19

Framework for Markov processes

Markov process: $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}^x)$, $X0=x$, Px-a.s.

We will consider a Markov process to be a pair $(X_t, \mathbb{P}^x)$
where X_t is a single stochastic process and $\{\mathbb{P}^x\}$ is a family of probability measures, one probability measure $\mathbb{P}^x$ corresponding to each element x of the state space.

19.1 Introduction

The idea that a Markov process consists of one process and many probabilities is one that takes some getting used to. To explain this, let us first look at an example. Suppose $X_1, X_2, \ldots$ is a Markov chain with stationary transition probabilities with K states: $1, 2, \ldots, K$. Everything we want to know about X can be determined if we know $p(i, j) = \mathbb{P}(X_1 = j \mid X_0 = i)$ for each i and j and $\mu(i) = \mathbb{P}(X_0 = i)$ for each i. We sometimes think of having a different Markov chain for every choice of starting distribution $\mu = (\mu(1), \ldots, \mu(K))$. But instead let us define a new probability space by taking Ω' to be the collection of all sequences $\omega = (\omega_0, \omega_1, \ldots)$ such that each ω_n takes one of the values $1, \ldots, K$. Define $X_n(\omega) = \omega_n$. Define $\mathcal{F}_n$ to be the σ-field generated by $X_0, \ldots, X_n$; this is the same as the σ-field generated by sets of the form $\{\omega : \omega_0 = a_0, \ldots, \omega_n = a_n\}$, where $a_0, \ldots, a_n \in \{1, 2, \ldots, K\}$. For each $x = 1, 2, \ldots, K$, define a probability measure $\mathbb{P}^x$ on Ω' by

$$\begin{aligned} \mathbb{P}^x(X_0 = x_0, X_1 = x_1, \ldots X_n = x_n) \\ = 1_{\{x\}}(x_0) p(x_0, x_1) \cdots p(x_{n-1}, x_n). \end{aligned} \tag{19.1}$$

We have K different probability measures, one for each of $x = 1, 2, \ldots, K$, and we can start with an arbitrary probability distribution μ if we define $\mathbb{P}^\mu(A) = \sum_{i=1}^{K} \mathbb{P}^i(A)\mu(i)$. We have lost no information by this redefinition, and it turns out this works much better when doing technical details. $X* -> X$

The value of $X_0(\omega) = \omega_0$ can be any of $1, 2, \ldots, K$; the notion of starting at x is captured by $\mathbb{P}^x$, not by X_0. The probability measure $\mathbb{P}^x$ is concentrated on those ω's for which $\omega_0 = x$ and $\mathbb{P}^x$ gives no mass to any other ω.

Let us now look at Brownian motion, and see how this framework plays out there. Let $\mathbb{P}$ be a probability measure and let W_t be a D1 Brownian motion wrt $\mathbb{P}$ started at 0.
Then $W^x{}_t = x + W_t$ is a D1 Brownian motion started at x. Let $\Omega' = C[0, \infty)$

(We do not require that $\omega(0) = 0$ or that $\omega(0)$ take any particular value of x.) Define

$$\boxed{X_t(\omega) = \omega(t).} \qquad (19.2)$$

This will be our process. Let $\mathcal{F}$ be the σ-field on $\Omega' = C[0,\infty)$ generated by the cylindrical subsets of $C[0,\infty)$; define $\mathbb{P}^x$ to be the law of W^x:

$$\mathbb{P}^x(X \in A) = \mathbb{P}(W^x \in A), \qquad x \in \mathbb{R}, A \in \mathcal{F}. \qquad (19.3)$$

$\mathbb{P}^x$ is determined by the fact that if $n \geq 1$, $t_1 \leq \cdots \leq t_n$, and $B_1, \ldots, B_n$ are Borel subsets of $\mathbb{R}$, then

$$\mathbb{P}(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n) = \mathbb{P}(W^x_{t_1} \in B_1, \ldots, W^x_{t_n} \in B_n).$$

We call the pair $(X_t, \mathbb{P}^x)$, $x \in \mathbb{R}$, $t \geq 0$, a *Brownian motion*.

19.2 Definition of a Markov process

We want to allow our Markov processes to take values in spaces other than the Euclidean ones. For now, we take our state space $\mathcal{S}$ to be a separable metric space, furnished with the Borel σ-field. For the beginner, just think of $\mathbb{R}$ in place of $\mathcal{S}$.

To define a Markov process, we start with a measurable space $(\Omega, \mathcal{F})$ and suppose we have a filtration $\{\mathcal{F}_t\}$ (not necessarily satisfying the usual conditions).

Definition 19.1 A *Markov process* $(X_t, \mathbb{P}^x)$ is a stochastic process, adapt to Ft

$$X : [0,\infty) \times \Omega \to \mathcal{S}$$

and a family of probability measures $\{\mathbb{P}^x : x \in \mathcal{S}\}$ on $(\Omega, \mathcal{F})$ satisfying the following.
transition proba.

(1) For each t and each Borel subset A of $\mathcal{S}$, the map $x \to \mathbb{P}^x(X_t \in A)$ is Borel measurable.

(2) For each $s, t \geq 0$, each Borel subset A of $\mathcal{S}$, and each $x \in \mathcal{S}$, we have

$$\mathbb{P}^x(X_{s+t} \in A \mid \mathcal{F}_s) = \mathbb{P}^{X_s}(X_t \in A), \qquad \mathbb{P}^x - \text{a.s.} \qquad (19.4)$$

Ex(f(Xs+t)|Fs) = E{Xs}(f(Xt)), f: Cb (19.4*)

==> Px(Xt ∈A|F0) = P{x}(Xt∈A) (X0) (10.4-o)

$$\text{Let} \quad \varphi(x) := \mathbb{P}^x(X_t \in A), \qquad (19.5)$$

each $\mathcal{F}_t \subset \mathcal{F}$. $X_t : \mathcal{F}_t$ measurable ; φ is Borel measurable.

$\mathbb{P}^{X_s}(X_t \in A)$ in (19.4) is a rv: $\omega \in \Omega \mapsto \varphi(X_s(\omega))$.

Definition 19.1(2): for each s,t, each A, and each x, there is a null set $N_{s,t,x,A} \subset \Omega$ wrt $\mathbb{P}^x$ and for $\omega \notin N_{s,t,x,A}$, the conditional expectation $\mathbb{P}x(X_{s+t} \in A \mid \mathcal{F}_s) == \varphi(X_s)$.

Pt+s(x, A ∩ B) = ∫BPt(y,A) P(x,dy), B ~ Bo(S) (10.4+)

notation We use $\mathbb{E}^x$ for expectation wrt $\mathbb{P}^x$. As with $\mathbb{P}^{X_s}(X_t \in A)$, the notation $\mathbb{E}^{X_s} f(X_t)$, where f is bounded and Borel measurable, is to be taken to mean $\psi(X_s)$ with $\psi(x) = \mathbb{E}^x f(X_t)$.

If we want to talk about our Markov process started with distribution μ, we define

$$\mathbb{P}^\mu(B) := \int \mathbb{P}^x(B)\,\mu(dx),$$

and similarly for $\mathbb{E}^\mu$; here μ is a probability on $\mathcal{S}$.

Extenstion:
(.) ==> (x) ==> (A) ==> (f/X) ==> (μ/P)

19.3 Transition probabilities

If $\mathcal{B}$ is the Borel σ-field on a metric space $\mathcal{S}$, a *kernel* $Q(x, A)$ on $\mathcal{S}$ is a map from $\mathcal{S} \times \mathcal{B} \to \mathbb{R}$:

(1) For each $x \in \mathcal{S}$, $Q(x, \cdot)$ is a measure on $(\mathcal{S}, \mathcal{B})$. *transition measure*

(2) For each $A \in \mathcal{B}$, the function $x \to Q(x, A)$ is Borel measurable.

Definition 19.2 A collection of kernels $\{P_t(x, A); t \geq 0\}$ are ***Markov transition probabilities*** for a Markov process $(X_t, \mathbb{P}^x)$ if push forward

Pt(x, .): Probability
(push-forward mea. of Px under Xt)

(2) For each $x \in \mathcal{S}$, each Borel subset A of $\mathcal{S}$, and each $s, t \geq 0$,

Chapman–Kolmogorov equations

$$P_{t+s}(x, A) = \boxed{\int_{\mathcal{S}} P_t(y, A) P_s(x, dy).} \tag{19.6}$$

(1) For each $x \in \mathcal{S}$, each Borel subset A of $\mathcal{S}$, and each $t \geq 0$, μ(dx) == μ

$$\boxed{P_t(x, A) := \mathbb{P}^x(X_t \in A).} \tag{19.7}$$

Define

$$P_t f(x) := \int f(y) P_t(x, dy) \tag{19.8}$$

$\mu(f) := \int f \mu(dx)$

when $f : \mathcal{S} \to \mathbb{R}$ is Borel measurable and either bounded or non-negative.

