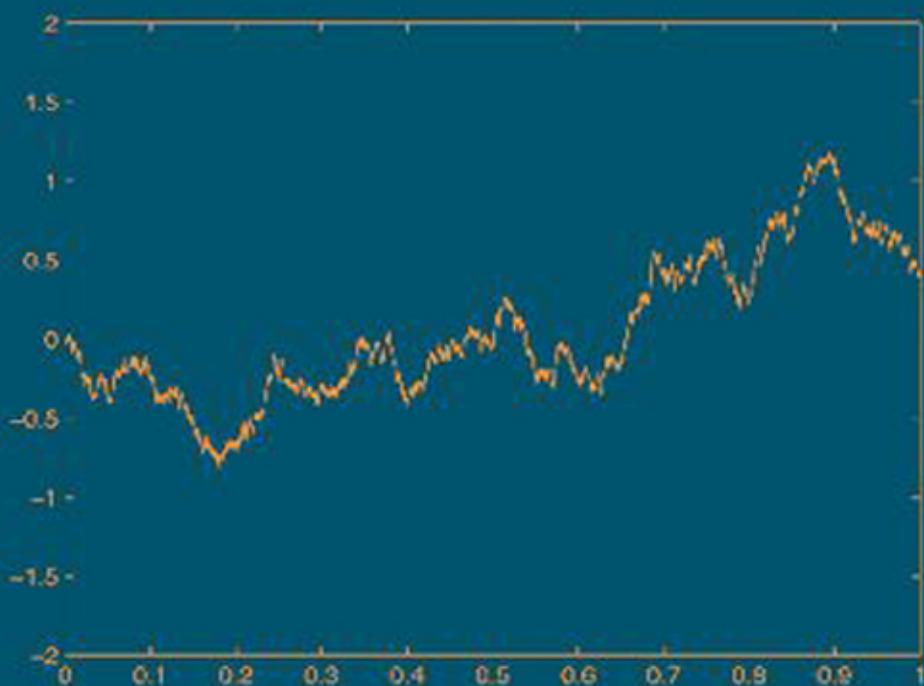


**Cambridge Series in Statistical  
and Probabilistic Mathematics**

# Stochastic Processes

**Richard F. Bass**



CAMBRIDGE

CAMBRIDGE

more information – [www.cambridge.org/9781107008007](http://www.cambridge.org/9781107008007)

## Stochastic Processes

This comprehensive guide to stochastic processes gives a complete overview of the theory and addresses the most important applications. Pitched at a level accessible to beginning graduate students and researchers from applied disciplines, it is both a course book and a rich resource for individual readers. Subjects covered include Brownian motion, stochastic calculus, stochastic differential equations, Markov processes, weak convergence of processes, and semigroup theory. Applications include the Black–Scholes formula for the pricing of derivatives in financial mathematics, the Kalman–Bucy filter used in the US space program, and also theoretical applications to partial differential equations and analysis. Short, readable chapters aim for clarity rather than for full generality. More than 350 exercises are included to help readers put their new-found knowledge to the test and to prepare them for tackling the research literature.

RICHARD F. BASS is Board of Trustees Distinguished Professor in the Department of Mathematics at the University of Connecticut.

CAMBRIDGE SERIES IN STATISTICAL AND  
PROBABILISTIC MATHEMATICS

---

*Editorial Board*

- Z. Ghahramani (Department of Engineering, University of Cambridge)  
R. Gill (Mathematical Institute, Leiden University)  
F. P. Kelly (Department of Pure Mathematics and Mathematical Statistics,  
University of Cambridge)  
B. D. Ripley (Department of Statistics, University of Oxford)  
S. Ross (Department of Industrial and Systems Engineering,  
University of Southern California)  
M. Stein (Department of Statistics, University of Chicago)

This series of high-quality upper-division textbooks and expository monographs covers all aspects of stochastic applicable mathematics. The topics range from pure and applied statistics to probability theory, operations research, optimization, and mathematical programming. The books contain clear presentations of new developments in the field and also of the state of the art in classical methods. While emphasizing rigorous treatment of theoretical methods, the books also contain applications and discussions of new techniques made possible by advances in computational practice.

A complete list of books in the series can be found at <http://www.cambridge.org/statistics>.  
Recent titles include the following:

11. *Statistical Models*, by A. C. Davison
12. *Semiparametric Regression*, by David Ruppert, M. P. Wand and R. J. Carroll
13. *Exercises in Probability*, by Loïc Chaumont and Marc Yor
14. *Statistical Analysis of Stochastic Processes in Time*, by J. K. Lindsey
15. *Measure Theory and Filtering*, by Lakhdar Aggoun and Robert Elliott
16. *Essentials of Statistical Inference*, by G. A. Young and R. L. Smith
17. *Elements of Distribution Theory*, by Thomas A. Severini
18. *Statistical Mechanics of Disordered Systems*, by Anton Bovier
19. *The Coordinate-Free Approach to Linear Models*, by Michael J. Wichura
20. *Random Graph Dynamics*, by Rick Durrett
21. *Networks*, by Peter Whittle
22. *Saddlepoint Approximations with Applications*, by Ronald W. Butler
23. *Applied Asymptotics*, by A. R. Brazzale, A. C. Davison and N. Reid
24. *Random Networks for Communication*, by Massimo Franceschetti and Ronald Meester
25. *Design of Comparative Experiments*, by R. A. Bailey
26. *Symmetry Studies*, by Marlos A. G. Viana
27. *Model Selection and Model Averaging*, by Gerda Claeskens and Nils Lid Hjort
28. *Bayesian Nonparametrics*, edited by Nils Lid Hjort *et al.*
29. *From Finite Sample to Asymptotic Methods in Statistics*, by Pranab K. Sen,  
Julio M. Singer and Antonio C. Pedrosa de Lima
30. *Brownian Motion*, by Peter Mörters and Yuval Peres
31. *Probability*, by Rick Durrett
33. *Stochastic Processes*, by Richard F. Bass
34. *Structured Regression for Categorical Data*, by Gerhard Tutz

CAMBRIDGE UNIVERSITY PRESS  
Cambridge, New York, Melbourne, Madrid, Cape Town,  
Singapore, São Paulo, Delhi, Tokyo, Mexico City

Cambridge University Press  
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)  
Information on this title: [www.cambridge.org/9781107008007](http://www.cambridge.org/9781107008007)

© R. F. Bass 2011

This publication is in copyright. Subject to statutory exception  
and to the provisions of relevant collective licensing agreements,  
no reproduction of any part may take place without the written  
permission of Cambridge University Press.

First published 2011

Printed in the United Kingdom at the University Press, Cambridge

*A catalogue record for this publication is available from the British Library*

*Library of Congress Cataloguing in Publication data*

Bass, Richard F.

Stochastic processes / Richard F. Bass.

p. cm. – (Cambridge series in statistical and probabilistic mathematics ; 33)

Includes index.

ISBN 978-1-107-00800-7 (hardback)

1. Stochastic analysis. I. Title.

QA274.2.B375 2011

519.2'32 – dc23 2011023024

ISBN 978-1-107-00800-7 Hardback

---

Cambridge University Press has no responsibility for the persistence or  
accuracy of URLs for external or third-party internet websites referred to  
in this publication, and does not guarantee that any content on such  
websites is, or will remain, accurate or appropriate.

---

---

# Contents

<i>Preface</i>	<i>page</i> xiii
<i>Frequently used notation</i>	xv
<b>1 Basic notions</b>	1
1.1 Processes and $\sigma$ -fields	1
1.2 Laws and state spaces	3
<b>2 Brownian motion</b>	6
2.1 Definition and basic properties	6
<b>3 Martingales</b>	13
3.1 Definition and examples	13
3.2 Doob's inequalities	14
3.3 Stopping times	15
3.4 The optional stopping theorem	17
3.5 Convergence and regularity	17
3.6 Some applications of martingales	20
<b>4 Markov properties of Brownian motion</b>	25
4.1 Markov properties	25
4.2 Applications	27
<b>5 The Poisson process</b>	32
<b>6 Construction of Brownian motion</b>	36
6.1 Wiener's construction	36
6.2 Martingale methods	39
<b>7 Path properties of Brownian motion</b>	43
<b>8 The continuity of paths</b>	49

<b>9</b>	<b>Continuous semimartingales</b>	54
9.1	Definitions	54
9.2	Square integrable martingales	55
9.3	Quadratic variation	57
9.4	The Doob–Meyer decomposition	58
<b>10</b>	<b>Stochastic integrals</b>	64
10.1	Construction	64
10.2	Extensions	69
<b>11</b>	<b>Itô’s formula</b>	71
<b>12</b>	<b>Some applications of Itô’s formula</b>	77
12.1	Lévy’s theorem	77
12.2	Time changes of martingales	78
12.3	Quadratic variation	79
12.4	Martingale representation	79
12.5	The Burkholder–Davis–Gundy inequalities	82
12.6	Stratonovich integrals	84
<b>13</b>	<b>The Girsanov theorem</b>	89
13.1	The Brownian motion case	89
13.2	An example	92
<b>14</b>	<b>Local times</b>	94
14.1	Basic properties	94
14.2	Joint continuity of local times	96
14.3	Occupation times	97
<b>15</b>	<b>Skorokhod embedding</b>	100
15.1	Preliminaries	100
15.2	Construction of the embedding	105
15.3	Embedding random walks	108
<b>16</b>	<b>The general theory of processes</b>	111
16.1	Predictable and optional processes	111
16.2	Hitting times	115
16.3	The debut and section theorems	117
16.4	Projection theorems	119
16.5	More on predictability	120
16.6	Dual projection theorems	122
16.7	The Doob–Meyer decomposition	124
16.8	Two inequalities	126

<b>17</b>	<b>Processes with jumps</b>	130
17.1	Decomposition of martingales	130
17.2	Stochastic integrals	133
17.3	Itô's formula	135
17.4	The reduction theorem	139
17.5	Semimartingales	141
17.6	Exponential of a semimartingale	143
17.7	The Girsanov theorem	144
<b>18</b>	<b>Poisson point processes</b>	147
<b>19</b>	<b>Framework for Markov processes</b>	152
19.1	Introduction	152
19.2	Definition of a Markov process	153
19.3	Transition probabilities	154
19.4	An example	156
19.5	The canonical process and shift operators	158
<b>20</b>	<b>Markov properties</b>	160
20.1	Enlarging the filtration	160
20.2	The Markov property	162
20.3	Strong Markov property	164
<b>21</b>	<b>Applications of the Markov properties</b>	167
21.1	Recurrence and transience	167
21.2	Additive functionals	169
21.3	Continuity	170
21.4	Harmonic functions	171
<b>22</b>	<b>Transformations of Markov processes</b>	177
22.1	Killed processes	177
22.2	Conditioned processes	178
22.3	Time change	180
22.4	Last exit decompositions	181
<b>23</b>	<b>Optimal stopping</b>	184
23.1	Excessive functions	184
23.2	Solving the optimal stopping problem	187
<b>24</b>	<b>Stochastic differential equations</b>	192
24.1	Pathwise solutions of SDEs	192
24.2	One-dimensional SDEs	196
24.3	Examples of SDEs	198

<b>25</b>	<b>Weak solutions of SDEs</b>	204
<b>26</b>	<b>The Ray–Knight theorems</b>	209
<b>27</b>	<b>Brownian excursions</b>	214
<b>28</b>	<b>Financial mathematics</b>	218
28.1	Finance models	218
28.2	Black–Scholes formula	220
28.3	The fundamental theorem of finance	223
28.4	Stochastic control	226
<b>29</b>	<b>Filtering</b>	229
29.1	The basic model	229
29.2	The innovation process	230
29.3	Representation of $\mathcal{F}^Z$ -martingales	231
29.4	The filtering equation	232
29.5	Linear models	234
29.6	Kalman–Bucy filter	234
<b>30</b>	<b>Convergence of probability measures</b>	237
30.1	The portmanteau theorem	237
30.2	The Prohorov theorem	239
30.3	Metrics for weak convergence	241
<b>31</b>	<b>Skorokhod representation</b>	244
<b>32</b>	<b>The space <math>C[0, 1]</math></b>	247
32.1	Tightness	247
32.2	A construction of Brownian motion	248
<b>33</b>	<b>Gaussian processes</b>	251
33.1	Reproducing kernel Hilbert spaces	251
33.2	Continuous Gaussian processes	254
<b>34</b>	<b>The space <math>D[0, 1]</math></b>	259
34.1	Metrics for $D[0, 1]$	259
34.2	Compactness and completeness	262
34.3	The Aldous criterion	264
<b>35</b>	<b>Applications of weak convergence</b>	269
35.1	Donsker invariance principle	269
35.2	Brownian bridge	273
35.3	Empirical processes	275

<b>36</b>	<b>Semigroups</b>	279
36.1	Constructing the process	279
36.2	Examples	283
<b>37</b>	<b>Infinitesimal generators</b>	286
37.1	Semigroup properties	286
37.2	The Hille–Yosida theorem	292
37.3	Nondivergence form elliptic operators	296
37.4	Generators of Lévy processes	297
<b>38</b>	<b>Dirichlet forms</b>	302
38.1	Framework	303
38.2	Construction of the semigroup	304
38.3	Divergence form elliptic operators	307
<b>39</b>	<b>Markov processes and SDEs</b>	312
39.1	Markov properties	312
39.2	SDEs and PDEs	314
39.3	Martingale problems	315
<b>40</b>	<b>Solving PDEs</b>	319
40.1	Poisson's equation	319
40.2	Dirichlet problem	320
40.3	Cauchy problem	321
40.4	Schrödinger operators	323
<b>41</b>	<b>One-dimensional diffusions</b>	326
41.1	Regularity	326
41.2	Scale functions	327
41.3	Speed measures	329
41.4	The uniqueness theorem	333
41.5	Time change	334
41.6	Examples	336
<b>42</b>	<b>Lévy processes</b>	339
42.1	Examples	339
42.2	Construction of Lévy processes	340
42.3	Representation of Lévy processes	344
<b>Appendices</b>		
<b>A</b>	<b>Basic probability</b>	348
A.1	First notions	348
A.2	Independence	353
A.3	Convergence	355
A.4	Uniform integrability	356

A.5	Conditional expectation	357
A.6	Stopping times	359
A.7	Martingales	359
A.8	Optional stopping	360
A.9	Doob's inequalities	361
A.10	Martingale convergence theorem	362
A.11	Strong law of large numbers	364
A.12	Weak convergence	367
A.13	Characteristic functions	370
A.14	Uniqueness and characteristic functions	372
A.15	The central limit theorem	372
A.16	Gaussian random variables	374
<b>B</b>	<b>Some results from analysis</b>	378
B.1	The monotone class theorem	378
B.2	The Schwartz class	379
<b>C</b>	<b>Regular conditional probabilities</b>	380
<b>D</b>	<b>Kolmogorov extension theorem</b>	382
	<i>References</i>	385
	<i>Index</i>	387

---

## Preface

Why study stochastic processes? This branch of probability theory offers sophisticated theorems and proofs, such as the existence of Brownian motion, the Doob–Meyer decomposition, and the Kolmogorov continuity criterion. At the same time stochastic processes also have far-reaching applications: the explosive growth in options and derivatives in financial markets throughout the world derives from the Black–Scholes formula, while NASA relies on the Kalman–Bucy method to filter signals from satellites and probes sent into outer space.

A graduate student taking a year-long course in probability theory first learns about sequences of random variables and topics such as laws of large numbers, central limit theorems, and discrete time martingales. In the second half of the course, the student will then turn to stochastic processes, which is the subject of this text. Topics covered here are Brownian motion, stochastic integrals, stochastic differential equations, Markov processes, the Black–Scholes formula of financial mathematics, the Kalman–Bucy filter, as well as many more.

The 42 chapters of this book can be grouped into seven parts. The first part consists of Chapters 1–8, where some of the basic processes and ideas are introduced, including Brownian motion. The next group of chapters, Chapters 9–15, introduce the theory of stochastic calculus, including stochastic integrals and Itô’s formula. Chapters 16–18 explore jump processes. This requires a study of the foundations of stochastic processes, which is also known as the general theory of processes. Next we take up Markov processes in Chapters 19–23. A formidable obstacle to the study of Markov processes is the notation, and I have attempted to make this as accessible as possible. Chapters 24–29 involve stochastic differential equations. Two very important applications, to financial mathematics and to filtering, appear in Chapters 28 and 29, respectively. Probability measures on metric spaces and the weak convergence of random variables taking values in a metric space prove to be relevant to the study of stochastic processes. These and related topics are treated in Chapters 30–35. We then return to Markov processes, namely, their construction and some important examples, in Chapters 36–42. Tools used in the construction include infinitesimal generators, Dirichlet forms, and solutions to stochastic differential equations, while two important examples that we consider are diffusions on the real line and Lévy processes.

The prerequisites to this book are a sound knowledge of basic measure theory and a course in the classical aspects of probability. The probability topics needed are provided (with proofs) in an appendix.

There is far too much material in this book to cover in a single semester, and even too much for a full year. I recommend that as a minimum the following chapters be studied: Chapters 1–5, Chapters 9–13, Chapters 19–21, and Chapter 24. If possible, include either

Chapter 28 or Chapter 29. In Chapter 11, the statement and corollaries of Itô's formula are very important, but the proof of Itô's formula may be omitted.

I would like to thank the many students who patiently sat through my lectures, pointed out errors, and made suggestions. I especially would like to thank my colleague Sasha Teplyaev who taught a course from a preliminary version of this book and made a great number of useful suggestions.

# 1

---

## Basic notions

In a first course on probability one typically works with a sequence of random variables  $X_1, X_2, \dots$ . For stochastic processes, instead of indexing the random variables by the positive integers, we index them by  $t \in [0, \infty)$  and we think of  $X_t$  as being the value at time  $t$ . The random variable could be the location of a particle on the real line, the strength of a signal, the price of a stock, and many other possibilities as well.

We will also work with increasing families of  $\sigma$ -fields  $\{\mathcal{F}_t\}$ , known as filtrations. The  $\sigma$ -field  $\mathcal{F}_t$  is supposed to represent what we know up to time  $t$ .

### 1.1 Processes and $\sigma$ -fields

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. A real-valued *stochastic process* (or simply a process) is a map  $X$  from  $[0, \infty) \times \Omega$  to the reals. We write  $X_t = X_t(\omega) = X(t, \omega)$ . We will impose stronger measurability conditions shortly, but for now we require that the rvs  $X_t$  be measurable wrt  $\mathcal{F}$  for each  $t \geq 0$ .

A collection of  $\sigma$ -fields  $\mathcal{F}_t$  such that  $\mathcal{F}_t \subset \mathcal{F}$  for each  $t$  and  $\mathcal{F}_s \subset \mathcal{F}_t$  if  $s \leq t$  is called a *filtration*. Define  $\mathcal{F}_{t+} = \cap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$ . A filtration is *right continuous* if  $\mathcal{F}_{t+} = \mathcal{F}_t$  for all  $t \geq 0$ . The  $\sigma$ -field  $\mathcal{F}_{t+}$  is supposed to represent what one knows if one looks ahead an infinitesimal amount. Most of the filtrations we will come across will be right continuous, but see Exercise 1.1.

A *null set*  $N$  is one that has outer probability 0:

$$\inf\{\mathbb{P}(A) : N \subset A, A \in \mathcal{F}\} = 0.$$

A filtration is *complete* if each  $\mathcal{F}_t$  contains every null set. A filtration that is right continuous and complete is said to satisfy the *usual conditions*.

Given a filtration  $\{\mathcal{F}_t\}$ , whether or not it satisfies the usual conditions, we define  $\mathcal{F}_\infty$  to be the  $\sigma$ -field generated by  $\cup_{t \geq 0} \mathcal{F}_t$ , that is, the smallest  $\sigma$ -field containing  $\cup_{t \geq 0} \mathcal{F}_t$ , and we write

$$\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t.$$

Recall that the arbitrary intersection of  $\sigma$ -fields is a  $\sigma$ -field, but the union of even two  $\sigma$ -fields need not be a  $\sigma$ -field.

a stochastic process  $X$  is *adapted* to a filtration  $\{\mathcal{F}_t\}$  if  $X_t$  is  $\mathcal{F}_t$  measurable for each  $t$ . Often one starts with a stochastic process  $X$  and wants to define a filtration wrt which  $X$  is adapted.

We define the *minimal augmented filtration* generated by  $X$  to be the smallest filtration that is right continuous and complete and wrt which the process  $X$  is adapted. For each  $t$ ,  $\mathcal{F}_t$  is in general strictly larger than the smallest  $\sigma$ -field wrt which  $\{X_s : s \leq t\}$  is measurable because of the inclusion of the null sets. It is important to include the null sets; see Exercise 1.5. There is no widely accepted name for what we call the minimal augmented filtration; I like this nomenclature because it is descriptive and sufficiently different from “filtration generated by  $X$ ” to avoid confusion.

The minimal augmented filtration generated by the process  $X_t$  can be constructed in three steps. 1, let  $\{\mathcal{F}_t^{00}\}$  be the smallest filtration wrt which  $X$  is adapted, that is,

$$\mathcal{F}_t^{00} = \sigma(X_s : s \leq t). \quad (1.1)$$

Let  $\mathbb{P}^*$  be the outer probability corresponding to  $\mathbb{P}$ : for  $A \subset \Omega$ ,

$$\mathbb{P}^*(A) = \inf\{\mathbb{P}(B) : B \in \mathcal{F}, A \subset B\}.$$

Let  $\mathcal{N}$  be the collection of null sets, so that  $\mathcal{N} = \{A \subset \Omega : \mathbb{P}^*(A) = 0\}$ .

$$2, \quad \mathcal{F}_t^0 = \sigma(\mathcal{F}_t^{00} \cup \mathcal{N}). \quad \text{F00\_t VN} \quad (1.2)$$

$$3, \quad \mathcal{F}_t = \cap_{s > t} \mathcal{F}_s^0 \quad \text{F0_{\{t+\}}}$$

Exercise 1.2 asks you to check that  $\{\mathcal{F}_t\}$  is the minimal augmented filtration generated by  $X$ . We will refer to  $\{\mathcal{F}_t^{00}\}$  as the *filtration generated by  $X$* .

*Two stochastic processes  $X$  and  $Y$  are said to be *indistinguishable* if  $\mathbb{P}(X_t = Y_t \text{ for } t \geq 0) = 1$*

*$X$  and  $Y$  are *versions* of each other if for each  $t \geq 0$ , we have  $\mathbb{P}(X_t = Y_t) = 1$ .*

*Example.* of two processes that are versions of each other but are not indistinguishable let  $\Omega = [0, 1]$ ,  $\mathcal{F}$  the Borel  $\sigma$ -field on  $[0, 1]$ ,  $\mathbb{P}$  Lebesgue measure on  $[0, 1]$ ,  $X(t, \omega) = 0$  for all  $t$  and  $\omega$ , and  $Y(t, \omega)$  equal to 1 if  $t = \omega$  and 0 otherwise . Note that the functions  $t \mapsto X(t, \omega)$  are continuous for each  $\omega$ , but the functions  $t \mapsto Y(t, \omega)$  are not continuous for any  $\omega$ .

If  $X$  is a stochastic process, the functions  $t \mapsto X(t, \omega)$  are called the *paths* or *trajectories* of  $X$ . There will be one path for each  $\omega$ . If the paths of  $X$  are continuous functions, except for a set of  $\omega$ 's in a null set, then  $X$  is called a *continuous process*, or is said to be continuous. We similarly define right continuous process, left continuous process, etc. “path-valued rv”

A function  $f(t)$  is *right continuous with left limits* if  $\lim_{h \rightarrow 0, h \downarrow 0} f(t + h) = f(t)$  for all  $t$  and  $\lim_{h \rightarrow 0, h \uparrow 0} f(t + h)$  exists for all  $t > 0$ . Almost all our stochastic processes will have the property that except for a null set of  $\omega$ 's the function  $t \mapsto X(t, \omega)$  is right continuous and has left limits. *cadlag* abbreviates the French “continue a droite, limite à gauche.”

## 1.2 Laws and state spaces

Let  $\mathcal{S}$  be a topological space. The Borel  $\sigma$ -field on  $\mathcal{S}$  is defined to be the  $\sigma$ -field generated by the open sets of  $\mathcal{S}$ . A function  $f: \mathcal{S} \rightarrow \mathbb{R}$  is Borel measurable if  $f^{-1}(G)$  is in the Borel  $\sigma$ -field of  $\mathcal{S}$  whenever  $G$  is an open subset of  $\mathbb{R}$ . A rv  $Y: \Omega \rightarrow \mathcal{S}$  is measurable wrt a  $\sigma$ -field  $\mathcal{F}$  of subsets of  $\Omega$  if  $\{\omega \in \Omega : Y(\omega) \in A\}$  is in  $\mathcal{F}$  whenever  $A$  is in the Borel  $\sigma$ -field on  $\mathcal{S}$ .

A stochastic process taking values in a topological space  $\mathcal{S}$  is a map  $X: [0, \infty) \times \Omega \rightarrow \mathcal{S}$ , where for each  $t$ , the rv  $X_t$  is measurable wrt  $\mathcal{F}$ .

Recall that if we have a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and  $Y: \Omega \rightarrow \mathbb{R}$  is a rv, then the *law* of  $Y$  is the probability measure  $\mathbb{P}_Y$  on the Borel subsets of  $\mathbb{R}$  defined by  $\mathbb{P}_Y(A) = \mathbb{P}(Y \in A)$ . Similarly, if  $Y: \Omega \rightarrow \mathbb{R}^d$  is a  $d$ -dimensional random vector, then the law of  $Y$  is the probability measure  $\mathbb{P}_Y$  on the Borel subsets of  $\mathbb{R}^d$  defined by  $\mathbb{P}_Y(A) = \mathbb{P}(Y \in A)$ . We extend this definition to rvs  $Y$  taking values in a topological space  $\mathcal{S}$ . In this case  $\mathbb{P}_Y$  is a probability measure on the Borel subsets of  $\mathcal{S}$  with the same definition:

$\mathbb{P}_Y(A) = \mathbb{P}(Y \in A)$ . In particular, if  $Y$  and  $Z$  are two rvs with the same state space  $\mathcal{S}$ , then  $Y$  and  $Z$  will have the *same law* if  $\mathbb{P}(Y \in A) = \mathbb{P}(Z \in A)$  for all Borel subsets  $A$  of  $\mathcal{S}$ .

The relevance of the preceding paragraph to stochastic processes is this. Suppose  $X$  and  $Y$  are stochastic processes with continuous paths. Let  $\mathcal{S} = C[0, \infty)$  be the collection of real-valued continuous functions on  $[0, \infty)$  together with the usual metric defined in terms of the supremum norm:

$$d(f, g) = \sup_{0 \leq t} |f(t) - g(t)|.$$

(Strictly speaking, we should write  $C([0, \infty))$ , but we follow the usual convention and drop the outside parentheses.) Let the random variable  $\bar{X}$  taking values in  $\mathcal{S}$  be defined by setting  $\bar{X}(\omega)$  to be the continuous function  $t \mapsto X(t, \omega)$ , and define  $\bar{Y}$  similarly. More precisely,  $\bar{X}: \Omega \rightarrow \mathcal{S}$  with

$$\bar{X}(\omega)(t) = X(t, \omega), \quad t \geq 0. \quad \text{Xt} \in M, t > 0$$

Then  $\bar{X}$  and  $\bar{Y}$  are random variables taking values in the metric space  $\mathcal{S}$ , and saying that  $\bar{X}$  and  $\bar{Y}$  have the same law means that  $\mathbb{P}(\bar{X} \in A) = \mathbb{P}(\bar{Y} \in A)$  for all Borel subsets  $A$  of  $\mathcal{S}$ . When this happens, we also say that the stochastic processes  $X$  and  $Y$  have the same law.

Two stochastic processes  $X$  and  $Y$  have the same *finite-dimensional distributions* if for every  $n \geq 1$  and every  $t_1 < \dots < t_n$ , the laws of  $(X_{t_1}, \dots, X_{t_n})$  and  $(Y_{t_1}, \dots, Y_{t_n})$  are equal.

Most often the topological spaces we will consider will also be metric spaces, but there will be a few occasions when we want to consider topological spaces that are not metric spaces. Suppose  $\mathcal{S} = \mathbb{R}^{[0, \infty)}$ . We furnish  $\mathcal{S}$  with the product topology.  $\mathcal{S}$  can be identified with the collection of real-valued functions on  $[0, \infty)$ , but the topology is not given by the supremum norm nor by any other metric. We use  $f$  for elements of  $\mathcal{S}$ , where  $f(t)$  is the  $t$ th coordinate of  $f$ . We call a subset  $A$  of  $\mathcal{S}$  a *cylindrical set* if there exist  $n \geq 1$ , non-negative reals  $t_1, t_2, \dots, t_n$ , and a Borel subset  $B$  of  $\mathbb{R}^n$  such that B: super-cube

$$A = \{f \in \mathcal{S} : (f(t_1), \dots, f(t_n)) \in B\}. \quad \square \quad \times R^*$$

$$Xt_1 \in I_1, \dots, Xtn \in I_n$$

The appropriate  $\sigma$ -field to use on  $\mathcal{S}$  is the one generated by the collection of cylindrical sets.

We want to generalize this notion slightly by allowing more general index sets and by allowing for the possibility of considering only a subset of the product space.

**Definition 1.1** Let  $\mathcal{U}$  be a topological space,  $T$  an arbitrary index set, and  $B$  a subset of  $\mathcal{U}_T$ . We say a set  $C$  is a *cylindrical subset* of  $B$  if there exist  $n \geq 1$ ,  $t_1, \dots, t_n \in T$ , and a Borel subset  $A$  of  $\mathbb{R}^n$  such that

$$C = \{f \in B : (f(t_1), \dots, f(t_n)) \in A\}.$$

## Exercises

- 1.1 This exercise gives an example where  $\{\mathcal{F}_t^{00}\}$  defined by (1.1) is not right continuous. Let  $\Omega = \{a, b\}$ , let  $\mathcal{F}$  be the collection of all subsets of  $\Omega$ , and let  $\mathbb{P}(\{a\}) = \mathbb{P}(\{b\}) = \frac{1}{2}$ . Define

$$X_t(\omega) = \begin{cases} 0, & t \leq 1; \\ 0, & t > 1 \text{ and } \omega = a; \\ t - 1, & t > 1 \text{ and } \omega = b. \end{cases}$$

Calculate  $\mathcal{F}_t^{00} = \sigma(X_s; s \leq t)$  and show  $\{\mathcal{F}_t^{00}\}$  is not right continuous.

- 1.2 If  $X$  is a stochastic process, let  $\mathcal{F}_t^{00}$ ,  $\mathcal{F}_t^0$ , and  $\mathcal{F}_t$  be defined by (1.1), (1.2), and (1.3), respectively. Show that  $\{\mathcal{F}_t\}$  is the minimal augmented filtration generated by  $X$ .
- 1.3 Let  $\{\mathcal{F}_t\}$  be a filtration satisfying the usual conditions and let  $\mathcal{B}[0, t]$  be the Borel  $\sigma$ -field on  $[0, t]$ . A real-valued stochastic process  $X$  is *progressively measurable* if for each  $t \geq 0$ , the map  $(s, \omega) \rightarrow X(s, \omega)$  from  $[0, t] \times \Omega$  to  $\mathbb{R}$  is measurable wrt the product  $\sigma$ -field  $\mathcal{B}[0, t] \times \mathcal{F}_t$ .
- (1) If  $X$  is adapted to  $\{\mathcal{F}_t\}$  and define

Fact.  $\sum_k X_k(\omega) 1\{\mathbf{Ak}\}(t)$

:  $[0, t] \times \mathbb{F}_s$ , if  $X_k \sim \mathbb{F}_s$  for all  $k$   
that  $t$  in  $\mathbf{Ak}$

$$X_t^{(n)}(\omega) := \sum_{k=0}^{\infty} X_{k/2^n}(\omega) 1_{[k/2^n, (k+1)/2^n)}(t),$$

staire-wise path/stoch. p.

show that  $X^{(n)}$  is progressively measurable for each  $n \geq 1$ .

(2) Use (1) to show that if  $X$  is adapted to  $\{\mathcal{F}_t\}$  and has left continuous paths, then  $X$  is progressively measurable.

(3) If  $X$  is adapted to  $\{\mathcal{F}_t\}$  and we define

$$Y_t^{(n)}(\omega) = \sum_{k=0}^{\infty} X_{(k+1)/2^n}(\omega) 1_{[k/2^n, (k+1)/2^n)}(t),$$

show that for each  $t \geq 0$ , the map  $(s, \omega) \rightarrow Y^{(n)}(s, \omega)$  from  $[0, t] \times \Omega$  to  $\mathbb{R}$  is measurable wrt  $\mathcal{B}[0, t] \times \mathcal{F}_{t+2^{-n}}$ .

(4) Show that if  $X$  is adapted to  $\{\mathcal{F}_t\}$  and has right continuous paths, then  $X$  is progressively measurable.

- 1.4 Let  $\mathcal{S} = \mathbb{R}^{[0,1]}$ , the set of functions from  $[0, 1]$  to  $\mathbb{R}$ , and let  $\mathcal{F}$  be the  $\sigma$ -field generated by the cylindrical sets. The purpose of this exercise is to show that the elements of  $\mathcal{F}$  depend on only countably many coordinates.

Let  $\mathcal{S}_0 = \{(x_1, x_2, \dots)\}$ , the set of sequences taking values in  $\mathbb{R}$ . Let  $\mathcal{F}_0$  be the  $\sigma$ -field generated by the cylindrical subsets of  $\mathbb{R}^{\mathbb{N}}$ , where  $\mathbb{N} = \{1, 2, \dots\}$ .

Show that  $B \in \mathcal{F}$  iff there exist  $t_1, t_2, \dots$  in  $[0, 1]$  and a set  $C \in \mathcal{F}_0$  such that

$$B = \{f \in \mathcal{S} : (f(t_1), f(t_2), \dots) \in C\}.$$

- 1.5 Null sets are sometimes important! Let  $\mathcal{S}$  and  $\mathcal{F}$  be as in Exercise 1.4. Show that  $D \notin \mathcal{F}$ , where

$$D = \{f \in \mathcal{S} : f \text{ is a continuous function on } [0, 1]\}.$$

- 1.6 Suppose  $X$  is a stochastic process,  $\{\mathcal{F}_t\}$  its minimal augmented filtration, and  $\mathcal{F}_{\infty} = \vee_{t \geq 0} \mathcal{F}_t$ . Suppose with probability one, the paths of  $X$  are right continuous with left limits. Let  $X_{t-} = \lim_{s < t, s \rightarrow t} X_s$ , the left-hand limit at time  $t$ , and  $\Delta X_t = X_t - X_{t-}$ , the size of the jump at time  $t$ . If

$$A = \{\exists t \geq 0 : \Delta X_t > 1\},$$

prove  $A \in \mathcal{F}_{\infty}$ .  $\text{At} := \{\triangle X_t > 1\}; A = \cup \text{At}$

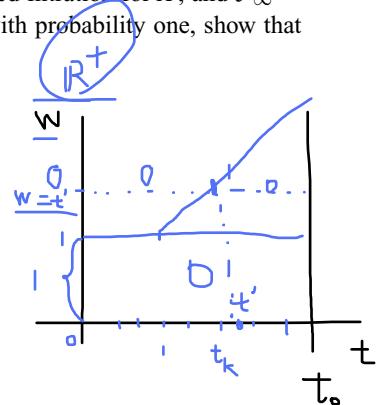
- 1.7 Suppose  $X$  is a stochastic process,  $\{\mathcal{F}_t\}$  is the minimal augmented filtration for  $X$ , and  $\mathcal{F}_{\infty} = \vee_{t \geq 0} \mathcal{F}_t$ . If the paths of  $X$  are right continuous with left limits with probability one, show that the event

$$A = \{X \text{ has continuous paths}\}$$

is in  $\mathcal{F}_{\infty}$ .

$$\text{At} := \{X \text{ has cont. paths on } [0, t]\}; A = \cup \text{At}$$

to prove that  $\text{At} \in \mathcal{F}_t$



Lemma.  $A \in \mathcal{F}_t$  iff  $A = \{(X_{t1}, X_{t2}, \dots) \in B, t_k <= t\}$

where  $B$ : Borel. ( $\mathcal{F}_t$ : generated by  $X$ )

construct an event  $w$  that:

$X_{tk}(w) = 0$ ,  $w$  not in  $A$ ,  $w = t' \Rightarrow \{0\}$  not in  $B$

$X_t(w) = 0$ ,  $w$  in  $A \Rightarrow \{0\}$  in  $B$

# 2

---

## Brownian motion

Brownian motion is by far the most important stochastic process. It is the archetype of Gaussian processes, of continuous time martingales, and of Markov processes. It is basic to the study of SDEs, financial mathematics, and filtering, to name only a few of its applications.

In this chapter we define Brownian motion and consider some of its elementary aspects. Later chapters will take up the construction of Brownian motion and properties of Brownian motion paths.

### 2.1 Definition and basic properties

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and let  $\{\mathcal{F}_t\}$  be a filtration, not necessarily satisfying the usual conditions.

**Definition 2.1**  $W_t = W_t(\omega)$  is a one-dimensional *Brownian motion* wrt  $\{\mathcal{F}_t\}$  and the probability measure  $\mathbb{P}$ , started at 0, if

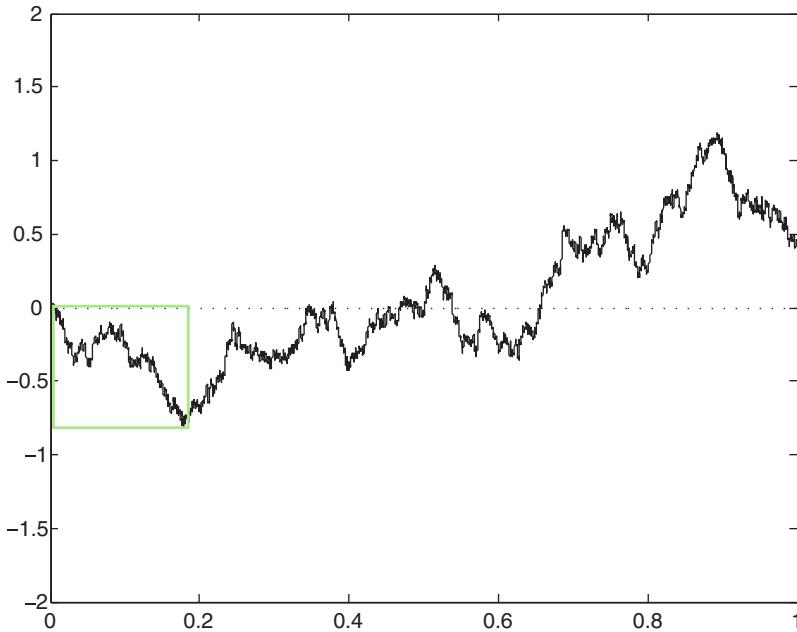
- (1)  $W_t$  is  $\mathcal{F}_t$  measurable for each  $t \geq 0$ . Wt: Stoch. p.
- (2)  $W_0 = 0$ , a.s.
- (3)  $W_t - W_s$  is a normal rv with mean 0 and variance  $t - s$  whenever  $s < t$ .
- (4)  $W_t - W_s$  is independent of  $\mathcal{F}_s$  whenever  $s < t$ . Wt: Markov chain
- (5)  $W_t$  has continuous paths.

If instead of (2) we have  $W_0 = x$ , we say we have a Brownian motion started at  $x$ . Definition 2.1(4) is referred to as the *independent increments* property of Brownian motion. The fact that  $W_t - W_s$  has the same law as  $W_{t-s}$ , which follows from Definition 2.1(3), is called the *stationary increments* property. When no filtration is specified, we assume the filtration is the filtration generated by  $W$ , i.e.,  $\mathcal{F}_t = \sigma(W_s; s \leq t)$ . Sometimes a one-dimensional Brownian motion started at 0 is called a *standard Brownian motion*.

Figure 2.1 is a simulation of a typical Brownian motion path.

We define  $d$ -dimensional Brownian motion wrt a filtration  $\{\mathcal{F}_t\}$  and started at  $x = (x_1, \dots, x_d)$  to be  $(W_t^{(1)}, \dots, W_t^{(d)})$ , where the  $W^{(i)}$  are each one-dimensional Brownian motions wrt  $\{\mathcal{F}_t\}$  started at  $x_i$ , respectively, and  $W^{(1)}, \dots, W^{(n)}$  are all independent.

The law of a Brownian motion is called *Wiener measure*. More precisely, given a Brownian motion  $W$ , we can view it as a random variable taking values in  $C[0, \infty)$ , the space of real-valued continuous functions on  $[0, \infty)$ . The law of  $W$  is the measure  $\mathbb{P}_W$  on



**Figure 2.1** Simulation of a typical Brownian motion path.

$C[0, \infty)$  defined by  $\mathbb{P}_W(A) = \mathbb{P}(W \in A)$  for all Borel subsets  $A$  of  $C[0, \infty)$ . The measure  $\mathbb{P}_W$  is Wiener measure.

There are a number of transformations one can perform on a Brownian motion that yield a new Brownian motion. The first one is called the *scaling property of Brownian motion*, or simply *scaling*.

**Proposition 2.2** *If  $W$  is a Brownian motion started at 0,  $a > 0$ , and  $Y_t = aW_{t/a^2}$ , then  $Y_t$  is a Brownian motion started at 0.*

*Proof* We use  $\mathcal{G}_t = \mathcal{F}_{t/a^2}$  for the filtration for  $Y$ . Clearly  $Y_t$  has continuous paths,  $Y_0 = 0$ , a.s., and  $Y_t$  is  $\mathcal{G}_t$  measurable. If  $s < t$ ,

$$Y_t - Y_s = a(W_{t/a^2} - W_{s/a^2})$$

is independent of  $\mathcal{F}_{s/a^2} = \mathcal{G}_s$ . Finally,  $Y_t - Y_s$  will be a normal random variable with mean zero and

$$\text{Var}(Y_t - Y_s) = a^2 \text{Var}(W_{t/a^2} - W_{s/a^2}) = a^2 \left( \frac{t}{a^2} - \frac{s}{a^2} \right) = t - s.$$

□

see Exercises 2.3 and 2.5.

Recall what it means for a finite collection of rvs to be jointly normal ; see (A.29). A stochastic process  $X$  is *Gaussian* or *jointly normal* if all its finite-dimensional distributions are jointly normal, that is, if for each  $n \geq 1$  and  $t_1 < \dots < t_n$ , the collection of rvs  $X_{t_1}, \dots, X_{t_n}$  is a jointly normal collection.

**Proposition 2.3** If  $W$  is a Brownian motion, then  $W$  is a Gaussian process.

*Proof* Suppose  $W$  is a Brownian motion and let  $0 = t_0 < t_1 < \dots < t_n$ . Define

$$Z_i := \frac{W_{t_i} - W_{t_{i-1}}}{\sqrt{t_i - t_{i-1}}}, \quad i = 1, 2, \dots, n.$$

By Definition 2.1(4),  $Z_i$  is independent of  $\mathcal{F}_{t_{i-1}}$ , and hence independent of  $Z_1, \dots, Z_{j-1}$ . By Definition 2.1(3),  $Z_i$  is a mean-zero rv with variance one. ==>

$$W_{t_j} = \sum_{i=1}^j (t_i - t_{i-1})^{1/2} Z_i, \quad j = 1, \dots, n,$$

and so  $(W_{t_1}, \dots, W_{t_n})$  is jointly normal. It follows that Brownian motion is a Gaussian process.  $\square$

Since the law of a finite collection of jointly normal rvs is determined by their means and covariances, let's calculate the covariance of  $W_s$  and  $W_t$  when  $W$  is a Brownian motion. If  $s \leq t$ , then

$$\begin{aligned} t - s &= \text{Var}(W_t - W_s) = \text{Var} W_t + \text{Var} W_s - 2 \text{Cov}(W_s, W_t) \\ &= t + s - 2 \text{Cov}(W_s, W_t) \end{aligned}$$

from Definition 2.1(2) and (3). Hence  $\text{Cov}(W_s, W_t) = s$  if  $s \leq t$ . written as

$$\text{Cov}(W_s, W_t) = s \wedge t. \quad (2.1)$$

We have the following converse.

**Theorem 2.4** If  $W$  is a process such that all the finite-dimensional distributions are jointly normal,  $\mathbb{E} W_s = 0$  for all  $s$ ,  $\text{Cov}(W_s, W_t) = s$  when  $s \leq t$ , and the paths of  $W_t$  are continuous, then  $W$  is a Brownian motion.

*Proof* For  $\mathcal{F}_t$  we take the filtration generated by  $W$ . If we take  $s = t$ , then  $\text{Var} W_t = \text{Cov}(W_t, W_t) = t$ . In particular,  $\text{Var} W_0 = 0$ , and since  $\mathbb{E} W_0 = 0$ , then  $W_0 = 0$ , a.s. We have

$$\begin{aligned} \text{Var}(W_t - W_s) &= \text{Var} W_t - 2 \text{Cov}(W_s, W_t) + \text{Var} W_s \\ &= t - 2s + s = t - s. \end{aligned}$$

We have thus established all the parts of Definition 2.1 except for the independence of  $W_t - W_s$  from  $\mathcal{F}_s$ .

If  $r \leq s < t$ , then

$$\text{Cov}(W_t - W_s, W_r) = 0,$$

and so  $W_t - W_s$  is independent of  $W_r$  by Proposition A.55. This shows that  $W_t - W_s$  is independent of  $\mathcal{F}_s$ .  $\square$

We now look at two results that are more technical. These should only be skimmed on the first reading of the book: read the statements, but not the proofs. The first result says that if  $W$  is a Brownian motion wrt the filtration generated by  $W$ , then it is also a Brownian motion wrt the minimal augmented filtration.

**Proposition 2.5** Let  $W_t$  be a Brownian motion wrt  $\{\mathcal{F}_t^{00}\}$ , where  $\mathcal{F}_t^{00} = \sigma(\mathcal{F}_t^0 \cup \mathcal{N})$ , and  $\mathcal{F}_t = \cap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}^0$ .  
 $\sigma(W_s; s \leq t)$ . Let  $\mathcal{N}$  be the collection of null sets,  $\mathcal{F}_t^0 = \sigma(\mathcal{F}_t^{00} \cup \mathcal{N})$ , and  $\mathcal{F}_t = \cap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}^0$ .

- (1)  $W$  is a Brownian motion wrt the filtration  $\{\mathcal{F}_t\}$ .
- (2)  $\mathcal{F}_t = \mathcal{F}_t^0$  for each  $t$ .

*Proof* (1) The only property we need to check is Definition 2.1(4). If  $f$  is a continuous bounded function on  $\mathbb{R}$ ,  $A \in \mathcal{F}_s^{00}$ , and  $s < t$ , then because  $W$  is a Brownian motion wrt  $\{\mathcal{F}_t^{00}\}$ , the independent increments property shows that

$$\mathbb{E}[f(W_t - W_s); A] = \mathbb{E}[f(W_t - W_s)] = \mathbb{E}[f(W_t - W_s)] \mathbb{P}(A). \quad (2.2)$$

If  $A$  is such that  $A \setminus B$  and  $B \setminus A$  are null sets for some  $B \in \mathcal{F}_s^{00}$ , it is easy to see that (2.2) continues to hold. By linearity, it also holds if  $A$  is a finite disjoint union of such sets. If  $\mathcal{C}_1$  is the collection of subsets of  $\mathcal{F}_s^0$  that are finite disjoint unions of such sets, then  $\mathcal{C}_1$  is an algebra of subsets of  $\mathcal{F}_s^0$ . Let  $\mathcal{M}_1$  be the collection of subsets of  $\mathcal{F}_s^0$  for which (2.2) holds. It is readily checked that  $\mathcal{M}_1$  is a monotone class. By the monotone class theorem (Theorem B.2),  $\mathcal{M}_1$  is equal to the smallest  $\sigma$ -field containing  $\mathcal{C}_1$ , which is  $\mathcal{F}_s^0$ . Therefore (2.2) holds for all  $A \in \mathcal{F}_s^0$ . /

Now suppose  $A \in \mathcal{F}_s = \mathcal{F}_{s+}^0$ . Then for each  $\varepsilon > 0$ ,  $A \in \mathcal{F}_{s+\varepsilon}^0$ , and so using (2.2) with  $s$  replaced by  $s + \varepsilon$  and  $t$  replaced by  $t + \varepsilon$ , we have

$$\mathbb{E}[f(W_{t+\varepsilon} - W_{s+\varepsilon}); A] = \mathbb{E}[f(W_{t+\varepsilon} - W_{s+\varepsilon})] \mathbb{P}(A). \quad (2.3)$$

Letting  $\varepsilon \rightarrow 0$  and using the facts that  $f$  is bounded and continuous and  $W$  has continuous paths, the DCT ==>

$$\mathbb{E}[f(W_t - W_s); A] = \mathbb{E}[f(W_t - W_s)] \mathbb{P}(A). \quad (2.4)$$

This equation holds whenever  $f$  is continuous and  $A \in \mathcal{F}_s$ . By a limit argument, (2.4) holds whenever  $f$  is the indicator of a Borel subset of  $\mathbb{R}$ . That says that  $W_t - W_s$  and  $\mathcal{F}_s$  are independent.

(2) Fix  $t$  and choose  $t_0 > t$ . Let  $\mathcal{M}_2$  be the collection of subsets of  $\mathcal{F}_{t_0}^{00}$  whose conditional expectation wrt  $\mathcal{F}_t$  is  $\mathcal{F}_t^0$  measurable, that is,  $A \in \mathcal{M}_2$  if  $A \in \mathcal{F}_{t_0}^{00}$  and  $\mathbb{E}[1_A | \mathcal{F}_t]$  is  $\mathcal{F}_t^0$  measurable. Let  $\mathcal{C}_2$  be the collection of events  $A$  for which there exist  $n \geq 1$ ,  $0 \leq s_0 < s_1 < \dots < s_n \leq t_0$  with  $t = s_i$ , and Borel subsets  $B_1, \dots, B_n$  of  $\mathbb{R}$  such that

$$A = (W_{s_1} - W_{s_0} \in B_1, \dots, W_{s_n} - W_{s_{n-1}} \in B_n).$$

Suppose  $A$  is of this form, and suppose  $t = s_i$ . Then by the independence result that we proved in (1),

$$\mathbb{E}[1_A | \mathcal{F}_t] = 1_{(W_{s_1} - W_{s_0} \in B_1, \dots, W_{s_i} - W_{s_{i-1}} \in B_i)} \mathbb{P}(W_{s_{i+1}} - W_{s_i} \in B_{i+1}, \dots, W_{s_n} - W_{s_{n-1}} \in B_n),$$

which is  $\mathcal{F}_t^0$  measurable. Thus  $\mathcal{C}_2 \subset \mathcal{M}_2$ . Finite unions of sets in  $\mathcal{C}_2$  form an algebra of subsets of  $\mathcal{F}_{t_0}^{00}$  that generate  $\mathcal{F}_t^{00}$ . It is easy to check that  $\mathcal{M}_2$  is a monotone class, so by the monotone class theorem,  $\mathcal{M}_2 = \mathcal{F}_t^{00}$ . By linearity and taking monotone limits, if  $Y$  is non-negative and  $\mathcal{F}_t^{00}$  measurable, then  $\mathbb{E}[Y | \mathcal{F}_t]$  is  $\mathcal{F}_t^0$  measurable.

To finish, suppose  $A \in \mathcal{F}_t$ . Then since  $t < t_0$ , we see that  $A \in \mathcal{F}_{t_0}^0$ . By Exercise 2.7, there exists  $Y \in \mathcal{F}_{t_0}^{00}$  such that  $1_A = Y$ , a.s. Then  $\mathbb{E}[Y | \mathcal{F}_t]$  is  $\mathcal{F}_t^0$  measurable. Since  $\mathcal{F}_t^0$  contains all the null sets,  $1_A = \mathbb{E}[1_A | \mathcal{F}_t]$  is also  $\mathcal{F}_{t_0}$  measurable, or  $A \in \mathcal{F}_{t_0}$ . □

□

**The question:** if  $W$  and  $W'$  are both Brownian motions, do they have all the same

properties? revisit the example of Chapter 1 where  $\Omega = [0, 1]$ ,  $\mathcal{F}$  is the Borel  $\sigma$ -field on  $[0, 1]$ ,  $\mathbb{P}$  is Lebesgue measure on  $[0, 1]$ ,  $X(t, \omega) = 0$  for all  $t$  and  $\omega$ , and  $Y(t, \omega)$  is 1 if  $t = \omega$  and 0 otherwise.

For each  $t$ ,  $\mathbb{P}(X_t = Y_t) = 1$ , so  $X$  and  $Y$  have the same finite-dimensional distributions. However, if

$$A = \{f : f \text{ is not a continuous function on } [0, 1]\},$$

then  $(X \in A)$  is a null set but  $(Y \in A)$  is not. Even though  $X$  and  $Y$  have the same finite-dimensional distributions,  $X$  has continuous paths but  $Y$  does not.

To rephrase our question, is it true that  $\mathbb{P}(W \in A) = \mathbb{P}(W' \in A)$  for every Borel subset  $A$  of  $C[0, \infty)$ ? We know  $W$  and  $W'$  have the same finite-dimensional distributions because each is jointly normal with zero means and  $\text{Cov}(W_s, W_t) = s \wedge t = \text{Cov}(W'_s, W'_t)$ . The fact that the answer to our question is yes then comes from the following theorem. We look at  $C[0, t_0]$  instead of  $C[0, \infty)$  for the sake of simplicity.

**Theorem 2.6** Let  $t_0 > 0$  and let  $X, Y$  be rv's taking values in  $C[0, t_0]$  which have the same finite-dimensional distributions. Then the laws of  $X$  and  $Y$  are equal.

*Proof* Let  $\mathcal{M}$  be the collection of Borel subsets  $A$  of  $C[0, t_0]$  for which  $\mathbb{P}(X \in A)$  equals  $\mathbb{P}(Y \in A)$ . We will show that  $\mathcal{M}$  is a monotone class and then use the monotone class theorem to show that  $\mathcal{M} =$  the Borel  $\sigma$ -field on  $C[0, t_0]$ .

First, let  $\mathcal{C}$  be the collection of all cylindrical subsets of  $C[0, t_0]$  (defined by Definition 1.1). Since the finite-dimensional distributions of  $X$  and  $Y$  are equal, then  $\mathcal{M}$  contains  $\mathcal{C}$ . It is easy to check that  $\mathcal{C}$  is an algebra of subsets of  $C[0, t_0]$ . If  $A_1 \supset A_2 \supset \dots$  are elements of  $\mathcal{M}$ , then

$$\mathbb{P}(X \in \cap_n A_n) = \lim_n \mathbb{P}(X \in A_n) = \lim_n \mathbb{P}(Y \in A_n) = \mathbb{P}(Y \in \cap_n A_n)$$

since  $\mathbb{P}$  is a finite measure. Therefore  $\cap_n A_n \in \mathcal{M}$ . A very similar argument shows that if  $A_1 \subset A_2 \subset \dots$  are elements of  $\mathcal{M}$ , then  $\cup_n A_n \in \mathcal{M}$ . Therefore  $\mathcal{M}$  is a monotone class. By the monotone class theorem,  $\mathcal{M}$  contains the smallest  $\sigma$ -field containing  $\mathcal{C}$ . We will show that  $\mathcal{M}$  contains all the open sets; then  $\mathcal{M}$  will contain the smallest  $\sigma$ -field containing the open sets, and we will be done.

Since  $C[0, t_0]$  is separable, every open set is the countable union of open balls. Because  $\mathcal{M}$  is a  $\sigma$ -field, it suffices to show that  $\mathcal{M}$  contains the open balls in  $C[0, t_0]$ , that is, all sets of the form

$$B(f_0, r) = \{f \in C[0, t_0] : \sup_{0 \leq t \leq t_0} |f(t) - f_0(t)| < r\}$$

where  $r > 0$  and  $f_0 \in C[0, t_0]$ . For each  $m$  and  $n$ ,

$$\{f \in C[0, t_0] : \sup_{0 \leq k \leq 2^n t_0} |f(k/2^n) - f_0(k/2^n)| \leq r - (1/m)\}$$

is a set in  $\mathcal{C}$ , and so is in  $\mathcal{M}$ . As  $n \rightarrow \infty$ , these sets decrease to

$$D_m = \{f \in C[0, t_0] : \sup_{0 \leq t \leq t_0} |f(t) - f_0(t)| \leq r - (1/m)\},$$

since all the functions we are considering are continuous. Finally,  $D_m$  increases to  $B(f_0, r)$  as  $m \rightarrow \infty$ , so  $B(f_0, r)$  is in  $\mathcal{M}$  as desired.  $\square$

## Exercises

- 2.1 Suppose  $W$  is a Brownian motion on  $[0, 1]$ . Let

$$Y_t = W_{1-t} - W_1.$$

Show that  $Y_t$  is a Brownian motion on  $[0, 1]$ .

- 2.2 This exercise shows that the projection of a  $d$ -dimensional Brownian motion onto a hyperplane yields a one-dimensional Brownian motion. Suppose  $(W_t^{(1)}, \dots, W_t^{(d)})$  is a  $d$ -dimensional Brownian motion started from 0 and  $\lambda_1, \dots, \lambda_d \in \mathbb{R}$  with  $\sum_{i=1}^d \lambda_i^2 = 1$ . Show that  $X_t = \sum_{i=1}^d \lambda_i W_t^{(i)}$  is a one-dimensional Brownian motion started from 0.
- 2.3 This exercise shows that rotating a Brownian motion about the origin yields another Brownian motion. Let  $W$  be a  $d$ -dimensional Brownian motion started at 0 and let  $A$  be a  $d \times d$  orthogonal matrix, that is,  $A^{-1} = A^T$ . Show that  $Y_t = AW_t$  is again a  $d$ -dimensional Brownian motion.
- 2.4 Here is a converse to Exercise 2.2: roughly speaking, if all the projections of a  $d$ -dimensional process  $X$  onto hyperplanes are one-dimensional Brownian motions, then  $X$  is a  $d$ -dimensional Brownian motion.

Suppose  $(X_t^1, \dots, X_t^d)$  is a  $d$ -dimensional continuous process, i.e., one taking values in  $\mathbb{R}^d$ . Let  $\{\mathcal{F}_t\}$  be the minimal augmented filtration generated by  $X$ . Suppose that whenever  $\lambda_1, \dots, \lambda_d \in \mathbb{R}$  with  $\sum_{i=1}^d \lambda_i^2 = 1$ , then  $\sum_{i=1}^d \lambda_i X_t^i$  is a one-dimensional Brownian motion started at 0 with respect to the filtration  $\{\mathcal{F}_t\}$ .

(1) If  $u = (u_1, \dots, u_d)$ , let  $\|u\| = (\sum u_j^2)^{1/2}$  and let  $\lambda_j = u_j/\|u\|$ . Calculate

$$\mathbb{E} \exp \left( i \sum_{j=1}^d u_j X_t^j \right) = \mathbb{E} \exp \left( i \|u\| \sum_{j=1}^d \lambda_j X_t^j \right),$$

the joint characteristic function of  $X_t$ .

(2) If  $t_0 < t_1 < \dots < t_n$ , use independence and (1) to calculate

$$\mathbb{E} \exp \left( i \sum_{k=0}^{n-1} \sum_{j=1}^d u_j^k (X_{t_{k+1}}^j - X_{t_k}^j) \right).$$

(3) Prove that  $(X_t^1, \dots, X_t^d)$  is a  $d$ -dimensional Brownian motion started from 0.

(Some care is needed with the filtrations. If we only know that  $Y^\lambda = \sum_i \lambda_i X^i$  is a Brownian motion wrt the filtration generated by  $Y^\lambda$  for each  $\lambda = (\lambda_1, \dots, \lambda_d)$ , the assertion is not true.

See Revuz and Yor (1999), Exercise I.1.19.)

- 2.5 Let  $W_t$  be a Brownian motion and suppose

$$\lim_{t \rightarrow \infty} W_t/t = 0, \quad \text{a.s.} \tag{2.5}$$

Let  $Z_t = tW_{1/t}$  if  $t > 0$  and set  $Z_0 = 0$ . (*time inversion*.) Show that  $Z$  is a Brownian motion. (We will see later that the assumption (2.5) is superfluous; see Theorem 7.2.)

- 2.6 Let  $X$  and  $Y$  be two independent Brownian motions started at 0 and let  $t_0 > 0$ . Let

$$Z_t = \begin{cases} X_t, & t \leq t_0, \\ X_{t_0} + Y_{t-t_0}, & t > t_0. \end{cases}$$

Prove that  $Z$  is also a Brownian motion.

- 2.7 Let  $\mathcal{F}_t^{00}$  and  $\mathcal{F}_t^0$  be defined as in (1.1) and (1.2). Prove that if  $X$  is  $\mathcal{F}_t^0$  measurable, there exists  $Z$  such that  $Z$  is  $\mathcal{F}_t^{00}$  measurable and  $X = Z$ , a.s.
- 2.8 The *symmetric difference* of two sets  $A$  and  $B$  is defined by  $A \Delta B = (A \setminus B) \cup (B \setminus A)$ . Prove that  $\mathcal{F}_t^0 = \{A \subset \Omega : A \Delta B \in \mathcal{N} \text{ for some } B \in \mathcal{F}_t^{00}\}$ .

### Notes

Brownian motion is named for Robert Brown, a botanist who observed the erratic motion of colloidal particles in suspension in the 1820s. Brownian motion was used by Bachelier in 1900 in his PhD thesis to model stock prices and was the subject of an important paper by Einstein in 1905. The rigorous mathematical foundations for Brownian motion were first given by Wiener in 1923.

- 2.7+ Prove that  $f$  is  $\mathcal{F}$  measurable (comp. of  $F_0$ ) iff there exists  $g$  such that  $g$  is  $\mathcal{F}_0$  measurable and  $f=g$ , a.s.

# 3

---

## Martingales

Although discrete-time martingales are useful in a first course on probability, they are nowhere near as useful as continuous-time martingales are in the study of stochastic processes. The whole theory of stochastic integrals and SDEs is based on martingales indexed by times  $t \in [0, \infty)$ . After giving the definition and some examples, we extend Doob's inequalities, the optional stopping theorem , and the martingale convergence theorem to continuous -time martingales . We then derive some estimates for Brownian motion using martingale techniques.

### 3.1 Definition and examples

Let  $\{\mathcal{F}_t\}$  be a filtration, not necessarily satisfying the usual conditions.

**Definition 3.1**  $M_t$  is a continuous -time *martingale* wrt the filtration  $\{\mathcal{F}_t\}$  and the probability measure  $\mathbb{P}$ , if  $M_t$ : Stoch. p.

- (1)  $\mathbb{E} |M_t| < \infty$  for each  $t$ ; (integrable)
- (2)  $\mathbb{E} [M_t | \mathcal{F}_s] = M_s$ , a.s., if  $s < t$ . (stab.)       $\mathbb{E}(M_t; A) = \mathbb{E}(M_s; A), A: \mathcal{F}_s$

If in part (2) “=” is replaced by “ $\geq$ ,” then  $M_t$  is a *submartingale*, and if it is replaced by “ $\leq$ ,” then we have a *supermartingale*.

Taking expectations in Definition 3.1(2), we see that if  $s < t$ , then  $\mathbb{E} M_s \leq \mathbb{E} M_t$  is  $M$  is a submartingale and  $\mathbb{E} M_s \geq \mathbb{E} M_t$  if  $M$  is a supermartingale. Thus submartingales tend to increase, on average, and supermartingales tend to decrease, on average.

There are many martingales associated with Brownian motion. Here are three examples.

**Example 3.2** Let  $M_t = W_t$ , where  $W_t$  is a Brownian motion. Then  $M_t$  is a martingale. To verify Definition 3.1(2), we write

$$\mathbb{E} [M_t | \mathcal{F}_s] = M_s + \mathbb{E} [W_t - W_s | \mathcal{F}_s] = M_s + \mathbb{E} [W_t - W_s] = M_s$$

$W_t - W_s$  is independent of  $\mathcal{F}_s$ .

**Example 3.3** Let  $M_t = W_t^2 - t$ , where  $W_t$  is a Brownian motion. To show  $M_t$  is a martingale, we write

$$\begin{aligned}\mathbb{E}[M_t | \mathcal{F}_s] &= \mathbb{E}[(W_t - W_s + W_s)^2 | \mathcal{F}_s] - t \\ &= W_s^2 + \mathbb{E}[(W_t - W_s)^2 | \mathcal{F}_s] + 2\mathbb{E}[W_s(W_t - W_s) | \mathcal{F}_s] - t \\ &= W_s^2 + \mathbb{E}[(W_t - W_s)^2] + 2W_s\mathbb{E}[W_t - W_s | \mathcal{F}_s] - t \\ &= W_s^2 + \mathbb{E}[(W_t - W_s)^2] + 2W_s\mathbb{E}[W_t - W_s] - t \\ &= W_s^2 + (t - s) - t = M_s.\end{aligned}$$

**Example 3.4** Again let  $W_t$  be a Brownian motion, let  $a \in \mathbb{R}$ , and let  $M_t = e^{aW_t - a^2t/2}$ . Since  $W_t - W_s$  is normal with mean zero and variance  $t - s$ , we know  $\mathbb{E}e^{a(W_t - W_s)} = e^{a^2(t-s)/2}$ ; see (A.6). Then

$$\begin{aligned}\mathbb{E}[M_t | \mathcal{F}_s] &= e^{-a^2t/2}e^{aW_s}\mathbb{E}[e^{a(W_t - W_s)} | \mathcal{F}_s] \\ &= e^{-a^2t/2}e^{aW_s}\mathbb{E}[e^{a(W_t - W_s)}] \\ &= e^{-a^2t/2}e^{aW_s}e^{a^2(t-s)/2} = M_s.\end{aligned}$$

We give one more example of a martingale, although not one derived from Brownian motion.

**Example 3.5** Recall that given a filtration  $\{\mathcal{F}_t\}$ , each  $\mathcal{F}_t$  is contained in  $\mathcal{F}$ . Let  $X$  be an integrable  $\mathcal{F}$  measurable rv, and let  $M_t = \mathbb{E}[X | \mathcal{F}_t]$ . Then

$$\mathbb{E}[M_t | \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[X | \mathcal{F}_t] | \mathcal{F}_s] = \mathbb{E}[X | \mathcal{F}_s] = M_s,$$

and  $M$  is a martingale.

### 3.2 Doob's inequalities

We derive the analogs of Doob's inequalities in the stochastic process context.

**Theorem 3.6** Suppose  $M_t$  is a martingale or non-negative submartingale with paths that are right continuous with left limits. Then

$$(1) \quad \mathbb{P}(\sup_{s \leq t} |M_s| \geq \lambda) \leq \mathbb{E}|M_t|/\lambda. \quad \text{vs.t } \{ |M_s| > \lambda \}$$

(2) If  $1 < p < \infty$ , then

$$\mathbb{E}[\sup_{s \leq t} |M_s|]^p \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_t|^p.$$

*Proof* We will do the case where  $M_t$  is a martingale, the submartingale case being nearly identical. Let  $\mathcal{D}_n = \{kt/2^n : 0 \leq k \leq 2^n\}$ . If we set  $N_k^{(n)} = M_{kt/2^n}$  and  $\mathcal{G}_k^{(n)} = \mathcal{F}_{kt/2^n}$ , it is clear that  $\{N_k^{(n)}\}$  is a discrete-time martingale wrt  $\{\mathcal{G}_k^{(n)}\}$ . Let

Discretization trick

$$A_n = \{ \sup_{s \leq t, s \in \mathcal{D}_n} |M_s| > \lambda \}.$$

Dn={0,t1,...,tn,t}: [0,t]-partition

Nk=Mtk ~ Ftk: martingale

By Doob's inequality for discrete-time martingales (Theorem A.32),

$$\mathbb{P}(A_n) = \mathbb{P}\left(\max_{k \leq 2^n} |N_k^{(n)}| > \lambda\right) \leq \frac{\mathbb{E}|N_{2^n}^{(n)}|}{\lambda} = \frac{\mathbb{E}|M_t|}{\lambda}. \quad /$$

Note that the  $A_n$  are increasing, and since  $M_t$  is right continuous,

$$\cup_n A_n = \{\sup_{s \leq t} |M_s| > \lambda\}.$$

Then

$$\mathbb{P}(\sup_{s \leq t} |M_s| > \lambda) = \mathbb{P}(\cup_n A_n) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n) \leq \mathbb{E}|M_t|/\lambda.$$

If we apply this with  $\lambda$  replaced by  $\lambda - \varepsilon$  and let  $\varepsilon \rightarrow 0$ , we obtain (1).

The proof of (2) is similar. By Doob's inequality for discrete-time martingales (Theorem A.33),

$$\mathbb{E}\left[\sup_{k \leq 2^n} |N_k^{(n)}|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|N_{2^n}^{(n)}|^p = \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_t|^p.$$

Since  $\sup_{k \leq 2^n} |N_k^{(n)}|^p$  increases to  $\sup_{s \leq t} |M_s|^p$  by the right continuity of  $M$ , (2) follows by Fatou's lemma.  $\square$

### 3.3 Stopping times

Throughout this section we suppose we have a filtration  $\{\mathcal{F}_t\}$  satisfying the usual conditions.

**Definition 3.7** A random variable  $T : \Omega \rightarrow [0, \infty]$  is a *stopping time* if for all  $t$ ,  $(T \leq t) \in \mathcal{F}_t$ . We say  $T$  is a finite stopping time if  $T < \infty$ , a.s. We say  $T$  is a bounded stopping time if there exists  $K \in [0, \infty)$  such that  $T \leq K$ , a.s.

Note that  $T$  can take the value infinity. Stopping times are also known as *optional times*.

Given a stochastic process  $X$ ,  $X_T(\omega) := X(T(\omega), \omega)$ : for each  $\omega$  we evaluate  $t = T(\omega)$  and then look at  $X(\cdot, \omega)$  at this time.

**Proposition 3.8** Suppose  $\mathcal{F}_t$  satisfies the usual conditions. Then

- (1)  $T$  is a stopping time iff  $(T \leq t) \in \mathcal{F}_t$  for all  $t$ .  $T = t \in \mathcal{F}_t$
- (2) If  $T = t$ , a.s., then  $T$  is a stopping time.
- (3) If  $S$  and  $T$  are stopping times, then so are  $S \vee T$  and  $S \wedge T$ .
- (4) If  $T_n$ ,  $n = 1, 2, \dots$ , are stopping times with  $T_1 \leq T_2 \leq \dots$ , then so is  $\sup_n T_n$ .
- (5) If  $T_n$ ,  $n = 1, 2, \dots$ , are stopping times with  $T_1 \geq T_2 \geq \dots$ , then so is  $\inf_n T_n$ .
- (6) If  $s \geq 0$  and  $S$  is a stopping time, then so is  $S + s$ .

$T \leq t$  contains in  $T < s$ :  $F_s < t$

For a Borel measurable set  $A$ , let **Hitting time**

$$T_A := \inf\{t > 0 : X_t \in A\}. \quad X(TA) : \text{cl}(A) \quad (3.1)$$

$$\begin{aligned} TA < t &= \exists q < t, X_t \in A \\ &= \cup_{q < t} \{X_t \in A\} \end{aligned}$$

At: increasing events  $=>$  At+  $= \cap_{s > t} A_s$ :  $F_t+$

$\square$

**Proposition 3.9** Suppose  $\mathcal{F}_t$  satisfies the usual conditions and  $X_t$  has continuous paths.

- (1) If  $A$  is open, then  $T_A$  is a stopping time.
- (2) If  $A$  is closed, then  $T_A$  is a stopping time.

*Proof* (1)  $(T_A < t) = \bigcap_{q \in \mathbb{Q}_+, q < t} (X_q \in A)$ , where  $\mathbb{Q}_+$  denotes the set of non-negative rationals. Since  $(X_q \in A) \in \mathcal{F}_q \subset \mathcal{F}_t$ , then  $(T_A < t) \in \mathcal{F}_t$ .

(2) Let  $A_n = \{x : \text{dist}(x, A) < 1/n\}$ , the set of points within a distance  $1/n$  from  $A$ . Each  $A_n$  is open and thus by (1),  $T_{A_n}$  is a stopping time. Moreover, the  $A_n$  decrease, so the  $T_{A_n}$  increase. Let  $T = \sup_n T_{A_n}$ , a stopping time by Proposition 3.8(4). Since  $A \subset A_n$ , then  $T_A \geq T_{A_n}$ , so  $T_A \geq T$ . Because  $X$  has continuous paths, on  $(T < \infty)$ ,  $X_T = \lim_n X_{T_{A_n}}$ . If  $n \geq m$ , then  $X(T_{A_n}) \in \overline{A_n} \subset \overline{A_m}$ . Therefore  $X_T \in \overline{A_m}$  for each  $m$ . Since  $A = \bigcap_m \overline{A_m}$ , then  $X_T \in A$ . Therefore  $T_A \leq T$ , and hence  $T = T_A$ .  $\square$

It is true that under the hypotheses of the preceding proposition,  $T_A$  is a stopping time for every Borel set  $A$ , but that is much harder to prove; see Section 16.2.

approximate stopping times from the right. If  $T$  is a finite stopping time, define

$$\begin{aligned} T' := t\{k+1\}, \quad tk \leq T < t\{k+1\} \\ T': \text{stopping time} \quad T_n(\omega) = (k+1)/2^n \quad \text{if } k/2^n \leq T(\omega) < (k+1)/2^n. \end{aligned} \quad (3.2)$$

$T' = tk \sim Ftk$  Exercise 3.5 asks you to prove that the  $T_n$  are stopping times decreasing to  $T$ .

Define

$$\begin{aligned} &= \cap \{A \in \mathcal{F}: A \cap (T \leq t) \in \mathcal{F}_t\} \\ \mathcal{F}_T = \{A \in \mathcal{F} : \text{for each } t > 0, A \cap (T \leq t) \in \mathcal{F}_t\}. \end{aligned} \quad (3.3)$$

This definition of  $\mathcal{F}_T$ , which is supposed to be the collection of events that are “known” by time  $T$ , is not very intuitive. But it turns out that this definition works well in applications.

**Proposition 3.10** Suppose  $\{\mathcal{F}_t\}$  is a filtration satisfying the usual conditions.

- (1)  $\mathcal{F}_T$  is a  $\sigma$ -field.
- (2) If  $S \leq T$ , then  $\mathcal{F}_S \subset \mathcal{F}_T$ .
- (3) If  $\mathcal{F}_{T+} = \cap_{\varepsilon > 0} \mathcal{F}_{T+\varepsilon}$ , then  $\mathcal{F}_{T+} = \mathcal{F}_T$ .
- (4) If  $X_t$  has right-continuous paths, then  $X_T$  is  $\mathcal{F}_T$  measurable.

*Proof* If  $A \in \mathcal{F}_T$ , then  $A^c \cap (T \leq t) = (T \leq t) \setminus [A \cap (T \leq t)] \in \mathcal{F}_t$ , so  $A^c \in \mathcal{F}_T$ . The rest of the proof of (1) is easy.

Suppose  $A \in \mathcal{F}_S$  and  $S \leq T$ . Then  $A \cap (T \leq t) = [A \cap (S \leq t)] \cap (T \leq t)$ . We have  $A \cap (S \leq t) \in \mathcal{F}_t$  because  $A \in \mathcal{F}_S$ , while  $(T \leq t) \in \mathcal{F}_t$  because  $T$  is a stopping time. Therefore  $A \cap (T \leq t) \in \mathcal{F}_t$ , which proves (2).

For (3), if  $A \in \mathcal{F}_{T+}$ , then  $A \in \mathcal{F}_{T+\varepsilon}$  for every  $\varepsilon$ , and so  $A \cap (T + \varepsilon \leq t) \in \mathcal{F}_t$  for all  $t$ . Hence  $A \cap (T \leq t - \varepsilon) \in \mathcal{F}_t$  for all  $t$ , or equivalently  $A \cap (T \leq t) \in \mathcal{F}_{t-\varepsilon}$  for all  $t$ . This is true for all  $\varepsilon$ , so  $A \cap (T \leq t) \in \mathcal{F}_{t-} = \mathcal{F}_t$ . This says  $A \in \mathcal{F}_T$ .

(4) Define  $T_n$  by (3.2). Note

$$(X_{T_n} \in B) \cap (T_n = k/2^n) = (X_{k/2^n} \in B) \cap (T_n = k/2^n) \in \mathcal{F}_{k/2^n}.$$

Since  $T_n$  only takes values in  $\{k/2^n : k \geq 0\}$ , we conclude  $(X_{T_n} \in B) \cap (T_n \leq t) \in \mathcal{F}_t$  and so  $(X_{T_n} \in B) \in \mathcal{F}_{T_n} \subset \mathcal{F}_{T+1/2^n}$ .

Hence  $X_{T_n}$  is  $\mathcal{F}_{T+1/2^n}$  measurable. If  $n \geq m$ , then  $X_{T_n}$  is measurable wrt  $\mathcal{F}_{T+1/2^n} \subset \mathcal{F}_{T+1/2^m}$ . Since  $X_{T_n} \rightarrow X_T$ , then  $X_T$  is  $\mathcal{F}_{T+1/2^m}$  measurable for each  $m$ . Therefore  $X_T$  is measurable wrt  $\mathcal{F}_{T+} = \mathcal{F}_T$ .  $\square$

### 3.4 The optional stopping theorem

We will need Doob's optional stopping theorem for continuous-time martingales. An example to keep in mind is  $M_t = W_{t \wedge t_0}$ , where  $W$  is a Brownian motion and  $t_0$  is some fixed time. Exercise 3.12 is a version of the optional stopping time with slightly weaker hypotheses that is often useful.

**Theorem 3.11** *Let  $\{\mathcal{F}_t\}$  be a filtration satisfying the usual conditions. If  $M_t$  is a martingale or non-negative submartingale whose paths are right continuous,  $\sup_{t \geq 0} \mathbb{E} M_t^2 < \infty$ , and  $T$  is a finite stopping time, then  $\mathbb{E} M_T \geq \mathbb{E} M_0$ .*

*Proof* We do the submartingale case, the martingale case being very similar. By Doob's inequality (Theorem 3.6(1)),

$$\mathbb{E} [\sup_{s \leq t} M_s^2] \leq 4\mathbb{E} M_t^2.$$

Letting  $t \rightarrow \infty$ , we have  $\mathbb{E} [\sup_{t \geq 0} M_t^2] < \infty$  by Fatou's lemma.  $\square$

Let us first suppose that  $T < K$ , a.s., for some real number  $K$ . Define  $T_n$  by (3.2). Let  $N_k^{(n)} = M_{k/2^n}$ ,  $\mathcal{G}_k^{(n)} = \mathcal{F}_{k/2^n}$ , and  $S_n = 2^n T_n$ . By Doob's optional stopping theorem applied to the submartingale  $N_k^{(n)}$ , we have

$$\mathbb{E} M_0 = \mathbb{E} N_0^{(n)} \leq \mathbb{E} N_{S_n}^{(n)} = \mathbb{E} M_{T_n}.$$

Since  $M$  is right continuous,  $M_{T_n} \rightarrow M_T$ , a.s. The rvs  $|M_{T_n}|$  are bounded by  $1 + \sup_{t \geq 0} M_t^2$ , so by dominated convergence,  $\mathbb{E} M_{T_n} \rightarrow \mathbb{E} M_T$ .

We apply the above to the stopping time  $T \wedge K$  to get  $\mathbb{E} M_{T \wedge K} \geq \mathbb{E} M_0$ . The rvs  $M_{T \wedge K}$  are bounded by  $1 + \sup_{t \geq 0} M_t^2$ , so by dominated convergence, we get  $\mathbb{E} M_T \geq \mathbb{E} M_0$  when we let  $K \rightarrow \infty$ .  $\square$

### 3.5 Convergence and regularity

We present the continuous-time version of Doob's martingale convergence theorem. We will see that not only do we get limits as  $t \rightarrow \infty$ , but also a regularity result.

Let  $\mathcal{D}_n = \{k/2^n : k \geq 0\}$ ,  $\mathcal{D} = \bigcup_n \mathcal{D}_n$ .

**Theorem 3.12** *Let  $\{M_t : t \in \mathcal{D}\}$  be either a martingale, a submartingale, or a supermartingale wrt  $\{\mathcal{F}_t : t \in \mathcal{D}\}$  and suppose  $\sup_{t \in \mathcal{D}} \mathbb{E} |M_t| < \infty$ . Then*

(1)  $\lim_{t \rightarrow \infty} M_t$  exists, a.s.

(2) With probability one  $M_t$  has left and right limits along  $\mathcal{D}$ .

The second conclusion says that except for a null set, if  $t_0 \in [0, \infty)$ , then both  $\lim_{t \in \mathcal{D}, t \uparrow t_0} M_t$  and  $\lim_{t \in \mathcal{D}, t \downarrow t_0} M_t$  exist and are finite. The null set does not depend on  $t_0$ .

*Proof* Martingales are also submartingales and if  $M_t$  is a supermartingale, then  $-M_t$  is a submartingale, so we may wlog restrict our attention to submartingales.

$$\lim_{t \rightarrow 0} M_t(w) \text{ iff } P(\cup_m n > m \{ |M_t - M| < 1/k \}) \rightarrow 0 +$$

By Doob's inequality (Theorem 3.6(1)),

$$\mathbb{P}(\sup_{t \in \mathcal{D}_n, t \leq n} |M_t| > \lambda) \leq \frac{1}{\lambda} \mathbb{E} |M_n|.$$

Letting  $n \rightarrow \infty$  and using Fatou's lemma,

$$\mathbb{P}(\sup_{t \in \mathcal{D}} |M_t| > \lambda) \leq \frac{1}{\lambda} \sup_t \mathbb{E} |M_t|.$$

This is true for all  $\lambda$ , so with probability one,  $\{|M_t| : t \in \mathcal{D}\}$  is a bounded set. /

Therefore the only way either (1) or (2) can fail is that if for some pair of rationals  $a < b$  the number of upcrossings of  $[a, b]$  by  $\{M_t : t \in \mathcal{D}\}$  is infinite.

**define upcrossings**

i-th upcrossing time

Given an interval  $[a, b]$  and a submartingale  $M$ , if  $S_1 = \inf\{t : M_t \leq a\}$ ,  $T_i = \inf\{t > S_i : M_t \geq b\}$ , and  $S_{i+1} = \inf\{t > T_i : M_t \leq a\}$ , then the number of upcrossings up to time  $u$  is  $\sup\{k : T_k \leq u\}$ . cond. hitting time

**Doob's upcrossing lemma** (Theorem A.34) if  $V_n$  is the number of upcrossings by  $\{M_t : t \in \mathcal{D}_n \cap [0, n]\}$ , then

$$\mathbb{E} V_n \leq \frac{\mathbb{E} |M_n|}{b - a}.$$

Letting  $n \rightarrow \infty$  and using Fatou's lemma, the number of upcrossings of  $[a, b]$  by  $\{M_t : t \in \mathcal{D}\}$  has finite expectation, hence is finite, a.s. If  $N_{a,b}$  is the null set where the number of upcrossings of  $[a, b]$  by  $\{M_t : t \in \mathcal{D}\}$  is infinite and  $N = \bigcup_{\substack{\text{upcrossing} \\ \text{event}}} N_{a,b}$ , then  $\mathbb{P}(N) = 0$ . If  $\omega \notin N$ , then (1) and (2) hold. □

**Corollary 3.13** Let  $\{\mathcal{F}_t\}$  be a filtration satisfying the usual conditions, and let  $M_t$  be a martingale wrt  $\{\mathcal{F}_t\}$ . Then  $M$  has a version that is also a martingale and has paths that are right continuous with left limits.

*Proof* For each integer  $N \geq 1$ ,  $\mathbb{E} |M_t| \leq \mathbb{E} |M_N| < \infty$  for  $t \leq N$  since  $|M_t|$  is a submartingale by the conditional expectation form of Jensen's inequality (Proposition A.21). Therefore  $M_{t \wedge N}$  has left and right limits when taking limits along  $t \in \mathcal{D}$ . Since  $N$  is arbitrary,  $M_t$  has left and right limits when taking limits along  $t \in \mathcal{D}$ , except for a null set.

Let

$$\tilde{M}_t = \lim_{u \in \mathcal{D}, u > t, u \rightarrow t} M_u. \quad u \rightarrow t+$$

It is clear that  $\tilde{M}$  has paths that are right continuous with left limits. Since  $\mathcal{F}_{t+} = \mathcal{F}_t$  and  $\tilde{M}_t$  is  $\mathcal{F}_{t+}$  measurable, then  $\tilde{M}_t$  is  $\mathcal{F}_t$  measurable. /

Let  $N$  be fixed. We will show  $\{M_t ; t \leq N\}$  is a uniformly integrable family of rvs; see Section A.4. Let  $\varepsilon > 0$ . Since  $M_N$  is integrable, there exists  $\delta$  such that if  $\mathbb{P}(A) < \delta$ , then  $\mathbb{E} [|M_N|; A] < \varepsilon$ . If  $L$  is large enough,  $\mathbb{P}(|M_t| > L) \leq \mathbb{E} |M_t|/L \leq \mathbb{E} |M_N|/L < \delta$ . Then

$$\mathbb{E} [|M_t|; |M_t| > L] \leq \mathbb{E} [|M_N|; |M_t| > L] < \varepsilon,$$

since  $|M_t|$  is a submartingale and  $(|M_t| > L) \in \mathcal{F}_t$ . Uniform integrability is proved. /

Now let  $t < N$ . If  $B \in \mathcal{F}_t$ ,

$$\mathbb{E}[\tilde{M}_t; B] = \lim_{u \in \mathcal{D}, u > t, u \rightarrow t} \mathbb{E}[M_u; B] = \mathbb{E}[M_t; B].$$

Here we used the Vitali convergence theorem (Theorem A.19) and the fact that  $M_t$  is a martingale. Since  $\tilde{M}_t$  is  $\mathcal{F}_t$  measurable, this proves that  $\tilde{M}_t = M_t$ , a.s. Since  $N$  was arbitrary, we have this for all  $t$ . We thus have found a version of  $M$  that has paths that are right continuous with left limits. That  $\tilde{M}_t$  is a martingale is easy.  $\square$

A process  $A_t$  has **increasing paths** if the function  $t \rightarrow A_t(\omega)$  is increasing a.s.

**Proposition 3.14** Suppose  $\{\mathcal{F}_t\}$  is a filtration satisfying the usual conditions and suppose  $A_t$  is an adapted process with paths that are increasing, are right continuous with left limits, distr. and  $A_\infty = \lim_{t \rightarrow \infty} A_t$  exists, a.s. Suppose  $X$  is a non-negative integrable rv, and  $M_t$  is a version of the martingale  $\mathbb{E}[X | \mathcal{F}_t]$  which has paths that are right continuous with left limits. Suppose  $\mathbb{E}[XA_\infty] < \infty$ . Then

$$\mathbb{E} \left[ \int_0^\infty X dA_s \right] = \mathbb{E} \int_0^\infty M_s dA_s. \quad (3.4)$$

it is not stoch. int. defined as  $w \rightarrow \text{int } M_s(w) d A_s(w)$  where  $A_t(w)$  is increasing right cont. (distr.)

$$E f(.) = \int E(Ms(w) d As(w)) ???$$

*Proof* First suppose  $X$  and  $A$  are bounded. Let  $n > 1$  and write  $\mathbb{E} \int_0^\infty X dA_s$  as

$$\begin{aligned} & \sum_{k=1}^{\infty} \mathbb{E} [X(A_{k/2^n} - A_{(k-1)/2^n})]. \quad \text{discretize} \\ &= \mathbb{E} \left[ \sum_{k=1}^{\infty} \mathbb{E} [X | \mathcal{F}_{k/2^n}] (A_{k/2^n} - A_{(k-1)/2^n}) \right]. \quad \text{cond. exp.} \end{aligned}$$

Given  $s$  and  $n$ , define  $s_n$  to be that value of  $k/2^n$  such that  $(k-1)/2^n < s \leq k/2^n$ . We then have

$$\mathbb{E} \int_0^\infty X dA_s = \mathbb{E} \int_0^\infty M_{s_n} dA_s. \quad (3.5)$$

For any value of  $s$ ,  $s_n \downarrow s$  as  $n \rightarrow \infty$ , and since  $M$  has right-continuous paths,  $M_{s_n} \rightarrow M_s$ . Since  $X$  is bounded, so is  $M$ . By dominated convergence, the rhs of (3.5) converges to

$$\mathbb{E} \int_0^\infty M_s dA_s.$$

This completes the proof when  $X$  and  $A$  are bounded. We apply this to  $X \wedge N$  and  $A \wedge N$ , let  $N \rightarrow \infty$ , and use monotone convergence for the general case.  $\square$

The only reason we assume  $X$  is non-negative is so that the integrals make sense. The equation (3.4) ==>

$$\mathbb{E} \int_0^\infty X dA_s = \mathbb{E} \int_0^\infty \mathbb{E}[X | \mathcal{F}_s] dA_s. \quad (3.6)$$

We also have

$$\mathbb{E} \int_0^t X dA_s = \mathbb{E} \int_0^t \mathbb{E}[X | \mathcal{F}_s] dA_s \quad (3.7)$$

for each  $t$ . This follows either by following the above proof or by applying Proposition 3.14 to  $A_{s \wedge t}$ .

### 3.6 Some applications of martingales

**Proposition 3.15** *If  $W_t$  is a Brownian motion, then*

$$\mathbb{P}(\sup_{s \leq t} W_s \geq \lambda) \leq e^{-\lambda^2/2t}, \quad \lambda > 0, \quad \text{Hoeffding/Chenov ineq.} \quad (3.8)$$

and

$$\mathbb{P}(\sup_{s \leq t} |W_s| \geq \lambda) \leq 2e^{-\lambda^2/2t}, \quad \lambda > 0. \quad (3.9)$$

*Proof* For any  $a$  the process  $\{e^{aW_t}\}$  is a submartingale. To see this, since  $x \rightarrow e^{ax}$  is convex, the conditional expectation form of Jensen's inequality (Proposition A.21) implies

$$\mathbb{E}[e^{aW_t} | \mathcal{F}_s] \geq e^{a\mathbb{E}[W_t | \mathcal{F}_s]} = e^{aW_s}.$$

By Doob's inequality (Theorem 3.6(1)),

$$\mathbb{P}(\sup_{s \leq t} W_s \geq \lambda) = \mathbb{P}(\sup_{s \leq t} e^{aW_s} \geq e^{a\lambda}) \leq \frac{\mathbb{E} e^{aW_t}}{e^{a\lambda}}. \quad (3.10)$$

Since  $\mathbb{E} e^{aY} = e^{a^2 \text{Var } Y/2}$  if  $Y$  is Gaussian with mean 0 by (A.6), it follows that the right side of (3.10) is bounded by  $e^{-a\lambda} e^{a^2 t/2}$ . If we now set  $a = \lambda/t$ , we obtain (3.8). Inequality (3.9) follows by applying (3.8) to  $W$  and to  $-W$  and adding.  $\square$

Let us use martingales to calculate some probabilities. Let us suppose  $a, b > 0$  and set  $T = \inf\{t > 0 : W_t = -a \text{ or } W_t = b\}$ , the first time Brownian motion exits the interval  $[-a, b]$ . By Proposition 3.9,  $T$  is a stopping time.

**Proposition 3.16** *Let  $W$  be a Brownian motion, let  $T = \inf\{t > 0 : W_t \notin [-a, b]\}$ , and let  $a, b > 0$ . Then*

$$\mathbb{P}(W_T = -a) = \frac{b}{a+b}, \quad \mathbb{P}(W_T = b) = \frac{a}{a+b}, \quad (3.11)$$

WT = -a or b

and

$$\mathbb{E} T = ab. \quad (3.12)$$

*Proof* Since  $W_t^2 - t$  is a martingale with  $W_0 = 0$ , it is easy to check that for each  $u$ ,  $W_{t \wedge u}^2 - (t \wedge u)$  is also a martingale. Applying Theorem 3.11, we see that  $\mathbb{E} W_{u \wedge T}^2 = \mathbb{E}[u \wedge T]$ . As  $u \rightarrow \infty$ , the rhs  $\rightarrow \mathbb{E} T$  by monotone convergence.  $|W_{u \wedge T}|^2$  is bounded

by  $(a+b)^2$ , so by dominated convergence the lhs  $\rightarrow \mathbb{E} W_T^2 \leq (a+b)^2$  as  $u \rightarrow \infty$ . Therefore

$$\mathbb{E} T = \mathbb{E} W_T^2. \quad (3.13)$$

In particular,  $\mathbb{E} T < \infty$ , so we know  $T < \infty$ , a.s.

We use that  $T$  is finite, a.s., to conclude that  $\mathbb{P}(W_T \in \{-a, b\}) = 1$ , or

$$1 = \mathbb{P}(W_T = -a) + \mathbb{P}(W_T = b). \quad (3.14)$$

Since  $W_t$  is a martingale, then so is  $W_{t \wedge u}$  for each  $u$ , and therefore  $\mathbb{E} W_{u \wedge T} = 0$ . Letting  $u \rightarrow \infty$  and using dominated convergence (noting  $|W_{u \wedge T}|$  is bounded by  $a+b$ ), we have  $\mathbb{E} W_T = 0$ , or

$$0 = (-a)\mathbb{P}(W_T = -a) + b\mathbb{P}(W_T = b). \quad (3.15)$$

We get (3.11) by solving (3.14) and (3.15) for the unknowns  $\mathbb{P}(W_T = -a)$  and  $\mathbb{P}(W_T = b)$ .

We get (3.12) by (3.13), writing

$$\mathbb{E} T = \mathbb{E} W_T^2 = (-a)^2 \mathbb{P}(W_T = -a) + b^2 \mathbb{P}(W_T = b),$$

and substituting the values from (3.11).

□

In proving Proposition 3.16, we used the fact that  $W_{t \wedge T}$  is a martingale and  $\mathbb{P}(T < \infty) = 1$ . The same proof shows

**Corollary 3.17** Suppose  $M_t$  is a martingale with continuous paths and with  $M_0 = 0$ , a.s.,  $T = \inf\{t \geq 0 : M_t \notin [-a, b]\}$ , and  $T < \infty$ , a.s. Then

$$\mathbb{P}(M_T = -a) = \frac{b}{a+b}, \quad \mathbb{P}(M_T = b) = \frac{a}{a+b}.$$

We can also use martingales to get more subtle results. Suppose  $r > 0$ . Since  $e^{rW_t - r^2 t/2}$  is a martingale, as above

$$\mathbb{E} e^{rW_{T \wedge t} - r^2(T \wedge t)/2} = 1.$$

The exponent is bounded by  $rb$  if  $r > 0$ , so we can let  $t \rightarrow \infty$  and use dominated convergence to get

$$\mathbb{E} e^{rW_T - r^2 T/2} = 1.$$

This can be written as

$$e^{-ra} \mathbb{E}[e^{-r^2 T/2}; W_T = -a] + e^{rb} \mathbb{E}[e^{-r^2 T/2}; W_T = b] = 1.$$

Since  $e^{-rW_t - r^2 t/2}$  is also a martingale, similar reasoning gives us

$$e^{ra} \mathbb{E}[e^{-r^2 T/2}; W_T = -a] + e^{-rb} \mathbb{E}[e^{-r^2 T/2}; W_T = b] = 1.$$

We can solve those two equations to obtain

$$\mathbb{E}[e^{-r^2 T/2}; W_T = -a] = \frac{e^{rb} - e^{-ra}}{e^{r(a+b)} - e^{-r(a+b)}} \quad (3.16)$$

and

$$\mathbb{E}[e^{-r^2 T/2}; W_T = b] = \frac{e^{ra} - e^{-rb}}{e^{r(a+b)} - e^{-r(a+b)}}. \quad (3.17)$$

The left-hand sides of (3.16) and (3.17) are the Laplace transforms of the quantities  $\mathbb{P}(T \in dt; W_T = -a)/dt$  and  $\mathbb{P}(T \in dt; W_T = b)/dt$ , respectively, and finding the inverse Laplace transforms of the right-hand sides of (3.16) and (3.17) gives us formulas for  $\mathbb{P}(T \in dt; W_T = -a)/dt$  and  $\mathbb{P}(T \in dt; W_T = b)/dt$ . If we add the two formulas, we get an expression for  $\mathbb{P}(T \in dt)/dt$ , and integrating over  $t$  from 0 to  $t_0$  gives an expression for  $\mathbb{P}(T \leq t_0)$ .

We sketch how to invert the Laplace transform and leave the detailed calculations and justification for inverting a Laplace transform term by term to the interested reader. See also Karatzas and Shreve (1991), Section 2.8. The right-hand side of (3.16) is equal to

$$\frac{e^{-ra} - e^{-ra-2rb}}{1 - e^{-2r(a+b)}}.$$

Since  $e^{-2r(a+b)} < 1$ , we can use

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$$

to expand the denominator as a power series; if we set  $\lambda = r^2/2$ , then

$$\begin{aligned} E\left[e^{-\lambda T}; W_T = -a\right] \\ = \sum_{n=0}^{\infty} \left( e^{-(2n+1)\sqrt{2\lambda}a - 2n\sqrt{2\lambda}b} - e^{-(2n+1)\sqrt{2\lambda}a - (2n+2)\sqrt{2\lambda}b} \right). \end{aligned} \quad (3.18)$$

We then use the fact that the Laplace transform of

$$\frac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$$

is  $e^{-k\sqrt{\lambda}}$  to find the inverse Laplace transform of the right-hand side of (3.18) by inverting term by term.

Similarly (see Exercises 3.15 and 3.16), if  $b > 0$ ,  $W$  is a Brownian motion, and  $S = \inf\{t > 0 : W_t = b\}$ , then  $\mathbb{E} e^{-\lambda S} = e^{-\sqrt{2\lambda}b}$ . Inverting the Laplace transform,

$$\mathbb{P}(S \in dt) = \frac{b}{\sqrt{2\pi t^3}} e^{-b^2/2t}, \quad t \geq 0. \quad (3.19)$$

## Exercises

- 3.1 If  $W$  is a Brownian motion, show that

$$W_t^3 - 3 \int_0^t W_s ds$$

is a martingale.

- 3.2 Suppose  $\{\mathcal{F}_t\}$  is a filtration satisfying the usual conditions. Show that if  $M_t$  is a submartingale and  $\mathbb{E} M_t = \mathbb{E} M_0$  for all  $t$ , then  $M$  is a martingale.
- 3.3 Let  $X$  be a submartingale. Show that  $\sup_{t \geq 0} \mathbb{E} |X_t| < \infty$  if and only if  $\sup_{t \geq 0} \mathbb{E} X_t^+ < \infty$ .
- 3.4 Prove all parts of Proposition 3.8.

- 3.5 If  $T_n$  is defined by (3.2), show  $T_n$  is a stopping time for each  $n$  and  $T_n \downarrow T$ .
- 3.6 ***alternate definition of  $\mathcal{F}_T$  more appealing, but not as useful.***  $\text{FT} := \sigma\{\text{YT}; Y \sim \text{Ft}, Y: \text{bd, RCLL}\}$   
 Suppose that  $\{\mathcal{F}_t\}$  satisfies the usual conditions. Show that  $\mathcal{F}_T$  is equal to the  $\sigma$ -field generated by the collection of rvs  $\{Y_T\}$  such that  $Y$  is a bounded process with paths that are right continuous with left limits and  $Y$  is adapted to the filtration  $\{\mathcal{F}_t\}$ .
- 3.7 Suppose  $\{\mathcal{F}_t\}$  is a filtration satisfying the usual conditions. Show that if  $T$  is a stopping time, then  $T$  is  $\mathcal{F}_T$  measurable.
- 3.8 Suppose  $\{\mathcal{F}_t\}$  is a filtration satisfying the usual conditions and  $T$  is a stopping time. Show that if  $S$  is a  $\mathcal{F}_T$  measurable random variable with  $S \geq T$ , then  $S$  is a stopping time.
- 3.9 This exercise demonstrates that the conclusion of Corollary 3.13 cannot be extended to submartingales. Find a filtration  $\{\mathcal{F}_t\}$  satisfying the usual conditions and a submartingale  $X$  wrt  $\{\mathcal{F}_t\}$  such that  $X$  does not have a version with paths that are right continuous with left limits.
- 3.10 Suppose  $\{\mathcal{F}_t\}$  is a filtration satisfying the usual conditions. Show that if  $S$  and  $T$  are stopping times and  $X$  is a bounded  $\mathcal{F}_\infty$  measurable rv, then

$$\mathbb{E}[\mathbb{E}[X | \mathcal{F}_S] | \mathcal{F}_T] = \mathbb{E}[X | \mathcal{F}_{S \wedge T}].$$

*Hint:* Let  $Y_t = \mathbb{E}[X | \mathcal{F}_t]$  and  $Z_t = Y_{t \wedge S}$ . Show the lhs =  $Y_{S \wedge T}$ .

- 3.11 A martingale or submartingale  $M_t$  is uniformly integrable if the family  $\{M_t : t \geq 0\}$  is a uniformly integrable family of random variables. Show that if  $M_t$  is a uniformly integrable martingale with paths that are right continuous with left limits, then  $\{M_T; T \text{ a finite stopping time}\}$  is a uniformly integrable family of random variables. Show this also holds if  $M_t$  is a non-negative submartingale with paths that are right continuous with left limits.
- 3.12 This exercise weakens the conditions on the optional stopping theorem. Show that if  $M_t$  is a uniformly integrable martingale that is right continuous with left limits and  $T$  is a finite stopping time, then  $\mathbb{E} M_T = \mathbb{E} M_0$ .
- 3.13 Let  $W$  be a Brownian motion and let  $T$  be a stopping time with  $\mathbb{E} T < \infty$ . Prove that  $\mathbb{E} W_T = 0$  and  $\mathbb{E} W_T^2 = \mathbb{E} T$ . This is not an easy application of the optional stopping theorem because we do not know that  $W_{t \wedge T}$  is necessarily a uniformly integrable martingale.
- 3.14 Suppose that  $(W_t^1, \dots, W_t^d)$  is a  $d$ -dimensional Brownian motion. Show that if  $i \neq j$ , then  $W_t^i W_t^j$  is a martingale.
- 3.15 Let  $W_t$  be a Brownian motion,  $b > 0$ , and  $T = \inf\{t > 0 : W_t = b\}$ . Show  $T < \infty$ , a.s. Show  $\mathbb{E} T = \infty$ .  
*Hint:* Take a limit in (3.11).
- 3.16 Suppose  $W$  is a Brownian motion and  $b > 0$ . If  $S = \inf\{t > 0 : W_t = b\}$ , show that the Laplace transform of the density of  $S$  is given by

$$\mathbb{E} e^{-\lambda S} = e^{-\sqrt{2\lambda}b}.$$

- 3.17 Let  $W_t$  be a Brownian motion. Show that if  $\alpha > 1/2$ , then

$$\lim_{t \rightarrow \infty} \frac{W_t}{t^\alpha} = 0, \quad \text{a.s.}$$

*Hint:* Let  $\alpha_0 \in (1/2, \alpha)$ , estimate

$$\mathbb{P}\left(\sup_{2^n \leq s \leq 2^{n+1}} |W_s| \geq (2^n)^{\alpha_0}\right)$$

using (3.9), and then use the Borel–Cantelli lemma.

- 3.18 Let  $W_t$  be a one-dimensional Brownian motion and  $\alpha \in (0, 1/2]$ . Prove that

$$\limsup_{t \rightarrow \infty} \frac{|W_t|}{t^\alpha} > 0, \quad \text{a.s.}$$

- 3.19 If  $W$  is a Brownian motion and  $b$  is a constant, then the process  $X_t = W_t + bt$  is a *Brownian motion with drift*. Prove that if  $b > 0$ , then

$$\lim_{t \rightarrow \infty} X_t = \infty, \quad \text{a.s.}$$

3.7+ At  $\sim$  Ft, increasing events ,  $F\{At\} := \{A \sim F, A \cap At \sim F_t, t \geq 0\}$  ( $\sigma$ -algebra)  
 $\implies At \sim F\{At\}, t \geq 0$

AT := {w  $\in$  A(T(w))};  $1_{AT} = 1\{w \in AT(w)\} = (1\{At\})_T$  event on stopping time

Fact.  $(At \cap (T \leq t))T = AT$ ,  $(At \cap Bt)T = AT \cap BT$

FT =  $\sigma\{AT, A \sim F\}$

At =  $A \cap (T \leq t)$

# 4

---

## Markov properties of Brownian motion

Markov property, strong Markov property

### 4.1 Markov properties

**Theorem 4.1** Let  $\{\mathcal{F}_t\}$  be a filtration, not necessarily satisfying the usual conditions, and let  $W$  be a Brownian motion wrt  $\{\mathcal{F}_t\}$ . If  $u$  is a fixed time, then  $Y_t = W_{t+u} - W_u$  is a Brownian motion independent of  $\mathcal{F}_u$ .

*Proof* Let  $\mathcal{G}_t = \mathcal{F}_{t+u}$ . It is clear that  $Y$  has continuous paths, is zero at time 0, and is adapted to  $\{\mathcal{G}_t\}$ . Since  $Y_t - Y_s = W_{t+u} - W_{s+u}$ , then  $Y_t - Y_s$  is a mean zero normal rv with variance  $(t+u) - (s+u) = t - s$  that is independent of  $\mathcal{F}_{s+u} = \mathcal{G}_s$ .  $\square$

The strong Markov property is the Markov property extended by replacing fixed times  $u$  by finite stopping times.

**Theorem 4.2** Let  $\{\mathcal{F}_t\}$  be a filtration, not necessarily satisfying the usual conditions, and let  $W$  be a Brownian motion adapted to  $\{\mathcal{F}_t\}$ . If  $T$  is a finite stopping time, then  $Y_t = W_{T+t} - W_T$  is a Brownian motion independent of  $\mathcal{F}_T$ .

*Proof* We will first show that whenever  $m \geq 1$ ,  $t_1 < \dots < t_m$ ,  $f$  is a bounded continuous function on  $\mathbb{R}^m$ , and  $A \in \mathcal{F}_T$ , then

$$\mathbb{E}[f(Y_{t_1}, \dots, Y_{t_m}); A] = \mathbb{E}[f(W_{t_1}, \dots, W_{t_m})] \mathbb{P}(A). \quad (4.1)$$

Once we have done this, we will then show how (4.1) implies our theorem.

Define  $T_n$  by (3.2). We have

Discretization trick

$$\begin{aligned} \mathbb{E}[f(W_{T_n+t_1} - W_{T_n}, \dots, W_{T_n+t_m} - W_{T_n}); A] \\ = \sum_{k=1}^{\infty} \mathbb{E}[f(W_{t_1+k/2^n} - W_{k/2^n}, \dots, W_{t_m+k/2^n} - W_{k/2^n}); A, T_n = k/2^n]. \end{aligned} \quad (4.2)$$

Formula.  $E(f(T_n)) = \sum_k E(f(t_k); T_n = t_k)$

“ $\mathbb{E}[\dots; A, T_n = k/2^n]$ ” as an abbreviation for “ $\mathbb{E}[\dots; A \cap (T_n = k/2^n)]$ .” Since  $A \in \mathcal{F}_T$ ,

then  $A \cap (T_n = k/2^n) = A \cap ((T < k/2^n) \setminus (T < (k-1)/2^n)) \in \mathcal{F}_{k/2^n}$ . We use the independent increments property of Brownian motion and the fact that  $W_t - W_s$  has the same law as  $W_{t-s}$  to see that (4.2) =

$$\begin{aligned} & \sum_{k=1}^{\infty} \mathbb{E}[f(W_{t_1+k/2^n} - W_{k/2^n}, \dots, W_{t_m+k/2^n} - W_{k/2^n})] \mathbb{P}(A, T_n = k/2^n) \\ &= \sum_{k=1}^{\infty} \mathbb{E}[f(W_{t_1}, \dots, W_{t_m})] \mathbb{P}(A, T_n = k/2^n) \\ &= \mathbb{E}[f(W_{t_1}, \dots, W_{t_m})] \mathbb{P}(A) \end{aligned}$$

which is the rhs of (4.1). Thus

$$\mathbb{E}[f(W_{T_n+t_1} - W_{T_n}, \dots, W_{T_n+t_m} - W_{T_n}); A] = \mathbb{E}[f(W_{t_1}, \dots, W_{t_m})] \mathbb{P}(A). \quad (4.3)$$

Now let  $n \rightarrow \infty$ . By the right continuity of the paths of  $W$ , the boundedness and continuity off, and the dominated convergence theorem, the lhs of (4.3) converges to the lhs of (4.1).

If we take  $A = \Omega$  in (4.1), we obtain

$$\mathbb{E}[f(Y_{t_1}, \dots, Y_{t_m})] = \mathbb{E}[f(W_{t_1}, \dots, W_{t_m})]$$

whenever  $m \geq 1$ ,  $t_1, \dots, t_m \in [0, \infty)$ , and  $f$  is a bounded continuous function on  $\mathbb{R}^m$ . This implies that the finite-dimensional distributions of  $Y$  and  $W$  are the same. Since  $Y$  has continuous paths,  $Y$  is a Brownian motion.

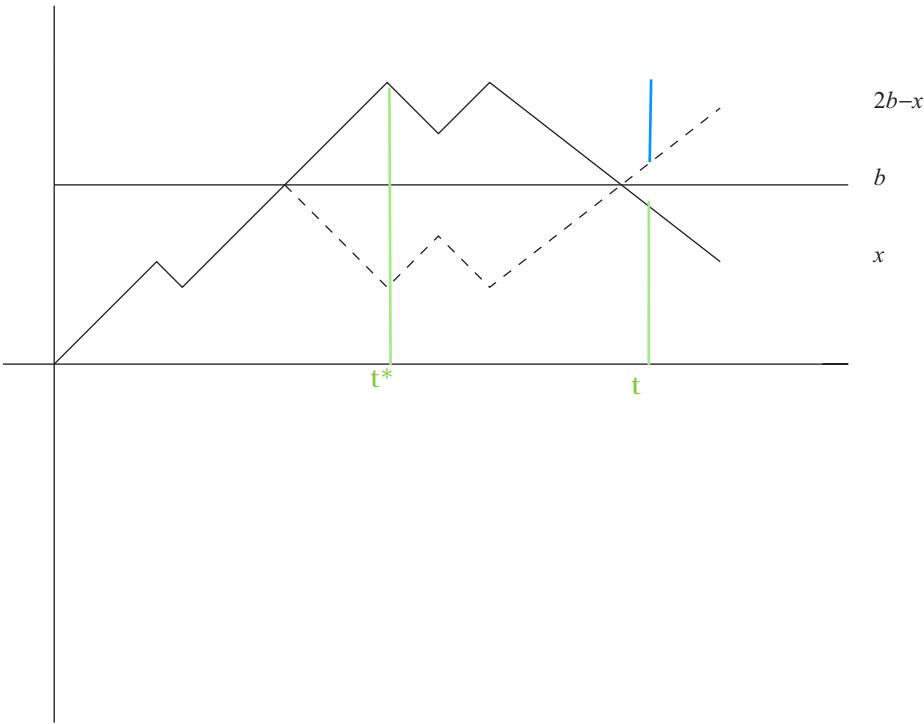
Next take  $A \in \mathcal{F}_T$ . By using a limit argument, (4.1) holds whenever  $f$  is the indicator of a Borel subset  $B$  of  $\mathbb{R}^d$ , i.e.

$$\mathbb{P}(Y \in B, A) = \mathbb{P}(Y \in B) \mathbb{P}(A) \quad (4.4)$$

whenever  $B$  is a cylindrical set. Let  $\mathcal{M}$  be the collection of all Borel subsets  $B$  of  $C[0, \infty)$  for which (4.4) holds. Let  $\mathcal{C}$  be the collection of all cylindrical subsets of  $C[0, \infty)$ . Then we observe that  $\mathcal{M}$  is a monotone class containing  $\mathcal{C}$  and  $\mathcal{C}$  is an algebra of subsets of  $C[0, \infty)$  generating the Borel  $\sigma$ -field of  $C[0, \infty)$ . By the monotone class theorem (Theorem B.2),  $\mathcal{M}$  is the Borel  $\sigma$ -field on  $C[0, \infty)$ , and since (4.4) holds for all sets  $B \in \mathcal{M}$ , this establishes the independence of  $Y$  and  $\mathcal{F}_T$ .  $\square$

Observe that what is needed for the above proof to work is not that  $W$  be a Brownian motion, but that the process  $W$  have right continuous paths and that  $W_t - W_s$  be independent of  $\mathcal{F}_s$  and have the same distribution as  $W_{t-s}$ .

**Corollary 4.3** *Let  $\{\mathcal{F}_t\}$  be a filtration, not necessarily satisfying the usual conditions, and let  $X$  be a process adapted to  $\{\mathcal{F}_t\}$ . Suppose  $X$  has paths that are right continuous with left limits and suppose  $X_t - X_s$  is independent of  $\mathcal{F}_s$  and has the same law as  $X_{t-s}$  whenever  $s < t$ . If  $T$  is a finite stopping time, then  $Y_t = X_{T+t} - X_T$  is a process that is independent of  $\mathcal{F}_T$  and  $X$  and  $Y$  have the same law.*



**Figure 4.1** The reflection principle.

## 4.2 Applications

The first application is known as the reflection principle and allows us to get control of the maximum of a Brownian motion. *idea*. Suppose that  $W_t$  is a Brownian motion and for some path, the Brownian motion goes above a level  $b$  before time  $t$  but that at time  $t$  the value of  $W_t$  is less than  $x$ , where  $x < b$ . We could take the graph of this path and reflect it across the horizontal line at level  $b$  the first time the path crosses the level  $b$  (Figure 4.1). This will give us a new path that ends up above  $2b-x$ . Thus there is a one-to-one correspondence between paths where the maximum up to time  $t$  is above  $b$  and  $W_t$  is below  $x$  and the paths where  $W_t$  is above  $2b-x$ .

**Theorem 4.4** Let  $W_t$  be a Brownian motion,  $b > 0$ ,  $T = \inf\{t : W_t \geq b\}$ , and  $x < b$ . Then

$$\mathbb{P}(\sup_{s \leq t} W_s \geq b, W_t < x) = \mathbb{P}(W_t > 2b-x). \quad (4.5)$$

*Proof* Let  $T_n$  be defined by (3.2). We first show that

$$\mathbb{P}(T_n \leq t, W_t - W_{T_n} < x - b) = \mathbb{P}(T_n \leq t, W_t - W_{T_n} > b - x). \quad (4.6)$$

Writing  $[x]$  for the integer part of  $x$ , the lhs of (4.6) ==

$$\begin{aligned} &= \sum_{k=0}^{[2^n t]} \mathbb{P}(T_n = k/2^n, W_t - W_{k/2^n} < x - b) \\ &= \sum_{k=0}^{[2^n t]} \mathbb{P}(T_n = k/2^n) \mathbb{P}(W_t - W_{T_n} < x - b), \end{aligned}$$

using the independent increments property of Brownian motion and the fact that we have  $(T_n = k/2^n) \in \mathcal{F}_{k/2^n}$ . Using the symmetry of the normal distribution, that is, that  $W_t - W_s$  and  $W_s - W_t$  have the same law, this is the same as

$$\sum_{k=0}^{[2^n t]} \mathbb{P}(T_n = k/2^n) \mathbb{P}(W_t - W_{T_n} > b - x),$$

and reversing the steps above, this equals the rhs of (4.6). Since  $W$  has continuous paths,

$W_T = b$ , so  $(T = t) \subset (W_t = b)$ . Because  $W_t$  is a normal rv, then  $\mathbb{P}(T = t) = 0$ .  
Also,  $\mathbb{P}(W_t - W_T = b - x) = \mathbb{P}(W_t - W_T = x - b) = 0$

If we now let  $n \rightarrow \infty$  in (4.6), we obtain

$$\mathbb{P}(T \leq t, W_t - W_T < x - b) = \mathbb{P}(T \leq t, W_t - W_T > b - x). \quad |$$

$\Leftrightarrow$

$$\mathbb{P}(T \leq t, W_t < x) = \mathbb{P}(T \leq t, W_t > 2b - x). \quad (4.7)$$

By the definition of  $T$  and the continuity of the paths of  $W$ , the lhs (4.7) = the lhs of (4.5). If  $W_t > 2b - x$ , then automatically  $T \leq t$ , so the rhs of (4.7) = the rhs of (4.5).

□

Our second application will be useful when studying local time in Chapter 14.

**Proposition 4.5** Let  $W_t$  be a Brownian motion wrt a filtration  $\{\mathcal{F}_t\}$  satisfying the usual conditions. Let  $T$  be a finite stopping time and  $s > 0$ . If  $a < b$ , then

$$\mathbb{P}(W_{T+s} \in [a, b] \mid \mathcal{F}_T) \leq \frac{|b - a|}{\sqrt{2\pi s}}.$$

*Proof* If  $A \in \mathcal{F}_T$ , let  $k > 0$  and write

$$\begin{aligned} &\mathbb{P}(W_{T+s} \in [a, b], A) \quad (\text{WT} + s - \text{WT}) + \text{WT} \\ &= \sum_{j=-\infty}^{\infty} \mathbb{P}(W_{T+s} \in [a, b], A, j/k \leq W_T < (j+1)/k) \\ &\leq \sum_{j=-\infty}^{\infty} \mathbb{P}(W_{T+s} - W_T \in [a - (j+1)/k, b - j/k], \\ &\quad A, j/k \leq W_T \leq (j+1)/k). \end{aligned}$$

Using the fact that  $W_{T+s} - W_T$  is a Brownian motion independent of  $\mathcal{F}_T$ , this is less than or equal to

$$\begin{aligned} & \sum_{j=-\infty}^{\infty} \mathbb{P}(W_s \in [a - (j+1)/k, b - j/k]) \mathbb{P}(A, j/k \leq W_T \leq (j+1)/k) \\ & \leq \sum_{j=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{b-a+1/k}{\sqrt{s}} \mathbb{P}(A, j/k \leq W_T \leq (j+1)/k) \\ & \leq \frac{1}{\sqrt{2\pi}} \frac{b-a+1/k}{\sqrt{s}} \mathbb{P}(A). \end{aligned}$$

We used here the formula for the density of a normal rv with mean zero and variance  $s$ . This is true for all  $k$ , so letting  $k \rightarrow \infty$  yields our result.  $\square$

### Exercises

$\max\{s: D_n\} W_s$

- 4.1 If  $W$  is a Brownian motion, let  $S_t = \sup_{s \leq t} W_s$ . Find the density for  $S_t$ .
- 4.2 With  $W$  and  $S$  as in Exercise 4.1, find the joint density of  $(S_t, W_t)$ .
- 4.3 Let  $W$  be a Brownian motion started at  $a > 0$  and let  $T_0$  be the first time  $W$  hits 0. Find the law of  $\sup_{t \leq T_0} W_t$ .
- 4.4 Use the reflection principle to prove that if  $W$  is a Brownian motion and  $T = \inf\{t > 0 : W_t \in (0, \infty)\}$ , then

$$\mathbb{P}(T = 0) = 1.$$

In other words, Brownian motion enters the interval  $(0, \infty)$  immediately. By symmetry it enters the interval  $(-\infty, 0)$  immediately. Conclude that Brownian motion hits 0 infinitely often in every time interval  $[0, t]$ .

- 4.5 Let  $W_t$  be a Brownian motion and  $\{\mathcal{F}_t\}$  be the minimal augmented filtration generated by  $W$ . Let

$$T = \inf\{t > 0 : W_t = \sup_{0 \leq s \leq 1} W_s\}.$$

Show that  $T$  is *not* a stopping time wrt  $\{\mathcal{F}_t\}$ .

- 4.6 Let  $W$  and  $S$  be as in Exercise 4.1.

(1) Let  $0 < s < t < u$  and let  $a < b$  with  $b - a \leq 1$ . Show that there exists a constant  $c$ , depending on  $s, t$ , and  $u$ , but not  $a$  or  $b$ , such that

$$\mathbb{P}(S_s \in [a, b], \sup_{t \leq r \leq u} W_r \in [a, b]) \leq c(b - a)^2.$$

(2) Show that the path of a Brownian motion does not take on the same value as a local maximum twice. That is, if  $S$  and  $T$  are times when  $W$  has a local maximum, then  $W_S \neq W_T$ , a.s.

- 4.7 Let  $V_t$  be the number of upcrossings of  $[0, 1]$  by a Brownian motion  $W$  up to time  $t$ . This means we let  $S_1 = 0$ ,  $T_i = \inf\{t > S_i : W_t \geq 1\}$ , and  $S_{i+1} = \inf\{T > T_i : W_t \leq 0\}$  for  $i = 1, 2, \dots$ , and we set  $V_t = \sup\{k : T_k \leq t\}$ . Show that  $V_t \rightarrow \infty$ , a.s., as  $t \rightarrow \infty$ .



4.8 Let  $W$  be a Brownian motion. The *zero set* of Brownian motion is the random set

$$Z(\omega) = \{t \in [0, 1] : W_t(\omega) = 0\}.$$

(1) Show that  $Z(\omega)$  is a closed set for each  $\omega$ .

(2) Show that with probability one, every point of  $Z(\omega)$  is a limit point of  $Z(\omega)$ . Conclude that  $Z(\omega)$  is an uncountable set.

4.9 Let  $W$  be a one-dimensional Brownian motion and  $\delta > 0$ .

(1) Prove that there exists  $\gamma$  such that if  $t \leq \gamma$ , then

$$\mathbb{P}(0 \leq W_t \leq \delta/2) \geq 1/4 \quad \text{and} \quad \mathbb{P}(-\delta/2 \leq W_t \leq 0) \geq 1/4.$$

(2) Prove there exists  $\gamma$  such that

$$\mathbb{P}(\sup_{s \leq \gamma} |W_s| > \delta/2) \leq 1/8.$$

(3) Prove that if  $m \geq 1$ , then

$$\begin{aligned} \mathbb{P}\left(\sup_{m\gamma \leq s \leq (m+1)\gamma} |W_s - W_{m\gamma}| \leq \delta/2, W_{m\gamma} \in [0, \delta/2], |W_{(m+1)\gamma}| \leq \delta/2 \mid \mathcal{F}_{m\gamma}\right) \\ \geq \frac{1}{8} \mathbb{P}\left(\sup_{m\gamma \leq s \leq (m+1)\gamma} |W_s - W_{m\gamma}| \leq \delta/2, W_{m\gamma} \in [0, \delta/2]\right) \end{aligned}$$

and the same with  $W_{m\gamma} \in [-\delta/2, 0]$  in place of  $W_{m\gamma} \in [0, \delta/2]$ . Conclude that

$$\begin{aligned} \mathbb{P}\left(\sup_{m\gamma \leq s \leq (m+1)\gamma} |W_s - W_{m\gamma}| \leq \delta/2, |W_{m\gamma}| \leq \delta/2, |W_{(m+1)\gamma}| \leq \delta/2 \mid \mathcal{F}_{m\gamma}\right) \\ \geq \frac{1}{8} \mathbb{P}\left(\sup_{m\gamma \leq s \leq (m+1)\gamma} |W_s - W_{m\gamma}| \leq \delta/2, |W_{m\gamma}| \leq \delta/2\right). \end{aligned}$$

(4) Use induction to prove that if  $t_0 > 0$ , there exists  $c_1 > 0$  such that

$$\mathbb{P}(\sup_{s \leq t_0} |W_s| \leq \delta) > c_1.$$

(5) Prove that if  $W$  is a  $d$ -dimensional Brownian motion,  $t_0 > 0$ , and  $\delta > 0$ , there exists  $c_2$  such that

$$\mathbb{P}(\sup_{s \leq t_0} |W_s| \leq \delta) > c_2.$$

4.10 The *p-variation* of a function  $f$  on the interval  $[0, 1]$  is

$$V^p(f) = \sup \left\{ \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|^p : n \geq 1, 0 = t_0 < t_1, \dots < t_n = 1 \right\};$$

the supremum is over all partitions  $P$  of  $[0, 1]$ . In this exercise we will prove that if  $p < 2$  and  $W$  is a Brownian motion, then  $V^p(W) = \infty$ , a.s.

(1) Let  $X_i$  be an i.i.d. sequence of rvs with finite mean. Use the strong law of large numbers to prove that if  $K > \mathbb{E} X_1$ , then

$$\mathbb{P}\left(\sum_{i=1}^n X_i > Kn\right) \rightarrow 0$$

as  $n \rightarrow \infty$ .

(2) If  $p < 2$ , take  $r \in (p, 2)$ , and let  $\varepsilon_n = n^{-1/r}$ . Let  $S_0 = 0$  and for  $i \geq 0$ , set  $S_{i+1} = \inf\{t > S_i : |W_t - W_{S_i}| > \varepsilon_n\}$ . Set  $X_i = \varepsilon_n^{-2}(S_i - S_{i-1})$ . Prove that the  $X_i$  are i.i.d. with finite mean.

(3) Use (1) to show that

$$\mathbb{P}(S_n > 1) = \mathbb{P}\left(\sum_{i=1}^n X_i > \varepsilon_n^{-2}\right) \rightarrow 0$$

as  $n \rightarrow \infty$ .

(4) Using the partition  $\{S_0, S_1, \dots, S_n\}$ , show that  $V^p(W) \geq n\varepsilon_n^p$  on the event  $(S_n \leq 1)$ .

(5) Conclude  $V^p(W) = \infty$ , a.s.

# 5

---

## The Poisson process

At the opposite extreme from Brownian motion is the Poisson process. This is a process that only changes value by means of jumps, and even then, the jumps are nicely spaced. The Poisson process is the prototype of a pure jump process, and later we will see that it is the building block for an important class of stochastic processes known as Lévy processes.

**Definition 5.1** Let  $\{\mathcal{F}_t\}$  be a filtration, not necessarily satisfying the usual conditions. A *Poisson process* with parameter  $\lambda > 0$  is a stochastic process  $X$  satisfying the following properties:

- (1)  $X_0 = 0$ , a.s.
- (2) The paths of  $X_t$  are right continuous with left limits.
- (3) If  $s < t$ , then  $X_t - X_s$  is a Poisson rv with parameter  $\lambda(t - s)$ .
- (4) If  $s < t$ , then  $X_t - X_s$  is independent of  $\mathcal{F}_s$ .

Define  $X_{t-} = \lim_{s \rightarrow t, s < t} X_s$ , the left-hand limit at time  $t$ , and  $\Delta X_t = X_t - X_{t-}$ , the size of the jump at time  $t$ . We say a function  $f$  is increasing if  $s < t$  implies  $f(s) \leq f(t)$ . We use “strictly increasing” when  $s < t$  implies  $f(s) < f(t)$ . We have the following proposition.

**Proposition 5.2** *Let  $X$  be a Poisson process. With probability one, the paths of  $X_t$  are increasing and are constant except for jumps of size 1. There are only finitely many jumps in each finite time interval.*

*Proof* For any fixed  $s < t$ , we have that  $X_t - X_s$  has the distribution of a Poisson random variable with parameter  $\lambda(t - s)$ , hence is non-negative, a.s.; let  $N_{s,t}$  be the null set of  $\omega$ 's where  $X_t(\omega) < X_s(\omega)$ . The set of pairs  $(s, t)$  with  $s$  and  $t$  rational is countable, and so  $N = \cup_{s,t \in \mathbb{Q}_+} N_{s,t}$  is also a null set, where we write  $\mathbb{Q}_+$  for the non-negative rationals. For  $\omega \notin N$ ,  $X_t \geq X_s$  whenever  $s < t$  are rational. In view of the right continuity of the paths of  $X$ , this shows the paths of  $X$  are increasing with probability one.

Similarly, since Poisson random variables only take values in the non-negative integers,  $X_t$  is a non-negative integer, a.s. Using this fact for every  $t$  rational shows that with probability one,  $X_t$  takes values only in the non-negative integers when  $t$  is rational, and the right continuity of the paths implies this is also the case for all  $t$ . Since the paths have left limits, there can only be finitely many jumps in finite time.

It remains to prove that  $\Delta X_t$  is either 0 or 1 for all  $t$ . Let  $t_0 > 0$ . If there were a jump of size 2 or larger at some time  $t$  strictly less than  $t_0$ , then for each  $n$  sufficiently large there

exists  $0 \leq k_n \leq 2^n$  such that  $X_{(k_n+1)t_0/2^n} - X_{k_nt_0/2^n} \geq 2$ . Therefore

$$\begin{aligned}\mathbb{P}(\exists s < t_0 : \Delta X_s \geq 2) &\leq \mathbb{P}(\exists k \leq 2^n : X_{(k+1)t_0/2^n} - X_{kt_0/2^n} \geq 2) \\ &\leq 2^n \sup_{k \leq 2^n} \mathbb{P}(X_{(k+1)t_0/2^n} - X_{kt_0/2^n} \geq 2) \\ &= 2^n \mathbb{P}(X_{t_0/2^n} \geq 2^n) \\ &\leq 2^n (1 - \mathbb{P}(X_{t_0/2^n} = 0) - \mathbb{P}(X_{t_0/2^n} = 1)) \\ &= 2^n \left(1 - e^{-\lambda t_0/2^n} - (\lambda t_0/2^n) e^{-\lambda t_0/2^n}\right).\end{aligned}\tag{5.1}$$

We used property 5.1(3) for the two equalities. By l'Hôpital's rule,  $(1 - e^{-x} - xe^{-x})/x \rightarrow 0$  as  $x \rightarrow 0$ . We apply this with  $x = \lambda t_0/2^n$ , and see that the last line of (5.1) tends to 0 as  $n \rightarrow \infty$ . Since the left-hand side of (5.1) does not depend on  $n$ , it must be 0. This holds for each  $t_0$ .  $\square$

Another characterization of the Poisson process is as follows. Let  $T_1 = \inf\{t : \Delta X_t = 1\}$ , the time of the first jump. Define  $T_{i+1} = \inf\{t > T_i : \Delta X_t = 1\}$ , so that  $T_i$  is the time of the  $i$ th jump.

**Proposition 5.3** *The random variables  $T_1, T_2 - T_1, \dots, T_{i+1} - T_i, \dots$  are independent exponential random variables with parameter  $\lambda$ .*

*Proof* In view of Corollary 4.3 it suffices to show that  $T_1$  is an exponential random variable with parameter  $\lambda$ . If  $T_1 > t$ , then the first jump has not occurred by time  $t$ , so  $X_t$  is still zero. Hence

$$\mathbb{P}(T_1 > t) = \mathbb{P}(X_t = 0) = e^{-\lambda t},$$

using the fact that  $X_t$  is a Poisson random variable with parameter  $\lambda t$ .  $\square$

We can reverse the characterization in Proposition 5.3 to construct a Poisson process. We do one step of the construction, leaving the rest as Exercise 5.4.

Let  $U_1, U_2, \dots$  be independent exponential random variables with parameter  $\lambda$  and let  $T_j = \sum_{i=1}^j U_i$ . Define

$$X_t(\omega) = k \quad \text{if } T_k(\omega) \leq t < T_{k+1}(\omega).\tag{5.2}$$

An examination of the densities shows that an exponential random variable has a gamma distribution with parameters  $\lambda$  and  $r = 1$ , so by Proposition A.49,  $T_j$  is a gamma random variable with parameters  $\lambda$  and  $j$ . Thus

$$\mathbb{P}(X_t < k) = \mathbb{P}(T_k > t) = \int_t^\infty \frac{\lambda e^{-\lambda x} (\lambda x)^{k-1}}{\Gamma(k)} dx.$$

Performing the integration by parts repeatedly shows that

$$\mathbb{P}(X_t < k) = \sum_{i=0}^{k-1} e^{-\lambda t} \frac{(\lambda t)^i}{i!},$$

and so  $X_t$  is a Poisson random variable with parameter  $\lambda t$ .

We will use the following proposition later.

**Proposition 5.4** Let  $\{\mathcal{F}_t\}$  be a filtration satisfying the usual conditions. Suppose  $X_0 = 0$ , a.s.,  $X$  has paths that are right continuous with left limits,  $X_t - X_s$  is independent of  $\mathcal{F}_s$  if  $s < t$ , and  $X_t - X_s$  has the same law as  $X_{t-s}$  whenever  $s < t$ . If the paths of  $X$  are piecewise constant, increasing, all the jumps of  $X$  are of size 1, and  $X$  is not identically 0, then  $X$  is a Poisson process.

*Proof* Let  $T_0 = 0$  and  $T_{i+1} = \inf\{t > T_i : \Delta X_t = 1\}$ ,  $i = 1, 2, \dots$ . We will show that if we set  $U_i = T_i - T_{i-1}$ , then the  $U_i$  are i.i.d. exponential random variables and then appeal to Exercise 5.4.

By Corollary 4.3, the  $U_i$  are independent and have the same law. Hence it suffices to show  $U_1$  is an exponential random variable. We observe

$$\begin{aligned}\mathbb{P}(U_1 > s + t) &= \mathbb{P}(X_{s+t} = 0) = \mathbb{P}(X_{s+t} - X_s = 0, X_s = 0) \\ &= \mathbb{P}(X_{t+s} - X_s = 0)\mathbb{P}(X_s = 0) = \mathbb{P}(X_t = 0)\mathbb{P}(X_s = 0) \\ &= \mathbb{P}(U_1 > t)\mathbb{P}(U_1 > s).\end{aligned}$$

Setting  $f(t) = \mathbb{P}(U_1 > t)$ , we thus have  $f(t+s) = f(t)f(s)$ . Since  $f(t)$  is decreasing and  $0 < f(t) < 1$ , we conclude  $\mathbb{P}(U_1 > t) = f(t) = e^{-\lambda t}$  for some  $\lambda > 0$ , or  $U_1$  is an exponential random variable.  $\square$

## Exercises

- 5.1 Suppose  $P_t$  is a Poisson process and we write  $X_t = P_{t-}$ . Is  $P_1 - X_{1-t}$  a Poisson process on  $[0, 1]$ ? Why or why not?
- 5.2 Let  $P$  be a Poisson process with parameter  $\lambda$ . Show that

$$\lim_{n \rightarrow \infty} \sup_{t \leq 1} \left| \frac{P_{nt}}{n} - \lambda t \right| = 0, \quad \text{a.s.}$$

- 5.3 Show that if  $P^{(1)}$  and  $P^{(2)}$  are independent Poisson processes with parameters  $\lambda_1$  and  $\lambda_2$ , respectively, then  $P_t^{(1)} + P_t^{(2)}$  is a Poisson process with parameter  $\lambda_1 + \lambda_2$ .
- 5.4 If  $X$  is defined by (5.2), show that  $X$  is a Poisson process.
- 5.5 Let  $X_t$  be a stochastic process and let  $\{\mathcal{F}_t^{00}\}$  be the filtration generated by  $X$ . Suppose  $X$  is a Poisson process with respect to the filtration  $\{\mathcal{F}_t^{00}\}$ . Show that  $X$  is a Poisson process with respect to the minimal augmented filtration generated by  $X$ .

*Hint:* Imitate the proof of Proposition 2.5.

- 5.6 Suppose  $P_t$  is a Poisson process and  $f$  and  $g$  are non-negative bounded deterministic functions with compact support. Find necessary and sufficient conditions on  $f$  and  $g$  so that  $\int_0^\infty f(s) dP_s$  and  $\int_0^\infty g(s) dP_s$  are independent.

*Hint:* First show that the characteristic function of  $F = \int_0^\infty f(s) dP_s$  is

$$\mathbb{E} e^{iuF} = \exp \left( \int_0^\infty (e^{iuf(s)} - 1) ds \right).$$

- 5.7 We will talk about weak convergence in general metric spaces in Chapters 30–35. This exercise is concerned with the weak convergence of real-valued random variables as defined in Section A.12.

Suppose for each  $n$ ,  $P_n$  is a Poisson random variable with parameter  $\lambda_n$  and  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Prove that

$$\frac{P_n - \lambda_n}{\sqrt{\lambda_n}}$$

converges weakly to a normal random variable with mean zero and variance one.

*Hint:* Imitate the proof of Theorem A.51.

# 6

---

## Construction of Brownian motion

There are several ways of constructing Brownian motion, none of them easy. Here we give two constructions. The first is the one that Wiener used, which is based on Fourier series. The second uses martingale techniques. A method due to Lévy can be found in [Bass \(1995\)](#); see also Exercises [6.4](#) and [6.5](#). We will see several other constructions in later chapters.

### 6.1 Wiener's construction

For any of the constructions of Brownian motion, the main step is to construct  $W_t$  for  $t \in [0, 1]$ . Once we have done this, we get Brownian motion for all  $t$  rather easily. More specifically, suppose we have a Brownian motion  $Y^{(0)}$  started at 0 on the time interval  $[0, 1]$ . Take independent copies  $Y^{(1)}, Y^{(2)}, \dots$ , each on  $[0, 1]$ . We have  $Y_0^{(i)} = 0$  for each  $i$ , and now to get Brownian motion started at 0, define  $W_t$  to be equal to  $Y_t^{(0)}$  if  $t \leq 1$ , equal to  $Y_1^{(0)} + Y_{t-1}^{(1)}$  if  $1 < t \leq 2$ , and more generally

$$W_t = \left( \sum_{i=0}^{[t]-1} Y_1^{(i)} \right) + Y_{t-[t]}^{[t]}$$

if  $t \geq 1$ , where  $[t]$  is the largest integer less than or equal to  $t$ . This will give Brownian motion started at 0 on the time interval  $[0, \infty)$ .

Therefore the crux of the problem is to construct Brownian motion on  $[0, 1]$ . Because we are working with Fourier series, it is more convenient to look at Brownian motion on  $[0, \pi]$ ; we can just disregard times between 1 and  $\pi$  when we are done.

Throughout this chapter we make the supposition that we can find a countable sequence  $Z_1, Z_2, \dots$  of independent and identically distributed mean zero normal random variables with variance one that are  $\mathcal{F}$  measurable, where  $(\Omega, \mathcal{F}, \mathbb{P})$  is our probability space. This is an extremely mild condition.

**Theorem 6.1** *There exists a process  $\{W_t; 0 \leq t \leq 1\}$  that is Brownian motion.*

*Proof* If we fix  $t \in [0, \pi]$  and compute the Fourier series for the function  $f(s) = s \wedge t$ , it is an exercise in calculus to get the Fourier coefficients. We end up with

$$s \wedge t = \frac{st}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k^2}. \quad (6.1)$$

This suggests letting  $Z_0, Z_1, \dots$  be i.i.d. normal random variables with mean 0 and variance 1 and setting

$$W_t = \frac{t}{\sqrt{\pi}} Z_0 + \sum_{k=1}^{\infty} \left( \sqrt{\frac{2}{\pi}} \frac{\sin kt}{k} \right) Z_k. \quad (6.2)$$

Assuming there is no problem with convergence, we see that  $W_t$  has mean zero, since each of the  $Z_i$  does, and that

$$\mathbb{E}[W_s W_t] = \frac{st}{\pi} + \sum_{k=1}^{\infty} \frac{2}{\pi} \frac{\sin ks \sin kt}{k^2} = s \wedge t \quad (6.3)$$

as required. We used the independence of the  $Z_i$  here to show that  $\mathbb{E}[Z_i Z_j] = 0$  if  $i \neq j$ .

We argue that there is in fact no difficulty with the convergence. Note that  $\sum_{k=1}^m \frac{\sin^2 kt}{k^2}$  increases as  $m$  increases to a finite limit. Therefore

$$\mathbb{E} \left[ \left( \sum_{k=m}^n Z_k \frac{\sin kt}{k} \right)^2 \right] = \sum_{k=m}^n \frac{\sin^2 kt}{k^2} \rightarrow 0$$

in  $L^2$  as  $m, n \rightarrow \infty$ . This means that the sum on the right of (6.2) is a Cauchy sequence in  $L^2$ . By the completeness of  $L^2$ , the sum on the right of (6.2) converges in  $L^2$ . A use of the Cauchy–Schwarz inequality allows us to justify the formula for the expectation of  $W_s W_t$ .

If we let

$$W_t^j = \frac{t}{\sqrt{\pi}} Z_0 + \sum_{k=1}^j \left( \sqrt{\frac{2}{\pi}} \frac{\sin kt}{k} \right) Z_k,$$

then  $(W_{t_1}^j, \dots, W_{t_m}^j)$  is a jointly normal collection of random variables for each  $j$  whenever  $t_1, \dots, t_n \in [0, \pi]$ . By Remark A.56, it follows that  $(W_{t_1}, \dots, W_{t_m})$  is a jointly normal collection of random variables. Therefore  $W_t$  is a Gaussian process. Since each  $W_t$  has mean zero and  $\text{Cov}(W_s, W_t) = s \wedge t$ , then  $W_t$  has the correct finite-dimensional distributions to be a Brownian motion.

The only part remaining to the construction is to show that  $W_t$  as constructed above has continuous paths, for we can then use Theorem 2.4. In what follows, pay attention to where the absolute values are placed. If one is cavalier about placing them, one will very likely run into trouble.

Define

$$S_m(t) = \sum_{k=m}^{2m-1} \frac{\sin kt}{k} Z_k$$

and let  $T_m = \sup_{0 \leq t \leq \pi} |S_m(t)|$ . We write

$$W_t = \frac{t}{\sqrt{\pi}} Z_0 + \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} S_{2^n}(t).$$

We will show

$$\mathbb{E} T_m^2 \leq \frac{c}{m^{1/2}}. \quad (6.4)$$

Once we have this, then by the Fubini theorem and then Jensen's inequality,

$$\mathbb{E} \sum_{n=0}^{\infty} T_{2^n} = \sum_{n=0}^{\infty} \mathbb{E} T_{2^n} \leq \sum_{n=0}^{\infty} (\mathbb{E} [T_{2^n}^2])^{1/2} < \infty.$$

Therefore  $\sum_{n=0}^{\infty} T_{2^n} < \infty$ , a.s., and by the Weierstrass  $M$ -test (see, e.g., Rudin, 1976), we have that with probability 1,  $\sum_{n=0}^{\infty} S_{2^n}(t)$  converges uniformly in  $t$ . Since each  $S_{2^n}(t)$  is a continuous function of  $t$ , we see that the uniform limit is also continuous and we are done.

We therefore have to prove (6.4). Using  $|\sum_k a_k|^2 = \sum_{j,k} a_k \bar{a}_j$  for  $a_k$  complex valued, we have

$$\begin{aligned} T_m^2 &\leq \sup_{0 \leq t \leq \pi} \left| \sum_{k=m}^{2m-1} \frac{e^{ikt}}{k} Z_k \right|^2 \\ &\leq \sup_{0 \leq t \leq \pi} \left| \sum_{j,k=m}^{2m-1} \frac{e^{ikt} e^{-ijt}}{jk} Z_j Z_k \right| \\ &\leq \sum_{k=m}^{2m-1} \frac{1}{k^2} Z_k^2 + 2 \sup_{0 \leq t \leq \pi} \left| \sum_{\ell=1}^{m-1} \sum_{j=m}^{2m-\ell-1} \frac{e^{i\ell t}}{j(j+\ell)} Z_j Z_{j+\ell} \right| \\ &\leq \sum_{k=m}^{2m-1} \frac{1}{k^2} Z_k^2 + 2 \sum_{\ell=1}^{m-1} \left| \sum_{j=m}^{2m-\ell-1} \frac{1}{j(j+\ell)} Z_j Z_{j+\ell} \right|. \end{aligned} \quad (6.5)$$

In the third inequality we wrote

$$\sum_{j,k=m}^{2m-1} = \sum_{m \leq j=k \leq 2m-1} + 2 \sum_{m \leq j < k \leq 2m-1},$$

and then set  $\ell = k - j$ . Write  $I$  for the first sum on the last line of (6.5) and  $J_\ell$  for  $\sum_{j=m}^{2m-\ell-1} \frac{1}{j(j+\ell)} Z_j Z_{j+\ell}$ . The expectation of  $I$  is equal to

$$\sum_{k=m}^{2m-1} \frac{1}{k^2} \leq \frac{c}{m}.$$

We next look at the expectation of the  $J_\ell$ . Since the  $Z_i$  are mean zero and independent,  $\mathbb{E}[Z_{i_1} Z_{i_2} Z_{i_3} Z_{i_4}]$  is zero unless either all four subscripts are equal or else two subscripts are equal and the other two subscripts are also equal. By Jensen's inequality,

$$\begin{aligned} \mathbb{E} |J_\ell| &\leq \left( \mathbb{E} \left[ \sum_{j=m}^{2m-\ell-1} \frac{1}{j(j+\ell)} Z_j Z_{j+\ell} \right]^2 \right)^{1/2} \\ &= \left( \sum_{j=m}^{2m-\ell-1} \frac{1}{j^2(j+\ell)^2} \right)^{1/2}. \end{aligned} \quad (6.6)$$

The last equality follows by multiplying out

$$\left( \sum_j \frac{Z_j Z_{j+\ell}}{j(j+\ell)} \right)^2$$

and noting that expectations of the cross-product terms are zero. Since  $j \geq m$  in the last line of (6.6) and there are at most  $m$  terms in the sum, the last line of (6.6) is bounded by  $(cm/m^4)^{1/2} = cm^{-3/2}$ . Therefore

$$\mathbb{E} \sum_{\ell=1}^{m-1} |J_\ell| \leq c/m^{1/2}.$$

Substituting in (6.5) completes the proof of (6.4).  $\square$

By Proposition 2.5, the Brownian motion that we constructed is a Brownian motion with respect to the minimal augmented filtration.

## 6.2 Martingale methods

Here, we use martingale methods to take care of the continuity of the paths. We proceed as in the previous section to construct  $\{W_t; 0 \leq t \leq \pi\}$ , where  $W_t$  is a Gaussian process with  $\mathbb{E} W_t = 0$  and  $\text{Cov}(W_s, W_t) = s \wedge t$ , and we need to show that  $W$  has a version with continuous paths. We show that  $W$  is a martingale, and so has a version with paths that are right continuous with left limits. We use Doob's inequalities to control the oscillation of  $W$  over short time intervals, and then use the Borel–Cantelli lemma to show continuity.

**Theorem 6.2** *If  $\{W_t; t \leq 1\}$  is a Gaussian process with  $\mathbb{E} W_t = 0$  for all  $t \leq 1$  and  $\text{Cov}(W_s, W_t) = s \wedge t$  for all  $s, t \leq 1$ , then there is a version of  $W$  that is a Brownian motion on  $[0, 1]$ .*

*Proof* As in the proof of Theorem 6.1, we need to show that  $W$  has a version with continuous paths. Since  $\text{Cov}(W_t - W_s, W_r) = r - r = 0$  if  $r \leq s < t$ , we see by Proposition A.55 that  $W_t - W_s$  is independent of  $\mathcal{F}_s^{00} = \sigma(W_r; r \leq s)$ . Then

$$\mathbb{E}[W_t - W_s | \mathcal{F}_s^{00}] = \mathbb{E}[W_t - W_s] = 0,$$

so  $W_t$  is a martingale. By Theorem 3.12, with probability one,  $W$  has left and right limits along  $\mathcal{D}$ , the dyadic rationals. Let  $W'_t = \lim_{u>t, u \in \mathcal{D}, u \rightarrow t} W_u$ . Since  $\mathbb{E}(W_u - W_t)^2 = u - t \rightarrow 0$  as  $u \rightarrow t$ , then  $W'_t = W_t$ , a.s., or  $W'$  is a version of  $W$  with paths that are right continuous with left limits. We now drop the primes. Set  $W_t = W_1$  if  $t \geq 1$ .

For any  $t_0 \in [0, 1]$ ,  $W_{t+t_0} - W_{t_0}$  is also a martingale, and by Jensen's inequality for conditional expectations (Proposition A.21),  $|W_{t+t_0} - W_{t_0}|^4$  is a submartingale. Using Doob's inequalities (Theorem 3.6), if  $\lambda > 0$  and  $t_0, \delta \in [0, 1]$ ,

$$\begin{aligned} \mathbb{P}(\sup_{t_0 \leq t \leq t_0 + \delta} |W_t - W_{t_0}| \geq \lambda) &= \mathbb{P}(\sup_{t_0 \leq t \leq t_0 + \delta} |W_t - W_{t_0}|^4 \geq \lambda^4) \\ &\leq c \frac{\mathbb{E} |W_{t_0 + \delta} - W_{t_0}|^4}{\lambda^4}. \end{aligned}$$

Since  $W_{t_0 + \delta} - W_{t_0}$  is a mean zero normal random variable with variance  $\delta$  if  $t_0 + \delta \leq 1$ , we have

$$\mathbb{P}(\sup_{t_0 \leq t \leq t_0 + \delta} |W_t - W_{t_0}| \geq \lambda) \leq c \frac{\delta^2}{\lambda^4}. \quad (6.7)$$

Let

$$A_n = \{\exists k \leq 2^n : \sup_{k/2^n \leq t \leq (k+2)/2^n} |W_t - W_{k/2^n}| > 2^{-n/8}\}.$$

From (6.7) with  $\delta = 2^{-n+1}$  and  $\lambda = 2^{-n/8}$ ,

$$\begin{aligned} \mathbb{P}(A_n) &\leq 2^n \max_{k \leq 2^n} \mathbb{P}\left(\sup_{k/2^n \leq t \leq (k+2)/2^n} |W_t - W_{k/2^n}| > 2^{-n/8}\right) \\ &\leq \frac{c2^n 2^{-2n}}{2^{-n/2}} = c2^{-n/2}, \end{aligned}$$

which is summable. By the Borel–Cantelli lemma,  $\mathbb{P}(A_n \text{ i.o.}) = 0$ . (The event ( $A_n$  i.o.) is the event where  $\omega$  is in infinitely many of the  $A_n$ .)

Except for a set of  $\omega$ 's in a null set, there exists a positive integer  $N$  (which will depend on  $\omega$ ) such that if  $n \geq N$ , then  $\omega \notin A_n$ . Given  $\varepsilon > 0$ , take  $n \geq N$  such that  $2^{-n/8} < \varepsilon/2$ . If  $|t - s| \leq 2^{-n}$  with  $s, t \in [0, 1]$ , then  $s, t \in [k/2^n, (k+2)/2^n]$  for some  $k \leq 2^n$ . Since  $\omega \notin A_n$ ,

$$|W_t - W_s| \leq |W_t - W_{k/2^n}| + |W_s - W_{k/2^n}| \leq 2 \cdot 2^{-n/8} < \varepsilon.$$

This proves the continuity of  $W_t$ .  $\square$

There is nothing special about the trigonometric polynomials in this second construction. Let  $\langle f, g \rangle = \int_0^1 f(r)g(r) dr$  be the inner product for the Hilbert space  $L^2[0, 1]$ ; we consider only real-valued functions for simplicity. Let  $\{\varphi_n\}$  be a complete orthonormal system for  $L^2[0, 1]$ : we have  $\langle \varphi_m, \varphi_n \rangle = 0$  if  $m \neq n$ ,  $\langle \varphi_n, \varphi_n \rangle = 1$  for each  $n$ , and  $f = 0$ , a.e., if  $\langle f, \varphi_n \rangle = 0$  for all  $n$ . One property of a complete orthonormal system is Parseval's identity, which says that

$$\langle f, f \rangle = \sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2;$$

see Folland (1999). If we replace  $f$  by  $g$  and then by  $f + g$  and use

$$\langle f, g \rangle = \frac{1}{2}[\langle f + g, f + g \rangle - \langle f, f \rangle - \langle g, g \rangle],$$

we obtain

$$\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \langle g, \varphi_n \rangle.$$

Now let

$$a_n(t) = \langle 1_{[0,t]}, \varphi_n \rangle = \int_0^t \varphi_n(r) dr.$$

If  $Z_1, Z_2, \dots$  are independent mean zero normal random variables with variance one, let

$$W_t = \sum_{n=1}^{\infty} a_n(t) Z_k. \tag{6.8}$$

Assuming there is no difficulty with the convergence, we have

$$\begin{aligned}\text{Cov}(W_s, W_t) &= \sum_{n=1}^{\infty} a_n(s)a_n(t) = \sum_{n=1}^{\infty} \langle 1_{[0,s]}, \varphi_n \rangle \langle 1_{[0,t]}, \varphi_n \rangle \\ &= \langle 1_{[0,s]}, 1_{[0,t]} \rangle = s \wedge t.\end{aligned}$$

Exercise 6.2 asks you to verify that the process  $W$  defined by (6.8) is a mean zero Gaussian process on  $[0, 1]$  with the same covariances as a Brownian motion.

## Exercises

- 6.1 Let  $Z_0, Z_1, Z_2, \dots$  be a sequence of independent identically distributed mean zero normal random variables with variance one. Define

$$X_t = \frac{t^2}{2\sqrt{\pi}}Z_0 + \sum_{k=1}^{\infty} \left( \sqrt{\frac{2}{\pi}} \frac{\cos kt}{k^2} \right) Z_k. \quad (6.9)$$

- (1) Show that the convergence in (6.9) is absolute and uniform over  $t \in [0, 1]$ .
- (2) Show that  $X_t$  is a Gaussian process.
- (3) If  $W_t$  is a Brownian motion and

$$Y_t = \int_0^t W_r dr, \quad t \in [0, 1],$$

show that  $X$  and  $Y$  have the same finite-dimensional distributions. Show that  $X$  and  $Y$  have the same law when viewed as random variables taking values in  $C[0, 1]$ . (The process  $X$  is sometimes known as *integrated Brownian motion*.)

- (4) Find  $\text{Cov}(X_s, X_t)$ .

- 6.2 Let  $\{\varphi_n\}$  be a complete orthonormal system for  $L^2[0, 1]$ . Show that the sum (6.8) converges in  $L^2$  and give the details of the proof that the resulting process  $W$  is a mean zero Gaussian process with  $\text{Cov}(W_s, W_t) = s \wedge t$  if  $s, t \in [0, 1]$ .
- 6.3 Let  $\mathcal{D} = \{k/2^n : n \geq 1, k = 0, 1, \dots, 2^n\}$  be the dyadic rationals. Suppose the collection of random variables  $\{V_t : t \in \mathcal{D}\}$  is jointly normal, each  $V_t$  has mean zero, and  $\text{Cov}(V_s, V_t) = s \wedge t$ .
- (1) Prove that the paths of  $V$  are uniformly continuous over  $t \in \mathcal{D}$ .
  - (2) If we define  $W_t = \lim_{s \in \mathcal{D}, s \rightarrow t} V_s$ , prove that  $W$  is a Brownian motion.
- 6.4 In this and the next exercise we give the Haar function construction of Brownian motion. Let  $\varphi_{00} = 1$  on  $[0, 1]$  and for  $i = 1, 2, \dots$ , and  $1 \leq j \leq 2^{i-1}$ , set

$$\varphi_{ij}(x) = \begin{cases} 2^{(i-1)/2}, & (2j-2)/2^i \leq x < (2j-1)/2^i, \\ -2^{(i-1)/2}, & (2j-1)/2^i \leq x < 2j/2^i, \\ 0, & \text{otherwise.} \end{cases}$$

It is a well-known and easily proved result from analysis (see, e.g., Bass (1995), Section I.2) that the collection  $\{\varphi_{ij}\}$  is a complete orthonormal system for  $L^2[0, 1]$ .

For each  $i, j$ , define

$$\psi_{ij}(t) = \int_0^t \varphi_{ij}(s) ds,$$

for each  $i$  and  $j$ , let  $Y_{ij}$  be independent mean zero normal random variables with variance one, and let

$$V_i(t) = \sum_{j=1}^{2^{i-1}} Y_{ij} \varphi_{ij}(t)$$

for  $i \geq 1$ . Set  $V_0 = Y_{00}\varphi_{00}$ .

- (1) Fix  $i \geq 1$ . Prove that each  $\psi_{ij}$  is bounded by  $2^{(-i-1)/2}$ . Prove that the sets  $\{t : \psi_{ij}(t) > 0\}$ ,  $j = 1, \dots, 2^{i-1}$ , are disjoint.
- (2) Fix  $i \geq 1$ . Write

$$\mathbb{P}(\exists t \in [0, 1] : |V_i(t)| > i^{-2}) \leq \mathbb{P}(\exists j \leq 2^{i-1} : |Y_{ij}| 2^{(-i-1)/2} > i^{-2}),$$

use Proposition A.52 to estimate this, and conclude that

$$\sum_{i=1}^{\infty} \mathbb{P}(\sup_{0 \leq t \leq 1} |V_i(t)| > i^{-2}) < \infty. \quad (6.10)$$

6.5 This is a continuation of Exercise 6.4. With  $\varphi_{ij}$ ,  $\psi_{ij}$ ,  $Y_{ij}$ , and  $V_i$  as in that problem, let

$$W_t = \sum_{i=0}^{\infty} V_i(t).$$

- (1) Prove that  $W$  is a jointly normal Gaussian process with mean zero and  $\text{Cov}(W_s, W_t) = s \wedge t$ .
- (2) Use (6.10) and the Borel–Cantelli lemma to show that  $\sum_{i=1}^n |V_i(t)|$  converges uniformly over  $[0, 1]$ . Conclude that  $W$  is a Brownian motion.

---

## Path properties of Brownian motion

The paths of Brownian motion are continuous, but we will see that they are not differentiable. How continuous are they? We will see that the paths satisfy what is known as a Hölder continuity condition. A precise description of the oscillatory behavior of Brownian motion will be given by the law of the iterated logarithm.

A function  $f: [0, 1] \rightarrow \mathbb{R}$  is said to be *Hölder continuous of order  $\alpha$*  if there exists a constant  $M$  such that

$$|f(t) - f(s)| \leq M|t - s|^\alpha, \quad s, t \in [0, 1]. \quad (7.1)$$

We show that the paths of Brownian motion are Hölder continuous of order  $\alpha$  if  $\alpha < \frac{1}{2}$ . (They are also not Hölder continuous of order  $\alpha$  if  $\alpha \geq \frac{1}{2}$ ; we will see this from the law of the iterated logarithm.)

