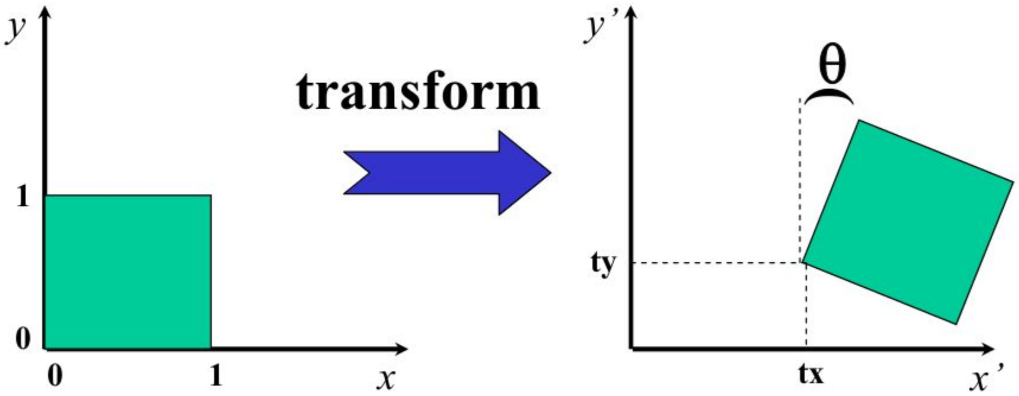
equations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

matrix form

Illustration: Robert Collins

Euclidean (rigid)



$$x' = x_i \cos\theta - y_i \sin\theta + t_x$$
$$y' = x_i \sin\theta + y_i \cos\theta + t_y$$

equations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

matrix form

Illustration: Robert Collins

Notation: Partitioned matrices

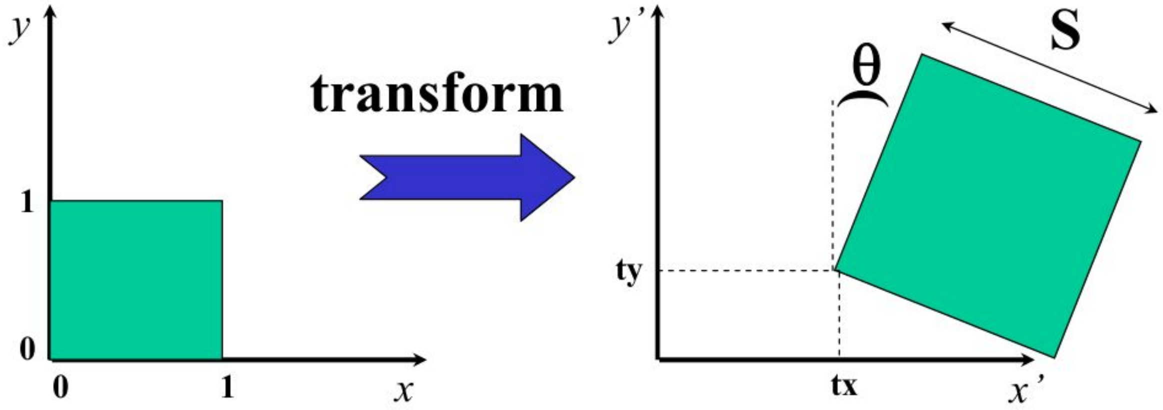
$$\begin{bmatrix} x' \\ y' \\ \hline 1 \end{bmatrix} = \left[\begin{array}{cc|c} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ \hline 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x \\ y \\ \hline 1 \end{bmatrix}$$

$$\begin{bmatrix} \overset{\text{2x1}}{p'} \\ \overset{\text{1x1}}{1} \end{bmatrix} = \begin{bmatrix} \overset{\text{2x2}}{R} & \overset{\text{2x1}}{t} \\ \overset{\text{1x2}}{0} & \overset{\text{1x1}}{1} \end{bmatrix} \begin{bmatrix} \overset{\text{2x1}}{p} \\ \overset{\text{1x1}}{1} \end{bmatrix}$$ **matrix form**

$$p' = Rp + t$$ **equation form**

Illustration: Robert Collins

Similarity (scaled Euclidean)



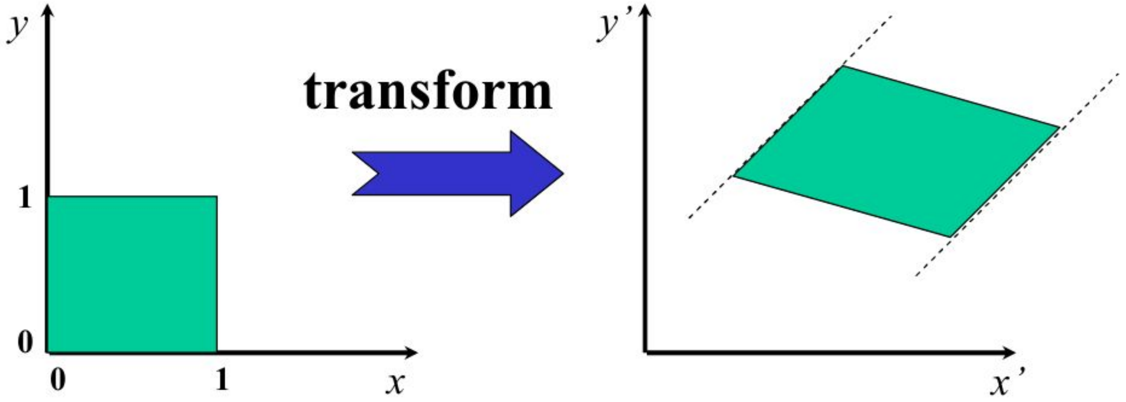
$$p' = sRp + t$$

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} = \begin{bmatrix} sR & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