Lemma 19.3 *Suppose P_t are Markov transition probabilities. If f is Borel measurable and either non-negative or bounded, then $P_t f$ is non-negative (respectively, bounded) and Borel measurable and*

$$P_t f(x) = \mathbb{E}^x f(X_t), \qquad x \in \mathcal{S}. \tag{19.9}$$

□

∫μ(x)dx (A) := ∫μ(x)(A)dx
∫μ(x)dx (f) := ∫μ(x)(f)dx = dx(μ(x)(f))

(19.6) be rephrased in terms of equality of measures: for each x

$$\boxed{P_{s+t}(x, dz) = \int_{y\in\mathcal{S}} P_t(y, dz) P_s(x, dy).} \tag{19.10}$$

Multiplying (19.10) by a bounded Borel measurable function $f(z)$ and integrating gives

$$P_{s+t} f(x) = \int P_t f(y) P_s(x, dy). \tag{19.11}$$

The rhs $= P_s(P_t f)(x)$, ==>

$$P_{s+t} f(x) = P_s P_t f(x), \tag{19.12}$$

The equation (19.12) is known as the *semigroup property*.

By Lemma 19.3, P_t is a linear operator on the space of bounded Borel measurable functions on $\mathcal{S}$. (19.12) <==>

$$P_{s+t} = P_s P_t. \quad P_0 = I \tag{19.13}$$

Operators satisfying (19.13) are called a *semigroup*, and are much studied in functional analysis.

One more observation about semigroups: if we take expectations in (19.4), we obtain

$$\mathbb{P}^x(X_{s+t} \in A) = \mathbb{E}^x\Big[\mathbb{P}^{X_s}(X_t \in A)\Big].$$

The lhs is $P_{s+t} 1_A(x)$ and the rhs is

$$\mathbb{E}^x[P_t 1_A(X_s)] = P_s P_t 1_A(x),$$

and so (19.4) encodes the semigroup property.

The resolvent / λ-potential *of a semigroup* P_t

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x)\, dt, \qquad \lambda \ge 0, \quad x \in \mathcal{S}.$$

This can be recognized as the Laplace transform of P_t. By Lemma 19.3 and the Fubini theorem, we see that

$$R_\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t)\, dt.$$

Resolvents are useful because they are typically easier to work with than semigroups.

When practitioners of stochastic calculus tire of a martingale, they "stop" it. Markov process theorists are a harsher lot and they "kill" their processes. To be precise, attach an isolated point Δ to $\mathcal{S}$. Thus one looks at $\widehat{\mathcal{S}} = \mathcal{S} \cup \Delta$, and the topology on $\widehat{\mathcal{S}}$ is the one generated by the open sets of $\mathcal{S}$ and $\{\Delta\}$. Δ is called the *cemetery point*. All functions on $\mathcal{S}$ are extended to $\widehat{\mathcal{S}}$ by defining them to be 0 at Δ. At some random time ζ the Markov process is killed, which means that $X_t = \Delta$ for all $t \geq \zeta$. The time ζ is called the *lifetime* of the Markov process.

19.4 An example

Example, Brownian motion. Let X_t and $\mathbb{P}^x$ be defined by (19.2) and (19.3). Define $\mathcal{F}_t = \sigma(X_r; r \leq t)$. Observe that since, under $\mathbb{P}$, $W_t \sim \mathrm{N}(0, t)$

$$\boxed{\mathbb{P}^x(X_t \in A) = \mathbb{P}(W_t^x \in A)} = \mathbb{P}(x + W_t \in A) \qquad (19.14)$$
$$= \frac{1}{\sqrt{2\pi t}} \int_A e^{-(y-x)^2/2t}\, dy.$$

By DCT, $x \to \mathbb{P}^x(X_t \in A)$ is continuous/ measurable.

Proposition 19.4 *Let W be a Brownian motion as defined by Definition 2.1, let $W_t^x = x + W_t$, and let $(X_t, \mathbb{P}^x)$ be defined by (19.2) and (19.3). If f is bounded and Borel measurable,*

$$\boxed{\mathbb{E}^x[f(X_{t+s}) \mid \mathcal{F}_s] = \mathbb{E}^{X_s} f(X_t), \qquad \mathbb{P}^x\text{-a.s.}} \qquad (19.15)$$

Proof We will first prove

test basis:

$$\mathbb{E}^x[f(X_{t+s}) \mid \mathcal{F}_s] = \mathbb{E}^{X_s} f(X_t) \qquad (19.16)$$

when $\boxed{f(x) = e^{iux}}$. Using independent increments and the fact that $W_{t+s} - W_s$ has the same law as W_t, we see that under each $\mathbb{P}^x$, $X_{t+s} - X_s$ is independent of $\mathcal{F}_s$ and has the same law as a mean zero normal rv with variance t. We conclude that

$$\mathbb{E}^x e^{iu(X_{t+s} - X_s)} = e^{-u^2 t/2};$$

see (A.25). We then write

$$\mathbb{E}^x\Big[e^{iuX_{t+s}} | \mathcal{F}_s\Big] = \mathbb{E}^x\Big[e^{iu(X_{t+s} - X_s)} | \mathcal{F}_s\Big] e^{iuX_s}$$
$$= \mathbb{E}^x\Big[e^{iu(X_{t+s} - X_s)}\Big] e^{iuX_s}$$
$$= e^{-u^2 t/2} e^{iuX_s}.$$

1. Ex(f(Xt+s)|Fs) = h(Xs)
2. Exf(Xt) = h(x)

=h(Xt)

On the other hand, for any y,

$$\mathbb{E}^y e^{iuX_t} = \mathbb{E}\, e^{iuW_t^y} = \mathbb{E}\, e^{iuW_t} e^{iuy} = e^{-u^2 t/2} e^{iuy}.$$ =h(y)

It proves (19.16) for this f. /

Now suppose that $f \in C^\infty$ with compact support and let $\widehat{f}$ be the Fourier transform of f. In (19.16) we replace u by $-u$, multiply both sides by $\widehat{f}(u)$, and integrate over $u \in \mathbb{R}$. Using the Fourier inversion formula, we then have

$$\begin{aligned}
\mathbb{E}^x[f(X_{t+s}) \mid \mathcal{F}_s] &= (2\pi)^{-1}\mathbb{E}^x\Big[\int e^{-iuX_{t+s}}\widehat{f}(u)\,du \mid \mathcal{F}_s\Big] \\
&= (2\pi)^{-1}\mathbb{E}^{X_s}\int e^{-iuX_t}\widehat{f}(u)\,du \\
&= \mathbb{E}^{X_s}f(X_t).
\end{aligned}$$

We used the Fubini theorem several times to interchange expectation and integration; this is justified because f in C^∞ with compact support implies $\widehat{f}$ is in the Schwartz class; see Section B.2. This proves the proposition for f in C^∞ with compact support, and a limit argument gives it for all bounded and measurable f. □

The same proof works for d-dimensional Brownian motion.

Set

$$P_t(x,A) := \mathbb{P}^x(X_t \in A) = \mathbb{P}(W_t + x \in A) = \frac{1}{\sqrt{2\pi t}}\int_A e^{-(y-x)^2/2t}\,dy. \tag{19.17}$$

Clearly for each x and t, $P_t(x,\cdot)$ is a measure with total mass 1. As we mentioned earlier, the function $x \to P_t(x,A)$ is continuous, hence Borel measurable. We will show the Chapman–Kolmogorov equations. These follow from the next proposition.

