# Topological Ring Theory

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## 1 Basic Concepts

We show the definition of topological rings.

**Definition 1.1 (Topological Ring)** *A ring $R$ with a topological structure is a topological ring if the operators $-, * : R \times R \to R$ are continuous. It is separated, when its topology is separated (Hausdorff space).*

We show some obvious facts.

**Fact 1.1**
1. *$R$ is an Abelian group with $+$;*
2. *A subring $M$ and quotient ring $R/J$ of $R$ is also topological rings with topological structure inherited from R; $R/J$ is separated iff $J$ is closed;*
3. *The closure $\bar{M}$ of a subring $M$ is also a topological ring. A direct product of topological rings is a topological ring in a natural way;*
4. *For a separated ring, we define completion $\tilde{R}$ (also separated) of $R$ regarding $R$ as a uniform space. And we have an extension $\tilde{R}/R$.*

**Example 1.1** $\mathbb{R}$ *is a topological ring (complete, separated), and $\mathbb{R} = \tilde{\mathbb{Q}}$. So $\mathbb{C}$ is.*

**Example 1.2** $\mathbb{R}^{n\times n}$ *is a topological ring (complete, separated). So $\mathbb{C}^{n\times n}$ is.*

**Definition 1.2 ($J$-adic topology)** *$J$ is an ideal of a communative tring $R$, then set $\{J^m, m \in \mathbb{N}\}$ forms a fundamental system of neighbourhoods of 0 that generates so-called $J$-adic topology. It is separated if $\bigcap_m J^m = \{0\}$.*

**Example 1.3** *The $p$-adic topology on the integers is an example of an $(p)$-adic topology.*

$\mathrm{TopRing}$ denotes the category of topological rings with continuous homomorphisms as its morphisms.

## 2 Top on Homomorphism

Homomorphism set $\mathrm{Hom}(R, S)$ is a ring, $\phi\psi(a) = \phi(a)\psi(a), (\phi + \psi)(a) = \phi(a) + \psi(a)$. Let $M(A, V) = \{\phi \in \mathrm{Hom}(R, S) \,|\, \phi[A] \subset V\}$. If $A \in \mathcal{G}, V \in \mathcal{B}$, then $M(A, V)$ form a top basis of $\mathrm{Hom}(R, S)$, where $\mathcal{G}$ is directed (e.g. $\mathcal{G} = \mathcal{K}$).