**Theorem 7.1** *If  $\alpha < \frac{1}{2}$ , the paths of Brownian motion are Hölder continuous of order  $\alpha$  on  $[0, 1]$ .*

*Proof* *Step 1.* First we apply the Borel–Cantelli lemma to a certain sequence of sets. Let  $W$  be a Brownian motion and set

$$A_n = \{\exists k \leq 2^n - 1 : \sup_{k/2^n \leq t \leq (k+1)/2^n} |W_t - W_{k/2^n}| > 2^{-n\alpha}\}.$$

Since  $W_{t+k/2^n} - W_{k/2^n}$  is a Brownian motion,

$$\begin{aligned} \mathbb{P}(A_n) &\leq 2^n \sup_{k \leq 2^n} \mathbb{P}(\sup_{t \leq 1/2^n} |W_{t+k/2^n} - W_{k/2^n}| > 2^{-n\alpha}) \\ &\leq 2^n \mathbb{P}(\sup_{t \leq 1/2^n} |W_t| > 2^{-n\alpha}) \\ &\leq 2 \cdot 2^n \exp(-2^{-2n\alpha}/2(2^{-n})). \end{aligned} \quad (7.2)$$

Here we used Proposition 3.15. Since  $\alpha < \frac{1}{2}$ , then  $2^{n(1-2\alpha)} > 2n$  for  $n$  large, and the last line of (7.2) is less than

$$2^{n+1} \exp(-2^{n(1-2\alpha)}/2) \leq 2^{n+1} e^{-n}$$

if  $n$  is large. Hence  $\sum \mathbb{P}(A_n) < \infty$ , and  $\mathbb{P}(A_n \text{ i.o.}) = 0$  by the Borel–Cantelli lemma.

*Step 2.* Next we show that this implies the Hölder continuity. For almost every  $\omega$  there exists  $N$  (depending on  $\omega$ ) such that if  $n \geq N$ , then  $\omega \notin A_n$ . Let  $s \leq t$  be two points in  $[0, 1]$ . If  $2^{-(n+2)} \leq t - s \leq 2^{-(n+1)}$  for some  $n \geq N$  and  $k$  is the largest integer such that

$k/2^{n+2} \leq s$ , then

$$\begin{aligned}|W_t - W_s| &\leq |W_t - W_{t \wedge ((k+1)/2^{n+2})}| + |W_{t \wedge ((k+1)/2^{n+2})} - W_{k/2^{n+2}}| \\&\quad + |W_s - W_{k/2^{n+2}}| \\&\leq 3 \cdot 2^{-n\alpha} \leq 3 \cdot 4^\alpha |t - s|^\alpha.\end{aligned}$$

We know  $|W_t(\omega)|$  is bounded on  $[0, 1]$  since the paths are continuous; let  $K$  (depending on  $\omega$ ) be the bound. If  $|t - s| \geq 2^{-(N+1)}$ , then

$$|W_t - W_s| \leq 2K \leq (2K)(2^{N+1})|t - s| \leq (2K)(2^{N+1})|t - s|^\alpha.$$

Thus, no matter whether  $|t - s|$  is small or large, there exists  $L$  (depending on  $\omega$ ) such that  $|W_t(\omega) - W_s(\omega)| \leq L|t - s|^\alpha$  for all  $s, t \in [0, 1]$ .  $\square$

One of the most beautiful theorems in probability theory is the law of the iterated logarithm (LIL). It describes precisely how Brownian motion oscillates.

**Theorem 7.2** *Let  $W$  be a Brownian motion. We have*

$$\limsup_{t \rightarrow \infty} \frac{|W_t|}{\sqrt{2t \log \log t}} = 1, \quad \text{a.s.}$$

and

$$\limsup_{t \rightarrow 0} \frac{|W_t|}{\sqrt{2t \log \log(1/t)}} = 1, \quad \text{a.s.}$$

*Proof* The second assertion follows from the first by time inversion; see Exercise 2.5. Thus we only need to prove the first assertion.

*Proof of upper bound:* We use the Borel–Cantelli lemma. Let  $\varepsilon > 0$  and then choose  $q$  larger than 1 but close enough to 1 so that  $(1 + \varepsilon)^2/q > 1$ . Let

$$A_n = (\sup_{s \leq q^n} |W_s| > (1 + \varepsilon)\sqrt{2q^{n-1} \log \log q^{n-1}}).$$

By Proposition 3.15,

$$\begin{aligned}\mathbb{P}(A_n) &\leq 2 \exp\left(-\frac{(1 + \varepsilon)^2 2q^{n-1} \log \log q^{n-1}}{2q^n}\right) \\&= 2 \exp\left(-\frac{(1 + \varepsilon)^2}{q} (\log(n-1) + \log \log q)\right) = \frac{c}{(n-1)^{(1+\varepsilon)^2/q}},\end{aligned}$$

where we are using our convention that the letter  $c$  denotes a constant whose exact value is unimportant. This is summable in  $n$ , so  $\sum \mathbb{P}(A_n) < \infty$ .

By the Borel–Cantelli lemma,  $\mathbb{P}(A_n \text{ i.o.}) = 0$ . Hence, except for a null set, there exists  $N = N(\omega)$  such that  $\omega \notin A_n$  if  $n \geq N(\omega)$ . If  $t \geq q^N$ , then for some  $n \geq N + 1$  we have  $q^{n-1} \leq t \leq q^n$ , and

$$|W_t| \leq \sup_{s \leq q^n} |W_s| \leq (1 + \varepsilon)\sqrt{2q^{n-1} \log \log q^{n-1}} \leq (1 + \varepsilon)\sqrt{2t \log \log t}.$$

Therefore

$$\limsup_{t \rightarrow \infty} \frac{|W_t|}{\sqrt{2t \log \log t}} \leq 1 + \varepsilon, \quad \text{a.s.} \quad (7.3)$$

Since  $\varepsilon > 0$  is arbitrary, the upper bound is proved.

*Proof of lower bound:* We start with the second half of the Borel–Cantelli lemma. Let  $\varepsilon > 0$  and then take  $q > 1$  very large so that

$$\frac{(1 - \varepsilon)^2(1 + \varepsilon)}{1 - q^{-1}} < 1$$

and  $2/\sqrt{q} < \varepsilon/2$ . This is possible because  $(1 - \varepsilon)^2(1 + \varepsilon) = (1 - \varepsilon^2)(1 - \varepsilon) < 1$ . Let

$$B_n = (W_{q^{n+1}} - W_{q^n} > (1 - \varepsilon)\sqrt{2q^{n+1} \log \log q^{n+1}}).$$

Since Brownian motion has independent increments, the events  $B_n$  are independent. Let

$$Z = \frac{W_{q^{n+1}} - W_{q^n}}{\sqrt{q^{n+1} - q^n}}.$$

Then  $Z$  is a mean zero normal random variable with variance one. By Proposition A.52, we see that

$$\begin{aligned} \mathbb{P}(B_n) &= \mathbb{P}(Z > (1 - \varepsilon)\sqrt{2q^{n+1} \log \log q^{n+1}}/\sqrt{q^{n+1} - q^n}) \\ &\geq \exp\left(-\frac{(1 - \varepsilon)^2(1 + \varepsilon)2q^{n+1} \log \log q^{n+1}}{2(q^{n+1} - q^n)}\right) \\ &= c \exp\left(-(1 - \varepsilon)^2(1 + \varepsilon)\frac{\log(n+1) + \log \log q}{1 - q^{-1}}\right) \end{aligned}$$

for  $n$  large. Hence

$$\sum_n \mathbb{P}(B_n) \geq c \sum_n \frac{1}{(n+1)^{(1-\varepsilon)^2(1+\varepsilon)/(1-q^{-1})}} = \infty.$$

By the Borel–Cantelli lemma, with probability one,  $\omega$  is in infinitely many  $B_n$ . Consequently, with probability one, infinitely often

$$W_{q^{n+1}} - W_{q^n} > (1 - \varepsilon)\sqrt{2q^{n+1} \log \log q^{n+1}}. \quad (7.4)$$

The inequality (7.4) is not exactly what we want, as we want a lower bound for  $W_{q^{n+1}}$ , but we can derive the desired lower bound by using the upper bound we proved in Step 1. We know from (7.3) that for  $n$  large enough,

$$|W_{q^n}| \leq 2\sqrt{2q^n \log \log q^n} \leq \frac{2}{\sqrt{q}}\sqrt{2q^{n+1} \log \log q^{n+1}} < \frac{\varepsilon}{2}\sqrt{2q^{n+1} \log \log q^{n+1}}.$$

Thus infinitely often

$$W_{q^{n+1}} > (1 - 3\varepsilon/2)\sqrt{2q^{n+1} \log \log q^{n+1}}.$$

This proves

$$\limsup_{n \rightarrow \infty} \frac{W_{q^{n+1}}}{\sqrt{2q^{n+1} \log \log q^{n+1}}} \geq 1 - \frac{3\varepsilon}{2}, \quad \text{a.s.}$$

Since  $\varepsilon$  is arbitrary, the lower bound follows.  $\square$

The law of the iterated logarithm show that the paths of  $W_t$  are not differentiable at time 0, a.s. Applying this to  $W_{s+t} - W_t$ , we see that for each  $t$ ,  $W$  is not differentiable at time  $t$ , a.s. But the null set  $N_t$  might depend on  $t$ , and it is even conceivable that  $\cup_{t \in [0,1]} N_t$  is not a null set. We have the following stronger result, which says that except for a set of  $\omega$ 's that form a null set,  $t \rightarrow W_t(\omega)$  is a function that does not have a derivative at any time  $t \in [0, 1]$ .

**Theorem 7.3** *With probability one, the paths of Brownian motion are nowhere differentiable.*

*Proof* Note that if  $Z$  is a normal random variable with mean 0 and variance 1, then

$$\mathbb{P}(|Z| \leq r) = \frac{1}{\sqrt{2\pi}} \int_{-r}^r e^{-x^2/2} dx \leq 2r. \quad (7.5)$$

Let  $M, h > 0$  and let

$$\begin{aligned} A_{M,h} &= \{\exists s \in [0, 1] : |W_t - W_s| \leq M|t - s| \text{ if } |t - s| \leq h\}, \\ B_n &= \{\exists k \leq 2n : |W_{k/n} - W_{(k-1)/n}| \leq 4M/n, \\ &\quad |W_{(k+1)/n} - W_{k/n}| \leq 4M/n, |W_{(k+2)/n} - W_{(k+1)/n}| \leq 4M/n\}. \end{aligned}$$

We check that  $A_{M,h} \subset B_n$  if  $n \geq 2/h$ . To see this, if  $\omega \in A_{M,h}$ , there exists an  $s$  such that  $|W_t - W_s| \leq M|t - s|$  if  $|t - s| \leq 2/n$ ; let  $k/n$  be the largest multiple of  $1/n$  less than or equal to  $s$ . Then

$$|(k+2)/n - s| \leq 2/n \quad \text{and} \quad |(k+1)/n - s| \leq 2/n,$$

and therefore

$$\begin{aligned} |W_{(k+2)/n} - W_{(k+1)/n}| &\leq |W_{(k+2)/n} - W_s| + |W_s - W_{(k+1)/n}| \\ &\leq 2M/n + 2M/n < 4M/n. \end{aligned}$$

Similarly  $|W_{(k+1)/n} - W_{k/n}|$  and  $|W_{k/n} - W_{(k-1)/n}|$  are less than  $4M/n$ .

Using the independent increments property, the stationary increments property, and (7.5),

$$\begin{aligned} \mathbb{P}(B_n) &\leq 2n \sup_{k \leq 2n} \mathbb{P}(|W_{k/n} - W_{(k-1)/n}| < 4M/n, |W_{(k+1)/n} - W_{k/n}| < 4M/n, \\ &\quad |W_{(k+2)/n} - W_{(k+1)/n}| < 4M/n) \\ &\leq 2n \mathbb{P}(|W_{1/n}| < 4M/n, |W_{2/n} - W_{1/n}| < 4M/n, \\ &\quad |W_{3/n} - W_{2/n}| < 4M/n) \\ &= 2n \mathbb{P}(|W_{1/n}| < 4M/n) \mathbb{P}(|W_{2/n} - W_{1/n}| < 4M/n) \\ &\quad \times \mathbb{P}(|W_{3/n} - W_{2/n}| < 4M/n) \\ &= 2n (\mathbb{P}(|W_{1/n}| < 4M/n))^3 \\ &\leq cn \left( \frac{4M}{\sqrt{n}} \right)^3, \end{aligned}$$

which tends to 0 as  $n \rightarrow \infty$ . Hence for each  $M$  and  $h$ ,

$$\mathbb{P}(A_{M,h}) \leq \limsup_{n \rightarrow \infty} \mathbb{P}(B_n) = 0.$$

This implies that the probability that there exists  $s \leq 1$  such that

$$\limsup_{h \rightarrow 0} \frac{|W_{s+h} - W_s|}{|h|} \leq M$$

is zero. Since  $M$  is arbitrary, this proves the theorem.  $\square$

## Exercises

- 7.1 Here you are asked to find a more precise description of the modulus of continuity of Brownian paths. Prove that

$$\lim_{\delta \rightarrow 0} \sup_{s,t \in [0,1], 0 < |t-s| < \delta} \frac{|W_t - W_s|}{\sqrt{\delta \log(1/\delta)}} < \infty, \quad \text{a.s.}$$

*Hint:* Imitate the proof of Theorem 7.1.

- 7.2 The following is part of what is known as *Chung's law of the iterated logarithm*. We will see in Section 40.3 that there exists  $c_1$  such that

$$\mathbb{P}(\sup_{s \leq t} |W_s| \leq \lambda) \leq c_1 e^{-\pi^2 t / 8\lambda^2}$$

for  $t/\lambda^2$  sufficiently large. Prove that

$$\liminf_{t \rightarrow \infty} \frac{\sup_{s \leq t} |W_s|}{\sqrt{t / \log \log t}} < \infty, \quad \text{a.s.}$$

- 7.3 Let  $W_t$  be a one-dimensional Brownian motion. We will see in Section 40.3 that there exists  $c_2$  such that

$$\mathbb{P}(\sup_{s \leq t} |W_s| \leq \lambda) \geq c_2 e^{-\pi^2 t / 8\lambda^2}$$

if  $t/\lambda^2$  is sufficiently large. Prove that

$$\liminf_{t \rightarrow \infty} \frac{\sup_{s \leq t} |W_s|}{\sqrt{t / \log \log t}} > 0, \quad \text{a.s.}$$

This is the other half of Chung's law of the iterated logarithm. In fact,

$$\liminf_{t \rightarrow \infty} \frac{\sup_{s \leq t} |W_s|}{\sqrt{t / \log \log t}} = c, \quad \text{a.s.} \tag{7.6}$$

Identify  $c$  and prove (7.6).

- 7.4 A function  $f$  is Hölder continuous of order  $\alpha$  at a point  $t$  if there exists  $c$  such that  $|f(u) - f(t)| \leq c|u - t|^\alpha$  for all  $u$ . Suppose  $\alpha > 1/2$  and  $W_t$  is a Brownian motion. Show that the event

$$A = \{\exists t \in [0, 1] : W \text{ is Hölder continuous of order } \alpha \text{ at } t\}$$

has probability 0.

*Hint:* Imitate the proof of nowhere differentiability, but use more than three time intervals.

- 7.5 Let  $W$  be a one-dimensional Brownian motion and let  $M_t = \sup_{s \leq t} W_s$  (with no absolute value signs). Prove that if  $\varepsilon > 0$ , then

$$\liminf_{t \rightarrow \infty} \frac{M_t}{\sqrt{t}/(\log t)^{1+\varepsilon}} > 0, \quad \text{a.s.}$$

- 7.6 This is a complement to Exercise 4.10. Prove that if  $p > 2$  and  $W$  is a Brownian motion, then the  $p$ -variation of  $W$ , defined in Exercise 4.10, is finite, a.s.

*Hint:* Use the fact that the paths of Brownian motion are Hölder continuous of order  $\alpha$  if  $\alpha < 1/2$ .

- 7.7 Let  $W$  be a Brownian motion and let  $Z$  be the zero set:  $Z = \{t \in [0, 1] : W_t = 0\}$ .

(1) Show there exists a constant  $c$  not depending on  $x$  or  $\delta$  such that

$$\mathbb{P}(\exists s \leq \delta : W_s = -x) \leq \mathbb{P}(\sup_{s \leq \delta} |W_s| \geq |x|) \leq ce^{-x^2/2\delta}.$$

(2) Use the Markov property of Brownian motion to show that there exists a constant  $c$  not depending on  $s$  or  $t$  such that

$$\mathbb{P}(Z \cap [s, t] \neq \emptyset) \leq c \left(1 \wedge \sqrt{\frac{t-s}{s}}\right).$$

- 7.8 Given a Borel measurable subset  $A$  of  $[0, 1]$ , define

$$H_\gamma(A) = \limsup_{\delta \rightarrow 0} \left[ \inf \left\{ \sum_{i=1}^{\infty} [b_i - a_i]^\gamma : A \subset \bigcup_{i=1}^{\infty} [a_i, b_i], \sup_i |b_i - a_i| \leq \delta \right\} \right].$$

In other words, cover  $A$  by the union of intervals  $[a_i, b_i]$  and define the analog of Lebesgue measure. The differences are that we look at  $|b_i - a_i|^\gamma$  but do not require that  $\gamma$  be one, and we require that none of the intervals be longer than  $\delta$ . The quantity  $H_\gamma(A)$  is called the *Hausdorff measure* of  $A$  with respect to the function  $x^\gamma$ . The *Hausdorff dimension* of a set  $A$  is defined to be

$$\inf\{\gamma : H_\gamma(A) > 0\} = \sup\{\gamma : H_\gamma(A) = \infty\}.$$

(For subsets of  $\mathbb{R}^d$ , we replace the intervals  $[a_i, b_i]$  by balls of radius  $r_i$ .) As a warm-up to this exercise, prove that the Hausdorff dimension of the standard Cantor set in  $[0, 1]$  is  $\log 2 / \log 3$ .

The purpose of this exercise is to show that if  $W$  is a Brownian motion and  $Z = \{t \in [0, 1] : W_t = 0\}$  is the zero set, then the Hausdorff dimension of  $Z$  is no more than  $1/2$ .

(1) For each  $n$ , let  $\mathcal{C}_n$  be the collection of intervals  $[i/2^n, (i+1)/2^n]$  contained in  $[0, 1]$  that intersect  $Z$ . ( $\mathcal{C}_n$  is random.) If  $\#\mathcal{C}_n$  is the cardinality of  $\mathcal{C}_n$ , use Exercise 7.7 to show

$$\mathbb{E}[\#\mathcal{C}_n] \leq \sum_{i=0}^{2^n-1} \mathbb{P}(Z \cap [i/2^n, (i+1)/2^n] \neq \emptyset) \leq c2^{n/2}.$$

(2) Write

$$\sum_{[i/2^n, (i+1)/2^n] \in \mathcal{C}_n} |2^{-n}|^\gamma = 2^{-n\gamma} \#\mathcal{C}_n.$$

Use the Chebyshev inequality and (1) to conclude that the Hausdorff dimension of  $Z$  is less than or equal to  $1/2$ , a.s. (We will show that it is at least  $1/2$  in Exercise 14.10.)

# 8

---

## The continuity of paths

It is often important to know whether a stochastic path has continuous paths. An important sufficient condition is the Kolmogorov continuity criterion. This criterion is also useful in showing the continuity of a family of random variables  $X^a$  in the variable  $a$ , where  $a$  is a parameter other than time. Kolmogorov's continuity criterion is part (2) of Theorem 8.1.

Let  $\mathcal{D}_n = \{k/2^n : k \leq 2^n\}$  and let  $\mathcal{D} = \cup_n \mathcal{D}_n$ . The set  $\mathcal{D}$  is known as the set of *dyadic rationals* in  $[0, 1]$ . We will use

$$\sum_{i=1}^{\infty} i^{-2} \leq 1 + \int_1^{\infty} x^{-2} dx = 2.$$

(In fact by a standard exercise using Parseval's identity in the theory of Fourier series,  $\sum_{i=1}^{\infty} i^{-2}$  is actually equal to  $\pi^2/6$ .)

We will be considering at first a real-valued process  $\{X_t : t \in \mathcal{D}\}$ . To show continuity by considering  $X_t - X_s$  for all pairs  $(s, t)$  doesn't work – there are too many pairs. Kolmogorov's proof circumvents this problem by considering only a restricted collection of pairs. To bound  $X_{15/32} - X_{11/32}$ , for example, we compare  $X_{15/32}$  to  $X_{7/16}$ , compare  $X_{7/16}$  to  $X_{3/8}$ , and compare  $X_{3/8}$  to  $X_{1/4}$ , and we also compare  $X_{11/32}$  to  $X_{5/16}$  and compare  $X_{5/16}$  to  $X_{1/4}$ . The advantage of this complicated way of matching pairs is that each comparison, say, for example  $X_{3/8}$  to  $X_{1/4}$ , is used for a great many of the possible pairs  $(s, t)$ .

The proof of Theorem 8.1 has three main steps. Step 1 is to reduce the problem to proving the bound (8.3). The second step is to set up the comparisons that we need, and the third is to obtain estimates on all the comparisons.

**Theorem 8.1** Suppose  $\{X_t : t \in \mathcal{D}\}$  is a real-valued process and there exist  $c_1, \varepsilon$ , and  $p > 0$  such that

$$\mathbb{E}[|X_t - X_s|^p] \leq c_1 |t - s|^{1+\varepsilon}, \quad s, t \in \mathcal{D}. \quad (8.1)$$

Then the following hold.

(1) There exists  $c_2$  depending only on  $c_1, p$ , and  $\varepsilon$  such that for  $M > 0$ ,

$$\mathbb{P}\left(\sup_{s, t \in \mathcal{D}, s \neq t} \frac{|X_t - X_s|}{|t - s|^{\varepsilon/4p}} \geq M\right) \leq c_1/M^p. \quad (8.2)$$

(2) With probability one,  $X_t$  is uniformly continuous on  $\mathcal{D}$ .

*Proof* Step 1. Let  $\lambda_n = M2^{-(n+1)\varepsilon/4p}$  and

$$A_n = \{|X_t - X_s| \geq \lambda_n \text{ for some } s, t \in \mathcal{D} \text{ with } |t - s| \leq 2^{-n}\}.$$

Recall our convention that the letter  $c$  denotes unimportant constants which can change from line to line. We will show

$$\mathbb{P}(A_n) \leq c2^{-n\varepsilon/4}M^{-p}. \quad (8.3)$$

This implies (1) and (2) as follows. If  $|X_t - X_s| \geq M|t - s|^{\varepsilon/4p}$  for some  $s, t \in \mathcal{D}$  with  $s \neq t$ , choose  $n$  such that  $2^{-(n+1)} < |t - s| \leq 2^{-n}$ , and then  $A_n$  holds. The event on the left-hand side of (8.2) is contained in  $\cup_n A_n$ , and using (8.3) shows that

$$\mathbb{P}(\cup_n A_n) \leq cM^{-p} \sum_{n=1}^{\infty} 2^{-n\varepsilon/4} = cM^{-p},$$

which implies (1). Let

$$B_M = \{ \sup_{s, t \in \mathcal{D}, s \neq t} |X_t - X_s| / |t - s|^{\varepsilon/4p} \geq M \}.$$

Note  $B_M$  decreases as  $M$  increases and from (1) we have  $\mathbb{P}(\cap_{M=1}^{\infty} B_M) = 0$ . Thus except for an event of probability zero, each  $\omega$  is in  $B_M^c$  for some  $M$  (where  $M$  depends on  $\omega$ ), and this implies (2). Thus we must show (8.3).

*Step 2.* Define  $a(j, t)$  to be the integer multiple of  $2^{-j}$  that is closest to  $t$  (if there are two different multiples that are equally close, we use some convention to break the tie). If  $t \in \mathcal{D}_m$ , then  $a(m, t) = t$ . If  $|t - s| \leq 2^{-n}$ , then  $|a(n, t) - a(n, s)| \leq 2^{-n+2}$ .

Now if  $s, t \in \mathcal{D}_m$  and  $m \geq n$ , we use the triangle inequality to write

$$\begin{aligned} |X_t - X_s| &= |X_{a(m,t)} - X_{a(m,s)}| \\ &\leq |X_{a(n,t)} - X_{a(n,s)}| \\ &\quad + |X_{a(n+1,t)} - X_{a(n,t)}| + \cdots + |X_{a(m,t)} - X_{a(m-1,t)}| \\ &\quad + |X_{a(n+1,s)} - X_{a(n,s)}| + \cdots + |X_{a(m,s)} - X_{a(m-1,s)}|. \end{aligned} \quad (8.4)$$

If  $|X_{a(n,t)} - X_{a(n,s)}| < \lambda_n/2$  and for each  $i$

$$|X_{a(n+i+1,t)} - X_{a(n+i,t)}| < \frac{\lambda_n}{8(i+1)^2}$$

and the same with  $t$  replaced by  $s$ , then by (8.4)

$$|X_t - X_s| < \frac{\lambda_n}{2} + 2 \sum_{i=0}^{\infty} \frac{\lambda_n}{8(i+1)^2} \leq \lambda_n.$$

Hence if  $|X_t - X_s| \geq \lambda_n$  for some  $s, t \in \mathcal{D}_m$ , then at least one of the events  $E, F_i$ , or  $G_i$ ,  $i \geq 0$ , must hold, where

$$\begin{aligned} E &= \{|X_{a(n,t)} - X_{a(n,s)}| \geq \lambda_n/2 \text{ for some } s, t \in \mathcal{D}_n \text{ with } |s - t| \leq 2^{-n}\}, \\ F_i &= \{|X_{a(n+i+1,t)} - X_{a(n+i,t)}| \geq \lambda_n/8(i+1)^2 \text{ for some } t\}, \\ G_i &= \{|X_{a(n+i+1,s)} - X_{a(n+i,s)}| \geq \lambda_n/8(i+1)^2 \text{ for some } s\}. \end{aligned}$$

*Step 3.* For the event  $E$  to hold, we must have  $|X_r - X_q| \geq \lambda_n/2$  for some  $q, r \in \mathcal{D}_n$  with  $|q - r| \leq 2^{-n+2}$ . There are at most  $c2^n$  such pairs  $(q, r)$ , so the probability of  $E$  is bounded, using Chebyshev's inequality and (8.1), by

$$\begin{aligned} (c2^n) & \sup_{q \in \mathcal{D}_n, r \in \mathcal{D}_{n+1}, |r-q| \leq 2^{-n+2}} \mathbb{P}(|X_r - X_q| \geq \lambda_n/2) \\ & \leq c2^n \frac{\sup_{q \in \mathcal{D}_n, r \in \mathcal{D}_{n+1}, |r-q| \leq 2^{-n+2}} \mathbb{E}[|X_r - X_q|^p]}{(\lambda_n/2)^p} \\ & \leq \frac{c2^n}{\lambda_n^p} (2^{-n+2})^{1+\varepsilon} \\ & \leq \frac{c2^{-n\varepsilon}}{\lambda_n^p}. \end{aligned}$$

For  $F_i$  to hold, that is, for  $|X_{a(n+i+1,t)} - X_{a(n+i,t)}|$  to be greater than  $\lambda_n/8(i+1)^2$  for some  $t$ , we must have  $|X_r - X_q| \geq \lambda_n/8(i+1)^2$  for some  $r \in \mathcal{D}_{n+i}$ ,  $q \in \mathcal{D}_{n+i+1}$  with  $|r-q| \leq 2^{-n-i+2}$ . There are at most  $c2^{n+i}$  such pairs, and so the probability of  $F_i$  is bounded by

$$\begin{aligned} (c2^{n+i}) & \sup_{r \in \mathcal{D}_{n+i}, q \in \mathcal{D}_{n+i+1}, |r-q| \leq 2^{-n-i+2}} \mathbb{P}\left(|X_r - X_q| \geq \frac{\lambda_n}{8(i+1)^2}\right) \\ & \leq c \frac{2^{n+i} 2^{(-n-i+2)(1+\varepsilon)} (8(i+1)^2)^p}{\lambda_n^p} \\ & \leq \frac{c2^{-n\varepsilon} 2^{-i\varepsilon/2}}{\lambda_n^p}. \end{aligned}$$

Here we used the fact that  $2^{-i\varepsilon}(i+1)^{2p} \leq c2^{-i\varepsilon/2}$  for some constant  $c$  depending on  $p$  and  $\varepsilon$  but not  $i$ . We have the same bound for  $G_i$ . Therefore

$$\mathbb{P}(\cup_i (F_i \cup G_i) \cup E) \leq \sum_{i=0}^{\infty} \frac{c2^{-n\varepsilon/2} 2^{-i\varepsilon/2}}{\lambda_n^p} + \frac{c2^{-n\varepsilon/2}}{\lambda_n^p} \leq c2^{-n\varepsilon/2} \lambda_n^{-p}.$$

Letting  $m \rightarrow \infty$  we have

$$\mathbb{P}(A_n) \leq c2^{-n\varepsilon/2} \lambda_n^{-p} = c2^{-n\varepsilon/4} M^{-p}$$

as required.  $\square$

The proof of Theorem 8.1 is an example of what is known as a *metric entropy* or *chaining* argument.

In the above, the only place we relied on the fact that we were using real-valued processes was in using the triangle inequality. Therefore with only slight changes in notation, we have the following theorem.

**Theorem 8.2** Suppose  $X$  takes values in some metric space  $\mathcal{S}$  with metric  $d_{\mathcal{S}}$  and there exist  $c_1, \varepsilon$ , and  $p > 0$  such that

$$\mathbb{E}[d_{\mathcal{S}}(X_s, X_t)^p] \leq c_1|t - s|^{1+\varepsilon}, \quad s, t \in \mathcal{D}. \quad (8.5)$$

Then the following hold.

(1) There exists  $c_2$  depending only on  $c_1, p$ , and  $\varepsilon$  such that for  $M > 0$ ,

$$\mathbb{P}\left(\sup_{s, t \in \mathcal{D}, s \neq t} \frac{d_{\mathcal{S}}(X_s, X_t)}{|t - s|^{\varepsilon/2p}} \geq M\right) \leq c_2/M^p.$$

(2) With probability one,  $X_t$  is uniformly continuous on  $\mathcal{D}$ .

**Remark 8.3** Theorem 8.2 holds for random variables indexed by time, but the analogous result holds for the continuity in  $a$  of random variables  $X^a$  indexed by some parameter  $a$  running through  $\mathcal{D}$ . We may also let the parameter  $a$  run instead through the dyadic rationals in  $[b_1, b_2]$  for any  $b_1 < b_2$ .

The proof of the following corollary is an adaptation of the proof of Theorem 8.1 and is left as Exercise 8.1.

**Corollary 8.4** Suppose there exist  $c_1, \varepsilon, N$ , and  $p > 0$  such that if  $n \leq N$ ,

$$\mathbb{E}[d_{\mathcal{S}}(X_s, X_t)^p] \leq c|t - s|^{1+\varepsilon}, \quad s, t \in \mathcal{D}_n.$$

Then there exists  $c_2$  depending on  $c_1, \varepsilon$ , and  $p$  but not  $N$  such that for  $M > 0$  and  $n \leq N$  we have

$$\mathbb{P}\left(\sup_{s, t \in \mathcal{D}_n, s \neq t} \frac{d_{\mathcal{S}}(X_s, X_t)}{|t - s|^{\varepsilon/2p}} \geq M\right) < c_2M^{-p}.$$

Recall the definition of Hölder continuity from (7.1).

**Proposition 8.5** If  $\alpha < 1/2$ , then the paths of a one-dimensional Brownian motion  $\{W_t; 0 \leq t \leq 1\}$  are Hölder continuous of order  $\alpha$  with probability one.

*Proof* By the stationary increments property and scaling,

$$\mathbb{E}|W_t - W_s|^p = \mathbb{E}|W_{t-s}|^p = |t - s|^{p/2}\mathbb{E}|W_1|^p.$$

If  $\alpha < 1/2$ , choose  $p$  large enough so that  $((p/2) - 1)/p > \alpha$  and then take  $\varepsilon = (p/2) - 1$ . (Here  $\varepsilon$  is large!) Take  $\gamma$  sufficiently small that  $(\varepsilon/p) - \gamma > \alpha$ . Then by Exercise 8.2 the paths of  $W_t$  are Hölder continuous of order  $\alpha$ , with probability one, provided we restrict  $t$  to  $\mathcal{D}$ . But the paths of Brownian motion are continuous, so we see that we have Hölder continuity of order  $\alpha$  when  $t \in [0, 1]$ .  $\square$

## Exercises

8.1 Prove Corollary 8.4.

8.2 If the hypothesis of Theorem 8.1 holds and  $\gamma < \varepsilon/p$ , show that there exists  $c_2$  depending only on  $c_1, \varepsilon, \gamma$ , and  $p$  such that for  $M > 0$

$$\mathbb{P}\left(\sup_{s, t \in \mathcal{D}, s \neq t} \frac{d_{\mathcal{S}}(X_s, X_t)}{|t - s|^{(\varepsilon/p)-\gamma}} \geq M\right) \leq cM^{-p}.$$

- 8.3 Suppose  $X$  is a real-valued process and there exist constants  $c_1, c_2$  such that

$$\mathbb{P}(|X_t - X_s| > \lambda) \leq c_1 e^{-c_2 \lambda \log^4(1/|t-s|)}, \quad s, t \in [0, 1].$$

Prove that with probability one,  $X$  has a version which is uniformly continuous on the dyadic rationals in  $[0, 1]$ .

- 8.4 Suppose  $(X_t, t \in [0, 1])$  is a mean zero Gaussian process and there exist  $c$  and  $\varepsilon$  such that

$$\text{Var}(X_t - X_s) \leq c|t - s|^\varepsilon, \quad s, t \in [0, 1].$$

Prove that there is a version of  $X$  that has continuous paths on  $[0, 1]$ .

- 8.5 Let  $X$  be as in Exercise 8.4. For what values  $\alpha$  will  $X$  have paths that are Hölder continuous of order  $\alpha$ ? ( $\alpha$  will depend on  $\varepsilon$ .)

- 8.6 Let  $\{X_{s,t} : s, t \in [0, 1]\}$  be a collection of random variables. Suppose there exist  $c, p$ , and  $\varepsilon > 0$  such that

$$\mathbb{E} |X_{s',t'} - X_{s,t}|^p \leq c(|t' - t| + |s' - s|)^{2+\varepsilon}.$$

Prove that with probability one, the map  $(s, t) \rightarrow X_{s,t}(\omega)$  is uniformly continuous on  $\mathcal{D} \times \mathcal{D} = \{(s, t); s, t \in \mathcal{D}\}$ .

# 9

---

## Continuous semimartingales

Roughly speaking, a semimartingale is the sum of a martingale and a process whose paths are of bounded variation. In this chapter we consider semimartingales whose paths are continuous. We will give definitions, and then investigate in more detail the class of martingales that are square integrable. Finally we present a proof of the Doob–Meyer decomposition for continuous supermartingales. The Doob–Meyer decomposition used to be considered a very hard theorem, but at least in the continuous case, an elementary proof is possible. For a proof for the general case, see Chapter 16.

### 9.1 Definitions

Let  $\{\mathcal{F}_t\}$  be a filtration satisfying the usual conditions and let

$$\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t = \sigma\left(\bigcup_{t \geq 0} \mathcal{F}_t\right).$$

We say a process  $X$  has *increasing paths* or that  $X$  is an *increasing process* if the functions  $t \rightarrow X_t(\omega)$  are increasing with probability one. Throughout this book saying  $f$  is “increasing” means that  $s < t$  implies  $f(s) \leq f(t)$ , while saying  $f$  is “strictly increasing” means that  $s < t$  implies  $f(s) < f(t)$ . A process  $X$  with *paths of bounded variation* is just what one would expect: with probability one, the functions  $t \rightarrow X_t(\omega)$  are of bounded variation. We say  $X$  has *paths locally of bounded variation* if there exist stopping times  $R_n \rightarrow \infty$  such that the process  $X_{t \wedge R_n}$  has paths of bounded variation for each  $n$ .

We turn to martingales. A martingale  $M$  is a *uniformly integrable martingale* if the family of random variables  $\{M_t\}$  is uniformly integrable. A process  $X$  is a *local martingale* if there exist stopping times  $R_n \rightarrow \infty$  such that  $M_t^n = X_{t \wedge R_n}$  is a uniformly integrable martingale for each  $n$ . A martingale whose paths are continuous is called a *continuous martingale* and we similarly define a *right-continuous martingale*.

A *semimartingale* is a process  $X$  of the form  $X_t = M_t + A_t$ , where  $M_t$  is a local martingale and  $A_t$  is a process whose paths are locally of bounded variation. As a consequence of the Doob–Meyer decomposition we will see that submartingales and supermartingales are semimartingales.

As an example, a Brownian motion  $W_t$  is a martingale and is a local martingale (let  $R_n$  be identically equal to  $n$ ), but is not a uniformly integrable martingale. We will define what it means to be a square integrable martingale in the next section; Brownian motion is not a square integrable martingale.

## 9.2 Square integrable martingales

**Definition 9.1** A martingale is a *square integrable martingale* if there exists a  $\mathcal{F}_\infty$  measurable random variable  $M_\infty$  such that  $\mathbb{E} M_\infty^2 < \infty$  and  $M_t = \mathbb{E}[M_\infty | \mathcal{F}_t]$  for all  $t$ .

An example of a square integrable martingale would be  $M_t = W_{t \wedge t_0}$ , where  $W_t$  is a Brownian motion and  $t_0$  is a fixed time; in this case  $M_\infty = W_{t_0}$ .

**Proposition 9.2** Let  $\{\mathcal{F}_t\}$  be a filtration satisfying the usual conditions and  $M$  a right continuous process. The following are equivalent:

- (1)  $M_t$  is a square integrable martingale.
- (2)  $M$  is a martingale with  $\sup_{t \geq 0} \mathbb{E} M_t^2 < \infty$ .
- (3)  $M$  is a martingale with  $\mathbb{E}[\sup_{t \geq 0} M_t^2] < \infty$ .

*Proof* To show (1) implies (2), suppose  $M$  is a square integrable martingale. Then by Jensen's inequality for conditional expectations (Proposition A.21),

$$\mathbb{E} M_t^2 = \mathbb{E}[(\mathbb{E}[M_\infty | \mathcal{F}_t])^2] \leq \mathbb{E}[\mathbb{E}[M_\infty^2 | \mathcal{F}_t]] = \mathbb{E} M_\infty^2.$$

To show (2) implies (3), for each  $N$ ,

$$\mathbb{E}[\sup_{0 \leq t \leq N} M_t^2] \leq 4\mathbb{E} M_N^2$$

by Doob's inequalities. That (2) implies (3) follows by letting  $N \rightarrow \infty$  and using Fatou's lemma.

Now suppose (3) holds, and we will show (1) holds. Since  $\mathbb{E} M_n^2$  is uniformly bounded in  $n$ , the martingale convergence theorem (Theorem A.35) implies that  $M_n$  converges almost surely and in  $L^2$ . Let us call the limit  $M_\infty$ ; we have  $\mathbb{E} M_\infty^2 < \infty$  by the  $L^2$  convergence. Since  $\mathbb{E} M_n^2$  is uniformly bounded, then  $M_n$  is a uniformly integrable martingale, and by Proposition A.37,  $M_n = \mathbb{E}[M_\infty | \mathcal{F}_n]$ . If  $n - 1 \leq t \leq n$ , we have

$$M_t = \mathbb{E}[M_n | \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[M_\infty | \mathcal{F}_n] | \mathcal{F}_t] = \mathbb{E}[M_\infty | \mathcal{F}_t],$$

as required.  $\square$

For the remainder of this section all our martingales will have paths that are right continuous with left limits.

**Proposition 9.3** If  $M$  is a square integrable martingale and  $S \leq T$  are finite stopping times, then  $\mathbb{E}[M_T | \mathcal{F}_S] = M_S$ .

*Proof* Let  $A \in \mathcal{F}_S$  and define  $U(\omega) = S(\omega)1_A(\omega) + T(\omega)1_{A^c}(\omega)$ . Thus  $U$  is equal to  $S$  if  $\omega \in A$  and otherwise is equal to  $T$ . Since  $A \in \mathcal{F}_S \subset \mathcal{F}_T$ , then we have  $(U \leq t) = [(S \leq t) \cap A] \cup [(T \leq t) \cap A^c]$  is in  $\mathcal{F}_t$ , and therefore  $U$  is a stopping time. By Proposition 3.11,

$$\mathbb{E} M_0 = \mathbb{E} M_U = \mathbb{E}[M_S; A] + \mathbb{E}[M_T; A^c]$$

and

$$\mathbb{E} M_0 = \mathbb{E} M_T = \mathbb{E}[M_T; A] + \mathbb{E}[M_T; A^c].$$

These two equations imply that  $\mathbb{E}[M_S; A] = \mathbb{E}[M_T; A]$ , which is what we needed to prove.  $\square$

By Exercise 3.11, the conclusion is valid if  $M$  is a uniformly integrable martingale.

As an immediate corollary we have

**Corollary 9.4** *Suppose  $M$  is a square integrable martingale and  $T$  is a stopping time. Then  $X_t = M_{t \wedge T}$  is a martingale with respect to  $\{\mathcal{F}_{t \wedge T}\}$ .*

The proof of the following proposition is similar to that of Proposition 9.3. It may be viewed as a converse of the optional stopping theorem.

**Proposition 9.5** *Suppose  $\{\mathcal{F}_t\}$  is a filtration satisfying the usual conditions and  $M$  is a process that is adapted to  $\{\mathcal{F}_t\}$  such that  $M_t$  is integrable for each  $t$ . If  $\mathbb{E} M_T = 0$  for every bounded stopping time  $T$ , then  $M_t$  is a martingale.*

*Proof* Suppose  $s < t$  and  $A \in \mathcal{F}_s$ . Define  $T$  to be equal to  $s$  if  $\omega \in A$  and equal to  $t$  if  $\omega \notin A$ . As in the proof of Proposition 9.3, but even more simply,  $T$  is a stopping time, so

$$0 = \mathbb{E} M_T = \mathbb{E}[M_s; A] + \mathbb{E}[M_t; A^c].$$

The fixed time  $t$  is a stopping time, hence

$$0 = \mathbb{E} M_t = \mathbb{E}[M_t; A] + \mathbb{E}[M_t; A^c].$$

Comparing,  $\mathbb{E}[M_t; A] = \mathbb{E}[M_s; A]$ , which proves  $M$  is a martingale.  $\square$

**Proposition 9.6** *Suppose  $M_t$  is a square integrable martingale. Then*

$$\mathbb{E}[(M_T - M_S)^2 | \mathcal{F}_S] = \mathbb{E}[M_T^2 - M_S^2 | \mathcal{F}_S]. \quad (9.1)$$

*Proof* By Proposition 9.3

$$\begin{aligned} \mathbb{E}[(M_T - M_S)^2 | \mathcal{F}_S] &= \mathbb{E}[M_T^2 | \mathcal{F}_S] - 2M_S \mathbb{E}[M_T | \mathcal{F}_S] + M_S^2 \\ &= \mathbb{E}[M_T^2 | \mathcal{F}_S] - M_S^2 \\ &= \mathbb{E}[M_T^2 - M_S^2 | \mathcal{F}_S] \end{aligned}$$

and we are done.  $\square$

If we take expectations in (9.1), we obtain

$$\mathbb{E}[(M_T - M_S)^2] = \mathbb{E} M_T^2 - \mathbb{E} M_S^2. \quad (9.2)$$

**Theorem 9.7** *Suppose  $M_0 = 0$ ,  $M_t$  is a continuous local martingale, and the paths of  $M_t$  are locally of bounded variation. Then  $M$  is identically 0, a.s., that is,  $\mathbb{P}(M_t = 0 \text{ for all } t) = 1$ .*

*Proof* Using the definition of local martingale, it suffices to suppose  $M$  is a continuous uniformly integrable martingale. Let  $t_0$  be fixed and let  $A_t$  denote the total variation of the paths of  $M$  up to time  $t$ . If  $T_N = \inf\{t : A_t \geq N\}$ , we look at  $M_t^N = M_{T_N \wedge t \wedge t_0}$ . Using Proposition 9.3 and the remark following it, we see that  $M^N$  is also a continuous martingale with paths of bounded variation, and if  $M^N$  is identically zero, then letting  $N \rightarrow \infty$  and  $t_0 \rightarrow \infty$ , we obtain our result. Therefore it suffices to suppose the total variation of  $M_t$  is bounded by  $N$ , a.s. In particular,  $M_t$  is bounded by  $N$ .

Let  $n \geq 1$  and set

$$V_n = \sup_{k \leq 2^n - 1} |M_{(k+1)t_0/2^n} - M_{kt_0/2^n}|.$$

Note  $V_n \leq 2N$ , a.s., and  $V_n \rightarrow 0$ , a.s., as  $n \rightarrow \infty$  by the uniform continuity of the paths of  $M$  on  $[0, t_0]$ . By dominated convergence,  $\mathbb{E} V_n \rightarrow 0$  as  $n \rightarrow \infty$ . We write

$$\begin{aligned} \mathbb{E} M_{t_0}^2 &= \mathbb{E} \left[ \sum_{k=0}^{2^n-1} (M_{(k+1)t_0/2^n}^2 - M_{kt_0/2^n}^2) \right] \\ &= \mathbb{E} \left[ \sum_{k=0}^{2^n-1} (M_{(k+1)t_0/2^n} - M_{kt_0/2^n})^2 \right] \\ &\leq \mathbb{E} \left[ V_n \sum_{k=0}^{2^n-1} |M_{(k+1)t_0/2^n} - M_{kt_0/2^n}| \right] \\ &\leq N \mathbb{E} V_n. \end{aligned}$$

The second equality follows by (9.2). Since  $n$  is arbitrary and  $\mathbb{E} V_n \rightarrow 0$ , then  $\mathbb{E} M_{t_0}^2 = 0$ . By Doob's inequalities,  $\mathbb{E} [\sup_{s \leq t_0} M_s^2] = 0$ . Hence  $M$  is identically 0 up to time  $t_0$ .  $\square$

### 9.3 Quadratic variation

**Definition 9.8** A continuous square integrable martingale  $M_t$  has *quadratic variation*  $\langle M \rangle_t$  (sometimes written  $\langle M, M \rangle_t$ ) if  $M_t^2 - \langle M \rangle_t$  is a martingale, where  $\langle M \rangle_t$  is a continuous adapted increasing process with  $\langle M \rangle_0 = 0$ .

In the case where  $W$  is a Brownian motion,  $t_0$  is fixed, and  $M_t = W_{t \wedge t_0}$  the quadratic variation of  $M$  is just  $\langle M \rangle_t = t \wedge t_0$  by Example 3.3. Brownian motion itself does not fit perfectly into the framework of stochastic integration because it is not a square integrable martingale, although it is a martingale; we will be dealing with this point several times in what follows.

We will show existence and uniqueness of  $\langle M \rangle_t$  by means of the Doob–Meyer decomposition, Theorem 9.12, below. However we defer the proof of the Doob–Meyer decomposition until the next section. A process  $Z$  is of *class D* if  $\{Z_T : T \text{ a finite stopping time}\}$  is a uniformly integrable family of random variables.

**Theorem 9.9** *Let  $M_t$  be a continuous square integrable martingale. There exists a continuous adapted increasing process  $\langle M \rangle_t$  with  $\langle M \rangle_0 = 0$  and with increasing paths such that  $M_t^2 - \langle M \rangle_t$  is a martingale.*

*If  $A_t$  is a continuous adapted increasing process such that  $M_t^2 - A_t$  is a martingale, then  $\mathbb{P}(A_t \neq \langle M \rangle_t \text{ for some } t) = 0$ .*

*Proof* By Jensen's inequality for conditional expectations,

$$\mathbb{E} [M_t^2 | \mathcal{F}_s] \geq (\mathbb{E} [M_t | \mathcal{F}_s])^2 = M_s^2$$

if  $s < t$ , and so  $M_t^2$  is a submartingale. Since  $M_\infty$  is square integrable, given  $\varepsilon$  there exists  $\delta$  such that  $\mathbb{E} [M_\infty^2; A] < \varepsilon$  if  $\mathbb{P}(A) < \delta$ . Since  $M_t^2$  is a submartingale, if  $K > \mathbb{E} M_\infty^2 / \delta$ , then

$$\mathbb{P}(M_t^2 > K) \leq \mathbb{E} M_t^2 / K \leq \mathbb{E} M_\infty^2 / K < \delta,$$

and consequently

$$\mathbb{E}[M_t^2; M_t^2 > K] \leq \mathbb{E}[M_\infty^2; M_t^2 > K] < \varepsilon.$$

By Exercise 3.11,  $M_t^2$  is of class  $D$ . Applying the Doob–Meyer decomposition (Theorem 9.12) to  $-M_t^2$ , we write  $-M_t^2 = N_t - B_t$ , where  $N_t$  is a martingale and  $B_t$  has increasing paths. We then set  $\langle M \rangle_t = B_t$ . The uniqueness follows from the uniqueness part of the Doob–Meyer decomposition.  $\square$

In view of Proposition 9.3 and the definition of  $\langle M \rangle$ , we have

$$\begin{aligned} \mathbb{E}[(M_T - M_S)^2 - (\langle M \rangle_T - \langle M \rangle_S) | \mathcal{F}_S] \\ = \mathbb{E}[M_T^2 - M_S^2 - (\langle M \rangle_T - \langle M \rangle_S) | \mathcal{F}_S] = 0 \end{aligned} \quad (9.3)$$

if  $S$  and  $T$  are finite stopping times and  $M$  is a continuous square integrable martingale.

If  $M$  and  $N$  are two square integrable martingales, we define  $\langle M, N \rangle_t$  by

$$\langle M, N \rangle_t = \frac{1}{2}[\langle M + N \rangle_t - \langle M \rangle_t - \langle N \rangle_t]. \quad (9.4)$$

This is sometimes called the *covariation* of  $M$  and  $N$ .

An alternative representation of  $\langle M \rangle_t$  is the following. A proof could be given now, but it is a bit messy. After we have Itô's formula this will be easier.

**Theorem 9.10** *Let  $M$  be a square integrable martingale and let  $t_0 > 0$ . Then  $\langle M \rangle_t$  is the limit in probability of*

$$\sum_{k=0}^{[2^n t_0]} (M_{(k+1)/2^n} - M_{k/2^n})^2,$$

where  $[2^n t_0]$  is the largest integer less than or equal to  $2^n t_0$ .

## 9.4 The Doob–Meyer decomposition

In this section we give a proof of the Doob–Meyer decomposition for continuous supermartingales. First we need the following inequality, which has many other uses as well.

**Proposition 9.11** *Suppose  $A^1$  and  $A^2$  are two increasing adapted continuous processes starting at zero with  $A_\infty^i = \lim_{t \rightarrow \infty} A_t^i < \infty$ , a.s.,  $i = 1, 2$ , and suppose there exists a positive real  $K$  such that for all  $t$ ,*

$$\mathbb{E}[A_\infty^i - A_t^i | \mathcal{F}_t] \leq K, \quad \text{a.s.}, \quad i = 1, 2. \quad (9.5)$$

*Let  $B_t = A_t^1 - A_t^2$ . Suppose there exists a non-negative random variable  $V$  with  $\mathbb{E} V^2 < \infty$  such that for all  $t$ ,*

$$|\mathbb{E}[B_\infty - B_t | \mathcal{F}_t]| \leq \mathbb{E}[V | \mathcal{F}_t], \quad \text{a.s.} \quad (9.6)$$

*Then*

$$\mathbb{E} \sup_{t \geq 0} B_t^2 \leq 8\mathbb{E} V^2 + 8\sqrt{2}K(\mathbb{E} V^2)^{1/2}. \quad (9.7)$$

*Proof* We start by showing

$$\mathbb{E}(A_\infty^i)^2 \leq 2K^2, \quad i = 1, 2. \quad (9.8)$$

First suppose  $A_\infty^i$  is bounded by a positive real number  $L$ . Note that we have  $\mathbb{E} A_\infty^i = \mathbb{E} [\mathbb{E} [A_\infty^i - A_0^i | \mathcal{F}_0]] \leq K$ . A simple calculation shows that

$$(A_\infty^i)^2 = 2 \int_0^\infty (A_\infty^i - A_t^i) dA_t^i.$$

We then have, using Proposition 3.14,

$$\begin{aligned} \mathbb{E} (A_\infty^i)^2 &= 2\mathbb{E} \int_0^\infty (A_\infty^i - A_t^i) dA_t^i \\ &= 2\mathbb{E} \int_0^\infty (\mathbb{E} [A_\infty^i | \mathcal{F}_t] - A_t^i) dA_t^i \\ &= 2\mathbb{E} \int_0^\infty \mathbb{E} [A_\infty^i - A_t^i | \mathcal{F}_t] dA_t^i \\ &\leq 2K\mathbb{E} \int_0^\infty dA_t^i = 2K\mathbb{E} A_\infty^i \leq 2K^2. \end{aligned}$$

If we let  $T_L = \inf\{t : A_t^1 + A_t^2 \geq L\}$  and  $A_t^{i,L} = A_{t \wedge T_L}^i$ , then (9.5) still holds if we replace  $A_t^i$  by  $A_t^{i,L}$ . We obtain  $\mathbb{E} (A_\infty^{i,L})^2 \leq 2K^2$ , and then letting  $L \rightarrow \infty$  and using Fatou's lemma proves (9.8).

We next write

$$B_\infty^2 = 2 \int_0^\infty (B_\infty - B_t) dB_t,$$

and hence

$$\begin{aligned} \mathbb{E} B_\infty^2 &= 2\mathbb{E} \int_0^\infty \mathbb{E} [B_\infty - B_t | \mathcal{F}_t] dB_t \\ &\leq \mathbb{E} \int_0^\infty \mathbb{E} [V | \mathcal{F}_t] d(A_t^1 + A_t^2) \\ &= \mathbb{E} \int_0^\infty V d(A_t^1 + A_t^2) \\ &= \mathbb{E} [V(A_\infty^1 + A_\infty^2)]. \end{aligned}$$

The bound (9.8) takes care of the integrability concerns. By the Cauchy–Schwarz inequality we obtain

$$\mathbb{E} B_\infty^2 \leq (\mathbb{E} [(A_\infty^1 + A_\infty^2)^2])^{1/2} (\mathbb{E} V^2)^{1/2} \leq 2\sqrt{2}K(\mathbb{E} V^2)^{1/2}.$$

Now let  $M_t = \mathbb{E} [B_\infty | \mathcal{F}_t]$ ,  $N_t = \mathbb{E} [V | \mathcal{F}_t]$ , where we take the right-continuous versions (see Corollary 3.13), and let  $X_t = M_t - B_t$ . We have

$$|X_t| = |\mathbb{E} [B_\infty - B_t | \mathcal{F}_t]| \leq N_t,$$

and using Doob's inequalities,

$$\mathbb{E} \sup_{t \geq 0} X_t^2 \leq \mathbb{E} \sup_{t \geq 0} N_t^2 \leq 4\mathbb{E} N_\infty^2 = 4\mathbb{E} V^2.$$

Also by Doob's inequalities,

$$\mathbb{E} \sup_{t \geq 0} M_t^2 \leq 4\mathbb{E} M_\infty^2 = 4\mathbb{E} B_\infty^2.$$

Since  $\sup_{t \geq 0} |B_t| \leq \sup_{t \geq 0} |X_t| + \sup_{t \geq 0} |M_t|$ , our result follows.  $\square$

We now prove the Doob–Meyer decomposition for continuous supermartingales. In view of the proof of Proposition A.30, we would like to let

$$A_t = \int_0^t \mathbb{E} \left[ \frac{dZ_s}{ds} \mid \mathcal{F}_s \right] ds,$$

but this doesn't make sense. We instead define an approximation  $A_t^h$  by (9.9) and show that  $A_t^h$  converges to what we want as  $h \rightarrow 0$ .

**Theorem 9.12** *Suppose  $Z_t$  is a continuous adapted supermartingale of class D. Then there exists an increasing adapted continuous process  $A_t$  with paths locally of bounded variation started at 0 and a continuous local martingale  $M_t$  such that*

$$Z_t = M_t - A_t.$$

If  $M'$  and  $A'$  are two other such processes with  $Z_t = M'_t - A'_t$ , then  $M_t = M'_t$  and  $A_t = A'_t$  for all  $t$ , a.s.

*Proof* Let us prove the second assertion first. Let  $S_N$  be the first time that  $|M_t| + |M'_t|$  exceeds  $N$ . If

$$Z_t = M_t - A_t = M'_t - A'_t,$$

then  $M_{t \wedge S_N} - M'_{t \wedge S_N} = A_{t \wedge S_N} - A'_{t \wedge S_N}$  is a martingale whose paths are locally of bounded variation. By Theorem 9.7,  $M_{t \wedge S_N} = M'_{t \wedge S_N}$ , a.s. Since this is true for all  $N$ , then  $M_t = M'_t$ .

Now let us prove the existence of  $M$  and  $A$ . Let  $T_N = \inf\{t : |Z_t| \geq N\} \wedge N$  and  $Z_t^N = Z_{t \wedge T_N}$ . By Exercise 9.2,  $Z^N$  is a supermartingale. If we prove the decomposition  $Z_t^N = M_t^N - A_t^N$  for each  $N$ , then by the uniqueness assertion, if  $N_1 < N_2$ , we have  $A_t^{N_1}$  and  $M_t^{N_1}$  agreeing with  $A_t^{N_2}$  and  $M_t^{N_2}$ , respectively, for  $t \leq T_{N_1}$ . Hence given  $t$ , we can choose  $N$  large enough so that  $t \leq T_N$  and then define  $M_t = M_t^N$ ,  $A_t = A_t^N$ . Clearly this gives the desired decomposition. Thus we may suppose that  $Z_t$  is bounded by some  $N$  and that  $Z_t$  is constant for  $t \geq N$ .

Let  $V_\delta = \sup_{|t-s| \leq \delta} |Z_t - Z_s|$ . Since  $Z$  has continuous paths,

$$V_\delta = \sup_{s,t \in \mathbb{Q}_+, |t-s| \leq \delta} |Z_t - Z_s|,$$

and therefore  $V_\delta$  is measurable with respect to  $\mathcal{F}_\infty$ . Since the paths of  $Z$  are uniformly continuous,  $V_\delta \rightarrow 0$ , a.s., as  $\delta \rightarrow 0$ , and since  $|V_\delta| \leq 2N$ , we have by dominated convergence that  $\mathbb{E} V_\delta^2 \rightarrow 0$  as  $\delta \rightarrow 0$ .

We define

$$A_t^h = \frac{1}{h} \int_0^t (Z_s - \mathbb{E}[Z_{s+h} \mid \mathcal{F}_s]) ds. \quad (9.9)$$

At this point we do not know even that  $\mathbb{E}[Z_{s+h} \mid \mathcal{F}_s]$  has any nice measurability properties (it is not a martingale, for example); let us assume that it has a version that has continuous paths, is adapted, and is jointly measurable in  $t$  and  $\omega$ , and prove this

fact a bit later on. Because  $Z$  is a supermartingale,  $A^h$  is increasing. We have (note Exercise 9.6)

$$\begin{aligned} E[A_\infty^h - A_t^h \mid \mathcal{F}_t] &= \frac{1}{h} \mathbb{E} \left[ \int_t^\infty \mathbb{E}[Z_s - Z_{s+h} \mid \mathcal{F}_s] ds \mid \mathcal{F}_t \right] \\ &= \frac{1}{h} \int_t^\infty \mathbb{E}[Z_s - Z_{s+h} \mid \mathcal{F}_t] ds \\ &= \frac{1}{h} \mathbb{E} \left[ \int_t^\infty Z_s ds - \int_{t+h}^\infty Z_s ds \mid \mathcal{F}_t \right] \\ &= \frac{1}{h} \mathbb{E} \left[ \int_t^{t+h} Z_s ds \mid \mathcal{F}_t \right] \\ &= \mathbb{E} \left[ \int_0^1 Z_{t+uh} du \mid \mathcal{F}_t \right]. \end{aligned}$$

Since  $Z$  is bounded by  $N$ , it follows that  $A^h$  satisfies (9.5). If  $k < h$ , then

$$\begin{aligned} |\mathbb{E}[(A_\infty^h - A_t^h) - (A_\infty^k - A_t^k) \mid \mathcal{F}_t]| &= \left| \mathbb{E} \left[ \int_0^1 (Z_{t+uh} - Z_{t+uk}) du \mid \mathcal{F}_t \right] \right| \\ &\leq \mathbb{E}[V_h \mid \mathcal{F}_t]. \end{aligned}$$

Now apply Proposition 9.11 to see that  $\mathbb{E} \sup_{t \geq 0} (A_t^h - A_t^k)^2 \rightarrow 0$  as  $k, h \rightarrow 0$ . This shows that whenever  $h_n$  decreases to 0, then  $A^{h_n}$  is a Cauchy sequence in a normed linear space, where the norm is given by

$$\|X\| = (\mathbb{E} \sup_{t \geq 0} |X_t|^2)^{1/2}, \quad (9.10)$$

which is complete by Exercise 9.5. Therefore there exists a limit  $A$ . Since

$$\mathbb{E} \sup_{t \geq 0} (A_t^h - A_t)^2 \rightarrow 0$$

as  $h \rightarrow 0$ , there exists a subsequence  $h_n \rightarrow 0$  such that  $\sup_{t \geq 0} (A_t^{h_n} - A_t)^2 \rightarrow 0$ , a.s., which proves that  $A_t$  is continuous and increasing.

We calculate

$$\begin{aligned} \mathbb{E}[A_\infty - A_t \mid \mathcal{F}_t] &= \lim_{h \rightarrow 0} \mathbb{E}[A_\infty^h - A_t^h \mid \mathcal{F}_t] \\ &= \lim_{h \rightarrow 0} \mathbb{E} \left[ \int_0^1 Z_{t+uh} du \mid \mathcal{F}_t \right] \\ &= \mathbb{E} \left[ \int_0^1 Z_t du \mid \mathcal{F}_t \right] \\ &= Z_t. \end{aligned}$$

Therefore

$$Z_t = \mathbb{E}[A_\infty \mid \mathcal{F}_t] - A_t,$$

which is the decomposition of  $Z$  into a martingale minus an increasing process.

Fix  $h$ . It remains to show that there is a version of  $\mathbb{E}[Z_{s+h} \mid \mathcal{F}_s]$  that is a continuous jointly measurable adapted process. Define  $Y_t = Z_{t+h}$  and define  $Y_t^n$  to be equal to  $Y_{k/2^n}$  if  $k/2^n \leq t < (k+1)/2^n$ . Take the right-continuous version  $\tilde{Y}_t^{k,n}$  of the martingale  $\mathbb{E}[Y_{k/2^n} \mid \mathcal{F}_t]$  (see Corollary 3.13) and let

$$\tilde{Y}_t^n(\omega) = \sum_{k=0}^{\infty} 1_{[k/2^n, (k+1)/2^n)}(t) \tilde{Y}_t^{k,n}(\omega).$$

Note that  $\tilde{Y}_t^n = \mathbb{E}[Y_t^n \mid \mathcal{F}_t]$ , a.s., for all  $t$ . Moreover,  $\tilde{Y}_t^n$  is right continuous, so we see that it is jointly measurable in  $t$  and  $\omega$ . Now for  $n > m$ ,

$$\sup_{t \geq 0} |\tilde{Y}_t^n - \tilde{Y}_t^m| \leq \sup_{t \geq 0} \mathbb{E}[V_{2^{-m}} \mid \mathcal{F}_t]. \quad (9.11)$$

We have already seen that there exists a subsequence such that the right-hand side of (9.11) converges to 0 almost surely. Hence along the appropriate subsequence,  $\tilde{Y}_t^n$  converges uniformly. If we call the limit  $\tilde{Y}_t$ , we see that  $\tilde{Y}_t$  is right continuous, adapted, and jointly measurable. If  $k/2^n \leq t \leq (k+1)/2^n$ , then  $|Y_t^n - Y_{k/2^n}^n| \leq V_{2^{-n}}$ , so

$$|\tilde{Y}_t^n - \tilde{Y}_{k/2^n}^n| = |\mathbb{E}[Y_t^n - Y_{k/2^n}^n \mid \mathcal{F}_t]| \leq \mathbb{E}[V_{2^{-n}} \mid \mathcal{F}_t].$$

By the triangle inequality,

$$|\tilde{Y}_t^n - \tilde{Y}_s^n| \leq 2 \sup_{t \geq 0} \mathbb{E}[V_{2^{-n}} \mid \mathcal{F}_t]$$

if  $k/2^n \leq s, t \leq (k+1)/2^n$ . Therefore the largest jump of  $\tilde{Y}_t^n$  is bounded by  $2 \sup_{t \geq 0} \mathbb{E}[V_{2^{-n}} \mid \mathcal{F}_t]$ , and we conclude the limit  $\tilde{Y}$  has continuous paths. Finally,  $Y_t^n$  differs from  $Y_t$  by at most  $V_{2^{-n}}$ , so we see by passing to the limit that  $\tilde{Y}_t$  is a version of  $\mathbb{E}[Z_{t+h} \mid \mathcal{F}_t]$ .  $\square$

## Exercises

- 9.1 Let  $W_t$  be a Brownian motion started at 1 and  $T_0 = \inf\{t > 0 : W_t = 0\}$ . Is  $M_t = W_{t \wedge T_0}$  a square integrable martingale? A locally square integrable martingale? A uniformly integrable martingale? A martingale? A local martingale? A semimartingale?
- 9.2 Prove that if  $M$  is a submartingale such that the paths of  $M$  are continuous,  $\sup_t |M_t|$  is integrable, and  $S \leq T$  are finite stopping times, then  $\mathbb{E}[M_T \mid \mathcal{F}_S] \geq M_S$ . Note that the last part of the proof of Proposition 9.3 breaks down here.
- 9.3 Suppose  $M_t$  is a local martingale with continuous paths. Show that if  $N > 0$ ,  $T_N = \inf\{t : |M_t| \geq N\}$ , and  $M_t^N = M_{t \wedge T_N}$ , then  $M^N$  is a uniformly integrable martingale.
- 9.4 Suppose  $W_t^1$  and  $W_t^2$  are two independent Brownian motions,  $t_0 > 0$ , and  $M_t^i = W_{t \wedge t_0}$ ,  $i = 1, 2$ . Show  $\langle M^1, M^2 \rangle_t = 0$ .
- 9.5 Show that the norm defined in (9.10) is complete.
- 9.6 Let  $Z_t$  be a bounded supermartingale with continuous paths that is constant from some time  $t_0$  on. Show that for each  $t$

$$\mathbb{E}\left[\int_t^\infty \mathbb{E}[Z_s - Z_{s+h} \mid \mathcal{F}_s] ds \mid \mathcal{F}_t\right] = \int_t^\infty \mathbb{E}[Z_s - Z_{s+h} \mid \mathcal{F}_t] ds, \quad \text{a.s.}$$

- 9.7 We mentioned that one can prove the existence of  $\langle M \rangle$  without using the Doob–Meyer theorem. Here is how that argument starts. Let  $M$  be a bounded continuous martingale and for each  $n$ , define

$$I_n(t) = \sum_{i=0}^{\lceil t2^n \rceil} (M_{(i+1)/2^n} - M_{i/2^n})^2.$$

Here  $[x]$  is the integer part of  $x$ . Prove that for each  $t > 0$ ,  $\mathbb{E} |I_n(t) - I_m(t)|^2 \rightarrow 0$  as  $n, m \rightarrow \infty$ . One can then define  $\langle M \rangle_t$  as the  $L^2$  limit of  $I_n(t)$ .

*Hint:* If  $n > m$ , note that

$$M_{(i+1)/2^m} - M_{i/2^m} = \sum_{j=2^{n-m}i}^{2^{n-m}(i+1)-1} (M_{(j+1)/2^n} - M_{j/2^n}).$$

## Notes

The first proof of the Doob–Meyer decomposition was by Meyer in the early 1960s and was a major breakthrough. There are now a number of alternate proofs. The proof we give here for continuous supermartingales is new.

# 10

---

## Stochastic integrals

This chapter is devoted to the construction of stochastic integrals, primarily with respect to continuous square integrable martingales. The motivating example is  $\int_0^t H_s dW_s$ , where  $W$  is a Brownian motion and  $H$  is an adapted process satisfying certain conditions. We cannot define this integral as a Lebesgue–Stieltjes integral because the paths of Brownian motion are nowhere differentiable (Theorem 7.3).

One way to visualize a stochastic integral is to think of  $dW_s$  as “white noise,” on a radio and  $H_s$  as the volume control which increases or decreases the white noise by a factor. For another model, if  $W_s$  is supposed to represent a stock price at time  $s$  (of course, stock prices can’t be negative, while Brownian motion can!) and  $H_s$  is the number of shares held at time  $s$ , then the stochastic integral represents the net profit.

### 10.1 Construction

Let  $M_t$  be a continuous square integrable martingale with respect to a filtration  $\{\mathcal{F}_t\}$  satisfying the usual conditions, and suppose  $H_t$  is an adapted process. Under appropriate additional assumptions on  $H$ , we want to define

$$N_t = \int_0^t H_s dM_s, \quad (10.1)$$

the stochastic integral of  $H$  with respect to  $M$ .

We impose two conditions on the integrand  $H_t$ , a measurability one and an integrability one. First we define the *predictable*  $\sigma$ -field  $\mathcal{P}$  on  $[0, \infty) \times \Omega$ . This is the smallest  $\sigma$ -field of subsets of  $[0, \infty) \times \Omega$  with respect to which all left continuous, bounded, and adapted processes are measurable. In symbols,

$$\mathcal{P} = \sigma(X : X \text{ is left continuous, bounded, and adapted to } \{\mathcal{F}_t\}).$$

This can be rephrased by saying  $\mathcal{P}$  is the  $\sigma$ -field on  $[0, \infty) \times \Omega$  generated by the collection of all sets of the form

$$\{(t, \omega \in [0, \infty) \times \Omega : X_t(\omega) > a\},$$

where  $a \in \mathbb{R}$  and  $X$  is a bounded, adapted, left continuous process. We require  $H : [0, \infty) \times \Omega \rightarrow \mathbb{R}$  to be measurable with respect to  $\mathcal{P}$ . When this happens, we say  $H$  is *predictable*. The integrability is easier to state: we require

$$\mathbb{E} \int_0^\infty H_s^2 d\langle M \rangle_s < \infty. \quad (10.2)$$

Observe that  $H$  will meet both requirements if  $H$  is bounded, adapted, and has continuous paths.

We define  $\int_0^t H_s dM_s$  in three steps:

Step 1. When  $H_s(\omega) = K(\omega)1_{(a,b]}(s)$ , where  $K$  is bounded and  $\mathcal{F}_a$  measurable.

Step 2. When  $H_s$  is a sum of processes of the form in Step 1.

Step 3. When  $H$  is predictable and satisfies (10.2).

If  $M_t = W_{t \wedge t_0}$ , where  $W$  is a Brownian motion and  $t_0$  is a fixed time, then  $\langle M \rangle_t = t \wedge t_0$ , and it might help the reader to work through the proofs in this special case. Even in this situation, all the elements of the general construction are present.

We will need the following easy lemma.

**Lemma 10.1** *The predictable  $\sigma$ -field  $\mathcal{P}$  is generated by the collection  $\mathcal{C}$  of processes of the form  $X_t(\omega) = \sum_{i=1}^n K_i(\omega)1_{(a_i, b_i]}(t)$ , where for each  $i$ ,  $K_i$  is a bounded  $\mathcal{F}_{a_i}$  measurable random variable.*

*Proof* If  $X \in \mathcal{C}$ , then  $X$  is bounded, adapted, and left continuous, hence  $X$  is a predictable process. Thus  $\mathcal{C} \subset \mathcal{P}$ .

On the other hand, if  $Y$  is a bounded, adapted, left-continuous process, we can approximate  $Y$  by the processes

$$Y_t^n(\omega) = \sum_{i=0}^{n2^n} Y_{i/2^n}(\omega)1_{(i/2^n, (i+1)/2^n]}(t).$$

Each such  $Y^n$  is in  $\mathcal{C}$ . Therefore the  $\sigma$ -field generated by  $\mathcal{C}$  contains  $\mathcal{P}$ .  $\square$

**Proposition 10.2** *Suppose  $H$  is as in Step 1 above. Then*

$$N_t = K(M_{t \wedge b} - M_{t \wedge a})$$

is a continuous martingale,

$$\mathbb{E} N_\infty^2 = \mathbb{E} \int_0^\infty K^2 1_{(a,b]}(s) d\langle M \rangle_s = \mathbb{E} [K^2 (\langle M \rangle_b - \langle M \rangle_a)],$$

and

$$\langle N \rangle_t = \int_0^t K^2 1_{(a,b]}(s) d\langle M \rangle_s.$$

*Proof* The continuity of the paths of  $N$  is clear. Set  $N_\infty = K(M_b - M_a)$ . Since  $K$  is bounded and  $M$  is square integrable,  $\mathbb{E} N_\infty^2 < \infty$ . We will show  $N_t = \mathbb{E}[N_\infty | \mathcal{F}_t]$ , which will prove that  $N_t$  is a martingale.

If  $t \geq b$ , then since  $K$ ,  $M_b$ , and  $M_a$  are  $\mathcal{F}_t$  measurable,

$$\mathbb{E}[N_\infty | \mathcal{F}_t] = K(M_b - M_a) = N_t.$$

If  $a \leq t \leq b$ ,  $K$  is  $\mathcal{F}_t$  measurable, and

$$\mathbb{E}[K(M_b - M_a) | \mathcal{F}_t] = K\mathbb{E}[M_b - M_a | \mathcal{F}_t] = K(M_t - M_a) = N_t.$$

In particular,  $N_a = \mathbb{E}[N_\infty | \mathcal{F}_a] = 0$ . Finally, if  $t \leq a$ ,

$$\mathbb{E}[N_\infty | \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[N_\infty | \mathcal{F}_a] | \mathcal{F}_t] = 0 = N_t.$$

For  $\mathbb{E} N_\infty^2$ , we have by (9.2) with  $S = a$  and  $T = b$ ,

$$\begin{aligned}\mathbb{E} N_\infty^2 &= \mathbb{E} [K^2(M_b - M_a)^2] = \mathbb{E} [K^2 \mathbb{E} [(M_b - M_a)^2 | \mathcal{F}_a]] \\ &= \mathbb{E} [K^2 \mathbb{E} [\langle M \rangle_b - \langle M \rangle_a | \mathcal{F}_a]] = \mathbb{E} [K^2 (\langle M \rangle_b - \langle M \rangle_a)].\end{aligned}$$

To verify the formula for  $\langle N \rangle_t$ , let

$$\begin{aligned}L_\infty &= K^2(M_b - M_a)^2 - K^2(\langle M \rangle_b - \langle M \rangle_a), \\ L_t &= K^2(M_{b \wedge t} - M_{a \wedge t})^2 - K^2(\langle M \rangle_{b \wedge t} - \langle M \rangle_{a \wedge t}).\end{aligned}$$

Then

$$L_t = N_t^2 - \int_0^t K^2 1_{(a,b]}(s) d\langle M \rangle_s,$$

and we must show that  $L_t$  is a martingale. To do this, it suffices to show  $L_t = \mathbb{E} [L_\infty | \mathcal{F}_t]$ .

If  $t \geq b$ , then  $L_\infty$  is  $\mathcal{F}_t$  measurable, so  $\mathbb{E} [L_\infty | \mathcal{F}_t] = L_\infty = L_t$ . If  $a \leq t \leq b$ , then

$$\begin{aligned}\mathbb{E} [L_\infty | \mathcal{F}_t] &= K^2 \mathbb{E} [(M_b - M_a)^2 - (\langle M \rangle_b - \langle M \rangle_a) | \mathcal{F}_t] \\ &= K^2 \mathbb{E} [M_b^2 - M_a^2 - (\langle M \rangle_b - \langle M \rangle_a) | \mathcal{F}_t] \\ &= K^2 \mathbb{E} [M_t^2 - M_a^2 - (\langle M \rangle_t - \langle M \rangle_a) | \mathcal{F}_t] \\ &= K^2 \mathbb{E} [(M_t - M_a)^2 - (\langle M \rangle_t - \langle M \rangle_a) | \mathcal{F}_t] \\ &= L_t,\end{aligned}$$

using (9.1) and (9.3) with the stopping times there being fixed positive real numbers. In particular,  $\mathbb{E} [L_\infty | \mathcal{F}_a] = L_a = 0$ . Finally, if  $t \leq a$ ,

$$\mathbb{E} [L_\infty | \mathcal{F}_t] = \mathbb{E} [\mathbb{E} [L_\infty | \mathcal{F}_a] | \mathcal{F}_t] = 0 = L_a$$

as required.  $\square$

Next suppose

$$H_s(\omega) = \sum_{j=1}^J K_j 1_{(a_j, b_j]}(s), \quad (10.3)$$

where each  $K_j$  is  $\mathcal{F}_{a_j}$  measurable and bounded. We may rewrite  $H$  so that the intervals  $(a_j, b_j]$  satisfy  $a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_J < b_J$ . For example, if  $H_s = K_1 1_{(a_1, b_1]} + K_2 1_{(a_2, b_2]}$  with  $a_1 < a_2 < b_1 < b_2$ , we may rewrite  $H_s$  as

$$K_1 1_{(a_1, a_2]} + (K_1 + K_2) 1_{(a_2, b_1]} + K_2 1_{(b_1, b_2]}.$$

Define

$$N_t = \sum_{j=1}^J K_j (M_{t \wedge b_j} - M_{t \wedge a_j}). \quad (10.4)$$

We need to check that rewriting  $H_s$  so that  $a_1 < b_1 \leq a_2 < \dots < b_J$  does not affect the value of  $N_t$ , but this is routine.

**Proposition 10.3** *With  $H$  as in (10.3) and  $N$  defined by (10.4),  $N_t$  is a continuous martingale,*

$$\mathbb{E} N_\infty^2 = \mathbb{E} \int_0^\infty H_s^2 d\langle M \rangle_s,$$

and

$$\langle N \rangle_t = \int_0^t H_s^2 d\langle M \rangle_s. \quad (10.5)$$

*Proof* By linearity,  $N_t$  is a continuous martingale. We have

$$\begin{aligned} \mathbb{E} N_\infty^2 &= \mathbb{E} \left[ \sum_j H_j^2 (M_{b_j} - M_{a_j})^2 \right] \\ &\quad + 2\mathbb{E} \left[ \sum_{i < j} H_i H_j (M_{b_i} - M_{a_i})(M_{b_j} - M_{a_j}) \right]. \end{aligned} \quad (10.6)$$

The cross terms vanish, because when  $i < j$  and we condition on  $\mathcal{F}_{a_j}$ , we have

$$\mathbb{E} [H_i H_j (M_{b_i} - M_{a_i}) \mathbb{E} [(M_{b_j} - M_{a_j}) | \mathcal{F}_{a_j}]] = 0.$$

For the terms in the first sum in (10.6), by (9.3)

$$\begin{aligned} \mathbb{E} [H_j^2 (M_{b_j} - M_{a_j})^2] &= \mathbb{E} [H_j^2 \mathbb{E} [(M_{b_j} - M_{a_j})^2 | \mathcal{F}_{a_j}]] \\ &= \mathbb{E} [H_j^2 \mathbb{E} [\langle M \rangle_{b_j} - \langle M \rangle_{a_j} | \mathcal{F}_{a_j}]] \\ &= \mathbb{E} [H_j^2 (\langle M \rangle_{b_j} - \langle M \rangle_{a_j})]. \end{aligned}$$

Therefore

$$\mathbb{E} N_\infty^2 = \mathbb{E} \int_0^\infty H_s^2 d\langle M \rangle_s. \quad (10.7)$$

The argument for  $\langle N \rangle_t$  is similar.  $\square$

Now suppose  $H_s$  is predictable and (10.2) holds. Choose  $H_s^n$  of the form given in (10.3) above such that

$$\mathbb{E} \int_0^\infty (H_s^n - H_s)^2 d\langle M \rangle_s \rightarrow 0.$$

To see that this can be done, define

$$\|Y\|_2 = \left( \mathbb{E} \int_0^\infty Y_t^2 d\langle M \rangle_t \right)^{1/2}$$

for  $Y$  predictable. Then  $\|Y\|_2$  is an  $L^2$  norm on functions on  $[0, \infty) \times \Omega$ , so by Lemma 10.1 we can approximate  $H$  in this norm by processes of the form given in (10.3). (When  $H$  is bounded, adapted, and has continuous paths, taking  $H_s^n$  equal to  $H_{k/2^n}$  if  $k/2^n < s \leq (k+1)/2^n$  for  $s < n$  and  $H_s^n = 0$  if  $s \geq n$  will work.)

By Doob's inequalities we have

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \geq 0} \left( \int_0^t (H_s^n - H_s^m) dM_s \right)^2 \right] &\leq 4\mathbb{E} \left( \int_0^\infty (H_s^n - H_s^m)^2 d\langle M \rangle_s \right)^2 \\ &= 4\mathbb{E} \int_0^\infty (H_s^n - H_s^m)^2 d\langle M \rangle_s \rightarrow 0. \end{aligned}$$

The norm

$$\|Y\|_\infty = (\mathbb{E} [\sup_t |Y_t|^2])^{1/2} \quad (10.8)$$

is complete; this was shown in Exercise 9.5. Thus there exists a process  $N_t$  such that  $\sup_{t \geq 0} |N_t - \int_0^t H_s^n dM_s| \rightarrow 0$  in  $L^2$ .

If  $H_s^n$  and  $\bar{H}_s^n$  are two sequences converging in the  $\|\cdot\|_2$  norm to  $H$ , then

$$\mathbb{E} \left( \int_0^t (H_s^n - \bar{H}_s^n) dM_s \right)^2 = \mathbb{E} \int_0^t (H_s^n - \bar{H}_s^n)^2 d\langle M \rangle_s \rightarrow 0,$$

or the limit is independent of which sequence  $H^n$  we choose.

It is easy to see, because of the  $L^2$  convergence, that  $N_t$  is a martingale: if  $A \in \mathcal{F}_s$ , then

$$\mathbb{E} \left[ \int_0^t H_r^n dM_r; A \right] = \mathbb{E} \left[ \int_0^s H_r^n dM_r; A \right]$$

by Proposition 10.3. Now use that

$$\begin{aligned} \left| \mathbb{E} \left[ \int_0^t H_r^n dM_r - N_t; A \right] \right| &\leq \mathbb{E} \left| \int_0^t H_r^n dM_r - N_t \right| \\ &\leq \left( \mathbb{E} \left( \int_0^t H_r^n dM_r - N_t \right)^2 \right)^{1/2} \rightarrow 0 \end{aligned}$$

and similarly with  $t$  replaced by  $s$ .

Similar arguments using the  $L^2$  convergence show that

$$\mathbb{E} N_t^2 = \mathbb{E} \int_0^t H_s^2 d\langle M \rangle_s, \quad (10.9)$$

and

$$\langle N \rangle_t = \int_0^t H_s^2 d\langle M \rangle_s. \quad (10.10)$$

Because  $\sup_{t \geq 0} |N_t - \int_0^t H_s^n dM_s| \rightarrow 0$  in  $L^2$ , there exists a subsequence  $\{n_k\}$  such that the convergence takes place almost surely, that is

$$\sup_{t \geq 0} \left| \int_0^t H_s^{n_k} dM_s - N_t \right| \rightarrow 0, \quad \text{a.s.}$$

Since each  $\int_0^t H_s^n dM_s$  has continuous paths, with probability one,  $N_t$  has continuous paths. We write  $N_t = \int_0^t H_s dM_s$  and call  $N_t$  the stochastic integral of  $H$  with respect to  $M$ .

We summarize our construction as follows.

**Theorem 10.4** Suppose the filtration  $\{\mathcal{F}_t\}$  satisfies the usual conditions and  $M_t$  is a square integrable martingale with continuous paths. Suppose  $H$  is of the form

$$\sum_{j=1}^J K_j(\omega) 1_{(a_j, b_j]}(s), \quad (10.11)$$

where each  $K_j$  is bounded and  $\mathcal{F}_{a_j}$  measurable. In this case define

$$\int_0^t H_s dM_s = \sum_{j=1}^J K_j (M_{t \wedge b_j} - M_{t \wedge a_j}).$$

If  $H$  is predictable and

$$\mathbb{E} \int_0^\infty H_s^2 d\langle M \rangle_s < \infty,$$

choose  $H^n$  of the form given in (10.11) with  $\mathbb{E} \int_0^\infty (H_s^n - H_s)^2 d\langle M \rangle_s \rightarrow 0$ , and define

$$N_t = \int_0^t H_s dM_s$$

to be the limit with respect to the norm (10.8) of  $\int_0^t H_s^n dM_s$ . Then  $N_t$  is a continuous martingale,

$$\mathbb{E} N_\infty^2 = \mathbb{E} \int_0^\infty H_s^2 d\langle M \rangle_s,$$

and

$$\langle N \rangle_t = \int_0^t H_s^2 d\langle M \rangle_s.$$

Moreover the definition of  $N_t$  is independent of the particular choice of the  $H^n$ .

## 10.2 Extensions

There are some extensions of the definition that are fairly routine.

*Extension 1.* If

$$\int_0^\infty H_s^2 d\langle M \rangle_s < \infty, \quad \text{a.s.,}$$

but without the expectation being finite, let

$$T_N = \inf \left\{ t : \int_0^t H_s^2 d\langle M \rangle_s > N \right\}.$$

$M'_t = M_{t \wedge T_N}$  is a square integrable martingale with  $\langle M' \rangle_t = \langle M \rangle_{t \wedge T_N}$ , so  $\int_0^t H_s^2 d\langle M' \rangle_s \leq N$ . Define  $\int_0^t H_s dM_s$  to be the quantity  $\int_0^t H_s dM_{s \wedge T_N}$  if  $t \leq T_N$ . If  $t \leq T_K \leq T_N$ , we need to check that  $\int_0^t H_s d\langle M \rangle_{s \wedge T_K} = \int_0^t H_s d\langle M \rangle_{s \wedge T_N}$ , so that our definition is consistent. This is part of Exercise 10.2.

*Extension 2.* If  $M_t$  is a continuous local martingale (see Section 9.1 for the definition), let  $S_n = \inf\{t : |M_t| \geq n\}$ . By Exercise 9.3,  $M_{t \wedge S_n}$  will be a uniformly integrable martingale, and in fact, since  $M_{t \wedge S_n}$  is bounded, it is square integrable. For  $t \leq S_n$  we set

$$\int_0^t H_s dM_s = \int_0^t H_s dM_{s \wedge S_n}$$

and  $\langle M \rangle_t = \langle M \rangle_{t \wedge S_n}$ . Again there is consistency to check, which is also part of Exercise 10.2.

*Extension 3.* Suppose that  $X_t = M_t + A_t$  is a semimartingale with continuous paths, so that  $M$  is a local martingale and  $A$  is a process with paths locally of bounded variation. If  $\int_0^\infty H_s^2 d\langle M \rangle_s + \int_0^\infty |H_s| |dA_s| < \infty$ , we define

$$\int_0^t H_s dX_s = \int_0^t H_s dM_s + \int_0^t H_s dA_s,$$

where the first integral on the right is a stochastic integral and the second is a Lebesgue–Stieltjes integral.

For a semimartingale, we define

$$\langle X \rangle_t = \langle M \rangle_t. \quad (10.12)$$

Given two semimartingales  $X$  and  $Y$  we define  $\langle X, Y \rangle_t$  by:

$$\langle X, Y \rangle_t = \frac{1}{2} [\langle X + Y \rangle_t - \langle X \rangle_t - \langle Y \rangle_t].$$

## Exercises

10.1 Prove (10.5) in Proposition 10.3.

10.2 Check the consistency of the first two extensions of the definition of stochastic integrals.

10.3 Show that if  $M$  is a continuous square integrable martingale, and  $T$  a finite stopping time, then

$$\int_0^\infty 1_{[0,T]} dM_s = M_T.$$

10.4 Show that if  $N_t = \int_0^t H_s dM_s$  where  $M$  is a continuous square integrable martingale,  $H$  is predictable, and  $\mathbb{E} \int_0^\infty H_s^2 d\langle M \rangle_s < \infty$ , and  $L_t = \int_0^t K_s dN_s$ , where  $K$  is predictable and  $\mathbb{E} \int_0^\infty K_s^2 d\langle N \rangle_s < \infty$ , then

$$L_t = \int_0^t H_s K_s dM_s.$$

10.5 Show that if  $M$ ,  $H$ , and  $N$  are as in Exercise 10.4, then  $\langle M, N \rangle_t = \int_0^t H_s d\langle M \rangle_s$ .

*Hint:* Derive a formula for  $\langle N + M \rangle_t$  from the fact that

$$N_t + M_t = \int_0^t (1 + H_s) dM_s.$$

10.6 Suppose that  $M$  and  $L$  are square integrable martingales,  $H$  is predictable and satisfies (10.2), and  $N_t = \int_0^t H_s dM_s$ . Show that

$$\langle N, L \rangle_t = \int_0^t H_s d\langle M, L \rangle_s. \quad (10.13)$$

Sometimes the stochastic integral of  $H$  with respect to  $M$  is defined to be the square integrable martingale  $N$  for which (10.13) holds for all square integrable martingales  $L$ .

10.7 Show that if  $M$  and  $N$  are square integrable martingales with continuous paths, then

$$\langle M, N \rangle_t \leq (\langle M \rangle_t)^{1/2} (\langle N \rangle_t)^{1/2}.$$

*Hint:* Imitate an appropriate proof of the Cauchy–Schwarz inequality. This result is a special case of the inequality of Kunita–Watanabe.

# 11

---

## Itô's formula

The most important result in the theory of stochastic integration is Itô's formula. This is also known as the *change of variables formula*.

Let  $C^k$  be the functions that are  $k$  times continuously differentiable and  $C_b^k$  those functions  $C^k$  such that the function and its  $i$ th-order derivatives are bounded for  $i \leq k$ .

**Theorem 11.1** *Let  $X_t$  be a semimartingale with continuous paths and suppose  $f \in C^2$ . Then for almost every  $\omega$*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X \rangle_s, \quad t \geq 0. \quad (11.1)$$

Step 1 will be to reduce to the case when  $f \in C_b^3$  and  $X$  has appropriate boundedness conditions. Step 2 is a use of Taylor's formula; see (11.2). Step 3 shows that each term converges to the appropriate quantity, and Step 4 removes the restriction that  $f$  be in  $C_b^3$ .

*Proof* Step 1. If  $X_t = M_t + A_t$  is the decomposition of  $X$  into a local martingale  $M$  and a process  $A$  that has paths locally of bounded variation, let  $V_t$  be the total variation of  $A$  up to time  $t$ :  $V_t = \int_0^t |dA_s|$ . Let

$$T_N = \inf\{t : |M_t| > N \text{ or } \langle M \rangle_t > N \text{ or } V_t > N\}.$$

By the continuity of paths,  $T_N \rightarrow \infty$ , a.s., as  $N \rightarrow \infty$ , so for almost every  $\omega$  and for each  $t$ ,  $t \wedge T_N = t$  for  $N$  large enough. Since Itô's formula is a path-by-path result, it suffices to prove Itô's formula for  $X_{t \wedge T_N}$  for each  $N$ , or what amounts to the same thing, we may take  $N$  arbitrary and assume  $M_t$ ,  $\langle M \rangle_t$ ,  $A_t$ , and  $V_t$  are all bounded by  $N$ . In this case,  $X_t$  is bounded by  $2N$ .

Since  $X$  is bounded, we may modify  $f$ ,  $f'$ , and  $f''$  outside of  $[-2N, 2N]$  without affecting the validity of Itô's formula. Therefore we will also assume  $f \in C^2$  with compact support. Let us temporarily assume in addition that  $f'''$  exists and is continuous; we will remove this last assumption later on.

Let  $t_0 > 0$ ,  $\varepsilon > 0$ ,  $S_0 = 0$ , and define

$$\begin{aligned} S_{i+1} = S_{i+1}(\varepsilon) = \inf\{t > S_i : |M_t - M_{S_i}| > \varepsilon \text{ or } \langle M \rangle_t - \langle M \rangle_{S_i} > \varepsilon \\ \text{or } V_t - V_{S_i} > \varepsilon\} \wedge t_0. \end{aligned}$$

Note  $S_i = t_0$  for  $i$  sufficiently large (how large depends on  $\omega$ ) by the continuity of the paths.

*Step 2.* The key idea to proving Itô's formula is Taylor's theorem. We write

$$\begin{aligned} f(X_{t_0}) - f(X_0) &= \sum_{i=0}^{\infty} [f(X_{S_{i+1}}) - f(X_{S_i})] \\ &= \sum_{i=0}^{\infty} f'(X_{S_i})(X_{S_{i+1}} - X_{S_i}) + \frac{1}{2} \sum_{i=0}^{\infty} f''(X_{S_i})(X_{S_{i+1}} - X_{S_i})^2 \\ &\quad + \sum_{i=0}^{\infty} R_i, \end{aligned} \tag{11.2}$$

where  $R_i$  is the remainder term. We have  $|R_i| \leq c \|f''\|_\infty |X_{S_{i+1}} - X_{S_i}|^3$ .

*Step 3.* Let us first look at the terms with  $f'$  in them. Let  $H_s^\varepsilon = f'(X_{S_i})$  if  $S_i \leq s < S_{i+1}$ . By the continuity of  $f'$  and  $X_s$ , we see that  $H_s^\varepsilon$  converges boundedly and pointwise to  $f'(X_s)$ . In particular,  $\int_0^{t_0} |H_s^\varepsilon - f'(X_s)| dV_s \rightarrow 0$  boundedly, hence

$$\mathbb{E} \int_0^{t_0} |H_s^\varepsilon - f'(X_s)| dV_s \rightarrow 0.$$

Also,

$$\mathbb{E} \left( \int_0^{t_0} (H_s^\varepsilon - f'(X_s)) dM_s \right)^2 = \mathbb{E} \int_0^{t_0} |H_s^\varepsilon - f'(X_s)|^2 d\langle M \rangle_s \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ . We then have

$$\sum_i f'(X_{S_i})(X_{S_{i+1}} - X_{S_i}) = \int_0^{t_0} H_s^\varepsilon (dM_s + dA_s) \rightarrow \int_0^{t_0} f'(X_s) (dM_s + dA_s),$$

which leads to the  $f'$  term in Itô's formula.

Next let us look at the  $f''$  terms. We can write

$$(X_{S_{i+1}} - X_{S_i})^2 = (M_{S_{i+1}} - M_{S_i})^2 + 2(M_{S_{i+1}} - M_{S_i})(A_{S_{i+1}} - A_{S_i}) + (A_{S_{i+1}} - A_{S_i})^2.$$

Note  $\sum_i f''(X_{S_i})(M_{S_{i+1}} - M_{S_i})(A_{S_{i+1}} - A_{S_i})$  is bounded in absolute value by

$$\sum_i \varepsilon \|f''\|_\infty |A_{S_{i+1}} - A_{S_i}| \leq \varepsilon \|f''\|_\infty \int_0^{t_0} dV_s \leq \varepsilon \|f''\|_\infty N,$$

which goes to 0 as  $\varepsilon \rightarrow 0$ ; this follows from the definition of  $S_i$ . Similarly the expression  $\sum_i f''(X_{S_i})(A_{S_{i+1}} - A_{S_i})^2$  also goes to 0. Therefore we need to show

$$\sum_i f''(X_{S_i})(M_{S_{i+1}} - M_{S_i})^2 \rightarrow \int_0^{t_0} f''(X_s) d\langle X \rangle_s.$$

By an argument very similar to the one for the  $f'$  terms,

$$\frac{1}{2} \sum_i f''(X_{S_i})(\langle M \rangle_{S_{i+1}} - \langle M \rangle_{S_i}) \rightarrow \frac{1}{2} \int_0^{t_0} f''(X_s) d\langle M \rangle_s, \tag{11.3}$$

and since  $\langle X \rangle_t = \langle M \rangle_t$  for semimartingales (see (10.12)), the right-hand side of (11.3) is the correct  $f''$  term. We thus need to show that

$$\sum_i f''(X_{S_i})[(M_{S_{i+1}} - M_{S_i})^2 - (\langle M \rangle_{S_{i+1}} - \langle M \rangle_{S_i})] \rightarrow 0 \tag{11.4}$$

as  $\varepsilon \rightarrow 0$ .

We will show

$$\mathbb{E} \left( \sum_{i=0}^{\infty} B_i \right)^2 \rightarrow 0, \quad (11.5)$$

where

$$B_i = f''(X_{S_i})[(M_{S_{i+1}} - M_{S_i})^2 - (\langle M \rangle_{S_{i+1}} - \langle M \rangle_{S_i})].$$

We have

$$\mathbb{E} \left( \sum_i B_i \right)^2 = \mathbb{E} \sum_i B_i^2 + 2 \sum_{i < j} B_i B_j.$$

If  $i < j$ , then

$$\mathbb{E}[B_i B_j] = \mathbb{E}[B_i \mathbb{E}[B_j | \mathcal{F}_{S_{i+1}}]].$$

By (9.2) and the fact that  $S_{i+1} \leq S_j$ , we see that

$$\mathbb{E}[B_j | \mathcal{F}_{S_{i+1}}] = f''(X_{S_j}) \mathbb{E}[(M_{S_{j+1}} - M_{S_j})^2 - (\langle M \rangle_{S_{j+1}} - \langle M \rangle_{S_j}) | \mathcal{F}_{S_{i+1}}] = 0,$$

so the cross-products vanish.

Therefore to prove (11.5) it remains to show  $\mathbb{E} \sum_i B_i^2 \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . We use the easy inequality  $(x+y)^2 \leq 2x^2 + 2y^2$ . Since  $f''$  is bounded,

$$\begin{aligned} \mathbb{E} \sum_i B_i^2 &\leq 2 \|f''\|_{\infty}^2 \sum_i \mathbb{E}[(M_{S_{i+1}} - M_{S_i})^4] \\ &\quad + 2 \|f''\|_{\infty}^2 \sum_i \mathbb{E}[(\langle M \rangle_{S_{i+1}} - \langle M \rangle_{S_i})^2]. \end{aligned} \quad (11.6)$$

The first sum on the right-hand side of (11.6) is bounded by

$$\begin{aligned} 2\varepsilon^2 \|f''\|_{\infty}^2 \sum_i \mathbb{E}[(M_{S_{i+1}} - M_{S_i})^2] &= 2\varepsilon^2 \|f''\|_{\infty}^2 \mathbb{E}[M_{t_0}^2 - M_0^2] \\ &\leq 8\varepsilon^2 \|f''\|_{\infty}^2 N^2. \end{aligned}$$

The second sum on the right-hand side of (11.6) is bounded by

$$2\varepsilon \|f''\|_{\infty}^2 \sum_i \mathbb{E}[(\langle M \rangle_{S_{i+1}} - \langle M \rangle_{S_i})] \leq 2\varepsilon \|f''\|_{\infty}^2 \mathbb{E}\langle M \rangle_{t_0} \leq 2\varepsilon \|f''\|_{\infty}^2 N.$$

Both of these tend to 0 as  $\varepsilon \rightarrow 0$ . Therefore  $\mathbb{E} \sum_i B_i^2 \rightarrow 0$ , and the proof of the convergence for the  $f''$  term is complete.

The final terms to examine are the remainder terms. We have shown that  $\mathbb{E} \sum_i (X_{S_{i+1}} - X_{S_i})^2$  remains bounded as  $\varepsilon \rightarrow 0$ . Since

$$|R_i| \leq c\varepsilon \|f'''\|_\infty (X_{S_{i+1}} - X_{S_i})^2,$$

we see  $\mathbb{E} \sum_i |R_i| \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

*Step 4.* To finish up, we remove the assumption that  $f \in C^3$ . (We still assume that  $f \in C^2$  with compact support.) Take a sequence  $\{f_m\}$  of  $C^3$  functions such that  $f_m$ ,  $f'_m$ , and  $f''_m$  converge uniformly to  $f$ ,  $f'$ , and  $f''$ , respectively. Apply Itô's formula with  $f_m$  and then let  $m \rightarrow \infty$ . The terms  $f_m(X_t)$  and  $f_m(X_0)$  clearly converge to  $f(X_t)$  and  $f(X_0)$ . The  $f'_m$  terms converge because

$$\mathbb{E} \left( \int_0^{t_0} (f'_m(X_s) - f'(X_s)) dM_s \right)^2 = \mathbb{E} \int_0^{t_0} |f'_m(X_s) - f'(X_s)|^2 d\langle M \rangle_s \rightarrow 0$$

and

$$\mathbb{E} \left| \int_0^{t_0} (f'_m(X_s) - f'(X_s)) dA_s \right| \leq \mathbb{E} \int_0^{t_0} |f'_m(X_s) - f'(X_s)| dV_s \rightarrow 0$$

as  $m \rightarrow \infty$ . The  $f''_m$  terms converge by dominated convergence. This shows that (11.1) holds for each  $t_0$ , except for a null set  $N_{t_0}$  depending on  $t_0$ . Let  $N = \cup_{t \in \mathbb{Q}_+} N_t$ , where  $\mathbb{Q}_+$  denotes the non-negative rationals. If  $\omega \notin N$ , then (11.1) holds for every  $t_0$  rational. Each term in (11.1) is continuous, a.s. (with a null set  $N'$  independent of  $t_0$ ). Therefore if  $\omega \notin N \cup N'$ , (11.1) holds for all  $t_0$ .  $\square$

There is a multivariate version of Itô's formula, which is proved in a very similar way:

**Theorem 11.2** Suppose  $X_t^1, \dots, X_t^d$  are continuous semimartingales,  $X_t = (X_t^1, \dots, X_t^d)$ , and  $f$  is a  $C^2$  function on  $\mathbb{R}^d$ . Then with probability one,

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t \sum_{i=1}^d \frac{\partial f}{\partial x_i}(X_s) dX_s^i \\ &\quad + \frac{1}{2} \int_0^t \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s) d\langle X^i, X^j \rangle_s \end{aligned} \tag{11.7}$$

for all  $t \geq 0$ .

The following is known as the *integration by parts formula* or *Itô's product formula*, and is very useful.

**Corollary 11.3** If  $X$  and  $Y$  are semimartingales with continuous paths, then

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t.$$

*Proof* By Itô's formula,

$$X_t^2 = X_0^2 + 2 \int_0^t X_s dX_s + \langle X \rangle_t.$$

The analogous formula holds when  $X$  is replaced by  $Y$  and when  $X$  is replaced by  $X + Y$ . We then use

$$X_t Y_t = \frac{1}{2}[(X_t + Y_t)^2 - X_t^2 - Y_t^2];$$

substituting the formulas for  $X_t^2$ ,  $Y_t^2$ , and  $(X_t + Y_t)^2$  that we obtained from Itô's formula and doing some algebra yields our result.  $\square$

## Exercises

- 11.1 Suppose  $W_t$  is a Brownian motion and  $a \in \mathbb{R}$ . Show that the amount of time Brownian motion spends at the point  $a$  is zero, i.e., that

$$\int_0^t 1_{\{a\}}(W_s) ds = 0, \quad \text{a.s.}$$

for all  $t > 0$ .

- 11.2 Let  $a < b$  and let  $f_{a,b}$  be the  $C^1$  function such that  $f_{a,b}(0) = f'_{a,b}(0) = 0$  and

$$f'_{a,b}(x) = \int_0^x 1_{[a,b]}(y) dy, \quad x \in \mathbb{R}.$$

In other words,  $f_{a,b}$  is the function whose second derivative is  $1_{[a,b]}$ , except that the second derivative is not defined at  $a$  and  $b$ . Show Itô's formula holds for  $f_{a,b}$ :

$$f_{a,b}(W_t) = \int_0^t f'_{a,b}(W_s) dW_s + \frac{1}{2} \int_0^t 1_{[a,b]}(W_s) ds.$$

- 11.3 If  $W_t$  is a Brownian motion,  $a > 0$ , and  $T = \inf\{t > 0 : |W_t| = a\}$ , calculate  $\mathbb{E} \int_0^T (W_s)^k ds$  for each non-negative integer  $k$ . Also calculate

$$\mathbb{E} \int_0^T 1_{[b_1, b_2]}(W_s) ds$$

if  $[b_1, b_2] \subset [-a, a]$ .

- 11.4 Let  $W$  be a Brownian motion, let  $t_0 < t_1 < \dots < t_n = 1$ , and let

$$B_i = (W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1}).$$

Show there exists a constant  $c_1$  not depending on  $\{t_0, \dots, t_n\}$  such that

$$\mathbb{E} \left( \sum_{i=1}^n B_i \right)^2 \leq c_1 \max_{1 \leq i \leq n} |t_i - t_{i-1}|.$$

- 11.5 Use Exercise 11.4 and the Borel–Cantelli lemma to prove that if  $W$  is a Brownian motion, then

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} (W_{k/2^n} - W_{(k-1)/2^n})^2 = 1, \quad \text{a.s.}$$

- 11.6 In our proof of Itô's formula, the use of stopping times simplifies the proof considerably. This exercise considers a proof of Itô's formula using fixed times. Suppose  $M$  is a bounded continuous martingale,  $A$  is a continuous process whose paths have total variation bounded by  $N > 0$ , a.s., and  $X_t = M_t + A_t$ .

(1) Writing  $[x]$  for the integer part of  $x$ , prove that for each  $t$ ,

$$\sum_{i=1}^{[2^n t] + 1} (X_{(i+1)/2^n} - X_{i/2^n})^2$$

converges in probability to  $\langle X \rangle_t$ .

(2) Prove that if  $f$  is a  $C^2$  function whose second derivative is bounded, then

$$\sum_{i=1}^{[2^n t] + 1} f''(X_{i/2^n})(X_{(i+1)/2^n} - X_{i/2^n})^2$$

converges in probability to

$$\int_0^t f''(X_s) d\langle X_s \rangle.$$

Since the increments of  $M$  and  $A$  are not uniformly bounded by something small, this is much harder than the proof of Theorem 11.1 given in this chapter.

- 11.7 Here is an alternate way to prove Itô's formula.

(1) Suppose  $X = M + A$ , where  $M$  and  $A$  are as in Exercise 11.6. Write

$$\begin{aligned} X_t^2 - X_0^2 &= \sum_{i=0}^{[t2^n] - 1} (X_{(i+1)/2^n}^2 - X_{i/2^n}^2) \\ &= \sum_{i=0}^{[t2^n] - 1} 2X_{i/2^n}(X_{(i+1)/2^n} - X_{i/2^n}) + \sum_{i=0}^{[t2^n] - 1} (X_{(i+1)/2^n} - X_{i/2^n})^2. \end{aligned}$$

Use Exercise 11.6 to show that Itô's formula holds when  $f(x) = x^2$ .

(2) Derive the Itô product formula. Then use induction to show that Itô's formula holds when  $f(x) = x^n$ ,  $n$  a positive integer.

(3) Given  $f \in C^2$ , find polynomials  $P_m$  such that  $P_m, P'_m, P''_m$  converge uniformly to  $f, f', f''$ , respectively, on a compact interval as  $m \rightarrow \infty$ . Apply Itô's formula for  $P_m$  and show that one can take limits to derive Itô's formula for  $f$ .

# 12

---

## Some applications of Itô's formula

We will be using Itô's formula throughout the book. In this chapter we give some applications, each of which will turn out to be quite useful.

### 12.1 Lévy's theorem

The following is known as Lévy's theorem. Recall that if  $M$  is a local martingale with continuous paths and  $T_N = \inf\{t : |M_t| \geq N\}$ , we defined  $\langle M \rangle_t$  to be equal to  $\langle M \rangle_{t \wedge T_N}$  if  $t \leq T_N$ ; see Section 10.2. Moreover, by Exercise 9.3,  $M_{t \wedge N}$  is a square integrable martingale for each  $N$ .

**Theorem 12.1** *Let  $M_t$  be a continuous local martingale with respect to a filtration  $\{\mathcal{F}_t\}$  satisfying the usual conditions such that  $M_0 = 0$  and  $\langle M \rangle_t = t$ . Then  $M_t$  is a Brownian motion with respect to  $\{\mathcal{F}_t\}$ .*

*Proof* Fix  $t_0$  and let  $N_t = M_{t+t_0} - M_{t_0}$ ,  $\mathcal{F}'_t = \mathcal{F}_{t+t_0}$ . It is routine to check that  $N_t$  is a martingale with respect to  $\mathcal{F}'_t$  and that  $\langle N \rangle_t = t$ . Note  $\mathcal{F}'_0$  will not be the trivial  $\sigma$ -field in general. We see that

$$\mathbb{E} N_t^2 = \mathbb{E} M_{t+t_0}^2 - \mathbb{E} M_{t_0}^2 = t < \infty.$$

If  $f$  is a function mapping the reals to the complex numbers, we may still use Itô's formula: just apply Itô's formula to the real and imaginary parts of  $f$ . Doing this for  $f(x) = e^{iux}$ , where  $u$  and  $x$  are real, we have

$$e^{iuN_t} = 1 + iu \int_0^t e^{iuN_s} dN_s - \frac{u^2}{2} \int_0^t e^{iuN_s} ds. \quad (12.1)$$

If we take  $T_K = \inf\{t : |N_t| \geq K\}$ , then

$$e^{iuN_{t \wedge T_K}} = 1 + iu \int_0^{t \wedge T_K} e^{iuN_s} dN_s - \frac{u^2}{2} \int_0^{t \wedge T_K} e^{iuN_s} ds. \quad (12.2)$$

Take  $A \in \mathcal{F}'_0$ , multiply (12.2) by  $1_A$ , and take expectations. The stochastic integral is a martingale, so this term will have 0 expectation. Then let  $K \rightarrow \infty$ , and we are left with

$$\mathbb{E}[e^{iuN_t}; A] = \mathbb{P}(A) - \frac{u^2}{2} \int_0^t \mathbb{E}[e^{iuN_s}; A] ds. \quad (12.3)$$

We used the Fubini theorem here. (The reason we introduced the stopping time  $T_K$  is that  $N_{t \wedge T_K}$  is a square integrable martingale, and hence the stochastic integral is a martingale. We might run into integrability problems if we worked with (12.1) instead of (12.2).)

Write  $J(t) = \mathbb{E}[e^{iuN_t}; A]$ , so we have

$$J(t) = \mathbb{P}(A) - \frac{u^2}{2} \int_0^t J(s) ds. \quad (12.4)$$

Since  $J$  is bounded, (12.4) shows that  $J$  is continuous. Since  $J$  is continuous, using (12.4) again shows that  $J$  is differentiable. Hence  $J'(t) = -\frac{u^2}{2}J(t)$  with  $J(0) = \mathbb{P}(A)$ . The only solution to this ordinary differential equation is

$$J(t) = \mathbb{P}(A)e^{-u^2 t/2}. \quad (12.5)$$

If we set  $A = \Omega$ , this tells us that  $\mathbb{E}[e^{iuN_t}] = e^{-u^2 t/2}$ , and by the uniqueness theorem for characteristic functions (Theorem A.48),  $M_{t+t_0} - M_{t_0}$  is a mean zero normal random variable with variance  $t$ . Equation (12.5) also tells us that

$$\mathbb{E}[e^{iuN_t}; A] = \mathbb{E}[e^{iuN_t}]\mathbb{P}(A) \quad (12.6)$$

when  $A \in \mathcal{F}'_0$ . Let  $f$  be a  $C^\infty$  function with compact support. The Fourier transform  $\widehat{f}(u)$  will be in the Schwartz class; see Section B.2. Replacing  $u$  by  $-u$  in (12.6), multiplying the resulting equation by  $\widehat{f}(u)$ , and integrating over  $u \in \mathbb{R}$ , we have

$$\int \widehat{f}(u)\mathbb{E}[e^{-iuN_t}; A] du = \int \widehat{f}(u)\mathbb{E}[e^{-iuN_t}] du \mathbb{P}(A).$$

Using the Fubini theorem and the Fourier inversion theorem, and dividing by a constant, we conclude

$$\mathbb{E}[f(N_t); A] = \mathbb{E}[f(N_t)]\mathbb{P}(A).$$

Since  $\widehat{f}$  is in the Schwartz class, integrability is not a problem when applying the Fubini theorem. A limit argument shows that this equation holds with  $f$  equal to  $1_B$ , where  $B$  is a Borel subset of  $\mathbb{R}$ , hence

$$\mathbb{P}(M_{t+t_0} - M_{t_0} \in B, A) = \mathbb{P}(M_{t+t_0} - M_{t_0} \in B)\mathbb{P}(A).$$

This shows the independence of  $M_{t+t_0} - M_{t_0}$  and  $\mathcal{F}_{t_0}$ . We thus see that  $M_t$  is a continuous process starting at 0 with  $M_{t+t_0} - M_{t_0}$  being a mean zero normal random variable with variance  $t$  independent of  $\mathcal{F}_{t_0}$ , and therefore  $M$  is a Brownian motion.  $\square$

## 12.2 Time changes of martingales

The next theorem says that most continuous martingales arise from Brownian motion via a time change. That is, the paths are the same, but the rate at which one moves along the paths varies. In fact, it is possible to show that all continuous martingales arise from a time change of a Brownian motion that is possibly stopped at a random time.

**Theorem 12.2** Suppose  $M_t$  is a continuous local martingale,  $M_0 = 0$ ,  $\langle M \rangle_t$  is strictly increasing, and  $\lim_{t \rightarrow \infty} \langle M \rangle_t = \infty$ , a.s. Let

$$\tau(t) = \inf\{u : \langle M \rangle_u \geq t\}.$$

Then  $W_t = M_{\tau(t)}$  is a Brownian motion with respect to  $\mathcal{F}'_t = \mathcal{F}_{\tau(t)}$ .

*Proof* Let us first suppose that  $W_t^2$  is integrable. We have by Proposition 9.3 that

$$\mathbb{E}[W_t | \mathcal{F}_s] = \mathbb{E}[M_{\tau(t)} | \mathcal{F}_{\tau(s)}] = M_{\tau(s)} = W_s,$$

or  $W_t$  is a continuous martingale. Similarly,  $W_t^2 - t$  is a martingale. Now apply Lévy's theorem, Theorem 12.1. Removing the assumption that  $W_t^2$  is integrable is left as Exercise 12.1.  $\square$

### 12.3 Quadratic variation

Itô's formula allows us to prove Theorem 9.10 fairly simply.

*Proof of Theorem 9.10* If  $T_K = \inf\{t : |M_t| \geq K\}$ , we will show that

$$\sum_{k=0}^{[t_0 2^n]} (M_{T_K \wedge (k+1)/2^n} - M_{T_K \wedge k/2^n})^2$$

converges to  $\langle M \rangle_{t_0 \wedge T_K}$ . Since  $T_K \rightarrow \infty$  as  $K \rightarrow \infty$ , this will prove the proposition. Thus we may assume  $M$  is bounded by  $K$ .

If  $s > 0$  and we let  $N_t = M_{s+t} - M_s$ , then  $N_t$  is a martingale with respect to the filtration  $\mathcal{F}'_t = \mathcal{F}_{s+t}$  and we can check that  $\langle N \rangle_t = \langle M \rangle_{t+s} - \langle M \rangle_s$ . By Itô's formula applied to the process  $N$ , we obtain

$$(M_{t+s} - M_s)^2 = 2 \int_0^t (M_{r+s} - M_r) dM_r + (\langle M \rangle_{t+s} - \langle M \rangle_s).$$

Applying this with  $t = 1/2^n$  and  $s = k/2^n$  and summing, we see that

$$\sum_{k=0}^{[t_0 2^n]} (M_{(k+1)/2^n} - M_{k/2^n})^2 - \langle M \rangle_t = 2 \int_0^{t_0} L_r^n dM_r + R, \quad (12.7)$$

where  $L_r^n = M_r - M_{k/2^n}$  for  $k/2^n \leq r < (k+1)/2^n$  and

$$R = \langle M \rangle_{([t_0 2^n]+1)/2^n} - \langle M \rangle_{t_0}.$$

Note

$$\mathbb{E} \left( 2 \int_0^{t_0} L_r^n dM_r \right)^2 = 4 \mathbb{E} \int_0^{t_0} (L_r^n)^2 d\langle M \rangle_r. \quad (12.8)$$

The integrand  $(L_r^n)^2$  is bounded by  $4K^2$ ,  $\mathbb{E} \langle M \rangle_t = \mathbb{E} M_t^2 \leq K^2$  is finite, and  $L_r^n$  tends to 0 as  $n \rightarrow \infty$ . By dominated convergence, the right-hand side of (12.8) tends to 0 as  $n \rightarrow \infty$ . As for the remainder term,  $R$  goes to 0 by the continuity of the paths of  $\langle M \rangle_t$ . The reason we only have convergence in probability rather than in  $L^2$  is due to the stopping time argument involving  $T_K$ .  $\square$

### 12.4 Martingale representation

The next theorem says that every martingale adapted to the filtration of a Brownian motion can be expressed as a stochastic integral with respect to the Brownian motion. This

used to be a rather arcane result that was of interest only to probabilists specializing in martingales. But then it turned out that this theorem is the basis for showing the completeness of the market in the theory of financial mathematics; see Chapter 28. The martingale representation theorem is also key to the innovations approach to stochastic filtering; see Chapter 29.

**Theorem 12.3** *Let  $\mathcal{F}_t$  be the minimal augmented filtration generated by a one-dimensional Brownian motion  $W_t$ , let  $t_0 > 0$ , and let  $Y$  be  $\mathcal{F}_{t_0}$  measurable with  $\mathbb{E} Y^2 < \infty$ . There exists a predictable process  $H_s$  with  $\mathbb{E} \int_0^{t_0} H_s^2 ds < \infty$  such that*

$$Y = \mathbb{E} Y + \int_0^{t_0} H_s dW_s, \quad \text{a.s.} \quad (12.9)$$

The proof consists of showing (12.9) holds for successively larger classes of random variables. Step 1 of the proof shows that the equation holds for random variables of the form  $e^{iu(W_t - W_s)}$  and Step 2 shows that (12.9) holds for products of such random variables. In Step 3, it is shown that if the equation holds for a set of random variables, it holds for the closure of that set with respect to the  $L^2$  norm.

*Proof* Step 1. Let  $X_t = iuW_t + u^2 t/2$ . Note  $\langle X \rangle_t = (iu)^2 \langle W \rangle_t$ . By Itô's formula applied with  $f(x) = e^x$ ,

$$\begin{aligned} e^{iuW_t + u^2 t/2} &= 1 + \int_0^t e^{X_r} d(iuW_r - u^2 r/2) + \frac{1}{2} \int_0^t (-u^2) e^{X_r} dr \\ &= 1 + \int_0^t iue^{iuW_r + u^2 r/2} dW_r. \end{aligned}$$

Therefore

$$e^{iuW_t} = e^{-u^2 t/2} + \int_0^t iue^{iuW_r + u^2 r/2 - u^2 t/2} dW_r. \quad (12.10)$$

The integrand in the stochastic integral in (12.10) is  $e^{iuW_r}$  times a deterministic function, hence is predictable. Therefore (12.9) holds when  $Y = e^{iuW_t}$  and moreover, the support of  $H$  in this case is contained in  $[0, t]$ , that is,  $H_r = 0$  if  $r \notin [0, t]$ . Similarly, (12.9) holds when  $Y = e^{iu(W_t - W_s)}$ , and in this case the support of the corresponding  $H$  is  $[s, t]$ .

Step 2. Suppose now that  $Y_1$  and  $Y_2$  are two random variables for which (12.9) holds with the supports of the corresponding  $H_1$  and  $H_2$  overlapping by at most finitely many points. To be more precise, if  $Y_i = \mathbb{E} Y_i + \int_0^{t_0} H_i(s) dW_s$ ,  $i = 1, 2$ , then we suppose that, with probability one,  $H_1(s)H_2(s) = 0$  except for finitely many points  $s$ . This implies

$$\int_0^{t_0} H_1(s)H_2(s) ds = 0.$$

Let  $Z_i(t) = \mathbb{E} Y_i + \int_0^t H_i(s) dW_s$ ,  $i = 1, 2$ . Note  $Z_i(0) = \mathbb{E} Y_i$  and  $Z_i(t_0) = Y_i$ . Then by the product formula (Corollary 11.3),

$$\begin{aligned} Y_1 Y_2 &= (\mathbb{E} Y_1)(\mathbb{E} Y_2) + \int_0^{t_0} Z_1(s) dZ_2(s) + \int_0^{t_0} Z_2(s) dZ_1(s) + \langle Z_1, Z_2 \rangle_{t_0} \\ &= (\mathbb{E} Y_1)(\mathbb{E} Y_2) + \int_0^{t_0} Z_1(s) H_2(s) dW_s + \int_0^{t_0} Z_2(s) H_1(s) dW_s \\ &\quad + \int_0^{t_0} H_1(s) H_2(s) ds \\ &= (\mathbb{E} Y_1)(\mathbb{E} Y_2) + \int_0^{t_0} K_s dW_s, \end{aligned} \tag{12.11}$$

where  $K_s = Z_1(s) H_2(s) + Z_2(s) H_1(s)$ , and so the support of  $K_s$  is contained in the union of the supports of  $H_1(s)$  and  $H_2(s)$ . Taking an expectation in (12.11),  $\mathbb{E}[Y_1 Y_2] = (\mathbb{E} Y_1)(\mathbb{E} Y_2)$ . Thus (12.9) holds for  $Y_1 Y_2$ . Using induction, (12.9) will hold for the product of  $n$  random variables  $Y_i$ ,  $i = 1, \dots, n$ , provided the supports of any two of the corresponding  $H_i$  overlap by at most finitely many values of  $s$ . Combining this with Step 1, we see that if  $s_1 < s_2 < \dots < s_{n+1} \leq t_0$ , then the random variables of the form

$$Y = \exp \left( i \sum_{j=1}^n u_j (W_{s_{j+1}} - W_{s_j}) \right) \tag{12.12}$$

satisfy (12.9).

*Step 3.* We claim that random variables of the form (12.12) generate  $\sigma(W_s; s \leq t_0)$ . To see this, we proceed as in the last paragraph of the proof of Theorem 12.1, namely, we replace each  $u_j$  by  $-u_j$ , multiply by  $\hat{f}(u_1, \dots, u_n)$ , the Fourier transform of a  $C^\infty$  function  $f$  with compact support, integrate over  $(u_1, \dots, u_n) \in \mathbb{R}^n$ , use the Fubini theorem and the Fourier inversion theorem, and we obtain random variables of the form

$$f(W_{s_2} - W_{s_1}, \dots, W_{s_{n+1}} - W_{s_n})$$

for  $f$  in  $C^\infty$  with compact support. By a limit argument, such random variables generate  $\sigma(W_s; s \leq t_0)$ . We will prove that whenever  $Y_n$  satisfies (12.9) and  $Y_n \rightarrow Y$  in  $L^2$ , then  $Y$  satisfies (12.9). By Exercise 2.7 and Proposition 2.5, this will prove our theorem.

Suppose each  $Y_n$  satisfies (12.9) with integrand  $H_n(s)$  and suppose  $Y_n \rightarrow Y$  in  $L^2$ . Then  $\mathbb{E} Y_n \rightarrow \mathbb{E} Y$ , and  $Y_n - \mathbb{E} Y_n$  converges in  $L^2$  to  $Y - \mathbb{E} Y$ . Since

$$\mathbb{E} \int_0^{t_0} (H_n(s) - H_m(s))^2 ds = \mathbb{E} ((Y_n - \mathbb{E} Y_n) - (Y_m - \mathbb{E} Y_m))^2 \rightarrow 0,$$

the sequence  $H_n$  is a Cauchy sequence with respect to the norm  $\|X\| = (\mathbb{E} \int_0^{t_0} X_s^2 ds)^{1/2}$ , which is an  $L^2$  norm and hence complete. Therefore there exists  $H_s$  (which is predictable because each  $H_n(s)$  is predictable) such that  $\mathbb{E} \int_0^{t_0} H_s^2 ds < \infty$  and  $\mathbb{E} \int_0^{t_0} (H_n(s) - H_s)^2 ds \rightarrow 0$ . Hence

$$\mathbb{E} \left( (Y_n - \mathbb{E} Y_n) - \int_0^{t_0} H_s dW_s \right)^2 = \mathbb{E} \int_0^{t_0} (H_n(s) - H_s)^2 ds \rightarrow 0.$$

Since  $Y_n - \mathbb{E} Y_n$  converges in  $L^2$  to  $Y - \mathbb{E} Y$ , it follows that  $Y - \mathbb{E} Y = \int_0^{t_0} H_s dW_s$ , a.s.  $\square$

**Corollary 12.4** Suppose  $M_t$  is a right-continuous square integrable martingale with respect to the minimal augmented filtration  $\{\mathcal{F}_t\}$  generated by a one-dimensional Brownian motion and suppose  $M_0 = 0$ . Let  $t_0 > 0$ . Then there exists a predictable process  $H_s$  with  $\mathbb{E} \int_0^{t_0} H_s^2 ds < \infty$  such that with probability one

$$M_t = \int_0^t H_s dW_s$$

for all  $t \leq t_0$ .

*Proof* Since  $M_t$  is a martingale,  $\mathbb{E}[M_{t_0} \mid \mathcal{F}_0] = M_0$ , and taking expectations,  $\mathbb{E} M_{t_0} = \mathbb{E} M_0 = 0$ . By Theorem 12.3, there exists a predictable process  $H$  with  $\mathbb{E} \int_0^{t_0} H_s^2 ds < \infty$  such that  $M_{t_0} = \int_0^{t_0} H_s dW_s$ .

Taking conditional expectations with respect to  $\mathcal{F}_t$ , we obtain  $M_t = \int_0^t H_s dW_s$ . This holds almost surely for each  $t$ . Thus except for a null set of  $\omega$ 's, it holds for all  $t$  rational. Since  $M_t$  is right continuous, it holds for all  $t$ .  $\square$

**Corollary 12.5** If  $M_t$  is a square integrable martingale with respect to the minimal augmented filtration of a one-dimensional Brownian motion  $W$ , then  $M_t$  has a version with continuous paths.

*Proof* By Corollary 3.13,  $M$  has a version with right continuous paths. By Corollary 12.4,  $M$  can be written as a stochastic integral with respect to  $W$ . But such stochastic integrals have continuous paths by Theorem 10.4.  $\square$

It is important for the martingale representation theorem that  $M_t$  be a martingale with respect to the minimal augmented filtration of  $W$  and not a larger filtration. For example, let  $(X, Y)$  be a two-dimensional Brownian motion and let  $\{\mathcal{F}_t\}$  be the minimal augmented filtration generated by  $(X, Y)$ . We show that we cannot write  $Y_1$  as a stochastic integral with respect to  $X_t$ . If it were possible to do so, since  $Y_1$  has mean zero, we would have

$$Y_1 = \int_0^1 H_s dX_s.$$

Taking conditional expectations,  $Y_t = \int_0^t H_s dX_s$ . Then  $\langle X, Y \rangle_t = \int_0^t H_s ds$  by Exercise 10.5. But if  $(X, Y)$  is two-dimensional Brownian motion, then  $X$  and  $Y$  are independent, and so  $\langle X, Y \rangle_t = 0$  by Exercise 9.4, a contradiction. (However, it is true, by a proof similar to that of Theorem 12.3, if  $\{\mathcal{F}_t\}$  is the minimal augmented filtration of a  $d$ -dimensional Brownian motion  $(W^1, \dots, W^d)$  and  $Y$  is square integrable and  $\mathcal{F}_{t_0}$  measurable, then there exist suitable processes  $H_s^i$  such that  $Y = \mathbb{E} Y + \sum_{i=1}^d \int_0^{t_0} H_s^i dW_s^i$ .)

## 12.5 The Burkholder–Davis–Gundy inequalities

Next we turn to a pair of basic inequalities, those of Burkholder, Davis, and Gundy. In both of the following theorems, the constant depends on  $p$ , the exponent. As stated and proved below, we require  $p \geq 2$  for Theorems 12.6 and 12.7; in fact, the two theorems are true (with a different proof) as long as  $p > 0$ ; see Bass (1995), pp. 62–4, or Exercise 12.12. The proof we present here is a nice application of Itô's formula.

Define

$$M_t^* = \sup_{s \leq t} |M_s|.$$

**Theorem 12.6** Let  $M_t$  be a continuous local martingale with  $M_0 = 0$ , a.s., and suppose  $2 \leq p < \infty$ . There exists a constant  $c_1$  depending on  $p$  such that for any finite stopping time  $T$ ,

$$\mathbb{E} (M_T^*)^p \leq c_1 \mathbb{E} \langle M \rangle_T^{p/2}.$$

*Proof* There is nothing to prove if the left-hand side is zero, so we may assume it is positive. First suppose  $M_T^*$  is bounded by a positive constant  $K$ . Note for  $p \geq 2$  the function  $x \rightarrow |x|^p$  is  $C^2$ . By Doob's inequalities and then Itô's formula (and the fact that  $|M_s| \geq 0$ ), we have

$$\begin{aligned} \mathbb{E} |M_T^*|^p &\leq c \mathbb{E} |M_T|^p \\ &= c \mathbb{E} \int_0^T p|M_s|^{p-1} dM_s + \frac{1}{2} c \mathbb{E} \int_0^T p(p-1)|M_s|^{p-2} d\langle M \rangle_s \\ &\leq c \mathbb{E} \int_0^T (M_T^*)^{p-2} d\langle M \rangle_s \\ &= c \mathbb{E} [(M_T^*)^{p-2} \langle M \rangle_T]. \end{aligned}$$

(Recall our convention about constants and the letter  $c$ .) Using Hölder's inequality with exponents  $p/(p-2)$  and  $p/2$ , we obtain

$$\mathbb{E} (M_T^*)^p \leq c (\mathbb{E} (M_T^*)^p)^{\frac{p-2}{p}} (\mathbb{E} (\langle M \rangle_T^{\frac{p}{2}}))^{\frac{2}{p}}.$$

Dividing both sides by  $(\mathbb{E} (M_T^*)^p)^{(p-2)/p}$  and then taking both sides to the power  $p/2$  gives our result.

We then apply the above to  $T \wedge U_K$ , where  $U_K = \inf\{t : |M_t| \geq K\}$ , let  $K \rightarrow \infty$ , and use Fatou's lemma.  $\square$

**Theorem 12.7** Let  $M_t$  be a continuous local martingale with  $M_0 = 0$ , a.s., and suppose  $2 \leq p < \infty$ . There exists a constant  $c_2$  depending on  $p$  such that for any finite stopping time  $T$ ,

$$\mathbb{E} \langle M \rangle_T^{p/2} \leq c_2 \mathbb{E} (M_T^*)^p.$$

*Proof* As in the previous theorem, we may assume the left-hand side is positive. Set  $r = p/2$ . Let us first suppose  $\langle M \rangle_T$  and  $M_T^*$  are bounded by a positive constant  $K$ . Let  $N_t = M_{t \wedge T}$ , so that  $\langle N \rangle_\infty = \langle M \rangle_T$ , and let  $A_t = \langle M \rangle_{t \wedge T}^{r-1}$ . Using integration by parts,

$$\begin{aligned} \int_0^\infty \langle N \rangle_s dA_s &= \langle N \rangle_\infty A_\infty - \int_0^\infty A_s d\langle N \rangle_s \\ &= \langle N \rangle_\infty^r - \frac{1}{r} \langle N \rangle_\infty^r. \end{aligned}$$

Since

$$\int_0^\infty \langle N \rangle_\infty dA_s = \langle N \rangle_\infty^r,$$

we then have

$$\langle N \rangle_\infty^r = r \int_0^\infty (\langle N \rangle_\infty - \langle N \rangle_s) dA_s.$$

Using Propositions 3.14 and 9.6,

$$\begin{aligned} \mathbb{E} \langle M \rangle_T^r &= \mathbb{E} \langle N \rangle_\infty^r = r \mathbb{E} \int_0^\infty (\langle N \rangle_\infty - \langle N \rangle_s) dA_s \\ &= r \mathbb{E} \int_0^\infty (\mathbb{E} [\langle N \rangle_\infty | \mathcal{F}_s] - \langle N \rangle_s) dA_s \\ &= r \mathbb{E} \int_0^\infty \mathbb{E} [\langle N \rangle_\infty - \langle N \rangle_s | \mathcal{F}_s] dA_s \\ &= r \mathbb{E} \int_0^\infty \mathbb{E} [N_\infty^2 - N_s^2 | \mathcal{F}_s] dA_s \\ &\leq c \mathbb{E} \int_0^\infty \mathbb{E} [(N_\infty^*)^2 | \mathcal{F}_s] dA_s \\ &= c \mathbb{E} [(N_\infty^*)^2 A_\infty] \\ &= c \mathbb{E} [(M_T^*)^2 \langle M \rangle_T^{r-1}]. \end{aligned}$$

We use Hölder's inequality with exponents  $r$  and  $r/(r-1)$ , divide both sides by the quantity  $(\mathbb{E} \langle M \rangle_T^{r-1})^{r/(r-1)}$ , and then take both sides to the  $r$ th power. We then get

$$\mathbb{E} \langle M \rangle_T^r \leq c \mathbb{E} (M_T^*)^{2r},$$

which is what we wanted.

To remove the restriction that  $\langle M \rangle$  and  $M^*$  are bounded, we apply the above to  $T \wedge V_K$  in place of  $T$ , where  $V_K = \inf\{t : \langle M \rangle_t + M_t^* \geq K\}$ , let  $K \rightarrow \infty$ , and use Fatou's lemma.  $\square$

## 12.6 Stratonovich integrals

For stochastic differential geometry and also many other purposes, the Stratonovich integral is more convenient than the Itô integral. If  $X$  and  $Y$  are continuous semimartingales, the *Stratonovich integral*, denoted  $\int_0^t X_s \circ dY_s$ , is defined by

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2} \langle X, Y \rangle_t.$$

Both the beauty and the difficulty of Itô's formula are due to the quadratic variation term. The change of variables formula for the Stratonovich integral avoids this.

**Theorem 12.8** Suppose  $f \in C^3$  and  $X$  is a continuous semimartingale. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s.$$

*Proof* By Itô's formula applied to the function  $f$  and the definition of the Stratonovich integral, it suffices to show that

$$\langle f'(X), X \rangle_t = \int_0^t f''(X_s) d\langle X \rangle_s. \quad (12.13)$$

Applying Itô's formula to the function  $f'$ , which is in  $C^2$ ,

$$f'(X_t) = f'(X_0) + \int_0^t f''(X_s) dX_s + \frac{1}{2} \int_0^t f'''(X_s) d\langle X \rangle_s,$$

from which (12.13) follows.  $\square$

If  $X$  and  $Y$  are continuous semimartingales and we apply the change of variables formula with  $f(x) = x^2$  to  $X + Y$  and  $X - Y$ , we obtain

$$(X_t + Y_t)^2 = (X_0 + Y_0)^2 + 2 \int_0^t (X_s + Y_s) \circ d(X_s + Y_s)$$

and

$$(X_t - Y_t)^2 = (X_0 - Y_0)^2 + 2 \int_0^t (X_s - Y_s) \circ d(X_s - Y_s).$$

Taking the difference and then dividing by 4, we have the *product formula for Stratonovich integrals*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s \circ dY_s + \int_0^t Y_s \circ dX_s. \quad (12.14)$$

The Stratonovich integral  $\int H_s \circ dX_s$  can be represented as a limit of Riemann sums.

**Proposition 12.9** Suppose  $H$  and  $X$  are continuous semimartingales and  $t_0 > 0$ . Then  $\int_0^t H_s \circ dX_s$  is the limit in probability as  $n \rightarrow \infty$  of

$$\sum_{k=0}^{2^n-1} \frac{H_{kt_0/2^n} + H_{(k+1)t_0/2^n}}{2} (X_{(k+1)t_0/2^n} - X_{kt_0/2^n}).$$

*Proof* We write the sum as

$$\begin{aligned} & \sum H_{kt_0/2^n} (X_{(k+1)t_0/2^n} - X_{kt_0/2^n}) \\ & + \frac{1}{2} \sum (H_{(k+1)t_0/2^n} - H_{kt_0/2^n}) (X_{(k+1)t_0/2^n} - X_{kt_0/2^n}). \end{aligned}$$

The first sum tends to  $\int_0^t H_s dX_s$  while by Exercise 12.10 the second sum tends to  $\frac{1}{2} \langle H, X \rangle_t$ . This proves the proposition.  $\square$

## Exercises

- 12.1 Show that  $W_t$  and  $W_t^2 - t$  are local martingales, where  $W$  is defined in the statement of Theorem 12.2.
- 12.2 Suppose  $\{\mathcal{F}_t\}$  is a filtration satisfying the usual conditions,  $X$  is a Brownian motion with respect to  $\{\mathcal{F}_t\}$ , and  $T$  is a finite stopping time with respect to this same filtration. Let  $Y$  be another Brownian motion that is independent of  $\{\mathcal{F}_t\}$  and define

$$Z_t = \begin{cases} X_t, & t < T \\ X_T + Y_{t-T}, & t \geq T. \end{cases}$$

Show that  $Z$  is a Brownian motion (although not necessarily with respect to  $\{\mathcal{F}_t\}$ ).

- 12.3 Suppose  $M_t$  is a continuous local martingale with respect to a filtration  $\{\mathcal{F}_t\}$  satisfying the usual conditions,  $T$  is a stopping time with respect to  $\{\mathcal{F}_t\}$ , and  $\langle M \rangle_t = t \wedge T$ . Prove that  $M_{t \wedge T}$  has the same law as a Brownian motion stopped at time  $T$ .
- 12.4 Here is a multidimensional version of Lévy's theorem. Let  $\{\mathcal{F}_t\}$  be a filtration satisfying the usual conditions. Suppose  $(M_t^1, \dots, M_t^d)$  is a  $d$ -dimensional process such that each component  $M_t^i$  is a continuous martingale with respect to  $\{\mathcal{F}_t\}$  with  $\langle M^i \rangle_t = t$ . Suppose that  $\langle M^i, M^j \rangle_t = 0$  if  $i \neq j$ . Prove that  $(M_t^1, \dots, M_t^d)$  is a  $d$ -dimensional Brownian motion.
- 12.5 Let  $\{\mathcal{F}_t\}$  be a filtration satisfying the usual conditions. Let  $A_t$  be a strictly increasing continuous process adapted to  $\{\mathcal{F}_t\}$  with  $\lim_{t \rightarrow \infty} A_t = \infty$ , a.s. Suppose  $(M_t^1, \dots, M_t^d)$  is a  $d$ -dimensional process such that each component  $M_t^i$  is a continuous martingale with respect to  $\{\mathcal{F}_t\}$  and  $\langle M^i \rangle_t = A_t$ . Suppose that  $\langle M^i, M^j \rangle_t = 0$  if  $i \neq j$ . Prove that  $(M_t^1, \dots, M_t^d)$  is a time change of  $d$ -dimensional Brownian motion.
- 12.6 Suppose  $M$  is a continuous local martingale such that  $\langle M \rangle_t$  is deterministic. Prove that  $M$  is a Gaussian process.
- 12.7 Suppose  $M$  is a continuous local martingale with  $M_0 = 0$ , a.s. Show that there exists a Brownian motion  $W$ , an increasing process  $\tau_t$ , and a stopping time  $T$  such that  $M_t = W_{\tau_t \wedge T}$  for all  $t$ .
- 12.8 Let  $M_t$  be a continuous local martingale. Show that the events  $(M_\infty^* < \infty)$  and  $(\langle M \rangle_\infty < \infty)$  differ by at most a null set.
- 12.9 Let  $M_t$  be a continuous local martingale. Prove that

$$\mathbb{P}(\sup_{t \geq 0} |M_t| > x, \langle M \rangle_\infty < y) \leq 2e^{-x^2/2y}.$$

- 12.10 Suppose  $X$  and  $Y$  are continuous semimartingales and  $t_0 > 0$ . Prove that

$$\sum_{k=0}^{2^n-1} (X_{(k+1)t_0/2^n} - X_{kt_0/2^n})(Y_{(k+1)t_0/2^n} - Y_{kt_0/2^n})$$

converges to  $\langle X, Y \rangle_{t_0}$  in probability.

- 12.11 Let  $p > 0$ . Suppose  $X$  and  $Y$  are non-negative random variables,  $\beta > 1$ ,  $\delta \in (0, 1)$ , and  $\varepsilon \in (0, \beta^{-p}/2)$  such that

$$\mathbb{P}(X > \beta\lambda, Y < \delta\lambda) \leq \varepsilon \mathbb{P}(X \geq \lambda)$$

for all  $\lambda > 0$ . This inequality is known as a *good- $\lambda$  inequality*. Prove that there exists a constant  $c$  (depending on  $\beta$ ,  $\delta$ ,  $\varepsilon$ , and  $p$  but not  $X$  or  $Y$ ) such that

$$\mathbb{E} X^p \leq c \mathbb{E} Y^p.$$

*Hint:* First assume  $X$  is bounded. Write

$$\begin{aligned} \mathbb{P}(X/\beta > \lambda) &= \mathbb{P}(X > \beta\lambda, Y < \delta\lambda) + \mathbb{P}(Y \geq \delta\lambda) \\ &\leq \varepsilon \mathbb{P}(X \geq \lambda) + \mathbb{P}(Y/\delta \geq \lambda). \end{aligned}$$

Multiply by  $p\lambda^{p-1}$ , integrate over  $\lambda$ , and use the fact that  $\varepsilon < \beta^{-P}/2$ .

- 12.12 Use Exercise 12.11 to prove that the Burkholder–Davis–Gundy inequalities hold for all  $p > 0$ .  
*Hint:* Use time change to reduce to the case of a Brownian motion  $W$ . If  $T$  is a stopping time and  $U = \inf\{t : W_T^* > \lambda\}$ , write

$$\begin{aligned}\mathbb{P}(W_T^* > \beta\lambda, T^{1/2} < \delta\lambda) &= \mathbb{P}(W_T^* > \beta\lambda, T < \delta^2\lambda^2, U < \infty) \\ &\leq \mathbb{P}\left(\sup_{U \leq t \leq U + \delta^2\lambda^2} |W_t - W_U| > (\beta - 1)\lambda, U < \infty\right).\end{aligned}$$

Condition on  $\mathcal{F}_U$ , use Theorem 4.2, and notice that  $\mathbb{P}(U < \infty) = \mathbb{P}(W_T^* > \lambda)$ .

- 12.13 Define the  $H^1$  norm of a martingale by

$$\|M\|_{H^1} = \mathbb{E} [\sup_{t \geq 0} |M_t|].$$

Prove that this is a norm. Does there exist a uniformly integrable continuous martingale that is not in  $H^1$ ?

- 12.14 Let  $W$  be a Brownian motion and let  $T$  be a stopping time. Prove that if  $\mathbb{E} T^{1/2} < \infty$ , then  $\mathbb{E} W_T = 0$ .

- 12.15 Suppose  $W = (W^1, \dots, W^d)$  is a  $d$ -dimensional Brownian motion started at 0, and let  $\{\mathcal{F}_t\}$  be the minimal augmented filtration of  $W$ . Suppose  $Y$  is a  $\mathcal{F}_1$  measurable random variable with mean zero and finite variance. Prove there exist predictable processes  $H^1, \dots, H^d$  such that  $\mathbb{E} \int_0^1 (H_s^i)^2 ds < \infty$  for each  $i$  and

$$Y = \sum_{i=1}^d \int_0^1 H_s^i dW_s^i.$$

- 12.16 Suppose  $W$  is a Brownian motion and  $H$  is adapted, bounded, and right continuous. Let  $t \geq 0$ . Show

$$\frac{1}{W_{t+h} - W_t} \int_t^{t+h} H_s dW_s$$

converges in probability to  $H_t$ .

- 12.17 Let  $W$  be a Brownian motion and  $\alpha > 0$ . Show that

$$\int_0^t \frac{1}{|W_s|^\alpha} ds$$

is infinite almost surely if  $\alpha \geq 1$  but finite almost surely if  $\alpha < 1$ .

- 12.18 Here is a useful inequality. Suppose  $A$  is an increasing process with  $A_0 = 0$ , a.s., and suppose there exists a non-negative random variable  $B$  such that for each  $t$ ,

$$\mathbb{E}[A_\infty - A_t | \mathcal{F}_t] \leq \mathbb{E}[B | \mathcal{F}_t], \quad \text{a.s.}$$

Prove that for each integer  $p \geq 1$ , there exists a constant  $c_p$  depending only on  $p$  such that

$$\mathbb{E} A_\infty^p \leq c_p \mathbb{E} B^p.$$

*Hint:* Write

$$A_\infty = p! \int_0^\infty (A_\infty - A_t) dA_t,$$

take expectations, and use Proposition 3.14.

12.19 Let  $W$  be a one-dimensional Brownian motion with filtration  $\{\mathcal{F}_t\}$  and let  $f(r, s)$  be a deterministic function. Define the *multiple stochastic integral* by

$$\int_0^t \int_0^s f(r, s) dW_r dW_s = \int_0^t \left( \int_0^s f(r, s) dW_r \right) dW_s,$$

provided

$$\int_0^t \int_0^s f(r, s)^2 dr ds < \infty,$$

and similarly for higher-order multiple stochastic integrals.

(1) If  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  are bounded and deterministic,  $n \neq m$ ,

$$M_t^f = \int_0^t \cdots \int_0^{r_{m-1}} f dW_{r_1} \cdots dW_{r_m},$$

and  $M_t^g$  is defined similarly, show that  $\mathbb{E}[M_t^f M_t^g] = 0$  for all  $t$ .

(2) Show that the collection of random variables

$$\{M_1^f : f \text{ has domain } \mathbb{R}^m \text{ for some } m \text{ and is bounded and deterministic}\}$$

is dense in the set of mean zero  $\mathcal{F}_1$  measurable random variables with respect to the  $L^2(\mathbb{P})$  norm.

# 13

---

## The Girsanov theorem

We look at what happens to a Brownian motion when we change  $\mathbb{P}$  to another probability measure  $\mathbb{Q}$ . This may seem strange, but there are many applications of this, including to financial mathematics and to filtering; see Chapters 28 and 29. Another application we will give (at the end of this chapter in Section 13.2) is to determine the probability a Brownian motion  $W_s$  crosses a line  $a + bs$  before time  $t$ .

### 13.1 The Brownian motion case

We start with an observation. Suppose  $Y_t$  is a continuous local martingale with  $Y_0 = 0$  and let  $Z_t = e^{Y_t - \langle Y \rangle_t / 2}$ . Applying Itô's formula to  $X_t = Y_t - \frac{1}{2} \langle Y \rangle_t$ , with the function  $e^x$  yields

$$\begin{aligned} Z_t &= e^{Y_t - \langle Y \rangle_t / 2} = 1 + \int_0^t e^{X_s} d\left(Y_s - \frac{1}{2} \langle Y \rangle_s\right) + \frac{1}{2} \int_0^t e^{X_s} d\langle Y \rangle_s \\ &= 1 + \int_0^t Z_s dY_s. \end{aligned} \tag{13.1}$$

This can be abbreviated by  $dZ_t = Z_t dY_t$ .  $Z_t$  is called the *exponential of the martingale*  $Y$ , and since  $Z$  is the stochastic integral with respect to a local martingale, it is itself a local martingale.

Before stating the Girsanov theorem, we need two technical lemmas.

**Lemma 13.1** *Suppose  $Y$  is a continuous local martingale with  $Y_0 = 0$  and  $Z_t = e^{Y_t - \langle Y \rangle_t / 2}$ . If  $\langle Y \rangle_t$  is a bounded random variable for each  $t$ , then  $\mathbb{E} |Z_t|^p < \infty$  for each  $p > 1$  and each  $t$ .*

*Proof* Let us first suppose  $Y$  is bounded in absolute value by  $N$ . Since  $Z_t \geq 0$ , we have by the Cauchy–Schwarz inequality

$$\begin{aligned} \mathbb{E} Z_t^p &= \mathbb{E} e^{pY_t - p\langle Y \rangle_t / 2} \\ &= \mathbb{E} \left[ e^{pY_t - p^2 \langle Y \rangle_t} e^{(p^2 - (p/2)) \langle Y \rangle_t} \right] \\ &\leq \left( \mathbb{E} e^{2pY_t - 2p^2 \langle Y \rangle_t} \right)^{1/2} \left( \mathbb{E} e^{(2p^2 - p) \langle Y \rangle_t} \right)^{1/2}. \end{aligned} \tag{13.2}$$

By the exact same calculation as in (13.1) but with  $Y$  replaced by  $2pY$ , we see  $e^{2pY_t - 2p^2 \langle Y \rangle_t}$  is a stochastic integral of a bounded integrand with respect to a bounded martingale, and hence is a martingale. This shows that the first factor on the last line of (13.2) is 1. By our assumption that  $\langle Y \rangle_t$  is bounded, the second factor on this line is finite and does not depend on  $N$ .

If  $Y$  is not bounded, let  $T_N = \inf\{s : |Y_s| \geq N\}$ , apply the above argument to  $Y_{t \wedge T_N}$ , and let  $N \rightarrow \infty$ .  $\square$

The second lemma is the following.

**Lemma 13.2** Suppose  $A_t$  is a continuous increasing process adapted to a filtration  $\{\mathcal{F}_t\}$  satisfying the usual conditions. Let  $X$  be a bounded random variable,  $H$  a bounded adapted process,  $s < t$ , and  $B \in \mathcal{F}_s$ . Then

$$\mathbb{E} \left[ \int_s^t X H_r dA_r; B \right] = \mathbb{E} \left[ \int_s^t \mathbb{E}[X | \mathcal{F}_r] H_r dA_r; B \right].$$

*Proof* By linearity, it suffices to suppose  $X$  and  $H$  are non-negative. Let  $A'_r = A_{r+s}$ ,  $H'_r = H_{r+s}$ , and  $\mathcal{F}'_r = \mathcal{F}_{r+s}$ . Let  $C_r = \int_0^r H'_s 1_B dA'_s$ , and so we must show

$$\mathbb{E} \int_0^{t-s} X dC_r = \mathbb{E} \int_0^{t-s} \mathbb{E}[X | \mathcal{F}'_r] dC_r.$$

This follows by Proposition 3.14.  $\square$

Let  $M_t$  be a non-negative continuous martingale with  $M_0 = 1$ , a.s. Define a new probability measure  $\mathbb{Q}$  by  $\mathbb{Q}(A) = \mathbb{E}[M_t; A]$  if  $A \in \mathcal{F}_t$ . Note  $\mathbb{Q}$  is a probability measure because  $\mathbb{Q}(\Omega) = \mathbb{E} M_t = \mathbb{E} M_0 = 1$ .  $\mathbb{Q}$  is well-defined because if  $A \in \mathcal{F}_s \subset \mathcal{F}_t$ , then since  $M$  is a martingale, we have  $\mathbb{E}[M_t; A] = \mathbb{E}[M_s; A]$ .

A more general version of the Girsanov theorem is possible (see Exercise 13.5), but the Girsanov theorem is most frequently used with Brownian motion.

**Theorem 13.3** Suppose  $W_t$  is a Brownian motion with respect to  $\mathbb{P}$ ,  $H$  is bounded and predictable,

$$M_t = \exp \left( \int_0^t H_r dW_r - \frac{1}{2} \int_0^t H_r^2 dr \right), \quad (13.3)$$

and

$$\mathbb{Q}(B) = \mathbb{E}_{\mathbb{P}}[M_t; B] \quad \text{if } B \in \mathcal{F}_t. \quad (13.4)$$

Then  $W_t - \int_0^t H_r dr$  is a Brownian motion with respect to  $\mathbb{Q}$ .

*Proof* We prove the theorem by showing  $W_t - \int_0^t H_r dr$  satisfies the hypotheses of Lévy's theorem (Theorem 12.1). We first show  $W_t - \int_0^t H_r dr$  is a martingale with respect to  $\mathbb{Q}$ . By (13.1) with  $Y_t = \int_0^t H_r dW_r$  and  $Z_t = M_t$ ,

$$M_t = 1 + \int_0^t M_r H_r dW_r.$$

By Exercise 10.5,

$$\langle M, W \rangle_t = \int_0^t M_r H_r dr. \quad (13.5)$$

We want to show that if  $B \in \mathcal{F}_s$ , then

$$\mathbb{E}_{\mathbb{Q}} \left[ W_t - \int_0^t H_r dr; B \right] = \mathbb{E}_{\mathbb{Q}} \left[ W_s - \int_0^s H_r dr; B \right]. \quad (13.6)$$

If  $B \in \mathcal{F}_s$ , then using the definition of  $\mathbb{Q}$  and the product formula (Corollary 11.3),

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[W_t; B] &= \mathbb{E}_{\mathbb{P}}[M_t W_t; B] \\ &= \mathbb{E}_{\mathbb{P}}\left[\int_0^t M_r dW_r; B\right] + E_{\mathbb{P}}\left[\int_0^t W_r dM_r; B\right] \\ &\quad + \mathbb{E}_{\mathbb{P}}[\langle M, W \rangle_t; B]\end{aligned}\tag{13.7}$$

and

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[W_s; B] &= \mathbb{E}_{\mathbb{P}}[M_s W_s; B] \\ &= \mathbb{E}_{\mathbb{P}}\left[\int_0^s M_r dW_r; B\right] + E_{\mathbb{P}}\left[\int_0^s W_r dM_r; B\right] \\ &\quad + \mathbb{E}_{\mathbb{P}}[\langle M, W \rangle_s; B].\end{aligned}\tag{13.8}$$

Since  $H$  is bounded,  $\langle \int_0^\cdot H_r dW_r \rangle_t \leq ct$ . By Lemma 13.1,  $M_t$  is a martingale and  $\mathbb{E} |M_t|^p < \infty$  for each  $t$  and each  $p \geq 1$ . Since stochastic integrals with respect to martingales are martingales,

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^t M_r dW_r; B\right] = \mathbb{E}_{\mathbb{P}}\left[\int_0^s M_r dW_r; B\right]\tag{13.9}$$

and

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^t W_r dM_r; B\right] = \mathbb{E}_{\mathbb{P}}\left[\int_0^s W_r dM_r; B\right].\tag{13.10}$$

Combining (13.7), (13.8), (13.9), and (13.10), we see that (13.6) will follow if we show

$$\mathbb{E}_{\mathbb{P}}[\langle M, W \rangle_t - \langle M, W \rangle_s; B] = \mathbb{E}_{\mathbb{Q}}\left[\int_s^t H_r dr; B\right].\tag{13.11}$$

Using Lemma 13.2 and (13.5), we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\int_s^t H_r dr; B\right] &= \mathbb{E}_{\mathbb{P}}\left[M_t \int_s^t H_r dr; B\right] = \mathbb{E}_{\mathbb{P}}\left[\int_s^t M_t H_r dr; B\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[\int_0^t \mathbb{E}[M_t | \mathcal{F}_r] H_r dr; B\right] = \mathbb{E}_{\mathbb{P}}\left[\int_s^t M_r H_r dr; B\right] \\ &= \mathbb{E}_{\mathbb{P}}[\langle M, W \rangle_t - \langle M, W \rangle_s; B],\end{aligned}$$

which proves (13.11).

A similar proof shows that  $(W_t - \int_0^t H_r dr)^2 - t$  is a martingale with respect to  $\mathbb{Q}$ , and hence the quadratic variation of  $W_t - \int_0^t H_r dr$  under  $\mathbb{Q}$  is still  $t$  (or see Exercise 13.2). Since the process  $W_t - \int_0^t H_r dr$  has continuous paths, by Lévy's theorem,  $W_t - \int_0^t H_r dr$  is a Brownian motion under  $\mathbb{Q}$ .  $\square$

The assumption that  $H$  be bounded can be weakened, but in practice it is more common to use a stopping time argument; for an example, see the proof of Theorem 29.3.

### 13.2 An example

Let us give an example of the use of the Girsanov theorem, namely, to compute the probability that Brownian motion crosses a line  $a + bt$  by time  $t_0$ ,  $a > 0$ . We want to find an exact expression for  $\mathbb{P}(\exists t \leq t_0 : W_t = a + bt)$ , where  $W$  is a Brownian motion.

Let  $W_t$  be a Brownian motion under  $\mathbb{P}$ . Define  $\mathbb{Q}$  on  $\mathcal{F}_{t_0}$  by

$$d\mathbb{Q}/d\mathbb{P} = M_t = e^{-bW_t - b^2 t/2}.$$

By the Girsanov theorem, under  $\mathbb{Q}$ ,  $\tilde{W}_t = W_t + bt$  is a Brownian motion, and  $W_t = \tilde{W}_t - bt$ .

Let  $A = (\sup_{s \leq t_0} W_s \geq a)$ . If we set  $S = \inf\{t > 0 : W_t = a\}$ , then  $A = (S \leq t_0)$  and  $A \in \mathcal{F}_{S \wedge t_0}$ . We write

$$\begin{aligned} \mathbb{P}(\exists t \leq t_0 : W_t = a + bt) &= \mathbb{P}(\exists t \leq t_0 : W_t - bt = a) \\ &= \mathbb{P}(\sup_{s \leq t_0} (W_s - bs) \geq a). \end{aligned} \quad (13.12)$$

$W_t$  is a Brownian motion under  $\mathbb{P}$  while  $\tilde{W}_t$  is a Brownian motion under  $\mathbb{Q}$ . Therefore the last line of (13.12) is equal to

$$\mathbb{Q}(\sup_{s \leq t_0} (\tilde{W}_s - bs) \geq a).$$

This in turn is equal to

$$\mathbb{Q}(\sup_{s \leq t_0} W_s \geq a) = \mathbb{Q}(A).$$

To evaluate  $\mathbb{Q}(A)$ , note  $M_S = e^{-ab - b^2 S/2}$  and by (3.19) with  $b$  replaced by  $a$ ,

$$\mathbb{P}(S \in ds) = \frac{a}{\sqrt{2\pi s^3}} e^{-a^2/2s}.$$

Now we use optional stopping to obtain

$$\begin{aligned} \mathbb{P}(\exists t \leq t_0 : W_t = a + bt) &= \mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[M_{t_0}; A] \\ &= \mathbb{E}_{\mathbb{P}}[M_{S \wedge t_0}; S \leq t_0] \\ &= \mathbb{E}_{\mathbb{P}}[M_S; S \leq t_0] \\ &= \int_0^{t_0} e^{-ab - b^2 s/2} \frac{a}{\sqrt{2\pi s^3}} e^{-a^2/2s} ds. \end{aligned} \quad (13.13)$$

### Exercises

- 13.1 Whether a filtration satisfies the usual conditions depends on the class of null sets and hence the probability measure involved matters. Suppose  $\{\mathcal{F}_t\}$  satisfies the usual conditions with respect to  $\mathbb{P}$ ,  $H$  is a bounded predictable process,  $W$  a Brownian motion with respect to  $\mathbb{P}$ ,  $M$  defined by (13.3), and  $\mathbb{Q}$  defined by (13.4). If  $t_0 > 0$  and  $A \in \sigma(W_s; s \leq t_0)$ , show  $\mathbb{P}(A) = 0$  if and only if  $\mathbb{Q}(A) = 0$ .