equations **matrix form**

Illustration: Robert Collins

Affine



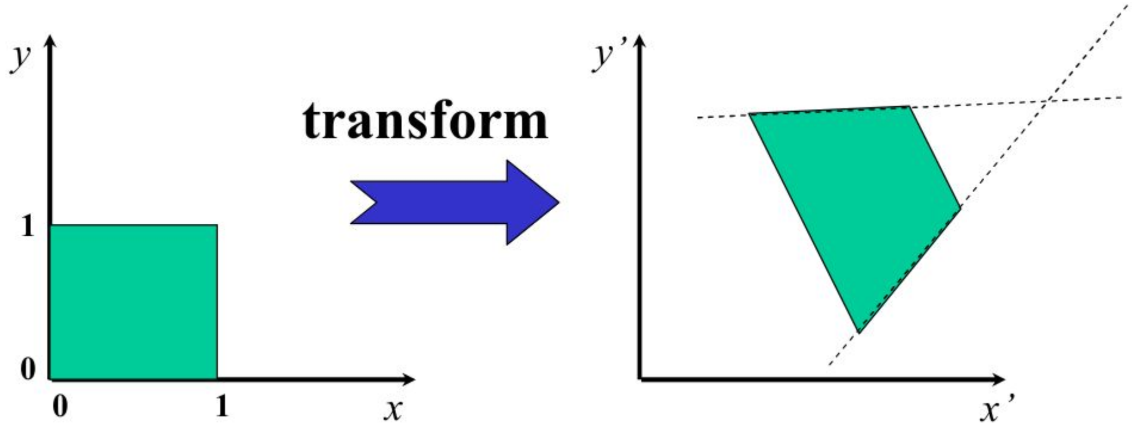
$$p' = Ap + b$$

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

equations **matrix form**

Illustration: Robert Collins

Projective



Note!

$$p' = \frac{Ap + b}{c^T p + 1}$$

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} \sim \begin{bmatrix} A & b \\ c^T & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

equations **matrix form**

Illustration: Robert Collins

More on projective transform

Each point in 2D is actually a vector in 3D

Equivalent up to scaling factor

α***H** ~ **H**

Have to normalize to get back to 2D

$$\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix} \qquad \bar{\mathbf{x}} = \tilde{\mathbf{x}} \,/\, \tilde{w}$$

Why does this make sense?

Pinhole camera model:

- Every point in 3D projects onto our viewing plane through our aperture
- Points along a vector are indistinguishable

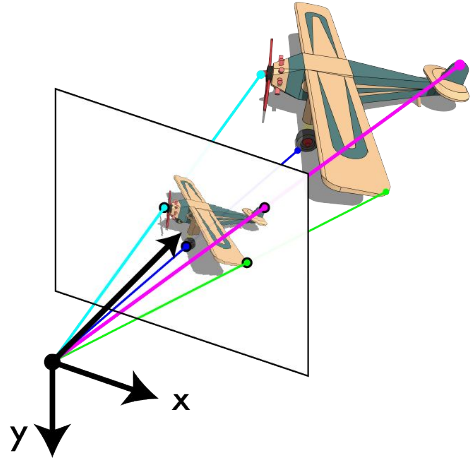


Illustration by JP. Reddie

More on projective transform

Using homography to project point

Multiply $\tilde{x}$ by $\tilde{H}$ to get $\tilde{x'}$

Convert to $\tilde{x'}$ by dividing by $\tilde{w'}$

$$\begin{bmatrix} \tilde{x}' \\ \tilde{y}' \\ \tilde{w}' \end{bmatrix} = \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix}$$

