INNER PRODUCT SPACES

Three weeks: from Monday, 6 February to Friday, 10 March 2023

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Context

When studying the vector spaces, we have until now neglected something: the idea of "distance" from a vector to another. All of us understand this notion in the physical space, whose dimension is 2 or 3. We can imagine that this can be extended to other spaces. Indeed, in many applications of linear algebra, this notion is required.

Here are some examples:

- In data analysis, we have observations of some random variables X, Y, Z, etc. Statisticians use them to estimate the distributions of these variables and to find some dependence relations. For example, can we find a relation of the form $Z = aX = bY + c$? In real life, such a relation is never exactly satisfied. There is always an error term and the statisticians have to minimize this error. They find a value $(a, b, c) \in \mathbb{R}^3$ which minimizes the distance from Z to $aX + bY + c$.

  Most of the time, this distance is the expectation of the quadratic error $\mathrm{E}\left((Z - aX - bY - c)^2\right)$.

- A time signal is a mapping $t \longmapsto s(t)$ which is defined for each t as the value at this time t of a given physical variable. We will see later in the semester that, provided some conditions are fulfilled, the signal can be written as an infinite sum of trigonometric functions: this is the Fourier representation of the signal. This representation is widely used to analyze or to transform the signal (filtering, etc.). But in a computer program we cannot use an infinite sum. We replace it with a finite sum containing many terms.

  Quantifying the resulting error consists in finding a distance from the function s to the finite sum of trigonometric functions.

In the physical space, the distance from a vector to another is computed with an ***inner product***. In other spaces, it is also possible to define some inner products. These products enable one to define some distance functions.

Learning outcomes

Bilinear symmetric forms and inner products

- Ability to recognize an inner product.
- Represent a bilinear form with a matrix.
- Express a symmetric bilinear form with the coordinates of the vectors in another basis.
- Use the Cauchy-Schwarz and Minkowsi theorems to prove some inequalities.

Orthogonality

- Build an orthogonal (or orthonormal) basis (Gram-Schmidt process).
- Analyze a symmetric bilinear form by diagonalization of its matrix.

Orthogonal projection

- Knowledge of the orthogonal supplementary theorem.
- Ability to find an orthogonal projection onto a linear subspace F.
- Ability to build and to use an orthogonal basis of a linear subspace F in order to find an orthogonal projection onto F.
- Find the distance from a vector u to a linear subspace F.

1 Symmetric bilinear forms and inner products

1.1 Summary

An ***inner product*** defined over a vector space E is a mapping $\varphi : E\times E \longrightarrow \mathbb{R}$ which satisfies some basic properties. An ***inner product space*** is a vector space on which an inner product has been defined. A ***euclidean space*** is a ***finite-dimensional*** vector space on which an inner product has been defined. The inner product gives a structure to the vector space E: first by defining orthogonal vectors, second by defining a "norm" over E and, consequently, a "distance" from a vector to another.

Specifically, the basic properties that an inner product φ must satisfy are the followings:

1. for all $u_0 \in E$, the mapping $v \longmapsto \varphi(u_0, v)$ is linear and, similarly, for all $v_0 \in E$ the mapping $u \longmapsto \varphi(u, v_0)$ is linear;
2. for all $(u, v) \in E^2$, $\varphi(u, v) = \varphi(v, u)$;
3. for all $u \in E$, $\varphi(u, u) \geqslant 0$ and $\varphi(u, u) = 0 \Longrightarrow u = 0_E$.

Properties 1 and 2 define a "symmetric bilinear form", property 3 specify that this form is "positive non-degenerate". Then the norm of a vector $u \in E$ is $\|u\| = \sqrt{\varphi(u, u)}$ and the distance from a vector u to a vector v is the norm $\|u - v\|$.