- 13.2 Theorem 9.10 allows us to avoid some calculations in the last paragraph of the proof of Theorem 13.3. Suppose  $X$  is a continuous semimartingale under  $\mathbb{P}$  and  $\mathbb{Q}$  is a probability measure equivalent to  $\mathbb{P}$ . That is, a set is a null set for  $\mathbb{P}$  if and only if it is a null set for  $\mathbb{Q}$ . Show  $X$  is a semimartingale under  $\mathbb{Q}$  and the quadratic variation of  $X$  under  $\mathbb{P}$  equals the quadratic variation of  $X$  under  $\mathbb{Q}$ .
- 13.3 Let  $W = (W^1, \dots, W^d)$  be a  $d$ -dimensional Brownian motion with minimal augmented filtration  $\{\mathcal{F}_t\}$  and let  $H_1, \dots, H_d$  be bounded predictable processes. Let

$$M_t = \exp \left( \sum_{i=1}^d \int_0^t H_i(s) dW_s^i - \frac{1}{2} \sum_{i=1}^d \int_0^t |H_i(s)|^2 ds \right).$$

Define a probability measure  $\mathbb{Q}$  by setting  $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[M_t; A]$  if  $A \in \mathcal{F}_t$ . Let  $\tilde{W}_t^i = W_t^i - \int_0^t H_i(s) ds$  for each  $i$ . Prove that  $\tilde{W} = (\tilde{W}^1, \dots, \tilde{W}^d)$  is a  $d$ -dimensional Brownian motion under  $\mathbb{Q}$ .

- 13.4 Let  $W_t$  be a  $d$ -dimensional Brownian motion and let  $\delta, t_0 > 0$ . Let  $f : [0, t_0] \rightarrow \mathbb{R}^d$  be a continuous function. Prove that there exists a constant  $c$  such that

$$\mathbb{P}(\sup_{s \leq t_0} |W_s - f(s)| < \delta) > c.$$

This is known as the *support theorem* for Brownian motion.

*Hint:* First assume that  $f$  has a bounded derivative. Use Exercise 4.9 and the Girsanov theorem.

- 13.5 Here is a more general form of the Girsanov theorem. Suppose  $L_t$  is a bounded continuous martingale under  $\mathbb{P}$ ,  $M_t = e^{L_t - \langle L \rangle_t / 2}$ , and  $\mathbb{Q}$  is a probability measure defined by  $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[M_{t_0}; A]$  if  $A \in \mathcal{F}_{t_0}$ . Suppose  $\{\mathcal{F}_t\}$  is a filtration satisfying the usual conditions with respect to both  $\mathbb{P}$  and  $\mathbb{Q}$ . Show that if  $X$  is a martingale under  $\mathbb{P}$ , then  $X_t - \langle X, L \rangle_t$  is a martingale under  $\mathbb{Q}$ .

# 14

---

## Local times

Let  $W_t$  be a one-dimensional Brownian motion. Although the Lebesgue measure of the random set  $\{t : W_t = 0\}$  is 0, a.s., nevertheless there is an increasing continuous process which grows only when the Brownian motion is at 0. This increasing process is known as local time at 0. We want to derive some of its properties.

### 14.1 Basic properties

Let  $W$  be a Brownian motion. By Jensen's inequality for conditional expectations (Proposition A.21),  $|W_t|$  is a submartingale, and by the Doob–Meyer decomposition (Theorem 9.12), it can be written as a martingale plus an increasing process. Since  $W_t$  is itself a martingale, the increasing process grows only at times when the Brownian motion is at 0.

Rather than appealing to the Doob–Meyer decomposition, we give the explicit decomposition of  $|W_t|$ . We define

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0; \\ 0, & x = 0; \\ -1, & x < 0. \end{cases}$$

**Theorem 14.1** *Let  $W_t$  be a one-dimensional Brownian motion.*

(1) *There exists a non-negative increasing continuous adapted process  $L_t^0$  such that*

$$|W_t| = \int_0^t \operatorname{sgn}(W_s) dW_s + L_t^0. \quad (14.1)$$

(2)  *$L_t^0$  increases only when  $W$  is at 0. More precisely, if  $W_s(\omega) \neq 0$  for  $r \leq s \leq t$ , then  $L_r^0(\omega) = L_t^0(\omega)$ .*

$L_t^0$  is called the *local time at 0*. The equation (14.1) is called the *Tanaka formula*.

*Proof* Define

$$f_\varepsilon(x) = \begin{cases} x^2/2\varepsilon, & |x| < \varepsilon; \\ |x| - (\varepsilon/2), & |x| \geq \varepsilon. \end{cases}$$

The function  $f_\varepsilon$  is an approximation to the function  $|\cdot|$ , and note that  $f_\varepsilon(0) = f'_\varepsilon(0)$ , while  $f''_\varepsilon(x) = \varepsilon^{-1}1_{[-\varepsilon, \varepsilon]}(x)$ , except at  $x = \pm\varepsilon$ .

We apply the extension of Itô's formula given in Exercise 11.2 to  $f_\varepsilon(W_t)$  and obtain

$$f_\varepsilon(W_t) = \int_0^t f'_\varepsilon(W_s) dW_s + \frac{1}{2} \int_0^t f''_\varepsilon(W_s) ds.$$

As we let  $\varepsilon \rightarrow 0$ , we see that  $f_\varepsilon(x) \rightarrow |x|$  uniformly, and  $f'_\varepsilon(x) \rightarrow \text{sgn}(x)$  boundedly. By Doob's inequalities, if  $t_0 > 0$ ,

$$\mathbb{E} \sup_{t \leq t_0} \left| \int_0^t f'_\varepsilon(W_s) dW_s - \int_0^t \text{sgn}(W_s) dW_s \right|^2 \rightarrow 0, \quad (14.2)$$

while  $\sup_{t \leq t_0} |f_\varepsilon(W_t) - |W_t|| \rightarrow 0$ , a.s. Therefore there exists an increasing process  $L_t^0$  and a subsequence  $\varepsilon_n \rightarrow 0$  such that

$$\sup_{t \leq t_0} \left| \frac{1}{2\varepsilon_n} \int_0^t 1_{[-\varepsilon_n, \varepsilon_n]}(W_s) ds - L_t^0 \right| \rightarrow 0, \quad \text{a.s.} \quad (14.3)$$

Hence for almost every  $\omega$  there is convergence uniformly over  $t$  in finite intervals, so  $L_t^0$  is continuous in  $t$ . Since  $\frac{1}{2\varepsilon_n} \int_0^t 1_{[-\varepsilon_n, \varepsilon_n]}(W_s) ds$  increases only for those times  $t$  where  $|W_t| \leq \varepsilon_n$ , then  $L_t^0$  increases only on the set of times when  $W_t = 0$ .  $\square$

In the Tanaka formula, the stochastic integral term is a martingale, say  $N_t$ . Note  $\langle N \rangle_t = t$ , since  $\text{sgn}(x)^2 = 1$  unless  $x = 0$ , and we have seen that Brownian motion spends 0 time at 0 (Exercise 11.1). Hence we have exhibited reflecting Brownian motion, namely  $|W_t|$ , as the sum of another Brownian motion,  $N_t$ , and a continuous process that increases only when  $W$  is at zero.

Let  $M_t$  denote  $\sup_{s \leq t} W_s$ . Note we do not have an absolute value here. The following, due to Lévy, is often useful.

**Theorem 14.2** *The two-dimensional processes  $(|W|, L^0)$  and  $(M - W, M)$  have the same law.*

*Proof* Let  $V_t = -N_t$  in the Tanaka formula, so that

$$|W_t| = -V_t + L_t^0. \quad (14.4)$$

Let  $S_t = \sup_{s \leq t} V_s$ . We will show  $S_t = L_t^0$ . This will prove the result, since  $V$  is a Brownian motion, and hence  $(M - W, M)$  is equal in law to  $(S - V, S) = (|W|, L^0)$ .

From (14.4),  $V_t = L_t^0 - |W_t|$ , or  $V_t \leq L_t^0$  for all  $t$ , hence  $S_t \leq L_t^0$ , since  $L^0$  is increasing.  $L_t^0$  increases only when  $W_t = 0$  and at those times

$$L_t^0 = V_t + |W_t| = V_t \leq S_t.$$

Given two increasing functions with  $f \leq g$ , if  $f(t) = g(t)$  at those times when  $f$  increases, a little thought shows that  $f$  and  $g$  are equal for all  $t$ . Hence  $L_t^0 = S_t$  for all  $t$ .  $\square$

Just as we defined  $L_t^0$  via the Tanaka formula, we can construct local time at the level  $a$  by the formula

$$|W_t - a| - |W_0 - a| = \int_0^t \text{sgn}(W_s - a) dW_s + L_t^a, \quad (14.5)$$

and the same proof as above shows that  $L_t^a$  is the limit in  $L^2$  of

$$\frac{1}{2\varepsilon} \int_0^t 1_{[a-\varepsilon, a+\varepsilon]}(W_s) ds.$$

## 14.2 Joint continuity of local times

Next we will prove that  $L_t^a$  can be taken to be jointly continuous in both  $t$  and  $a$ .

**Theorem 14.3** *Let  $W$  be a one-dimensional Brownian motion and let  $L_t^a$  be the local time of  $W$  at level  $a$ . For each  $a \in \mathbb{R}$  there exists a version  $\tilde{L}_t^a$  of  $L_t^a$  so that with probability one,  $\tilde{L}_t^a$  is jointly continuous in  $t$  and  $a$ .*

Recall that two processes  $X$  and  $Y$  are versions of each other if for each  $t$ ,  $X_t = Y_t$ , a.s. We will use the Kolmogorov continuity criterion, Corollary 8.2, together with Remark 8.3. We will obtain an estimate on  $\tilde{N}_t^a - \tilde{N}_t^b$ , where  $\tilde{N}_t^a = \int_0^t \operatorname{sgn}(W_s - a) dW_s$ , by means of the Burkholder–Davis–Gundy inequalities.

*Proof* Let  $M > 0$  be arbitrary. It suffices to show the joint continuity for times less than or equal to  $M$  and for  $|a| \leq M$ . Let

$$N_t^a = \int_0^{M \wedge t} \operatorname{sgn}(W_s - a) dW_s.$$

Since  $|W_t - a|$  is uniformly continuous in  $t$  and  $a$  for  $|t| \leq M$ ,  $|a| \leq M$ , by the Tanaka formula (14.5) it suffices to establish the same fact for  $N_t^a$ .

Let  $T$  be a stopping time bounded by  $M$  and  $a < b$ . Since  $(N_t^a - N_t^b)^2 - \langle N^a - N^b \rangle_t$  is a martingale,

$$\begin{aligned} \mathbb{E} [((N_M^a - N_M^b) - (N_T^a - N_T^b))^2 | \mathcal{F}_T] \\ = \mathbb{E} \left[ \int_T^M (\operatorname{sgn}(W_s - a) - \operatorname{sgn}(W_s - b))^2 ds | \mathcal{F}_T \right] \\ = 4\mathbb{E} \left[ \int_T^M 1_{[a,b]}(W_s) ds | \mathcal{F}_T \right] \\ \leq 4\mathbb{E} \left[ \int_T^{M+T} 1_{[a,b]}(W_s) ds | \mathcal{F}_T \right] \\ = 4\mathbb{E} \left[ \int_0^M 1_{[a,b]}(W_{s+T}) ds | \mathcal{F}_T \right]; \end{aligned}$$

recall Exercise 11.1. From Proposition 4.5 we deduce

$$\mathbb{E} \left[ \int_0^M 1_{[a,b]}(W_{s+T}) ds | \mathcal{F}_T \right] \leq \int_0^M \frac{c(b-a)}{\sqrt{s}} ds \leq c(b-a).$$

Thus

$$\mathbb{E} [((N_M^a - N_M^b) - (N_T^a - N_T^b))^2 | \mathcal{F}_T] \leq c|b-a|,$$

and so by (9.3)

$$\mathbb{E} [\langle N^a - N^b \rangle_M - \langle N^a - N^b \rangle_T | \mathcal{F}_T] \leq c|b-a|.$$

If we write  $A_t = \langle N^a - N^b \rangle_t$ , then we have by Proposition 3.14

$$\begin{aligned}\mathbb{E} A_M^2 &= 2\mathbb{E} \int_0^M (A_M - A_t) dA_t \\ &= 2\mathbb{E} \left[ \int_0^M (\mathbb{E}[A_M | \mathcal{F}_t] - A_t) dA_t \right] \\ &= 2\mathbb{E} \left[ \int_0^M \mathbb{E}[A_M - A_t | \mathcal{F}_t] dA_t \right] \\ &\leq c|b-a|\mathbb{E} \int_0^M dA_t \leq c|b-a|^2.\end{aligned}$$

Applying the Burkholder–Davis–Gundy inequalities,

$$\mathbb{E} [\sup_{t \leq M} |N_t^a - N_t^b|^4] \leq c|b-a|^2. \quad (14.6)$$

By the Kolmogorov continuity criterion applied on the Banach space of continuous functions with the metric  $d(f, g) = \sup_{t \leq M} |f(t) - g(t)|$ , we see  $N_t^a$  is continuous as a function of  $a$  for  $a$  in the dyadic rationals in  $[-M, M]$ , uniformly over  $t \leq M$ . Therefore  $L_a^t$  is continuous over  $a$  in the dyadic rationals in  $[-M, M]$ , uniformly for  $t \leq M$ . Also, (14.5) and (14.6) imply

$$\mathbb{E} [\sup_{t \leq M} |L_t^a - L_t^b|^4] \leq c(|a-b| \wedge 1)^2. \quad (14.7)$$

Note that if we define  $\tilde{L}_t^a = \lim L_t^{b_n}$  where the limit is as  $b_n \rightarrow a$  and  $b_n$  is in the dyadic rationals, then (14.7) implies that  $\tilde{L}_t^a = L_t^a$ , a.s. The uniform continuity of  $L_t^a$  over  $a$  in the dyadic rationals and  $t \leq M$  implies the joint continuity of  $\tilde{L}_t^a$ .  $\square$

### 14.3 Occupation times

If we integrate local times over a set, we obtain occupation times. More precisely, we have the following.

**Theorem 14.4** *Let  $W_t$  be a Brownian motion and  $L_t^y$  the local time at the level  $y$ , where we take  $L_t^y$  to be jointly continuous in  $t$  and  $y$ . If  $f$  is non-negative and Borel measurable,*

$$\int f(y) L_t^y dy = \int_0^t f(W_s) ds, \quad \text{a.s.} \quad (14.8)$$

*with the null set independent of  $f$  and  $t$ .*

*Proof* Suppose we prove the above equality for each  $C^2$  function  $f$  with compact support and denote the null set by  $N_f$ . Taking a countable collection  $\{f_i\}$  of non-negative  $C^2$  functions with compact support that are dense in the set of non-negative continuous functions on  $\mathbb{R}$  with compact support and letting  $N = \cup_i N_{f_i}$ , then if  $\omega \notin N$  we have the above equality for all  $f_i$ . By taking limits, we have (14.8) for all bounded and continuous  $f$ . A further limiting procedure implies our result.

Suppose  $f$  is bounded and  $C^2$  with compact support. Notice that the process  $\int f(y)L_t^y dy$  is increasing and continuous. Define

$$g(x) = \int f(y)|x - y| dy. \quad (14.9)$$

By Exercise 14.1,  $g$  is  $C^2$  with  $\frac{1}{2}g'' = f$ . If we take the Tanaka formula (14.5), replace  $a$  by  $y$ , multiply by  $f(y)$ , and integrate over  $\mathbb{R}$  with respect to  $y$ , we see that

$$g(W_t) - g(W_0) = \text{martingale} + \int_0^t f(y)L_t^y dy.$$

Using Itô's formula,

$$\begin{aligned} g(W_t) - g(W_0) &= \text{martingale} + \frac{1}{2} \int_0^t g''(W_s) ds \\ &= \text{martingale} + \int_0^t f(W_s) ds. \end{aligned}$$

Thus

$$\int_0^t f(y)L_t^y dy - \int_0^t f(W_s) ds$$

is a continuous martingale with paths locally of bounded variation, hence by Theorem 9.7 it is identically 0.  $\square$

## Exercises

14.1 Suppose  $f$  is  $C^2$  with compact support and

$$g(x) = \int f(y)|x - y| dy.$$

Show that  $g$  is  $C^2$  and  $g'' = 2f$ .

14.2 Let  $L_t^y$  be the jointly continuous local times of a Brownian motion  $W$ . Show

$$\frac{1}{2\varepsilon} \int_0^t 1_{[y-\varepsilon, y+\varepsilon]}(W_s) ds \rightarrow L_t^y, \quad \text{a.s.}$$

Show the null set can be taken to be independent of  $y$ . Thus there is no need to take a subsequence  $\varepsilon_n$  to get almost sure convergence to  $L_t^y$ .

14.3 Let  $W$  be a Brownian motion and fix  $t$ . Show that the function  $x \rightarrow \int_0^t 1_{(-\infty, x]}(W_s) ds$  is continuous, a.s., but that the function  $x \rightarrow 1_{(-\infty, x]}(W_t)$  is not continuous.

14.4 Let  $\{\mathcal{F}_t\}$  be a filtration satisfying the usual conditions. Suppose  $W_t$  is a Brownian motion and  $X_t = W_t + A_t$ , where  $X_t \geq 0$  for all  $t$ , a.s., and  $A_t$  is an increasing continuous adapted process such that  $A$  increases only at those times when  $X_t = 0$ . Suppose also that  $X'_t = W_t + A'_t$ , where  $X'_t \geq 0$  for all  $t$ , a.s., and  $A'_t$  is an increasing continuous adapted process that increases only when  $X'_t = 0$ . Show that  $X'_t = X_t$  and  $A_t = A'_t$ , a.s., for all  $t \geq 0$ .

14.5 Let  $W$  be a Brownian motion and  $L_t^0$  the local time at 0. Since  $L_t^0$  is increasing, for each  $\omega$  there is a Lebesgue–Stieltjes measure  $dL_t^0$ . Show that the support of  $dL_t^0$  is equal to  $\{t : W_t = 0\}$ .

Since Theorem 14.1(2) states that  $L_t^0$  does not increase when  $W_t$  is not equal to 0, what you need to show is that with probability one, if  $W_u(\omega) = 0$  and  $t < u < v$ , then  $L_v^0(\omega) > L_t^0(\omega)$ .

- 14.6 Use Tanaka's formula to show that if  $L_t^y$  is the local time of Brownian motion at level  $y$ ,  $a \leq x \leq y \leq b$ , and  $T = \inf\{t > 0 : W_t \notin [a, b]\}$ , then

$$\mathbb{E}^x L_T^y = \frac{2(x-a)(b-y)}{b-a}.$$

- 14.7 If  $L_t^0$  is the local time of a Brownian motion at 0, show that  $L_{at}^0$  has the same law as  $\sqrt{a}L_t^0$ .

- 14.8 Let  $W$  be a Brownian motion with local times  $L_t^y$ . Set  $L_t^* = \sup_y L_t^y$ . Let  $p > 0$ . Prove that there exist constants  $c_1, c_2$  such that if  $T$  is any finite stopping time,

$$c_1 \mathbb{E} T^{p/2} \leq EL_T^* \leq c_2 \mathbb{E} T^{p/2}.$$

The constants  $c_1, c_2$  can depend on  $p$ , but not on  $T$ .

*Hint:* Use Exercise 12.11.

- 14.9 This exercise defines the local time of a continuous martingale. If  $M$  is a continuous martingale, then  $M_t^2$  is a submartingale and so equals a martingale plus an increasing process. The increasing process  $L_t^0$  is called the local time of  $M$  at 0.

(1) Prove the analog of Tanaka's formula.

(2) Define the local time  $L_t^a$  of  $M$  at  $a$ . Prove that  $L_t^a$  is jointly continuous in  $t$  and  $a$ .

(3) Prove that

$$\int_0^t f(M_s) d\langle M \rangle_s = \int_{\mathbb{R}} L_t^a f(a) da, \quad \text{a.s.}$$

if  $f$  is non-negative and measurable.

- 14.10 This exercise is a complement to Exercise 7.8. Let  $W$  be a Brownian motion and let us define  $Z = \{t \in [0, 1] : W_t = 0\}$ , the zero set. Let  $\varepsilon \in (0, 1/2)$  and let  $\delta > 0$ . Fix  $\omega$  and let  $\{B_i\}$  be any countable covering of  $Z(\omega)$  by closed intervals such that the interiors of the  $B_i$ 's are pairwise disjoint and the length of each  $B_i$  is less than or equal to  $\delta$ . We write  $B_i = [a_i, b_i]$ .

Let  $\varepsilon > 0$ . Since  $L^0$  has the same law of the maximum of Brownian motion, there exists a  $c$  (depending on  $\omega$ ) such that

$$L_t^0 - L_s^0 \leq c(t-s)^{\frac{1}{2}-\frac{\varepsilon}{2}}$$

for each  $0 \leq s \leq t \leq 0$ . Write

$$\begin{aligned} \sum_i |b_i - a_i|^{\frac{1}{2}-\varepsilon} &\geq \frac{\delta^{-\varepsilon/2}}{c} c \sum_i |b_i - a_i|^{\frac{1}{2}-\frac{\varepsilon}{2}} \\ &\geq \frac{\delta^{-\varepsilon/2}}{c} \sum_i (L_{b_i}^0 - L_{a_i}^0) \\ &= \frac{\delta^{-\varepsilon/2}}{c} [L_1^0 - L_0^0]. \end{aligned}$$

Show that this implies that the Hausdorff dimension of  $Z$  is at least  $1/2$ .

# 15

---

## Skorokhod embedding

Suppose  $Y$  is a random variable with mean zero and finite variance. Skorokhod proved the remarkable fact that if  $W$  is a Brownian motion, there exists a stopping time  $T$  such that  $W_T$  has the same law as  $Y$ . Without any restrictions on  $T$ , there is a trivial solution (see Exercise 15.1), so one wants to require that  $\mathbb{E} T < \infty$ . Skorokhod's construction required an additional random variable that is independent of the Brownian motion, but since that time there have been 15 or 20 other constructions, most of which don't require the extra randomization, that is,  $T$  is a stopping time for the minimal augmented filtration generated by  $W$ .

Although conceptually some constructions are easier than others, none is easy from the point of view of technical details. We will give a construction that doesn't have any optimality properties, but is a nice example of stochastic calculus. Then we will use this to prove an embedding for random walks.

### 15.1 Preliminaries

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a *Lipschitz function* if there exists a constant  $k$  such that

$$|f(y) - f(x)| \leq k|y - x|, \quad x, y \in \mathbb{R}. \quad (15.1)$$

By the mean value theorem, if  $f$  has a bounded derivative, then  $f$  is a Lipschitz function.

We will need the following well-known theorem from the theory of ordinary differential equations.

**Theorem 15.1** Suppose  $F : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  is a bounded function and there exists a positive real  $k$  such that

$$|F(t, x) - F(t, y)| \leq k|x - y|$$

for all  $t \geq 0$  and all  $x, y \in \mathbb{R}$ . Let  $y_0 \in \mathbb{R}$ , define the function  $y^0$  by  $y^0(t) = y_0$  for all  $t \geq 0$ , and define the function  $y^i$  inductively by

$$y^{i+1}(t) = y_0 + \int_0^t F(s, y^i(s)) ds, \quad t \geq 0. \quad (15.2)$$

Then the functions  $y^i$  converge uniformly on bounded intervals to a function  $y$  that satisfies

$$y(t) = y_0 + \int_0^t F(s, y(s)) ds. \quad (15.3)$$

For any  $s$  such that  $F(s, y(s))$  is continuous at  $s$ ,  $y$  satisfies

$$\frac{dy}{ds} = F(s, y(s)). \quad (15.4)$$

The solution to (15.3) is unique.

This inductive procedure for obtaining the solution to (15.4) is known as *Picard iteration*.

*Proof* Note each  $y^i(t)$  is bounded in absolute value by  $|y_0| + t \sup |F|$ . Let  $g_i(t) = \sup_{s \leq t} |y^{i+1}(s) - y^i(s)|$ . If  $s \leq t$ , then

$$\begin{aligned} |y^{i+1}(s) - y^i(s)| &= \left| \int_0^s [F(r, y^i(r)) - F(r, y^{i-1}(r))] dr \right| \\ &\leq \int_0^t |F(r, y^i(r)) - F(r, y^{i-1}(r))| dr \\ &\leq k \int_0^t |y^i(r) - y^{i-1}(r)| dr \\ &\leq k \int_0^t g_{i-1}(r) dr. \end{aligned}$$

Taking the supremum over  $s \leq t$ , we have

$$g_i(t) \leq k \int_0^t g_{i-1}(r) dr.$$

Fix  $t_0$ . Now  $g_1(t)$  is bounded for  $t \leq t_0$ , say by  $L$ . Then  $g_2(t) \leq k \int_0^t L dr = kLt$  for each  $t \leq t_0$ , and then  $g_3(t) \leq k \int_0^t (kLr) dr = k^2 Lt^2/2$  and  $g_4(t) \leq k \int_0^t (k^2 Lr^2/2) dr = k^3 Lt^3/3!$  By induction  $g_i(t) \leq k^{i-1} Lt^{i-1}/(i-1)!$  We conclude  $\sum_{i=1}^{\infty} g_i(t_0) < \infty$ .

Then

$$\sup_{s \leq t_0} |y^n(s) - y^m(s)| \leq \sum_{i=m}^{n-1} g_i(t_0),$$

which tends to zero as  $m$  and  $n$  tend to infinity. By the completeness of the space  $C[0, t_0]$ , there exists a continuous function  $y$  such that  $\sup_{s \leq t_0} |y^n(s) - y(s)| \rightarrow 0$  as  $n \rightarrow \infty$ .

$F$  is continuous in the  $x$  variable, so taking the limit in (15.2) shows that  $y$  solves (15.3). If  $F$  is continuous at a particular value of  $s$ , then (15.4) holds by the fundamental theorem of calculus.

To prove uniqueness, suppose  $x$  and  $y$  are solutions to (15.4) and let us set  $g(t) = \sup_{s \leq t} |x(s) - y(s)|$ . If  $s \leq t$ , then

$$\begin{aligned} |x(s) - y(s)| &\leq \int_0^s |F(r, x(r)) - F(r, y(r))| dr \\ &\leq k \int_0^t |x(r) - y(r)| dr \\ &\leq k \int_0^t g(r) dr. \end{aligned}$$

Taking the supremum over  $s \leq t$ , we obtain

$$g(t) \leq k \int_0^t g(r) dr.$$

For  $t \leq t_0$ , we have  $|x(t)|$  and  $|y(t)|$  bounded by a constant, say  $L$ , so  $g(t)$  is bounded for  $t \leq t_0$ . We then have  $g(t) \leq k \int_0^t L dr = kLt$  for each  $t \leq t_0$  and then  $g(t) \leq k \int_0^t kLr dr = k^2 Lt^2/2$ . Iterating, we have  $g(t) \leq k^i t^i L/i!$  for each  $i$ , and hence  $g(t) = 0$ . This is true for each  $t$ , hence  $x(s) = y(s)$  for all  $s \leq t_0$ .  $\square$

If the random variable  $Y$  that we are considering is equal to 0, a.s., we can just let our stopping time  $T$  equal 0, a.s., and then  $W_T = 0 = Y$  if  $W$  is a Brownian motion. In the remainder of this section and the next we assume  $\mathbb{E} Y = 0$ ,  $\mathbb{E} Y^2 < \infty$ , but that  $Y$  is not identically zero.

Define

$$p_s(y) = \frac{1}{\sqrt{2\pi s}} e^{-y^2/2s},$$

the density of a mean zero normal random variable with variance  $s$ . Use  $p'_s(x)$  to denote the derivative of  $p_s$  with respect to  $x$ .

**Lemma 15.2** *Suppose  $W$  is a Brownian motion and  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\mathbb{E}[g(W_1)^2] < \infty$ . For  $0 < s < 1$ , let*

$$a(s, x) = - \int p'_{1-s}(z - x) g(z) dz \quad (15.5)$$

and

$$b(s, x) = \int p_{1-s}(z - x) g(z) dz. \quad (15.6)$$

We have

$$g(W_1) = \mathbb{E} g(W_1) + \int_0^1 a(s, W_s) dW_s, \quad \text{a.s.} \quad (15.7)$$

and

$$\mathbb{E}[g(W_1) | \mathcal{F}_s] = b(s, W_s), \quad \text{a.s.} \quad (15.8)$$

*Proof* We will first prove (15.7), and we will first look at the case when  $g(x) = e^{ix}$ .

By Ito's formula with the function  $f(x) = e^x$  applied to the semimartingale  $X_t = iuW_t + u^2 t/2$

$$\begin{aligned} e^{iuW_t + u^2 t/2} &= 1 + \int_0^t e^{X_s} d(iuW_s + u^2 s/2) + \frac{1}{2} \int_0^t (-u^2) e^{X_s} ds \\ &= 1 + iu \int_0^t e^{iuW_s + u^2 s/2} dW_s, \end{aligned}$$

so

$$e^{iuW_1} = e^{-u^2/2} + \int_0^1 iue^{iuW_s} e^{u^2(s-1)/2} dW_s.$$

We need to check that

$$iue^{iux}e^{u^2(s-1)/2} = a(s, x).$$

Using integration by parts,

$$\begin{aligned} a(s, x) &= - \int p'_{1-s}(z - x)g(z) dz = \int p_{1-s}(z - x)g'(z) dz \\ &= iu \int \frac{1}{\sqrt{2\pi(1-s)}} e^{-(z-x)^2/2(1-s)} e^{iuz} dz. \end{aligned}$$

This is  $iu$  times the characteristic function of a normal random variable with mean  $x$  and variance  $1 - s$ , and so by (A.25) equals

$$iue^{iux}e^{-u^2(1-s)/2},$$

as desired. We therefore have

$$e^{iuW_1} = \mathbb{E} e^{iuW_1} - \int_0^1 \int p'_{1-s}(z - W_s) e^{iuz} dz dW_s. \quad (15.9)$$

Now suppose  $g$  is in the Schwartz class (see Section B.2), replace  $u$  by  $-u$  in (15.9), multiply by the Fourier transform of  $g$ , and integrate over  $u \in \mathbb{R}$ . We then obtain

$$\begin{aligned} (2\pi)^{-1}g(W_1) &= (2\pi)^{-1}\mathbb{E} g(W_1) \\ &\quad - \int \int_0^1 \int p'_{1-s}(z - W_s) e^{-iuz} \widehat{g}(u) dz dW_s du, \end{aligned} \quad (15.10)$$

where  $\widehat{g}$  is the Fourier transform of  $g$ . Using the Fubini theorem (check that there is no trouble with the stochastic integral; see Exercise 15.2) and the inversion formula for Fourier transforms, the triple integral on the right-hand side of (15.10) is equal to

$$- (2\pi)^{-1} \int_0^1 \int p'_{1-s}(z - W_s) g(z) dz dW_s, \quad (15.11)$$

which gives us (15.7) when  $g$  is the Schwartz class. A limit argument gives us (15.7) for all  $g$  that we are interested in.

To prove (15.8) we again start with the case  $g(x) = e^{iux}$ . We have

$$\begin{aligned} \mathbb{E}[e^{iuW_1} | \mathcal{F}_s] &= e^{iuW_s} \mathbb{E}[e^{iu(W_1 - W_s)} | \mathcal{F}_s] = e^{iuW_s} \mathbb{E}[e^{iu(W_1 - W_s)}] \\ &= e^{iuW_s} e^{-u^2(1-s)/2}, \end{aligned}$$

using the independent increments property of Brownian motion and (A.25). On the other hand, the definition of  $b(s, x)$  shows that when  $g(x) = e^{iux}$ ,  $b(s, x)$  is the characteristic function of a normal random variable with mean  $x$  and variance  $1 - s$ , so

$$b(s, x) = e^{iux} e^{-u^2(1-s)/2}.$$

Replacing  $x$  by  $W_s$  proves (15.8) in the case  $g(x) = e^{iux}$ . We extend this to general  $g$  in the same way as in the proof of (15.7).  $\square$

Next, we want to find a reasonable function  $g$  such that  $g(W_1)$  is equal in law to  $Y$ , where again  $W$  is a Brownian motion. Let  $F_Y(x) = \mathbb{P}(Y \leq x)$ , the distribution function of  $Y$  and let  $\Phi(x) = \mathbb{P}(W_1 \leq x)$ . Then

$$\mathbb{P}(\Phi(W_1) \leq x) = \mathbb{P}(W_1 \leq \Phi^{-1}(x)) = \Phi(\Phi^{-1}(x)) = x$$

for  $x \in [0, 1]$ , so  $\Phi(W_1)$  is a uniform random variable on  $[0, 1]$ . Define

$$g(x) = F_Y^{-1}(\Phi(x)). \quad (15.12)$$

We use the right-continuous version of  $F_Y^{-1}$  if  $F_Y^{-1}$  is not continuous. Then

$$\mathbb{P}(g(W_1) \leq x) = \mathbb{P}(\Phi(W_1) \leq F_Y(x)) = F_Y(x),$$

or  $Y$  is equal in law to  $g(W_1)$  as desired. Note  $g$  is an increasing function.

We will need the following estimates.

**Proposition 15.3** *Let  $g$  be defined by (15.12) and define  $a$  and  $b$  by (15.5) and (15.6).*

(1) *For each  $L > 0$  and  $s_0 < 1$ ,  $a$  is continuously differentiable on  $[0, s_0] \times [-L, L]$ . Also, for each  $L > 0$  and  $s_0 < 1$ ,  $a$  is bounded below by a positive constant on  $[0, s_0] \times [-L, L]$ .*

(2) *For each  $L > 0$  and  $s_0 < 1$ ,  $b$  is continuously differentiable on  $[0, s_0] \times [-L, L]$ .*

(3) *For each  $s \in [0, s_0]$ , the function  $x \rightarrow b(s, x)$  is strictly increasing. For each fixed  $s$ , let  $B(s, x)$  be the inverse of  $b(s, x)$  (so that  $B(s, b(s, x)) = x$  and  $b(s, B(s, x)) = x$ ). For each  $L > 0$  and  $s_0 < 1$ ,  $B$  is continuously differentiable on  $[0, s_0] \times [-L, L]$ .*

*Proof* To start, we observe that for every  $r > 0$ ,

$$\mathbb{E} e^{r|W_1|} \leq \mathbb{E} e^{rW_1} + \mathbb{E} e^{-rW_1} < \infty.$$

Since  $|z|^m \leq m!e^{|z|}$  if  $m$  is a non-negative integer, then by the Cauchy–Schwarz inequality and the fact that  $\mathbb{E} Y^2 < \infty$ ,

$$\begin{aligned} \int |z|^m e^{r|z|} e^{-z^2/2} |g(z)| dz &\leq m! \int e^{(r+1)|z|} e^{-z^2/2} |g(z)| dz \\ &= m! \mathbb{E} \left[ e^{(r+1)|W_1|} |g(W_1)| \right] \\ &\leq m! \left( \mathbb{E} e^{2(r+1)|W_1|} \right)^{1/2} (\mathbb{E} |g(W_1)|^2)^{1/2} \\ &\leq m! \left( \mathbb{E} e^{2(r+1)|W_1|} \right)^{1/2} (\mathbb{E} Y^2)^{1/2} < \infty. \end{aligned} \quad (15.13)$$

We now turn to (1).

$$\begin{aligned} |p'_{1-s}(z-x)| &\leq c \frac{|z-x|}{(1-s)^{3/2}} e^{-(z-x)^2/2(1-s)} \\ &\leq c |z-x| e^{-x^2/2(1-s)} e^{zx/2(1-s)} e^{-z^2/2(1-s)} \\ &\leq c (|z| + L) e^{|z|L/2(1-s_0)} e^{-z^2/2} \\ &\leq c |z| e^{c'|z|} e^{-z^2/2} + c e^{c'|z|} e^{-z^2/2}. \end{aligned}$$

Therefore

$$|a(s, x)| \leq \int c |z| e^{c'|z|} e^{-z^2/2} |g(z)| dz + \int c e^{c'|z|} e^{-z^2/2} |g(z)| dz,$$

which is bounded by (15.13). This gives an upper bound for  $a$ .

By the mean value theorem,

$$|p'_{1-s}(z-x) - p'_{1-s}(z-(x+h))| \leq c|h|(1+|z|^2+L^2)e^{-(z-x)^2/2(1-s)}$$

if  $s \leq s_0$ ,  $|x| \leq L$ , and  $|h| \leq 1$ , so

$$\left| \frac{1}{h} (p'_{1-s}(z-x) - p'_{1-s}(z-(x+h))) \right| \leq c(1+|z|^2)e^{c|z|}e^{-z^2/2}.$$

In view of (15.13), we can use dominated convergence to conclude that

$$\frac{\partial a}{\partial x}(s, x) = \int p''_{1-s}(z-x)g(z) dz$$

and that  $|\partial a(s, x)/\partial x|$  is bounded above on  $[0, s_0] \times [-L, L]$ .

By a similar argument we obtain that  $|\partial a(s, x)/\partial s|$  is also bounded above on  $[0, s_0] \times [-L, L]$ . The same argument shows that the second partial derivatives of  $a$  are bounded, and hence the first partial derivatives are continuous.

Using integration by parts,

$$a(s, x) = \int p_{1-s}(z-x) dg(z),$$

where the integral is a Lebesgue–Stieltjes integral; recall that  $g$  is an increasing function. Since we are working under the assumption that  $Y$  is not identically zero, then  $g$  is not identically zero, which implies that  $a$  is bounded below for  $s \leq s_0$  and  $|x| \leq L$ .

The proof of (2) is quite similar. To prove (3), as above, we can use a dominated convergence argument to prove

$$\frac{\partial b(s, x)}{\partial x} = a(s, x).$$

Since  $a(s, x) > 0$  for each  $x$  and for each  $s < s_0$ , we conclude that  $x \rightarrow b(s, x)$  is strictly increasing. The estimates for  $B$  follow from the implicit function theorem applied to  $f(s, x, y) = 0$ , where  $f(s, x, y) = b(s, x) - y$ .  $\square$

## 15.2 Construction of the embedding

**Theorem 15.4** Suppose  $Y$  is a random variable with  $\mathbb{E} Y = 0$  and  $\mathbb{E} Y^2 < \infty$ . There exists a Brownian motion  $N$  and a stopping time  $T$  with respect to the minimal augmented filtration of  $N$  such that  $N_T$  is equal in law to  $Y$ . Moreover  $\mathbb{E} T = \mathbb{E} Y^2$ .

*Proof* The idea is to define  $M$  by (15.14) below and do a time change so that  $N_T = M_1 = g(W_1)$ . To show that  $T$  is a stopping time relative to the minimal augmented filtration for  $N$ , we set up an ordinary differential equation that the time change solves and use Picard iteration to show that the solution can be obtained in a constructive way.

The case where  $Y$  is identically zero is trivial for we take  $T = 0$ , so we suppose  $Y$  is not identically zero. Let  $W_t$  be a Brownian motion and let  $\{\mathcal{F}_t\}$  be its minimal augmented filtration. Define the function  $g$  by (15.12) and define  $a$  and  $b$  for  $s < 1$  by (15.5) and (15.6). Define  $a(s, x) = 1$  and  $b(s, x) = x$  if  $s \geq 1$ .

Now let

$$M_t = \int_0^t a(s, W_s) dW_s, \quad (15.14)$$

and hence

$$\langle M \rangle_t = \int_0^t a(s, W_s)^2 ds.$$

Note  $\langle M \rangle_t \rightarrow \infty$ , a.s., as  $t \rightarrow \infty$ . Since  $\mathbb{E} Y = 0$ , then  $\mathbb{E} g(W_1) = 0$ , so  $M_1 = g(W_1)$  by (15.7). Let

$$\tau_t = \inf\{s : \langle M \rangle_s \geq t\},$$

the inverse of  $\langle M \rangle$ . By Theorem 12.2, if we set  $N_t = M_{\tau_t}$ , then  $N$  is a Brownian motion. Let  $\{\mathcal{G}_t\}$  be the minimal augmented filtration generated by  $N$ .

We let  $T = \langle M \rangle_1$ . Then

$$N_T = N_{\langle M \rangle_1} = M_{\tau_{\langle M \rangle_1}} = M_1 = g(W_1),$$

and  $N_T$  has the same law as  $Y$ .

For the integrability of  $T$  we have

$$\mathbb{E} T = \mathbb{E} \langle M \rangle_1 = \mathbb{E} M_1^2 = \mathbb{E} [g(W_1)^2] = \mathbb{E} Y^2 = \text{Var } Y < \infty. \quad (15.15)$$

It remains to show that  $T$  is a stopping time with respect to  $\{\mathcal{G}_t\}$ . Since  $T = \lim_{s \uparrow 1} \langle M \rangle_s$ , it suffices to show that  $\langle M \rangle_s$  is a stopping time with respect to  $\{\mathcal{G}_t\}$  for each  $s < 1$ . Fix  $K$ . We will show

$$(\tau_t \leq s, \sup_{s \leq t} |N_s| \leq K) \in \mathcal{G}_t, \quad s < 1. \quad (15.16)$$

Letting  $K \rightarrow \infty$  will then show  $(\langle M \rangle_s \geq t) = (\tau_t \leq s) \in \mathcal{G}_t$  for  $s < 1$ .

Since  $\tau$  is the inverse of  $\langle M \rangle$ , then

$$\frac{d\tau_t}{dt} = \frac{1}{d\langle M \rangle_{\tau_t}/d\tau_t} = \frac{1}{a(\tau_t, W_{\tau_t})^2}$$

with  $\tau_0 = 0$ , a.s. With  $B(s, x)$  being the inverse of  $b(s, x)$  in the  $x$  variable,

$$M_s = \mathbb{E} [M_1 \mid \mathcal{F}_s] = \mathbb{E} [g(W_1) \mid \mathcal{F}_s] = b(s, W_s),$$

or

$$W_s = B(s, M_s), \quad s < 1.$$

Therefore

$$W_{\tau_t} = B(\tau_t, M_{\tau_t}) = B(\tau_t, N_t)$$

on the event  $(\tau_t \leq s)$  if  $s < 1$ . Thus  $\tau_t$  solves the equation

$$\frac{d\tau_t}{dt} = \frac{1}{a(\tau_t, B(\tau_t, N_t))^2}, \quad \tau_0 = 0,$$

or

$$\tau_t = \int_0^t \frac{1}{a(\tau_u, B(\tau_u, N_u))^2} du.$$

Fix  $s$  and  $t$  and choose  $s_0 \in (s, 1)$ . Let  $S_K = \inf\{t : |N_t| \geq K\}$  and let  $N_t^K = N_{t \wedge S_K}$ . Define

$$\Phi(q, r) = \frac{1}{(a(r, B(r, N_q^K(\omega))))^2}$$

if  $r \leq s_0$ . Observe that  $\Phi$  depends on  $\omega$ . Define  $\Phi(q, r) = 1$  for  $r \geq 1$  and define  $\Phi(q, r)$  by linear interpolation for  $r \in (s_0, 1)$ . Note that by Proposition 15.3,  $\Phi$  is continuous, bounded, and there exists  $k > 0$  such that

$$|\Phi(q, r) - \Phi(q, r')| \leq k|r - r'|, \quad r \in \mathbb{R}, q \in [0, \infty).$$

$\tau_t$  solves the equation

$$\tau_t = \int_0^t \Phi(u, \tau_u) du.$$

We solve the differential equation

$$y(t) = \int_0^t \Phi(u, y(u)) du \tag{15.17}$$

using Theorem 15.1. The function  $y^0(t)$  in the statement of Theorem 15.1 is identically zero, and the function  $y^1(t) = \int_0^t \Phi(u, y^0(u)) du$  (which depends on  $\omega$  because  $\Phi$  does) will be  $\mathcal{G}_t$  measurable, and by induction, the functions  $y^i(t)$  will be  $\mathcal{G}_t$  measurable. Therefore the limit,  $y(t)$ , will be  $\mathcal{G}_t$  measurable. Since  $|N_q^K(\omega)| \leq K$  for all  $q$  and we are only interested in the solution to (15.17) for  $y(t) \leq s$ , then  $\tau_t = y(t)$  as long as  $\tau_t \leq s$ ; therefore (15.16) holds and the proof is complete.  $\square$

In the above theorem, we started with a Brownian motion  $W$ , constructed a new Brownian motion  $N$ , and then defined our stopping time  $T$  in terms of  $N$ . We can actually start with a Brownian motion  $W$  and define a stopping time that is a stopping time with respect to the minimal augmented filtration of  $W$ .

**Corollary 15.5** *Let  $W$  be a Brownian motion and let  $\{\mathcal{F}_t\}$  be the minimal augmented filtration for  $W$ . Let  $Y$  be a random variable with  $\mathbb{E} Y = 0$  and  $\text{Var } Y < \infty$ . There exists a stopping time  $V$  with respect to  $\{\mathcal{F}_t\}$  such that  $W_V$  has the same law as  $Y$ .*

*Proof* We sketch the proof and ask you to give the details in Exercise 15.3. Define

$$\bar{\Phi}(q, r) = \frac{1}{(a(r, B(r, W_q(\omega))))^2}$$

and solve the equation

$$\frac{d\bar{\tau}_t}{dt} = \bar{\Phi}(t, \bar{\tau}_t), \quad \bar{\tau}_0 = 0$$

by Picard iteration. The proof of Theorem 15.4 shows that the solution  $\bar{\tau}_t$  will satisfy  $(\bar{\tau}_t \leq s) \in \mathcal{F}_t$  for every  $t$  as long as  $s < 1$ . Let  $A$  be the inverse of  $\bar{\tau}$ , and define  $V = \lim_{s \uparrow 1} A_s$ . Then  $V$  will be the desired stopping time.  $\square$

### 15.3 Embedding random walks

Let us give an application of Skorokhod embedding to show that we can find a Brownian motion that is relatively close to a random walk. Suppose  $Y_1, Y_2, \dots$  is an i.i.d. sequence of real-valued random variables with mean zero and variance one. Given a Brownian motion  $W_t$  we can find a stopping time  $T_1$  such that  $W_{T_1}$  has the same distribution as  $Y_1$ . We use the strong Markov property at time  $T_1$  and find a stopping time  $T_2$  for  $W_{T_1+t} - W_{T_1}$  so that  $W_{T_1+T_2} - W_{T_1}$  has the same distribution as  $Y_2$  and is independent of  $\mathcal{F}_{T_1}$ . We continue. We see that the  $T_i$  are i.i.d. and by Theorem 15.4,  $\mathbb{E} T_i = \mathbb{E} Y_i^2 = 1$ . Let  $U_k = \sum_{i=1}^k T_i$ . Then for each  $n$ ,  $S_n = \sum_{i=1}^n Y_i$  has the same distribution as  $W_{U_n}$ .

#### Theorem 15.6

$$\sup_{i \leq n} |W_{U_i} - W_i| / \sqrt{n}$$

tends to 0 in probability as  $n \rightarrow \infty$ .

*Proof* We will show that for each  $\varepsilon > 0$

$$\limsup_{n \rightarrow \infty} \mathbb{P}(\sup_{k \leq n} |W_{U_k} - W_k| > \varepsilon \sqrt{n}) \leq \varepsilon. \quad (15.18)$$

Since the paths of Brownian motion are continuous, we can find  $\delta \leq 1$  small such that

$$\mathbb{P}(\sup_{s, t \leq 2, |t-s| \leq \delta} |W_t - W_s| > \varepsilon) < \varepsilon/2.$$

By scaling,

$$\mathbb{P}(\sup_{s, t \leq 2n, |t-s| \leq \delta n} |W_t - W_s| > \varepsilon \sqrt{n}) < \varepsilon/2. \quad (15.19)$$

The strong law of large numbers (Theorem A.38) says that  $U_n/n \rightarrow \mathbb{E} T_1 = 1$ , a.s., and in fact, by Proposition A.39, we even have

$$\frac{\max_{k \leq n} |U_k - k|}{n} \rightarrow 0, \quad \text{a.s.} \quad (15.20)$$

Therefore

$$\begin{aligned} & \mathbb{P}(\max_{k \leq n} |W_{U_k} - W_k| > \varepsilon \sqrt{n}) \\ & \leq \mathbb{P}(\max_{k \leq n} |U_k - k| > \delta n) + \mathbb{P}(\sup_{s, t \leq 2n, |t-s| \leq \delta n} |W_t - W_s| > \varepsilon \sqrt{n}) \\ & \leq \mathbb{P}\left(\max_{k \leq n} \frac{|U_k - k|}{n} > \delta\right) + \frac{\varepsilon}{2}. \end{aligned}$$

By (15.20) this will be less than  $\varepsilon$  if we take  $n$  sufficiently large.  $\square$

### Exercises

- 15.1 Without some supplemental conditions on  $T$ , the problem of Skorokhod embedding is trivial. Suppose  $W$  is a Brownian motion with respect to a filtration  $\{\mathcal{F}_t\}$  satisfying the usual conditions. Suppose  $Y$  is a finite random variable and suppose  $h$  is a real-valued function such that  $h(W_1)$  has the same law as  $Y$ .

- (1) Show that if  $T = \inf\{t > 1 : W_t = h(W_1)\}$ , then  $W_T$  and  $Y$  have the same law.  
 (2) Give an example of a mean zero random variable  $Y$  with finite variance such that if  $T$  is defined as in (1), then  $\mathbb{E} T = \infty$ .
- 15.2 Show that the triple integral on the right-hand side of (15.10) is equal to the expression in (15.11).
- 15.3 A sketch was given for the proof of Corollary 15.5. Provide a detailed proof.
- 15.4 Here is another approach to proving Corollary 15.5. Let  $Y, N, T$ , and  $\{\mathcal{G}_t\}$  be as in the proof of Theorem 15.4.
- (1) Show that there is a random variable  $U$  that is measurable with respect to  $\sigma(N_s : 0 \leq s < \infty)$  such that  $U = T$ , a.s.
  - (2) Show there is a Borel measurable map  $H : C[0, \infty) \rightarrow [0, \infty)$  such that  $U = H(N)$ .
  - (3) If  $W$  is a Brownian motion, define  $V = H(W)$ . Show  $V$  is a stopping time with respect to the minimal augmented filtration generated by  $W$  such that  $W_V$  has the same law as  $Y$ .
- 15.5 Suppose  $p \in (0, 1/2)$  and  $Y$  is a random variable such that  $\mathbb{P}(Y = 1) = \mathbb{P}(Y = -1) = p$  and  $\mathbb{P}(Y = 0) = 1 - 2p$ . Let  $W$  be a Brownian motion. Let  $S_x = \inf\{t > 0 : W_t = x\}$  and let  $T = \inf\{t > S_x \wedge S_{-x} : W_t \in \{-1, 0, 1\}\}$ . Determine  $x$  such that  $W_T$  and  $Y$  have the same law.
- 15.6 Suppose  $Y$  is a mean zero random variable and there exists a real number  $K > 0$  such that  $|Y| \leq K$ , a.s. Let  $W$  be a Brownian motion and let  $T$  be a stopping time with  $\mathbb{E} T < \infty$  such that  $W_T$  and  $Y$  have the same law. (We do not necessarily assume that  $T$  was constructed by the method of Section 15.2.) Let  $S_K = \inf\{t : |W_t| \geq K\}$ . Prove that  $T \leq S_K$ , a.s.
- 15.7 Let  $Y_i$  be a sequence of i.i.d. random variables with  $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = \frac{1}{2}$ , and let  $S_n = \sum_{i=1}^n Y_i$ .  $S_n$  is called a *simple symmetric random walk*. Let  $T_1, T_2, \dots$  and  $U_1, U_2, \dots$  be as in Section 15.3.
- (1) Prove that  $\mathbb{E} T_1^p < \infty$  for all  $p \geq 1$ .
  - (2) Prove that if  $\varepsilon > 0$ ,
- $$\lim_{n \rightarrow \infty} \frac{\sup_{k \leq n} |U_k - k|}{n^{(1/2)+\varepsilon}} = 0, \quad \text{a.s.}$$
- Hint:* Use Doob's inequalities to estimate
- $$\mathbb{P}(\sup_{k \leq n} |U_k - k| \geq \delta n^{(1/2)+\varepsilon}).$$
- (3) Show that
- $$\sup_{i \leq n} |W_{U_i} - W_i| / n^{(1/4)+(\varepsilon/2)}$$
- tends to zero in probability as  $n \rightarrow \infty$ .
- 15.8 Let  $S_n, T_i$ , and  $U_i$  be as in Exercise 15.7. Prove that
- $$\lim_{n \rightarrow \infty} \frac{\sup_{i \leq n} |W_{U_i} - W_i|}{\sqrt{n}} = 0, \quad \text{a.s.}$$
- 15.9 Let  $S_n$  be a simple symmetric random walk; see Exercise 15.7. Let  $Y$  be a bounded symmetric random variable that takes values only in  $\mathbb{Z}$ . ( $Y$  being symmetric means that  $Y$  and  $-Y$  have the same law.) Does there necessarily exist a stopping time  $N$  such that  $S_N$  and  $Y$  have the same law? Why or why not?

**Notes**

The survey article [Obłój \(2004\)](#) summarizes many different methods of Skorokhod embedding. The embedding presented here is from [Bass \(1983\)](#); see also [Stroock \(2003\)](#), pp. 213–17.

# 16

---

## The general theory of processes

The name “general theory of processes” refers to the foundations of stochastic processes. Specific topics include measurability issues and classifications of stopping times. This chapter is fairly technical and abstract and should only be skimmed on the first reading of this book: read the definitions and statements of theorems, propositions, and lemmas, but not the proofs.

The two main results we discuss are the measurability of hitting times, and the Doob–Meyer decomposition of submartingales, Theorem 16.29.

### 16.1 Predictable and optional processes

Suppose  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space. The *outer probability*  $\mathbb{P}^*$  associated with  $\mathbb{P}$  is given by

$$\mathbb{P}^*(A) = \inf\{\mathbb{P}(B) : A \subset B, B \in \mathcal{F}\}. \quad (16.1)$$

A set  $A$  is a  $\mathbb{P}$ -null set if  $\mathbb{P}^*(A) = 0$ . We suppose throughout this chapter that  $\{\mathcal{F}_t\}$  is a filtration satisfying the usual conditions; recall from Chapter 1 that this means that each  $\mathcal{F}_t$  contains all the  $\mathbb{P}$ -null sets and that  $\cap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$  for each  $t$ . Let  $\pi : [0, \infty) \times \Omega \rightarrow \Omega$  be defined by

$$\pi(t, \omega) = \omega. \quad (16.2)$$

We define the *predictable*  $\sigma$ -field  $\mathcal{P}$  to be the  $\sigma$ -field on  $[0, \infty) \times \Omega$  generated by the collection of all bounded left continuous processes adapted to  $\mathcal{F}_t$ . That is,  $\mathcal{P}$  is the  $\sigma$ -field on  $[0, \infty) \times \Omega$  generated by the collection of all sets of the form

$$\{(t, \omega) \in [0, \infty) \times \Omega : X_t(\omega) > a\},$$

where  $a \in \mathbb{R}$  and  $X$  is a bounded, adapted, left-continuous process. The *optional*  $\sigma$ -field  $\mathcal{O}$  is the  $\sigma$ -field on  $[0, \infty) \times \Omega$  generated by the collection of all bounded right-continuous processes adapted to  $\mathcal{F}_t$ . The word for predictable in French is “prévisible.” The older literature uses “well measurable” in place of the word “optional.”

If  $S$  and  $T$  are random variables taking values in  $[0, \infty]$ , let  $[S, T] = \{(t, \omega) \in [0, \infty) \times \Omega : S(\omega) \leq t < T(\omega)\}$ , and define  $(S, T]$ ,  $(S, T)$ , etc. similarly. With this notation,  $[T, T]$ , the graph of  $T$ , is equal to  $\{(t, \omega) \in [0, \infty) \times \Omega : T(\omega) = t < \infty\}$ . Note that  $[T, T]$  is a subset of  $[0, \infty) \times \Omega$ , so  $\pi([T, T]) = (T < \infty)$ .

Recall that a stopping time can take the value  $\infty$ . A stopping time  $T$  is *predictable* if there exists a sequence of stopping times  $T_n$  such that for all  $\omega$

- (1)  $T_1(\omega) \leq T_2(\omega) \leq \dots$ ,
- (2)  $\lim_{n \rightarrow \infty} T_n(\omega) = T(\omega)$ , and
- (3) if  $T(\omega) > 0$ , then  $T_n(\omega) < T(\omega)$  for each  $n$ .

In this case, the stopping times  $T_n$  predict  $T$  or announce  $T$ . If  $T$  is a stopping time satisfying (1)–(3) above and  $S = T$ , a.s., then we call  $S$  a predictable stopping time as well. A stopping time  $T$  is *totally inaccessible* if  $\mathbb{P}(T = S < \infty) = 0$  for every predictable stopping time  $S$ .

For an example of a predictable stopping time, let  $W_t$  be a Brownian motion started at 0 and let  $T = \inf\{t > 0 : W_t = 1\}$ . The stopping time  $T$  is predicted by the stopping times  $T_n = \inf\{t > 0 : W_t = 1 - (1/n)\}$ .

For an example of a totally inaccessible stopping time, let  $P_t$  be a Poisson process with parameter 1 and let  $T = \inf\{t : P_t = 1\}$ , the first time the Poisson process jumps. Since  $P_t$  has independent increments,  $P_t - t$  is a martingale, just as in Example 3.2. By (A.8),  $\mathbb{E}[(P_t - t)^2] < \infty$ . If  $S$  is a bounded predictable stopping time, by the optional stopping theorem,  $\mathbb{E}P_S = \mathbb{E}S$ . If  $S_n$  are stopping times predicting  $S$ , then by monotone convergence

$$\mathbb{E}P_{S-} = \lim_{n \rightarrow \infty} \mathbb{E}P_{S_n} = \lim_{n \rightarrow \infty} \mathbb{E}S_n = \mathbb{E}S.$$

Therefore  $\mathbb{E}[P_S - P_{S-}] = 0$ , and since  $P_t$  is an increasing process, this says that  $P$  does not jump at time  $S$ . Applying this to  $S \wedge M$  and letting  $M \rightarrow \infty$ , we see that  $P$  does not jump at any predictable time  $S$ , whether or not  $S$  is bounded. Therefore  $\mathbb{P}(T = S < \infty) = 0$ , so  $T$  is totally inaccessible.

The proof of the following proposition is reminiscent of that of the Vitali covering theorem from measure theory.

**Proposition 16.1** *Let  $T$  be a stopping time. There exist predictable stopping times  $S_1, S_2, \dots$  and a totally inaccessible stopping time  $U$  such that  $[T, T] = [U, U] \cup (\cup_{i=1}^{\infty} [S_i, S_i])$ .*

*Proof* Let

$$a_1 = \sup\{\mathbb{P}(S = T < \infty) : S \text{ is a predictable stopping time}\}$$

and choose  $S_1$  to be a predictable stopping time such that  $\mathbb{P}(S_1 = T < \infty) \geq \frac{1}{2}a_1$ . Given  $S_1, \dots, S_n$ , let

$$\begin{aligned} a_{n+1} &= \sup\{\mathbb{P}(S = T < \infty, S \neq S_1, \dots, S \neq S_n) : \\ &\quad S \text{ is a predictable stopping time}\} \end{aligned}$$

and choose  $S_{n+1}$  such that  $\mathbb{P}(S_{n+1} = T < \infty, S_{n+1} \neq S_1, \dots, S_{n+1} \neq S_n) \geq \frac{1}{2}a_{n+1}$ .

If this procedure stops after  $n$  steps, set  $U(\omega)$  equal to  $T(\omega)$  if  $T(\omega)$  is not equal to any of  $S_1(\omega), \dots, S_n(\omega)$  and equal to infinity otherwise. It is easy to check that  $U$  is a stopping time that is totally inaccessible.

The other alternative is that this procedure continues indefinitely. In this case define

$$U(\omega) = \begin{cases} T(\omega), & T(\omega) \neq S_1(\omega), S_2(\omega), \dots, \\ \infty, & \text{otherwise.} \end{cases}$$

There is no problem checking that  $U$  is a stopping time, but we need to show that  $U$  is totally inaccessible. Since probabilities are bounded by one, we have  $a_n \rightarrow 0$ . If there exists a predictable stopping time  $S$  such that  $b = \mathbb{P}(S = U < \infty) > 0$ , then  $b > 2a_n$  for some  $n$ , and in our construction we would have then chosen  $S$  in place of the  $S_n$  we did choose. Therefore such a stopping time  $S$  cannot exist.  $\square$

**Proposition 16.2** (1) *The optional  $\sigma$ -field  $\mathcal{O}$  is generated by the collection of sets*

$$\{[S, T) : S, T \text{ stopping times}\}.$$

(2)  $\mathcal{O}$  is generated by the collection of sets of the form  $[a, b) \times C$ , where  $a < b$  and  $C \in \mathcal{F}_a$ .

(3) *The predictable  $\sigma$ -field  $\mathcal{P}$  is generated by the collection of sets*

$$\{(S, T] : S, T \text{ stopping times}\}.$$

(4)  $\mathcal{P}$  is generated by the collection of sets

$$\{[S, T) : S, T \text{ predictable stopping times}\}.$$

(5)  $\mathcal{P}$  is generated by the collection of sets of the form  $[b, c) \times C$ , where  $a < b < c$  and  $C \in \mathcal{F}_a$ .

*Proof* (1) Since  $1_{[S,T)}$  is a bounded right-continuous process that is adapted to  $\{\mathcal{F}_t\}$ , sets of the form  $[S, T)$  are optional. Now suppose  $X$  is a bounded adapted process with right-continuous paths. Let  $\varepsilon > 0$ , let  $U_0 = 0$ , a.s., and let

$$U_{i+1} = \inf\{t > U_i : |X_t - X_{U_i}| > \varepsilon\}, \quad i \geq 0. \quad (16.3)$$

Since  $X$  has right-continuous paths,

$$(U_1 < t) = \cap_{q \in \mathbb{Q}_+, q < t} \{|X_q - X_0| > \varepsilon\},$$

where  $\mathbb{Q}_+$  denotes the positive rationals, and it follows that  $U_1$  is a stopping time. Similarly  $U_i$  is a stopping time for each  $i$ ; Exercise 16.4 asks you to prove this. If we set

$$X_t^\varepsilon(\omega) = \sum_{i=0}^{\infty} X_{U_i}(\omega) 1_{[U_i(\omega), U_{i+1}(\omega))}(t),$$

then  $\sup_t |X_t - X_t^\varepsilon| \leq \varepsilon$ . Therefore it suffices to show that each process  $X^\varepsilon$  is measurable with respect to the  $\sigma$ -field  $\widehat{\mathcal{O}}$  generated by the collection of sets of the form  $[S, T)$ .

To do that, it suffices to show that processes of the form

$$Y_t(\omega) = 1_A(\omega) 1_{[U_i(\omega), U_{i+1}(\omega))}(t),$$

where  $A \in \mathcal{F}_{U_i}$ , are measurable with respect to  $\widehat{\mathcal{O}}$ . If we set  $S(\omega)$  equal to  $U_i(\omega)$  if  $\omega \in A$  and equal to  $\infty$  otherwise and we set  $T(\omega)$  equal to  $U_{i+1}(\omega)$  if  $\omega \in A$  and  $\infty$  otherwise, then  $Y_t(\omega) = 1_{[S(\omega), T(\omega))}$ .

(2) If  $C \in \mathcal{F}_a$ , then  $1_C(\omega) 1_{[a,b)}(t)$  is a bounded right-continuous adapted process, so it is optional. By (1), every bounded right-continuous adapted process can be approximated by linear combinations of processes of the form  $1_{[S,T)}$ . Now  $1_{[S,T)} = 1_{[S,\infty)} - 1_{[T,\infty)}$ , and  $1_{[S,\infty)}$

is the limit of  $1_{[S_n, \infty)}$ , where  $S_n = k/2^n$  if  $(k-1)/2^n \leq S < k/2^n$ , and we can similarly approximate  $1_{[T, \infty)}$ . Note

$$1_{[S_n(\omega), \infty)}(t) = \sum_{k=1}^{\infty} 1_{((k-1)/2^n \leq S(\omega) < k/2^n)} 1_{[k/2^n, \infty)}(t).$$

Since  $((k-1)/2^n \leq S(\omega) < k/2^n) \in \mathcal{F}_{k/2^n}$  and  $1_{[k/2^n, \infty)}(t)$  is the limit of  $1_{[k/2^n, m)}(t)$  as  $m \rightarrow \infty$ , we see that every bounded right-continuous adapted process is measurable with respect to the  $\sigma$ -field generated by processes of the form  $1_A(\omega)1_{[a, b)}(t)$ , where  $A$  is  $\mathcal{F}_a$  measurable.

For (3),  $1_{(S, T]}$  is left continuous, bounded, and adapted, hence predictable. Any left-continuous adapted bounded process can be approximated by processes of the form

$$\sum_{k=0}^{n2^n-1} X_{k/2^n}(\omega) 1_{(k/2^n, (k+1)/2^n]}(t),$$

which in turn can be approximated by linear combinations of processes of the form  $Y = 1_A(\omega)1_{(a, b]}(t)$ , where  $A$  is  $\mathcal{F}_a$  measurable. Such a process  $Y$  is of the form  $1_{(S, T]}$  if we define  $S$  and  $T$  by

$$S(\omega) = \begin{cases} a, & \omega \in A, \\ \infty, & \omega \notin A, \end{cases} \quad T(\omega) = \begin{cases} b, & \omega \in A, \\ \infty, & \omega \notin A. \end{cases}$$

To prove (4), note that  $S + \frac{1}{k}$  is always a predictable stopping time (predicted by the stopping times  $S_n = S + \frac{1}{k} - \frac{1}{n}$  for  $n > k$ ). We have

$$(S, T] = \cup_k \{\cap_m [S + \frac{1}{k}, T + \frac{1}{m})\}.$$

On the other hand, if  $S$  and  $T$  are predictable and are predicted by sequences  $S_n$  and  $T_m$ , respectively, then

$$[S, T] = \cap_n \{\cup_m (S_n, T_m)\}.$$

(4) now follows by using (3).

(5) As long as  $a + (1/n) < b$ , the processes  $1_C(\omega)1_{(b-(1/n), c-(1/n))}(t)$  are left continuous, bounded, and adapted, hence predictable. The process  $1_C(\omega)1_{[b, c]}(t)$  is the limit of these processes as  $n \rightarrow \infty$ , so is predictable. On the other hand, if  $X_t$  is a bounded adapted left-continuous process, it can be approximated by

$$\sum_{k=1}^{n2^n-1} X_{(k-1)/2^n}(\omega) 1_{(k/2^n, (k+1)/2^n]}(t).$$

Each summand can be approximated by linear combinations of processes of the form  $1_C(\omega)1_{(b, c]}(t)$ , where  $C \in \mathcal{F}_a$  and  $a < b < c$ . Finally,  $1_C(\omega)1_{(b, c]}(t)$  is the limit of  $1_C(\omega)1_{[b+(1/n), c+(1/n))}(t)$  as  $n \rightarrow \infty$ .  $\square$

A consequence of Proposition 16.2(1) and (4) is that  $\mathcal{P} \subset \mathcal{O}$ .

## 16.2 Hitting times

Let  $\mathcal{S}$  be a separable metric space. Suppose  $\{\mathcal{F}_t\}$  is a filtration satisfying the usual conditions and  $X$  is a stochastic process taking values  $\mathcal{S}$  whose paths are right continuous and such that the jump times are totally inaccessible. Saying the jump times are totally inaccessible means that if  $T$  is a predictable stopping time, then  $X_{T-} = X_T$ , a.s., where  $X_{T-} = \lim_{s < T, s \rightarrow T} X_s$ .

If  $B$  is a Borel subset of a metric space  $\mathcal{S}$  and  $X$  is a  $\mathcal{S}$ -valued process, let

$$U_B = \inf\{t \geq 0 : X_t \in B\}$$

and

$$T_B = \inf\{t > 0 : X_t \in B\}.$$

$T_B$  is known as the first *hitting time* of  $B$  and  $U_B$  as the first *entry time* of  $B$ .

**Proposition 16.3** (1) If  $A$  is an open set, then  $T_A$  and  $U_A$  are stopping times.

(2) If  $A$  is a compact set, then  $T_A$  and  $U_A$  are stopping times.

*Proof* (1) Since the paths of  $X_t$  are right continuous and  $A$  is open, for each  $t$ ,

$$(T_A < t) = \bigcup_{q \in \mathbb{Q}_+, q < t} (X_q \in A) \in \mathcal{F}_t,$$

where  $\mathbb{Q}_+$  denotes the non-negative rationals. Thus  $T_A$  is a stopping time. Since

$$(U_A < t) = (T_A < t) \cup (X_0 \in A) \in \mathcal{F}_t, \quad (16.4)$$

then  $U_A$  is also a stopping time.

(2) Now suppose  $A$  is compact and let  $A_n = \{x \in \mathcal{S} : d(x, A) < 1/n\}$ . Each set  $A_n$  is open, hence  $T_{A_n}$  is a stopping time for each  $n$ . The  $T_{A_n}$  increase; let  $T$  be the limit. If we show  $T = T_A$ , a.s., this will prove  $T_A$  is a stopping time.

Since  $A \subset A_n$ , then  $T_{A_n} \leq T_A$  for each  $n$ . Therefore  $T \leq T_A$ . On the other hand, if  $n > m$ , then  $X_{T_{A_n}} \in \overline{A}_n \subset \overline{A}_m$ , the closure of  $A_m$ . Either  $T_{A_n}(\omega) = T(\omega)$  for all  $n$  sufficiently large, in which case  $X_T(\omega) \in \overline{A}_m$ , or else  $T_{A_n}(\omega) < T(\omega)$  for all  $n$ . In the latter case,  $X_T(\omega) = \lim_{n \rightarrow \infty} X_{T_{A_n}}(\omega) \in \overline{A}_m$  except for  $\omega$ 's in a null set since the jump times of  $X$  are totally inaccessible. In either case,  $X_T \in \overline{A}_m$ . This is true for all  $m$ , so  $X_T \in \cap_m \overline{A}_m = A$ , and therefore  $T_A \leq T$ .

We conclude  $T_A$  is a stopping time. To prove  $U_A$  is a stopping time, we argue using (16.4) as above.  $\square$

For the proof of the following, which uses Choquet's capacity theorem, we refer the reader to Blumenthal and Getoor (1968), Section I.10. Fix  $t$  and define

$$R_t(A) = \{\omega : X_s(\omega) \in A \text{ for some } s \in [0, t]\} = (U_A \leq t). \quad (16.5)$$

**Theorem 16.4** If  $A$  is a Borel subset of  $\mathcal{S}$ , then  $R_t(A) \in \mathcal{F}_t$  and there exists an increasing sequence of compact sets  $K_n$  contained in  $A$  such that  $\mathbb{P}(R_t(K_n)) \uparrow \mathbb{P}(R_t(A))$ .

Since  $(U_A \leq t) = R_t(A)$ , we have the following as an immediate corollary.

**Theorem 16.5** For all Borel sets  $A$ ,  $U_A$  is a stopping time.

Here is the main theorem of this section.

**Theorem 16.6** Suppose  $\{\mathcal{F}_t\}$  is a filtration satisfying the usual conditions and  $X$  is a right continuous process whose jump times are totally inaccessible. If  $B$  is a Borel subset of  $S$ , then  $T_B$  is a stopping time.

*Proof* If we let  $Y_t^\delta = X_{t+\delta}$  and  $U_B^\delta = \inf\{t \geq 0 : Y_t^\delta \in B\}$ , then by the above,  $U_B^\delta$  is a stopping time with respect to the filtration  $\{\mathcal{F}_t^\delta\}$ , where  $\mathcal{F}_t^\delta = \mathcal{F}_{t+\delta}$ . It follows that  $\delta + U_B^\delta$  is a stopping time with respect to the filtration  $\{\mathcal{F}_t\}$ . Since  $(1/m) + U_B^{1/m} \downarrow T_B$ , then  $T_B$  is a stopping time with respect to  $\{\mathcal{F}_t\}$ .  $\square$

We now show that the hitting times of Borel sets can be approximated by the hitting times of compact sets.

**Proposition 16.7** There exists an increasing sequence of compact sets  $K_n$  contained in  $B$  such that  $U_{K_n} \downarrow U_B$  on  $(U_B < \infty)$ ,  $\mathbb{P}$ -a.s.

*Proof* For each  $t$  we can find an increasing sequence of compact sets  $L_n^t$  contained in  $B$  with  $\mathbb{P}(R_t(L_n^t)) \uparrow \mathbb{P}(R_t(B))$ . Let  $q_j$  be an enumeration of the non-negative rationals. Let  $K_n = L_n^{q_1} \cup \dots \cup L_n^{q_n}$ . Then the  $K_n$  are compact, form an increasing sequence, and are all contained in  $B$ . Thus  $U_{K_n}$  decreases, say to  $S$ , and since  $U_{K_n} \geq U_B$  for all  $n$ , then  $S \geq U_B$ . If we prove  $S \leq U_B$ ,  $\mathbb{P}$ -a.s., then  $S = U_B$ , and we have our result.

If  $U_B < S$ , there exists a rational  $q_j$  with  $U_B < q_j < S$ . Hence it suffices to prove  $\mathbb{P}(U_B < q_j < S) = 0$  for all  $j$ . If  $U_B < q_j$ , then  $\omega \in R_{q_j}(B)$ . Since  $R_{q_j}(L_n^{q_j}) \uparrow R_{q_j}(B)$ , a.s., then except for a null set,  $\omega$  will be in  $R_{q_j}(L_n^{q_j})$  for all  $n$  large enough, hence in  $R_{q_j}(K_n)$  if  $n$  is large enough. Then  $U_{K_n}(\omega) \leq q_j < U_B$  or  $S \leq q_j$ . Therefore  $\mathbb{P}(U_B < q_j < S) = 0$ .  $\square$

**Theorem 16.8** There exists an increasing sequence of compacts  $K_n$  contained in  $B$  such that  $T_{K_n} \downarrow T_B$ .

*Proof* Let  $Y_t^\delta = X_{t+\delta}$  and  $U_B^\delta = \inf\{t \geq 0 : Y_t^\delta \in B\}$ . Applying the above proposition to  $Y_t^{1/m}$ , for each  $m$  there exist compact sets  $L_n^m$ , increasing in  $n$  and contained in  $B$ , such that  $U_{L_n^m}^{1/m} \downarrow U_B^{1/m}$ . Let  $K_n = L_n^1 \cup \dots \cup L_n^n$ . Then  $K_n$  is an increasing sequence of compact sets contained in  $B$ , and  $U_{K_n}^{1/m} \downarrow U_B^{1/m}$ . Also, for each  $n$ ,  $1/m + U_{K_n}^{1/m} \downarrow T_{K_n}$  and  $1/m + U_B^{1/m} \downarrow T_B$ . We write

$$\begin{aligned} T_B &= \lim_m (1/m + U_B^{1/m}) = \lim_m \lim_n (1/m + U_{K_n}^{1/m}) \\ &= \lim_n \lim_m (1/m + U_{K_n}^{1/m}) = \lim_n T_{K_n}. \end{aligned}$$

Since  $1/m + U_{K_n}^{1/m}$  is decreasing in both  $m$  and  $n$ , the change in the order of taking limits is justified. Since  $T_{K_n}$  is decreasing, this completes the proof.  $\square$

### 16.3 The debut and section theorems

If  $E \subset [0, \infty) \times \Omega$ , let  $D_E = \inf\{t \geq 0 : (t, \omega) \in E\}$ , the *debut* of  $E$ . An important generalization of Theorem 16.6 is the following, known as the *debut theorem*.

**Theorem 16.9** *If  $E \in \mathcal{O}$ , then  $D_E$  is a stopping time.*

The proof of this theorem is beyond the scope of this book, and we refer the reader to Dellacherie and Meyer (1978) for a proof.

Using Theorem 16.9, we can weaken the assumptions on  $X$  in Theorem 16.6.

**Theorem 16.10** *If  $X$  is an optional process taking values in  $\mathcal{S}$  and  $B$  is a Borel subset of  $\mathcal{S}$ , then  $U_B$  and  $T_B$  are stopping times.*

*Proof* Since  $B$  is a Borel subset of  $\mathcal{S}$  and  $X$  is an optional process, then  $1_B(X_t)$  is also an optional process.  $U_B$  is then the debut of the set  $E = \{(s, \omega) : 1_B(X_s(\omega)) = 1\}$ , and therefore is a stopping time.

To prove that  $T_B$  is a stopping time, we argue exactly as in the proof of Theorem 16.6.  $\square$

**Remark 16.11** In the theory of Markov processes, the notion of completion of a  $\sigma$ -field is a bit different. However it is still the case that the hitting times of Borel sets by right continuous processes are stopping times. See Remark 20.4.

The *optional section theorem* is the following.

**Theorem 16.12** *If  $E$  is an optional set and  $\varepsilon > 0$ , there exists a stopping time  $T$  such that  $[T, T] \subset E$  and  $\mathbb{P}(\pi(E)) \leq \mathbb{P}(T < \infty) + \varepsilon$ .*

The statement of the *predictable section theorem* is very similar.

**Theorem 16.13** *If  $E$  is a predictable set and  $\varepsilon > 0$ , there exists a predictable stopping time  $T$  such that  $[T, T] \subset E$  and  $\mathbb{P}(\pi(E)) \leq \mathbb{P}(T < \infty) + \varepsilon$ .*

Again we refer to Dellacherie and Meyer (1978) for proofs. We note that Proposition 16.7 is a precursor of the optional section theorem. To see this, let  $A$  be a Borel set and let  $E = \{(t, \omega) : X_t \in A\}$ . Then  $D_E = U_A$ . If the process is right continuous, then  $X_{U_{K_n}} \in K_n \subset A$ , where the  $K_n$  are as in Proposition 16.7, and the graphs of the  $U_{K_n}$  are contained in  $E$ .

Here is a corollary of Theorems 16.12 and 16.13.

**Corollary 16.14** (1) *If  $X$  and  $Y$  are optional processes such that  $\mathbb{P}(X_T = Y_T) = 1$  for every finite stopping time  $T$ , then  $X$  and  $Y$  are indistinguishable:  $\mathbb{P}(X_t = Y_t \text{ for all } t) = 1$ .*

(2) *If  $X$  and  $Y$  are predictable processes with  $\mathbb{P}(X_T = Y_T) = 1$  for every finite predictable stopping time  $T$ , then  $X$  and  $Y$  are indistinguishable.*

*Proof* We prove (1), the proof of (2) being similar. Let  $F = \{(t, \omega) : X_t(\omega) \neq Y_t(\omega)\}$ . Then  $F$  is an optional set, and if  $\mathbb{P}(\pi(F)) > 0$ , there exists a stopping time  $U$  with  $[U, U] \subset F$

and  $\mathbb{P}(U < \infty) > 0$ . By looking at  $T = U \wedge N$  for sufficiently large  $N$ , we obtain a contradiction.  $\square$

Another application of the section theorems is the following.

**Proposition 16.15** *Suppose  $[T, T]$  is a predictable set. Then  $T$  is a predictable stopping time.*

*Proof* Since  $T$  is the debut of  $[T, T]$ , then  $T$  is a stopping time. By the predictable section theorem, Theorem 16.13, for each  $n$  there exists a predictable stopping time  $S_n$  such that  $[S_n, S_n] \subset [T, T]$  and

$$\mathbb{P}(\pi([S_n, S_n])) \geq \mathbb{P}(\pi([T, T])) - 2^{-n}.$$

Saying  $[S_n, S_n] \subset [T, T]$  implies that for each  $\omega$ , either  $S_n(\omega) = T(\omega)$  or else  $S_n(\omega) = \infty$ . The set of  $\omega$ 's for which  $T(\omega) < \infty$  but  $S_n(\omega) = \infty$  has probability at most  $2^{-n}$ .

Let  $Q_n = S_1 \wedge \dots \wedge S_n$ . Then the  $Q_n$ 's are predictable stopping times by Exercise 16.1, they decrease,  $[Q_n, Q_n] \subset [T, T]$ , and  $\mathbb{P}(\pi([Q_n, Q_n])) \geq \mathbb{P}(\pi([T, T])) - 2^{-n+1}$ . Let  $Q = \lim_n Q_n$ . If  $Q(\omega) < \infty$ , then  $Q_n(\omega) < \infty$  for all  $n$  sufficiently large (how large depends on  $\omega$ ); since  $Q_n(\omega)$  is either equal to  $T(\omega)$  or to  $\infty$ ,  $Q_n(\omega) = Q(\omega)$  for all  $n$  sufficiently large, and hence  $Q(\omega) = T(\omega)$ . If  $T(\omega) < \infty$ , then except for a set of  $\omega$ 's of probability zero,  $Q_n(\omega) = T(\omega)$  for  $n$  sufficiently large. Therefore  $Q = T$ , a.s.

Choose  $R_{nm}$  predicting  $Q_n$  as  $m \rightarrow \infty$ . Choose  $m_n$  large enough such that

$$\mathbb{P}(R_{nm_n} + 2^{-n} < Q_n < \infty) < 2^{-n} \quad \text{and} \quad \mathbb{P}(R_{nm_n} < n, Q_n = \infty) < 2^{-n}.$$

Let  $U_n = n \wedge R_{nm_n} \wedge R_{n+1,m_{n+1}} \wedge \dots$ . Fix  $n$  for the moment. If  $0 < Q(\omega) < \infty$ , then  $R_{jm_j}(\omega) < Q_j(\omega) = Q(\omega)$  for all  $j$  sufficiently large. Choosing  $j > n$  sufficiently large,  $U_n(\omega) \leq R_{jm_j}(\omega) < Q(\omega)$ .

The  $U_n$  increase; let  $T$  be the limit. By the Borel–Cantelli lemma, if  $Q(\omega) < \infty$ , then  $R_{nm_n}(\omega) \geq Q_n(\omega) - 2^{-n} = Q(\omega) - 2^{-n}$  for all  $n$  sufficiently large, except for a set of  $\omega$ 's of probability zero. Therefore  $U_n(\omega) \geq Q(\omega) - 2^{-n+1}$  for  $n$  sufficiently large, and we conclude that  $U_n(\omega) \uparrow Q(\omega)$ , except for a set of  $\omega$ 's of probability zero.

If  $Q(\omega) = \infty$ , then  $Q_n(\omega) = \infty$  for all  $n$ . By the Borel–Cantelli lemma, except for a set of probability zero,  $R_{nm_n} \geq n$  for  $n$  sufficiently large. Hence  $U_n(\omega) = n$  for  $n$  sufficiently large, so  $U_n(\omega) < Q(\omega)$  and  $U_n(\omega) \uparrow Q(\omega)$ . Thus  $Q$  is predictable and  $T = Q$ , a.s. (We leave consideration of those  $\omega$  for which  $Q(\omega) = 0$  to the reader.)  $\square$

**Proposition 16.16** *Let  $X_t$  be a predictable process with paths that are right continuous with left limits. If  $a \in \mathbb{R}$  and  $T = \inf\{t > 0 : X_t \geq a\}$ , then  $T$  is a predictable stopping time.*

*Proof* The set  $A = \{(t, \omega) : X_t(\omega) \geq a\}$  is a predictable set. Since  $X_t$  is right continuous,  $[T, \infty) = A \cup (T, \infty) \in \mathcal{P}$  by Proposition 16.2, and so  $[T, T] = [T, \infty) \setminus (T, \infty) \in \mathcal{P}$ . Now apply Proposition 16.15.  $\square$

## 16.4 Projection theorems

Let  $\mathcal{B}[0, \infty)$  be the Borel  $\sigma$ -field on  $[0, \infty)$ , let  $\mathcal{F}_\infty = \vee_{t \geq 0} \mathcal{F}_t$ , and let  $\mathcal{H}$  be the product  $\sigma$ -field

$$\mathcal{H} = \mathcal{B}[0, \infty) \times \mathcal{F}_\infty. \quad (16.6)$$

The following is the optional projection theorem.

**Theorem 16.17** *Let  $X$  be a bounded process that is  $\mathcal{H}$  measurable. There exists a unique optional process  ${}^oX$  such that*

$${}^oX_T 1_{(T < \infty)} = \mathbb{E}[X_T 1_{(T < \infty)} | \mathcal{F}_T] \quad (16.7)$$

for all stopping times  $T$ , including those taking infinite values. If  $X \geq 0$ , then  ${}^oX \geq 0$ .

${}^oX$  is called the *optional projection* of  $X$ . If  $X$  is already optional, then by the uniqueness result, Corollary 16.14,  ${}^oX = X$ .

If we take our stopping time  $T$  in (16.7) equal to a fixed time  $t$ , we have

$${}^oX_t = \mathbb{E}[X_t | \mathcal{F}_t], \quad \text{a.s.} \quad (16.8)$$

This observation is sometimes useful when  $X$  is not an adapted process and one wants a version of  $\mathbb{E}[X_t | \mathcal{F}_t]$  that is jointly measurable in  $t$  and  $\omega$ .

If (16.7) holds, then taking expectations shows that

$$\mathbb{E}[{}^oX_T; T < \infty] = \mathbb{E}[X_T; T < \infty] \quad (16.9)$$

for all stopping times  $T$ . Conversely, suppose (16.9) holds for all stopping times  $T$ . If  $S$  is a stopping time and  $A \in \mathcal{F}_S$ , let  $S_A$  be defined by

$$S_A(\omega) = \begin{cases} S(\omega) & \omega \in A; \\ \infty & \omega \notin A. \end{cases} \quad (16.10)$$

Then (16.9) with  $T$  replaced by  $S_A$  implies that

$$\mathbb{E}[{}^oX_S 1_{(S < \infty)}; A] = \mathbb{E}[X_S 1_{(S < \infty)}; A].$$

Since  ${}^oX_S 1_{(S < \infty)}$  is  $\mathcal{F}_S$  measurable, this implies (16.7) holds for the stopping time  $S$ . Consequently (16.7) holding for all stopping times  $T$  is equivalent to (16.9) holding for all stopping times  $T$ .

*Proof of Theorem 16.17* The uniqueness is immediate from Corollary 16.14. We look at existence. If  $X_t(\omega) = 1_F(\omega)1_{[a,b]}(t)$  where  $F \in \mathcal{F}_\infty$ , we set  ${}^oX_t$  equal to  $\mathbb{E}[1_F | \mathcal{F}_t]1_{[a,b]}(t)$ , where we use Corollary 3.13 to take the right continuous version of the martingale  $\mathbb{E}[1_F | \mathcal{F}_t]$ . We check:

$$\begin{aligned} \mathbb{E}[{}^oX_T; T < \infty] &= \mathbb{E}[\mathbb{E}[1_F | \mathcal{F}_T]1_{[a,b]}(T); T < \infty] \\ &= \mathbb{E}[1_F 1_{[a,b]}(T); T < \infty] \\ &= \mathbb{E}[X_T; T < \infty] \end{aligned}$$

since  $(T < \infty)$  and  $1_{[a,b]}(T)$  are both  $\mathcal{F}_T$  measurable. We then use linearity and limits to define  ${}^oX$  for bounded measurable  $X$ . The positivity of  ${}^oX$  when  $X \geq 0$  is clear from the construction.  $\square$

Almost the same proof gives

**Theorem 16.18** *Let  $X$  be a bounded measurable process. There exists a unique predictable process  ${}^pX$ , called the predictable projection of  $X$ , such that*

$$\mathbb{E}[{}^pX_T; T < \infty] = \mathbb{E}[X_T; T < \infty]$$

for every predictable stopping time  $T$ . If  $X \geq 0$ , then  ${}^pX \geq 0$ .

*Proof* Uniqueness is as before. If  $X_t = 1_F(\omega)1_{(a,b]}(t)$ , we let  ${}^pX_t = 1_{(a,b]}(t)Z_{t-}(\omega)$ , where  $Z_{t-}$  denotes the left-hand limit of  $Z_t$  at time  $t$  and  $Z_t$  is the right-continuous version of the martingale  $\mathbb{E}[1_F | \mathcal{F}_t]$ . We use linearity and limits to define  ${}^pX$  for bounded measurable  $X$ . The positivity of  ${}^pX$  when  $X \geq 0$  is clear.  $\square$

## 16.5 More on predictability

If  $U$  is a random time, i.e., a  $\mathcal{F}_\infty$  measurable map from  $\Omega$  to  $[0, \infty]$ , define

$$\mathcal{F}_{U-} = \sigma\{X_U : X \text{ is bounded and predictable}\}.$$

**Lemma 16.19** *Suppose  $T$  is a predictable stopping time predicted by stopping times  $T_n$ . Then  $\mathcal{F}_{T-} = \bigvee_{n=1}^\infty \mathcal{F}_{T_n}$ .*

*Proof* If  $X$  is left continuous, adapted, and bounded, then  $X_T = \lim X_{T_m}$  and  $X_{T_m} \in \mathcal{F}_{T_m} \subset \bigvee_n \mathcal{F}_{T_n}$ , so  $X_T \in \bigvee_n \mathcal{F}_{T_n}$ . An argument using the monotone class theorem shows  $\mathcal{F}_{T-} \subset \bigvee_n \mathcal{F}_{T_n}$ .

On the other hand, suppose  $A \in \mathcal{F}_{T_n}$  for some  $n$ . Define  $X = 1_{(U_n, \infty)}$ , where  $U_n = T_n$  if  $\omega \in A$  and  $\infty$  otherwise. Since  $T_n < T$  on  $(T > 0)$ , then  $X_T = 1_A$ . (We leave consideration of what happens on the event  $(T = 0)$  to the reader.)  $X$  is predictable since it is left continuous, adapted, and bounded, so  $A \in \mathcal{F}_{T-}$  measurable. Therefore  $\mathcal{F}_{T_n} \subset \mathcal{F}_{T-}$  for all  $n$ , and we conclude  $\bigvee_n \mathcal{F}_{T_n} \subset \mathcal{F}_{T-}$ .  $\square$

**Corollary 16.20** *Suppose  $T$  is a predictable stopping time. If  $M$  is a uniformly integrable martingale with right-continuous paths, then*

$$\mathbb{E}[M_T | \mathcal{F}_{T-}] = M_{T-}.$$

*Proof* If  $X_t = M_{t-}$ , then  $X$  is left continuous, hence predictable, so  $M_{T-} = X_T$  is  $\mathcal{F}_{T-}$  measurable by the definition of  $\mathcal{F}_{T-}$  and a limit argument. Suppose the sequence  $T_n$  predicts  $T$ . If  $A \in \mathcal{F}_{T_m}$  and  $n > m$ , then  $A \in \mathcal{F}_{T_m} \subset \mathcal{F}_{T_n}$ , and by optional stopping (see Exercise 3.12),

$$\mathbb{E}[M_T; A] = \mathbb{E}[M_{T_n}; A] \rightarrow \mathbb{E}[M_{T-}; A]$$

as  $n \rightarrow \infty$ . Since  $\mathcal{F}_{T-} = \bigvee_m \mathcal{F}_{T_m}$ , we have  $\mathbb{E}[M_T; A] = \mathbb{E}[M_{T-}; A]$  for all  $A \in \mathcal{F}_{T-}$ . Now use the definition of conditional expectation.  $\square$

**Corollary 16.21** *Let  $S$  be a predictable stopping time,  $M$  a square integrable martingale, and  $N_t = \Delta M_S 1_{(t \geq S)}$ . Then  $N_t$  is a square integrable martingale.*

*Proof* Since  $|N_t| \leq 2 \sup_{s \geq 0} |M_s|$ ,  $N$  is square integrable. We will show  $N$  is a martingale by showing  $\mathbb{E} N_T = 0$  for all bounded stopping times  $T$ , and then appealing to Proposition 9.5.

If  $T$  is a bounded stopping time, then  $(T \geq S) \in \mathcal{F}_{S-}$ ; to see this, if  $S_m$  is a sequence of stopping times predicting  $S$ , then  $(T \geq S) = \cap_m (T \geq S_m) \in \vee_m \mathcal{F}_{S_m}$ . Using Corollary 16.20,

$$\mathbb{E} N_T = \mathbb{E} \Delta M_S 1_{(T \geq S)} = \mathbb{E} [M_S; T \geq S] - \mathbb{E} [M_{S-}; T \geq S] = 0,$$

and we are done.  $\square$

We now show that every stopping time for Brownian motion is predictable.

**Proposition 16.22** *Let  $\{\mathcal{F}_t\}$  be the minimal augmented filtration of a Brownian motion. If  $T$  is a stopping time with respect to  $\{\mathcal{F}_t\}$ , then  $T$  is a predictable stopping time.*

*Proof* Let  $T$  be a stopping time for Brownian motion. Let  $g$  be a continuous strictly increasing function from  $[0, \infty]$  to  $[0, 1]$ , e.g.,  $g(s) = (2/\pi) \arctan s$ . Let  $M_t$  be the right-continuous modification of the martingale  $\mathbb{E}[g(T) | \mathcal{F}_t]$ . The property of Brownian motion that is key here is that every martingale adapted to the filtration of a Brownian motion is continuous; see Corollary 12.5. Hence  $M_t$  can be taken to be continuous.

Let  $V_t = M_t - g(T \wedge t)$ . Then  $V_t$  has continuous paths and since  $g(T \wedge t)$  increases with  $t$ ,  $V$  is a supermartingale. We have

$$V_t = \mathbb{E}[g(T) - g(T \wedge t) | \mathcal{F}_t],$$

so  $V$  is non-negative. Clearly  $V_T = 0$ . If  $S$  is the first time that  $V_t$  is 0, then  $S \leq T$ . Also,

$$0 = \mathbb{E} V_S = \mathbb{E}[g(T) - g(T \wedge S)],$$

so  $S \geq T$ .

We let  $T_n = \inf\{t : V_t = 1/n\}$ . By the continuity of  $V$ , it is clear that each  $T_n$  is strictly less than  $T$  if  $T > 0$  and the  $T_n$  increase up to  $T$ . Hence  $T$  is predictable.  $\square$

Now let us suppose that  $A_t$  is a right-continuous adapted process whose paths are increasing. We call such a process an *increasing process*.  $\Delta A_t$  denotes the jump of  $A$  at time  $t$ , that is,  $\Delta A_t = A_t - A_{t-}$ .

**Proposition 16.23** *Suppose  $A_t$  is an increasing process such that*

- (1)  $\Delta A_T = 0$  whenever  $T$  is a totally inaccessible stopping time, and
- (2)  $\Delta A_T$  is  $\mathcal{F}_{T-}$  measurable whenever  $T$  is a predictable stopping time.

*Then  $A$  is predictable.*

*Proof* Let  $U_{mi}$  be the  $i$ th time  $|\Delta A_t| \in (2^{-m}, 2^{-m+1}]$ . The  $U_{mi}$  are predictable stopping times by Exercise 16.5. We decompose each  $U_{mi}$  as in Proposition 16.1. Since  $A$  does not jump at totally inaccessible times, none of the  $U_{mi}$  has a totally inaccessible part.

We do this for each  $m$  and  $i$  and obtain a countable collection of predictable stopping times, the union of whose graphs contains all the jump times of  $A$ . We order them in some way as  $R_1, R_2, \dots$ . Define  $T_1 = R_1$ , define  $T_2$  by setting  $T_2(\omega) = R_2(\omega)$  if  $R_2(\omega) \neq R_1(\omega)$  and infinity otherwise. Set  $T_n(\omega) = R_n(\omega)$  if  $R_n(\omega) \neq R_1(\omega), \dots, R_{n-1}(\omega)$  and  $T_n(\omega) = \infty$  otherwise. We thus get a sequence of predictable stopping times  $T_n$  with disjoint graphs and

$\cup_n [T_n, T_n]$  includes all the jumps of  $A$ , except for the set of  $\omega$ 's of probability zero. The  $T_n$  are predictable stopping times by Exercise 16.6.

Since  $A$  jumps only at the predictable stopping times  $T_n$ , we see that we can write  $A_t = A_t^c + \sum_i (\Delta A_{T_n}) 1_{[T_n, \infty)}(t)$ , where  $A^c$  is a continuous increasing process. By hypothesis,  $\Delta A_{T_n}$  is  $\mathcal{F}_{T_n^-}$  measurable. Therefore the proof will be complete once we show  $(\Delta A_{T_n}) 1_{[T_n, \infty)}$  is a predictable process.

It therefore suffices to show that the process  $Y_t = 1_B(\omega) 1_{[T, \infty)}(t)$  is predictable if  $T$  is a predictable stopping time and  $B \in \mathcal{F}_{T^-}$ . Since  $Y_t = 1_{[T_B, \infty)}(t)$ , where  $T_B$  is equal to  $T$  if  $\omega \in B$  and equal to infinity otherwise, the predictability of  $Y$  follows by Exercise 16.3.  $\square$

## 16.6 Dual projection theorems

In this section  $A_t$  is a right-continuous increasing process with  $A_0 = 0$ , a.s. We do not necessarily assume that  $A_t$  is adapted, only that  $A$  is measurable with respect to  $\mathcal{H}$  defined by (16.6). Define  $\mu_A$  on elements of  $\mathcal{H}$  by

$$\mu_A(B) = \mathbb{E} \int_0^\infty 1_B(t, \omega) dA_t(\omega).$$

We define  $\mu_A(X)$  by  $\mathbb{E} \int_0^\infty X_t dA_t$  if  $X$  is bounded and  $\mathcal{H}$  measurable. Note that if  $X = 0$ , then  $\mu_A(X) = 0$ .

**Theorem 16.24** Suppose  $\mu$  is a bounded positive measure on  $\mathcal{H}$  such that  $\mu(X) = 0$  whenever  $X = 0$ . Then there exists a unique right-continuous increasing process  $A$  with  $A_0 = 0$ , a.s., such that  $\mu = \mu_A$ .

*Proof* First, uniqueness. If  $\mu = \mu_A = \mu_B$ , let  $t > 0$  and let  $C$  be the set of  $\omega$ 's where  $A_t(\omega) > B_t(\omega) + \varepsilon$ . Then  $\mu_A([0, t] \times C) \geq \mu_B([0, t] \times C) + \varepsilon \mathbb{P}(C)$ , which implies  $\mathbb{P}(C) = 0$ . Since  $\varepsilon$  is arbitrary, then  $A_t = B_t$ , a.s. Since  $A$  and  $B$  are right continuous, we conclude  $A = B$ .

To prove existence, for each rational  $q$ , define  $v_q(C) = \mu([0, q] \times C)$ . Clearly  $v_q$  is absolutely continuous with respect to  $\mathbb{P}$ . Let  $\tilde{A}_q$  be the Radon–Nikodym derivative of  $v_q$  with respect to  $\mathbb{P}$ . Since  $\mu$  is positive,  $\tilde{A}$  is increasing in  $q$ . Let  $A_t = \limsup_{q \rightarrow t, q > t} \tilde{A}_q$ . It is easy to check that  $\mu_A = \mu$ .  $\square$

**Theorem 16.25** Suppose  $A$  is right continuous,  $A_0 = 0$ , a.s., and  $\mu_A(X) = \mu_A({}^oX)$  for every bounded  $\mathcal{H}$  measurable process  $X$ . Then  $A_t$  is optional.

*Proof* Since  $A_t$  is right continuous, we need only show that  $A_t$  is adapted. Fix  $t$  and let  $Y$  be a bounded  $\mathcal{F}_\infty$  measurable random variable,

$$Z = Y - \mathbb{E}[Y | \mathcal{F}_t],$$

and  $X_s(\omega) = 1_{[0,t]}(s)Z(\omega)$ . If  $T$  is a stopping time, then  $(T \leq t) \in \mathcal{F}_t$ , and so by the definitions of  $X$  and  $Z$ ,

$$\mathbb{E}[{}^oX_T; T < \infty] = \mathbb{E}[X_T; T < \infty] = \mathbb{E}[Z; T \leq t] = 0.$$

This implies  ${}^oX = 0$  by the definition of  ${}^oX$ . Hence

$$\mathbb{E}[A_t Z] = \mathbb{E} \left[ \int_0^\infty X_s dA_s \right] = \mu_A(X) = \mu_A({}^oX) = 0.$$

Thus  $\mathbb{E}[A_t Y] = \mathbb{E}[A_t \mathbb{E}[Y | \mathcal{F}_t]]$ . We write

$$\begin{aligned}\mathbb{E}[A_t Y] &= \mathbb{E}[A_t \mathbb{E}[Y | \mathcal{F}_t]] = \mathbb{E}[\mathbb{E}[(A_t \mathbb{E}[Y | \mathcal{F}_t]) | \mathcal{F}_t]] \\ &= \mathbb{E}[\mathbb{E}[A_t | \mathcal{F}_t] \mathbb{E}[Y | \mathcal{F}_t]] = \mathbb{E}[\mathbb{E}[(Y \mathbb{E}[A_t | \mathcal{F}_t]) | \mathcal{F}_t]] \\ &= \mathbb{E}[Y \mathbb{E}[A_t | \mathcal{F}_t]].\end{aligned}$$

Hence  $\mathbb{E}[A_t Y] = \mathbb{E}[Y \mathbb{E}[A_t | \mathcal{F}_t]]$  for all bounded  $Y$ , or  $A_t = \mathbb{E}[A_t | \mathcal{F}_t]$ , a.s., which says that  $A_t$  is  $\mathcal{F}_t$  measurable.  $\square$

**Theorem 16.26** *If  $\mu_A(X) = \mu_A({}^p X)$  for all bounded  $X$ , then  $A$  is predictable and can be taken to be right continuous.*

*Proof* By hypothesis, together with Exercise 16.8,

$$\mu_A({}^o X) = \mu_A({}^p({}^o X)) = \mu_A({}^p X) = \mu_A(X).$$

By Theorem 16.25,  $A_t$  is right continuous and optional. We need to show that  $A$  does not jump at totally inaccessible times and that  $\Delta A_T$  is  $\mathcal{F}_{T-}$  measurable at predictable times  $T$ ; we then use Proposition 16.23.

Let  $T$  be a totally inaccessible stopping time and let  $B = (\Delta A_T > 0)$ . Set  $T_B$  equal to  $T$  on  $B$  and equal to infinity otherwise. It is easy to check that  $T_B$  is also totally inaccessible. Let  $X = 1_{[T_B, T_B]}$ . If  $U$  is a predictable stopping time,  $\mathbb{E}[X_U; U < \infty] = \mathbb{P}(T_B = U < \infty) = 0$ . By the definition of predictable projection,  ${}^p X = 0$ . Hence

$$\mathbb{E}[\Delta A_T; \Delta A_T > 0] = \mathbb{E}[\Delta A_{T_B}] = \mu_A(X) = \mu_A({}^p X) = 0.$$

Now suppose  $T$  is a predictable stopping time. Let  $Y$  be a bounded  $\mathcal{H}$  measurable random variable, set

$$Z = Y - \mathbb{E}[Y | \mathcal{F}_{T-}],$$

and  $X = Z 1_{[T, T]}$ . Let  $S$  be any predictable stopping time. Then if  $W = 1_{[S, S]}$ ,  $W = \lim_{n \rightarrow \infty} 1_{[S, S + (1/n)]}$  is a predictable process by Proposition 16.2(4). By the definition of  $\mathcal{F}_{T-}$ ,  $W_T$  is  $\mathcal{F}_{T-}$  measurable. This is the same as saying  $(S = T < \infty) \in \mathcal{F}_{T-}$ . Therefore

$$\mathbb{E}[X_S; S < \infty] = \mathbb{E}[Z; S = T < \infty] = 0.$$

This implies  ${}^p X = 0$ , and then

$$0 = \mu_A({}^p X) = \mu_A(X) = \mathbb{E}[Z \Delta A_T].$$

Similarly to the proof of Theorem 16.25,

$$\begin{aligned}\mathbb{E}[\Delta A_T Y] &= \mathbb{E}[\Delta A_T \mathbb{E}[Y | \mathcal{F}_{T-}]] \\ &= \mathbb{E}[\mathbb{E}[\Delta A_T | \mathcal{F}_{T-}] \mathbb{E}[Y | \mathcal{F}_{T-}]] \\ &= \mathbb{E}[Y \mathbb{E}[\Delta A_T | \mathcal{F}_{T-}]].\end{aligned}$$

Since this holds for all  $Y$ , then  $\Delta A_T = \mathbb{E}[\Delta A_T | \mathcal{F}_{T-}]$  is  $\mathcal{F}_{T-}$  measurable.  $\square$

We now define the dual optional projection and the dual predictable projection of an increasing process. Given a right-continuous increasing, not necessarily adapted process  $A_t$  with  $A_0 = 0$ , a.s., define  $\mu_o$  by

$$\mu_o(X) = \mu_A({}^o X) \tag{16.11}$$

for bounded  $\mathcal{H}$  measurable  $X$ . Exercise 16.11 asks you to prove that  $\mu_o$  is a measure. Clearly  $\mu_o(^oX) = \mu_A(^o(^oX)) = \mu_A(^oX) = \mu_o(X)$ . By Theorem 16.17, we see that  ${}^oX \geq 0$  if  $X \geq 0$ , hence  $\mu_o$  is a positive measure. If  $X = 0$ , then  ${}^oX = 0$ , so  $\mu_o(X) = \mu_A({}^oX) = 0$ . Therefore by Theorems 16.24 and 16.25,  $\mu_o$  corresponds to an optional increasing process  $A^o$ , called the *dual optional projection* of  $A$ .

The dual optional projection is used in excursion theory. More commonly used is the *dual predictable projection*, which is defined in a very similar way. Define  $\mu_p(X) = \mu_A({}^pX)$ , and let  $A^p$  be the predictable increasing process associated with  $\mu_p$ . We often denote  $A^p$  by  $\tilde{A}$  and call it the *compensator* of  $A$ . The reason for this terminology is the following proposition.

**Proposition 16.27** *Let  $A_t$  be an adapted increasing process with  $A_0 = 0$ , a.s. Then  $A_t - \tilde{A}_t$  is a martingale.*

*Proof* Let  $s < t$ , let  $B \in \mathcal{F}_s$ , define

$$S(\omega) = \begin{cases} s, & \omega \in B, \\ \infty, & \omega \notin B, \end{cases} \quad \text{and} \quad T(\omega) = \begin{cases} t, & \omega \in B, \\ \infty, & \omega \notin B. \end{cases}$$

Let  $X = 1_{(S,T]}$ . Then

$$\mathbb{E}[A_t - A_s; B] = \mu_A(X) = \mu_A({}^pX) = \mu_{A^p}(X) = \mathbb{E}[A_t^p - A_s^p; B],$$

which does it.  $\square$

## 16.7 The Doob–Meyer decomposition

**Proposition 16.28** *If  $M$  is a predictable uniformly integrable martingale with paths that are right continuous with left limits, then  $M$  is continuous.*

*Proof* Let  $\varepsilon > 0$  and let  $T = \inf\{t : |\Delta M_t| > \varepsilon\}$ .  $T$  is a predictable stopping time by Exercise 16.2. By Corollary 16.20,  $\mathbb{E}[M_T | \mathcal{F}_{T-}] = M_{T-}$ . By the definition of  $\mathcal{F}_{T-}$  and a limit argument,  $M_T$  is  $\mathcal{F}_{T-}$  measurable, and thus  $\mathbb{E}[M_T | \mathcal{F}_{T-}] = M_T$ . Hence  $M_T = M_{T-}$  at all predictable stopping times, and in particular at time  $T$ . But  $\varepsilon$  is arbitrary, so  $M$  has no jumps.  $\square$

We say a process  $X$  is of *class D* if the family  $\{X_T : T \text{ a stopping time}\}$  is uniformly integrable. The *Doob–Meyer decomposition* is the following. If  $Z_t$  is a supermartingale, then  $-Z_t$  is a submartingale, and it is a matter only of convenience whether we state the Doob–Meyer decomposition in terms of submartingales or supermartingales.

**Theorem 16.29** *Suppose  $Z_t$  is a submartingale of class D with paths that are right continuous with left limits and such that  $Z_0 = 0$ , a.s. Then  $Z_t = M_t + A_t$ , where  $M_t$  is a uniformly integrable right-continuous martingale with  $M_0 = 0$ , a.s., and  $A_t$  is a predictable increasing process with  $A_0 = 0$ , a.s. The decomposition is unique.*

The existence is the hard part. We define a measure  $\mu$  by  $\mu((S, T]) = E[Z_T - Z_S]$  for stopping times  $S \leq T$ , and then let  $A$  be the increasing process such that  $\mu_A(X) = \mu({}^pX)$ .

*Proof* We start with uniqueness. If  $Z_t = M_t + A_t = N_t + B_t$ , then  $M_t - N_t = B_t - A_t$ , and so  $M_t - N_t$  is a predictable uniformly integrable martingale. By Proposition 16.28,  $M_t - N_t$  is

a continuous martingale. Since  $M_t - N_t = B_t - A_t$ , then  $M_t - N_t$  is a continuous martingale whose paths are of bounded variation on each finite time interval, hence  $M_t - N_t = 0$  by Theorem 9.7. This proves uniqueness.

We turn to existence. By the martingale convergence theorem (Theorem 3.12),  $Z_\infty = \lim_{t \rightarrow \infty} Z_t$  exists, a.s. By Fatou's lemma,  $\mathbb{E}[|Z_\infty|] < \infty$ .

Let  $\mathcal{I}$  denote the collection of finite unions of subsets of  $[0, \infty) \times \Omega$  of the form  $(S, T]$ , where  $S \leq T$  are stopping times. Define  $\mu((S, T]) = \mathbb{E}[Z_T - Z_S]$ . Since  $Z$  is a submartingale, then  $\mu$  is non-negative. We note that  $\mathcal{I}$  is an algebra and that  $\mu$  is finitely additive on  $\mathcal{I}$ .

If  $K = (S_1, T_1] \cup \dots \cup (S_n, T_n]$  with  $S_1 \leq T_1 \leq S_2 \leq \dots \leq T_n$ , set  $\bar{K} = [S_1, T_1] \cup \dots \cup [S_n, T_n]$ .

If  $H = (S, T]$  and  $\varepsilon > 0$ , let

$$S_n(\omega) = \begin{cases} S(\omega) + (1/n), & S(\omega) + (1/n) < T(\omega), \\ \infty, & \text{otherwise,} \end{cases}$$

and

$$T_n(\omega) = \begin{cases} T(\omega), & S(\omega) + (1/n) < T(\omega), \\ \infty, & \text{otherwise.} \end{cases}$$

Then  $[S_n, T_n] \subset (S, T]$  and  $S_n \downarrow S$ ,  $T_n \downarrow T$ . Since  $Z$  is right continuous and of class  $D$ , then  $\mu(S_n, T_n) = \mathbb{E}[Z_{T_n} - Z_{S_n}] \rightarrow \mathbb{E}[Z_T - Z_S] = \mu(H)$ . Thus if  $n$  is sufficiently large and we take  $K = (S_n, T_n]$ , then  $\bar{K} \subset H$  and  $\mu(K) > \mu(H) - \varepsilon$ .

We now prove that  $\mu$  is countably additive on  $\mathcal{I}$ . Suppose  $H_n \in \mathcal{I}$  with  $H_n \downarrow \emptyset$ . We need to show that  $\mu(H_n) \downarrow 0$ .

Let  $\varepsilon > 0$  and choose  $K_n \in \mathcal{I}$  such that  $\bar{K}_n \subset H_n$  with  $\mu(K_n) > \mu(H_n) - \varepsilon/2^n$ . Let  $L_n = \bar{K}_1 \cap \dots \cap \bar{K}_n$ . Then for each  $n$  we have  $\mu(H_n) \leq \mu(L_n) + \varepsilon$ . Since  $L_n \subset \bar{K}_n \subset H_n$ , we have  $L_n \downarrow \emptyset$ .

Let  $D_{L_n}$  be the debut of  $L_n$ . The stopping times  $D_{L_n}$  increase; let  $R$  be the limit. Let  $F_n = F_n(\omega) = \{t : (t, \omega) \in L_n\}$ . This is a closed subset of  $[0, \infty)$ , and  $D_{L_n}(\omega) \in F_n \subset F_m$  whenever  $n \geq m$  and  $D_{L_n}(\omega) < \infty$ . If  $R(\omega) < \infty$ , then  $R(\omega) \in F_m$  for each  $m$ , which contradicts  $\cap_m F_m = \emptyset$ . Therefore  $R = \infty$ . Since  $Z$  is of class  $D$ , then  $Z_{D_{L_n}}$  converges almost surely and in  $L^1$  to  $Z_\infty$ . Thus  $\mu(L_n) \leq \mathbb{E}[Z_\infty - Z_{D_{L_n}}] \rightarrow 0$ . Hence  $\limsup \mu(H_n) < \varepsilon$ , and since  $\varepsilon$  is arbitrary,  $\mu(H_n) \rightarrow 0$ .

This proves that  $\mu$  is countably additive on  $\mathcal{I}$ . By the Carathéodory extension theorem,  $\mu$  may be extended to a measure on  $\mathcal{P}$ .

Define  $\tilde{\mu}(X) = \mu({}^p X)$ . Then  $\tilde{\mu}({}^p X) = \mu({}^p ({}^p X)) = \mu({}^p X) = \tilde{\mu}(X)$ , and so there exists a predictable right-continuous increasing process  $A_t$  such that  $\tilde{\mu} = \mu_A$ . Since

$$\mathbb{E} A_\infty = \mu_A(1_{(0, \infty)}) = \mu({}^p 1_{(0, \infty)}) = \mu(1_{(0, \infty)}) = \mathbb{E}[Z_\infty - Z_0] < \infty,$$

$A_\infty$  is integrable, and since  $A_t$  is an increasing process, the collection of random variables  $\{A_t\}$  is uniformly integrable.

If  $S$  is any stopping time, then by Proposition 16.2,  $(S, \infty)$  is a predictable set, hence  ${}^p 1_{(S, \infty)} = 1_{(S, \infty)}$ . We thus have

$$\mathbb{E}[A_\infty - A_S] = \tilde{\mu}((S, \infty)) = \mu({}^p 1_{(S, \infty)}) = \mu(1_{(S, \infty)}) = \mathbb{E}[Z_\infty - Z_S].$$

Letting  $t > 0$  and  $B \in \mathcal{F}_t$ , define  $S = t$  if  $\omega \in B$  and equal to infinity otherwise. Then

$$\mathbb{E}[A_\infty - A_t; B] = \mathbb{E}[A_\infty - A_S] = \mathbb{E}[Z_\infty - Z_S] = \mathbb{E}[Z_\infty - Z_t; B],$$

or  $M_t = Z_t - A_t$  is a martingale. Proposition A.17 tells us that  $M$  is a uniformly integrable martingale.  $\square$

A process  $X$  is of *class DL* if there exist stopping times  $V_n \rightarrow \infty$  such that  $X_{t \wedge V_n}$  is of class  $D$  for each  $n$ . It is clear that there is a version of the Doob–Meyer decomposition for submartingales of class  $DL$ .

**Proposition 16.30** *The process  $A$  is continuous if and only if  $\mathbb{E} Z_{T_n} \rightarrow \mathbb{E} Z_T$  whenever  $T_n \uparrow T$  and  $T_n < T$  on  $(T > 0)$ .*

*Proof* Let  $T$  be a predictable stopping time predicted by the sequence  $T_n$ . Since we know  $\mathbb{E}[A_\infty - A_{T_n}] = \mathbb{E}[Z_\infty - Z_{T_n}]$ , then taking limits,

$$\mathbb{E}[A_\infty - A_{T_-}; T < \infty] = \mathbb{E}[Z_\infty - Z_{T_-}; T < \infty],$$

using the fact that  $Z$  is of class  $D$ . Also  $\mathbb{E}[A_\infty - A_T] = \mathbb{E}[Z_\infty - Z_T]$ . Thus  $\mathbb{E}[A_T - A_{T_-}] = \mathbb{E}[Z_T - Z_{T_-}]$ . Then  $\mathbb{E}[A_T - A_{T_-}] = 0$  if and only if  $\mathbb{E} Z_T = \mathbb{E} Z_{T_-}$ .  $\square$

**Corollary 16.31** *Let  $S$  be a totally inaccessible stopping time,  $Y$  a non-negative bounded random variable that is  $\mathcal{F}_S$  measurable, and  $A_t = Y1_{(t \geq S)}$ . Let  $\tilde{A}$  be the compensator of  $A$ . Then  $\tilde{A}$  has continuous paths.*

*Proof* Let  $T$  be a stopping time and let  $T_n$  be stopping times increasing to  $T$ . If we have  $\mathbb{P}(T = S) = 0$ , then  $\lim_{n \rightarrow \infty} A_{T_n} = A_T$ , a.s., since  $A$  jumps only at time  $S$ . If  $\mathbb{P}(T = S) > 0$ , then  $[T, T]$  cannot contain the graph of a predictable stopping time since  $S$  is totally inaccessible. Therefore we cannot have  $T_n < T$  for all  $n$  with positive probability, hence  $T_n(\omega) = T(\omega)$  for all  $n$  sufficiently large (depending on  $\omega$ ). Thus again  $\lim_{n \rightarrow \infty} A_{T_n} = A_T$ , a.s. By Proposition 16.30,  $\tilde{A}$  is continuous.  $\square$

## 16.8 Two inequalities

**Proposition 16.32** *Suppose  $Z_t = M_t - A_t$ , where  $M_t$  is a uniformly integrable martingale and  $A_t$  is an increasing predictable process with  $A_0 = 0$ , a.s. Suppose  $Z$  is bounded, that is, there exists  $K > 0$  such that  $\mathbb{P}(|Z_t| > K \text{ for some } t) = 0$ . If  $p$  is any positive integer;*

$$\mathbb{E} A_\infty^p < \infty.$$

*Proof* Let  $\lambda > 0$  and let  $M = 4K$ . Let  $T = \inf\{t : A_t \geq \lambda\}$ . Because  $A_{T_-} \leq \lambda$ ,

$$\begin{aligned} \mathbb{P}(A_\infty \geq \lambda + M) &= \mathbb{P}(A_\infty \geq \lambda + M, T < \infty) \\ &\leq \mathbb{P}(A_\infty - A_{T_-} \geq M, T < \infty) \\ &\leq \mathbb{E}\left[\frac{A_\infty - A_{T_-}}{M}; A_\infty - A_{T_-} \geq M, T < \infty\right] \\ &\leq \frac{1}{M} \mathbb{E}[A_\infty - A_{T_-}; T < \infty]. \end{aligned}$$

We will show

$$\frac{1}{M} \mathbb{E}[A_\infty - A_{T-}; T < \infty] \leq \frac{1}{2} \mathbb{P}(T < \infty), \quad (16.12)$$

which, since  $\mathbb{P}(T < \infty) = \mathbb{P}(A_\infty \geq \lambda)$ , implies

$$\mathbb{P}(A_\infty \geq \lambda + M) \leq \frac{1}{2} \mathbb{P}(A_\infty \geq \lambda). \quad (16.13)$$

Taking  $\lambda = kM$  in (16.13) yields

$$\mathbb{P}(A_\infty \geq (k+1)M) \leq \frac{1}{2} \mathbb{P}(A_\infty \geq kM).$$

Since  $\mathbb{P}(A_\infty \geq M) \leq 1$ , induction tells us

$$\mathbb{P}(A_\infty \geq kM) \leq \frac{1}{2^{k-1}},$$

which implies our conclusion.

Therefore we need to prove (16.12).  $T$  is a predictable stopping time by Proposition 16.16. Let  $T_n$  be stopping times with  $T_n \uparrow T$  and  $T_n < T$  on ( $T > 0$ ). Let  $n$  be fixed for the moment and let  $N > 0$ . If  $j > n$ ,

$$\begin{aligned} \mathbb{E}[A_\infty - A_{T_j}; T_n < N] &= \mathbb{E}[\mathbb{E}[A_\infty - A_{T_j} | \mathcal{F}_{T_j}]; T_n < N] \\ &= -\mathbb{E}[\mathbb{E}[Z_\infty - Z_{T_j} | \mathcal{F}_{T_j}]; T_n < N] \\ &\leq 2K\mathbb{P}(T_n < N) \end{aligned}$$

since  $Z_t + A_t$  is a martingale,  $(T_n < N) \in \mathcal{F}_{T_n} \subset \mathcal{F}_{T_j}$ , and  $|Z|$  is bounded by  $K$ . Letting  $j \rightarrow \infty$  and using Fatou's lemma, we get

$$\mathbb{E}[A_\infty - A_{T-}; T_n < N] \leq 2K\mathbb{P}(T_n < N).$$

Letting  $n \rightarrow \infty$ , by Fatou's lemma again,

$$\mathbb{E}[A_\infty - A_{T-}; T < N] \leq 2K\mathbb{P}(T < N).$$

Finally, letting  $N \rightarrow \infty$ , by monotone convergence,

$$\mathbb{E}[A_\infty - A_{T-}; T < \infty] \leq 2K\mathbb{P}(T < \infty).$$

By our choice of  $M$ , this gives (16.12). □

For use in the reduction theorem in Chapter 17, we will need a variation of the preceding proposition.

**Proposition 16.33** *Let  $U$  be a stopping time,  $Y$  a non-negative integrable random variable that is  $\mathcal{F}_U$  measurable. Let  $N_t$  be the right-continuous version of  $\mathbb{E}[Y | \mathcal{F}_t]$ . Suppose there exists  $K > 0$  such that  $N_t \leq K$  if  $t < U$ . Let  $Z_t = Y1_{(t \geq U)}$ , which is an increasing process, and let  $A_t$  be its compensator. If  $p$  is a positive integer, then  $\mathbb{E}A_\infty^p < \infty$ .*

*Proof* As in the proof of Proposition 16.32, it suffices to show

$$\mathbb{E}[A_\infty - A_{T-}; T < \infty] \leq K\mathbb{P}(T < \infty), \quad (16.14)$$

where  $\lambda > 0$  and  $T = \inf\{t : A_t \geq \lambda\}$ . Since  $A$  is a predictable process, then  $T$  is a predictable stopping time by Proposition 16.16. Let  $T_n$  be stopping times predicting  $T$ .

Let  $N, n \geq 1$ . If  $j > n$ , then  $(T_n < N) \in \mathcal{F}_{T_n} \subset \mathcal{F}_{T_j}$  and

$$\mathbb{E}[A_\infty - A_{T_j}; T_n < N] = \mathbb{E}[Z_\infty - Z_{T_j}; T_n < N]. \quad (16.15)$$

We observe that  $Z_\infty - Z_{T_j} = 0$  on the event  $(T_j \geq U)$ , while  $Z_\infty - Z_{T_j} = Y$  on the event  $(T_j < U)$ . Therefore

$$\begin{aligned} \mathbb{E}[Z_\infty - Z_{T_j}; T_n < N] &= \mathbb{E}[Y; T_j < U, T_n < N] \\ &= \mathbb{E}[\mathbb{E}[Y | \mathcal{F}_{T_j}]; T_j < U, T_n < N] \\ &= \mathbb{E}[N_{T_j}; T_j < U, T_n < N] \\ &\leq K\mathbb{P}(T_j < U, T_n < N) \\ &\leq K\mathbb{P}(T_n < N). \end{aligned}$$

With this and (16.15), we can now proceed as in the proof of Proposition 16.32 to obtain (16.14).  $\square$

### Exercises

- 16.1 Show that if  $S_1, \dots, S_n$  are predictable stopping times, then so are  $S_1 \wedge \dots \wedge S_n$  and  $S_1 \vee \dots \vee S_n$ .
- 16.2 If  $A_t$  is a predictable process with paths that are right continuous with left limits and  $a > 0$ , show  $T = \inf\{t > 0 : \Delta A_t > a\}$  is a predictable stopping time.
- 16.3 Show that if  $T$  is a predictable stopping time,  $B \in \mathcal{F}_{T-}$ , and  $T_B(\omega)$  is defined to be equal to  $T(\omega)$  if  $\omega \in B$  and equal to  $\infty$  otherwise, then  $T_B$  is a predictable stopping time.
- 16.4 Let  $X$  be a bounded adapted right-continuous process, let  $\varepsilon > 0$ , let  $U_0 = 0$ , a.s., and define  $U_i$  by (16.3) for  $i \geq 1$ . Show each  $U_i$  is a stopping time.
- 16.5 Let  $A$  be a predictable increasing process and let  $S_k$  be the  $k$ th time  $A$  jumps more than  $\varepsilon$ . Thus  $S_0 = 0$ , a.s., and  $S_{k+1} = \inf\{t > S_k : \Delta A_t > \varepsilon\}$ . Show each  $S_k$  is a predictable stopping time.
- 16.6 Show that the stopping times  $T_n$  defined in the proof of Proposition 16.23 are predictable.
- 16.7 Show that if  $P_t$  is a Poisson process, then  $(^p P)_t = P_{t-}$ .
- 16.8 Show that if  $X$  is bounded and measurable with respect to the product  $\sigma$ -field  $\mathcal{B}[0, \infty) \times \mathcal{F}_\infty$ , then  ${}^p(^o X) = {}^p X$ .
- 16.9 Suppose  $T$  is a totally inaccessible stopping time. Show that if  $X = 1_{[T, T]}$ , then  ${}^p X = 0$ .
- 16.10 If  $P$  is a Poisson process with parameter  $\lambda$ , determine  $P_t^o$  and  $P_t^p$ .
- 16.11 Show that  $\mu_o$  defined in (16.11) is a measure.
- 16.12 Let  $X_t$  be a continuous process and suppose there exists  $K > 0$  such that for all  $t$ ,

$$\mathbb{E}[|X_\infty - X_t| | \mathcal{F}_t] \leq K, \quad \text{a.s.}$$

Let  $X_\infty^* = \sup_{t \geq 0} |X_t|$ . Prove that there exists  $a$  depending only on  $K$  such that

$$\mathbb{E} e^{aX_\infty^*} < \infty.$$

This is sometimes called the *John–Nirenberg inequality* after the inequality of the same name in analysis.

*Hint:* Imitate the proof of Proposition 16.32. This exercise is somewhat easier than the proof of that proposition because  $X$  has continuous paths.

16.13 A martingale  $M$  is said to be in the space  $BMO$  if

$$\sup_{t \geq 0} \mathbb{E} [M_\infty^2 - M_t^2 \mid \mathcal{F}_t] < \infty, \quad \text{a.s.}$$

Let  $M_t^* = \sup_{s \leq t} |M_s|$ . Show that if  $M$  is in  $BMO$ , then there exists  $a > 0$  such that

$$\mathbb{E} e^{aM_\infty^*} < \infty.$$

The name  $BMO$  comes from the “bounded mean oscillation” spaces of harmonic analysis.

*Hint:* Use Exercise 16.12.

## Notes

A progressively measurable set is one whose indicator is a progressively measurable process, which is defined in Exercise 1.3. In fact, the debut of a progressively measurable set is a stopping time; see [Dellacherie and Meyer \(1978\)](#).

An elementary proof of the general Doob–Meyer theorem along the lines of the proof given in Chapter 9 can be found in [Bass \(1996\)](#).

See [Dellacherie and Meyer \(1978\)](#) for more on the general theory of processes.

## Processes with jumps

In this chapter we investigate the stochastic calculus for processes which may have jumps as well as a continuous component. If  $X$  is not a continuous process, it is no longer true that  $X_{t \wedge T_N}$  is a bounded process when  $T_N = \inf\{t : |X_t| \geq N\}$ , since there could be a large jump at time  $T_N$ . We investigate stochastic integrals with respect to square integrable (not necessarily continuous) martingales, Itô's formula, and the Girsanov transformation. We prove the reduction theorem that allows us to look at semimartingales that are not necessarily bounded.

Since I encouraged you to skim Chapter 16 on the first reading of this book, it is only fair that I tell you the facts that we will need from that chapter. We will need the Doob–Meyer decomposition (Theorem 16.29), Proposition 16.1, Corollaries 16.21 and 16.31, and the two inequalities in Propositions 16.32 and 16.33.

### 17.1 Decomposition of martingales

We assume throughout this chapter that  $\{\mathcal{F}_t\}$  is a filtration satisfying the usual conditions. This means that each  $\mathcal{F}_t$  contains every  $\mathbb{P}$ -null set and  $\cap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$  for each  $t$ .

Let us begin by recalling a few definitions and facts. The *predictable  $\sigma$ -field* is the  $\sigma$ -field of subsets of  $[0, \infty) \times \Omega$  generated by the collection of bounded, left-continuous processes that are adapted to  $\{\mathcal{F}_t\}$ ; see Section 10.1. A stopping time  $T$  is *predictable* and predicted by the sequence of stopping times  $T_n$  if  $T_n \uparrow T$ , and  $T_n < T$  on the event  $(T > 0)$ . A stopping time  $T$  is *totally inaccessible* if  $\mathbb{P}(T = S) = 0$  for every predictable stopping time  $S$ . The graph of a stopping time  $T$  is  $[T, T] = \{(t, \omega) : t = T(\omega) < \infty\}$ ; see Section 16.1. If  $X_t$  is a process that is right continuous with left limits, we set  $X_{t-} = \lim_{s \rightarrow t, s < t} X_s$  and  $\Delta X_t = X_t - X_{t-}$ . Thus  $\Delta X_t$  is the size of the jump of  $X_t$  at time  $t$ .

Suppose  $A_t$  is a bounded increasing process whose paths are right continuous with left limits. Recall that a function  $f$  is increasing if  $s < t$  implies  $f(s) \leq f(t)$ . Then trivially  $A_t$  is a submartingale, and by the Doob–Meyer decomposition, Theorem 16.28, there exists a predictable increasing process  $\tilde{A}_t$  such that  $A_t - \tilde{A}_t$  is a martingale. We call  $\tilde{A}_t$  the *compensator* of  $A_t$ .

If  $A_t = B_t - C_t$  is the difference of two increasing processes  $B_t$  and  $C_t$ , then we can use linearity to define  $\tilde{A}_t$  as  $\tilde{B}_t - \tilde{C}_t$ . We can even extend the notion of compensator to the case where  $A_t$  is complex valued and has paths that are locally of bounded variation by looking at the real and imaginary parts.

We will use the following lemma.

**Lemma 17.1** *If  $A_t = B_t - C_t$ , where  $B_t$  and  $C_t$  are increasing right-continuous processes with  $B_0 = C_0 = 0$ , a.s., and in addition  $B$  and  $C$  are bounded, then*

$$\mathbb{E} \sup_{t \geq 0} \tilde{A}_t^2 < \infty.$$

*Proof* By Proposition 16.32,  $\mathbb{E} \tilde{B}_\infty^2 < \infty$  and  $\mathbb{E} \tilde{C}_\infty^2 < \infty$ , and so

$$\mathbb{E} \sup_{t \geq 0} \tilde{A}_t^2 \leq \mathbb{E} [2 \sup_{t \geq 0} \tilde{B}_t^2 + 2 \sup_{t \geq 0} \tilde{C}_t^2] \leq 2\mathbb{E} \tilde{B}_\infty^2 + 2\mathbb{E} \tilde{C}_\infty^2 < \infty.$$

We are done.  $\square$

A key result is the following *orthogonality lemma*.

**Lemma 17.2** *Suppose  $A_t$  is a bounded increasing right-continuous process with  $A_0 = 0$ , a.s.,  $\tilde{A}_t$  is the compensator of  $A$ , and  $M_t = A_t - \tilde{A}_t$ . Suppose  $N_t$  is a right continuous square integrable martingale such that  $(\Delta N_t)(\Delta M_t) = 0$  for all  $t$ . Then  $\mathbb{E} M_\infty N_\infty = 0$ .*

*Proof* By Lemma 17.1,  $M$  is square integrable. Suppose

$$H(s, \omega) = K(\omega)1_{(a,b]}(s)$$

with  $K$  being  $\mathcal{F}_a$  measurable. Since  $M_t$  is of bounded variation, we have (this is a Lebesgue–Stieltjes integral here)

$$\mathbb{E} \int_0^\infty H_s dM_s = \mathbb{E}[K(M_b - M_a)] = \mathbb{E}[K\mathbb{E}[M_b - M_a | \mathcal{F}_a]] = 0.$$

We saw in Lemma 10.1 that linear combinations of such  $H$ 's generate the predictable  $\sigma$ -field. Thus by linearity and taking limits,  $\mathbb{E} \int_0^\infty H_s dM_s = 0$  if  $H_s$  is a predictable process such that  $\mathbb{E} \int_0^\infty |H_s| |dM_s| < \infty$ . In particular, since  $N_{s-}$  is left continuous and hence predictable,  $\mathbb{E} \int_0^\infty N_{s-} dM_s = 0$ , provided we check integrability:

$$\begin{aligned} \mathbb{E} \left| \int_0^\infty |N_{s-}| |dM_s| \right| &\leq \mathbb{E} \int_0^\infty (\sup_r |N_r|) |dM_s| \\ &= \mathbb{E}[(\sup_r |N_r|) (A_\infty + \tilde{A}_\infty)] < \infty \end{aligned}$$

by the Cauchy–Schwarz inequality.

By hypothesis,  $\mathbb{E} \int_0^\infty \Delta N_s dM_s = 0$ , so  $\mathbb{E} \int_0^\infty N_s dM_s = 0$ . On the other hand, using Proposition 3.14, we see

$$\mathbb{E} M_\infty N_\infty = \mathbb{E} \int_0^\infty N_\infty dM_s = \mathbb{E} \int_0^\infty N_s dM_s = 0.$$

The proof is complete.  $\square$

If we apply the above to  $N_{t \wedge T}$ , we have  $\mathbb{E} M_\infty N_T = 0$ . If we then condition on  $\mathcal{F}_T$ ,

$$\mathbb{E}[M_T N_T] = \mathbb{E}[N_T \mathbb{E}[M_\infty | \mathcal{F}_T]] = \mathbb{E}[N_T M_\infty] = 0. \quad (17.1)$$

The reason for the name ‘‘orthogonality lemma’’ is that by (17.1) and Proposition 9.5,  $M_t N_t$  is a martingale. This implies that  $\langle M, N \rangle_t$  (which we will define soon, and is defined similarly to the case of continuous martingales) is identically equal to 0.

Let  $M_t$  be a square integrable martingale with paths that are right continuous and left limits, so that  $\mathbb{E} M_\infty^2 < \infty$ . For each  $i \in \mathbb{Z}$ , let  $T_{i1} = \inf\{t : |\Delta M_t| \in [2^i, 2^{i+1})\}$ ,  $T_{i2} = \inf\{t > T_{i1} : |\Delta M_t| \in [2^i, 2^{i+1})\}$ , and so on;  $i$  can be both positive and negative. Since  $M_t$  is right continuous with left limits, for each  $i$ ,  $T_{ij} \rightarrow \infty$  as  $j \rightarrow \infty$ . We conclude that  $M_t$  has at most countably many jumps. Next we decompose each  $T_{ij}$  into predictable and totally inaccessible parts by Proposition 16.1. We relabel the jump times as  $S_1, S_2, \dots$  so that each  $S_k$  is either predictable or totally inaccessible, the graphs of the  $S_k$  are disjoint,  $M$  has a jump at each time  $S_k$  and only at these times, and  $|\Delta M_{S_k}|$  is bounded for each  $k$ ; of the proof of Proposition 16.23. We do not assume that  $S_{k_1} \leq S_{k_2}$  if  $k_1 \leq k_2$ , and in general it would not be possible to arrange this.

If  $S_i$  is a totally inaccessible stopping time, let

$$A_i(t) = \Delta M_{S_i} 1_{(t \geq S_i)} \quad (17.2)$$

and

$$M_i(t) = A_i(t) - \tilde{A}_i(t), \quad (17.3)$$

where  $\tilde{A}_i$  is the compensator of  $A_i$ .  $A_i(t)$  is the process that is 0 up to time  $S_i$  and then jumps an amount  $\Delta M_{S_i}$ ; thereafter it is constant. By Corollary 16.31,  $\tilde{A}$  is continuous. If  $S_i$  is a predictable stopping time, let

$$M_i(t) = \Delta M_{S_i} 1_{(t \geq S_i)}. \quad (17.4)$$

By Corollary 16.21,  $M_i$  is a martingale. Note that in either case,  $M - M_i$  has no jump at time  $S_i$ .

**Theorem 17.3** Suppose  $M$  is a square integrable martingale and we define  $M_i$  as in (17.3) and (17.4).

- (1) Each  $M_t$  is square integrable.
- (2)  $\sum_{i=1}^{\infty} M_i(\infty)$  converges in  $L^2$ .
- (3) If  $M_t^c = M_t - \sum_{i=1}^{\infty} M_i(t)$ , then  $M^c$  is square integrable and we can find a version that has continuous paths.
- (4) For each  $i$  and each stopping time  $T$ ,  $\mathbb{E}[M_T^c M_i(T)] = 0$ .

*Proof* (1) If  $S_i$  is a totally inaccessible stopping time and we let  $B_t = (\Delta M_{S_i})^+ 1_{(t \geq S_i)}$  and  $C_t = (\Delta M_{S_i})^- 1_{(t \geq S_i)}$ , then (1) follows by Lemma 17.1. If  $S_i$  is predictable, (1) follows by Corollary 16.21.

(2) Let  $V_n(t) = \sum_{i=1}^n M_i(t)$ . By the orthogonality lemma (Lemma 17.2),  $\mathbb{E}[M_i(\infty) M_j(\infty)] = 0$  if  $i \neq j$  and  $\mathbb{E}[M_i(\infty)(M_\infty - V_n(\infty))] = 0$  if  $i \leq n$ . We thus

have

$$\begin{aligned} \sum_{i=1}^n \mathbb{E} M_i(\infty)^2 &= \mathbb{E} V_n(\infty)^2 \\ &\leq \mathbb{E} \left[ M_\infty - V_n(\infty) \right]^2 + \mathbb{E} V_n(\infty)^2 \\ &= \mathbb{E} \left[ M_\infty - V_n(\infty) + V_n(\infty) \right]^2 \\ &= \mathbb{E} M_\infty^2 < \infty. \end{aligned}$$

Therefore the series  $\mathbb{E} \sum_{i=1}^n M_i(\infty)^2$  converges. If  $n > m$ ,

$$\mathbb{E} [(V_n(\infty) - V_m(\infty))^2] = \mathbb{E} \left[ \sum_{i=m+1}^n M_i(\infty) \right]^2 = \sum_{i=m+1}^n \mathbb{E} M_i(\infty)^2.$$

This tends to 0 as  $n, m \rightarrow \infty$ , so  $V_n(\infty)$  is a Cauchy sequence in  $L^2$ , and hence converges.

(3) From (2), Doob's inequalities, and the completeness of  $L^2$ , the random variables  $\sup_{t \geq 0} [M_t - V_n(t)]$  converge in  $L^2$  as  $n \rightarrow \infty$ . Let  $M_t^c = \lim_{n \rightarrow \infty} [M_t - V_n(t)]$ . There is a sequence  $n_k$  such that

$$\sup_{t \geq 0} |(M_t - V_{n_k}(t)) - M_t^c| \rightarrow 0, \quad \text{a.s.}$$

We conclude that the paths of  $M_t^c$  are right continuous with left limits. By the construction of the  $M_i$ ,  $M - V_{n_k}$  has jumps only at times  $S_i$  for  $i > n_k$ . We therefore see that  $M^c$  has no jumps, i.e., it is continuous.

(4) By the orthogonality lemma and (17.1),

$$\mathbb{E} [M_i(T)(M_T - V_n(T))] = 0$$

if  $T$  is a stopping time and  $i \leq n$ . Letting  $n$  tend to infinity proves (4).  $\square$

## 17.2 Stochastic integrals

If  $M_t$  is a square integrable martingale, then  $M_t^2$  is a submartingale by Jensen's inequality for conditional expectations. Just as in the case of continuous martingales, we can use the Doob–Meyer decomposition (this time, we use Theorem 16.29 instead of Theorem 9.12) to find a predictable increasing process starting at 0, denoted  $\langle M \rangle_t$ , such that  $M_t^2 - \langle M \rangle_t$  is a martingale.

Let us define

$$[M]_t = \langle M^c \rangle_t + \sum_{s \leq t} |\Delta M_s|^2. \quad (17.5)$$

Here  $M^c$  is the continuous part of the martingale  $M$  as defined in Theorem 17.3. As an example, if  $M_t = P_t - t$ , where  $P_t$  is a Poisson process with parameter 1, then  $M_t^c = 0$  and

$$[M]_t = \sum_{s \leq t} \Delta P_s^2 = \sum_{s \leq t} \Delta P_s = P_t,$$

because all the jumps of  $P_t$  are of size one. In this case  $\langle M \rangle_t = t$ ; this follows from Proposition 17.4 below.

In defining stochastic integrals, one could work with  $\langle M \rangle_t$ , but the process  $[M]_t$  is the one that shows up naturally in many formulas, such as the product formula.

**Proposition 17.4**  $M_t^2 - [M]_t$  is a martingale.

*Proof* By the orthogonality lemma and (17.1) it is easy to see that

$$\langle M \rangle_t = \langle M^c \rangle_t + \sum_i \langle M_i \rangle_t.$$

Since  $M_t^2 - \langle M \rangle_t$  is a martingale, we need only show  $[M]_t - \langle M \rangle_t$  is a martingale. Since

$$[M]_t - \langle M \rangle_t = \left( \langle M^c \rangle_t + \sum_{s \leq t} |\Delta M_s|^2 \right) - \left( \langle M^c \rangle_t + \sum_i \langle M_i \rangle_t \right),$$

it suffices to show that  $\sum_i \langle M_i \rangle_t - \sum_i \sum_{s \leq t} |\Delta M_i(s)|^2$  is a martingale.

By Exercise 17.1

$$M_i(t)^2 = 2 \int_0^t M_i(s-) dM_i(s) + \sum_{s \leq t} |\Delta M_i(s)|^2, \quad (17.6)$$

where the first term on the right-hand side is a Lebesgue–Stieltjes integral. If we approximate this integral by a Riemann sum and use the fact that  $M_i$  is a martingale, we see that the first term on the right in (17.6) is a martingale. Thus  $M_i^2(t) - \sum_{s \leq t} |\Delta M_i(s)|^2$  is a martingale. Since  $M_i^2(t) - \langle M_i \rangle_t$  is a martingale, summing over  $i$  completes the proof.  $\square$

If  $H_s$  is of the form

$$H_s(\omega) = \sum_{i=1}^n K_i(\omega) 1_{(a_i, b_i]}(s), \quad (17.7)$$

where each  $K_i$  is bounded and  $\mathcal{F}_{a_i}$  measurable, define the stochastic integral by

$$N_t = \int_0^t H_s dM_s = \sum_{i=1}^n K_i [M_{b_i \wedge t} - M_{a_i \wedge t}].$$

Very similar proofs to those in Chapter 10 show that the left-hand side will be a martingale and (with  $[\cdot]$  instead of  $\langle \cdot \rangle$ ),  $N_t^2 - [N]_t$  is a martingale.

If  $H$  is  $\mathcal{P}$  measurable and  $\mathbb{E} \int_0^\infty H_s^2 d[M]_s < \infty$ , approximate  $H$  by integrands  $H_s^n$  of the form (17.7) so that

$$\mathbb{E} \int_0^\infty (H_s - H_s^n)^2 d[M]_s \rightarrow 0$$

and define  $N_t^n$  as the stochastic integral of  $H^n$  with respect to  $M_t$ . By almost the same proof as that of Theorem 10.4, the martingales  $N_t^n$  converge in  $L^2$ . We call the limit  $N_t = \int_0^t H_s dM_s$  the *stochastic integral* of  $H$  with respect to  $M$ . A subsequence of the  $N^n$  converges uniformly over  $t \geq 0$ , a.s., and therefore the limit has paths that are right continuous with left limits. The same arguments as those of Theorem 10.4 apply to prove that the stochastic integral is a martingale and

$$[N]_t = \int_0^t H_s^2 d[M]_s.$$

A consequence of this last equation is that

$$\mathbb{E} \left( \int_0^t H_s dM_s \right)^2 = \mathbb{E} \int_0^t H_s^2 d[M]_s. \quad (17.8)$$

### 17.3 Itô's formula

We will first prove Itô's formula for a special case, namely, we suppose  $X_t = M_t + A_t$ , where  $M_t$  is a square integrable martingale and  $A_t$  is a process of bounded variation whose total variation is integrable. The extension to semimartingales without the integrability conditions will be done later in the chapter (in Section 17.5) and is easy. Define  $\langle X^c \rangle_t$  to be  $\langle M^c \rangle_t$ .

**Theorem 17.5** Suppose  $X_t = M_t + A_t$ , where  $M_t$  is a square integrable martingale and  $A_t$  is a process with paths of bounded variation whose total variation is integrable. Suppose  $f$  is  $C^2$  on  $\mathbb{R}$  with bounded first and second derivatives. Then

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_{s-}) dX_s + \frac{1}{2} \int_0^t f''(X_{s-}) d\langle X^c \rangle_s \\ &\quad + \sum_{s \leq t} [f(X_s) - f(X_{s-}) - f'(X_{s-}) \Delta X_s]. \end{aligned} \quad (17.9)$$

*Proof* The proof will be given in several steps. Set

$$S(t) = \int_0^t f'(X_{s-}) dX_s, \quad Q(t) = \frac{1}{2} \int_0^t f''(X_{s-}) d\langle X^c \rangle_s,$$

and

$$J(t) = \sum_{s \leq t} [f(X_s) - f(X_{s-}) - f'(X_{s-}) \Delta X_s].$$

We use these letters as mnemonics for “stochastic integral term,” “quadratic variation term,” and “jump term,” respectively.

*Step 1.* Suppose  $X_t$  has a single jump at time  $T$  which is either a predictable stopping time or a totally inaccessible stopping time and there exists  $N > 0$  such that  $|\Delta M_T| + |\Delta A_T| \leq N$  a.s.

If  $T$  is totally inaccessible, let  $C_t = \Delta M_T 1_{(t \geq T)}$  and let  $\tilde{C}_t$  be the compensator. If we replace  $M_t$  by  $M_t - C_t + \tilde{C}_t$  and  $A_t$  by  $A_t + C_t - \tilde{C}_t$ , we may assume that  $M_t$  is continuous. If  $T$  is a predictable stopping time, replace  $M_t$  by  $M_t - \Delta M_T 1_{(t \geq T)}$  and  $A_t$  by  $A_t + \Delta M_T 1_{(t \geq T)}$ , and again we may assume  $M$  is continuous.

Let  $B_t = \Delta X_T 1_{(t \geq T)}$ . Set  $\hat{X}_t = X_t - B_t$  and  $\hat{A}_t = A_t - B_t$ . Then  $\hat{X}_t = M_t + \hat{A}_t$  and  $\hat{X}_t$  is a continuous process that agrees with  $X_t$  up to but not including time  $T$ . We have  $\hat{X}_{s-} = \hat{X}_s$  and  $\Delta \hat{X}_s = 0$  if  $s \leq T$ . By Theorem 11.1

$$\begin{aligned} f(\hat{X}_t) &= f(\hat{X}_0) + \int_0^t f'(\hat{X}_s) d\hat{X}_s + \frac{1}{2} \int_0^t f''(\hat{X}_s) d\langle M \rangle_s \\ &= f(\hat{X}_0) + \int_0^t f'(\hat{X}_{s-}) d\hat{X}_s + \frac{1}{2} \int_0^t f''(\hat{X}_{s-}) d\langle \tilde{X}^c \rangle_s \\ &\quad + \sum_{s \leq t} [f(\hat{X}_s) - f(\hat{X}_{s-}) - f'(\hat{X}_{s-}) \Delta \hat{X}_s], \end{aligned}$$

since the sum on the last line is zero. For  $t < T$ ,  $\widehat{X}_t$  agrees with  $X_t$ . At time  $T$ ,  $f(X_t)$  has a jump of size  $f(X_T) - f(X_{T-})$ . The integral with respect to  $\widehat{X}$ ,  $S(t)$ , will jump  $f'(X_{T-})\Delta X_T$ ,  $Q(t)$  does not jump at all, and  $J(t)$  jumps  $f(X_T) - f(X_{T-}) - f'(X_{T-})\Delta X_T$ . Therefore both sides of (17.9) jump the same amount at time  $T$ , and hence in this case we have (17.9) holding for  $t \leq T$ .

*Step 2.* Suppose there exist times  $T_1 < T_2 < \dots$  with  $T_n \rightarrow \infty$ , each  $T_i$  is either a totally inaccessible stopping time or a predictable stopping time, for each  $i$ , there exists  $N_i > 0$  such that  $|\Delta M_{T_i}|$  and  $|\Delta A_{T_i}|$  are bounded by  $N_i$ , and  $X_t$  is continuous except at the times  $T_1, T_2, \dots$ . Let  $T_0 = 0$ .

Fix  $i$  for the moment. Define  $X'_t = X_{(t-T_i)^+}$ , define  $A'_t$  and  $M'_t$  similarly, and apply Step 1 to  $X'$  at time  $T_i + t$ . We have for  $T_i \leq t \leq T_{i+1}$

$$\begin{aligned} f(X_t) &= f(X_{T_i}) + \int_{T_i}^t f'(X_{s-}) dX_s + \frac{1}{2} \int_{T_i}^t f''(X_{s-}) d\langle X^c \rangle_s \\ &\quad + \sum_{T_i < s \leq t} [f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s]. \end{aligned}$$

Thus for any  $t$  we have

$$\begin{aligned} f(X_{T_{i+1} \wedge t}) &= f(X_{T_i \wedge t}) + \int_{T_i \wedge t}^{T_{i+1} \wedge t} f'(X_{s-}) dX_s + \frac{1}{2} \int_{T_i \wedge t}^{T_{i+1} \wedge t} f''(X_{s-}) d\langle X^c \rangle_s \\ &\quad + \sum_{T_i \wedge t < s \leq T_{i+1} \wedge t} [f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s]. \end{aligned}$$

Summing over  $i$ , we have (17.9) for each  $t$ .

*Step 3.* We now do the general case. As in the paragraphs preceding Theorem 17.3, we can find stopping times  $S_1, S_2, \dots$  such that each jump of  $X$  occurs at one of the times  $S_i$  and so that for each  $i$ , there exists  $N_i > 0$  such that  $|\Delta M_{S_i}| + |\Delta A_{S_i}| \leq N_i$ . Moreover each  $S_i$  is either a predictable stopping time or a totally inaccessible stopping time. Let  $M$  be decomposed into  $M^c$  and  $M_i$  as in Theorem 17.3 and let

$$A_t^c = A_t - \sum_{i=1}^{\infty} \Delta A_{S_i} 1_{(t \geq S_i)}.$$

Since  $A_t$  is of bounded variation, then  $A^c$  will be finite and continuous. Define

$$M_t^n = M_t^c + \sum_{i=1}^n M_i(t)$$

and

$$A_t^n = A_t^c + \sum_{i=1}^n \Delta A_{S_i} 1_{(t \geq S_i)},$$

and let  $X_t^n = M_t^n + A_t^n$ . We already know that  $M^n$  converges uniformly over  $t \geq 0$  to  $M$  in  $L^2$ . If we let  $B_t^n = \sum_{i=1}^n (\Delta A_{S_i})^+ 1_{(t \geq S_i)}$  and  $C_t^n = \sum_{i=1}^n (\Delta A_{S_i})^- 1_{(t \geq S_i)}$  and let  $B_t = \sup_n B_t^n$ ,  $C_t = \sup_n C_t^n$ , then the fact that  $A$  has paths of bounded variation implies that with probability

one,  $B_t^n \rightarrow B_t$  and  $C_t^n \rightarrow C_t$  uniformly over  $t \geq 0$  and  $A_t = B_t - C_t$ . In particular, we have convergence in total variation norm:

$$\mathbb{E} \int_0^\infty |d(A_t^n) - A_t| \rightarrow 0.$$

We define  $S^n(t)$ ,  $Q^n(t)$ , and  $J^n(t)$  analogously to  $S(t)$ ,  $Q(t)$ , and  $J(t)$ , respectively. By applying Step 2 to  $X^n$ , we have

$$f(X_t^n) = f(X_0^n) + S^n(t) + Q^n(t) + J^n(t),$$

and we need to show convergence of each term. We now examine the various terms.

Uniformly in  $t$ ,  $X_t^n$  converges to  $X_t$  in probability, that is,

$$\mathbb{P}(\sup_{t \geq 0} |X_t^n - X_t| > \varepsilon) \rightarrow 0$$

as  $n \rightarrow \infty$  for each  $\varepsilon > 0$ . Since  $\int_0^t d\langle M^c \rangle_s < \infty$ , by dominated convergence

$$\int_0^t f''(X_{s-}^n) d\langle M^c \rangle_s \rightarrow \int_0^t f''(X_{s-}) d\langle M^c \rangle_s$$

in probability. Therefore  $Q^n(t) \rightarrow Q(t)$  in probability. Also,  $f(X_t^n) \rightarrow f(X_t)$  and  $f(X_0) \rightarrow f(X_0)$ , both in probability.

We now show  $S^n(t) \rightarrow S(t)$ . Write

$$\begin{aligned} & \int_0^t f'(X_{s-}^n) dA_s^n - \int_0^t f'(X_{s-}) dA_s \\ &= \left[ \int_0^t f'(X_{s-}^n) dA_s^n - \int_0^t f'(X_{s-}^n) dA_s \right] \\ &\quad + \left[ \int_0^t f'(X_{s-}) dA_s - \int_0^t f'(X_{s-}) dA_s \right] \\ &= I_1^n + I_2^n. \end{aligned}$$

We see that

$$|I_1^n| \leq \|f'\|_\infty \int_0^t |dA_s^n - dA_s| \rightarrow 0$$

as  $n \rightarrow \infty$ , while by dominated convergence,  $|I_2^n|$  also tends to 0.

We next look at the stochastic integral part of  $S^n(t)$ .

$$\begin{aligned} & \int_0^t f'(X_{s-}^n) dM_s^n - \int_0^t f'(X_{s-}) dM_s \\ &= \left[ \int_0^t f'(X_{s-}^n) dM_s^n - \int_0^t f'(X_{s-}) dM_s^n \right] \\ &\quad + \left[ \int_0^t f'(X_{s-}) dM_s^n - \int_0^t f'(X_{s-}) dM_s \right] \\ &= I_3^n + I_4^n. \end{aligned}$$

The  $L^2$  norm of  $I_3^n$  is bounded by

$$\mathbb{E} \int_0^t |f'(X_{s-}^n) - f'(X_{s-})|^2 d[M^n]_s \leq \mathbb{E} \int_0^t |f'(X_{s-}^n) - f'(X_{s-})|^2 d[M]_s,$$

which goes to zero by dominated convergence. Also

$$I_4^n = \int_0^t f'(X_{s-}) \sum_{i=n+1}^{\infty} dM_i(s),$$

so using the orthogonality lemma (Lemma 17.2), the  $L^2$  norm of  $I_4^n$  is less than

$$\|f'\|_{\infty}^2 \sum_{i=n+1}^{\infty} \mathbb{E}[M_i]_{\infty} \leq \|f'\|_{\infty}^2 \sum_{i=n+1}^{\infty} \mathbb{E} M_i(\infty)^2,$$

which goes to zero as  $n \rightarrow \infty$ .