Proposition 19.5 *If $s,t > 0$ and $x,z \in \mathbb{R}$, then*

$$\begin{aligned}
\int_{y\in\mathbb{R}} \frac{1}{\sqrt{2\pi t}}e^{-(y-x)^2/2t}\frac{1}{\sqrt{2\pi s}}e^{-(z-y)^2/2s}\,dy \\
= \frac{1}{\sqrt{2\pi(s+t)}}e^{-(z-x)^2/2(s+t)}.
\end{aligned} \tag{19.18}$$

Proof This is a well-known property of the Gaussian density, but we can derive (19.18) from Proposition 19.4. Let f be continuous with compact support. Taking expectations in (19.15),

$$\boxed{\begin{aligned}\mathbb{E}^x f(X_{t+s}) = \mathbb{E}^x[\mathbb{E}^{X_s}f(X_t)],\\[2em] P_{t+s}f(x) = P_sP_tf(x).\end{aligned}}$$

or

Using Lemma 19.3 and (19.17),

$$\begin{aligned}
&\int f(x)\frac{1}{\sqrt{2\pi(s+t)}}e^{-(z-x)^2/2(s+t)}\,dx \\
&\quad = \int f(x)\int \frac{1}{\sqrt{2\pi t}}e^{-(y-x)^2/2t}\frac{1}{\sqrt{2\pi s}}e^{-(z-y)^2/2s}\,dy\,dx.
\end{aligned}$$

Since this holds for all continuous f with compact support, (19.18) holds for a.e. x. Since both sides of (19.18) are continuous in x, then (19.18) holds for all x. □

lemma Pt(x,dy) = pt(x,y)dy ==> ∫ps(x,z)pt(z,y)dz = ps+t(x,y)

19.5 The canonical process and shift operators

Suppose we have a Markov process $(X_t, \mathbb{P}^x)$ where $\mathcal{F}_t = \sigma(X_s; s \leq t)$. Suppose for the moment that X_t has continuous paths. For this to even make sense, it is necessary that the set $\{t \to X_t$ is not continuous$\}$ to be in $\mathcal{F}$, and then we require this event to be $\mathbb{P}^x$-null for each x. Define $\widetilde{\Omega}$ to be the set of continuous functions on $[0, \infty)$. If $\widetilde{\omega} \in \widetilde{\Omega}$, set $\widetilde{X}_t = \widetilde{\omega}(t)$. Define $\widetilde{\mathcal{F}}_t = \sigma(\widetilde{X}_s; s \leq t)$ and $\widetilde{\mathcal{F}}_\infty = \vee_{t\geq 0}\widetilde{\mathcal{F}}_t$. Finally define $\widetilde{\mathbb{P}}^x$ on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}_\infty)$ by $\widetilde{\mathbb{P}}^x(\widetilde{X} \in \cdot) = \mathbb{P}^x(X \in \cdot)$. Thus $\widetilde{\mathbb{P}}^x$ is specified uniquely by

$$\widetilde{\mathbb{P}}^x(\widetilde{X}_{t_1} \in A_1, \ldots, \widetilde{X}_{t_n} \in A_n) = \mathbb{P}^x(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n)$$

for $n \geq 1$, $A_1, \ldots, A_n$ Borel subsets of $\mathcal{S}$, and $t_1 < \cdots < t_n$. Clearly there is so far no loss (or gain) by looking at the Markov process $(\widetilde{X}_t, \widetilde{\mathbb{P}}^x)$, which is called the *canonical process*.

Suppose we are working with the canonical process, and drop the tildes everywhere.

Define the *shift operators* $\theta_t : \Omega \to \Omega$. $\theta_t(\omega) \in \Omega$: continuous $[0, \infty)$ to $\mathcal{S}$.

$$\theta_t(\omega)(s) := \omega(t + s).$$

Then

$$\boxed{X_s \circ \theta_t(\omega) = X_s(\theta_t(\omega)) = \theta_t(\omega)(s) = \omega(t + s) = X_{t+s}(\omega).}$$

The shift operator θ_t takes the path of X and chops off and discards the part of the path before time t.

We will use expressions like $f(X_s) \circ \theta_t$:

$$(f(X_s) \circ \theta_t)(\omega) = f(X_s(\theta_t(\omega))) = f(X_{s+t}(\omega)),$$
$$\text{or } f(X_s) \circ \theta_t = f(X_{s+t}).$$

If the paths of X are not continuous, but instead only RCLL, we can follow exactly the above procedure, except we start with $\Omega^\sim$ being the collection of RCLL functions from $[0, \infty)$ to $\mathcal{S}$.

Even if we are not in this canonical setup, from now on we will suppose there exist shift operators mapping Ω into itself so that

$$\boxed{X_s \circ \theta_t = X_{s+t}.}$$

Exercises

19.1 Suppose $(X_t, \mathbb{P}^x)$ is a Brownian motion and $S_t = \sup_{s\leq t} X_s$. Show that $((X_t, S_t), \mathbb{P}^x)$ is a Markov process and determine the transition probabilities.

19.2 Suppose $(X_t, \mathbb{P}^x)$ is a Brownian motion, f a non-negative, bounded, Borel measurable function, and $A_t = \int_0^t f(X_s)\, ds$. Show that $((X_t, A_t), \mathbb{P}^x)$ is a Markov process.

19.3 Suppose P_t is a Poisson process with parameter λ. Let Ω' be the collection of functions on $[0, \infty)$ which are right continuous and which have left limits, let $\mathcal{F}$ be the σ-field on Ω' generated by the cylindrical subsets of Ω', let $P_t^x = x + P_t$, and let $\mathbb{P}^x$ be the law of $x + P$. Show that $(X_t, \mathbb{P}^x)$ is a Markov process and determine the transition probabilities.

19.4 Suppose m is a measure on the Borel subsets $\mathcal{B}$ of a metric space $\mathcal{S}$. Suppose for each $t > 0$ there exist jointly measurable non-negative functions $p_t : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ such that $\int p_t(x, y)\, m(dy) = 1$ for each x and t and define

$$P_t(x, A) = \int_A p_t(x, y)\, m(dy).$$

Show that the kernels P_t satisfy the Chapman–Kolmogorov equations iff

$$\int p_s(x, y) p_t(y, z)\, m(dy) = p_{s+t}(x, z)$$

for every $s, t \geq 0$, every $x \in \mathcal{S}$, and m-a.e. z.

19.5 The Ornstein–Uhlenbeck process Y started at x is a continuous Gaussian process with $\mathbb{E}\, Y_t = e^{-t/2} x$ and covariance

$$\mathrm{Cov}\,(Y_s, Y_t) = e^{-(s+t)/2}(e^{s \wedge t} - 1).$$

If X is the canonical process and $\mathbb{P}^x$ is the law of an Ornstein–Uhlenbeck process started at x, show that $(X_t, \mathbb{P}^x)$ is a Markov process and determine the transition probabilities.

Notes

For more, see Blumenthal and Getoor (1968).

20

Markov properties

1, we want to talk about what it means in the Markov process context for a filtration to satisfy the usual conditions . This is now more complicated than in Chapter 1 because we have more than one probability measure . 2, we want to extend the Markov property to expressions that are more complicated than $\mathbb{E}^x[f(X_{s+t}) \mid \mathcal{F}_s]$. 3, we want to look at the strong Markov property, which means we look at expressions like $\mathbb{E}^x[f(X_{T+t}) \mid \mathcal{F}_T]$, where T is a stopping time.

Assume that X has paths that are RCLL. To be more precise, if

$$N = \{\omega : \text{ the function } t \to X_t(\omega) \text{ is not RCLL}\},$$

then $N \in \mathcal{F}$ and N is $\mathbb{P}_x$-null for every $x \in \mathcal{S}$.

20.1 Enlarging the filtration

Notation. Define

$$\mathcal{F}_t^{00} = \sigma(X_s; s \leq t), \qquad t \geq 0. \tag{20.1}$$

This is the smallest σ-field wrt which each X_s is measurable for $s \leq t$. We let $\mathcal{F}_t^0$ be the completion of $\mathcal{F}_t^{00}$, but we need to be careful what we mean by completion here, because we have more than one probability measure present. Let $\mathcal{N}$ be the collection of sets that are $\mathbb{P}^x$-null for every $x \in \mathcal{S}$. Thus $N \in \mathcal{N}$ if $(\mathbb{P}^x)^*(N) = 0$ for each $x \in \mathcal{S}$, where $(\mathbb{P}^x)^*$ is the outer probability corresponding to $\mathbb{P}^x$. The outer probability

$$(\mathbb{P}^x)^*(S) := \inf\{\mathbb{P}^x(B) : A \subset B, B \in \mathcal{F}\}.$$

Let

$$\mathcal{F}_t^0 = \sigma(\mathcal{F}_t^{00} \cup \mathcal{N}). \tag{20.2}$$

Finally, let

$$\mathcal{F}_t = \mathcal{F}_{t+}^0 = \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}^0. \tag{20.3}$$

We call $\{\mathcal{F}_t\}$ the *minimal augmented filtration* generated by X. Ultimately, we will work only with $\{\mathcal{F}_t\}$, but we need the other two filtrations at intermediate stages. The reason for worrying about which filtrations to use is that $\{\mathcal{F}_t^{00}\}$ is too small to include many interesting sets (such as those arising in the law of the iterated logarithm, for example), while if the filtration is too large, the Markov property will not hold for that filtration.

The filtration matters when defining a Markov process; see Definition 19.1(3). We will **assume** throughout this section that $(X_t, \mathbb{P}^x)$ is a Markov process wrt the filtration $\{\mathcal{F}_t^{00}\}$,

$$\boxed{\mathbb{P}^x(X_{s+t} \in A \mid \mathcal{F}_s^{00}) = \mathbb{P}^{X_s}(X_t \in A), \qquad \mathbb{P}^x\text{-a.s.}} \tag{20.4}$$

whenever A is a Borel subset of $\mathcal{S}$ and $s, t \geq 0$.