As a first example, consider the standard inner product on $\mathbb{R}^3$, defined for all $\big(u{=}(x_1, y_1, z_1),\ v = (x_2, y_2, z_2)\big) \in \mathbb{R}^3 \times \mathbb{R}^3$ by:

$$\varphi(u, v) = x_1x_2 + y_1y_2 + z_1z_2$$

It is easy to check that φ satisfies the three properties above. Furthermore, if U and V are the column matrices containing the coordinates of the vectors u and v in the standard basis, then

$$\varphi(u, v) \ = \ {}^tUV = \begin{pmatrix} x_1 & y_1 & z_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

A general property states that, if E admits a basis $\mathcal{B}$, then all bilinear form (and all inner product) can be expressed as a matrix product[1]

$$\varphi(u, v) \ = \ {}^tUAV$$

where the columns U and V contain the coordinates of vectors u and v in the basis $\mathcal{B}$. A is the matrix of φ in this basis. Furthermore, the bilinear form φ is ***symmetric*** if and only if the matrix A is ***symmetric*** too (that is, ${}^tA = A$).

1.2 Exercises

Exercise 1.1

In the vector space $E = \mathbb{R}^2$, consider the three functions φ_1, φ_2 and φ_3 from E^2 to $\mathbb{R}$, defined for all $\big(u{=}(x_1, y_1), v{=}(x_2, y_2)\big) \in E^2$ by:

$$\varphi_1(u, v) = x_1x_2{+}x_1y_2{+}2y_1x_2{+}y_1y_2, \quad \varphi_2(u, v) = x_1y_2{+}y_1x_2 \quad \text{and} \quad \varphi_3(u, v) = x_1x_2{-}x_1y_2{-}y_1x_2{+}3y_1y_2$$

For each of them:

1. Show that φ is a bilinear form and find its matrix in the standard basis.
2. Is it an inner product over E?
3. If it is, express $\|u\|$ for any $u = (x, y) \in \mathbb{R}^2$.

[1] To be fully rigorous, the result of this matrix product is a matrix of size $1{\times}1$, whose unique coefficient is $\varphi(u, v)$.

Exercise 1.2

On $E=\mathbb{R}_2[X]$, consider the bilinear form φ defined for all $(P,Q)\in E^2$ by:

$$\varphi(P,Q)=\int_0^1 P(x)Q(x)\,\mathrm{d}x$$

1. Show that φ is an inner product on E.
2. Find its matrix in the standard basis of E.
3. Let $P=a_0+a_1X+a_2X^2\in E$. Give a matrix expression of $\|P\|^2$.
4. **(Personal work)** Answer to the previous questions with

$$\varphi(P,Q)=\int_{-1}^1 P(x)Q(x)\,\mathrm{d}x$$

Exercise 1.3

Let $(E,\langle\,,\rangle)$ be an inner product space, $(u,v,w,x)\in E^4$ and $(a,b,c,d)\in\mathbb{R}^4$.

1. Expand $\langle au+bv\,,cw+dx\rangle$
2. Expand $\|au+bv\|^2$.
3. Is the mapping $u\longmapsto\|u\|$ a linear map?
4. Discuss the following property:

$$\forall(u,v)\in E^2,\quad \langle u\,,v\rangle=0\iff u=0_E \text{ or } v=0_E$$

Exercise 1.4 *Change of basis*

Consider the bilinear form on $E=\mathbb{R}^3$ defined for all $\big(u{=}(x_1,y_1,z_1),v{=}(x_2,y_2,z_2)\big)\in E^2$ by:

$$\varphi(u,v)=5x_1x_2+6y_1y_2+3z_1z_2-5x_1y_2-5y_1x_2-3y_1z_2-3z_1y_2+2x_1z_2+2z_1x_2$$

1. Find the matrix of φ in the standard basis of $\mathbb{R}^3$ and check that this form is symmetric.
2. Consider the family of E: $\mathcal{B}'=\big(\varepsilon_1{=}(1,1,0),\varepsilon_2{=}(1,1,1),\varepsilon_3{=}(0,1,1)\big)$.
   Show that $\mathcal{B}'$ is a basis of E and find the transition matrix from the standard basis to $\mathcal{B}'$.
3. Find the matrix of φ in this basis $\mathcal{B}'$.
   Deduce an expression of $\varphi(u,v)$ using the coordinates of u and v in the basis $\mathcal{B}'$.
4. Show that φ is an inner product on E.

