Inequalities

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1 Analysis 1

Lemma 1.1 (Triebel) *If $f \in \mathscr{S}_\Omega$, then there exist $\phi \in \mathscr{S}_{\Omega'}, \Omega \subset \Omega'$ that $D^\alpha f = f * \phi$, so*

$$D^\alpha f \leq C|f| * \langle x \rangle^{-N}$$

where Ω, Ω' are compact and C depends on the size of Ω. (N can be chosen arbitrarily large)

The proof is based on "the $\mathscr{S}_\Omega$-inserting method".

Proof. Notice $f = \mathscr{F}^{-1}(\mathscr{F} f \mathscr{F} g), g \in \mathscr{S}_{\Omega'}$ and $g = 1$ on Ω. Let $\phi = D^\alpha g$, then $D^\alpha f = \mathscr{F}^{-1}(\mathscr{F} f \mathscr{F} \phi)$. $\square$

Assume Ω is a compact subset of $\mathbb{R}^n$, with diameter R.

Lemma 1.2 (Nikol'skij-Triebel) *If $f \in L^p_\Omega, 0 < p \leq q \leq \infty$, then*

$$\|f\|_{L^q} \leq C\|f\|_{L^p}$$

where $C \sim R^{n(\frac{1}{p}-\frac{1}{q})}$.

Following inequality is an analogue of Young's inequality.

Theorem 1.1 *If $f, g \in L^p_\Omega, 0 < p \leq 1$, then*

$$\|f * g\|_{L^q} \leq C\|f\|_{L^p}\|g\|_{L^p}$$

where $C \sim R^{n(\frac{1}{p}-1)}$.

Proof. Let $h_x(y) = f(y)g(x-y)$ with support Ω' such that $|\Omega'| \sim |\Omega|$. Then with Lemma 1.2, we have $f * g(x) = \|h_x\|_1 \leq \|h_x\|_p$, namely

$$|f * g(x)|^p \lesssim C|f|^p * |g|^p$$

where $C \sim R^{n(\frac{1}{p}-1)}$. Integrate two side to obtain the inequality. $\square$

Lemma 1.3 (Soblev type inequality) *Assume (polynomial)* $P(\xi)^{-1} \in L^p$*,*

$$\|f\|_{\mathscr{F}L^r} \lesssim \|P(D)f\|_{L^q}$$

where $\frac{1}{r} = \frac{1}{p} - \frac{1}{q} + 1, 1 \le q \le 2, r, p \ge 1$.

Proof. Apply Holder inequality to

$$f(x) = P^{-1}(x)P(x)f(x).$$

Then use Yang-Hausdorff inequality. $\square$

As an example, $P(x) = \langle x \rangle^d, d > \frac{n}{p}$, i.e. $H^{d,q} \hookrightarrow \mathscr{F}L^r$.

Corollary 1.1 *(case* $q = 1$*)*

$$\|f\|_{\mathscr{F}L^p} \lesssim \|P(D)f\|_{L^1}$$

where $P(\xi)^{-1} \in L^p, p \ge 1$.

Define the smoothness degree of X under one-parameter group $\mathbb{R}^+$,

$$\deg(X) = s, \text{ for } f \neq 0, \|(\lambda)f\| \sim \lambda^s.$$

Lemma 1.4 *If normed spaces* $X \hookrightarrow Y$*, then* $\deg(X) \ge \deg(Y)$.

Generally,

Lemma 1.5 *If normed spaces* $X \hookrightarrow Y$*, then* $\|(\lambda)\|_X \gtrsim \|(\lambda)\|_Y$.

Note $\|(\lambda)\|_X \sim \lambda^{\deg(X)}$ if $\deg(X)$ exists. The optimal s_1, s_2 such that $\lambda^{s_1} \lesssim \|(\lambda)\| \lesssim \lambda^{s_2}, \lambda \ge 1$ are named the lower smoothness degree ($\deg^-(X)$) and upper smoothness degree ($\deg^+(X)$) respectively.

Lemma 1.6 *If normed spaces* $X \hookrightarrow Y$*, then* $\deg^+(X) \ge \deg^-(Y)$.