Finally, we look at the convergence of  $J^n$ . The idea here is to break both  $J(t)$  and  $J^n(t)$  into two parts, the jumps that might be relatively large (jumps at times  $S_i$  for  $i \leq N$  where  $N$  will be chosen appropriately) and the remaining jumps. Let  $N > 1$  be chosen later.

$$\begin{aligned} J(t) - J^n(t) &= \sum_{s \leq t} [f(X_s) - f(X_{s-}) - f'(X_{s-}) \Delta X_s] \\ &\quad - \sum_{s \leq t} [f(X_s^n) - f(X_{s-}^n) - f'(X_{s-}^n) \Delta X_s^n] \\ &= \sum_{\{i: S_i \leq t\}} [f(X_{S_i}) - f(X_{S_i-}) - f'(X_{S_i-}) \Delta X_{S_i}] \\ &\quad - \sum_{\{i: S_i \leq t\}} [f(X_{S_i}^n) - f(X_{S_i-}^n) - f'(X_{S_i-}^n) \Delta X_{S_i}^n] \\ &= \sum_{\{i > N: S_i \leq t\}} [f(X_{S_i}) - f(X_{S_i-}) - f'(X_{S_i-}) \Delta X_{S_i}] \\ &\quad - \sum_{\{i > N: S_i \leq t\}} [f(X_{S_i}^n) - f(X_{S_i-}^n) - f'(X_{S_i-}^n) \Delta X_{S_i}^n] \\ &\quad + \sum_{\{i \leq N, S_i \leq t\}} \left\{ [f(X_{S_i}) - f(X_{S_i-}) - f'(X_{S_i-}) \Delta X_{S_i}] \right. \\ &\quad \left. - [f(X_{S_i}^n) - f(X_{S_i-}^n) - f'(X_{S_i-}^n) \Delta X_{S_i}^n] \right\} \\ &= I_5^N - I_6^{n,N} + I_7^{n,N}. \end{aligned}$$

By the fact that  $M$  and  $A$  are right continuous with left limits,  $|\Delta M_{S_i}| \leq 1/2$  and  $|\Delta A_{S_i}| \leq 1/2$  if  $i$  is large enough (depending on  $\omega$ ), and then  $|\Delta X_{S_i}| \leq 1$ , and also

$$\begin{aligned} |\Delta X_{S_i}|^2 &\leq 2|\Delta M_{S_i}|^2 + 2|\Delta A_{S_i}|^2 \\ &\leq 2|\Delta M_{S_i}|^2 + |\Delta A_{S_i}|. \end{aligned}$$

We have

$$|I_5^N| \leq \|f''\|_\infty \sum_{i>N, S_i \leq t} (\Delta X_{S_i})^2$$

and

$$|I_6^{n,N}| \leq \|f''\|_\infty \sum_{n \geq i > N, S_i \leq t} (\Delta X_{S_i})^2.$$

Since  $\sum_{i=1}^\infty |\Delta M_{S_i}|^2 \leq [M]_\infty < \infty$  and  $\sum_{i=1}^\infty |\Delta A_{S_i}| < \infty$ , then given  $\varepsilon > 0$ , we can choose  $N$  large such that

$$\mathbb{P}(|I_5^N| + |I_6^{n,N}| > \varepsilon) < \varepsilon.$$

Once we choose  $N$ , we then see that  $I_7^{n,N}$  tends to zero in probability as  $n \rightarrow \infty$ , since  $X_t^n$  converges in probability to  $X_t$  uniformly over  $t \geq 0$ . We conclude that  $J^n(t)$  converges to  $J(t)$  in probability as  $n \rightarrow \infty$ .

This completes the proof.  $\square$

## 17.4 The reduction theorem

Let  $M$  be a process adapted to  $\{\mathcal{F}_t\}$ . If there exist stopping times  $T_n$  increasing to  $\infty$  such that each process  $M_{t \wedge T_n}$  is a uniformly integrable martingale, we say  $M$  is a *local martingale*. If each  $M_{t \wedge T_n}$  is a square integrable martingale, we say  $M$  is a *locally square integrable martingale*. We say a stopping time  $T$  *reduces* a process  $M$  if  $M_{t \wedge T}$  is a uniformly integrable martingale.

**Lemma 17.6** (1) *The sum of two local martingales is a local martingale.*

(2) *If  $S$  and  $T$  both reduce  $M$ , then so does  $S \vee T$ .*

(3) *If there exist times  $T_n \rightarrow \infty$  such that  $M_{t \wedge T_n}$  is a local martingale for each  $n$ , then  $M$  is a local martingale.*

*Proof* (1) If the sequence  $S_n$  reduces  $M$  and the sequence  $T_n$  reduces  $N$ , then  $S_n \wedge T_n$  will reduce  $M + N$ .

(2)  $M_{t \wedge (S \vee T)}$  is bounded in absolute value by  $|M_{t \wedge T}| + |M_{t \wedge S}|$ . Both  $\{|M_{t \wedge T}|\}$  and  $\{|M_{t \wedge S}|\}$  are uniformly integrable families of random variables. Now use Proposition A.17.

(3) Let  $S_{nm}$  be a family of stopping times reducing  $M_{t \wedge T_n}$  and let  $S'_{nm} = S_{nm} \wedge T_n$ . Renumber the stopping times into a single sequence  $R_1, R_2, \dots$  and let  $H_k = R_1 \vee \dots \vee R_k$ . Note  $H_k \uparrow \infty$ . To show that  $H_k$  reduces  $M$ , we need to show that  $R_i$  reduces  $M$  and use (2). But  $R_i = S'_{nm}$  for some  $m, n$ , so  $M_{t \wedge R_i} = M_{t \wedge S_{nm} \wedge T_n}$  is a uniformly integrable martingale.  $\square$

Let  $M$  be a local martingale with  $M_0 = 0$ . We say that a stopping time  $T$  *strongly reduces*  $M$  if  $T$  reduces  $M$  and the martingale  $\mathbb{E}[|M_T| \mid \mathcal{F}_s]$  is bounded on  $[0, T)$ , that is, there exists  $K > 0$  such that

$$\sup_{0 \leq s < T} \mathbb{E}[|M_T| \mid \mathcal{F}_s] \leq K, \quad \text{a.s.}$$

**Lemma 17.7** (1) *If  $T$  strongly reduces  $M$  and  $S \leq T$ , then  $S$  strongly reduces  $M$ .*

(2) *If  $S$  and  $T$  strongly reduce  $M$ , then so does  $S \vee T$ .*

(3) *If  $Y_\infty$  is integrable, then  $\mathbb{E}[\mathbb{E}[Y_\infty \mid \mathcal{F}_T] \mid \mathcal{F}_S] = \mathbb{E}[Y_\infty \mid \mathcal{F}_{S \wedge T}]$ .*

*Proof* (1) Note  $\mathbb{E}[|M_S| \mid \mathcal{F}_s] \leq \mathbb{E}[|M_T| \mid \mathcal{F}_s]$  by Jensen's inequality, hence  $S$  strongly reduces  $M$ .

(2) It suffices to show that  $\mathbb{E}[|M_{S \vee T}| \mid \mathcal{F}_t]$  is bounded for  $t < T$ , since by symmetry the same will hold for  $t < S$ . For  $t < T$  this expression is bounded by

$$\mathbb{E}[|M_T| \mid \mathcal{F}_t] + \mathbb{E}[|M_S|1_{(S>T)} \mid \mathcal{F}_t].$$

The first term is bounded since  $T$  strongly reduces  $M$ . For the second term, if  $t < T$ ,

$$\begin{aligned} 1_{(t < T)} \mathbb{E}[|M_S|1_{(S>T)} \mid \mathcal{F}_t] &= \mathbb{E}[|M_S|1_{(S>T)}1_{(t < T)} \mid \mathcal{F}_t] \\ &\leq \mathbb{E}[|M_S|1_{(t < S)} \mid \mathcal{F}_t] \\ &= \mathbb{E}[|M_S| \mid \mathcal{F}_t]1_{(t < S)}, \end{aligned}$$

which in turn is bounded since  $S$  strongly reduces  $M$ .

(3) This is Exercise 3.10. □

**Lemma 17.8** *If  $M$  is a local martingale with  $M_0 = 0$ , then there exist stopping times  $T_n \uparrow \infty$  that strongly reduce  $M$ .*

*Proof* Let  $R_n \uparrow \infty$  be a sequence reducing  $M$ . Let

$$S_{nm} = R_n \wedge \inf\{t : \mathbb{E}[|M_{R_n}| \mid \mathcal{F}_t] \geq m\}.$$

Arrange the stopping times  $S_{nm}$  into a single sequence  $\{U_n\}$  and let  $T_n = U_1 \vee \dots \vee U_n$ . In view of the preceding lemmas, we need to show  $U_i$  strongly reduces  $M$ , which will follow if  $S_{nm}$  does for each  $n$  and  $m$ .

Let  $Y_t = \mathbb{E}[|M_{R_n}| \mid \mathcal{F}_t]$ , where we take a version whose paths are right continuous with left limits.  $Y$  is bounded by  $m$  on  $[0, S_{nm}]$ . By Jensen's inequality for conditional expectations and Lemma 17.7

$$\begin{aligned} \mathbb{E}[|M_{S_{nm}}|1_{(t < S_{nm})} \mid \mathcal{F}_t] &\leq \mathbb{E}[|\mathbb{E}[|M_{R_n}| \mid \mathcal{F}_{S_{nm}}]|1_{(t < S_{nm})} \mid \mathcal{F}_t] \\ &= \mathbb{E}[\mathbb{E}[|M_{R_n}|1_{(t < S_{nm})} \mid \mathcal{F}_{S_{nm}}] \mid \mathcal{F}_t] \\ &= \mathbb{E}[|M_{R_n}|1_{(t < S_{nm})} \mid \mathcal{F}_{S_{nm} \wedge t}] \\ &= Y_{S_{nm} \wedge t}1_{(t < S_{nm})} \\ &= Y_t1_{(t < S_{nm})} \leq m. \end{aligned}$$

We are done. □

Our main theorem of this section is the following.

**Theorem 17.9** *Suppose  $M$  is a local martingale. Then there exist stopping times  $T_n \uparrow \infty$  such that  $M_{t \wedge T_n} = U_t^n + V_t^n$ , where each  $U^n$  is a square integrable martingale and each  $V^n$  is a martingale whose paths are of bounded variation and such that the total variation of the paths of  $V_n$  is integrable. Moreover,  $U_t = U_T$  and  $V_t = V_T$  for  $t \geq T$ .*

The last sentence of the statement of the theorem says that  $U$  and  $V$  are both constant from time  $T$  on.

*Proof* It suffices to prove that if  $M$  is a local martingale with  $M_0 = 0$  and  $T$  strongly reduces  $M$ , then  $M_{t \wedge T}$  can be written as  $U + V$  with  $U$  and  $V$  of the described form. Thus we

may assume  $M_t = M_T$  for  $t \geq T$ ,  $|M_T|$  is integrable, and  $\mathbb{E}[|M_T| \mid \mathcal{F}_t]$  is bounded, say by  $K$ , on  $[0, T]$ .

Let  $A_t = M_T 1_{(t \geq T)} = M_t 1_{(t \geq T)}$ , let  $\tilde{A}$  be the compensator of  $A$ , let  $V = A - \tilde{A}$ , and let  $U = M - A + \tilde{A}$ . Then  $V$  is a martingale of bounded variation. We compute the expectation of the total variation of  $V$ . Let  $B_t = M_T^+ 1_{(t \geq T)}$  and  $C_t = M_T^- 1_{(t \geq T)}$ . Then the expectation of the total variation of  $A$  is bounded by  $\mathbb{E}|M_T| < \infty$  and the expectation of the total variation of  $\tilde{A}$  is bounded by

$$\mathbb{E} \tilde{B}_\infty + \mathbb{E} \tilde{C}_\infty = \mathbb{E} B_\infty + \mathbb{E} C_\infty \leq \mathbb{E}|M_T| < \infty.$$

We need to show  $U$  is square integrable. Note

$$\begin{aligned} |M_t - A_t| &= |M_t| 1_{(t < T)} = |\mathbb{E}[M_\infty \mid \mathcal{F}_t]| 1_{(t < T)} \\ &= |\mathbb{E}[\mathbb{E}[M_\infty \mid \mathcal{F}_{T \vee t}] \mid \mathcal{F}_t]| 1_{(t < T)} = |\mathbb{E}[M_{T \vee t} \mid \mathcal{F}_t]| 1_{(t < T)} \\ &= |\mathbb{E}[M_T \mid \mathcal{F}_t]| 1_{(t < T)} \leq \mathbb{E}[|M_T| \mid \mathcal{F}_t] 1_{(t < T)} \leq K. \end{aligned}$$

Therefore it suffices to show  $\tilde{A}$  is square integrable.

Our hypotheses imply that  $\mathbb{E}[M_T^+ \mid \mathcal{F}_t]$  is bounded by  $K$  on  $[0, T]$ , and by Proposition 16.33,  $\mathbb{E} \tilde{B}_\infty^2 < \infty$ . Similarly,  $\mathbb{E} \tilde{C}_\infty^2 < \infty$ . Since  $A = B - C$ , then  $\tilde{A} = \tilde{B} - \tilde{C}$ , and it follows that  $\sup_{t \geq 0} \tilde{A}_t$  is square integrable.  $\square$

## 17.5 Semimartingales

We define a *semimartingale* to be a process of the form  $X_t = X_0 + M_t + A_t$ , where  $X_0$  is finite, a.s., and is  $\mathcal{F}_0$  measurable,  $M_t$  is a local martingale, and  $A_t$  is a process whose paths have bounded variation on  $[0, t]$  for each  $t$ .

If  $M_t$  is a local martingale, let  $T_n$  be a sequence of stopping times as in Theorem 17.9. We set  $M_{t \wedge T_n}^c = (U^n)_t^c$  for each  $n$  and

$$[M]_{t \wedge T_n} = \langle M^c \rangle_{t \wedge T_n} + \sum_{s \leq t \wedge T_n} \Delta M_s^2.$$

It is easy to see that these definitions are independent of how we decompose  $M$  into  $U^n + V^n$  and of which sequence of stopping times  $T_n$  strongly reducing  $M$  we choose. We define  $\langle X^c \rangle_t = \langle M^c \rangle_t$  and define

$$[X]_t = \langle X^c \rangle_t + \sum_{s \leq t} \Delta X_s^2.$$

We say an adapted process  $H$  is *locally bounded* if there exist stopping times  $S_n \uparrow \infty$  and constants  $K_n$  such that on  $[0, S_n]$  the process  $H$  is bounded by  $K_n$ . If  $X_t$  is a semimartingale and  $H$  is a locally bounded predictable process, define  $\int_0^t H_s dX_s$  as follows. Let  $X_t = X_0 + M_t + A_t$ . If  $R_n = T_n \wedge S_n$ , where the  $T_n$  are as in Theorem 17.9 and the  $S_n$  are the stopping times used in the definition of locally bounded, set  $\int_0^{t \wedge R_n} H_s dM_s$  to be the stochastic integral as defined in Section 17.2. Define  $\int_0^{t \wedge R_n} H_s dA_s$  to be the usual Lebesgue–Stieltjes integral. Define the stochastic integral with respect to  $X$  as the sum of these two. Since  $R_n \uparrow \infty$ , this defines  $\int_0^t H_s dX_s$  for all  $t$ . One needs to check that the definition does not depend on the decomposition of  $X$  into  $M$  and  $A$  nor on the choice of stopping times  $R_n$ .

We now state the general Itô formula.

**Theorem 17.10** Suppose  $X$  is a semimartingale and  $f$  is  $C^2$ . Then

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_{s-}) dX_s + \frac{1}{2} \int_0^t f''(X_{s-}) d\langle X^c \rangle_s \\ &\quad + \sum_{s \leq t} [f(X_s) - f(X_{s-}) - f'(X_{s-}) \Delta X_s]. \end{aligned}$$

*Proof* First suppose  $f$  has bounded first and second derivatives. Let  $T_n$  be stopping times strongly reducing  $M_t$ , let  $S_n = \inf\{t : \int_0^t |dA_s| \geq n\}$ , let  $R_n = T_n \wedge S_n$ , and let  $X_t^n = X_{t \wedge R_n} - \Delta A_{R_n}$ . Since the total variation of  $A_t$  is bounded on  $[0, R_n]$ , it follows that  $X^n$  is a semimartingale which is the sum of a square integrable martingale and a process whose total variation is integrable. We apply Theorem 17.5 to this process.  $X_t^n$  agrees with  $X_t$  on  $[0, R_n]$ . As in the proof of Theorem 17.5, by looking at the jump at time  $R_n$ , both sides of Itô's formula jump the same amount at time  $R_n$ , and so Itô's formula holds for  $X_t$  on  $[0, R_n]$ . If we now only assume that  $f$  is  $C^2$ , we approximate  $f$  by a sequence  $f_m$  of functions that are  $C^2$  and whose first and second derivatives are bounded, and then let  $m \rightarrow \infty$ ; we leave the details to the reader. Thus Itô's formula holds for  $t$  in the interval  $[0, R_n]$  and for  $f$  without the assumption of bounded derivatives. Finally, we observe that  $R_n \rightarrow \infty$ , so except for a null set, Itô's formula holds for each  $t$ .  $\square$

The proof of the following corollary is similar to the proof of Itô's formula.

**Corollary 17.11** If  $X_t = (X_t^1, \dots, X_t^d)$  is a process taking values in  $\mathbb{R}^d$  such that each component is a semimartingale, and  $f$  is a  $C^2$  function on  $\mathbb{R}^d$ , then

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t \sum_{i=1}^d \frac{\partial f}{\partial x_i}(X_{s-}) dX_s^i \\ &\quad + \frac{1}{2} \int_0^t \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-}) d\langle (X^i)^c, (X^j)^c \rangle_s \\ &\quad + \sum_{s \leq t} \left[ f(X_s) - f(X_{s-}) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}(X_{s-}) \Delta X_s^i \right]. \end{aligned}$$

If  $X$  and  $Y$  are real-valued semimartingales, define

$$[X, Y]_t = \frac{1}{2}([X + Y]_t - [X]_t - [Y]_t). \quad (17.10)$$

The following corollary is the product formula for semimartingales with jumps.

**Corollary 17.12** If  $X$  and  $Y$  are semimartingales of the above form,

$$X_t Y_t = X_0 Y_0 + \int_0^t X_{s-} dY_s + \int_0^t Y_{s-} dX_s + [X, Y]_t.$$

*Proof* Apply Theorem 17.10 with  $f(x) = x^2$ . Since in this case

$$f(X_s) - f(X_{s-}) - f'(X_{s-}) \Delta X_s = \Delta X_s^2,$$

we obtain

$$X_t^2 = X_0^2 + 2 \int_0^t X_{s-} dX_s + [X]_t. \quad (17.11)$$

Applying (17.11) with  $X$  replaced by  $Y$  and by  $X + Y$  and using

$$X_t Y_t = \frac{1}{2} [(X_t + Y_t)^2 - X_t^2 - Y_t^2]$$

gives our result.  $\square$

## 17.6 Exponential of a semimartingale

A function with finite total variation is *purely discontinuous* if it has no continuous part, i.e.,  $a(t) = \sum_{s \leq t} \Delta a(s)$ .

**Theorem 17.13** *Let  $X_t$  be a semimartingale. Define*

$$Z_t = Z_0 \exp \left( X_t - \frac{1}{2} \langle X^c \rangle_t \right) \prod_{0 \leq s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}. \quad (17.12)$$

*Then  $Z_t$  is a semimartingale,  $\prod_{0 \leq s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}$  is a process of bounded variation whose paths are purely discontinuous, and  $Z_t$  satisfies*

$$Z_t = Z_0 + \int_0^t Z_{s-} dX_s. \quad (17.13)$$

*Proof* Since the product of finitely many functions of bounded variation which are purely discontinuous will give a function of the same type and in each finite interval there are only finitely many jumps of  $X_t$  of size larger in absolute value than  $1/2$ , it suffices to consider

$$V'_t = \prod_{0 \leq s \leq t} (1 + \Delta X_s) e^{-\Delta X_s} 1_{(|\Delta X_s| \leq 1/2)}.$$

Note

$$\log V'_t = \sum_{s \leq t} (\log(1 + \Delta X_s) - \Delta X_s) 1_{(|\Delta X_s| \leq 1/2)},$$

which is bounded in absolute value by a constant times  $\sum_{s \leq t} \Delta X_s^2 < \infty$ . Exercise 17.4 tells us that  $V'_t = \exp(\log V'_t)$  is a purely discontinuous process, and consequently  $V$  is also.

We apply the multivariate version of Itô's formula (Corollary 17.11). Let  $f(x, y) = e^x y$  and let  $Z_t = f(K_t, V_t)$  where  $K_t = X_t - \frac{1}{2} \langle X^c \rangle_t$ . We obtain

$$\begin{aligned} Z_t - Z_0 &= \int_0^t Z_{s-} dK_s + \int_0^t e^{K_{s-}} dV_s + \frac{1}{2} \int_0^t Z_{s-} d\langle K^c \rangle_t \\ &\quad + \sum_{s \leq t} [Z_s - Z_{s-} - Z_{s-} \Delta K_s - e^{-K_{s-}} \Delta V_s] \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We have

$$I_1 = \int_0^t Z_{s-} dX_s - \frac{1}{2} \int_0^t Z_{s-} d\langle X^c \rangle_s.$$

Since  $V_t$  is purely discontinuous,

$$I_2 = \sum_{s \leq t} e^{K_{s-}} \Delta V_s.$$

Since  $K^c = X^c$ ,

$$I_3 = \frac{1}{2} \int_0^t Z_{s-} d\langle X^c \rangle_s.$$

Note that  $Z_s = Z_{s-}(1 + \Delta X_s)$  and  $Z_{s-} \Delta K_s = Z_{s-} \Delta X_s$ , so

$$I_4 = - \sum_{s \leq t} e^{K_{s-}} \Delta V_s.$$

Summing yields (17.13).  $\square$

The solution  $Z$  of (17.13) is called the *exponential of the semimartingale  $X$* .

## 17.7 The Girsanov theorem

Let  $\mathbb{P}$  and  $\mathbb{Q}$  be two equivalent probability measures, that is,  $\mathbb{P}$  and  $\mathbb{Q}$  are mutually absolutely continuous. Let  $M_\infty$  be the Radon–Nikodym derivative of  $\mathbb{Q}$  with respect to  $\mathbb{P}$  and let  $M_t = \mathbb{E}[M_\infty | \mathcal{F}_t]$ . The martingale  $M_t$  is uniformly integrable since  $M_\infty \in L^1(\mathbb{P})$ . Once a non-negative martingale hits zero, it is easy to see that it must be zero from then on; this is Exercise 17.5. Since  $\mathbb{Q}$  and  $\mathbb{P}$  are equivalent, then  $M_\infty > 0$ , a.s., and so  $M_t$  never equals zero, a.s. Observe that  $M_T$  is the Radon–Nikodym derivative of  $\mathbb{Q}$  with respect to  $\mathbb{P}$  on  $\mathcal{F}_T$ .

Let  $L_t$  be the local martingale defined by

$$L_t = \int_0^t \frac{1}{M_{s-}} dM_s,$$

so that

$$dM_t = M_{t-} dL_t,$$

or  $M$  is the exponential of  $L$ .

**Theorem 17.14** Suppose  $X$  is a local martingale with respect to  $\mathbb{P}$ . Then  $X_t - D_t$  is a local martingale with respect to  $\mathbb{Q}$ , where

$$D_t = \int_0^t \frac{1}{M_s} d[X, M]_s = \int_0^t \frac{M_{s-}}{M_s} d[X, L]_s.$$

Note that in the formula for  $D$ , we are using a Lebesgue–Stieltjes integral.

*Proof* Exercise 17.6 tells us that it suffices to show that  $M_t(X_t - D_t)$  is a local martingale with respect to  $\mathbb{P}$ . By Corollary 17.12,

$$\begin{aligned} d(M(X - D))_t &= (X - D)_{t-} dM_t + M_{t-} dX_t - M_{t-} dD_t \\ &\quad + d[M, X - D]_t. \end{aligned}$$

The first two terms on the right are local martingales with respect to  $\mathbb{P}$ . Since  $D$  is of bounded variation, the continuous part of  $D$  is zero, hence

$$[M, D]_t = \sum_{s \leq t} \Delta M_s \Delta D_s = \int_0^t \Delta M_s dD_s.$$

Thus

$$M_t(X_t - D_t) = \text{local martingale} + [M, X]_t - \int_0^t M_s dD_s.$$

Using the definition of  $D$  shows that  $M_t(X_t - D_t)$  is a local martingale.  $\square$

## Exercises

- 17.1 Suppose  $a(t)$  is a deterministic right-continuous nondecreasing function of  $t$  with  $a(0) = 0$ . Prove the following formulas:

$$a(t)^2 = \int_0^t [(a(t) - a(s)) + (a(t) - a(s-))] da(s), \quad (17.14)$$

$$\begin{aligned} \text{and } a(t)^2 &= \int_0^t (2a(s-) + \Delta a(s)) da(s) \\ &= 2 \int_0^t a(s-) da(s) + \sum_s (\Delta a(s))^2. \end{aligned} \quad (17.15)$$

*Hint:* First do the case where  $a$  has only finitely many discontinuities.

- 17.2 If  $A_t$  is an increasing process and  $\tilde{A}_t$  is its compensator, show that  $\tilde{A}$  jumps only when  $A$  does.
- 17.3 Let  $P_t^j$ ,  $j \in \mathbb{Z}$ , be independent Poisson processes with parameter  $\lambda_j$ . Suppose  $\lambda_j = \lambda_{-j}$  for each  $j \neq 0$ . Suppose  $\lambda_j$  decreases as  $j$  increases for  $j \geq 1$ . Let

$$X_t = \sum_{j \in \mathbb{Z}} P_t^j.$$

Determine reasonable conditions on the sequence  $\lambda_j$  so that  $X$  is a semimartingale. A local martingale. A martingale. A locally square integrable martingale.

- 17.4 Show that if  $f(t)$  is a purely discontinuous function, then  $e^{f(t)}$  is also.
- 17.5 Suppose  $M$  is a non-negative right-continuous martingale and  $T = \inf\{t > 0 : M_t = 0\}$ . Show that  $M_t = 0$  on  $(t > T)$ .
- 17.6 Suppose  $\mathbb{P}$  and  $\mathbb{Q}$  are two equivalent probability measures,  $M_\infty$  is the Radon–Nikodym derivative of  $\mathbb{Q}$  with respect to  $\mathbb{P}$ , and  $M_t = \mathbb{E}[M_\infty | \mathcal{F}_t]$ . Show that  $Y_t$  is a local martingale with respect to  $\mathbb{Q}$  if and only if  $Y_t M_t$  is a local martingale with respect to  $\mathbb{P}$ .
- 17.7 Suppose  $X_t$  is an increasing process with paths that are right continuous with left limits,  $X_0 = 0$ , a.s.,  $X$  is purely discontinuous, and all jumps are of size  $+1$  only. Suppose  $X_t - t$  is a martingale. Prove that  $X$  is a Poisson process.

*Hint:* Imitate the proof of Theorem 12.1. When using Itô's formula, it is important to use the fact that  $\Delta X_t$  is always 0 or 1.

- 17.8 Suppose  $X_t$  is an increasing process with paths that are right continuous with left limits,  $X_0 = 0$ , a.s.,  $X$  is purely discontinuous, and all jumps are of size +1 only. Suppose  $\lim_{t \rightarrow \infty} X_t = \infty$ , a.s. Prove that  $X$  is a time change of a Poisson process.
- 17.9 Suppose  $P_t$  is a Poisson process with parameter  $\lambda$ ,  $\{\mathcal{F}_t\}$  is the minimal augmented filtration for  $P$ , and  $M_t = P_t - \lambda t$ . Suppose  $Y$  is a  $\mathcal{F}_1$  measurable random variable with finite mean and variance. Prove that there exists a predictable process  $H$  such that

$$Y = \mathbb{E} Y + \int_0^1 H_s dM_s.$$

- 17.10 Let  $P_1$  and  $P^2$  be two independent Poisson processes with the same parameter. Let  $X_t = P_t^1 - P_t^2$  and let  $\{\mathcal{F}_t\}$  be the minimal augmented filtration for  $X$ . Find a bounded mean zero random variable  $Y$  that is  $\mathcal{F}_1$  measurable which does not satisfy

$$Y = \int_0^1 H_s dX_s$$

for any predictable process  $H$ .

# 18

---

## Poisson point processes

Poisson point processes are random measures that are related to Poisson processes. We will use them when we study Lévy processes in Chapter 42. Poisson point processes are also useful in the study of excursions, even excursions of a continuous process such as Brownian motion (see Chapter 27), and they arise when studying stochastic differential equations with jumps.

Let  $\mathcal{S}$  be a metric space,  $\mathcal{G}$  the collection of Borel subsets of  $\mathcal{S}$ , and  $\lambda$  a measure on  $(\mathcal{S}, \mathcal{G})$ .

**Definition 18.1** We say a map

$$N : \Omega \times [0, \infty) \times \mathcal{G} \rightarrow \{0, 1, 2, \dots\}$$

(writing  $N_t(A)$  for  $N(\omega, t, A)$ ) is a *Poisson point process* if

- (1) for each Borel subset  $A$  of  $\mathcal{S}$  with  $\lambda(A) < \infty$ , the process  $N_t(A)$  is a Poisson process with parameter  $\lambda(A)$ , and
- (2) for each  $t$  and  $\omega$ ,  $N(t, \cdot)$  is a measure on  $\mathcal{G}$ .

A model to keep in mind is where  $\mathcal{S} = \mathbb{R}$  and  $\lambda$  is a Lebesgue measure. For each  $\omega$  there is a collection of points  $\{(s, z)\}$  (where the collection depends on  $\omega$ ). The number of points in this collection with  $s \leq t$  and  $z$  in a subset  $A$  is  $N_t(A)(\omega)$ . Since  $\lambda(\mathbb{R}) = \infty$ , there are infinitely many points in every time interval.

A consequence of the definition is that since  $\lambda(\emptyset) = 0$ , then  $N_t(\emptyset)$  is a Poisson process with parameter 0; in other words,  $N_t(\emptyset)$  is identically zero.

Our main result is that  $N_t(A)$  and  $N_t(B)$  are independent if  $A$  and  $B$  are disjoint.

**Theorem 18.2** Let  $\{\mathcal{F}_t\}$  be a filtration satisfying the usual conditions. Let  $\mathcal{S}$  be a metric space furnished with a positive measure  $\lambda$ . Suppose that  $N_t(A)$  is a Poisson point process with respect to the measure  $\lambda$ . If  $A_1, \dots, A_n$  are pairwise measurable subsets of  $\mathcal{S}$  with  $\lambda(A_k) < \infty$  for  $k = 1, \dots, n$ , then the processes  $N_t(A_1), \dots, N_t(A_n)$  are mutually independent.

*Proof* We first make the observation that because  $N(t, \cdot)$  is a measure and the  $A_1, A_2, \dots, A_n$  are disjoint, then  $\sum_{k=1}^n N_t(A_k) = N_t(\bigcup_{k=1}^n A_k)$  is a Poisson process with finite parameter. A Poisson process has jumps of size one only, hence no two of the  $N_t(A_k)$  have jumps at the same time.

To prove the theorem, it suffices to let  $0 = r_0 < r_1 < \dots < r_m$  and show that the random variables

$$\{N_{r_j}(A_k) - N_{r_{j-1}}(A_k) : 1 \leq j \leq m, 1 \leq k \leq n\}$$

are independent. Since for each  $j$  and each  $k$ ,  $N_{r_j}(A_k) - N_{r_{j-1}}(A_k)$  is independent of  $\mathcal{F}_{r_{j-1}}$ , it suffices to show that for each  $j \leq m$ , the random variables

$$\{N_{r_j}(A_k) - N_{r_{j-1}}(A_k) : 1 \leq k \leq n\}$$

are independent. We will do the case  $j = m = 1$  and write  $r$  for  $r_j$  for simplicity; the case when  $j, m > 1$  differs only in notation.

We will prove this using induction. We start with the case  $n = 2$  and show the independence of  $N_r(A_1)$  and  $N_r(A_2)$ . Each  $N_t(A_k)$  is a Poisson process, and so  $N_t(A_k)$  has moments of all orders. Let  $u_1, u_2 \in \mathbb{R}$  and set

$$\phi_k = \lambda(A_k)(e^{iu_k} - 1), \quad k = 1, 2.$$

Let

$$M_t^k = e^{iu_k N_t(A_k) - t\phi_k}.$$

We see that  $M_t^k$  is a martingale because  $\mathbb{E} e^{iu_k N_t(A_k)} = e^{t\phi_k}$ , and therefore

$$\begin{aligned} \mathbb{E}[M_t^k | \mathcal{F}_s] &= M_s^k \mathbb{E}[e^{iu(N_t(A_k) - N_s(A_k)) - (t-s)\phi_k} | \mathcal{F}_s] \\ &= M_s^k e^{-(t-s)\phi_k} \mathbb{E}[e^{iu(N_t(A_k) - N_s(A_k))}] = M_s^k, \end{aligned}$$

using the independence and stationarity of the increments of a Poisson process.

Since we have argued that no two of the  $N_t(A_k)$  jump at the same time, the same is true for the  $M_t^k$  and so  $[M_t^j, M_t^k]_t = 0$  if  $j \neq k$ . By the product formula (Corollary 17.12) and Itô's formula (Theorem 17.10)

$$\begin{aligned} M_t^k &= 1 - \phi_k \int_0^t e^{iu_k N_{s-}(A_k) - s\phi_k} ds + iu_k \int_0^t e^{iu_k N_{s-}(A_k) - s\phi_k} dN_s(A_k) \\ &\quad + \sum_{s \leq t} e^{iu_k N_{s-}(A_k) - s\phi_k} [e^{iu_k \Delta N_s(A_k)} - 1 - iu_k \Delta N_s(A_k)] \\ &= 1 - \phi_k \int_0^t e^{iu_k N_{s-}(A_k) - s\phi_k} ds + \sum_{s \leq t} e^{iu_k N_{s-}(A_k) - s\phi_k} [e^{iu_k \Delta N_s(A_k)} - 1] \\ &= 1 - \tilde{B}_t^k + B_t^k. \end{aligned}$$

We see therefore that  $M_t^k - 1$  is of the form  $B_t^k - \tilde{B}_t^k$ , where  $B_t^k$  is a complex-valued process whose paths are locally of bounded variation, and  $\tilde{B}_t^k$  is the compensator of  $B_t^k$ .

Let  $\bar{M}_t^k = M_{t \wedge r}^k - 1$ . Since the  $M_t^k$  do not jump at the same time, by the orthogonality lemma (Lemma 17.2),  $\mathbb{E} \bar{M}_\infty^1 \bar{M}_\infty^2 = 0$ , which translates to

$$\mathbb{E} M_r^1 M_r^2 = 1.$$

This implies

$$\mathbb{E} \left[ e^{i(u_1 N_r(A_1) + u_2 N_r(A_2))} \right] = e^{r\phi_1} e^{r\phi_2} = \mathbb{E} \left[ e^{iu_1 N_r(A_1)} \right] \mathbb{E} \left[ e^{iu_2 N_r(A_2)} \right].$$

Since this holds for all  $u_1, u_2$ , then  $N_r(A_1)$  and  $N_r(A_2)$  are independent. We conclude that the processes  $N_t(A_1)$  and  $N_t(A_2)$  are independent.

To handle the case  $n = 3$ , we first show that  $M_t^1 M_t^2$  is a martingale. We write

$$\begin{aligned} \mathbb{E}[M_t^1 M_t^2 | \mathcal{F}_s] &= M_s^1 M_s^2 e^{-(t-s)(\phi_1 + \phi_2)} \mathbb{E}[e^{i(u_1(N_t(A_1) - N_s(A_1)) + u_2(N_t(A_2) - N_s(A_2)))} | \mathcal{F}_s] \\ &= M_s^1 M_s^2 e^{-(t-s)(\phi_1 + \phi_2)} \mathbb{E}[e^{i(u_1(N_t(A_1) - N_s(A_1)) + u_2(N_t(A_2) - N_s(A_2)))}] \\ &= M_s^1 M_s^2, \end{aligned}$$

using the fact that  $N_t(A_1)$  and  $N_t(A_2)$  are independent of each other and each have stationary and independent increments.

Note that  $M_t^3 = e^{iu_3 N_t(A_3) - t\phi_3}$  has no jumps in common with  $M_t^1$  or  $M_t^2$ . Therefore if  $\bar{M}_t^3 = M_{t \wedge r}^3$ , then

$$\mathbb{E}[\bar{M}_\infty^3 (\bar{M}_\infty^1 \bar{M}_\infty^2)] = 0,$$

and as before this leads to

$$\mathbb{E}[M_r^3 (M_r^1 M_r^2)] = 1.$$

As above this implies that  $N_r(A_1)$ ,  $N_r(A_2)$ , and  $N_r(A_3)$  are independent. To prove the general induction step is similar.  $\square$

We will also need the following corollary.

**Corollary 18.3** *Let  $\mathcal{F}_t$  and  $N_t(A_k)$  be as in Theorem 18.2. Suppose  $Y_t$  is a process with paths that are right continuous with left limits such that  $Y_t - Y_s$  is independent of  $\mathcal{F}_s$  whenever  $s < t$  and  $Y_t - Y_s$  has the same law as  $Y_{t-s}$  for each  $s < t$ . Suppose moreover that  $Y$  has no jumps in common with any of the  $N_t(A_k)$ . Then the processes  $N_t(A_1), \dots, N_t(A_n)$ , and  $Y_t$  are independent.*

*Proof* The law of  $Y_0$  is the same as that of  $Y_t - Y_t$ , so  $Y_0 = 0$ , a.s. By the fact that  $Y$  has stationary and independent increments,

$$\mathbb{E} e^{iuY_{s+t}} = \mathbb{E} e^{iuY_s} \mathbb{E} e^{iu(Y_{s+t} - Y_s)} = \mathbb{E} e^{iuY_s} \mathbb{E} e^{iuY_t},$$

which implies that the characteristic function of  $Y$  is of the form  $\mathbb{E} e^{iuY_t} = e^{t\psi(u)}$  for some function  $\psi(u)$ .

We fix  $u \in \mathbb{R}$  and define

$$M_t^Y = e^{iuY_t - t\psi(u)}.$$

As in the proof of Theorem 18.2, we see that  $M_t^Y$  is a martingale. Since  $M^Y$  has no jumps in common with any of the  $M_t^k$ , if  $\bar{M}_t^Y = M_{t \wedge r}^Y$ , we see by Lemma 17.2 that

$$\mathbb{E}[\bar{M}_\infty^Y (\bar{M}_\infty^1 \cdots \bar{M}_\infty^n)] = 1,$$

or

$$\mathbb{E}[M_r^Y M_r^1 \cdots M_r^n] = 1.$$

This leads as above to the independence of  $Y$  from all the  $N_t(A_k)$ 's.  $\square$

We now turn to stochastic integrals with respect to Poisson point processes. In the same way that a nondecreasing function on the reals gives rise to a measure, so  $N_t(A)$  gives rise

to a random measure  $\mu(dt, dz)$  on the product  $\sigma$ -field  $\mathcal{B}[0, \infty) \times \mathcal{G}$ , where  $\mathcal{B}[0, \infty)$  is the Borel  $\sigma$ -field on  $[0, \infty)$ ;  $\mu$  is determined by

$$\mu([0, t] \times A)(\omega) = N_t(A)(\omega).$$

Define a nonrandom measure  $\nu$  on  $\mathcal{B}[0, \infty) \times \mathcal{G}$  by  $\nu([0, t] \times A) = t\lambda(A)$  for  $A \in \mathcal{G}$ . If  $\lambda(A) < \infty$ , then  $\mu([0, t] \times A) - \nu([0, t] \times A)$  is the same as a Poisson process minus its mean, hence is locally a square integrable martingale.

We can define a stochastic integral with respect to the random measure  $\mu - \nu$  as follows. Suppose  $H(\omega, s, z)$  is of the form

$$H(\omega, s, z) = \sum_{i=1}^n K_i(\omega) 1_{(a_i, b_i]}(s) 1_{A_i}(z), \quad (18.1)$$

where for each  $i$  the random variable  $K_i$  is bounded and  $\mathcal{F}_{a_i}$  measurable and  $A_i \in \mathcal{G}$  with  $\lambda(A_i) < \infty$ . For such  $H$  we define

$$\begin{aligned} N_t &= \int_0^t \int H(\omega, s, z) d(\mu - \nu)(ds, dz) \\ &= \sum_{i=1}^n K_i(\mu - \nu)((a_i, b_i] \cap [0, t]) \times A_i. \end{aligned} \quad (18.2)$$

Let us assume without loss of generality that the  $A_i$  are disjoint. It is not hard to see (Exercise 18.3) that  $N_t$  is a martingale, that  $N^c = 0$ , and that

$$[N]_t = \int_0^t \int H(\omega, s, z)^2 \mu(ds, dz). \quad (18.3)$$

Since  $\langle N \rangle_t$  must be predictable and all the jumps of  $N$  are totally inaccessible, it follows from Proposition 16.30 that  $\langle N \rangle_t$  is continuous. Since  $[N]_t - \langle N \rangle_t$  is a martingale, we conclude

$$\langle N \rangle_t = \int_0^t \int H(\omega, s, z)^2 \nu(ds, dz). \quad (18.4)$$

Suppose  $H(s, z)$  is a predictable process in the following sense:  $H$  is measurable with respect to the  $\sigma$ -field generated by all processes of the form (18.1). Suppose also that

$$\mathbb{E} \int_0^\infty \int_S H(s, z)^2 \nu(ds, dz) < \infty.$$

Take processes  $H^n$  of the form (18.1) converging to  $H$  in the space  $L^2$  with norm  $(\mathbb{E} \int_0^\infty \int_S H^2 d\nu)^{1/2}$ . The corresponding  $N_t^n = \int_0^t H^n(s, z) d(\mu - \nu)$  are easily seen to be a Cauchy sequence in  $L^2$ , and the limit  $N_t$  we call the *stochastic integral of  $H$  with respect to  $\mu - \nu$* . As in the continuous case, we may prove that  $\mathbb{E} N_t^2 = \mathbb{E} [N]_t = \mathbb{E} \langle N \rangle_t$ , and it follows from this, (18.3), and (18.4) that

$$[N]_t = \int_0^t \int_S H(s, z)^2 \mu(ds, dz), \quad \langle N \rangle_t = \int_0^t \int_S H(s, z)^2 \nu(ds, dz). \quad (18.5)$$

One may think of the stochastic integral as follows: if  $\mu$  gives unit mass to a point at time  $t$  with value  $z$ , then  $N_t$  jumps at this time  $t$  and the size of the jump is  $H(t, z)$ .

## Exercises

- 18.1 Suppose  $\{\mathcal{F}_t\}$  is a filtration satisfying the usual conditions and  $P_t^1$  and  $P_t^2$  are Poisson processes with respect to  $\{\mathcal{F}_t\}$  with parameters  $\lambda_1, \lambda_2$ , respectively. Suppose  $P_t^1 + P_t^2$  is a Poisson process with parameter  $\lambda_1 + \lambda_2$ . Prove that  $P^1$  and  $P^2$  are independent processes.
- 18.2 Suppose  $\{\mathcal{F}_t\}$  is a filtration satisfying the usual conditions,  $P_t$  is a Poisson process with respect to  $\{\mathcal{F}_t\}$ , and  $W_t$  is a Brownian motion with respect to  $\{\mathcal{F}_t\}$ . Show that if  $W_t + P_t$  has stationary and independent increments, then  $P$  and  $W$  are independent processes.
- 18.3 If  $H$  is as in (18.1) and  $N$  is defined by (18.2), show that  $N$  is a martingale,  $N^c = 0$ , and  $[N]_t$  is given by (18.3).
- 18.4 Suppose  $\{A_s, 0 < s < \infty\}$  is a collection of subsets of  $\mathcal{S}$  such that  $\lambda(A_s) \rightarrow \infty$  as  $s \rightarrow \infty$ . Show that  $N_t(A_s)/\lambda(A_s)$  converges to  $t$  uniformly over finite intervals, where the convergence is in probability.
- 18.5 Suppose  $\{A_s, 0 < s < \infty\}$  is a collection of subsets of  $\mathcal{S}$  such that  $A_r \subset A_s$  if  $r \leq s$  and  $\lambda(A_s) \rightarrow \infty$  as  $s \rightarrow \infty$ . Show that for each  $t$ ,

$$\sup_{u \leq t} \left| \frac{N_u(A_s)}{\lambda(A_s)} - u \right|$$

tends to zero almost surely as  $s \rightarrow \infty$ .

- 18.6 Let  $\mathcal{S}$  be a metric space and  $\lambda$  a  $\sigma$ -finite measure on  $\mathcal{S}$ . Construct a Poisson point process which has  $\lambda$  as the corresponding measure.
- 18.7 Let  $P_t^j, j = 1, 2, \dots$  be independent Poisson processes with parameter  $\beta_j$ . Let  $X_t = \sum_{j=1}^{\infty} a_j P_t^j$ , where  $a_j$  is a sequence such that  $X_t$  is finite, a.s. For  $A \subset \mathbb{R} \setminus \{0\}$ , define  $N_t(A)$  to be the number of times before time  $t$  that  $X$  has a jump whose size is in  $A$ :

$$N_t(A) = \sum_{s \leq t} 1_A(X_s - X_{s-}).$$

Prove that  $N_t$  is a Poisson point process and determine  $\lambda$ .

# 19

---

## Framework for Markov processes

It is not uncommon for a Markov process to be defined as a sextuple  $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}^x)$ , and for additional notation (e.g.,  $\zeta, \Delta, \mathcal{S}, P_t, R_\lambda$ , etc.) to be introduced rather rapidly. This can be intimidating for the beginner. We will explain all this notation in as gentle a manner as possible. We will consider a Markov process to be a pair  $(X_t, \mathbb{P}^x)$  (rather than a sextuple), where  $X_t$  is a single stochastic process and  $\{\mathbb{P}^x\}$  is a family of probability measures, one probability measure  $\mathbb{P}^x$  corresponding to each element  $x$  of the state space.

### 19.1 Introduction

The idea that a Markov process consists of one process and many probabilities is one that takes some getting used to. To explain this, let us first look at an example. Suppose  $X_1, X_2, \dots$  is a Markov chain with stationary transition probabilities with  $K$  states:  $1, 2, \dots, K$ . Everything we want to know about  $X$  can be determined if we know  $p(i, j) = \mathbb{P}(X_1 = j | X_0 = i)$  for each  $i$  and  $j$  and  $\mu(i) = \mathbb{P}(X_0 = i)$  for each  $i$ . We sometimes think of having a different Markov chain for every choice of starting distribution  $\mu = (\mu(1), \dots, \mu(K))$ . But instead let us define a new probability space by taking  $\Omega'$  to be the collection of all sequences  $\omega = (\omega_0, \omega_1, \dots)$  such that each  $\omega_n$  takes one of the values  $1, \dots, K$ . Define  $X_n(\omega) = \omega_n$ . Define  $\mathcal{F}_n$  to be the  $\sigma$ -field generated by  $X_0, \dots, X_n$ ; this is the same as the  $\sigma$ -field generated by sets of the form  $\{\omega : \omega_0 = a_0, \dots, \omega_n = a_n\}$ , where  $a_0, \dots, a_n \in \{1, 2, \dots, K\}$ . For each  $x = 1, 2, \dots, K$ , define a probability measure  $\mathbb{P}^x$  on  $\Omega'$  by

$$\begin{aligned} \mathbb{P}^x(X_0 = x_0, X_1 = x_1, \dots, X_n = x_n) \\ = 1_{\{x\}}(x_0)p(x_0, x_1) \cdots p(x_{n-1}, x_n). \end{aligned} \tag{19.1}$$

We have  $K$  different probability measures, one for each of  $x = 1, 2, \dots, K$ , and we can start with an arbitrary probability distribution  $\mu$  if we define  $\mathbb{P}^\mu(A) = \sum_{i=1}^K \mathbb{P}^i(A)\mu(i)$ . We have lost no information by this redefinition, and it turns out this works much better when doing technical details.

The value of  $X_0(\omega) = \omega_0$  can be any of  $1, 2, \dots, K$ ; the notion of starting at  $x$  is captured by  $\mathbb{P}^x$ , not by  $X_0$ . The probability measure  $\mathbb{P}^x$  is concentrated on those  $\omega$ 's for which  $\omega_0 = x$  and  $\mathbb{P}^x$  gives no mass to any other  $\omega$ .

Let us now look at Brownian motion, and see how this framework plays out there. Let  $\mathbb{P}$  be a probability measure and let  $W_t$  be a D1 Brownian motion wrt  $\mathbb{P}$  started at 0. Then  $W_t^x = x + W_t$  is a D1 Brownian motion started at  $x$ . Let  $\Omega' = C[0, \infty)$

(We do not require that  $\omega(0) = 0$  or that  $\omega(0)$  take any particular value of  $x$ .) Define

$$X_t(\omega) = \omega(t). \quad (19.2)$$

This will be our process. Let  $\mathcal{F}$  be the  $\sigma$ -field on  $\Omega' = C[0, \infty)$  generated by the cylindrical subsets of  $C[0, \infty)$ ; define  $\mathbb{P}^x$  to be the law of  $W^x$ :

$$\mathbb{P}^x(X \in A) = \mathbb{P}(W^x \in A), \quad x \in \mathbb{R}, A \in \mathcal{F}. \quad (19.3)$$

$\mathbb{P}^x$  is determined by the fact that if  $n \geq 1$ ,  $t_1 \leq \dots \leq t_n$ , and  $B_1, \dots, B_n$  are Borel subsets of  $\mathbb{R}$ , then

$$\mathbb{P}(X_{t_1} \in B_1, \dots, X_{t_n} \in B_n) = \mathbb{P}(W_{t_1}^x \in B_1, \dots, W_{t_n}^x \in B_n).$$

We call the pair  $(X_t, \mathbb{P}^x)$ ,  $x \in \mathbb{R}$ ,  $t \geq 0$ , a *Brownian motion*.

## 19.2 Definition of a Markov process

We want to allow our Markov processes to take values in spaces other than the Euclidean ones. For now, we take our state space  $\mathcal{S}$  to be a separable metric space, furnished with the Borel  $\sigma$ -field. For the beginner, just think of  $\mathbb{R}$  in place of  $\mathcal{S}$ .

To define a Markov process, we start with a measurable space  $(\Omega, \mathcal{F})$  and suppose we have a filtration  $\{\mathcal{F}_t\}$  (not necessarily satisfying the usual conditions).

**Definition 19.1** A *Markov process*  $(X_t, \mathbb{P}^x)$  is a stochastic process

$$X : [0, \infty) \times \Omega \rightarrow \mathcal{S}$$

and a family of probability measures  $\{\mathbb{P}^x : x \in \mathcal{S}\}$  on  $(\Omega, \mathcal{F})$  satisfying the following.

- (1) For each  $t$ ,  $X_t$  is  $\mathcal{F}_t$  measurable.
- (2) For each  $t$  and each Borel subset  $A$  of  $\mathcal{S}$ , the map  $x \rightarrow \mathbb{P}^x(X_t \in A)$  is Borel measurable.
- (3) For each  $s, t \geq 0$ , each Borel subset  $A$  of  $\mathcal{S}$ , and each  $x \in \mathcal{S}$ , we have

$$\mathbb{P}^x(X_{s+t} \in A \mid \mathcal{F}_s) = \mathbb{P}^{X_s}(X_t \in A), \quad \mathbb{P}^x - \text{a.s.} \quad (19.4)$$

Let

$$\varphi(x) := \mathbb{P}^x(X_t \in A), \quad (19.5)$$

so that  $\varphi$  is a function mapping  $\mathcal{S}$  to  $\mathbb{R}$ . Part of the definition of filtration given in Chapter 1 is that each  $\mathcal{F}_t \subset \mathcal{F}$ . Since we are requiring  $X_t$  to be  $\mathcal{F}_t$  measurable, that means  $X_t \in A$  is in  $\mathcal{F}$  and it makes sense to talk about  $\mathbb{P}^x(X_t \in A)$ . Definition 19.1(2) says that the function  $\varphi$  is Borel measurable.

The expression  $\mathbb{P}^{X_s}(X_t \in A)$  on the rhs of (19.4) is a rv and its value at  $\omega \in \Omega$  is defined to be  $\varphi(X_s(\omega))$ , with  $\varphi$  given by (19.5). Note that the randomness in  $\mathbb{P}^{X_s}(X_t \in A)$  is thus all due to the  $X_s$  term and not the  $X_t$  term. Definition 19.1(3) can be rephrased as saying that for each  $s, t$ , each  $A$ , and each  $x$ , there is a set  $N_{s,t,x,A} \subset \Omega$  that is a null set with respect to  $\mathbb{P}^x$  and for  $\omega \notin N_{s,t,x,A}$ , the conditional expectation  $\mathbb{P}^x(X_{s+t} \in A \mid \mathcal{F}_s) = \varphi(X_s)$ .

We have now explained all the terms in the sextuple  $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}^x)$  except for  $\theta_t$ . These are called shift operators and are maps from  $\Omega \rightarrow \Omega$  such that  $X_s \circ \theta_t = X_{s+t}$ . We defer the precise meaning of the  $\theta_t$  and the rationale for them until Section 19.5, where they will appear in a natural way.

In the remainder of the section and in Section 19.3 we define some of the additional notation commonly used for Markov processes. The first one is almost self-explanatory. We use  $\mathbb{E}^x$  for expectation with respect to  $\mathbb{P}^x$ . As with  $\mathbb{P}^x(X_t \in A)$ , the notation  $\mathbb{E}^{X_s} f(X_t)$ , where  $f$  is bounded and Borel measurable, is to be taken to mean  $\psi(X_s)$  with  $\psi(y) = \mathbb{E}^y f(X_t)$ .

If we want to talk about our Markov process started with distribution  $\mu$ , we define

$$\mathbb{P}^\mu(B) = \int \mathbb{P}^x(B) \mu(dx),$$

and similarly for  $\mathbb{E}^\mu$ ; here  $\mu$  is a probability on  $\mathcal{S}$ .

### 19.3 Transition probabilities

If  $\mathcal{B}$  is the Borel  $\sigma$ -field on a metric space  $\mathcal{S}$ , a *kernel*  $Q(x, A)$  on  $\mathcal{S}$  is a map from  $\mathcal{S} \times \mathcal{B} \rightarrow \mathbb{R}$ :

- (1) For each  $x \in \mathcal{S}$ ,  $Q(x, \cdot)$  is a measure on  $(\mathcal{S}, \mathcal{B})$ .
- (2) For each  $A \in \mathcal{B}$ , the function  $x \rightarrow Q(x, A)$  is Borel measurable.

**Definition 19.2** A collection of kernels  $\{P_t(x, A); t \geq 0\}$  are *Markov transition probabilities* for a Markov process  $(X_t, \mathbb{P}^x)$  if

- (1)  $P_t(x, \mathcal{S}) = 1$  for each  $t \geq 0$  and each  $x \in \mathcal{S}$ . Pt(x, .): Probability
- (2) For each  $x \in \mathcal{S}$ , each Borel subset  $A$  of  $\mathcal{S}$ , and each  $s, t \geq 0$ ,

$$P_{t+s}(x, A) = \int_{\mathcal{S}} P_t(y, A) P_s(x, dy). \quad (19.6)$$

- (3) For each  $x \in \mathcal{S}$ , each Borel subset  $A$  of  $\mathcal{S}$ , and each  $t \geq 0$ ,

$$P_t(x, A) = \mathbb{P}^x(X_t \in A). \quad (19.7)$$

Definition 19.2(3) can be rephrased as saying that for each  $x$ , the measures  $P_t(x, dy)$  and  $\mathbb{P}^x(X_t \in dy)$  are the same. We define

$$P_t f(x) := \int f(y) P_t(x, dy) \quad (19.8)$$

when  $f : \mathcal{S} \rightarrow \mathbb{R}$  is Borel measurable and either bounded or non-negative.

**Lemma 19.3** Suppose  $P_t$  are Markov transition probabilities. If  $f$  is Borel measurable and either non-negative or bounded, then  $P_t f$  is non-negative (respectively, bounded) and Borel measurable and

$$P_t f(x) = \mathbb{E}^x f(X_t), \quad x \in \mathcal{S}. \quad (19.9)$$

*Proof* Using (19.7) and Definition 19.1(2), the Borel measurability and (19.9) hold when  $f$  is the indicator of a set  $A$ . By linearity they hold for simple functions, and then using monotone convergence they hold for non-negative functions. Using linearity again, we have measurability and (19.9) holding for  $f$  bounded and Borel measurable. The non-negativity (respectively, the boundedness) of  $f$  follows from (19.9).  $\square$

The equations (19.6) are known as the *Chapman–Kolmogorov equations*. They can be rephrased in terms of equality of measures: for each  $x$

$$P_{s+t}(x, dz) = \int_{y \in \mathcal{S}} P_t(y, dz) P_s(x, dy). \quad (19.10)$$

Multiplying (19.10) by a bounded Borel measurable function  $f(z)$  and integrating gives

$$P_{s+t}f(x) = \int P_t f(y) P_s(x, dy). \quad (19.11)$$

The right-hand side is the same as  $P_s(P_t f)(x)$ , so we have

$$P_{s+t}f(x) = P_s P_t f(x), \quad (19.12)$$

i.e., the functions  $P_{s+t}f$  and  $P_s P_t f$  are the same. The equation (19.12) is known as the *semigroup property*.

By Lemma 19.3,  $P_t$  is a linear operator on the space of bounded Borel measurable functions on  $\mathcal{S}$ . We can then rephrase (19.12) simply as

$$P_{s+t} = P_s P_t. \quad (19.13)$$

Operators satisfying (19.13) are called a *semigroup*, and are much studied in functional analysis. We will show in Chapter 36 how to construct the Markov process corresponding to a given semigroup  $P_t$ . More about semigroups can also be found in Chapter 37.

One more observation about semigroups: if we take expectations in (19.4), we obtain

$$\mathbb{P}^x(X_{s+t} \in A) = \mathbb{E}^x \left[ \mathbb{P}^{X_s}(X_t \in A) \right].$$

The lhs is  $P_{s+t}1_A(x)$  and the rhs is

$$\mathbb{E}^x[P_t 1_A(X_s)] = P_s P_t 1_A(x),$$

and so (19.4) encodes the semigroup property.

The *resolvent* or  $\lambda$ -*potential* of a semigroup  $P_t$  is defined by

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt, \quad \lambda \geq 0, \quad x \in \mathcal{S}.$$

This can be recognized as the Laplace transform of  $P_t$ . By Lemma 19.3 and the Fubini theorem, we see that

$$R_\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t) dt.$$

Resolvents are useful because they are typically easier to work with than semigroups.

When practitioners of stochastic calculus tire of a martingale, they “stop” it. Markov process theorists are a harsher lot and they “kill” their processes. To be precise, attach an

isolated point  $\Delta$  to  $\mathcal{S}$ . Thus one looks at  $\widehat{\mathcal{S}} = \mathcal{S} \cup \Delta$ , and the topology on  $\widehat{\mathcal{S}}$  is the one generated by the open sets of  $\mathcal{S}$  and  $\{\Delta\}$ .  $\Delta$  is called the *cemetery point*. All functions on  $\mathcal{S}$  are extended to  $\widehat{\mathcal{S}}$  by defining them to be 0 at  $\Delta$ . At some random time  $\zeta$  the Markov process is killed, which means that  $X_t = \Delta$  for all  $t \geq \zeta$ . The time  $\zeta$  is called the *lifetime* of the Markov process.

## 19.4 An example

Let us give an example, that of Brownian motion, of course. Let  $X_t$  and  $\mathbb{P}^x$  be defined by (19.2) and (19.3). Define  $\mathcal{F}_t = \sigma(X_r; r \leq t)$ . Clearly Definition 19.1(1) holds. Observe that since, under  $\mathbb{P}$ ,  $W_t$  is a mean zero normal rv with variance  $t$ ,

$$\begin{aligned}\mathbb{P}^x(X_t \in A) &= \mathbb{P}(W_t^x \in A) = \mathbb{P}(x + W_t \in A) \\ &= \frac{1}{\sqrt{2\pi t}} \int_A e^{-(y-x)^2/2t} dy.\end{aligned}\tag{19.14}$$

By dominated convergence,  $x \rightarrow \mathbb{P}^x(X_t \in A)$  is continuous, therefore measurable. This proves Definition 19.1(2). It remains to prove Definition 19.1(3), which is the following proposition.

**Proposition 19.4** *Let  $W$  be a Brownian motion as defined by Definition 2.1, let  $W_t^x = x + W_t$ , and let  $(X_t, \mathbb{P}^x)$  be defined by (19.2) and (19.3). If  $f$  is bounded and Borel measurable,*

$$\mathbb{E}^x[f(X_{t+s}) \mid \mathcal{F}_s] = \mathbb{E}^{X_s} f(X_t), \quad \mathbb{P}^x\text{-a.s.}\tag{19.15}$$

*Proof* We will first prove

$$\mathbb{E}^x[f(X_{t+s}) \mid \mathcal{F}_s] = \mathbb{E}^{X_s} f(X_t)\tag{19.16}$$

when  $f(x) = e^{iux}$ . Using independent increments and the fact that  $W_{t+s} - W_s$  has the same law as  $W_t$ , we see that under each  $\mathbb{P}^x$ ,  $X_{t+s} - X_s$  is independent of  $\mathcal{F}_s$  and has the same law as a mean zero normal rv with variance  $t$ . We conclude that

$$\mathbb{E}^x e^{iux(X_{t+s}-X_s)} = e^{-u^2 t/2},$$

see (A.25). We then write

$$\begin{aligned}\mathbb{E}^x \left[ e^{iux_{t+s}} \mid \mathcal{F}_s \right] &= \mathbb{E}^x \left[ e^{iux(X_{t+s}-X_s)} \mid \mathcal{F}_s \right] e^{iux_s} \\ &= \mathbb{E}^x \left[ e^{iux(X_{t+s}-X_s)} \right] e^{iux_s} \\ &= e^{-u^2 t/2} e^{iux_s}.\end{aligned}$$

On the other hand, for any  $y$ ,

$$\mathbb{E}^y e^{iux_t} = \mathbb{E} e^{iux_t^y} = \mathbb{E} e^{iux_t} e^{iuy} = e^{-u^2 t/2} e^{iuy}.$$

Replacing  $y$  by  $X_s$  proves (19.16) for this  $f$ .

Now suppose that  $f \in C^\infty$  with compact support and let  $\widehat{f}$  be the Fourier transform of  $f$ . In (19.16) we replace  $u$  by  $-u$ , multiply both sides by  $\widehat{f}(u)$ , and integrate over  $u \in \mathbb{R}$ . Using

the Fourier inversion formula, we then have

$$\begin{aligned}\mathbb{E}^x[f(X_{t+s}) \mid \mathcal{F}_s] &= (2\pi)^{-1} \mathbb{E}^x \left[ \int e^{-iuX_{t+s}} \widehat{f}(u) du \mid \mathcal{F}_s \right] \\ &= (2\pi)^{-1} \mathbb{E}^{X_s} \int e^{-iuX_t} \widehat{f}(u) du \\ &= \mathbb{E}^{X_s} f(X_t).\end{aligned}$$

We used the Fubini theorem several times to interchange expectation and integration; this is justified because  $f$  in  $C^\infty$  with compact support implies  $\widehat{f}$  is in the Schwartz class; see Section B.2. This proves the proposition for  $f$  in  $C^\infty$  with compact support, and a limit argument gives it for all bounded and measurable  $f$ .  $\square$

The same proof works for  $d$ -dimensional Brownian motion.

Set

$$P_t(x, A) = \mathbb{P}^x(X_t \in A) = \mathbb{P}(W_t + x \in A) = \frac{1}{\sqrt{2\pi t}} \int_A e^{-(y-x)^2/2t} dy. \quad (19.17)$$

Clearly for each  $x$  and  $t$ ,  $P_t(x, \cdot)$  is a measure with total mass 1. As we mentioned earlier, the function  $x \rightarrow P_t(x, A)$  is continuous, hence Borel measurable. We will show the Chapman–Kolmogorov equations. These follow from the next proposition.

**Proposition 19.5** *If  $s, t > 0$  and  $x, z \in \mathbb{R}$ , then*

$$\begin{aligned}&\int_{y \in \mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t} \frac{1}{\sqrt{2\pi s}} e^{-(z-y)^2/2s} dy \\ &= \frac{1}{\sqrt{2\pi(s+t)}} e^{-(z-x)^2/2(s+t)}.\end{aligned} \quad (19.18)$$

*Proof* This is a well-known property of the Gaussian density, but we can derive (19.18) from Proposition 19.4. Let  $f$  be continuous with compact support. Taking expectations in (19.15),

$$\mathbb{E}^x f(X_{t+s}) = \mathbb{E}^x [\mathbb{E}^{X_s} f(X_t)],$$

or

$$P_{t+s} f(x) = P_s P_t f(x).$$

Using Lemma 19.3 and (19.17),

$$\begin{aligned}&\int f(x) \frac{1}{\sqrt{2\pi(s+t)}} e^{-(z-x)^2/2(s+t)} dx \\ &= \int f(x) \int \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t} \frac{1}{\sqrt{2\pi s}} e^{-(z-y)^2/2s} dy dx.\end{aligned}$$

Since this holds for all continuous  $f$  with compact support, (19.18) holds for almost every  $x$ . Since both sides of (19.18) are continuous in  $x$ , then (19.18) holds for all  $x$ .  $\square$

## 19.5 The canonical process and shift operators

Suppose we have a Markov process  $(X_t, \mathbb{P}^x)$  where  $\mathcal{F}_t = \sigma(X_s; s \leq t)$ . Suppose for the moment that  $X_t$  has continuous paths. For this to even make sense, it is necessary that the set  $\{t \rightarrow X_t \text{ is not continuous}\}$  to be in  $\mathcal{F}$ , and then we require this event to be  $\mathbb{P}^x$ -null for each  $x$ . Define  $\tilde{\Omega}$  to be the set of continuous functions on  $[0, \infty)$ . If  $\tilde{\omega} \in \tilde{\Omega}$ , set  $\tilde{X}_t = \tilde{\omega}(t)$ . Define  $\tilde{\mathcal{F}}_t = \sigma(\tilde{X}_s; s \leq t)$  and  $\tilde{\mathcal{F}}_\infty = \cup_{t \geq 0} \tilde{\mathcal{F}}_t$ . Finally define  $\tilde{\mathbb{P}}^x$  on  $(\tilde{\Omega}, \tilde{\mathcal{F}}_\infty)$  by  $\tilde{\mathbb{P}}^x(\tilde{X} \in \cdot) = \mathbb{P}^x(X \in \cdot)$ . Thus  $\tilde{\mathbb{P}}^x$  is specified uniquely by

$$\tilde{\mathbb{P}}^x(\tilde{X}_{t_1} \in A_1, \dots, \tilde{X}_{t_n} \in A_n) = \mathbb{P}^x(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n)$$

for  $n \geq 1$ ,  $A_1, \dots, A_n$  Borel subsets of  $\mathcal{S}$ , and  $t_1 < \dots < t_n$ . Clearly there is so far no loss (or gain) by looking at the Markov process  $(\tilde{X}_t, \tilde{\mathbb{P}}^x)$ , which is called the *canonical process*.

Let us now suppose we are working with the canonical process, and we drop the tildes everywhere. We define the *shift operators*  $\theta_t : \Omega \rightarrow \Omega$  as follows.  $\theta_t(\omega)$  will be an element of  $\Omega$  and therefore is a continuous function from  $[0, \infty)$  to  $\mathcal{S}$ . Define

$$\theta_t(\omega)(s) = \omega(t + s).$$

Then

$$X_s \circ \theta_t(\omega) = X_s(\theta_t(\omega)) = \theta_t(\omega)(s) = \omega(t + s) = X_{t+s}(\omega).$$

The shift operator  $\theta_t$  takes the path of  $X$  and chops off and discards the part of the path before time  $t$ .

We will use expressions like  $f(X_s) \circ \theta_t$ . If we apply this to  $\omega \in \Omega$ , then

$$(f(X_s) \circ \theta_t)(\omega) = f(X_s(\theta_t(\omega))) = f(X_{s+t}(\omega)),$$

or  $f(X_s) \circ \theta_t = f(X_{s+t})$ .

If the paths of  $X$  are not continuous, but instead only right continuous with left limits, we can follow exactly the above procedure, except we start with  $\tilde{\Omega}$  being the collection of functions from  $[0, \infty)$  to  $\mathcal{S}$  that are right continuous with left limits.

Even if we are not in this canonical setup, from now on we will suppose there exist shift operators mapping  $\Omega$  into itself so that

$$X_s \circ \theta_t = X_{s+t}.$$

### Exercises

- 19.1 Suppose  $(X_t, \mathbb{P}^x)$  is a Brownian motion and  $S_t = \sup_{s \leq t} X_s$ . Show that  $((X_t, S_t), \mathbb{P}^x)$  is a Markov process and determine the transition probabilities.
- 19.2 Suppose  $(X_t, \mathbb{P}^x)$  is a Brownian motion,  $f$  a non-negative, bounded, Borel measurable function, and  $A_t = \int_0^t f(X_s) ds$ . Show that  $((X_t, A_t), \mathbb{P}^x)$  is a Markov process.
- 19.3 Suppose  $P_t$  is a Poisson process with parameter  $\lambda$ . Let  $\Omega'$  be the collection of functions on  $[0, \infty)$  which are right continuous and which have left limits, let  $\mathcal{F}$  be the  $\sigma$ -field on  $\Omega'$  generated by the cylindrical subsets of  $\Omega'$ , let  $P_t^x = x + P_t$ , and let  $\mathbb{P}^x$  be the law of  $x + P$ . Show that  $(X_t, \mathbb{P}^x)$  is a Markov process and determine the transition probabilities.

- 19.4 Suppose  $m$  is a measure on the Borel subsets  $\mathcal{B}$  of a metric space  $\mathcal{S}$ . Suppose for each  $t > 0$  there exist jointly measurable non-negative functions  $p_t : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$  such that  $\int p_t(x, y) m(dy) = 1$  for each  $x$  and  $t$  and define

$$P_t(x, A) = \int_A p_t(x, y) m(dy).$$

Show that the kernels  $P_t$  satisfy the Chapman–Kolmogorov equations if and only if

$$\int p_s(x, y) p_t(y, z) m(dy) = p_{s+t}(x, z)$$

for every  $s, t \geq 0$ , every  $x \in \mathcal{S}$ , and  $m$ -almost every  $z$ .

- 19.5 The Ornstein–Uhlenbeck process  $Y$  started at  $x$  is a continuous Gaussian process with  $\mathbb{E} Y_t = e^{-t/2}x$  and covariance

$$\text{Cov}(Y_s, Y_t) = e^{-(s+t)/2}(e^{s+t} - 1).$$

If  $X$  is the canonical process and  $\mathbb{P}^x$  is the law of an Ornstein–Uhlenbeck process started at  $x$ , show that  $(X_t, \mathbb{P}^x)$  is a Markov process and determine the transition probabilities.

## Notes

For more, see [Blumenthal and Getoor \(1968\)](#).

# 20

---

## Markov properties

We want to accomplish three things in this chapter. 1, we want to talk about what it means in the Markov process context for a filtration to satisfy the usual conditions. This is now more complicated than in Chapter 1 because we have more than one probability measure. 2, we want to extend the Markov property to expressions that are more complicated than  $\mathbb{E}^x[f(X_{s+t}) | \mathcal{F}_s]$ . 3, we want to look at the strong Markov property, which means we look at expressions like  $\mathbb{E}^x[f(X_{T+t}) | \mathcal{F}_T]$ , where  $T$  is a stopping time.

Throughout this chapter we assume that  $X$  has paths that are right continuous with left limits. To be more precise, if

$$N = \{\omega : \text{the function } t \rightarrow X_t(\omega) \text{ is not right continuous with left limits}\},$$

then we assume  $N \in \mathcal{F}$  and  $N$  is  $\mathbb{P}^x$ -null for every  $x \in \mathcal{S}$ .

### 20.1 Enlarging the filtration

Let us first introduce some notation. Define

$$\mathcal{F}_t^{00} = \sigma(X_s; s \leq t), \quad t \geq 0. \quad (20.1)$$

This is the smallest  $\sigma$ -field with respect to which each  $X_s$  is measurable for  $s \leq t$ . We let  $\mathcal{F}_t^0$  be the completion of  $\mathcal{F}_t^{00}$ , but we need to be careful what we mean by completion here, because we have more than one probability measure present. Let  $\mathcal{N}$  be the collection of sets that are  $\mathbb{P}^x$ -null for every  $x \in \mathcal{S}$ . Thus  $N \in \mathcal{N}$  if  $(\mathbb{P}^x)^*(N) = 0$  for each  $x \in \mathcal{S}$ , where  $(\mathbb{P}^x)^*$  is the outer probability corresponding to  $\mathbb{P}^x$ . The outer probability  $(\mathbb{P}^x)^*$  is defined by

$$(\mathbb{P}^x)^*(S) = \inf\{\mathbb{P}^x(B) : A \subset B, B \in \mathcal{F}\}.$$

Let

$$\mathcal{F}_t^0 = \sigma(\mathcal{F}_t^{00} \cup \mathcal{N}). \quad (20.2)$$

Finally, let

$$\mathcal{F}_t = \mathcal{F}_{t+}^0 = \cap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}^0. \quad (20.3)$$

We call  $\{\mathcal{F}_t\}$  the *minimal augmented filtration* generated by  $X$ . Ultimately, we will work only with  $\{\mathcal{F}_t\}$ , but we need the other two filtrations at intermediate stages. The reason for worrying about which filtrations to use is that  $\{\mathcal{F}_t^{00}\}$  is too small to include many interesting sets (such as those arising in the law of the iterated logarithm, for example), while if the filtration is too large, the Markov property will not hold for that filtration.

The filtration matters when defining a Markov process; see Definition 19.1(3). We will assume throughout this section that  $(X_t, \mathbb{P}^x)$  is a Markov process with respect to the filtration  $\{\mathcal{F}_t^{00}\}$ , that is,

$$\mathbb{P}^x(X_{s+t} \in A \mid \mathcal{F}_s^{00}) = \mathbb{P}^{X_s}(X_t \in A), \quad \mathbb{P}^x\text{-a.s.} \quad (20.4)$$

whenever  $A$  is a Borel subset of  $\mathcal{S}$  and  $s, t \geq 0$ .

We will also make the following assumption, which will be needed here and also in Section 20.3.

**Assumption 20.1** Suppose  $P_t f$  is continuous on  $\mathcal{S}$  whenever  $f$  is bounded and continuous on  $\mathcal{S}$ .

Markov processes satisfying Assumption 20.1 are called *Feller processes* or *weak Feller processes*. If  $P_t f$  is continuous whenever  $f$  is bounded and Borel measurable, then the Markov process is said to be a *strong Feller process*.

We show that we can replace  $\mathcal{F}_t^{00}$  in (20.4) by  $\mathcal{F}_t^0$ .

**Proposition 20.2** Let  $(X_t, \mathbb{P}^x)$  be a Markov process and suppose that (20.4) holds. If  $A$  is a Borel subset of  $\mathcal{S}$ ,  $x \in \mathcal{S}$ , and  $s, t \geq 0$ , then

$$\mathbb{P}^x(X_{s+t} \in A \mid \mathcal{F}_s^0) = \mathbb{P}^{X_s}(X_t \in A), \quad \mathbb{P}^x\text{-a.s.} \quad (20.5)$$

*Proof* Since the right-hand side is a function of  $X_s$  and hence  $\mathcal{F}_s^0$  measurable, we need to show that if  $B \in \mathcal{F}_s^0$ , then

$$\mathbb{P}^x(X_{s+t} \in A, B) = \mathbb{E}^x \left[ \mathbb{P}^{X_s}(X_t \in A); B \right]. \quad (20.6)$$

This holds for  $B \in \mathcal{F}_s^{00}$  by (20.4). It holds for sets  $B \in \mathcal{N}$ , the class of null sets, since both sides are 0. Therefore it holds for sets  $B$  such that there exists  $B_1 \in \mathcal{F}_s^{00}$  with  $B \Delta B_1$  being a null set. By linearity it holds for finite disjoint unions of sets of the form just described. The class of such finite disjoint unions is a monotone class that generates  $\mathcal{F}_s^0$ , and our result follows by the monotone class theorem, Theorem B.2.  $\square$

The next step is to go from  $\mathcal{F}_s^0$  to  $\mathcal{F}_s$ .

**Proposition 20.3** Let  $(X_t, \mathbb{P}^x)$  be a Markov process and suppose that (20.4) holds. If Assumption 20.1 holds and  $f$  is a bounded Borel measurable function, then

$$\mathbb{E}^x[f(X_{s+t}) \mid \mathcal{F}_s] = \mathbb{E}^{X_s} f(X_t), \quad \mathbb{P}^x\text{-a.s.} \quad (20.7)$$

It will turn out (see Proposition 20.7 below) that  $\mathcal{F}_s^0$  is equal to  $\mathcal{F}_s$ , but we do not know this yet.

*Proof* We start with (20.5). By linearity, we have

$$\mathbb{E}^x[f(X_{s+t}) \mid \mathcal{F}_s^0] = \mathbb{E}^{X_s} f(X_t), \quad \mathbb{P}^x\text{-a.s.}, \quad (20.8)$$

when  $f$  is a simple random variable, then by monotone convergence when  $f$  is non-negative, and then by linearity again, when  $f$  is bounded and Borel measurable. In particular, we have this when  $f$  is bounded and continuous.

If  $B \in \mathcal{F}_s = \mathcal{F}_{s+}^0$ , then  $B \in \mathcal{F}_{s+\varepsilon}^0$  for every  $\varepsilon > 0$ . Hence by (20.8) with  $s$  replaced by  $s + \varepsilon$ , if  $f$  is bounded and continuous,

$$\mathbb{E}^x[f(X_{s+t+\varepsilon}); B] = \mathbb{E}^x\left[\mathbb{E}^{X_{s+\varepsilon}}f(X_t); B\right]. \quad (20.9)$$

The right-hand side is equal to

$$\mathbb{E}^x[P_t f(X_{s+\varepsilon}); B];$$

since  $P_t f$  is continuous and  $X_t$  has paths that are right continuous with left limits, this converges to

$$\mathbb{E}^x[P_t f(X_s); B] = \mathbb{E}^x\left[\mathbb{E}^{X_s}f(X_t); B\right]$$

by dominated convergence. The left-hand side of (20.9) converges, using dominated convergence, the continuity of  $f$ , and the fact that  $X$  has paths that are right continuous with left limits, to

$$\mathbb{E}^x[f(X_{s+t}); B].$$

We therefore have

$$\mathbb{E}^x[f(X_{s+t}); B] = \mathbb{E}^x\left[\mathbb{E}^{X_s}f(X_t); B\right]. \quad (20.10)$$

A limit argument shows this holds whenever  $f$  is bounded and measurable. Since  $B$  is an arbitrary event in  $\mathcal{F}_s$ , that completes the proof.  $\square$

**Remark 20.4** In Chapter 16, we discussed the fact that the first time a right continuous process whose jump times are totally inaccessible hits a Borel set is a stopping time, provided the filtration satisfies the usual conditions. Even though the notion of completion of a filtration is a bit different in the context of Markov processes, the result is still true. See Blumenthal and Getoor (1968).

## 20.2 The Markov property

We start with the Markov property given by Proposition 20.3:

$$\mathbb{E}^x[f(X_{s+t}) \mid \mathcal{F}_s] = \mathbb{E}^{X_s}[f(X_t)], \quad \mathbb{P}^x\text{-a.s.} \quad (20.11)$$

Since  $f(X_{s+t}) = f(X_t) \circ \theta_s$ , if we write  $Y$  for the random variable  $f(X_t)$ , we have

$$\mathbb{E}^x[Y \circ \theta_s \mid \mathcal{F}_s] = \mathbb{E}^{X_s}Y, \quad \mathbb{P}^x\text{-a.s.} \quad (20.12)$$

We wish to generalize this to other random variables  $Y$ .

**Proposition 20.5** Let  $(X_t, \mathbb{P}^x)$  be a Markov process and suppose (20.11) holds. Suppose  $Y = \prod_{i=1}^n f_i(X_{t_i-s})$ , where the  $f_i$  are bounded, Borel measurable, and  $s \leq t_1 \leq \dots \leq t_n$ . Then (20.12) holds.

*Proof* We will prove this by induction on  $n$ . The case  $n = 1$  is (20.11), so we suppose the equality holds for  $n$  and prove it for  $n + 1$ .

Let  $V = \prod_{j=2}^{n+1} f_j(X_{t_j - t_1})$  and  $h(y) = \mathbb{E}^y V$ . By the induction hypothesis,

$$\begin{aligned}\mathbb{E}^x \left[ \prod_{j=1}^{n+1} f_j(X_{t_j}) \mid \mathcal{F}_s \right] &= \mathbb{E}^x \left[ \mathbb{E}^x [V \circ \theta_{t_1} \mid \mathcal{F}_{t_1}] f_1(X_{t_1}) \mid \mathcal{F}_s \right] \\ &= \mathbb{E}^x \left[ (\mathbb{E}^{X_{t_1}} V) f_1(X_{t_1}) \mid \mathcal{F}_s \right] \\ &= \mathbb{E}^x [(h f_1)(X_{t_1}) \mid \mathcal{F}_s].\end{aligned}$$

By (20.11) this is  $\mathbb{E}^{X_s} [(h f_1)(X_{t_1 - s})]$ . For any  $y$ ,

$$\begin{aligned}\mathbb{E}^y [(h f_1)(X_{t_1 - s})] &= \mathbb{E}^y [(\mathbb{E}^{X_{t_1 - s}} V) f_1(X_{t_1 - s})] \\ &= \mathbb{E}^y \left[ \mathbb{E}^y [V \circ \theta_{t_1 - s} \mid \mathcal{F}_{t_1 - s}] f_1(X_{t_1 - s}) \right] \\ &= \mathbb{E}^y [(V \circ \theta_{t_1 - s}) f_1(X_{t_1 - s})].\end{aligned}$$

If we replace  $V$  by its definition, replace  $y$  by  $X_s$ , and use the definition of  $\theta_{t_1 - s}$ , we get the desired equality for  $n + 1$  and hence the induction step.  $\square$

We now come to the general version of the Markov property. As usual,  $\mathcal{F}_\infty = \vee_{t \geq 0} \mathcal{F}_t$ . The expression  $Y \circ \theta_t$  for general  $Y$  may seem puzzling at first. We will give some examples when we get to applications of the strong Markov property in Chapter 21.

**Theorem 20.6** *Let  $(X_t, \mathbb{P}^x)$  be a Markov process and suppose (20.11) holds. Suppose  $Y$  is bounded and measurable with respect to  $\mathcal{F}_\infty$ . Then*

$$\mathbb{E}^x [Y \circ \theta_s \mid \mathcal{F}_s] = \mathbb{E}^{X_s} Y, \quad \mathbb{P}^x\text{-a.s.} \quad (20.13)$$

*Proof* If in Proposition 20.5 we take  $f_j(x) = 1_{A_j}(x)$  for Borel measurable  $A_j$ , we have

$$\mathbb{E}^x [1_B \circ \theta_s \mid \mathcal{F}_s] = \mathbb{E}^{X_s} 1_B \quad (20.14)$$

when  $B = \{\omega : \omega(t_1) \in A_1, \dots, \omega(t_n) \in A_n\}$ . It is easy to see that the set of  $B$ 's for which (20.14) holds is a monotone class. By an argument using the monotone class theorem, (20.14) holds for all  $B$  that are measurable with respect to  $\mathcal{F}_\infty$ . Taking linear combinations, (20.13) holds for  $Y$ 's that are simple random variables. Using monotone convergence, (20.13) holds for non-negative  $Y$ 's, and then by linearity for bounded  $Y$ 's.  $\square$

**Proposition 20.7** *Let  $(X_t, \mathbb{P}^x)$  be a Markov process with respect to  $\{\mathcal{F}_t\}$ . Let  $\mathcal{F}_t^0$  and  $\mathcal{F}_t$  be defined by (20.2) and (20.3). Then  $\mathcal{F}_t = \mathcal{F}_t^0$  for each  $t \geq 0$ .*

*Proof* Let  $Y_1 = \prod_{i=1}^n f_i(X_{t_i})$  and  $Y_2 = \prod_{j=1}^m g_j(X_{u_j})$ , where  $t_1 < \dots < t_n \leq s$  and  $0 \leq u_1 < \dots < u_m$  and the  $f_j$  and  $g_j$  are bounded Borel measurable functions. Then by Proposition 20.5,

$$\mathbb{E}^x [(Y_1)(Y_2 \circ \theta_s) \mid \mathcal{F}_s] = Y_1 \mathbb{E}^{X_s} Y_2.$$

Since  $\mathbb{E}^{X_s} Y_2$  is a function of  $X_s$ , then  $(Y_1)(\mathbb{E}^{X_s} Y_2)$  is  $\mathcal{F}_s^0$  measurable. Using a monotone class argument, we conclude that if  $Y$  is bounded and  $\mathcal{F}_\infty$  measurable, then  $\mathbb{E}^x [Y \mid \mathcal{F}_s]$  is  $\mathcal{F}_s^0$

measurable. Now apply this to  $Y = 1_A$  for  $A \in \mathcal{F}_s$  to obtain that  $1_A = \mathbb{E}^x[1_A \mid \mathcal{F}_s]$  is  $\mathcal{F}_s^0$  measurable.  $\square$

The following is known as the *Blumenthal 0–1 law*.

**Proposition 20.8** *Let  $(X_t, \mathbb{P}^x)$  be a Markov process with respect to  $\{\mathcal{F}_t\}$ . If  $A \in \mathcal{F}_0$ , then for each  $x$ ,  $\mathbb{P}^x(A)$  is equal to 0 or 1.*

*Proof* Suppose  $A \in \mathcal{F}_0$ . Under  $\mathbb{P}^x$ ,  $X_0 = x$ , a.s., and then

$$\mathbb{P}^x(A) = \mathbb{E}^{X_0} 1_A = \mathbb{E}^x[1_A \circ \theta_0 \mid \mathcal{F}_0] = 1_A \circ \theta_0 = 1_A \in \{0, 1\}, \quad \mathbb{P}^x\text{-a.s.}$$

since  $1_A \circ \theta_0$  is  $\mathcal{F}_0$  measurable. Our result follows because  $\mathbb{P}^x(A)$  is a real number and not random.  $\square$

### 20.3 Strong Markov property

Given a stopping time  $T$ , recall that the  $\sigma$ -field of events known up to time  $T$  is defined to be

$$\mathcal{F}_T = \{A \in \mathcal{F}_\infty : A \cap (T \leq t) \in \mathcal{F}_t \text{ for all } t > 0\}.$$

We define  $\theta_T(\omega)(t) = \omega(T(\omega) + t)$ . Thus, for example,  $X_t \circ \theta_T(\omega) = X_{T(\omega)+t}(\omega)$  and  $X_T(\omega) = X_{T(\omega)}(\omega)$ .

Now we can state the strong Markov property. The notation and definition are admittedly a bit opaque at this stage – be patient until we reach the examples in the next chapter.

**Theorem 20.9** *Suppose  $(X_t, \mathbb{P}^x)$  is a Markov process with respect to  $\{\mathcal{F}_t\}$ , that Assumption 20.1 holds, and that  $T$  is finite stopping time. If  $Y$  is bounded and measurable with respect to  $\mathcal{F}_\infty$ , then*

$$\mathbb{E}^x[Y \circ \theta_T \mid \mathcal{F}_T] = \mathbb{E}^{X_T} Y, \quad \mathbb{P}^x\text{-a.s.}$$

*Proof* Following the proofs of Section 20.2, it is enough to prove

$$\mathbb{E}^x[f(X_{T+t}) \mid \mathcal{F}_T] = \mathbb{E}^{X_T} f(X_t) \tag{20.15}$$

for  $f$  bounded. We can obtain this by a limit argument if we have (20.15) for  $f$  bounded and continuous. Define  $T_n$  to be equal to  $(k+1)/2^n$  on the event  $(k/2^n \leq T < (k+1)/2^n)$ .

If  $A \in \mathcal{F}_T$ , then  $A \in \mathcal{F}_{T_n}$ . Therefore  $A \cap (T_n = k/2^n) \in \mathcal{F}_{k/2^n}$  and we have by the Markov property, Theorem 20.6,

$$\begin{aligned} \mathbb{E}^x[f(X_{T_n+t}); A, T_n = k/2^n] &= \mathbb{E}^x[f(X_{t+k/2^n}); A, T = k/2^n] \\ &= \mathbb{E}^x[\mathbb{E}^{X_{k/2^n}} f(X_t); A, T_n = k/2^n] \\ &= \mathbb{E}^x[\mathbb{E}^{X_{T_n}} f(X_t); A, T_n = k/2^n]. \end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}^x[f(X_{T_n+t}); A] &= \sum_{k=1}^{\infty} \mathbb{E}^x[f(X_{T_n+t}); A, T_n = k/2^n] \\ &= \sum_{k=1}^{\infty} \mathbb{E}^x[\mathbb{E}^{X_{T_n}} f(X_t); A, T_n = k/2^n] \\ &= \mathbb{E}^x[\mathbb{E}^{X_{T_n}} f(X_t); A].\end{aligned}$$

Now let  $n \rightarrow \infty$ .  $\mathbb{E}^x[f(X_{T_n+t}); A] \rightarrow \mathbb{E}^x[f(X_{T+t}); A]$  by dominated convergence and the continuity of  $f$  and the right continuity of  $X_t$ . On the other hand, using the continuity of  $P_t f$ ,  $\mathbb{E}^{X_{T_n}} f(X_t) = P_t f(X_{T_n}) \rightarrow P_t f(X_T) = \mathbb{E}^{X_T} f(X_t)$ . Therefore

$$\mathbb{E}^x[f(X_{T+t}); A] = \mathbb{E}^x[\mathbb{E}^{X_T} f(X_t); A]$$

for all  $A \in \mathcal{F}_T$ , and hence (20.15) holds.  $\square$

Recall that we are restricting our attention to Markov processes whose paths are right continuous with left limits. If we have a Markov process  $(X_t, \mathbb{P}^x)$  whose paths are right continuous with left limits, which has shift operators  $\{\theta_t\}$ , and which satisfies the conclusion of Theorem 20.9, whether or not Assumption 20.1 holds, then we say that  $(X_t, \mathbb{P}^x)$  is a *strong Markov process*. A strong Markov process is said to be *quasi-left continuous* if  $X_{T_n} \rightarrow X_T$ , a.s., on  $\{T < \infty\}$  whenever  $T_n$  are stopping times increasing up to  $T$ . Unlike in the definition of predictable stopping times given in Chapter 16, we are not requiring the  $T_n$  to be strictly less than  $T$ . A *Hunt process* is a strong Markov process that is quasi-left continuous. Quasi-left continuity does not imply left continuity; consider the Poisson process.

**Proposition 20.10** *If  $(X_t, \mathbb{P}^x)$  is a strong Markov process and Assumption 20.1 holds, then  $X_t$  is quasi-left continuous.*

*Proof* First suppose  $T$  is bounded,  $T_n$  increases to  $T$ ,  $Y = \lim_{n \rightarrow \infty} X_{T_n}$ , and  $f$  and  $g$  are bounded and continuous. If  $T_n = T$  for some  $n$ , then  $\lim_{n \rightarrow \infty} g(X_{T_n+t}) = g(X_{T+t})$ , and if  $T_n < T$  for all  $n$ , then  $\lim_{n \rightarrow \infty} g(X_{T_n+t}) = g(X_{(T+t)-})$ , where  $X_{s-}$  is the left-hand limit at time  $s$ . In either case,

$$\lim_{t \rightarrow 0} \lim_{n \rightarrow \infty} g(X_{T_n+t}) = g(X_T).$$

Then

$$\begin{aligned}\mathbb{E}^x[f(Y)g(X_T)] &= \lim_{t \rightarrow 0} \lim_{n \rightarrow \infty} \mathbb{E}^x[f(X_{T_n})g(X_{T_n+t})] \\ &= \lim_{t \rightarrow 0} \lim_{n \rightarrow \infty} \mathbb{E}^x[f(X_{T_n})P_t g(X_{T_n})] \\ &= \lim_{t \rightarrow 0} \mathbb{E}^x[f(Y)P_t g(Y)] = \mathbb{E}^x[f(Y)g(Y)].\end{aligned}$$

By a limit argument we have

$$\mathbb{E}^x[h(Y, X_T)] = \mathbb{E}^x[h(Y, Y)] \tag{20.16}$$

for all bounded measurable functions  $h$  on  $\mathcal{S} \times \mathcal{S}$ . Now take  $h(x, y)$  to be zero if  $x = y$  and one otherwise. The right-hand side of (20.16) is 0, so the left-hand side is also.

If  $T$  is not bounded, apply the argument in the preceding paragraph to the stopping time  $T \wedge M$ , where  $M$  is a positive real, and then let  $M \rightarrow \infty$ .  $\square$

## Exercises

- 20.1 Suppose that  $\mathcal{S}$  is a locally compact separable metric space and  $C_0$  is the set of continuous functions on  $\mathcal{S}$  that vanish at infinity. To say a continuous function  $f$  vanishes at infinity means that given  $\varepsilon > 0$  there exists a compact set  $K$  such that  $|f(x)| < \varepsilon$  if  $x \notin K$ . Show that if Assumption 20.1 is replaced by the assumptions that  $P_t f \in C_0$  whenever  $f \in C_0$  and  $P_t f \rightarrow f$  uniformly as  $t \rightarrow 0$  whenever  $f \in C_0$ , then the conclusion of Theorem 20.9 still holds.
- 20.2 Suppose  $(X_t, \mathbb{P}^x)$  is a Markov process with respect to a filtration  $\{\mathcal{F}_t\}$ . Suppose that  $\mathcal{E}_t \subset \mathcal{F}_t$  for each  $t$  and that  $X_t$  is  $\mathcal{E}_t$  measurable for each  $t$ . Show that  $(X_t, \mathbb{P}^x)$  is a Markov process with respect to the filtration  $\{\mathcal{E}_t\}$ .
- 20.3 Give an example of a Markov process that is not a strong Markov process.  
*Hint:* Let the state space be  $[0, \infty)$  and starting from  $x \in (0, \infty)$ , let  $X$  move deterministically at constant speed to the right. Starting at 0, let  $X$  wait an exponential length of time, and then begin moving at constant speed to the right.
- 20.4 Let  $(X_t, \mathbb{P}^x)$  be Brownian motion and let  $\{\mathcal{F}_t\}$  be the minimal augmented filtration. Suppose  $B \in \vee_{t \geq 0} \mathcal{F}_t$  and for some  $s > 0$  is of the form  $1_B = 1_A \circ \theta_s$ . Show that if  $B$  is a  $\mathbb{P}^x$ -null set for some  $x$ , then it is a  $\mathbb{P}^x$ -null set for every  $x$ .
- 20.5 Let  $P_t$  be transition probabilities for a Poisson process with parameter  $\lambda$ . These are defined in Exercise 19.3. Show that Assumption 20.1 holds.
- 20.6 Suppose  $(X_t, \mathbb{P}^x)$  is a Markov process with transition probabilities  $P_t$ ,  $f$  is a bounded Borel measurable function,  $t_0 > 0$ , and we define  $M_t = P_{t_0-t}f(X_t)$  for  $t \leq t_0$ . Show that  $(M_t, t \leq t_0)$  is a  $\mathbb{P}^x$ -martingale for each  $x$ .
- 20.7 Use the Blumenthal 0–1 law to show that if  $W$  is a one-dimensional Brownian motion and  $T = \inf\{t > 0 : W_t > 0\}$  is the first time Brownian motion hits  $(0, \infty)$ , then  $\mathbb{P}(T = 0) = 1$ .
- 20.8 Let  $A$  be a Borel subset of a metric space  $\mathcal{S}$ . Let  $T_A = \inf\{t : X_t \in A\}$ , where  $(X_t, \mathbb{P}^x)$  is a strong Markov process. Show that  $\mathbb{P}^x(T_A = 0)$  is either 0 or 1 for each  $x$ .
- 20.9 Let  $(X_t, \mathbb{P}^x)$  be a strong Markov process and let  $A$  be a Borel subset of  $\mathcal{S}$ . We define  $A^r$  by setting  $A^r = \{x : \mathbb{P}^x(T_A = 0) = 1\}$ , where  $T_A$  is the first hitting time of  $A$ . Thus  $A^r$  is the set of points that are regular for  $A$ . Prove that for each  $x$ ,

$$\mathbb{P}^x(X_{T_A} \in A \cup A^r) = 1.$$

# 21

---

## Applications of the Markov properties

We give some applications of the Markov property and the strong Markov property. In the first application, we show that  $d$ -dimensional Brownian motion is transient if  $d \geq 3$ . Next we consider estimates on additive functionals. (An example of an additive functional is  $A_t = \int_0^t f(X_s) ds$ , where  $f$  is a non-negative function on the state space of the Markov process  $X$ .) Third is a sufficient criterion for a Markov process to have continuous paths. Finally, we discuss harmonic functions and show how to solve the classical Dirichlet problem of analysis and partial differential equations.

### 21.1 Recurrence and transience

Let  $W_t = (W_1(t), \dots, W_d(t))$  be a  $d$ -dimensional Brownian motion started at 0 with  $d \geq 3$  and let  $W_t^x = x + W_t$  be Brownian motion started at  $x$ . Let  $h(y) = |y|^{2-d}$ . A direct calculation of derivatives shows that

$$\Delta h(x) = \sum_{i=1}^d \frac{\partial^2 h}{\partial x_i^2}(x) = 0, \quad x \neq 0.$$

(Noting that

$$\frac{\partial}{\partial y_i} |y| = \frac{\partial}{\partial y_i} (y_1^2 + \dots + y_d^2)^{1/2} = \frac{y_i}{|y|}$$

helps with the calculation.) By Exercise 9.4,  $\langle W_i, W_j \rangle_t$  equals 0 if  $i \neq j$  and we saw in Section 9.3 that it equals  $t$  if  $i = j$ . Suppose  $r < |x| < R$ , and let

$$S = \inf\{t : |W_t^x| \leq r \text{ or } |W_t^x| \geq R\}.$$

$S$  is finite, a.s., because  $|W_t^x| \geq |W_1(t)| - |x|$  and  $W_1(t)$  exits  $[-2R, 2R]$  in finite time by Theorem 7.2. By Itô's formula,

$$\begin{aligned} h(W_{t \wedge S}^x) &= h(W_0^x) + \text{martingale} + \frac{1}{2} \int_0^{t \wedge S} \sum_{i=1}^d \frac{\partial^2 h}{\partial x_i^2}(W_s^x) ds \\ &= h(x) + \text{martingale}. \end{aligned}$$

Therefore  $h(W_{t \wedge S}) - h(x)$  is a martingale started at 0. The function  $h$  is equal to  $r^{2-d}$  on  $\partial B(0, r)$ , the boundary of  $B(0, r)$ , and equal to  $R^{2-d}$  on  $\partial B(0, R)$ , the boundary of  $B(0, R)$ .

By Corollary 3.17, we deduce

$$\begin{aligned} & \mathbb{P}(W_t^x \text{ hits } B(0, r) \text{ before } B(0, R)) \\ &= \mathbb{P}(h(W_t^x) - h(x) \text{ hits } r^{2-d} - |x|^{2-d} \text{ before } R^{2-d} - |x|^{2-d}) \\ &= \frac{|x|^{2-d} - R^{2-d}}{r^{2-d} - R^{2-d}}. \end{aligned}$$

If we let  $R \rightarrow \infty$  and recall that  $2 - d < 0$ , we see that

$$\mathbb{P}(W_t^x \text{ ever hits } \partial B(0, r)) = \left(\frac{r}{|x|}\right)^{d-2}. \quad (21.1)$$

We want to use the strong Markov property to go from (21.1) to

$$\lim_{t \rightarrow \infty} |W_t^x| = \infty.$$

(There are other ways besides the strong Markov property of showing this.) The first step in doing this is to convert to the Markov process notation. Let  $(X_t, \mathbb{P}^x)$  be a Brownian motion. What we have shown is that

$$\mathbb{P}^x(X_t \text{ ever hits } \partial B(0, r)) = \left(\frac{r}{|x|}\right)^{d-2}. \quad (21.2)$$

Let  $M > 0$  and let

$$\begin{aligned} S_1 &= \inf\{t : |X_t| \geq 2M\}, \\ T_1 &= \inf\{t > S_1 : |X_t| \leq M\}, \\ S_2 &= \inf\{t > T_1 : |X_t| \geq 2M\}, \\ T_2 &= \inf\{t > S_2 : |X_t| \leq M\}, \end{aligned}$$

and so on. Another way of writing this is to define

$$S = \inf\{t > 0 : |X_t| \geq 2M\}, \quad T = \inf\{t > 0 : |X_t| \leq M\},$$

and then to let  $S_1 = S$ , and for each  $i \geq 1$ ,

$$T_i = S_i + T \circ \theta_{S_i}, \quad S_{i+1} = T_i + S \circ \theta_{T_i}.$$

Let us explain what is going on. Given a path  $\omega$ , which is a continuous function from  $[0, \infty)$  to  $\mathbb{R}^d$ ,  $T \circ \theta_{S_i}$  means to proceed along the path until time  $S_i$ , disregard this piece, and then see how long it takes after time  $S_i$  to first enter  $B(0, M)$ . If we add the quantity  $S_i$  to  $T \circ \theta_{S_i}$ , we then get the amount of time for  $X_t$  to first enter  $B(0, M)$  after time  $S_i$ . Thus  $T_i$  with the shift notation is the same as  $\inf\{t > S_i : X_t \in B(0, M)\}$ . The shift notation interpretation of  $S_{i+1}$  is similar.

Now we can apply the strong Markov property. Since  $T_{i+1} = S_{i+1} + T \circ \theta_{S_{i+1}}$ , we can write

$$\begin{aligned} \mathbb{P}^x(T_{i+1} < \infty) &= \mathbb{P}^x(S_{i+1} < \infty, T \circ \theta_{S_{i+1}} < \infty) \\ &= \mathbb{E}^x \left[ \mathbb{P}^x(T \circ \theta_{S_{i+1}} < \infty \mid \mathcal{F}_{S_{i+1}}); S_{i+1} < \infty \right] \\ &= \mathbb{E}^x \left[ \mathbb{P}^{X_{S_{i+1}}} (T < \infty); S_{i+1} < \infty \right]. \end{aligned}$$

At time  $S_{i+1}$ , we have  $|X_{S_{i+1}}| = 2M$ , and by (21.1)

$$\mathbb{P}^{X_{S_{i+1}}}(T < \infty) = (\frac{1}{2})^{d-2}.$$

Therefore

$$\mathbb{P}^x(T_{i+1} < \infty) \leq 2^{2-d} \mathbb{P}^x(S_{i+1} < \infty) \leq 2^{2-d} \mathbb{P}^x(T_i < \infty).$$

The last inequality is simply the fact that  $S_{i+1} \geq T_i$ . Since  $\mathbb{P}^x(T_1 < \infty) \leq 1$ , induction tells us that

$$\mathbb{P}^x(T_i < \infty) \leq 2^{(2-d)(i-1)} \rightarrow 0$$

as  $i \rightarrow \infty$ . Hence  $\mathbb{P}^x(T_i < \infty \text{ for all } i) = 0$ . Since  $T_i$  increases as  $i$  increases, for almost all  $\omega$ ,  $T_i$  will be infinite for  $i$  sufficiently large (how large will depend on  $\omega$ ). Hence  $X_t$  returns to  $B(0, M)$  for a last time, a.s. Since  $M$  is arbitrary, this proves that  $X_t$  tends to  $\infty$  as  $t \rightarrow \infty$ .

We have thus proved

**Proposition 21.1** *If  $(X_t, \mathbb{P}^x)$  is a  $d$ -dimensional Brownian motion and  $d \geq 3$ , then  $|X_t| \rightarrow \infty$  as  $t \rightarrow \infty$  with  $\mathbb{P}^x$ -probability one for each  $x$ .*

## 21.2 Additive functionals

Let  $D$  be a closed subset of  $\mathcal{S}$ , let  $f : D \rightarrow [0, \infty)$ , let  $S = \tau_D$ , and let

$$A = \sup_{x \in D} \mathbb{E}^x \int_0^S f(X_s) ds,$$

where  $\tau_D = \inf\{t > 0 : X_t \notin D\}$  is the first time  $X$  exits  $D$ .

**Proposition 21.2** *If  $A < \infty$ , then*

$$\sup_{x \in D} \mathbb{P}^x \left( \int_0^S f(X_s) ds \geq 2kA \right) \leq 2^{-k}. \quad (21.3)$$

This is rather remarkable: as soon as one gets a bound on the expectation, although it must be uniform in  $x$ , one gets exponential tails for the distribution. A use of Chebyshev's inequality would only give the bound  $(2k)^{-1}$ .

*Proof* Let  $B_t = \int_0^{t \wedge S} f(X_s) ds$ . This is a special case of what is known as an additive functional; see Section 22.3. Let  $U_1 = \inf\{t : B_t \geq 2A\}$ , and let  $U_{i+1} = U_i + U_1 \circ \theta_{U_i}$ . To explain this formula, composing  $\omega$  with  $\theta_{U_i}$  means we disregard the path before time  $U_i$ . Thus  $U_1 \circ \theta_{U_i}$  is the length of time after time  $U_i$  until  $B_t$  has increased an amount  $2A$  over its value at  $U_i$ . Therefore  $U_i + U_1 \circ \theta_{U_i}$  is the  $(i+1)$ st time  $B$  has increased by  $2A$ . The event  $\mathbb{P}^x(B_S \geq 2kA)$  is bounded by

$$\begin{aligned} \mathbb{P}^x(U_k \leq S) &= \mathbb{P}^x(U_{k-1} \leq S, U_1 \circ \theta_{U_{k-1}} \leq S \circ \theta_{U_{k-1}}) \\ &= \mathbb{E}^x [\mathbb{P}^x(U_1 \circ \theta_{U_{k-1}} \leq S \circ \theta_{U_{k-1}} | \mathcal{F}_{U_{k-1}}); U_{k-1} \leq S] \\ &= \mathbb{E}^x [\mathbb{P}^{X_{U_{k-1}}}(U_1 \leq S); U_{k-1} \leq S]. \end{aligned}$$

If  $U_{k-1} \leq S$ , then  $X_{U_{k-1}} \in D$ . If  $y \in D$ ,

$$\mathbb{P}^y(U_1 \leq S) \leq \mathbb{P}^y\left(\int_0^S f(X_s)ds \geq 2A\right) \leq \frac{\mathbb{E}^y \int_0^S f(X_s)ds}{2A} \leq \frac{1}{2}$$

by Chebyshev's inequality. Then

$$\mathbb{P}^x(U_k \leq S) \leq \frac{1}{2}\mathbb{P}^x(U_{k-1} \leq S)$$

and (21.3) follows by induction.  $\square$

We give another proof of Proposition 4.5.

**Proposition 21.3** *Let  $W$  be a one-dimensional Brownian motion. If  $T$  is a finite stopping time and  $a < b$ , then*

$$\mathbb{P}(W_{T+t} \in [a, b] \mid \mathcal{F}_T) \leq \frac{b-a}{\sqrt{2\pi t}}, \quad \text{a.s.}$$

*Proof* Let  $(X_t, \mathbb{P}^x)$  be a one-dimensional Brownian motion. If  $y \in \mathbb{R}$ , then

$$\begin{aligned} \mathbb{P}^y(X_t \in [a, b]) &= \mathbb{P}^0(X_t \in [a-y, b-y]) \\ &= \frac{1}{\sqrt{2\pi t}} \int_{a-y}^{b-y} e^{-z^2/2t} dz \leq \frac{b-a}{\sqrt{2\pi t}}. \end{aligned} \tag{21.4}$$

By the strong Markov property,

$$\begin{aligned} \mathbb{P}(W_{T+t} \in [a, b] \mid \mathcal{F}_T) &= \mathbb{P}^0(X_{T+t} \in [a, b] \mid \mathcal{F}_T) = \mathbb{E}^0[1_{[a,b]}(X_t) \circ \theta_T \mid \mathcal{F}_T] \\ &= \mathbb{E}^{X_T}[1_{[a,b]}(X_t)] = \mathbb{P}^{X_T}(X_t \in [a, b]). \end{aligned}$$

Now use (21.4) with  $y$  replaced by  $X_T$ .  $\square$

### 21.3 Continuity

Let us now come up with a criterion for a Markov process to have continuous paths. We assume we have a strong Markov process  $(X_t, \mathbb{P}^x)$  whose paths are right continuous with left limits. Let  $d(\cdot, \cdot)$  be the metric for the state space  $\mathcal{S}$ .

**Lemma 21.4** *Let  $(X_t, \mathbb{P}^x)$  be a strong Markov process with state space  $\mathcal{S}$ . For all  $x \in \mathcal{S}$  and all  $\lambda \geq 0$ ,*

$$\mathbb{P}^x(\sup_{s \leq t} d(X_s, x) \geq \lambda) \leq 2 \sup_{s \leq t} \sup_{y \in \mathcal{S}} \mathbb{P}^y(d(X_s, X_0) \geq \lambda/2).$$

Note that the left-hand side has the supremum inside while the right-hand side has the suprema outside the probability.

*Proof* Let us use the notation

$$F(t, \lambda) = \sup_{s \leq t} \sup_{y \in \mathcal{S}} \mathbb{P}^y(d(X_s, X_0) \geq \lambda). \tag{21.5}$$

Write  $S = \inf\{t : d(X_t, X_0) \geq \lambda\}$ . Then by the strong Markov property,

$$\begin{aligned}\mathbb{P}^x(\sup_{s \leq t} d(X_s, x) \geq \lambda) &\leq \mathbb{P}^x(d(X_t, x) \geq \lambda/2) + \mathbb{P}^x(S < t, d(X_t, X_0) \leq \lambda/2) \\ &\leq F(t, \lambda/2) + \mathbb{E}^x \left[ \mathbb{P}^{X_S}(d(X_{t-S}, X_0) \geq \lambda/2) \right] \\ &\leq 2F(t, \lambda/2);\end{aligned}\tag{21.6}$$

see Exercise 21.2.  $\square$

**Proposition 21.5** *Let  $(X_t, \mathbb{P}^x)$  be a strong Markov process. With  $F(t, \lambda)$  defined as in (21.5), suppose*

$$\frac{F(t, \lambda)}{t} \rightarrow 0 \tag{21.7}$$

as  $t \rightarrow 0$  for each  $\lambda > 0$ . Then  $X_t$  has continuous paths with  $\mathbb{P}^x$ -probability one for each  $x$ .

For  $X$  a Brownian motion,  $F(t, \lambda) \leq 2e^{-\lambda^2/8t}$  by Proposition 3.15, and hence  $F(t, \lambda)/t \rightarrow 0$  as  $t \rightarrow 0$ . Thus Brownian motion satisfies (21.7). On the other hand, (21.7) is not satisfied for the Poisson process; see Exercise 21.3.

*Proof* Suppose  $\lambda, t_0 > 0$  and  $X$  has a jump of size larger than  $4\lambda$  at some time before  $t_0$  with positive probability, that is,

$$\mathbb{P}^x(\sup_{t \leq t_0} d(X_{t-}, X_t) \geq 4\lambda) > 0,$$

where  $X_{t-} = \lim_{s \uparrow t, s < t} X_s$ . Then for each  $n$  there exists  $k \leq [t_0 2^n] + 1$  such that

$$\sup_{s, t \in [k/2^n, (k+1)/2^n]} d(X_s, X_t) \geq 4\lambda;$$

$[x]$  is the largest integer less than or equal to  $x$ . Therefore there exists  $k \leq [t_0 2^n] + 1$  such that

$$\sup_{s \in [k/2^n, (k+1)/2^n]} d(X_s, X_{k/2^n}) \geq 2\lambda.$$

But by Lemma 21.4

$$\begin{aligned}\mathbb{P}^x(\exists k \leq [t_0 2^n] + 1 : \sup_{k/2^n \leq s \leq (k+1)/2^n} d(X_s, X_{k/2^n}) \geq 2\lambda) \\ &\leq ([t_0 2^n] + 1) \sup_y \mathbb{P}^y(\sup_{s \leq 2^{-n}} d(X_s, X_0) \geq 2\lambda) \\ &\leq 2([t_0 2^n] + 1)F(2^{-n}, \lambda)\end{aligned}$$

for every  $n$ . In the first inequality we used the Markov property at time  $k/2^n$  and the fact that there are at most  $[t_0 2^n] + 1$  intervals. Letting  $n \rightarrow \infty$ , we see the probability of a jump of size larger than  $4\lambda$  before time  $t_0$  must be zero. Since  $\lambda$  and  $t_0$  are arbitrary, the paths of  $X$  are continuous.  $\square$

## 21.4 Harmonic functions

Suppose  $(X_t, \mathbb{P}^x)$  is a continuous Markov process satisfying the strong Markov property, and for each  $x$ , the sets of paths are right continuous with left limits with  $\mathbb{P}^x$ -probability one. Let

$D$  be an open subset of  $\mathcal{S}$ , and suppose that  $\tau_D < \infty$ , a.s., with respect to each  $\mathbb{P}^x$ , where  $\tau_D = \inf\{t : X_t \notin D\}$  is the time of the first exit from  $D$ . Let  $f$  be a bounded measurable function on  $\partial D$ , the boundary of  $D$ .

**Proposition 21.6** Define

$$h(x) = \mathbb{E}^x f(X_{\tau_D})$$

and  $\mathcal{F}'_s = \mathcal{F}_{s \wedge \tau_D}$ . Then for each  $x$ ,  $h(X_{t \wedge \tau_D})$  is a martingale under  $\mathbb{P}^x$  with respect to the filtration  $\{\mathcal{F}'_t\}$ .

*Proof* Let  $s < t$ . Consider a path  $\omega$  starting at  $x$  and continuing until it exits  $D$  at time  $\tau_D(\omega)$ . If we have  $u \leq \tau_D$  and we cut off the first  $u$  time units of the path, we have a path going from  $X_u(\omega)$  and proceeding until it exits  $D$ . But note that the point at which it exits will not be changed by cutting off a piece from the beginning of the path. Therefore  $X_{\tau_D} \circ \theta_u = X_{\tau_D}$  if  $u \leq \tau_D$ . Using this,

$$\begin{aligned} \mathbb{E}^x[h(X_{t \wedge \tau_D}) \mid \mathcal{F}_{s \wedge \tau_D}] &= \mathbb{E}^x[\mathbb{E}^{X_{t \wedge \tau_D}} f(X_{\tau_D}) \mid \mathcal{F}_{s \wedge \tau_D}] \\ &= \mathbb{E}^x[\mathbb{E}^x[f(X_{\tau_D}) \circ \theta_{t \wedge \tau_D} \mid \mathcal{F}_{t \wedge \tau_D}] \mid \mathcal{F}_{s \wedge \tau_D}] \\ &= \mathbb{E}^x[f(X_{\tau_D}) \mid \mathcal{F}_{s \wedge \tau_D}] \\ &= \mathbb{E}^x[f(X_{\tau_D}) \circ \theta_{s \wedge \tau_D} \mid \mathcal{F}_{s \wedge \tau_D}] \\ &= \mathbb{E}^{X_{s \wedge \tau_D}} f(X_{\tau_D}) = h(X_{s \wedge \tau_D}), \end{aligned}$$

as required.  $\square$

This becomes particularly interesting in the case when  $X_t$  is a  $d$ -dimensional Brownian motion. Suppose  $D$  is a bounded domain (i.e., a bounded open subset) in  $\mathbb{R}^d$ . There exists  $M$  such that  $D \subset B(0, M)$ . We know  $X_t^1$ , the first component of  $X_t$  is a one-dimensional Brownian motion, and by Theorem 7.2,  $X_t^1$  will exit  $[-M, M]$  in finite time, no matter what  $X_0^1$  is. Therefore the time for  $X_t$  to exit  $D$  will be finite almost surely with respect to each  $\mathbb{P}^x$ . Take  $x \in D$  and take  $\delta$  smaller than the distance from  $x$  to the boundary of  $D$ . If  $S = \inf\{t : |X_t - x| \geq \delta\}$ , the first time  $X$  leaves the ball of radius  $\delta$  about  $x$ , then by Proposition 21.6 and optional stopping, we have

$$h(x) = \mathbb{E}^x h(X_S). \quad (21.8)$$

By Exercise 2.3 we know that  $d$ -dimensional Brownian motion is rotationally invariant. We conclude from this that the location where a Brownian motion hits the boundary of a ball of radius  $\delta$  about the starting point must have a uniform distribution. Hence  $X_S$  will be uniformly distributed on  $\partial B(x, \delta)$ . Thus (21.8) can be rewritten as

$$h(x) = \int_{\partial B(x, \delta)} h(y) \sigma_{x, \delta}(dy),$$

where  $\sigma_{x, \delta}$  is a surface measure on  $\partial B(x, \delta)$  normalized to have total mass one. This holds for every  $\delta$  small enough, and since  $h$  is bounded (because  $f$  is), it can be shown that  $h$  is  $C^2$

in  $D$  and is harmonic there:

$$\Delta h(x) = \sum_{i=1}^d \frac{\partial^2 h}{\partial x_i^2}(x) = 0;$$

the proof is not obvious – see Bass (1995), Section II.1.

We can use Proposition 21.6 to give a solution to the Dirichlet problem. In the Dirichlet problem one is given a domain in  $\mathbb{R}^d$  and a continuous function  $f$  on the boundary of  $D$ . One wants to find a continuous function  $h$  that is harmonic inside  $D$ , that is,  $\Delta h(x) = 0$  for  $x \in D$ , and that agrees with  $f$  on  $\partial D$ . There are domains for which one cannot solve the Dirichlet problem, but a solution can be found provided the domain is moderately nice. We explain how to solve the Dirichlet problem probabilistically; the class of domains where one can do this is the same as the class where one can solve the Dirichlet problem analytically.

Let us say that a point  $x$  is *regular* for a Borel subset  $A$  if  $\mathbb{P}^x(T_A = 0) = 1$ , where  $T_A = \inf\{t > 0 : X_t \in A\}$ . Thus a point  $x$  is regular for a set  $A$  if starting at  $x$  the Brownian motion enters  $A$  immediately. For example, a consequence of Theorem 7.2 is that the point 0 is regular for the set  $A = (0, \infty)$  when we have a one-dimensional Brownian motion.

**Theorem 21.7** Suppose  $D$  is a bounded open domain in  $\mathbb{R}^d$  and  $f$  is a function on  $\partial D$  that is continuous on  $\partial D$ . Let  $(X_t, \mathbb{P}^x)$  be a  $d$ -dimensional Brownian motion and  $\tau_D = \inf\{t : X_t \in D^c\}$ . If each point of  $\partial D$  is regular for  $D^c$ , then  $h(x) = \mathbb{E}^x f(X_{\tau_D})$  is a solution to the Dirichlet problem.

The regularity condition says that starting at any point of  $\partial D$ , Brownian motion enters  $D^c$  immediately. Uniqueness of the solution to the Dirichlet problem is easy, and we do not address this here.

*Proof* We have already seen in Proposition 21.6 and the remarks immediately following the proof of that proposition that  $h$  is harmonic in  $D$ . This implies that  $h$  is continuous in  $D$ . Thus we only need to show that  $h$  agrees with  $f$  on  $\partial D$ .

Our first step is to fix  $t$  and  $\varepsilon$  and to show that the set

$$\{x : \mathbb{P}^x(\tau_D \leq t) > 1 - \varepsilon\}$$

is an open set. Let  $s < t$ , define  $\varphi_s(x) = \mathbb{P}^x(\tau_D \leq t - s)$ , and let

$$w_s(x) = \mathbb{P}^x(X_u \in D^c \text{ for some } u \in [s, t]).$$

By the Markov property at time  $s$ ,

$$\begin{aligned} w_s(x) &= \mathbb{E}^x \mathbb{P}^{X_s}(X_u \in D^c \text{ for some } u \in [0, t - s]) = \mathbb{E}^x [\mathbb{P}^{X_s}(\tau_D \leq t - s)] \\ &= \mathbb{E}^x \varphi_s(X_s) = (2\pi s)^{-d/2} \int \varphi_s(y) e^{-|x-y|^2/2s} dy. \end{aligned}$$

By dominated convergence, the last integral is a continuous function of  $x$ . If

$$w_0(x) = \mathbb{P}^x(X_u \in D^c \text{ for some } u \in [0, t]),$$

then  $w_s(x) \uparrow w_0(x)$ , so  $\{x : w_0(x) > 1 - \varepsilon\} = \cup_{s \in (0, t)} \{x : w_s(x) > 1 - \varepsilon\}$  is open.

Let  $z \in \partial D$ . Let  $\varepsilon > 0$  and choose  $\eta$  such that  $|f(w) - f(z)| < \varepsilon$  if  $|w - z| < \eta$  and  $w \in \partial D$ . Pick  $t$  small so that  $\mathbb{P}^0(\sup_{s \leq t} |X_s| > \eta/2) < \varepsilon$ ; this is possible because Brownian

motion has continuous paths. Because  $z \in \partial D$  and every point of  $\partial D$  is regular for  $D^c$ ,  $\mathbb{P}^z(\tau_D \leq t) = 1$ . Finally choose  $\delta < (\eta/2) \wedge \varepsilon$  so that if  $|w - z| < \delta$  and  $w \in D$ , then  $\mathbb{P}^w(\tau_D \leq t) > 1 - \varepsilon$ .

Now if  $|w - z| < \delta$  and  $w \in D$ , then

$$\begin{aligned}\mathbb{P}^w(|X_{\tau_D} - z| < \eta) &\geq \mathbb{P}^w(\tau_D \leq t, \sup_{s \leq t} |X_s - w| \leq \eta/2) \\ &\geq \mathbb{P}^w(\tau_D \leq t) - \mathbb{P}^0(\sup_{s \leq t} |X_s| > \eta/2) \\ &\geq (1 - \varepsilon) - \varepsilon.\end{aligned}$$

The set  $\partial D$  is a bounded and closed subset of  $\mathbb{R}^d$ , hence compact, and since  $f$  is continuous on  $\partial D$ , there exists  $M$  such that  $|f'|$  is bounded by  $M$ . If  $|w - z| < \delta$  and  $w \in D$ ,

$$\begin{aligned}|h(w) - f(z)| &= |\mathbb{E}^w f(X_{\tau_D}) - f(z)| \\ &\leq |\mathbb{E}^w [f(X_{\tau_D}); |X_{\tau_D} - z| < \eta] - f(z)\mathbb{P}^w(|X_{\tau_D} - z| < \eta)| \\ &\quad + 2M\mathbb{P}^w(|X_{\tau_D} - z| \geq \eta) \\ &\leq \varepsilon\mathbb{P}^w(|X_{\tau_D} - z| < \eta) + 4M\varepsilon \leq (1 + 4M)\varepsilon.\end{aligned}$$

We used the fact that  $|f(X_{\tau_D}) - f(z)| < \varepsilon$  if  $|X_{\tau_D} - z| < \eta$ . Since  $\varepsilon$  is arbitrary, this proves that  $h(w) \rightarrow f(z)$  as  $w \rightarrow z$  inside  $D$ .  $\square$

Let us give a sufficient condition for a point to be regular for a domain  $D$ . Let  $\tilde{V}_a = \{(x_1, \dots, x_d) : x_1 > 0, (x_2^2 + \dots + x_d^2) < a^2 x_1^2\}$ . The vertex of  $\tilde{V}_a$  is the origin. A cone  $V$  in  $\mathbb{R}^d$  is a translation and rotation of  $\tilde{V}_a$  for some  $a$ .

The following is known as the *Poincaré cone condition*.

**Proposition 21.8** *Suppose there exists a cone  $V$  with vertex  $y \in \partial D$  such that  $V \cap B(y, r) \subset D^c$  for some  $r > 0$ . Then  $y$  is regular for  $D^c$ .*

*Proof* By translation and rotation of the coordinates, we may suppose  $y = 0$  and  $V = \tilde{V}_a$  for some  $a$ . Then for each  $t$ ,

$$\begin{aligned}\mathbb{P}^0(\tau_D \leq t) &\geq \mathbb{P}^0(X_t \in D^c) \geq \mathbb{P}^0(X_t \in V \cap B(0, r)) \\ &\geq \mathbb{P}^0(X_t \in V) - \mathbb{P}^0(X_t \notin B(0, r)).\end{aligned}$$

By scaling, the last term is  $\mathbb{P}^0(X_1 \in V) - \mathbb{P}^0(X_1 \notin B(0, r/\sqrt{t}))$ , which converges to

$$\mathbb{P}^0(X_1 \in V) = (2\pi)^{-d/2} \int_V e^{-|z|^2/2} dz > 0$$

as  $t \rightarrow 0$ . Observe  $\mathbb{P}^0(\tau_D \leq t)$  converges to  $\mathbb{P}^0(\tau_D = 0)$ . By the Blumenthal 0–1 law (Proposition 20.8),  $\mathbb{P}^0(\tau_D = 0) = 1$ .  $\square$

Continue to suppose  $(X_t, \mathbb{P}^x)$  is a  $d$ -dimensional Brownian motion and  $D$  is a bounded domain, but now we suppose  $d \geq 3$ . Define

$$U(x, A) = \mathbb{E}^x \int_0^\infty 1_A(X_s) ds, \quad x \in D.$$

This is the same as the  $\lambda$ -resolvent of  $1_A$  with  $\lambda = 0$ . We write

$$\begin{aligned} U(x, A) &= \mathbb{E}^x \int_0^\infty 1_A(X_s), ds \\ &= \int_0^\infty \mathbb{P}^x(X_s \in A) ds \\ &= \int_0^\infty \int_A \frac{1}{(2\pi s)^{d/2}} e^{-|y-x|^2/2s} dy ds \\ &= \int_A \int_0^\infty \frac{1}{(2\pi s)^{d/2}} e^{-|y-x|^2/2s} ds dy. \end{aligned}$$

Some calculus shows that the inside integral is equal to  $c|x - y|^{2-d}$ . If we denote  $c|x - y|^{2-d}$  by  $u(x, y)$ , we then have that

$$U(x, A) = \int_A u(x, y) dy. \quad (21.9)$$

The expression  $u(x, y)$  is called the *Newtonian potential density*. Note that  $u(x, y)$  is a function only of  $|x - y|$ , it blows up as  $|x - y| \rightarrow 0$ , and tends to 0 as  $|x - y| \rightarrow \infty$ .

If  $x$  is in the interior of  $D$ , then  $u(x, \cdot)$  will be bounded on  $\partial D$ . Define  $h_x(z) = \mathbb{E}^z u(x, X_{\tau_D})$ ; we saw above that  $h_x$  is harmonic. Now define  $g_D(x, y) = u(x, y) - h_x(y)$ ; this function of two variables is called the *Green's function* or *Green function* for  $D$  with pole at  $x$ . This is a well-known object in analysis – let us give a probabilistic interpretation. Since  $u(x, y)$  is symmetric in  $x$  and  $y$ , if  $A \subset D$  we have

$$\begin{aligned} \int_A g_D(x, y) dx &= \int_A u(x, y) dx - \int_A \mathbb{E}^y u(x, X_{\tau_D}) dx \\ &= \mathbb{E}^y \int_0^\infty 1_A(X_s) ds - \mathbb{E}^y \int_A u(x, X_{\tau_D}) dx \\ &= \mathbb{E}^y \int_0^\infty 1_A(X_s) ds - \mathbb{E}^y \left[ \mathbb{E}^{X_{\tau_D}} \int_0^\infty 1_A(X_s) ds \right]. \end{aligned} \quad (21.10)$$

Using the strong Markov property and then a change of variables,

$$\begin{aligned} \mathbb{E}^y \left[ \mathbb{E}^{X_{\tau_D}} \int_0^\infty 1_A(X_s) ds \right] &= \mathbb{E}^y \left[ \mathbb{E}^y \left[ \int_0^\infty 1_A(X_s) \circ \theta_{\tau_D} ds \mid \mathcal{F}_{\tau_D} \right] \right] \\ &= \mathbb{E}^y \int_0^\infty 1_A(X_s) \circ \theta_{\tau_D} ds \\ &= \mathbb{E}^y \int_0^\infty 1_A(X_{\tau_D+s}) ds \\ &= \mathbb{E}^y \int_{\tau_D}^\infty 1_A(X_s) ds. \end{aligned}$$

Substituting this in (21.10) we have

$$\int_A g_D(x, y) dx = \mathbb{E}^y \int_0^{\tau_D} 1_A(X_s) ds.$$

For this reason  $g_D$  is sometimes called the *occupation time density* for  $D$ .

## Exercises

- 21.1 Suppose  $d = 2$ ,  $(X_t, \mathbb{P}^x)$  is a two-dimensional Brownian motion, and  $r > 0$ . Imitate the argument of Proposition 21.1 but with  $h(x) = \log(|x|)$  to show that  $\mathbb{P}^x(X_t \text{ hits } B(0, r)) = 1$  when  $|x| > r$ . Then use the strong Markov property to show that there are times  $T_i \rightarrow \infty$  with  $X_{T_i} \in B(0, r)$ . That is, two-dimensional Brownian motion is neighborhood recurrent.
- 21.2 In the proof of Lemma 21.4, justify each inequality in (21.6).
- 21.3 Let  $(X_t, \mathbb{P}^x)$  be a Poisson process with parameter  $a$  and let  $F$  be defined by (21.5). Show  $F(t, 1/2)/t$  does not converge to 0 as  $t \rightarrow 0$ .
- 21.4 Suppose  $d \geq 3$ ,  $(X_t, \mathbb{P}^x)$  is a  $d$ -dimensional Brownian motion, and

$$Uf(x) = \mathbb{E}^x \int_0^\infty f(X_s) ds.$$

Show that if  $f$  is bounded and measurable with compact support, then  $Uf$  is continuous and  $|Uf(x)| \rightarrow 0$  as  $|x| \rightarrow \infty$ . Show that if  $f \in C^2$  with compact support, then  $Uf$  is  $C^2$ . Show that  $\frac{1}{2}\Delta Uf = -f$ .

- 21.5 Let  $W_t$  be a Brownian motion and  $f$  a continuous function. Prove that if  $f(W_t)$  is a submartingale, then  $f$  must be convex.
- 21.6 Prove the *maximum principle* for harmonic functions. This says that if  $h$  is harmonic in a bounded domain  $D$ , then

$$\sup_{x \in \bar{D}} |h(x)| \leq \sup_{x \in \partial D} |h(x)|.$$

- 21.7 If  $W$  is a  $d$ -dimensional Brownian motion started at 0, find  $\mathbb{E} T$ , where  $T$  is the first time  $W$  exits the ball of radius  $r$  centered at the origin.

*Hint:* Use the fact that  $|W_t|^2 - dt$  is a martingale.

- 21.8 Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded function with  $|f(x) - f(y)| \leq |x - y|$  for all  $x, y \in \mathbb{R}$ . Let  $D_\varepsilon = \{(x, y) \in \mathbb{R}^2 : f(x) < y < f(x) + \varepsilon\}$  for  $\varepsilon \in (0, 1)$ . Let  $(X_t, \mathbb{P}^x)$  be a two-dimensional Brownian motion and let  $\tau_\varepsilon = \inf\{t : X_t \notin D_\varepsilon\}$ . Prove that there exists a constant  $c$  not depending on  $\varepsilon$  such that  $\mathbb{E}^0 \tau_\varepsilon \leq c\varepsilon^2$ .

*Hint:* By Exercise 21.7 the expected time for two-dimensional Brownian motion to leave a ball of radius  $2\varepsilon$  is less than  $c\varepsilon^2$ . Then use the strong Markov property repeatedly at the times  $S_i$ , where  $S_i$  is the first time after time  $S_{i-1}$  that Brownian motion has moved at least  $2\varepsilon$  from  $X_{S_{i-1}}$ .

## Transformations of Markov processes

There are a number of interesting transformations that make new Markov processes out of old. We will look at four: killing, conditioning, changing time, and stopping at a last exit time. These are only a few of the possible transformations.

### 22.1 Killed processes

One sometimes wants to consider a Markov process up until a stopping time  $\zeta$ , called the *lifetime* of the process. We affix to our state space  $\mathcal{S}$  an isolated point  $\Delta$ , called the *cemetery* state, and the topology on  $\mathcal{S}_\Delta = \mathcal{S} \cup \{\Delta\}$  is the one generated by the collection of open sets of  $\mathcal{S}$  together with the set  $\{\Delta\}$ . We define the *killed process*  $\widehat{X}$  by

$$\widehat{X}_t = \begin{cases} X_t, & t < \zeta; \\ \Delta, & t \geq \zeta, \end{cases} \quad (22.1)$$

and we say we *kill* the process  $X$  at time  $\zeta$ . Every function  $f$  on  $\mathcal{S}$  is defined to be 0 at  $\Delta$ .

One example of this situation would be to let  $\zeta = \tau_D$ , where  $D$  is a subset of  $\mathcal{S}$  and  $\tau_D = \inf\{t > 0 : X_t \notin D\}$ , the first exit from the set  $D$ . Another common occurrence is to let  $\zeta = S$ , where  $S$  is a random variable with an exponential distribution with parameter  $\lambda$ , i.e.,  $\mathbb{P}(S > t) = e^{-\lambda t}$ , such that  $S$  is independent of  $X$ . A third possibility would be to let  $\zeta = \inf\{t : \int_0^t f(X_s) ds \geq 1\}$ , where  $f$  is a non-negative function. The crucial property of  $\zeta$  is that it be a *terminal time*:

$$\zeta = s + \zeta \circ \theta_s \quad \text{if } s < \zeta. \quad (22.2)$$

**Proposition 22.1** *If  $(X_t, \mathbb{P}^x)$  is a strong Markov process and (22.2) holds, then  $(\widehat{X}_t, \mathbb{P}^x)$  satisfies the Markov and strong Markov properties.*

*Proof* As in Section 20.2, we need to show

$$\mathbb{E}^x[f(\widehat{X}_t) \circ \theta_T | \mathcal{F}_T] = \mathbb{E}^{\widehat{X}_T} f(\widehat{X}_t), \quad \mathbb{P}^x\text{-a.s.}$$

If  $A \in \mathcal{F}_T$ ,

$$\mathbb{E}^x[f(\widehat{X}_t) \circ \theta_T; A] = \mathbb{E}^x[f(X_{t+T}); A, T + t < \zeta].$$

On the other hand,

$$\begin{aligned}\mathbb{E}^{\widehat{X}_T} f(\widehat{X}_t) &= \mathbb{E}^{X_T}[f(X_t); t < \zeta]1_{(T < \zeta)} \\ &= \mathbb{E}^x[f(X_t) \circ \theta_T; t \circ \theta_T < \zeta \circ \theta_T | \mathcal{F}_T]1_{(T < \zeta)} \\ &= \mathbb{E}^x[f(X_{t+T}); T + t \circ \theta_T < T + \zeta \circ \theta_T, T < \zeta | \mathcal{F}_T] \\ &= \mathbb{E}^x[f(X_{t+T}); T + t < \zeta | \mathcal{F}_T],\end{aligned}$$

since  $T + t \circ \theta_T = T + t$  and  $T + \zeta \circ \theta_T = \zeta$  on  $(T < \zeta)$ . Hence

$$\mathbb{E}^x[\mathbb{E}^{\widehat{X}_T} f(\widehat{X}_t); A] = \mathbb{E}^x[f(X_{t+T}); T + t < \zeta, A],$$

as required.  $\square$

## 22.2 Conditioned processes

Another type of transformation of a Markov process is by conditioning, also known as *Doob's h-path transform*. To motivate this, let  $D$  be a domain in  $\mathbb{R}^d$  and let  $X_t$  be a Brownian motion killed on exiting the domain. One would like to give a precise meaning to the intuitive notion of Brownian motion conditioned to exit the domain at a certain point. Let  $h$  be a positive harmonic function in  $D$  (i.e.,  $h$  is  $C^2$  in  $D$ , and  $\Delta h = 0$  there) and suppose that  $h$  is 0 everywhere on the boundary of  $D$  except at one point  $z$ . The Poisson kernel for the ball or for the half-space gives examples of such harmonic functions. Then, heuristically, we have by the Markov property at time  $t$ ,

$$\begin{aligned}\mathbb{P}^x(X_t \in dy | X_{\tau_D} = z) &= \frac{\mathbb{P}^x(X_t \in dy, X_{\tau_D} = z)}{\mathbb{P}^x(X_{\tau_D} = z)} \\ &= \frac{\mathbb{P}^x(X_t \in dy) \mathbb{P}^y(X_{\tau_D} = z)}{\mathbb{P}^x(X_{\tau_D} = z)}.\end{aligned}$$

If  $p^0(t, x, dy)$  represents the probability that Brownian motion started at  $x$  and killed on leaving  $D$  is in  $dy$  at time  $t$ , we then expect that the analogous probability for Brownian motion conditioned to exit  $D$  at  $z$  ought to be  $h(y)p^0(t, x, dy)/h(x)$ . We now make this precise.

Let us look at a strong Markov process  $X$ . We say a function  $h$  is *invariant* with respect to  $X$  if  $P_t h(x) = h(x)$  for all  $t$  and  $x$ , where  $P_t$  is the semigroup associated with  $X$ . If  $h$  is invariant, by the Markov property,

$$\begin{aligned}\mathbb{E}^x[h(X_t) | \mathcal{F}_s] &= \mathbb{E}^x[h(X_{t-s}) \circ \theta_s | \mathcal{F}_s] = \mathbb{E}^{X_s} h(X_{t-s}) \\ &= P_{t-s} h(X_s) = h(X_s),\end{aligned}$$

and so for each  $x$ ,  $h(X_t)$  is a martingale with respect to  $\mathbb{P}^x$ . Conversely, if  $h(X_t)$  is a martingale with respect to  $\mathbb{P}^x$  for all  $x$ ,

$$P_t h(x) = \mathbb{E}^x h(X_t) = h(x)$$

by the definition of martingale, and so  $h$  is invariant. In the case of Brownian motion killed on leaving a domain, the invariant functions are thus the harmonic ones.

Now let  $h$  be a non-negative invariant function for a strong Markov process  $X$ . Letting  $M_t = h(X_t)/h(X_0)$ ,  $M_t$  is a non-negative continuous martingale with  $M_0 = 1$ ,  $\mathbb{P}^x$ -a.s., as long as  $h(x) > 0$ .

We define the  $h$ -path transform of the Markov process  $X$  by setting

$$\mathbb{P}_h^x(A) = \mathbb{E}^x[M_t; A], \quad A \in \mathcal{F}_t. \quad (22.3)$$

Since  $M_0 = 1$ ,  $\mathbb{P}_h^x(\Omega) = 1$ . Observe that  $P_h^x$  gives more mass to paths where  $h(X_t)$  is big and less to where it is small. Note the similarity to the Girsanov theorem.

We have the following.

**Proposition 22.2** *Suppose  $(X_t, \mathbb{P}^x)$  is a strong Markov process and that  $h$  is non-negative and invariant. Then  $(X_t, \mathbb{P}_h^x)$  forms a strong Markov process.*

*Proof* Suppose  $A \in \mathcal{F}_s$  and  $h(x) \neq 0$ . (We leave consideration of the case where  $h(x) = 0$  to the reader.) Then

$$\begin{aligned} \mathbb{E}_h^x[f(X_{t+s}); A] &= \frac{\mathbb{E}^x[f(X_{t+s})h(X_{t+s}); A]}{h(x)} \\ &= \frac{\mathbb{E}^x[\mathbb{E}^{X_s}[f(X_t)h(X_t)]; A]}{h(x)} \\ &= \mathbb{E}^x\left[\frac{1}{h(X_s)}\mathbb{E}^{X_s}[f(X_t)h(X_t)]h(X_s); A\right] \end{aligned}$$

by the Markov property for  $X$ . This is equal to

$$\mathbb{E}^x[\mathbb{E}_h^{X_s}[f(X_t)]h(X_s); A]/h(x) = E_h^x[\mathbb{E}_h^{X_s}f(X_t); A].$$

The Markov property follows from this. The strong Markov property is proved in almost identical fashion.  $\square$

Let us consider an example. Let  $(X_t, \mathbb{P}^x)$  be a Brownian motion on the non-negative axis killed on first hitting 0. This is the same as a Brownian motion killed on exiting  $(0, \infty)$ . This will be a strong Markov process. Since the second derivative of the function  $h(x) = x$  is 0, then  $h$  is harmonic on  $(0, \infty)$ , and so is invariant for the killed Brownian motion. Let us now condition using the function  $h$  to get Brownian motion conditioned to hit infinity before hitting zero.

To identify the resulting process, we argue as follows. Fix  $x$  and let  $T_\varepsilon = \inf\{t > 0 : X_t < \varepsilon\}$ . The Radon–Nikodym derivative of the law of  $\mathbb{P}_h^x$  with respect to  $\mathbb{P}^x$  on  $\mathcal{F}_{t \wedge T_\varepsilon}$  is  $M_{t \wedge T_\varepsilon} = h(X_{t \wedge T_\varepsilon})/h(x)$ . We can rewrite  $M_{t \wedge T_\varepsilon}$  as

$$M_{t \wedge T_\varepsilon} = \exp(\log X_{t \wedge T_\varepsilon} - \log x) = \exp\left(\int_0^{t \wedge T_\varepsilon} \frac{1}{X_s} dX_s - \frac{1}{2} \int_0^{t \wedge T_\varepsilon} \left(\frac{1}{X_s}\right)^2 ds\right),$$

using Itô's formula. By the Girsanov theorem, under  $\mathbb{P}_h^x$ ,

$$W_{t \wedge T_\varepsilon} = X_{t \wedge T_\varepsilon} - \int_0^{t \wedge T_\varepsilon} \frac{1}{X_s} ds$$

is a martingale. By Exercise 13.2, its quadratic variation is  $t \wedge T_\varepsilon$ , and so by Exercise 12.3,  $W_{t \wedge T_\varepsilon}$  is a Brownian motion stopped at time  $T_\varepsilon$ . We have

$$X_{t \wedge T_\varepsilon} = x + W_{t \wedge T_\varepsilon} + \int_0^{t \wedge T_\varepsilon} \frac{1}{X_s} ds,$$

or  $X$  satisfies the stochastic differential equation

$$dX_t = dW_t + \frac{1}{X_t} dt$$

for  $t \leq T_\varepsilon$ . We will see later (Section 24.3) that this is the stochastic differential equation defining the Bessel process of order 3. The same argument shows that Brownian motion killed on exiting  $(0, a)$  and then conditioned to hit  $a$  before 0 is also a Bessel process of order 3 up until the time of first hitting  $a$ .

### 22.3 Time change

An *additive functional* is an increasing adapted process with  $A_0 = 0$ , a.s., such that

$$A_t = A_s + A_{t-s} \circ \theta_s \quad (22.4)$$

if  $s < t$ . The simplest examples are what are known as *classical additive functionals*:  $A_t = \int_0^t f(X_r) dr$ , where  $f$  is a non-negative measurable function. We have

$$A_t - A_s = \int_s^t f(X_r) dr = \int_0^{t-s} f(X_r) dr \circ \theta_s = A_{t-s} \circ \theta_s.$$

If we have the uniform limit of additive functionals, we again get an additive functional, and thus, for example, the local times  $L_t^x$  of a one-dimensional Brownian motion are also additive functionals.

Given a Markov process  $X$  and an additive functional  $A$ , let

$$B_t = \inf\{u : A_u > t\}$$

and

$$X'_t = X_{B_t}.$$

Let  $\mathcal{F}'_t = \mathcal{F}_{B_t}$ . Thus  $X'$  is a time change of  $X$ .

**Proposition 22.3** *Let  $(X_t, \mathbb{P}^x)$  be a strong Markov process and  $A_t$  an additive functional. With  $B$  defined as above,  $(X'_t, \mathbb{P}^x)$  is also a strong Markov process.*

*Proof* We verify the strong Markov property. Let  $\mathcal{F}'_t = \mathcal{F}_{B_t}$ . Then if  $T$  is a stopping time for  $\mathcal{F}'_t$ , we have

$$\mathbb{E}^x[f(X'_{T+t}) \mid \mathcal{F}'_T] = \mathbb{E}^x[f(X(B_{T+t})) \mid \mathcal{F}_{B_T}].$$

$B_T$  can be seen to be a stopping time with respect to  $\{\mathcal{F}_t\}$  and  $B_{T+t} = B_t \circ \theta_{B_T}$  where the  $\theta_t$  are the shift operators, so this is

$$\mathbb{E}^x \mathbb{E}^{X(B_T)} f(X_{B_t}) = \mathbb{E}^x \mathbb{E}^{X'_T} f(X'_t).$$

This suffices to show that  $X'$  is a strong Markov process. □

## 22.4 Last exit decompositions

Let  $A$  be a Borel set, and let  $L$  be the last visit to  $A$ :

$$L = \sup\{s : X_s \in A\}.$$

We define  $L$  to be 0 if  $X$  never hits  $A$ . The random time  $L$  is not a stopping time, but we can nevertheless kill the process  $X$  at time  $L$ . It turns out the resulting process  $Y$  is the process  $X$  conditioned by the function  $h(x) = \mathbb{P}^x(T_A < \infty)$ . The intuitive meaning of this is that  $Y$  is  $X$  conditioned to hit the set  $A$ .

Let  $T = \inf\{t : X_t \in A\}$ , and set

$$Y_t = \begin{cases} X_t, & t < L, \\ \Delta, & t \geq L. \end{cases}$$

Let  $\mathcal{H}_t = \sigma(Y_s; s \leq t)$ .

**Proposition 22.4** *If  $(X_t, \mathbb{P}^x)$  is a strong Markov process, then  $(Y_t, \mathbb{P}^x)$  is a Markov process with respect to  $\{\mathcal{H}_t\}$ .*

*Proof* If  $B \subset \mathcal{S}$  (so that  $\Delta \notin B$ ), then

$$(Y_t \in B) = (X_t \in B, L > t) = (X_t \in B, T \circ \theta_t < \infty),$$

since  $L$ , the last time that  $X$  is in  $A$ , will be larger than  $t$  if and only if  $X$  hits  $A$  at some time after time  $t$ . We conclude that the function  $x \rightarrow \mathbb{P}^x(Y_t \in B)$  is Borel measurable. Since

$$\mathbb{P}^x(Y_t = \Delta) = \mathbb{P}^x(L \leq t) = 1 - \mathbb{P}^x(L > t) = 1 - \mathbb{P}^x(T \circ \theta_t < \infty),$$

then the function  $x \rightarrow \mathbb{P}^x(Y_t = \Delta)$  is also Borel measurable.

We need to show that if  $C \in \mathcal{H}_s$ ,

$$\mathbb{E}^x[f(Y_t); C] = \mathbb{E}^x[Q_{t-s}f(Y_s); C], \quad (22.5)$$

where  $f$  is bounded and measurable,  $h(x) = \mathbb{P}^x(L > 0)$ , and

$$Q_t g(x) = \frac{1}{h(x)} P_t(g h)(x)$$

when  $h(x) \neq 0$ . (Set  $Q_t g(x) = 0$  if  $h(x) = 0$ .)

It suffices to show (22.5) when  $C = (Y_{r_1} \in B_1, \dots, Y_{r_n} \in B_n)$  for  $r_1 \leq \dots \leq r_n \leq s$  and the  $B_1, \dots, B_n$  are Borel sets. If we set

$$C_s = (X_{r_1} \in B_1, \dots, X_{r_n} \in B_n),$$

then  $C_s \in \mathcal{F}_s$ ,  $C \cap (L > s) = C_s \cap (L > s)$ , and  $C \cap (L > t) = C_s \cap (L > t)$ .

We start with

$$\begin{aligned} \mathbb{E}^x[f(Y_t); C] &= \mathbb{E}^x[f(X_t); C, L > t] = \mathbb{E}^x[f(X_t); C_s, L > t] \\ &= \mathbb{E}^x[f(X_t); C_s, L \circ \theta_t > 0]. \end{aligned}$$

Conditioning on  $\mathcal{F}_t$ , this is equal to

$$\mathbb{E}^x[f(X_t)\mathbb{P}^{X_t}(L > 0); C_s] = \mathbb{E}^x[f(X_t)h(X_t); C_s].$$

Conditioning on  $\mathcal{F}_s$ , this in turn is equal to

$$\begin{aligned}\mathbb{E}^x[P_{t-s}(fh)(X_{t-s}); C_s] &= \mathbb{E}^x[h(X_s)Q_{t-s}f(X_s); C_s] \\ &= \mathbb{E}^x[\mathbb{P}^{X_s}(L > 0)Q_{t-s}f(X_s); C_s] \\ &= \mathbb{E}^x[Q_{t-s}f(X_s); C_s, L \circ \theta_s > 0],\end{aligned}\tag{22.6}$$

where we used the Markov property for the last equality. Continuing, we have that the last line of (22.6) is equal to

$$\begin{aligned}\mathbb{E}^x[Q_{t-s}f(X_s); C_s, L > s] &= \mathbb{E}^x[Q_{t-s}f(X_s); C, L > s] \\ &= \mathbb{E}^x[Q_{t-s}f(Y_s); C],\end{aligned}$$

as desired.  $\square$

We can also look at  $X_{L+t}$ , where  $L$  is as above. This new process is again a strong Markov process, and this time is the process  $X$  conditioned by the function  $h(x) = \mathbb{P}^x(T_A = \infty)$ . The intuitive meaning of this is that  $X_{L+t}$  is  $X$  conditioned never to hit  $A$ . Since we are looking at the process after the last visit to  $A$ , this is entirely plausible. For a proof of the Markov property of  $X_{L+t}$ , see Meyer *et al.* (1972).

## Exercises

- 22.1 Let  $(X_t, \mathbb{P}^x)$  be a one-dimensional Brownian motion,  $L_t^x$  the local time of Brownian motion at  $x$ , and  $m$  a positive finite measure on  $\mathbb{R}$ . Show that  $A_t = \int L_t^x m(dx)$  is an additive functional.
- 22.2 We consider the *space-time process*. Let  $V_t = V_0 + t$ . The process  $V_t$  is simply the process that increases deterministically at unit speed. Thus  $V_t$  can represent time. If  $(X_t, \mathbb{P}^x)$  is a Markov process, show that  $((X_t, V_t), \mathbb{P}^{(x,v)})$  is also a Markov process. Is  $((X_t, V_t), \mathbb{P}^{(x,v)})$  necessarily a strong Markov process if  $(X_t, \mathbb{P}^x)$  is a strong Markov process?
- For some applications, one lets  $V_t = V_0 - t$ , and one thinks of time running backwards. Space-time processes are useful when considering parabolic partial differential equations.
- 22.3 Suppose  $(X_t, \mathbb{P}^x)$  is a strong Markov process and  $f$  is a non-negative invariant function for  $(X_t, \mathbb{P}^x)$ . Write  $\mathbb{Q}^x$  for  $\mathbb{P}_f^x$ . Suppose  $g$  is a non-negative invariant function for  $(X_t, \mathbb{Q}^x)$ . Show that  $fg$  is a non-negative invariant function for  $(X_t, \mathbb{P}^x)$  and that  $\mathbb{Q}_g^x = \mathbb{P}_{fg}^x$ .
- 22.4 Suppose  $A$  and  $B$  are additive functionals for a Markov process and  $A$  and  $B$  have continuous paths. Prove that if  $\mathbb{E}^x A_t = \mathbb{E}^x B_t$  for all  $x$  and  $t$ , then

$$\mathbb{P}^x(A_t \neq B_t \text{ for some } t \geq 0) = 0$$

for all  $x$ .

*Hint:* Show  $A_t - B_t$  is a martingale.

- 22.5 Suppose  $A$  and  $B$  are additive functionals with continuous paths and suppose  $\mathbb{E}^x A_\infty = \mathbb{E}^x B_\infty < \infty$  for each  $x$ . Show

$$\mathbb{P}^x(A_t \neq B_t \text{ for some } t \geq 0) = 0$$

for each  $x$ .

*Hint:* If  $f(x) = \mathbb{E}^x A_\infty$ , then

$$\mathbb{E}^x[A_\infty | \mathcal{F}_t] - A_t = \mathbb{E}^x A_\infty - f(X_t),$$

and similarly with  $B$  in place of  $A$ . Then  $A - B$  is a  $\mathbb{P}^x$  martingale for each  $x$ .

- 22.6 Let  $A$  be an additive functional with continuous paths. Suppose there exists  $K > 0$  such that  $\mathbb{E}^x A_\infty \leq K$  for each  $x$ . Prove that there exists a constant  $c$  depending only on  $K$  such that

$$\mathbb{E} e^{cA_\infty} < \infty, \quad x \in \mathcal{S}.$$

- 22.7 Here is an argument that the law of a Brownian motion conditioned to have a maximum at a certain level is a Bessel process of order 3.

Let  $W$  be a one-dimensional Brownian motion killed on hitting 0. Let  $S_t = \sup_{s \leq t} W_s$  be the maximum. By Exercise 19.1,  $X = (W, S)$  is a Markov process. Determine the law of  $X$  for  $t \leq L$ , where  $L$  is the last time  $X$  hits the diagonal. To define  $L$  more precisely, let  $D = \{(w, s) : w = s, w > 0\}$  and  $L = \sup\{t \geq 0 : X_t \in D\}$ .  $L$  is finite, a.s., because  $W$  will hit 0 in finite time with probability one.

## Notes

Markov processes are in some sense supposed to have the property that the past and the future are independent given the present. From this point of view, one might hope that a Markov process run backwards is again a Markov process. This is, more or less, the case; see [Chung and Walsh \(1969\)](#) or [Rogers and Williams \(2000a\)](#).

# 23

---

## Optimal stopping

A nice application of Markov process theory is optimal stopping. Suppose we have a *reward function*  $g \geq 0$  and we want to find the stopping time  $T$  that maximizes the value of  $\mathbb{E}^x g(X_T)$  and we also want to find the value of this expectation. This is the *optimal stopping problem*.

An important example of an optimal stopping problem is pricing the American put. (See Chapter 28 for more on options.) A European put is an option to sell a share of stock at a fixed price  $K$  at a certain time  $t_0$ . If at time  $t_0$  the price  $S_{t_0}$  of the stock is lower than  $K$ , one can make a profit by buying a share of stock on the stock exchange for  $S_{t_0}$  dollars, exercising the put (which means selling a share of stock for  $K$  dollars), and taking home a profit of  $K - S_{t_0}$ . If the price of the stock is above  $K$  at time  $t_0$ , it would be silly to exercise the put, and thus the put is worthless. An American put is almost the same, but one has the option to sell a share of stock at price  $K$  at any time before time  $t_0$ . An American put is more valuable than a European put because if one exercises the option early, that is, sells the share of stock before time  $t_0$ , then one can put the money in a risk-free asset such as a bond or in the bank and earn interest on the money. When should one exercise an American put to maximize the expected return? One cannot look into the future, so the time should be a stopping time. The stopping time should depend on the stock price, the exercise price, and also the time until time  $t_0$ . Thus one is in the optimal stopping context with  $X_t = (t, S_t)$ , where  $S_t$  is the stock price, and one wants to find a stopping time  $T$  that maximizes a certain reward function.

### 23.1 Excessive functions

A solution to the optimal stopping problem can be given in the Markov case through the use of excessive functions. A non-negative function  $f$  is *excessive* for a Markov process  $X$  if  $P_t f(x) \leq f(x)$  for all  $t$  and  $x$  and  $P_t f(x)$  increases up to  $f(x)$  pointwise as  $t \rightarrow 0$ . Here  $P_t$  is the semigroup associated with the Markov process  $X$ . If  $g \geq 0$ , define

$$Ug(x) = \int_0^\infty P_s g(x) ds = \mathbb{E}^x \int_0^\infty g(X_s) ds. \quad (23.1)$$

When  $g \geq 0$ ,  $Ug$  is excessive. To see this, using the semigroup property and a change of variables,

$$\begin{aligned} P_t f(x) &= P_t \left( \int_0^\infty P_s g(x) ds \right) = \int_0^\infty P_{s+t} g(x) ds \\ &= \int_t^\infty P_s g(x) ds. \end{aligned}$$

This is certainly less than the integral from 0 to  $\infty$ , hence is less than  $f(x)$ , and  $P_t f(x)$  increases up to  $f(x)$  by monotone convergence. (It is possible that  $f$  is infinite for some or all  $x$ .)

The theory of excessive functions is an important part of Markov process theory and we refer the reader to [Blumenthal and Getoor \(1968\)](#), a book which has inspired a generation of Markov process theorists.

We have the following.

**Lemma 23.1** *If  $f$  is excessive, there exist functions  $g_n \geq 0$  such that  $Ug_n$  increases up to  $f$ , where  $Ug_n$  is defined by (23.1).*

*Proof* Let  $g_n = n(f - P_{1/n}f)$ . Since  $f$  is excessive, then  $g_n \geq 0$ . We have

$$\begin{aligned} Ug_n &= n \int_0^\infty P_s f \, ds - n \int_0^\infty P_{s+(1/n)} f \, ds \\ &= n \int_0^{1/n} P_s f \, ds, \end{aligned}$$

which is less than  $f$  and increases to  $f$ . □

Next we have

**Proposition 23.2** (1) *If  $f$  is excessive,  $T$  is a finite stopping time, and  $h(x) = \mathbb{E}^x f(X_T)$ , then  $h$  is excessive.*

(2) *If  $f$  is excessive and  $T$  is a finite stopping time, then  $f(x) \geq \mathbb{E}^x f(X_T)$ .*

(3) *If  $f$  is excessive, then  $f(X_t)$  is a supermartingale*

*Proof* (1) First suppose  $f = Ug$  for some non-negative function  $g$ . Then

$$\begin{aligned} h(x) &= \mathbb{E}^x Ug(X_T) = \mathbb{E}^x \mathbb{E}^{X_T} \int_0^\infty g(X_s) \, ds \\ &= \mathbb{E}^x \int_0^\infty g(X_{s+T}) \, ds = \mathbb{E}^x \int_T^\infty g(X_s) \, ds \end{aligned} \tag{23.2}$$

by the strong Markov property and a change of variables. The same argument shows that

$$P_t h(x) = \mathbb{E}^x h(X_t) = \mathbb{E}^x \mathbb{E}^{X_t} \int_T^\infty g(X_s) \, ds = \mathbb{E}^x \int_{T+t}^\infty g(X_s) \, ds.$$

This is less than  $\mathbb{E}^x \int_T^\infty g(X_s) \, ds = h(x)$  and increases up to  $h(x)$  as  $t \downarrow 0$ .

Now let  $f$  be excessive but not necessarily of the form  $Ug$ . In the paragraph above, replace  $g$  by the  $g_n$  that were defined in Lemma 23.1 to conclude

$$P_t h(x) = \lim_{n \rightarrow \infty} P_t U g_n(x) \leq \lim_{n \rightarrow \infty} U g_n(x) = h(x).$$

That  $P_t h$  increases up to  $h$  is proved similarly; there is no difficulty interchanging the limit as  $n$  tends to infinity and the limit as  $t$  tends to 0 since  $P_t U g_n$  increases both as  $n$  increases and as  $t$  decreases.