We will also make the following assumption, which will be needed here and also in Section 20.3.

Assumption 20.1 *Suppose $P_t f$: Cb on $\mathcal{S}$ whenever f:Cb on $\mathcal{S}$.*

Markov processes satisfying Assumption 20.1 are called (*weak*) *Feller processes*. If $P_t f$ is continuous whenever f is bounded and Borel measurable, then the Markov process is said to be a *strong Feller process*.

We show that we can replace $\mathcal{F}_t^{00}$ in (20.4) by $\mathcal{F}_t^0$.

Proposition 20.2 *Let $(X_t, \mathbb{P}^x)$ be a Markov process and suppose that (20.4) holds. If A is a Borel subset of $\mathcal{S}$, $x \in \mathcal{S}$, and $s, t \geq 0$, then*

$$\mathbb{P}^x(X_{s+t} \in A \mid \mathcal{F}_s^0) = \mathbb{P}^{X_s}(X_t \in A), \qquad \mathbb{P}^x\text{-a.s.} \tag{20.5}$$

Proof Since the rhs is a function of X_s and hence $\mathcal{F}_s^0$ measurable, we need to show that if $B \in \mathcal{F}_s^0$, then

$$\boxed{\mathbb{P}^x(X_{s+t} \in A, B) = \mathbb{E}^x\Big[\mathbb{P}^{X_s}(X_t \in A); B\Big].} \tag{20.6}$$

This holds for $B \in \mathcal{F}_s^{00}$ by (20.4). It holds for sets $B \in \mathcal{N}$, the class of null sets, since both sides are 0. Therefore it holds for sets B such that there exists $B_1 \in \mathcal{F}_s^{00}$ with $B \Delta B_1$ being a null set. By linearity it holds for finite disjoint unions of sets of the form just described. The class of such finite disjoint unions is a monotone class that generates $\mathcal{F}_s^0$, and our result follows by the monotone class theorem, Theorem B.2. □

The next step is to go from $\mathcal{F}_s^0$ to $\mathcal{F}_s$.

Proposition 20.3 *Let $(X_t, \mathbb{P}^x)$ be a Markov process and suppose that (20.4) holds. If Assumption 20.1 holds and f is a bounded Borel measurable function, then*

$$\mathbb{E}^x[f(X_{s+t}) \mid \mathcal{F}_s] = \mathbb{E}^{X_s} f(X_t), \qquad \mathbb{P}^x\text{-a.s.} \tag{20.7}$$

It will turn out (see Proposition 20.7 below) that $\mathcal{F}_s^0 = \mathcal{F}_s$, but we do not know this yet.

Proof We start with (20.5). By linearity, we have monotone class thm for functions

$$\mathbb{E}^x[f(X_{s+t}) \mid \mathcal{F}_s^0] = \mathbb{E}^{X_s} f(X_t), \qquad \mathbb{P}^x\text{-a.s.}, \tag{20.8}$$

when f is simple, then by monotone convergence when f is non-negative, and then by linearity again, when f is bounded and Borel measurable, in particular, when f: Cb.

If $B \in \mathcal{F}_s = \mathcal{F}^0_{s+}$, then $B \in \mathcal{F}^0_{s+\varepsilon}$ for every $\varepsilon > 0$. Hence by (20.8) with s replaced by $s+\varepsilon$, if f: Cb,

$$\begin{aligned}\mathbb{E}^x[f(X_{s+t+\varepsilon}); B] &= \mathbb{E}^x\Big[\mathbb{E}^{X_{s+\varepsilon}} f(X_t); B\Big]\\ &= \mathbb{E}^x[P_t f(X_{s+\varepsilon}); B];\end{aligned} \tag{20.9}$$

since $P_t f$ is continuous and X_t has paths that are right continuous with left limits, this converges to

$$\mathbb{E}^x[P_t f(X_s); B] = \mathbb{E}^x\Big[\mathbb{E}^{X_s} f(X_t); B\Big]$$

by dominated convergence . The lhs of (20.9) converges , using dominated convergence , the continuity of f, and the fact that X has paths that are right continuous with left limits, to

$$\mathbb{E}^x[f(X_{s+t}); B].$$

We therefore have

$$\mathbb{E}^x[f(X_{s+t}); B] = \mathbb{E}^x\Big[\mathbb{E}^{X_s} f(X_t); B\Big]. \tag{20.10}$$

A **limit argument** shows this holds whenever f is bounded and measurable.

□

Remark 20.4 In Chapter 16, we discussed the fact that the first time a right continuous process whose jump times are totally inaccessible hits a Borel set is a stopping time, provided the filtration satisfies the usual conditions. Even though the notion of completion of a filtration is a bit different in the context of Markov processes, the result is still true. See Blumenthal and Getoor (1968).

20.2 The Markov property

We start with the Markov property given by Proposition 20.3:

$$\mathbb{E}^x[f(X_{s+t}) \mid \mathcal{F}_s] = \mathbb{E}^{X_s}[f(X_t)], \qquad \mathbb{P}^x\text{-a.s.} \tag{20.11}$$

Since $f(X_{s+t}) = f(X_t) \circ \theta_s$, if we write Y for $f(X_t)$, we have

$$\mathbb{E}^x[Y \circ \theta_s \mid \mathcal{F}_s] = \mathbb{E}^{X_s} Y, \qquad \mathbb{P}^x\text{-a.s.} \tag{20.12}$$

We wish to generalize this to other rvs Y. Ex(Y0s; B) = Ex(E{Xs}Y;B) , for all B: Fs

Proposition 20.5 *Let* $(X_t, \mathbb{P}^x)$ *be a Markov process and suppose* (20.11) *holds. Suppose* $Y = \prod_{i=1}^n f_i(X_{t_i - s})$, *where the* f_i *are bounded, Borel measurable, and* $s \le t_1 \le \cdots \le t_n$. *Then* (20.12) *holds.*

Proof We will prove this by induction on n. The case $n = 1$ is (20.11), so we suppose the equality holds for n and prove it for $n+1$.

Let $V = \prod_{j=2}^{n+1} f_j(X_{t_j - t_1})$ and $h(y) = \mathbb{E}^y V$. By the induction hypothesis,

$$\mathbb{E}^x\Big[\boxed{\prod_{j=1}^{n+1} f_j(X_{t_j})}\Big|\mathcal{F}_s\Big] = \mathbb{E}^x\Big[\boxed{\mathbb{E}^x[V\circ\theta_{t_1}|\mathcal{F}_{t_1}] f_1(X_{t_1})}\Big|\mathcal{F}_s\Big]$$
$$= \mathbb{E}^x\Big[(\mathbb{E}^{X_{t_1}} V) f_1(X_{t_1})\Big|\mathcal{F}_s\Big]$$
$$= \mathbb{E}^x[(hf_1)(X_{t_1})|\mathcal{F}_s].$$
$$= \mathbb{E}^{X_s}[(hf_1)(X_{t_1-s})].$$
$$\mathbb{E}^y[(hf_1)(X_{t_1-s})] = \mathbb{E}^y[(\mathbb{E}^{X_{t_1-s}} V) f_1(X_{t_1-s})]$$
$$= \mathbb{E}^y\Big[\mathbb{E}^y[V\circ\theta_{t_1-s}|\mathcal{F}_{t_1-s}] f_1(X_{t_1-s})\Big]$$
$$= \mathbb{E}^y[(V\circ\theta_{t_1-s}) f_1(X_{t_1-s})].$$

If we replace V by its definition, replace y by X_s, and use the definition of θ_{t_1-s}, we get the desired equality for $n+1$ and hence the induction step. □

We now come to the general version of the Markov property. As usual, $\mathcal{F}_\infty = \vee_{t\ge 0}\mathcal{F}_t$. The expression $Y\circ\theta_t$ for general Y may seem puzzling at first.