The default one-parameter group is $(\lambda)f = f(\lambda\cdot)$ for function spaces.

2 Analysis 2

Bootstrap principle (continuity method): In connected space X, if subset $A \neq \emptyset$ is open and closed, then $A = X$.

Example 2.1 *If* $f \in C(X), f(x_0) = 0, \{f = 0\}$ *is open, then* $f = 0$.

Lemma 2.1 (Gap principle 1) *Let* $f(x) \in C(\Omega), f(x_0) < \epsilon (> \epsilon)$ *where* Ω *is a connected space. If* $f(x) < \epsilon$ *or* $> \epsilon$ *for all* $x \in \Omega$*, then* $f(x) < \epsilon (> \epsilon)$ *for all* $x \in \Omega$.

Proof. $\{f < \epsilon\} = \{f \le \epsilon\}$. $\square$

We say ϵ is a gap of f. Moreover, if any $\epsilon > 0$ can be a gap of f, then $f = 0$.

Lemma 2.2 (Gap principle 2) *Let $f(x) \in C(\Omega), |f(x_0)| < \epsilon$ where Ω is a connected space. Whenever $|f(x)| \le \epsilon$ where $\epsilon > 0$, $|f(x)| \le \epsilon$ for all $x \in \Omega$. Then $|f(x)| \le \epsilon, \forall x \in \Omega$.*

Lemma 2.3 (Gap principle 3) *Let $f(x) \in C(\Omega), |f(x_0)| < \epsilon$ where Ω is a connected space. Whenever $|f(x)| \le \eta(\epsilon)$ where $\epsilon > 0$ and $\eta(x) > x, \forall x > 0$, $|f(x)| \le \epsilon$ for all $x \in \Omega$. Then $|f(x)| \le \epsilon, \forall x \in \Omega$.*

$(\epsilon, \eta(\epsilon)]$ is an impenetrable barrier of f.

Corollary 2.1 (Terence's Bootstrap principle) *Let $f(x) \in C^+(\Omega)$ where Ω is a connected space. $f(x) \le H(f(x_0)) + \eta(f(x))$, for all $x \in \Omega$ where $\eta(x) = o(x), H(x) \to 0, x \to 0, H(0) = 0$. We have*

1) $\forall(\mathrm{sml})\epsilon > 0 \exists(\mathrm{sml})\delta > 0, f(x_0) < \delta \to f \le \epsilon$; ($f(x_0) \neq 0$)

2) if $f(x_0) \ll 1$, then f is bounded;

3) if $f(x_0) = 0$, then $f = 0$.

Proof. Select small $\epsilon > 0$ and $f(x) \le \epsilon$. Let $f(x_0) = \delta > 0$ that $H(\delta) < \frac{\epsilon}{2}$.

$$\begin{aligned} f(x) &\le H(f(x_0)) + \eta(f(x)) \\ &\le \frac{\epsilon}{2} + o(\epsilon) < \epsilon. \end{aligned}$$

$\square$

Example 2.2 *It holds when $f(x) \lesssim f(x_0)^\beta + f(x)^\alpha, \alpha > 1, \beta > 0$.*

Following is an important application of bootstrap principle.

Proposition 2.1 *$V \in C^2(\mathbb{R}^n), V(0) = V'(0) = 0, H_V > 0$. $\mathrm{sml} u_0, u_1 \in \mathbb{R}^n \exists! u : \mathbb{R} \to \mathbb{R}^n$ is the bounded solution to the Cauchy problem*

$$u''(t) = V'(u(t)), u(0) = u_0, u'(0) = u_1.$$

Proof. $f(t) = \|u'(t)\|^2 + \|u(t)\|^2$. We have $f(t) \lesssim f(0) + f(t)^3$. $\square$

Theorem 2.1 (Acausal Gronwall inequality) *Assume that the measurable function $K'(t,s)$ satisfies that*

$$\begin{cases} K'(t,s)K'(s,r) \le K'(t,r), K'(t,s) \ge 0, K'(t,t) = 1, \\ \sup_t \int K(t,s)K'(t,s)^{-1}\mathrm{d}s < C, \sup_s K'(t,s)\phi(s) \text{ is locally bounded.} \end{cases}$$