$$\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\,\tilde{\mathbf{x}}$$

$$\bar{\mathbf{x}} = \tilde{\mathbf{x}} \,/\, \tilde{w}$$

Summary of 2D transformations

Euclidean

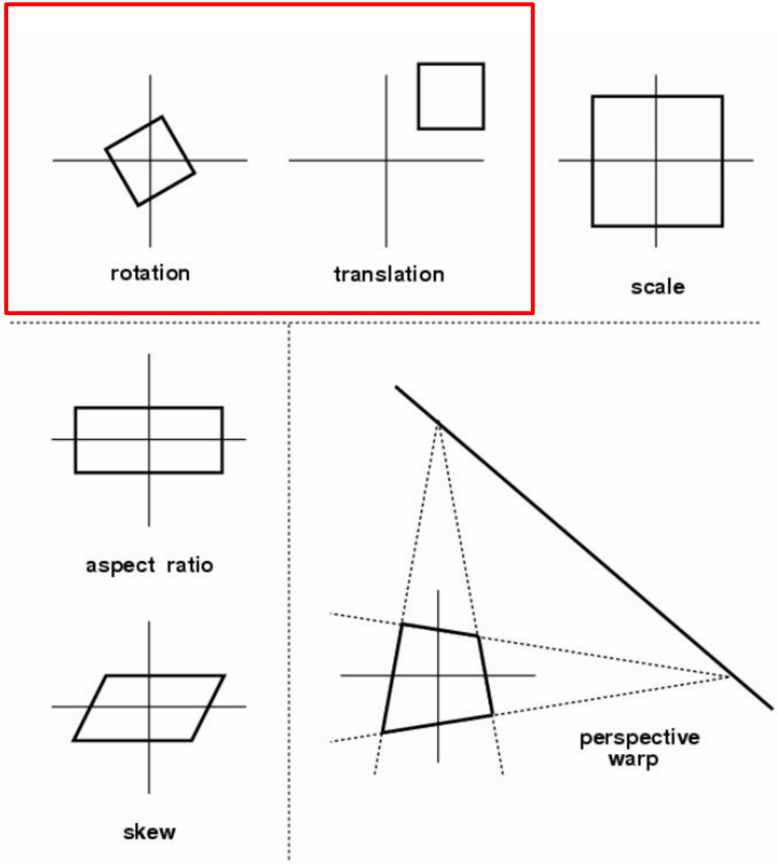


Illustration: Robert Collins

Summary of 2D transformations

Similarity

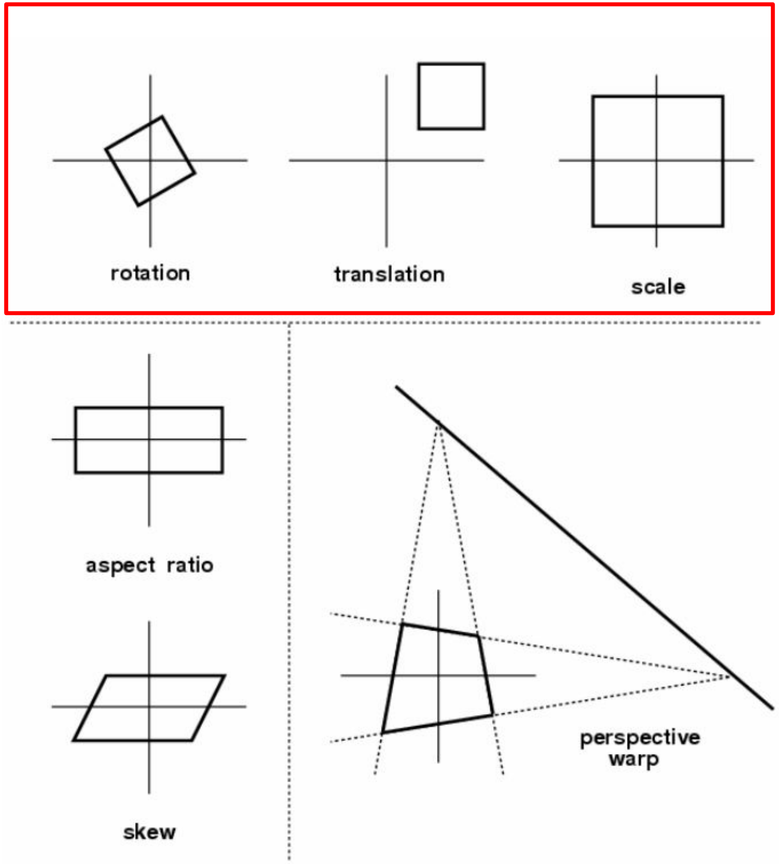


Illustration: Robert Collins

Summary of 2D transformations

Affine

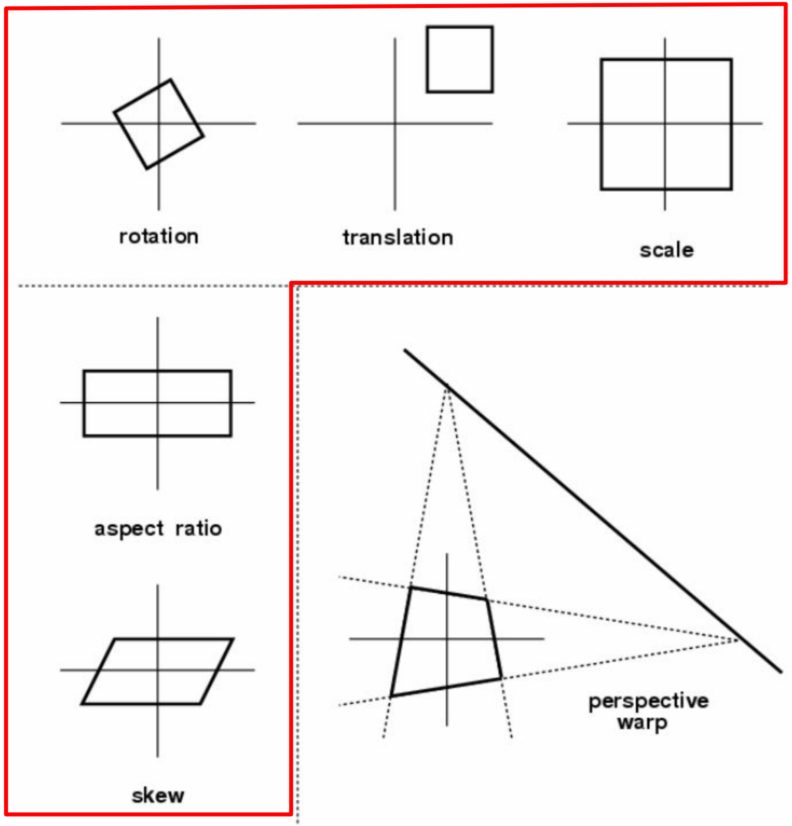


Illustration: Robert Collins

Summary of 2D transformations

Projective

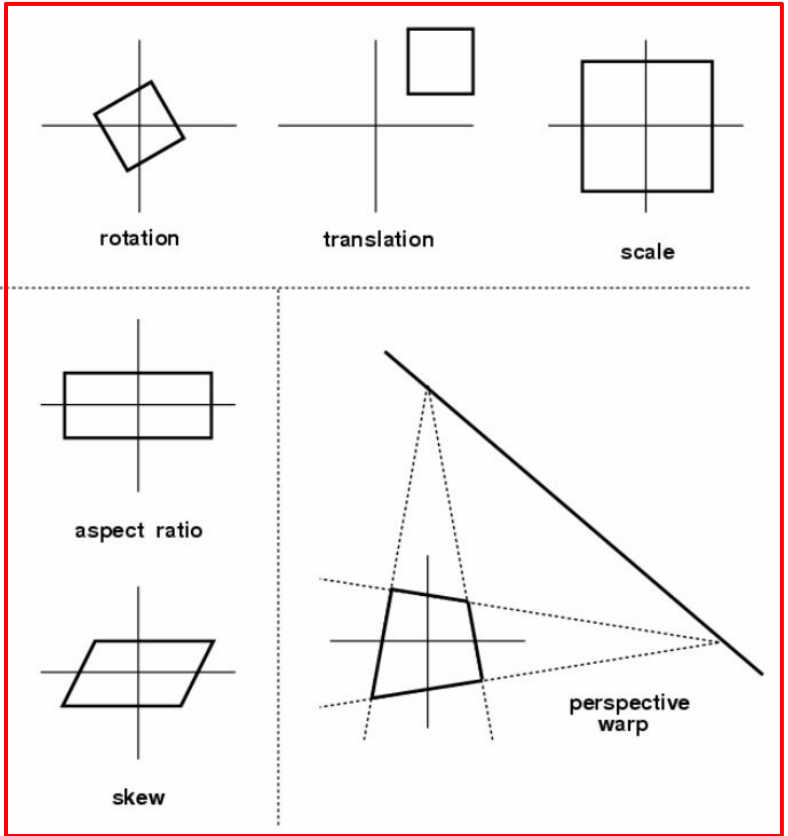
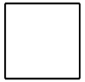
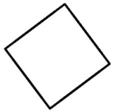
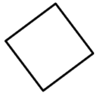
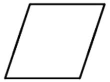
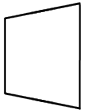