Exercise 1.5 *Another change of basis*

In $E=\mathbb{R}_2[X]$, consider the inner product defined for all $(P,Q)\in E^2$ by:

$$\langle P\,,Q\rangle=\int_0^2 P(x)Q(x)\,\mathrm{d}x$$

Express $\langle P\,,Q\rangle$ with the coordinates of P and Q:

1. in the standard basis $\mathcal{B}=(1,X,X^2)$,
2. in the basis $\mathcal{B}'=(1,(X-1),(X-1)^2)$.

Exercise 1.6
Let $(E, \langle\,,\rangle)$ be an inner product space and $\|.\|$ the norm associated to $\langle\,,\rangle$. Let $f \in \mathcal{L}(E)$. For all $(u, v) \in E^2$, we set

$$\varphi(u, v) = \langle f(u)\,, f(v)\rangle$$

Provide a necessary and sufficient condition on f for φ to be an inner product over E.

Exercise 1.7
In the euclidean space $E = \mathbb{R}^2$ together with its standard inner product, consider the vectors

$$u = (2, 1) \qquad \text{and} \qquad v = (1, 3)$$

1. Represent graphically $\langle u\,, v\rangle$ and explain the Cauchy-Schwarz's relation.
2. Explain geometrically the Minkowski's relation.

Exercise 1.8
Let $E = \mathcal{C}([0, 1])$ be the set of the continuous functions from $[0, 1]$ to $\mathbb{R}$ and consider the mapping $\varphi : E \times E \to \mathbb{R}$ defined by

$$\varphi(f, g) = \int_0^1 f(x)g(x)\,\mathrm{d}x$$

1. Show that φ is an inner product over E.
2. Show that for all $f \in E$,
$$\left|\int_0^1 f(x)\,\mathrm{d}x\right|^2 \leqslant \int_0^1 f^2(x)\,\mathrm{d}x$$
3. Let $f \in E$ of class $\mathcal{C}^1$ such that $f(0) = 0$. Show that
$$f^2(x) \leqslant \int_0^1 \left(f'(x)\right)^2 \mathrm{d}x$$

2 Orthogonality

2.1 Summary

In an inner product space, two vectors are ***orthogonal*** if their inner product is zero. In the physical space of dimension 2 or 3, the vectors are perpendicular. Keep in mind this geometrical representation: it enables one to develop intuitions in other spaces, which can be formalized in a second step with mathematical proofs and computations.

For instance, Pythagoras theorem is true in all the inner product spaces. Yet, it is better understood when represented geometrically.

Furthermore, the concept of orthogonal (or orthonormal) basis can be extended to all the euclidean spaces. Building such a basis facilitates many computations. For example, the inner product has a simple formula when expressed with the coordinates of the vectors in an orthogonal basis. We will see at the next section that finding an orthogonal projection onto a linear subspace F is very simple if we have an orthonormal basis of F.

Finally, the spectral theorem states that any ***symmetric*** matrix $A \in \mathcal{M}_n(\mathbb{R})$ admits an eigenbasis. The latter can be chosen such that it is orthonormal with the standard inner product of $\mathbb{R}^n$. Then this eigenbasis is both orthonormal for the standard inner product, and orthogonal for the inner product defined the matrix A. Furthermore, the transition matrix is easy to inverse: finding the coordinates in the eigenbasis of an arbitrary vector is straightforward.

2.2 Exercises

Exercise 2.9

1. In the vector space $E = \mathbb{R}^2$ together with the standard inner product, consider the basis $\mathcal{B} = ((1,1),(1,2))$.

   (a) Using the Gram-Schmidt process and starting from this basis $\mathcal{B}$, build an orthonormal basis of E.

   (b) Let $u = (x,y) \in E$. Find the coordinates of u in this new basis.

   (c) Let $(u,v) \in E^2$. Express $\langle u \,, v\rangle$ with the coordinates of u and v in this basis.

