<u>Workshop session 1</u> (September 2023)

Surfaces and levels sets

Exercice 4. We consider the functions

$$f_1(x_1, x_2) = x_1^2 - x_2^2 \qquad \text{et} \qquad f_2(x_1, x_2) = 2x_1x_2$$

Sketch the level sets associated with $f_1(x_1, x_2) = 12$ and $f_2(x_1, x_2) = 16$ on the same diagram. Indicate on the diagram the values of $\boldsymbol{x} = [x_1, x_2]^\intercal$ for which $\boldsymbol{f}(\boldsymbol{x}) = [f_1(x_1, x_2), f_2(x_1, x_2)]^\intercal = [12, 16]^\intercal$ where the exponent $^\intercal$ indicate a column vector.

Limits and continuity

Exercice 6. In the following cases, use algebraic techniques to evaluate the limit.

(1) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^4}{x^2 + 2y^2}$ **(2)** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ **(3)** $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 - y^2 - z^2}{x^2 + y^2 - z^2}$

Exercice 10. We consider the function $\langle\cdot|\cdot\rangle_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, defined by

$$\langle \boldsymbol{x}|\boldsymbol{y}\rangle_2 = 2x_1y_1 + 3x_2y_1 + 3x_1y_2 + 5x_2y_2$$

where $\boldsymbol{x} = (x_1, x_2)^\intercal$ et $\boldsymbol{y} = (y_1, y_2)^\intercal$ are column vectors. After determining the matrix $\boldsymbol{Q}$ such that $\langle \boldsymbol{x}|\boldsymbol{y}\rangle_2 = \boldsymbol{x}^\intercal\boldsymbol{Q}^2\boldsymbol{y}$, show the following results ($\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$) :

(1) $\langle \boldsymbol{x}|\boldsymbol{x}\rangle_2 \geq 0$

(2) $\langle \boldsymbol{x}|\boldsymbol{y}\rangle_2 = \langle \boldsymbol{y}|\boldsymbol{x}\rangle_2$

(3) $\langle \boldsymbol{x} + \boldsymbol{y}|\boldsymbol{z}\rangle_2 = \langle \boldsymbol{x}|\boldsymbol{z}\rangle_2 + \langle \boldsymbol{y}|\boldsymbol{z}\rangle_2$

(4) $\langle \boldsymbol{x}|\lambda\boldsymbol{y}\rangle_2 = \lambda\langle \boldsymbol{x}|\boldsymbol{y}\rangle_2$

The partial derivative

Exercice 13. Compute $f_{xx}, f_{xy} = f_{yx}$ and f_{yy} for the following functions.

(1) $x^2 + 3xy + 2y^2$

(2) $(x + 3y)^2$

(3) $(x + iy)^3$

(4) e^{ax+by}

(5) $1/\sqrt{x^2 + y^2}$

(6) $(x + y)^n$

(7) $\cos ax \sin by$

(8) $1/(x + iy)$

Tangent planes and linear approximation

Exercice 17. Find the tangent plane and the normal vector at P.

(1) $z = \sqrt{x^2 + y^2}$, $P(0, 1, 1)$

(2) $x + y + z = 17$, $P(3, 4, 10)$

(3) $z = x/y$, $P(6, 3, 2)$

(4) $z = e^{x+2y}$, $P(0, 0, 1)$

(5) $x^2 + y^2 + z^2 = 6$, $P(1, 2, 1)$

(6) $x^2 + y^2 + 2z^2 = 7$, $P(1, 2, 1)$

(7) $z = x^y$, $P(1, 1, 1)$

(8) $V = \pi r^2 h$, $P(2, 2, 8\pi)$