Note for Fourier Multipliers

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1 Lemmata of Inequality

Lemma 1.1 *Assume* $f \in \dot{H}^{m,p}(\mathbb{R}^n)$*,*

$$\| \triangle_h^m f\|_{L^p} \lesssim |h|^m \|f\|_{\dot{H}^{m,p}}.$$

Proof. Take case $m = 1$ for example. Set an auxiliary function

$$F(t) = \frac{\mathrm{d}f(x+th)}{\mathrm{d}t} = \nabla f(x+th) \cdot h$$

for any $x, h \in \mathbb{R}^n$. We complete the proof, integrating F form 0 to 1. □

The lemma bridges difference and differential calculus (see Miao's book and [3]). It appears in the proof of the equivalency of the norms of the function spaces.

Sequence space $\ell^{s,q}$ is defined as $\{\|a\| = (\sum_k \langle k\rangle^{sq}|a_k|^q)^{\frac{1}{q}} < \infty\}$ in the context of modulation spaces.

Lemma 1.2 *(see [2])for any* $R > 0$*,*

$$(\int_{|x|>R} |K|^p)^{\frac{1}{p}} \lesssim \langle R\rangle^{-s}\|K\|_{W(L^p,\ell^{s,1})}, s \geq 0.$$

Proof. Let $\Delta_k = \mathrm{supp}\phi_k$ and B_R is the ball with radius R about the origin. Obviously $\Delta_k \cap B_R \neq \emptyset$ implies $|k| > \frac{R}{2}$ for large R. Thus

$$\begin{aligned}
(\int_{|x|>R} |K|^p \mathrm{d}x)^{\frac{1}{p}} &\leq \sum_k (\int_{|x|>R} |\phi(x-k)K(x)|^p \mathrm{d}x)^{\frac{1}{p}} \\
&\lesssim \sum_{|k|>R/2} \|\phi_k(x)K(x)\|_p \qquad (1)\\
&= \sum_{|k|>R/2} \langle k\rangle^{-s}\langle k\rangle^s \|\phi_k(x)K(x)\|_p \\
&\lesssim \langle R\rangle^{-s} \sum_k \langle k\rangle^s \|\phi_k(x)K(x)\|_p \\
&= \langle R\rangle^{-s}\|K\|_{W(L^p,\ell^{s,1})}.
\end{aligned}$$

Since

$$W(L^p, \ell^1) \hookrightarrow L^p, \qquad (2)$$

for finite R bounded by a certain constant,

$$\begin{aligned}(\int_{|x|>R}|K|^p\mathrm{d}x)^{\frac{1}{p}} &\leq \|K\|_p\\ &\leq \|K\|_{W(L^p,\ell^1)}.\end{aligned}$$

Since $W(L^p,\ell^{s,1}) \hookrightarrow W(L^p,\ell^1)$, we complete the proof. $\square$

Remark 1.1 *One of more general forms of (2) is*

$$W(X,\ell^1) \hookrightarrow X,$$

where X is Banach function space.

Remark 1.2 *$X^{s,p}$ denotes the function space with the norm*

$$\sup_{R>0}\langle R\rangle^s(\int_{|x|>R}|K|^p)^{\frac{1}{p}}, s\geq 0,$$

that is extremely like Murrey spaces. We have

$$W(L^p,\ell^{s,1}) \hookrightarrow X^{s,p}.$$

Remark 1.3 *The key of the proof is (1). It is true for any unitary partition consisting of compactly supported functions $\{\psi_k\}$ and the distances between supports $\Delta_k = \mathrm{supp}\psi_k$ and the origin tents to infinity as $|k| \to \infty$.*

Corollary 1.1

$$M^s_{2,1} \hookrightarrow \mathscr{F}X^{s,1}.$$

With same method, for diadic decomposition (see [3]) we have

Lemma 1.3 *for any $r>0$,*

$$(\int_{|x|>2^r}|K|^p)^{\frac{1}{p}} \lesssim 2^{-rs}\|K\|_{W(L^p,\ell^{s,1})}, s\geq 0.$$

Remark 1.4 *In the fact, it is just*

$$W(L^p,\ell^{s,1}) \hookrightarrow X^{s,p},$$

where $\ell^{s,q}$ is the sequence space with norm $(\sum_k 2^{ksq}|a_k|^q)^{\frac{1}{q}}$.

Corollary 1.2

$$\dot{B}^s_{2,1} \hookrightarrow \mathscr{F}X^{s,1}.$$

Fact 1.1 $\mathfrak{M}^1 = \mathscr{F}M_{1,\infty} \hookrightarrow \mathfrak{M}^p \hookrightarrow \mathscr{F}M_{p,\infty}$.

Notice $\mathfrak{M}^p = W(\mathcal{M}^p,\ell^\infty)$. (see [1])

Fact 1.2 $1 \in M^s_{\infty,1}, \mathscr{F}M_{1,\infty}$.

2 representation of Multipliers

$\mathcal{D}(X,Y)$ represents the spaces of bounded operators $Tf(x) = d(x)f(x) : X \to Y$, so $\mathcal{D}(X,Y) = \mathcal{M}(\mathscr{F}X, \mathscr{F}Y)$.

Theorem 2.1 *If $q_1 \leq q_2$, then*

$$\mathcal{M}(M(X_1, \ell^{s_1,q_1}), M(X_2, \ell^{s_2,q_2})) = W(\mathcal{M}(X_1, X_2), \ell^{s_2-s_1,\infty})$$

where $\|\psi_k f\|_{X_i} \lesssim \|f\|_{X_i}$. Similarly,

$$\mathcal{D}(W(X_1, \ell^{s_1,q_1}), W(X_2, \ell^{s_2,q_2})) = W(\mathcal{D}(X_1, X_2), \ell^{s_2-s_1,\infty}),$$

where X_1, X_2 are solid.

References

[1] A. Miyachi, F. Nicloa, S. Rivetti, A. Tabacco, and N. Tomita. Estimates for unimodular fourier multipliers on modulation space. *Proc. Amer. Math. Soc.*, 137, 2009.

[2] Naohito Tomita. On the hormander multipier theorem and modulation spaces. *Appl. Comput. Harmon. Anal.*, 26:408–415, 2009.

[3] Hans Triebel. *Theory of Function Spaces*. Birkhauser Verlag, 1983.