(2) As in the proof of (1), it suffices to consider the case where  $f = Ug$  and then take limits. By (23.2),

$$\mathbb{E}^x Ug(X_T) = \mathbb{E}^x \int_T^\infty g(X_s) ds \leq \mathbb{E}^x \int_0^\infty g(X_s) ds = Ug(x).$$

(3) By the Markov property,

$$\mathbb{E}^x[f(X_t) | \mathcal{F}_s] = \mathbb{E}^{X_s}[f(X_{t-s})] = P_{t-s}f(X_s) \leq f(X_s).$$

The proof is complete.  $\square$

By Proposition 23.2,  $f(X_t)$  is a supermartingale and therefore has left and right limits along the dyadic rationals. We could take a version of  $f(X_t)$  that is right continuous, but there is the danger that doing so would result in a version of  $X$  that is not right continuous with left limits. We want to have  $X$  fixed and then conclude that  $f(X_t)$  is right continuous with left limits without needing to take a version.

**Proposition 23.3** *Let  $(X_t, \mathbb{P}^x)$  be a strong Markov process. If  $f$  is excessive, then for each  $x$ ,  $f(X_t)$  is right continuous with left limits  $\mathbb{P}^x$  almost surely.*

For a proof, we refer the reader to Blumenthal and Getoor (1968), Theorem II.2.12 or to Exercise 23.8.

Given a function  $g$ , the function  $G$  is an *excessive majorant* for  $g$  if  $G$  is excessive and  $G \geq g$  pointwise.  $G$  is the *least excessive majorant* for  $g$  if (1)  $G$  is an excessive majorant, and (2) if  $\tilde{G}$  is any other excessive majorant, then  $G \leq \tilde{G}$  pointwise.

It turns out, which we will prove below, that an optimal stopping time is to stop the first time  $X_t$  leaves the set where  $g(x) < G(x)$ . Therefore it is important to be able to calculate the least excessive majorant of a function.

Here is one method of constructing the least excessive majorant. We say a function  $f : \mathcal{S} \rightarrow \mathbb{R}$  is *lower semicontinuous* if  $\{x : f(x) > a\}$  is an open set for every real number  $a$ . See Exercise 23.9 for information about lower semicontinuous functions.

**Proposition 23.4** *Suppose that  $g$  is non-negative, bounded, and continuous and that Assumption 20.1 holds. Let  $g_0 = g$ , let  $T_n = \{k/2^n : 0 \leq k \leq n2^n\}$ , and define*

$$g_n(x) = \max_{t \in T_n} P_t g_{n-1}(x)$$

for  $n = 1, 2, \dots$ . Then  $g_n(x)$  increases pointwise to  $G(x)$ , the least excessive majorant of  $g$ .

*Proof* Since  $g_n(x) \geq P_0 g_{n-1}(x) = \mathbb{E}^x g_{n-1}(X_0) = g_{n-1}(x)$ , the sequence  $g_n(x)$  is increasing. Call the limit  $H(x)$ .

We first show  $H$  is lower semicontinuous. If  $g_{n-1}$  is bounded and continuous, then  $P_t g_{n-1}$  is bounded and continuous for each  $t$  by Assumption 20.1. Since the maximum of a finite number of continuous functions is continuous, then  $g_n$  is bounded and continuous. By an induction argument, each  $g_n$  is continuous. By Exercise 23.9,  $H$  is lower semicontinuous.

We next show that  $H$  is excessive. If  $t \in T_m$  and  $n \geq m$ , then

$$H(x) \geq g_n(x) \geq P_t g_{n-1}(x) = \mathbb{E}^x g_{n-1}(X_t).$$



Letting  $n$  tend to infinity,  $H(x) \geq \mathbb{E}^x H(X_t)$  if  $t \in T_m$  for some  $m$ . Now take  $t_k \in \cup_m T_m$  with  $t_k \rightarrow t$ . Since  $H$  is lower semicontinuous, then using Exercise 23.9 and Fatou's lemma,

$$H(x) \geq \liminf_{k \rightarrow \infty} \mathbb{E}^x H(X_{t_k}) \geq \mathbb{E}^x [\liminf_{k \rightarrow \infty} H(X_{t_k})] \geq \mathbb{E}^x H(X_t).$$

If  $a \in \mathbb{R}$ , let  $E_a = \{y : H(y) > a\}$ , which is open. If  $a < H(x)$ , then

$$P_t H(x) = \mathbb{E}^x H(X_t) \geq a \mathbb{P}^x(X_t \in E_a) \rightarrow a$$

as  $t \rightarrow 0$ . Therefore  $\liminf_{t \rightarrow 0} P_t H(x) \geq a$  for all  $a < H(x)$ , hence

$$\liminf_{t \rightarrow 0} P_t H(x) \geq H(x),$$

and we conclude  $P_t H(x) \rightarrow H(x)$  as  $t \rightarrow 0$ . Thus  $H$  is excessive.

Suppose now that  $F$  is excessive and  $F \geq g$  pointwise. If  $F \geq g_{n-1}$ , then  $F(x) \geq P_t F(x) \geq P_t g_{n-1}(x)$  for every  $t \in T_n$ , hence  $F(x) \geq g_n(x)$ . By an induction argument,  $F(x) \geq g_n(x)$  for all  $n$ , hence  $F(x) \geq H(x)$ . Therefore  $H$  is the least excessive majorant of  $g$ .  $\square$

In one case, at least, finding the least excessive majorant is easy. Suppose we have a one-dimensional Brownian motion killed on leaving an interval  $[a, b]$  and a non-negative function  $g$  defined on  $[a, b]$ . Then the least excessive majorant is the smallest concave function  $G$  that is larger than or equal to  $g$  everywhere. To see this, if  $G$  is the smallest concave function, by Jensen's inequality

$$P_t G(x) = \mathbb{E}^x G(X_t) \leq G(\mathbb{E}^x X_t) \leq G(x).$$

Because  $G$  is concave, it is continuous, and so  $P_t G(x) = \mathbb{E}^x G(X_t) \rightarrow G(x)$  as  $t \rightarrow 0$ . Therefore  $G$  is excessive. If  $\tilde{G}$  is another excessive function larger than  $g$  and  $a \leq c < x < d \leq b$ , we have  $\tilde{G}(x) \geq \mathbb{E}^x \tilde{G}(X_S)$ , where  $S$  is the first time the process leaves  $[c, d]$  by Proposition 23.2(1). Since  $X$  is equal to a Brownian motion up to time  $S$ , we know the exact distribution of  $X_S$ ; see Proposition 3.16. Therefore

$$\tilde{G}(x) \geq \mathbb{E}^x \tilde{G}(X_S) = \frac{d-x}{d-c} \tilde{G}(c) + \frac{x-c}{d-c} \tilde{G}(d).$$

Rearranging this inequality shows that  $\tilde{G}$  is concave. Recall that the minimum of two concave functions is concave, so  $G \wedge \tilde{G}$  is a concave function larger than  $g$  that is less than or equal to  $G$ . But  $G$  is the smallest concave function larger than or equal to  $g$ , hence  $G = G \wedge \tilde{G}$ , or  $G \leq \tilde{G}$ . Thus  $G$  is the least excessive majorant of  $g$ .

## 23.2 Solving the optimal stopping problem

Now let us turn to proving that an optimal stopping time can be given in terms of the least excessive majorant. For simplicity we will suppose that  $g$  is non-negative, continuous, and bounded. We will assume that our Markov process and  $g$  are such that a least excessive majorant  $G$  exists. Let  $g^*$  be the *optimal reward*:

$$g^*(x) = \sup\{\mathbb{E}^x g(X_T) : T \text{ a stopping time}\}.$$

Let  $D = \{x : g(x) < G(x)\}$ , the *continuation region* and let  $\tau_D = \inf\{t : X_t \notin D\}$ .

**Theorem 23.5** *Let  $(X_t, \mathbb{P}^x)$  be a strong Markov process and  $g$ ,  $g^*$ ,  $G$ , and  $D$  as above. If  $\tau_D < \infty$ ,  $\mathbb{P}^x$ -a.s., then  $g^*(x) = G(x) = \mathbb{E}^x g(X_{\tau_D})$ .*

In other words, an optimal stopping time is to stop the first time the process hits  $\{x : G(x) = g(x)\}$ .

*Proof* Let  $D_\varepsilon = \{x : g(x) < G(x) - \varepsilon\}$ , and write  $\tau_\varepsilon$  for  $\tau_{D_\varepsilon}$ . Let  $H_\varepsilon(x) = \mathbb{E}^x[G(X_{\tau_\varepsilon})]$ , which is excessive by Proposition 23.2(2).

The first step of the proof is to prove (23.3) below. Second, we prove  $G(x) \leq g^*(x)$ . The third step is to prove that  $G(x) = g^*(x)$  and the fourth that  $g^*(x) = \mathbb{E}^x g(X_{\tau_D})$ .

*Step 1.* Let  $\varepsilon > 0$ . We claim

$$g(x) \leq H_\varepsilon(x) + \varepsilon, \quad x \in D. \quad (23.3)$$

To prove this, we suppose not, that is, we let

$$b = \sup_{x \in D}(g(x) - H_\varepsilon(x))$$

and suppose  $b > \varepsilon$ . Choose  $\eta < \varepsilon$ , and then choose  $x_0$  such that

$$g(x_0) - H_\varepsilon(x_0) \geq b - \eta. \quad (23.4)$$

Since  $H_\varepsilon + b$  is an excessive majorant of  $g$  by the definition of  $b$ , and  $G$  is the least excessive majorant, then

$$G(x_0) \leq H_\varepsilon(x_0) + b. \quad (23.5)$$

From (23.4) and (23.5) we conclude

$$G(x_0) \leq g(x_0) + \eta. \quad (23.6)$$

By the Blumenthal 0–1 law (Proposition 20.8), either  $\tau_\varepsilon$  is strictly positive with  $\mathbb{P}^{x_0}$  probability one or else zero with  $\mathbb{P}^{x_0}$  probability one. In the first case, for each  $t > 0$ ,

$$\begin{aligned} g(x_0) + \eta &\geq G(x_0) \\ &\geq \mathbb{E}^x[G(X_{t \wedge \tau_\varepsilon})] \\ &\geq \mathbb{E}^{x_0}[g(X_t) + \varepsilon; \tau_\varepsilon > t]. \end{aligned}$$

The first inequality is (23.6), the second is due to  $G$  being excessive, and the third because  $G > g + \varepsilon$  up until the time  $\tau_\varepsilon$ . If we let  $t \rightarrow 0$  and use the fact that  $g$  is continuous, we get  $g(x_0) + \eta \geq g(x_0) + \varepsilon$ , a contradiction to the way we chose  $\eta$ .

In the second case, where  $\tau_\varepsilon = 0$  with  $\mathbb{P}^{x_0}$ -probability one, we have

$$H_\varepsilon(x_0) = \mathbb{E}^{x_0} G(X_{\tau_\varepsilon}) = \mathbb{E}^{x_0} G(X_0) = G(x_0) \geq g(x_0) \geq H_\varepsilon(x_0) + b - \eta,$$

a contradiction since we chose  $\eta < b$ .

In either case we reach a contradiction, so (23.3) must hold.

*Step 2.* A conclusion we reach from (23.3) is that  $H_\varepsilon + \varepsilon$  is an excessive majorant of  $g$ . Therefore

$$\begin{aligned} G(x) &\leq H_\varepsilon(x) + \varepsilon \\ &= \mathbb{E}^x[G(X_{\tau_\varepsilon})] + \varepsilon \\ &\leq \mathbb{E}^x[g(X_{\tau_\varepsilon}) + \varepsilon] + \varepsilon \\ &\leq g^*(x) + 2\varepsilon. \end{aligned} \quad (23.7)$$

The first inequality holds because  $G$  is the least excessive majorant, the second inequality because  $g(X_{\tau_\varepsilon}) + \varepsilon = G(X_{\tau_\varepsilon})$  by the definition of  $\tau_\varepsilon$ , and the third by the definition of  $g^*$ . Since  $\varepsilon$  is arbitrary, we see that  $G(x) \leq g^*(x)$ .

*Step 3.* For any stopping time  $T$ , because  $G$  is excessive and majorizes  $g$ ,

$$G(x) \geq \mathbb{E}^x G(X_T) \geq \mathbb{E}^x g(X_T).$$

Taking the supremum over all stopping times  $T$ ,  $G(x) \geq g^*(x)$ , and therefore  $G(x) = g^*(x)$ .

*Step 4.* Because  $\tau_D$  is finite almost surely, the continuity of  $g$  tells us that  $\mathbb{E}^x g(X_{\tau_\varepsilon}) \rightarrow \mathbb{E}^x g(X_{\tau_D})$  as  $\varepsilon \rightarrow 0$ . By the definition of  $g^*$ , we know that  $\mathbb{E}^x g(X_{\tau_\varepsilon}) \leq g^*(x)$ .

On the other hand, by the definitions of  $\tau_\varepsilon$  and  $H_\varepsilon$ ,

$$\mathbb{E}^x g(X_{\tau_\varepsilon}) = \mathbb{E}^x G(X_{\tau_\varepsilon}) - \varepsilon = H_\varepsilon(x) - \varepsilon.$$

By the first inequality in (23.7), the right-hand side is greater than or equal to  $G(x) - 2\varepsilon = g^*(x) - 2\varepsilon$ . Letting  $\varepsilon \rightarrow 0$  we obtain

$$\mathbb{E}^x g(X_{\tau_D}) \geq g^*(x)$$

as desired.  $\square$

The following two corollaries are useful in applications.

**Corollary 23.6** Suppose there exists a Borel set  $A$  such that  $h$  is an excessive majorant of  $g$ , where  $h(x) = \mathbb{E}^x g(X_{\tau_A})$  and  $\tau_A = \inf\{t : X_t \notin A\}$ . Then  $g^*(x) = h(x)$ .

*Proof* Let  $G$  be the least excessive majorant of  $g$ . Then  $h(x) \geq G(x)$ . However,

$$h(x) = \mathbb{E}^x g(X_{\tau_A}) \leq \sup_T \mathbb{E}^x g(X_T) = g^*(x) = G(x)$$

by Theorem 23.5.  $\square$

**Corollary 23.7** Suppose  $g$  is continuous and  $G$ , the least excessive majorant of  $g$ , is lower semicontinuous. Let  $D$  be the continuation region, suppose  $\tau_D < \infty$ , a.s., and let  $h(x) = \mathbb{E}^x g(X_{\tau_D})$ . If  $h \geq g$ , then  $h = g^*$ .

*Proof* Note  $D = \{x : g(x) < G(x)\} = \cup_{a < b} [(g(x) < a) \cap (G(x) > b)]$ , where the union is over all pairs of real numbers  $a < b$ . Since  $G$  is lower semicontinuous and  $g$  is continuous, then  $D$  is open. This implies  $X_{\tau_D} \notin D$ , and so  $g(X_{\tau_D}) \geq G(X_{\tau_D})$ , a.s. Since  $g \leq G$ , we see that

$$h(x) = \mathbb{E}^x g(X_{\tau_D}) = \mathbb{E}^x G(X_{\tau_D}).$$

Since  $G$  is excessive, then  $h$  is also excessive by Proposition 23.2. Therefore  $h$  is an excessive majorant of  $g$  and we can apply Corollary 23.6.  $\square$

## Exercises

- 23.1 Show that if  $f$  is excessive, then  $1 - e^{-f}$  is excessive. Thus, for some purposes it is enough to look at bounded excessive functions.
- 23.2 Show that if  $f$  and  $g$  are excessive, then  $f \wedge g$  is excessive.

- 23.3 Let  $A_t$  be an additive functional (defined in (22.4)) and let  $f(x) = \mathbb{E}^x A_\infty$ . Show that  $f$  is excessive.
- 23.4 Let  $f$  be an excessive function for a strong Markov process  $(X_t, \mathbb{P}^x)$ . Let  $\varepsilon > 0$  and  $S_1 = \inf\{t : |f(X_t) - f(X_0)| \geq \varepsilon\}$ . Let  $S_{i+1} = S_i + S_1 \circ \theta_{S_i}$ . Prove that  $f(X_{S_i})$  is a supermartingale with respect to the  $\sigma$ -fields  $\mathcal{F}_{S_i}$  and with respect to  $\mathbb{P}^x$  for each  $x$ .
- 23.5 For each  $n$ , let  $A_t^n$  be an additive functional with continuous paths and suppose that  $f_n(x) = \mathbb{E}^x A_\infty^n$  is finite for every  $x$ . Suppose  $A_t$  is a continuous additive functional with  $f(x) = \mathbb{E}^x A_\infty$  also finite for each  $x$ . Suppose  $f_n$  converges to  $f$  uniformly. Prove that for each  $x$ , with  $\mathbb{P}^x$ -probability one,  $A_t^n$  converges to  $A_t$ , uniformly over  $t \geq 0$ .

*Hint:* Use Proposition 9.11.

- 23.6 Suppose  $f$  is bounded and excessive,  $\lambda \geq 0$ , and  $A = \{y : f(y) \leq \lambda\}$ . Prove that if  $x \in A^r$  (i.e.,  $x$  is regular for  $A$ ), then  $f(x) \leq \lambda$ .

*Hint:* Use the optional section theorem (Theorem 16.12) to find stopping times  $T_m$  whose graphs are contained in  $\{(t, \omega) : t \leq 1/m, f(X_t) \leq \lambda\}$  with  $\mathbb{P}^x$ -probability at least  $1 - (1/m)$ .

If the  $g_n$  are as in Lemma 23.1, write

$$\begin{aligned} U g_n(x) &= \mathbb{E}^x \int_0^{T_m} g_n(X_s) ds + \mathbb{E}^x U g_n(X_{T_m}) \\ &\leq \mathbb{E}^x \int_0^{T_m} g_n(X_s) ds + \mathbb{E}^x f(X_{T_m}) \\ &\leq \mathbb{E}^x \int_0^{T_m} g_n(X_s) ds + \lambda + \|f\|_\infty / m. \end{aligned}$$

Let  $m \rightarrow \infty$ , then  $n \rightarrow \infty$ .

- 23.7 Suppose  $f$  is bounded and excessive,  $\lambda \geq 0$ , and  $B = \{y : f(y) \geq \lambda\}$ . Prove that if  $x \in B^r$ , then  $f(x) \leq \lambda$ .

*Hint:* Use the optional stopping theorem as in Exercise 23.6 to find stopping times  $R_m$  analogous to the  $T_m$ . Write

$$f(x) \geq \mathbb{E}^x f(X_{R_m}) \geq \lambda - \|f\|_\infty / m,$$

and then let  $m \rightarrow \infty$ .

- 23.8 (1) Suppose  $f$  is bounded and excessive,  $x \in \mathcal{S}$ ,  $\varepsilon > 0$ , and  $C = \{y : |f(y) - f(x)| \geq \varepsilon\}$ . Use Exercises 23.6 and 23.7 to show that if  $z \in C^r$ , then  $|f(z) - f(x)| \geq \varepsilon$ .

(2) Let  $f$ ,  $\varepsilon$ , and  $x$  be as in (1) and set  $S = \inf\{t > 0 : |f(X_t) - f(x)| \geq \varepsilon\}$ . Use Exercise 20.9 to show that  $|f(X_S) - f(x)| \geq \varepsilon$  with  $\mathbb{P}^x$ -probability one.

(3) Let  $f$ ,  $\varepsilon$ ,  $x$ , and  $S$  be as in (2). Define  $S = 0$  and  $S_{i+1} = S_i + S \circ \theta_{S_i}$ . By Exercise 23.4,  $f(X_{S_i})$  is a positive supermartingale. Use Corollary A.36 to show  $S_i \rightarrow \infty$ ,  $\mathbb{P}^x$ -a.s. Deduce that with  $\mathbb{P}^x$ -probability one,  $f(X_t)$  has paths that are right continuous with left limits.

(4) Use Exercise 23.1 to show that if  $f$  is excessive but not necessarily bounded, then  $f(X_t)$  has paths that are right continuous with left limits.

- 23.9 (1) Show that every continuous function is lower semicontinuous.

(2) Show that if  $f$  is lower semicontinuous and  $x \in \mathcal{S}$ , then

$$\liminf_{y \rightarrow x} f(y) \geq f(x).$$

(3) Show that if  $f_n$  is a sequence of continuous functions increasing to  $f$ , then  $f$  is lower semicontinuous.

23.10 Suppose  $g$  is non-negative, bounded, and continuous, and Assumption 20.1 holds. Let  $g_0 = g$  and define  $g_n(x) = \sup_{t \geq 0} P_t g_{n-1}(x)$  for  $n \geq 1$ . Prove that  $g_n$  increases to the least excessive majorant of  $g$ .

### Notes

See Øksendal (2003) for further information on optimal stopping.

Exercise 23.3 shows that  $\mathbb{E}^x A_\infty$  is an excessive function if  $A$  is an additive functional. To a large extent the converse is true: given an excessive function  $f$  and some mild conditions, there exists an additive functional  $A$  such that  $f(x) = \mathbb{E}^x A_\infty$  for all  $x$ . The proof is a modification of the Doob–Meyer decomposition of  $f(X_t)$  that takes into account the fact there is a family of probabilities instead of just one; see Blumenthal and Getoor (1968).

The optimal stopping problem involving American puts has a theoretical solution: look at the least excessive majorant for a certain reward function. The reward function is not just  $(K - s)^+$  because the interest earned on the money obtained after the sale of a share of stock needs to be taken into account. Moreover, the excessive functions here are relative to the space-time process  $(S_t, t)$ , not those relative to  $S_t$ . Finding a satisfactory solution to this optimal stopping problem is still open and is important.

# 24

---

## Stochastic differential equations

Stochastic differential equations are used in modeling a wide variety of physical and economic situations, and are one of the main reasons for the interest in stochastic integrals.

We consider stochastic differential equations (SDEs) of the form

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt,$$

where  $\sigma$  and  $b$  are real-valued functions and  $W$  is a one-dimensional Brownian motion. We also consider multidimensional analogs of this equation. If  $X$  represents the position of a particle, the  $\sigma(X_t) dW_t$  term says that the particle  $X$  diffuses like a multiple of Brownian motion, but how strong the diffusivity depends on the location of the particle. The  $b(X_t) dt$  term represents a push in one direction or another, the size of the push depending on the location of the particle.

### 24.1 Pathwise solutions of SDEs

Let  $W_t$  be a one-dimensional Brownian motion with respect to a filtration  $\{\mathcal{F}_t\}$  satisfying the usual conditions; see Chapter 1. We want to consider SDEs of the form

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt, \quad X_0 = x_0. \quad (24.1)$$

This means that  $X_t$  satisfies the equation

$$X_t = x_0 + \int_0^t \sigma(X_s) dW_s + \int_0^t b(X_s) ds, \quad t \geq 0. \quad (24.2)$$

Here  $\sigma$  and  $b$  are Borel measurable functions, the first integral in (24.2) is a stochastic integral with respect to the Brownian motion  $W_t$ , and (24.2) holds almost surely, that is, we can find versions of  $\int_0^t \sigma(X_s) dW_s$  such that for almost all  $\omega$ , (24.2) holds for all  $t$ . In order to be able to define the stochastic integral, we require that any solution  $X_t$  to (24.2) be adapted to the filtration  $\{\mathcal{F}_t\}$ . If  $X$  satisfies (24.2), then  $X$  will automatically have continuous paths. We want to consider existence and uniqueness of solutions to the equation (24.2).

**Definition 24.1** A stochastic process  $X$  will be a *pathwise solution* to (24.1) if  $X$  is adapted to the filtration  $\{\mathcal{F}_t\}$  and (24.2) holds almost surely, where the null set does not depend on  $t$ . We say the solution to (24.1) is *pathwise unique* if whenever  $X'_t$  is another solution, then

$$\mathbb{P}(X_t \neq X'_t \text{ for some } t \geq 0) = 0. \quad (24.3)$$

Sometimes pathwise uniqueness is used for a slightly stronger concept: one can let  $W$  be a Brownian motion with respect to each of two filtrations  $\{\mathcal{F}_t\}$  and  $\{\mathcal{F}'_t\}$ , which are possibly

different, and one can let  $X'_t$  be adapted to  $\{\mathcal{F}'_t\}$ . One then requires (24.3) to hold. We won't need to use this modification of the definition, and in any case our proof of uniqueness will be equally valid in this situation.

The function  $\sigma$  in (24.1) is called the *diffusion coefficient* and the function  $b$  is called the *drift coefficient*.  $\sigma$  tells us the intensity of the noise at a point, and  $b$  tells us if there is a push in any direction at a given point.

We will suppose that  $\sigma$  and  $b$  are *Lipschitz functions*: there exists a constant  $c$  such that

$$|\sigma(x) - \sigma(y)| \leq c|x - y|, \quad |b(x) - b(y)| \leq c|x - y|. \quad (24.4)$$

We also suppose for now that  $\sigma$  and  $b$  are bounded.

**Theorem 24.2** Suppose  $\sigma$  and  $b$  are bounded Lipschitz functions. Then there exists a pathwise solution to (24.1) and this solution is pathwise unique.

*Proof* Existence. Let  $X_0(t) = x_0$  for all  $t$  and define  $X_i(t)$  recursively by

$$X_{i+1}(t) = x_0 + \int_0^t \sigma(X_i(s)) dW_s + \int_0^t b(X_i(s)) ds. \quad (24.5)$$

Note that  $X_0(t)$  is trivially adapted to  $\{\mathcal{F}_t\}$ , and an induction argument shows that  $X_i$  is adapted to  $\{\mathcal{F}_t\}$  for each  $i$ .

Fix  $t_0$ . We will show existence (and uniqueness) up to time  $t_0$ ; since  $t_0$  is arbitrary, this will achieve the theorem.

Since  $(x + y)^2 \leq 2x^2 + 2y^2$ , then

$$\begin{aligned} \mathbb{E} \sup_{r \leq t} |X_{i+1}(r) - X_i(r)|^2 &= \mathbb{E} \left[ \sup_{r \leq t} \left( \int_0^r [\sigma(X_i(s)) - \sigma(X_{i-1}(s))] dW_s \right. \right. \\ &\quad \left. \left. + \int_0^r [b(X_i(s)) - b(X_{i-1}(s))] ds \right)^2 \right] \\ &\leq 2\mathbb{E} \left[ \sup_{r \leq t} \left( \int_0^r [\sigma(X_i(s)) - \sigma(X_{i-1}(s))] dW_s \right)^2 \right] \\ &\quad + 2\mathbb{E} \left[ \sup_{r \leq t} \left( \int_0^r [b(X_i(s)) - b(X_{i-1}(s))] ds \right)^2 \right]. \end{aligned}$$

By Doob's inequalities (Theorem 3.6) and the fact that  $\sigma$  is a Lipschitz function, the first term after the inequality is bounded by

$$\begin{aligned} 8\mathbb{E} \left[ \left( \int_0^t [\sigma(X_i(s)) - \sigma(X_{i-1}(s))] dW_s \right)^2 \right] &= 8\mathbb{E} \int_0^t [\sigma(X_i(s)) - \sigma(X_{i-1}(s))]^2 ds \\ &\leq c\mathbb{E} \int_0^t |X_i(s) - X_{i-1}(s)|^2 ds. \end{aligned}$$

By the Cauchy–Schwarz inequality, the fact that  $t \leq t_0$ , and the fact that  $b$  is a Lipschitz function, the second term is bounded by

$$\begin{aligned} 2\mathbb{E} \left( \int_0^t |b(X_i(s)) - b(X_{i-1}(s))| ds \right)^2 &\leq 2t_0\mathbb{E} \int_0^t |b(X_i(s)) - b(X_{i-1}(s))|^2 ds \\ &\leq c\mathbb{E} \int_0^t |X_i(s) - X_{i-1}(s)|^2 ds. \end{aligned}$$

Therefore

$$\mathbb{E} \sup_{r \leq t} |X_{i+1}(r) - X_i(r)|^2 \leq c \mathbb{E} \int_0^t |X_i(s) - X_{i-1}(s)|^2 ds. \quad (24.6)$$

Let  $g_i(t) = \mathbb{E} \sup_{r \leq t} |X_i(r) - X_{i-1}(r)|^2$ . Thus provided we choose  $A$  big enough,  $g_1(t) \leq A$  for  $t \leq t_0$  and

$$g_{i+1}(t) \leq A \int_0^t g_i(s) ds, \quad t \leq t_0.$$

(Clearly  $|X_{i+1}(t) - X_i(t)|^2 \leq \sup_{r \leq t} |X_{i+1}(r) - X_i(r)|^2$ .) Thus

$$g_2(t) \leq A \int_0^t g_1(s) ds \leq A \int_0^t A ds = A^2 t,$$

$$g_3(t) \leq A \int_0^t g_2(s) ds \leq A \int_0^t A^2 s ds = A^3 t^2 / 2,$$

and continuing by induction,

$$g_i(t) \leq A^i t^{i-1} / (i-1)!$$

Exercise 24.1 asks you to show that if we define

$$\|Y\|_t = (\mathbb{E} \sup_{r \leq t} |Y_r|^2)^{1/2} \quad (24.7)$$

when  $Y$  is a stochastic process, then  $\|Y\|_t$  is a norm and the corresponding metric is complete. Hence

$$\begin{aligned} (\mathbb{E} \sup_{r \leq t_0} |X_n(s) - X_m(s)|^2)^{1/2} &= \|X_n - X_m\|_{t_0} \\ &\leq \sum_{i=m}^{n-1} \|X_{i+1} - X_i\|_{t_0} \\ &\leq \sum_{i=m}^{n-1} (g_i(t_0))^{1/2} \end{aligned}$$

can be made small by taking  $m, n$  large. (We use the ratio test to show that the sum  $\sum (A^i t_0^{i-1} / (i-1)!)^{1/2}$  converges.) We have  $\mathbb{E} X_0(t)^2 < \infty$ . By the completeness of  $\|\cdot\|_{t_0}$  there exists  $X_t$  such that  $\mathbb{E} \sup_{s \leq t_0} |X_n(s) - X_s|^2 \rightarrow 0$  as  $n \rightarrow \infty$ . This implies there exists a subsequence  $\{n_j\}$  such that  $\sup_{s \leq t_0} |X_{n_j}(s) - X_s|^2 \rightarrow 0$  almost surely; since each  $X_{n_j}$  is continuous in  $t$ , then  $X_t$  is also. Taking a limit in (24.5) as  $n \rightarrow \infty$  shows  $X_t$  satisfies (24.2).

*Uniqueness.* Suppose  $X_t$  and  $X'_t$  are two solutions to (24.2). Let

$$g(t) = \mathbb{E} \sup_{r \leq t} |X_r - X'_r|^2.$$

Very similarly to the existence proof,  $\mathbb{E} \sup_{r \leq t} |X_r|^2 < \infty$ , the same with  $X$  replaced by  $X'$ , and

$$\begin{aligned}\mathbb{E} \sup_{r \leq t} |X_r - X'_r|^2 &\leq 2\mathbb{E} \left[ \sup_{r \leq t} \left( \int_0^r [\sigma(X_s) - \sigma(X'_s)] dW_s \right)^2 \right] \\ &\quad + 2\mathbb{E} \left[ \sup_{r \leq t} \left( \int_0^r [b(X_s) - b(X'_s)] ds \right)^2 \right] \\ &\leq c\mathbb{E} \int_0^t |X_s - X'_s|^2 ds.\end{aligned}$$

Therefore there exists  $A > 0$  such that  $g(t)$  is bounded by  $A$  and  $g(t) \leq A \int_0^t g(s) ds$ .

Then  $g(t) \leq A \int_0^t A ds = A^2 t$ ,  $g(t) \leq A \int_0^t A^2 s ds = A^3 t^2 / 2$ , etc. Thus we have  $g(t) \leq A^i t^{i-1} / (i-1)!$  for all  $i$ , which is only possible if  $g(t) = 0$ . This implies that  $X_t = X'_t$  for all  $t \leq t_0$ , except for a null set.  $\square$

We also want to consider the SDE (24.1) when  $\sigma$  and  $b$  are Lipschitz functions, but not necessarily bounded. Note  $|\sigma(x)| \leq |\sigma(0)| + c|x|$ , so that  $|\sigma(x)|$  is less than or equal to  $c(1 + |x|)$ , and the same for  $b$ .

**Theorem 24.3** *Suppose  $\sigma$  and  $b$  are Lipschitz functions, but not necessarily bounded. Then there exists a pathwise solution to (24.1) and this solution is pathwise unique.*

*Proof* Let  $\sigma_n$  and  $b_n$  be bounded Lipschitz functions that agree with  $\sigma$  and  $b$ , respectively, on  $[-n, n]$ . Let  $X_n$  be the unique pathwise solution to (24.1) with  $\sigma$  and  $b$  replaced by  $\sigma_n$  and  $b_n$ , respectively. Let  $T_n = \inf\{t : |X_n(t)| \geq n\}$ . We claim  $X_n(t) = X_m(t)$  if  $t \leq T_n \wedge T_m$ ; to prove this, let  $g(t) = \mathbb{E} \sup_{s \leq t \wedge T_n \wedge T_m} |X_n(s) - X_m(s)|^2$ , and proceed as in the uniqueness part of the proof of Theorem 24.2. We then have existence and uniqueness of the SDE for  $t \leq T_n$  for each  $n$ .

To complete the proof, it suffices to show  $T_n \rightarrow \infty$ . Let

$$h_n(t) = \mathbb{E} \sup_{s \leq t \wedge T_n} |X_n(s)|^2.$$

Then

$$\begin{aligned}h_n(t) &\leq c|x_0|^2 + c\mathbb{E} \left( \int_0^t \sigma_n(X_n(s)) dW_s \right)^2 + c\mathbb{E} \int_0^t b_n(X_n(s))^2 ds \\ &\leq c|x_0|^2 + c\mathbb{E} \int_0^t \sigma_n(X_n(s))^2 ds + ct_0 \mathbb{E} \int_0^t b_n(X_n(s))^2 ds \\ &\leq c|x_0|^2 + c + c\mathbb{E} \int_0^t |X_n(s)|^2 ds \\ &\leq c + c \int_0^t h_n(s) ds,\end{aligned}$$

using estimates very similar to those of the proof of Theorem 24.2. By Exercise 24.2,  $h_n(t) \leq ce^{ct}$  if  $t \leq t_0$ . Note the constant  $c$  can be chosen to be independent of  $n$ . Then

$$\mathbb{P}(T_n < t_0) = \mathbb{P}(\sup_{s \leq t_0} |X_n(s)| \geq n) \leq \frac{\mathbb{E} \sup_{s \leq t_0} |X_n(s)|^2}{n^2} \leq \frac{h_n(t_0)}{n^2} \rightarrow 0$$

as  $n \rightarrow \infty$ . Since  $t_0$  is arbitrary,  $T_n \rightarrow \infty$ , a.s.  $\square$

Although we considered one-dimensional SDEs for simplicity, the same arguments apply when we have higher-dimensional SDEs. Let

$$W = (W^1, \dots, W^d)$$

be a  $d$ -dimensional Brownian motion, let  $\sigma_{ij}(x)$  be bounded Lipschitz functions for  $i = 1, \dots, n$  and  $j = 1, \dots, d$ , and let  $b_i(x)$  be bounded Lipschitz functions for  $i = 1, \dots, n$ . Consider the system of equations

$$dX_t^i = \sum_{j=1}^d \sigma_{ij}(X_t) dW_t^j + b_i(X_t) dt, \quad i = 1, \dots, n. \quad (24.8)$$

This is frequently written in matrix form

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt \quad (24.9)$$

where we view  $X = (X^1, \dots, X^n)$  as a  $n \times 1$  matrix,  $b = (b_1, \dots, b_n)$  as a  $n \times 1$  matrix-valued function,  $W$  as a  $d \times 1$  matrix, and  $\sigma$  as a  $n \times d$  matrix-valued function. We have existence and uniqueness to the system (24.8). Exercise 24.5 asks you to prove this in the case when  $n = d$ , although there is nothing at all special about requiring  $n = d$ .

## 24.2 One-dimensional SDEs

Although our proof of pathwise existence and uniqueness was for SDEs in one dimension, as is pointed out in Exercise 24.5, almost the same proof works in higher dimensions. In this section we look at a pathwise uniqueness result that is valid only for SDEs on  $\mathbb{R}$ . The equation we look at is the same as the one in the last section, namely,

$$X_t = x_0 + \int_0^t \sigma(X_s) dW_s + \int_0^t b(X_s) ds. \quad (24.10)$$

**Theorem 24.4** Suppose  $b$  is bounded and Lipschitz. Suppose there exists a continuous function  $\rho : [0, \infty) \rightarrow [0, \infty)$  such that  $\rho(0) = 0$ ,

$$\int_0^\varepsilon \rho^{-2}(u) du = \infty \quad (24.11)$$

for all  $\varepsilon > 0$ , and  $\sigma$  is bounded and satisfies

$$|\sigma(x) - \sigma(y)| \leq \rho(|x - y|)$$

for all  $x$  and  $y$ . Then the solution to (24.10) is pathwise unique.

For an example, let  $b(x) = 0$  for all  $x$ , and let  $\sigma$  be Hölder continuous of order  $\alpha$ , that is, there exists  $c$  such that  $|\sigma(x) - \sigma(y)| \leq c|x - y|^\alpha$ . Then we take  $\rho(x) = x^\alpha$ , and the integral condition in the theorem is satisfied if and only if  $\alpha \geq 1/2$ . If (24.11) holds for all  $\varepsilon > 0$ , we say the Yamada–Watanabe condition holds.

Instead of proving this theorem right away and then essentially repeating the proof to give a comparison theorem, we will state and prove a comparison theorem (Theorem 24.5) and then obtain Theorem 24.4 as a corollary of Theorem 24.5.

We only prove the uniqueness of the solution to (24.10) here. The existence is a consequence of some measure-theoretic magic; see Revuz and Yor (1999), Theorem IX.1.7.

**Theorem 24.5** Suppose  $\sigma$  satisfies the conditions in Theorem 24.4. Suppose  $X_t$  satisfies (24.10) with  $b$  a Lipschitz function. Suppose  $Y_t$  is a continuous semimartingale satisfying

$$Y_t \geq Y_0 + \int_0^t \sigma(Y_s) dW_s + \int_0^t B(Y_s) ds,$$

where  $B$  is a Borel measurable function and  $B(z) \geq b(z)$  for all  $z$ . If  $Y_0 \geq x$ , a.s., then  $Y_t \geq X_t$  almost surely for all  $t$ .

*Proof* Let  $a_n \downarrow 0$  be selected so that

$$\int_{a_n}^{a_{n-1}} (\rho(u))^{-2} du = n.$$

This can be done inductively. Choose  $a_0$  arbitrarily. Since  $\int_r^{a_0} \rho(x)^{-2} dx$  increases to infinity as  $r \rightarrow 0$ , we can choose  $a_1$  such that  $\int_{a_1}^{a_0} \rho(x)^{-2} dx = 1$ ; in a similar manner we choose  $a_2, a_3, \dots$ . Let  $h_n$  be continuous, supported in  $(a_n, a_{n-1})$ ,  $0 \leq h_n(u) \leq 2/n\rho^2(u)$ , and  $\int_{a_n}^{a_{n-1}} h_n(u) du = 1$  for each  $n$ . The idea here is to start with the function  $(1 + \varepsilon)1_{(a_n, a_{n-1})}(u)/(n\rho(u)^2)$  for some small  $\varepsilon$ , and then modify this near  $a_n$  and  $a_{n-1}$  to get a function that is continuous, is supported in  $(a_n, a_{n-1})$ , and integrates to 1. Let  $f_n$  be such that  $f_n(0) = f'_n(0) = 0$  and  $f''_n = h_n$ . Note

$$f'_n(u) = \int_0^u f''_n(s) ds = \int_0^u h_n(s) ds \leq 1$$

and  $f'_n(u) \geq 0$ , so  $0 \leq f'_n(u) \leq 1$  and  $f'_n(u) = 1$  if  $u \geq a_{n-1}$ . Hence  $f_n(u) \uparrow u$  as  $n \rightarrow \infty$  for each  $u \geq 0$ .

Since  $x \leq y$ , then  $f_n(x - y) = 0$ , and we have by Itô's formula

$$\begin{aligned} f_n(X_t - Y_t) &= \text{martingale} + \int_0^t f'_n(X_s - Y_s)[b(X_s) - B(Y_s)] ds \\ &\quad + \frac{1}{2} \int_0^t f''_n(X_s - Y_s)[\sigma(X_s) - \sigma(Y_s)]^2 ds. \end{aligned} \tag{24.12}$$

We take expectations of both sides. The martingale term has 0 expectation. The final term on the right-hand side is bounded in expectation by

$$\frac{1}{2} \mathbb{E} \int_0^t \frac{2}{n(\rho|X_s - Y_s|)^2} (\rho|X_s - Y_s|)^2 ds \leq \frac{t}{n}$$

by the assumptions on  $\sigma$  and the bound on  $f''_n = h_n$ , and so goes to 0 as  $n \rightarrow \infty$ . The expectation of the second term on the right of (24.12) is bounded above by

$$\begin{aligned} &\mathbb{E} \int_0^t f'_n(X_s - Y_s)[b(X_s) - b(Y_s)] ds + \mathbb{E} \int_0^t f'_n(X_s - Y_s)[b(Y_s) - B(Y_s)] ds \\ &\leq c \mathbb{E} \int_0^t (1_{[0,\infty)}(X_s - Y_s)) |X_s - Y_s| ds \\ &= c \mathbb{E} \int_0^t (X_s - Y_s)^+ ds. \end{aligned}$$

Letting  $n \rightarrow \infty$ ,

$$\mathbb{E} (X_t - Y_t)^+ \leq c \int_0^t \mathbb{E} (X_s - Y_s)^+ ds.$$

If we set  $g(t) = \mathbb{E} (X_t - Y_t)^+$ , we have

$$g(t) \leq c \int_0^t g(s) ds,$$

and by Exercise 24.2 we conclude  $g(t) = 0$  for each  $t$ . Using the continuity of the paths of  $X_t$  and  $Y_t$  completes the proof.  $\square$

We now prove Theorem 24.4.

*Proof of Theorem 24.4* Suppose  $X$  and  $X'$  are two solutions to (24.10). Then by Theorem 24.5 with  $Y = X'$  and  $B = b$ , we have  $X_t \leq X'_t$  for all  $t$ . Applying this argument with  $X$  and  $X'$  reversed yields  $X'_t \leq X_t$  for all  $t$ , which completes the proof.  $\square$

### 24.3 Examples of SDEs

#### Ornstein–Uhlenbeck process

The Ornstein–Uhlenbeck process is the solution to the SDE

$$dX_t = dW_t - \frac{X_t}{2} dt, \quad X_0 = x. \quad (24.13)$$

The existence and uniqueness follow by Theorem 24.3. Note that the drift coefficient is not bounded, so Theorem 24.2 is not sufficient. The process behaves like a Brownian motion, with a drift that pushes the process towards the origin; the farther the process gets from the origin, the stronger the push.

The equation (24.13) can be solved explicitly. Rearranging, multiplying by  $e^{t/2}$ , and using the product rule,

$$d[e^{t/2} X_t] = e^{t/2} dX_t + e^{t/2} \frac{X_t}{2} dt = e^{t/2} dW_t,$$

so

$$e^{t/2} X_t = X_0 + \int_0^t e^{s/2} dW_s,$$

or

$$X_t = e^{-t/2} x + e^{-t/2} \int_0^t e^{s/2} dW_s. \quad (24.14)$$

We used here the fact that the martingale part of the semimartingale  $Z_t = e^{t/2}$  is zero, and therefore  $\langle Z, W \rangle_t = 0$ . By Exercise 24.6,  $X_t$  is a Gaussian process and the distribution of  $X_t$  is that of a normal random variable with mean  $e^{-t/2} x$  and variance equal to  $e^{-t} \int_0^t (e^{s/2})^2 ds = 1 - e^{-t}$ .

If we let  $Y_t = \int_0^t e^{s/2} dW_s$  and  $V_t = Y_{\log(t+1)}$ , then  $Y_t$  is a mean-zero continuous Gaussian process with independent increments, and hence so is  $V_t$ . Since

$$\text{Var}(V_u - V_t) = \int_{\log(t+1)}^{\log(u+1)} e^s ds = u - t,$$

then  $V_t$  is a Brownian motion. Hence

$$X_t = e^{-t/2}x + e^{-t/2}V(e^t - 1).$$

This representation of an Ornstein–Uhlenbeck process in terms of a Brownian motion is useful for, among other things, calculating the exit probabilities of a square root boundary.

### Linear equations

We consider the linear equation

$$dX_t = AX_t dW_t + BX_t dt, \quad X_0 = x_0, \quad (24.15)$$

where  $A$  and  $B$  are constants. One place this comes up is in models of stock prices in financial mathematics; see Chapter 28. We have pathwise existence and uniqueness by Theorem 24.3; here both the diffusion and drift coefficients are unbounded.

We will give a candidate for the solution, and verify that it solves (24.15). By the pathwise uniqueness, this will then be the only solution. Our candidate is

$$X_t = x_0 e^{AW_t + (B - A^2/2)t}.$$

To verify that this is a solution, we use Itô's formula with the process  $AW_t + (B - A^2/2)t$  and the function  $e^x$ :

$$\begin{aligned} X_t &= x_0 + \int_0^t e^{AW_s + (B - A^2/2)s} A dW_s + \int_0^t e^{AW_s + (B - A^2/2)s} (B - A^2/2) ds \\ &\quad + \frac{1}{2} \int_0^t e^{AW_s + (B - A^2/2)s} A^2 ds \\ &= x_0 + \int_0^t e^{AW_s + (B - A^2/2)s} A dW_s + \int_0^t e^{AW_s + (B - A^2/2)s} B ds \\ &= x_0 + \int_0^t AX_s dW_s + \int_0^t BX_s ds. \end{aligned}$$

Let us summarize our discussion.

**Proposition 24.6** *The unique pathwise solution to*

$$dX_t = AX_t dW_t + BX_t dt$$

is

$$X_t = X_0 e^{AW_t + (B - A^2/2)t}.$$

If we write  $Z_t = AW_t + Bt$ , then (24.15) becomes

$$dX_t = X_t dZ_t, \quad X_0 = x_0. \quad (24.16)$$

The equation (24.16) makes sense for arbitrary continuous semimartingales  $Z$ , and by using Itô's formula as above, one can see that a solution is  $X_t = x_0 e^{Z_t - \langle Z \rangle_t / 2}$ .

### Bessel processes

We consider Bessel processes and the squares of Bessel processes. The reason for the name is that these processes turn out to be Markov processes and the infinitesimal generator of the semigroup (see Chapter 37) is related to Bessel's equation, a type of differential equation.

A Bessel process of order  $\nu \geq 2$  is defined to be a solution of the SDE

$$dX_t = dW_t + \frac{\nu - 1}{2X_t} dt, \quad X_0 = x. \quad (24.17)$$

Bessel processes of order  $0 \leq \nu < 2$  can also be defined using (24.17), but only up until the first time the process  $X$  reaches 0; some extra information needs to be given as to what the process does at 0. The square of a Bessel process of order  $\nu \geq 0$  is defined to be the solution to the SDE

$$dY_t = 2\sqrt{|Y_t|} dW_t + \nu dt, \quad Y_0 = y. \quad (24.18)$$

There is no difficulty defining the square of a Bessel process for  $0 \leq \nu < 2$ .

By Theorem 24.4 we have pathwise uniqueness for the solution to (24.18), because  $|y|^{1/2} - |x|^{1/2} \leq |y - x|^{1/2}$ , and we can thus take  $\rho(u) = 2u^{1/2}$  in Theorem 24.4. The solution to (24.18) when  $\nu = 0$  and  $y = 0$  is clearly  $Y_t = 0$  for all  $t$ . By Theorem 24.5 with  $b(x) = \nu$  and  $B(x) = 0$ , we see that the solution to (24.18) is greater than or equal to 0 for all  $t$ . We may thus omit the absolute value in (24.18) and rewrite it as

$$dY_t = 2\sqrt{Y_t} dW_t + \nu dt, \quad Y_0 = y. \quad (24.19)$$

If we apply Itô's formula to the solution  $Y_t$  of (24.19) with the function  $\sqrt{x}$ , we see that  $X_t = \sqrt{Y_t}$  solves (24.17) for  $t$  up until the first time  $Y$  reaches 0; the function  $\sqrt{x}$  is twice continuously differentiable as long as we stay away from 0. We will see shortly that the square of a Bessel process started away from 0 never hits 0 if and only if  $\nu \geq 2$ .

Using Itô's formula with a  $d$ -dimensional process  $W_t$  and the function  $|x|^2$  shows that the square of the modulus of a  $d$ -dimensional Brownian motion is the square of a Bessel process of order  $d$ ; this is Exercise 24.7.

Bessel processes have the same scaling properties as Brownian motion. That is, if  $X_t$  is a Bessel process of order  $\nu$  started at  $x$ , then  $aX_{a^{-2}t}$  is a Bessel process of order  $\nu$  started at  $ax$ . In fact, from (24.17),

$$d(aX_{a^{-2}t}) = a dW_{a^{-2}t} + a^2 \frac{\nu - 1}{2aX_{a^{-2}t}} d(a^{-2}t),$$

and the assertion follows from the uniqueness of the solution to (24.17) and the fact that  $aW(a^{-2}t)$  is again a Brownian motion.

Bessel processes are useful for comparison purposes, and so the following is worthwhile.

**Proposition 24.7** Suppose  $Y_t$  is the square of a Bessel process of order  $\nu$ . Suppose  $Y_0 = y$ . The following hold with probability one.

- (1) If  $\nu > 2$  and  $y > 0$ ,  $Y_t$  never hits 0.
- (2) If  $\nu = 2$  and  $y > 0$ ,  $Y_t$  hits every neighborhood of 0, but never hits the point 0.
- (3) If  $0 < \nu < 2$ ,  $Y_t$  hits 0.
- (4) If  $\nu = 0$ , then  $Y_t$  hits 0. If started at 0, then  $Y_t$  remains at 0 forever.

When we say that  $Y_t$  hits 0, we consider only times  $t > 0$ . We define  $T_0 = \inf\{t > 0 : Y_t = 0\}$  and say that  $Y_t$  hits 0 if  $T_0 < \infty$ .

*Proof* We prove (2). An application of Itô's formula with the process being the square of a Bessel process of order 2 and the function being  $\log x$  shows that  $\log Y_t$  is a martingale up until the first hitting time of 0; cf. Exercise 21.1. The quadratic variation of  $\log Y_t$  is  $\int_0^t Y_s^{-2} ds$  for  $t$  less than the hitting time of 0. Suppose  $0 < a < y < b$ .

We claim that  $Y_t$  leaves the interval  $[a, b]$ , a.s. If not,  $\langle \log Y \rangle_t \geq b^{-2}t \rightarrow \infty$  as  $t \rightarrow \infty$ . Since  $\log Y_t$  is a martingale, it is a time change of Brownian motion, and Brownian motion leaves  $[\log a, \log b]$  with probability one, a contradiction.

Then by Corollary 3.17,

$$\mathbb{P}(Y_t \text{ hits } a \text{ before } b) = \frac{\log b - \log y}{\log b - \log a}. \quad (24.20)$$

Letting  $b \rightarrow \infty$ , we see that  $\mathbb{P}(Y_t \text{ hits } a) = 1$ , and since  $a$  is arbitrary,  $Y_t$  hits every neighborhood of 0. If in (24.20) we hold  $b$  fixed instead and let  $a \rightarrow 0$ , we see  $\mathbb{P}(Y_t \text{ hits } 0 \text{ before } b) = 0$ ; since  $b$  is arbitrary, this proves that  $Y_t$  never hits the point 0.

Parts (1), (3), and (4) are similar, but instead of  $\log |x|$  we use  $|x|^{(2-\nu)/2}$ . The details are left as Exercise 24.8.  $\square$

## Exercises

- 24.1 Show that  $\|\cdot\|_l$ , defined by (24.7) gives rise to a complete normed linear space.
- 24.2 Suppose  $g(t)$  is non-negative and bounded on each finite subinterval of  $[0, \infty)$ . Suppose there exist constants  $A$  and  $B$  such that

$$g(t) \leq A + B \int_0^t g(s) ds \quad (24.21)$$

for each  $t \geq 0$ . Prove that  $g(t) \leq Ae^{Bt}$  for all  $t \geq 0$ . This result is known as *Gronwall's lemma*.

*Hint:* Write

$$g(t) \leq A + B \int_0^t \left[ A + B \int_0^s g(r) dr \right] ds,$$

use (24.21) to substitute for  $g(r)$ , and iterate.

- 24.3 The starting point in (24.1) can be random. Suppose  $Y$  is a random variable that is measurable with respect to  $\mathcal{F}_0$ ,  $Y$  is square integrable, and  $\sigma$  and  $b$  are bounded and Lipschitz. Prove pathwise existence and uniqueness for the equation

$$X_t = Y + \int_0^t \sigma(X_s) dW_s + \int_0^t b(X_s) ds.$$

- 24.4 The functions  $\sigma$  and  $b$  in (24.1) can depend on time as well as space. Suppose  $\sigma : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $b : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  are bounded and uniformly Lipschitz in the second variable: there exists  $c$  independent of  $s$  such that  $|\sigma(s, x) - \sigma(s, y)| \leq c|x - y|$  and similarly for  $b$ . Prove pathwise existence and uniqueness for the equation

$$X_t = x_0 + \int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds.$$

- 24.5 Here is a multidimensional analog of (24.1). Suppose the functions  $\sigma_{ij} : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $1 \leq i, j \leq d$ , are bounded and Lipschitz, and  $b_i : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $i = 1, \dots, d$ , are bounded and Lipschitz,  $W^j$  are independent one-dimensional Brownian motions,  $x_0 = (x_0^{(1)}, \dots, x_0^{(d)})$ , and  $X_t = (X_t^{(1)}, \dots, X_t^{(d)})$  satisfies

$$X_t^{(i)} = x_0^{(i)} + \int_0^t \sum_{j=1}^d \sigma_{ij}(X_s) dW_s^j + \int_0^t b_i(X_s) ds \quad (24.22)$$

for  $i = 1, \dots, d$ . Prove pathwise existence and uniqueness for this system of equations.

- 24.6 Suppose  $f$  and  $g$  map  $[0, \infty) \rightarrow \mathbb{R}$  with  $\int_0^\infty f(t)^2 dt < \infty$  and  $\int_0^\infty g(t)^2 dt < \infty$ . Show that  $\int_0^\infty f(t) dW_t$  is a mean zero Gaussian random variable, the same with  $f$  replaced by  $g$ , and

$$\text{Cov} \left( \int_0^\infty f(t) dW_t, \int_0^\infty g(t) dW_t \right) = \int_0^\infty f(t)g(t) dt.$$

*Hint:* Approximate  $f$  and  $g$  by piecewise constant deterministic functions.

- 24.7 Show that if  $W_t$  is a  $d$ -dimensional Brownian motion, then  $|W_t|^2$  is the square of a Bessel process of order  $d$ .

- 24.8 Prove (1), (3), and (4) of Proposition 24.7.

- 24.9 Let  $X$  be the solution to  $dX_t = \sigma(X_t) dW_t + b(X_t) dt$ , where  $W$  is a one-dimensional Brownian motion,  $\sigma$  and  $b$  are Lipschitz continuous real-valued functions, and  $|\sigma(x)| \leq c(1 + |x|)$  and  $|b(x)| \leq c(1 + |x|)$ . Let  $t_0 > 0$ . Prove that if  $p \geq 2$ , then

$$\mathbb{E} [\sup_{s \leq t_0} |X_s|^p] \leq c(1 + |x_0|^p).$$

- 24.10 Let  $W$  be a one-dimensional Brownian motion and let  $X_t^x$  be the solution to

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt, \quad X_0 = x.$$

Suppose  $\sigma$  and  $b$  are  $C^\infty$  functions and that  $\sigma$  and  $b$  and all their derivatives are bounded. Show that for each  $t$  the map  $x \rightarrow X_t^x$  is continuous in  $x$  with probability one. Show that the map is differentiable in  $x$ .

- 24.11 Suppose  $A(t)$  and  $B(t)$  are deterministic functions of  $t$ . Find an explicit solution to the one-dimensional SDE

$$dX_t = A(t) dW_t + B(t) dt, \quad X_0 = x.$$

## Notes

If one wants to have a stochastic differential equation with jumps, besides a Brownian motion, one integrates with respect to a Poisson point process, which is defined in Chapter 18. Using the notation of that chapter, one considers the stochastic differential equation

$$\begin{aligned} dX_t &= \sigma(X_{t-}) dW_t + b(X_{t-}) dt \\ &\quad + \int_{\mathcal{S}} F(X_{t-}, z) (\mu(dt, dz) - \nu(dt, dz)), \quad X_0 = x_0, \end{aligned}$$

which means that we want a solution to

$$\begin{aligned} X_t = x_0 + \int_0^t \sigma(X_{s-}) dW_s + \int_0^t b(X_{s-}) ds \\ + \int_0^t \int_S F(X_{s-}, z) (\mu(ds, dz) - \nu(ds, dz)). \end{aligned}$$

There is pathwise existence and uniqueness to this SDE provided  $F$  satisfies a suitable Lipschitz-like condition; see [Skorokhod \(1965\)](#).

# 25

---

## Weak solutions of SDEs

In Chapter 24 we considered SDEs of the form

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt, \quad (25.1)$$

where  $W$  is a Brownian motion and  $\sigma$  and  $b$  are Lipschitz functions, or in one dimension, where  $\sigma$  has a modulus of continuity satisfying an integral condition. When the coefficients  $\sigma$  and  $b$  fail to be sufficiently smooth, it is sometimes the case that (25.1) may not have a pathwise solution at all, or it may not be unique. We define another notion of existence and uniqueness that is useful.

**Definition 25.1** A *weak solution*  $(X, W, \mathbb{P})$  to (25.1) exists if there exists a probability measure  $\mathbb{P}$  and a pair of processes  $(X_t, W_t)$  such that  $W_t$  is a Brownian motion under  $\mathbb{P}$  and (25.1) holds. There is *weak uniqueness* holding for (25.1) if whenever  $(X, W, \mathbb{P})$  and  $(X', W', \mathbb{P}')$  are two weak solutions, then the joint law of  $(X, W)$  under  $\mathbb{P}$  and the joint law of  $(X', W')$  under  $\mathbb{P}'$  are equal. When this happens, we also say that the solution to (25.1) is *unique in law*.

Let us discuss the relationship between weak solutions and pathwise solutions. If the solution to (25.1) is pathwise unique, then weak uniqueness holds. For a proof of this result under very general hypotheses, see [Revuz and Yor \(1999\)](#), theorem IX.1.7. In the case that  $\sigma$  and  $b$  are Lipschitz functions, the proof is much simpler.

**Proposition 25.2** Suppose  $\sigma$  and  $b$  are bounded Lipschitz functions and  $x_0 \in \mathbb{R}^d$ . Then weak uniqueness holds for (25.1).

*Proof* For notational simplicity we consider the case of dimension one. Suppose  $(X, W, \mathbb{P})$  and  $(X', W', \mathbb{P}')$  are two weak solutions to (25.1). Let  $X_0(t) = x_0$  and define  $X_{i+1}(t)$  by

$$X_{i+1}(t) = x_0 + \int_0^t \sigma(X_i(s)) dW_s + \int_0^t b(X_i(s)) ds. \quad (25.2)$$

We saw by the proof of Theorem 24.2 that the limit of the  $X_i$  exists, uniformly over finite time intervals, and solves (25.1), and the solution is pathwise unique. Since  $X$  also solves (25.1), we conclude that  $X_i$  converges (uniformly over finite time intervals) to  $X$ , a.s., with respect to  $\mathbb{P}$ . Similarly, if we let  $X'_0(t) = x_0$  and define  $X'_{i+1}(t)$  by

$$X'_{i+1}(t) = x_0 + \int_0^t \sigma(X'_i(s)) dW'_s + \int_0^t b(X'_i(s)) ds, \quad (25.3)$$

then  $X'_i$  converges, uniformly over finite time intervals, to  $X'$ .

Now since  $W$  is a Brownian motion under  $\mathbb{P}$  and  $W'$  is a Brownian motion under  $\mathbb{P}'$ , then the law of  $(X_0, W)$  under  $\mathbb{P}$  equals the law of  $(X'_0, W')$  under  $\mathbb{P}'$ . By (25.2) and (25.3), the law of  $(X_1, W)$  under  $\mathbb{P}$  equals the law of  $(X'_1, W')$  under  $\mathbb{P}'$ , and iterating, the law of  $(X_i, W)$  under  $\mathbb{P}$  equals the law of  $(X'_i, W')$  under  $\mathbb{P}'$ . Passing to the limit, the law of  $(X, W)$  under  $\mathbb{P}$  equals the law of  $(X', W')$  under  $\mathbb{P}'$ .  $\square$

We now give an example to show that weak uniqueness might hold even if pathwise uniqueness does not. Let  $\sigma(x)$  be equal to 1 if  $x \geq 0$  and  $-1$  otherwise. We take  $b$  to be identically 0. We consider solutions to

$$X_t = \int_0^t \sigma(X_s) dW_s. \quad (25.4)$$

Weak uniqueness holds since if  $W$  is a Brownian motion under  $\mathbb{P}$ , then  $X_t$  must be a martingale, and the quadratic variation of  $X$  is  $d\langle X \rangle_t = \sigma(X_t)^2 dt = dt$ ; by Lévy's theorem (Theorem 12.1),  $X_t$  is a Brownian motion. Given a Brownian motion  $X_t$  and letting  $W_t = \int_0^t \frac{1}{\sigma(X_s)} dX_s$ , then again by Lévy's theorem,  $W_t$  is a Brownian motion and  $X_t = \int_0^t \sigma(X_s) dW_s$ ; thus weak solutions exist.

On the other hand, pathwise uniqueness does not hold. To see this, let  $Y_t = -X_t$ . We have

$$Y_t = \int_0^t \sigma(Y_s) dW_s - 2 \int_0^t 1_{\{0\}}(X_s) dW_s. \quad (25.5)$$

The second term on the right has quadratic variation  $4 \int_0^t 1_{\{0\}}(X_s) ds$ ; this is 0 almost surely because we showed in Exercise 11.1 that the amount of time Brownian motion spends at 0 has Lebesgue measure 0. Therefore  $Y$  is another pathwise solution to (25.4).

This example is not satisfying because one would like  $\sigma$  to be positive and even continuous if possible. Such examples exist, however. For each  $\beta < 1/2$ , Barlow (1982) has constructed functions  $\sigma$  that are Hölder continuous of order  $\beta$  and bounded above and below by positive constants and for which

$$dX_t = \sigma(X_t) dW_t, \quad X_0 = x_0, \quad (25.6)$$

has a unique weak solution but no pathwise solution exists.

Let us show how the technique of time change can be used to study weak uniqueness. We consider the SDE (25.6).

**Proposition 25.3** *If  $\sigma$  is Borel measurable and there exist  $c_2 > c_1 > 0$  such that  $c_1 \leq \sigma(x) \leq c_2$  for all  $x$ , then weak existence and weak uniqueness hold for (25.6).*

*Proof* We consider only uniqueness, leaving existence as Exercise 25.1. Suppose  $(X, W, \mathbb{P})$  and  $(X', W', \mathbb{P}')$  are two weak solutions. Then  $X_t$  is a martingale, and as in Section 12.2, if we set

$$A_t = \int_0^t \sigma(X_s)^2 ds, \quad \tau_t = \inf\{s : A_s \geq t\},$$

then  $M_t = X_{\tau_t}$  is a Brownian motion under  $\mathbb{P}$ . Define  $A'$ ,  $\tau'$ , and  $M'$  analogously. The law of  $M$  under  $\mathbb{P}$  is that of a Brownian motion, as is that of  $M'$  under  $\mathbb{P}'$ .

Now let

$$B_t = \int_0^t \frac{1}{\sigma(M_s)^2} ds, \quad \rho_t = \inf\{s : B_s \geq t\}. \quad (25.7)$$

Since  $M_t$  is a Brownian motion and  $\sigma$  is bounded above and below by positive constants, then  $B_t$  is continuous, strictly increasing, and increases to infinity as  $t \rightarrow \infty$ , and the same is therefore true of  $\rho_t$ . By a change of variables,

$$\begin{aligned} B_t &= \int_0^t \frac{1}{\sigma(X_{\tau_s})^2} ds = \int_0^{\tau_t} \frac{1}{\sigma(X_u)^2} dA_u \\ &= \int_0^{\tau_t} \frac{1}{\sigma(X_u)^2} \sigma(X_u)^2 du = \tau_t. \end{aligned}$$

Therefore  $M_{\rho_t} = X_{\tau(\rho_t)} = X_t$ . We have the analogous formulas with primes.

The law of  $M$  under  $\mathbb{P}$  equals the law of  $M'$  under  $\mathbb{P}'$  since both are Brownian motions, so by (25.7) the law of  $(M, B)$  under  $\mathbb{P}$  equals the law of  $(M', B')$  under  $\mathbb{P}'$ , and consequently the law of  $(M, \rho)$  under  $\mathbb{P}$  equals the law of  $(M', \rho')$  under  $\mathbb{P}'$ . Since  $X_t = M_{\rho_t}$  and similarly for  $X'$ , we conclude the law of  $X$  under  $\mathbb{P}$  equals the law of  $X'$  under  $\mathbb{P}'$ . Finally, since  $W_t = \int_0^t \frac{1}{\sigma(X_s)} dX_s$  and similarly for  $W'$ , the joint law of  $(X, W)$  under  $\mathbb{P}$  equals the joint law of  $(X', W')$  under  $\mathbb{P}'$ .  $\square$

We point out that in the above proof it is essential that one can reconstruct  $X$  from  $M$  in a measurable way.

We now use the Girsanov theorem to prove weak uniqueness for (25.1).

**Proposition 25.4** *Suppose  $\sigma$  and  $b$  are measurable and bounded above and  $\sigma$  is bounded below by a positive constant. Then weak existence and uniqueness holds for (25.1).*

*Proof* We prove the weak uniqueness, leaving it as Exercise 25.2 to prove existence. Define  $\{\mathcal{F}_t\}$  to be the minimal augmented filtration generated by  $X$ ,

$$M_t = \exp \left( - \int_0^t \frac{b}{\sigma}(X_s) dW_s - \frac{1}{2} \int_0^t \left( \frac{b}{\sigma}(X_s) \right)^2 ds \right),$$

and  $\mathbb{Q}$  the probability measure defined by  $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[M_t; A]$  if  $A \in \mathcal{F}_t$ . By Theorem 13.3, under  $\mathbb{Q}$ , the process  $\tilde{W}_t = W_t + \int_0^t (b/\sigma)(X_s) ds$  is a Brownian motion, and

$$dX_t = \sigma(X_t) \left( dW_t + \frac{b}{\sigma}(X_t) dt \right) = \sigma(X_t) d\tilde{W}_t.$$

Define  $M'$ ,  $\mathbb{Q}'$ , and  $\tilde{W}'$  analogously. By Proposition 25.3 the law of  $(X, \tilde{W})$  under  $\mathbb{Q}$  is equal to the law of  $(X', \tilde{W}')$  under  $\mathbb{Q}'$ . Let  $n \geq 1$ ,  $t_1 < \dots < t_n$ , and let  $A_1, \dots, A_n$  be Borel subsets of  $\mathbb{R}$ . Set  $B = \{X_{t_1} \in A_1, \dots, X_{t_n} \in A_n\}$  and define  $B'$  analogously. We have

$$\begin{aligned} \mathbb{P}(B) &= \int_B \frac{d\mathbb{P}}{d\mathbb{Q}} d\mathbb{Q} = \int_B \exp \left( \int_0^t \frac{b}{\sigma}(X_s) dW_s + \frac{1}{2} \int_0^t \left( \frac{b}{\sigma}(X_s) \right)^2 ds \right) d\mathbb{Q} \\ &= \int_B \exp \left( \int_0^t \frac{b}{\sigma}(X_s) d\tilde{W}_s - \frac{1}{2} \int_0^t \left( \frac{b}{\sigma}(X_s) \right)^2 ds \right) d\mathbb{Q}. \end{aligned}$$

Using the analogous formula for  $\mathbb{P}'(B')$  and the fact that the law of  $(X, \tilde{W})$  under  $\mathbb{Q}$  is the same as that of  $(X', \tilde{W}')$  under  $\mathbb{Q}'$ , we see that  $\mathbb{P}(B) = \mathbb{P}'(B')$ ; thus the finite-dimensional distributions of  $X$  under  $\mathbb{P}$  and of  $X'$  under  $\mathbb{P}'$  are the same. Since both  $X$  and  $X'$  are continuous processes, we conclude from Theorem 2.6 that the law of  $X$  under  $\mathbb{P}$  equals the law of  $X'$  under  $\mathbb{P}'$ . Defining  $Y_t = X_t - \int_0^t b(X_s) ds$  and similarly for  $Y'$ , the joint law of  $(X, Y)$  under  $\mathbb{P}$  equals the joint law of  $(X', Y')$  under  $\mathbb{P}'$ . Finally,  $W_t = \int_0^t \frac{1}{\sigma(X_s)} dY_s$  and similarly for  $W'$ , so we obtain our conclusion.  $\square$

The procedure of using the Girsanov theorem to get rid of the drift also works in higher dimensions. However the time change procedure of Proposition 25.3 is not nearly as useful in higher dimensions as in one dimension. The question of weak uniqueness for the system of equations in Exercise 24.5 is quite an interesting one; see Bass (1997) and Stroock and Varadhan (1977).

## Exercises

- 25.1 Show weak existence holds under the hypotheses of Proposition 25.3.
- 25.2 Show weak existence holds under the hypotheses of Proposition 25.4.
- 25.3 Here is an example of an SDE where weak uniqueness does not hold. Suppose  $W$  is a one-dimensional Brownian motion and  $\alpha \in (0, \frac{1}{2})$ . Let  $\sigma(x) = 1 \wedge |x|^\alpha$ . Find two solutions to

$$dX_t = \sigma(X_t) dW_t$$

that are not equal in law.

*Hint:* One is the solution that is identically zero. The other can be constructed by time changing a Brownian motion by the inverse of the increasing process

$$2 \int_0^t (1 \wedge |X_s|^{2\alpha})^{-1} ds.$$

- 25.4 (1) Suppose  $a_s$  and  $b_s$  are bounded predictable processes with  $a_s$  bounded below by a positive constant. Let  $W$  be a one-dimensional Brownian motion. Suppose  $Y$  is a one-dimensional semimartingale such that

$$dY_t = a_t Y_t dW_t + b_t dt, \quad Y_0 = 0.$$

Prove that if  $t_0 > 0$  and  $\varepsilon > 0$ , there exists a constant  $c > 0$  depending only on  $t_0$ ,  $\varepsilon$ , and the bounds on  $a_s$  and  $b_s$  such that

$$\mathbb{P}(\sup_{s \leq t_0} |Y_s| < \varepsilon) > c.$$

(2) Now let  $W$  be  $d$ -dimensional Brownian motion, let  $x \in \mathbb{R}^d$ , and let  $\sigma$  be a  $d \times d$  matrix-valued function that is bounded and such that  $\sigma \sigma^T(x)$  is positive definite, uniformly in  $x$ . That is, there exists  $\Lambda > 0$  such that for all  $x$ ,

$$\sum_{i,j=1}^d y_i y_j (\sigma(x) \sigma^T(x))_{ij} \geq \Lambda \sum_{i=1}^d y_i^2, \quad (y_1, \dots, y_d) \in \mathbb{R}^d.$$

Let  $b$  be a  $d \times 1$  matrix-valued function that is bounded. Let  $X$  be the solution to

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt, \quad X_0 = x.$$

Use Itô's formula to find an equivalent expression for  $|X_t - x|^2$ . Then use (1) to prove that if  $t_0 > 0$  and  $\varepsilon > 0$ , there exists a constant  $c > 0$  not depending on  $x$  such that

$$\mathbb{P}^x(\sup_{s \leq t_0} |X_s - x| < \varepsilon) > c.$$

25.5 This is the *support theorem* for solutions to SDEs. Let  $X$ ,  $x$ ,  $\varepsilon$ , and  $t_0$  be as in (2) of Exercise 25.4. Suppose  $\psi : [0, t_0] \rightarrow \mathbb{R}^d$  is a continuous function with  $\psi(0) = x$ . Use the Girsanov theorem to prove that there exists  $c > 0$  such that

$$\mathbb{P}^x(\sup_{s \leq t_0} |X_s - \psi(s)| < \varepsilon) > c.$$

25.6 Suppose weak uniqueness holds for the one-dimensional stochastic differential equation

$$dX_t = \sigma(X_t) dW_t, \quad X_0 = x, \quad (25.8)$$

where  $W$  is a one-dimensional Brownian motion. Suppose also that there exists a process  $X'$  that is adapted to the minimal augmented filtration of  $W$  with  $X'_0 = x$  and  $dX'_t = \sigma(X'_t) dW_t$ . Prove that pathwise uniqueness holds for (25.8).

*Hint:* Show there exists a measurable map  $F$  from  $C[0, \infty) \rightarrow C[0, \infty)$  such that  $X' = F(W)$ . If  $X''$  is another solution to (25.8), then weak uniqueness shows that the laws of  $(X'', W)$  and  $(X', W)$  are equal, hence  $X'' = F(W) = X'$ .

# 26

---

## The Ray–Knight theorems

The local time of Brownian motion,  $L_t^x$ , is parameterized by space and time:  $x$  and  $t$ . Ray and Knight independently discovered that at certain stopping times  $T$ , the process  $x \rightarrow L_T^x$  is a Markov process.

Times that work are (1) the first time local time at 0 reaches a level  $r$ ; (2) an exponential random variable  $T$  that is independent of the Brownian motion; and (3) the first time  $T$  that Brownian motion reaches the level one. We will prove the version of the Ray–Knight theorems in the last case. We will show that if  $W$  is a Brownian motion with local times  $L_t^x$  and

$$T = \inf\{t > 0 : W_t = 1\},$$

then the process  $L_T^{1-x}$  indexed by  $x$  has the same law as the square of a Bessel process of order 2. We will see in Chapter 39 that the square of a Bessel process is a Markov process.

We will use the following lemma.

**Lemma 26.1** Suppose  $X_t^{(j)}$ ,  $j = 1, 2$ , are two continuous processes such that

$$\mathbb{E} \exp \left( - \int_0^1 f(s) X_s^{(1)} ds \right) = \mathbb{E} \exp \left( - \int_0^1 f(s) X_s^{(2)} ds \right)$$

whenever  $f$  is a non-negative continuous function with support in  $(0, 1)$ . Then the laws of  $\{X_t^{(j)} ; 0 \leq t \leq 1\}$ ,  $j = 1, 2$ , are equal.

*Proof* Let  $\varphi$  be a non-negative continuous function with support in  $[0, 1]$  such that  $\int_0^1 \varphi(x) dx = 1$ , and let  $\varphi_\varepsilon(x) = \varepsilon^{-1} \varphi(x/\varepsilon)$ , so that the sequence  $\{\varphi_\varepsilon\}$  is an approximation to the identity. If  $g$  is a continuous function and  $t \neq 0$ , then  $\int g(s) \varphi_\varepsilon(s-t) ds \rightarrow g(t)$ . Now let  $t_1, \dots, t_n \in (0, 1)$ ,  $a_1, \dots, a_n > 0$ , and set  $f_\varepsilon(x) = \sum_{i=1}^n a_i \varphi_\varepsilon(x - t_i)$ . Using the hypothesis and letting  $\varepsilon \rightarrow 0$ , we obtain

$$\mathbb{E} \exp \left( - \sum_{i=1}^n a_i X_{t_i}^{(1)} \right) = \mathbb{E} \exp \left( - \sum_{i=1}^n a_i X_{t_i}^{(2)} \right).$$

The left-hand side is the joint Laplace transform of  $(X_{t_1}^{(1)}, \dots, X_{t_n}^{(1)})$  and the right-hand side is the same for  $X^{(2)}$ . By the uniqueness of the Laplace transform, the finite-dimensional distributions of  $X^{(1)}$  and  $X^{(2)}$  are equal. Both processes have continuous paths, and the conclusion now follows from Theorem 2.6.  $\square$

Let  $B_t$  be a Brownian motion, not necessarily the same as  $W_t$ , and let  $Z_t$  be the non-negative solution to

$$dZ_t = 2\sqrt{Z_t} dB_t + 2 dt, \quad Z_0 = 0, \quad 0 \leq t \leq 1. \quad (26.1)$$

The solution to this equation is unique by Theorem 24.4, and  $Z_t$  is the square of a Bessel process of order 2.

**Theorem 26.2** *The processes  $\{L_T^{1-x}; 0 \leq x \leq 1\}$  and  $\{Z_x; 0 \leq x \leq 1\}$  have the same law.*

*Proof* Let  $f \geq 0$  be a continuous function whose support  $[a, b]$  is a subset of  $(0, 1)$ . Let  $F$  be the solution to

$$F''(x) = 2F(x)f(x), \quad F(1) = 1, \quad F'(1) = 0;$$

see Exercise 26.1. Define  $g(x) = f(1-x)$  and  $G(x) = F(1-x)$ , so that  $G'' = 2Gg$ ,  $G'(0) = 0$ , and  $G(0) = 1$ . We will show

$$\mathbb{E} \exp \left( - \int_0^1 f(x) L_T^{1-x} dx \right) = \mathbb{E} \exp \left( - \int_0^1 f(t) Z_t dt \right), \quad (26.2)$$

and then apply Lemma 26.1.

The left-hand side of (26.2) is equal to

$$\begin{aligned} \mathbb{E} \exp \left( - \int_0^1 f(1-x) L_T^x dx \right) &= \mathbb{E} \exp \left( - \int_0^1 g(x) L_T^x dx \right) \\ &= \mathbb{E} \exp \left( - \int_0^T g(X_s) ds \right), \end{aligned}$$

where the last equality follows from the occupation time formula (Theorem 14.4). Let

$$M_t = G(W_t) e^{- \int_0^t g(W_s) ds}.$$

By Itô's formula and the product formula,

$$\begin{aligned} dM_t &= -G(W_t)g(W_t)e^{- \int_0^t g(W_s) ds} dt + G'(W_t)e^{- \int_0^t g(W_s) ds} dW_t \\ &\quad + \frac{1}{2}G''(W_t)e^{- \int_0^t g(W_s) ds} dt \\ &= G'(W_t)e^{- \int_0^t g(W_s) ds} dW_t, \end{aligned}$$

since  $\frac{1}{2}G'' - Gg = 0$ . Therefore  $M_t$  is a martingale. Since  $G$  is bounded on  $(-\infty, 1]$ , then  $M_{t \wedge T}$  is bounded and we then have

$$1 = G(0) = \mathbb{E} M_0 = \mathbb{E} M_T = \mathbb{E} G(1) e^{- \int_0^T g(W_s) ds},$$

so

$$\mathbb{E} \exp \left( - \int_0^T g(W_s) ds \right) = \frac{1}{G(1)}. \quad (26.3)$$

Now look at the right-hand side of (26.2). Let

$$N_t = \frac{1}{F(t)} \exp \left( Z_t \frac{F'(t)}{2F(t)} - \int_0^t f(s) Z_s ds \right).$$

Let

$$Y_t = Z_t \frac{F'(t)}{2F(t)},$$

so using (26.1),

$$dY_t = Z_t \frac{2F(t)F''(t) - 2F'(t)^2}{4F(t)^2} dt + 2 \frac{F'(t)}{2F(t)} \sqrt{Z_t} dB_t + 2 \frac{F'(t)}{2F(t)} dt.$$

If

$$X_t = Y_t - \int_0^t f(s)Z_s ds,$$

then the martingale part of  $X$  is

$$\int_0^t \frac{F'(s)}{F(s)} \sqrt{Z_s} dB_s,$$

and hence

$$d\langle X \rangle_t = \left( \frac{F'(t)}{F(t)} \right)^2 Z_t dt.$$

By Itô's formula and the product formula and using  $F'' = 2Ff$ ,

$$\begin{aligned} dN_t &= -\frac{F'(t)}{F(t)^2} e^{X_t} dt + \frac{1}{F(t)} e^{X_t} \left\{ Z_t \frac{F''(t)}{2F(t)} dt - Z_t \frac{F'(t)^2}{2F(t)^2} dt \right. \\ &\quad \left. + \frac{F'(t)}{F(t)} \sqrt{Z_t} dB_t + \frac{F'(t)}{F(t)} dt - f(t)Z_t dt \right\} \\ &\quad + \frac{1}{2} \frac{1}{F(t)} e^{X_t} \frac{F'(t)^2}{F(t)^2} Z_t dt \\ &= \frac{F'(t)}{F(t)} \sqrt{Z_t} dB_t. \end{aligned}$$

Observe that  $F$  is continuous and positive on  $[0, 1]$ , hence bounded below on  $[0, 1]$  by a positive constant. Also  $F'$  is bounded above on  $[0, 1]$ . We see that  $N_{t \wedge 1}$  is a martingale. Then  $\mathbb{E} N_0 = 1/F(0) = 1/G(1)$ , while

$$\begin{aligned} \mathbb{E} N_1 &= \frac{1}{F(1)} \mathbb{E} \exp \left( Z_1 \frac{F'(1)}{2F(1)} - \int_0^1 f(s)Z_s ds \right) \\ &= \mathbb{E} e^{- \int_0^1 f(s)Z_s ds}. \end{aligned}$$

Therefore

$$\mathbb{E} \exp \left( - \int_0^1 f(s)Z_s ds \right) = \mathbb{E} N_1 = \mathbb{E} N_0 = \frac{1}{G(1)}.$$

Combining with (26.3), we conclude the two sides of (26.2) are equal.  $\square$

You may wonder how the function  $F$  was arrived at. Exercises 26.2 and 26.3 may shed some light on this.

### Exercises

- 26.1 Suppose  $f$  is a non-negative continuous function whose support  $[a, b]$  is a subset of  $(0, 1)$ . Show that there is a unique solution to the ordinary differential equation  $F''(x) = 2F(x)f(x)$ ,  $F(1) = 1$ ,  $F'(1) = 0$ , that  $F$  is everywhere positive, and  $F$  is bounded on  $[0, \infty)$ .

*Hint:* Since  $f$  is zero in  $(b, \infty)$ , then  $F''$  is zero there, and hence is of the form  $F(x) = Ax + B$  for some  $A$  and  $B$  for  $x \geq b$ . Since  $F'(1) = 0$ , conclude that  $A$  is 0.

- 26.2 Suppose  $X_t$  is a solution to the one-dimensional SDE

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt.$$

Suppose  $\sigma$  and  $b$  are bounded and continuous and  $f$  is a bounded and continuous function. What ordinary differential equation must  $H(x)$  satisfy (in terms of  $\sigma$ ,  $b$ , and  $f$ ) in order that

$$M_t = H(X_t) e^{\int_0^t f(X_s) ds}$$

be a martingale?

- 26.3 Suppose  $X_t$  is a solution to the one-dimensional SDE

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt.$$

Suppose  $\sigma$  and  $b$  are bounded and continuous and  $f$  is a bounded and continuous function. What partial differential equation must  $K(x, t)$  satisfy (in terms of  $\sigma$ ,  $b$ , and  $f$ ) in order that

$$N_t = K(X_t, t) e^{\int_0^t f(s) X_s ds}$$

be a martingale?

- 26.4 Let  $W$  be a Brownian motion and  $L_t^y$  the local times at level  $y$ . Prove that local times at a fixed time  $t$  are not a Markov process. That is, let  $t > 0$  be fixed and show that  $(L_t^y, y \geq 0)$  is not a Markov process in the variable  $y$ .
- 26.5 Let  $S$  be the first time two-dimensional Brownian motion exits the unit ball and let  $\psi(\lambda) = \mathbb{P}^0(S > \lambda)$ . If  $W$  is a one-dimensional Brownian motion with local times  $L_t^x$  and  $T = \inf\{t > 0 : W_t = 1\}$ , find the distribution of  $Y = \sup_{0 \leq x \leq 1} L_T^x$  in terms of  $\psi$ , i.e., write  $\mathbb{P}(Y \leq \lambda)$  in terms of the function  $\psi$ .
- 26.6 Suppose  $x \in (0, 1)$ . With  $W$  and  $T$  as in Exercise 26.5, find the distribution of  $L_T^x$ .
- 26.7 Let  $W$  be a one-dimensional Brownian motion with local times  $L_t^x$ . Let  $T_r = \inf\{t > 0 : L_t^0 = r\}$ . The law of the process  $x \rightarrow L_{T_r}^x$  can be described as follows:
- (1) The law of  $\{L_{T_r}^x, x \geq 0\}$  is the same as the law of  $\{X_x, x \geq 0\}$  started at  $r$ , where  $X$  is the square of a Bessel process of order 0.
  - (2) The law of  $\{L_{T_r}^{-x}, x \geq 0\}$  is also the same as the law of  $\{X_x, x \geq 0\}$  started at  $r$ , where  $X$  is the square of a Bessel process of order 0.
  - (3) The processes  $\{L_{T_r}^x, x \geq 0\}$  and  $\{L_{T_r}^{-x}, x \geq 0\}$  are independent of each other.

This is proved in [Revuz and Yor \(1999\)](#), Section XI.2, or for a challenge, try to prove (1) for yourself using the techniques of this chapter. Using this description of  $L_{T_r}^x$ , find the distribution of  $L_{T_r}^* = \sup_x L_{T_r}^x$ .

### Notes

There are several other proofs of the Ray–Knight theorems. One by Walsh ([Rogers and Williams, 2000b](#); [Walsh, 1978](#)) uses excursion theory. In the next chapter we will indicate some ideas used in that proof.

## Brownian excursions

The paths of a Brownian motion  $W_t$  are continuous, so the zero set  $Z(\omega) = \{t : W_t(\omega) = 0\}$  is a closed set. The complement of  $Z(\omega)$  is an open subset of the reals, hence is the countable union of disjoint open intervals. If  $(a, b)$  is one of those intervals (depending on  $\omega$ , of course), then  $\{W_t(\omega) : a \leq t \leq b\}$  is a continuous function of  $t$  that is zero at  $t = a$  and  $t = b$  but is never 0 for any  $t \in (a, b)$ . We call this piece of the path of  $W_t(\omega)$  an *excursion*.

To be more formal, let  $\mathcal{E}$  be the collection of continuous functions  $f$  with domain  $[0, \infty)$  such that the following hold: there exists a positive real  $\sigma_f$  such that  $f(0) = 0$ ,  $f(\sigma_f) = 0$ ,  $f(t) \neq 0$  if  $t \in (0, \sigma_f)$ , and  $f(t) = 0$  if  $t > \sigma_f$ . We make  $\mathcal{E}$  into a metric space by furnishing it with the supremum norm. Given a Borel subset  $A$  of  $\mathcal{E}$ , we say that the Brownian motion  $W$  has had an excursion in  $A$  by time  $t$  if there exists a time  $u$  and a function  $f \in A$  such that  $u + \sigma_f \leq t$  and  $W_{u+s}(\omega) = f(s)$  for all  $s \leq \sigma_f$ . Let  $K_t(A)$  be the number of excursions of  $W$  in  $A$  by time  $t$ . Let  $L_t^0$  be Brownian local time at 0, and let

$$T_r = \inf\{t > 0 : L_t^0 \geq r\} \quad (27.1)$$

be the inverse of Brownian local time at zero.

Set

$$N_r(A) = K_{T_r}(A).$$

Although  $N_t(A)$  might be identically infinite for some sets  $A$ , it will be finite for others. For example, let  $\delta > 0$  and suppose that every function in  $A$  has a supremum greater than  $\delta$ . The continuity of the paths of  $W$  implies that  $N_t(A)$  is finite for every  $t$ .

The main result of this section is the following.

**Theorem 27.1**  $N_t(\cdot)$  is a Poisson point process.

*Proof* If  $N_t(B)$  is not infinite, then it has right-continuous paths that increase at most 1 at any given time. The main step will be to show that  $N_t(B)$  has stationary increments and  $N_t(B) - N_s(B)$  is independent of the  $\sigma$ -field generated by the random variables

$$\{N_r(A) : r \leq s, A \text{ a Borel subset of } \mathcal{E}\}.$$

If  $r_1 \leq \dots \leq r_n \leq s < t$ ,  $k \geq 0$ ,  $j_1, \dots, j_n \geq 0$ , and  $B$  and  $A_1, \dots, A_n$  are Borel subsets of  $\mathcal{E}$ , then

$$\begin{aligned} \mathbb{P}(N_t(B) - N_s(B) = k; N_{r_1}(A_1) = j_1, \dots, N_{r_n}(A_n) = j_n) \\ = \mathbb{P}(K_{T_s}(B) - K_{T_s}(B) = k; K_{T_{r_1}}(A_1) = j_1, \dots, K_{T_{r_n}}(A_n) = j_n) \\ = \mathbb{E} [\mathbb{P}^{W_{T_s}}(K_{T_{t-s}}(B) - K_{T_0}(B) = k; K_{T_{r_1}}(A_1) = j_1, \dots, K_{T_{r_n}}(A_n) = j_n)], \end{aligned} \quad (27.2)$$

where we used the strong Markov property at time  $T_s$ . Since  $T_s$  is the first time that local time of Brownian motion at 0 exceeds  $s$  and  $L_t^0$  increases only when  $W$  is at 0, then at time  $T_s$  the process  $W$  is at 0, so  $W_{T_s} = 0$ . Therefore the last expression in (27.2) equals

$$\mathbb{P}^0(K_{T_{t-s}}(B) - K_0(B) = k) \mathbb{P}(K_{T_{r_1}}(A_1) = j_1, \dots, K_{T_{r_n}}(A_n) = j_n),$$

which can be rewritten as

$$\mathbb{P}^0(N_{t-s}(B) - N_0(B) = k) \mathbb{P}(N_{r_1}(A_1) = j_1, \dots, N_{r_n}(A_n) = j_n).$$

This shows that the law of  $N_t(B) - N_s(B)$  is the same as the law of  $N_{t-s}(B) - N_0(B)$  and is independent of  $\sigma(N_r(A) : r \leq s, A \subset \mathcal{E})$ , which is what we wanted.

Observe that  $N_t(B)$  is constant except for jumps of size one. By Proposition 5.4,  $N_t(B)$  is a Poisson process. It is clear that  $N_t(B)$  is a measure in  $B$ , which completes the proof.  $\square$

Let  $m(A) = \mathbb{E}^0 N_1(A)$ . The measure  $A$  is called the *excursion measure*. We can say a few things about  $m$ .

**Proposition 27.2** *If*

$$A = \{f \in \mathcal{E} : \sup_t |f(t)| > a\},$$

*then  $m(A) = 1/a$ .*

*Proof* Let  $U = \inf\{t : |W_t| = a\}$  and  $V = \inf\{t > U : W_t = 0\}$ . Since  $|W_t| - L_t^0$  is a martingale by Theorem 14.1, then  $\mathbb{E}^0 |W_{t \wedge U}| = \mathbb{E}^0 L_{t \wedge U}^0$ . Letting  $t \rightarrow \infty$  and using dominated convergence on the left and monotone convergence on the right,

$$\mathbb{E} L_U^0 = \mathbb{E}^0 |W_U| = a.$$

Set  $R = \inf\{r : N_r(A) = 1\}$ . Because  $N_r(A)$  is a Poisson process, then  $R$  is an exponential random variable with parameter  $\mathbb{E} N_1(A) = m(A)$ . It therefore suffices to show  $\mathbb{E}^0 R = a$ ; see (A.9).

We have  $R = \inf\{r : K_{T_r}(A) = 1\}$ , and because  $K$  can only increase at times when  $W_t = 0$ , then

$$R = \inf\{L_t^0 : K_t(A) = 1\}.$$

Now  $K_t(A)$  will first equal one when  $t = V$ . But because local time at 0 does not increase when  $W$  is not at 0,  $L_V^0 = L_U^0$ . Therefore

$$\mathbb{E}^0 R = \mathbb{E}^0 L_V^0 = \mathbb{E}^0 L_U^0 = a.$$

We conclude that  $m(A) = 1/a$ .  $\square$

By symmetry, if  $B = \{f \in \mathcal{E} : \sup_t f(t) > a\}$ , then  $m(B) = 1/(2a)$ . One can say more about  $m$ . Consider those excursions whose maximum is some fixed value  $b$ . Starting at any point other than 0, the excursion can be viewed as a Brownian motion killed at 0 and conditioned to have maximum  $b$ . Such a path can be decomposed into the part before the maximum, which is a Brownian motion conditioned to hit  $b$  before 0, and the part after the maximum, which is Brownian motion conditioned to hit 0 before  $b$ . The former can be shown to have the same law as a three-dimensional Bessel process, up until it hits the level  $b$  (see the example in Section 22.2), and the latter the same law as  $b - X_t$ , where  $X_t$  is also a three-dimensional Bessel process up until it hits the level  $b$ . Moreover, the part of the path before the maximum can be taken to be independent of the part of the path after the maximum. See [Rogers and Williams \(2000b\)](#) for details.

Let us briefly revisit the Ray–Knight theorems and indicate how Brownian excursions can be used to obtain information about local times at different levels. Fix  $r$  and let  $T_r = \inf\{t > 0 : L_t^0 \geq r\}$ . If  $x > 0$  and  $y_1, \dots, y_n < 0$ , then the local time at  $x$  is a function of the excursions from 0 that hit  $x$  and the local times at  $y_1, \dots, y_n$  are functions of the excursions that go below zero. Since the set of excursions that take positive values and those that take negative values are independent, then  $L_{T_r}^x$  should be independent of  $L_{T_r}^{y_1}, \dots, L_{T_r}^{y_n}$ . To find the distribution of  $L_{T_r}^x$ , there are a Poisson number of excursions that reach the level  $x$ . Each excursion that reaches  $x$  contributes an amount to the local time at  $x$  that is an exponential random variable; see Exercise 27.1. After proving some additional independence, namely, that the amount each excursion contributes to local time at  $x$  is independent of the amount any other excursion contributes and that the amount contributed by an excursion is independent of the number of excursions reaching  $x$ , we see that  $L_{T_r}^x$  should have the same distribution as a Poisson number of independent exponential random variables.

## Exercises

- 27.1 Let  $W$  be a Brownian motion,  $x > 0$ , and  $T = \inf\{t > 0 : W_t = x\}$ . If  $L_t^x$  is the local time at  $x$ , show that the distribution of  $L_T^x$  is an exponential random variable. Determine the parameter of this exponential random variable.
- 27.2 Let  $W$  be a one-dimensional Brownian motion. This exercise asks you to prove that the normalized number of downcrossings by time  $t$  converges to local time at 0. If  $a > 0$ , let  $S_0 = 0$ ,  $T_0 = \inf\{t : W_t = a\}$ , and for  $i \geq 1$ ,

$$S_i = \inf\{t > T_{i-1} : W_t = 0\}, \quad T_i = \inf\{t > S_i : W_t = a\}.$$

Then  $D_t(a)$ , the number of downcrossings up to time  $t$ , is defined to be  $\sup\{k : S_k \leq t\}$ . Prove that there exists a constant  $c$  such that

$$\lim_{a \rightarrow 0} a D_t(a) = c L_t^0, \quad \text{a.s.,}$$

where  $L_t^0$  is local time at 0 of  $W$ . Determine  $c$ .

*Hint:* Use Exercise 18.5.

- 27.3 Let  $(X_t, \mathbb{P}^x)$  be a Brownian motion.

(1) Use the reflection principle to find

$$\mathbb{P}^0(X_s > -a \text{ for all } s \leq r).$$

This is the same as  $\mathbb{P}^a(T_0 > r)$ , where  $T_0$  is the first time the Brownian motion hits 0.

(2) Let

$$A(a, r) = \{f \in \mathcal{E} : \sup f(t) > a, \sigma_f > r\},$$

$$B(r) = \{f \in \mathcal{E} : \sigma_f > r, \sup f(t) > 0\},$$

and

$$C(a) = \{f \in \mathcal{E} : \sup f(t) > a\}.$$

Prove that

$$m(B(r)) = \lim_{a \rightarrow 0} m(A(a, r)) = \lim_{a \rightarrow 0} [m(C(a)) \times \mathbb{P}^a(T_0 > r)]$$

and use this and (1) to compute  $m(B(r))$ . By symmetry,  $m(\{f \in \mathcal{E} : \sigma_f > r\})$  will be twice the value of  $m(B(r))$ .

- 27.4 Let  $W$  be a Brownian motion. Let  $E_t(r)$  be the number of excursions of length larger than  $r$  that have been completed by time  $t$ . An excursion of length larger than  $r$  means that  $\sigma_f > r$ . Show that there exists a constant  $c$  such that

$$\lim_{r \rightarrow 0} \sqrt{r} E_t(r) = c L_t^0, \quad \text{a.s.}$$

Determine  $c$ .

One interesting point here is that this shows that  $L_t^0$  is determined entirely by the zero set  $Z(\omega) = \{t : W_t(\omega) = 0\}$ .

- 27.5 Let  $\delta > 0$  and  $A_\delta = \{f \in \mathcal{E} : \sup_t |f(t)| > \delta\}$ . Let  $S_1 = \inf\{t : K_t(A_\delta) = 1\}$  and  $S_2 = \inf\{t > S_1 : K_t(A_\delta) = 2\}$ . Thus  $S_1$  and  $S_2$  are the times the first and second excursions in  $A_\delta$  have been completed. Let  $Y_1(t)$  be the excursion completed at time  $S_1$  and define  $Y_2(t)$  similarly. To be more precise, if  $R_1 = \sup\{t < S_1 : W_t = 0\}$ , then  $Y_1(s) = W_{R_1+s}$  if  $s \leq S_1 - R_1$  and  $Y_1(s)$  is equal to 0 for all  $s \geq S_1 - R_1$ .

Prove that  $Y_1$  and  $Y_2$  are independent.

*Hint:* Use the strong Markov property at time  $S_1$ .

## Notes

Besides its use in the Ray–Knight theorems (Rogers and Williams, 2000b), excursion theory is useful in many other contexts. See Rogers and Williams (2000b) for applications to Skorokhod embedding and to the arc sine law.

# 28

---

## Financial mathematics

A European call option is the option to buy a share of stock at a given price at some particular time in the future. For example, I might buy a call option to purchase one share of Company X for \$40 three months from today. When the three months is up, I check the price of Company X. If, say, it is \$35, then my option is worthless, because why would I buy a share for \$40 using the option when I could buy it on the open market for \$35? But if three months from now, the share price is, say, \$45, then I can exercise my option, which means I buy a share for \$40, and I can then turn around immediately and sell that share for \$45 and make a profit of \$5. Thus, today, there is a potential for a profit if I have a call option, and so I should pay something to purchase that option. A significant part of financial mathematics is devoted to the question of what is the fair price I should pay for a call option.

Options originated in the commodities market, where farmers wanted to hedge their risks. Since then many types of options have been developed (options are also known as derivatives), and the amount of money invested in options has for the past several years exceeded the amount of money invested in stocks.

In 1973 Black and Scholes, using the reasonable principle that you can't get something for nothing, came up with a convincing formula for the price of an option. This chapter gives two derivations of the Black–Scholes formula, proves the fundamental theorem of finance, and finishes by considering a stochastic control problem. The Black–Scholes formula is a beautiful example of applied stochastic processes.

### 28.1 Finance models

Let  $W_t$  be a Brownian motion. We assume that  $S_t$  is the price of a stock or other risky security. If we have \$2,000 and we buy 100 shares in a stock that sells for \$20 per share and it goes up \$2, or if we buy 10 shares in a stock selling for \$200 per share that goes up \$20, we are equally happy; it is the percentage increase that matters. With this in mind, we assume that  $S_t$  satisfies

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \quad (28.1)$$

This is plausible, since then  $dS_t/S_t = \mu dt + \sigma dW_t$ , that is, we are assuming the relative change in price is a multiple of Brownian motion with drift. The quantity  $\mu$  is known as the *mean rate of return* and  $\sigma$  is called the *volatility*. The solution to this SDE is

$$S_t = S_0 e^{\sigma W_t + (\mu - (\sigma^2/2))t} \quad (28.2)$$

by Proposition 24.6.

We also assume the existence of a bond with price  $B_t$ , which is assumed to be riskless, and the equation for  $B_t$  is

$$dB_t = rB_t dt,$$

which implies

$$B_t = B_0 e^{rt}.$$

Suppose at time  $t$  one buys  $A$  shares of stock. The cost is  $AS_t$ . If one sells the shares at time  $t+h$ , one receives  $AS_{t+h}$ , and the net gain is  $A(S_{t+h} - S_t)$ . One can also sell short, i.e., let  $A$  be negative. The formula for the gain is the same.

Suppose at time  $t_i$  one holds  $A_i$  shares, up until time  $t_{i+1}$ . The total net gain over the whole period  $t_0$  to  $t_n$  is  $\sum_{i=0}^{n-1} A_i (S_{t_{i+1}} - S_{t_i})$ . This is the same as the stochastic integral  $\int_0^t a_t dS_t$  if  $a_t$  equals  $A_i$  when  $t_i \leq t < t_{i+1}$ .

One should allow  $A_i$  to depend on the entire past  $\mathcal{F}_{t_i}$ . Idealizing, one allows continuous trading, and if  $a_s$  is the number of shares held at time  $s$ , the net gain through trading the stock is  $\int_0^t a_s dS_s$ . One has a similar net gain of  $\int_0^t b_s dB_s$  when trading bonds if  $b_s$  is the number of bonds held at time  $s$ .

Although  $a_t$  can depend on the entire past  $\mathcal{F}_t$ , one does not want to let  $a_t$  depend on the future. This helps explain why the class of predictable integrands is the appropriate one to use.

The pair  $(a, b)$  is called a *trading strategy*. Set

$$V_t = a_t S_t + b_t B_t, \quad (28.3)$$

the amount of wealth one has at time  $t$ . The strategy is *self-financing* if

$$V_t = V_0 + \int_0^t a_s dS_s + \int_0^t b_s dB_s \quad (28.4)$$

for all  $t$ . The first integral represents the net gain from trading in the stock, the second integral the net gain from trading in the bond, and (28.4) says that one's wealth at time  $t$  is equal to what one starts with plus what one has realized through trading in the stock and bond. We assume throughout that there are no transaction costs (i.e., no brokerage fees).

A European call gives the buyer the option of buying a share of the stock at a fixed time  $t_E$  at price  $K$ . The time  $t_E$  is called the *exercise time*. After time  $t_E$ , the option has expired and is worthless.

What is the option worth? At time  $t_E$ , if  $S_{t_E} \leq K$ , the option is worth nothing, for who would pay  $K$  dollars for a share of stock when it sells for  $S_{t_E}$  dollars? If  $S_{t_E} > K$ , one can use the option to buy a share of the stock at price  $K$  and immediately sell it at price  $S_{t_E}$ , to make a profit of  $S_{t_E} - K$ . Thus the value of the option at time  $t_E$  is  $(S_{t_E} - K)^+$ . An important question is: how much should the option sell for? What is a fair price for the option at time 0?

There are a myriad of types of options. The American call is almost the same as the European call, except that one is allowed to buy a share of the stock at price  $K$  at any time in the interval  $[0, t_E]$ . The European put gives the buyer the option to sell a share of the stock at price  $K$  at time  $t_E$ , while the American put gives the buyer the option to sell a share at price  $K$  anytime before time  $t_E$ .

## 28.2 Black–Scholes formula

In 1973 Black and Scholes came up with their formula for the price of a European call. We will give two derivations of this formula.

*Derivation 1.* First of all, the interest rate  $r$  on the bond may be considered to be the same as the rate of inflation. Thus the value of the option  $(S_{t_E} - K)^+$  in today's dollars is

$$C = e^{-rt_E} (S_{t_E} - K)^+. \quad (28.5)$$

In this first derivation we work in today's dollars. Therefore the present-day value of the stock is  $P_t = e^{-rt} S_t$ . Note  $P_0 = S_0$  and the present-day value of our option at time  $t_E$  is then

$$C = e^{-rt_E} (S_{t_E} - K)^+ = (P_{t_E} - e^{-rt_E} K)^+. \quad (28.6)$$

By the product formula,

$$\begin{aligned} dP_t &= e^{-rt} dS_t - re^{-rt} S_t dt \\ &= e^{-rt} \sigma S_t dW_t + e^{-rt} \mu S_t dt - re^{-rt} S_t dt \\ &= \sigma P_t dW_t + (\mu - r) P_t dt. \end{aligned}$$

The solution to this stochastic differential equation (see Proposition 24.6) is

$$P_t = P_0 e^{\sigma W_t + (\mu - r - \sigma^2/2)t}.$$

Also, the net gain or loss in present-day dollars when holding  $a_s$  shares of stock at time  $s$  is  $\int_0^t a_s dP_s$ .

Define  $\mathbb{Q}$  on  $\mathcal{F}_{t_E}$  by

$$d\mathbb{Q}/d\mathbb{P} = M_{t_E} = \exp \left( -\frac{\mu - r}{\sigma} W_{t_E} - \frac{(\mu - r)^2}{2\sigma^2} t_E \right).$$

Under  $\mathbb{Q}$ ,  $\tilde{W}_t = W_t + \frac{\mu - r}{\sigma} t$  is a Brownian motion by the Girsanov theorem.

Now

$$dP_t = \sigma P_t dW_t + (\mu - r) P_t dt = \sigma P_t \left( dW_t + \frac{\mu - r}{\sigma} dt \right) = \sigma P_t d\tilde{W}_t.$$

Therefore under  $\mathbb{Q}$ ,  $P_t$  is a martingale since stochastic integrals with respect to martingales are martingales. The solution to the SDE

$$dP_t = \sigma P_t d\tilde{W}_t$$

is

$$P_t = P_0 e^{\sigma \tilde{W}_t - (\sigma^2/2)t}, \quad (28.7)$$

so  $P_t$  and  $\tilde{W}_t$  have the same filtration.

$C$  is  $\mathcal{F}_{t_E}$  measurable. By the martingale representation theorem (Theorem 12.3), there exists an adapted process  $A_s$  such that

$$C = \mathbb{E}_{\mathbb{Q}} C + \int_0^{t_E} A_s d\tilde{W}_s = \mathbb{E}_{\mathbb{Q}} C + \int_0^{t_E} D_s dP_s,$$

where  $D_s = A_s / (\sigma P_s)$ .

Therefore, if one follows the trading strategy of buying and selling the stock  $S_t$ , where one holds  $D_s$  shares of stock at time  $s$ , one can obtain  $C - \mathbb{E}_{\mathbb{Q}}C$  dollars at time  $t_E$ . Or, starting with  $\mathbb{E}_{\mathbb{Q}}C$  dollars and buying and selling stock, one can get the identical output as  $C$ , almost surely. A standard assumption in finance is that of no arbitrage, which means you cannot make a profit without taking some risk. To avoid riskless profits,  $C$  must sell for  $\mathbb{E}_{\mathbb{Q}}C$ .

To explain this in more detail, suppose you could sell the European call for  $C'$  dollars. If  $C' > \mathbb{E}_{\mathbb{Q}}C$ , you could sell a call for  $C'$  dollars, use the money and invest in the trading strategy of holding  $D_s$  shares of stock at time  $s$ , and at time  $t_E$  have  $C' + C - \mathbb{E}_{\mathbb{Q}}C$  worth of stocks and options. The buyer of the option decides whether to exercise the option, and it costs you  $C$  dollars to meet that obligation. With probability one, you have gained  $C' - \mathbb{E}_{\mathbb{Q}}C$  dollars, a riskless profit. If  $C' < \mathbb{E}_{\mathbb{Q}}C$ , simply reverse the roles of buying and selling. The only way to avoid making a riskless profit is if  $C' = \mathbb{E}_{\mathbb{Q}}C$ .

To find  $\mathbb{E}_{\mathbb{Q}}C$ , using (28.6) and (28.7) we write

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}C &= \mathbb{E}_{\mathbb{Q}}[(S_0 e^{\sigma \tilde{W}_{t_E} - \sigma^2 t_E / 2} - e^{-rt_E} K)^+] \\ &= \frac{1}{\sqrt{2\pi t_E}} \int (S_0 e^{\sigma y - \sigma^2 t_E / 2} - e^{-rt_E} K)^+ e^{-y^2 / 2t_E} dy,\end{aligned}\tag{28.8}$$

which is the Black–Scholes formula. One can, if one wishes, perform some calculations to find alternate expressions for the right-hand side.

It is noteworthy that  $\mu$  does not appear in (28.8)! You and I might have different opinions as to what  $\mu$ , the mean rate of return, is equal to, but we should agree on the price of the call. This was a shock to economists when this was first discovered. The value of  $\sigma$ , the volatility, does enter into the formula.

Until we evaluated  $\mathbb{E}_{\mathbb{Q}}C$  in (28.8), the actual form of  $C$  was unimportant. For any type of option expiring at time  $t_E$ , Derivation 1 tells us that its price at time zero should be its expectation under  $\mathbb{Q}$ .

*Derivation 2.* In this approach, which is the one used by Black and Scholes, we use the actual values of the securities, not the present-day values. Let  $V_t$  be the value of the option at time  $t$  and assume

$$V_t = f(S_t, t_E - t)\tag{28.9}$$

for all  $t$ , where  $f$  is some function that is sufficiently smooth. We also want  $V_{t_E} = (S_{t_E} - K)^+$ .

Recall the multivariate version of Itô's formula (Theorem 11.2). We apply this with  $d = 2$  and  $X_t = (S_t, t_E - t)$ . From (28.1),

$$\langle S \rangle_t = \sigma^2 S_t^2 dt,$$

$\langle t_E - t \rangle_t = 0$  since  $t_E - t$  is of bounded variation and hence has no martingale part, and  $\langle S, t_E - t \rangle_t = 0$ . Also,  $d(t_E - t) = -dt$ . Then

$$\begin{aligned}V_t - V_0 &= f(S_t, t_E - t) - f(S_0, t_E) \\ &= \int_0^t f_x(S_u, t_E - u) dS_u - \int_0^t f_t(S_u, t_E - u) du \\ &\quad + \frac{1}{2} \int_0^t \sigma^2 S_u^2 f_{xx}(S_u, t_E - u) du.\end{aligned}\tag{28.10}$$

Here  $f_x$  is the partial derivative with respect to  $x$ , the first variable,  $f_{xx}$  is the second partial derivative with respect to  $x$ , and  $f_t$  is the partial derivative with respect to  $t$ , the second variable. On the other hand,

$$V_t - V_0 = \int_0^t a_u dS_u + \int_0^t b_u dB_u. \quad (28.11)$$

By (28.3) and (28.9),

$$b_t = \frac{V_t - a_t S_t}{B_t} = \frac{f(S_t, t_E - t) - a_t S_t}{B_t}. \quad (28.12)$$

Also, recall  $B_t = B_0 e^{rt}$ . Comparing (28.10) with (28.11), we must therefore have

$$a_t = f_x(S_t, t_E - t) \quad (28.13)$$

and

$$-f_t(S_t, t_E - t) + \frac{1}{2}\sigma^2 S_t^2 f_{xx}(S_t, t_E - t) = b_t B_0 r e^{rt}. \quad (28.14)$$

Substituting for  $b_t$  using (28.12),

$$\begin{aligned} & r[f(S_t, t_E - t) - S_t f_x(S_t, t_E - t)] \\ &= -f_t(S_t, t_E - t) + \frac{1}{2}\sigma^2 S_t^2 f_{xx}(S_t, t_E - t) \end{aligned} \quad (28.15)$$

for almost all  $t$  and all  $S_t$ . Since  $S_t$  is a continuous process, (28.15) leads to the parabolic partial differential equation (PDE)

$$f_t = \frac{1}{2}\sigma^2 x^2 f_{xx} + rx f_x - rf, \quad (x, s) \in (0, \infty) \times [0, t_E],$$

and

$$f(x, 0) = (x - K)^+.$$

Solving this equation for  $f$ ,  $f(x, t_E)$  tells us what  $V_0$  should be, i.e., the cost of setting up the equivalent portfolio. This partial differential equation can be solved and the solution is the Black–Scholes formula. Equation (28.13) shows what the trading strategy should be.

Let us now briefly discuss American calls. Recall that these are ones where the holder can buy the security at price  $K$  at any time up to time  $t_E$ . Since the holder of an American call can always wait up to time  $t_E$ , which is equivalent to having a European call, the value of an American call should always be at least as large as the value of the corresponding European call.

Suppose one exercises an American call early. If  $S_{t_E} > K$  and one exercised early, at time  $t_E$  one has one share of stock, for which one paid  $K$ , and one has a profit of  $(S_{t_E} - K)$ . However, because one purchased the stock before time  $t_E$ , one lost the interest  $Ke^{r(t_E - t)}$  that would have accrued by waiting to exercise the option. (We are supposing  $r \geq 0$ .) Thus in this case it would have been better to wait until time  $t_E$  to exercise the option.

On the other hand, if  $S_{t_E} < K$ , exercising the option early would mean that one has lost  $|S_{t_E} - K|$ , whereas for the European option, one would have not exercised at all, and lost nothing (other than the price of the option).

In either case, exercising early gains nothing, hence the price of an American call should be the same as that of a European call.

One can equally well price the European put, the option to sell a share of stock at price  $K$  at time  $t_E$ , by either Derivation 1 or Derivation 2 of the Black–Scholes formula. However this analysis breaks down for American puts (sell a share of stock anytime up to time  $t_E$ ), because in this case one gains by selling early: one can earn interest on the money received.

### 28.3 The fundamental theorem of finance

In the preceding section, we showed there was a probability measure  $\mathbb{Q}$  under which  $P_t$  was a martingale. This is true very generally. Let  $S_t$  be the price of a security in present-day dollars. We will suppose  $S_t$  is a continuous semimartingale, and can be written  $S_t = M_t + A_t$ .

The *NFLVR condition* (“no free lunch with vanishing risk”) is that one cannot find fixed positive real numbers  $t_0, \varepsilon, b > 0$ , and predictable processes  $H_n$  with  $\int_0^{t_0} |H_n(s)| |dA_s| + \int_0^{t_0} H_n^2 d\langle M \rangle_s < \infty$ , a.s., for each  $n$  such that

$$\int_0^{t_0} H_n(s) dS_s > -\frac{1}{n}, \quad \text{a.s.},$$

for all  $n$  and

$$\mathbb{P}\left(\int_0^{t_0} H_n(s) dS_s > b\right) > \varepsilon.$$

Here  $t_0, b, \varepsilon$  do not depend on  $n$ . The condition says that one can with positive probability  $\varepsilon$  make a profit of  $b$  and with a loss no larger than  $1/n$ .  $\mathbb{Q}$  is an *equivalent martingale measure* if  $\mathbb{Q}$  is a probability measure,  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$  (which means they have the same null sets), and  $S_t$  is a local martingale under  $\mathbb{Q}$ .

**Theorem 28.1** *If  $S_t$  is a continuous semimartingale and the NFLVR condition holds, then there exists an equivalent martingale measure  $\mathbb{Q}$ .*

*Proof* Let us prove first of all that  $dA_t$  is absolutely continuous with respect to  $d\langle M \rangle_t$ . We suppose not and obtain a contradiction. Consider the measures  $\mu_A$  and  $\mu_{\langle M \rangle}$  on the predictable  $\sigma$ -field defined by

$$\mu_A(D) = \mathbb{E} \int_0^\infty 1_D dA_t, \quad \mu_{\langle M \rangle}(D) = \mathbb{E} \int_0^\infty 1_D d\langle M \rangle_t. \quad (28.16)$$

Since  $A$  is of bounded variation and continuous, it is a predictable process, and we can write  $A_t = B_t - C_t$ , where  $B$  and  $C$  are continuous increasing processes and where  $\mu_B$  and  $\mu_C$  are mutually singular measures on the predictable  $\sigma$ -field; we define  $\mu_B$  and  $\mu_C$  analogously to (28.16). To give a few more details on how to do this, we write  $A_t = B'_t - C'_t$ , where  $B'$  and  $C'$  are continuous increasing processes, we find non-negative predictable processes  $b_t$  and  $c_t$  such that  $B'_t = \int_0^t b_s d(B'_s + C'_s)$  and  $C'_t = \int_0^t c_s d(B'_s + C'_s)$ , and then let  $B_t = \int_0^t (b_s - (b_s \wedge c_s)) d(B'_s + C'_s)$  and  $C_t = \int_0^t (c_s - (b_s \wedge c_s)) d(B'_s + C'_s)$ . We leave it to the reader to check that  $B$  and  $C$  are the desired processes. Since  $\mu_B$  and  $\mu_C$  are mutually singular, there exists a set  $E$  in the predictable  $\sigma$ -field such that  $\mu_B(D) = \mu_B(D \cap E)$  and  $\mu_C(D) = \mu_C(D \cap E^c)$  for all sets  $D$  in the predictable  $\sigma$ -field.

If  $\mu_A$  is not absolutely continuous with respect to  $\mu_{\langle M \rangle}$ , then at least one of  $\mu_B$  and  $\mu_C$  is not absolutely continuous. We assume that  $\mu_B$  is not, for otherwise we can look at  $-S_t$  instead of  $S_t$ . Therefore there exists a predictable set  $F$  and a fixed time  $t_0$  such that  $\int_0^{t_0} 1_F dB_s$  is almost

surely non-negative, is strictly positive with positive probability, and  $\int_0^{t_0} 1_F d\langle M \rangle_s = 0$ . We can replace  $F$  by  $F \cap E$  and so assume that  $F \subset E$ , and hence  $\mu_C(F) = \mu_C(F \cap E^c) = 0$ . Then

$$\int_0^{t_0} 1_F dS_s = \int_0^{t_0} 1_F dM_s + \int_0^{t_0} 1_F dB_s + \int_0^{t_0} 1_F dC_s.$$

The stochastic integral term is 0 because  $\int_0^{t_0} (1_F)^2 d\langle M \rangle_s = 0$ . The integral with respect to  $C_s$  is zero because  $\mu_C(F) = 0$ . We then have the NFLVR condition violated with  $H_n = 1_F$  for all  $n$ . Hence absolute continuity is established, and by the Radon–Nikodym theorem,  $A_t = \int_0^t h_s d\langle M \rangle_s$  for some predictable process  $h_s$ .

Our next goal is to show  $\int_0^t h_s^2 d\langle M \rangle_s < \infty$  for all  $t$ . Let

$$U = \inf \left\{ t : \int_0^t h_s^2 d\langle M \rangle_s = \infty \right\}.$$

On  $(U < \infty)$  there are two possibilities:

(1)  $\int_0^t h_s^2 d\langle M \rangle_s < \infty$  if  $t < U$  but  $\int_0^U h_s^2 d\langle M \rangle_s = \infty$ , and

(2)  $\int_0^U h_s^2 d\langle M \rangle_s < \infty$  but  $\int_U^{U+\varepsilon} h_s^2 d\langle M \rangle_s = \infty$  for all  $\varepsilon$ .

(For a real variable analog, consider the two functions  $f_1(t) = \int_{-1}^t \frac{1}{|x|} dx$  and  $f_2(t) = \int_{-1}^t 1_{(x>0)} \frac{1}{x} dx$  at  $t = 0$ .)

Let us investigate case (1) and show that it cannot happen. Choose a fixed time  $t_0$  such that  $\mathbb{P}(U < t_0) > 0$ . Let

$$R_1 = R_1(n) = \inf \left\{ t : \int_0^t h_s^2 d\langle M \rangle_s \geq n^4 \right\} \wedge U \wedge t_0.$$

We suppose

$$\inf_n \mathbb{P}(R_1(n) < U \wedge t_0) > 0 \quad (28.17)$$

and obtain a contradiction. Let  $H_t = h_t 1_{[0, R_1]} / n^4$ . Then

$$\int_0^{t_0} H_s dA_s = \int_0^{R_1} \frac{h_s^2}{n^4} d\langle M \rangle_s \geq 1$$

on  $(R_1 < U < t_0)$ . On the other hand,

$$\mathbb{E} \left( \sup_{t \leq t_0} \left| \int_0^t H_s dM_s \right| \right)^2 \leq 4 \mathbb{E} \int_0^{t_0} H_s^2 d\langle M \rangle_s \leq \frac{4}{n^8} n^4 = \frac{4}{n^4}$$

by Doob's inequalities. Therefore

$$\mathbb{P} \left( \sup_{t \leq t_0} \left| \int_0^t H_s dM_s \right| > \frac{1}{n} \right) \leq \frac{\mathbb{E} \sup_{t \leq t_0} \left| \int_0^t H_s dM_s \right|^2}{n^{-2}} \leq \frac{4/n^4}{n^{-2}} = \frac{4}{n^2}.$$

Let

$$R_2 = R_2(n) = \inf \left\{ t : \left| \int_0^t H_s dM_s \right| \geq 1/n \right\}$$

and let  $\tilde{H}_t = H_t 1_{[0, R_2]}$ . We then have

$$\begin{aligned}\mathbb{P}(R_2 < R_1) &\leq \mathbb{P}(R_2 \leq t_0) \leq 4/n^2, \\ \int_0^{t_0} \tilde{H}_s dS_s &= \int_0^{R_2} \tilde{H}_s dM_s + \int_0^{R_2} \tilde{H}_s dA_s \\ &\geq -\frac{1}{n} + \int_0^{R_2} \frac{h_s^2}{n^4} d\langle M \rangle_s \geq -1/n\end{aligned}$$

almost surely, and

$$\begin{aligned}\mathbb{P}\left(\int_0^{t_0} \tilde{H}_s dS_s \geq \frac{1}{2}\right) &\geq \mathbb{P}(R_1 < U < t_0) - \mathbb{P}(R_2 < R_1) \\ &\geq \mathbb{P}(R_1 < U < t_0) - \frac{4}{n^2}.\end{aligned}$$

We do this for each  $n$ , and thus obtain a contradiction to the NFLVR condition, so (28.17) cannot hold.

Case (2) is similar: choose  $\delta_n$  such that  $\int_{U+\delta_n}^{U+1} h_s^2 d\langle M \rangle_s \geq n^4$  with positive probability, let  $H_t = h_t 1_{[U+\delta_n, U+1]} / n^4$ , and proceed as above. We leave the details as Exercise 28.3.

We thus have  $\int_0^t h_s^2 d\langle M \rangle_s < \infty$ , a.s., for each  $t$ . Consequently the quantity  $\sup_{s \leq t} |\int_0^s h_r dM_r|$  is also finite. Let

$$V_n = \inf \left\{ t : \left| \int_0^t h_s dM_s \right| \geq n \text{ or } \int_0^t h_s^2 d\langle M \rangle_s \geq n \right\}.$$

We conclude  $V_n \uparrow \infty$ .

Define  $\mathbb{Q}$  on  $\mathcal{F}_{V_n}$  by

$$d\mathbb{Q}/d\mathbb{P} = \exp \left( - \int_0^{V_n} h_s dM_s - \frac{1}{2} \int_0^{V_n} h_s^2 d\langle M \rangle_s \right).$$

The exponent is bounded, so  $\mathbb{Q}$  is well defined. Under  $\mathbb{Q}$ , if  $t \leq V_n$ , then

$$M_t - \left\langle - \int_0^{\cdot} h_s dM_s, M \right\rangle_t = M_t + \int_0^t h_s d\langle M \rangle_s = M_t + A_t$$

is a martingale by the Girsanov theorem (Exercise 13.5). Therefore  $S_t = M_t + A_t$  is a local martingale.

Finally,  $e^{-\int_0^t h_s dM_s - \frac{1}{2} \int_0^t h_s^2 d\langle M \rangle_s}$  is never zero nor infinite, so  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$ .  $\square$

Let us give two examples to clarify the proof. Let  $C$  be the standard Cantor set and let  $g(t)$  be the Cantor function. Suppose  $S_t = W_t + g(t)$ , where  $W$  is a Brownian motion. We then let  $H_t = 1_C(t)$ . Since the Cantor function increases only on the Cantor set,  $\int_0^1 H_s dg(s) = 1$ . Since the Cantor set has Lebesgue measure 0, then  $\int_0^1 H_s^2 ds = 0$ . But this is the quadratic variation of  $\int_0^1 H_s dW_s$ , so this stochastic integral is also 0. It follows that

$$\int_0^1 H_s dS_s = \int_0^1 H_s dW_s + \int_0^1 H_s dg(s) = 1,$$

which says that with the trading strategy  $H$  we make a profit of 1 almost surely, that is, without any risk. Therefore the NFLVR condition is violated. This example indicates why we must have  $dA_t$  absolutely continuous with respect to  $d\langle M \rangle_t$ .

Suppose now that  $W$  is a Brownian motion and  $S_t = W_t + \int_0^t H_s ds$  with  $H_s$  bounded. Let

$$M_t = e^{-\int_0^t H_s dW_s - \frac{1}{2} \int_0^t H_s^2 ds},$$

and define  $\mathbb{Q}$  on  $\mathcal{F}_1$  by  $d\mathbb{Q}/d\mathbb{P} = M_1$ . By the Girsanov theorem,  $S_t = W_t + \int_0^t H_s ds$  is a martingale under  $\mathbb{Q}$ . This example shows that if the Radon–Nikodym derivative of  $dA_t$  with respect to  $d\langle M \rangle_t$  is not too bad, we can apply the Girsanov theorem.

## 28.4 Stochastic control

The theory of stochastic control, which includes a study of the Hamilton–Jacobi–Bellman (HJB) equation and requires some knowledge of partial differential equations, is beyond the scope of this book. However, we can consider one simple useful example. Suppose we have available to us a stock which satisfies the SDE

$$dS_t = \sigma S_t dW_t + \mu S_t dt,$$

where  $W_t$  is a Brownian motion, and a risk-free asset which satisfies the equation

$$dB_t = rB_t dt.$$

We want to put a proportion  $u$  of our wealth  $Z_t$  into the stock and the remainder into the risk-free asset. We will restrict  $0 \leq u \leq 1$ , so that we do not borrow nor have short selling. Also, we take  $\mu > r$ , for if the mean rate of return on the stock is less than the risk-free rate, we simply put all our money in the risk-free asset. How do we choose  $u$  in order to maximize our return?

First of all, what do we mean by maximizing our return? Typically one chooses ahead of time a deterministic function  $U$ , called the utility function, and one wants to maximize  $\mathbb{E} U(Z_{t_0})$  at some fixed time  $t_0$ . Usually utility functions are taken to be increasing and concave. The function is chosen to be increasing because more money is considered better. It is chosen concave because one assumes that twice the amount of money will give increased pleasure, but not twice as much pleasure.

Let us work out the optimal control problem when  $U(x) = x^p$  for some  $p \in (0, 1)$ . If  $Z_t$  (depending on  $u$ ) is our wealth, we have  $Z_t = S_t + B_t$  and  $S_t = uZ_t$ ,  $B_t = (1-u)Z_t$ . We will allow  $u$  to depend on  $t$  and  $\omega$ , but our answer will turn out to be deterministic and independent of  $t$ , i.e.,  $u$  is a constant.

We have seen (Proposition 24.6) that

$$S_t = S_0 e^{\sigma W_t - \sigma^2 t/2 + \mu t}$$

and  $\langle S \rangle_t = \sigma^2 S_t^2 dt$  and that the equation for  $B_t$  has the solution

$$B_t = B_0 e^{rt}.$$

Therefore neither  $S_t$  nor  $B_t$  can ever be 0 or negative, and so  $Z_t > 0$  for all  $t$ . Applying Itô's formula to  $Z_t^p$  and noting that  $\langle Z \rangle_t = \langle S \rangle_t$ , we have

$$\begin{aligned} dZ_t^p &= pZ_t^{p-1} dZ_t + \frac{1}{2}p(p-1)Z_t^{p-2} d\langle Z \rangle_t \\ &= pZ_t^{p-1}\sigma S_t dW_t + pZ_t^{p-1}\mu S_t dt + pZ_t^{p-1}rB_t dt \\ &\quad + \frac{1}{2}p(p-1)Z_t^{p-2}\sigma^2 S_t^2 dt \\ &= puZ_t^p\sigma dW_t + puZ_t^p\mu dt + p(1-u)rZ_t^p dt \\ &\quad + \frac{1}{2}p(p-1)Z_t^p\sigma^2 u^2 dt. \end{aligned}$$

Therefore

$$\mathbb{E} Z_{t_0}^p = \mathbb{E} Z_0^p + p\mathbb{E} \int_0^{t_0} Z_t^p [u\mu + (1-u)r + \frac{1}{2}(p-1)\sigma^2 u^2] dt.$$

This will be largest if the expression

$$F(u) = u\mu + (1-u)r + \frac{1}{2}(p-1)\sigma^2 u^2$$

is largest, which by elementary calculus is largest when

$$u = \frac{\mu - r}{(1-p)\sigma^2}.$$

## Exercises

28.1 Let

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy,$$

the *cumulative normal distribution function*. Rewrite the Black–Scholes formula for the value of a European call in terms of  $\Phi$ . This is the way the Black–Scholes formula is written in finance books.

- 28.2 A European put that gives one the option to sell a share of stock at price  $K$  at time  $t_E$  has value  $(K - S_{t_E})^+$  at time  $t_E$ . Find the present-day value of the European put at time 0.
- 28.3 Carry out the details of the proof of Theorem 28.1 for Case 2.
- 28.4 If the utility function in Section 28.4 is  $U(x) = \log x$  instead of  $U(x) = x^p$ , what is the optimal choice for  $u$ ?
- 28.5 Let  $a, b > 0$ , let  $Y_i$  be i.i.d. random variables that take only the values  $b$  and  $-a$ , and let  $S_n = \sum_{i=1}^n Y_i$ . Show that if  $\mathbb{P}(Y_1 = b) > 0$  and  $\mathbb{P}(Y_1 = -a) > 0$ , there exists a probability measure  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  under which  $S_n$  is a martingale. Describe the Radon–Nikodym derivative of  $\mathbb{Q}$  with respect to  $\mathbb{P}$ .
- 28.6 Suppose the interest rate  $r$  is equal to 0 and an option  $V$  has payoff

$$\sup_{s \leq t_e} S_s$$

at time  $t_e$ . What is the price of  $V$  at time 0?

- 28.7 Suppose the interest rate  $r$  is equal to 0. Let  $U$  be the option that pays off  $-\inf_{s \leq t_e} S_s$  at time  $t_e$ . What is the price of  $U$  at time 0?

If  $V$  is as in Exercise 28.6, then  $U + V$  is the option that pays off the maximum of the stock price minus the minimum of the stock price, in other words, “buy low, sell high.” Naturally such an option would be expensive. It is remarkable that there exists a trading strategy that can duplicate this payoff, even though the times when the maximum and minimum occur are not stopping times.

---

## Filtering

Stochastic filtering is a nice example of nontrivial interesting mathematics that is extremely useful. For example, it has been used extensively in NASA's space program.

The method we use is called the *innovations approach* to filtering, and uses Lévy's theorem, the martingale representation theorem, and other results from stochastic calculus.

We will start with a fairly general model, except for simplicity we will assume our observation process is one-dimensional. The extension to the  $d$ -dimensional case is mostly routine. Later on we will look at a specific model, the linear model, where one can give fairly explicit solutions to the filtering equation for real-life problems.

### 29.1 The basic model

We start with a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , together with a filtration  $\{\mathcal{F}_t\}$  satisfying the usual conditions. In filtering theory, there are a number of filtrations present, and we will need to be careful about which ones are which.

We have a *signal process*  $X_t$  taking values in a complete separable metric space  $\mathcal{S}$  and we let  $\{\mathcal{F}_t^X\}$  be the minimal augmented filtration generated by  $X$ . We have a function  $f$  mapping  $\mathcal{S}$  to the reals, we suppose  $\mathbb{E} |f(X_t)|^2 < \infty$  for all  $t$ , and we suppose that there exists a process  $A_s$  adapted to the filtration  $\{\mathcal{F}_s^X\}$  such that

$$M_t = f(X_t) - f(X_0) - \int_0^t A_s ds$$

is a martingale with respect to the filtration  $\{\mathcal{F}_t^X\}$ . Next we discuss the observation process. Let  $W_t$  be a one-dimensional Brownian motion with respect to the filtration  $\{\mathcal{F}_t^X\}$ , let  $h_t$  be a real-valued process adapted to  $\{\mathcal{F}_t^X\}$ , and suppose

$$Z_t = W_t + \int_0^t h_s ds. \quad (29.1)$$

The process  $Z_t$  is called the *observation process* and is what we observe. Let  $\{\mathcal{F}_t^Z\}$  be the filtration generated by the process  $Z$ . In practice one does not necessarily want to assume that  $\{\mathcal{F}_t^Z\}$  is right continuous, but let us assume that it is for simplicity. Requiring the filtration to be complete is not a serious issue.

For an example, suppose that  $dX_t = \sigma(X_t) d\bar{W}_t + b(X_t) dt$  as in Chapter 24, where  $\bar{W}_t$  is a  $d$ -dimensional Brownian motion and  $\sigma$  and  $b$  are matrix valued, and suppose  $f \in C^2(\mathbb{R}^d)$  is bounded or has linear growth. Then Itô's formula shows that such an  $f$  will satisfy our

assumptions. In this case  $h_s$  in (29.1) is of the form  $g(X_s)$  for a particular function  $g$ ; see Section 39.3.

The goal of filtering is to get the best estimate of  $f(X_t)$  from the observations  $\{Z_t\}$ . We want to find the best estimate for  $f(X_t)$  in the following sense. We want to minimize the mean square error  $\mathbb{E} |f(X_t) - Y|^2$  over all random variables  $Y$  that are  $\mathcal{F}_t^Z$  measurable, i.e., over all random variables that can be determined by the observations up to time  $t$ . The rationale is that since  $\mathcal{F}_t^Z$  is the information we have observed up to time  $t$ , we want our estimate to be  $\mathcal{F}_t^Z$  measurable, and among all random variables that are  $\mathcal{F}_t^Z$  measurable, we want the one closest to  $f(X_t)$  in  $L^2$  norm, which means we minimize the mean square error.

**Lemma 29.1** *The best mean square error estimate of  $f(X_t)$  over the class of  $\mathcal{F}_t^Z$  measurable random variables is*

$$Y = \mathbb{E}[f(X_t) \mid \mathcal{F}_t^Z].$$

*Proof* By our assumptions on  $f$ , the random variable  $V = f(X_t)$  is in  $L^2(\mathbb{P})$ . Let  $Y$  be the best mean square estimator. The collection  $\mathcal{M}$  of  $L^2$  random variables which are  $\mathcal{F}_t^Z$  measurable is a linear subspace of  $L^2$ , and the element of a Hilbert space that minimizes the distance from  $V$  to this subspace  $\mathcal{M}$  is the projection onto  $\mathcal{M}$ . Therefore  $Y$  is the projection of  $V$  onto  $\mathcal{M}$ . Hence  $V - Y$  is orthogonal (in the  $L^2$  sense) to every element of  $\mathcal{M}$ . In particular, if  $E \in \mathcal{F}_t^Z$ ,

$$\mathbb{E}[(V - Y)1_E] = 0,$$

which implies  $\mathbb{E}[V; E] = \mathbb{E}[Y; E]$ . This holds for every  $E \in \mathcal{F}_t^Z$  and  $Y$  is  $\mathcal{F}_t^Z$  measurable, hence  $Y = \mathbb{E}[V \mid \mathcal{F}_t^Z]$ .  $\square$

Given any process  $H_t$  that is  $\{\mathcal{F}_t\}$  adapted, we use the notation  $\widehat{H}_t = \mathbb{E}[H_t \mid \mathcal{F}_t^Z]$ . We will look at expressions like  $\int_0^t \widehat{H}_s ds$ , and you might wonder about the joint measurability of  $\widehat{H}$  in  $\omega$  and  $t$ , since  $\widehat{H}_t$  is only defined almost surely for each  $t$ . The way to deal with this is to let  $\widehat{H}_t$  be the optional projection of  $H$  with respect to the optional  $\sigma$ -field generated by  $\{\mathcal{F}_t^Z\}$ ; see (16.8) in Chapter 16.

## 29.2 The innovation process

We next define the *innovation process*

$$N_t = Z_t - \int_0^t \widehat{h}_s ds. \quad (29.2)$$

(Following our convention on notation,  $\widehat{h}_s = \mathbb{E}[h_s \mid \mathcal{F}_s^Z]$ .) Note that although  $N_t$  is  $\mathcal{F}_t^Z$  measurable, we cannot determine it from (29.2) because it contains the unknown  $\widehat{h}_s$  on the right-hand side.

**Proposition 29.2**  *$N_t$  is a Brownian motion with respect to the filtration  $\{\mathcal{F}_t^Z\}$ .*

*Proof* We will show that  $N_t$  is a continuous martingale with respect to the filtration  $\{\mathcal{F}_t^Z\}$  whose quadratic variation is  $t$ , and then our result follows from Lévy's theorem (Theorem 12.1). That  $N_t$  is continuous is obvious, and  $\langle N \rangle_t = \langle Z \rangle_t = \langle W \rangle_t = t$  from the definitions of  $Z$  and  $W$ . Thus we need to show that  $N$  is a martingale with respect to  $\{\mathcal{F}_t^Z\}$ .

If  $r \geq s$ , we have

$$\mathbb{E}[\widehat{h}_r | \mathcal{F}_s^Z] = \mathbb{E}[\mathbb{E}[h_r | \mathcal{F}_r^Z] | \mathcal{F}_s^Z] = \mathbb{E}[h_r | \mathcal{F}_s^Z]. \quad (29.3)$$

Then using Exercise 29.1,

$$\begin{aligned} \mathbb{E}[N_t - N_s | \mathcal{F}_s^Z] &= \mathbb{E}[Z_t - Z_s | \mathcal{F}_s^Z] - \int_s^t \mathbb{E}[\widehat{h}_r | \mathcal{F}_s^Z] dr \\ &= \mathbb{E}[W_t - W_s | \mathcal{F}_s^Z] + \int_s^t \mathbb{E}[h_r - \widehat{h}_r | \mathcal{F}_s^Z] dr \\ &= \mathbb{E}[\mathbb{E}[W_t - W_s | \mathcal{F}_s^X] | \mathcal{F}_s^Z] = 0, \end{aligned} \quad (29.4)$$

since  $\mathcal{F}_s^Z \subset \mathcal{F}_s^X$ .  $\square$

### 29.3 Representation of $\mathcal{F}^Z$ -martingales

In this section we prove that if  $Y_t$  is a martingale with respect to  $\{\mathcal{F}_t^Z\}$ , then  $Y$  can be represented as a stochastic integral with respect to  $N$ . This is not an immediate consequence of Theorem 12.3 because we do not know that  $N_t$  generates  $\{\mathcal{F}_t^Z\}$ ; the filtration generated by  $N$  could conceivably be strictly smaller than the one generated by  $Z$ .

**Theorem 29.3** Suppose  $Y_t$  is a square integrable martingale with respect to  $\{\mathcal{F}_t^Z\}$ . Let  $\mathcal{P}^Z$  be the predictable  $\sigma$ -field defined on  $[0, \infty) \times \Omega$  in terms of  $\{\mathcal{F}_t^Z\}$ . Then there exists  $H_s$  which is  $\mathcal{P}^Z$  measurable and with  $\mathbb{E} \int_0^\infty H_s^2 ds < \infty$  such that

$$Y_t = Y_0 + \int_0^t H_s dN_s \quad (29.5)$$

for all  $t$ .

To clarify,  $\mathcal{P}^Z$  is the  $\sigma$ -field generated by all bounded left-continuous processes that are adapted to  $\{\mathcal{F}_t^Z\}$ .

*Proof* First let us treat the case where  $\int_0^t \widehat{h}_s dN_s$ ,  $\int_0^t |\widehat{h}_s|^2 ds$ , and  $Y_t$  are each bounded. Define  $\mathbb{Q}$  on  $\mathcal{F}_t^Z$  by  $d\mathbb{Q}/d\mathbb{P}|_{\mathcal{F}_t^Z} = M_t$ , where

$$M_t = \exp \left( - \int_0^t \widehat{h}_s dN_s - \frac{1}{2} \int_0^t |\widehat{h}_s|^2 ds \right).$$

Then by the Girsanov theorem (Theorem 13.3)

$$Z_t = N_t + \int_0^t \widehat{h}_s ds$$

is a martingale under  $\mathbb{Q}$  with respect to  $\{\mathcal{F}_t^Z\}$ . Since  $\langle Z \rangle_t = \langle N \rangle_t = \langle W \rangle_t = t$ , then  $Z$  is a Brownian motion under  $\mathbb{Q}$  with respect to  $\{\mathcal{F}_t^Z\}$ .

Let  $\widetilde{Y}_t = M_t^{-1} Y_t$ . If  $A \in \mathcal{F}_s^Z$ , then  $A \in \mathcal{F}_s^X$  and

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[\widetilde{Y}_t; A] &= \mathbb{E}_{\mathbb{P}}[M_t(M_t^{-1} Y_t); A] = \mathbb{E}_{\mathbb{P}}[Y_t; A] = \mathbb{E}_{\mathbb{P}}[Y_s; A] \\ &= \mathbb{E}_{\mathbb{P}}[M_s(M_s^{-1} Y_s); A] = \mathbb{E}_{\mathbb{Q}}[\widetilde{Y}_s; A]. \end{aligned}$$

Therefore  $\tilde{Y}_t$  is a martingale under  $\mathbb{Q}$  with respect to  $\{\mathcal{F}_t^Z\}$ . By the martingale representation theorem (Theorem 12.3) there exists  $K_s \in \mathcal{P}^Z$  such that

$$\tilde{Y}_t = \tilde{Y}_0 + \int_0^t K_s dZ_s = \tilde{Y}_0 + \int_0^t K_s dN_s + \int_0^t K_s \hat{h}_s ds.$$

On the other hand,  $dM_t = -M_t \hat{h}_t dN_t$  and  $Y_t = M_t \tilde{Y}_t$ . We have  $d\langle M, Y \rangle_t = -M_t \hat{h}_t K_t dt$ . By the product formula,

$$\begin{aligned} Y_t &= M_0 \tilde{Y}_0 + \int_0^t \tilde{Y}_s dM_s + \int_0^t M_s d\tilde{Y}_s + \langle M, \tilde{Y} \rangle_t \\ &= Y_0 - \int_0^t \tilde{Y}_s M_s \hat{h}_s dN_s + \int_0^t K_s M_s dN_s + \int_0^t K_s \hat{h}_s M_s ds - \int_0^t M_s \hat{h}_s K_s ds, \end{aligned}$$

which is of the desired form if we set  $H_s = K_s M_s - \tilde{Y}_s M_s \hat{h}_s$ .

In the general case, let

$$T_K = \inf \left\{ t : \left| \int_0^t \hat{h}_s dN_s \right| + \int_0^t |\hat{h}_s|^2 ds + |Y_t| \geq K \right\}.$$

We apply the above argument to  $Y_{t \wedge T_K}$  and use Exercise 29.3 to get

$$Y_{t \wedge T_K} = Y_0 + \int_0^t H_s^K dN_s,$$

where  $H_s^K$  is predictable with respect to the  $\sigma$ -fields  $\{\mathcal{F}_{t \wedge T_K}^Z\}$  and is 0 from time  $T_K$  on. Since  $Y_t$  is square integrable,  $Y_{T_K} \rightarrow Y_\infty$  almost surely and in  $L^2(\mathbb{P})$  as  $K \rightarrow \infty$ , and

$$\mathbb{E} \left[ \int_0^\infty |H_s^K - H_s^L|^2 ds \right] = \mathbb{E} [ |Y_{T_K} - Y_{T_L}|^2 ] \rightarrow 0$$

as  $K, L \rightarrow \infty$ . Using the completeness of  $L^2$ , there exists  $H_s$  such that  $\mathbb{E} \int_0^\infty H_s^2 ds < \infty$  and  $\mathbb{E} \int_0^\infty |H_s - H_s^K|^2 ds \rightarrow 0$  as  $K \rightarrow \infty$ . It is routine to check that  $H_s$  is  $\mathcal{P}^Z$  measurable and that (29.5) holds.  $\square$

## 29.4 The filtering equation

We now derive the general filtering equation. First we need a lemma.

**Lemma 29.4** *If  $Y_t - \int_0^t H_s ds$  is a martingale with respect to  $\{\mathcal{F}_t^X\}$ , then  $\widehat{Y}_t - \int_0^t \widehat{H}_s ds$  is a martingale with respect to  $\{\mathcal{F}_t^Z\}$ .*

*Proof* Since  $\mathcal{F}_s^Z \subset \mathcal{F}_s^X$ ,

$$\begin{aligned} & \mathbb{E} \left[ \widehat{Y}_t - \widehat{Y}_s - \int_s^t \widehat{H}_r dr \mid \mathcal{F}_s^Z \right] \\ &= \mathbb{E} \left[ \mathbb{E} [Y_t \mid \mathcal{F}_t^Z] - \mathbb{E} [Y_s \mid \mathcal{F}_s^Z] - \int_s^t \mathbb{E} [H_r \mid \mathcal{F}_r^Z] dr \mid \mathcal{F}_s^Z \right] \\ &= \mathbb{E} \left[ Y_t - Y_s - \int_s^t H_r dr \mid \mathcal{F}_s^Z \right] \\ &= \mathbb{E} \left[ \mathbb{E} \left[ Y_t - Y_s - \int_s^t H_r dr \mid \mathcal{F}_s^X \right] \mid \mathcal{F}_s^Z \right] = 0. \end{aligned}$$

The first equality is proved in a fashion similar to the one you were asked to prove in Exercise 29.1.  $\square$

Here is the filtering equation.

**Theorem 29.5** Let  $M_t = f(X_t) - f(X_0) - \int_0^t A_s ds$  be a martingale with respect to  $\{\mathcal{F}_t^X\}$  and write  $F_s$  for  $f(X_s)$ . Suppose  $\langle M, W \rangle_t = \int_0^t D_s ds$ . Then

$$\widehat{F}_t = \widehat{F}_0 + \int_0^t \widehat{A}_s ds + \int_0^t (\widehat{F}_s \widehat{h}_s - \widehat{F}_s \widehat{h}_s + \widehat{D}_s) dN_s. \quad (29.6)$$

*Proof* By Lemma 29.4,

$$L_t = \widehat{F}_t - \widehat{F}_0 - \int_0^t \widehat{A}_s ds \quad (29.7)$$

is a martingale with respect to  $\{\mathcal{F}_t^Z\}$  and by Theorem 29.3, there exists  $H_s$  such that

$$L_t = \int_0^t H_s dN_s. \quad (29.8)$$

By the product formula

$$\begin{aligned} F_t Z_t &= \int_0^t F_s dZ_s + \int_0^t Z_s dF_s + \int_0^t D_s ds \\ &= \int_0^t F_s dN_s + \int_0^t F_s h_s ds + \int_0^t Z_s dM_s + \int_0^t Z_s A_s ds + \int_0^t D_s ds \\ &= \mathcal{F}^X\text{-martingale} + \int_0^t [F_s h_s + Z_s A_s + D_s] ds. \end{aligned}$$

By Lemma 29.4 and the obvious fact that  $Z$  is adapted to  $\{\mathcal{F}_t^Z\}$ ,

$$\widehat{F}_t Z_t = \widehat{F}_t \widehat{Z}_t = \mathcal{F}^Z\text{-martingale} + \int_0^t (\widehat{F}_s \widehat{h}_s + Z_s \widehat{A}_s + \widehat{D}_s) ds.$$

Again using the product formula,

$$\begin{aligned} \widehat{F}_t Z_t &= \int_0^t \widehat{F}_s dZ_s + \int_0^t Z_s d\widehat{F}_s + \int_0^t H_s ds \\ &= \mathcal{F}^Z\text{-martingale} + \int_0^t [\widehat{F}_s \widehat{h}_s + Z_s \widehat{A}_s + H_s] ds. \end{aligned}$$

Therefore

$$\int_0^t (\widehat{F_s h}_s + Z_s \widehat{A}_s + \widehat{D}_s - \widehat{F}_s \widehat{h}_s - Z_s \widehat{A}_s - H_s) ds$$

is a continuous  $\mathcal{F}^Z$ -martingale that has paths that are locally of bounded variation and which is zero at time zero, hence is identically zero by Theorem 9.7. Hence with probability one,  $H_s = \widehat{F}_s \widehat{h}_s - \widehat{F}_s \widehat{h}_s + \widehat{D}_s$  for almost every  $s$ . Substituting this in (29.8) and combining with (29.7) gives our result.  $\square$

## 29.5 Linear models

The filtering equation (29.6) is difficult to apply in most cases. However, in the linear model, we can get a much simpler representation. To define the *linear model* in  $d$  dimensions, let  $X_t$  solve

$$dX_t = A(t) d\overline{W}_t + B(t)X_t dt, \quad (29.9)$$

where  $\overline{W}_t$  is a  $d$ -dimensional Brownian motion and  $A(t)$  and  $B(t)$  are deterministic  $d \times d$  matrices that are continuous in  $t$ . Let

$$dZ_t = dW_t + C(t)X_t dt, \quad (29.10)$$

where  $C$  is a deterministic  $d \times d$  matrix-valued function that is continuous in  $t$  and  $W_t$  is a  $d$ -dimensional Brownian motion independent of  $\overline{W}$  and  $X$ .

Why is this model useful? Suppose  $X_t$  is two-dimensional with  $X_t^{(1)}$  being the position of a particle and  $X_t^{(2)}$  its velocity. Suppose the position and the velocity have some randomness and that our observations of the position and velocity are noisy. This fits into the model (29.9)–(29.10) if we take

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C(t) = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}.$$

For another example, suppose a particle has a fixed unknown velocity and the position is observed, but obscured by noise. Let  $X_t^{(1)}$  and  $X_t^{(2)}$  be the position and velocity and let  $A(t)$  be the zero matrix,

$$B(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The solution of the filtering problem modeled by (29.9)–(29.10) is known as the *Kalman–Bucy filter*. For simplicity we will consider the special case where the dimension  $d$  is 1 and  $A, B, C$  are constant in  $t$ ; the general case is done in exactly the same way, but the notation becomes much more complicated (see Kallianpur, 1980). We will further assume  $\mathbb{E} X_0$  and  $\text{Var } X_0$  are known.

## 29.6 Kalman–Bucy filter

Let

$$V_t = \widehat{X}_t^2 - (\widehat{X}_t)^2,$$

the conditional variance of  $X_t$  given  $\mathcal{F}_t^Z$ .

**Theorem 29.6**  $V_t$  solves the deterministic ordinary differential equation

$$\frac{dV_t}{dt} = 1 + 2BV_t - C^2V_t^2, \quad V_0 = \text{Var } X_0 \quad (29.11)$$

In particular,  $V_t$  is deterministic.  $\widehat{X}_t$  solves

$$d\widehat{X}_t = CV_t dZ_t + (B - CV_t)\widehat{X}_t dt, \quad \widehat{X}_0 = \mathbb{E} X_0. \quad (29.12)$$

The equation (29.11) is an example of what is known as a Riccati equation. We get a similar equation when  $d > 1$  or when  $A, B$ , and  $C$  depend on  $t$ , but in general one cannot solve the Riccati equation explicitly. However, when  $d = 1$  and  $A, B, C$  do not depend on  $t$ , one can solve (29.11) by separation of variables. Write

$$\frac{dV}{1 + 2BV - C^2V^2} = dt,$$

and integrate both sides.

When  $d = 1$  (and even if  $A, B$ , and  $C$  depend on time), we can solve (29.12). Let  $G_t = B - CV_t$  so that we have

$$d\widehat{X}_t = CV_t dZ_t + G_t\widehat{X}_t dt,$$

or by the product formula

$$d\left[e^{-\int_0^t G_r dr}\widehat{X}_t\right] = e^{-\int_0^t G_r dr}CV_t dZ_t,$$

and hence

$$\widehat{X}_t = \mathbb{E} X_0 + \int_0^t e^{\int_s^t G_r dr}CV_s dZ_s.$$

(Cf. the solution of (24.15).)

*Proof of Theorem 29.6* By Itô's formula, if  $f \in C^2$ ,

$$f(X_t) - f(X_0) = \mathcal{F}^X\text{-martingale} + \int_0^t [\frac{1}{2}f''(X_s) + BX_s f'(X_s)] ds.$$

By the filtering equation applied with  $f(x) = x$ ,

$$\widehat{X}_t = \mathbb{E} X_0 + B \int_0^t \widehat{X}_s ds + C \int_0^t V_s dN_s. \quad (29.13)$$

By Exercises 29.4(2) and 29.5(3),

$$\widehat{X}_t^3 - \widehat{X}_t \widehat{X}_t^2 = 2\widehat{X}_t V_t. \quad (29.14)$$

With the filtering equation applied with  $f(x) = x^2$  and (29.14),

$$\begin{aligned} \widehat{X}_t^2 &= \mathbb{E} X_0^2 + C \int_0^t (1 + 2B\widehat{X}_s^2) ds + C \int_0^t (\widehat{X}_s^3 - \widehat{X}_s \widehat{X}_s^2) dN_s \\ &= \mathbb{E} X_0^2 + C \int_0^t (1 + 2B\widehat{X}_s^2) ds + 2C \int_0^t V_s \widehat{X}_s dN_s. \end{aligned}$$

Therefore

$$\begin{aligned} dV_t &= d(\widehat{X}_t^2 - (\widehat{X}_t)^2) \\ &= 2CV_t \widehat{X}_t dN_t + (1 + 2B\widehat{X}_t^2 dt) - 2\widehat{X}_t(CV_t dN_t + B\widehat{X}_t dt) - C^2 V_t^2 dt \\ &= (1 + 2BV_t - C^2 V_t^2) dt. \end{aligned} \tag{29.15}$$

This shows that  $V_t$  solves the deterministic ordinary differential equation (29.15). This equation has a unique solution (cf. Theorem 15.1), so  $V_t$  is deterministic. We obtain (29.12) from (29.2), (29.10), and (29.13).  $\square$

## Exercises

- 29.1 Justify the first equality in (29.4).
- 29.2 Show that if  $M_t$  is a martingale with respect to  $\{\mathcal{F}_t^X\}$ , then  $\widehat{M}_t$  is a martingale with respect to  $\{\mathcal{F}_t^Z\}$ .
- 29.3 Suppose  $W$  is a Brownian motion and  $\{\mathcal{F}_t\}$  is its minimal augmented filtration. Let  $T$  be a bounded stopping time with respect to  $\{\mathcal{F}_t\}$ . Suppose  $Y$  is a  $\mathcal{F}_T$  measurable random variable with  $\mathbb{E} Y^2 < \infty$ . Show that there exists a predictable process  $H_s$  with  $\mathbb{E} \int_0^T H_s^2 ds < \infty$  such that  $Y = \mathbb{E} Y + \int_0^T H_s dW_s$ , a.s.
- 29.4 (1) Show that the solution to (29.9) is a Gaussian process.  
(2) Show that the solutions  $(X_t, Z_t)$  to (29.9)–(29.10) form a Gaussian process.
- 29.5 (1) Show that if  $X$  is a normal random variable with mean  $\mu$  and variance  $\sigma^2$ , then

$$\mathbb{E} X^3 = \mu(\mu^2 + 3\sigma^2).$$

- (2) Show that if  $X, Y_1, \dots, Y_n$  are jointly normal random variables, then

$$\begin{aligned} \mathbb{E}[X^3 | Y_1, \dots, Y_n] &= \mathbb{E}[X | Y_1, \dots, Y_n](\mathbb{E}[X | Y_1, \dots, Y_n]^2 \\ &\quad + 3\text{Var}[X | Y_1, \dots, Y_n]), \end{aligned}$$

where

$$\text{Var}[X | Y_1, \dots, Y_n] = \mathbb{E}[(X - \mathbb{E}[X | Y_1, \dots, Y_n])^2 | Y_1, \dots, Y_n].$$

- (3) Show that

$$\widehat{X}_t^3 = \widehat{X}_t((\widehat{X}_t)^2 + 3\text{Var}(X_t | \mathcal{F}_t^Z)),$$

where

$$\text{Var}(X_t | \mathcal{F}_t^Z) = \mathbb{E}[(X_t - \widehat{X}_t)^2 | \mathcal{F}_t^Z] = \widehat{X}_t^2 - (\widehat{X}_t)^2.$$

## Notes

For more on filtering, see Kallianpur (1980) and Øksendal (2003).

# 30

---

## Convergence of probability measures

Suppose we have a sequence of probabilities on a metric space  $\mathcal{S}$  and we want to define what it means for the sequence to converge weakly. Alternately, we may have a sequence of random variables and want to say what it means for the random variables to converge weakly. We will apply the results we obtain here in later chapters to the case where  $\mathcal{S}$  is a function space such as  $C[0, 1]$  and obtain theorems on the convergence of stochastic processes.

For now our state space is assumed to be an arbitrary metric space, although we will soon add additional assumptions on  $\mathcal{S}$ . We use the Borel  $\sigma$ -field on  $\mathcal{S}$ , which is the  $\sigma$ -field generated by the open sets in  $\mathcal{S}$ . We write  $A^0$ ,  $\bar{A}$ , and  $\partial A$  for the interior, closure, and boundary of  $A$ , respectively.

### 30.1 The portmanteau theorem

Clearly the definition of weak convergence of real-valued random variables in terms of distribution functions (see Section A.12) has no obvious analog. The appropriate generalization is the following; cf. Proposition A.41.

**Definition 30.1** A sequence of probabilities  $\{\mathbb{P}_n\}$  on a metric space  $\mathcal{S}$  furnished with the Borel  $\sigma$ -field is said to *converge weakly* to  $\mathbb{P}$  if  $\int f d\mathbb{P}_n \rightarrow \int f d\mathbb{P}$  for every bounded and continuous function  $f$  on  $\mathcal{S}$ . A sequence of random variables  $\{X_n\}$  taking values in  $\mathcal{S}$  converges weakly to a random variable  $X$  taking values in  $\mathcal{S}$  if  $\mathbb{E} f(X_n) \rightarrow \mathbb{E} f(X)$  whenever  $f$  is a bounded and continuous function.