$\cup_t F_t$

Theorem 20.6 *Let $(X_t, \mathbb{P}^x)$ be a Markov process and suppose (20.11) holds. Suppose Y is bounded and measurable wrt $\mathcal{F}_\infty$. Then*

$$\boxed{\mathbb{E}^x[Y\circ\theta_s \mid \mathcal{F}_s] = \mathbb{E}^{X_s} Y,} \qquad \mathbb{P}^x\text{-a.s.} \tag{20.13}$$

Proof If in Proposition 20.5 we take $f_j(x) = 1_{A_j}(x)$ for Borel measurable A_j, we have

$$\text{Ft} <: \ \{B:\ \mathbb{E}^x[1_B\circ\theta_s \mid \mathcal{F}_s] = \mathbb{E}^{X_s} 1_B\ \} \tag{20.14}$$

when $\boxed{B = \{\omega : \omega(t_1)\in A_1, \dots, \omega(t_n)\in A_n\}}$. It is easy to see that the set of B's for which (20.14) holds is a monotone class. By an argument using the monotone class theorem, (20.14) holds for all B that are measurable wrt $\mathcal{F}_\infty$. Taking linear combinations, (20.13) holds for Y's that are simple rvs. Using monotone convergence, (20.13) holds for non-negative Y's, and then by linearity for bounded Y's. monotone class thm for functions □

Monotone argument

Proposition 20.7 *Let $(X_t, \mathbb{P}^x)$ be a Markov process wrt $\{\mathcal{F}_t\}$. Let $\mathcal{F}_t^0$ and $\mathcal{F}_t$ be defined by (20.2) and (20.3). Then $\mathcal{F}_t = \mathcal{F}_t^0$ for each $t \ge 0$.*

Proof Let $Y_1 = \prod_{i=1}^n f_i(X_{t_i})$ and $Y_2 = \prod_{j=1}^m g_j(X_{u_j})$, where $t_1 < \cdots < t_n \le s$ and $0 \le u_1 < \cdots < u_m$ and the f_j and g_j are bounded Borel measurable functions. Then by Proposition 20.5,

$$\boxed{\mathbb{E}^x[(Y_1)(Y_2\circ\theta_s) \mid \mathcal{F}_s] = Y_1\mathbb{E}^{X_s} Y_2.}$$

Since $\mathbb{E}^{X_s} Y_2$ is a function of X_s, then $(Y_1)(\mathbb{E}^{X_s} Y_2)$ is $\mathcal{F}_s^0$ measurable. Using a monotone class argument, we conclude that if Y is bounded and $\mathcal{F}_\infty$ measurable, then $\mathbb{E}^x[Y \mid \mathcal{F}_s]$ is $\mathcal{F}_s^0$ measurable.

Now apply this to $Y = 1_A$ for $A \in \mathcal{F}_s$ to obtain that $1_A = \mathbb{E}^x[1_A \mid \mathcal{F}_s]$ is $\mathcal{F}_s^0$ measurable.

□

Blumenthal 0–1 law

Proposition 20.8 *Let $(X_t, \mathbb{P}^x)$ be a Markov process wrt $\{\mathcal{F}_t\}$. If $A \in \mathcal{F}_0$, then for each x, $\mathbb{P}^x(A) = 0$ or 1.*

Proof Suppose $A \in \mathcal{F}_0$. Under $\mathbb{P}^x$, $X_0 = x$, a.s., and then

$$\boxed{\mathbb{P}^x(A) = \mathbb{E}^{X_0} 1_A} = \mathbb{E}^x[1_A \circ \theta_0 \mid \mathcal{F}_0] = 1_A \circ \theta_0 = 1_A \in \{0, 1\}, \qquad \mathbb{P}^x\text{-a.s.}$$

since $1_A \circ \theta_0$ is $\mathcal{F}_0$ measurable. Our result follows because $\mathbb{P}^x(A)$ is a real number and not random.

P{x}(X0∈A)(X0) = P0(X0, A) = 1(X0∈A), a.s.

□

20.3 Strong Markov property

Given a stopping time T, recall that the σ-field of events known up to time T:

$$\mathcal{F}_T = \big\{A \in \mathcal{F}_\infty : A \cap (T \le t) \in \mathcal{F}_t \text{ for all } t > 0\big\}.$$

We define θ_T by $\boxed{\theta_T(\omega)(t) = \omega(T(\omega) + t)}$. Thus, for example, $X_t \circ \theta_T(\omega) = X_{T(\omega)+t}(\omega)$ and $X_T(\omega) = X_{T(\omega)}(\omega)$.

strong Markov property. shift theorem

Theorem 20.9 *Suppose $(X_t, \mathbb{P}^x)$ is a Markov process wrt $\{\mathcal{F}_t\}$, that Assumption 20.1 holds, and that T is finite stopping time. If Y is bounded and measurable wrt $\mathcal{F}_\infty$, then*

$$\mathbb{E}^x[Y \circ \theta_T | \mathcal{F}_T] = \mathbb{E}^{X_T} Y, \qquad \mathbb{P}^x\text{-a.s.}$$

Proof Following the proofs of Section 20.2, it is enough to prove

$$\boxed{\mathbb{E}^x[f(X_{T+t}) | \mathcal{F}_T] = \mathbb{E}^{X_T} f(X_t), \text{ for } f \text{ bounded.}} \tag{20.15}$$

We can obtain this by a limit argument if we have (20.15) for f bounded and continuous. Define $T_n = (k+1)/2^n$ on the event $(k/2^n \le T < (k+1)/2^n)$.

If $A \in \mathcal{F}_T$, then $A \in \mathcal{F}_{T_n}$. Therefore $A \cap (T_n = k/2^n) \in \mathcal{F}_{k/2^n}$ and we have by the Markov property, Theorem 20.6,

$$\begin{aligned}
\mathbb{E}^x[f(X_{T_n+t}); A, T_n = k/2^n] &= \mathbb{E}^x[f(X_{t+k/2^n}); A, T_n = k/2^n] \\
&= \mathbb{E}^x[\mathbb{E}^{X_{k/2^n}} f(X_t); A, T_n = k/2^n] \\
&= \mathbb{E}^x[\mathbb{E}^{X_{T_n}} f(X_t); A, T_n = k/2^n].
\end{aligned}$$

$$\Longrightarrow \qquad \mathbb{E}^x[f(X_{T_n+t}); A] = \mathbb{E}^x[\mathbb{E}^{X_{T_n}} f(X_t); A].$$

Discretization Trick:

gen. decomp./discretize

∫wf(t,w) — > ∫wf(g(w),w) — > {f(g(w);Ak)} ~ {f(gk; Ak)}

(— >{f(gn; gn=gk)}, Ak={gn=gk})

Now let $n \to \infty$. $\mathbb{E}^x[f(X_{T_n+t}); A] \to \mathbb{E}^x[f(X_{T+t}); A)]$ by dominated convergence and the continuity of f and the right continuity of X_t. On the other hand, using the continuity of $P_t f$, $\mathbb{E}^{X_{T_n}} f(X_t) = P_t f(X_{T_n}) \to P_t f(X_T) = \mathbb{E}^{X_T} f(X_t)$. Therefore

$$\mathbb{E}^x[f(X_{T+t}); A] = \mathbb{E}^x[\mathbb{E}^{X_T} f(X_t); A]$$

for all $A \in \mathcal{F}_T$, and hence (20.15) holds. □

If we have a Markov process $(X_t, \mathbb{P}^x)$ whose paths RCLL which has shift operators $\{\theta_t\}$, and which satisfies the conclusion of Theorem 20.9, whether or not Assumption 20.1 holds, then we say that $(X_t, \mathbb{P}^x)$ is a *strong Markov process.*

shift theorem

A strong Markov process is said to be *quasi-left continuous* if $X_{T_n} \to X_T$, a.s., on $\{T < \infty\}$ whenever T_n are stopping times increasing up to T. Unlike in the definition of predictable stopping times given in Chapter 16, we are not requiring the T_n to be strictly less than T. A *Hunt process* is a strong Markov process that is quasi-left continuous. Quasi-left continuity does not imply left continuity; consider the Poisson process.

Proposition 20.10 *If $(X_t, \mathbb{P}^x)$ is a strong Markov process and Assumption 20.1 holds, then X_t is quasi-left continuous.*

Proof First suppose T is bounded, T_n increases to T, $\boxed{Y = \lim_{n\to\infty} X_{T_n}}$, and f and g are bounded and continuous. If $T_n = T$ for some n, then $\lim_{n\to\infty} g(X_{T_n+t}) = g(X_{T+t})$, and if $T_n < T$ for all n, then $\lim_{n\to\infty} g(X_{T_n+t}) = g(X_{(T+t)-})$, where X_{s-} is the left-hand limit at time s. In either case,

$$\lim_{t\to 0} \lim_{n\to\infty} g(X_{T_n+t}) = g(X_T).$$

Then

$$\begin{aligned}
\mathbb{E}^x[f(Y)g(X_T)] &= \lim_{t\to 0} \lim_{n\to\infty} \mathbb{E}^x[f(X_{T_n})g(X_{T_n+t})] \\
&= \lim_{t\to 0} \lim_{n\to\infty} \mathbb{E}^x[f(X_{T_n})P_t g(X_{T_n})] \\
&= \lim_{t\to 0} \mathbb{E}^x[f(Y)P_t g(Y)] = \mathbb{E}^x[f(Y)g(Y)].
\end{aligned}$$

By a limit argument we have

$$\mathbb{E}^x[h(Y, X_T)] = \mathbb{E}^x[h(Y, Y)] \tag{20.16}$$

for all bounded measurable functions h on $\mathcal{S} \times \mathcal{S}$. Now take $h(x, y) = 0$ if $x = y$ and 1, otherwise. The rhs of (20.16) = 0, so the lhs = 0 also.

