Signal classification

Guillaume Tochon

LRE

Mathematical definition of a signal

A **signal** is defined as a function $\boxed{\begin{array}{rcl} x : I \subseteq \mathbb{R} & \to & \mathbb{C} \\ t & \mapsto & x(t) \end{array}}$ that satisfies:

$\to$ x is bounded (in magnitude): $\exists\ 0 < M < +\infty$ such that $|x(t)| < M\ \forall t \in I$
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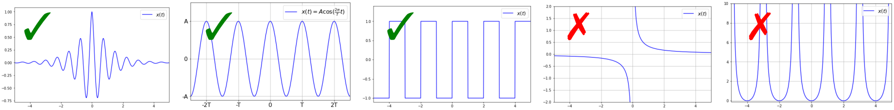

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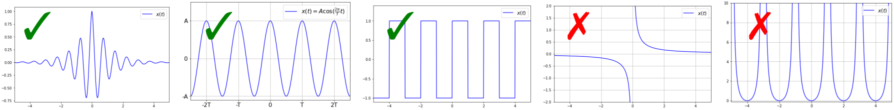
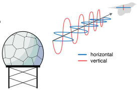
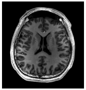


Remarks:

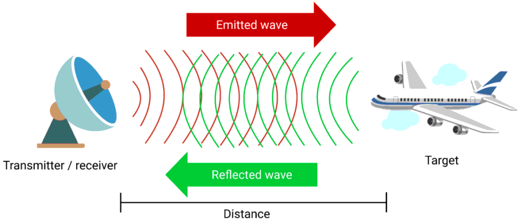
- We restrict ourselves to univariate and one-dimensional signals.
- x can take complex values.
- In general, $I = \mathbb{R}$.
- By abuse of language, we will refer to t as the *time* variable.
- The graph of x is called the *time representation*.
- The set of signals is a vector space in which we can define a basis, inner product, and norm.





The radar example

Principle of radar detection (by the way, radar is the acronym of *radio detection and ranging*):



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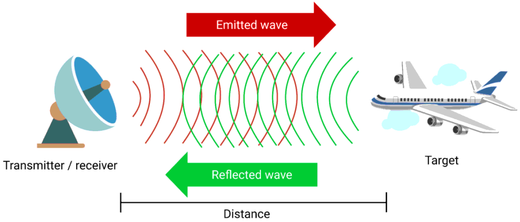

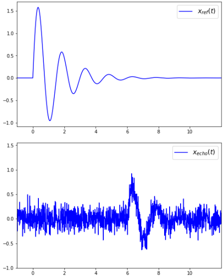

1. Emission of a reference signal x_{ref} that propagates to the target.
2. The echo returns to the receiver, which records x_{echo}.
3. Measurement of the echo delay $\Rightarrow$ distance to the target.

$\Rightarrow$ How to calculate the echo delay most reliably?

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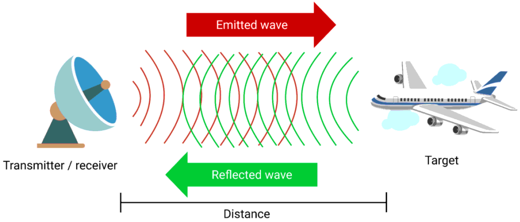

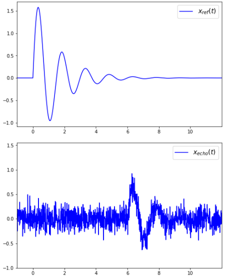

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Idea: Find the optimal translation factor to superimpose the pattern of x_{echo} on x_{ref} to maximize the similarity between these two signals.

The dot product

The similarity between two vectors x and y is given by their dot product $\langle x, y \rangle$.

Recalls on the dot product for discrete vectors

$\rightarrow$ in $\mathbb{R}^2$: $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \Rightarrow$

$$\begin{aligned} \langle x, y \rangle &= x_1 y_1 + x_2 y_2 \\ &= \|x\| \|y\| \cos\theta \end{aligned}$$

$\rightarrow$ in $\mathbb{R}^n$: $x, y \in \mathbb{R}^n \Rightarrow \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$

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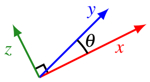

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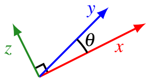

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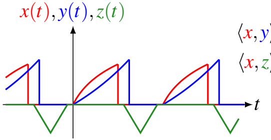
Here, the manipulated vectors are signals, thus functions (in other words, vectors from a vector space of infinite dimension...)

We can also define a dot product for such vectors.

$\Rightarrow$ The symbol $\sum$ is replaced by its continuous equivalent $\int$, up to some precautions to be taken...



