Function of several variables
Session 1

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Themes :

- Surfaces and level sets
- Limits and continuity
- The partial derivative
- Tangent plane and linear approximations

Surfaces and level sets

- We can define a **2D surface** within a (euclidian) 3D space by an equation of the form

$$z = f(x,y) \qquad \text{or} \qquad 0 = F(x,y,z)$$

- To study more finely and systematically such surface we can use the concept of **level set** defined by the equation

$$c = f(x,y)$$

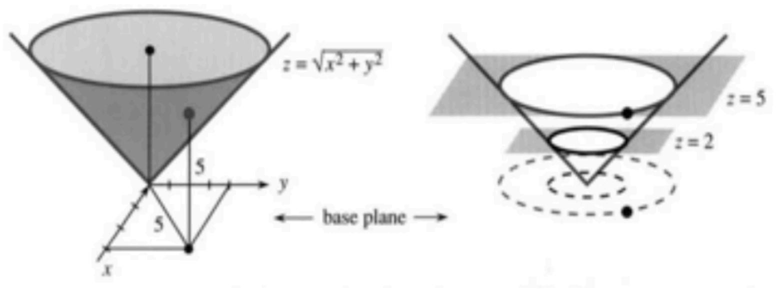

Figure – The surface for $z = f(x, y) = \sqrt{x^2 + y^2}$ is a cone. The level lines (or curves) are circles.

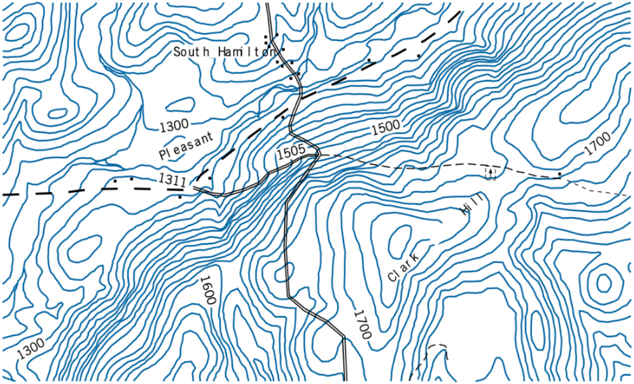

Figure – Topological map of the region located around the town of South Hamilton in New York state (US).

- **Limit of a two variables functions**

  - New : We use the notion of **disk** δ to define the *neighbourhood* of a point $\boldsymbol{a} = (a, b)$ corresponding to the set

    $$\left\{\boldsymbol{x} \in \mathbb{R}^2 \,|\, \|\boldsymbol{x} - \boldsymbol{a}\| < \delta\right\}$$

  - From that we can establish a **first definition** of the limit allowing to build fundamental (expected) **properties** (sum, multiplication, etc..).

  - A important difference, with respect to the 1D case, is that you can now approach a point via **several directions** (an infinity in fact...)

• Interior points and boundary points

- Problem : the notion of disk is not **not adapted** when we evaluate the limit of a function on its boundary...
- Indeed, some points of the disk might not belong to the **domain** of the functio $f(x, y)$.
- In this case, we propose an **improved definition** so that all the points of the disk belong to the domain of $f(x, y)$.

- **Continuity for functions of two variables**

    - Once the notion of limit properly defined, we can easily **generalise** the definition of continuity to higher dimensions, to wit, if $f(\boldsymbol{x})$ is continuous at $\boldsymbol{x}_0$:

$$\textbf{(1)}\ f(\boldsymbol{x}_0) \text{ exists}$$

$$\textbf{(2)}\ \lim_{\boldsymbol{x}\to\boldsymbol{x}_0} f(\boldsymbol{x}) \text{ exists}$$

$$\textbf{(3)}\ \lim_{\boldsymbol{x}\to\boldsymbol{x}_0} f(\boldsymbol{x}) = f(\boldsymbol{x}_0)$$

    - Following this, the usual (expected) **properties** (sum, product, composition) are insured.

- **Functions of three variables and more**

  - The passage to higher dimensions supÃ©rieures is easily done by generalising the concept of disk to a **ball** :

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{a}\| < \delta\}$$

    with a euclidean norm (for instance) given by

$$\|\boldsymbol{x} - \boldsymbol{a}\| = \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2}$$

  - Several **norms** exist and allow to define a « distance » between two elements belonging to a given space depending on our needs...

  - The definitions and concepts that follow stay unchanged !

The partial derivative

- The essential : for a function $f(x, y)$, we compute

$$\frac{\partial f}{\partial x} = f_x \qquad \text{or} \qquad \frac{\partial f}{\partial y} = f_y$$

  by considering the other variables, with respect to which we do not differentiate, as **constants**.