Lemma. E{x}(h(Y,XT)) = 0 iff Y = XT a.s.
<== E{x}(f(Y)g(XT))=E{x}(f(Y)g(Y)) for all f, g : Cb

If T is not bounded, apply the argument in the preceding paragraph to the stopping time $T \wedge M$, where M is a positive real, and then let $M \to \infty$. $\square$

Exercises

20.1 Suppose that $\mathcal{S}$ is a locally compact separable metric space and C_0 is the set of continuous functions on $\mathcal{S}$ that vanish at infinity. To say a continuous function f vanishes at infinity means that given $\varepsilon > 0$ there exists a compact set K such that $|f(x)| < \varepsilon$ if $x \notin K$. Show that if Assumption 20.1 is replaced by the assumptions that $P_t f \in C_0$ whenever $f \in C_0$ and $P_t f \to f$ uniformly as $t \to 0$ whenever $f \in C_0$, then the conclusion of Theorem 20.9 still holds.

20.2 Suppose $(X_t, \mathbb{P}^x)$ is a Markov process wrt a filtration $\{\mathcal{F}_t\}$. Suppose that $\mathcal{E}_t \subset \mathcal{F}_t$ for each t and that X_t is $\mathcal{E}_t$ measurable for each t. Show that $(X_t, \mathbb{P}^x)$ is a Markov process wrt the filtration $\{\mathcal{E}_t\}$.

20.3 Give an example of a Markov process that is not a strong Markov process.
Hint: Let the state space be $[0, \infty)$ and starting from $x \in (0, \infty)$, let X move deterministically at constant speed to the right. Starting at 0, let X wait an exponential length of time, and then begin moving at constant speed to the right.

20.4 Let $(X_t, \mathbb{P}^x)$ be Brownian motion and let $\{\mathcal{F}_t\}$ be the minimal augmented filtration. Suppose $B \in \vee_{t \geq 0} \mathcal{F}_t$ and for some $s > 0$ is of the form $1_B = 1_A \circ \theta_s$. Show that if B is a $\mathbb{P}^x$-null set for some x, then it is a $\mathbb{P}^x$-null set for every x.

20.5 Let P_t be transition probabilities for a Poisson process with parameter λ. These are defined in Exercise 19.3. Show that Assumption 20.1 holds.

20.6 Suppose $(X_t, \mathbb{P}^x)$ is a Markov process with transition probabilities P_t, f is a bounded Borel measurable function, $t_0 > 0$, and we define $M_t = P_{t_0 - t} f(X_t)$ for $t \leq t_0$. Show that $(M_t, t \leq t_0)$ is a $\mathbb{P}^x$-martingale for each x.

20.7 Use the Blumenthal 0–1 law to show that if W is a one-dimensional Brownian motion and $T = \inf\{t > 0 : W_t > 0\}$ is the first time Brownian motion hits $(0, \infty)$, then $\mathbb{P}(T = 0) = 1$.

20.8 Let A be a Borel subset of a metric space $\mathcal{S}$. Let $T_A = \inf\{t : X_t \in A\}$, where $(X_t, \mathbb{P}^x)$ is a strong Markov process. Show that $\mathbb{P}^x(T_A = 0)$ is either 0 or 1 for each x.

20.9 Let $(X_t, \mathbb{P}^x)$ be a strong Markov process and let A be a Borel subset of $\mathcal{S}$. We define A^r by setting $A^r = \{x : \mathbb{P}^x(T_A = 0) = 1\}$, where T_A is the first hitting time of A. Thus A^r is the set of points that are regular for A. Prove that for each x,

$$\mathbb{P}^x(X_{T_A} \in A \cup A^r) = 1.$$

21

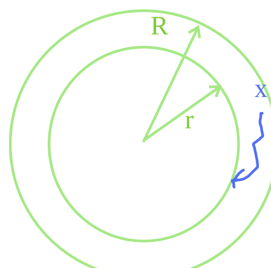
Applications of the Markov properties

1, we show that d-dimensional Brownian motion is transient if $d \geq 3$.
2, we consider estimates on additive functionals. An example of an additive functional is $A_t = \int_0^t f(X_s)\,ds$, where f is a non-negative function
3, a sufficient criterion for a Markov process to have continuous paths.
4, we discuss harmonic functions and show how to solve the classical Dirichlet problem of analysis and PDEs.

21.1 Recurrence and transience

Let $W_t = (W_1(t), \ldots, W_d(t))$ be a d-dimensional Brownian motion started at 0 with $d \geq 3$ and let $W_t^x = x + W_t$ be Brownian motion started at x. Let $h(y) = |y|^{2-d}$.

$$\Delta h(x) = \sum_{i=1}^{d} \frac{\partial^2 h}{\partial x_i^2}(x) = 0, \qquad x \neq 0.$$



By Exercise 9.4, $\langle W_i, W_j \rangle_t = 0$ if $i \neq j$ and we saw in Section 9.3 that it $= t$ if $i = j$. Suppose $\boxed{r < |x| < R,}$ and let (hitting time leave from the ring)

$$S = \inf\{t : |W_t^x| \leq r \text{ or } |W_t^x| \geq R\}.$$

S is finite, a.s., because $|W_t^x| \geq |W_1(t)| - |x|$ and $W_1(t)$ exits $[-2R, 2R]$ in finite time by Theorem 7.2. By Itô's formula,

$$\begin{aligned} h(W_{t\wedge S}^x) &= h(W_0^x) + \text{martingale} + \tfrac{1}{2} \int_0^{t\wedge S} \sum_{i=1}^{d} \frac{\partial^2 h}{\partial x_i^2}(W_s^x)\,ds \\ &= h(x) + \text{martingale}. \end{aligned}$$

Therefore $h(W_{t\wedge S}) - h(x)$ is a martingale started at 0. The function $h = r^{2-d}$ on $\partial B(0, r)$ and $= R^{2-d}$ on $\partial B(0, R)$.

By Corollary 3.17, we deduce

$$\begin{aligned}
&\mathbb{P}(W_t^x \text{ hits } B(0,r) \text{ before } B(0,R))\\
&\quad = \mathbb{P}(h(W_t^x) - h(x) \text{ hits } r^{2-d} - |x|^{2-d} \text{ before } R^{2-d} - |x|^{2-d})\\
&\quad = \frac{|x|^{2-d} - R^{2-d}}{r^{2-d} - R^{2-d}}.
\end{aligned}$$

If we let $R \to \infty$ and recall that $2 - d < 0$, we see that

$$\mathbb{P}(W_t^x \text{ ever hits } \partial B(0,r)) = \left(\frac{r}{|x|}\right)^{d-2}. \tag{21.1}$$

We want to use the strong Markov property to go from (21.1) to

$$\lim_{t\to\infty} |W_t^x| = \infty.$$

(There are other ways besides the strong Markov property of showing this.) The first step in doing this is to convert to the Markov process notation. Let $(X_t, \mathbb{P}^x)$ be a Brownian motion. What we have shown is that

$$\mathbb{P}^x(\boxed{X_t \text{ ever hits } \partial B(0,r))} = \left(\frac{r}{|x|}\right)^{d-2}. \tag{21.2}$$

Let $M > 0$ and let

$$\begin{aligned}
S_1 &= \inf\{t : |X_t| \ge 2M\},\\
T_1 &= \inf\{t > S_1 : |X_t| \le M\},\\
S_2 &= \inf\{t > T_1 : |X_t| \ge 2M\},\\
T_2 &= \inf\{t > S_2 : |X_t| \le M\}, \ldots
\end{aligned}$$

Another way

$$S := \inf\{t > 0 : |X_t| \ge 2M\}, \qquad T := \inf\{t > 0 : |X_t| \le M\},$$

and then to let $S_1 = S$, and for each $i \ge 1$,

$$T_i = S_i + T \circ \theta_{S_i}, \qquad S_{i+1} = T_i + S \circ \theta_{T_i}.$$

Explain . Given a path ω: C$[0,\infty)$ to $\mathbb{R}^d$, $T \circ \theta_{S_i}$ T(w{t+ Si}) means to proceed along the path until time S_i, disregard this piece, and then see how long it takes after time S_i to first enter $B(0,M)$. If we add the quantity S_i to $T \circ \theta_{S_i}$, we then get the amount of time for X_t to first enter $B(0,M)$ after time S_i. Thus T_i =inf$\{t > S_i : X_t \in B(0,M)\}$. The shift notation interpretation of S_{i+1} is similar.