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Integrability

Before writing expressions like $\int_{-\infty}^{+\infty} x(t)dt$, it is important to ensure that it **can** actually be done...

Integrability of a function

We say that a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$)

- is **integrable** over I if $\int_I |f(t)|dt < +\infty$
- is **p-integrable** over I (for $p \in \mathbb{N}^\star$) if $\int_I |f(t)|^p dt < +\infty$

We usually denote as $\mathcal{L}^p(I)$ the vector space of p-integrable functions over I.

$\Rightarrow \mathcal{L}^2(I)$ is the space of square-integrable functions over I: $\int_I |f(t)|^2 dt < +\infty$

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In signal processing, the quantity $\boxed{E_x = \int_I |x(t)|^2 dt}$ is the **energy** of the signal over I.

$\Rightarrow$ $\mathcal{L}^2(I)$ is the space of signals with finite energy over I.
In practice, we work in $\mathcal{L}^2(\mathbb{R})$, the space of signals with finite energy over $\mathbb{R}$.

The space $\mathcal{L}^2(\mathbb{R})$

A dot product can be defined in the vector space (we call it a functional space) $\mathcal{L}^2(\mathbb{R})$.

Dot product (Hermitian product, to be precise...) between two signals of finite energy

The mapping
$$\begin{array}{rcl} \langle\ ,\ \rangle : \quad \mathcal{L}^2(\mathbb{R}) \times \mathcal{L}^2(\mathbb{R}) & \rightarrow & \mathbb{C} \\ (x, y) & \mapsto & \boxed{\langle x, y \rangle = \int_{\mathbb{R}} x(t)\overline{y(t)}dt} \end{array}$$

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Remarks:

- If the signals are real-valued ($x(t), y(t) \in \mathbb{R}$), $\langle x, y\rangle = \int_{\mathbb{R}} x(t)y(t)dt$
- Although it's an abuse of notation (by the way, why?), we will allow ourselves to write $\langle x(t), y(t)\rangle$ instead of $\langle x, y\rangle$ for the dot product between signals x and y.

Exercise: You can check yourself that this definition satisfies the axioms of the dot product.

The space $\mathcal{L}^2(\mathbb{R})$

From any dot product $\langle\ ,\ \rangle$ can be defined a norm $\|\cdot\| : x \mapsto \|x\| = \sqrt{\langle x, x\rangle}$

Norm of a signal with finite energy

Let $x \in \mathcal{L}^2(\mathbb{R})$,

$\|x\|^2 = \langle x, x\rangle = \int_{\mathbb{R}} x(t)\overline{x(t)}dt = \int_{\mathbb{R}} |x(t)|^2 dt = E_x < +\infty$ (since $x \in \mathcal{L}^2(\mathbb{R})$)

$\boxed{E_x = \|x\|^2}$ $\Rightarrow$ energy (signal) = square of the norm (mathematics).

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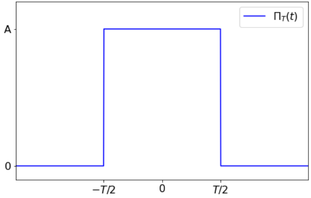
Example: Let's consider the *window* function with width T: $\Pi_T(t) = \begin{cases} 1 & t \in [-\frac{T}{2}, \frac{T}{2}] \\ 0 & \text{otherwise} \end{cases}$



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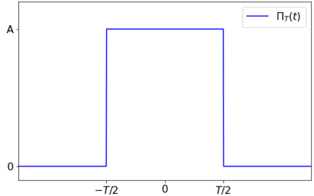

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$$E_x = \int_{\mathbb{R}} |x(t)|^2 dt = \int_{-T/2}^{T/2} A^2 dt = \boxed{A^2 T}$$

The importance of signal support for energy calculation

In general, any signal with $\underbrace{\text{bounded support}}_{\exists B>0 \text{ such that } |x(t)|=0 \ \ \forall |t|>B}$ has finite energy (since by definition, the signal has bounded magnitude).

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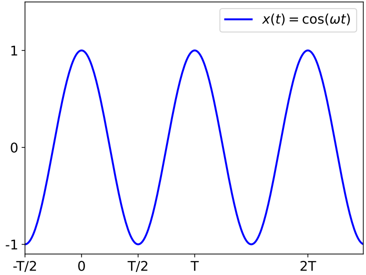
For example, $x : t \mapsto \cos(\omega t)$ (recall: angular frequency $\omega = \dfrac{2\pi}{T} = 2\pi\nu$ with T period and ν frequency)