Illustration: Robert Collins

Summary of 2D transformations

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\left[\begin{array}{c\|c} \mathbf{I} & \mathbf{t} \end{array}\right]_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\left[\begin{array}{c\|c} \mathbf{R} & \mathbf{t} \end{array}\right]_{2\times 3}$	3	lengths	
similarity	$\left[\begin{array}{c\|c} s\mathbf{R} & \mathbf{t} \end{array}\right]_{2\times 3}$	4	angles	
affine	$\left[\begin{array}{c} \mathbf{A} \end{array}\right]_{2\times 3}$	6	parallelism	
projective	$\left[\begin{array}{c} \tilde{\mathbf{H}} \end{array}\right]_{3\times 3}$	8	straight lines	

from R.Szeliski

Image Mappings Overview

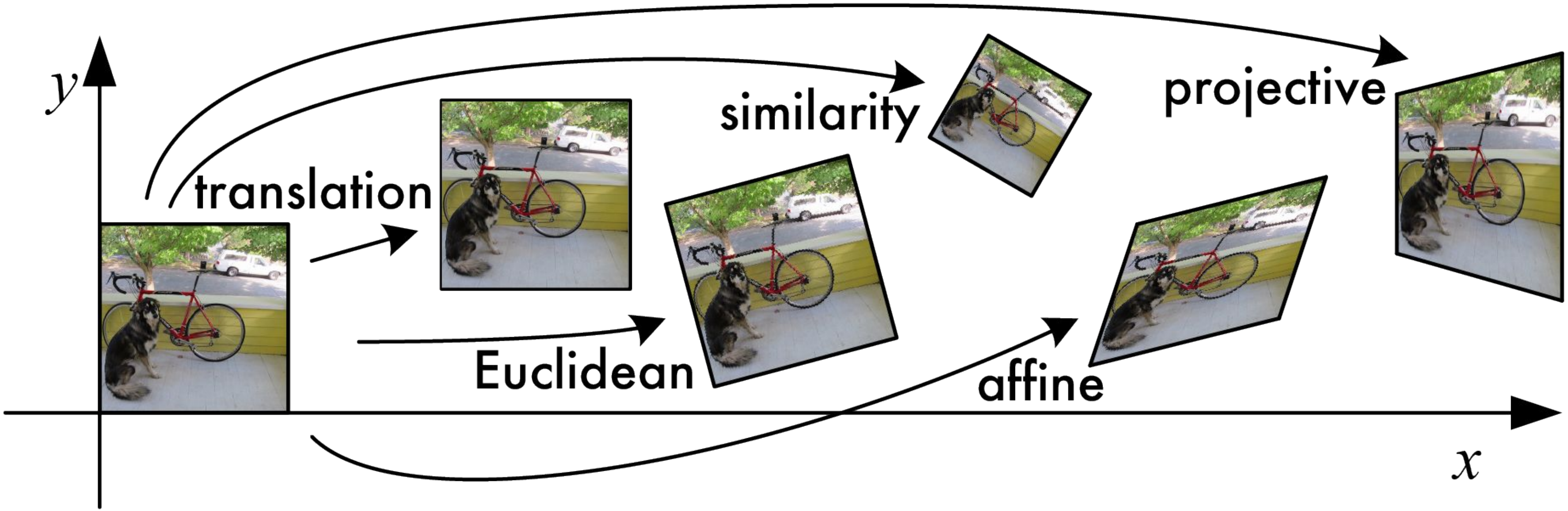
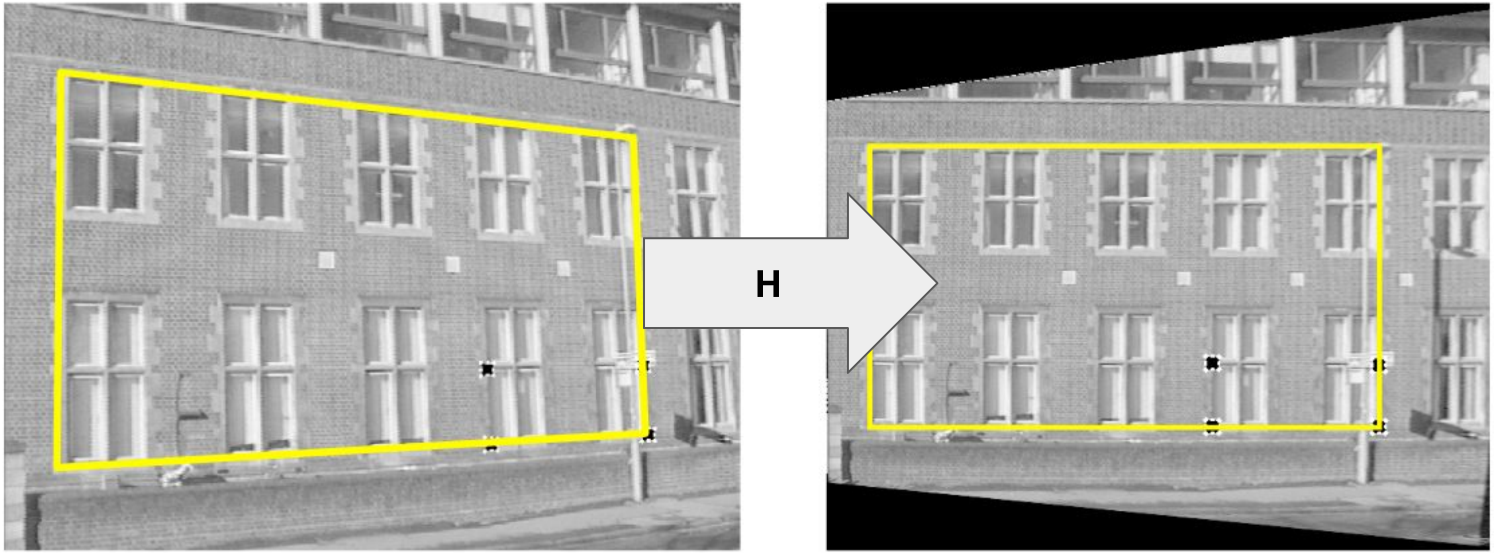