Now we can apply the strong Markov property. Since $T_{i+1} = S_{i+1} + T \circ \theta_{S_{i+1}}$, we can write

$$\begin{aligned}
\mathbb{P}^x(T_{i+1} < \infty) &= \mathbb{P}^x(S_{i+1} < \infty, T \circ \theta_{S_{i+1}} < \infty)\\
&= \mathbb{E}^x\Big[\mathbb{P}^x(T \circ \theta_{S_{i+1}} < \infty \mid \mathcal{F}_{S_{i+1}}); S_{i+1} < \infty\Big]\\
&= \mathbb{E}^x\Big[\mathbb{P}^{X_{S_{i+1}}}(T < \infty); S_{i+1} < \infty\Big].
\end{aligned}$$

At time S_{i+1}, we have $|X_{S_{i+1}}| = 2M$, and by (21.1)

$$\mathbb{P}^{X_{S_{i+1}}}(T < \infty) = (\tfrac{1}{2})^{d-2}.$$

Therefore

$$\mathbb{P}^x(T_{i+1} < \infty) \le 2^{2-d}\mathbb{P}^x(S_{i+1} < \infty) \le 2^{2-d}\mathbb{P}^x(T_i < \infty).$$

The last inequality is simply the fact that $S_{i+1} \ge T_i$. Since $\mathbb{P}^x(T_1 < \infty) \le 1$, induction tells us that

$$\mathbb{P}^x(T_i < \infty) \le 2^{(2-d)(i-1)} \to 0$$

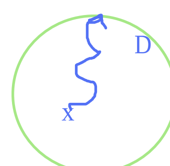
as $i \to \infty$. Hence $\mathbb{P}^x(T_i < \infty$ for all $i) = 0$. Since T_i increases as i increases, a.s. T_i will be infinite for i sufficiently large (how large will depend on ω). Hence X_t returns to $B(0, M)$ for a last time, a.s. Since M is arbitrary, this proves that $X_t \to \infty$ as $t \to \infty$.
===>

Proposition 21.1 *If $(X_t, \mathbb{P}^x)$ is a d-dimensional Brownian motion and $d \ge 3$, then $|X_t| \to \infty$ as $t \to \infty$ with $\mathbb{P}^x$-probability one for each x.*

21.2 Additive functionals

Let D be a closed subset of $\mathcal{S}$, let $f : D \to [0, \infty)$, let $S = \tau_D$, and let

$$A = \sup_{x\in D} \mathbb{E}^x \int_0^S f(X_s)\, ds,$$



where $\tau_D = \inf\{t > 0 : X_t \notin D\}$ is the first time X exits D.

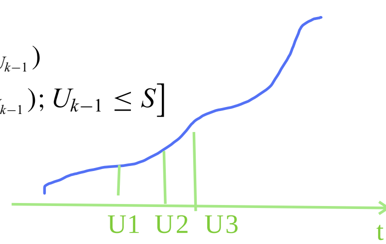
Proposition 21.2 *If $A < \infty$, then*

$$\mathbb{P}^x\left(\boxed{\int_0^S f(X_s)ds \ge 2kA}\right) \le 2^{-k}. \tag{21.3}$$

Remark. as soon as one gets a bound on the expectation, although it must be uniform in x, one gets exponential tails for the distribution . A use of Chebyshev 's inequality would only give the bound $(2k)^{-1}$.

Proof Let $\boxed{B_t = \int_0^{t\wedge S} f(X_s)\, ds}$. This is a special case of what is known as an additive functional; see Section 22.3. Let $U_1 = \inf\{t : B_t \ge 2A\}$, and let $\boxed{U_{i+1} = U_i + U_1 \circ \theta_{U_i}}$. To explain this formula, composing ω with θ_{U_i} means we disregard the path before time U_i. Thus $U_1 \circ \theta_{U_i}$ is the length of time after time U_i until B_t has increased an amount $2A$ over its value at U_i. Therefore $U_i + U_1 \circ \theta_{U_i}$ is the $(i+1)$st time B has increased by $2A$. The event $\mathbb{P}^x(B_S \ge 2kA)$ is bounded by

$$\begin{aligned}\mathbb{P}^x(U_k \le S) &= \mathbb{P}^x(U_{k-1} \le S, U_1 \circ \theta_{U_{k-1}} \le S \circ \theta_{U_{k-1}})\\ &= \mathbb{E}^x\big[\mathbb{P}^x(U_1 \circ \theta_{U_{k-1}} \le S \circ \theta_{U_{k-1}} | \mathcal{F}_{U_{k-1}}); U_{k-1} \le S\big]\\ &= \mathbb{E}^x\big[\mathbb{P}^{X_{U_{k-1}}}(U_1 \le S); U_{k-1} \le S\big].\end{aligned}$$



If $U_{k-1} \le S$, then $X_{U_{k-1}} \in D$. If $y \in D$,

$$\mathbb{P}^y(U_1 \le S) \le \mathbb{P}^y\Big(\int_0^S f(X_s)ds \ge 2A\Big) \le \frac{\mathbb{E}^y \int_0^S f(X_s)ds}{2A} \le \tfrac{1}{2}$$

by Chebyshev's inequality. Then

$$\boxed{\mathbb{P}^x(U_k \le S) \le \tfrac{1}{2}\mathbb{P}^x(U_{k-1} \le S)}$$

and (21.3) follows by induction. □

We give another proof of Proposition 4.5.

Proposition 21.3 *Let W be a one-dimensional Brownian motion. If T is a finite stopping time and $a < b$, then*

$$\mathbb{P}(W_{T+t} \in [a,b] \mid \mathcal{F}_T) \le \frac{b-a}{\sqrt{2\pi t}}, \qquad \text{a.s.}$$

Proof Let $(X_t, \mathbb{P}^x)$ be a one-dimensional Brownian motion. If $y \in \mathbb{R}$, then

$$\boxed{\mathbb{P}^y(X_t \in [a,b]) = \mathbb{P}^0(X_t \in [a-y, b-y])} \tag{21.4}$$
$$= \frac{1}{\sqrt{2\pi t}} \int_{a-y}^{b-y} e^{-z^2/2t}\, dz \le \frac{b-a}{\sqrt{2\pi t}}.$$

By the strong Markov property, $\mathbb{P}(W_{T+t} \in [a,b] \mid \mathcal{F}_T) = \mathbb{P}^0(X_{T+t} \in [a,b] \mid \mathcal{F}_T)$
$$= \mathbb{P}^{X_T}(X_t \in [a,b]).$$

Now use (21.4) with y replaced by X_T.

□

21.3 Continuity

Let us now come up with a criterion for a Markov process to have continuous paths.
Assume we have a strong Markov process $(X_t, \mathbb{P}^x)$ whose paths are RCLL . Let $d(\cdot,\cdot)$ be the metric for the state space $\mathcal{S}$.

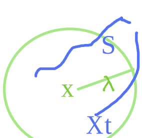
Lemma 21.4 *Let $(X_t, \mathbb{P}^x)$ be a strong Markov process with state space $\mathcal{S}$. For all $x \in \mathcal{S}$ and all $\lambda \ge 0$,*

$$\mathbb{P}^x\big(\boxed{\sup_{s\le t} d(X_s, x) \ge \lambda}\big) \le 2 \sup_{s \le t} \sup_{y\in\mathcal{S}} \mathbb{P}^y(d(X_s, X_0) \ge \lambda/2).$$

Note the lhs has the supremum inside while the rhs has the suprema outside the probability.

Proof Let

$$F(t,\lambda) = \sup_{s\le t} \sup_{y \in \mathcal{S}} \mathbb{P}^y(d(X_s, X_0) \ge \lambda). \tag{21.5}$$



Write $\boxed{S = \inf\{t : d(X_t, X_0) \geq \lambda\}}$. Then by the strong Markov property, hitting time decomp. of event

$$
\begin{aligned}
\mathbb{P}^x(\sup_{s \leq t} d(X_s, x) \geq \lambda) &\leq \mathbb{P}^x(\boxed{d(X_t, x) \geq \lambda/2}) + \mathbb{P}^x(S < t, \boxed{d(X_t, X_0) \leq \lambda/2}) \\
&\leq F(t, \lambda/2) + \mathbb{E}^x\Big[\mathbb{P}^{X_S}(d(X_{t-S}, X_0) \geq \lambda/2)\Big] \\
&\leq 2F(t, \lambda/2);
\end{aligned}
\tag{21.6}
$$

see Exercise 21.2. □

Proposition 21.5 *Let $(X_t, \mathbb{P}^x)$ be a strong Markov process. With $F(t, \lambda)$ defined as in (21.5), suppose*

$$
\frac{F(t, \lambda)}{t} \to 0 \tag{21.7}
$$

as $t \to 0$ for each $\lambda > 0$. Then X_t has continuous paths with $\mathbb{P}^x$-probability one for each x.

For X a Brownian motion, $F(t, \lambda) \leq 2e^{-\lambda^2/8t}$ by Proposition 3.15, and hence $F(t, \lambda)/t \to 0$ as $t \to 0$. Thus Brownian motion satisfies (21.7). On the other hand, (21.7) is not satisfied for the Poisson process; see Exercise 21.3.