2. **(Personal work)** In the vector space $E = \mathbb{R}^3$ together with the standard inner product, consider the basis $\mathcal{B} = ((1,1,1),(1,2,1),(1,1,2))$.

   (a) Using the Gram-Schmidt process and starting from this basis $\mathcal{B}$, build an orthonormal basis of E.

   (b) Let $u = (x,y,z) \in E$. Find the coordinates of u in this new basis.

   (c) Let $(u,v) \in E^2$. Express $\langle u \,, v\rangle$ with the coordinates of u and v in this basis.

Exercise 2.10

Consider the vector space $E = \mathbb{R}^3$ together with the inner product defined for every $(u{=}(x_1,y_1,z_1), v = (x_2,y_2,z_2)) \in E^2$ by:

$$\langle u \,, v\rangle = x_1x_2 + 2y_1y_2 + 2z_1z_2 + x_1y_2 + y_1x_2 + y_1z_2 + z_1y_2$$

1. Using the Gram-Schmidt process and starting from the standard basis of E, build an orthonormal basis of E.
2. Let $u = (x,y,z) \in E$. Find the coordinates of u in this new basis.
3. Let $(u,v) \in E^2$. Express $\langle u \,, v\rangle$ with the coordinates of u and v in this basis.

Exercise 2.11

Let $E = \mathbb{R}_2[X]$ together with the inner product defined for every $(P,Q) \in E^2$ by:

$$\langle P \,, Q\rangle = \int_0^1 P(x)Q(x)\,\mathrm{d}x$$

1. Using the Gram-Schmidt process and starting from the standard basis of E, build an orthonormal basis of E.
2. Let $P = a_0 + a_1X + a_2X^2 \in E$. Find the coordinates of P in this new basis.
3. Let $(P,Q) \in E^2$. Express $\langle P \,, Q\rangle$ with the coordinates of P and Q in this basis.

Exercise 2.12

Let $E = \mathbb{R}^2$ together with the standard inner product. Consider the matrices

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$$

1. Study of the endomorphisms f, g and h defined by these three matrices in the standard basis.

   (a) Diagonalize A, B and C and check that they admit a same eigenbasis. Show that we can choose an orthonormal eigenbasis $\mathcal{B}'$.

(b) Let P be the transition matrix from the standard basis to $\mathscr{B}'$. Check that ${}^tP = P^{-1}$.
(c) Let $u = (x, y) \in E$. Find the coordinates of u in basis $\mathcal{B}'$.
(d) By reasoning in $\mathcal{B}'$, determine the images by f, g and h of the unit circle centered at 0_E.

2. Let φ_A, φ_B and φ_C be the bilinear forms defined by the matrices A, B and C in the standard basis.
   (a) Check that these forms are symmetric.
   (b) Let $(u, v) \in E^2$. Express $\varphi_A(u, v)$, $\varphi_B(u, v)$ and $\varphi_C(u, v)$ with the coordinates of u and v in basis $\mathcal{B}'$.
   (c) Which of these bilinear symmetric forms is(are) positive non-degenerate?

3 Orthogonal projection

3.1 Summary

If $(E, \langle\,,\rangle)$ is a inner product space and F a finite-dimensional linear subspace of E, then the orthogonal supplementary theorem states that F and $F^\perp$ are supplementary. Thus, for every $u \in E$, there exists a unique $(v, w) \in F \times F^\perp$ such that

$$u = \underbrace{v}_{\in F} + \underbrace{w}_{\in F^\perp}$$

The vector v is the projection of u onto F, parallel to $F^\perp$. This defines the ***orthogonal projection*** of u onto F.

A property of this projection is that $v = p_F(u)$ is the F-vector the closest to u. It is the vector in F which minimizes $\|u-v\|^2$. Many optimization problem are solved using an orthogonal projection: first, define an inner product corresponding to the function to be optimized, then compute an orthogonal projection.

