PROBABILITIES AND STATISTICS 1

I. Continuous distributions

Guillaume Euvrard

Exercise 1

In each of the examples below, discuss whether the function f is a density or not, depending on $k \in \mathbb{R}$.

If it is: consider a random variable X having a density f.

- Find its cumulative distribution function.
- Find a bilateral prediction interval at the precision 95% that is, find an interval $[a, b]$ such that

$$P(X \in [a,b]) = 0.95, \qquad P(X < a) = 0.025 \qquad \text{and} \qquad P(X > b) = 0.025$$

1. $f(x) = \begin{cases} \frac{k}{x} & \text{if } x \geqslant 1 \\ 0 & \text{otherwise} \end{cases}$

2. $f(x) = \begin{cases} \frac{k}{x^2} & \text{if } x \geqslant 1 \\ 0 & \text{otherwise} \end{cases}$

3. $f(x) = \begin{cases} \frac{k}{x^3} & \text{if } x \geqslant 2 \\ 0 & \text{otherwise} \end{cases}$

4. $f(x) = k\, e^{-2x}$

5. $f(x) = \begin{cases} k\, e^{-2x} & \text{if } x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$

6. $f(x) = \begin{cases} k & \text{if } A \leqslant x \leqslant B \\ 0 & \text{otherwise} \end{cases}$ where $[A, B]$ is a given interval of $\mathbb{R}$.

Exercise 2

Let $(a, b) \in \mathbb{R}^2$ such that $a < b$ and X a random variable admitting as density the function

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

We say that X is uniformly distributed on $[a, b]$, this is denoted by: $X \rightsquigarrow \text{Unif}\,(a, b)$.

1. What is the cumulative distribution function of X?

2. Let $Z = \dfrac{X-a}{b-a}$. Find the cumulative distribution function of Z, then find its density.

3. Suppose that, in a programming project, you must randomly pick a variable which is uniformly distributed on $[25, 30]$.

   Assume that the language has a function `rand()` which returns a random number uniformly distributed on $[0, 1]$. How can you use this function?

Exercise 3

A server receives requests from clients. Consider the random variable

$$T = \text{«Time delay until the next request»}$$

1. In this question, we suppose that there exists $\lambda > 0$ such that a density of T is:

$$f(t) = \begin{cases} \lambda\, e^{-\lambda t} & \text{if } t \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

   We say that T is exponential-distributed with the parameter λ: $T \rightsquigarrow \operatorname{Exp}(\lambda)$.

   (a) Show that f is a density and find the cumulative distribution function of T.

   (b) Find $a \in \mathbb{R}$ such that $P(X \leqslant a) = 0.95$.

   (c) Let $t_0 > 0$. For a given $\Delta t \in \mathbb{R}$, compute the conditional probability $P(T > t_0 + \Delta t \,|\, T > t_0)$. Compare with $P(T > \Delta t)$.
   We say that the variable T is «memoryless». Explain this expression.

2. **(Bonus)** Suppose that T is memoryless. Furthermore, suppose that T can only take positive values and that its cumulative distribution function is continuous and differentiable on $\mathbb{R}^+$.

   (a) Let $t \in \mathbb{R}^+$. Express the hypothesis:

$$\forall\, \Delta t \in \mathbb{R}_+^*, \quad P(T \leqslant t + \Delta t \,|\, T > t) = P(T \leqslant \Delta t)$$

   using the cumulative distribution function F.

   (b) By dividing this relation by Δt and studying the limit as Δt approaches 0, find a differential equation which is satisfied by F.

   (c) Deduce that X is exponential-distributed.

Exercise 4

Let Z be a random variable admitting as density the function $\varphi(x) = \dfrac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$.

1. Express its cumulative distribution function as an integral. This function is denoted by Φ (don't try to find an explicit formula).

2. Explain why, for all $\beta > 0$, $\Phi(-\beta) = 1 - \Phi(\beta)$.

   We accept without proof that, for $\beta = 1.96$, we get $\Phi(-\beta) = 1 - \Phi(\beta) = 0.025$.

3. Let $(m, \sigma) \in \mathbb{R} \times \mathbb{R}_+^*$ and $X = m + \sigma Z$. Express the cumulative distribution function of X using Φ, then find a density of X.

   Provide a 95% prediction interval for X.

   We say that X is normal-distributed with the parameters m and σ^2: $X \rightsquigarrow \mathcal{N}(m, \sigma^2)$.

★ **Exercise 5**

Consider a random variable X such that $X \rightsquigarrow \operatorname{Unif}(-1, 2)$.

1. Find the cumulative distribution function of $|X|$.

2. Deduce a density of $|X|$.