Image adapted from R. Szeliski by JP. Reddie

Warping images

Warping Example



Source Image

Destination image

from Hartley & Zisserman

Warping & Bilinear Interpolation

Given a transformation between two images (coordinate systems)
we want to **“warp” one image** into the **coordinate system** of the **other**.

We will call the coordinate system
where we are **mapping from**
the **“source”** image.

We will call the coordinate system
we are **mapping to**
the **“destination”** image.

Forward Warping

For each pixel x in the source image

Determine where it goes as H(x)

Color the destination pixel

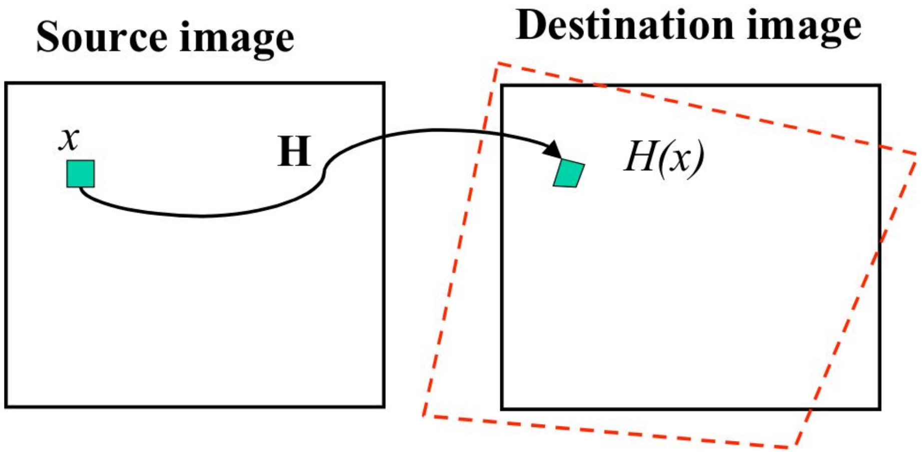


Illustration: Robert Collins

Forward Warping

For each pixel x in the source image

Determine where it

Color the

PROBLEM

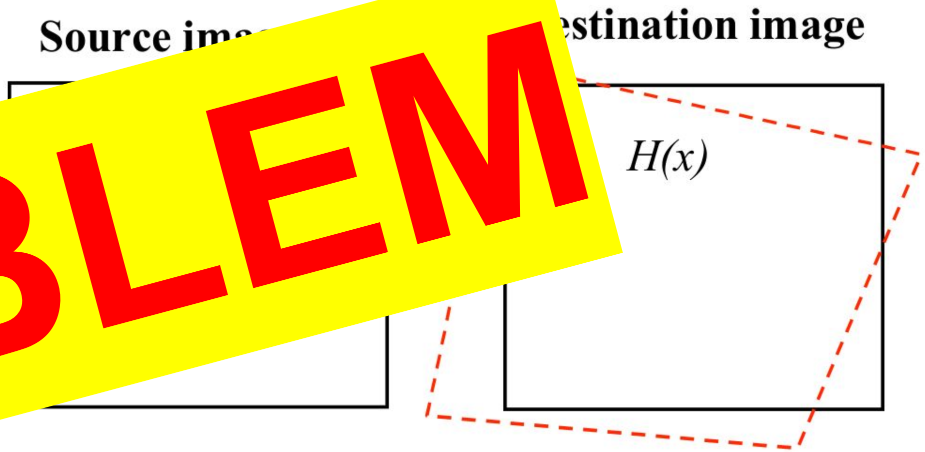


Illustration: Robert Collins

Forward Warping Problem

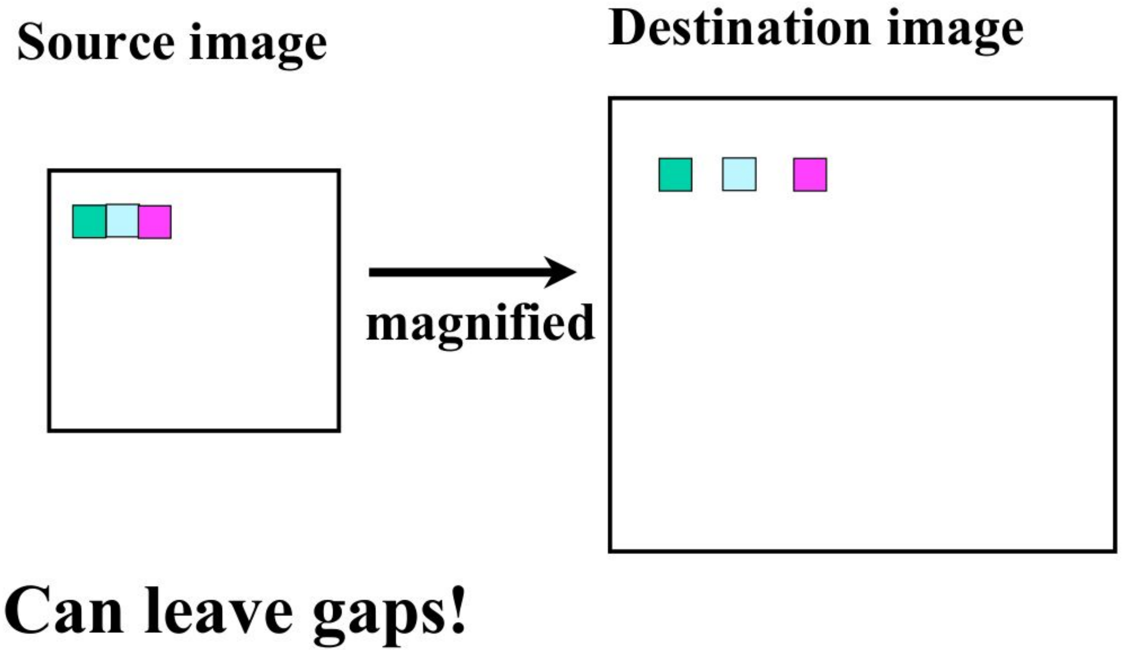
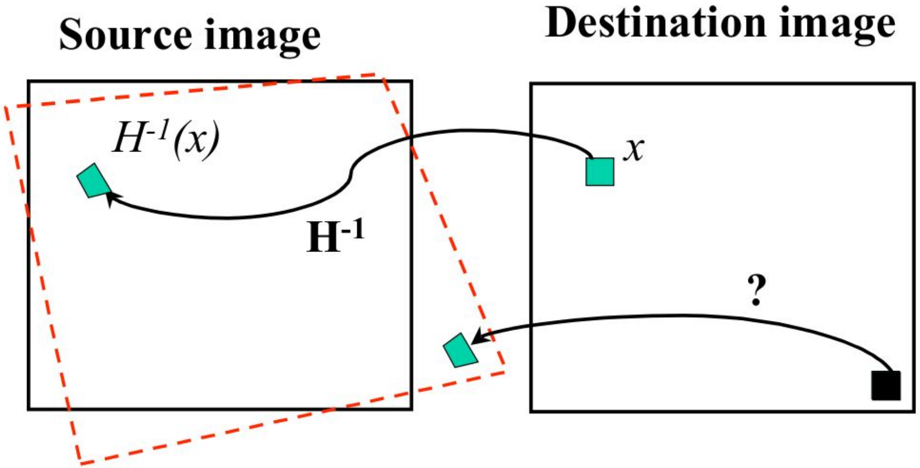


Illustration: Robert Collins

Backward Warping — No gap



For each pixel x in the destination image

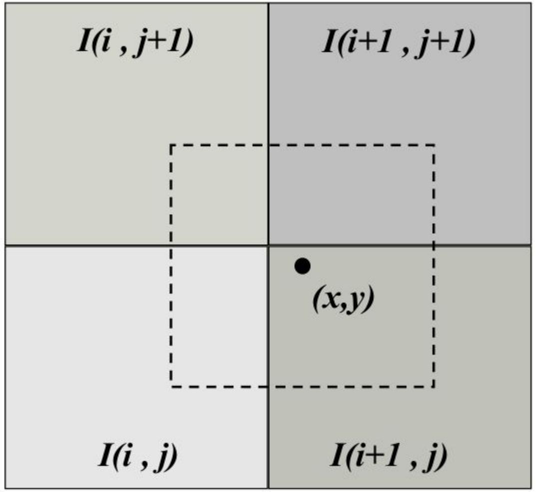
Determine where it comes from as $\mathbf{H}^{-1}$ **(x)**