Saying  $X_n$  converges weakly to  $X$  is the same as saying that the laws of  $X_n$  converge weakly to the law of  $X$ . To see this, if  $\mathbb{P}_n$  is the law of  $X_n$ , that is,  $\mathbb{P}_n(A) = \mathbb{P}(X_n \in A)$  for each Borel subset  $A$  of  $\mathcal{S}$ , then  $\mathbb{E} f(X_n) = \int f d\mathbb{P}_n$  and  $\mathbb{E} f(X) = \int f d\mathbb{P}$ . (This holds when  $f$  is an indicator by the definition of the law of  $X_n$  and  $X$ , then for simple functions by linearity, then for non-negative measurable functions by monotone convergence, and then for arbitrary bounded and Borel measurable  $f$  by linearity.)

What might cause a bit of confusion is that weak convergence in probability is not the same as weak convergence in functional analysis, but rather is equivalent to what is known as weak-\* convergence in functional analysis. Feel free to skip the remainder of this paragraph where we explain this. Recall that if  $B$  is a Banach space and  $B^*$  is its dual, then  $x_n \in B$  converges weakly to  $x \in B$  if  $f(x_n) \rightarrow f(x)$  for all  $f \in B^*$ .  $f_n \in B^*$  converges with respect to the weak-\* topology to  $f \in B^*$  if  $f_n(x) \rightarrow f(x)$  for all  $x \in B$ . By the Riesz representation theorem, there is a one-to-one correspondence between positive bounded linear functionals on  $B = C(X)$ , the continuous functions on  $X$ , where  $X$  is compact, and the set  $\mathcal{M}$  of finite

measures on  $X$ . When  $B = C(X)$ ,  $B^*$  can be identified with  $\mathcal{M}$ , and measures  $\mathbb{P}_n$  with mass 1 in  $\mathcal{M}$  converge to  $\mathbb{P} \in \mathcal{M}$  with respect to the weak-\* topology if  $\mathbb{P}_n(g) \rightarrow \mathbb{P}(g)$  for every  $g \in B = C(X)$ . Interpreting  $\mathbb{P}_n(g)$  as  $\int g d\mathbb{P}_n$  shows the connection.

Returning to weak convergence in the probability sense, the following theorem, known as the portmanteau theorem, gives some other characterizations. For this chapter we let

$$F_\delta = \{x : d(x, F) < \delta\} \quad (30.1)$$

for closed sets  $F$ , the set of points within  $\delta$  of  $F$ , where  $d(x, F) = \inf\{d(x, y) : y \in F\}$ .

**Theorem 30.2** Suppose  $\{\mathbb{P}_n, n = 1, 2, \dots\}$  and  $\mathbb{P}$  are probabilities on a metric space. The following are equivalent.

- (1)  $\mathbb{P}_n$  converges weakly to  $\mathbb{P}$ .
- (2)  $\limsup_n \mathbb{P}_n(F) \leq \mathbb{P}(F)$  for all closed sets  $F$ .
- (3)  $\liminf_n \mathbb{P}_n(G) \geq \mathbb{P}(G)$  for all open sets  $G$ .
- (4)  $\lim_n \mathbb{P}_n(A) = \mathbb{P}(A)$  for all Borel sets  $A$  such that  $\mathbb{P}(\partial A) = 0$ .

*Proof* The equivalence of (2) and (3) is easy because if  $F$  is closed, then  $G = F^c$  is open and  $\mathbb{P}_n(G) = 1 - \mathbb{P}_n(F)$ .

To see that (2) and (3) imply (4), suppose  $\mathbb{P}(\partial A) = 0$ . Then

$$\begin{aligned} \limsup_n \mathbb{P}_n(A) &\leq \limsup_n \mathbb{P}_n(\bar{A}) \leq \mathbb{P}(\bar{A}) \\ &= \mathbb{P}(A^0) \leq \liminf_n \mathbb{P}_n(A^0) \leq \liminf_n \mathbb{P}_n(A). \end{aligned}$$

Next, let us show (4) implies (2). Let  $F$  be closed. If  $y \in \partial F_\delta$ , then  $d(y, F) = \delta$ . The sets  $\partial F_\delta$  are disjoint for different  $\delta$ . At most countably many of them can have positive  $\mathbb{P}$ -measure, hence there exists a sequence  $\delta_k \downarrow 0$  such that  $\mathbb{P}(\partial F_{\delta_k}) = 0$  for each  $k$ . Then

$$\limsup_n \mathbb{P}_n(F) \leq \limsup_n \mathbb{P}_n(\bar{F}_{\delta_k}) = \mathbb{P}(\bar{F}_{\delta_k}) = \mathbb{P}(F_{\delta_k})$$

for each  $k$ . Since  $\mathbb{P}(F_{\delta_k}) \downarrow \mathbb{P}(F)$  as  $\delta_k \rightarrow 0$ , this gives (2).

We show now that (1) implies (2). Suppose  $F$  is closed. Let  $\varepsilon > 0$ . Take  $\delta > 0$  small enough so that  $\mathbb{P}(\bar{F}_\delta) - \mathbb{P}(F) < \varepsilon$ . Then take  $f$  continuous, to be equal to 1 on  $F$ , to have support in  $\bar{F}_\delta$ , and to be bounded between 0 and 1. For example,  $f(x) = 1 - (1 \wedge \delta^{-1}d(x, F))$  would do. Then

$$\begin{aligned} \limsup_n \mathbb{P}_n(F) &\leq \limsup_n \int f d\mathbb{P}_n = \int f d\mathbb{P} \\ &\leq \mathbb{P}(\bar{F}_\delta) \leq \mathbb{P}(F) + \varepsilon. \end{aligned}$$

Since this is true for all  $\varepsilon$ , (2) follows.

Finally, let us show (2) implies (1). Let  $f$  be bounded and continuous. If we show

$$\limsup_n \int f d\mathbb{P}_n \leq \int f d\mathbb{P}, \quad (30.2)$$

for every such  $f$ , then applying this inequality to both  $f$  and  $-f$  will give (1). By adding a sufficiently large positive constant to  $f$  and then multiplying by a suitable constant, without

loss of generality we may assume  $f$  is bounded and takes values in  $(0, 1)$ . We define  $F_i = \{x : f(x) \geq i/k\}$ , which is closed.

$$\begin{aligned} \int f d\mathbb{P}_n &\leq \sum_{i=1}^k \frac{i}{k} \mathbb{P}_n\left(\frac{i-1}{k} \leq f(x) < \frac{i}{k}\right) \\ &= \sum_{i=1}^k \frac{i}{k} [\mathbb{P}_n(F_{i-1}) - \mathbb{P}_n(F_i)] \\ &= \sum_{i=0}^{k-1} \frac{i+1}{k} \mathbb{P}_n(F_i) - \sum_{i=1}^k \frac{i}{k} \mathbb{P}_n(F_i) \\ &\leq \frac{1}{k} + \frac{1}{k} \sum_{i=1}^k \mathbb{P}_n(F_i). \end{aligned}$$

Similarly,

$$\int f d\mathbb{P} \geq \frac{1}{k} \sum_{i=1}^k \mathbb{P}(F_i).$$

Then

$$\begin{aligned} \limsup_n \int f d\mathbb{P}_n &\leq \frac{1}{k} + \frac{1}{k} \sum_{i=1}^k \limsup_n \mathbb{P}_n(F_i) \\ &\leq \frac{1}{k} + \frac{1}{k} \sum_{i=1}^k \mathbb{P}(F_i) \leq \frac{1}{k} + \int f d\mathbb{P}. \end{aligned}$$

Since  $k$  is arbitrary, this gives (30.2).  $\square$

If  $x_n \rightarrow x$ ,  $\mathbb{P}_n = \delta_{x_n}$ , and  $\mathbb{P} = \delta_x$ , it is easy to see  $\mathbb{P}_n$  converges weakly to  $\mathbb{P}$ . Letting  $A = \{x\}$  shows that one cannot, in general, have  $\lim_n \mathbb{P}_n(F) = \mathbb{P}(F)$  for all closed sets  $F$ .

## 30.2 The Prohorov theorem

It turns out there is a simple condition that ensures that a sequence of probability measures has a weakly convergent subsequence.

**Definition 30.3** A sequence of probabilities  $\mathbb{P}_n$  on a metric space  $\mathcal{S}$  is *tight* if for every  $\varepsilon$  there exists a compact set  $K$  (depending on  $\varepsilon$ ) such that  $\sup_n \mathbb{P}_n(K^c) \leq \varepsilon$ .

The important result here is Prohorov's theorem.

**Theorem 30.4** *If a sequence of probability measures on a metric space  $\mathcal{S}$  is tight, there is a subsequence that converges weakly to a probability measure on  $\mathcal{S}$ .*

*Proof* Suppose first that the metric space  $\mathcal{S}$  is compact. Then  $C(\mathcal{S})$ , the collection of continuous functions on  $\mathcal{S}$ , is a separable metric space when furnished with the supremum norm; this is Exercise 30.1. Let  $\{f_i\}$  be a countable collection of non-negative elements of  $C(\mathcal{S})$  whose linear span is dense in  $C(\mathcal{S})$ . For each  $i$ ,  $\int f_i d\mathbb{P}_n$  is a bounded sequence, so we

have a convergent subsequence. By a diagonalization procedure, we can find a subsequence  $n'$  such that  $\int f_i d\mathbb{P}_{n'}$  converges for all  $i$ . By the term “diagonalization procedure” we are referring to the well-known method of proof of the Ascoli–Arzelà theorem; see any book on real analysis for a detailed explanation. Call the limit  $Lf_i$ . Clearly  $0 \leq Lf_i \leq \|f_i\|_\infty$ ,  $L$  is linear, and so we can extend  $L$  to a bounded linear functional on  $\mathcal{S}$ . By the Riesz representation theorem (Rudin, 1987), there exists a measure  $\mathbb{P}$  such that  $Lf = \int f d\mathbb{P}$ . Since  $\int f_i d\mathbb{P}_{n'} \rightarrow \int f_i d\mathbb{P}$  for all  $f_i$ , it is not hard to see, since each  $\mathbb{P}_{n'}$  has total mass 1, that  $\int f d\mathbb{P}_{n'} \rightarrow \int f d\mathbb{P}$  for all  $f \in C(\mathcal{S})$ . Therefore  $\mathbb{P}_{n'}$  converges weakly to  $\mathbb{P}$ . Since  $Lf \geq 0$  if  $f \geq 0$ , then  $\mathbb{P}$  is a positive measure. The function that is identically equal to 1 is bounded and continuous, so  $1 = \mathbb{P}_{n'}(\mathcal{S}) = \int 1 d\mathbb{P}_{n'} \rightarrow \int 1 d\mathbb{P}$ , or  $\mathbb{P}(\mathcal{S}) = 1$ .

Next suppose that  $\mathcal{S}$  is a Borel subset of a compact metric space  $\mathcal{S}'$ . Extend each  $\mathbb{P}_n$ , initially defined on  $\mathcal{S}$ , to  $\mathcal{S}'$  by setting  $\mathbb{P}_n(\mathcal{S}' \setminus \mathcal{S}) = 0$ . By the first paragraph of the proof, there is a subsequence  $\mathbb{P}_{n'}$  that converges weakly to a probability  $\mathbb{P}$  on  $\mathcal{S}'$  (the definition of weak convergence here is relative to the topology on  $\mathcal{S}'$ ). Since the  $\mathbb{P}_n$  are tight, there exist compact subsets  $K_m$  of  $\mathcal{S}$  such that  $\mathbb{P}_n(K_m) \geq 1 - 1/m$  for all  $n$ . The  $K_m$  will also be compact relative to the topology on  $\mathcal{S}'$ , so by Theorem 30.2,

$$\mathbb{P}(K_m) \geq \limsup_{n'} \mathbb{P}_{n'}(K_m) \geq 1 - 1/m.$$

Since  $\cup_m K_m \subset \mathcal{S}$ , we conclude  $\mathbb{P}(\mathcal{S}) = 1$ .

If  $G$  is open in  $\mathcal{S}$ , then  $G = H \cap \mathcal{S}$  for some  $H$  open in  $\mathcal{S}'$ . Then

$$\liminf_{n'} \mathbb{P}_{n'}(G) = \liminf_{n'} \mathbb{P}_{n'}(H) \geq \mathbb{P}(H) = \mathbb{P}(H \cap \mathcal{S}) = \mathbb{P}(G).$$

Thus by Theorem 30.2,  $\mathbb{P}_{n'}$  converges weakly to  $\mathbb{P}$  relative to the topology on  $\mathcal{S}$ .

Now let  $\mathcal{S}$  be an arbitrary metric space. Since all the  $\mathbb{P}_n$ ’s are supported on  $\cup_m K_m$ , we can replace  $\mathcal{S}$  by  $\cup_m K_m$ , or we may as well assume that  $\mathcal{S}$  is  $\sigma$ -compact, and hence separable. It remains to embed the separable metric space  $\mathcal{S}$  into a compact metric space  $\mathcal{S}'$ . If  $d$  is the metric on  $\mathcal{S}$ ,  $d \wedge 1$  will also be an equivalent metric, that is, one that generates the same collection of open sets, so we may assume  $d$  is bounded by 1. Now  $\mathcal{S}$  can be embedded in  $\mathcal{S}' = [0, 1]^\mathbb{N}$  as follows. We define a metric on  $\mathcal{S}'$  by

$$d'(a, b) = \sum_{i=1}^{\infty} 2^{-i} (|a^i - b^i| \wedge 1), \quad a = (a^1, a^2, \dots), b = (b^1, b^2, \dots). \quad (30.3)$$

Being the product of compact spaces,  $\mathcal{S}'$  is itself compact. If  $\{z_j\}$  is a countable dense subset of  $\mathcal{S}$ , let  $I : \mathcal{S} \rightarrow [0, 1]^\mathbb{N}$  be defined by

$$I(x) = (d(x, z_1), d(x, z_2), \dots).$$

We leave it to the reader to check that  $I$  is a one-to-one continuous open map of  $\mathcal{S}$  to a subset of  $\mathcal{S}'$ . Since  $\mathcal{S}$  is  $\sigma$ -compact, and the continuous image of compact sets is compact, then  $I(\mathcal{S})$  is a Borel set.  $\square$

Clearly, Prohorov’s theorem is easily modified to handle the case of finite measures on  $\mathcal{S}$ .

### 30.3 Metrics for weak convergence

Since we have defined a notion of convergence of probability measures, one might wonder if one can make the set of probability measures  $\mathcal{M}$  on  $\mathcal{S}$  into a metric space so that weak convergence is equivalent to convergence in  $\mathcal{M}$ . This is indeed possible and in fact there are a number of metrics on the space of probability measures that work. We will focus on the *Prohorov metric*.

**Definition 30.5** If  $\mathbb{P}$  and  $\mathbb{Q}$  are probability measures on a separable metric space  $\mathcal{S}$ , define

$$d_{\mathcal{M}}(\mathbb{P}, \mathbb{Q}) = \inf\{\varepsilon : \mathbb{P}(F) \leq \mathbb{Q}(F_\varepsilon) + \varepsilon \text{ for all } F \text{ closed}\}. \quad (30.4)$$

It is not immediately obvious that  $d_{\mathcal{M}}$  is even a metric, so the first task is to show that it is.

**Proposition 30.6**  $d_{\mathcal{M}}$  is a metric on  $\mathcal{M}$ .

*Proof* We start with symmetry, that is, that  $d_{\mathcal{M}}(\mathbb{Q}, \mathbb{P}) = d_{\mathcal{M}}(\mathbb{P}, \mathbb{Q})$ . Let  $\alpha$  be any real number larger than  $d_{\mathcal{M}}(\mathbb{P}, \mathbb{Q})$ . If  $H$  is closed, then  $H_\alpha = \{x : d(x, H) < \alpha\}$  is open and  $K = \mathcal{S} \setminus H_\alpha$  is closed. Note that  $H \subset \mathcal{S} \setminus K_\alpha$ , where  $K_\alpha = \{x : d(x, K) < \alpha\}$ , because if  $x \in H$ , then  $d(x, K) \geq \alpha$ , so  $x \notin K_\alpha$  and hence  $x \in \mathcal{S} \setminus K_\alpha$ . Since  $K$  is closed, by the definition of  $d_{\mathcal{M}}(\mathbb{P}, \mathbb{Q})$ ,

$$\mathbb{P}(H_\alpha) = 1 - \mathbb{P}(K) \geq 1 - \mathbb{Q}(K_\alpha) - \alpha = \mathbb{Q}(\mathcal{S} \setminus K_\alpha) - \alpha \geq \mathbb{Q}(H) - \alpha,$$

or  $\mathbb{Q}(H) \leq \mathbb{P}(H_\alpha) + \alpha$ . Since  $H$  was an arbitrary closed set,  $d_{\mathcal{M}}(\mathbb{Q}, \mathbb{P}) \leq \alpha$ , and it follows that  $d_{\mathcal{M}}(\mathbb{Q}, \mathbb{P}) \leq d_{\mathcal{M}}(\mathbb{P}, \mathbb{Q})$ . Reversing the roles of  $\mathbb{P}$  and  $\mathbb{Q}$  shows symmetry.

Clearly  $d_{\mathcal{M}}(\mathbb{P}, \mathbb{Q}) \geq 0$ . If  $d_{\mathcal{M}}(\mathbb{P}, \mathbb{Q}) = 0$ , then  $\mathbb{P}(F) = \mathbb{Q}(F) = 0$  for all closed sets  $F$ . Since the collection of closed sets generates the Borel  $\sigma$ -field, it is not hard to see that  $\mathbb{P}(A) = \mathbb{Q}(A)$  for all Borel subsets  $A$ , and hence  $\mathbb{P} = \mathbb{Q}$ .

Finally we prove the triangle inequality. Suppose  $\mathbb{P}, \mathbb{Q}, \mathbb{R} \in \mathcal{M}$ . If  $\alpha$  is any real larger than  $d_{\mathcal{M}}(\mathbb{P}, \mathbb{Q})$  and  $\beta$  any real larger than  $d_{\mathcal{M}}(\mathbb{Q}, \mathbb{R})$ , then for any  $\varepsilon > 0$  and any closed set  $F$

$$\begin{aligned} \mathbb{P}(F) &\leq \mathbb{Q}(F_\alpha) + \alpha \leq \mathbb{Q}(\overline{F_\alpha}) + \alpha \\ &\leq \mathbb{R}((\overline{F_\alpha})_\beta) + \alpha + \beta \\ &\leq \mathbb{R}(F_{\alpha+\beta+\varepsilon}) + (\alpha + \beta + \varepsilon). \end{aligned}$$

Therefore  $d_{\mathcal{M}}(\mathbb{P}, \mathbb{R}) \leq \alpha + \beta + \varepsilon$ , and since  $\varepsilon$  is arbitrary, the triangle inequality follows.  $\square$

Now we show that weak convergence is equivalent to convergence in the topology generated by  $d_{\mathcal{M}}$ , at least if  $\mathcal{S}$  is separable. ( $L^\infty[0, 1]$  is an example of a nonseparable metric space.)

**Proposition 30.7** Suppose  $\mathcal{S}$  is a separable metric space. A sequence of probability measures  $\mathbb{P}_n$  on  $\mathcal{S}$  converges weakly to a probability  $\mathbb{P}$  if and only if  $d_{\mathcal{M}}(\mathbb{P}_n, \mathbb{P}) \rightarrow 0$ .

*Proof* We first suppose  $d_{\mathcal{M}}(\mathbb{P}_n, \mathbb{P}) \rightarrow 0$  and show that  $\mathbb{P}_n$  converges weakly to  $\mathbb{P}$ . Separability is not used in this part of the proof. Suppose  $F$  is closed and set  $\varepsilon_n = d_{\mathcal{M}}(\mathbb{P}_n, \mathbb{P}) + 1/n$ . Since  $\mathbb{P}_n(F) \leq \mathbb{P}(F_{\varepsilon_n}) + \varepsilon_n$ , then

$$\limsup_n \mathbb{P}_n(F) \leq \limsup_n \mathbb{P}(F_{\varepsilon_n}) = \mathbb{P}(F),$$

and we now apply Theorem 30.2(2).

We now suppose  $\mathbb{P}_n$  converges weakly to  $\mathbb{P}$ . Let  $\varepsilon > 0$ . Cover  $\mathcal{S}$  with countably many balls  $\{B_i\}$  of diameter less than  $\varepsilon/2$  (separability is used here) and let  $A_1 = B_1$ ,  $A_2 = B_2 \setminus B_1$ ,  $A_3 = B_3 \setminus (B_1 \cup B_2)$ ,  $A_4 = B_4 \setminus (B_1 \cup B_2 \cup B_3)$ , and so on. Hence the  $A_n$  form a collection of disjoint sets which cover  $\mathcal{S}$  and each  $A_n$  has diameter less than  $\varepsilon/2$ . Choose  $N$  large enough so that  $\mathbb{P}(\cup_{i=1}^N A_i) > 1 - \varepsilon/2$ . Let  $\mathcal{G}$  be the collection of open sets of the form  $(A_{i_1} \cup \dots \cup A_{i_j})_{\varepsilon/2}$  such that  $i_1, \dots, i_j \leq N$ . That is, we look at all finite unions of  $A_1, \dots, A_N$ , and then take the  $(\varepsilon/2)$ -enlargements. The collection  $\mathcal{G}$  is finite. This fact and Theorem 30.2(3) imply that we can find  $n_0$  such that  $\mathbb{P}(G) \leq \mathbb{P}_n(G) + \varepsilon/2$  if  $n \geq n_0$  and  $G \in \mathcal{G}$ .

Suppose  $F$  is closed. Let  $G = (\cup\{A_i : i \leq N, A_i \cap F \neq \emptyset\})_{\varepsilon/2}$ . Then  $G \in \mathcal{G}$  and if  $n \geq n_0$

$$\begin{aligned}\mathbb{P}(F) &\leq \mathbb{P}(G) + \mathbb{P}(\cup_{i=N+1}^{\infty} A_i) \leq \mathbb{P}(G) + \varepsilon/2 \\ &\leq \mathbb{P}_n(G) + \varepsilon \leq \mathbb{P}_n(F_\varepsilon) + \varepsilon.\end{aligned}$$

In the last inequality we used the definition of  $G$  and the fact that the  $A_i$  have diameters less than  $\varepsilon/2$ . This shows  $d_{\mathcal{M}}(\mathbb{P}, \mathbb{P}_n) \leq \varepsilon$  if  $n \geq n_0$ , which in turn implies  $d_{\mathcal{M}}(\mathbb{P}, \mathbb{P}_n) \rightarrow 0$ .  $\square$

## Exercises

- 30.1 If  $\mathcal{S}$  is a metric space, then it is well known that  $C(\mathcal{S})$ , the collection of continuous functions with the metric

$$d(f, g) = \sup_{x \in \mathcal{S}} |f(x) - g(x)|$$

is a metric space. Show that if  $\mathcal{S}$  is compact, then  $C(\mathcal{S})$  is separable.

- 30.2 Suppose  $X_n$  converges weakly to  $X$  and the random variables  $Z_n$  are such that  $d(X_n, Z_n)$  converges to 0 in probability. Prove that  $Z_n$  converges weakly to  $X$ . This is known as *Slutsky's theorem*.

*Hint:* Start with  $\mathbb{P}(Z_n \in F) \leq \mathbb{P}(X_n \in \bar{F}_\delta) + \mathbb{P}(d(X_n, Z_n) \geq \delta)$ .

- 30.3 Suppose  $X_n$  take values in a normed linear space and converge weakly to  $X$ . Suppose  $c_n$  are scalars converging to  $c$ . Show  $c_n X_n$  converges weakly to  $cX$ .

- 30.4 Give an example of a sequence  $\mathbb{P}_n$  converging weakly to  $\mathbb{P}$  and a function  $f$  that is continuous but not bounded such that  $\int f d\mathbb{P}_n$  does not converge to  $\int f d\mathbb{P}$ .

- 30.5 Give an example of a sequence  $\mathbb{P}_n$  converging weakly to  $\mathbb{P}$  and a function  $f$  that is bounded but not continuous such that  $\int f d\mathbb{P}_n$  does not converge to  $\int f d\mathbb{P}$ .

- 30.6 Show that if  $X_n$  converges weakly to  $X$  and  $Y_n$  converges in probability to 0, then  $X_n Y_n$  converges in probability to 0.

- 30.7 This exercise considers a sequence of probability measures that have densities. Suppose  $\mathcal{S}$  is furnished with the Borel  $\sigma$ -field and  $\mu$  is a measure on  $\mathcal{S}$ . Suppose that  $f_n : \mathcal{S} \rightarrow [0, \infty)$  and  $f : \mathcal{S} \rightarrow [0, \infty)$  are measurable functions, each of whose integral over  $\mathcal{S}$  is one, and define  $\mathbb{P}_n(A) = \int_A f_n(x) \mu(dx)$  for each  $n$  and  $\mathbb{P}(A) = \int_A f(x) \mu(dx)$ .

(1) Show that if  $f_n \rightarrow f$ ,  $\mu$ -a.e., then  $\mathbb{P}_n$  converges weakly to  $\mathbb{P}$ .

(2) Give an example where  $\mathbb{P}_n$  and  $\mathbb{P}$  are as above,  $\mathbb{P}_n$  converges weakly to  $\mathbb{P}$ , but  $f_n$  does not converge almost everywhere to  $f$ .

- 30.8 Give an example of continuous processes  $X_n$  and  $X$  such that all the finite-dimensional distributions of  $X_n$  converge weakly to the corresponding finite-dimensional distributions of  $X$ , but where  $X_n$  does not converge weakly to  $X$  with respect to the topology of  $C[0, 1]$ .
- 30.9 Suppose  $X$  is a random variable taking values in a complete separable metric space. If  $\varepsilon > 0$ , show there exists a compact set  $K$  such that  $\mathbb{P}(X \notin K) < \varepsilon$ .

*Hint:* For each  $n$  choose closed balls  $\{B_{ni}, i = 1, \dots, N_n\}$  such that

$$\mathbb{P}(X \notin \bigcup_{i=1}^{N_n} B_{ni}) < \varepsilon/2^{n+1}.$$

Then  $K = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{N_n} B_{ni}$  is totally bounded, hence compact.

- 30.10 Suppose  $X_n$  converges weakly to  $X$  and the metric space  $\mathcal{S}$  is complete and separable. Prove that the sequence  $\{X_n\}$  is tight.
- 30.11 Let  $\mathcal{L}$  be the collection of continuous functions on  $\mathcal{S}$  such that

- (1)  $\sup_{x \in \mathcal{S}} |f(x)| \leq 1$ .
- (2)  $|f(x) - f(y)| \leq d(x, y)$  for all  $x, y \in \mathcal{S}$ .

Define

$$d_{\mathcal{L}}(\mathbb{P}, \mathbb{Q}) = \sup_{f \in \mathcal{L}} \left| \int f \, d\mathbb{P} - \int f \, d\mathbb{Q} \right|.$$

Show that  $d_{\mathcal{L}}$  is a metric on the collection of probability measures on the Borel  $\sigma$ -field of  $\mathcal{S}$ . Prove that a sequence of probability measures  $\mathbb{P}_n$  converges weakly to  $\mathbb{P}$  if and only if  $d_{\mathcal{L}}(\mathbb{P}_n, \mathbb{P}) \rightarrow 0$ .

- 30.12 Suppose  $\mathcal{S}$  is a separable metric space. Show that  $\mathcal{M}$  is separable.

## Notes

For more information, see Billingsley (1968) and Ethier and Kurtz (1986).

# 31

---

## Skorokhod representation

Suppose  $\mathcal{S}$  is a complete separable metric space furnished with the Borel  $\sigma$ -field. We are going to show that if  $X_n$  are random variables taking values in  $\mathcal{S}$  converging weakly to a random variable  $X$ , then we can find another probability space and other random variables  $X'_n, X'$  such that the law of  $X'_n$  equals the law of  $X_n$  for each  $n$ , the law of  $X'$  equals the law of  $X$ , and  $X'_n$  converges to  $X'$  almost surely.

Let  $\Omega' = [0, 1]$ ,  $\mathcal{F}'$  the Borel  $\sigma$ -field on  $[0, 1]$ , and  $\mathbb{P}'$  Lebesgue measure. We first prove

**Theorem 31.1** *Let  $\mathbb{P}$  be a probability measure on  $\mathcal{S}$ . Then there exists a random variable  $X$  mapping  $\Omega'$  to  $\mathcal{S}$  such that the law of  $X'$  under  $\mathbb{P}'$  is equal to  $\mathbb{P}$ .*

*Proof* For each  $k \geq 1$ , let  $\{A_{ki}\}$  be a countable disjoint covering of  $\mathcal{S}$  by Borel sets of diameter less than  $1/k$ , such that  $\mathbb{P}(\partial A_{ki}) = 0$ , and  $\{A_{ki}\}$  is a refinement of  $\{A_{k-1,i}\}$ . We can construct these families inductively. To start, cover  $\mathcal{S}$  with countably many balls of radius less than 1. Since for each  $x_0$ ,  $\mathbb{P}(\{x : |x - x_0| = r\})$  can be nonzero for at most countably many values of  $r$ , we can arrange matters so that the  $\mathbb{P}$ -measure of the boundary of these balls is 0. We order the balls  $B_1, B_2, \dots$ , and then let  $A_{11} = B_1, A_{12} = B_2 \setminus B_1, A_{13} = B_3 \setminus (B_1 \cup B_2)$ , and so on. To construct  $\{A_{2i}\}$ , we first find a similar covering of  $\mathcal{S}$  by sets  $\{A'_{2i}\}$  of diameter less than  $1/2$ , and then take all intersections of sets in  $\{A'_{2i}\}$  with sets in  $\{A_{1j}\}$ .

We inductively define closed subintervals of  $[0, 1]$  by choosing  $I_{11}$  to have left endpoint at 0 and length equal to  $\mathbb{P}(A_{11})$ , then  $I_{12}$  to have left endpoint equal to the right endpoint of  $I_{11}$  and length equal to  $\mathbb{P}(A_{12})$ , and so forth. We then decompose  $I_{11}$  into subintervals  $\{I_{21}\}$  in an analogous way so that the lengths of the subintervals match the probabilities of the  $A_{2i}$ 's contained in  $A_{11}$ . We then subdivide  $I_{12}$ , and so on. We observe that  $\{I_{ki}\}$  is a refinement of  $\{I_{k-1,i}\}$  for all  $k \geq 2$  and  $\mathbb{P}'(I_{ki}) = \mathbb{P}(A_{ki})$  for all  $k$  and  $i$ .

Pick a point  $x_{ki} \in A_{ki}$  for each  $k$  and  $i$ . We define  $X^k$  by setting  $X^k(\omega')$  equal to  $x_{ki}$  if  $\omega' \in I_{ki}$ . (The set of endpoints of the  $I_{ki}$  is countable, hence has Lebesgue measure 0, and it doesn't matter how we define  $X^k$  at those points.) For each  $\omega'$  except those that are endpoints of some  $I_{ki}$ , if  $n \geq m$ , then  $X^n(\omega')$  and  $X^m(\omega')$  are in the same  $A_{mi}$  for some  $i$ . Since the diameter of  $A_{mi}$  is less than  $1/m$ , we see that  $d(X^n(\omega'), X^m(\omega')) \leq 1/m$ . That is,  $X^n(\omega')$  is a Cauchy sequence. The space  $\mathcal{S}$  is complete, so we can define  $X(\omega')$  to be the limit of the  $X^n(\omega')$ . The collection of endpoints of the  $I_{mi}$  is countable, so the limit exists for almost every  $\omega'$ .

It remains to show that the law of  $X$  under  $\mathbb{P}'$  is  $\mathbb{P}$ . Let  $F$  be a closed set, let  $F_k = \{x : d(x, F) < 1/k\}$ , and let  $J_k = \{i : A_{ki} \cap F \neq \emptyset\}$ . We have

$$\begin{aligned}\mathbb{P}'(X^k \in F) &\leq \mathbb{P}'(X^k \in \cup_{i \in J_k} A_{ki}) \leq \sum_{i \in J_k} \mathbb{P}'(X^k \in A_{ki}) \\ &= \sum_{i \in J_k} \mathbb{P}'(I_{ki}) = \sum_{i \in J_k} \mathbb{P}(A_{ki}) \leq \mathbb{P}(\bar{F}_k).\end{aligned}$$

We used the fact that each  $A_{ki}$  has diameter less than  $1/k$ . Hence

$$\limsup_k \mathbb{P}'(X^k \in F) \leq \mathbb{P}(F).$$

Therefore the laws of  $X^k$  under  $\mathbb{P}'$  converge weakly to  $\mathbb{P}$ . But we know  $d(X^k(\omega'), X(\omega')) \leq 1/k$ , so  $X^k$  converges to  $X$ , a.s., with respect to  $\mathbb{P}'$ . If  $f$  is continuous and bounded,  $\mathbb{E}'f(X^k) \rightarrow \mathbb{E}'f(X)$  by dominated convergence, so  $X^k \rightarrow X$  weakly. Therefore the law of  $X$  under  $\mathbb{P}'$  is equal to  $\mathbb{P}$ .  $\square$

We did not need the fact that the  $A_{ki}$  were continuity sets, i.e., that the probability of the boundary of  $A_{ki}$  is zero, but this will be used in the next theorem, which is known as the *Skorokhod representation*.

**Theorem 31.2** Suppose  $\mathbb{P}_n$  are probability measures on  $\mathcal{S}$  converging weakly to  $\mathbb{P}$ . Then there exist random variables  $X_n$  mapping  $\Omega'$  to  $\mathcal{S}$  with laws  $\mathbb{P}_n$  and a random variable  $X$  mapping  $\Omega'$  to  $\mathcal{S}$  with law  $\mathbb{P}$  such that  $X_n \rightarrow X$ , a.s.

Equivalently, if  $X'_n$  converges to  $X'$  weakly, there exist random variables  $X_n$  and  $X$  mapping  $\Omega'$  to  $\mathcal{S}$  with laws equal to  $X'_n$  and  $X$ , respectively, such that  $X_n \rightarrow X$ , a.s.

*Proof* Let the  $A_{ki}$  be as in the proof of the previous theorem, and for each  $\mathbb{P}_n$  define intervals  $I_{ki}^n$  and random variables  $X_n^k$  as was done above, and let  $X_n$  be the limit of the  $X_n^k$ 's. Let  $K_{kn} = \{i : \mathbb{P}(A_{ki}) > \mathbb{P}_n(A_{ki})\}$  and  $K_{kn}^c = \{i : \mathbb{P}(A_{ki}) \leq \mathbb{P}_n(A_{ki})\}$ . Since

$$\sum_i [\mathbb{P}(A_{ki}) - \mathbb{P}_n(A_{ki})] = 1 - 1 = 0,$$

we have

$$\sum_{K_{kn}^c} [\mathbb{P}(A_{ki}) - \mathbb{P}_n(A_{ki})] = - \sum_{K_{kn}} [\mathbb{P}(A_{ki}) - \mathbb{P}_n(A_{ki})].$$

Hence

$$\begin{aligned}\sum_i |\mathbb{P}'(I_{ki}) - \mathbb{P}'(I_{ki}^n)| &= \sum_i |\mathbb{P}(A_{ki}) - \mathbb{P}_n(A_{ki})| \\ &= \sum_{K_{kn}} [\mathbb{P}(A_{ki}) - \mathbb{P}_n(A_{ki})] - \sum_{K_{kn}^c} [\mathbb{P}(A_{ki}) - \mathbb{P}_n(A_{ki})] \\ &= 2 \sum_{K_{kn}} [\mathbb{P}(A_{ki}) - \mathbb{P}_n(A_{ki})] \\ &= 2 \sum_i [\mathbb{P}(A_{ki}) - \mathbb{P}_n(A_{ki})]^+.\end{aligned}\tag{31.1}$$

Each term in the sum on the last line goes to 0 as  $n \rightarrow \infty$  by Theorem 30.2 because the  $A_{ki}$  are  $\mathbb{P}$ -continuity sets, that is,  $\mathbb{P}(\partial A_{ki}) = 0$ ; also each term is dominated by  $\mathbb{P}(A_{ki})$ , and

$\sum_i \mathbb{P}(A_{ki}) = 1$ . Therefore by dominated convergence the sum on the last line of (31.1) goes to 0.

Fix  $k$  and  $j$  and let  $\alpha, \alpha_n$  be the left-hand endpoints of  $I_{kj}, I_{kj}^n$ , respectively. Then (31.1) allows us to use dominated convergence to conclude that

$$\alpha = \sum_{i \in J} \mathbb{P}'(I_{ki}) = \lim_{n \rightarrow \infty} \sum_{i \in J} \mathbb{P}'(I_{ki}^n) = \lim_{n \rightarrow \infty} \alpha_n,$$

where  $J$  consists of those  $i$  such that  $I_{ki}$  is to the left of  $I_{kj}$ ; note that for  $i \in J$  we have that  $I_{ki}^n$  is to the left of  $I_{kj}^n$  and conversely, if  $I_{ki}^n$  is to the left of  $I_{kj}^n$ , then  $i \in J$ . Similarly the right-hand endpoint of  $I_{kj}^n$  converges to the right-hand endpoint of  $I_{kj}$ .

If  $\omega'$  is in the interior of  $I_{kj}$ , then it will be in the interior of  $I_{kj}^n$  for all sufficiently large  $n$ . This means that for  $n$  sufficiently large,

$$d(X(\omega'), X_n(\omega')) \leq 2/k.$$

This implies our result.  $\square$

## Exercises

- 31.1 Suppose  $f$  is bounded,  $X_n$  converges to  $X$  weakly, and also that  $\mathbb{P}(X \in D_f) = 0$ , where  $D_f = \{x : f \text{ is not continuous at } x\}$ . Show that  $f(X_n)$  converges weakly to  $f(X)$ .
- 31.2 Suppose a sequence  $\{X_n\}$  is uniformly integrable and  $X_n$  converges to  $X$  weakly. Show  $\mathbb{E} X_n \rightarrow \mathbb{E} X$ .
- 31.3 Give an example of a sequence of random variables  $X_n$  converging weakly to  $X$  and where each  $X_n$  is integrable, but  $X$  is not integrable.
- 31.4 Suppose  $X_n$  converges weakly to  $X$  and each  $X_n$  is non-negative. Prove that

$$\mathbb{E} X \leq \liminf_{n \rightarrow \infty} \mathbb{E} X_n.$$

- 31.5 Suppose  $X_n$  converges weakly to  $X$  and each  $X_n$  has the property that with probability one,

$$|X_n(t) - X_n(s)| \leq |t - s|, \quad s, t \leq 1.$$

(This might arise, for example, if each  $X_n$  is of the form  $X_n(t) = \int_0^t Y_n(s) ds$  and each  $Y_n$  is bounded by 1.) Prove that  $X$  has this same property, that is, with probability one,

$$|X(t) - X(s)| \leq |t - s|, \quad s, t \leq 1.$$

- 31.6 Here is a way to prove one direction of Lebesgue's theorem on Riemann integrable functions.

(1) For each  $n \geq 1$  and each  $i \leq n$ , let  $x_{in}$  be a point in  $[(i-1)/n, i/n]$ . Let  $\mathbb{P}_n$  be the probability measure that assigns mass  $1/n$  to each point  $x_{in}$ ,  $i = 1, 2, \dots, n$ . Show that  $\mathbb{P}_n$  converges weakly to  $\mathbb{P}$ , where  $\mathbb{P}$  is a Lebesgue measure on  $[0, 1]$ .

(2) Suppose  $f$  is a bounded function which is continuous at almost every point of  $[0, 1]$ . Show that  $\int f d\mathbb{P}_n \rightarrow \int f d\mathbb{P}$ . Note that  $\int f d\mathbb{P}_n$  is a Riemann sum approximation to  $\int_0^1 f(x) dx$ .

# 32

---

## The space $C[0, 1]$

We examine weak convergence for the space  $C[0, 1]$ , the set of continuous real-valued functions on  $[0, 1]$ . We give a criterion for the laws of a sequence of continuous stochastic processes to be tight. We apply these results to show that a simple symmetric random walk converges weakly to a Brownian motion, which in particular gives another construction of Brownian motion.

### 32.1 Tightness

Let  $C[0, 1]$  be the collection of continuous real-valued functions from  $[0, 1]$  into  $\mathbb{R}$ . We make  $C[0, 1]$  into a metric space by defining

$$d(f, g) = \sup_{t \in [0, 1]} |f(t) - g(t)|,$$

and it is well known that  $C[0, 1]$  is separable and complete. We recall the Ascoli–Arzelà theorem: if a family  $\mathcal{F}$  of functions on a compact set is equicontinuous and uniformly bounded at one point, then every subsequence in  $\mathcal{F}$  has a further subsequence in  $\mathcal{F}$  that converges. Rephrased another way, if the family  $\mathcal{F}$  is equicontinuous and uniformly bounded at one point, then the closure of  $\mathcal{F}$  is compact. We furnish  $C[0, 1]$  with the Borel  $\sigma$ -field.

Given a continuous function  $f$  on  $[0, 1]$ , we define  $\omega_f$ , the *modulus of continuity* of  $f$ , by

$$\omega_f(\delta) = \sup_{s, t \in [0, 1], |t-s| < \delta} |f(t) - f(s)|.$$

We have the following criterion for a sequence of continuous processes to be tight.

**Theorem 32.1** *Suppose the  $X_n$  are continuous real-valued processes. Suppose for each  $\varepsilon$  and  $\eta > 0$  there exist  $n_0$ ,  $A$ , and  $\delta$  (depending on  $\varepsilon$  and  $\eta$ ) such that if  $n \geq n_0$ , then*

$$\mathbb{P}(\omega_{X_n}(\delta) \geq \varepsilon) \leq \eta \tag{32.1}$$

and

$$\mathbb{P}(|X_n(0)| \geq A) \leq \eta. \tag{32.2}$$

Then the  $X_n$  are tight.

*Proof* Since each  $X_i$  is a continuous process, then for each  $i$ ,  $\mathbb{P}(\omega_{X_i}(\delta) \geq \varepsilon) \rightarrow 0$  as  $\delta \rightarrow 0$  by dominated convergence. Hence, given  $\varepsilon$  and  $\eta$  we can, by taking  $\delta$  smaller if necessary, assume that (32.1) holds for all  $n$ .

Choose  $\varepsilon_m = \eta_m = 2^{-m}$  and consider the  $\delta_m$  and  $A_m$  so that

$$\sup_n \mathbb{P}(\omega_{X_n}(\delta_m) \geq 2^{-m}) \leq 2^{-m}$$

and

$$\sup_n \mathbb{P}(|X_n(0)| \geq A_m) \leq 2^{-m}.$$

Let

$$K_{m_0} = \{f \in C[0, 1] : \sup_{s, t \in [0, 1], |t-s| \leq \delta_m} |f(t) - f(s)| \leq 2^{-m} \text{ for all } m \geq m_0, \\ |f(0)| \leq A_{m_0}\}.$$

Each  $K_{m_0}$  is an equicontinuous family, and by the Ascoli–Arzelá theorem, each  $K_{m_0}$  is a compact subset of  $C[0, 1]$ . We have

$$\mathbb{P}(X_n \notin K_{m_0}) \leq \mathbb{P}(|X_n(0)| \geq A_{m_0}) + \sum_{m=m_0}^{\infty} \mathbb{P}(\omega_{X_n}(\delta_m) \geq \varepsilon_m) \\ \leq 2^{-m_0} + \sum_{m=m_0}^{\infty} 2^{-m} = 3 \cdot 2^{-m_0}.$$

This proves tightness.  $\square$

We have given one criterion for a process to have continuous paths, namely, Theorem 8.1. In the case of Markov processes, we have given another: Theorem 21.5.

## 32.2 A construction of Brownian motion

We will now use the results of Section 32.1 to give a construction of Brownian motion, quite different from that of Chapter 6.

Let  $Y_i$  be i.i.d. random variables with  $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = \frac{1}{2}$ . Then  $S_n = \sum_{i=1}^n Y_i$  is a *simple symmetric random walk*. Let  $Z_n(t) = S_{nt}/\sqrt{n}$  for  $t$  a multiple of  $1/n$  and define  $Z_t^n$  by linear interpolation for other  $t$ . That is, if  $k/n \leq t \leq (k+1)/n$ , then

$$Z_t^n = \frac{(k+1)-nt}{\sqrt{n}} S_k + \frac{nt-k}{\sqrt{n}} S_{k+1}. \quad (32.3)$$

The  $Z_n$  are continuous processes. Let  $\mathbb{P}_n$  be the law of  $Z_n$ , which will be a probability measure on  $C[0, 1]$ .

**Theorem 32.2** *The sequence  $\mathbb{P}_n$  converges weakly to a probability measure  $\mathbb{P}_\infty$  on  $C[0, 1]$ , and  $\mathbb{P}_\infty$  is the law of a Brownian motion.*

*Proof* The main step is to prove that the  $\mathbb{P}_n$  are tight. We then show that any subsequential limit point is a Wiener measure, that is, the law of a Brownian motion. We can then appeal to Theorem 31.1 to obtain the process  $X$ , which will be a Brownian motion.

A computation shows that

$$\mathbb{E} S_n^4 = \sum_{i=1}^n \mathbb{E} Y_i^4 + \sum_{i \neq j} (\mathbb{E} Y_i^2)(\mathbb{E} Y_j^2) \leq cn^2, \quad (32.4)$$

since  $\mathbb{E} Y_i$  and  $\mathbb{E} Y_i^3$  are both 0, the  $Y_i$ 's are independent, and the second sum has  $n(n-1) \leq n^2$  terms.

If  $s$  and  $t$  are multiples of  $1/n$ , then

$$\begin{aligned}\mathbb{E} |Z_t - Z_s|^4 &= \frac{1}{n^2} \mathbb{E} \left( \sum_{i=ns+1}^{nt} Y_i \right)^4 = \frac{1}{n^2} \mathbb{E} \left( \sum_{i=1}^{nt-ns} Y_i \right)^4 \\ &\leq \frac{c}{n^2} n^2 |t-s|^2 \leq c|t-s|^2.\end{aligned}\quad (32.5)$$

If we tried to get by with only the second moment, we would only end up with  $c|t-s|$ , which is not good enough for Theorem 8.1.

At this point we would like to apply Theorem 32.1, but we have the technical nuisance that  $s$  and  $t$  might not be multiples of  $1/n$ . If  $|t-s| \leq 2/n$ , then by the construction of  $Z_n$  using linear interpolation and the fact that the  $Y_i$ 's are bounded by one in absolute value, we have  $|Z_n(t) - Z_n(s)| \leq c|t-s|\sqrt{n}$  and then

$$\mathbb{E} |Z_n(t) - Z_n(s)|^4 \leq c|t-s|^4 n^2 \leq c|t-s|^2. \quad (32.6)$$

Suppose  $|t-s| > 2/n$ . Let  $s'$  be the largest multiple of  $1/n$  less than or equal to  $s$  and  $t'$  the largest multiple of  $1/n$  larger than or equal to  $t$ . Using (32.5) and (32.6),

$$\begin{aligned}\mathbb{E} |Z_n(t) - Z_n(s)|^4 &\leq c\mathbb{E} |Z_n(t) - Z_n(t')|^4 + c\mathbb{E} |Z_n(t') - Z_n(s')|^4 + \mathbb{E} |Z_n(s') - Z_n(s)|^4 \\ &\leq c|t-t'|^2 + c|t'-s'|^2 + c|s'-s|^2 \\ &\leq c|t-s|^2,\end{aligned}$$

since  $|t-t'|$ ,  $|t'-s'|$ , and  $|s'-s|$  are all less than  $c|t-s|$ . Note  $Z_n(0) = 0$  for all  $n$ . We now apply Theorems 8.1 and 32.1 to obtain the tightness.

Any subsequential limit point is a probability measure on  $C[0, 1]$ , so to show that the limit is a Brownian motion, it is enough by Theorem 2.6 to show that the finite-dimensional distributions under the limit law  $\mathbb{P}_\infty$  agree with those of Brownian motion. Fix  $t$ . Then  $Z_n(t)$  differs from  $S_{[nt]}/\sqrt{n}$  by at most  $1/\sqrt{n}$ , where  $[nt]$  is the largest integer less than or equal to  $nt$ . By the central limit theorem (Theorem A.51),  $S_{[nt]}/\sqrt{[nt]}$  converges weakly (with respect to the topology of  $\mathbb{R}$ ) to a mean zero normal random variable with variance one. By Exercise 30.3,  $S_{[nt]}/\sqrt{n}$  converges weakly to a mean zero normal random variable with variance  $t$ , and by Exercise 30.2,  $Z_n(t)$  converges weakly to a mean zero normal random variable with variance  $t$ . This shows that the one-dimensional distributions of  $Z_n$  converge weakly to the one-dimensional distributions of a Brownian motion. We leave the analogous argument for the higher-dimensional distributions to the reader.  $\square$

One can also use Doob's inequalities to obtain the necessary tightness estimate. If  $s$  and  $t$  are multiples of  $1/n$ , we have

$$\begin{aligned}\mathbb{P}(\max_{ns \leq k \leq nt} |S_k - S_{ns}| > \lambda\sqrt{n}) &\leq c \frac{\mathbb{E} |S_{nt} - S_{ns}|^4}{\lambda^4 n^2} \\ &\leq c \frac{|t-s|^2}{\lambda^4}.\end{aligned}\quad (32.7)$$

### Exercises

- 32.1 The support of a measure  $\lambda$  is the smallest closed set  $F$  such that  $\lambda(F^c) = 0$ . Let  $\mathbb{P}$  be a Wiener measure on  $C[0, 1]$ , i.e., the law of a Brownian motion on  $[0, 1]$ . Use Exercise 13.4 to prove that the support of  $\mathbb{P}$  is all of  $C[0, 1]$ .
- 32.2 Let  $(\mathcal{S}, d)$  be a complete separable metric space and let  $\mathcal{R}$  be a subset of  $\mathcal{S}$ . Then  $(\mathcal{R}, d)$  is also a metric space. If  $X_n$  converges weakly to  $X$  with respect to the topology of  $(\mathcal{S}, d)$  and each  $X_n$  and  $X$  take values in  $\mathcal{R}$ , does  $X_n$  converge weakly to  $X$  with respect to the topology of  $(\mathcal{R}, d)$ ? Does the answer change if  $\mathcal{R}$  is a closed subset of  $\mathcal{S}$ ?  
If  $X_n$  and  $X$  take values in  $\mathcal{R}$  and  $X_n$  converges weakly to  $X$  with respect to the topology of  $(\mathcal{R}, d)$ , does  $X_n$  converge weakly to  $X$  with respect to the topology of  $(\mathcal{S}, d)$ ? What if  $\mathcal{R}$  is a closed subset of  $\mathcal{S}$ ?
- 32.3 Give a proof of Theorem 32.2 using (32.7) in place of Theorem 8.1.
- 32.4 Suppose  $(X, W, \mathbb{P})$  is a weak solution to

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt, \quad X_0 = x, \quad (32.8)$$

where  $W$  is a one-dimensional Brownian motion and  $\sigma$  and  $b$  are bounded and continuous, but we do not assume that  $\sigma$  is bounded below by a positive constant. Suppose the solution to (32.8) is unique in law.

Suppose  $\sigma_n$  and  $b_n$  are Lipschitz functions which are uniformly bounded and which converge uniformly to  $\sigma$  and  $b$ , respectively. Let  $X_t(n)$  be the unique pathwise solution to

$$dY_t = \sigma_n(Y_t) dW_t + b_n(Y_t) dt, \quad Y_0 = x;$$

the probability measure here is  $\mathbb{P}$ . Prove that  $X_t(n)$  converges weakly to  $X$  with respect to  $C[0, 1]$ .

- 32.5 Let  $W$  be a  $d$ -dimensional Brownian motion and let  $\{X_t, t \in [0, 1]\}$  be the solution to (24.22). If  $x \in \mathbb{R}^d$ , prove that the support of  $\mathbb{P}^x$  is all of  $C[0, 1]$ .

## Gaussian processes

A Gaussian process is a stochastic process where each of the finite-dimensional distributions is jointly normal. We will primarily, but not exclusively, be concerned with Gaussian processes that have continuous paths. For much of what we consider, it is not essential that the index set of times be  $[0, \infty)$ , and can in fact be almost any set. We will thus consider  $\{X_t : t \in T\}$  for some index set  $T$ , and where for every finite subset  $S$  of  $T$ , the collection  $\{X_s : s \in S\}$  is jointly normal.

### 33.1 Reproducing kernel Hilbert spaces

We define the covariance function  $\Gamma$  by

$$\Gamma(s, t) = \mathbb{E}[(X_s - \mathbb{E}X_s)(X_t - \mathbb{E}X_t)], \quad s, t \in T. \quad (33.1)$$

For our purposes, having a non-zero mean just complicates formulas without adding anything interesting, so in this chapter we will assume  $\mathbb{E}X_t = 0$  for all  $t \in T$ , and (33.1) becomes

$$\Gamma(s, t) = \mathbb{E}[X_s X_t], \quad s, t \in T. \quad (33.2)$$

We first show how  $\Gamma$  can be used to construct a Hilbert space called the reproducing kernel Hilbert space (RKHS).

When we write  $\Gamma(s, \cdot)$ , we mean that we fix an element  $s \in T$  and then consider the function  $g : T \rightarrow \mathbb{R}$  defined by  $g(t) = \Gamma(s, t)$  for  $t \in T$ . Let  $\mathcal{K}$  be the collection of finite linear combinations of the functions  $\Gamma(s, \cdot)$ ,  $s \in T$ . Thus each element of  $\mathcal{K}$  has the form

$$\sum_{j=1}^m a_j \Gamma(s_j, \cdot),$$

where  $m \geq 1$ , the  $a_j$ 's are real, and each  $s_j$ ,  $j = 1, \dots, m$ , is an element of  $T$ . If  $f = \sum_{j=1}^m a_j \Gamma(s_j, \cdot)$  and  $g = \sum_{k=1}^n b_k \Gamma(t_k, \cdot)$ , define

$$\langle f, g \rangle_{RKHS} = \sum_{j=1}^m \sum_{k=1}^n a_j b_k \Gamma(s_j, t_k).$$

We define  $\mathcal{H}$  to be the closure of  $\mathcal{K}$  with respect to the norm induced by the inner product  $\langle \cdot, \cdot \rangle_{RKHS}$ .

We need to show that this bilinear form is indeed an inner product, that what is known as the reproducing property holds, and that  $\mathcal{H}$  is a Hilbert space.

We start with the reproducing property. If  $f = \sum_{j=1}^m a_j \Gamma(s_j, \cdot)$ , then the *reproducing property* applied to  $f$  is the formula

$$\langle f, \Gamma(t, \cdot) \rangle_{RKHS} = f(t). \quad (33.3)$$

This follows from

$$\langle f, \Gamma(t, \cdot) \rangle_{RKHS} = \sum_{j=1}^m a_j \Gamma(s_j, t) = f(t).$$

By taking limits, (33.3) holds for all  $f \in \mathcal{H}$ .

To show that  $\langle \cdot, \cdot \rangle_{RKHS}$  is an inner product, notice that when

$$f = \sum a_j \Gamma(s_j, \cdot) \in \mathcal{K},$$

then

$$\begin{aligned} \langle f, f \rangle_{RKHS} &= \sum_{j=1}^m \sum_{k=1}^m a_j a_k \Gamma(s_j, s_k) = \sum_{j,k=1}^m a_j a_k \mathbb{E}[X_{s_j} X_{s_k}] \\ &= \mathbb{E} \left( \sum_{j=1}^m a_j X_{s_j} \right)^2 \geq 0. \end{aligned}$$

The Cauchy–Schwarz inequality holds for  $\langle \cdot, \cdot \rangle_{RKHS}$  (the standard proof of the Cauchy–Schwarz inequality applies), and so if  $\langle f, f \rangle_{RKHS} = 0$ , then

$$|f(t)|^2 = \langle f, \Gamma(t, \cdot) \rangle_{RKHS}^2 \leq \langle f, f \rangle_{RKHS} \langle \Gamma(t, \cdot), \Gamma(t, \cdot) \rangle_{RKHS} = 0,$$

and thus  $f$  is zero.

If  $f_n$  is a Cauchy sequence with respect to the norm

$$\|g\|_{RKHS} = \langle g, g \rangle_{RKHS}^{1/2},$$

then

$$\begin{aligned} |f_n(t) - f_m(t)|^2 &= \langle f_n - f_m, \Gamma(t, \cdot) \rangle_{RKHS}^2 \\ &\leq \langle f_n - f_m, f_n - f_m \rangle_{RKHS} \langle \Gamma(t, \cdot), \Gamma(t, \cdot) \rangle_{RKHS}, \end{aligned}$$

which tends to 0 as  $n, m \rightarrow \infty$ . Thus  $f_n$  converges pointwise. This is enough to prove  $\mathcal{H}$  is complete; this is Exercise 33.1.

We summarize.

**Proposition 33.1**  $\mathcal{H}$  with the inner product  $\langle \cdot, \cdot \rangle_{RKHS}$  is a Hilbert space. Moreover, if  $f \in \mathcal{H}$  and  $t \in T$ , then

$$\langle f, \Gamma(t, \cdot) \rangle_{RKHS} = f(t).$$

We consider another Hilbert space  $\mathcal{M}$ , the closure of the linear span of  $\{X_t : t \in T\}$  with respect to  $L^2(\mathbb{P})$ . We define

$$\langle Y, Z \rangle_{\mathcal{M}} = \mathbb{E}[YZ]$$

if  $Y$  and  $Z$  are both finite linear combinations of the  $X_t$ 's. Thus if  $m, n \geq 1$ ,  $a_j, b_k \in \mathbb{R}$ , we set

$$\left\langle \sum_{j=1}^m a_j X_{s_j}, \sum_{k=1}^n b_k X_{t_k} \right\rangle_{\mathcal{M}} = \sum_{j=1}^m \sum_{k=1}^n a_j b_k \mathbb{E}[X_{s_j} X_{t_k}], \quad (33.4)$$

and we let  $\mathcal{M}$  be the closure of the collection of random variables of the form  $\sum_{j=1}^m a_j X_{s_j}$  with respect to  $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ . Since  $\Gamma(s_j, t_k) = \mathbb{E}[X_{s_j} X_{t_k}]$ , from (33.4) we see that  $\mathcal{H}$  and  $\mathcal{M}$  are isomorphic, where we have a one-to-one correspondence between  $\sum_{j=1}^m a_j \Gamma(s_j, \cdot)$  and  $\sum_{j=1}^m a_j X_{s_j}$ .

Let  $\{e_n\}$  be a complete orthonormal system for  $\mathcal{H}$ . Let  $Y_n$  be the element of  $\mathcal{M}$  corresponding to  $e_n$ . Then

$$\mathbb{E}[Y_n Y_m] = \langle Y_n, Y_m \rangle_{\mathcal{M}} = \langle e_n, e_m \rangle_{RKHS} = \delta_{nm},$$

where  $\delta_{nm}$  is 0 if  $n \neq m$  and 1 if  $n = m$ . This implies that the  $Y_n$  are independent normal random variables with mean zero and variance one; see Proposition A.55. (Recall that we are assuming that all the  $X_t$ 's have mean zero.)

Since  $\Gamma(s, \cdot)$  is an element of  $\mathcal{H}$ , we can write

$$\Gamma(s, \cdot) = \sum_{n=1}^{\infty} \langle \Gamma(s, \cdot), e_n \rangle_{RKHS} e_n(\cdot) = \sum_{n=1}^{\infty} e_n(s) e_n(\cdot).$$

Using the correspondence between  $\mathcal{H}$  and  $\mathcal{M}$ , we have

$$X_s = \sum_{n=1}^{\infty} e_n(s) Y_n,$$

where the  $Y_n$  are i.i.d. standard normal variables. This is known as the *Karhunen–Loève expansion* of a Gaussian process.

**Example 33.2** Let's see what this expansion is in the case of Brownian motion. If we define

$$\langle f, g \rangle_{CM} = \int_0^1 f'(r) g'(r) dr \quad (33.5)$$

for  $f$  and  $g$  whose first derivatives are in  $L^2([0, 1])$  and such that  $f(0) = g(0) = 0$ , then because  $\Gamma(s, t) = s \wedge t$ ,

$$\begin{aligned} \langle \Gamma(s, \cdot), \Gamma(t, \cdot) \rangle_{CM} &= \int_0^1 1_{[0,s]}(r) 1_{[0,t]}(r) dr = s \wedge t \\ &= \Gamma(s, t), \end{aligned}$$

and we see that we have identified the reproducing kernel Hilbert space for Brownian motion on  $[0, 1]$ . The notation  $\langle \cdot, \cdot \rangle_{CM}$  is used because the Hilbert space with this inner product is called the *Cameron–Martin space*, a space that has many connections with Brownian motion.

If  $e_n(s) = \sqrt{2} \sin(n\pi s)/n\pi$ , then the sequence  $\{e_n\}$  is a complete orthonormal sequence for the Cameron–Martin space. The Karhunen–Loève expansion is equivalent to the formula (6.2) that we used in our first construction of Brownian motion.

### 33.2 Continuous Gaussian processes

We now turn to the construction of Gaussian processes with continuous paths. Suppose we have an index set  $T$  and a non-negative definite kernel  $\Gamma(\cdot, \cdot)$ . Saying  $\Gamma$  is *non-negative definite* means that for each  $n$  and each  $t_1, \dots, t_n \in T$ , the matrix whose  $(i, j)$  entry is  $\Gamma(t_i, t_j)$  is a non-negative definite matrix. We define a metric on  $T$  by defining

$$d(s, t) = (\text{Var}(X_t - X_s))^{1/2}.$$

Actually,  $d$  is a pseudo-metric because  $d(s, t) = 0$  does not necessarily imply  $t = s$ . An  $\varepsilon$ -ball is a set of the form  $\{t \in T : d(t, t_0) < \varepsilon\}$  for some  $t_0$ . Let  $N(\varepsilon)$  be the minimum number of  $\varepsilon$ -balls needed to cover  $T$ .

**Theorem 33.3** *Let  $\Gamma : T \times T \rightarrow \mathbb{R}$  be continuous with respect to the pseudo-metric  $d$ , symmetric, and non-negative definite. If for some  $\beta < 1$  and some constant  $c$  we have*

$$\log N(\varepsilon) \leq c\varepsilon^{-\beta}, \quad \varepsilon \in (0, 1), \tag{33.6}$$

*then there exists a continuous Gaussian process  $\{X_t : t \in T\}$  with covariance kernel  $\Gamma$ .*

One can in fact be more precise than (33.6) and give an integral condition that  $N(x)$  must satisfy for  $x$  small.

Before proving Theorem 33.3, let us look at a number of examples.

**Example 33.4** In the case of Brownian motion,  $\text{Var}(X_t - X_s) = |t - s|$ , so that  $d(s, t) = |s - t|^{1/2}$ . If  $T$  is the interval  $[0, 1]$ , then the set of intervals of length  $\varepsilon^2$  and centers  $k\varepsilon^2/4$ ,  $k = 0, 1, \dots, 4/\varepsilon^2$ , is a collection of  $\varepsilon$ -balls covering  $[0, 1]$ . Therefore  $N(\varepsilon) \leq c/\varepsilon^2$ , implying  $\log N(\varepsilon) \leq c \log(1/\varepsilon)$ , which satisfies (33.6). This and Theorem 2.4 gives a construction of Brownian motion.

**Example 33.5** We look at *fractional Brownian motion*. Let  $H \in (0, 2)$ .  $H$  is known as the *Hurst index*, where  $H = 1$  corresponds to Brownian motion. Define

$$\Gamma(s, t) = |s|^H + |t|^H - |s - t|^H.$$

This leads to  $d(s, t) = c|t - s|^{H/2}$ . Open intervals of length  $\varepsilon^{2/H}$  are  $\varepsilon$ -balls, and it takes  $c\varepsilon^{-2/H}$  of them to cover  $[0, 1]$ . Therefore again  $N(\varepsilon) \leq c \log(1/\varepsilon)$ , and (33.6) applies. One use of fractional Brownian motion is to model stock prices where there is more or less memory of the past than a Brownian motion has.

**Example 33.6** Here is our first example of a Gaussian process where  $T$  is not a subset of  $[0, \infty)$ . We construct a *Brownian sheet*,  $X(t_1, t_2)$ , where the points  $(t_1, t_2) \in [0, 1]^2$ . More generally we can consider  $X(t)$ , where  $t \in [0, 1]^d$ . This is no harder, but for simplicity of notation we consider only the case  $d = 2$ . If  $s = (s_1, s_2)$  and  $t = (t_1, t_2)$ , define

$$\Gamma(s, t) = (s_1 \wedge t_1)(s_2 \wedge t_2).$$

One motivation for this formula is to identify the point  $(t_1, t_2)$  with the rectangle  $R_t$  whose lower left corner is at the origin and whose upper right corner is at  $(t_1, t_2)$ . Then the covariance of  $X_s$  and  $X_t$  is the area of  $R_s \cap R_t$ .

Some simple geometry shows that if we put  $\varepsilon$ -balls centered at the points  $(c_1 j \varepsilon^2, c_1 k \varepsilon^2)$  for an appropriate  $c_1$  and with  $j, k \leq c_2 \varepsilon^{-2}$ , we cover  $T$ . Therefore  $N(\varepsilon) \leq c\varepsilon^{-4}$ , and so  $\log N(\varepsilon) \leq c \log(1/\varepsilon)$ .

**Example 33.7** We can generalize the last example. For every Borel subset  $A$  of  $[0, 1]^d$ , let  $X_A$  be a Gaussian random variable. We want the covariance of  $X_A$  and  $X_B$  to be the Lebesgue measure of  $A \cap B$ . This is known as a *set-indexed process*. If we let  $T$  be the collection of all Borel subsets of  $[0, 1]^d$ , one cannot get a continuous Gaussian process. In order to get a continuous process  $X$  one must restrict  $T$  to be a subcollection of sets whose boundaries are sufficiently smooth; see [Dudley \(1973\)](#).

**Example 33.8** Our last example has a more complicated index set. Let  $W$  be a one-dimensional Brownian motion. If  $f \in L^2[0, 1]$ , define

$$X_f = \int_0^1 f(s) dW_s.$$

By Exercise 24.6,  $X_f$  is a Gaussian random variable with mean 0 and variance  $\int_0^1 f(s)^2 ds$  and the covariance of  $X_f$  and  $X_g$  is  $\int_0^1 f(s)g(s) ds$ . It follows that

$$d(f, g)^2 = \int_0^1 (f(s) - g(s))^2 ds.$$

The process  $X_f$  is known as a *Gaussian field*.

For what subsets  $T$  of  $L^2([0, 1])$  can one define a process  $X_f$  that has continuous paths with respect to  $d$ ? This means that the map  $f \rightarrow X_f(\omega)$  is continuous for almost all  $\omega$ , where we use the pseudo-metric  $d$  to define open sets in  $T$ . It turns out  $T = \{f \in L^2([0, 1]) : \|f\|_2 \leq 1\}$  is too large to obtain a continuous Gaussian process, but, for example,  $T = \{f \in C^2([0, 1]) : \|f\|_\infty \leq 1, \|f'\|_\infty \leq 1, \|f''\|_\infty \leq 1\}$  is small enough to apply Theorem 33.3.

We now proceed to the proof of Theorem 33.3.

*Proof of Theorem 33.3* Since  $T$  can be covered by finitely many  $\varepsilon$ -balls for each  $\varepsilon$ , it follows that if  $\mathcal{A}(\varepsilon)$  is the collection of centers for the cover by  $\varepsilon$ -balls, then  $\mathcal{A} = \cup_{n=1}^\infty \mathcal{A}(2^{-n})$  is a countable dense subset of  $T$ . We first label the elements of  $\mathcal{A}$  by  $t_1, t_2, \dots$ . For each  $n$ , we construct the law of  $(X_{t_1}, \dots, X_{t_n})$ . We then use the Kolmogorov extension theorem to construct the law of  $\{X_t : t \in \mathcal{A}\}$ . Next we prove that  $t \rightarrow X_t$  is uniformly continuous on  $\mathcal{A}$ , almost surely. Finally we define  $X_t$  for all  $t \in T$  by continuity.

*Step 1.* We construct the law of  $(X_{t_1}, \dots, X_{t_n})$ . Let  $n$  be fixed, and let  $B$  be an  $n \times n$  matrix whose  $(i, j)$  entry is  $\Gamma(t_i, t_j)$ . The matrix  $B$  is symmetric, and non-negative definite by hypothesis. Let  $Y_1, \dots, Y_n$  be independent normal random variables with mean zero and variance one. If we let  $C$  be the non-negative definite square root of  $B$  and

$$X = CY$$

(viewed as vectors), or equivalently,

$$X_{t_i} = \sum_{j=1}^n C_{ij} Y_j,$$

a simple calculation shows that  $\mathbb{E}[X_{t_k} X_{t_m}] = B_{km} = \Gamma(t_k, t_m)$ . The  $X_{t_j}$ 's are jointly normal and this gives the first step of the construction.

*Step 2.* We apply the Kolmogorov extension theorem. Let  $\mathbb{P}_n$  be the law of  $(X_{t_1}, \dots, X_{t_n})$ . It is easy to see the consistency property holds for the  $\mathbb{P}_n$ , so by the Kolmogorov extension theorem, there exists a probability  $\mathbb{P}$  on  $\mathbb{R}^{\mathbb{N}}$  such that if we define  $X_t(\omega)$  by  $\omega(t)$  for  $t \in \mathcal{A}$ , the law of  $(X_{t_1}, \dots, X_{t_n})$  is  $\mathbb{P}_n$  for each  $n$ .

*Step 3.* We show that except for a null set of probability zero, the map  $t \rightarrow X_t(\omega)$  is uniformly continuous on  $\mathcal{A}$ .

To prove the uniform continuity, we proceed similarly to Theorem 8.1. For each point  $t \in \mathcal{A}$ , let  $t_j$  be the element of  $\mathcal{A}(2^{-j})$  closest to  $t$ , with some convention for breaking ties. We will fix  $J$  in a moment, and write

$$X_t = X_{t_j} + (X_{t_{j+1}} - X_{t_j}) + (X_{t_{j+2}} - X_{t_{j+1}}) + \dots,$$

where the sum is finite because  $t \in \mathcal{A}$ . Let  $\lambda > 0$ . If  $|X_t - X_s| > \lambda$  for some  $s, t \in \mathcal{A}$  with  $d(s, t) < 2^{-J}$ , then  $\omega$  is in one or more of the following events:

(a) the event

$$E_J = \{|X_{t_j} - X_{s_J}| > \lambda/2 \text{ for some } s_J, t_J \in \mathcal{A}(2^{-J}) \text{ with } d(s_J, t_J) \leq 3 \cdot 2^{-J}\};$$

(b) the event

$$F_j = \left\{ |X_{t_{j+1}} - X_{t_j}| > \frac{\lambda}{8j^2} \text{ for some } t_j \in \mathcal{A}(2^{-j}), t_{j+1} \in \mathcal{A}(2^{-(j+1)}) \right. \\ \left. \text{with } d(t_j, t_{j+1}) < 3 \cdot 2^{-j+1} \right\}$$

for some  $j \geq J$ ;

(c) the event

$$G_j = \left\{ |X_{s_{j+1}} - X_{s_j}| > \frac{\lambda}{8j^2} \text{ for some } s_j \in \mathcal{A}(2^{-j}), s_{j+1} \in \mathcal{A}(2^{-(j+1)}) \right. \\ \left. \text{with } d(s_j, s_{j+1}) < 3 \cdot 2^{-j+1} \right\}$$

for some  $j \geq J$ .

First we bound the probability of  $E_J$ . There are  $N(2^{-J})$  elements of  $\mathcal{A}(2^{-J})$ , so there are at most  $\exp(2c(2^J)^{\beta})$  pairs  $(s_J, t_J)$ . If  $d(t_J, s_J) < 3 \cdot 2^{-J}$ , then

$$\mathbb{P}(|X_{s_J} - X_{t_J}| > \lambda/2) \leq 2 \exp\left(-\frac{(\lambda/2)^2}{2 \cdot 3 \cdot 2^{-J}}\right).$$

Therefore the probability of  $E_J$  is bounded by

$$\mathbb{P}(E_J) \leq e^{c2^{\beta J}} e^{-c\lambda^2 2^J}.$$

Since  $\beta < 1$ , this can be made as small as we like by taking  $J$  large enough.

For any  $t_j$  and  $t_{j+1}$  with  $d(t_j, t_{j+1}) < 3 \cdot 2^{-j+1}$ ,

$$\mathbb{P}(|X_{t_j} - X_{t_{j+1}}| > \lambda/(8j^2)) \leq 2 \exp\left(\frac{\lambda^2/64j^4}{6 \cdot 2^{-j+1}}\right).$$

There are less than  $e^{c2^{\beta j}}$  points in  $\mathcal{A}(2^{-j})$  and  $e^{c2^{\beta(j+1)}}$  points in  $\mathcal{A}(2^{-(j+1)})$ , so less than  $e^{c2^{\beta j}}$  pairs. Thus the probability of  $F_j$  is bounded by

$$\mathbb{P}(F_j) \leq ce^{c2^{\beta j}} e^{-c\lambda^2 2^j/j^4}.$$

Since  $\beta < 1$ , this is summable in  $j$ , and  $\sum_{j=J}^{\infty} \mathbb{P}(F_j)$  can be made as small as we like if we take  $J$  large enough. We handle the bound for  $G_j$  similarly.

Thus, given  $\varepsilon$ , we have

$$\mathbb{P}\left(\sup_{s,t \in \mathcal{A}, d(s,t) < 2^{-J}} |X_t - X_s| > \lambda\right) \leq \varepsilon$$

if we take  $J$  large enough, where  $J$  depends on  $\varepsilon$  and  $\lambda$ . This suffices to prove the uniform continuity.

*Step 4.* We use continuity to complete the proof. Define  $X_t = \lim_{s \in \mathcal{A}, s \rightarrow t} X_s$ . The limit exists and will be a continuous function of  $t$  by virtue of the uniform continuity. By Remark A.56,  $X_t$  will have the desired covariance function.  $\square$

We have been considering Gaussian processes taking values in  $\mathbb{R}$ , but it is also of interest to look at Brownian motion taking values in a Hilbert space or a Banach space. There are three steps to constructing such a process:

- (1) constructing Gaussian measures on Banach (or Hilbert) spaces;
- (2) getting a suitable estimate on  $\|X_t - X_s\|$ ;
- (3) constructing a Brownian motion.

Of these three steps, the third follows along the lines we used for real-valued processes. Steps (1) and (2) require considerable work, and we refer the reader to [Bogachev \(1998\)](#) or [Kuo \(1975\)](#). A measure  $\mu$  on a Banach space is called Gaussian if  $\mu \circ L^{-1}$  is a Gaussian measure on  $\mathbb{R}$  for every linear functional  $L$  on the Banach space.

## Exercises

33.1 Finish the proof that  $\mathcal{H}$  as defined in Section 33.1 is complete.

33.2 Show that if in Example 33.8 we let

$$T = \{f \in C^1([0, 1]); \|f\|_\infty \leq 1, \|f'\|_\infty \leq 1\},$$

then  $N(\varepsilon)$  is bounded above by  $c_1 \varepsilon^{-1}$  and bounded below by  $c_2 \varepsilon^{-1}$ .

33.3 Suppose  $X^i$  and  $Y^i$  are two sequences of Brownian motions with all of the Brownian motions independent of each other. Let

$$Z_{(s,t)}^n = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_s^i Y_t^i.$$

Prove that  $Z^n$  converges weakly with respect to the topology of  $C([0, 1]^2)$  as  $n \rightarrow \infty$  to a Brownian sheet.

- 33.4 Let  $X$  be a Brownian bridge. (This will be studied further in Section 35.2.) This means that  $X$  is a mean zero Gaussian process with

$$\text{Cov}(X_s, X_t) = s \wedge t - st, \quad 0 \leq s, t \leq 1.$$

Identify the reproducing kernel Hilbert space for  $X$ .

- 33.5 Let  $X$  be the Ornstein–Uhlenbeck process started at 0. This was defined in Exercise 19.5. Identify the reproducing kernel Hilbert space for  $X$ .

# 34

---

## The space $D[0, 1]$

We define the space  $D[0, 1]$  to be the collection of real-valued functions on  $[0, 1]$  which are right continuous with left limits. We will introduce a topology on  $D = D[0, 1]$ , the Skorokhod topology, which makes  $D$  into a complete separable metric space. We will give a criterion for a subset of  $D$  to be compact, which will lead to some criteria for a family of probability measures on  $D$  to be tight.

### 34.1 Metrics for $D[0, 1]$

We write  $f(t-)$  for  $\lim_{s < t, s \rightarrow t} f(s)$ . We will need the following observation. If  $f$  is in  $D$  and  $\varepsilon > 0$ , let  $t_0 = 0$ , and for  $i > 0$  let  $t_{i+1} = \inf\{t > t_i : |f(t) - f(t_i)| > \varepsilon\} \wedge 1$ . Because  $f$  is right continuous with left limits, then from some  $i$  on,  $t_i$  must be equal to 1.

Our first try at a metric,  $\rho$ , makes  $D$  into a separable metric space, but one that is not complete. Let's start with  $\rho$  anyway, since we need it on the way to the metric  $d$  we end up with.

Let  $\Lambda$  be the set of functions  $\lambda$  from  $[0, 1]$  to  $[0, 1]$  that are continuous, strictly increasing, and such that  $\lambda(0) = 0$ ,  $\lambda(1) = 1$ . Define

$$\begin{aligned} \rho(f, g) = \inf\{\varepsilon > 0 : \exists \lambda \in \Lambda \text{ such that } \sup_{t \in [0, 1]} |\lambda(t) - t| < \varepsilon, \\ \sup_{t \in [0, 1]} |f(t) - g(\lambda(t))| < \varepsilon\}. \end{aligned}$$

Since the function  $\lambda(t) = t$  is in  $\Lambda$ , then  $\rho(f, g)$  is finite if  $f, g \in D$ . Clearly  $\rho(f, g) \geq 0$ . If  $\rho(f, g) = 0$ , then either  $f(t) = g(t)$  or else  $f(t) = g(t-)$  for each  $t$ ; since elements of  $D$  are right continuous with left limits, it follows that  $f = g$ . If  $\lambda \in \Lambda$ , then so is  $\lambda^{-1}$  and we have, setting  $s = \lambda^{-1}(t)$  and noting both  $s$  and  $t$  range over  $[0, 1]$ ,

$$\sup_{t \in [0, 1]} |\lambda^{-1}(t) - t| = \sup_{s \in [0, 1]} |s - \lambda(s)|$$

and

$$\sup_{t \in [0, 1]} |f(\lambda^{-1}(t)) - g(t)| = \sup_{s \in [0, 1]} |f(s) - g(\lambda(s))|,$$

and we conclude  $\rho(f, g) = \rho(g, f)$ . The triangle inequality follows from

$$\sup_{t \in [0, 1]} |\lambda_2 \circ \lambda_1(t) - t| \leq \sup_{t \in [0, 1]} |\lambda_1(t) - t| + \sup_{s \in [0, 1]} |\lambda_2(s) - s|$$

and

$$\begin{aligned} \sup_{t \in [0,1]} |f(t) - h(\lambda_2 \circ \lambda_1(t))| &\leq \sup_{t \in [0,1]} |f(t) - g(\lambda_1(t))| \\ &\quad + \sup_{s \in [0,1]} |g(s) - h(\lambda_2(s))|. \end{aligned}$$

Look at the set of  $f$  in  $D$  for which there exists an integer  $k$  such that  $f$  is constant and equal to a rational on each interval  $[(i-1)/k, i/k]$ . It is not hard to check (Exercise 34.1) that the collection of such  $f$ 's is dense in  $D$  with respect to  $\rho$ , which shows  $(D, \rho)$  is separable.

The space  $D$  with the metric  $\rho$  is not, however, complete; see Exercise 34.2. We therefore introduce a slightly different metric  $d$ . Define

$$\|\lambda\| = \sup_{s \neq t, s, t \in [0,1]} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right|$$

and let

$$d(f, g) = \inf\{\varepsilon > 0 : \exists \lambda \in \Lambda \text{ such that } \|\lambda\| \leq \varepsilon, \sup_{t \in [0,1]} |f(t) - g(\lambda(t))| \leq \varepsilon\}.$$

Note  $\|\lambda^{-1}\| = \|\lambda\|$  and  $\|\lambda_2 \circ \lambda_1\| \leq \|\lambda_1\| + \|\lambda_2\|$ . The symmetry of  $d$  and the triangle inequality follow easily from this, and we conclude  $d$  is a metric.

**Lemma 34.1** *There exists  $\varepsilon_0$  such that*

$$\rho(f, g) \leq 2d(f, g)$$

if  $d(f, g) < \varepsilon_0$ .

(It turns out  $\varepsilon_0 = 1/4$  will do.)

*Proof* Since  $\log(1 + 2x)/(2x) \rightarrow 1$  as  $x \rightarrow 0$ , we have

$$\log(1 - 2\varepsilon) < -\varepsilon < \varepsilon < \log(1 + 2\varepsilon)$$

if  $\varepsilon$  is small enough. Suppose  $d(f, g) < \varepsilon$  and  $\lambda$  is the element of  $\Lambda$  such that  $d(f, g) < \|\lambda\| < \varepsilon$  and  $\sup_{t \in [0,1]} |f(t) - g(\lambda(t))| < \varepsilon$ . Since  $\lambda(0) = 0$ , we have

$$\log(1 - 2\varepsilon) < -\varepsilon < \log \frac{\lambda(t)}{t} < \varepsilon < \log(1 + 2\varepsilon), \quad (34.1)$$

or

$$1 - 2\varepsilon < \frac{\lambda(t)}{t} < 1 + 2\varepsilon, \quad (34.2)$$

which implies  $|\lambda(t) - t| < 2\varepsilon$ , and hence  $\rho(f, g) \leq 2d(f, g)$ .  $\square$

We define the analog  $\xi_f$  of the modulus of continuity for a function in  $D$  as follows. Define  $\theta_f[a, b] = \sup_{s, t \in [a, b]} |f(t) - f(s)|$  and

$$\begin{aligned} \xi_f(\delta) &= \inf\{ \max_{1 \leq i \leq n} \theta_f[t_{i-1}, t_i] : \exists n \geq 1, 0 = t_0 < t_1 < \dots < t_n = 1 \\ &\quad \text{such that } t_i - t_{i-1} > \delta \text{ for all } i \leq n \}. \end{aligned}$$

Observe that if  $f \in D$ , then  $\xi_f(\delta) \downarrow 0$  as  $\delta \downarrow 0$ .

**Lemma 34.2** Suppose  $\delta < 1/4$ . Let  $f \in D$ . If  $\rho(f, g) \leq \delta^2$ , then  $d(f, g) \leq 4\delta + \xi_f(\delta)$ .

*Proof* Choose  $t_i$ 's such that  $t_i - t_{i-1} > \delta$  and  $\theta_f[t_{i-1}, t_i] < \xi_f(\delta) + \delta$  for each  $i$ . Pick  $\mu \in \Lambda$  such that  $\sup_t |f(t) - g(\mu(t))| < \delta^2$  and  $\sup_t |\mu(t) - t| < \delta^2$ . Then  $\sup_t |f(\mu^{-1}(t)) - g(t)| < \delta^2$ . Set  $\lambda(t_i) = \mu(t_i)$  and let  $\lambda$  be linear in between. Since  $\mu^{-1}(\lambda(t_i)) = t_i$  for all  $i$ , then  $t$  and  $\mu^{-1} \circ \lambda(t)$  always lie in the same subinterval  $[t_{i-1}, t_i]$ . Consequently

$$\begin{aligned} |f(t) - g(\lambda(t))| &\leq |f(t) - f(\mu^{-1}(\lambda(t)))| + |f(\mu^{-1}(\lambda(t))) - g(\lambda(t))| \\ &\leq \theta_f(\delta) + \delta^2 \\ &\leq \xi_f(\delta) + \delta + \delta^2 < \xi_f(\delta) + 4\delta. \end{aligned}$$

We have

$$\begin{aligned} |\lambda(t_i) - \lambda(t_{i-1}) - (t_i - t_{i-1})| &= |\mu(t_i) - \mu(t_{i-1}) - (t_i - t_{i-1})| \\ &\leq 2\delta^2 < 2\delta(t_i - t_{i-1}). \end{aligned}$$

Since  $\lambda$  is defined by linear interpolation,

$$|\lambda(t) - \lambda(s) - (t - s)| \leq 2\delta|t - s|, \quad s, t \in [0, 1],$$

which leads to

$$\left| \frac{\lambda(t) - \lambda(s)}{t - s} - 1 \right| \leq 2\delta,$$

or

$$\log(1 - 2\delta) \leq \log\left(\frac{\lambda(t) - \lambda(s)}{t - s}\right) \leq \log(1 + 2\delta).$$

Since  $\delta < \frac{1}{4}$ , we have  $\|\lambda\| \leq 4\delta$ . □

**Proposition 34.3** The metrics  $d$  and  $\rho$  are equivalent, i.e., they generate the same topology.

In particular,  $(D, d)$  is separable.

*Proof* Let  $B_\rho(f, r)$  denote the ball with center  $f$  and radius  $r$  with respect to the metric  $\rho$  and define  $B_d(f, r)$  analogously. Let  $\varepsilon > 0$  and let  $f \in D$ . If  $d(f, g) < \varepsilon/2$  and  $\varepsilon$  is small enough, then  $\rho(f, g) \leq 2d(f, g) < \varepsilon$ , and so  $B_d(f, \varepsilon/2) \subset B_\rho(f, \varepsilon)$ .

To go the other direction, what we must show is that given  $f$  and  $\varepsilon$ , there exists  $\delta$  such that  $B_\rho(f, \delta) \subset B_d(f, \varepsilon)$ .  $\delta$  may depend on  $f$ ; in fact, it has to in general, for otherwise a Cauchy sequence with respect to  $d$  would be a Cauchy sequence with respect to  $\rho$ , and vice versa. Choose  $\delta$  small enough that  $4\delta^{1/2} + \xi_f(\delta^{1/2}) < \varepsilon$ . By Lemma 34.2, if  $\rho(f, g) < \delta$ , then  $d(f, g) < \varepsilon$ , which is what we want.

Finally, suppose  $G$  is open with respect to the topology generated by  $\rho$ . For each  $f \in G$ , let  $r_f$  be chosen so that  $B_\rho(f, r_f) \subset G$ . Hence  $G = \bigcup_{f \in G} B_\rho(f, r_f)$ . Let  $s_f$  be chosen so that  $B_d(f, s_f) \subset B_\rho(f, r_f)$ . Then  $\bigcup_{f \in G} B_d(f, s_f) \subset G$ , and in fact the sets are equal because if  $f \in G$ , then  $f \in B_d(f, s_f)$ . Since  $G$  can be written as the union of balls which are open with respect to  $d$ , then  $G$  is open with respect to  $d$ . The same argument with  $d$  and  $\rho$  interchanged shows that a set that is open with respect to  $d$  is open with respect to  $\rho$ . □

### 34.2 Compactness and completeness

We now show completeness for  $(D, d)$ .

**Theorem 34.4** *The space  $D$  with the metric  $d$  is complete.*

*Proof* Let  $f_n$  be a Cauchy sequence with respect to the metric  $d$ . If we can find a subsequence  $n_j$  such that  $f_{n_j}$  converges, say, to  $f$ , then it is standard that the whole sequence converges to  $f$ . Choose  $n_j$  such that  $d(f_{n_j}, f_{n_{j+1}}) < 2^{-j}$ . For each  $j$  there exists  $\lambda_j$  such that

$$\sup_t |f_{n_j}(t) - f_{n_{j+1}}(\lambda_j(t))| \leq 2^{-j}, \quad \|\lambda_j\| \leq 2^{-j}.$$

As in (34.1) and (34.2),

$$|\lambda_j(t) - t| \leq 2^{-j+1}.$$

Then

$$\begin{aligned} & \sup_t |\lambda_{n+m+1} \circ \lambda_{m+n} \circ \cdots \circ \lambda_n(t) - \lambda_{n+m} \circ \cdots \circ \lambda_n(t)| \\ &= \sup_s |\lambda_{n+m+1}(s) - s| \\ &\leq 2^{-(n+m)} \end{aligned}$$

for each  $n$ . Hence for each  $n$ , the sequence  $\lambda_{m+n} \circ \cdots \circ \lambda_n$  (indexed by  $m$ ) is a Cauchy sequence of functions on  $[0, 1]$  with respect to the supremum norm on  $[0, 1]$ . Let  $v_n$  be the limit. Clearly  $v_n(0) = 0$ ,  $v_n(1) = 1$ ,  $v_n$  is continuous, and nondecreasing. We also have

$$\begin{aligned} & \left| \log \frac{\lambda_{n+m} \circ \cdots \circ \lambda_n(t) - \lambda_{n+m} \circ \cdots \circ \lambda_n(s)}{t - s} \right| \\ &\leq \|\lambda_{n+m} \circ \cdots \circ \lambda_n\| \\ &\leq \|\lambda_{n+m}\| + \cdots + \|\lambda_n\| \\ &\leq \frac{1}{2^{n-1}}. \end{aligned}$$

If we then let  $m \rightarrow \infty$ , we obtain

$$\left| \log \frac{v_n(t) - v_n(s)}{t - s} \right| \leq \frac{1}{2^{n-1}},$$

which implies  $v_n \in \Lambda$  with  $\|v_n\| \leq 2^{1-n}$ .

We see that  $v_n = v_{n+1} \circ \lambda_n$ . Consequently

$$\sup_t |f_{n_j}(v_j^{-1}(t)) - f_{n_{j+1}}(v_{j+1}^{-1}(t))| = \sup_s |f_{n_j}(s) - f_{n_{j+1}}(\lambda_j(s))| \leq 2^{-j}.$$

Therefore  $f_{n_j} \circ v_j^{-1}$  is a Cauchy sequence on  $[0, 1]$  with respect to the supremum norm. Let  $f$  be the limit. Since

$$\sup_t |f_{n_j}(v_j^{-1}(t)) - f(t)| \rightarrow 0$$

and  $\|v_j\| \rightarrow 0$  as  $j \rightarrow \infty$ , then  $d(f_{n_j}, f) \rightarrow 0$ .  $\square$

We next show that if  $f_n \rightarrow f$  with respect to  $d$  and  $f \in C[0, 1]$ , the convergence is in fact uniform.

**Proposition 34.5** Suppose  $f_n \rightarrow f$  in the topology of  $D[0, 1]$  with respect to  $d$  and  $f \in C[0, 1]$ . Then  $\sup_{t \in [0, 1]} |f_n(t) - f(t)| \rightarrow 0$ .

*Proof* Let  $\varepsilon > 0$ . Since  $f$  is uniformly continuous on  $[0, 1]$ , there exists  $\delta$  such that  $|f(t) - f(s)| < \varepsilon/2$  if  $|t - s| < \delta$ . For  $n$  sufficiently large there exists  $\lambda_n \in \Lambda$  such that  $\sup_t |f_n(t) - f(\lambda_n(t))| < \varepsilon/2$  and  $\sup_t |\lambda_n(t) - t| < \delta$ . Therefore  $|f(\lambda_n(t)) - f(t)| < \varepsilon/2$ , and so  $|f_n(t) - f(t)| < \varepsilon$ .  $\square$

We turn to compactness.

**Theorem 34.6** A set  $A$  has compact closure in  $D[0, 1]$  if

$$\sup_{f \in A} \sup_t |f(t)| < \infty$$

and

$$\lim_{\delta \rightarrow 0} \sup_{f \in A} \xi_f(\delta) = 0.$$

The converse of this theorem is also true, but we won't need this. See Billingsley (1968) or Exercise 34.9.

*Proof* A complete and totally bounded set in a metric space is compact, and  $D[0, 1]$  is a complete metric space. Hence it suffices to show that  $A$  is totally bounded: for each  $\varepsilon > 0$  there exist finitely many balls of radius  $\varepsilon$  that cover  $A$ .

Let  $\eta > 0$  and choose  $k$  large such that  $1/k < \eta$  and  $\xi_f(1/k) < \eta$  for each  $f \in A$ . Let  $M = \sup_{f \in A} \sup_t |f(t)|$  and let  $H = \{-M + j/k : j \leq 2kM\}$ , so that  $H$  is an  $\eta$ -net for  $[-M, M]$ . Let  $B$  be the set of functions  $f \in D[0, 1]$  that are constant on each interval  $[(i-1)/k, i/k]$  and that take values only in the set  $H$ . In particular,  $f(1) \in H$ .

We first prove that  $B$  is a  $2\eta$ -net for  $A$  with respect to  $\rho$ . If  $f \in A$ , there exist  $t_0, \dots, t_n$  such that  $t_0 = 0$ ,  $t_n = 1$ ,  $t_i - t_{i-1} > 1/k$  for each  $i$ , and  $\theta_f[t_{i-1}, t_i] < \eta$  for each  $i$ . Note we must have  $n \leq k$ . For each  $i$  choose integers  $j_i$  such that  $j_i/k \leq t_i < (j_i + 1)/k$ . The  $j_i$  are distinct since the  $t_i$  are at least  $1/k$  apart. Define  $\lambda$  so that  $\lambda(j_i/k) = t_i$  and  $\lambda$  is linear on each interval  $[j_i/k, j_{i+1}/k]$ . Choose  $g \in B$  such that  $|g(m/k) - f(\lambda(m/k))| < \eta$  for each  $m \leq k$ . Observe that each  $[m/k, (m+1)/k]$  lies inside some interval of the form  $[j_i/k, j_{i+1}/k]$ . Since  $\lambda$  is increasing,  $[\lambda(m/k), \lambda((m+1)/k)]$  is contained in  $[\lambda(j_i/k), \lambda(j_{i+1}/k)] = [t_i, t_{i+1}]$ . The function  $f$  does not vary more than  $\eta$  over each interval  $[t_i, t_{i+1}]$ , so  $f(\lambda(t))$  does not vary more than  $\eta$  over each interval  $[m/k, (m+1)/k]$ .  $g$  is constant on each such interval, and hence

$$\sup_t |g(t) - f(\lambda(t))| < 2\eta.$$

We have

$$|\lambda(j_i/k) - j_i/k| = |t_i - j_i/k| < 1/k < \eta$$

for each  $i$ . By the piecewise linearity of  $\lambda$ ,  $\sup_t |\lambda(t) - t| < \eta$ . Thus  $\rho(f, g) < 2\eta$ . We have proved that given  $f \in A$ , there exists  $g \in B$  such that  $\rho(f, g) < 2\eta$ , or  $B$  is a  $2\eta$ -net for  $A$  with respect to  $\rho$ .

Now let  $\varepsilon > 0$  and choose  $\delta > 0$  small so that  $4\delta + \xi_f(\delta) < \varepsilon$  for each  $f \in A$ . Set  $\eta = \delta^2/4$ . Choose  $B$  as above to be a  $2\eta$ -net for  $A$  with respect to  $\rho$ . By Lemma 34.2, if  $\rho(f, g) < 2\eta < \delta^2$ , then  $d(f, g) \leq 4\delta + \xi_f(\delta) < \varepsilon$ . Therefore  $B$  is an  $\varepsilon$ -net for  $A$  with respect to  $d$ .  $\square$

The following corollary is proved exactly similarly to Theorem 32.1.

**Corollary 34.7** *Suppose  $X_n$  are processes whose paths are right continuous with left limits. Suppose for each  $\varepsilon$  and  $\eta$  there exists  $n_0, R$ , and  $\delta$  such that*

$$\mathbb{P}(\xi_{X_n}(\delta) \geq \varepsilon) \leq \eta$$

and

$$\mathbb{P}(\sup_{t \in [0, 1]} |X_n(t)| \geq R) \leq \eta.$$

Then the  $X_n$  are tight with respect to the topology of  $D[0, 1]$ .

### 34.3 The Aldous criterion

A very useful criterion for tightness is the following one due to Aldous (1978).

**Theorem 34.8** *Let  $\{X_n\}$  be a sequence in  $D[0, 1]$ . Suppose*

$$\lim_{R \rightarrow \infty} \sup_n \mathbb{P}(|X_n(t)| \geq R) = 0 \quad (34.3)$$

for each  $t \in [0, 1]$  and that whenever  $\tau_n$  are stopping times for  $X_n$  and  $\delta_n \rightarrow 0$  are reals,

$$|X_n(\tau_n + \delta_n) - X_n(\tau_n)| \quad (34.4)$$

converges to 0 in probability as  $n \rightarrow \infty$ .

*Proof* We will set  $X_n(t) = X_n(1)$  for  $t \in [1, 2]$  to simplify notation. The proof of this theorem comprises four steps.

*Step 1.* We claim that (34.4) implies the following: given  $\varepsilon$  there exist  $n_0$  and  $\delta$  such that

$$\mathbb{P}(|X_n(\tau_n + s) - X_n(\tau_n)| \geq \varepsilon) \leq \varepsilon \quad (34.5)$$

for each  $n \geq n_0$ ,  $s \leq 2\delta$ , and  $\tau_n$  a stopping time for  $X_n$ . For if not, we choose an increasing subsequence  $n_k$ , stopping times  $\tau_{n_k}$ , and  $s_{n_k} \leq 1/k$  for which (34.5) does not hold. Taking  $\delta_{n_k} = s_{n_k}$  gives a contradiction to (34.4).

*Step 2.* Let  $\varepsilon > 0$ , fix  $n \geq n_0$ , and let  $T \leq U \leq 1$  be two stopping times for  $X_n$ . We will prove

$$\mathbb{P}(U \leq T + \delta, |X_n(U) - X_n(T)| \geq 2\varepsilon) \leq 16\varepsilon. \quad (34.6)$$

To prove this, we start by letting  $\lambda$  be Lebesgue measure. If

$$A_T = \{(\omega, s) \in \Omega \times [0, 2\delta] : |X_n(T + s) - X_n(T)| \geq \varepsilon\},$$

then for each  $s \leq 2\delta$  we have  $\mathbb{P}(\omega : (\omega, s) \in A_T) \leq \varepsilon$  by (34.5) with  $\tau_n$  replaced by  $T$ . Writing  $\mathbb{P} \times \lambda$  for the product measure, we then have

$$\mathbb{P} \times \lambda(A_T) \leq 2\delta\varepsilon. \quad (34.7)$$

Set  $B_T(\omega) = \{s : (\omega, s) \in A_T\}$  and  $C_T = \{\omega : \lambda(B_T(\omega)) \geq \frac{1}{4}\delta\}$ . From (34.7) and the Fubini theorem,

$$\int \lambda(B_T(\omega)) \mathbb{P}(d\omega) \leq 2\delta\varepsilon,$$

so

$$\mathbb{P}(C_T) \leq 8\varepsilon.$$

We similarly define  $B_U$  and  $C_U$ , and obtain  $\mathbb{P}(C_T \cup C_U) \leq 16\varepsilon$ .

If  $\omega \notin C_T \cup C_U$ , then  $\lambda(B_T(\omega)) \leq \frac{1}{4}\delta$  and  $\lambda(B_U(\omega)) \leq \frac{1}{4}\delta$ . Suppose  $U \leq T + \delta$ . Then

$$\lambda\{t \in [T, T + 2\delta] : |X_n(t) - X_n(T)| \geq \varepsilon\} \leq \frac{1}{4}\delta,$$

and

$$\lambda\{t \in [U, U + \delta] : |X_n(t) - X_n(U)| \geq \varepsilon\} \leq \frac{1}{4}\delta.$$

Hence there exists  $t \in [T, T + 2\delta] \cap [U, U + \delta]$  such that  $|X_n(t) - X_n(T)| < \varepsilon$  and  $|X_n(t) - X_n(U)| < \varepsilon$ ; this implies  $|X_n(U) - X_n(T)| < 2\varepsilon$ , which proves (34.6).

*Step 3.* We obtain a bound on  $\xi_{X_n}$ . Let  $T_{n0} = 0$  and

$$T_{n,i+1} = \inf\{t > T_{ni} : |X_n(t) - X_n(T_{ni})| \geq 2\varepsilon\} \wedge 2.$$

Note we have  $|X_n(T_{n,i+1}) - X_n(T_{ni})| \geq 2\varepsilon$  if  $T_{ni} < 2$ . We choose  $n_0, \delta$  as in Step 1. By Step 2 with  $T = T_{ni}$  and  $U = T_{n,i+1}$ ,

$$\mathbb{P}(T_{n,i+1} - T_{ni} < \delta, T_{ni} < 2) \leq 16\varepsilon. \quad (34.8)$$

Let  $K = [2/\delta] + 1$  and apply (34.5) with  $\varepsilon$  replaced by  $\varepsilon/K$  to see that there exist  $n_1 \geq n_0$  and  $\zeta \leq \delta \wedge \varepsilon$  such that if  $n \geq n_1$ ,  $s \leq 2\zeta$ , and  $\tau_n$  is a stopping time, then

$$\mathbb{P}(|X_n(\tau_n + s) - X_n(\tau_n)| > \varepsilon/K) \leq \varepsilon/K. \quad (34.9)$$

By (34.6) with  $T = T_{ni}$  and  $U = T_{n,i+1}$  and  $\delta$  replaced by  $\zeta$ ,

$$\mathbb{P}(T_{n,i+1} \leq T_{ni} + \zeta) \leq 16\varepsilon/K \quad (34.10)$$

for each  $i$  and hence

$$\mathbb{P}(\exists i \leq K : T_{n,i+1} \leq T_{ni} + \zeta) \leq 16\varepsilon. \quad (34.11)$$

We have

$$\begin{aligned} \mathbb{E}[T_{ni} - T_{n,i-1}; T_{nK} < 1] &\geq \delta \mathbb{P}(T_{ni} - T_{n,i-1} \geq \delta, T_{nK} < 1) \\ &\geq \delta [\mathbb{P}(T_{nK} < 1) - \mathbb{P}(T_{ni} - T_{n,i-1} < \delta, T_{nK} < 1)] \\ &\geq \delta [\mathbb{P}(T_{nK} < 1) - 16\varepsilon], \end{aligned}$$

where we used (34.8) in the last step. Summing over  $i$  from 1 to  $K$ ,

$$\begin{aligned} \mathbb{P}(T_{nK} < 1) &\geq \mathbb{E}[T_{nK}; T_{nK} < 1] = \sum_{i=1}^K \mathbb{E}[T_{ni} - T_{n,i-1}; T_{nK} < 1] \\ &\geq K\delta [\mathbb{P}(T_{nK} < 1) - 16\varepsilon] \geq 2[\mathbb{P}(T_{nK} < 1) - 16\varepsilon], \end{aligned}$$

or  $\mathbb{P}(T_{nK} < 1) \leq 32\varepsilon$ . Hence except for an event of probability at most  $32\varepsilon$ , we have  $\xi_{X_n}(\zeta) \leq 4\varepsilon$ .

*Step 4.* The last step is to obtain a bound on  $\sup_t |X_n(t)|$ . Let  $\varepsilon > 0$  and choose  $\delta$  and  $n_0$  as in Step 1. Define

$$D_{Rn} = \{(\omega, s) \in \Omega \times [0, 1] : |X_n(s)(\omega)| > R\}$$

for  $R > 0$ . The measurability of  $D_{Rn}$  with respect to the product  $\sigma$ -field  $\mathcal{F} \times \mathcal{B}[0, 1]$ , where  $\mathcal{B}[0, 1]$  is the Borel  $\sigma$ -field on  $[0, 1]$ , follows by the fact that  $X_n$  is right continuous with left limits. Let

$$G(R, s) = \sup_n \mathbb{P}(|X_n(s)| > R).$$

By (34.3),  $G(R, s) \rightarrow 0$  as  $R \rightarrow \infty$  for each  $s$ . Pick  $R$  large so that

$$\lambda(\{s : G(R, s) > \varepsilon\delta\}) < \varepsilon\delta.$$

Then

$$\int 1_{D_{Rn}}(\omega, s) \mathbb{P}(d\omega) = \mathbb{P}(|X_n(s)| > R) \leq \begin{cases} 1, & G(r, s) > \varepsilon\delta, \\ \varepsilon\delta, & \text{otherwise.} \end{cases}$$

Integrating over  $s \in [0, 1]$ ,

$$\mathbb{P} \times \lambda(D_{Rn}) < 2\varepsilon\delta.$$

If  $E_{Rn}(\omega) = \{s : (\omega, s) \in D_{Rn}\}$  and  $F_{Rn} = \{\omega : \lambda(E_{Rn}) > \delta/4\}$ , we have

$$\frac{1}{4}\delta \mathbb{P}(F_{Rn}) = \int_{F_{Rn}} \frac{1}{4}\delta \mathbb{P}(d\omega) \leq \int \int_0^1 1_{D_{Rn}}(\omega, s) \lambda(ds) \mathbb{P}(d\omega) \leq 2\varepsilon\delta,$$

so  $\mathbb{P}(F_{Rn}) \leq 8\varepsilon$ .

Define  $T = \inf\{t : |X_n(t)| \geq R + 2\varepsilon\} \wedge 2$  and define  $A_T$ ,  $B_T$ , and  $C_T$  as in Step 2. We have

$$\mathbb{P}(C_T \cup F_{Rn}) \leq 16\varepsilon.$$

If  $\omega \notin C_T \cup F_{Rn}$  and  $T < 2$ , then  $\lambda(E_{Rn}(\omega)) \leq \delta/4$ . Hence there exists  $t \in [T, T + 2\delta]$  such that  $|X_n(t)| \leq R$  and  $|X_n(t) - X_n(T)| \leq \varepsilon$ . Therefore  $|X_n(T)| \leq R + \varepsilon$ , which contradicts the definition of  $T$ . We conclude that  $T$  must equal 2 on the complement of  $C_T \cup F_{Rn}$ , or in other words, except for an event of probability at most  $16\varepsilon$ , we have  $\sup_t |X_n(t)| \leq R + 2\varepsilon$ , provided, of course, that  $n \geq n_0$ .

An application of Corollary 34.7 completes the proof.  $\square$

Aldous's criterion is particularly well suited for strong Markov processes.

**Proposition 34.9** *Suppose  $X_n$  is a sequence of real-valued strong Markov processes and there exists  $c$ ,  $p$ , and  $\gamma > 0$  such that*

$$\mathbb{E}^x |X_n(t) - X_n(0)|^p \leq ct^\gamma, \quad x \in \mathbb{R}, \quad t \in [0, 1]. \quad (34.12)$$

*Then for each  $x \in \mathbb{R}$ , the sequence of  $\mathbb{P}^x$ -laws of  $\{X_n\}$  is tight with respect to the space  $D[0, 1]$ .*

Unlike the Kolmogorov continuity criterion, we do not require  $\gamma > 1$ .

*Proof* Fix  $x$ . For each  $t$ ,

$$\begin{aligned}\mathbb{P}^x(|X_n(t)| \geq R + |x|) &\leq \mathbb{P}^x(|X_n(t) - X_n(0)| \geq R) \\ &\leq \frac{\mathbb{E}^x |X_n(t) - X_n(0)|^p}{R^p} \\ &\leq \frac{ct^\gamma}{R^p},\end{aligned}$$

which tends to 0 as  $R \rightarrow \infty$ . We used Chebyshev's inequality here.

Suppose  $\tau_n$  are stopping times for  $X_n$  and  $\delta_n \rightarrow 0$ . By the strong Markov property, for each  $\varepsilon > 0$

$$\begin{aligned}\mathbb{P}^x(|X_n(\tau_n + \delta_n) - X_n(\tau_n)| > \varepsilon) &\leq \frac{\mathbb{E}^x |X_n(\tau_n + \delta_n) - X_n(\tau_n)|^p}{\varepsilon^p} \\ &= \varepsilon^{-p} \mathbb{E}^x [\mathbb{E}^{X_n(\tau_n)} |X_n(\delta_n) - X_n(0)|^\gamma] \\ &\leq c\varepsilon^{-p} \delta_n^\gamma,\end{aligned}$$

which tends to 0 as  $n \rightarrow \infty$ . Now apply Theorem 34.8. □

## Exercises

- 34.1 Show that the space  $D$  with the metric  $\rho$  is separable.
  - 34.2 Let  $f_n = 1_{[1/2, 1/2+1/n]}$ . Show that this is a Cauchy sequence with respect to  $\rho$ , but does not converge to an element of  $D$ . Show  $\{f_n\}$  is not a Cauchy sequence with respect to  $d$ .
  - 34.3 Show that (with respect to the topology on  $D$ ) the subset  $C[0, 1]$  of  $D$  is nowhere dense.
  - 34.4 Consider  $D$  with the metric  $d_{\sup}(f, g) = \sup_{t \in [0, 1]} |f(t) - g(t)|$ . Show that  $D$  is not separable with respect to the metric  $d_{\sup}$ .
  - 34.5 Suppose  $\mathbb{P}$  and  $\mathbb{P}'$  are measures supported on  $D[0, 1]$  that agree on all cylindrical subsets of  $D[0, 1]$ . In other words, all the finite-dimensional distributions agree. Prove that  $\mathbb{P} = \mathbb{P}'$  on  $D[0, 1]$ .
  - 34.6 Show that the following are continuous functions on the space  $D[0, 1]$ .
    - (1)  $f(x) = \sup_{t \leq 1} x(t)$ .
    - (2)  $f(x) = \int_0^1 x(t) dt$ .
    - (3)  $f(x) = \sup_{t \leq 1} (x(t) - x(t-))$ .
  - 34.7 Let  $P$  be a Poisson process with parameter  $\lambda$ . Prove that
- $$\frac{P_{nt} - n\lambda t}{\sqrt{n\lambda}}$$
- converges weakly with respect to the topology of  $D[0, 1]$  as  $n \rightarrow \infty$  to a Brownian motion.
- 34.8 Suppose  $X_n$  converges weakly to  $X$  with respect to the topology of  $C[0, 1]$ . Prove that  $X_n$  converges weakly to  $X$  with respect to the topology of  $D[0, 1]$ .

- 34.9 This is the converse to Theorem 34.6. Let  $A$  be an index set, and suppose the collection of functions  $\{f_\alpha, f \in A\}$  is precompact in  $D[0, 1]$ , i.e., its closure is compact.

- (1) Prove  $\sup_{\alpha \in A} \sup_{0 \leq t \leq 1} |f(t)| < \infty$ .  
(2) Prove

$$\limsup_{\delta \rightarrow 0} \sup_{\alpha \in A} \xi_{f_\alpha}(\delta) = 0.$$

### Notes

See Billingsley (1968) for more information.

## Applications of weak convergence

In Chapter 32 we showed how weak convergence of stochastic processes could be used to give another construction of Brownian motion by showing that a simple symmetric random walk converges to a Brownian motion. In the first section of this chapter, we show that the sum of independent, identically distributed mean zero random variables with variance one also converges to a Brownian motion, which is known as the Donsker invariance principle.

We then consider a Brownian bridge, which is a Brownian motion conditioned to return to zero at time one. We prove in Section 35.3 that a Brownian bridge is the limit process for a sequence of normalized empirical processes.

### 35.1 Donsker invariance principle

Suppose the  $Y_i$  are i.i.d. real-valued random variables with mean zero and variance one,  $S_n = \sum_{i=1}^n Y_i$ , and  $Z_n(t)$  is defined to be equal to  $S_{nt}/\sqrt{n}$  if  $nt$  is an integer and defined by linear interpolation for other values of  $t$ . The Donsker invariance principle says that the  $Z_n$  converge weakly with respect to the space  $C[0, 1]$  to a Brownian motion. This is a bit more delicate than in Section 32.2 because here our  $Y_i$  only have second moments.

The statement of the *Donsker invariance principle* is the following.

**Theorem 35.1** *Let the  $Y_i$  and  $Z_n$  be as above. Then  $Z_n$  converges weakly to the law of Brownian motion on  $[0, 1]$  with respect to the metric of  $C[0, 1]$ .*

Before we prove this, we give an application and explain the name “invariance principle.” An example of how the Donsker invariance principle can be used is the following.

**Corollary 35.2** *Let  $M = \sup_{s \leq 1} W_s$  and  $M_n = \sup_{s \leq 1} Z_n(s)$ , where  $W$  is a Brownian motion. Then  $M_n$  converges weakly to  $M$ .*

*Proof* Let  $g$  be a bounded and continuous function on the reals and define a function  $F$  on  $C[0, 1]$  by

$$F(f) = g(\sup_{s \leq 1} f(s)).$$

Notice  $|\sup_{s \leq 1} f_2(s) - \sup_{s \leq 1} f_1(s)| \leq \sup_{s \leq 1} |f_2(s) - f_1(s)|$  and therefore  $F : C[0, 1] \rightarrow \mathbb{R}$  is bounded and continuous. Since  $Z_n$  converges weakly to  $W$  with respect to the topology on  $C[0, 1]$ , then  $\mathbb{E} F(Z_n) \rightarrow \mathbb{E} F(W)$ . This is equivalent to  $\mathbb{E} g(M_n) \rightarrow \mathbb{E} g(M)$ . Because  $g$  is an arbitrary bounded continuous function on the reals, we conclude  $M_n \rightarrow M$  weakly.  $\square$

This corollary says that the distribution of  $\max_{i \leq n} S_i / \sqrt{n}$  converges to the supremum of a Brownian motion. We can actually use this to derive the distribution of the maximum of a Brownian motion: first determine the distribution of the maximum of  $S_n$  when the  $Y_i$ 's are particularly simple, such as when they are a simple symmetric random walk. (That is,  $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = \frac{1}{2}$ .) Then take the limit as  $n \rightarrow \infty$ . In the case of a simple symmetric random walk, we can find the distribution of the maximum using the reflection principle, and there are no technical difficulties with the proof, unlike using the reflection principle with Brownian motion.

Another useful example is where  $I_n = \int_0^1 |Z_n(t)|^2 dt$  and  $I = \int_0^1 |W_t|^2 dt$ . Here the distribution of  $I$  can be found by an eigenvalue argument (Kuo, 1975), and this is then an approximation to the distribution of  $I_n$ .

If  $f$  is a continuous function from  $C[0, 1]$  to  $\mathbb{R}$ , an argument similar to the proof of Corollary 35.2 shows that  $f(Z_n)$  converges weakly to  $f(W)$ . We get the same limit process, regardless of the distribution of the  $Y_i$ 's, provided only that they are i.i.d. with mean zero and variance one. This is where the name “invariance principle” comes from – the limit is invariant with respect to changing the distribution of the  $Y_i$ 's.

**Lemma 35.3** Suppose we have a sequence  $Y_i$  of i.i.d. random variables with mean zero and variance one and  $S_n = \sum_{i=1}^n Y_i$ . Suppose  $\lambda > 4$ . Then

$$\mathbb{P}(\max_{i \leq n} |S_i| \geq \lambda\sqrt{n}) \leq \frac{4}{3}\mathbb{P}(|S_n| \geq \lambda\sqrt{n}/2).$$

*Proof* Let  $N = \min\{i : |S_i| \geq \lambda\sqrt{n}\}$ , the first time  $S_i$  is bigger than  $\lambda\sqrt{n}$ .  $N$  is a stopping time and  $(N = i)$  is in the  $\sigma$ -field generated by  $Y_1, \dots, Y_i$ . We have

$$\begin{aligned} \mathbb{P}(\max_{i \leq n} |S_i| \geq \lambda\sqrt{n}) &\leq \mathbb{P}(|S_n| \geq \lambda\sqrt{n}/2) + \mathbb{P}(N < n, |S_n| < \lambda\sqrt{n}/2) \\ &\leq \mathbb{P}(|S_n| \geq \lambda\sqrt{n}/2) \\ &\quad + \sum_{i=1}^{n-1} \mathbb{P}(N = i, |S_n| < \lambda\sqrt{n}/2). \end{aligned} \tag{35.1}$$

If  $N = i$  with  $i < n$  and  $|S_n| < \lambda\sqrt{n}/2$ , then  $|S_n - S_i| \geq \lambda\sqrt{n}/2$ , and moreover the event  $\{|S_n - S_i| \geq \lambda\sqrt{n}/2\}$  is in the  $\sigma$ -field generated by  $Y_{i+1}, \dots, Y_n$ , and hence is independent of the event  $\{N = i\}$ . Using Chebyshev's inequality, the sum on the last line of (35.1) is bounded by

$$\begin{aligned} \sum_{i=1}^{n-1} \mathbb{P}(N = i) \mathbb{P}(|S_n - S_i| \geq \lambda\sqrt{n}/2) &\leq \sum_{i=1}^{n-1} \mathbb{P}(N = i) \frac{\mathbb{E} |S_n - S_i|^2}{\lambda^2 n/4} \\ &= \sum_{i=1}^{n-1} \mathbb{P}(N = i) \frac{n-i}{\lambda^2 n/4} \\ &\leq \frac{1}{4} \mathbb{P}(N < i) \\ &\leq \frac{1}{4} \mathbb{P}(\max_{i \leq n} |S_i| \geq \lambda\sqrt{n}), \end{aligned}$$

since  $\lambda > 4$ . Therefore

$$\mathbb{P}(\max_{i \leq n} |S_i| \geq \lambda\sqrt{n}) \leq \mathbb{P}(|S_n| \geq \lambda\sqrt{n}/2) + \frac{1}{4} \mathbb{P}(\max_{i \leq n} |S_i| \geq \lambda\sqrt{n}).$$

Subtracting the second term on the right from both sides and multiplying by 4/3 proves the lemma.  $\square$

Note that the central limit theorem tells us that for any  $\beta > 0$

$$\mathbb{P}(|S_n| \geq \beta\sqrt{n}) \rightarrow \mathbb{P}(|Z| \geq \beta) \leq e^{-\beta^2/2},$$

where  $Z$  is a mean zero normal random variable with variance one, and hence for  $n$  large (depending on  $\beta$ ),

$$\mathbb{P}(|S_n| \geq \beta\sqrt{n}) \leq 2e^{-\beta^2/2}. \quad (35.2)$$

**Lemma 35.4** *For each  $\varepsilon, \eta > 0$ , there exist  $n_0$  and  $\delta$  such that if  $n \geq n_0$  and  $s \in [0, 1 - \delta]$ , then*

$$\mathbb{P}\left(\sup_{s \leq t \leq s+\delta} |Z_n(t) - Z_n(s)| > \varepsilon\right) \leq \eta\delta.$$

*Proof* Let  $\varepsilon, \eta > 0$ , and choose  $\delta$  small enough that  $2e^{-\varepsilon^2/128\delta} \leq \delta\eta/2$ . Then choose  $j_0$  large enough so that, using (35.2),

$$\mathbb{P}\left(|S_j| > \frac{\varepsilon\sqrt{j}}{8\sqrt{\delta}}\right) \leq 2e^{-\varepsilon^2/128\delta} \leq \delta\eta/2$$

if  $j \geq j_0$ . Finally, choose  $n_0 \geq j_0/\delta + 2$ , so that if  $n \geq n_0$ , then  $[n\delta] + 2 \geq j_0$  and  $n\delta \geq ([n\delta] + 2)/2$ , where  $[x]$  is the largest integer less than or equal to  $x$ .

Let  $n \geq n_0$  and set  $J = [n\delta] + 2$ . Suppose there exists  $s$  such that for some  $t \in [s, s + \delta]$  we have  $|Z_n(t) - Z_n(s)| > \varepsilon$ . Then there exists  $j \leq n$  such that for some  $i$  between  $j$  and  $j + J$  we have  $|S_i - S_j| \geq \varepsilon\sqrt{n}/2$ . Therefore  $n \geq J/2\delta$  and by Lemma 35.3

$$\begin{aligned} \mathbb{P}\left(\sup_{s \leq t \leq s+\delta} |Z_n(t) - Z_n(s)| > \varepsilon\right) &\leq \mathbb{P}\left(\max_{j \leq i \leq j+J} |S_i - S_j| > \sqrt{n}\varepsilon/2\right) \\ &\leq \mathbb{P}\left(\max_{j \leq i \leq j+J} |S_i - S_j| > \frac{\sqrt{J}\varepsilon}{4\sqrt{\delta}}\right) \\ &\leq \frac{4}{3}\mathbb{P}\left(|S_{j+J} - S_j| > \frac{\sqrt{J}\varepsilon}{8\sqrt{\delta}}\right) \\ &\leq \frac{4}{3}\mathbb{P}\left(|S_J| > \frac{\sqrt{J}\varepsilon}{8\sqrt{\delta}}\right) \\ &\leq \delta\eta. \end{aligned}$$

The proof is complete.  $\square$

**Lemma 35.5** *For each  $\varepsilon, \eta > 0$  there exist  $n_0$  and  $\delta$  such that if  $n \geq n_0$ ,*

$$\mathbb{P}(\omega_{Z_n}(\delta) \geq \varepsilon) \leq 2\eta.$$

*Proof* We will take  $\delta = 1/K$  for some large  $K$ . If  $|t - s| \leq 1/K$ , then either both  $s, t$  are in the same interval  $[(i-1)/K, i/K]$  or they are in adjoining intervals. Thus they both lie in some interval of the form  $[(i-2)/K, i/K]$ . Since

$$|Z_n(t) - Z_n(s)| \leq |Z_n(t) - Z_n((i-2)/K)| + |Z_n(s) - Z_n((i-2)/K)|,$$

then using Lemma 35.4 with  $\delta = 2/K$

$$\begin{aligned} & \mathbb{P}(\exists s, t \in [0, 1] : |Z_n(t) - Z_n(s)| \geq \varepsilon, |t - s| < \delta) \\ & \leq \mathbb{P}(\exists i \leq K : \sup_{(i-2)/K \leq s \leq i/K} |Z_n(s) - Z_{(i-2)/K}| \geq \varepsilon/2) \\ & \leq K \sup_i \mathbb{P}\left(\sup_{(i-2)/K \leq s \leq i/K} |Z_n(s) - Z_{(i-2)/K}| \geq \varepsilon/2\right) \\ & \leq K\eta(2/K) = 2\eta, \end{aligned}$$

which proves the lemma.  $\square$

We can now prove the Donsker invariance principle.

*Proof of Theorem 35.1* By Lemma 35.5, Theorem 32.1, and the fact that  $Z_n(0) = 0$  for all  $n$ , the laws of the  $Z_n$  are tight. Therefore by Prohorov's theorem (Theorem 30.4), every subsequence has a further subsequence which converges weakly with respect to the topology on  $C[0, 1]$ . We therefore only need to show that every subsequential limit point of the  $Z_n$  with respect to weak convergence is a Brownian motion. Since our processes lie in  $C[0, 1]$ , the paths of any subsequential limit point are continuous, so it suffices by Theorem 2.6 to show that the finite-dimensional distributions of  $Z_n$  converge weakly to the corresponding finite-dimensional distributions of a Brownian motion  $W$ . We will show the one-dimensional distributions converge, and leave the analogous argument for the higher-dimensional distributions to the reader.

We have

$$\mathbb{P}(\max_{i \leq n} |Y_i|/\sqrt{n} \geq \varepsilon) \leq n\mathbb{P}(|Y_1| \geq \sqrt{n}\varepsilon) \leq n\mathbb{P}(|Y_1|^2/\varepsilon^2 \geq n). \quad (35.3)$$

For any integrable non-negative random variable  $X$ ,

$$n\mathbb{P}(X \geq n) = \mathbb{E}[n; X \geq n] \leq \mathbb{E}[X; X \geq n],$$

which tends to zero by dominated convergence. Therefore

$$\mathbb{P}(\max_{i \leq n} |Y_i|/\sqrt{n} \geq \varepsilon) \rightarrow 0. \quad (35.4)$$

Fix  $t \in [0, 1]$ . By the central limit theorem,  $S_{[nt]}/\sqrt{[nt]}$  converges weakly on  $\mathbb{R}$  to a mean zero normal random variable with variance one, and by Exercise 30.3, we see that  $S_{[nt]}/\sqrt{n}$  converges weakly to a mean zero normal random variable with variance  $t$ . From the preceding paragraph we conclude that for each  $t$ ,  $|Z_n(t) - S_{[nt]}/\sqrt{n}|$  converges to zero in probability. By Exercise 30.2,  $Z_n(t)$  has the same weak limit as  $S_{[nt]}/\sqrt{n}$ , namely, a mean zero normal random variable with variance  $t$ , which is the distribution of  $W_t$ .  $\square$

There is an elegant proof of the Donsker invariance principle using Skorokhod embedding. Unlike the proof above, however, this second proof does not extend to random variables taking values in  $\mathbb{R}^d$ .

By Theorem 15.6 we can find a Brownian motion  $W$  and a random walk  $S_n$  such that

$$\sup_{i \leq n} \frac{|S_i - W_i|}{\sqrt{n}} \rightarrow 0$$

in probability. By the continuity of paths of  $W$ ,

$$\mathbb{P}\left(\sup_{|t-s|\leq 1/n, s,t\leq 1} |W_t - W_s| > \varepsilon\right) \rightarrow 0.$$

If we let  $W^n(t) = W_{nt}/\sqrt{n}$ , we then have that  $\sup_{i\leq n} |Z_n(i/n) - W_n(i/n)|$  tends to zero in probability as  $n \rightarrow \infty$  and also, because  $W_n$  is again a Brownian motion,

$$\mathbb{P}\left(\sup_{|t-s|\leq 1/n, s,t\leq 1} |W_n(t) - W_n(s)| > \varepsilon\right) \rightarrow 0.$$

We conclude that

$$\sup_{t\leq 1} |Z_n(t) - W_n(t)| \rightarrow 0.$$

The law of  $W_n$  is that of a Brownian motion and does not depend on  $n$ . By Exercise 30.2 we obtain that  $Z_n$  converges weakly to the law of a Brownian motion.

If the above proof seems too simple, remember that we used Theorem 15.6, which in turn relies on Skorokhod embedding.

One might ask about the weak convergence of  $\tilde{Z}_n(t) = S_{[nt]}/\sqrt{n}$ ; these are the normalized partial sums without the linear interpolation. Rather than being continuous and piecewise linear like the  $Z_n(t)$ , the  $\tilde{Z}_n(t)$  are piecewise constant and have jumps.

**Proposition 35.6** *Suppose the  $Y_i$  are independent with mean zero and variance one. The  $\tilde{Z}_n$  converge weakly with respect to the topology of  $D[0, 1]$  to Brownian motion.*

*Proof* The  $Z_n$  converge weakly with respect to the topology of  $C[0, 1]$  to a Brownian motion. By the Skorokhod representation (Theorem 31.2), we can find a probability space and random variables  $Z'_n$  having the same law as  $Z_n$  that converge almost surely with respect to the supremum norm. Therefore the  $Z'_n$  converge almost surely with respect to the metric of  $D[0, 1]$ , and hence the  $Z_n$  converge weakly to a Brownian motion with respect to the topology of  $D[0, 1]$ . If we show that  $\sup_{t\leq 1} |Z_n(t) - \tilde{Z}_n(t)|$  converges to zero in probability, then our result will follow by Exercise 30.2.

Now  $Z_n(t)$  and  $\tilde{Z}_n(t)$  will differ by more than  $\varepsilon$  for some  $t$  only if some  $Y_i$  is larger than  $\sqrt{n}\varepsilon$  in absolute value. But by (35.4), the probability of this tends to zero as  $n \rightarrow \infty$ .  $\square$

## 35.2 Brownian bridge

A Brownian bridge  $W_t^0$  is the process defined by

$$W_t^0 = W_t - tW_1, \quad 0 \leq t \leq 1,$$

where  $W$  is a Brownian motion.  $W^0$  has continuous paths, is jointly normal, is zero at time 0 and at time 1, has mean zero, and we calculate its covariance by

$$\begin{aligned} \text{Cov}(W_s^0, W_t^0) &= \text{Cov}(W_s, W_t) - s \text{Cov}(W_1, W_t) - t \text{Cov}(W_s, W_1) + st \text{Var}(W_1) \\ &= s \wedge t - st, \end{aligned}$$

recalling (2.1).

A Brownian bridge can be characterized as a Brownian motion conditioned to be zero at time 1. To make this precise, let  $W$  be a Brownian motion started at zero under  $\mathbb{P}$ , and for  $A$  a Borel subset of  $C[0, 1]$ , define

$$\mathbb{P}_\varepsilon(A) = \mathbb{P}(W \in A \mid |W_1| \leq \varepsilon);$$

cf. (A.13). Set  $\mathbb{P}_0(A) = \mathbb{P}(W^0 \in A)$ , the law of  $W^0$ .

**Proposition 35.7**  $\mathbb{P}_\varepsilon$  converges weakly to  $\mathbb{P}_0$  with respect to the topology of  $C[0, 1]$  as  $\varepsilon \rightarrow 0$ .

*Proof* Since  $W$  is a jointly normal process and

$$\text{Cov}(W_t - tW_1, W_1) = \text{Cov}(W_t, W_1) - t\text{Var}(W_1) = 0,$$

then the process  $W_t^0 = W_t - tW_1$  and the random variable  $W_1$  are independent by Proposition A.55. Let  $F$  be any closed subset of  $C[0, 1]$  and let  $F_\delta = \{g \in C[0, 1] : d(g, F) < \delta\}$ , where  $d(g, F) = \inf\{d(g, f) : f \in F\}$  and  $d$  here is the supremum norm. Note  $\sup_{t \leq 1} |W_t - W_t^0| \leq \varepsilon$  on the event  $\{|W_1| \leq \varepsilon\}$ . If  $\delta > \varepsilon$ ,

$$\begin{aligned}\mathbb{P}_\varepsilon(F) &= \mathbb{P}(W \in F \mid |W_1| \leq \varepsilon) \leq \mathbb{P}(W^0 \in F_\delta \mid |W_1| \leq \varepsilon) \\ &= \mathbb{P}(W^0 \in F_\delta) = \mathbb{P}_0(F_\delta).\end{aligned}$$

Thus  $\limsup_{\varepsilon \rightarrow 0} \mathbb{P}_\varepsilon(F) \leq \mathbb{P}_0(F_\delta)$ . Since  $F$  is closed,  $\mathbb{P}_0(F_\delta) \rightarrow \mathbb{P}_0(F)$  as  $\delta \rightarrow 0$ , so  $\limsup \mathbb{P}_\varepsilon(F) \leq \mathbb{P}_0(F)$ . An application of Theorem 30.2 completes the proof.  $\square$

We show that a Brownian bridge can also be represented as the solution  $X$  of the stochastic differential equation

$$dX_t = dW_t - \frac{X_t}{1-t} dt, \quad X_0 = 0, \tag{35.5}$$

where  $W$  is a Brownian motion. This is plausible:  $X$  behaves much like a Brownian motion until  $t$  is close to 1, when there is a strong push toward the origin. The existence and uniqueness theory of Chapter 24 shows uniqueness and existence for the solution of (35.5) for  $s \leq t$  for any  $t < 1$ ; see Exercise 24.4. We can solve (35.5) explicitly. We have

$$dW_t = dX_t + \frac{X_t}{1-t} dt = (1-t) d\left[\frac{X_t}{1-t}\right],$$

or

$$X_t = (1-t) \int_0^t \frac{dW_s}{1-s}.$$

Thus  $X_t$  is a continuous Gaussian process with mean zero. The variance of  $X_t$  is

$$(1-t)^2 \int_0^t (1-s)^{-2} ds = t - t^2,$$

the same as the variance of a Brownian bridge. A similar calculation shows that the covariance of  $X_t$  and  $X_s$  is the same as the covariance of  $W_t - tW_1$  and  $W_s - sW_1$ ; see Exercise 24.6. Hence the finite-dimensional distributions of  $X_t$  and a Brownian bridge are the same. We now appeal to Theorem 2.6.

### 35.3 Empirical processes

In this section we will consider empirical processes, which are useful in statistics in estimating distribution functions. Let  $X_i$ ,  $i = 1, \dots, n$ , be i.i.d. random variables that are uniformly distributed on the interval  $[0, 1]$ . Define the *empirical process*

$$F_n(t) = \frac{1}{n} \sum_{i=1}^n 1_{[0,t]}(X_i). \quad (35.6)$$

The Glivenko–Cantelli theorem (Theorem A.40) says that

$$\sup_{t \in [0,1]} |F_n(t) - t| \rightarrow 0, \quad \text{a.s.}$$

Our goal in this section is to obtain the corresponding weak limit theorem. Let

$$Z_n(t) = \sqrt{n}(F_n(t) - t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (1_{[0,t]}(X_i) - t). \quad (35.7)$$

We will show that  $Z_n$  converges weakly with respect to  $D[0, 1]$  to a Brownian bridge.

Let

$$\omega_{Z_n}(\delta) = \sup_{s,t \in [0,1], |t-s|<\delta} |Z_n(t) - Z_n(s)|.$$

The paths of  $Z_n$  are not continuous: they have a jump of size  $1/n$  at every time  $X_i$ . Thus  $\omega_{Z_n}(\delta)$  does not tend to zero as  $\delta \rightarrow 0$ . Nevertheless we can get reasonable estimates on  $\omega_{Z_n}(\delta)$ .

We need an elementary lemma on binomial random variables, the proof of which is Exercise 35.1.

**Lemma 35.8** *Suppose  $S_n$  is a binomial random variable with parameters  $n$  and  $p$ . Then there exists a constant  $c$  not depending on  $n$  or  $p$  such that*

$$\mathbb{E} |S_n - \mathbb{E} S_n|^4 \leq cnp + cn^2 p^2 \quad (35.8)$$

and

$$\mathbb{E} |S_n|^4 \leq cnp + cn^4 p^4. \quad (35.9)$$

**Proposition 35.9** *Let  $\varepsilon, \eta > 0$ . There exists  $\delta$  and  $n_0$  such that if  $n \geq n_0$ , then*

$$\mathbb{P}(\omega_{Z_n}(\delta) > \varepsilon) \leq \eta.$$

The idea of the proof is to use Corollary 8.4 to estimate  $Z_n(t) - Z_n(s)$  when  $|t - s|$  is small and use estimates on binomials when  $|t - s|$  is large.

*Proof* Let  $\varepsilon, \eta > 0$ . We will choose  $n_0, \delta$  later. Assuming that they have been chosen, suppose  $n \geq n_0$  and choose  $k$  such that  $n \leq 2^k < 2n$ . If  $t \in [0, 1]$ , let  $t(k)$  be the largest multiple of  $2^{-k}$  less than or equal to  $t$  and similarly define  $s(k)$ . Let  $\mathcal{D}_k = \{i/2^k : 0 \leq i \leq 2^k\}$ . We will show there exists  $\delta > 0$  such that

$$\mathbb{P}\left(\sup_{s,t \in \mathcal{D}_k, |t-s|<2\delta} |Z_n(t) - Z_n(s)| > \varepsilon/3\right) < \eta/3 \quad (35.10)$$

and

$$\mathbb{P}(\sup_{s \in [0, 1]} |Z_n(s) - Z_n(s(k))| > \varepsilon/3) < \eta/3. \quad (35.11)$$

*Step 1.* We first prove (35.10) by using Corollary 8.4. Suppose  $s, t \in \mathcal{D}_k$  with  $|t - s| < 2\delta$ . Then either  $s = t$ , in which case  $Z_n(t) - Z_n(s) = 0$ , or else  $|t - s| \geq 2^{-k} \geq 1/(2n)$ . Take  $p = t - s$  and note that  $1_{(s,t]}(X_i)$  is a Bernoulli random variable with parameter  $p$ . Using (35.7) and Lemma 35.8,

$$\begin{aligned} \mathbb{E} |Z_n(t) - Z_n(s)|^4 &\leq \frac{c}{n^2} (np + n^2 p^2) \\ &= c \left( \frac{p}{n} + p^2 \right) \leq c|t - s|^2, \end{aligned}$$

where in the last line we used  $1/n \leq 2|t - s|$ . By Corollary 8.4,

$$\begin{aligned} \mathbb{P}(\sup_{s, t \in \mathcal{D}_k, |t-s|<2\delta} |Z_n(t) - Z_n(s)| > \varepsilon/3) &\leq \mathbb{P}\left(\sup_{s, t \in \mathcal{D}_k, |t-s|<2\delta} \frac{|Z_n(t) - Z_n(s)|}{|t - s|^{1/8}} > c \frac{\varepsilon}{\delta^{1/8}}\right) \\ &\leq c(\varepsilon/\delta^{1/8})^{-4} = c\delta^{1/2}/\varepsilon^4. \end{aligned}$$

We choose  $\delta$  small enough so that the last term is less than  $\eta/3$ .

*Step 2.* We now prove (35.11). Let

$$T_n(t) = \sum_{i=1}^n 1_{[0,t]}(X_i).$$

Observe that  $T_n(t)$  is nondecreasing in  $t$ . If there exists  $s \in [0, 1]$  such that  $T_n(s) - T_n(s(k)) > \varepsilon\sqrt{n}/3$ , then there exists  $j \leq 2^k - 1$  such that  $T_n((j+1)/2^k) - T_n(j/2^k) > \varepsilon\sqrt{n}/3$ . Therefore, using (35.9),

$$\begin{aligned} &\mathbb{P}\left(\sup_{s \in [0, 1]} \frac{T_n(s) - T_n(s(k))}{\sqrt{n}} > \varepsilon/3\right) \\ &\leq \mathbb{P}(\exists j \leq 2^k - 1 : T_n((j+1)/2^k) - T_n(j/2^k) > \varepsilon\sqrt{n}/3) \\ &\leq 2^k \sup_{j \leq 2^k - 1} \mathbb{P}(T_n((j+1)/2^k) - T_n(j/2^k) > \varepsilon\sqrt{n}/3) \\ &\leq c2^k \frac{\sup_j \mathbb{E} |T_n((j+1)/2^k) - T_n(j/2^k)|^4}{\varepsilon^4 n^2} \\ &\leq c2^k \frac{n2^{-k} + (n2^{-k})^4}{\varepsilon^4 n^2}. \end{aligned}$$

Since  $n2^{-k} \leq 2$ , the last line is less than or equal to

$$c2^k n2^{-k} / \varepsilon^4 n^2 = c_1 / \varepsilon^4 n.$$

We choose  $n_0 > 1/\delta$  large enough so that if  $n \geq n_0$ , then  $c_1/\varepsilon^4 n$  is less than  $\eta/3$ .

Also,

$$\mathbb{E}[T_n(s) - T_n(s(k))] \leq n(s - s(k)) \leq n2^{-k} \leq 2$$

will be less than  $\varepsilon\sqrt{n}/3$  if  $n \geq 36/\varepsilon^2$  and we choose  $n_0$  larger if necessary so that  $n_0 > 36/\varepsilon^2$ . Since

$$Z_n(t) - Z_n(s) = \frac{T_n(t) - T_n(s)}{\sqrt{n}} - \frac{\mathbb{E}[T_n(t) - T_n(s)]}{\sqrt{n}},$$

(35.11) follows.

*Step 3.* Now that we have (35.10) and (35.11), we write

$$|Z_n(t) - Z_n(s)| \leq |Z_n(t) - Z_n(t(k))| + |Z_n(t(k)) - Z_n(s(k))| + |Z_n(s(k)) - Z_n(s)|.$$

If  $|t - s| < \delta$ , then  $|t(k) - s(k)| \leq \delta + 2^{-k} \leq \delta + 1/n$ . Provided  $n \geq n_0 > 1/\delta$ , combining (35.10) and (35.11) gives

$$\mathbb{P}\left(\sup_{s,t \in [0,1], |t-s|<\delta} |Z_n(t) - Z_n(s)| > \varepsilon\right) < \eta$$

as required.  $\square$

**Theorem 35.10** *The  $Z_n$  converge weakly to a Brownian bridge with respect to the topology of  $D[0, 1]$ .*

*Proof* We smooth  $Z_n$  to get a continuous process  $V_n$ . Set  $Z_n(t) = Z_n(1)$  for  $t \in [1, 2]$  and set

$$V_n(t) = n \int_0^{n^{-1}} Z_n(u+t) du.$$

We have

$$\begin{aligned} |V_n(t_2) - V_n(t_1)| &\leq n \int_0^{n^{-1}} |Z_n(t_2+u) - Z_n(t_1+u)| du \\ &\leq n \int_0^{n^{-1}} \omega_{Z_n}(|t_2 - t_1|) du = \omega_{Z_n}(|t_2 - t_1|). \end{aligned}$$

Note also that by (35.8) with  $p = t - s$  and using Jensen's inequality with the measure  $n1_{[0,n^{-1}]}(u) du$ ,

$$\mathbb{E} |V_n(0)|^4 \leq n \int_0^{n^{-1}} \mathbb{E} |Z_n(u)|^4 du \leq c.$$

Hence

$$\mathbb{P}(|V_n(0)| \geq A) \leq \frac{\mathbb{E} |V_n(0)|^4}{A^4} \leq \frac{c}{A^4}.$$

Therefore by Theorem 8.1, the  $V_n$  are tight with respect to weak convergence on  $C[0, 1]$ . If the  $V_{n_j}$  converges weakly (with respect to  $C[0, 1]$ ), by the Skorokhod representation we may find  $V'_{n_j}$  with the same law as  $V_{n_j}$  that converge almost surely. Then the  $V'_{n_j}$  will also converge almost surely in the space  $D[0, 1]$ . This proves that the  $V_n$  are tight in  $D[0, 1]$  by Exercise 30.10.

Given  $\varepsilon$  and  $\eta$ , choose  $\delta$  and  $n_0$  such that  $\mathbb{P}(\omega_{Z_n}(\delta) > \varepsilon) < \eta$  if  $n \geq n_0$ . We have

$$|V_n(t) - Z_n(t)| \leq n \int_0^{n^{-1}} |Z_n(u+t) - Z_n(t)| du \leq \omega_{Z_n}(n^{-1}).$$

If  $n$  is large enough so that  $n^{-1} < \delta$ , then

$$\mathbb{P}(\sup_t |V_n(t) - Z_n(t)| > \varepsilon) \leq \mathbb{P}(\omega_{Z_n}(n^{-1}) > \varepsilon) \leq \mathbb{P}(\omega_{Z_n}(\delta) > \varepsilon) < \eta.$$

Therefore  $V_n - Z_n$  converges to 0 in probability, and by Exercise 30.2 the subsequential limit points of  $V_n$  are the same as those of  $Z_n$ .

It remains to show that any subsequential limit point of the  $Z_n$  is a Brownian bridge. This follows from the multidimensional central limit theorem for multinomials (see Remark A.57) and is left as Exercise 35.2.  $\square$

## Exercises

- 35.1 Prove Lemma 35.8.
- 35.2 Prove that the finite-dimensional distributions of  $Z_n$  in Theorem 35.10 converge to those of a Brownian bridge.
- 35.3 If  $W_t^0$  is a Brownian bridge, prove that  $Y_t = W_{1-t}^0$  is also a Brownian bridge.
- 35.4 Let  $t_0 < 1$ . The SDE (35.5) has a unique solution when  $X_0 = 0$  is replaced by  $X_0 = x$ . Let  $\mathbb{P}^x$  be the law of the solution when  $X_0 = x$  and let  $Z_t$  be the canonical process. Show that  $(Z_t, \mathbb{P}^x)$  is not a Markov process.
- 35.5 Let  $N_t(A)$  be a Poisson point process with respect to the measure space  $(\mathcal{S}, m)$  and let  $A_s, s > 0$ , be an increasing sequence of subsets of  $\mathcal{S}$  with  $m(A_s) \rightarrow \infty$  as  $s \rightarrow \infty$ . Does

$$\frac{N_t(A_s) - m(A_s)}{\sqrt{m(A_s)}}$$

converge weakly with respect to  $D[0, 1]$  as  $s \rightarrow \infty$ ? What is the limit?

This can be applied to get central limit theorems for the number of downcrossings of a Brownian motion, for example.

- 35.6 This exercise asks you to prove that the Poisson process conditioned to be equal to  $n$  at time 1 has the same law as  $n$  times the empirical process. Here is the precise statement. Suppose  $P_t$  is a Poisson process with parameter  $\lambda > 0$ . Let  $\mathbb{Q}$  be the law of  $\{P_t, t \in [0, 1]\}$  conditioned so that  $P_1 = n$ . Thus  $\mathbb{Q}$  is a probability on  $D[0, 1]$  with

$$\mathbb{Q}(P \in A) = \mathbb{P}(P \in A \mid P_1 = n).$$

Since  $(P_1 = n)$  is an event with positive probability, there is no difficulty defining these conditional probabilities. Prove that  $\mathbb{Q}$  is also the law of  $\{nF_n(t), t \in [0, 1]\}$ , where  $F_n$  is defined in Section 35.3.

# 36

---

## Semigroups

In this chapter we suppose we have a semigroup of positive contraction operators  $\{P_t\}$ , and we show how to construct a Markov process  $X$  corresponding to this semigroup. In Chapters 37 and 38, we will show how such semigroups might arise.

We suppose that we have a state space  $\mathcal{S}$  that is a separable locally compact metric space  $\mathcal{S}$ . Let  $C_0$  be the set of continuous functions on  $\mathcal{S}$  that vanish at infinity. Recall that  $f \in C_0$  if  $f$  is continuous, and given  $\varepsilon$ , there exists a compact set  $K$  depending on  $\varepsilon$  and  $f$  such that

$$|f(x)| < \varepsilon, \quad x \notin K.$$

We use the usual supremum norm on  $C_0$ . We assume we have a semigroup  $\{P_t\}$  of positive contractions mapping  $C_0$  to  $C_0$ . More precisely, we assume

**Assumption 36.1** There exists a family  $\{P_t\}$ ,  $t \geq 0$ , of operators on  $C_0$  such that

- (1) If  $f \in C_0$ , then

$$P_t(P_s f)(x) = P_{t+s} f(x), \quad x \in \mathcal{S}, \quad s, t \geq 0.$$

- (2) If  $f(x) \geq 0$  for all  $x$  and if  $t \geq 0$ , then  $P_t f(x) \geq 0$  for all  $x$ .

- (3) For all  $t$ ,  $\|P_t f\| \leq \|f\|$ .

- (4) If  $f \in C_0$ , then  $P_t f \rightarrow f$  uniformly as  $t \rightarrow 0$ .

Our goal in this section is to construct a process  $X$  corresponding to the semigroup  $P_t$ . The steps we use are the following.

(1) We temporarily assume each  $P_t$  maps the function 1 into itself. We define  $X_t$  for  $t$  in the dyadic rationals and define  $\mathbb{P}^x$  using the Kolmogorov extension theorem.

- (2) We verify a preliminary version of the Markov property.

(3) We use the regularity theorem for supermartingales to show that  $X$  has left and right limits along the dyadic rationals, and then define  $X_t$  for all  $t$ .

- (4) We verify that our process  $(X_t, \mathbb{P}^x)$  corresponds to the semigroup  $P_t$ .

- (5) We remove the assumption that  $P_t 1 = 1$ .

### 36.1 Constructing the process

Let us assume the following for now. We will remove this assumption at the end of this section.

**Assumption 36.2**  $P_t 1(x) = 1$  for all  $x$  and all  $t \geq 0$ .

We now begin the construction of  $(X_t, \mathbb{P}^x)$ .

*Step 1.* Let  $\mathcal{D}_n = \{k/2^n : k \geq 0\}$  and let  $\mathcal{D} = \cup_n \mathcal{D}_n$ , the dyadic rationals. Let  $\Omega$  be the set of functions from  $\mathcal{D}$  to  $\mathcal{S}$ . Define

$$X_t(\omega) = \omega(t), \quad t \in \mathcal{D}, \quad \omega \in \Omega.$$

We let  $\mathcal{F}$  be the  $\sigma$ -field on  $\Omega$  generated by the collection of cylindrical subsets of  $\Omega$ .

By the Riesz representation theorem (see [Rudin \(1987\)](#)), for each  $t > 0$  there exists a measure  $P_t(x, dy)$  such that

$$P_t f(x) = \int f(y) P_t(x, dy), \quad f \in C_0. \quad (36.1)$$

(The Riesz representation theorem is most often phrased for continuous functions on compact spaces; since we are working with  $C_0$ , we can let the state space satisfy slightly weaker hypotheses; see [Folland \(1999\)](#), p. 223.) We can use (36.1) to define  $P_t f$  for all bounded Borel measurable functions  $f$ . Since  $P_t$  maps  $C_0$  to  $C_0$ , and continuous functions are Borel measurable, a limit argument shows that  $P_t f$  is Borel measurable whenever  $f$  is bounded and Borel measurable.

Our main task in this step is to define  $\mathbb{P}^x$ .  $\mathcal{D}$  is countable and we fix a labeling  $\mathcal{D} = \{t_1, t_2, \dots\}$ . Let  $E_n = \{t_1, \dots, t_n\}$ . Let  $s_1 \leq \dots \leq s_n$  be the ordering of  $E_n$  according to the usual ordering of the reals, so that  $s_1$  is the smallest element of the set  $\{t_1, \dots, t_n\}$ ,  $s_2$  is the next smallest, and so on. Define

$$\begin{aligned} \mathbb{P}_n^x(X_{s_1} \in A_1, \dots, X_{s_n} \in A_n) \\ = \int_{A_n} \cdots \int_{A_1} P_{s_1}(x, dx_1) P_{s_2-s_1}(x_1, dx_2) \cdots P_{s_n-s_{n-1}}(x_{n-1}, dx_n) \end{aligned} \quad (36.2)$$

for  $A_1, \dots, A_n$  Borel subsets of  $\mathcal{S}$ . The  $\mathbb{P}_n^x$  are consistent in the sense of [Appendix D](#). The key to checking this is to observe that if  $s_1, \dots, s_n$  is the ordering of  $E_n$  and we temporarily write  $s_1, \dots, s_i, \bar{s}, s_{i+1}, \dots, s_n$  for the ordering of  $E_{n+1}$ , then

$$\int_{\mathcal{S}} P_{\bar{s}-s_i}(x_{i-1}, d\bar{x}) P_{s_{i+1}-\bar{s}}(\bar{x}, dx_i) = P_{s_{i+1}-s_i}(x_{i-1}, dx_i)$$

by the semigroup property; cf. (19.10).

By the Kolmogorov extension theorem ([Theorem D.1](#)), for each  $x$  there exists a probability  $\mathbb{P}^x$  such that

$$\mathbb{P}^x(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n) = \mathbb{P}_n^x(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n)$$

for each  $n$  whenever  $A_1, \dots, A_n$  are Borel subsets of  $\mathcal{S}$ .

If  $\mathbb{E}^x$  is the expectation corresponding to  $\mathbb{P}^x$ , (36.2) can be rewritten as

$$\begin{aligned} \mathbb{E}^x[f_1(X_{s_1}) \cdots f_n(X_{s_n})] \\ = \int \cdots \int f_1(x_1) \cdots f_n(x_n) P_{s_1}(x, dx_1) P_{s_2-s_1}(x_1, dx_2) \cdots \\ \times P_{s_n-s_{n-1}}(x_{n-1}, dx_n) \end{aligned} \quad (36.3)$$

when  $f_i = 1_{A_i}$  for each  $i$ . To see this, by linearity we have (36.3) when the functions  $f_i$  are simple functions; by a limit argument we have (36.3) when the  $f_i$  are Borel measurable and non-negative, and by linearity, (36.3) holds when the  $f_i$  are bounded and Borel measurable.

By (36.2) we have

$$\mathbb{P}^x(X_t \in A) = \mathbb{E} 1_A(X_t) = \int_A P_t(x, dy) = P_t 1_A(x).$$

Using linearity and a limit argument, we have  $\mathbb{E}^x f(X_t) = P_t f(x)$  when  $f$  is bounded and Borel measurable.

**Proposition 36.3** *If  $f$  is bounded and Borel measurable,  $s, t > 0$ , and  $x \in \mathcal{S}$ , then*

$$\mathbb{E}^x [\mathbb{E}^{X_t} f(X_s)] = \mathbb{E}^x f(X_{s+t}). \quad (36.4)$$

*Proof* The proof of (36.4) is mainly a matter of sorting out notation. Let  $\varphi(x) = \mathbb{E}^x f(X_s) = P_s f(x)$ . Hence  $\mathbb{E}^{X_t} f(X_s) = \varphi(X_t) = P_s f(X_t)$ . Then the left-hand side is  $\mathbb{E}^x (P_s f)(X_t) = P_t (P_s f)(x)$ . The right-hand side of (36.4) is  $P_{s+t} f(x)$ , and so the two sides agree by the semigroup property.  $\square$

*Step 2.* We so far only have  $X_t$  constructed for  $t \in \mathcal{D}$ . To extend the definition to all  $t$ , we want to let  $X_t = \lim_{u>t, u \in \mathcal{D}, u \rightarrow t} X_u$ . But before we can make that definition, we need to know that the limits exist. We will use the regularity of supermartingales to show this, so we need to look at conditional expectations. Let

$$\mathcal{F}'_s = \sigma(X_r; r \leq s, r \in \mathcal{D}).$$

**Proposition 36.4** *If  $s < t$  with  $s, t \in \mathcal{D}$  and  $f$  is bounded and Borel measurable, then*

$$\mathbb{E}^x [f(X_t) \mid \mathcal{F}'_s] = \mathbb{E}^{X_s} f(X_{t-s}), \quad \mathbb{P}^x\text{-a.s.} \quad (36.5)$$

*Proof* Take  $n \geq 1$ ,  $r_1 \leq r_2 \leq \dots \leq r_n \leq s$  with each  $r_j$  in  $\mathcal{D}$ , and  $A_1, \dots, A_n$  Borel subsets of  $\mathcal{S}$ . It suffices to show that

$$\mathbb{E}^x [f(X_t) 1_{A_1}(X_{r_1}) \dots 1_{A_n}(X_{r_n})] = \mathbb{E}^x [(\mathbb{E}^{X_s} f(X_{t-s})) 1_{A_1}(X_{r_1}) \dots 1_{A_n}(X_{r_n})], \quad (36.6)$$

since the events  $(X_{r_1} \in A_1, \dots, X_{r_n} \in A_n)$  generate  $\mathcal{F}'_s$ . The right-hand side of (36.6) is equal to

$$\mathbb{E}^x [P_{t-s} f(X_s) 1_{A_1}(X_{r_1}) \dots 1_{A_n}(X_{r_n})]. \quad (36.7)$$

From (36.3)

$$\begin{aligned} \mathbb{E}^x [P_{t-s} f(X_s) 1_{A_1}(X_{r_1}) \dots 1_{A_n}(X_{r_n})] &= \int \dots \int P_{t-s} f(y) 1_{A_1}(x_1) \dots 1_{A_n}(x_n) \\ &\quad \times P_{r_1}(x, dx_1) \dots P_{r_n-r_{n-1}}(x_{n-1}, x_n) P_{s-r_n}(x_n, dy). \end{aligned} \quad (36.8)$$

But  $P_{t-s} f(y) = \int f(z) P_{t-s}(y, dz)$ . Substituting this in (36.8) and using (36.3) again, we obtain the left-hand side of (36.6).  $\square$

*Step 3.* We define  $R_\lambda$ , the resolvent or  $\lambda$ -resolvent of  $P_t$ , by

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt. \quad (36.9)$$

**Lemma 36.5** *If  $f \geq 0$  is bounded and Borel measurable and  $x \in \mathcal{S}$ , then  $M_t = e^{-\lambda t} R_\lambda f(X_t)$ ,  $t \in \mathcal{D}$ , is a supermartingale with respect to the filtration  $\{\mathcal{F}'_t; t \in \mathcal{D}\}$  and the probability measure  $\mathbb{P}^x$ .*

*Proof* What we need to show is that if  $s < t \in \mathcal{D}$ , then

$$\mathbb{E}^x[e^{-\lambda t} R_\lambda f(X_t) | \mathcal{F}'_s] \leq e^{-\lambda s} R_\lambda f(X_s), \quad \mathbb{P}^x\text{-a.s.}$$

By Proposition 36.3 the left-hand side is

$$e^{-\lambda t} \mathbb{E}^X R_\lambda f(X_{t-s}),$$

so what we need to show is that

$$\mathbb{E}^y R_\lambda f(X_{t-s}) \leq e^{\lambda(t-s)} R_\lambda f(y) \tag{36.10}$$

for all  $y$ . The left-hand side of (36.10) is

$$\begin{aligned} P_{t-s} R_\lambda f(y) &= \int_0^\infty e^{-\lambda r} P_{t-s} P_r f(y) dr \\ &= \int_0^\infty e^{-\lambda r} P_{r+t-s} f(y) dr \\ &= e^{\lambda(t-s)} \int_{t-s}^\infty e^{-\lambda r} P_r f(y) dr \\ &\leq e^{\lambda(t-s)} \int_0^\infty e^{-\lambda r} P_r f(y) dr \\ &= e^{\lambda(t-s)} R_\lambda f(y). \end{aligned}$$

The first equality is the Fubini theorem, the second the semigroup property, and the third equality comes from a change of variables.  $\square$

Next, if  $f$  is non-negative and bounded, by Theorem 3.12 with  $\mathbb{P}$  replaced by  $\mathbb{P}^x$ , we see that  $e^{-\lambda t} R_\lambda f(X_t)$  has left and right limits along  $t \in \mathcal{D}$ . Therefore the same is true for  $R_\lambda f(X_t)$ .

By Assumption 36.1 and dominated convergence, we have that if  $f \in C_0$ ,

$$\begin{aligned} \lambda R_\lambda f(x) - f(x) &= \int_0^\infty e^{-\lambda t} (P_t f(x) - f(x)) dt \\ &= \int_0^\infty e^{-t} (P_{t/\lambda} f(x) - f(x)) dt \end{aligned}$$

tends to zero uniformly in  $x$  as  $\lambda \rightarrow 0$ . Take a countable dense subset  $\{f_i\}$  of  $C_0$  and look at  $jR_j f_i(X_t)$  for all positive integers  $j$ . Since  $jR_j f_i(X_t)$  has left and right limits along  $\mathcal{D}$ , a.s., letting  $j \rightarrow \infty$ , we see that  $f_i(X_t)$  does also. We conclude that  $X_t$  has left and right limits along  $\mathcal{D}$ .