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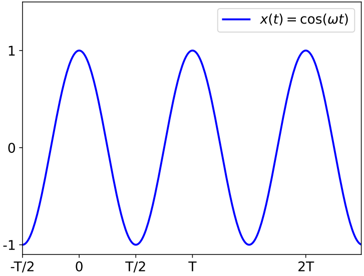
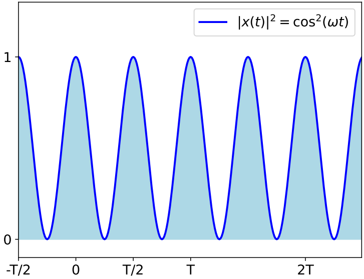


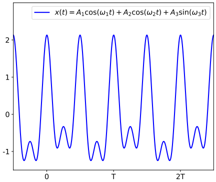


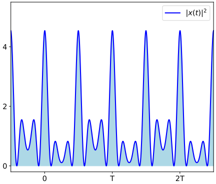
$$E_x = \int_{\mathbb{R}} |x(t)|^2 dt = \int_{-\infty}^{+\infty} \cos^2(\omega t) dt = \boxed{+\infty}$$

What about signals with unbounded support?

→ $t \mapsto \cos(\omega t)$ is not of finite energy. The same goes for $t \mapsto \sin(\omega t)$...

→ ...and any linear combination of cos / sin, regardless of their amplitude and frequency/angular frequency/period.





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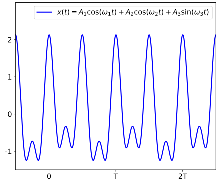
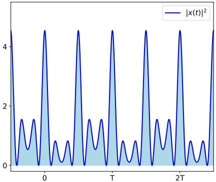
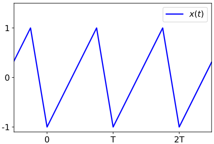
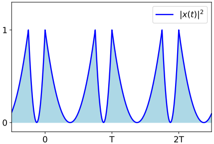




- $\rightarrow$ Therefore, (see the lecture on Fourier Series) for any T-periodic signal (except $x(t) = 0\ \forall t$...)





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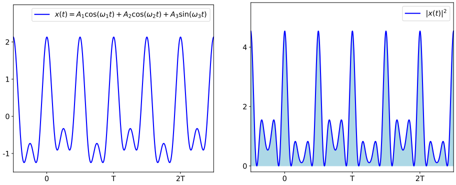

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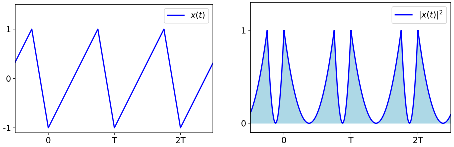

The space of signals with finite energy $\mathcal{L}^2(\mathbb{R})$ is not sufficiently exhaustive to allow a general description of the signals commonly encountered in signal processing.

Mean power

Mean power of a signal

The **mean power** of a signal x is the temporal average of its energy:

$$\boxed{P_x = \lim_{T \to +\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt}$$

We say that x has **finite mean power** if $P_x < +\infty$, and we denote $\mathcal{L}^{pm}(\mathbb{R})$ the space of signals with finite mean power (it is also a vector space).

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Remarks:

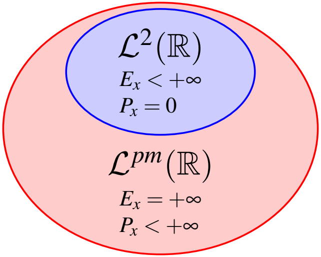
- This relates to the interpretation of power in physics ($1\text{W} = 1\ \text{J.s}^{-1}$).
- $\langle\ ,\ \rangle : (x, y) \mapsto \langle x, y\rangle = \lim_{T \to +\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\overline{y(t)} dt$ is a dot product in $\mathcal{L}^{pm}(\mathbb{R})$.
- In $\mathcal{L}^{pm}(\mathbb{R})$ endowed with this dot product, we have $\langle x, x\rangle = \boxed{\|x\|^2 = P_x}$
- If x is T-periodic, then $P_x = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \frac{1}{T} \int_0^T |x(t)|^2 dt$

  $\rightarrow$ the mean power is calculated over a single period.

Relationship between $\mathcal{L}^2(\mathbb{R})$ and $\mathcal{L}^{pm}(\mathbb{R})$

Relationship between energy and mean power

Any signal x with finite energy $E_x < +\infty$ has zero mean power $P_x = 0$ (hence finite).
Any signal x with positive finite mean power $0 < P_x < +\infty$ has infinite energy $E_x = +\infty$.



The space of signals with finite mean power $\mathcal{L}^{pm}(\mathbb{R})$ includes the space of signals with finite energy $\mathcal{L}^2(\mathbb{R})$, while providing a broader framework that also includes T-periodic signals.