If $\begin{cases} u \le A + \epsilon \int K(t,s)u(s)\mathrm{d}s, \\ u \le \phi, \end{cases}$ *then* $u \le \sup_s K'(t,s)A(s)$*, where* $\epsilon \ll 1$ *depends on* K, K'*.*

Proof. Let the auxiliary function $B(t) = \sup_s K'(t,s)(A(s) + \sigma\phi(s))$, we have

$$\begin{aligned} B(t) &\ge A(t) + \sigma\phi(t)\,(\ge \sigma u(t)), \\ B(s) &\le K'(t,s)^{-1}B(t). \end{aligned}$$

Assume $u(t) \le MB(t)$ for smallest $M > 0$, then

$$\begin{aligned} u(t) &\le A(t) + \int K(t,s)u(s)\mathrm{d}s \\ &\le B(t) + \epsilon M \int K(t,s)K'(t,s)^{-1}B(t)\mathrm{d}s \\ &\le B(t) + \frac{1}{2}MB(t). \end{aligned}$$

Thus $M \le 1 + \frac{1}{2}M$, namely $M \le 2$. In a word, we have $u(t) \le 2B(t)$. It follows that

$$u(t) \le 2\sup_s K'(t,s)A(s),$$

since σ is arbitrarily small. $\square$

It is trivial to replace $u \le \psi$ with $u \lesssim \phi$.

Corollary 2.2 (Acausal Gronwall inequality (convolution type)) *Assume that the measurable function $K'(t)$ satisfies that*

$$\begin{cases} K'(s)K'(t) \le K'(s+t), K'(t) \ge 0, K'(0) = 1, \\ \sup_t \int K(t-s)K'(t-s)^{-1}\mathrm{d}s < C, \sup_s K'(t-s)\phi(s) \text{ is locally bounded.} \end{cases}$$

If $\begin{cases} u \le A + \epsilon \int K(t-s)u(s)\mathrm{d}s, \\ u \le \phi, \end{cases}$ *then* $u \le \sup_s K'(t-s)A(s)$*, where* $\epsilon \ll 1$ *depends on* K, K'*.*

Example 2.3 $k(t) = e^{\alpha t} \wedge e^{-\beta t}, k'(t) = e^{\alpha' t} \wedge e^{-\beta' t}, 0 < \alpha' < \alpha, 0 < \beta' < \beta.$

3 Elementary

Lemma 3.1 *In norm space, we have*

$$|x - x_1| + |x - x_2| \gtrsim |x| + 1, x_1 \neq x_2.$$

The implicit constant depends on x_1, x_2

Proof. Consider it on two parts $|x| < C, |x| \geq C$. $\square$

4 Operator

Lemma 4.1 *In a Banach space* X*, If* $S \in \mathcal{G}(X)$*, then*

$$\frac{\|T\|}{\|S^{-1}\|} \leq \|TS\|, \|ST\| \leq \|T\|\|S\|.$$

If $\|S^{-1}\| = \|S\| = 1$*, then* $\|TS\| = \|ST\| = \|T\|$.

5 Lattice theory

Theorem 5.1 (Gronwall) *Let* $G_f := \{x \in L | x \leq f(x)\}$ *where* L *is a completed lattice and* f *is order preserving, then there exists unique*

$$m = \sup G_f \in \mathrm{fix}(f) \subseteq G_f$$

and

$$\mathrm{fix}(f) = \{m\} \Rightarrow G_f \leq m.$$

Proof. For any poset $(L, \leq)$, we have the following facts,

i) $x \in G_f \Rightarrow x \leq f(x) \leq \cdots \leq f^{(n)}(x) \leq \cdots \in G_f$,

ii) $m = \sup G_f \Rightarrow m \in \mathrm{fix}(f) \subseteq G_f$.

Let $(x)_f = \{x, f(x), \cdots\}$ that is contained in G_f when $x \in G_f$. $\square$