Now define  $X_t = \lim_{u>t, u \in \mathcal{D}, u \rightarrow t} X_u$ . Then  $X_u$  is right continuous with left limits. We check that

$$\mathbb{P}^x(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n) = \int_{A_1} \cdots \int_{A_n} P_{t_1}(x, dx_1) \cdots P_{t_n-t_{n-1}}(x_{n-1}, dx_n).$$

To see this, we know this holds when the  $t_i$  are in  $\mathcal{D}$ . By linearity and a limit argument, we conclude

$$\mathbb{E}^x[f_1(X_{t_1}) \cdots f_n(X_{t_n})] = \int \cdots \int f(x_1) \cdots f(x_n) P_{t_1}(x, dx_1) \cdots P_{t_n-t_{n-1}}(x_{n-1}, dx_n) \quad (36.11)$$

when the  $f_i$  are bounded and continuous. Using a limit argument, we know (36.11) holds when the  $t_i$  are arbitrary non-negative real numbers. Using a limit argument again, (36.11) holds for all bounded and measurable  $f$ , in particular, when  $f_i = 1_{A_i}$ .

*Step 4.* It remains to show that  $(X_t, \mathbb{P}^x)$  satisfies Definition 19.1 and that  $P_t$  is the semigroup of this process. Let  $\mathcal{F}_t^{00} = \sigma(X_s; s \leq t)$ . Then we have already shown that  $(X_t, \mathbb{P}^x)$  is a Markov process with respect to the filtration  $\{\mathcal{F}_t^{00}\}$ , except for showing that

$$\mathbb{P}^x(X_{s+t} \in A \mid \mathcal{F}_s^{00}) = \mathbb{P}^{X_s}(X_t \in A).$$

However, this can be proved almost identically to the way we proved Proposition 36.4.

*Step 5.* Sometimes the semigroup is a contraction semigroup and satisfies Assumption 36.1 but not Assumption 36.2. In this case the  $P_t(x, A)$  are called *sub-Markov transition probability kernels*. The missing probability is due to the process being killed, and we can handle this situation as follows. Let  $\mathcal{S}_\Delta = \mathcal{S} \cup \{\Delta\}$ , where we introduce an isolated point  $\{\Delta\}$ . The topology on  $\mathcal{S}_\Delta$  is the one generated by the open sets on  $\mathcal{S}$  together with the set  $\{\Delta\}$ . Given a function  $f$  on  $\mathcal{S}$ , we extend it to  $\mathcal{S}_\Delta$  by setting  $f(\Delta) = 0$ . We replace  $P_t(x, A)$  by  $\bar{P}_t(x, A)$ , where

$$\begin{cases} \bar{P}_t(x, A) = P_t(x, A), & x \in \mathcal{S}, \quad A \subset \mathcal{S}, \\ \bar{P}_t(x, \{\Delta\}) = 1 - P_t(x, \mathcal{S}), & x \in \mathcal{S}, \\ \bar{P}_t(\Delta, \{\Delta\}) = 1. \end{cases} \quad (36.12)$$

One can go through the above construction with  $\bar{P}_t$  and obtain a strong Markov process  $X_t$  whose state space is  $\mathcal{S}_\Delta$ . It is not hard to show that starting at  $\Delta$ , the process stays at  $\Delta$  forever; see Exercise 36.1.

We remark that by the results of Chapter 20 and also Exercise 20.1, we can expand the filtration from  $\{\mathcal{F}_t^{00}\}$  to  $\{\mathcal{F}_t\}$ , where  $\{\mathcal{F}_t\}$  is right continuous and each  $\mathcal{F}_t$  contains all the sets that are null with respect to each  $\mathbb{P}^x$ . In addition, the strong Markov property will hold for  $(X_t, \mathbb{P}^x)$ .

## 36.2 Examples

**Example 36.6** Our first example is a Brownian motion. Let

$$p(t, x, y) = (2\pi t)^{d/2} e^{-|x-y|^2/2t},$$

and set

$$P_t(x, A) = \int_A p(t, x, y) dy.$$

We know

$$\int p(t, x, z) p(s, z, y) dz = p(t+s, x, y)$$

by Proposition 19.5, and so  $P_t$  satisfies the semigroup property. We showed in Section 19.4 that Assumption 36.1 is satisfied, except for the fact that  $P_t$  maps  $C_0$  to  $C_0$ ; this is Exercise 36.2. Therefore we have a strong Markov process associated with  $P_t$ . By Proposition 21.5, the paths of the strong Markov process can be taken to be continuous. This gives yet another construction of a Brownian motion.

**Example 36.7** We now use the machinery we have developed in this chapter to construct the Poisson process. Define transition probabilities by

$$P_t(x, A) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} 1_A(x+k),$$

where  $\lambda$  is some fixed parameter. If  $p(t, k) = e^{-\lambda t} (\lambda t)^k / k!$ , then

$$P_t f(x) = \sum_{k=0}^{\infty} f(x+k) p(t, k). \quad (36.13)$$

Thus

$$P_s(P_t f)(x) = \sum_{j=0}^{\infty} P_t f(x+j) p(s, j) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} f(x+j+k) p(t, k) p(s, j).$$

This is equal to

$$\sum_{m=0}^{\infty} f(x+m) \sum_{k=0}^m p(t, m-k) p(s, k), \quad (36.14)$$

which by Exercise 36.3 is equal to

$$\sum_{m=0}^{\infty} f(x+m) p(s+t, m) = P_{s+t} f(x). \quad (36.15)$$

Therefore the semigroup property holds.

We therefore have a strong Markov process  $X$  whose paths are right continuous with left limits. We want to show that the process  $X_t$  under the probability measure  $\mathbb{P}^0$  is a Poisson process. That  $\mathbb{P}^0(X_0 = 0) = 1$  is obvious. We need to show that Definition 5.1(3) and (4) hold. For the former,

$$\mathbb{P}^0(X_t - X_s = k) = \sum_{j=0}^{\infty} \mathbb{P}(X_t = j+k, X_s = j) \quad (36.16)$$

$$= \sum_{j=0}^{\infty} p(s, j) p(t-s, k) = p(t-s, k), \quad (36.17)$$

as desired. For Definition 5.1(4), suppose  $r_1 \leq r_2 \leq \dots \leq r_n \leq s < t$ ,  $a_1, \dots, a_n$  are integers, and let  $A = (X_{r_1} = a_1, \dots, X_{r_n} = a_n)$ . We will be done if we show

$$\mathbb{P}^0(X_t - X_s = k, A) = \mathbb{P}^0(X_t - X_s = k) \mathbb{P}^0(A). \quad (36.18)$$

The left-hand side of (36.18) is equal to

$$\begin{aligned}
 \sum_{j=0}^{\infty} \mathbb{P}^0(X_t = j+k, X_s = j, A) &= \sum_{j=0}^{\infty} \mathbb{E}^0[\mathbb{P}^0(X_t = j+k | \mathcal{F}_s); X_s = j, A] \\
 &= \sum_{j=0}^{\infty} \mathbb{E}^0[\mathbb{P}^{X_s}(X_{t-s} = j+k); X_s = j, A] \\
 &= \sum_{j=0}^{\infty} \mathbb{E}^0[\mathbb{P}^j(X_{t-s} = j+k); X_s = j, A] \\
 &= \sum_{j=0}^{\infty} \mathbb{E}^0[p(t-s, k); X_s = j, A] \\
 &= p(t-s, k)\mathbb{P}^0(A).
 \end{aligned}$$

Together with (36.16) this proves (36.18).

### Exercises

- 36.1 Suppose  $P_t$  is a family of sub-Markov transition probabilities and we define  $\bar{P}_t$  by (36.12). Show that  $\bar{P}_t$  is a family of Markov transition probabilities. Show that  $\mathbb{P}^\Delta(X_t \neq \Delta \text{ for some } t > 0) = 0$ , i.e., starting at  $\Delta$ , the process stays there forever.
- 36.2 Show that if  $P_t(x, A)$  is defined by (19.17), and  $P_t f(x) = \int f(y) P_t(x, dy)$ , then  $P_t$  maps  $C_0$  into  $C_0$ .
- 36.3 Show that (36.14) equals (36.15).
- 36.4 Show that  $P_t$  defined by (36.13) satisfies all the parts of Assumption 36.1.
- 36.5 Suppose  $\{\mu_t, t \geq 0\}$  is a tight family of probability measures on the real line. Suppose there exists a function  $\psi : \mathbb{R} \rightarrow \mathbb{C}$  such that the Fourier transforms of the  $\mu_t$  have the following form:

$$\int e^{iux} \mu_t(dx) = e^{t\psi(u)}, \quad t \geq 0, u \in \mathbb{R}.$$

(1) Prove that  $\mu_t$  converges weakly to  $\mu_0$  as  $t \rightarrow 0$ . Note that  $\mu_0$  is the same as point mass at 0.

(2) Define the operators  $P_t$  by

$$P_t f(x) = \int f(x-y) \mu_t(dy).$$

Prove that the  $P_t$  form a strongly continuous semigroup of contraction operators mapping  $C_0$  into  $C_0$ . Conclude that there exists a strong Markov process whose semigroup is given by the  $P_t$ .

This semigroup is called a *convolution semigroup* because  $\mu_{t+s} = \mu_t * \mu_s$ , in the sense of convolution of measures. We will see later that these are associated with Lévy processes.

### Notes

See Blumenthal and Getoor (1968) for further information.

## Infinitesimal generators

Often a Markov process is specified in terms of its behavior at each point, and one wants to form a global picture of the process. This means one is given the infinitesimal generator, which is a linear operator that is an unbounded operator in general, and one wants to come up with the semigroup for the Markov process.

We will begin by looking further at semigroups and resolvents, and then define the infinitesimal generator of a semigroup. We will prove the Hille–Yosida theorem, which is the primary tool for constructing semigroups from infinitesimal generators. Then we will look at two important examples: elliptic operators in nondivergence form and Lévy processes.

### 37.1 Semigroup properties

Let  $\mathcal{S}$  be a locally compact separable metric space. We will take  $\mathcal{B}$  to be a separable Banach space of real-valued functions on  $\mathcal{S}$ . For the most part, we will take  $\mathcal{B}$  to be the continuous functions on  $\mathcal{S}$  that vanish at infinity (with the supremum norm), although another common example is to let  $\mathcal{B}$  be the set of functions on  $\mathcal{S}$  that are in  $L^2$  with respect to some measure. We use  $\|\cdot\|$  for the norm on  $\mathcal{B}$ .

For the duration of this chapter we will make the following assumption.

**Assumption 37.1** Suppose that  $P_t$ ,  $t \geq 0$ , are operators acting on  $\mathcal{B}$  such that

- (1) the  $P_t$  are contractions:  $\|P_t f\| \leq \|f\|$  for all  $t \geq 0$  and all  $f \in \mathcal{B}$ ,
- (2) the  $P_t$  form a semigroup:  $P_s P_t = P_{t+s}$  for all  $s, t \geq 0$ , and
- (3) the  $P_t$  are strongly continuous: if  $f \in \mathcal{B}$ , then  $P_t f \rightarrow f$  as  $t \rightarrow 0$ .

Note that the semigroup property implies in particular that  $P_s$  and  $P_t$  commute. For a bounded operator  $A$  on  $\mathcal{B}$ ,  $\|A\| = \sup\{\|Af\| : \|f\| \leq 1\}$ , so saying  $P_t$  is a contraction is the same as saying  $\|P_t\| \leq 1$ .

Define the *resolvent* or  $\lambda$ -*resolvent* operator of a semigroup  $P_t$  by

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt. \quad (37.1)$$

The resolvent equation is

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu. \quad (37.2)$$

We show that the semigroup property implies the resolvent equation.

**Proposition 37.2** *The resolvent equation (37.2) holds.*

*Proof* We write

$$\begin{aligned}
 R_\lambda(R_\mu f)(x) &= \int_0^\infty e^{-\lambda t} P_t(R_\mu f)(x) dt \\
 &= \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s} P_t(P_s f)(x) ds dt \\
 &= \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s} P_{t+s} f(x) ds dt \\
 &= \int_0^\infty e^{-\lambda t} e^{\mu t} \int_t^\infty e^{-\mu s} P_s f(x) ds dt \\
 &= \int_0^\infty \int_0^s e^{-(\lambda-\mu)t} e^{-\mu s} P_s f(x) dt ds \\
 &= \int_0^\infty \frac{1 - e^{-(\lambda-\mu)s}}{\lambda - \mu} e^{-\mu s} P_s f(x) ds \\
 &= \frac{1}{\mu - \lambda} \left[ \int_0^\infty e^{-\lambda s} P_s f(x) ds - \int_0^\infty e^{-\mu s} P_s f(x) ds \right] \\
 &= \frac{1}{\mu - \lambda} [R_\lambda f(x) - R_\mu f(x)].
 \end{aligned}$$

The second equality uses Exercise 37.2, the fourth a change of variables, and the fifth the Fubini theorem.  $\square$

We have the following corollary to Proposition 37.2.

**Corollary 37.3** *If  $\mu, \lambda > 0$  and  $|\mu - \lambda| < \lambda$ , then*

$$R_\mu f = R_\lambda f + \sum_{i=1}^{\infty} (\lambda - \mu)^i R_\lambda^{i+1} f. \quad (37.3)$$

Here  $R_\lambda^2 f = R_\lambda(R_\lambda f)$ , and similarly for  $R_\lambda^i f$ .

*Proof* By Proposition 37.2, we have

$$R_\mu f = R_\lambda f + (\lambda - \mu) R_\lambda R_\mu f. \quad (37.4)$$

If we substitute for  $R_\mu f$  in the last term on the right-hand side of (37.4), we have

$$R_\mu f = R_\lambda f + (\lambda - \mu) R_\lambda R_\lambda f + (\lambda - \mu)^2 R_\lambda R_\lambda R_\mu f.$$

We again substitute for  $R_\mu f$ , and repeat. Since

$$\|(\lambda - \mu)R_\lambda\| \leq \frac{|\lambda - \mu|}{\lambda},$$

which is less than one in absolute value,  $(\lambda - \mu)^i R_\lambda^{i+1} R_\mu f$  converges to zero as  $i \rightarrow \infty$  and the series converges.  $\square$

**Remark 37.4** In particular, if  $R_\lambda$  and  $S_\lambda$  are two resolvents that agree at one value of  $\lambda$ , say  $\lambda_0$ , then Corollary 37.3 applied once with  $R_\lambda$  and once with  $S_\lambda$  implies that if  $\lambda < 2\lambda_0$ , then

$$\begin{aligned} R_\lambda f &= R_{\lambda_0} f + \sum_{i=1}^{\infty} (\lambda_0 - \lambda)^i (R_{\lambda_0})^{i+1} f \\ &= S_{\lambda_0} f + \sum_{i=1}^{\infty} (\lambda_0 - \lambda)^i (S_{\lambda_0})^{i+1} f = S_\lambda f, \end{aligned}$$

or  $R_\lambda$  and  $S_\lambda$  agree for  $\lambda < 2\lambda_0$ . Applying this observation again with  $\lambda_0$  replaced by  $3\lambda_0/2$ , then  $R_\lambda$  and  $S_\lambda$  agree for  $\lambda < 3\lambda_0$ . Continuing this argument, we see that  $R_\lambda$  and  $S_\lambda$  must agree for each positive value of  $\lambda$ .

If for some  $f \in \mathcal{B}$ ,

$$\left\| \frac{P_h f - f}{h} - g \right\| \rightarrow 0$$

as  $h \rightarrow 0$ , we say that  $f$  is in the domain of the *infinitesimal generator* of the semigroup, we write  $g = \mathcal{L}f$  and write  $f \in \mathcal{D} = \mathcal{D}(\mathcal{L})$ . Generally  $\mathcal{D}(\mathcal{L})$  is a proper subset of  $\mathcal{B}$ . If  $f \in \mathcal{D}$  and  $t > 0$ , then

$$\frac{P_h P_t f - P_t f}{h} = \frac{P_t P_h f - P_t f}{h} = P_t \left( \frac{P_h f - f}{h} \right) \rightarrow P_t \mathcal{L} f, \quad (37.5)$$

since  $P_t$  is a contraction. Therefore  $P_t f \in \mathcal{D}$  when  $f \in \mathcal{D}$  and  $\mathcal{L}(P_t f) = P_t(\mathcal{L}f)$ .

**Proposition 37.5** Fix  $\lambda > 0$  and let  $C = \{R_\lambda f : f \in \mathcal{B}\}$ . Then  $C = \mathcal{D}(\mathcal{L})$  and for  $f \in \mathcal{B}$ ,

$$\mathcal{L}R_\lambda f = \lambda R_\lambda f - f.$$

*Proof* Suppose that  $g \in C$ , so that  $g = R_\lambda f$  for some  $f \in \mathcal{B}$ . Then

$$P_h R_\lambda f = \int_0^\infty e^{-\lambda t} P_{h+t} f dt = e^{\lambda h} \int_h^\infty e^{-\lambda t} P_t f dt, \quad (37.6)$$

and so

$$P_h g - g = P_h R_\lambda f - R_\lambda f = (e^{\lambda h} - 1) \int_h^\infty e^{-\lambda t} P_t f dt - \int_0^h e^{-\lambda t} P_t f dt. \quad (37.7)$$

Dividing by  $h$  and letting  $h \rightarrow 0$ , the first term on the right of (37.7) converges (use Exercise 37.2) to

$$\lambda \int_0^\infty e^{-\lambda t} P_t f dt = R_\lambda f.$$

Since  $f \in \mathcal{B}$ , then  $P_t f \rightarrow f$  as  $t \rightarrow 0$ . After dividing by  $h$ , the second term on the right-hand side of (37.7) converges to  $f$ . Thus

$$\mathcal{L}(R_\lambda f) = \lambda R_\lambda f - f, \quad (37.8)$$

as required.

We have shown that  $C \subset \mathcal{D}(\mathcal{L})$ , and we now show the opposite inclusion. Suppose  $f \in \mathcal{D}(\mathcal{L})$ . Let  $g = \lambda f - \mathcal{L}f$ , which is in  $\mathcal{B}$ . Since  $P_t$  and  $\mathcal{L}$  commute, then  $R_\lambda$  and  $\mathcal{L}$  commute, and by (37.8),

$$\begin{aligned} f &= \lambda R_\lambda f - (\lambda R_\lambda f - f) = \lambda R_\lambda f - R_\lambda \mathcal{L}f \\ &= R_\lambda g, \end{aligned}$$

which is in  $C$ .  $\square$

**Example 37.6** Let us compute the infinitesimal generator when  $(X_t, \mathbb{P}^x)$  is a one-dimensional Brownian motion. For our space  $\mathcal{B}$  we take the continuous functions on  $\mathbb{R}$  that vanish at infinity. Suppose  $f \in C^2$  with compact support. By a Taylor series expansion,

$$P_h f(x) = \mathbb{E}^x f(X_h) = f(x) + f'(x)\mathbb{E}^x(X_h - x) + \frac{1}{2}f''(x)\mathbb{E}^x(X_h - x)^2 + R_h,$$

where  $R_h$  is the remainder term. We know  $R_h$  is bounded by

$$\|f''\|_\infty \mathbb{E}^x[\varphi(X_h - x)],$$

where  $\varphi$  is bounded and  $|\varphi(y)/y^2| \rightarrow 0$  as  $y \rightarrow 0$ . Since  $W_h$  started at  $x$  has mean  $x$  and variance  $h$ , we have

$$P_h f(x) = f(x) + \frac{1}{2}f''(x)h + R_h,$$

where  $|R_h/h|$  tends to zero as  $h \rightarrow 0$ . Therefore

$$\frac{P_h f - f}{h} \rightarrow \frac{1}{2}f'',$$

the convergence being with respect to the supremum norm. Exactly the same argument holds in higher dimensions to show that  $\mathcal{L}f = \frac{1}{2}\Delta f$ . We have shown that  $\mathcal{D}(\mathcal{L})$  contains the  $C^2$  functions with compact support, but have not actually identified the domain of the infinitesimal generator. We refer the reader to [Knight \(1981\)](#) for a detailed discussion.

The domain of an infinitesimal generator is nearly as important as the operator itself. We will briefly discuss aspects of the domains of the infinitesimal generator for absorbing Brownian motion and for reflecting Brownian motion on  $[0, \infty)$ . Both have the same operator  $\mathcal{L}f = \frac{1}{2}f''$  but different domains.

Absorbing Brownian motion on  $[0, \infty)$  is Brownian motion killed on first hitting  $(-\infty, 0)$ . Let  $W_t$  be standard Brownian motion on  $\mathbb{R}$  and let  $X_t$  be  $W_t$  killed on first hitting  $(-\infty, 0)$ . If  $f \in C^2[0, \infty)$  with  $f$  and its first and second derivatives being bounded and uniformly continuous and  $x \neq 0$ ,  $(\mathbb{E}^x f(X_t) - f(x))/t$  differs from  $(\mathbb{E}^x f(W_t) - f(x))/t$  by at most

$$\|f\|_\infty \mathbb{P}^x(T_0 < t)/t,$$

where  $T_0$  is the first time  $W_t$  hits  $(-\infty, 0)$ . If  $x \neq 0$ ,

$$\frac{\mathbb{P}^x(T_0 < t)}{t} \leq \frac{\mathbb{P}^x(\sup_{s \leq t} |W_s - W_0| \geq x)}{t} \leq \frac{2}{t} e^{-x^2/2t} \rightarrow 0$$

as  $t \rightarrow 0$ . Therefore for  $x \neq 0$ , the infinitesimal generator of absorbing Brownian motion is the same as the infinitesimal generator of standard Brownian motion, namely,  $\frac{1}{2}f''(x)$ .

If  $f = R_\lambda g$  for  $g$  bounded and continuous, we have

$$f(0) = R_\lambda g(0) = \mathbb{E}^0 \int_0^{T_0} e^{-\lambda t} g(X_t) dt = 0.$$

We use the fact that starting at 0,  $T_0 = 0$ , a.s., by Theorem 7.2. Using Proposition 37.5, every function in the domain of the infinitesimal generator of absorbing Brownian motion must satisfy  $f(0) = 0$ .

We can define reflecting Brownian motion on  $[0, \infty)$  by  $X_t = |W_t|$ , where  $W$  is a one-dimensional Brownian motion on  $\mathbb{R}$ . As in the preceding paragraph, the infinitesimal generator for  $X$  agrees with  $\frac{1}{2}f''(x)$  if  $x \neq 0$ . For  $x = 0$ , an application of Taylor's theorem gives

$$\mathbb{E}^0 f(|W_t|) = f(0) + f'(0)\mathbb{E}^0 |W_t| + \frac{1}{2}f''(0)\mathbb{E}^0 |W_t|^2 + \mathbb{E}^0 R_t,$$

where  $R_t$  is a remainder term. Subtracting  $f(0)$  from both sides and dividing by  $t$ , and noting  $\mathbb{E}^0 |W_t|/t = c_1\sqrt{t}/t \rightarrow \infty$  as  $t \rightarrow 0$ , the only way we can get convergence is if  $f'(0) = 0$ . Thus every function in the domain of the infinitesimal generator of reflecting Brownian motion must satisfy  $f'(0) = 0$ .

In higher dimensions, the analogous restriction for reflecting Brownian motion is that the normal derivative  $\partial f / \partial n$  must equal zero on the boundary of the domain, where  $n$  is the inward-pointing unit normal vector. In the partial differential equations literature, this is known as the *Neumann boundary condition*, and models situations where there is no heat flow across the boundary. For absorbing Brownian motion the analogous restriction is that  $f = 0$  on the boundary of the domain, and this is called the *Dirichlet boundary condition*.

**Example 37.7** Next we compute the generator for a Poisson process with parameter  $\lambda$ . We can let  $\mathcal{B}$  be as in Example 37.6. We have

$$\begin{aligned} P_h f(x) &= \sum_{i=0}^{\infty} e^{-\lambda h} \frac{(\lambda h)^i}{i!} f(x+i) \\ &= e^{-\lambda h} f(x) + e^{-\lambda h} \lambda h f(x+1) + \sum_{i=2}^{\infty} e^{-\lambda h} \frac{(\lambda h)^i}{i!} f(x+i). \end{aligned}$$

Subtracting  $f(x)$  from both sides, dividing by  $h$ , and letting  $h \rightarrow 0$ , we obtain

$$\mathcal{L}f(x) = -\lambda f(x) + \lambda f(x+1) = \lambda[f(x+1) - f(x)].$$

In this case the domain of  $\mathcal{L}$  is all of  $\mathcal{B}$ .

A very useful result is Dynkin's formula.

**Theorem 37.8** Suppose  $P_t$  operating on the space  $\mathcal{B}$  of continuous functions vanishing at infinity is the semigroup of a Markov process  $(X_t, \mathbb{P}^x)$ ,  $f \in \mathcal{D}(\mathcal{L})$ , and  $f$  and  $\mathcal{L}f$  are bounded. If  $x \in \mathcal{S}$  and  $T$  is a stopping time with  $\mathbb{E}^x T < \infty$ , then

$$\mathbb{E}^x f(X_T) - f(x) = \mathbb{E}^x \int_0^T \mathcal{L}f(X_r) dr.$$

*Proof* If  $f \in \mathcal{D}(\mathcal{L})$ , then  $\mathcal{L}f \in \mathcal{B}$ , and so  $P_t \mathcal{L}f$  is continuous in  $t$ . Moreover, as we saw in (37.5),

$$\frac{\partial}{\partial t} P_t f(y) = P_t \mathcal{L}f(y).$$

By the fundamental theorem of calculus,

$$P_t f(y) - f(y) = \int_0^t P_r \mathcal{L}f(y) dr,$$

which can be rewritten as

$$\mathbb{E}^y f(X_t) - f(y) = \mathbb{E}^y \int_0^t \mathcal{L}f(X_r) dr; \quad (37.9)$$

we used the Fubini theorem here as well. This holds for each  $y \in \mathcal{S}$  and each  $t > 0$ .

Set  $M_t = f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_r) dr$ . What (37.9) says is that  $\mathbb{E}^y M_t = 0$  for all  $y$  and all  $t$ . By the Markov property,

$$\begin{aligned} \mathbb{E}^x [M_t - M_s \mid \mathcal{F}_s] &= \mathbb{E}^x \left[ f(X_t) - f(X_s) - \int_s^t \mathcal{L}f(X_r) dr \mid \mathcal{F}_s \right] \\ &= \mathbb{E}^x \left[ \left( f(X_{t-s}) - f(X_0) - \int_0^{t-s} \mathcal{L}f(X_r) dr \right) \circ \theta_s \mid \mathcal{F}_s \right] \\ &= \mathbb{E}^{X_s} M_{t-s} = 0. \end{aligned}$$

Therefore  $M_t$  is a martingale with respect to  $\mathbb{P}^x$  for each  $x$ . If  $T$  is a bounded stopping time, then by optional stopping,  $\mathbb{E}^x M_T = 0$ . If  $T$  is instead only integrable with respect to  $\mathbb{P}^x$ , we have  $\mathbb{E}^x M_{T \wedge n} = 0$  for each  $n$ . We then let  $n \rightarrow \infty$  and use the fact that  $f$  and  $\mathcal{L}f$  are bounded to conclude  $\mathbb{E}^x M_T = 0$ , which is what we want.  $\square$

We say a few words about the Kolmogorov backward and forward equations. Suppose the semigroup  $P_t$  can be written

$$P_t f(x) = \int f(y) p(t, x, y) dy,$$

for functions  $p(t, x, y)$ , which are called *transition densities*. Provided there are no difficulties interchanging integration and differentiation, the equation

$$\frac{\partial}{\partial t} P_t f(x) = \mathcal{L} P_t f(x)$$

can be rewritten as

$$\int f(y) \frac{\partial}{\partial t} p(t, x, y) dy = \int f(y) \mathcal{L} p(t, x, y) dy,$$

which leads to the *Kolmogorov backward equation*

$$\frac{\partial}{\partial t} p(t, x, y) = \mathcal{L} p(t, x, y),$$

where  $\mathcal{L}$  operates on the  $x$  variable and  $y$  is held fixed.

If  $\mathcal{L}$  has an adjoint operator  $\mathcal{L}^*$ , which means  $\int f(\mathcal{L}g) = \int (\mathcal{L}^*f)g$  for  $f$  and  $g$  in the domains of  $\mathcal{L}^*$  and  $\mathcal{L}$ , respectively, the equation

$$\frac{\partial}{\partial t} P_t f(x) = P_t \mathcal{L} f(x)$$

can be rewritten as

$$\int f(y) \frac{\partial}{\partial t} p(t, x, y) dy = \int \mathcal{L} f(y) p(t, x, y) dy = \int f(y) \mathcal{L}^* p(t, x, y) dy,$$

which leads to the *Kolmogorov forward equation*

$$\frac{\partial}{\partial t} p(t, x, y) = \mathcal{L}^* p(t, x, y),$$

where  $\mathcal{L}^*$  operates on the  $y$  variable and  $x$  is held fixed.

### 37.2 The Hille–Yosida theorem

We now show how to construct a semigroup given the infinitesimal generator. We start with a few preliminary observations. If  $A$  is a bounded operator, we can define

$$e^A = I + A + A^2/2! + \cdots = \sum_{i=0}^{\infty} A^i/i!$$

To see that the series converges, note that

$$\left\| \sum_{i=m}^n A^i/i! \right\| \leq \sum_{i=m}^n \|A^i\|/i! \leq \sum_{i=m}^{\infty} \|A\|^i/i!,$$

which will be small if  $m$  is large since  $\|A\|$  is a finite number. Similarly,

$$\|e^A\| \leq \sum_{i=0}^{\infty} \|A\|^i/i! = e^{\|A\|}.$$

**Proposition 37.9** Suppose  $\{R_\lambda\}$  is a family of bounded operators defined on  $\mathcal{B}$  such that

- (1) the resolvent equation holds,
- (2)  $\|R_\lambda\| \leq 1/\lambda$  for each  $\lambda > 0$ , and
- (3)  $\|\lambda R_\lambda f - f\| \rightarrow 0$  as  $\lambda \rightarrow \infty$  for each  $f \in \mathcal{B}$ .

Then there exists a strongly continuous semigroup  $P_t$  whose resolvent is  $R_\lambda$ .

*Proof* Let  $D_\lambda = \lambda(\lambda R_\lambda - I)$  and  $Q_t^\lambda = e^{tD_\lambda}$ . Note that the resolvent equation implies that  $D_\lambda$  and  $D_\mu$  commute and therefore all the operators  $D_\lambda$ ,  $Q_t^\lambda$ ,  $D_\mu$ , and  $Q_t^\mu$  commute. Since  $\|\lambda R_\lambda\| \leq 1$ , then

$$\|Q_t^\lambda\| = e^{-\lambda t} \|e^{t\lambda^2 R_\lambda}\| \leq e^{-\lambda t} e^{\|t\lambda^2 R_\lambda\|} \leq e^{-\lambda t} e^{\lambda t} = 1.$$

We first show that the set of  $f$  such that  $D_\lambda f$  converges as  $\lambda \rightarrow \infty$  is a dense subset of  $\mathcal{B}$ . If  $f = R_a g$  for some  $a > 0$  and some  $g \in \mathcal{B}$ , then by the resolvent equation

$$\begin{aligned} D_\lambda f &= \lambda(\lambda R_\lambda - I)(R_a g) = \lambda^2 R_\lambda R_a g - \lambda R_a g \\ &= \frac{\lambda^2}{\lambda - a} (R_a g - R_\lambda g) - \lambda R_a g. \end{aligned}$$

We have

$$\frac{\lambda^2}{\lambda - a} R_\lambda g = \frac{\lambda}{\lambda - a} \lambda R_\lambda g \rightarrow g$$

as  $\lambda \rightarrow \infty$  by hypothesis (3) and

$$\frac{\lambda^2}{\lambda - a} R_a g - \lambda R_a g = \frac{\lambda}{\lambda - a} a R_a g \rightarrow a R_a g$$

as  $\lambda \rightarrow \infty$ . Therefore

$$D_\lambda R_a g \rightarrow a R_a g - g. \quad (37.10)$$

Thus  $D_\lambda$  converges on  $E = \cup_{a>0} \{R_a f : f \in \mathcal{B}\}$ . But for any  $f \in \mathcal{B}$ ,  $aR_a f \rightarrow f$  as  $a \rightarrow \infty$  and  $aR_a f = R_a(af) \in E$ , which proves that  $E$  is a dense subset of  $\mathcal{B}$ .

Next we show that if  $D_\lambda f$  converges, then  $Q_t^\lambda f$  converges. Suppose  $D_\lambda f$  converges and  $\varepsilon > 0$ . Choose  $M$  such that if  $\lambda, \mu \geq M$ , then  $\|D_\lambda f - D_\mu f\| < \varepsilon$ . Since  $\partial Q_t^\lambda f / dt = D_\lambda Q_t^\lambda f$  and  $Q_0^\lambda, Q_0^\mu$  are both the identity operator, we have

$$\begin{aligned} Q_t^\lambda f - Q_t^\mu f &= \int_0^t \frac{\partial}{\partial s} (Q_s^\lambda Q_{t-s}^\mu f) ds \\ &= \int_0^t [Q_s^\lambda D_\lambda Q_{t-s}^\mu f - Q_s^\lambda D_\mu Q_{t-s}^\mu f] ds \\ &= \int_0^t [Q_s^\lambda Q_{t-s}^\mu (D_\lambda f - D_\mu f)] ds, \end{aligned}$$

so

$$\|Q_t^\lambda f - Q_t^\mu f\| \leq t \|D_\lambda f - D_\mu f\| < \varepsilon t,$$

using that  $Q_s^\lambda$  and  $Q_{t-s}^\mu$  are contractions.

Since  $\varepsilon$  is arbitrary, this proves that  $Q_t^\lambda f$  is a Cauchy sequence in  $\mathcal{B}$  and hence converges. Call the limit  $P_t f$ . We can easily check that  $Q_t^\lambda$  is a semigroup for each  $\lambda > 0$  and we saw that  $Q_t^\lambda$  is a contraction for each  $t$  and  $\lambda$ . It follows that  $P_t$  is a semigroup and that the norm of each  $P_t$  is bounded by 1. Each  $Q_t^\lambda$  is strongly continuous, and by the uniform convergence, it follows that  $P_t f \rightarrow f$  as  $t \rightarrow 0$  for  $f \in E$ . Since each  $P_t$  is a contraction and  $E$  is dense in  $\mathcal{B}$ , we can extend each  $P_t$  so as to have domain  $\mathcal{B}$  and so that the  $P_t$  will be a strongly continuous semigroup on  $\mathcal{B}$ .

Let  $S_\lambda$  be the resolvent for  $P_t$ . It remains to prove that  $S_\lambda = R_\lambda$ . Fix  $a$  and let  $f = R_a g$ . We saw in (37.10) that  $D_\lambda R_a g \rightarrow a R_a g - g$ . Now  $Q_t^\lambda$  is a semigroup for each  $\lambda$  and by Exercise 37.4, the infinitesimal generator of  $Q_t^\lambda$  is  $D_\lambda$ . By the fundamental theorem of calculus,

$$Q_t^\lambda (R_a g) - R_a g = \int_0^t \frac{\partial}{\partial s} (Q_s^\lambda R_a g) ds = \int_0^t Q_s^\lambda (D_\lambda R_a g) ds.$$

Letting  $\lambda \rightarrow \infty$ ,

$$P_t (R_a g) - R_a g = \int_0^t P_s (a R_a g - g) ds.$$

Let  $b < a$ . Multiply the above equation by  $e^{-bt}$  and integrate over  $t$  from 0 to  $\infty$ . Then

$$\begin{aligned} S_b(R_a g) - \frac{1}{b} R_a g &= \int_0^\infty e^{-bt} \int_0^t P_s(aR_a g - g) ds dt \\ &= \int_0^\infty \int_s^\infty e^{-bt} P_s(aR_a g - g) dt ds \\ &= \int_0^\infty \frac{1}{b} e^{-bs} P_s(aR_a g - g) ds \\ &= \frac{1}{b} S_b(aR_a g - g). \end{aligned}$$

Therefore

$$S_b g = R_a g + (a - b) S_b R_a g.$$

Applying this with  $g$  replaced by  $R_a g$ , iterating, and using Corollary 37.3, we obtain

$$S_b g = R_a g + (a - b) R_a^2 g + (a - b)^2 R_a^3 g + \dots = R_b g.$$

By Remark 37.4, this proves  $S_b = R_b$  for all  $b$ .  $\square$

We now show that under appropriate hypotheses on  $\mathcal{L}$ , there exists a semigroup whose infinitesimal generator is  $\mathcal{L}$ . This is known as the Hille–Yosida theorem. We say that an operator  $\mathcal{L}$  is *dissipative* if

$$\|(\lambda - \mathcal{L})f\| \geq \lambda \|f\|, \quad f \in \mathcal{D}(\mathcal{L}). \quad (37.11)$$

**Theorem 37.10** Suppose  $\mathcal{L}$  is an operator such that

- (1) the domain of  $\mathcal{L}$  is a dense subset of  $\mathcal{B}$ ,
- (2) the range of  $\lambda - \mathcal{L}$  is  $\mathcal{B}$  for each  $\lambda$ , and
- (3)  $\mathcal{L}$  is dissipative.

Then there exists a semigroup on  $\mathcal{B}$  which has  $\mathcal{L}$  as its infinitesimal generator.

*Proof* If  $(\lambda - \mathcal{L})f = (\lambda - \mathcal{L})g$ , then

$$\lambda \|f - g\| \leq \|(\lambda - \mathcal{L})(f - g)\| = 0,$$

or  $f = g$ . Thus  $\lambda - \mathcal{L}$  is a one-to-one map, hence is invertible because the range of  $\lambda - \mathcal{L}$  is  $\mathcal{B}$ . We let  $R_\lambda$  be the inverse, and thus the domain of  $R_\lambda$  is all of  $\mathcal{B}$ .

We first show that the resolvent equation holds. We observe

$$(\mu - \mathcal{L}) \frac{1}{\lambda - \mu} R_\mu f = \frac{1}{\lambda - \mu} f$$

and

$$\begin{aligned} (\mu - \mathcal{L}) \frac{1}{\lambda - \mu} R_\lambda f &= (\mu - \lambda) \frac{1}{\lambda - \mu} R_\lambda f + (\lambda - \mathcal{L}) \frac{1}{\lambda - \mu} R_\lambda f \\ &= -R_\lambda f + \frac{1}{\lambda - \mu} f. \end{aligned}$$

Combining,

$$(\mu - \mathcal{L}) R_\mu R_\lambda f = R_\lambda f = (\mu - \mathcal{L}) \frac{1}{\lambda - \mu} (R_\mu - R_\lambda) f.$$

Applying  $R_\mu$  to both sides yields the resolvent equation.

The hypothesis that  $\|\lambda - \mathcal{L}\|f \geq \lambda\|f\|$  immediately implies  $\|R_\lambda f\| \leq \|f\|/\lambda$ .

We next show  $\lambda R_\lambda f \rightarrow f$  as  $\lambda \rightarrow \infty$ . If  $f \in \mathcal{D}$ , then

$$R_\lambda \mathcal{L} f = \mathcal{L} R_\lambda f = \lambda R_\lambda f - f,$$

and so

$$\|\lambda R_\lambda f - f\| \leq \frac{1}{\lambda} \|\mathcal{L} f\| \rightarrow 0$$

as  $\lambda \rightarrow \infty$ . Since  $\|\lambda R_\lambda\| \leq 1$  and the domain of  $\mathcal{L}$  is dense in  $\mathcal{B}$ , we conclude  $\lambda R_\lambda f \rightarrow f$  for all  $f \in \mathcal{B}$ .

We use Proposition 37.9 to construct  $P_t$ . By Proposition 37.9,  $R_\lambda$  is the resolvent for  $P_t$ . If  $\mathcal{M}$  is the infinitesimal generator for  $P_t$ , then by Proposition 37.5, the domain of  $\mathcal{M}$  is  $\{R_\lambda f : f \in \mathcal{B}\}$ . Since we know  $\mathcal{L}(R_\lambda f) = \lambda R_\lambda f - f \in \mathcal{B}$ , then the domain of  $\mathcal{L}$  contains  $\{R_\lambda f : f \in \mathcal{B}\}$ . Since  $\mathcal{M}$  is the infinitesimal generator of  $P_t$ , by Proposition 37.5,  $\mathcal{M}(R_\lambda f) = \lambda R_\lambda f - f$ . Therefore  $\mathcal{L}$  is an extension of  $\mathcal{M}$ .

If  $f \in \mathcal{D}(\mathcal{L})$ , then  $g = (\lambda - \mathcal{L})f \in \mathcal{B}$ , and thus

$$(\lambda - \mathcal{M})^{-1}g \in \mathcal{D}(\mathcal{M}) \subset \mathcal{D}(\mathcal{L}).$$

Hence

$$(\lambda - \mathcal{L})f = g = (\lambda - \mathcal{M})(\lambda - \mathcal{M})^{-1}g = (\lambda - \mathcal{L})(\lambda - \mathcal{M})^{-1}g.$$

Since  $\lambda - \mathcal{L}$  is one-to-one, then  $f = (\lambda - \mathcal{M})^{-1}g$ , which implies  $f \in \mathcal{D}(\mathcal{M})$ . Therefore  $\mathcal{M} = \mathcal{L}$  and so  $\mathcal{L}$  is the generator of  $P_t$ .  $\square$

When applying the Hille–Yosida theorem, it is quite often the case that it is easier to show that the range of  $\lambda - \mathcal{L}$  is only dense in  $\mathcal{B}$ , rather than being all of  $\mathcal{B}$ . When that occurs, one needs to look at a closed extension  $\overline{\mathcal{L}}$  of  $\mathcal{L}$ . An operator  $\overline{\mathcal{L}}$  is *closed* if whenever  $f_n \rightarrow f$  and  $\overline{\mathcal{L}}f_n \rightarrow g$ , then  $f \in \mathcal{D}(\overline{\mathcal{L}})$  and  $\overline{\mathcal{L}}f = g$ . To construct the closed extension of  $\mathcal{L}$ , where we assume that  $\mathcal{L}$  is dissipative (defined by (37.11)), let  $R_\lambda g = f$  if  $(\lambda - \mathcal{L})f = g$ .  $\mathcal{L}$  being dissipative is equivalent to the norm of  $R_\lambda$  being bounded by  $1/\lambda$  on the range of  $\lambda - \mathcal{L}$ , and so we can extend the domain of  $R_\lambda$  uniquely to all of  $\mathcal{B}$ . Now define  $\mathcal{D}(\overline{\mathcal{L}})$  to be the range of  $R_\lambda$  and set

$$\overline{\mathcal{L}}R_\lambda g = \lambda R_\lambda g - g. \quad (37.12)$$

We will soon give two examples where infinitesimal generators can be used to construct very useful processes. The first is where the infinitesimal generator is an elliptic operator of second order in non-divergence form. The second case studies the infinitesimal generators of Lévy processes.

We should mention that there is another important example where infinitesimal generators are useful in constructing a process, that of *infinite particle systems*. The name “infinite particle systems” refers to a class of models with discrete space and continuous time that are useful in mathematical biology and in statistical mechanics. One of the simplest examples is the voter model. Suppose at every point in  $\mathbb{Z}^2$ , the integer lattice in the plane, there is a voter, who is leaning either toward the Democrat candidate or the Republican candidate. At each point, the voter waits a length of time that is exponential with parameter one, chooses

one of his four nearest neighbors at random, and then changes his view to agree with that neighbor. Other infinite particle systems include the contact process (modeling the spread of infection), Ising model (modeling ferromagnetism), and the exclusion model (used in solid state physics). See [Liggett \(2010\)](#) for how to construct these processes using infinitesimal generators, and for much more.

### 37.3 Nondivergence form elliptic operators

Let us consider the operator  $\mathcal{L}$  defined on  $C^2$  functions on  $\mathbb{R}^d$  by

$$\mathcal{L}f(x) = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d b_i(x) \frac{\partial f}{\partial x_i}(x).$$

We suppose  $a_{ij}(x) = a_{ji}(x)$  for all  $x$ . We assume the  $a_{ij}$  and  $b_i$  are bounded and Hölder continuous of order  $\alpha \in (0, 1)$ : there exists  $c$  such that

$$|a_{ij}(x) - a_{ij}(y)| \leq c|x - y|^\alpha, \quad |b_i(x) - b_i(y)| \leq c|x - y|^\alpha,$$

for  $i, j = 1, \dots, d$ . We also assume a *uniform ellipticity* condition on the  $a_{ij}$ : there exists  $\Lambda > 0$  such that

$$\sum_{i,j=1}^d a_{ij}(x) y_i y_j \geq \Lambda \sum_{i=1}^d y_i^2, \quad (y_1, \dots, y_d) \in \mathbb{R}^d.$$

Uniform ellipticity says that the matrix whose  $(i, j)$ th element is  $a_{ij}(x)$  is positive definite, uniformly in  $x$ .

If the  $a_{ij}$  and  $b_i$  were Lipschitz continuous, we can construct the Markov process with infinitesimal generator  $\mathcal{L}$  using stochastic differential equations (see Chapter 39), which is a more probabilistic way of doing it. Even when the  $a_{ij}$  are continuous and the  $b_i$  only measurable, it is possible to construct the Markov process via SDEs, although this is much harder. Here we illustrate how the Hille–Yosida theorem can be used in constructing these processes.

Let  $\mathcal{B}$  be the space of continuous functions that vanish at infinity. We will want the domain of  $\mathcal{L}$  to include the class  $\mathcal{C}$  of functions  $f$  such that  $f$  and its first and second partial derivatives are continuous and vanish at infinity. Then  $\mathcal{C}$  is dense in  $\mathcal{B}$  and  $\mathcal{L}$  maps  $\mathcal{C}$  into  $\mathcal{B}$ .

We show that  $\mathcal{L}$  is dissipative. Let  $f \in \mathcal{C}$  and let  $x_0$  be a point where  $|f(x_0)| = \|f\|$ . There is nothing to prove if  $f$  is identically zero. If  $f(x_0) < 0$ , we can look at  $-f$ , so let us suppose  $f(x_0) > 0$ . Such a point  $x_0$  exists because  $f$  is continuous and vanishes at infinity. It suffices to show that  $\mathcal{L}f(x_0) \leq 0$ , since then

$$\lambda \|f\| = \lambda f(x_0) \leq (\lambda - \mathcal{L})f(x_0) \leq \|(\lambda - \mathcal{L})f\|.$$

Let  $A$  be the matrix whose  $(i, j)$  element is  $a_{ij}(x_0)$  and let  $H$  be the Hessian at  $x_0$  so that

$$H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0).$$

Let  $y \in \mathbb{R}^d$  and consider the function  $f(x_0 + ty)$ ,  $t \in \mathbb{R}$ . Since  $x_0$  is the location of a local maximum for this function, its second derivative, which is  $\sum_{i,j=1}^d y_i y_j H_{ij}$ , will be less than or equal to 0. The first derivative of this function will be zero at  $x_0$ .

Since  $A$  is positive definite, there exists an orthogonal matrix  $P$  and a diagonal matrix  $D$  with positive entries such that  $A = P^T D P$ . Recall the trace of a square matrix is defined by  $\text{Trace}(C) = \sum_{i=1}^d C_{ii}$  and  $\text{Trace}(AB) = \text{Trace}(BA)$ . Note

$$\sum_{i,j=1}^d a_{ij}(x_0) \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) = \text{Trace}(AH).$$

We have

$$\text{Trace}(AH) = \text{Trace}(P^T DPH) = \text{Trace}(PHP^T D) = \sum_{i=1}^d (PHP^T)_{ii} D_{ii},$$

since  $D$  is a diagonal matrix. Thus to show that  $\text{Trace}(AH) \leq 0$ , it suffices to show that  $(PHP^T)_{ii} \leq 0$  for each  $i$ . If we let  $e_i$  be the unit vector in the  $x_i$  direction and  $y = P^T e_i$ , we have

$$(PHP^T)_{ii} = e_i^T PHP^T e_i = y^T H y = \sum_{i,j=1}^d y_i y_j H_{ij} \leq 0.$$

Since  $x_0$  is the location of a local maximum, then  $\frac{\partial f}{\partial x_i}(x_0) = 0$ , and we conclude  $\mathcal{L}f(x_0) \leq 0$ .

Since  $\mathcal{L}1 = 0$ , then  $P_t 1 = 1$  for all  $t$ . This and Exercise 37.1 imply that the  $P_t$  are non-negative operators.

To apply the Hille–Yosida theorem, it remains to show that the range of  $\lambda - \mathcal{L}$  is dense in  $\mathcal{B}$ . For this we refer the reader to the PDE literature, e.g., Bass (1997), Chapter 3 or Gilbarg and Trudinger (1983), Chapters 5,6.

## 37.4 Generators of Lévy processes

Let  $n$  be a measure on  $\mathbb{R} \setminus \{0\}$  satisfying

$$\int (h^2 \wedge 1) n(dh) < \infty.$$

Consider the operator  $\mathcal{L}$  defined on  $C^2$  functions by

$$\mathcal{L}f(x) = \int [f(x+h) - f(x) - 1_{(|h| \leq 1)} f'(x)h] n(dh).$$

We will show that  $\mathcal{L}$  is the infinitesimal generator of a Markov semigroup. We construct these processes, the Lévy processes, probabilistically in Chapter 42. We confine ourselves to the one-dimensional case, although the argument for higher dimensions is completely analogous.

We let  $\mathcal{B}$  be the continuous functions vanishing at infinity. We let  $\mathcal{C}$  be the class of Schwartz functions, which is the class of  $C^\infty$  functions, all of whose  $k$ th partial derivatives go to zero faster than  $|x|^{-m}$  as  $|x| \rightarrow \infty$  for every  $k = 0, 1, \dots$  and every  $m = 1, 2, \dots$ ; see Section B.2.

First we show that  $\mathcal{L}$  maps  $\mathcal{C}$  into  $\mathcal{B}$ , so that the domain of  $\mathcal{L}$  contains  $\mathcal{C}$ , and hence is dense in  $\mathcal{B}$ . Given  $M > 1$  and  $f \in \mathcal{C}$ , by Taylor's theorem

$$\begin{aligned} |\mathcal{L}f(x)| &\leq \int |f(x+h) - f(x) - 1_{(|h| \leq 1)} f'(x)h| n(dh) \\ &\leq \sup_{|y-x| \leq 1} (|f''(y)|) \int_{0 < |h| \leq 1} h^2 n(dh) + 2 \left( \sup_{|y-x| \leq M} |f(y)| \right) \int_{1 < |h| \leq M} n(dh) \\ &\quad + 2 \int_{|h| > M} \|f\|_\infty n(dh). \end{aligned} \tag{37.13}$$

This shows  $|\mathcal{L}f(x)|$  is finite. Given  $\varepsilon > 0$  and  $f \in \mathcal{C}$ , choose  $M$  large so that

$$\int_{|h| > M} n(dh) < \varepsilon / \|f\|_\infty.$$

Since the first two terms on the right-hand side of (37.13) tend to zero as  $|x| \rightarrow \infty$ , then  $\mathcal{L} : \mathcal{C} \rightarrow \mathcal{B}$ .

To show  $\mathcal{L}$  is dissipative, let  $f \in \mathcal{C}$  and choose  $x_0$  such that  $|f(x_0)| = \|f\|$ . There is nothing to prove if  $\|f\| = 0$ , so assume  $\|f\| > 0$ . Because  $f$  is in the Schwartz class, it takes on its maximum and its minimum. By looking at  $-f$  if necessary, we may suppose  $f(x_0) > 0$ . Since  $x_0$  is the location of a local maximum,  $f'(x_0) = 0$  and  $f(x_0 + h) - f(x_0) \leq 0$  for each  $h$ , hence  $\mathcal{L}f(x_0) \leq 0$ . Then

$$\lambda \|f\| = \lambda f(x_0) \leq (\lambda - \mathcal{L})f(x_0) \leq \|(\lambda - \mathcal{L})f\|.$$

Taking limits, this holds for every  $f$  in the domain of  $\mathcal{L}$ .

Finally we need to show that the range of  $\lambda - \mathcal{L}$  is dense in  $\mathcal{B}$ . This is the most complicated part and we break the argument into steps.

*Step 1.* We start by computing the Fourier transform of  $\mathcal{L}f$  if  $f \in \mathcal{C}$ . Let  $n_\delta(dh) = 1_{(|h| \geq \delta)} n(dh)$  and let

$$\mathcal{L}_\delta f(x) = \int [f(x+h) - f(x) - 1_{(|h| \leq 1)} f'(x)h] n_\delta(dh).$$

Then  $n_\delta$  is a finite measure. Using the Fubini theorem and the fact that the Fourier transform of the function  $x \rightarrow f(x+h)$  is  $e^{ihu} \widehat{f}(u)$  and the Fourier transform of  $f'(x)$  is  $-iu \widehat{f}(u)$ ,

$$\begin{aligned} \widehat{\mathcal{L}_\delta f}(u) &= \int \int [e^{iux} f(x+h) - e^{iux} f(x) - 1_{(|h| \leq 1)} e^{iux} f'(x)h] dx n_\delta(dh) \\ &= \widehat{f}(u) \int [e^{-ihu} - 1 + 1_{(|h| \leq 1)} iuh] n_\delta(dh) \\ &= \widehat{f}(u) \int [e^{-ihu} - 1 + 1_{(|h| \leq 1)} iuh] 1_{(|h| \geq \delta)} n(dh). \end{aligned} \tag{37.14}$$

The expression in brackets on the last line is bounded by  $c(h^2 \wedge 1)$  and by dominated convergence the last line converges to  $\widehat{f}(u)\psi(u)$  as  $\delta \rightarrow 0$ , where

$$\psi(u) = \int [e^{-ihu} - 1 + 1_{(|h| \leq 1)} iuh] n(dh). \tag{37.15}$$

Since

$$\begin{aligned} & |\widehat{\mathcal{L}}f(u) - \widehat{\mathcal{L}_\delta f}(u)| \\ &= \left| \int e^{iux} \int_{|h|<\delta} [f(x+h) - f(x) - 1_{(|h|\leq 1)} f'(x)h] n(dh) dx \right| \\ &\leq \int (\sup_{|y-x|<\delta} |f''(y)|) \int_{|h|<\delta} h^2 n(dh) dx, \end{aligned}$$

which tends to zero as  $\delta \rightarrow 0$  because  $f \in \mathcal{C}$ , we conclude

$$\widehat{\mathcal{L}}f(u) = \widehat{f}(u)\psi(u). \quad (37.16)$$

*Step 2.* Now let  $g \in \mathcal{C}$ , let  $\varepsilon > 0$ , choose  $K > 1$  such that  $\int_{|h|\geq K} n(dh) < \varepsilon$ , let  $m_K(dh) = 1_{(|h|\geq K)} n(dh)$ , and define  $\mathcal{L}_K$  and  $\psi_K$  in terms of  $m_K$ . We show there exists  $f \in \mathcal{C}$  such that  $g = (\lambda - \mathcal{L}_K)f = g$ .

We have

$$\psi_K(u) = \int_{|h|\leq K} [e^{-iuh} - 1 + iuh 1_{(|h|\leq 1)}] n(dh),$$

so using dominated convergence,

$$\psi'_K(u) = \int_{|h|\leq K} [-ihe^{-iuh} + ih 1_{(|h|\leq 1)}] n(dh),$$

$$\psi''_K(u) = \int_{|h|\leq K} [-h^2 e^{-iuh}] n(dh),$$

with similar formulas for the higher derivatives. Thus all the derivatives of  $\psi_K$  are bounded. Moreover the real part of  $\psi_K(u)$  is  $\int_{|h|\leq K} [\cos(uh) - 1] n(dh)$ , which is less than or equal to 0. Since  $g \in \mathcal{C}$ , by Section B.2,  $\widehat{g} \in \mathcal{C}$ . If we define  $f$  by

$$\widehat{f}(u) = \frac{1}{\lambda - \psi_K(u)} \widehat{g}(u), \quad (37.17)$$

we see that  $\widehat{f}$  and all its derivatives are continuous and tend to zero faster than  $|u|^{-m}$  for every  $m$ . Hence  $\widehat{f} \in \mathcal{C}$ , which implies  $f \in \mathcal{C}$  by Section B.2.

Notice  $(\lambda - \mathcal{L}_K)f = g$  because

$$\lambda \widehat{f}(u) - \widehat{\mathcal{L}_K f}(u) = \frac{\lambda - \psi_K(u)}{\lambda - \psi_K(u)} \widehat{g}(u) = \widehat{g}(u).$$

*Step 3.* We prove that  $\|\mathcal{L}f - \mathcal{L}_K f\| \leq c\varepsilon \|g\|$ .

Since  $g \in \mathcal{C}$ , then  $\widehat{g} \in L^1$ . From (37.17) we have  $|\widehat{f}(u)| \leq |\widehat{g}(u)|/\lambda$ . Then

$$\|f\|_\infty \leq c \|\widehat{f}\|_{L^1} \leq c \|\widehat{g}\|_{L^1}$$

and

$$\begin{aligned} |\mathcal{L}f(x) - \mathcal{L}_K f(x)| &\leq \int_{|h| \geq K} |f(x+h) - f(x)| n(dh) \\ &\leq 2\|f\|_\infty \int_{|h| \geq K} n(dh) \\ &\leq c\varepsilon \|\widehat{g}\|_{L^1}. \end{aligned}$$

*Step 4.* We complete the proof that the range of  $\lambda - \mathcal{L}$  is dense in  $\mathcal{B}$ . Since  $\|\mathcal{L}f - \mathcal{L}_K f\| \leq c\varepsilon \|g\|$  by Step 3 and  $(\lambda - \mathcal{L}_K)f = g$ , then

$$\|(\lambda - \mathcal{L})f - g\| \leq c\varepsilon \|g\|.$$

Because  $f \in \mathcal{C} \subset \mathcal{D}(\mathcal{L})$  and  $\varepsilon$  is arbitrary, this proves the range of  $\lambda - \mathcal{L}$  is dense in  $\mathcal{C}$ , hence in  $\mathcal{B}$ .

We thus have  $\mathcal{L}$  satisfying all the hypotheses of the Hille–Yosida theorem, and hence there exists a semigroup  $P_t$  mapping  $\mathcal{B}$  into  $\mathcal{B}$ . We again note that  $\mathcal{L}1 = 0$ , hence  $P_t = 1$  for all  $t$ , and so by Exercise 37.1, the  $P_t$  are non-negative operators.

## Exercises

- 37.1 Let  $\mathcal{B}$  be either the space  $L^2$  with respect to a finite measure or else the continuous functions vanishing at infinity for some locally compact separable metric space  $S$ . In the former case, we say  $f \geq 0$  if  $f(x) \geq 0$  for almost every  $x$ , in the latter case if  $f(x) \geq 0$  for all  $x$ . A semigroup is non-negative if  $f \geq 0$  implies  $P_t f \geq 0$  for all  $t \geq 0$ . Suppose that  $P_t$  is a semigroup, the space  $\mathcal{B}$  contains the constant functions, and  $P_t 1 = 1$  for all  $t$ . Show that  $P_t$  is a contraction if and only if  $P_t$  is non-negative.

- 37.2 Show that  $P_t$  and  $R_\lambda$  commute and that

$$P_t R_\lambda f = \int_0^\infty e^{-\lambda s} P_{s+t} f \, ds.$$

Show that for any  $a < b$  we have

$$\left\| \int_a^b e^{\lambda t} P_t f \, dt \right\| \leq \int_a^b e^{-\lambda t} \|P_t f\| \, dt.$$

*Hint:* Approximate  $R_\lambda f$  by a Riemann sum.

- 37.3 Show that if  $P_t$  is a contraction semigroup and  $R_\lambda$  is the resolvent, then

$$\|R_\lambda\| \leq 1/\lambda. \tag{37.18}$$

- 37.4 Show that if  $A$  is a bounded operator and  $T_t = e^{tA}$ , then  $T_t$  is a strongly continuous semigroup of operators with infinitesimal generator  $A$ . (We cannot assert that the  $T_t$  are contractions.)
- 37.5 Prove that if  $\mathcal{L}$  is dissipative, the domain of  $\mathcal{L}$  is dense in  $\mathcal{B}$ , and the range of  $\lambda - \mathcal{L}$  is dense in  $\mathcal{B}$ , then  $\overline{\mathcal{L}}$  defined in (37.12) is a closed extension of  $\mathcal{L}$  that is dissipative and the range of  $\lambda - \overline{\mathcal{L}}$  is equal to  $\mathcal{B}$ . Show there is only one such closed extension of  $\mathcal{L}$ .
- 37.6 If the range of  $\lambda - \mathcal{L}$  equals  $\mathcal{B}$  for a single value of  $\lambda$ , then the range of  $\lambda - \mathcal{L}$  equals  $\mathcal{B}$  for every value of  $\lambda$ .

*Hint:* Define  $R_\lambda$  as the inverse of  $\lambda - \mathcal{L}$ , then use (37.3) to define  $R_a$  for other values of  $a$ .

- 37.7 Let  $(X_t, \mathbb{P}^x)$  be a Markov process with transition probabilities given by  $P_t f(x) = f(x + t)$ . Determine  $\mathcal{L}$  and  $\mathcal{D}(\mathcal{L})$ .
- 37.8 Let  $P_t$  be a strongly continuous semigroup of contraction operators and let  $\mathcal{L}$  be the infinitesimal generator. Show that  $\mathcal{D}(\mathcal{L}^n)$  is dense in  $\mathcal{B}$  for every positive integer  $n$ .
- 37.9 This is a continuation of Exercise 36.5. Prove that if  $f \in C^2$  with compact support,  $P_t$  is the semigroup given in Exercise 36.5, and  $\mathcal{L}$  is the infinitesimal generator, then the Fourier transform of  $\mathcal{L}f$  is  $\hat{f}(u)\psi(u)$ .
- 37.10 Suppose that  $P_t$  is a strongly continuous semigroup, but not necessarily of contractions. Thus  $P_{t+s} = P_t P_s$  and  $P_t f \rightarrow f$  in norm if  $f \in \mathcal{B}$ , but we do not assume  $\|P_t\| \leq 1$ . Prove that there exist constants  $K, b > 0$  such that  $\|P_t\| \leq K e^{bt}$  for all  $t \geq 0$ .

*Hint:* Use the uniform boundedness principle from functional analysis to prove there exists  $c, t_0$  such that  $\|P_t\| \leq c$  if  $t \leq t_0$ . Then use the semigroup property.

## Dirichlet forms

When constructing semigroups, it is sometimes easier to start with a bilinear form, called the Dirichlet form, than to work with the infinitesimal generator, and to construct the semigroup from the form. For example, let  $\Delta$  be the Laplacian. If  $f, g \in C^2$  with compact support, then integration by parts shows

$$\int_{\mathbb{R}^d} f(x) (\frac{1}{2} \Delta g)(x) dx = - \int_{\mathbb{R}^d} \sum_{i=1}^d \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_i}(x) dx.$$

If we write

$$\mathcal{E}(f, g) = \frac{1}{2} \int \sum_{i=1}^d \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_i}(x) dx,$$

we thus have

$$\begin{aligned} \int_{\mathbb{R}^d} f(\frac{1}{2} \Delta g) dx &= -\mathcal{E}(f, g), \\ \int_{\mathbb{R}^d} f(\frac{1}{2} \Delta g) dx &= -\mathcal{E}(f, g) = -\mathcal{E}(g, f) = \int_{\mathbb{R}^d} g(\frac{1}{2} \Delta f) dx. \end{aligned} \tag{38.1}$$

If  $R_\lambda$  is the resolvent for Brownian motion, (38.1) and the fact that  $\frac{1}{2} \Delta R_\lambda f = \lambda R_\lambda f - f$  tells us that

$$\begin{aligned} \mathcal{E}(R_\lambda f, g) + \lambda \int (R_\lambda f) g dx &= - \int (\frac{1}{2} \Delta R_\lambda f) g dx + \lambda \int (R_\lambda f) g dx \\ &= - \int (\lambda R_\lambda f - f) g dx + \lambda \int (R_\lambda f) g dx \\ &= \int f g. \end{aligned} \tag{38.2}$$

The bilinear form  $\mathcal{E}(f, g)$  makes sense even if  $f, g$  are only in  $C^1$  with compact support, which is one major advantage of the Dirichlet form. Since  $\mathcal{E}$  is clearly linear in each variable, we have

$$\mathcal{E}(f, g) = \frac{1}{2} [\mathcal{E}(f+g, f+g) - \mathcal{E}(f, f) - \mathcal{E}(g, g)],$$

so to specify the Dirichlet form, it is only necessary to know  $\mathcal{E}(f, f)$ , a number, rather than  $\mathcal{L}f$ , a function. One disadvantage of Dirichlet forms is that one needs a self-adjoint operator, and not every infinitesimal generator is self-adjoint. Another disadvantage is that when working with Dirichlet forms,  $L^2$  is the natural space to work with, which means there are null sets one has to worry about. In particular, the construction of Chapter 36 is not directly applicable, because there we required our Banach space to be the set of continuous functions vanishing at infinity. (Modifications of the methods in Chapter 36 do work, however.)

### 38.1 Framework

suppose  $\mathcal{S}$  is a locally compact separable metric space together with a  $\sigma$ -finite measure  $m$  defined on the Borel subsets of  $\mathcal{S}$ .

We suppose there exists a dense subset  $\mathcal{D} = \mathcal{D}(\mathcal{E})$  of  $L_2(S, m)$  and a non-negative bilinear symmetric form  $\mathcal{E}$  on  $\mathcal{D} \times \mathcal{D}$ , ==>

$$\begin{aligned}\mathcal{E}(f, g) &= \mathcal{E}(g, f), & \mathcal{E}(f + g, h) &= \mathcal{E}(f, h) + \mathcal{E}(g, h) \\ \mathcal{E}(af, g) &= a\mathcal{E}(f, g), & \mathcal{E}(f, f) &\geq 0\end{aligned}$$

for  $f, g, h \in \mathcal{D}, a \in \mathbb{R}$ .

For  $a > 0$  define

$$\mathcal{E}_a(f, f) := \mathcal{E}(f, f) + a\langle f, f \rangle.$$

We can define a norm on  $\mathcal{D}$  using the inner product  $\mathcal{E}_a$ : the norm of  $f = (\mathcal{E}_a(f, f))^{1/2}$ .

so the norms induced by different  $a$ 's are all equivalent. We say  $\mathcal{E}$  is *closed* if  $\mathcal{D}$  is complete wrt  $\mathcal{E}_a$  for some  $a$ .

We say  $\mathcal{E}$  is *Markovian* if whenever  $u \in \mathcal{D}$ , then  $v = 0 \vee (u \wedge 1) \in \mathcal{D}$  and  $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$ . (A slightly weaker definition of Markovian is sometimes used.) A *Dirichlet form* is a non-negative bilinear symmetric form that is closed and Markovian.

Absorbing Brownian motion on  $[0, \infty)$  is a symmetric process. The corresponding Dirichlet form is

$$\mathcal{E}(f, f) = \frac{1}{2} \int_0^\infty |f'(x)|^2 dx,$$

and the appropriate domain turns out to be the completion of the set of  $C^1$  functions with compact support contained in  $(0, \infty)$  wrt  $\mathcal{E}_1$ . In particular, any function with compact support contained in  $(0, \infty)$  will be zero in a neighborhood of 0. In a domain  $D$  in higher dimensions, the Dirichlet form for absorbing Brownian motion becomes

$$\mathcal{E}(f, f) = \frac{1}{2} \int |\nabla f(x)|^2 dx, \tag{38.3}$$

with the domain of  $\mathcal{E}$  being the completion with respect to  $\mathcal{E}_1$  of the  $C^1$  functions whose support is contained in the interior of  $D$ .

Reflecting Brownian motion is also a symmetric process. For a domain  $D$ , the Dirichlet form is given by (38.3) and the domain  $\mathcal{D}(\mathcal{E})$  of the form is given by the completion with respect to the norm induced by  $\mathcal{E}_1$  of the  $C^1$  functions on  $\overline{D}$  with compact support, where  $\overline{D}$  is the closure of  $D$ . One might expect there to be some restriction on the normal derivative  $\partial f / \partial n$  on the boundary of  $D$ , but in fact there is no such restriction. To examine this further, consider the case of  $D = (0, \infty)$ . If one takes the class of functions  $f$  which are  $C^1$  with compact support and with  $f'(0) = 0$  and takes the closure with respect to the norm induced by  $\mathcal{E}_1$ , one gets the same class as  $\mathcal{D}(\mathcal{E})$ ; this is Exercise 38.1.

One nice consequence of the fact that we don't need to impose a restriction on the normal derivative in the domain of  $\mathcal{E}$  for reflecting Brownian motion is that this allows us to define reflecting Brownian motion in any domain, even when the boundary is not smooth enough for the notion of a normal derivative to be defined.

## 38.2 Construction of the semigroup

We now want to construct the resolvent corresponding to a Dirichlet form. The motivation given in (38.2) shows we should expect

$$\mathcal{E}_a(R_a f, g) = \langle f, g \rangle \quad (38.4)$$

for all  $a > 0$  and all  $f, g$  such that  $R_a f, g \in \mathcal{D}$ .  $\mathcal{B} = L^2(\mathcal{S}, m)$ .

**Theorem 38.1** *If  $\mathcal{E}$  is a Dirichlet form, there exists a family of resolvent operators  $\{R_\lambda\}$  such that*

- (1) *the  $R_\lambda$  satisfy the resolvent equation,*
- (2)  *$\|\lambda R_\lambda\| \leq 1$  for all  $\lambda > 0$ ,*
- (3)  *$\lambda R_\lambda f \rightarrow f$  as  $\lambda \rightarrow \infty$  for  $f \in \mathcal{B}$ , and*
- (4)  *$\mathcal{E}_a(R_a f, g) = \langle f, g \rangle$  if  $a > 0$ ,  $R_a f, g \in \mathcal{D}$ .*

Moreover, if  $f \in \mathcal{B}$  satisfies  $0 \leq f(x) \leq 1$ ,  $m$ -a.e., then for all  $a > 0$

$$0 \leq aR_a f \leq 1, \quad m\text{-a.e.} \quad (38.5)$$

*Proof*  $\langle f, g \rangle$  is a bounded on  $\mathcal{D}$  wrt  $\mathcal{E}_a$ ,

$\mathcal{D}$  is a Hilbert space wrt  $\mathcal{E}_a$ . set  $R_a f = u$ . In particular, (38.4) holds

We show the resolvent equation holds. If  $g \in \mathcal{D}$ ,

$$\begin{aligned}
 \mathcal{E}_a(R_a f - R_b f, g) &= \mathcal{E}_a(R_a f, g) - \mathcal{E}(R_b f, g) - a \langle R_b f, g \rangle \\
 &= \langle f, g \rangle - \mathcal{E}(R_b f, g) - b \langle R_b f, g \rangle + (b-a) \langle R_b f, g \rangle \\
 &= \langle f, g \rangle - \mathcal{E}_b(R_b f, g) + (b-a) \langle R_b f, g \rangle \\
 &= (b-a) \langle R_b f, g \rangle \\
 &= \mathcal{E}_a((b-a)R_a R_b f, g).
 \end{aligned}$$

Since this holds for all  $g \in \mathcal{D}$  and  $\mathcal{D}$  is dense in  $\mathcal{B}$ , then  $R_a - R_b = (b-a)R_a R_b$ .

$$\begin{aligned}
 b \langle bR_b f - f, bR_b f - f \rangle &\leq \mathcal{E}_b(bR_b f - f, bR_b f - f) \\
 &= b^2 \mathcal{E}_b(R_b f, R_b f) - 2b \mathcal{E}_b(R_b f, f) + \mathcal{E}_b(f, f) \\
 &= b^2 \langle R_b f, f \rangle - 2b \langle f, f \rangle + \mathcal{E}(f, f) + b \langle f, f \rangle \\
 &\leq \mathcal{E}(f, f).
 \end{aligned}$$

Now divide both sides by  $b$  to get  $\|bR_b f - f\|^2 \leq \mathcal{E}(f, f)/b \rightarrow 0$  as  $b \rightarrow \infty$ . Since  $\mathcal{D}$  is dense in  $\mathcal{B}$  and  $\|bR_b\| \leq 1$  for all  $b$ , we conclude  $bR_b f \rightarrow f$  for all  $f \in \mathcal{B}$ .

It remains to show  $0 \leq bR_b f \leq 1$ ,  $m$ -a.e., if  $0 \leq f \leq 1$ ,  $m$ -a.e. Fix  $f \in \mathcal{B}$  with  $0 \leq f \leq 1$ ,  $m$ -a.e., and let  $a > 0$ . Define a functional  $\psi$  on  $\mathcal{D}$  by

$$\psi(v) = \mathcal{E}(v, v) + a$$

$$\left\langle v - \frac{f}{a}, v - \frac{f}{a} \right\rangle.$$

We claim

$$\psi(R_a f) + \mathcal{E}_a(R_a f - v, R_a f - v) = \psi(v), \quad v \in \mathcal{D}. \quad (38.7)$$

$$\begin{aligned}
 aRa + E &= 1 \\
 Ib &= E + bRa \\
 IbRb &= Ra
 \end{aligned}$$

To see this, start with the lhs, which is equal to

$$\begin{aligned} & \mathcal{E}(R_af, R_af) + a \left\langle R_af - \frac{1}{a}f, R_af - \frac{1}{a}f \right\rangle + \mathcal{E}_a(R_af - v, R_af - v) \\ &= \mathcal{E}_a(R_af, R_af) - 2\langle R_af, f \rangle + \frac{1}{a}\langle f, f \rangle + \mathcal{E}_a(R_af, R_af) - 2\mathcal{E}_a(R_af, v) + \mathcal{E}_a(v, v) \\ &= \frac{1}{a}\langle f, f \rangle - 2\langle f, v \rangle + \mathcal{E}(v, v) + a\langle v, v \rangle \\ &= \psi(v). \end{aligned}$$

If follows from (38.7) and the fact that  $\mathcal{E}_a(g, g)$  is non-negative for any  $g \in \mathcal{D}$  that  $R_af$  is the function that minimizes  $\psi$ .

Set  $\phi(x) = 0 \vee (x \wedge (1/a))$  and let  $w = \phi(R_af)$ . Observe that  $|\phi(t) - s| \leq |t - s|$  for  $t \in \mathbb{R}$  and  $s \in [0, 1/a]$ , so

$$\left| w(x) - \frac{f(x)}{a} \right| \leq \left| R_af(x) - \frac{f(x)}{a} \right|,$$

and therefore

$$\left\langle w - \frac{f}{a}, w - \frac{f}{a} \right\rangle \leq \left\langle R_af - \frac{f}{a}, R_af - \frac{f}{a} \right\rangle. \quad (38.8)$$

Since  $\mathcal{E}$  is Markovian, then  $aw = 0 \vee ((aR_af) \wedge 1)$ , which leads to

$$\mathcal{E}(w, w) \leq \frac{1}{a^2} \mathcal{E}(aR_af, aR_af) = \mathcal{E}(R_af, R_af). \quad (38.9)$$

Adding (38.8) and (38.9), we conclude  $\psi(w) \leq \psi(R_af)$ . Since  $R_af$  is the minimizer for  $\psi$ , then  $w = R_af$ ,  $m$ -a.e. But  $0 \leq w \leq 1/a$ , and hence  $aR_af$  takes values in  $[0, 1]$ ,  $m$ -a.e.  $\square$

If we combine Proposition 37.9 and Theorem 38.1, we obtain a semigroup  $P_t$  whose resolvent satisfies (38.4). We would like to know that the analog of (38.5) holds for  $P_t$ .

**Corollary 38.2** *If  $0 \leq f \leq 1$ ,  $m$ -a.e., then  $0 \leq P_tf \leq 1$ ,  $m$ -a.e.*

*Proof* If  $0 \leq f \leq 1$ ,  $m$ -a.e., then  $0 \leq bR_b f \leq 1$ ,  $m$ -a.e., by Theorem 38.1, and iterating,  $0 \leq (bR_b)^i f \leq 1$ ,  $m$ -a.e., for every  $i$ . Using the notation of the proof of Proposition 37.9,

$$Q_t^b f(x) = e^{-bt} \sum_{i=0}^{\infty} (bt)^i (bR_b)^i f(x) / i!,$$

which will be non-negative,  $m$ -a.e., and bounded by  $e^{-bt} \sum_{i=0}^{\infty} (bt)^i / i!$ ,  $m$ -a.e. Passing to the limit as  $b \rightarrow \infty$ , we see that  $P_tf$  takes values in  $[0, 1]$ ,  $m$ -a.e.  $\square$

When it comes to using the semigroup  $P_t$  derived from a Dirichlet form to construct a Markov process  $X$ , there is a difficulty that we did not have before. Since  $P_t$  is constructed using an  $L^2$  procedure,  $P_tf$  is defined only up to almost everywhere equivalence. Without some continuity properties of  $P_tf$  for enough  $f$ 's, we must neglect some null sets. If the only null sets we could work with were sets of  $m$ -measure 0, we would be in trouble. For example, when  $\mathcal{S}$  is the plane and  $m$  is a two-dimensional Lebesgue measure, the  $x$  axis has measure zero, but a continuous process will (in general) hit the  $x$  axis. Fortunately there is a notion of sets of capacity zero, which are null sets that are smaller than sets of measure zero. It is

possible to construct a process  $X$  starting from all points  $x$  in  $\mathcal{S}$  except for those in a set  $\mathcal{N}$  of capacity zero and to show that starting from any point not in  $\mathcal{N}$ , the process never hits  $\mathcal{N}$ .

There is another difficulty when working with Dirichlet forms. In general, one must look at  $\tilde{\mathcal{S}}$ , a certain compactification of  $\mathcal{S}$ , which is a compact set containing  $\mathcal{S}$ . Even when our state space is a domain in  $\mathbb{R}^d$ ,  $\tilde{\mathcal{S}}$  is not necessarily equal to  $\overline{\mathcal{S}}$ , the Euclidean closure of  $\mathcal{S}$ , and one must work with  $\tilde{\mathcal{S}}$  instead of  $\overline{\mathcal{S}}$ . It can be shown that this problem will not occur if the Dirichlet form is regular. Let  $C_K$  be the set of continuous functions with compact support. A Dirichlet form  $\mathcal{E}$  is *regular* if  $\mathcal{D} \cap C_K$  is dense in  $\mathcal{D}$  with respect to the norm induced by  $\mathcal{E}_1$  and  $\mathcal{D} \cap C_K$  also is dense in  $C_K$  with respect to the supremum norm.

### 38.3 Divergence form elliptic operators

We want to show how to construct the Markov process corresponding to the operator

$$\mathcal{L}f(x) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( a_{ij}(\cdot) \frac{\partial f}{\partial x_j}(\cdot) \right)(x). \quad (38.10)$$

If the  $a_{ij}$ 's are smooth in  $x$ , this can be interpreted as first calculating the partial derivative of  $f$  with respect to  $x_j$ , multiplying the result by  $a_{ij}(x)$ , taking the partial derivative of the product with respect to  $x_i$ , and then summing over  $i$  and  $j$ . If, however, the  $a_{ij}$ 's are only bounded and measurable, one cannot even in general give any nontrivial examples of functions in the domain of  $\mathcal{L}$ . Here is where Dirichlet forms are the perfect tool. Operators of the form (38.10) are known as elliptic operators in divergence form or in variational form, and the study of their properties has a long history in PDE.

We assume  $a_{ij}(x) = a_{ji}(x)$  for each  $i$  and  $j$  and each  $x$ . We suppose the  $a_{ij}(x)$  are measurable functions and are uniformly bounded in  $x$  for each  $i$  and  $j$ . We also require *uniform ellipticity*: there exists  $\Lambda$  such that

$$\sum_{i,j=1}^d a_{ij}(x) y_i y_j \geq \Lambda \sum_{i=1}^d y_i^2, \quad (y_1, \dots, y_d) \in \mathbb{R}^d.$$

Just as in the nondivergence elliptic operator case, the matrix whose  $(i, j)$ th element is  $a_{ij}(x)$  is positive definite, uniformly in  $x$ .

We will shortly define a Dirichlet form, but let us first specify a domain. Let  $C_K^1$  be the collection of  $C^1$  functions with compact support, and define  $H^1$  to be the completion of  $C_K^1$  wrt the norm

$$\|f\|_{H^1} = \left( \int (|f(x)|^2 + |\nabla f(x)|^2) dx \right)^{1/2}. \quad (38.11)$$

One can show that  $H^1$  with this norm is a Banach space; this is Exercise 38.2.

Now for  $f \in C_K^1$  define

$$\mathcal{E}(f, f) = \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x) dx. \quad (38.12)$$

We can use the fact that  $C_K^1$  is dense in  $H^1$  to extend the definition of  $\mathcal{E}$  to all of  $H^1 \times H^1$ . The connection with the operator  $\mathcal{L}$  is that when the  $a_{ij}$  are smooth, integration by parts yields

$$\int (\mathcal{L}f)g \, dx = -\mathcal{E}(f, g)$$

if  $g$  is  $C^1$  with compact support; cf. (38.1).

Because of the boundedness and uniform ellipticity, there exist positive constants  $c_1$  and  $c_2$  not depending on  $f$  such that

$$c_1 \int |\nabla f(x)|^2 \, dx \leq \mathcal{E}(f, f) \leq c_2 \int |\nabla f(x)|^2 \, dx.$$

Therefore the norm induced by  $\mathcal{E}_1$  and the norm in  $H^1$  are equivalent. This implies  $\mathcal{E}$  is closed. By the definition of  $H^1$ ,  $\mathcal{E}$  is regular, and clearly  $\mathcal{E}$  is symmetric. Thus we need only to show that  $\mathcal{E}$  is Markovian.

Let  $\phi(x) = (0 \vee x) \wedge 1$ . For each  $\varepsilon > 0$  let  $\phi_\varepsilon$  be  $C^\infty$ , bounded, agreeing with  $\phi$  on  $[0, 1]$ , with  $\|\phi'_\varepsilon\|_\infty \leq 1$ , and such that  $\phi_\varepsilon(x) \rightarrow \phi(x)$  uniformly in  $x$  as  $\varepsilon \rightarrow 0$  and  $\phi'_\varepsilon(x) \rightarrow 1_{[0,1]}(x)$  pointwise as  $\varepsilon \rightarrow 0$ . Note  $\nabla \phi_\varepsilon(f) = \phi'_\varepsilon(f) \nabla f$ , so if  $f \in C_K^1$ ,

$$\mathcal{E}(\phi_\varepsilon(f), \phi_\varepsilon(f)) = \sum_{i,j=1}^d \int (\phi'_\varepsilon(f)(x))^2 a_{ij}(x) \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x) \, dx. \quad (38.13)$$

Since

$$\sum_{i,j=1}^d a_{ij}(x) \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x) \geq \Lambda |\nabla f(x)|^2 \geq 0$$

and  $|\phi'_\varepsilon(f)(x)| \leq 1$ , we see that

$$\mathcal{E}(\phi_\varepsilon(f), \phi_\varepsilon(f)) \leq \mathcal{E}(f, f).$$

Taking the limit as  $\varepsilon \rightarrow 0$  in (38.13) we obtain

$$\mathcal{E}(\phi(f), \phi(f)) \leq \mathcal{E}(f, f) < \infty. \quad (38.14)$$

In particular,  $\phi(f) \in H^1 = \mathcal{D}(\mathcal{E})$ . We now pass to the limit to show that (38.14) holds for all  $f \in H^1$ , which says that  $\mathcal{E}$  is Markovian.

We can therefore apply Theorem 38.1 to obtain a semigroup corresponding to the Dirichlet form  $\mathcal{E}$ . As mentioned earlier, there is potentially a problem in that the semigroup is only defined for points not in a certain null set. However, a famous result of Nash and of DeGiorgi shows that the semigroup  $P_t$  can be written as  $P_t f(x) = \int f(y) p(t, x, y) \, dy$  with  $p(t, x, y)$  Hölder continuous in  $x$  and  $y$ ; see Bass (1997), Chapter VII for a presentation of this result. This allows us to take the null set to be empty and to see that our semigroup satisfies the assumptions of Chapter 36. Therefore there exists a strong Markov process having  $P_t$  as its semigroup.

## Exercises

- 38.1 Let  $F_1 = \{f \in C^1[0, \infty) : f \text{ has compact support}\}$  and  $F_2 = F_1 \cap \{f \in C^1[0, \infty) : f \text{ has compact support, } f'(0) = 0\}$ . Show that the closures of  $F_1$  and  $F_2$  with respect to the norm  $(\int (|f(x)|^2 + |f'(x)|^2) dx)^{1/2}$  are the same.
- 38.2 If  $H^1$  is the completion of  $C_K^1$ , the  $C^1$  functions on  $\mathbb{R}^d$  with compact support, relative to the norm given by (38.11), show  $H^1$  is a Hilbert space.
- 38.3 Show that the resolvent operator  $R_\lambda$  defined in Theorem 38.1 is a symmetric operator, that is, if  $f, g \in \mathcal{B}$ , then  $\langle R_\lambda f, g \rangle = \langle f, R_\lambda g \rangle$ .
- 38.4 Show that if the resolvent operator  $R_\lambda$  is a symmetric operator, then the transition operators  $P_t$  are also symmetric: if  $f, g \in \mathcal{B}$ , then  $\langle P_t f, g \rangle = \langle f, P_t g \rangle$ .
- 38.5 To do the next few exercises, you will have to know some functional analysis, specifically, the spectral theorem for self-adjoint operators. See Lax (2002).

Let  $\mathcal{E}$  be a Dirichlet form with domain  $\mathcal{D}(\mathcal{E})$  and let  $\mathcal{L}$  be the infinitesimal generator of the semigroup  $P_t$  that corresponds to  $\mathcal{L}$ . Let  $E(d\lambda)$  be a spectral resolution of the identity for  $-\mathcal{L}$ . (The operator  $\mathcal{L}$  is a negative operator, so  $-\mathcal{L}$  is a positive one.) Then a consequence of the spectral theorem is that

$$P_t f = \int_0^\infty e^{-\lambda t} E(d\lambda) f$$

and

$$R_a f = \int_0^\infty \frac{1}{a + \lambda} E(d\lambda) f.$$

Also

$$\langle f, g \rangle = \int_0^\infty \langle E(d\lambda) f, g \rangle.$$

Show that if  $f, g \in \mathcal{D}$ , then

$$\mathcal{E}(f, g) = \int_0^\infty \lambda \langle E(d\lambda) f, g \rangle.$$

*Hint:* First prove it for  $f = R_a h$ . Write

$$\begin{aligned} \mathcal{E}(R_a h, g) &= \langle h, g \rangle - a \langle R_a h, g \rangle = \int_0^\infty \left(1 - \frac{a}{a + \lambda}\right) \langle E(d\lambda) h, g \rangle \\ &= \int_0^\infty \frac{\lambda}{a + \lambda} \langle E(d\lambda) h, g \rangle = \int_0^\infty \lambda \langle E(d\lambda) (R_a h), g \rangle. \end{aligned}$$

To extend this to all  $f$  in the domain of  $\mathcal{E}$ , use the fact that  $\mathcal{E}$  is closed.

- 38.6 If  $\mathcal{L}$  is the infinitesimal generator of the semigroup associated with the Dirichlet form  $\mathcal{E}$ , show that  $\mathcal{D}(\sqrt{-\mathcal{L}}) = \mathcal{D}(\mathcal{E})$ .
- 38.7 Show that if  $f \in \mathcal{D}(\mathcal{E})$ , then  $a R_a f$  converges to  $f$  with respect to the norm induced by  $\mathcal{E}_1$ .
- 38.8 Show that if  $b > 0$ , then  $\{R_b f : f \in L^2\}$  is a dense subset of  $\mathcal{D}(\mathcal{E})$  with respect to the norm induced by  $\mathcal{E}_1$ .
- 38.9 Show that  $\{P_t f : f \in L^2, t > 0\}$  is a dense subset of  $\mathcal{D}(\mathcal{E})$  with respect to the norm induced by  $\mathcal{E}_1$ .

38.10 This exercise shows how to approximate  $\mathcal{E}$  by forms whose domain is all of  $\mathcal{B}$ . Let

$$\mathcal{E}^{(t)}(f, g) = \frac{1}{t} \langle f - P_t f, g \rangle.$$

Show that if  $f \in \mathcal{D}(\mathcal{E})$ , then  $\mathcal{E}^{(t)}(f, f)$  increases to  $\mathcal{E}(f, f)$ . Show that if  $f, g \in \mathcal{D}(\mathcal{E})$ , then  $\mathcal{E}^{(t)}(f, g)$  converges to  $\mathcal{E}(f, g)$ .

38.11 Show that if  $u \in \mathcal{D}(\mathcal{E})$ , then  $|u| \in \mathcal{D}(\mathcal{E})$  and  $\mathcal{E}(|u|, |u|) \leq \mathcal{E}(u, u)$ .

*Hint:* Use Exercise 38.10.

38.12 Use Exercise 38.11 to show that if  $u \in \mathcal{D}(\mathcal{E})$ , then  $\mathcal{E}(u^+, u^-) \leq 0$ .

38.13 Suppose  $\{P_t\}$  are the transition probabilities corresponding to a Dirichlet form  $\mathcal{E}$ . Suppose there exist functions  $p_t(x, y)$  such that for each  $t$ ,

$$P_t f(x) = \int p_t(x, y) m(dy)$$

for almost every  $x$ . Prove that for almost every pair  $(x, y)$  with respect to the product measure  $m \times m$ ,  $p_t(x, y) = p_t(y, x)$ .

38.14 Let  $f \in L^2(m)$  and define the functional

$$\psi(u) = \mathcal{E}(u, u) + \lambda \langle u, u \rangle - 2 \langle f, u \rangle$$

for  $u$  in the domain of  $\mathcal{E}$ . Prove that  $\psi$  is minimized by  $u = R_\lambda f$ , and that this function is the unique minimizer.

38.15 Let  $P_t$  be the semigroup associated with a Dirichlet form and define

$$J(dx, dy) = P_t(x, dy) m(dx).$$

(1) Prove that if  $f, g$  are continuous with compact support, then

$$\int \int f(x)g(y) J(dx, dy) = \int \int g(x)f(y) J(dx, dy).$$

(2) With  $f$  and  $g$  continuous with compact support, prove that

$$\int f(x)g(y) J(dx, dy) = \langle f, P_t g \rangle$$

and

$$\int \int f(x)g(x) J(dx, dy) = \langle fg, P_t 1 \rangle.$$

(3) Let  $k(x) = 1 - P_t 1(x)$ . Prove that if  $\mathcal{E}^{(t)}$  is defined as in Exercise 38.10, then

$$2t\mathcal{E}^{(t)}(f, g) = \int \int (f(x) - f(y))(g(x) - g(y)) J(dx, dy) + \int f(x)g(x)k(x) m(dx).$$

(4) Is  $\mathcal{E}^{(t)}$  a Dirichlet form? A regular Dirichlet form?

- 38.16 This is a continuation of the previous exercise. If  $f$  is a function on the state space, we say that  $g$  is a normal contraction of  $f$  if  $|g(x)| \leq |f(x)|$  for all  $x$  and  $|g(x) - g(y)| \leq |f(x) - f(y)|$  for all  $x$  and  $y$ . As an example, note that if  $g(x) = -1 \vee (f(x) \wedge 1)$ , then  $g$  is a normal contraction of  $f$ . Prove that if  $f \in \mathcal{D}(\mathcal{E})$ , where  $\mathcal{E}$  is a Dirichlet form and  $g$  is a normal contraction of  $f$ , then for each  $t > 0$ ,

$$\mathcal{E}^{(t)}(g, g) \leq \mathcal{E}^{(t)}(f, f) \leq \mathcal{E}(f, f).$$

### Notes

See Fukushima *et al.* (1994) for further information.

# 39

---

## Markov processes and SDEs

One common way of constructing Markov processes is via stochastic differential equations. Roughly speaking, if there is uniqueness for every starting point, then one can create a strong Markov process. After proving this, we establish a connection between stochastic differential equations and partial differential equations, and then we describe what is known as the martingale problem.

### 39.1 Markov properties

Let  $\mathbb{P}$  be a probability and suppose  $W$  is a  $d$ -dimensional Brownian motion with respect to  $\mathbb{P}$ . Consider the SDE

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt. \quad (39.1)$$

Here  $\sigma$  is a  $d \times d$  matrix-valued function and  $b$  is a vector-valued function, both Borel measurable and bounded. This can be written in terms of components as

$$dX_t^i = \sum_{j=1}^d \sigma_{ij}(X_t) dW_t^j + b_i(X_t) dt, \quad i = 1, \dots, d,$$

where  $W = (W^1, \dots, W^d)$ . Let  $X_t^x$  be the solution to (39.1) when  $X_0 = x$ . Let  $\mathbb{P}^x$  be the law of  $X_t^x$ .

Let  $\Omega = C[0, \infty)$ , let  $\mathcal{F}$  be the cylindrical subsets of  $\Omega$ , and define  $Z_t(\omega) = \omega(t)$ . The main result of this section is that if weak existence and weak uniqueness hold for (39.1) for every starting point  $x$ , then the solutions  $(Z_t, \mathbb{P}^x)$  form a strong Markov process.

We begin by considering regular conditional probabilities.

**Definition 39.1** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, and let  $\mathcal{E}$  be a  $\sigma$ -field contained in  $\mathcal{F}$ . A *regular conditional probability* for  $\mathbb{E}[\cdot | \mathcal{E}]$  is a kernel  $Q(\omega, d\omega')$  such that

- (1)  $Q(\omega, \cdot)$  is a probability measure on  $(\Omega, \mathcal{E})$  for each  $\omega$ ;
- (2) for each  $A \in \mathcal{F}$ ,  $Q(\cdot, A)$  is a random variable that is measurable with respect to  $\mathcal{F}$ ;
- (3) for each  $A \in \mathcal{F}$  and each  $B \in \mathcal{E}$ ,

$$\int_B Q(\omega, A) \mathbb{P}(d\omega) = \mathbb{P}(A \cap B).$$

Regular conditional probabilities need not always exist, but if the probability space has sufficient structure, then they do. We provide a proof in the appendix; see Theorem C.1.  $Q(\omega, A)$  can be thought of as  $\mathbb{P}(A | \mathcal{E})(\omega)$ , regularized so as to have some joint measurability.

Recall that the definition of minimal augmented filtration for a Markov process was given in Section 20.1.

**Theorem 39.2** Suppose weak existence and weak uniqueness hold for the SDE (39.1) whenever  $X_0$  is a random variable that is in  $L^2$  and is measurable with respect to  $\mathcal{F}_0$ . Suppose the matrix  $\sigma(y)$  is invertible for each  $y$ . Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be defined as above. Let  $\mathbb{P}^x$  be the law of the weak solution when  $X_0$  is identically equal to  $x$ . Let  $\{\mathcal{F}_t\}$  be the minimal augmented filtration generated by  $Z$ . Then  $(\mathbb{P}^x, Z_t)$  is a strong Markov process.

*Proof* We will prove that if  $T$  is a bounded stopping time and  $f$  is a bounded and Borel measurable function on  $\mathbb{R}^d$ , then

$$\mathbb{E}^x[f(Z_{T+t}) \mid \mathcal{F}_T] = \mathbb{E}^{Z_T} f(Z_t), \quad \text{a.s.} \quad (39.2)$$

As in Section 20.3, this is sufficient to get the strong Markov property.

Fix  $x$ . Let

$$Y_t = Z_t - \int_0^t b(Z_r) dr \quad (39.3)$$

and

$$W'_t = \int_0^t \sigma^{-1}(Z_r) dY_r. \quad (39.4)$$

Since the  $\mathbb{P}^x$  law of  $Z_t$  is the same as the  $\mathbb{P}$  law of  $X_t^x$ , then the  $\mathbb{P}^x$  law of  $W'$  is the same as the  $\mathbb{P}$  law of  $W$ , or in other words,  $W'$  is a Brownian motion under  $\mathbb{P}^x$ . Rearranging (39.3) and (39.4), we have the equation

$$Z_t = Z_0 + \int_0^t \sigma(Z_r) dW'_r + \int_0^t b(Z_r) dr. \quad (39.5)$$

Let  $Q$  be a regular conditional probability for  $\mathbb{E}^x[\cdot \mid \mathcal{F}_T]$ . Let  $\tilde{Z}_t = Z_{T+t}$  and  $\tilde{W}_t = W'_{T+t} - W'_T$ . Using (39.5) with  $t$  replaced by  $T+t$  and then with  $t$  replaced by  $T$ , and taking the difference, we obtain

$$Z_{T+t} - Z_T = \int_T^{T+t} \sigma(Z_r) dW'_r + \int_T^{T+t} b(Z_r) dr,$$

and hence

$$\tilde{Z}_t = \tilde{Z}_0 + \int_0^t \sigma(\tilde{Z}_r) \tilde{W}_r + \int_0^t b(\tilde{Z}_r) dr. \quad (39.6)$$

We will show in a moment that  $\tilde{W}$  is a Brownian motion with respect to  $Q(\omega, \cdot)$  for  $\mathbb{P}^x$ -almost all  $\omega$ . Thus except for  $\omega$  in a  $\mathbb{P}^x$ -null set, (39.6) implies that under  $Q(\omega, \cdot)$ ,  $\tilde{Z}$  is a solution to (39.1) with starting point  $\tilde{Z}_0 = Z_T(\omega)$ . If  $\mathbb{E}_Q$  denotes the expectation with respect to  $Q$ , the weak uniqueness tells us that

$$\mathbb{E}_Q f(\tilde{Z}_t) = \mathbb{E}^{Z_T} f(Z_t), \quad \mathbb{P}^x(d\omega)\text{-a.s.} \quad (39.7)$$

On the other hand,

$$\mathbb{E}_Q f(\tilde{Z}_t) = \mathbb{E}_Q f(Z_{T+t}) = \mathbb{E}^x[f(Z_{T+t}) \mid \mathcal{F}_T], \quad \mathbb{P}^x(d\omega)\text{-a.s.} \quad (39.8)$$

Combining (39.7) and (39.8) proves (39.2).

It remains to prove that under  $\underline{Q}$  the process  $\tilde{W}$  is a Brownian motion.  $Q(\omega, \cdot)$  is a probability measure on  $\Omega'$ , so  $t \rightarrow \tilde{W}_t$  is continuous for every  $\omega'$ . Let  $t_1 < \dots < t_n$  and

$$\begin{aligned} N(u_2, \dots, u_n, t_1, \dots, t_n) &= \left\{ \omega : \mathbb{E}_{\underline{Q}} \exp \left( i \sum_{j=2}^n u_j (W'_{T+t_j} - W'_{T+t_{j-1}}) \right) \right. \\ &\quad \left. \neq \exp \left( - \sum_{j=2}^n |u_j|^2 (t_j - t_{j-1}) / 2 \right) \right\}. \end{aligned}$$

By the strong Markov property of the Brownian motion  $W'$  and the definition of  $Q$ ,

$$\begin{aligned} \mathbb{E}_{\underline{Q}} \exp \left( i \sum_{j=2}^n u_j (W'_{T+t_j} - W'_{T+t_{j-1}}) \right) &= \mathbb{E} \left[ \exp \left( i \sum_{j=2}^n u_j (W'_{T+t_j} - W'_{T+t_{j-1}}) \right) \mid \mathcal{F}_T \right] \\ &= \mathbb{E}^{W'_T} \exp \left( i \sum_{j=2}^n u_j (W'_{T+t_j} - W'_{T+t_{j-1}}) \right) \\ &= \exp \left( - \sum_{j=2}^n |u_j|^2 (t_j - t_{j-1}) / 2 \right), \end{aligned}$$

where the second equality holds almost surely, that is, except for a  $\mathbb{P}^x$ -null set of  $\omega$ 's. This shows that  $N(u_2, \dots, u_n, t_1, \dots, t_n)$  is a null set with respect to  $\mathbb{P}^x$ .

Let  $N$  be the union of all such  $N(u_1, \dots, u_n, t_1, \dots, t_n)$  for  $n \geq 1$ ,  $u_1, \dots, u_n$  rational, and  $t_1 < \dots < t_n$  rational. Therefore  $N$  is a  $\mathbb{P}^x$ -null set.

Suppose  $\omega \notin N$ . By the continuity of the paths of  $W'$ ,

$$\mathbb{E}_{\underline{Q}} \exp \left( i \sum_{j=2}^n u_j (W'_{T+t_j} - W'_{T+t_{j-1}}) \right) = \exp \left( - \sum_{j=2}^n |u_j|^2 (t_j - t_{j-1}) / 2 \right)$$

for all  $t, \dots, t_n \in [0, \infty)$  and  $u_2, \dots, u_n \in \mathbb{R}$ . Thus the finite-dimensional distributions of  $\tilde{W}$  under  $Q_T(\omega, \cdot)$  are those of a Brownian motion. By the continuity of  $\tilde{W}$  and Theorem 2.6, under  $Q_T$ ,  $\tilde{W}$  is a Brownian motion, except for a null set of  $\omega$ 's.  $\square$

By a slight abuse of notation, we will say  $(X_t, \mathbb{P}^x)$  is a strong Markov family when  $(Z_t, \mathbb{P}^x)$  is a strong Markov family.

## 39.2 SDEs and PDEs

The connection between stochastic differential equations and partial differential equations comes about through the following theorem, which is simply an application of Itô's formula. Let  $\mathcal{L}$  be the operator on functions in  $C^2$  defined by

$$\mathcal{L}f(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d b_i(x) \frac{\partial f}{\partial x_i}(x). \quad (39.9)$$

**Theorem 39.3** Suppose  $X_t$  is a solution to (39.1),  $\sigma$  and  $b$  are bounded and Borel measurable, and  $a = \sigma\sigma^T$ . Suppose  $f \in C^2$ . Then

$$f(X_t) = f(X_0) + M_t + \int_0^t \mathcal{L}f(X_s) ds, \quad (39.10)$$

where

$$M_t = \int_0^t \sum_{i,j=1}^d \frac{\partial f}{\partial x_i}(X_s) \sigma_{ij}(X_s) dW_s^j \quad (39.11)$$

is a local martingale.

*Proof* Since the components of the Brownian motion  $W_t$  are independent, we have  $d\langle W^k, W^\ell \rangle_t = 0$  if  $k \neq \ell$ ; see Exercise 9.4. Therefore

$$\begin{aligned} d\langle X^i, X^j \rangle_t &= \sum_k \sum_\ell \sigma_{ik}(X_t) \sigma_{jl}(X_t) d\langle W^k, W^\ell \rangle_t \\ &= \sum_k \sigma_{ik}(X_t) \sigma_{kj}^T(X_t) dt = a_{ij}(X_t) dt. \end{aligned}$$

We now apply Itô's formula:

$$\begin{aligned} f(X_t) &= f(X_0) + \sum_i \int_0^t \frac{\partial f}{\partial x_i}(X_s) dX_s^i + \frac{1}{2} \int_0^t \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s) d\langle X^i, X^j \rangle_s \\ &= f(X_0) + M_t + \sum_i \int_0^t \frac{\partial f}{\partial x_i}(X_s) b_i(X_s) ds + \frac{1}{2} \int_0^t \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s) a_{ij}(X_s) ds \\ &= f(X_0) + M_t + \int_0^t \mathcal{L}f(X_s) ds, \end{aligned}$$

and we are finished.  $\square$

### 39.3 Martingale problems

In this section we consider operators in *nondivergence form*, that is, operators of the form given by (39.9). We assume throughout this section that the coefficients  $a_{ij}$  and  $b_i$  are bounded and measurable and that  $a_{ij}(x) = a_{ji}(x)$  for all  $i, j = 1, \dots, d$  and all  $x \in \mathbb{R}^d$ . The coefficients  $a_{ij}$  are called the *diffusion coefficients* and the  $b_i$  are called the *drift coefficients*. We also assume that the operator  $\mathcal{L}$  is *uniformly elliptic*, which means that there exists  $\Lambda > 0$  such that

$$\sum_{i,j=1}^d y_i a_{ij}(x) y_j \geq \Lambda |y|^2, \quad y \in \mathbb{R}^d, x \in \mathbb{R}^d. \quad (39.12)$$

This says that the matrix  $a_{ij}(x)$  is positive definite, uniformly in  $x$ .

We saw in the previous section that if  $X_t$  is the solution to (39.1),  $a = \sigma\sigma^T$ , and  $f \in C^2$ , then

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s) ds \quad (39.13)$$

is a local martingale under  $\mathbb{P}$ . A very fruitful idea of Stroock and Varadhan is to phrase the association of  $X_t$  with  $\mathcal{L}$  in terms which use (39.13) as a key element. Let  $\Omega$  consist of all continuous functions  $\omega$  mapping  $[0, \infty)$  to  $\mathbb{R}^d$ . Let  $X_t(\omega) = \omega(t)$  and given a probability  $\mathbb{P}$ , let  $\{\mathcal{F}_t\}$  be the minimal augmented filtration generated by  $X$ . A probability measure  $\mathbb{P}$  is a solution to the *martingale problem for  $\mathcal{L}$  started at  $x_0$*  if

$$\mathbb{P}(X_0 = x_0) = 1 \quad (39.14)$$

and

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s) ds \quad (39.15)$$

is a local martingale under  $\mathbb{P}$  whenever  $f \in C^2(\mathbb{R}^d)$ . The martingale problem is *well posed* if there exists a solution  $\mathbb{P}$  and this solution is unique.

Uniqueness of the martingale problem for  $\mathcal{L}$  is closely connected to weak uniqueness or, equivalently, uniqueness in law of (39.1).

**Theorem 39.4** Suppose  $a = \sigma\sigma^T$  and suppose the matrix  $\sigma(x)$  is invertible for each  $x$ . Weak uniqueness for (39.1) holds if and only if the solution for the martingale problem for  $\mathcal{L}$  started at  $x$  is unique. Weak existence for (39.1) holds if and only if there exists a solution to the martingale problem for  $\mathcal{L}$  started at  $x$ .

*Proof* We prove the uniqueness assertion. Let  $\Omega$  be the continuous functions on  $[0, \infty)$  and  $Z_t$  the coordinate process:  $Z_t(\omega) = \omega(t)$ . First suppose the solution to the martingale problem is unique. If  $(X_t^1, W_t^1, \mathbb{P}_1)$  and  $(X_t^2, W_t^2, \mathbb{P}_2)$  are two weak solutions to (39.1), define  $\mathbb{P}_i^x$  on  $\Omega$  to be the law of  $X_i^i$  under  $\mathbb{P}_i$ ,  $i = 1, 2$ . Clearly  $\mathbb{P}_i^x(Z_0 = x) = \mathbb{P}_i(X_0^i = x) = 1$ . The expression in (39.13) is a local martingale under  $\mathbb{P}_i^x$  for each  $i$  and each  $f \in C^2$ . By the uniqueness for the solution of the martingale problem,  $\mathbb{P}_1^x = \mathbb{P}_2^x$ . This implies that the laws of  $X_t^1$  and  $X_t^2$  are the same, or weak uniqueness holds.

Now suppose weak uniqueness holds for (39.1). Let

$$Y_t = Z_t - \int_0^t b(Z_s) ds.$$

Let  $\mathbb{P}_1^x$  and  $\mathbb{P}_2^x$  be solutions to the martingale problem. If  $f(x) = x_k$ , the  $k$ th coordinate of  $x$ , then  $\partial f / \partial x_i(x) = \delta_{ik}$  and  $\partial^2 f / \partial x_i \partial x_j(x) = 0$ , where  $\delta_{ik}$  is 1 if  $i = k$  and 0 otherwise, and so  $\mathcal{L}f(Z_s) = b_k(Z_s)$ . We see from (39.13) that the  $k$ th coordinate of  $Y_t$  is a local martingale under  $\mathbb{P}_i^x$ .

Now let  $f(x) = x_k x_m$ . A simple computation shows that  $\mathcal{L}f(x) = a_{km}(x)$ , hence  $Y_t^k Y_t^m - \int_0^t a_{km}(Z_s) ds$  is a local martingale. We set

$$W_t = \int_0^t \sigma^{-1}(Z_s) dY_s.$$

The stochastic integral is finite since

$$\begin{aligned} & \mathbb{E} \int_0^t \sum_{j=1}^d (\sigma^{-1})_{ij}(Z_s) \sum_{k=1}^d (\sigma^{-1})_{ik}(Z_s) d\langle Y^j, Y^k \rangle_s \\ &= \mathbb{E} \int_0^t \sum_{i,k=1}^d (a^{-1})_{ik}(Z_s) a_{ik}(Z_s) ds = t < \infty. \end{aligned} \quad (39.16)$$

Since  $Y_t$  is a local martingale, it follows that  $W_t$  is a local martingale, and a calculation similar to (39.16) shows that  $W_t^k W_t^m - \delta_{km} t$  is also a martingale under  $\mathbb{P}_i^x$ . By Lévy's theorem (Exercise 12.4),  $W_t$  is a Brownian motion under both  $\mathbb{P}_1^x$  and  $\mathbb{P}_2^x$ , and  $(Z_t, W_t, \mathbb{P}_i^x)$  is a weak solution to (39.1). By the weak uniqueness hypothesis, the laws of  $Z_t$  under  $\mathbb{P}_1^x$  and  $\mathbb{P}_2^x$  agree, which is what we wanted to prove.

Exercise 39.1 asks you to prove that the existence of a weak solution to (39.1) is equivalent to the existence of a solution to the martingale problem.  $\square$

If the  $\sigma_{ij}$  and  $b_i$  are Lipschitz functions, the solution to (39.1) is pathwise unique; see Exercise 24.5. By Proposition 25.2, weak existence and uniqueness hold, and then the martingale problem for  $\mathcal{L}$  is well posed for every starting point.

A process that can be described in terms of a martingale problem (as well as other ways) is *super-Brownian motion*. Super-Brownian motion, also known as a *measure-valued branching diffusion process*, is a process whose state space is the set  $\mathcal{M}$  of finite positive measures on  $\mathbb{R}^d$ . The intuitive picture is as follows. Given an initial finite measure  $\mu$  as a starting point, let  $X_t^n$  be the process that starts with  $[n\mu(R^d)]$  particles, each with mass  $1/n$ , each distributed according to  $\mu(dx)/\mu(\mathbb{R}^d)$ , where  $[\cdot]$  denotes the integer part. Each particle moves as an independent Brownian motion for a time  $1/n$ , at which time each particle splits into two or dies, independently of the other particles. The particles that are now alive move as independent Brownian motions for time  $1/n$ , at which time each particle splits into two or dies, and so on.  $X_t^n$  is the measure that assigns mass  $1/n$  at each point at which there is a particle alive at time  $t$ . We take the right-continuous version of  $X_t^n$ . It turns out that the sequence converges weakly with respect to the topology of  $D[0, 1]$ , but where the state space is the set of right-continuous functions with left limits taking values in  $\mathcal{M}$  (rather than the set of real-valued functions) and the limit law can be characterized as the unique solution to a martingale problem. A solution to this martingale problem started at  $\mu \in \mathcal{M}$  is a probability measure on the space of continuous processes taking values in  $\mathcal{M}$  such that

- (1)  $\mathbb{P}(X_0 = \mu) = 1$ ;
- (2) if  $f \in C^\infty$  has compact support and we write  $v(f)$  for  $\int f d\nu$ , then

$$M_t^f = X_t(f) - \int_0^t X_r(\frac{1}{2}\Delta f) dr$$

is a continuous martingale with quadratic variation process given by

$$\langle M_t^f \rangle = \int_0^t X_r(f^2) dr.$$

See Dawson (1993) and Perkins (2002) for more on these processes.

## Exercises

- 39.1 Show that the existence of a weak solution to (39.1) is equivalent to the existence of a solution to the martingale problem for  $\mathcal{L}$ .
- 39.2 Suppose the  $a_{ij}$  are Lipschitz functions in  $x$  and the matrices  $a(x)$  are positive definite, uniformly in  $x$ ; see Exercise 25.4. Show that we can find matrices  $\sigma(x)$  so that each  $\sigma_{ij}$  is a Lipschitz function of  $x$  and  $a(x) = \sigma(x)\sigma^T(x)$  for each  $x$ .

- 39.3 If  $X$  is a solution to (39.1), give formulas for  $A_t$  and  $M_t$  in terms of  $\sigma$  and  $b$ , where  $M_t$  is a local martingale,  $A_t$  is a process whose paths are locally of bounded variation, and  $|X_t| = M_t + A_t$ .
- 39.4 Let  $A \in (-1, \infty)$  and let  $X$  be a solution to (39.1), where all the  $b_i$ 's are equal to 0,  $a = \sigma\sigma^T$ , and

$$a_{ij}(x) = \frac{\delta_{ij} + Ax_i x_j / |x|^2}{1 + A}$$

for  $x \neq 0$ , where  $\delta_{ij}$  is equal to 1 if  $i = j$  and 0 otherwise. Let  $a(0)$  be the identity matrix.

- (1) Prove that the matrices  $a(x)$  are uniformly elliptic.  
 (2) Show that  $|X_t|$  has the same law as a Bessel process of order

$$\frac{d + A}{1 + A}.$$

Conclude that if  $A$  is sufficiently close to  $-1$ , then  $X$  is transient, i.e.,  $\lim_{t \rightarrow \infty} |X_t| = \infty$ , a.s., while if  $A$  is sufficiently large, there exist arbitrarily large times  $t$  such that  $X_t = 0$ .

- 39.5 Suppose for each  $n \geq 1$ ,  $a_{ij}^n(x)$  is symmetric in  $i$  and  $j$ , is continuous in  $x$ , and the matrix whose  $(i, j)$ th entry is  $a_{ij}^n(x)$  is positive definite, uniformly in  $x$  and  $n$ . Let

$$\mathcal{L}^n f(x) = \sum_{i,j=1}^d a_{ij}^n(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \quad (39.17)$$

for  $f \in C^2$ . Suppose  $a_{ij}^n(x)$  converges to  $a_{ij}(x)$  uniformly in  $x$  as  $n \rightarrow \infty$ , and define  $\mathcal{L}$  analogously to (39.17). Fix  $x_0$  and let  $\mathbb{P}_n$  be a solution to the martingale problem for  $\mathcal{L}^n$  started at  $x_0$ .

- (1) Prove that  $\mathbb{P}_n$  converges weakly to a solution  $\mathbb{P}$  to the martingale problem for  $\mathcal{L}$  started at  $x_0$ .  
 (2) Prove that if the  $a_{ij}$  are continuously differentiable functions of  $x$  whose first partial derivatives are bounded, then there exists a solution to the martingale problem for  $\mathcal{L}$  started at  $x_0$ .  
 (3) Prove that if the  $a_{ij}$  are continuous functions of  $x$ , then there exists a solution to the martingale problem for  $\mathcal{L}$  started at  $x_0$ .

- 39.6 Suppose  $X$  is a solution to  $dX_t = \sigma(X_t) dW_t$ , where  $W$  is a  $d$ -dimensional Brownian motion,  $\sigma(x)$  is a  $d \times d$  matrix-valued function that is bounded, and  $\sigma^T \sigma$  is positive definite, uniformly in  $x$ . Prove the following estimate for the time to leave a ball: there exist constants  $c_1$  and  $c_2$  not depending on  $x_0$  such that

$$c_1 r^2 \leq \mathbb{E}^{x_0} \tau_{B(x_0, r)} \leq c_2 r^2, \quad r > 0,$$

where  $\tau_{B(x_0, r)} = \inf\{t > 0 : X_t \notin B(x_0, r)\}$ .

## Notes

See Bass (1997) for more information.

# 40

---

## Solving partial differential equations

We will be concerned with giving probabilistic representations of the solutions to certain PDEs. Throughout we will be assuming that the given PDE has a solution, the solution is unique, and the solution is sufficiently smooth. We will consider Poisson's equation, the Dirichlet problem, the Cauchy problem (with an application to Brownian passage times), and Schrödinger's equation.

We let  $X_t$  be the solution to

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt. \quad (40.1)$$

Here  $W$  is a  $d$ -dimensional Brownian motion,  $\sigma$  is a bounded Lipschitz continuous  $d \times d$  matrix-valued function,  $b$  is a bounded Lipschitz continuous  $d \times 1$  matrix-valued function, and  $X$  takes values in  $\mathbb{R}^d$ . We let  $a = \sigma\sigma^T$  and we consider the operator on  $C^2$  functions given by

$$\mathcal{L}f(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d b_i(x) \frac{\partial f}{\partial x_i}(x). \quad (40.2)$$

We suppose the operator  $\mathcal{L}$  is uniformly elliptic: there exists  $\Lambda > 0$  such that

$$\sum_{i,j=1}^d a_{ij}(x) y_i y_j \geq \Lambda \sum_{i=1}^d y_i^2, \quad y_1, \dots, y_d \in \mathbb{R}^d.$$

In fact, the uniform ellipticity of  $\mathcal{L}$  will be used only to guarantee that the exit times of bounded domains are finite, a.s.; see Exercise 40.1. For many non-uniformly elliptic operators, it is often the case that the finiteness of the exit times is known for other reasons, and the results then apply to equations involving these operators.

Let  $X_t^x$  be the solution to (40.1) when  $X_0 = x$  and let  $\mathbb{P}^x$  be the law of  $X_t^x$ . As in Chapter 39, we slightly abuse notation and say that  $(X_t, \mathbb{P}^x)$  is a strong Markov process.

### 40.1 Poisson's equation

We consider first *Poisson's equation* in  $\mathbb{R}^d$ . Suppose  $\lambda > 0$  and  $f$  is a  $C^1$  function with compact support. Poisson's equation is

$$\mathcal{L}u(x) - \lambda u(x) = -f(x), \quad x \in \mathbb{R}^d. \quad (40.3)$$

**Theorem 40.1** Suppose  $u$  is a  $C^2$  solution to (40.3) such that  $u$  and its first and second partial derivatives are bounded. Then

$$u(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t) dt.$$

*Proof* Let  $u$  be the solution to (40.3). By Theorem 39.3,

$$u(X_t) - u(X_0) = M_t + \int_0^t \mathcal{L}u(X_s) ds,$$

where  $M_t$  is a martingale. By the product formula,

$$e^{-\lambda t} u(X_t) - u(X_0) = \int_0^t e^{-\lambda s} dM_s + \int_0^t e^{-\lambda s} \mathcal{L}u(X_s) ds - \lambda \int_0^t e^{-\lambda s} u(X_s) ds.$$

Taking the expectation with respect to  $\mathbb{P}^x$  and letting  $t \rightarrow \infty$ ,

$$-u(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda s} (\mathcal{L}u - \lambda u)(X_s) ds.$$

Since  $\mathcal{L}u - \lambda u = -f$ , the result follows.  $\square$

Let us now let  $D$  be a nice bounded domain, e.g., a ball. Poisson's equation in  $D$  requires one to find a function  $u$  such that  $\mathcal{L}u - \lambda u = -f$  in  $D$  and  $u = 0$  on  $\partial D$ , where  $f \in C^2(\overline{D})$  and  $\lambda \geq 0$ . Here we can allow  $\lambda$  to be equal to 0.

**Theorem 40.2** Suppose  $u$  is a solution to Poisson's equation in a bounded domain  $D$  that is  $C^2$  in  $D$  and continuous on  $\overline{D}$ . Then

$$u(x) = \mathbb{E}^x \int_0^{\tau_D} e^{-\lambda s} f(X_s) ds.$$

*Proof* The proof is nearly identical to that of the previous theorem. We already mentioned that  $\tau_D < \infty$ , a.s.; see Exercise 40.1. Let  $S_n = \inf\{t : \text{dist}(X_t, \partial D) < 1/n\}$ . By Theorem 39.3,

$$u(X_{t \wedge S_n}) - u(X_0) = \text{martingale} + \int_0^{t \wedge S_n} \mathcal{L}u(X_s) ds.$$

By the product formula,

$$\begin{aligned} \mathbb{E}^x e^{-\lambda(t \wedge S_n)} u(X_{t \wedge S_n}) - u(x) &= \mathbb{E}^x \int_0^{t \wedge S_n} e^{-\lambda s} \mathcal{L}u(X_s) ds - \mathbb{E}^x \int_0^{t \wedge S_n} e^{-\lambda s} u(X_s) ds \\ &= -\mathbb{E}^x \int_0^{t \wedge S_n} e^{-\lambda s} f(X_s) ds. \end{aligned}$$

Now let  $n \rightarrow \infty$  and then  $t \rightarrow \infty$  and use the fact that  $u$  is zero on  $\partial D$ .  $\square$

## 40.2 Dirichlet problem

Let  $D$  be a ball (or other nice bounded domain) and let us consider the solution to the *Dirichlet problem*: given a continuous function  $f$  on  $\partial D$ , find  $u \in C(\overline{D})$  such that  $u$  is  $C^2$  in  $D$  and

$$\mathcal{L}u = 0 \text{ in } D, \quad u = f \text{ on } \partial D. \tag{40.4}$$

We considered the Dirichlet problem in the special case when  $\mathcal{L}$  is the Laplacian in Section 21.4.

**Theorem 40.3** Suppose  $u$  is a solution to the Dirichlet problem specified by (40.4). Then  $u$  satisfies

$$u(x) = \mathbb{E}^x f(X_{\tau_D}).$$

*Proof* As we mentioned above,  $\tau_D < \infty$ , a.s. Let  $S_n = \inf\{t : \text{dist}(X_t, \partial D) < 1/n\}$ . By Theorem 39.3,

$$u(X_{t \wedge S_n}) = u(X_0) + \text{martingale} + \int_0^{t \wedge S_n} \mathcal{L}u(X_s) ds.$$

Since  $\mathcal{L}u = 0$  inside  $D$ , taking expectations shows

$$u(x) = \mathbb{E}^x u(X_{t \wedge S_n}).$$

We let  $t \rightarrow \infty$  and then  $n \rightarrow \infty$ . By dominated convergence, we obtain  $u(x) = \mathbb{E}^x u(X_{\tau_D})$ . This is what we want since  $u = f$  on  $\partial D$ .  $\square$

If  $v \in C^2$  and  $\mathcal{L}v = 0$  in  $D$ , we say  $v$  is  $\mathcal{L}$ -harmonic in  $D$ .

### 40.3 Cauchy problem

The related parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \mathcal{L}u$$

is often of interest. Here  $u$  is a function of  $x \in \mathbb{R}^d$  and  $t \in [0, \infty)$ . When we write  $\mathcal{L}u$ , we mean

$$\mathcal{L}u(x, t) = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x, t) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i}(x, t).$$

We will sometimes write  $u_t$  for  $\partial u / \partial t$ .

Suppose for simplicity that the function  $f$  is a continuous function with compact support. The *Cauchy problem* is to find  $u$  such that  $u$  is bounded,  $u$  is  $C^2$  with bounded first and second partial derivatives in  $x$ ,  $u$  is  $C^1$  in  $t$  for  $t > 0$ , and

$$\begin{aligned} u_t(x, t) &= \mathcal{L}u(x, t), & t > 0, x \in \mathbb{R}^d, \\ u(x, 0) &= f(x), & x \in \mathbb{R}^d. \end{aligned} \tag{40.5}$$

**Theorem 40.4** Suppose there exists a solution to (40.5) that is  $C^2$  in  $x$  and  $C^1$  in  $t$  for  $t > 0$ . Then  $u$  satisfies

$$u(x, t) = \mathbb{E}^x f(X_t).$$

*Proof* Fix  $t_0$  and let  $M_t = u(X_t, t_0 - t)$ . Note

$$\frac{\partial}{\partial t} u(x, t_0 - t) = -u_t(x, t_0 - t).$$

Similarly to the proof of Theorem 39.3 (see Exercise 40.2) but using now the multivariate version of Itô's formula,

$$u(X_t, t_0 - t) = \text{martingale} + \int_0^t \mathcal{L}u(X_s, t_0 - s) ds - \int_0^t u_t(X_s, t_0 - s) ds. \quad (40.6)$$

Since  $u_t = \mathcal{L}u$ ,  $M_t$  is a martingale, and  $\mathbb{E}^x M_0 = \mathbb{E}^x M_{t_0}$ . On the one hand,

$$\mathbb{E}^x M_{t_0} = \mathbb{E}^x u(X_{t_0}, 0) = \mathbb{E}^x f(X_{t_0}),$$

while on the other hand,

$$\mathbb{E}^x M_0 = \mathbb{E}^x u(X_0, t_0) = u(x, t_0).$$

Since  $t_0$  is arbitrary, the result follows.  $\square$

A very similar proof allows one to represent the solution to the Cauchy problem in a bounded domain. Suppose  $u(x, t)$  is  $C^2$  in the  $x$  variable,  $C^1$  in the  $t$  variable, and satisfies

$$\frac{\partial u}{\partial t}(x, t) = \mathcal{L}u(x, t)$$

for  $(x, t) \in D \times (0, t_1]$ , where  $D$  is a bounded domain in  $\mathbb{R}^d$  and  $t_1 > 0$ . Suppose  $u(x, 0) = f(x)$  and  $u(x, t) = 0$  for all  $x \in \partial D$ . Exercise 40.3 asks you to show that in this case

$$u(x, t) = \mathbb{E}^x f(X_{t \wedge \tau_D}),$$

where again  $\tau_D$  is the first exit time of  $X$  from the domain  $D$ .

The Cauchy problem has an application to the passage times of Brownian motion. Suppose we look at the equation

$$u_x(x, t) = \frac{1}{2} u_{xx}(x, t), \quad 0 < x < b, \quad t > 0,$$

with

$$u(x, 0) = f(x) \text{ for all } x, \quad u(0, t) = u(b, t) = 0 \text{ for all } t,$$

where  $f$  is a bounded function on  $[0, b]$ . This is a partial differential equation (the *heat equation*) that is sometimes solved in undergraduate classes; see, e.g., Boyce and DiPrima (2009), Section 10.5. Using a combination of the technique of separation of variables and Fourier series expansions, the solution can then be shown to be

$$u(x, t) = \int f(y) p^0(t, x, y) dy,$$

where

$$p^0(t, x, y) = \frac{2}{b} \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t / 2b^2} \sin(n\pi x/b) \sin(n\pi y/b).$$

See also Knight (1981), p. 62. Since  $u(x, t)$  is also equal to  $\mathbb{E}^x f(X_{t \wedge \tau_D})$ , where  $D$  is the interval  $(0, b)$ , then the  $p^0(t, x, y)$  are the transition densities for Brownian motion killed on exiting  $(0, b)$ .

In particular, if we take  $f$  identically equal to 1 on  $(0, b)$ , we see that starting at  $x$  inside  $(0, b)$ ,  $\mathbb{P}^x(t < \tau_D)$  is asymptotically equal to  $ce^{-\pi^2 t / 2b^2}$ . If  $b$  is 2, this becomes  $ce^{-\pi^2 t / 8}$ .

Since the time for a Brownian motion started at 0 to leave  $(-1, 1)$  is the same as the time for a Brownian motion started at 1 to leave  $(0, 2)$ , we obtain the estimate that was used in Exercise 7.2.

## 40.4 Schrödinger operators

Finally we look at what happens when one adds a potential term, that is, when one considers the operator

$$\mathcal{L}u(x) + q(x)u(x). \quad (40.7)$$

This is known as the *Schrödinger operator*, and  $q(x)$  is known as the *potential*. Equations involving the operator in (40.7) are considerably simpler than the quantum mechanics Schrödinger equation because here all terms are real-valued.

If  $X_t$  is the diffusion corresponding to  $\mathcal{L}$ , then solutions to PDEs involving the operator in (40.7) can be expressed in terms of  $X_t$  by means of the *Feynman–Kac formula*. To illustrate, let  $D$  be a nice bounded domain, e.g., a ball,  $q$  a  $C^2$  function on  $\overline{D}$ , and  $f$  a continuous function on  $\partial D$ ;  $q^+$  denotes the positive part of  $q$ .

**Theorem 40.5** *Let  $D$ ,  $q$ ,  $f$  be as above. Let  $u$  be a  $C^2$  function on  $\overline{D}$  that agrees with  $f$  on  $\partial D$  and satisfies  $\mathcal{L}u + qu = 0$  in  $D$ . If*

$$\mathbb{E}^x \exp\left(\int_0^{\tau_D} q^+(X_s) ds\right) < \infty,$$

then

$$u(x) = \mathbb{E}^x \left[ f(X_{\tau_D}) e^{\int_0^{\tau_D} q(X_s) ds} \right]. \quad (40.8)$$

*Proof* Let  $B_t = \int_0^{t \wedge \tau_D} q(X_s) ds$ . By Itô's formula and the product formula,

$$e^{B(t \wedge \tau_D)} u(X_{t \wedge \tau_D}) = u(X_0) + \text{martingale} + \int_0^{t \wedge \tau_D} u(X_r) e^{B_r} dB_r + \int_0^{t \wedge \tau_D} e^{B_r} \mathcal{L}u(X_r) dr.$$

Taking the expectation with respect to  $\mathbb{P}^x$  and using Proposition 39.3,

$$\mathbb{E}^x e^{B(t \wedge \tau_D)} u(X_{t \wedge \tau_D}) = u(x) + \mathbb{E}^x \int_0^{t \wedge \tau_D} e^{B_r} u(X_r) q(X_r) dr + \mathbb{E}^x \int_0^{t \wedge \tau_D} e^{B_r} \mathcal{L}u(X_r) dr.$$

Since  $\mathcal{L}u + qu = 0$ ,

$$\mathbb{E}^x e^{B(t \wedge \tau_D)} u(X_{t \wedge \tau_D}) = u(x).$$

If we let  $t \rightarrow \infty$  and use the exponential integrability of  $q^+$ , the result follows.  $\square$

The existence of a solution to  $\mathcal{L}u + qu = 0$  in  $D$  depends on the finiteness of  $\mathbb{E}^x \exp(\int_0^{\tau_D} q^+(X_s) ds)$ , an expression that is sometimes known as the *gauge*.

Even in one dimension with  $D = (0, 1)$  and  $q$  a constant function, the gauge need not be finite. With  $x = 1/2$ ,  $\mathbb{P}^x(\tau_D > t)$  is asymptotically equal to  $ce^{-\pi^2 t/2}$  as  $t \rightarrow \infty$  by

Section 40.3. Hence

$$\begin{aligned}\mathbb{E}^x \exp\left(\int_0^{\tau_D} q \, ds\right) &= \mathbb{E}^x e^{q\tau_D} \\ &= \int_0^\infty q e^{qt} \mathbb{P}^x(\tau_D > t) \, dt;\end{aligned}$$

this is infinite if  $q \geq \pi^2/2$ .

## Exercises

40.1 This (lengthy) exercise is designed to guide you through a proof that solutions to (40.1) exit bounded sets in finite time, a.s.

(1) Suppose

$$X_t = W_t + \int_0^t a_s \, ds,$$

where  $W$  is a one-dimensional Brownian motion, and  $a_s$  is an adapted process bounded by  $K$ . Let  $L > K > 0$  and  $t_0 > 0$ . Show that there exists  $\varepsilon > 0$ , depending only on  $L, K$ , and  $t_0$  such that  $\mathbb{P}(|X_{t_0}| > 3L) > \varepsilon$ .

(2) Suppose  $X_t = M_t + \int_0^t a_s \, ds$ , where  $a_s$  is as in (1) and  $M$  is a continuous martingale with  $K^{-1} \leq d\langle M \rangle_t / dt \leq K$ , a.s. Use a time change argument to show that there exist  $L, \varepsilon > 0$  such that

$$\mathbb{P}(\sup_{s \leq 1} |X_s| \leq L) \leq 1 - \varepsilon.$$

(3) If now  $X$  is a solution to (40.1),  $a = \sigma\sigma^T$ , and  $\mathcal{L}$  given by (40.2) is uniformly elliptic, show by looking at the first coordinate of  $X$  that there exist  $L, \varepsilon$  such that

$$\mathbb{P}^x(\sup_{s \leq 1} |X_s| \leq L) \leq 1 - \varepsilon, \quad x \in B(0, L).$$

(4) What you have proved in (3) can be rephrased as saying that if  $(X_t, \mathbb{P}^x)$  is a strong Markov process that solves (40.1) for every starting point and  $\tau = \inf\{t : X_t \notin B(0, L)\}$ , then  $\mathbb{P}^x(\tau > 1) \leq 1 - \varepsilon$ , where  $\varepsilon$  does not depend on  $x$ . Now use the strong Markov property (cf. the proof of Proposition 21.2) to show  $\mathbb{P}^x(\tau > k) \leq (1 - \varepsilon)^k$ . Conclude that  $\tau < \infty$ ,  $\mathbb{P}^x$ -a.s., for each starting point  $x$ .

40.2 Prove (40.6).

40.3 Let  $D$  be a ball in  $\mathbb{R}^d$  and suppose  $u$  is the solution to the Cauchy problem in the domain  $D \times [0, t_1]$  as described in Section 40.3. Show that  $u(x, t) = \mathbb{E}^x f(X_{t \wedge \tau_D})$ .

40.4 Suppose  $f$  is such that the solution  $u$  to

$$u_t(x, t) = \mathcal{L}u(x, t) + q(x), \quad u(x, 0) = f(x),$$

is  $C^2$  in  $x$  and  $t$  and  $X$  is the diffusion associated with  $\mathcal{L}$ . Prove that

$$u(x, t) = \mathbb{E}^x \left[ f(X_t) e^{\int_0^t q(X_s) \, ds} \right].$$

- 40.5 Suppose  $(X_t, \mathbb{P}^x)$  is a Brownian motion on  $[0, b]$  with reflection at 0 and  $b$ . Find a series expansion for  $p(t, x, y)$ , the transition densities for  $X$ .

*Hint:* Imitate the argument for absorbing Brownian motion in Section 40.3, but now use the boundary conditions  $u_x(0, t) = u_x(b, t) = 0$ .

### Notes

See Bass (1997) for more on the connection between probability and PDEs.

# 41

---

## One-dimensional diffusions

Under very mild regularity conditions, every one-dimensional diffusion arises from first time-changing a one-dimensional Brownian motion and then making a transformation of the state space. We will prove this fact in this chapter.

### 41.1 Regularity

Throughout this chapter we suppose that we have a continuous process  $(X_t, \mathbb{P}^x)$  defined on an interval  $I$  contained in  $\mathbb{R}$ . For almost all of the chapter, we suppose for simplicity that the interval is in fact all of  $\mathbb{R}$ . We further suppose that  $(X_t, \mathbb{P}^x)$  is a strong Markov process with respect to a right-continuous filtration  $\{\mathcal{F}_t\}$  such that each  $\mathcal{F}_t$  contains all the sets that are  $\mathbb{P}^x$ -null for every  $x$ . We call such a process a *one-dimensional diffusion*.

Write

$$T_y = \inf\{t : X_t = y\}, \quad (41.1)$$

the first time the process  $X$  hits the point  $y$ . We will also assume that every point can be hit from every other point: for all  $x, y$ ,

$$\mathbb{P}^x(T_y < \infty) = 1. \quad (41.2)$$

When (41.2) holds, we say the diffusion is *regular*.

For any interval  $J$ , define  $\tau_J = \inf\{t : X_t \notin J\}$ , the first time the process leaves  $J$ . When  $X_t$  is a Brownian motion, we know (Proposition 3.16) that the distribution of  $X_t$  upon exiting  $[a, b]$  is

$$\mathbb{P}^x(X(\tau_{[a,b]}) = a) = \frac{b-x}{b-a}, \quad \mathbb{P}^x(X(\tau_{[a,b]}) = b) = \frac{x-a}{b-a}. \quad (41.3)$$

We say that a regular diffusion  $X_t$  is on *natural scale* if (41.3) holds for every interval  $[a, b]$ . We also say a regular diffusion  $X$  defined on an interval  $I$  properly contained in  $\mathbb{R}$  is on natural scale if (41.3) holds whenever  $[a, b] \subset I$  and  $x \in (a, b)$ .

If  $X_t$  is regular, then the process started at  $x$  must leave  $x$  immediately. That is, if  $S = \inf\{t > 0 : X_t \neq x\}$ , then  $\mathbb{P}^x(S = 0) = 1$ . To see this, let  $\varepsilon > 0$  and  $U = \inf\{t : |X_t - x| \geq \varepsilon\}$ . By the regularity of  $X$ ,  $\mathbb{E}^x e^{-U} > 0$ . Observe that  $U = S + U \circ \theta_S$ , where  $\theta_t$  is the shift operator. By the strong Markov property at time  $S$ ,

$$\mathbb{E}^x e^{-U} = \mathbb{E}^x [e^{-S} \mathbb{E}^x [e^{-U} \circ \theta_S \mid \mathcal{F}_S]] = \mathbb{E}^x [e^{-S} \mathbb{E}^{X_S} [e^{-U}]] = \mathbb{E}^x [e^{-S} \mathbb{E}^x e^{-U}],$$

since  $X_S = x$  by the continuity of the paths of  $X$ . The only way this can happen is if  $\mathbb{E}^x e^{-S} = 1$ , which implies  $S = 0$ ,  $\mathbb{P}^x$ -a.s.

## 41.2 Scale functions

We will show that given a regular diffusion, there exists a *scale function* that is continuous, strictly increasing, and such that  $s(X_t)$  is on natural scale.

We first look at a special case, when the diffusion is given as the solution to an SDE. Suppose  $X_t$  is given as the solution to

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt, \quad (41.4)$$

where we assume  $\sigma$  and  $b$  are real-valued, continuous and bounded above and  $\sigma$  is bounded below by a positive constant. Let  $a(x) = \sigma^2(x)$ . In this case we can give a formula for the scale function.

**Theorem 41.1** *The scale function  $s(x)$  is the solution to*

$$\frac{1}{2}a(x)s''(x) + b(x)s'(x) = 0,$$

and for some constants  $c_1, c_2$ , and  $x_0$  is given by

$$s(x) = c_1 + c_2 \int_{x_0}^x \exp\left(-\int_{x_0}^y \frac{2b(w)}{a(w)} dw\right) dy. \quad (41.5)$$

*Proof* To solve the differential equation, we write

$$\frac{s''(x)}{s'(x)} = -2 \frac{b(x)}{a(x)},$$

or  $(\log s'(x))' = -2b(x)/a(x)$ , from which (41.5) follows. Since we assumed that  $\sigma$  and  $b$  are continuous,  $s(x)$  given by (41.5) is  $C^2$ . Since  $\sigma$  is bounded below by a positive constant and  $b$  and  $\sigma$  are bounded,  $s$  given by (41.5) is strictly increasing. Applying Itô's formula,

$$s(X_t) - s(X_0) = \int_0^t s'(X_r) \sigma(X_r) dW_r \quad (41.6)$$

because

$$\int_0^t [\frac{1}{2}s''(X_r) \sigma(X_r)^2 + s'(X_r) b(X_r)] dr = 0.$$

This implies that  $s(X_t) - s(X_0)$  is a martingale, hence a time change of Brownian motion. Therefore the exit probabilities of  $s(X_t)$  for an interval  $[a, b]$  are the same as those of a Brownian motion, namely, those given by (41.3).  $\square$

From (41.6), if  $Y_t = s(X_t)$ , then

$$dY_t = (s'\sigma)(s^{-1}(Y_t)) dW_t. \quad (41.7)$$

Now we show there exists a scale function for general regular diffusions on  $\mathbb{R}$ . Let  $J$  be an interval  $[a, b]$ . We define

$$p(x) = p_J(x) = \mathbb{P}^x(X_{\tau_J} = b). \quad (41.8)$$

**Proposition 41.2** *Let  $J = [a, b]$  be a finite interval. Then  $p(X_{t \wedge \tau_J})$  is a regular diffusion on  $[0, 1]$  on natural scale.*

*Proof* First we show that  $p$  is increasing. To get to the point  $b$  starting from  $x$ , the process must first hit every point between  $x$  and  $b$  because  $X$  has continuous paths. If  $a < x < y < b$ , by the strong Markov property at time  $T_y$ ,  $p(x) \leq p(y)$ . We claim there is a positive probability that the process starting from  $x$  hits  $a$  before  $y$ , that is,

$$\mathbb{P}^x(T_a < T_y) > 0. \quad (41.9)$$

If (41.9) did not hold, then the process started at  $x$  must hit  $y$  before hitting  $a$ , then by the continuity of paths must hit  $x$  before hitting  $a$ , and once the process is again at  $x$ , it again hits  $y$  with probability one before  $a$  and so on. Therefore the process never hits  $a$ , a contradiction to the regularity; Exercise 41.2 asks you to make this argument precise. Therefore (41.9) does hold, and by the strong Markov property at  $T_y$ ,

$$p(x) = \mathbb{P}^x(T_y < T_a)p(y).$$

Since  $\mathbb{P}^x(T_y < T_a) = 1 - \mathbb{P}^x(T_a < T_y)$  is strictly less than 1,  $p$  is strictly increasing.

Next we show that  $p$  is continuous. We show continuity from the right; the proof of continuity from the left is similar. Suppose  $x_n \downarrow x$ . The process  $X_t$  has continuous paths, so given  $\varepsilon$  we can find  $t$  small enough so that  $\mathbb{P}^x(T_a < t) < \varepsilon$ . By the Blumenthal 0–1 law (Proposition 20.8),  $\mathbb{P}^x(T_{(x,b]} = 0)$  is zero or one, where  $T_{(x,b]}$  is the first time the process hits the interval  $(x, b]$ . If it is zero, the process immediately moves to the left from  $x$ , a.s., and by the strong Markov property at  $T_x$ , it never hits  $b$ , a contradiction. The probability must therefore be one. Thus by the continuity of paths, for  $n$  large enough,  $\mathbb{P}^x(T_{x_n} < t) \geq 1 - \varepsilon$ . Hence with probability at least  $1 - 2\varepsilon$ ,  $X_t$  hits  $x_n$  before  $a$ . Since

$$p(x) = \mathbb{P}^x(T_{x_n} < T_a) p(x_n) \geq (1 - 2\varepsilon)p(x_n)$$

and  $\varepsilon$  is arbitrary, we see that  $p(x) \geq \liminf_{n \rightarrow \infty} p(x_n)$ . Since  $p$  is strictly increasing,  $p(x_n)$  decreases, and therefore  $p(x) = \lim p(x_n)$ .

Finally, we show  $p(X_t)$  is on natural scale. Let  $[e, f] \subset (0, 1)$  and let

$$r(y) = \mathbb{P}^y(X_t \text{ hits } p^{-1}(f) \text{ before hitting } p^{-1}(e)).$$

Note that

$$\begin{aligned} \mathbb{P}^x(p(X_t) \text{ hits } f \text{ before } e) &= \mathbb{P}^{p^{-1}(x)}(X_t \text{ hits } p^{-1}(f) \text{ before } p^{-1}(e)) \\ &= r(p^{-1}(x)). \end{aligned} \quad (41.10)$$

For  $y \in [p^{-1}(a), p^{-1}(b)]$ , the strong Markov property tells us that

$$\begin{aligned} p(y) &= \mathbb{P}^y(X_t \text{ hits } p^{-1}(f) \text{ before } p^{-1}(e))p(p^{-1}(f)) \\ &\quad + \mathbb{P}^y(X_t \text{ hits } p^{-1}(e) \text{ before } p^{-1}(f))p(p^{-1}(e)) \\ &= r(y)f + (1 - r(y))e. \end{aligned} \quad (41.11)$$

Solving for  $r(y)$ , we obtain  $r(y) = (p(y) - e)/(f - e)$ . Substituting in (41.10),

$$\begin{aligned} \mathbb{P}^x(p(X_t) \text{ hits } f \text{ before } e) &= r(p^{-1}(x)) = (p(p^{-1}(x)) - e)/(f - e) \\ &= (x - e)/(f - e), \end{aligned}$$

which is the formula we wanted.  $\square$

Note that if  $X_t$  is on natural scale, then so is  $c_1 X_t + c_2$  for any constants  $c_1 > 0, c_2 \in \mathbb{R}$ .

**Theorem 41.3** *There exists a continuous strictly increasing function  $s$  such that  $s(X_t)$  is on natural scale on  $s(\mathbb{R})$ .*

*Proof* Let  $J_n$  be closed intervals increasing up to  $\mathbb{R}$ . Pick two points in  $J_1$ ; label them  $a$  and  $b$  with  $a < b$ . Choose  $A_n$  and  $B_n$  so that if  $s_n(x) = A_n p_{J_n}(x) + B_n$ , then  $s_n(a) = 0$  and  $s_n(b) = 1$ .

We will show that if  $n \geq m$ , then  $s_n = s_m$  on  $J_m$ . Once we have that, we can set  $s(x) = s_n(x)$  on  $J_n$ , and the theorem will be proved.

Suppose  $J_m = [e, f]$ . By Proposition 41.2, both  $s_m(X_t)$  and  $s_n(X_t)$  are on natural scale. For all  $x \in J_m$ ,

$$\begin{aligned} \frac{s_m(x) - s_m(e)}{s_m(f) - s_m(e)} &= \mathbb{P}^{s_m(x)}(s_m(X_t) \text{ hits } s_m(f) \text{ before } s_m(e)) \\ &= \mathbb{P}^x(X_t \text{ hits } f \text{ before } e). \end{aligned}$$

We have a similar equation with  $s_m$  replaced everywhere by  $s_n$ . It follows that

$$\frac{s_m(x) - s_m(e)}{s_m(f) - s_m(e)} = \frac{s_n(x) - s_n(e)}{s_n(f) - s_n(e)}$$

for all  $x$ , which implies that  $s_n(x) = C s_m(x) + D$  for some constants  $C$  and  $D$ . Since  $s_n$  and  $s_m$  are equal at both  $x = a$  and  $x = b$ , then  $C$  must be 1 and  $D$  must be 0.  $\square$

### 41.3 Speed measures

Suppose that  $(\mathbb{P}^x, X_t)$  is a regular diffusion on  $\mathbb{R}$  on natural scale. For each finite interval  $(a, b)$ , define

$$G_{ab}(x, y) = \begin{cases} \frac{2(x-a)(b-y)}{b-a}, & a < x \leq y < b, \\ \frac{2(y-a)(b-x)}{b-a}, & a < y \leq x < b, \end{cases} \quad (41.12)$$

and set  $G_{ab}(x, y) = 0$  if  $x$  or  $y$  is not in  $(a, b)$ . A measure  $m(dx)$  is the *speed measure* for the diffusion  $(X_t, \mathbb{P}^x)$  if

$$\mathbb{E}^x \tau_{(a,b)} = \int G_{ab}(x, y) m(dy) \quad (41.13)$$

for each finite interval  $(a, b)$  and each  $x \in (a, b)$ . As (41.13) indicates, the speed measure governs how quickly the diffusion moves through intervals.

As an example, let us argue that the speed measure for Brownian motion is a Lebesgue measure. By Proposition 3.16, if  $(X_t, \mathbb{P}^x)$  is a Brownian motion,

$$\mathbb{E}^x \tau_{(a,b)} = (x - a)(b - x).$$

On the other hand, a calculation shows that

$$\int G_{ab}(x, y) dy = (x - a)(b - x).$$

Since

$$\mathbb{E}^x \tau_{(a,b)} = \int G_{ab}(x, y) dy$$

and Brownian motion is on natural scale, we see that the speed measure  $m(dy)$  of Brownian motion is equal to a Lebesgue measure.

We will show that

- (1) a regular diffusion on natural scale has one and only one speed measure,
- (2) the law of the diffusion is determined by the speed measure, and
- (3) there exists a diffusion with a given speed measure.

We first want to show that any speed measure must satisfy  $0 < m(a, b) < \infty$  for any finite interval  $[a, b]$ . To start we have the following lemma.

**Lemma 41.4** *If  $[a, b]$  is a finite interval, then  $\sup_x \mathbb{E}^x \tau_{(a,b)}^k < \infty$  for each positive integer  $k$ .*

*Proof* Pick  $y \in (a, b)$ . Since  $X_t$  is a regular diffusion,  $\mathbb{P}^y(T_a < \infty) = 1$ , and hence there exists  $t_0$  such that  $\mathbb{P}^y(T_a > t_0) < 1/2$ . Similarly, taking  $t_0$  larger if necessary,  $\mathbb{P}^y(T_b > t_0) \leq 1/2$ . If  $a < x \leq y$ , then

$$\mathbb{P}^x(\tau_{(a,b)} > t_0) \leq \mathbb{P}^x(T_a > t_0) \leq \mathbb{P}^y(T_a > t_0) \leq 1/2,$$

and similarly,  $\mathbb{P}^x(\tau_{(a,b)} > t_0) \leq 1/2$  if  $y \leq x < b$ . By the Markov property,

$$\begin{aligned} \mathbb{P}^x(\tau_{(a,b)} > (n+1)t_0) &= \mathbb{E}^x[\mathbb{P}^{X(nt_0)}(\tau_{(a,b)} > t_0); \tau_{(a,b)} > nt_0] \\ &\leq \frac{1}{2} \mathbb{P}^x(\tau_{(a,b)} > nt_0), \end{aligned}$$

and by induction,  $\mathbb{P}^x(\tau_{(a,b)} > nt_0) \leq 2^{-n}$ . The lemma is now immediate.  $\square$

**Lemma 41.5** *If  $(X_t, \mathbb{P}^x)$  has a speed measure  $m$  and  $[a, b]$  is a non-empty finite interval, then  $0 < m(a, b) < \infty$ .*

*Proof* If  $m(a, b) = 0$ , then for  $x \in (a, b)$ , we have

$$\mathbb{E}^x \tau_{(a,b)} = \int G_{ab}(x, y) m(dy) = 0,$$

which implies  $\tau_{(a,b)} = 0$ ,  $\mathbb{P}^x$ -a.s., a contradiction to the continuity of the paths of  $X_t$ .

Next we show the finiteness of  $m(a, b)$ . Pick  $(e, f)$  such that  $[a, b] \subset (e, f)$ . There exists a constant  $c$  such that for  $x, y \in (a, b)$ ,  $G_{ef}(x, y)$  is bounded below by  $c$ , so

$$m(a, b) \leq c^{-1} \int_e^f G_{ef}(x, y) m(dy) = c^{-1} \mathbb{E}^x \tau_{(e,f)} < \infty.$$

This completes the proof.  $\square$

**Theorem 41.6** *A regular diffusion on natural scale on  $\mathbb{R}$  has one and only one speed measure.*

*Proof* First let  $I = (e, f)$  be a finite open interval. For  $n > 1$  let  $x_i = e + i(f - e)/2^n$ ,  $i = 0, 1, 2, \dots, 2^n$ . Let  $\mathcal{D}_n = \{x_i : 0 \leq i \leq 2^n\}$ . Let

$$m_n(dx) = 2^n \sum_{i=1}^{2^n-1} B(x_i) \delta_{x_i}, \quad (41.14)$$

where  $B(x_i) = \mathbb{E}^{x_i} \tau_{(x_{i-1}, x_{i+1})}$ . We first want to show that if  $[a, b]$  is a subinterval of  $I$  with  $a, b$  each in  $\mathcal{D}_n$  and  $x$  is also in  $\mathcal{D}_n$ , then

$$\mathbb{E}^x \tau_{(a,b)} = \int G_{ab}(x, y) m_n(dy). \quad (41.15)$$

To see this, let  $S_0 = 0$  and  $S_{j+1} = \inf\{t > S_j : |X_t - X_{S_j}| = 2^{-n}\} \wedge \tau_{(a,b)}$ . The  $S_j$ 's are the successive times that  $X$  moves  $2^{-n}$ , up until the time of leaving  $(a, b)$ . Because  $X$  is on natural scale,  $X_{S_{j+1}}$  is equal to  $X_{S_j} + 2^{-n}$  with probability  $\frac{1}{2}$  and equal to  $X_{S_j} - 2^{-n}$  with probability  $\frac{1}{2}$ , until leaving  $(a, b)$ . Therefore  $X_{S_j}$  is a simple symmetric random walk on the lattice with step size  $2^{-n}$ , stopped on leaving  $(a, b)$ .

Let  $J(x_i) = (x_i - 2^{-n}, x_i + 2^{-n})$  for  $x_i \neq a, b$ . Let  $J(a) = J(b) = \emptyset$ . By repeated use of the strong Markov property,

$$\begin{aligned} \mathbb{E}^x \tau_{(a,b)} &= \sum_{j=0}^{\infty} \mathbb{E}^x (S_{j+1} - S_j) \\ &= \mathbb{E}^x \sum_{j=0}^{\infty} \mathbb{E}^{X(S_j)} [\tau_{J(X_0)}] = \mathbb{E}^x \sum_{j=0}^{\infty} B(X_{S_j}) 1_{(a,b)}(X_{S_j}). \end{aligned}$$

Let  $N_i = \sum_{j=0}^{\infty} 1_{\{x_i\}}(X_{S_j})$ , the number of visits to  $x_i$  before exiting  $(a, b)$ . Then

$$\begin{aligned} \mathbb{E}^x \tau_{(a,b)} &= \mathbb{E}^x \sum_{j=0}^{\infty} B(X_{S_j}) 1_{(a,b)}(X_{S_j}) \\ &= \mathbb{E}^x \sum_{j=0}^{\infty} \sum_{i=1}^{2^n-1} B(X_{S_j}) 1_{\{x_i\}}(X_{S_j}) \\ &= \mathbb{E}^x \sum_{i=1}^{2^n-1} B(x_i) N_i. \end{aligned} \quad (41.16)$$

$\mathbb{E}^x N_i$  must equal 0 when  $x = a$  or  $x = b$  and satisfies the equation

$$\mathbb{E}^{x_j} N_i = \delta_{ij} + \frac{1}{2} (\mathbb{E}^{x_{j+1}} N_i + \mathbb{E}^{x_{j-1}} N_i), \quad (41.17)$$

where  $\delta_{ij}$  is 1 if  $i = j$  and 0 otherwise. This holds because for  $j \neq i$ , the process goes left or right, each with probability  $1/2$ , while if  $j = 1$ , we add one to  $N_i$  before going left or right. The function  $x \rightarrow \mathbb{E}^x N_i$  is hence piecewise linear on  $(a, x_i)$  and on  $(x_i, b)$ . Some algebra shows that we must have

$$\mathbb{E}^x N_i = 2^n G_{ab}(x, x_i). \quad (41.18)$$

Combining (41.16) and (41.18),

$$\begin{aligned} \mathbb{E}^x \tau_{(a,b)} &= \sum_{i=1}^{2^n-1} B(x_i) 2^n G_{ab}(x, x_i) \\ &= \int G_{ab}(x, y) m_n(dy), \end{aligned}$$

which is (41.15).

Using (41.15) and the same proof as that of Lemma 41.5,  $m_n(a, b)$  is bounded above by a constant independent of  $n$ . By a diagonalization procedure, there exists a subsequence  $n_k$  such that  $m_{n_k}$  converges weakly to  $m$ , where  $m$  is a measure that is finite on every subinterval  $(a, b)$  such that  $[a, b] \subset I$ . By the continuity of  $G_{ab}$ ,

$$\mathbb{E}^x \tau_{(a,b)} = \int G_{ab}(x, y) m(dy) \quad (41.19)$$

whenever  $a, b$ , and  $x$  are in  $\mathcal{D}_n$  for some  $n$ .

We now remove this last restriction. If  $a, b$  are not of this form, take  $a_r, b_r$  to be in  $\cup_n \mathcal{D}_n$  such that  $(a_r, b_r) \uparrow (a, b)$ . Then  $\tau_{(a_r, b_r)} \uparrow \tau_{(a,b)}$ , and by the continuity of  $G_{ab}$  in  $a, b, x$ , and  $y$ , we have (41.19) for all  $a$  and  $b$ . Take  $y_r \uparrow x, z_r \downarrow x$  such that  $y_r$  and  $z_r$  are in  $\mathcal{D}_n$  for some  $n$ . By the strong Markov property,

$$\begin{aligned} \mathbb{E}^x \tau_{(a,b)} &= \mathbb{E}^x \tau_{(y_r, z_r)} + \mathbb{E}^{y_r} \tau_{(a,b)} \mathbb{P}^x(X_{\tau_{(y_r, z_r)}} = y_r) \\ &\quad + \mathbb{E}^{z_r} \tau_{(a,b)} \mathbb{P}^x(X_{\tau_{(y_r, z_r)}} = z_r). \end{aligned}$$

By the continuity of  $G_{ab}$  in  $x$ , and the fact that  $\mathbb{E}^x \tau_{(y_r, z_r)} \rightarrow 0$  as  $r \rightarrow \infty$ , we obtain (41.19) for all  $x$ .

We leave the uniqueness as Exercise 41.3.

Finally, let  $I_k$  be finite subintervals increasing up to  $\mathbb{R}$ . Let  $m_k$  be the speed measure for  $X_t$  on the interval  $I_k$ . By the uniqueness result,  $m_k$  agrees with  $m_\ell$  on  $I_\ell$  if  $I_\ell \subset I_k$ . Setting  $m$  to be the measure whose restriction to  $I_k$  is  $m_k$  gives us the speed measure.  $\square$

The speed measure completely characterizes occupation times.

**Corollary 41.7** *Suppose  $X_t$  is a diffusion on natural scale on  $\mathbb{R}$ . If  $f$  is bounded and measurable, for each  $a < b$ ,*

$$\mathbb{E}^x \int_0^{\tau_{(a,b)}} f(X_s) ds = \int G_{ab}(x, y) f(y) m(dy). \quad (41.20)$$

*Proof* Suppose that  $f$  is continuous and bounded on  $[a, b]$ . Let  $x_i, S_j, B(x_i), N_i$ , and  $m_n$  be as in the proof of Theorem 41.6. Let

$$\varepsilon_n = \sup\{|f(x) - f(y)| : |x - y| \leq 2^{-n}\}.$$

Note that if  $(x - a)/(b - a)$  is a multiple of  $2^{-n}$ ,

$$\mathbb{E}^x \int_0^{\tau_{(a,b)}} f(X_s) ds = \sum_{j=0}^{\infty} \mathbb{E}^x \int_{S_j}^{S_{j+1}} f(X_s) ds \quad (41.21)$$

and

$$\begin{aligned} \mathbb{E}^x \sum_{j=0}^{\infty} f(X_{S_j})(S_{j+1} - S_j) &= \mathbb{E}^x \sum_{j=0}^{\infty} f(X_{S_j}) 1_{(a,b)}(X_{S_j}) \mathbb{E}^{X_{S_j}} S_1 \\ &= \sum_{i=1}^{2^n - 1} f(x_i) B(x_i) 1_{(a,b)}(x_i) \mathbb{E}^x N_i. \end{aligned} \quad (41.22)$$

Moreover, the right-hand side of (41.21) differs from the left-hand side of (41.22) by at most  $\varepsilon_n \mathbb{E}^x \tau_{(a,b)}$ . By (41.18) the right-hand side of (41.22) is equal to

$$\sum_{i=1}^{2^n-1} 2^n f(x_i) B(x_i) 1_{(a,b)}(x_i) G_{ab}(x, x_i) = \int G_{ab}(x, x_i) f(x_i) m_n(dx).$$

By weak convergence along an appropriate subsequence, the left-hand side and the right-hand side of (41.20) differ by at most  $\limsup_n \varepsilon_n \mathbb{E}^x \tau_{(a,b)}$ , which is zero. A limit argument then shows that (41.20) holds for all  $x \in [a, b]$ , and another limit argument shows that (41.20) holds for all bounded  $f$ .  $\square$

#### 41.4 The uniqueness theorem

We next turn to showing that the speed measure characterizes the law of a diffusion.