3.2 Exercises

Exercise 3.13
In the vector space $E = \mathbb{R}^2$ together with the standard inner product, consider the linear subspace

$$F = \{(x, y) \in E, x - y = 0\}$$

1. Find a basis of F. Let $\mathcal{B}$ denote this basis.
2. Compare $\mathcal{B}^\perp$ with $F^\perp$. Explain your result.
3. Find the orthogonal projection onto F of $u = (1, 2)$.
4. Determine $\underset{v\in F}{\text{Min}}\, \|u - v\|^2$. Provide a geometrical explanation.

Exercise 3.14
In the vector space $E = \mathbb{R}^3$ together with the standard inner product, consider the family

$$\mathcal{F} = \big((1, 1, 0),\ (0, 1, 1)\big)$$

1. Determine $\mathcal{F}^\perp$ (that is, provide a basis).
2. Determine $\big(\text{Span}(\mathcal{F})\big)^\perp$.

Exercise 3.15

In the vector space $E = \mathbb{R}^3$ together with the standard inner product, consider the linear subspaces

$$F_1 = \{(x, y, z) \in E, x + 2y - z = 0\} \qquad F_2 = \{(x, y, z) \in E, -x + y + z = 0\}$$

$$\text{and} \quad F_3 = \left\{ (x, y, z) \in E, \left| \begin{array}{lcl} x + y + z & = & 0 \\ x - 3y - z & = & 0 \end{array} \right. \right\}$$

For each of them:

1. Find $F^\perp$.
2. Check that $F^{\perp\perp} = F$ and explain why.
3. Find the orthogonal projection of $u = (1, 2, 3)$ onto F.
4. Determine $\underset{v \in F}{\text{Min}} \|u - v\|^2$.

Exercise 3.16

Let E be the vector space of the continuous functions on $[0, 1]$, together with the inner product

$$\langle f\,, g \rangle = \int_0^1 f(t)g(t)\ \mathrm{d}t$$

Consider the linear subspace $F = \{f \in E, f(0) = 0\}$.

1. Show that $F^\perp = \{0_E\}$.
2. Deduce $F^{\perp\perp}$.
3. Consider the function $x \longmapsto 1$. Can we define its orthogonal projection onto F?

Exercise 3.17

In the vector space $E = \mathbb{R}^4$ together with the standard inner product, consider the linear subspace

$$F = \left\{ (x, y, z, t) \in E, \left| \begin{array}{lcl} x - z + t & = & 0 \\ y - 2z + t & = & 0 \end{array} \right. \right\}$$

1. Find a basis of F.
2. Using the Gram-Schmidt process, build an orthogonal basis of F.
3. Find the orthogonal projection onto F of $u = (1, 1, 1, 1)$.

Exercise 3.18

Let $E = \mathbb{R}_2[X]$ together with the inner product

$$\langle P\,, Q \rangle = \int_0^1 P(t)Q(t)\ \mathrm{d}t$$

Let $F = \mathbb{R}_1[X]$ and $P = X^2$.

1. Determine the matrix of this inner product in the standard basis of E.
2. Calculate the orthogonal projection P_0 of P onto F.
3. Deduce $\underset{(a,b) \in \mathbb{R}^2}{\text{Min}} \displaystyle\int_0^1 (x^2 - ax - b)^2\,\mathrm{d}x$

Exercise 3.19

Let $E = \mathbb{R}_2[X]$. The mapping $\langle\,,\rangle :\ E \times E \to \mathbb{R}$ is defined by

$$\langle P\,, Q\rangle = \int_0^{+\infty} P(x)Q(x)e^{-x}\mathrm{d}x$$

1. Show that $\langle\,,\rangle$ is an inner product over E.

2. For every $n \in \mathbb{N}$, let us set $I_n = \displaystyle\int_0^{+\infty} x^n e^{-x}\,\mathrm{d}x$. Determine I_n as a function of $n \in \mathbb{N}$.

3. Deduce the matrix of the inner product $\langle\,,\rangle$ in the standard basis of E.

4. Calculate the orthogonal projection of X^2 onto $\mathbb{R}_1[X]$.

5. Deduce $\displaystyle\operatorname*{Min}_{(a,b)\in\mathbb{R}^2} \int_0^{+\infty} (x^2 - ax - b)^2 e^{-x}\mathrm{d}x$