Get color from that location

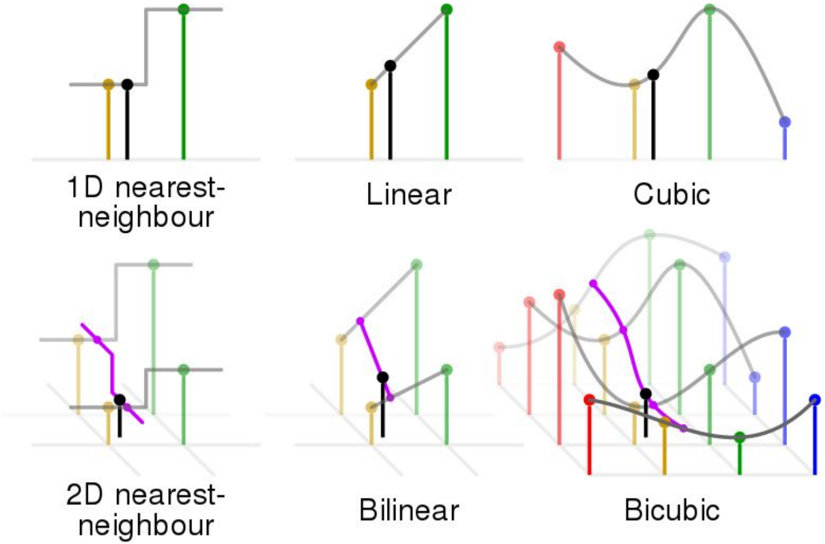
Interpolation

What do we mean by “get color from that location”?

Consider grey values. What is intensity at (x,y)?