**Theorem 41.8** *If  $(X_t, \mathbb{P}_i^x)$ ,  $i = 1, 2$ , are two diffusions on natural scale with the same speed measure  $m$ , then  $\mathbb{P}_1^x = \mathbb{P}_2^x$ .*

*Proof* We start by letting  $(a, b) \subset \mathbb{R}$  and considering the operator

$$R_\lambda^i f(x) = \mathbb{E}_i^x \int_0^{\tau_{(a,b)}} e^{-\lambda t} f(X_t) dt, \quad \lambda \geq 0, \quad (41.23)$$

for  $i = 1, 2$ . We show first that  $R_0^1 = R_0^2$ , that is, that

$$\mathbb{E}_1^x \int_0^{\tau_{(a,b)}} f(X_t) dt = \mathbb{E}_2^x \int_0^{\tau_{(a,b)}} f(X_t) dt$$

if  $f$  is bounded and Borel measurable. This is easy, because by Corollary 41.7, both sides are equal to

$$\int_a^b G_{ab}(x, y) m(dy).$$

Since  $(\widehat{X}_t, \mathbb{P}_i^x)$  is a Markov process, where  $\widehat{X}$  is the process  $X$  killed on exiting  $(a, b)$ , the resolvent equation (37.2) holds. We have

$$\begin{aligned} \|R_0^i f\|_\infty &\leq \|f\|_\infty \sup_x \mathbb{E}^x \tau_{(a,b)} \\ &= \|f\|_\infty \sup_x \int G_{ab}(x, y) m(dy) \\ &\leq c \|f\|_\infty m(a, b) < \infty. \end{aligned}$$

Since  $\|R_0^i\|_\infty < \infty$ , we can let  $\mu$  go to zero in (37.2). We can repeat the proof of Corollary 37.3 with  $\lambda = 0$  to see that

$$R_\mu^i f = R_0^i f + \sum_{j=1}^{\infty} (-\mu)^j (R_0^i)^{j+1} f$$

provided  $\mu < \|R_0^i\|_\infty$ . We can then use Remark 37.4 to obtain that  $R_\lambda^1 = R_\lambda^2$  for all  $\lambda > 0$ . We now take open intervals  $I_n$  increasing up to  $\mathbb{R}$ . Applying the above to  $I_n$  and letting  $n \rightarrow \infty$ ,

we have

$$\mathbb{E}_1^x \int_0^\infty e^{-\lambda t} f(X_t) dt = \mathbb{E}_2^x \int_0^\infty e^{-\lambda t} f(X_t) dt$$

whenever  $f$  is bounded and Borel measurable and  $x \in \mathbb{R}$ .

Suppose  $f$  is continuous as well. By the uniqueness of the Laplace transform, we see that  $\mathbb{E}_1^x f(X_t) = \mathbb{E}_2^x f(X_t)$  for almost every  $t$ , and since both terms are continuous in  $t$ , this equality holds for all  $t$ . By a limit argument, this equality holds for all bounded and Borel measurable  $f$ . Therefore the one-dimensional distributions of  $X$  under  $\mathbb{P}_1^x$  and  $\mathbb{P}_2^x$  agree.

If  $s < t$  and  $f$  and  $g$  are bounded and Borel measurable,

$$\begin{aligned} \mathbb{E}_1^x [f(X_s)g(X_t)] &= \mathbb{E}_1^x [f(X_s)P_{t-s}^1 g(X_s)] = \mathbb{E}_1^x [f(X_s)P_{t-s}^2 g(X_s)] \\ &= \mathbb{E}_1^x [(fP_{t-s}^2 g)(X_s)] = \mathbb{E}_2^x [(fP_{t-s}^2 g)(X_s)] \\ &= \mathbb{E}_2^x [f(X_s)P_{t-s}^2 g(X_s)] = \mathbb{E}_2^x [f(X_s)g(X_t)]. \end{aligned}$$

Here  $P_{t-s}^i$  is the semigroup for  $(X_t, \mathbb{P}_i^x)$ ; since the one-dimensional distributions agree,  $P_{t-s}^1 = P_{t-s}^2$ . We have thus shown the two-dimensional distributions of  $X$  under  $\mathbb{P}_1^x$  and  $\mathbb{P}_2^x$  agree. Continuing, we see that all the finite-dimensional distributions under  $\mathbb{P}_1^x$  and  $\mathbb{P}_2^x$  agree. By the continuity of the paths of  $X$  and Theorem 2.6, that is enough to show equality of  $\mathbb{P}_1^x$  and  $\mathbb{P}_2^x$ .  $\square$

## 41.5 Time change

We now want to show that if  $m$  is a measure such that  $0 < m(a, b) < \infty$  for all intervals  $[a, b]$ , then there exists a regular diffusion on natural scale on  $\mathbb{R}$  having  $m$  as a speed measure. If  $m(dx)$  had a density, say  $m(dx) = r(x) dx$ , we would proceed as follows. Let  $W_t$  be a one-dimensional Brownian motion and let

$$A_t = \int_0^t r(W_s) ds, \quad B_t = \inf\{u : A_t > u\}, \quad X_t = W_{B_t}.$$

In other words, we let  $X_t$  be a certain time change of Brownian motion. In general, where  $m(dx)$  does not have a density, we make use of the local times  $L_t^x$  of Brownian motion; see Chapter 14.

Let

$$A_t = \int L_t^x m(dx), \quad B_t = \inf\{u : A_u > t\}, \quad X_t = W_{B_t}. \quad (41.24)$$

**Theorem 41.9** *Let  $(W_t, \mathbb{P}^x)$  be a Brownian motion and  $m$  a measure on  $\mathbb{R}$  such that  $0 < m(a, b) < \infty$  for every finite interval  $(a, b)$ . Then, under  $\mathbb{P}^x$ ,  $X_t$  as defined by (41.24) is a regular diffusion on natural scale with speed measure  $m$ .*

*Proof* First we show that  $X_t$  is a continuous process. Fix  $\omega$ . If we choose  $a < \inf_{s \leq t} W_s$  and  $b > \sup_{s \leq t} W_s$ , then

$$A_t = \int L_t^x m(dx) = \int L_t^x 1_{[a,b]}(x) m(dx)$$

since  $L_t^x$  increases only for those times  $s$  when  $W_s = x$ . By the continuity of  $L_t^x$  and dominated convergence, we conclude that  $A_t(\omega)$  is continuous at time  $t$ . Next we show that  $A_t$  is strictly increasing. Fix  $\omega$ . If  $s < u$ , pick  $t \in (s, u)$ . Set  $x = W_t$ . Because the support of the measure

$dL_t^x$  is the set  $\{r : W_r = x\}$ , then  $L_u^x - L_s^x > 0$ . By the continuity of local times,  $L_u^y - L_s^y > 0$  for all  $y$  in a neighborhood of  $x$ , say  $(x - \delta, x + \delta)$ . Since  $m(x - \delta, x + \delta) > 0$ , then  $A_u - A_s > 0$ . Hence  $A_t$  is strictly increasing. This and the continuity of  $A_t$  imply that  $B_t$  is continuous, and therefore  $X_t$  is continuous.

Next we show that  $X_t$  is a regular diffusion on natural scale. By monotone convergence and the fact that  $L_t^x \rightarrow \infty$ , a.s., for each  $x$ ,  $A_t \uparrow \infty$ , hence  $B_t \uparrow \infty$ , so  $\tau_{(a,b)}^X < \infty$ ,  $\mathbb{P}^x$ -a.s., where  $\tau_{(a,b)}^X$  denotes the exit time of  $(a, b)$  by  $X_t$  and  $\tau_{(a,b)}^W$  denotes the corresponding exit time of  $W_t$ . Moreover,

$$\mathbb{P}^x(X(\tau_{(a,b)}^X) = b) = \mathbb{P}^x(W(\tau_{(a,b)}^W) = b) = \frac{x-a}{b-a},$$

since  $X_t$  is a time change of  $W_t$ .

To verify the strong Markov property, we repeat the argument of Section 22.3. Let  $\mathcal{F}'_t = \mathcal{F}_{B_t}$ . Then if  $T$  is a stopping time for  $\mathcal{F}'_t$ , we have

$$\mathbb{E}^x[f(X_{T+t}) \mid \mathcal{F}'_T] = \mathbb{E}^x[f(W(B_{T+t})) \mid \mathcal{F}_{B_T}].$$

$B_T$  can be seen to be a stopping time for  $\mathcal{F}_t$  and  $B_{T+t} = B_t \circ \theta_{B_T}$  where  $\theta_t$  are the shift operators, so this is

$$\mathbb{E}^x \mathbb{E}^{W(B_T)} f(W_{B_t}) = \mathbb{E}^x \mathbb{E}^{X_T} f(X_t).$$

As in Section 20.3, this suffices to show that  $X_t$  is a strong Markov process.

It remains to determine the speed measure of  $X_t$ . Fix  $(a, b)$  and write  $\tau_X$  for  $\tau_{(a,b)}^X$  and  $\tau_W$  for  $\tau_{(a,b)}^W$ . We have

$$\begin{aligned} \mathbb{E}^x \tau_X &= \mathbb{E}^x \int_0^\infty 1_{(a,b)}(X_{s \wedge \tau_X}) ds \\ &= \mathbb{E}^x \int_0^\infty 1_{(a,b)}(W_{B_{s \wedge \tau_X}}) ds \\ &= \mathbb{E}^x \int_0^\infty 1_{(a,b)}(W_{t \wedge \tau_W}) dA_t \\ &= \mathbb{E}^x \int \int_0^\infty 1_{(a,b)}(W_{t \wedge \tau_W}) L_t^y m(dy) \\ &= \mathbb{E}^x \int \int_0^{\tau_W} L_t^y m(dy) = \int \mathbb{E}^x L_{\tau_W}^y m(dy). \end{aligned}$$

We also have

$$\mathbb{E}^x L_{\tau_W}^y = \mathbb{E}^x |W_{\tau_W} - y| - |x - y|$$

by (14.5). This is equal to

$$\begin{aligned} &|a - y| \mathbb{P}^x(W_{\tau_W} = a) + |b - y| \mathbb{P}^x(W_{\tau_W} = b) - |x - y| \\ &= |a - y| \frac{b - x}{b - a} + |b - y| \frac{x - a}{b - a} - |x - y| = G_{ab}(x, y). \end{aligned}$$

We thus have

$$\mathbb{E}^x \tau_X = \int G_{ab}(x, y) m(dy),$$

as required.  $\square$

As a corollary to the proof, we see that a regular diffusion on natural scale is a local martingale, since it is a time change of Brownian motion.

## 41.6 Examples

Let us calculate the scale function and the speed measure for some examples of diffusions. First we need to connect the speed measure with the coefficients of an SDE.

Let us look at the solutions to the SDE (41.4), but now suppose  $b$  is identically zero, or  $dX_t = \sigma(X_t) dW_t$ . We again set  $a(x) = \sigma(x)^2$ .

**Theorem 41.10** *Suppose  $c_1 < \sigma(x) < c_2$  for all  $x$  and  $\sigma$  is continuous. The speed measure of  $X_t$  is given by*

$$m(dx) = \frac{1}{a(x)} dx.$$

*Proof* Since  $dX_t = \sigma(X_t) dW_t$ , then  $\langle X \rangle_t = \int_0^t a(X_s) ds$ . To obtain a Brownian motion  $\overline{W}_t$  by time-changing the martingale  $X_t$ , we must time-change by the inverse of  $\langle X \rangle_t$ . On the other hand, from Theorem 41.9,  $X_t$  is the time-change of a Brownian motion by  $B_t$ , where  $B_t$  is given by (41.24). Hence

$$B_t = \langle X \rangle_t = \int_0^t a(X_s) ds.$$

The inverse of  $B_t$ , namely,  $A_t$ , must then satisfy

$$\frac{dA_t}{dt} = \frac{1}{a(X_{A_t})} = \frac{1}{a(W_t)},$$

or

$$A_t = \int_0^t \frac{1}{a(W_s)} ds = \int L_t^y \frac{1}{a(y)} dy$$

for all  $t$ , using Theorem 14.4. However,  $A_t = \int L_t^y m(dy)$  by (41.24). Hence

$$\int L_t^y \frac{1}{a(y)} dy = \int L_t^y m(dy).$$

We know  $\mathbb{E}^x L_{\tau_{(c,d)}}^y = G_{cd}(x, y)$ . Therefore

$$\begin{aligned} \int G_{cd}(x, y) m(dy) &= \int \mathbb{E}^x L_{\tau_{(c,d)}}^y m(dy) \\ &= \int \mathbb{E}^x L_{\tau_{(c,d)}}^y \frac{1}{a(y)} dy \\ &= \int G_{cd}(x, y) \frac{1}{a(y)} dy \end{aligned}$$

for all  $c, d$ , and  $x$ , which implies  $m(dy) = (1/a(y)) dy$ . □

Now we can look at some examples and do calculations.

*Brownian motion with constant drift.* This process is the solution to the SDE  $dX_t = dW_t + b dt$ . From Theorem 41.1,  $s(x) = \exp(-2bx)$  is the scale function. If  $Y_t = s(X_t)$ , then

$(s'\sigma)(s^{-1}(y)) = -2by$ , or  $Y_t$  corresponds to the operator  $2b^2y^2f''$ , and the speed measure is  $(4b^2y^2)^{-1} dx$ .

*Bessel processes.* The process is only defined on the state space  $[0, \infty)$  instead of all of  $\mathbb{R}$  and there is a boundary condition at 0. We ignore this here and consider a Bessel process of order  $v$  up until the first hit of 0. Then  $X$  solves the SDE

$$dX_t = dW_t + \frac{v-1}{2X_t} dt.$$

If  $v \neq 2$ , a calculation using Theorem 41.1 shows that  $s(x) = x^{2-v}$ . Then  $Y_t = s(X_t)$  satisfies

$$dY_t = (2-v)Y_t^{(1-v)/(2-v)} dW_t,$$

and the speed measure is

$$m(dx) = (2-v)^{-2}x^{(2v-2)/(2-v)} dx, \quad x > 0.$$

## Exercises

41.1 In the proof of Proposition 41.2 we used the strong Markov property numerous times. Write out carefully in terms of shift operators and conditional expectations how the strong Markov property is applied in each case.

41.2 Give a rigorous proof of (41.9).

41.3 Show that if

$$\int G_{ab}(x, y) m_1(dy) = \int G_{ab}(x, y) m_2(dy)$$

for all  $x, a$ , and  $b$ , then  $m_1 = m_2$ .

41.4 Show that if  $X$  is a Bessel process of order 2, then the scale function is given by  $s(x) = \log x$ ,  $Y_t = s(X_t)$  satisfies  $dY_t = e^{-Y_t} dW_t$ , and the speed measure is  $m(dx) = e^{2x} dx$ .

41.5 Suppose  $X$  is a regular diffusion whose state space is  $\mathbb{R}$ . Prove that  $X$  is on natural scale if and only if

$$\mathbb{P}^{(a+b)/2}(T_a < T_b) = \frac{1}{2}$$

whenever  $a < b$ .

41.6 Let  $a > 0$  and let  $m(dx) = dx + a\delta_0(dx)$ , where  $\delta_0$  is the point mass at 0. Let  $(X_t, \mathbb{P}^x)$  be the diffusion on the line on natural scale whose speed measure is given by  $m$ . Show that under  $\mathbb{P}^0$ ,

$$\int_0^t 1_{\{0\}}(X_s) ds > 0$$

with probability one for each  $t > 0$ . Prove that for each  $t > 0$ ,  $Z_t = \{t : X_t = 0\}$  contains no intervals. Thus the zero set of the process  $X$  spends an amount of time at 0 that has positive Lebesgue measure, but the zero set contains no intervals.

41.7 Define

$$m_a(dx) = \begin{cases} dx, & x \geq 0, \\ a dx, & x < 0. \end{cases}$$

Let  $(X_t, \mathbb{P}_a^x)$  be the diffusion on natural scale on the line whose speed measure is given by  $m_a$ . Suppose  $x > 0$ .

Prove that if  $a \rightarrow \infty$ , then  $\mathbb{P}_a^x$  converges weakly to the law of Brownian motion absorbed (i.e., killed) at 0, started at  $x$ . What do you think happens when  $a \rightarrow 0$ ?

### Notes

We have considered diffusions on  $\mathbb{R}$  but most of what we discussed goes through for diffusions whose state space is an interval properly contained in  $\mathbb{R}$ . In this case, one must specify what the process does when it hits the boundary. Being absorbed (i.e., killed) or reflected are two options, but much more complicated behavior is possible. See [Itô and McKean \(1965\)](#) and [Knight \(1981\)](#) for the complete story.

# 42

---

## Lévy processes

A *Lévy process* is a process with stationary and independent increments whose paths are right continuous with left limits. Having *stationary increments* means that the law of  $X_t - X_s$  is the same as the law of  $X_{t-s} - X_0$  whenever  $s < t$ . Saying that  $X$  has *independent increments* means that  $X_t - X_s$  is independent of  $\sigma(X_r; r \leq s)$  whenever  $s < t$ .

We want to examine the structure of Lévy processes. We have three examples already: the Poisson process, Brownian motion, and the deterministic process  $X_t = t$ . It turns out that all Lévy processes can be built up out of these building blocks. We will show how to construct Lévy processes and give a representation of an arbitrary Lévy process.

Recall that we use  $X_{t-} = \lim_{s < t, s \rightarrow t} X_s$  and  $\Delta X_t = X_t - X_{t-}$ .

### 42.1 Examples

Let us begin by looking at some simple Lévy processes. Let  $P_t^j$ ,  $j = 1, \dots, J$ , be a sequence of independent Poisson processes with parameters  $\lambda_j$ , respectively. Each  $P_t^j$  is a Lévy process and the formula for the characteristic function of a Poisson random variable (see Section A.13) shows that the characteristic function of  $P_t^j$  is

$$\mathbb{E} e^{iuP_t^j} = \exp(t\lambda_j(e^{iu} - 1)).$$

Therefore the characteristic function of  $a_j P_t^j$  is

$$\mathbb{E} e^{iu a_j P_t^j} = \exp(t\lambda_j(e^{iu a_j} - 1))$$

and the characteristic function of  $a_j P_t^j - a_j \lambda_j t$  is

$$\mathbb{E} e^{iu a_j P_t^j - a_j \lambda_j t} = \exp(t\lambda_j(e^{iu a_j} - 1 - iua_j)).$$

If we let  $m_j$  be the measure on  $\mathbb{R}$  defined by  $m_j(dx) = \lambda_j \delta_{a_j}(dx)$ , where  $\delta_{a_j}(dx)$  is point mass at  $a_j$ , then the characteristic function for  $a_j P_t^j$  can be written as

$$\exp\left(t \int_{\mathbb{R}} [e^{iux} - 1] m_j(dx)\right) \tag{42.1}$$

and the one for  $a_j P_t^j - a_j \lambda_j t$  as

$$\exp\left(t \int_{\mathbb{R}} [e^{iux} - 1 - iux] m_j(dx)\right). \tag{42.2}$$

Now let

$$X_t = \sum_{j=1}^J a_j P_t^j.$$

It is clear that the paths of  $X_t$  are right continuous with left limits, and the fact that  $X$  has stationary and independent increments follows from the corresponding property of the  $P^j$ 's. Moreover, the characteristic function of a sum of independent random variables is the product of the characteristic functions, so the characteristic function of  $X_t$  is given by

$$\mathbb{E} e^{iuX_t} = \exp \left( t \int_{\mathbb{R}} [e^{iux} - 1] m(dx) \right) \quad (42.3)$$

with  $m(dx) = \sum_{j=1}^J \lambda_j \delta_{a_j}(dx)$ .

The process  $Y_t = X_t - t \sum_{j=1}^J a_j \lambda_j$  is also a Lévy process and its characteristic function is

$$\mathbb{E} e^{iuY_t} = \exp \left( t \int_{\mathbb{R}} [e^{iux} - 1 - iux] m(dx) \right), \quad (42.4)$$

again with  $m(dx) = \sum_{j=1}^J \lambda_j \delta_{a_j}(dx)$ .

**Remark 42.1** Recall from Proposition A.50 that if  $\varphi$  is the characteristic function of a random variable  $Z$ , then  $\varphi'(0) = i\mathbb{E} Z$  and  $\varphi''(0) = -\mathbb{E} Z^2$ . If  $Y_t$  is as in the paragraph above, then clearly  $\mathbb{E} Y_t = 0$ , and calculating the second derivative of  $\mathbb{E} e^{iuY_t}$  at 0, we obtain

$$\mathbb{E} Y_t^2 = t \int x^2 m(dx).$$

The following lemma is a restatement of Corollary 4.3.

**Lemma 42.2** *If  $X_t$  is a Lévy process and  $T$  is a finite stopping time, then  $X_{T+t} - X_T$  is a Lévy process with the same law as  $X_t - X_0$  and independent of  $\mathcal{F}_T$ .*

## 42.2 Construction of Lévy processes

A process  $X$  has *bounded jumps* if there exists a real number  $K > 0$  such that  $\sup_t |\Delta X_t| \leq K$ , a.s.

**Lemma 42.3** *If  $X_t$  is a Lévy process with bounded jumps and with  $X_0 = 0$ , then  $X_t$  has moments of all orders, that is,  $\mathbb{E} |X_t|^p < \infty$  for all positive integers  $p$ .*

*Proof* Suppose the jumps of  $X_t$  are bounded in absolute value by  $K$ . Since  $X_t$  is right continuous with left limits, there exists  $M > K$  such that  $\mathbb{P}(\sup_{s \leq t} |X_s| \geq 2M) \leq 1/2$ .

Let  $T_1 = \inf\{t : |X_t| \geq M\}$  and  $T_{i+1} = \inf\{t > T_i : |X_t - X_{T_i}| > M\}$ . For  $s < T_1$ ,  $|X_s| \leq M$ , and then  $|X_{T_1}| \leq |X_{T_1-}| + |\Delta X_{T_1}| \leq M + K \leq 2M$ . We have

$$\begin{aligned}\mathbb{P}(\sup_{s \leq t} |X_s| \geq 2(i+1)M) &\leq \mathbb{P}(T_{i+1} \leq t) \leq \mathbb{P}(T_i \leq t, T_{i+1} - T_i \leq t) \\ &= \mathbb{P}(\sup_{s \leq t} |X_{T_i+s} - X_{T_i}| \geq 2M, T_i \leq t) \\ &= \mathbb{P}(\sup_{s \leq t} |X_s| \geq 2M) \mathbb{P}(T_i \leq t) \\ &\leq \frac{1}{2} \mathbb{P}(T_i \leq t),\end{aligned}$$

using Lemma 42.2 in the last equality. By induction,  $\mathbb{P}(\sup_{s \leq t} |X_s| \geq 2iM) \leq 2^{-i}$ , and the lemma now follows immediately.  $\square$

A key lemma is the following.

**Lemma 42.4** Suppose  $I$  is a finite interval of the form  $(a, b)$ ,  $[a, b)$ ,  $(a, b]$ , or  $[a, b]$  with  $a > 0$  and  $m$  is a finite measure on  $\mathbb{R}$  giving no mass to  $I^c$ . Then there exists a Lévy process  $X_t$  satisfying (42.3).

*Proof* First let us consider the case where  $I = [a, b)$ . We approximate  $m$  by a discrete measure. If  $n \geq 1$ , let  $z_j = a + j(b-a)/n$ ,  $j = 0, \dots, n-1$ , and let

$$m_n(dx) = \sum_{j=0}^{n-1} m([z_j, z_{j+1})) \delta_{z_j}(dx),$$

where  $\delta_{z_j}$  is the point mass at  $z_j$ . The measures  $m_n$  converge weakly to  $m$  as  $n \rightarrow \infty$  in the sense that

$$\int f(x) m_n(dx) \rightarrow \int f(x) dx$$

whenever  $f$  is a bounded continuous function on  $\mathbb{R}$ . For each  $n$ , let  $P_t^{n,j}$ ,  $j = 0, \dots, n-1$ , be independent Poisson processes with parameters  $m([z_j, z_{j+1}))$  and let

$$X_t^n = \sum_{j=0}^{n-1} z_j P_t^{n,j}.$$

Then  $X^n$  is a Lévy process with jumps bounded by  $b$ .

By Lemma 42.2, if  $T_n$  is a stopping time for  $X^n$ ,  $\varepsilon > 0$ , and  $\delta > 0$ , then

$$\begin{aligned}\mathbb{P}(|X_{T_n+\delta}^n - X_{T_n}^n| > \varepsilon) &= \mathbb{P}(|X_\delta^n| > \varepsilon) \leq \mathbb{P}(X_\delta^n \neq 0) \\ &\leq \mathbb{P}\left(\sum_{j=0}^{n-1} P_s^{n,j} \neq 0\right).\end{aligned}\tag{42.5}$$

Since the sum of independent Poisson processes is a Poisson process, then  $\sum_{j=0}^{n-1} P_t^{n,j}$  is a Poisson process with parameter

$$\sum_{j=0}^{n-1} m([z_j, z_{j+1})) = m(I).$$

The last line of (42.5) is then bounded by

$$1 - e^{-\delta m(I)} \leq \delta m(I),$$

which tends to zero uniformly in  $n$  as  $\delta \rightarrow 0$ . Note  $X_0^n = 0$ , a.s. We can therefore apply the Aldous criterion (Theorem 34.8) to see that the  $X^n$  are tight with respect to weak convergence on the space  $D[0, t_0]$  for any  $t_0$ .

Any subsequential weak limit  $X$  will have paths that are right continuous with left limits. For any continuous bounded function  $f$  on  $\mathbb{R}$ ,

$$\mathbb{E} f(X_t^n - X_s^n) = \mathbb{E} f(X_{t-s}^n - X_0^n).$$

Passing to the limit along an appropriate subsequence,

$$\mathbb{E} f(X_t - X_s) = \mathbb{E} f(X_{t-s} - X_0).$$

Since  $f$  is an arbitrary bounded continuous function, we see that the laws of  $X_t - X_s$  and  $X_{t-s} - X_0$  are the same. Similarly we prove the increments are independent.

Since  $x \rightarrow e^{iux}$  is a bounded continuous function and  $m_n$  converges weakly to  $m$ , starting with

$$\mathbb{E} \exp(iuX_t^n) = \exp\left(t \int [e^{iux} - 1] m_n(dx)\right),$$

and passing to the limit, we obtain that the characteristic function of  $X$  under  $\mathbb{P}$  is given by (42.3).

If now the interval  $I$  contains the point  $b$ , we follow the above proof, except we let  $P_t^{n,n-1}$  be a Poisson random variable with parameter  $m([z_{n-1}, b])$ . Similarly, if  $I$  does not contain the point  $a$ , we change  $P_t^{n,0}$  to be a Poisson random variable with parameter  $m((a, z_1))$ . With these changes, the proof works for intervals  $I$ , whether or not they contain either of their endpoints.  $\square$

**Remark 42.5** If  $X$  is the Lévy process constructed in Lemma 42.4, then  $Y_t = X_t - \mathbb{E} X_t$  will be a Lévy process satisfying (42.4).

Here is the main theorem of this section.

**Theorem 42.6** Suppose  $m$  is a measure on  $\mathbb{R}$  with  $m(\{0\}) = 0$  and

$$\int (1 \wedge x^2)m(dx) < \infty.$$

Suppose  $b \in \mathbb{R}$  and  $\sigma \geq 0$ . There exists a Lévy process  $X_t$  such that

$$\mathbb{E} e^{iux} = \exp\left(t \left\{ iub - \sigma^2 u^2 / 2 + \int_{\mathbb{R}} [e^{iux} - 1 - iux 1_{(|x| \leq 1)}] m(dx) \right\}\right). \quad (42.6)$$

The above equation is called the *Lévy–Khintchine formula*. The measure  $m$  is called the *Lévy measure*. If we let

$$m(dx) = \frac{1+x^2}{x^2} m'(dx)$$

and

$$b = b' + \int_{(|x| \leq 1)} \frac{x^3}{1+x^2} m(dx) - \int_{(|x| > 1)} \frac{x}{1+x^2} m(dx),$$

then we can also write

$$\mathbb{E} e^{iux_t} = \exp \left( t \left\{ iub' - \sigma^2 u^2 / 2 + \int_{\mathbb{R}} \left[ e^{iux} - 1 - \frac{iux}{1+x^2} \right] \frac{1+x^2}{x^2} m'(dx) \right\} \right).$$

Both expressions for the Lévy–Khintchine formula are in common use.

*Proof* Let  $m(dx)$  be a measure supported on  $(0, 1]$  with  $\int x^2 m(dx) < \infty$ . Let  $m_n(dx)$  be the measure  $m$  restricted to  $(2^{-n}, 2^{-n+1}]$ . Let  $Y_t^n$  be independent Lévy processes whose characteristic functions are given by (42.4) with  $m$  replaced by  $m_n$ ; see Remark 42.5. Note  $\mathbb{E} Y_t^n = 0$  for all  $n$  by Remark 42.1. By the independence of the  $Y^n$ 's, if  $M < N$ ,

$$\mathbb{E} \left( \sum_{n=M}^N Y_t^n \right)^2 = \sum_{n=M}^N \mathbb{E} (Y_t^n)^2 = \sum_{n=M}^N t \int x^2 m_n(dx) = t \int_{2^{-N}}^{2^{-M}} x^2 m(dx).$$

By our assumption on  $m$ , this goes to zero as  $M, N \rightarrow \infty$ , and we conclude that  $\sum_{n=0}^N Y_t^n$  converges in  $L^2$  for each  $t$ . Call the limit  $Y_t$ . It is routine to check that  $Y_t$  has independent and stationary increments. Each  $Y_t^n$  has independent increments and is mean zero, so

$$\mathbb{E} [Y_t^n - Y_s^n \mid \mathcal{F}_s] = \mathbb{E} [Y_t^n - Y_s^n] = 0,$$

or  $Y^n$  is a martingale. By Doob's inequalities and the  $L^2$  convergence,

$$\mathbb{E} \sup_{s \leq t} \left| \sum_{n=M}^N Y_s^n \right|^2 \rightarrow 0$$

as  $M, N \rightarrow \infty$ , and hence there exists a subsequence  $M_k$  such that  $\sum_{n=1}^{M_k} Y_s^n$  converges uniformly over  $s \leq t$ , a.s. Therefore the limit  $Y_t$  will have paths that are right continuous with left limits.

If  $m$  is a measure supported in  $(1, \infty)$  with  $m(\mathbb{R}) < \infty$ , we do a similar procedure starting with Lévy processes whose characteristic functions are of the form (42.3). We let  $m_n(dx)$  be the restriction of  $m$  to  $(2^n, 2^{n+1}]$ , let  $X_t^n$  be independent Lévy processes corresponding to  $m_n$ , and form  $X_t = \sum_{n=0}^{\infty} X_t^n$ . Since  $m(\mathbb{R}) < \infty$ , for each  $t_0$ , the number of times  $t$  less than  $t_0$  at which any one of the  $X_t^n$  jumps is finite. This shows  $X_t$  has paths that are right continuous with left limits, and it is easy to then see that  $X_t$  is a Lévy process.

Finally, suppose  $\int x^2 \wedge 1 m(dx) < \infty$ . Let  $X_t^1, X_t^2$  be Lévy processes with characteristic functions given by (42.3) with  $m$  replaced by the restriction of  $m$  to  $(1, \infty)$  and  $(-\infty, -1)$ , respectively, let  $X_t^3, X_t^4$  be Lévy processes with characteristic functions given by (42.4) with  $m$  replaced by the restriction of  $m$  to  $(0, 1]$  and  $[-1, 0)$ , respectively, let  $X_t^5 = bt$ , and let  $X_t^6$  be  $\sigma$  times a Brownian motion. Suppose the  $X^i$ 's are all independent. Then their sum will be a Lévy process whose characteristic function is given by (42.6).  $\square$

A key step in the construction was the centering of the Poisson processes to get Lévy processes with characteristic functions given by (42.4). Without the centering one is forced to work only with characteristic functions given by (42.3).

### 42.3 Representation of Lévy processes

We now work toward showing that every Lévy process has a characteristic function of the form given by (42.6).

**Lemma 42.7** *If  $X_t$  is a Lévy process and  $A$  is a Borel subset of  $\mathbb{R}$  that is a positive distance from 0, then*

$$N_t(A) = \sum_{s \leq t} 1_A(\Delta X_s)$$

is a Poisson process.

Saying that  $A$  is a positive distance from 0 means that  $\inf\{|x| : x \in A\} > 0$ .

*Proof* Since  $X_t$  has paths that are right continuous with left limits and  $A$  is a positive distance from 0, then there can only be finitely many jumps of  $X$  that lie in  $A$  in any finite time interval, and so  $N_t(A)$  is finite and has paths that are right continuous with left limits. It follows from the fact that  $X_t$  has stationary and independent increments that  $N_t(A)$  also has stationary and independent increments. We now apply Proposition 5.4.  $\square$

**Theorem 42.8** *Let  $X_t$  be a Lévy process with  $X_0 = 0$  and let  $A_1, \dots, A_n$  be disjoint bounded Borel subsets of  $(0, \infty)$ , each a finite distance from 0. Set*

$$N_t(A_k) = \sum_{s \leq t} 1_{A_k}(\Delta X_s)$$

and

$$Y_t = X_t - \sum_{k=1}^n N_t(A_k).$$

Then the processes  $N_t(A_1), \dots, N_t(A_n)$ , and  $Y_t$  are mutually independent.

*Proof* Define  $\lambda(A) = \mathbb{E} N_1(A)$ . The previous lemma shows that if  $\lambda(A) < \infty$ , then  $N_t(A)$  is a Poisson process, and clearly its parameter is  $\lambda(A)$ . The result now follows from Theorem 18.3.  $\square$

Here is the representation theorem for Lévy processes.

**Theorem 42.9** *Suppose  $X_t$  is a Lévy process with  $X_0 = 0$ . Then there exists a measure  $m$  on  $\mathbb{R} - \{0\}$  with*

$$\int (1 \wedge x^2) m(dx) < \infty$$

and real numbers  $b$  and  $\sigma$  such that the characteristic function of  $X_t$  is given by (42.6).

*Proof* Define  $m(A) = \mathbb{E} N_1(A)$  if  $A$  is a bounded Borel subset of  $(0, \infty)$  that is a positive distance from 0. Since  $N_1(\cup_{k=1}^\infty A_k) = \sum_{k=1}^\infty N_1(A_k)$  if the  $A_k$  are pairwise disjoint and each is a positive distance from 0, we see that  $m$  is a measure on  $[a, b]$  for each  $0 < a < b < \infty$ , and  $m$  extends uniquely to a measure on  $(0, \infty)$ .

First we want to show that  $\sum_{s \leq t} \Delta X_s 1_{(\Delta X_s > 1)}$  is a Lévy process with characteristic function

$$\exp\left(t \int_1^\infty [e^{iux} - 1] m(dx)\right).$$

Since the characteristic function of the sum of independent random variables is equal to the product of the characteristic functions, it suffices to suppose  $0 < a < b$  and to show that

$$\mathbb{E} e^{iuz_t} = \exp\left(t \int_{(a,b]} [e^{iux} - 1] m(dx)\right),$$

where

$$Z_t = \sum_{s \leq t} \Delta X_s 1_{(a,b]}(\Delta X_s).$$

Let  $n > 1$  and  $z_j = a + j(b-a)/n$ . By Lemma 42.7,  $N_t((z_j, z_{j+1}])$  is a Poisson process with parameter

$$\ell_j = \mathbb{E} N_1((z_{j-1}, z_j]) = m((z_j, z_{j+1}]).$$

Thus  $\sum_{j=0}^{n-1} z_j N_t((z_j, z_{j+1}])$  has characteristic function

$$\prod_{j=0}^{n-1} \exp(t \ell_j (e^{iuz_j} - 1)) = \exp\left(t \sum_{j=0}^{n-1} (e^{iuz_j} - 1) \ell_j\right),$$

which is equal to

$$\exp\left(t \int (e^{iux} - 1) m_n(dx)\right), \quad (42.7)$$

where  $m_n(dx) = \sum_{j=0}^{n-1} \ell_j \delta_{z_j}(dx)$ . Since  $Z_t^n$  converges to  $Z_t$  as  $n \rightarrow \infty$ , passing to the limit shows that  $Z_t$  has a characteristic function of the form (42.6).

Next we show that  $m(1, \infty) < \infty$ . (We write  $m(1, \infty)$  instead of  $m((1, \infty))$  for esthetic reasons.) If not,  $m(1, K) \rightarrow \infty$  as  $K \rightarrow \infty$ . Then for each fixed  $L$  and each fixed  $t$ ,

$$\limsup_{K \rightarrow \infty} \mathbb{P}(N_t(1, K) \leq L) = \limsup_{K \rightarrow \infty} \sum_{j=0}^L e^{-tm(1,K)} \frac{m(1, K)^j}{j!} = 0.$$

This implies that  $N_t(1, \infty) = \infty$  for each  $t$ . However, this contradicts the fact that  $X_t$  has paths that are right continuous with left limits.

We define  $m$  on  $(-\infty, 0)$  similarly.

We now look at

$$Y_t = X_t - \sum_{s \leq t} \Delta X_s 1_{(|\Delta X_s| > 1)}.$$

This is again a Lévy process, and we need to examine its structure. This process has bounded jumps, hence has moments of all orders. By subtracting  $c_1 t$  for an appropriate constant  $c_1$ , we may suppose  $Y_t$  has mean 0. Let  $I_1, I_2, \dots$  be an ordering of the intervals  $\{[2^{-(m+1)}, 2^{-m}), (-2^{-m}, -2^{-(m+1)}] : m \geq 0\}$ . Let

$$\tilde{X}_t^k = \sum_{s \leq t} \Delta X_s 1_{(\Delta X_s \in I_k)}$$

and let  $X_t^k = \tilde{X}_t^k - \mathbb{E} \tilde{X}_t^k$ . By Corollary 18.3 and the fact that all the  $X^k$  have mean zero,

$$\sum_{k=1}^{\infty} \mathbb{E} (X_t^k)^2 \leq \mathbb{E} \left[ \left( Y_t - \sum_{k=1}^{\infty} X_t^k \right)^2 \right] + \mathbb{E} \left[ \left( \sum_{k=1}^{\infty} X_t^k \right)^2 \right] = \mathbb{E} (Y_t)^2 < \infty.$$

Hence

$$\mathbb{E} \left[ \sum_{k=M}^N X_t^k \right]^2 = \sum_{k=M}^N \mathbb{E} (X_t^k)^2$$

tends to zero as  $M, N \rightarrow \infty$ , and thus  $X_t - \sum_{k=1}^N X_t^k$  converges in  $L^2$ . The limit,  $X_t^c$ , say, will be a Lévy process independent of all the  $X_t^k$ . Moreover,  $X^c$  has no jumps, i.e., it is continuous. Since all the  $X^k$  have mean zero, then  $\mathbb{E} X_t^c = 0$ . By the independence of the increments,

$$\mathbb{E} [X_t^c - X_s^c | \mathcal{F}_s] = \mathbb{E} [X_t^c - X_s^c] = 0,$$

and we see  $X^c$  is a continuous martingale. Using the stationarity and independence of the increments,

$$\begin{aligned} \mathbb{E} [(X_{s+t}^c)^2] &= \mathbb{E} [(X_s^c)^2] + 2\mathbb{E} [X_s^c (X_{s+t}^c - X_s^c)] + \mathbb{E} [(X_{s+t}^c - X_s^c)^2] \\ &= \mathbb{E} [(X_s^c)^2] + \mathbb{E} [(X_t^c)^2], \end{aligned}$$

which implies that there exists a constant  $c_2$  such that  $\mathbb{E} (X_t^c)^2 = c_2 t$ . We then have

$$\begin{aligned} \mathbb{E} [(X_t^c)^2 - c_2 t | \mathcal{F}_s] &= (X_s^c)^2 - c_2 s + \mathbb{E} [(X_t^c - X_s^c)^2 | \mathcal{F}_s] - c_2(t-s) \\ &= (X_s^c)^2 - c_2 s + \mathbb{E} [(X_t^c - X_s^c)^2] - c_2(t-s) \\ &= (X_s^c)^2 - c_2 s. \end{aligned}$$

The quadratic variation process of  $X^c$  is therefore  $c_2 t$ , and by Lévy's theorem (Theorem 12.1),  $X_t^c / \sqrt{c_2}$  is a constant multiple of Brownian motion.

To complete the proof, it remains to show that  $\int_{-1}^1 x^2 m(dx) < \infty$ . But by Remark 42.1,

$$\int_{I_k} x^2 m(dx) = \mathbb{E} (X_1^k)^2,$$

and we have seen that

$$\sum_k \mathbb{E} (X_1^k)^2 \leq \mathbb{E} Y_1^2 < \infty.$$

Combining gives the finiteness of  $\int_{-1}^1 x^2 m(dx)$ . □

## Exercises

- 42.1 Let  $\alpha \in (0, 2)$  and let  $X$  be a Lévy process where  $b = \sigma = 0$  in the Lévy–Khintchine formula and the Lévy measure is  $m(dx) = c|x|^{-1-\alpha} dx$ . Show that if  $a > 0$  and  $Y_t = a^{1/\alpha} X_{at}$ , then  $Y$  has the same law as  $X$ . The process  $X$  is known as a *symmetric stable process of index  $\alpha$* .
- 42.2 Suppose  $W_t = (W_t^1, W_t^2)$  is a two-dimensional Brownian motion started at 0. Let  $\tau_s = \inf\{t > 0 : W_t^1 > s\}$ . Prove that  $W_{\tau_s}^2$  is a Lévy process and determine the Lévy measure.

*Hint:* Use scaling to make a guess.

- 42.3 Let  $W$  be a one-dimensional Brownian motion and let  $L^0$  be the local time at 0. Let  $T_t$  be the inverse of  $L^0$ , that is,  $T_t = \inf\{s : L_s^0 \geq t\}$ . Show  $T_t$  is a Lévy process and determine the Lévy measure.

*Hint:* Use scaling to get started.

- 42.4 Let  $W_t$  be a one-dimensional Brownian motion,  $L_t^y$  the local time at level  $y$ , and  $T_t$  the inverse local time at 0, that is,  $T_t = \inf\{s : L_s^0 \geq t\}$ . Let  $x > 0$  be fixed. Prove that  $L_{T_t}^x$  is a Lévy process.

- 42.5 Let  $X$  be a Lévy process with Lévy measure  $m$ . Prove that if  $A$  and  $B$  are disjoint closed sets, then

$$\mathbb{E}^x \sum_{s \leq t} 1_A(X_{s-}) 1_B(X_s) = \mathbb{E}^x \int_0^t 1_A(X_s) m(B - X_s) ds$$

for each  $x$ , where  $B - y = \{z - y : z \in B\}$ . This is the *Lévy system formula* in the case of Lévy processes. There is an analogous formula for Hunt processes.

- 42.6 A *stable subordinator*  $X$  of order  $\alpha \in (0, 1)$  is a Lévy process whose characteristic function is given by (42.6), where  $b = \sigma^2 = 0$  and  $m(dx) = c 1_{(x>0)} |x|^{-\alpha-1} dx$ . Suppose  $X$  is a stable subordinator of index  $\alpha$  and  $W$  is a Brownian motion. Show that, up to a deterministic time change, the process  $Z_t = W_{X_t}$  is a symmetric stable process of index  $2\alpha$ .

*Hint:* Start by using scaling.

- 42.7 Let  $Z_t$  be a symmetric stable process of order  $\alpha \in (0, 2)$ . Show that if  $\varepsilon > 0$ , then

$$\lim_{t \rightarrow \infty} \frac{|Z_t|}{t^{\alpha+\varepsilon}} = 0, \quad \text{a.s.}$$

□

### A.3 Convergence

In this section we consider three ways a sequence of random variables  $X_n$  can converge.

We say  $X_n$  converges to  $X$  almost surely if the event  $(X_n \not\rightarrow X)$  has probability zero.  $X_n$  converges to  $X$  in probability if for each  $\varepsilon$ ,  $\mathbb{P}(|X_n - X| > \varepsilon) \rightarrow 0$  as  $n \rightarrow \infty$ . For  $p \geq 1$ ,  $X_n$  converges to  $X$  in  $L^p$  if  $\mathbb{E}|X_n - X|^p \rightarrow 0$  as  $n \rightarrow \infty$ .

The following proposition shows some relationships among the types of convergence.

**Proposition A.14** (1) If  $X_n \rightarrow X$  almost surely, then  $X_n \rightarrow X$  in probability.

(2) If  $X_n \rightarrow X$  in  $L^p$ , then  $X_n \rightarrow X$  in probability.

(3) If  $X_n \rightarrow X$  in probability, there exists a subsequence  $n_j$  such that  $X_{n_j}$  converges to  $X$  almost surely.

*Proof* To prove (1), note  $X_n - X$  tends to zero almost surely, so  $1_{(-\varepsilon, \varepsilon)^c}(X_n - X)$  also converges to zero almost surely. Now apply the dominated convergence theorem.

(2) comes from Chebyshev's inequality:

$$\mathbb{P}(|X_n - X| > \varepsilon) = \mathbb{P}(|X_n - X|^p > \varepsilon^p) \leq \mathbb{E}|X_n - X|^p / \varepsilon^p \rightarrow 0$$

as  $n \rightarrow \infty$ .

To prove (3), choose  $n_j$  larger than  $n_{j-1}$  such that  $\mathbb{P}(|X_n - X| > 2^{-j}) < 2^{-j}$  whenever  $n \geq n_j$ . Thus if we let  $A_i = (|X_{n_j} - X| > 2^{-i}$  for some  $j \geq i$ ), then  $\mathbb{P}(A_i) \leq 2^{-i+1}$ . By the Borel–Cantelli lemma  $\mathbb{P}(A_i \text{ i.o.}) = 0$ . This implies  $X_{n_j} \rightarrow X$  almost surely on the complement of  $(A_i \text{ i.o.})$ . □

Let us give some examples to show there need not be any other implications among the three types of convergence.

Let  $\Omega = [0, 1]$ ,  $\mathcal{F}$  the Borel  $\sigma$ -field, and  $\mathbb{P}$  a Lebesgue measure. Let  $X_n = n^2 1_{(0,1/n)}$ . Then clearly  $X_n$  converges to zero almost surely and in probability, but  $\mathbb{E}X_n^p = n^{2p}/n \rightarrow \infty$  for any  $p \geq 1$ .

Let  $\Omega$  be the unit circle, and let  $\mathbb{P}$  be a Lebesgue measure on the circle normalized to have total mass 1. We use  $\theta$  to denote the angle that the ray from 0 through a point on the circle makes with the  $x$  axis. Let  $t_n = \sum_{i=1}^n i^{-1}$ , and let  $A_n = \{e^{i\theta} : t_{n-1} \leq \theta < t_n\}$ . Let  $X_n = 1_{A_n}$ .

Any point on the unit circle will be in infinitely many  $A_n$ , so  $X_n$  does not converge almost surely to zero. But  $\mathbb{P}(A_n) = 1/(2\pi n) \rightarrow 0$ , so  $X_n \rightarrow 0$  in probability and in  $L^p$ .

#### A.4 Uniform integrability

A sequence  $\{X_i\}$  of random variables is *uniformly integrable* if

$$\sup_i \int_{(|X_i|>M)} |X_i| d\mathbb{P} \rightarrow 0$$

as  $M \rightarrow \infty$ . This can be rephrased by saying: given  $\varepsilon > 0$  there exists  $M > 0$  such that  $\mathbb{E}[|X_i|; |X_i| > M] < \varepsilon$  for all  $i$ . Here  $M$  can be chosen independently of  $i$ .

**Lemma A.15** *If  $\{X_i\}$  is a uniformly integrable sequence of random variables, then  $\sup_i \mathbb{E}|X_i| < \infty$ .*

*Proof* There exists  $M$  such that  $\mathbb{E}[|X_i|; |X_i| > M] \leq 1$ . Then

$$\mathbb{E}|X_i| \leq \mathbb{E}[|X_i|; |X_i| \leq M] + \mathbb{E}[|X_i|; |X_i| > M] \leq M + 1,$$

and we are done.  $\square$

We say a sequence of random variables  $\{X_i\}$  is *uniformly absolutely continuous* if given  $\varepsilon$  there exists  $\delta$  such that  $\sup_i \mathbb{E}[|X_i|; A] \leq \varepsilon$  whenever  $\mathbb{P}(A) < \delta$ .

**Proposition A.16** *The following are equivalent.*

- (1) *The sequence  $\{X_i\}$  is uniformly integrable.*
- (2) *The sequence  $\{X_i\}$  is uniformly absolutely continuous and  $\sup_i \mathbb{E}|X_i| < \infty$ .*

*Proof* If (1) holds, we showed in Lemma A.15 that the expectations are uniformly bounded. Let  $\varepsilon > 0$  and choose  $M$  such that  $\sup_i \mathbb{E}[|X_i| : |X_i| > M] < \varepsilon/2$ . Then if  $\delta = \varepsilon/(2M)$  and  $\mathbb{P}(A) < \delta$ , we have

$$\mathbb{E}[|X_i|; A] \leq \mathbb{E}[|X_i|; |X_i| > M] + \mathbb{E}[|X_i|; |X_i| \leq M, A] < \frac{\varepsilon}{2} + M\mathbb{P}(A) \leq \varepsilon.$$

Now suppose (2) holds. Let  $\varepsilon > 0$  and choose  $\delta$  such that  $\mathbb{E}[|X_i|; A] < \varepsilon$  for all  $i$  if  $\mathbb{P}(A) \leq \delta$ . Let  $M = \sup_i \mathbb{E}|X_i|/\delta$ . Then by the Chebyshev inequality

$$\mathbb{P}(|X_i| > M) \leq \frac{\mathbb{E}|X_i|}{M} = \delta,$$

so  $\mathbb{E}[|X_i|; |X_i| > M] < \varepsilon$ .  $\square$

**Proposition A.17** *Suppose  $\{X_i\}$  and  $\{Y_i\}$  are each uniformly integrable sequences of random variables. Then  $\{X_i + Y_i\}$  is also a uniformly integrable sequence.*

*Proof* By Proposition A.16,

$$\sup_i \mathbb{E}|X_i + Y_i| \leq \sup_i \mathbb{E}|X_i| + \sup_i \mathbb{E}|Y_i| < \infty.$$

Using Proposition A.16 again, given  $\varepsilon$  there exists  $\delta$  such that  $\mathbb{E}[|X_i|; A] < \varepsilon/2$  and  $\mathbb{E}[|Y_i|; A] < \varepsilon/2$  if  $\mathbb{P}(A) < \delta$ . But then  $\mathbb{E}[|X_i + Y_i|; A] < \varepsilon$  and a third use of Proposition A.16 yields our result.  $\square$

**Proposition A.18** Suppose there exists  $\varphi : [0, \infty) \rightarrow [0, \infty)$  such that  $\varphi$  is increasing,  $\varphi(x)/x \rightarrow \infty$  as  $x \rightarrow \infty$ , and  $\sup_i \mathbb{E} \varphi(|X_i|) < \infty$ . Then the sequence  $\{X_i\}$  is uniformly integrable.

*Proof* Let  $\varepsilon > 0$  and choose  $x_0$  such that  $x/\varphi(x) < \varepsilon$  if  $x \geq x_0$ . If  $M \geq x_0$ ,

$$\int_{(|X_i|>M)} |X_i| = \int \frac{|X_i|}{\varphi(|X_i|)} \varphi(|X_i|) 1_{(|X_i|>M)} \leq \varepsilon \int \varphi(|X_i|) \leq \varepsilon \sup_i \mathbb{E} \varphi(|X_i|).$$

Since  $\varepsilon$  is arbitrary, we are done.  $\square$

### Vitali convergence theorem

**Theorem A.19** If  $X_n \rightarrow X$  a.s. and  $\{X_n\}$  is uniformly integrable, then  $\mathbb{E} |X_n - X| \rightarrow 0$ .

*Proof* By Proposition A.17 with  $Y_i = -X$  for each  $i$ , the sequence  $X_i - X$  is uniformly integrable. Let  $\varepsilon > 0$  and choose  $M$  such that

$$\int_{(|X_i-X|>M)} |X_i - X| < \varepsilon.$$

By dominated convergence,

$$\limsup_{i \rightarrow \infty} \mathbb{E} |X_i - X| \leq \limsup_{i \rightarrow \infty} \mathbb{E} [|X_i - X|; |X_i - X| \leq M] + \varepsilon = \varepsilon.$$

Since  $\varepsilon$  is arbitrary, then  $\mathbb{E} |X_i - X| \rightarrow 0$ .  $\square$

## A.5 Conditional expectation

If  $\mathcal{F} \subset \mathcal{G}$  are two  $\sigma$ -fields and  $X$  is an integrable  $\mathcal{G}$  measurable random variable, the *conditional expectation* of  $X$  given  $\mathcal{F}$ , written  $\mathbb{E}[X | \mathcal{F}]$  and read as “the expectation (or expected value) of  $X$  given  $\mathcal{F}$ ,” is any  $\mathcal{F}$  measurable random variable  $Y$  such that  $\mathbb{E}[Y; A] = \mathbb{E}[X; A]$  for every  $A \in \mathcal{F}$ . The *conditional probability* of  $A \in \mathcal{G}$  given  $\mathcal{F}$  is defined by  $\mathbb{P}(A | \mathcal{F}) = \mathbb{E}[1_A | \mathcal{F}]$ .

If  $Y_1, Y_2$  are two  $\mathcal{F}$  measurable random variables with  $\mathbb{E}[Y_1; A] = \mathbb{E}[Y_2; A]$  for all  $A \in \mathcal{F}$ , then  $Y_1 = Y_2$ , a.s., and so conditional expectation is unique up to almost sure equivalence.

In the case  $X$  is already  $\mathcal{F}$  measurable,  $\mathbb{E}[X | \mathcal{F}] = X$ . If  $X$  is independent from  $\mathcal{F}$ ,  $\mathbb{E}[X | \mathcal{F}] = \mathbb{E}X$ . Both of these facts follow immediately from the definition. For another example, if  $\{A_i\}$  is a finite collection of pairwise disjoint sets whose union is  $\Omega$ ,  $\mathbb{P}(A_i) > 0$  for all  $i$ , and  $\mathcal{F}$  is the  $\sigma$ -field generated by the  $A_i$ 's, then

$$\mathbb{P}(A | \mathcal{F}) = \sum_i \frac{\mathbb{P}(A \cap A_i)}{\mathbb{P}(A_i)} 1_{A_i}. \quad (\text{A.12})$$

This follows since the right-hand side is  $\mathcal{F}$  measurable and its expectation over any set  $A_i$  is  $\mathbb{P}(A \cap A_i)$ . Equation (A.12) provides the link with the definition of conditional probability from elementary probability: if  $\mathbb{P}(B) \neq 0$ , then

$$\mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}. \quad (\text{A.13})$$

We have

$$\mathbb{E}[\mathbb{E}[X | \mathcal{F}]] = \mathbb{E}X \quad (\text{A.14})$$

because  $\mathbb{E}[\mathbb{E}[X | \mathcal{F}]] = \mathbb{E}[\mathbb{E}[X | \mathcal{F}]; \Omega] = \mathbb{E}[X; \Omega] = \mathbb{E}X$ .

The following is easy to establish.

**Proposition A.20** (1) If  $X \geq Y$  are both integrable, then

$$\mathbb{E}[X | \mathcal{F}] \geq \mathbb{E}[Y | \mathcal{F}], \quad \text{a.s.}$$

(2) If  $X$  and  $Y$  are integrable and  $a \in \mathbb{R}$ , then

$$\mathbb{E}[aX + Y | \mathcal{F}] = a\mathbb{E}[X | \mathcal{F}] + \mathbb{E}[Y | \mathcal{F}].$$

It is easy to check that limit theorems such as monotone convergence and dominated convergence have conditional expectation versions, as do inequalities like Jensen's and Chebyshev's inequalities.

**Proposition A.21** If  $g$  is convex and  $X$  and  $g(X)$  are integrable,

$$\mathbb{E}[g(X) | \mathcal{F}] \geq g(\mathbb{E}[X | \mathcal{F}]), \quad \text{a.s.}$$

**Proposition A.22** If  $X$  and  $XY$  are integrable and  $Y$  is measurable wrt  $\mathcal{F}$ , then

$$\mathbb{E}[XY | \mathcal{F}] = Y\mathbb{E}[X | \mathcal{F}]. \quad (\text{A.15})$$

*Proof* If  $A \in \mathcal{F}$ , then for any  $B \in \mathcal{F}$ ,

$$\mathbb{E}[1_A \mathbb{E}[X | \mathcal{F}]; B] = \mathbb{E}[\mathbb{E}[X | \mathcal{F}]; A \cap B] = \mathbb{E}[X; A \cap B] = \mathbb{E}[1_AX; B].$$

Since  $1_A \mathbb{E}[X | \mathcal{F}]$  is  $\mathcal{F}$  measurable, this shows that (A.15) holds when  $Y = 1_A$  and  $A \in \mathcal{F}$ . Using linearity shows that (A.15) holds whenever  $Y$  is a simple  $\mathcal{F}$  measurable random variable. Taking limits, (A.15) holds whenever  $Y \geq 0$  is  $\mathcal{F}$  measurable and  $X$  and  $XY$  are integrable. Using linearity again completes the proof.  $\square$

Two other equalities are contained in the following.

**Proposition A.23** If  $\mathcal{E} \subset \mathcal{F} \subset \mathcal{G}$  are  $\sigma$ -fields, then

$$\mathbb{E}[\mathbb{E}[X | \mathcal{F}] | \mathcal{E}] = \mathbb{E}[X | \mathcal{E}] = \mathbb{E}[\mathbb{E}[X | \mathcal{E}] | \mathcal{F}].$$

*Proof* The right equality holds because  $\mathbb{E}[X | \mathcal{E}]$  is  $\mathcal{E}$  measurable, hence  $\mathcal{F}$  measurable. We then use the fact that if  $Y$  is  $\mathcal{F}$  measurable,  $\mathbb{E}[Y | \mathcal{F}] = Y$ .

To show the left equality, let  $A \in \mathcal{E}$ . Then since  $A$  is also in  $\mathcal{F}$ ,

$$\mathbb{E}[\mathbb{E}[\mathbb{E}[X | \mathcal{F}] | \mathcal{E}]; A] = \mathbb{E}[\mathbb{E}[X | \mathcal{F}]; A] = \mathbb{E}[X; A] = \mathbb{E}[\mathbb{E}[X | \mathcal{E}]; A].$$

Since both sides are  $\mathcal{E}$  measurable, the equality follows.  $\square$

To show the existence of  $\mathbb{E}[X | \mathcal{F}]$ , we proceed as follows.

**Proposition A.24** If  $X$  is integrable, then  $\mathbb{E}[X | \mathcal{F}]$  exists.

*Proof* Using linearity, we need only consider  $X \geq 0$ . Define a finite measure  $\mathbb{Q}$  on  $\mathcal{F}$  by  $\mathbb{Q}(A) = \mathbb{E}[X; A]$  for  $A \in \mathcal{F}$ . This is trivially absolutely continuous with respect to  $\mathbb{P}|_{\mathcal{F}}$ , the restriction of  $\mathbb{P}$  to  $\mathcal{F}$ . Let  $\mathbb{E}[X | \mathcal{F}]$  be the Radon–Nikodym derivative of  $\mathbb{Q}$  with respect to  $\mathbb{P}|_{\mathcal{F}}$ . Since  $\mathbb{Q}$  and  $\mathbb{P}|_{\mathcal{F}}$  are measures on  $\mathcal{F}$ , the Radon–Nikodym derivative is  $\mathcal{F}$  measurable, and so provides the desired random variable.  $\square$

When  $\mathcal{F} = \sigma(Y)$ , one usually writes  $\mathbb{E}[X | Y]$  for  $\mathbb{E}[X | \mathcal{F}]$ . Notation that is commonly used is  $\mathbb{E}[X | Y = y]$ . The definition is as follows. If  $A \in \sigma(Y)$ , then  $A = (Y \in B)$  for some Borel set  $B$  by the definition of  $\sigma(Y)$ , or  $1_A = 1_B(Y)$ . By linearity and taking limits, it follows that if  $Z$  is  $\sigma(Y)$  measurable, then  $Z = f(Y)$  for some Borel measurable function  $f$ . Set  $Z = \mathbb{E}[X | Y]$  and choose  $f$  Borel measurable so that  $Z = f(Y)$ . Then  $\mathbb{E}[X | Y = y]$  is defined to be  $f(y)$ .

If  $X \in L^2$  and  $\mathcal{M} = \{Y \in L^2 : Y \text{ is } \mathcal{F} \text{ measurable}\}$ , one can show that  $\mathbb{E}[X | \mathcal{F}]$  is equal to the projection of  $X$  onto the subspace  $\mathcal{M}$ .

## A.6 Stopping times

We next want to talk about stopping times. Suppose we have a sequence of  $\sigma$ -fields  $\mathcal{F}_i$  such that  $\mathcal{F}_i \subset \mathcal{F}_{i+1}$  for each  $i$ . An example would be if  $\mathcal{F}_i = \sigma(X_1, \dots, X_i)$ . A random mapping  $N$  from  $\Omega$  to  $\{0, 1, 2, \dots\}$  is called a *stopping time* if for each  $n$ ,  $(N \leq n) \in \mathcal{F}_n$ .

The proof of the following is immediate from the definitions.

**Proposition A.25** (1) *Fixed times  $n$  are stopping times.*

- (2) *If  $N_1$  and  $N_2$  are stopping times, then so are  $N_1 \wedge N_2$  and  $N_1 \vee N_2$ .*
- (3) *If  $N_n$  is an increasing sequence of stopping times, then so is  $N = \sup_n N_n$ .*
- (4) *If  $N_n$  is a decreasing sequence of stopping times, then so is  $N = \inf_n N_n$ .*
- (5) *If  $N$  is a stopping time, then so is  $N + n$ .*

We define

$$\mathcal{F}_N = \{A : A \cap (N \leq n) \in \mathcal{F}_n \text{ for all } n\}. \quad (\text{A.16})$$

## A.7 Martingales

In this section we consider martingales. Let  $\mathcal{F}_n$  be an increasing sequence of  $\sigma$ -fields. A sequence of random variables  $M_n$  is *adapted* to  $\mathcal{F}_n$  if for each  $n$ ,  $M_n$  is  $\mathcal{F}_n$  measurable.

$M_n$  is a *martingale* if  $M_n$  is adapted to  $\mathcal{F}_n$ ,  $M_n$  is integrable for all  $n$ , and

$$\mathbb{E}[M_n | \mathcal{F}_{n-1}] = M_{n-1}, \quad \text{a.s.,} \quad n = 2, 3, \dots \quad (\text{A.17})$$

If we have  $\mathbb{E}[M_n | \mathcal{F}_{n-1}] \geq M_{n-1}$ , a.s., for every  $n$ , then  $M_n$  is a *submartingale*. If we have  $\mathbb{E}[M_n | \mathcal{F}_{n-1}] \leq M_{n-1}$ , we have a *supermartingale*.

Let us look at some examples. If  $X_i$  is a sequence of mean zero independent random variables and  $S_n = \sum_{i=1}^n X_i$ , then  $M_n = S_n$  is a martingale, since

$$\begin{aligned} \mathbb{E}[M_n | \mathcal{F}_{n-1}] &= M_{n-1} + \mathbb{E}[M_n - M_{n-1} | \mathcal{F}_{n-1}] \\ &= M_{n-1} + \mathbb{E}[M_n - M_{n-1}] = M_{n-1}, \end{aligned}$$

using independence.

Another example is the following. If the  $X_i$ 's are independent and have mean zero and variance one,  $S_n$  is as in the previous example, and  $M_n = S_n^2 - n$ , then

$$\mathbb{E}[S_n^2 | \mathcal{F}_{n-1}] = \mathbb{E}[(S_n - S_{n-1})^2 | \mathcal{F}_{n-1}] + 2S_{n-1}\mathbb{E}[S_n | \mathcal{F}_{n-1}] - S_{n-1}^2 = 1 + S_{n-1}^2,$$

using independence. It follows that  $M_n$  is a martingale.

A third example is the following: if  $X \in L^1$  and  $M_n = \mathbb{E}[X | \mathcal{F}_n]$ , then  $M_n$  is a martingale. The proof of this is simple:

$$\mathbb{E}[M_{n+1} | \mathcal{F}_n] = \mathbb{E}[\mathbb{E}[X | \mathcal{F}_{n+1}] | \mathcal{F}_n] = \mathbb{E}[X | \mathcal{F}_n] = M_n.$$

If  $M_n$  is a martingale,  $g$  is convex, and  $g(M_n)$  is integrable for each  $n$ , then by Jensen's inequality for conditional expectations,

$$\mathbb{E}[g(M_{n+1}) | \mathcal{F}_n] \geq g(\mathbb{E}[M_{n+1} | \mathcal{F}_n]) = g(M_n), \quad (\text{A.18})$$

or  $g(M_n)$  is a submartingale. Similarly if  $g$  is convex and increasing on  $[0, \infty)$  and  $M_n$  is a positive submartingale, then  $g(M_n)$  is a submartingale because

$$\mathbb{E}[g(M_{n+1}) | \mathcal{F}_n] \geq g(\mathbb{E}[M_{n+1} | \mathcal{F}_n]) \geq g(M_n).$$

## A.8 Optional stopping

Note that if one takes expectations in (A.17), one has  $\mathbb{E}M_n = \mathbb{E}M_{n-1}$ , and by induction  $\mathbb{E}M_n = \mathbb{E}M_0$ . The theorem about martingales that lies at the basis of all other results is Doob's optional stopping theorem, which says that the same is true if we replace  $n$  by a stopping time  $N$ . There are various versions, depending on what conditions one puts on the stopping times.

**Theorem A.26** *If  $N$  is a stopping time wrt  $\mathcal{F}_n$  that is bounded by a positive real  $K$  and  $M_n$  a martingale, then  $\mathbb{E}M_N = \mathbb{E}M_0$ .*

*Proof* We write

$$\mathbb{E}M_N = \sum_{k=0}^K \mathbb{E}[M_N; N = k] = \sum_{k=0}^K \mathbb{E}[M_k; N = k].$$

Note ( $N = k$ ) is  $\mathcal{F}_j$  measurable if  $j \geq k$ , so

$$\begin{aligned} \mathbb{E}[M_k; N = k] &= \mathbb{E}[M_{k+1}; N = k] = \mathbb{E}[M_{k+2}; N = k] \\ &= \cdots = \mathbb{E}[M_K; N = k]. \end{aligned}$$

Hence

$$\mathbb{E}M_N = \sum_{k=0}^K \mathbb{E}[M_K; N = k] = \mathbb{E}M_K = \mathbb{E}M_0.$$

□

The same proof as that in Theorem A.26 gives the following corollary.

**Corollary A.27** If  $N$  is a stopping time bounded by  $K$  and  $M_n$  is a submartingale, then  $\mathbb{E} M_N \leq \mathbb{E} M_K$ .

The same proof also gives

**Corollary A.28** If  $N$  is a stopping time bounded by  $K$ ,  $A \in \mathcal{F}_N$ , and  $M_n$  is a submartingale, then  $\mathbb{E}[M_N; A] \leq \mathbb{E}[M_K; A]$ .

**Proposition A.29** If  $N_1 \leq N_2$  are stopping times bounded by  $K$  and  $M$  is a martingale, then  $\mathbb{E}[M_{N_2} | \mathcal{F}_{N_1}] = M_{N_1}$ , a.s.

*Proof* Suppose  $A \in \mathcal{F}_{N_1}$ . We need to show  $\mathbb{E}[M_{N_1}; A] = \mathbb{E}[M_{N_2}; A]$ . Define a new stopping time  $N_3$  by

$$N_3(\omega) = \begin{cases} N_1(\omega), & \omega \in A \\ N_2(\omega), & \omega \notin A. \end{cases}$$

It is easy to check that  $N_3$  is a stopping time, so  $\mathbb{E} M_{N_3} = \mathbb{E} M_K = \mathbb{E} M_{N_2}$  implies

$$\mathbb{E}[M_{N_1}; A] + \mathbb{E}[M_{N_2}; A^c] = \mathbb{E}[M_{N_2}].$$

Subtracting  $\mathbb{E}[M_{N_2}; A^c]$  from each side completes the proof.  $\square$

The following is known as the Doob decomposition for discrete time martingales.

**Proposition A.30** Suppose  $X_k$  is a submartingale with respect to an increasing sequence of  $\sigma$ -fields  $\mathcal{F}_k$ . Then we can write  $X_k = M_k + A_k$  such that  $M_k$  is a martingale adapted to the  $\mathcal{F}_k$  and  $A_k$  is a sequence of random variables with  $A_k$  being  $\mathcal{F}_{k-1}$  measurable and  $A_0 \leq A_1 \leq \dots$ .

*Proof* Let  $a_k = \mathbb{E}[X_k | \mathcal{F}_{k-1}] - X_{k-1}$  for  $k = 1, 2, \dots$ . Since  $X_k$  is a submartingale, each  $a_k \geq 0$ . Let  $A_k = \sum_{i=1}^k a_i$ . The fact that the  $A_k$  are increasing and measurable with respect to  $\mathcal{F}_{k-1}$  is clear. Set  $M_k = X_k - A_k$ . Then

$$\mathbb{E}[M_{k+1} - M_k | \mathcal{F}_k] = \mathbb{E}[X_{k+1} - X_k | \mathcal{F}_k] - a_{k+1} = 0,$$

or  $M_k$  is a martingale.  $\square$

Combining Propositions A.29 and A.30 we have

**Corollary A.31** Suppose  $X_k$  is a submartingale, and  $N_1 \leq N_2$  are bounded stopping times. Then

$$\mathbb{E}[X_{N_2} | \mathcal{F}_{N_1}] \geq X_{N_1}.$$

## A.9 Doob's inequalities

The first interesting consequences of the optional stopping theorems are Doob's inequalities.  $M_n^* := \max_{i \leq n} |M_i|$ .

**Theorem A.32** If  $M_n$  is a martingale or a positive submartingale,

$$\mathbb{P}(M_n^* \geq a) \leq \frac{1}{a} \mathbb{E}[|M_n|; M_n^* \geq a] \leq \frac{1}{a} \mathbb{E}|M_n|.$$

*Proof* Fix  $n$ . Set  $M_{n+1} = M_n$ . Let  $N = \min\{j : |M_j| \geq a\} \wedge (n+1)$ . Since the function  $|M_n|$  is a submartingale. If  $A = (M_n^* \geq a)$ , then  $A \in \mathcal{F}_N$  and we have

$$a\mathbb{P}(M_n^* \geq a) \leq \mathbb{E}[|M_N|; A] \leq \mathbb{E}[|M_n|; A] \leq \mathbb{E}|M_n|,$$

the first inequality by the definition of  $N$ , the second by Corollary A.28.  $\square$

**Theorem A.33** *If  $p > 1$ ,  $M$  is a martingale or positive submartingale, and  $\mathbb{E}|M_i|^p < \infty$  for  $i \leq n$ , then*

$$\mathbb{E}(M_n^*)^p \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_n|^p.$$

*Proof* Note  $M_n^* \leq \sum_{i=1}^n |M_i|$ , hence  $M_n^* \in L^p$ . We write, using Theorem A.32,

$$\begin{aligned} \mathbb{E}(M_n^*)^p &= \int_0^\infty pa^{p-1}\mathbb{P}(M_n^* > a) da \leq \int_0^\infty pa^{p-1}\mathbb{E}[|M_n|1_{(M_n^* \geq a)}/a] da \\ &= \mathbb{E} \int_0^{M_n^*} pa^{p-2}|M_n| da = \frac{p}{p-1}\mathbb{E}[(M_n^*)^{p-1}|M_n|] \\ &\leq \frac{p}{p-1}(\mathbb{E}(M_n^*)^p)^{(p-1)/p}(\mathbb{E}|M_n|^p)^{1/p}. \end{aligned}$$

The last inequality follows by Hölder's inequality. Now divide both sides by the quantity  $(\mathbb{E}(M_n^*)^p)^{(p-1)/p}$ .  $\square$

## A.10 Martingale convergence theorem

The martingale convergence theorem is another important consequence of optional stopping. The main step is the upcrossing lemma. The number of upcrossings of an interval  $[a, b]$  is the number of times a process  $M$  crosses from below  $a$  to above  $b$ .

To be more exact, let

$$S_1 = \min\{k : M_k \leq a\}, \quad T_1 = \min\{k > S_1 : M_k \geq b\},$$

and

$$S_{i+1} = \min\{k > T_i : M_k \leq a\}, \quad T_{i+1} = \min\{k > S_{i+1} : M_k \geq b\}.$$

The number of upcrossings  $U_n$  before time  $n$  is  $U_n = \max\{j : T_j \leq n\}$ .

**Theorem A.34** (Upcrossing lemma) *If  $M_k$  is a submartingale,*

$$\mathbb{E}U_n \leq \frac{1}{b-a}\mathbb{E}[(M_n - a)^+].$$

*Proof* The number of upcrossings of  $[a, b]$  by  $M_k$  is the same as the number of upcrossings of  $[0, b-a]$  by  $Y_k = (M_k - a)^+$ , where  $x^+ = x \vee 0$ . Moreover  $Y_k$  is still a submartingale. If we obtain the inequality for the number of upcrossings of the interval  $[0, b-a]$  by the process  $Y_k$ , we will have the desired inequality for upcrossings of  $M$ .

Thus we may assume  $a = 0$ . Fix  $n$  and define  $Y_{n+1} = Y_n$ . This will still be a submartingale. Define  $S_i, T_i$  as above, and let  $S'_i = S_i \wedge (n+1)$ ,  $T'_i = T_i \wedge (n+1)$ . Since  $T_{i+1} > S_{i+1} > T_i$ , then  $T'_{n+1} = n+1$ .

We write

$$\mathbb{E} Y_{n+1} = \mathbb{E} Y_{S'_1} + \sum_{i=0}^{n+1} \mathbb{E} [Y_{T'_i} - Y_{S'_i}] + \sum_{i=0}^{n+1} \mathbb{E} [Y_{S'_{i+1}} - Y_{T'_i}].$$

All the summands in the third term on the right are non-negative since  $Y_k$  is a submartingale. The first term on the right will be non-negative since  $Y$  is non-negative. For the  $j$ th upcrossing,  $Y_{T'_j} - Y_{S'_j} \geq b - a$ , while  $Y_{T'_j} - Y_{S'_j}$  is always greater than or equal to 0. Thus

$$\sum_{i=0}^{n+1} (Y_{T'_i} - Y_{S'_i}) \geq (b - a) U_n.$$

Hence

$$\mathbb{E} U_n \leq \frac{1}{b - a} \mathbb{E} Y_{n+1}. \quad (\text{A.19})$$

□

This leads to the martingale convergence theorem.

**Theorem A.35** *If  $M_n$  is a submartingale such that  $\sup_n \mathbb{E} M_n^+ < \infty$ , then  $M_n$  converges a.s.,  $n \rightarrow \infty$ .*

*Proof* For each  $a < b$ , let  $U_n(a, b)$  be the number of upcrossings of  $[a, b]$  by  $M$  up to time  $n$ , and let  $U(a, b) = \lim_{n \rightarrow \infty} U_n$ . For each pair  $a < b$  of rational numbers, by monotone convergence,

$$\mathbb{E} U(a, b) \leq \frac{1}{b - a} \sup_n \mathbb{E} (M_n - a)^+ < \infty.$$

Thus  $U(a, b) < \infty$ , a.s. If  $N_{a,b}$  is the set of  $\omega$ 's where  $U(a, b) = \infty$  and  $N = \cup_{a < b, a, b \in \mathbb{Q}_+} N_{a,b}$ , then  $\mathbb{P}(N) = 0$ . If  $\omega \notin N$ , we cannot have  $\limsup_{n \rightarrow \infty} M_n(\omega) > \liminf_{n \rightarrow \infty} M_n(\omega)$ . Therefore  $M_n$  converges almost surely, although we still have to rule out the possibility of the limit being infinite. Since  $M_n$  is a submartingale,  $\mathbb{E} M_n \geq \mathbb{E} M_0$ , and thus

$$\mathbb{E} |M_n| = \mathbb{E} M_n^+ + \mathbb{E} M_n^- = 2\mathbb{E} M_n^+ - \mathbb{E} M_n \leq 2\mathbb{E} M_n^+ - \mathbb{E} M_0.$$

By Fatou's lemma,

$$\mathbb{E} \lim_n |M_n| \leq \sup_n \mathbb{E} |M_n| \leq 2 \sup_n \mathbb{E} M_n^+ - \mathbb{E} M_0 < \infty,$$

or  $M_n$  converges a.s. to a finite limit. □

**Corollary A.36** *If  $X_n$  is a positive supermartingale or a martingale bounded above or below,  $X_n$  converges almost surely.*

*Proof* If  $X_n$  is a positive supermartingale,  $-X_n$  is a submartingale bounded above by 0. Now apply Theorem A.35.

If  $X_n$  is a martingale bounded above, by considering  $-X_n$ , we may assume  $X_n$  is bounded below. Looking at  $X_n + M$  for fixed  $M$  will not affect the convergence, so we may assume  $X_n$  is bounded below by 0. Now apply the first assertion of the corollary. □

$M_n$  is a *uniformly integrable martingale* if the collection of rvs  $\{M_n\}$  is uniformly integrable.

**Proposition A.37** (1) If  $M_n$  is a martingale with  $\sup_n \mathbb{E} |M_n|^p < \infty$  for some  $p > 1$ , then the convergence is in  $L^p$  as well as a.s.. This is also true when  $M_n$  is a submartingale.

(2) If  $M_n$  is a uniformly integrable martingale, then the convergence is in  $L^1$ .

(3) If  $M_n \rightarrow M_\infty$  in  $L^1$ , then  $M_n = \mathbb{E}[M_\infty | \mathcal{F}_n]$ .

*Proof* (1) If  $\sup_n \mathbb{E} |M_n|^p < \infty$ , then  $\sup_n \mathbb{E} M_n^+ < \infty$  and  $M_n$  converges almost surely. Let  $M_\infty$  be the limit. Then  $|M_n - M_\infty| \rightarrow 0$ , a.s., and

$$\begin{aligned}\mathbb{E} \sup_n |M_n - M_\infty|^p &\leq c \mathbb{E} \sup_n |M_n|^p + c \mathbb{E} |M_\infty|^p \\ &\leq c \mathbb{E} \sup_n |M_n|^p \\ &\leq c \sup_n \mathbb{E} |M_n|^p < \infty.\end{aligned}$$

The second inequality is by Fatou's lemma and the last by Doob's inequalities, Theorem A.33. The  $L^p$  convergence assertion now follows by dominated convergence.

(2) The  $L^1$  convergence assertion follows since a.s. convergence together with uniform integrability implies  $L^1$  convergence by the Vitali convergence theorem, Theorem A.19.

(3) Finally, if  $j < n$ , we have  $M_j = \mathbb{E}[M_n | \mathcal{F}_j]$ . If  $A \in \mathcal{F}_j$ ,

$$\mathbb{E}[M_j; A] = \mathbb{E}[M_n; A] \rightarrow \mathbb{E}[M_\infty; A]$$

by the  $L^1$  convergence of  $M_n$  to  $M_\infty$ . Since this is true for all  $A \in \mathcal{F}_j$ ,  $M_j = \mathbb{E}[M_\infty | \mathcal{F}_j]$ .  $\square$

## A.11 Strong law of large numbers

Suppose we have a sequence  $X_1, X_2, \dots$  of independent and identically distributed random variables. This means that the  $X_i$  are independent and each has the same law as  $X_1$ . This situation is very common, and we abbreviate this by saying the  $X_i$  are *i.i.d.*

Define

$$S_n = \sum_{i=1}^n X_i.$$

The  $S_n$  are called *partial sums*. In this section we suppose  $\mathbb{E}|X_1| < \infty$ . The strong law of large number is the precise version of the law of averages.

**Theorem A.38** If  $X_i$  is an i.i.d. sequence and  $\mathbb{E}|X_1| < \infty$ , then

$$\frac{S_n}{n} \rightarrow \mathbb{E} X_1, \quad \text{a.s.}$$

The proof we give is a mixture of the standard one and some martingale techniques. The standard proof (see, e.g., Chung (2001)) uses no martingale methods, while there is a proof (see Durrett (1996)) that is entirely martingale based.

*Proof* We may assume  $\mathbb{E} X_i = 0$ , for otherwise we replace  $X_i$  by  $X_i - \mathbb{E} X_i$ . Let  $Y_n = X_n 1_{(|X_n| \leq n)}$ ,  $Z_n = Y_n - \mathbb{E} Y_n$ , and

$$M_n = \sum_{i=1}^n \frac{Z_i}{i}.$$

Let  $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ . Note that the  $Z_i$  are independent but not identically distributed. Using the independence,  $M_n$  is a martingale:

$$\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n + \frac{1}{n+1} \mathbb{E}[Z_{n+1} | \mathcal{F}_n] = M_n + \frac{1}{n+1} \mathbb{E}[Z_{n+1}] = M_n.$$

We will need the estimate

$$\begin{aligned} \sum_{i=1}^{\infty} \mathbb{P}(|X_1| \geq i) &= \sum_{i=1}^{\infty} \int_{i-1}^i \mathbb{P}(|X_1| \geq i) dx \\ &\leq \int_0^{\infty} \mathbb{P}(|X_1| \geq x) dx = \mathbb{E}|X_1| < \infty, \end{aligned} \tag{A.20}$$

using Proposition A.4.

We show that  $\mathbb{E}|M_n|$  is bounded by a constant not depending on  $n$ . In fact, again using Proposition A.4,

$$\begin{aligned} \mathbb{E} M_n^2 &= \text{Var } M_n = \sum_{i=1}^n \frac{\text{Var } Z_i}{i^2} = \sum_{i=1}^n \frac{1}{i^2} \text{Var } Y_i \\ &\leq \sum_{i=1}^n \frac{1}{i^2} \mathbb{E} Y_i^2 \leq \sum_{i=1}^{\infty} \frac{1}{i^2} \int_0^i 2y \mathbb{P}(|X_i| \geq y) dy \\ &= 2 \sum_{i=1}^{\infty} \frac{1}{i^2} \int_0^{\infty} 1_{(y \leq i)} y \mathbb{P}(|X_1| \geq y) dy \\ &= 2 \int_0^{\infty} \sum_{i=1}^{\infty} \frac{1}{i^2} 1_{(y \leq i)} y \mathbb{P}(|X_1| \geq y) dy \\ &\leq c \int_0^{\infty} \frac{1}{y} \cdot y \mathbb{P}(|X_1| \geq y) dy \\ &= c \int_0^{\infty} \mathbb{P}(|X_1| \geq y) dy = c \mathbb{E}|X_1| < \infty. \end{aligned}$$

The uniform bound on  $\mathbb{E}|M_n|$  follows by Jensen's inequality.

By the martingale convergence theorem,  $M_n$  converges almost surely; let  $M_{\infty}$  be the limit. Some elementary calculus shows that  $\frac{1}{n} \sum_{i=1}^n M_i$  also converges to  $M_{\infty}$ , a.s. We now use summation by parts as follows. Since  $i(M_i - M_{i-1}) = Z_i$  and  $M_0 = 0$ , then

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n Z_i &= \frac{1}{n} \sum_{i=1}^n (iM_i - iM_{i-1}) = \frac{1}{n} \left( \sum_{i=1}^n iM_i - \sum_{i=1}^{n-1} (i+1)M_i \right) \\ &= M_n - \frac{n-1}{n} \left( \frac{1}{n-1} \sum_{i=1}^{n-1} M_i \right) \rightarrow M_{\infty} - M_{\infty} = 0. \end{aligned}$$

By dominated convergence and the fact that the  $X_i$  are identically distributed,

$$\mathbb{E} Y_n = \mathbb{E} [X_n 1_{(|X_n| \leq i)}] = \mathbb{E} [X_1 1_{(|X_1| \leq n)}] \rightarrow \mathbb{E} X_1 = 0$$

as  $n \rightarrow \infty$ , and this implies  $\frac{1}{n} \sum_{i=1}^n \mathbb{E} Y_i \rightarrow 0$ . Since  $Y_i = Z_i + \mathbb{E} Y_i$ , we conclude

$$\frac{1}{n} \sum_{i=1}^n Y_i \rightarrow 0, \quad \text{a.s.}$$

Finally,

$$\sum_{i=1}^{\infty} \mathbb{P}(X_i \neq Y_i) = \sum_{i=1}^{\infty} \mathbb{P}(|X_i| \geq i) = \sum_{i=1}^{\infty} \mathbb{P}(|X_1| \geq i) < \infty,$$

so by the Borel–Cantelli lemma, except for a set of probability zero,  $X_i = Y_i$  for all  $i$  greater than some positive integer  $I$  ( $I$  depends on  $\omega$ ). Hence

$$\left| \frac{1}{n} \sum_{i=1}^n X_i - \frac{1}{n} \sum_{i=1}^n Y_i \right| \leq \frac{1}{n} \sum_{i=1}^I |X_i - Y_i| \rightarrow 0, \quad \text{a.s.}$$

This completes the proof.  $\square$

The following extension of the strong law will be needed when comparing a random walk and a Brownian motion.

**Proposition A.39** *Suppose  $X_i$  is an i.i.d. sequence and  $\mathbb{E} |X_1| < \infty$ . Then*

$$\frac{\max_{k \leq n} |S_k - \mathbb{E} S_k|}{n} \rightarrow 0, \quad \text{a.s.}$$

*Proof* By looking at  $X_i - \mathbb{E} X_i$ , we may assume  $\mathbb{E} X_i = 0$ . Let  $j(n)$  be (one of) the value(s) of  $j$  such that  $|S_j| = \max_{k \leq n} |S_k|$ . Suppose  $S_n(\omega)/n \rightarrow 0$ . It suffices to show  $|S_{j(n)}(\omega)|/n \rightarrow 0$ , a.s.

If not, for this  $\omega$ , either

(1) there is a subsequence  $n_k \rightarrow \infty$  and  $\varepsilon > 0$  such that  $j(n_k) \rightarrow \infty$  and  $|S_{j(n_k)}|/n_k \geq \varepsilon$  for all  $k$ ; or

(2) there exists a subsequence  $n_k \rightarrow \infty$ ,  $\varepsilon > 0$ , and  $N > 1$  such that  $j(n_k) \leq N$  and  $|S_{j(n_k)}|/n_k \geq \varepsilon$  for all  $k$ .

In case (1), since  $j(n_k) \rightarrow \infty$ ,

$$\frac{|S_{j(n_k)}|}{n_k} = \frac{|S_{j(n_k)}|}{j(n_k)} \frac{j(n_k)}{n_k} \leq \frac{|S_{j(n_k)}|}{j(n_k)} \rightarrow 0,$$

a contradiction. In case (2),

$$\frac{|S_{j(n_k)}|}{n_k} \leq \frac{\max_{m \leq N} |S_m|}{n_k} \rightarrow 0,$$

also a contradiction.  $\square$

Another application of the strong law of large numbers is the *Glivenko–Cantelli theorem*. Let  $X_i$  i.i.d.  $\sim U[0, 1]$ ,

that is,  $\mathbb{P}(X_1 \leq t) = t$  if  $0 \leq t \leq 1$ . Let

$$F_n(t) = \frac{1}{n} \sum_{i=1}^n 1_{[0,t]}(X_i), \quad 0 \leq t \leq 1.$$

By the strong law,  $F_n(t) \rightarrow t$ , a.s., for each  $t$ . The Glivenko–Cantelli theorem says that the convergence is uniform over  $t$ .

**Theorem A.40** *With  $F_n$  as above,*

$$\sup_{0 \leq t \leq 1} |F_n(t) - t| \rightarrow 0, \quad \text{a.s.}$$

*Proof* For each  $t \in [0, 1]$ ,  $1_{[0,t]}(X_i)$  is a sequence of i.i.d. random variables with expectation  $\mathbb{P}(X_i \leq t) = t$ . By the strong law of large numbers, for each  $t$ ,  $F_n(t) \rightarrow t$ , a.s. Let  $N_t$  be the set of  $\omega$  such that  $F_n(t)(\omega)$  does not converge to  $t$ , and let  $N = \cup_{t \in [0,1]} N_t$ . Then  $\mathbb{P}(N) = 0$ .

Let  $\varepsilon > 0$  and take  $\omega \notin N$ . Take  $m > 2/\varepsilon$  and choose  $n_0$  large enough (depending on  $\omega$ ) such that

$$|F_n(k/m)(\omega) - (k/m)| < \varepsilon/2, \quad k = 0, 1, 2, \dots, m,$$

if  $n \geq n_0$ . Then if  $n \geq n_0$  and  $k/m \leq t < (k+1)/m$ ,

$$F_n(t) - t \leq F_n((k+1)/m) - k/m \leq F_n((k+1)/m) - (k+1)/m + \varepsilon/2 < \varepsilon,$$

and similarly  $F_n(t) - t > -\varepsilon$ . Hence for  $n \geq n_0$ ,

$$\sup_{t \in [0,1]} |F_n(t) - t| \leq \varepsilon.$$

Since  $\varepsilon$  is arbitrary, this proves the uniform convergence.  $\square$

## A.12 Weak convergence

We will see soon that if the  $X_i$  are i.i.d. with mean zero and variance one, then  $S_n/\sqrt{n}$  converges in the sense that

$$\mathbb{P}(S_n/\sqrt{n} \in [a, b]) \rightarrow \mathbb{P}(Z \in [a, b]),$$

where  $Z$  is a standard normal. We want to generalize the above type of convergence.

We say  $F_n$  converges weakly to  $F$  if  $F_n(x) \rightarrow F(x)$  for all  $x$  at which  $F$  is continuous. Here  $F_n$  and  $F$  are distribution functions. We say  $X_n$  converges weakly to  $X$  if  $F_{X_n}$  converges weakly to  $F_X$ . We also say  $X_n$  converges in distribution or converges in law to  $X$ . Probabilities  $\mu_n$  converge weakly if their corresponding distribution functions converge, that is, if  $F_{\mu_n}(x) = \mu_n(-\infty, x]$  converges weakly.

An example that illustrates why we restrict the convergence to continuity points of  $F$  is the following. Let  $X_n = 1/n$  with probability one, and  $X = 0$  with probability one.  $F_{X_n}(x)$  is 0 if  $x < 1/n$  and 1 otherwise. Note  $F_{X_n}(x)$  converges to  $F_X(x)$  for all  $x$  except  $x = 0$ .

**Proposition A.41**  *$X_n$  converges weakly to  $X$  if and only if  $\mathbb{E}g(X_n) \rightarrow \mathbb{E}g(X)$  for all  $g$  bounded and continuous.*

*Proof* Suppose  $\mathbb{E}g(X_n) \rightarrow \mathbb{E}g(X)$  whenever  $g$  is bounded and continuous. Let  $\varepsilon > 0$  and suppose  $x$  is a continuity point of  $F_X$ . Choose  $\delta$  such that  $F_X(x) - \varepsilon < F_X(x - \delta) \leq F_X(x + \delta) < F_X(x) + \varepsilon$ . Let  $g$  be a continuous function taking values in  $[0, 1]$  such that  $g$  equals 1 on  $(-\infty, x]$  and equals 0 on  $[x + \delta, \infty)$ . Then

$$\begin{aligned}\limsup_{n \rightarrow \infty} F_{X_n}(x) &\leq \limsup_{n \rightarrow \infty} \mathbb{E}g(X_n) \\ &= \mathbb{E}g(X) \leq F_X(x + \delta) < F_X(x) + \varepsilon.\end{aligned}$$

A similar argument shows that  $\liminf_{n \rightarrow \infty} F_{X_n} > F_X(x) - \varepsilon$ . Since  $\varepsilon$  is arbitrary,  $\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x)$ .

Now suppose  $X_n \rightarrow X$  weakly. Let  $\varepsilon > 0$  and choose  $M > 0$  such that  $M$  and  $-M$  are continuity points for  $F_X$  and also continuity points for each of the  $F_{X_n}$ ,  $F_X(-M) < \varepsilon$ , and  $F_X(M) > 1 - \varepsilon$ . Suppose  $g$  is bounded and continuous on  $\mathbb{R}$  and without loss of generality suppose  $g$  is bounded by 1. Then

$$\begin{aligned}\limsup_{n \rightarrow \infty} |\mathbb{E}[g(X_n); X_n \notin [-M, M]]| &\quad (\text{A.21}) \\ &\leq \limsup_{n \rightarrow \infty} \mathbb{P}(|X_n| \geq M) \\ &= \limsup_{n \rightarrow \infty} F_{X_n}(-M) + \limsup_{n \rightarrow \infty} (1 - F_{X_n}(M)) \\ &\leq 2\varepsilon.\end{aligned}$$

Similarly,

$$|\mathbb{E}[g(X); X \notin [-M, M]]| \leq 2\varepsilon. \quad (\text{A.22})$$

Take  $f$  to be a step function of the form  $\sum_{i=1}^m c_i 1_{(a_i, b_i]}$  such that  $|f(x) - g(x)| < \varepsilon$  for  $x \in [-M, M]$  and each  $a_i$  and  $b_i$  is a continuity point for  $F_X$  and also continuity points for each of the  $F_{X_n}$ . Then

$$\begin{aligned}\mathbb{E}f(X_n) &= \sum_{i=1}^m c_i (F_{X_n}(b_i) - F_{X_n}(a_i)) \\ &\rightarrow \sum_{i=1}^m c_i (F_X(b_i) - F_X(a_i)) = \mathbb{E}f(X).\end{aligned} \quad (\text{A.23})$$

Finally, since  $f$  differs from  $g$  by at most  $\varepsilon$  on  $[-M, M]$ , then

$$|\mathbb{E}f(X_n) - \mathbb{E}[g(X_n); X_n \in [-M, M]]| \leq \varepsilon \quad (\text{A.24})$$

and similarly when  $X_n$  is replaced by  $X$ . Combining (A.21), (A.22), (A.23), and (A.24) and using the fact that  $\varepsilon$  is arbitrary shows that  $\mathbb{E}g(X_n) \rightarrow \mathbb{E}g(X)$ .  $\square$

Let us examine the relationship between weak convergence and convergence in probability. If  $X_i$  is an i.i.d. sequence, then  $X_i$  converges weakly, in fact, to  $X_1$ , since all the  $X_i$ 's have the same distribution. But from the independence it is not hard to see that the sequence  $X_i$  does not converge in probability unless the  $X_i$ 's are identically constant. Therefore one can have weak convergence without convergence in probability.

- Proposition A.42** (1) If  $X_n$  converges to  $X$  in probability, then it converges weakly.  
 (2) If  $X_n$  converges weakly to a constant, it converges in probability.  
 (3) (Slutsky's theorem) If  $X_n$  converges weakly to  $X$  and  $Y_n$  converges weakly to a constant  $b$ , then  $X_n + Y_n$  converges weakly to  $X + b$  and  $X_n Y_n$  converges weakly to  $bX$ .

*Proof* To prove (1), let  $g$  be a bounded and continuous function. If  $n_j$  is any subsequence, then there exists a further subsequence such that  $X(n_{j_k})$  converges almost surely to  $X$ . Then by dominated convergence,  $\mathbb{E} g(X(n_{j_k})) \rightarrow \mathbb{E} g(X)$ . That suffices to show  $\mathbb{E} g(X_n)$  converges to  $\mathbb{E} g(X)$ .

For (2), if  $X_n$  converges weakly to  $b$ ,

$$\mathbb{P}(X_n - b > \varepsilon) = \mathbb{P}(X_n > b + \varepsilon) = 1 - \mathbb{P}(X_n \leq b + \varepsilon) \rightarrow 1 - \mathbb{P}(b \leq b + \varepsilon) = 0.$$

We use the fact that if  $Y$  is identically equal to  $b$ , then  $b + \varepsilon$  is a point of continuity for  $F_Y$ . A similar equation shows  $\mathbb{P}(X_n - b \leq -\varepsilon) \rightarrow 0$ , so  $\mathbb{P}(|X_n - b| > \varepsilon) \rightarrow 0$ .

We now prove the first part of (3), leaving the second part for the reader. Let  $x$  be a point such that  $x - b$  is a continuity point of  $F_X$ . Choose  $\varepsilon$  so that  $x - b + \varepsilon$  is again a continuity point. Then

$$\mathbb{P}(X_n + Y_n \leq x) \leq \mathbb{P}(X_n + b \leq x + \varepsilon) + \mathbb{P}(|Y_n - b| > \varepsilon) \rightarrow \mathbb{P}(X \leq x - b + \varepsilon).$$

Hence  $\limsup \mathbb{P}(X_n + Y_n \leq x) \leq \mathbb{P}(X + b \leq x + \varepsilon)$ . Since  $\varepsilon$  can be arbitrarily small and  $x - b$  is a continuity point of  $F_X$ , then  $\limsup \mathbb{P}(X_n + Y_n \leq x) \leq \mathbb{P}(X + b \leq x)$ . The  $\liminf$  is done similarly.  $\square$

We say a sequence of distribution functions  $\{F_n\}$  is *tight* if for each  $\varepsilon > 0$  there exists  $M$  such that  $F_n(M) \geq 1 - \varepsilon$  and  $F_n(-M) \leq \varepsilon$  for all  $n$ . A sequence of random variables  $\{X_n\}$  is tight if the corresponding distribution functions are tight; this is equivalent to  $\mathbb{P}(|X_n| \geq M) \leq \varepsilon$ .

**Theorem A.43** (Helly's theorem) *Let  $F_n$  be a sequence of distribution functions that is tight. There exists a subsequence  $n_j$  and a distribution function  $F$  such that  $F_{n_j}$  converges weakly to  $F$ .*

What could conceivably happen is that  $X_n$  is identically equal to  $n$ , so that  $F_{X_n} \rightarrow 0$ , but the function  $F$  that is identically equal to 0 is not a distribution function; the tightness precludes this.

*Proof* Let  $q_k$  be an enumeration of the rationals. Since  $F_n(q_k) \in [0, 1]$ , any subsequence has a further subsequence that converges. Use a diagonalization argument (as in the proof of the Ascoli–Arzelà theorem; see [Rudin \(1976\)](#)) so that  $F_{n_j}(q_k)$  converges for each  $q_k$  and call the limit  $F(q_k)$ .  $F$  is increasing, and define  $F(x) = \inf_{q_k \geq x} F(q_k)$ . Hence  $F$  is right continuous and increasing.

If  $x$  is a point of continuity of  $F$  and  $\varepsilon > 0$ , then there exist  $r$  and  $s$  rational such that  $r < x < s$  and  $F(s) - \varepsilon < F(x) < F(r) + \varepsilon$ . Then

$$F_{n_j}(x) \geq F_{n_j}(r) \rightarrow F(r) > F(x) - \varepsilon$$

and

$$F_{n_j}(x) \leq F_{n_j}(s) \rightarrow F(s) < F(x) + \varepsilon.$$

Since  $\varepsilon$  is arbitrary,  $F_{n_j}(x) \rightarrow F(x)$ .

Since the  $F_n$  are tight, there exists  $M$  such that  $F_n(-M) < \varepsilon$ . Then  $F(-M) \leq \varepsilon$ , which implies  $\lim_{x \rightarrow -\infty} F(x) = 0$ . Showing  $\lim_{x \rightarrow \infty} F(x) = 1$  is similar. Therefore  $F$  is in fact a distribution function.  $\square$

We conclude by giving an easily checked criterion for tightness.

**Proposition A.44** *Suppose there exists  $\varphi : [0, \infty) \rightarrow [0, \infty)$  that is increasing and  $\varphi(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . If  $a = \sup_n \mathbb{E} \varphi(|X_n|) < \infty$ , then the sequence  $\{X_n\}$  is tight.*

*Proof* Let  $\varepsilon > 0$ . Choose  $M$  such that  $\varphi(x) \geq a/\varepsilon$  if  $x > M$ . Then

$$\mathbb{P}(|X_n| > M) \leq \int \frac{\varphi(|X_n|)}{a/\varepsilon} 1_{(|X_n| > M)} d\mathbb{P} \leq \frac{\varepsilon}{a} \mathbb{E} \varphi(|X_n|) \leq \varepsilon.$$

The conclusion follows.  $\square$

In particular, if  $\sup_n \mathbb{E} |X_n|^2 < \infty$ , the sequence  $\{X_n\}$  is tight.

### A.13 Characteristic functions

We define the *characteristic function* of a random variable  $X$  by  $\varphi_X(t) = \mathbb{E} e^{itX}$  for  $t \in \mathbb{R}$ .

Note that  $\varphi_X(t) = \int e^{itx} \mathbb{P}_X(dx)$ . Thus if  $X$  and  $Y$  have the same law, they have the same characteristic function. Also, if the law of  $X$  has a density, that is,  $\mathbb{P}_X(dx) = f_X(x) dx$ , then  $\varphi_X(t) = \int e^{itx} f_X(x) dx$ , so in this case the characteristic function is the same as the definition of the Fourier transform of  $f_X$ .

**Proposition A.45**  $\varphi(0) = 1$ ,  $|\varphi(t)| \leq 1$ ,  $\varphi(-t) = \overline{\varphi(t)}$ , and  $\varphi$  is uniformly continuous.

*Proof* Since  $|e^{itx}| \leq 1$ , everything follows immediately from the definitions except the uniform continuity. For that we write

$$|\varphi(t+h) - \varphi(t)| = |\mathbb{E} e^{i(t+h)X} - \mathbb{E} e^{itX}| \leq \mathbb{E} |e^{itX} (e^{ihX} - 1)| = \mathbb{E} |e^{ihX} - 1|.$$

Since  $|e^{ihX} - 1|$  tends to zero almost surely as  $h \rightarrow 0$ , the right-hand side tends to zero by dominated convergence. Note that the right-hand side is independent of  $t$ .  $\square$

**Proposition A.46**  $\varphi_{aX}(t) = \varphi_X(at)$  and  $\varphi_{X+b}(t) = e^{ibt} \varphi_X(t)$ .

*Proof* The first follows from  $\mathbb{E} e^{it(aX)} = \mathbb{E} e^{it(at)X}$ , and the second is similar.  $\square$

**Proposition A.47** If  $X$  and  $Y$  are independent, then

$$\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t).$$

*Proof* From the multiplication theorem,

$$\mathbb{E} e^{it(X+Y)} = \mathbb{E} e^{itX} e^{itY} = \mathbb{E} e^{itX} \mathbb{E} e^{itY},$$

and we are done.  $\square$

Let us look at some examples of characteristic functions.

(1) *Bernoulli*: By direct computation,

$$\varphi_X(t) = pe^{it} + (1-p) = 1 - p(1 - e^{it}).$$

(2) *Binomial*: Write  $X$  as the sum of  $n$  independent Bernoulli random variables  $B_i$  with parameter  $p$ . Thus

$$\varphi_X(t) = \prod_{i=1}^n \varphi_{B_i}(t) = [\varphi_{B_i}(t)]^n = [1 - p(1 - e^{it})]^n.$$

(3) *Point mass at  $a$* :  $\mathbb{E} e^{itX} = e^{ita}$ . Note that when  $a = 0$ , then  $\varphi$  is identically equal to 1.

(4) *Poisson*:

$$\mathbb{E} e^{itX} = \sum_{k=0}^{\infty} e^{itk} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!} = e^{-\lambda} e^{\lambda e^{it}} = e^{\lambda(e^{it}-1)}.$$

(5) *Uniform on  $[a, b]$* :

$$\varphi(t) = \frac{1}{b-a} \int_a^b e^{itx} dx = \frac{e^{itb} - e^{ita}}{(b-a)it}.$$

Note that when  $a = -b$  this reduces to  $\sin(bt)/bt$ .

(6) *Exponential*:

$$\varphi(t) = \int_0^{\infty} \lambda e^{itx} e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{(it-\lambda)x} dx = \frac{\lambda}{\lambda - it}.$$

(7) *Standard normal*:

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} e^{-x^2/2} dx.$$

This can be done by completing the square and then doing a contour integration. Alternately,  $\varphi'(t) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} ix e^{itx} e^{-x^2/2} dx$ . (Do the real and imaginary parts separately, and use the dominated convergence theorem to justify taking the derivative inside.) Integrating by parts (do the real and imaginary parts separately),  $\varphi'(t) = -t\varphi(t)$ . The only solution to this differential equation with  $\varphi(0) = 1$  is  $\varphi(t) = e^{-t^2/2}$ .

(8) *Normal with mean  $\mu$  and variance  $\sigma^2$* : Writing  $X = \sigma Z + \mu$ , where  $Z$  is a standard normal, then

$$\varphi_X(t) = e^{i\mu t} \varphi_Z(\sigma t) = e^{i\mu t - \sigma^2 t^2/2}. \quad (\text{A.25})$$

(9) *Gamma*. If  $X$  has a gamma distribution with parameters  $\lambda$  and  $r$ , then its characteristic function is

$$\mathbb{E} e^{iuX} = \left( \frac{\lambda}{\lambda - it} \right)^r.$$

Formally, this comes from writing

$$\varphi(t) = \frac{1}{\Gamma(r)} \int_0^{\infty} e^{itx} \lambda e^{-\lambda x} (\lambda x)^{r-1} dx = \frac{\lambda^r}{\Gamma(r)} \int_0^{\infty} e^{-(\lambda-it)x} x^{r-1} dx$$

and performing a change of variables. To do it properly requires a contour integration around the boundary of the region in the complex plane that is bounded by the positive  $x$  axis, the ray  $\{(\lambda - it)r : r > 0\}$ ,  $\partial B(0, \varepsilon)$ , and  $\partial B(0, R)$ , and then letting  $\varepsilon \rightarrow 0$  and  $R \rightarrow \infty$ .

### A.14 Uniqueness and characteristic functions

**Theorem A.48** If  $\varphi_X = \varphi_Y$ , then  $\mathbb{P}_X = \mathbb{P}_Y$ .

*Proof* If  $f$  is in the Schwartz class, then so is  $\widehat{f}$ ; see Section B.2. We use the Fubini theorem and the Fourier inversion theorem to write

$$\mathbb{E} f(X) = (2\pi)^{-1} \mathbb{E} \left[ \int \widehat{f}(u) e^{-iuX} du \right] = (2\pi)^{-1} \int \widehat{f}(u) \varphi_X(-u) du,$$

and similarly for  $\mathbb{E} f(Y)$ . Since  $\varphi_X = \varphi_Y$ , we conclude  $\mathbb{E} f(X) = \mathbb{E} f(Y)$ . By a limit procedure, we have this equality for all bounded and measurable  $f$ , in particular, when  $f$  is the indicator of a set.  $\square$

The same proof works in higher dimensions: if

$$\mathbb{E} e^{i \sum_{j=1}^n u_j X_j} = \mathbb{E} e^{i \sum_{j=1}^n u_j Y_j}$$

for all  $(u_1, \dots, u_n) \in \mathbb{R}^n$ , then the joint laws of  $(X_1, \dots, X_n)$  and  $(Y_1, \dots, Y_n)$  are equal. The expression  $\mathbb{E} e^{i \sum_{j=1}^n u_j X_j}$  is called the *joint characteristic function* of  $(X_1, \dots, X_n)$ .

The following proposition can be proved directly, but the proof using characteristic functions is much easier.

**Proposition A.49** (1) If  $X$  and  $Y$  are independent,  $X$  is a normal random variable with mean  $a$  and variance  $b^2$ , and  $Y$  is a normal random variable with mean  $c$  and variance  $d^2$ , then  $X + Y$  is normal random variable with mean  $a + c$  and variance  $b^2 + d^2$ .

(2) If  $X$  and  $Y$  are independent,  $X$  is a Poisson random variable with parameter  $\lambda_1$ , and  $Y$  is a Poisson random variable with parameter  $\lambda_2$ , then  $X + Y$  is a Poisson random variable with parameter  $\lambda_1 + \lambda_2$ .

(3) If  $X$  and  $Y$  are independent random variables, where  $X$  has a gamma distribution with parameters  $\lambda$  and  $r_1$  and  $Y$  has a gamma distribution with parameters  $\lambda$  and  $r_2$ , then  $X + Y$  has a gamma distribution with parameters  $\lambda$  and  $r_1 + r_2$ .

*Proof* For (1),

$$\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t) = e^{iat-b^2t^2/2}e^{ict-c^2t^2/2} = e^{i(a+c)t-(b^2+d^2)t^2/2}.$$

Now use the uniqueness theorem.

Parts (2) and (3) are proved similarly.  $\square$

### A.15 The central limit theorem

We need the following estimate on moments.

**Proposition A.50** If  $\mathbb{E} |X|^k < \infty$  for an integer  $k$ , then  $\varphi_X$  has a continuous derivative of order  $k$  and

$$\varphi_X^{(k)}(t) = \int (ix)^k e^{itx} \mathbb{P}_X(dx).$$

In particular,  $\varphi_X^{(k)}(0) = i^k \mathbb{E} X^k$ .

*Proof* Write

$$\frac{\varphi_X(t+h) - \varphi_X(t)}{h} = \int \frac{e^{i(t+h)x} - e^{itx}}{h} \mathbb{P}(dx).$$

Since  $|e^{ihx} - 1| \leq |h| |x|$ , the integrand is bounded by  $|x|$ . Thus if  $\int |x| \mathbb{P}_X(dx) < \infty$ , we can use dominated convergence to obtain the desired formula for  $\varphi'_X(t)$ . As in the proof of Proposition A.45, we see  $\varphi'_X(t)$  is continuous. We do the case of general  $k$  by induction. Evaluating  $\varphi_X^{(k)}$  at 0 shows  $\varphi_X^{(k)}(0) = i^k \mathbb{E} X^k$ .  $\square$

By the above,

$$\mathbb{E} X^2 = -\varphi''_X(0). \quad (\text{A.26})$$

The simplest case of the central limit theorem (CLT) is the case when the  $X_i$ 's are i.i.d., with mean zero and variance one,  $S_n = \sum_{i=1}^n X_i$ , and then the CLT says that  $S_n/\sqrt{n}$  converges weakly to a standard normal. This is the case we prove.

We need the fact that if  $w_n$  are complex numbers converging to  $w$ , then  $(1 + (w_n/n))^n \rightarrow e^w$ . We leave the proof of this to the reader, with the warning that any proof using logarithms needs to be done with some care, since  $\log z$  is a multivalued function when  $z$  is complex.

**Theorem A.51** Suppose the  $X_i$ 's are i.i.d. random variables with mean zero and variance one. Then  $S_n/\sqrt{n}$  converges weakly to a standard normal.

*Proof* Since  $X_1$  has finite second moment, then  $\varphi_{X_1}$  has a continuous second derivative by Proposition A.50. By Taylor's theorem,

$$\varphi_{X_1}(t) = \varphi_{X_1}(0) + \varphi'_{X_1}(0)t + \varphi''_{X_1}(0)t^2/2 + R(t),$$

where  $|R(t)|/t^2 \rightarrow 0$  as  $|t| \rightarrow 0$ . Thus

$$\varphi_{X_1}(t) = 1 - t^2/2 + R(t).$$

Then

$$\varphi_{S_n/\sqrt{n}}(t) = \varphi_{S_n}(t/\sqrt{n}) = (\varphi_{X_1}(t/\sqrt{n}))^n = \left[ 1 - \frac{t^2}{2n} + R(t/\sqrt{n}) \right]^n.$$

Since  $t/\sqrt{n}$  converges to zero as  $n \rightarrow \infty$ , we have

$$\varphi_{S_n/\sqrt{n}}(t) \rightarrow e^{-t^2/2}.$$

Since  $\mathbb{E} S_n^2/n = 1$  for all  $n$ , Proposition A.44 tells us that the random variables  $S_n/\sqrt{n}$  are tight, and from Theorem A.43, subsequential weak limit points exist. By the preceding paragraph, any weak limit of a subsequence is a normal random variable with mean zero and variance one. Therefore the entire sequence converges weakly to a normal random variable with mean zero and variance one.  $\square$

### A.16 Gaussian random variables

A normal random variable is also known as a Gaussian random variable.

**Proposition A.52** *If  $Z$  is a mean zero normal random variable with variance one and  $x \geq 1$ , then*

$$\frac{1}{x} e^{-x^2/2} \leq \mathbb{P}(Z \geq x) \leq e^{-x^2/2}.$$

In particular, if  $\varepsilon > 0$ , there exists  $x_0$  such that

$$\mathbb{P}(Z \geq x) \geq e^{-(1+\varepsilon)x^2/2}$$

if  $x \geq x_0$ .

*Proof* For the right-hand inequality,

$$\mathbb{P}(Z \geq x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2} dy \leq \int_x^\infty \frac{y}{x} e^{-y^2/2} dy = \frac{1}{x} e^{-x^2/2}.$$

The left-hand inequality is left as an exercise.  $\square$

**Proposition A.53** *If  $X_n$  is a normal random variable with mean  $a_n$  and variance  $b_n^2$ ,  $X_n$  converges to  $X$  weakly,  $a_n \rightarrow a$ , and  $b_n \rightarrow b \neq 0$ , then  $X$  is a normal random variable with mean  $a$  and variance  $b^2$ .*

*Proof* Since

$$\mathbb{E} X_n^2 = \text{Var } X_n + (\mathbb{E} X_n)^2 = b_n^2 + a_n^2,$$

then  $\sup_n \mathbb{E} X_n^2 < \infty$ , and the  $X_n$  are tight. For each  $t$ , the characteristic functions converge:

$$\varphi_X(t) = \lim_{n \rightarrow \infty} \varphi_{X_n}(t) = \lim_{n \rightarrow \infty} e^{ita_n - t^2 b_n^2 / 2} = e^{ita - t^2 b^2 / 2},$$

and the last term is the characteristic function of a normal random variable with mean  $a$  and variance  $b^2$ . Therefore any weak subsequential limit point of the sequence  $X_n$  is a normal random variable with mean  $a$  and variance  $b^2$ .  $\square$

We next prove

**Proposition A.54** *If*

$$\mathbb{E} e^{i(uX+vY)} = \mathbb{E} e^{iuX} \mathbb{E} e^{ivY} \tag{A.27}$$

for all  $u$  and  $v$ , then  $X$  and  $Y$  are independent random variables.

*Proof* Let  $X'$  be a random variable with the same law as  $X$ ,  $Y'$  one with the same law as  $Y$ , and so that  $X'$  is independent of  $Y'$ . (We let  $\Omega = [0, 1]^2$ ,  $\mathbb{P}$  a Lebesgue measure,  $X'$  a function of the first variable, and  $Y'$  a function of the second variable defined as in Proposition A.2.) Then since  $e^{iuX'}$  and  $e^{ivY'}$  are independent,

$$\mathbb{E} e^{i(uX'+vY')} = \mathbb{E} e^{iuX'} \mathbb{E} e^{ivY'}. \tag{A.28}$$

Since  $X, X'$  have the same law,  $\mathbb{E} e^{iuX} = \mathbb{E} e^{iuX'}$ , and similarly for  $Y, Y'$ . Therefore, using (A.27) and (A.28),  $(X', Y')$  has the same joint characteristic function as  $(X, Y)$ . By the

uniqueness theorem for characteristic functions,  $(X', Y')$  has the same joint law as  $(X, Y)$ , which implies that  $X$  and  $Y$  are independent.  $\square$

A sequence of random variables  $X_1, \dots, X_n$  is said to be *jointly normal* if there exists a sequence of i.i.d. normal random variables  $Z_1, \dots, Z_m$  with mean zero and variance one and constants  $b_{ij}$  and  $a_i$  such that

$$X_i = \sum_{j=1}^m b_{ij} Z_j + a_i, \quad i = 1, \dots, n. \quad (\text{A.29})$$

In matrix notation,  $X = BZ + A$ . For simplicity, in what follows let us take  $A = 0$ ; the modifications for the general case are easy. The *covariance* of two random variables  $X$  and  $Y$  is defined to be  $\mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]$ . Since we are assuming our normal random variables are mean zero, we can omit the centering at expectations. Given a sequence of mean zero random variables, we can talk about the *covariance matrix*, which is

$$\text{Cov}(X) = \mathbb{E}XX^T,$$

where  $X^T$  denotes the transpose of the vector  $X$ . In the above case, we see  $\text{Cov}(X) = \mathbb{E}[(BZ)(BZ)^T] = \mathbb{E}[BZZ^TB^T] = BB^T$ , since  $\mathbb{E}ZZ^T = I$ , the identity.

Let us compute the joint characteristic function  $\mathbb{E}e^{iu^TX}$  of the vector  $X$ , where  $u$  is an  $n$ -dimensional vector. First, if  $v$  is an  $m$ -dimensional vector,

$$\mathbb{E}e^{iv^TZ} = \mathbb{E} \prod_{j=1}^m e^{iv_j Z_j} = \prod_{j=1}^m \mathbb{E}e^{iv_j Z_j} = \prod_{j=1}^m e^{-v_j^2/2} = e^{-v^Tv/2}$$

using the independence of the  $Z_j$ 's. Thus

$$\mathbb{E}e^{iu^TX} = \mathbb{E}e^{iu^TBZ} = e^{-u^TBB^Tu/2}.$$

By taking  $u = (0, \dots, 0, a, 0, \dots, 0)$  to be a constant times the unit vector in the  $j$ th coordinate direction, we deduce that  $X_j$  is indeed normal, and this is true for each  $j$ .

Note that the joint characteristic function of a jointly normal collection of random variables  $X = (X_1, \dots, X_n)$  is completely determined by  $BB^T$ , which is the covariance matrix of  $X$ . In the case when the  $X_i$ 's are not mean zero, we can readily check that the joint characteristic function is determined by the covariance matrix together with the vector of means  $\mathbb{E}X$ . Therefore the joint distribution of a jointly normal collection of random variables is determined by the covariance matrix and the means.

**Proposition A.55** *If the  $X_i$  are jointly normal and  $\text{Cov}(X_i, X_j) = 0$  for  $i \neq j$ , then the  $X_i$  are independent.*

*Proof* If  $\text{Cov}(X) = BB^T$  is a diagonal matrix, then the joint characteristic function of the  $X_i$ 's factors into the product of the characteristic functions of the  $X_i$ 's, and so by Proposition A.54, the  $X_i$ 's will in this case be independent.  $\square$

**Remark A.56** We note that the analog of Proposition A.53 holds for jointly normal random vectors. That is, if  $(X_j^1, \dots, X_j^n)$  is a jointly normal collection of random variables for each  $j$  and each  $X_j^i$  converges in probability to  $X^i$  and each  $X_i$  is nonconstant, then  $(X^1, \dots, X^n)$

is a jointly normal collection of random variables. This follows by looking at the joint characteristic functions as in the proof of Proposition A.53.

We present the multidimensional central limit theorem.

**Theorem A.57** *Let  $X_j = (X_j^1, \dots, X_j^d)$  be random vectors taking values in  $\mathbb{R}^d$  and suppose the  $X_1, X_2, \dots$  are independent and identically distributed. Suppose  $\mathbb{E} X_1^k = 0$  and  $\mathbb{E} (X_1^k)^2 < \infty$  for  $k = 1, \dots, d$  and let  $C_{k\ell} = \mathbb{E} [X_1^k X_1^\ell]$ . If  $S_n = \sum_{j=1}^n X_j$ , then  $S_n/\sqrt{n}$  converges weakly to a jointly normal random vector  $Z = (Z^1, \dots, Z^d)$  where each  $Z^k$  has mean zero and the covariance of  $Z^k$  and  $Z^\ell$  is  $C_{k\ell}$ .*

*Proof* Since

$$\mathbb{E} |S_n|^2/n = \sum_{j=1}^n \sum_{k=1}^d \mathbb{E} |X_j^k|^2/n$$

is bounded independently of  $n$ , the random vectors  $S_n/\sqrt{n}$  are tight, and therefore weak subsequential limit points exist. We need to show that any subsequential limit point is a jointly normal random vector with mean zero and covariance matrix  $C$ .

If  $u_1, \dots, u_d \in \mathbb{R}$ , then  $\sum_{k=1}^d u_k X_j^k$ ,  $j = 1, 2, \dots$ , will be a sequence of i.i.d. random variables with mean zero and variance  $\sum_{k,\ell=1}^d u_k u_\ell C_{k\ell}$ . By Theorem A.51,

$$\frac{\sum_{j=1}^n \sum_{k=1}^d u_k X_j^k}{\sqrt{n}}$$

converges weakly to a mean zero normal random variable with variance equal to  $\sum_{k,\ell=1}^d u_k u_\ell C_{k\ell}$ . If we write  $S_n = (S_n^1, \dots, S_n^d)$ , then

$$\mathbb{E} \exp \left( i \sum_{k=1}^d u_k S_n^k / \sqrt{n} \right) \rightarrow \exp \left( - \sum_{k,\ell=1}^d u_k u_\ell C_{k\ell} / 2 \right).$$

This shows that any subsequential limit point of the sequence  $S_n/\sqrt{n}$  has the required law.  $\square$

If  $(X, Y_1, \dots, Y_n)$  are jointly normal random variables, then the law of  $X$  given  $Y_1, \dots, Y_n$  is also Gaussian.

**Proposition A.58** *Suppose  $X, Y_1, \dots, Y_n$  are jointly normal random variables with mean zero. Let  $A$  be the  $n \times 1$  matrix whose  $i$ th entry is  $\text{Cov}(X, Y_i)$ ,  $B$  the  $n \times n$  matrix whose  $(i, j)$ th entry is  $\text{Cov}(Y_i, Y_j)$ , and  $D$  the  $n \times 1$  matrix whose  $i$ th entry is  $Y_i$ . Suppose  $B$  is invertible and let  $D = B^{-1}A$ . Then for  $u \in \mathbb{R}$ ,*

$$\mathbb{E} [e^{iuX} \mid Y_1, \dots, Y_n] = e^{iuD^T Y} e^{-(\text{Var } X - A^T B^{-1} A)/2}.$$

In particular, the law of  $X$  given  $Y_1, \dots, Y_n$  is that of a normal random variable with mean  $D^T Y$  and variance equal to  $\text{Var } X - A^T B^{-1} A$ .

*Proof* Note

$$\begin{aligned}\text{Cov}(X - D^T Y, Y_j) &= \text{Cov}(X, Y_j) - \sum_{i=1}^n D_i \text{Cov}(Y_i, Y_j) \\ &= A_j - \sum_{i=1}^n D_i B_{ij} = 0,\end{aligned}$$

so  $X - D^T Y$  is independent of each  $Y_j$ . Then

$$\begin{aligned}\mathbb{E}[e^{iuX} | Y_1, \dots, Y_n] &= e^{iuD^T Y} \mathbb{E}[e^{iu(X-D^T Y)} | Y_1, \dots, Y_n] \\ &= e^{iuD^T Y} \mathbb{E}[e^{iu(X-D^T Y)}] \\ &= e^{iuD^T Y} \mathbb{E} e^{-\text{Var}(X-D^T Y)/2}.\end{aligned}$$

To complete the proof, we calculate

$$\begin{aligned}\text{Var}(X - D^T Y) &= \text{Var } X - 2 \sum_i D_i A_i + \sum_{i,j} D_i B_{ij} D_j \\ &= \text{Var } X - A^T B^{-1} A,\end{aligned}$$

and we are done.  $\square$

# Appendix B

---

## Some results from analysis

### B.1 The monotone class theorem

The monotone class theorem is a result from measure theory used in the proof of the Fubini theorem.

**Definition B.1**  $\mathcal{M}$  is a *monotone class* if  $\mathcal{M}$  is a collection of subsets of  $X$  such that

- (1) if  $A_1 \subset A_2 \subset \dots, A = \cup_i A_i$ , and each  $A_i \in \mathcal{M}$ , then  $A \in \mathcal{M}$ ;
- (2) if  $A_1 \supset A_2 \supset \dots, A = \cap_i A_i$ , and each  $A_i \in \mathcal{M}$ , then  $A \in \mathcal{M}$ .

Recall that an algebra of sets is a collection  $\mathcal{A}$  of sets such that if  $A_1, \dots, A_n \in \mathcal{A}$ , then  $A_1 \cup \dots \cup A_n$  and  $A_1 \cap \dots \cap A_n$  are also in  $\mathcal{A}$ , and if  $A \in \mathcal{A}$ , then  $A^c \in \mathcal{A}$ .

The intersection of monotone classes is a monotone class, and the intersection of all monotone classes containing a given collection of sets is the smallest monotone class containing that collection.

**Theorem B.2** Suppose  $\mathcal{A}_0$  is an algebra of sets,  $\mathcal{A}$  is the smallest  $\sigma$ -field containing  $\mathcal{A}_0$ , and  $\mathcal{M}$  is the smallest monotone class containing  $\mathcal{A}_0$ . Then  $\mathcal{M} = \mathcal{A}$ .

*Proof* A  $\sigma$ -algebra is clearly a monotone class, so  $\mathcal{M} \subset \mathcal{A}$ . We must show  $\mathcal{A} \subset \mathcal{M}$ .

Let  $\mathcal{N}_1 = \{A \in \mathcal{M} : A^c \in \mathcal{M}\}$ . Note  $\mathcal{N}_1$  is contained in  $\mathcal{M}$ , contains  $\mathcal{A}_0$ , and is a monotone class. Since  $\mathcal{M}$  is the smallest monotone class containing  $\mathcal{A}_0$ , then  $\mathcal{N}_1 = \mathcal{M}$ , and therefore  $\mathcal{M}$  is closed under the operation of taking complements.

Let  $\mathcal{N}_2 = \{A \in \mathcal{M} : A \cap B \in \mathcal{M} \text{ for all } B \in \mathcal{A}_0\}$ .  $\mathcal{N}_2$  is contained in  $\mathcal{M}$ ;  $\mathcal{N}_2$  contains  $\mathcal{A}_0$  because  $\mathcal{A}_0$  is an algebra;  $\mathcal{N}_2$  is a monotone class because  $(\cup_{i=1}^{\infty} A_i) \cap B = \cup_{i=1}^{\infty} (A_i \cap B)$ , and similarly for intersections. Therefore  $\mathcal{N}_2 = \mathcal{M}$ ; in other words, if  $B \in \mathcal{A}_0$  and  $A \in \mathcal{M}$ , then  $A \cap B \in \mathcal{M}$ .

Let  $\mathcal{N}_3 = \{A \in \mathcal{M} : A \cap B \in \mathcal{M} \text{ for all } B \in \mathcal{M}\}$ . As in the preceding paragraph,  $\mathcal{N}_3$  is a monotone class contained in  $\mathcal{M}$ . By the last sentence of the preceding paragraph,  $\mathcal{N}_3$  contains  $\mathcal{A}_0$ . Hence  $\mathcal{N}_3 = \mathcal{M}$ .

We thus have that  $\mathcal{M}$  is a monotone class closed under the operations of taking complements and taking intersections. This shows  $\mathcal{M}$  is a  $\sigma$ -algebra, and so  $\mathcal{A} \subset \mathcal{M}$ .  $\square$

## B.2 The Schwartz class

A function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is in the Schwartz class if  $f$  is  $C^\infty$  and for each  $m, k \geq 0$  and each  $i_1, i_2, \dots, i_k \in \{1, 2, \dots, d\}$ ,

$$|x|^m \left| \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(x) \right| \rightarrow 0$$

as  $|x| \rightarrow \infty$ . (Here  $i_1, \dots, i_k$  need not be distinct.)

Suppose that  $f$  is in the Schwartz class. Suppose  $m, k \geq 0$  and  $i_1, \dots, i_k$  and  $j_1, \dots, j_n$  are each integers between 1 and  $d$  inclusive, and  $m_1, \dots, m_k$  are even positive integers. Let  $\widehat{f}$  be the Fourier transform of  $f$ :

$$\widehat{f}(u) = \int_{\mathbb{R}^d} e^{iu \cdot x} f(x) dx.$$

Then

$$u_{i_1}^{m_1} \cdots u_{i_k}^{m_k} \frac{\partial^{j_1 + \cdots + j_n} \widehat{f}}{\partial u_{j_1} \cdots \partial u_{j_n}}(u)$$

is bounded as a function of  $u$  because it is a constant times the Fourier transform of

$$x_{j_1} \cdots x_{j_n} \frac{\partial^{m_1 + \cdots + m_k} f}{\partial x_{i_1}^{m_1} \cdots \partial x_{i_k}^{m_k}},$$

which is in  $L^1(\mathbb{R}^d)$  since  $f$  is in the Schwartz class. We conclude that  $\widehat{f}$  is also in the Schwartz class.

# Appendix C

---

## Regular conditional probabilities

Let  $\mathcal{E} \subset \mathcal{F}$  be  $\sigma$ -fields, where  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space. A *regular conditional probability* for  $\mathbb{E}[\cdot | \mathcal{E}]$  is a map  $Q : \Omega \times \mathcal{F} \rightarrow [0, 1]$  such that

- (1)  $Q(\omega, \cdot)$  is a probability measure on  $(\Omega, \mathcal{F})$  for each  $\omega$ ;
- (2) for each  $A \in \mathcal{F}$ ,  $Q(\cdot, A)$  is an  $\mathcal{E}$  measurable random variable;
- (3) for each  $A \in \mathcal{F}$  and each  $B \in \mathcal{E}$ ,

$$\int_B Q(\omega, A) \mathbb{P}(d\omega) = \mathbb{P}(A \cap B).$$

$Q(\omega, A)$  can be thought of as  $\mathbb{P}(A | \mathcal{E})$ .

**Theorem C.1** Suppose  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space,  $\mathcal{E} \subset \mathcal{F}$ , and  $\Omega$  is in addition a complete and separable metric space. Then a regular conditional probability for  $\mathbb{P}(\cdot | \mathcal{E})$  exists.

*Proof* Since  $\Omega$  is a complete and separable metric space, we can embed  $\Omega$  as a subset of the compact set  $I = [0, 1]^{\mathbb{N}}$ , where we furnish  $I$  with the product topology. Let  $\{f_j\}$  be a countable collection of uniformly continuous functions on  $\Omega$  such that every finite subset of distinct elements is linearly independent and such that  $\mathcal{L}_0$ , the set of finite linear combinations of the  $f_j$ 's, is dense in the class of uniformly continuous functions on  $\Omega$ ; let us assume  $f_1$  is identically equal to 1.

For each  $j$ , let  $g_j = \mathbb{E}[f_j | \mathcal{E}]$ . (The random variables  $g_j$  are only defined up to almost sure equivalence. For each  $j$  we select an element  $g_j$  from the equivalence class and keep it fixed.) If  $r_1, \dots, r_n$  are rationals with

$$r_1 f_1(\omega) + \dots + r_n f_n(\omega) \geq 0$$

for all  $\omega$ , let

$$N(r_1, \dots, r_n) = \{\omega : r_1 g_1(\omega) + \dots + r_n g_n(\omega) < 0\}.$$

By the definition of  $g_j$ ,  $\mathbb{P}(N(r_1, \dots, r_n)) = 0$ . Let  $N_1$  be the union of all such  $N(r_1, \dots, r_n)$  with  $n \geq 1$ , the  $r_j$  rational. Then  $N_1 \in \mathcal{E}$  and  $\mathbb{P}(N_1) = 0$ .

Fix  $\omega \in \Omega \setminus N_1$ . Define a functional  $L_\omega$  on  $\mathcal{L}_0$  by

$$L_\omega(f) = t_1 g_1(\omega) + \dots + t_n g_n(\omega)$$

if

$$f = t_1 f_1 + \dots + t_n f_n.$$

We claim  $L_\omega$  is a positive linear functional. If  $f = t_1 f_1 + \dots + t_n f_n \geq 0$  and  $\varepsilon > 0$  is rational, then there exist rationals  $r_1, \dots, r_n$  such that  $r_1 f_1 + \dots + r_n f_n \geq -\varepsilon$  and  $|t_i - r_i| \leq \varepsilon$ ,  $i = 1, \dots, n$ , or

$$(r_1 + \varepsilon) f_1 + r_2 f_2 + \dots + r_n f_n \geq 0.$$

Since  $\omega \notin N_1$ , then

$$(r_1 + \varepsilon) g_1 + r_2 g_2 + \dots + r_n g_n \geq 0.$$

Letting  $\varepsilon \rightarrow 0$ , it follows that  $t_1 g_1 + \dots + t_n g_n \geq 0$ . This proves that  $L_\omega$  is positive.

Since  $L_\omega(f_1) = 1$ , this implies that  $L_\omega$  is a bounded linear functional, and by the Hahn–Banach theorem  $L_\omega$  can be extended to a positive linear functional on the closure of  $\mathcal{L}_0$ . Any uniformly continuous function on  $\Omega$  can be extended uniquely to  $\overline{\Omega}$ , the closure of  $\Omega$  in  $I$ , so  $L_\omega$  can be considered as a positive linear functional on  $C(\overline{\Omega})$ . By the Riesz representation theorem, there exists a probability measure  $Q(\omega, \cdot)$  such that

$$L_\omega(f) = \int f(\omega') Q(\omega, d\omega').$$

The mapping  $\omega \rightarrow L_\omega(f)$  is measurable with respect to  $\mathcal{E}$  for each  $f \in \mathcal{L}_0$ , hence for all uniformly continuous functions on  $\Omega$  by a limit argument. If  $B \in \mathcal{E}$  and  $f = t_1 f_1 + \dots + t_n f_n$ ,

$$\begin{aligned} \int_B \left[ \int f(\omega') Q(\omega, d\omega') \right] \mathbb{P}(d\omega) &= \int_B L_\omega f(\omega) \mathbb{P}(d\omega) \\ &= \int_B (t_1 g_1 + \dots + t_n g_n)(\omega) \mathbb{P}(d\omega) \\ &= \int_B \mathbb{E}[t_1 f_1 + \dots + t_n f_n | \mathcal{E}](\omega) \mathbb{P}(d\omega) \\ &= \int_B f(\omega) \mathbb{P}(d\omega) \end{aligned}$$

or  $\int f(\omega') Q(\omega, d\omega')$  is a version of  $\mathbb{E}[f | \mathcal{E}]$  if  $f \in \mathcal{L}_0$ . By a limit argument, the same is true for all  $f$  that are of the form  $f = 1_A$  with  $A \in \mathcal{F}$ .

Let  $G_{ni}$  be a sequence of balls of radius  $1/n$  (with respect to the metric on  $\Omega$ ) contained in  $\Omega$  and covering  $\Omega$ . Choose  $i_n$  such that  $\mathbb{P}(\cup_{i \leq i_n} G_{ni}) > 1 - 1/(n2^n)$ . The set  $H_n = \cap_{n \geq 1} \cup_{i \leq i_n} G_{ni}$  is totally bounded; let  $K_n$  be the closure of  $H_n$  in  $\Omega$ . Since  $\Omega$  is complete,  $K_n$  is complete and totally bounded, and hence compact, and  $\mathbb{P}(K_n) \geq 1 - 1/n$ . Hence

$$\mathbb{E}[Q(\cdot, \cup_{i=1}^\infty K_i); \Omega \setminus N_1] \geq \mathbb{E}[Q(\cdot, K_n); \Omega \setminus N_1] = \mathbb{P}(K_n) \geq 1 - (1/n)$$

for each  $n$ , or  $Q(\omega, \cup_{i=1}^\infty K_i) = 1$ , a.s. Let  $N_2$  be the null set for which this fails. Thus for  $\omega \in \Omega \setminus (N_1 \cup N_2)$ , we see that  $Q(\omega, d\omega')$  is a probability measure on  $\Omega$ . For  $\omega \in N_1 \cup N_2$ , set  $Q(\omega, \cdot) = \mathbb{P}(\cdot)$ . This  $Q$  is the desired regular conditional probability.  $\square$

# Appendix D

---

## Kolmogorov extension theorem

Suppose  $\mathcal{S}$  is a metric space. We use  $\mathcal{S}^{\mathbb{N}}$  for the product space  $\mathcal{S} \times \mathcal{S} \times \dots$  furnished with the product topology. We may view  $\mathcal{S}^{\mathbb{N}}$  as the set of sequences  $(x_1, x_2, \dots)$  of elements of  $\mathcal{S}$ . We use the  $\sigma$ -field on  $\mathcal{S}^{\mathbb{N}}$  generated by the cylindrical sets. Given an element  $x = (x_1, x_2, \dots)$  of  $\mathcal{S}^{\mathbb{N}}$ , we define  $\pi_n(x) = (x_1, \dots, x_n) \in \mathcal{S}^n$ .

We suppose we have a Radon probability measure  $\mu_n$  defined on  $\mathcal{S}^n$  for each  $n$ . (Being a Radon measure means that we can approximate  $\mu_n(A)$  from below by compact sets; see Folland (1999) for details.) The  $\mu_n$  are *consistent* if  $\mu_{n+1}(A \times \mathcal{S}) = \mu_n(A)$  whenever  $A$  is a Borel subset of  $\mathcal{S}^n$ . The *Kolmogorov extension theorem* is the following.

**Theorem D.1** *Suppose for each  $n$  we have a probability measure  $\mu_n$  on  $\mathcal{S}^n$ . Suppose the  $\mu_n$ 's are consistent. Then there exists a probability measure  $\mu$  on  $\mathcal{S}^{\mathbb{N}}$  such that  $\mu(A \times \mathcal{S}^{\mathbb{N}}) = \mu_n(A)$  for all  $A \subset \mathcal{S}^n$ .*

*Proof* Define  $\mu$  on cylindrical sets by  $\mu(A \times \mathcal{S}^{\mathbb{N}}) = \mu_n(A)$  if  $A \subset \mathcal{S}^n$ . By the consistency assumption,  $\mu$  is well defined. By the Carathéodory extension theorem, we can extend  $\mu$  to the  $\sigma$ -field generated by the cylindrical sets provided we show that whenever  $A_n$  are cylindrical sets decreasing to  $\emptyset$ , then  $\mu(A_n) \rightarrow 0$ .

Suppose  $A_n$  are cylindrical sets decreasing to  $\emptyset$  but  $\mu(A_n)$  does not tend to 0; by taking a subsequence we may assume without loss of generality that there exists  $\varepsilon > 0$  such that  $\mu(A_n) \geq \varepsilon$  for all  $n$ . We will obtain a contradiction.

We first want to arrange things so that each  $A_n = \pi_n(A_n) \times \mathcal{S}^{\mathbb{N}}$ . Suppose  $A_n$  is of the form

$$A_n = \{(x_1, x_2, \dots) : (x_1, \dots, x_{j_n}) \in B_n\},$$

where  $B_n$  is a Borel subset of  $\mathcal{S}^{j_n}$ . We choose  $m_n = n + \max(j_1, \dots, j_n)$ . Let  $A_0 = \mathcal{S}^{\mathbb{N}}$ . We then replace our original sequence  $A_1, A_2, \dots$  by the sequence  $A_0, \dots, A_0, A_1, \dots, A_1, A_2, \dots, A_2, A_3, \dots$ , where we have  $m_1$  occurrences of  $A_0$ ,  $m_2 - m_1$  occurrences of  $A_1$ ,  $m_3 - m_2$  occurrences of  $A_2$ , and so on. Therefore we may without loss of generality suppose  $j_n \leq n$ . We then have

$$A_n = \{(x_1, x_2, \dots) : (x_1, \dots, x_n) \in B_n \times \mathcal{S}^{n-j_n}\}.$$

Replacing  $B_n$  by  $B_n \times \mathcal{S}^{j_n-n}$ , we may without loss of generality suppose  $A_n = \pi_n(A_n) \times \mathcal{S}^{\mathbb{N}}$ .

We set  $\tilde{A}_n = \pi_n(A_n)$ . For each  $n$ , choose  $\tilde{B}_n \subset \tilde{A}_n$  so that  $\tilde{B}_n$  is compact and  $\mu(\tilde{A}_n \setminus \tilde{B}_n) \leq \varepsilon/2^{n+1}$ . Let  $B_n = \tilde{B}_n \times \mathcal{S}^{\mathbb{N}}$  and let  $C_n = B_1 \cap \dots \cap B_n$ . Hence  $C_n \subset B_n \subset A_n$ , and  $C_n \downarrow \emptyset$ , but

$$\mu(C_n) \geq \mu(A_n) - \sum_{i=1}^n \mu(A_i \setminus B_i) \geq \varepsilon/2,$$

and  $\tilde{C}_n = \pi_n(C_n)$ , the projection of  $C_n$  onto  $\mathcal{S}^n$ , is compact.

We will find  $x = (x_1, \dots, x_n, \dots) \in \cap_n C_n$  and obtain our contradiction. For each  $n$  choose a point  $y(n) \in C_n$ . The first coordinates of  $\{y(n)\}$ , namely,  $\{y_1(n)\}$ , form a sequence contained in  $\tilde{C}_1$ , which is compact, hence there is a convergent subsequence  $\{y_1(n_k)\}$ . Let  $x_1$  be the limit point. The first and second coordinates of  $\{y(n_k)\}$  form a sequence contained in the compact set  $\tilde{C}_2$ , so a further subsequence  $\{(y_1(n_{k_j}), y_2(n_{k_j}))\}$  converges to a point in  $\tilde{C}_2$ . Since  $\{n_{k_j}\}$  is a subsequence of  $\{n_k\}$ , the first coordinate of the limit is  $x_1$ . Therefore the limit point of  $\{(y_1(n_{k_j}), y_2(n_{k_j}))\}$  is of the form  $(x_1, x_2)$ , and this point is in  $\tilde{C}_2$ . We continue this procedure to obtain  $x = (x_1, x_2, \dots, x_n, \dots)$ . By our construction,  $(x_1, \dots, x_n) \in \tilde{C}_n$  for each  $n$ , hence  $x \in C_n$  for each  $n$ , or  $x \in \cap_n C_n$ , a contradiction.  $\square$

A typical application of this theorem is to construct a countable sequence of independent random variables. We construct  $X_1, \dots, X_n$  as in Proposition A.10. Here  $\mathcal{S} = [0, 1]$ . Let  $\mu_n$  be the law of  $(X_1, \dots, X_n)$ ; it is easy to check that the  $\mu_n$  form a consistent family. We use Theorem D.1 to obtain a probability measure  $\mu$  on  $[0, 1]^{\mathbb{N}}$ . To get random variables out of this, we let  $X_i(\omega) = \omega_i$  if  $\omega = (\omega_1, \omega_2, \dots)$ .



---

## References

- Aldous, D. 1978. Stopping times and tightness. *Ann. Probab.* **6**, 335–40.
- Barlow, M. T. 1982. One-dimensional stochastic differential equations with no strong solution. *J. London Math. Soc.* **26**, 335–47.
- Bass, R. F. 1983. Skorokhod imbedding via stochastic integrals. *Séminaire de Probabilités XVII*. New York: Springer-Verlag; 221–4.
- Bass, R. F. 1995. *Probabilistic Techniques in Analysis*. New York: Springer-Verlag.
- Bass, R. F. 1996. The Doob–Meyer decomposition revisited. *Can. Math. Bull.* **39**, 138–50.
- Bass, R. F. 1997. *Diffusions and Elliptic Operators*. New York: Springer-Verlag.
- Billingsley, P. 1968. *Convergence of Probability Measures*. New York: John Wiley & Sons, Ltd.
- Billingsley, P. 1971. *Weak Convergence of Measures: Applications in Probability*. Philadelphia: SIAM.
- Blumenthal, R. M. and Getoor, R. K. 1968. *Markov Processes and Potential Theory*. New York: Academic Press.
- Bogachev, V. I. 1998. *Gaussian Measures*. Providence, RI: American Mathematical Society.
- Boyce, W. E. and DiPrima, R. C. 2009. *Elementary Differential Equations and Boundary Value Problems*, 9th edn. New York: John Wiley & Sons, Ltd.
- Chung, K. L. 2001. *A Course in Probability Theory*, 3rd edn. San Diego: Academic Press.
- Chung, K. L. and Walsh, J. B. 1969. To reverse a Markov process. *Acta Math.* **123**, 225–51.
- Dawson, D. A. 1993. Measure-valued Markov processes. *Ecole d'Eté de Probabilités de Saint-Flour XXI–1991*. Berlin: Springer-Verlag.
- Dellacherie, C. and Meyer, P.-A. 1978. *Probability and Potential*. Amsterdam: North-Holland.
- Dudley, R. M. 1973. Sample functions of the Gaussian process. *Ann. Probab.* **1**, 66–103.
- Durrett, R. 1996. *Probability: Theory and Examples*. Belmont, CA: Duxbury Press.
- Ethier, S. N. and Kurtz, T. G. 1986. *Markov Processes: Characterization and Convergence*. New York: John Wiley & Sons, Ltd.
- Feller, W. 1971. *An Introduction to Probability Theory and its Applications*, 2nd edn. New York: John Wiley & Sons, Ltd.
- Folland, G. B. 1999. *Real Analysis: Modern Techniques and their Applications*, 2nd edn. New York: John Wiley & Sons, Ltd.
- Fukushima, M., Oshima, Y. and Takeda, M. 1994. *Dirichlet Forms and Symmetric Markov Processes*. Berlin: de Gruyter.
- Gilbarg, D. and Trudinger, N. S. 1983. *Elliptic Partial Differential Equations of Second Order*, 2nd edn. New York: Springer-Verlag.
- Itô, K. and McKean, Jr, H. P. 1965. *Diffusion Processes and their Sample Paths*. Berlin: Springer-Verlag.
- Kallianpur, G. 1980. *Stochastic Filtering Theory*. Berlin: Springer-Verlag.
- Karatzas, I. and Shreve, S. E. 1991. *Brownian Motion and Stochastic Calculus*, 2nd edn. New York: Springer-Verlag.
- Knight, F. B. 1981. *Essentials of Brownian Motion and Diffusion*. Providence, RI: American Mathematical Society.
- Kuo, H. H. 1975. *Gaussian Measures in Banach Spaces*. New York: Springer-Verlag.
- Lax, P. 2002. *Functional Analysis*. New York: John Wiley & Sons, Ltd.

- Liggett, T. M. 2010. *Continuous Time Markov Processes: An Introduction*. Providence, RI: American Mathematical Society.
- Meyer, P.-A., Smythe, R. T. and Walsh, J. B. 1972. Birth and death of Markov processes. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Vol. III*. Berkeley, CA: University of California Press; 295–305.
- Obłój, J. 2004. The Skorokhod embedding problem and its offspring. *Probab. Surv.* **1**, 321–90.
- Øksendal, B. 2003. *Stochastic Differential Equations: An Introduction with Applications*, 6th edn. Berlin: Springer-Verlag.
- Perkins, E. A. 2002. Dawson–Watanabe superprocesses and measure-valued diffusions. *Lectures on Probability Theory and Statistics (Saint-Flour; 1999)*. Berlin: Springer-Verlag; 125–324.
- Revuz, D. and Yor, M. 1999. *Continuous Martingales and Brownian Motion*, 3rd edn. Berlin: Springer-Verlag.
- Rogers, L. C. G. and Williams, D. 2000a. *Diffusions, Markov Processes, and Martingales, Vol. 1*. Cambridge: Cambridge University Press.
- Rogers, L. C. G. and Williams, D. 2000b. *Diffusions, Markov Processes, and Martingales, Vol. 2*. Cambridge: Cambridge University Press.
- Rudin, W. 1976. *Principles of Mathematical Analysis*, 3rd edn. New York: McGraw-Hill.
- Rudin, W. 1987. *Real and Complex Analysis*, 3rd edn. New York: McGraw-Hill.
- Skorokhod, A. V. 1965. *Studies in the Theory of Random Processes*. Reading, MA: Addison-Wesley.
- Stroock, D. W. 2003. *Markov Processes from K. Itô's Perspective*. Princeton, NJ: Princeton University Press.
- Stroock, D. W. and Varadhan, S. R. S. 1977. *Multidimensional Diffusion Processes*. Berlin: Springer-Verlag.
- Walsh, J. B. 1978. Excursions and local time. *Astérisque* **52–53**, 159–92.

---

# Index

- adapted, 1, 359  
additive functional, 169, 180  
  classical, 180  
Aldous criterion, 264  
almost surely, 348  
announce, 112  
  
Bessel processes, 200  
binomial, 349, 371  
Black–Scholes formula, 220  
Blumenthal 0–1 law, 164  
BMO, 129  
Borel–Cantelli lemma, 353, 354  
Brownian bridge, 273  
Brownian motion, 6, 153  
  covariance, 8  
  fractional, 254  
  integrated, 41  
  maximum, 27  
  standard, 6  
  with drift, 24  
  zero set, 30, 48, 99, 214, 217  
Brownian sheet, 254  
Burkholder–Davis–Gundy  
  inequalities, 82  
  
cadlag, 2  
Cameron–Martin space, 253  
canonical process, 158  
Cauchy problem, 321  
cemetary, 156, 177  
central limit theorem, 373  
chaining, 51  
change of variables formula, 71  
Chapman–Kolmogorov  
  equations, 155  
characteristic function, 370  
Chebyshev’s inequality, 352  
Chung’s law of the iterated  
  logarithm, 47  
class  $D$ , 57, 124  
class  $DL$ , 126  
closed form, 303  
closed operator, 295  
compensator, 124, 130  
  
complete filtration, 1  
conditional expectation, 357  
conditional probability, 357  
conditioned processes, 178  
consistent, 382  
construction of Brownian motion, 36, 248, 254, 284  
continuation region, 187  
continuous process, 2  
convergence  
  almost surely, 355  
  in  $L^p$ , 355  
  in distribution, 367  
  in law, 367  
  in probability, 355  
  weak, 367  
convolution semigroup, 285  
covariance, 375  
covariance matrix, 375  
covariation, 58  
cumulative normal distribution function, 227  
cylindrical set, 3  
  
 $D[0, 1]$   
  compactness, 263  
  completeness, 262  
  metrics, 259  
debut, 117  
debut theorem, 117  
density, 349  
diffusion coefficient, 193, 315  
Dirichlet boundary condition, 290  
Dirichlet form, 303  
Dirichlet problem, 320  
dissipative, 294  
distribution, 348  
distribution function, 348  
divergence form elliptic  
  operators, 307  
Donsker invariance principle, 269  
Doob decomposition, 361  
Doob’s  $h$ -path transform, 178  
Doob’s inequalities, 14, 361  
Doob–Meyer decomposition, 60, 124  
drift coefficient, 193, 315  
dual optional projection, 124

dual predictable projection, 124  
dyadic rationals, 49

empirical process, 275  
entry time, 115  
equivalent martingale measure, 223  
events, 348  
excessive, 184  
excessive majorant, 186  
exercise time, 219  
expected value, 348  
exponential, 349, 371  
martingale, 89  
semimartingale, 144  
exponential random variables, 33

Feller process, 161  
Feynman–Kac formula, 323  
filtration, 1, 2  
finite-dimensional distributions, 3  
Fourier series, 36

gamma, 350, 371  
gauge, 323  
Gaussian, 7, 374  
Gaussian field, 255  
Girsanov theorem, 89, 93, 144  
Glivenko–Cantelli theorem, 366  
good- $\lambda$  inequality, 86  
Green's function, 175  
Gronwall's lemma, 201

Hölder continuous, 43, 47  
harmonic, 173, 321  
Hausdorff dimension, 48, 99  
Hausdorff measure, 48  
heat equation, 322  
Helly's theorem, 369  
Hille–Yosida theorem, 292  
hitting time, 115  
Hunt process, 165  
Hurst index, 254

i.i.d., 364  
increasing process, 54, 121  
independent, 353  
independent increments, 6, 339  
indicator, 348  
indistinguishable, 2  
infinite particle systems, 295  
infinitesimal generator, 288  
innovation process, 230  
innovations approach, 229  
integration by parts formula, 74  
invariance principle, 108  
invariant, 178  
Itô's formula, 71  
multivariate, 74

Jensen's inequality, 352, 358  
John–Nirenberg inequality, 129  
joint characteristic function, 372  
jointly normal, 7, 375

Kalman–Bucy filter, 234  
Karhunen–Loève expansion, 253  
kernel, 154  
killed process, 177  
Kolmogorov backward equation, 291  
Kolmogorov continuity criterion, 49  
Kolmogorov extension theorem, 382  
Kolmogorov forward equation, 292  
Kunita–Watanabe inequality, 70

Lévy measure, 342  
Lévy process, 32, 297, 339  
Lévy system formula, 347  
Lévy's theorem, 77  
Lévy–Khintchine formula, 342  
last exit, 181  
law, 3, 10, 348  
law of the iterated logarithm, 44  
least excessive majorant, 186  
left continuous process, 2  
lifetime, 156, 177  
LIL, 44  
linear equations, 199  
linear model, 234  
Lipschitz function, 100, 193  
local time, 94, 209  
joint continuity, 96  
locally bounded, 141  
lower semicontinuous, 186

Markov property, 25  
Markov transition probabilities, 154  
Markovian, 303  
martingale, 13, 359  
continuous, 54  
convergence theorem, 363  
local, 54, 139  
locally square integrable, 139  
problem, 316  
representation theorem, 80, 81  
uniformly integrable, 54, 364  
maximum principle, 176  
mean, 350  
mean rate of return, 218  
measure-valued branching diffusion  
process, 317  
metric entropy, 51  
minimal augmented filtration, 2,  
160  
modulus of continuity, 247, 260  
moment, 350  
monotone class theorem, 378  
multiplication theorem, 354

- natural scale, 326  
 Neumann boundary condition, 290  
 Newtonian potential density, 175  
 NFLVR condition, 223  
 no free lunch, 223  
 nondivergence form, 296, 315  
 non-negative definite, 254  
 normal, 349, 371  
 nowhere differentiable, 46  
 null set, 1, 111
- observation process, 229  
 occupation time density, 175  
 occupation times, 97  
 one-dimensional diffusion, 326  
 optimal reward, 187  
 optimal stopping problem, 184  
 optional  $\sigma$ -field, 111  
 optional projection, 119  
 optional stopping theorem, 17, 360  
 optional time, 15  
 Ornstein–Uhlenbeck process, 159, 198  
 orthogonality lemma, 131  
 outer probability, 111
- $p$ -variation, 30, 48  
 partial sums, 364  
 paths, 2  
 paths locally of bounded variation, 54  
 Picard iteration, 101  
 Poincaré cone condition, 174  
 Poisson, 349, 371
  - point process, 147
  - process, 32, 171
 Poisson's equation, 319  
 portmanteau theorem, 237  
 potential, 155, 323  
 prévisible, 111  
 predict, 112  
 predictable, 64, 130  
 predictable  $\sigma$ -field, 64, 111  
 predictable projection, 120  
 probability, 348  
 process, 1  
 product formula, 74, 85  
 progressively measurable, 4  
 Prohorov metric, 241  
 Prohorov theorem, 239  
 purely discontinuous, 143
- quadratic variation, 57, 79  
 quasi-left continuous, 165
- random variables, 348  
 Ray–Knight theorem, 209  
 recurrence, 167  
 reduce, 139  
 reflection principle, 27
- regular, 173, 326  
 regular conditional probability, 312, 380  
 regular Dirichlet form, 307  
 reproducing property, 252  
 resolvent, 155, 286  
 reward function, 184  
 right continuous filtration, 1  
 right continuous process, 2  
 right continuous with left limits, 2
- scale function, 327  
 scaling, 7  
 Schrödinger operator, 323  
 Schwartz class, 379  
 section theorem
  - optional, 117
  - predictable, 117
 self-financing, 219  
 semigroup, 155  
 semigroup property, 155  
 semimartingale, 54, 141  
 set-indexed process, 255  
 shift operators, 158  
 signal process, 229  
 simple symmetric random walk, 109, 248  
 Skorokhod embedding, 100  
 Skorokhod representation, 245  
 Slutsky's theorem, 242, 369  
 space-time process, 182  
 spectral theorem, 309  
 speed measure, 329  
 square integrable martingale, 55  
 stable subordinator, 347  
 stationary increments, 6, 339  
 stochastic integral, 64, 134, 150
  - local martingales, 69
  - multiple, 88
  - semimartingales, 69
 stochastic process, 1  
 stopping time, 15, 359  
 Stratonovich integral, 84  
 strong Feller process, 161  
 strong law of large numbers, 364  
 strong Markov process, 165  
 strong Markov property, 25  
 strongly reduce, 139  
 sub-Markov transition probability
  - kernels, 283
 submartingale, 359  
 super-Brownian motion, 317  
 supermartingale, 359  
 support theorem, 93, 208  
 symmetric difference, 12  
 symmetric stable process, 346
- Tanaka formula, 94, 95  
 terminal time, 177  
 tight, 369

- time change, 78, 105, 180  
time inversion, 11  
totally inaccessible, 112, 130  
trading strategy, 219  
trajectories, 2  
transience, 167  
transition densities, 291  
transition probabilities, 154  
  
uniform ellipticity, 296, 307, 315  
uniformly absolutely continuous, 356  
uniformly integrable, 356  
unique in law, 204  
upcrossings, 18, 362  
usual conditions, 1  
  
variance, 350  
versions, 2  
Vitali convergence theorem,  
    357  
volatility, 218  
  
weak convergence, 367  
weak Feller process, 161  
weak solution, 204  
weak uniqueness, 204  
well posed, 316  
well measurable, 111  
Wiener measure, 6  
  
Yamada–Watanabe condition, 196