- Similarly to the on variable case, partial derivatives give informations on **variations** on the surface

$$z = f(x, y)$$

  in directions parallel to the axes Ox and Oy.

- We can also simply use the **partial functions**

$$f(x, y_0) \qquad \text{or} \qquad f(x_0, y)$$

to study the surface $z = f(x, y)$ in a given direction, parallel to $x = x_0$ or $y = y_0$.

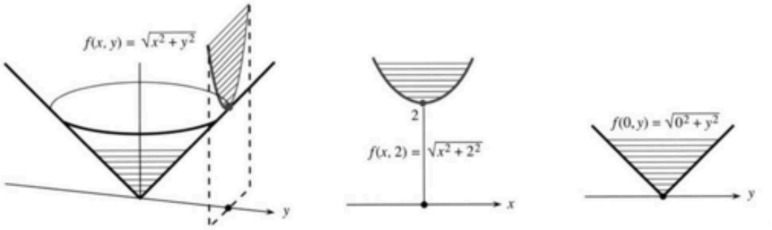

Figure – Partial functions $\sqrt{x^2+2^2}$ et $\sqrt{0^2+y^2}$ of the distance function $f = \sqrt{x^2+y^2}$.

- New : A **saddle point** configuration is possible if

$$f_x = f_y = 0$$

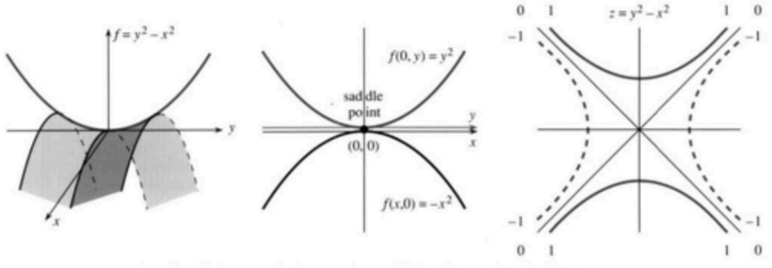

Figure – The function $f = y^2 - x^2$ showing a saddle point, its partial functions and its level sets.

- For a function of two variables $f(x, y)$ we compute four(!) partial **second derivatives**

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \quad , \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y} \quad , \quad f_{yx} = \frac{\partial^2 f}{\partial y \partial x} \quad , \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

- Luckily, if these second derivatives are **continuous** (true most of the time in usual applications) we have

$$f_{xy} = f_{yx}$$

also called the **Schwarz theorem**.

- The matrix of second derivatives is called the **Hessian**

$$\boldsymbol{H} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Tangent plane and linear approximations

- We can approximate an arbitrary surface, of equation $z = f(x, y)$, **locally** by a plane, at the base point (x_0, y_0), given by

$$z - z_0 = \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0)$$

- We can define a **normal vector** to this plane (and to the surface at the point (x_0, y_0)) by

$$\boldsymbol{N} = \begin{pmatrix} (f_x)_0 \\ (f_y)_0 \\ -1 \end{pmatrix}$$

It is oriented **towards the exterior** for a closed surface.

- If the surface is given by an equation of the form $c = F(x, y, z)$, where c is a constant, then the equation of the **tangent plane** becomes

$$\left(\frac{\partial F}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial F}{\partial y}\right)_0 (y - y_0) + \left(\frac{\partial F}{\partial z}\right)_0 (z - z_0) = 0$$

- The **normal vector** becomes then

$$\boldsymbol{N} = \begin{pmatrix} (F_x)_0 \\ (F_y)_0 \\ (F_z)_0 \end{pmatrix}$$

- We remark that by setting $f = F - z$ we can recover the previous formula.

- If we write the following equivalences (for small variations)

$$z - z_0 \approx dz = df \quad , \quad y - y_0 \approx dy \quad , \quad x - x_0 \approx dx$$

  in the equation of the tangent plane $z = f(x, y)$, we recover the **differential** of f, to wit

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy$$

- Here an **infinitesimal variation** of f is expressed (linearly) in terms of the (infinitesimal) variations of its parameters x et y.

- By resetting $df = f(x,y) - f(x_0,y_0)$, we can construct the **linear approximation** of f at the point (x_0, y_0)

$$f(x,y) \approx f(x_0,y_0) + \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0)$$

- The quadratic terms, i.e proportional to $(x-x_0)^2$ and $(y-y_0)^2$ are here **neglected** because (a priori) **smaller** than $(x-x_0)$ and $(y-y_0)$ not far from (x_0,y_0).

- The approximation becomes more and more **incorrect** as we are getting further from (x_0,y_0).