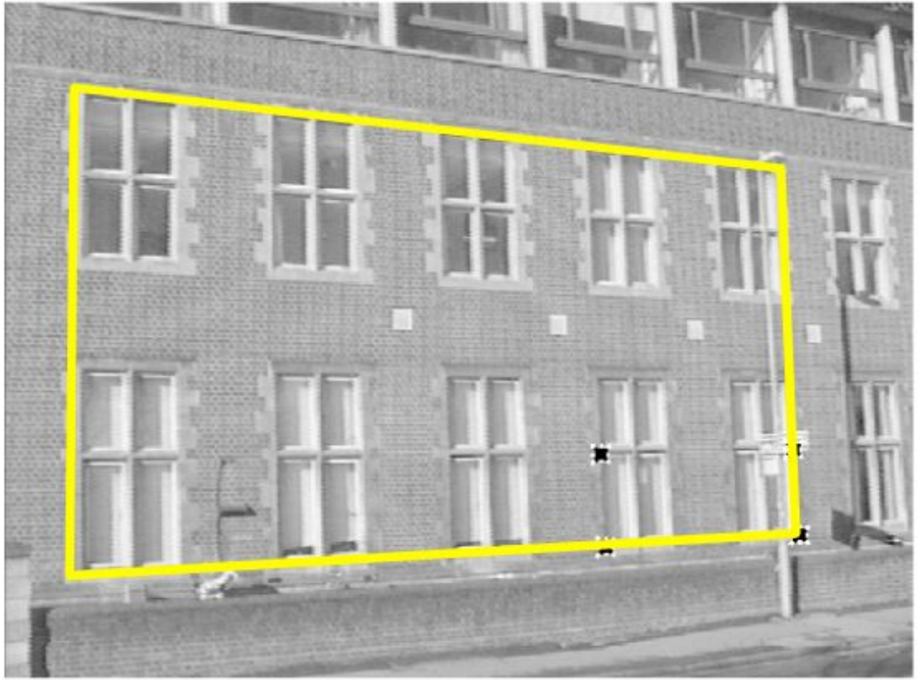


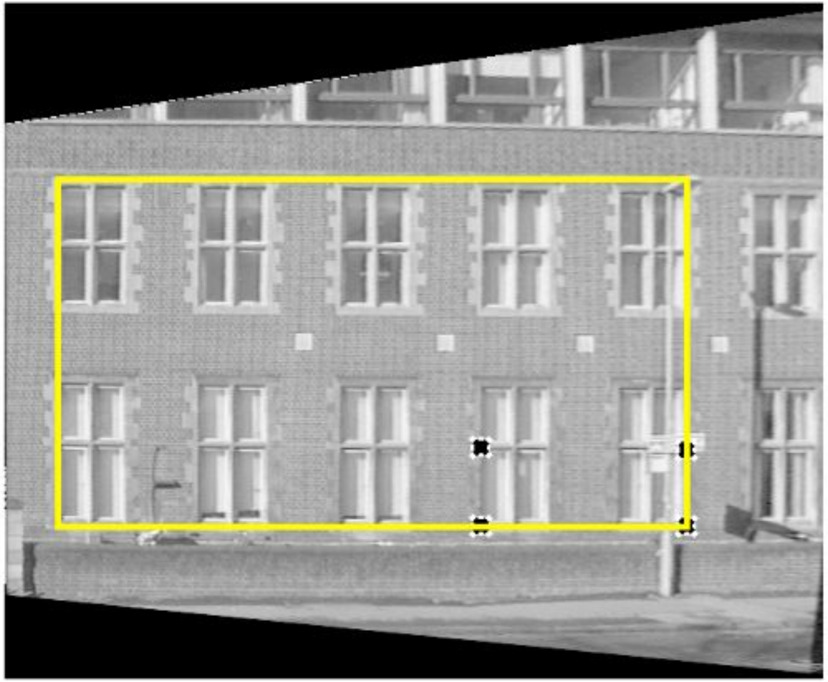
Need interpolation



Rigid vs Non rigid transform

Rigid vs non rigid

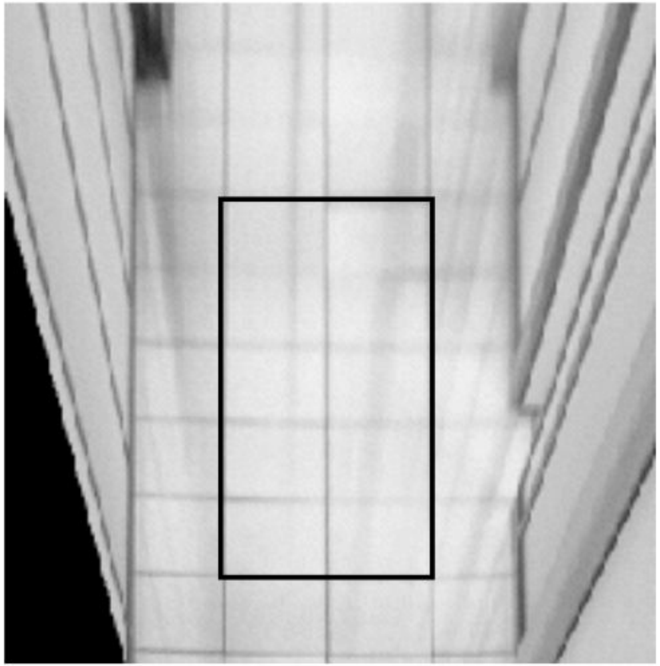




Rigid planar building facade

from Hartley & Zisserman

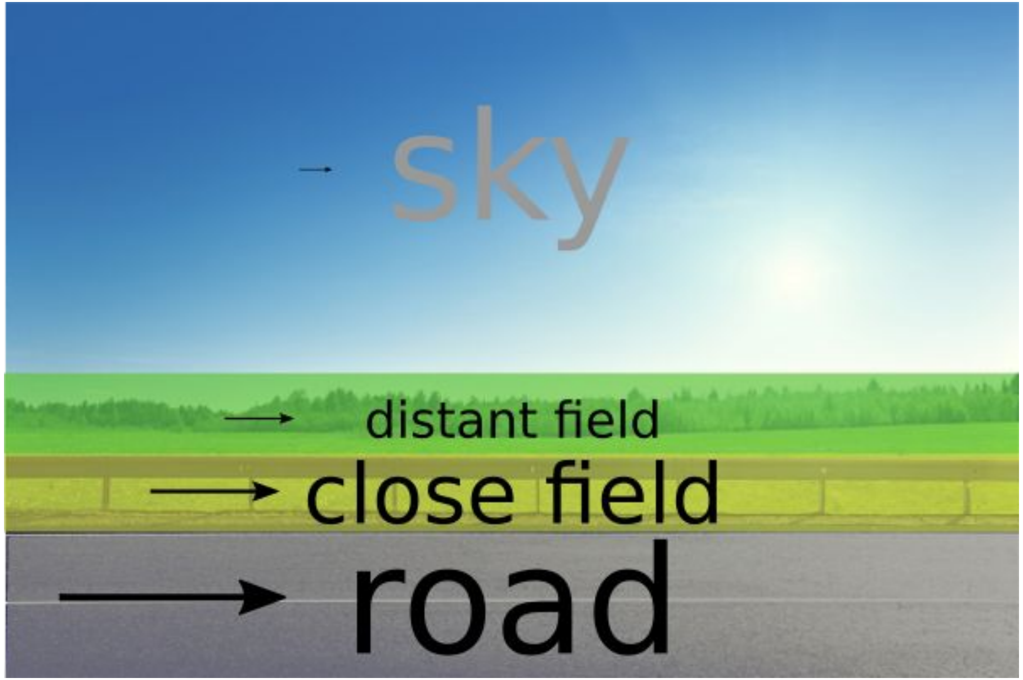
Rigid vs non rigid



Rigid planar floor

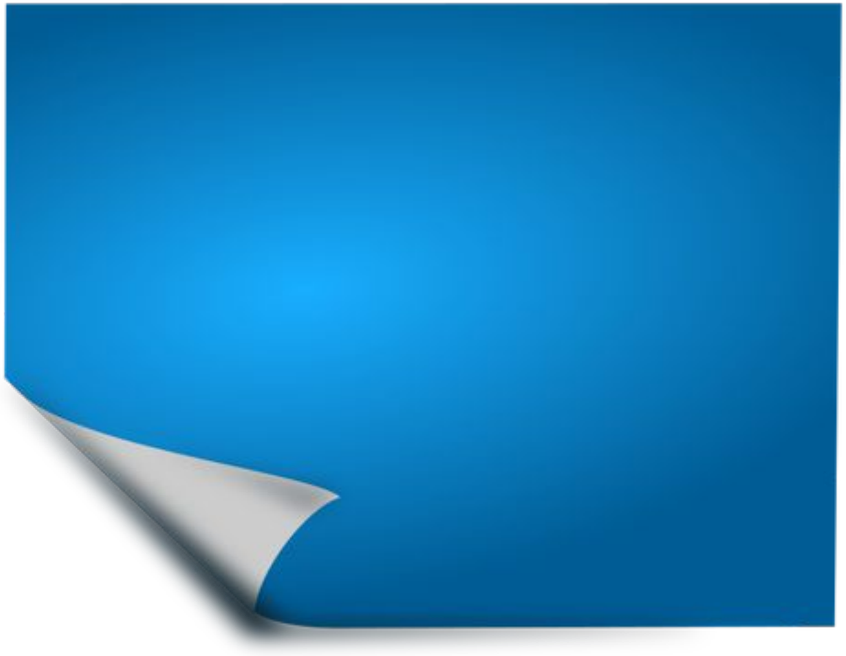
from Hartley & Zisserman

Rigid vs non rigid



Non rigid pixel motion

Rigid vs non rigid



Non rigid object deformation

Rigid vs non rigid

Rigid

A linear transformation of pixel coordinates

Summarized by a matrix.

Non rigid

Pixel displacement = vector field

(can also be a piecewise function)

Used in optical flow, medical image registration, shape from motion…

Way more complicated, would require a dedicated lecture.