Note of A. Weil Theorem

William

May 25, 2014

1 Basic Facts

Definition 1.1 *subgroup Γ is concrete subgroup of G, if there exists a neighbourhood U of 1 such that for all $a \neq 1 \in \Gamma$, $Ua \cap U = \emptyset$, equivalently $U^{-1}U \cap \Gamma = U \cap \Gamma = \{1\}$.*

Lemma 1.1 *If U is a separating neighbourhood, π is the nature mapping, then π is inject on xU, namely $\pi|_{xU} : xU \leftrightarrow \pi[xU]$. If for all $x \in X \subset G$ that $X \cap x\Gamma = \{x\}$, then $\sum_{\gamma\in\Gamma} \chi_{x\gamma} = 1$*

Theorem 1.1 *$1 \in X$ is a measurable subset of G, $\pi|_X : X \leftrightarrow G/\Gamma$. We have*

$$\int_G f \mathrm{d}\nu = \int_{G/\Gamma} \sum_{\gamma\in\Gamma} f(x\gamma)\mathrm{d}x^\circ = \int_X \sum_{\gamma\in\Gamma} f(x\gamma)\mathrm{d}\nu.$$

Notice that

$$\sum_{\gamma\in\Gamma} \chi_X(x\gamma) = \left\{ \begin{array}{ll} 1, & x \in X, \\ 0, & x \notin X \end{array} \right.$$

$$X \cap x\Gamma = \left\{ \begin{array}{ll} \{x\}, & x \in X, \\ \emptyset, & x \notin X \end{array} \right.$$

ν is an A. Weil measure: $\nu(X) = 1$.

2 Weil Theorem

Assume that M is a locally compact space and G is a locally compact group acting on M, and M is G-homogeneous space.

Definition 2.1 *G is purely noncontinuous, if for all $x \in M$, there exists the neighbourhood V of x for all $g \neq 1$, $V \cap gV = \emptyset$. It implies that $V \cap Gx = \{x\}$.*

Lemma 2.1 *We have the map $f^\circ(\pi(x)) = \int_G f(ax)\mathrm{d}a : C_C(M) \to C_C(M/G)$*

Lemma 2.2 *There exists* $f \in C_C(M)$ *such that* $\int_G f(ax)\mathrm{d}a = 1$ *for all* $x \in K$ *where* K *is any compact subset of* G.

Lemma 2.3 *For all* $f \in C_C(M)$ *there exists* $g \in C_C(M)$ *such that* $f = fg^\circ$.

Remark 2.1 *Assume that* $K \subset U$ *where* K *and* U *are compact subset and pre-compact subset of* M *respectively. We have* $K \prec g \prec U, g \in C_C(M)$. *Extent* $f = \begin{cases} \frac{g(x)}{g^\circ(x)}, & x \in GK, \\ \cdots, & \textit{else} \end{cases}$ *to* M *continuously.*

Theorem 2.1 (A. Weil Theorem) *We have*

i) ν *is a* Δ*-relatively invariant measure iff* $d\nu = \Delta d\mu$

ii) $\Delta_\nu^r = \Delta_\nu^l \Delta_{M,G}$

iii) There exists μ, *a* Δ*-relatively invariant measure iff* $\Delta_G^r = \Delta\Delta_{M,G}^r$

iv) $\int_M f(x)\mathrm{d}x = \int_{M/G} \mathrm{d}\mu \int_G f(gx)\mathrm{d}g$

v) $\int_{M/H} f(x)\mathrm{d}x = \int_{M/G} \mathrm{d}\mu \int_{G/H} f(gx)\mathrm{d}g$ *for all subgroup* H *of* G.