Proof Suppose $\lambda, t_0 > 0$ and X has a jump of size larger than 4λ at some time before t_0 with positive probability, that is,

$$
\mathbb{P}^x(\boxed{\sup_{t \leq t_0} d(X_{t-}, X_t) \geq 4\lambda}) > 0,
$$

P(sup{t≤t0}limε d(Xt+ε,Xt)>4λ)

where $X_{t-} = \lim_{s \uparrow t, s < t} X_s$. Then for each n there exists $k \leq [t_0 2^n] + 1$ such that

$$
\boxed{\sup_{s,t \in [k/2^n, (k+1)/2^n]} d(X_s, X_t) \geq 4\lambda;}
$$

Therefore there exists $k \leq [t_0 2^n] + 1$ such that

$$
\sup_{s \in [k/2^n, (k+1)/2^n]} d(X_s, X_{k/2^n}) \geq 2\lambda.
$$

But by Lemma 21.4

$$
\begin{aligned}
&\mathbb{P}^x(\exists k \leq [t_0 2^n] + 1 : \sup_{k/2^n \leq s \leq (k+1)/2^n} d(X_s, X_{k/2^n}) \geq 2\lambda) \\
&\quad \leq ([t_0 2^n] + 1) \sup_y \boxed{\mathbb{P}^y(\sup_{s \leq 2^{-n}} d(X_s, X_0) \geq 2\lambda)} \\
&\quad \leq 2([t_0 2^n] + 1) F(2^{-n}, \lambda)
\end{aligned}
$$

for every n. In the first inequality we used the Markov property at time $k/2^n$ and the fact that there are at most $[t_0 2^n] + 1$ intervals. Letting $n \to \infty$, we see the probability of a jump of size larger than 4λ before time t_0 must be zero. Since λ and t_0 are arbitrary, the paths of X are continuous. □

21.4 Harmonic functions

Suppose $(X_t, \mathbb{P}^x)$ is a continuous Markov process satisfying the strong Markov property, and for each x, the sets of paths are right continuous with left limits with $\mathbb{P}^x$-probability one.

Assume D be an open subset of $\mathcal{S}$, and $\tau_D < \infty$, a.s., wrt each $\mathbb{P}^x$, where $\tau_D = \inf\{t : X_t \notin D\}$. Let f be a bounded measurable function on ∂D.
hitting time of bdD

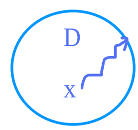
Proposition 21.6 *Define*

$$\boxed{h(x) := \mathbb{E}^x f(X_{\tau_D})}$$

and $\mathcal{F}'_s = \mathcal{F}_{s\wedge\tau_D}$. *Then for each* x, $h(X_{t\wedge\tau_D})$ *is a martingale under* $\mathbb{P}^x$ *wrt the filtration* $\{\mathcal{F}'_t\}$.

Proof Let $s < t$. Consider a path ω starting at x and continuing until it exits D at time $\tau_D(\omega)$. If we have $u \le \tau_D$ and we cut off the first u time units of the path, we have a path going from $X_u(\omega)$ and proceeding until it exits D. But note that the point at which it exits will not be changed by cutting off a piece from the beginning of the path. Therefore $\boxed{X_{\tau_D} \circ \theta_u = X_{\tau_D}}$ if $u \le \tau_D$. Using this,

$$\begin{aligned}
\mathbb{E}^x[h(X_{t\wedge\tau_D}) \mid \mathcal{F}_{s\wedge\tau_D}] &= \mathbb{E}^x\Big[\mathbb{E}^{X_{t\wedge\tau_D}} f(X_{\tau_D}) \mid \mathcal{F}_{s\wedge\tau_D}\Big] \\
&= \mathbb{E}^x[\mathbb{E}^x[f(X_{\tau_D}) \circ \theta_{t\wedge\tau_D} \mid \mathcal{F}_{t\wedge\tau_D}] \mid \mathcal{F}_{s\wedge\tau_D}] \\
&= \mathbb{E}^x[f(X_{\tau_D}) \mid \mathcal{F}_{s\wedge\tau_D}] \\
&= \mathbb{E}^x[f(X_{\tau_D}) \circ \theta_{s\wedge\tau_D} \mid \mathcal{F}_{s\wedge\tau_D}] \\
&= \mathbb{E}^{X_{s\wedge\tau_D}} f(X_{\tau_D}) = h(X_{s\wedge\tau_D}),
\end{aligned}$$



as required. □

This becomes particularly interesting in the case when X_t is a d-dimensional Brownian motion. Suppose D is a bounded domain (i.e., a bounded open subset) in $\mathbb{R}^d$. There exists M such that $D \subset B(0, M)$. We know X_t^1, the first component of X_t is a one-dimensional Brownian motion, and by Theorem 7.2, X_t^1 will exit $[-M, M]$ in finite time, no matter what X_0^1 is. Therefore the time for X_t to exit D will be finite a.s. wrt each $\mathbb{P}^x$. Take $x \in D$ and take δ smaller than the distance from x to the bd$D$.
If $\boxed{S = \inf\{t : |X_t - x| \ge \delta\}}$, the first time X leaves the ball of radius δ about x, then by Proposition 21.6 and optional stopping, we have

$$h(x) = \mathbb{E}^x h(X_S). \tag{21.8}$$

By Exercise 2.3 we know that d-dimensional Brownian motion is rotationally invariant. We conclude from this that the location where a Brownian motion hits the boundary of a ball of radius δ about the starting point must have a uniform distribution. Hence X_S will be uniformly distributed on $\partial B(x, \delta)$. Thus (21.8) ==

$$h(x) = \int_{\partial B(x,\delta)} h(y)\, \sigma_{x,\delta}(dy),$$

where $\sigma_{x,\delta}$ is a surface measure on $\partial B(x, \delta)$ normalized to have total mass one. This holds for every δ small enough, and since h is bounded (because f is), it can be shown that h is C^2

in D and is harmonic:

$$\Delta h(x) = \sum_{i=1}^{d} \frac{\partial^2 h}{\partial x_i^2}(x) = 0;$$

the proof is not obvious – see Bass (1995), Section II.1.

Dirichlet problem given a domain in $\mathbb{R}^d$ and a continuous function f on the boundary of D.to find a continuous function h that is harmonic inside D, that is, $\Delta h(x) = 0$ for $x \in D$, and that agrees with f on ∂D. There are domains for which one cannot solve the Dirichlet problem, but a solution can be found provided the domain is moderately nice. We explain how to solve the Dirichlet problem probabilistically; the class of domains where one can do this is the same as the class where one can solve the Dirichlet problem analytically. Let us say that a point x is *regular* for a Borel subset A if $\mathbb{P}^x(T_A = 0) = 1$, where $T_A = \inf\{t > 0 : X_t \in A\}$. Thus a point x is regular for a set A if starting at x the Brownian motion enters A immediately. For example, a consequence of Theorem 7.2 is that the point 0 is regular for the set $A = (0, \infty)$ when we have a 1D Brownian motion.

Theorem 21.7 *Suppose D is a bounded open domain in $\mathbb{R}^d$ and f is a function on ∂D that is continuous on ∂D. Let $(X_t, \mathbb{P}^x)$ be a d-dimensional Brownian motion and $\tau_D = \inf\{t : X_t \in D^c\}$. If each point of ∂D is regular for D^c, then $h(x) = \mathbb{E}^x f(X_{\tau_D})$ is a solution to the Dirichlet problem.*

The regularity condition says that starting at any point of ∂D, Brownian motion enters D^c immediately.

Proof We have already seen in Proposition 21.6 and the remarks immediately following the proof of that proposition that h is harmonic in D. This implies that h is continuous in D. Thus we only need to show that h agrees with f on ∂D.

Our first step is to fix t and ε and to show that the set

$$\{x : \mathbb{P}^x(\tau_D \le t) > 1 - \varepsilon\}$$

is an open set. Let $s < t$, define $\varphi_s(x) = \mathbb{P}^x(\tau_D \le t - s)$, and let

$$w_s(x) = \mathbb{P}^x(X_u \in D^c \text{ for some } u \in [s, t]).$$

By the Markov property at time s,

$$\begin{aligned} w_s(x) &= \mathbb{E}^x \mathbb{P}^{X_s}(X_u \in D^c \text{ for some } u \in [0, t - s]) = \mathbb{E}^x[\mathbb{P}^{X_s}(\tau_D \le t - s)] \\ &= \mathbb{E}^x \varphi_s(X_s) = (2\pi s)^{-d/2} \int \varphi_s(y) e^{-|x-y|^2/2s}\, dy. \end{aligned}$$

By dominated convergence, the last integral is a continuous function of x. If

$$w_0(x) = \mathbb{P}^x(X_u \in D^c \text{ for some } u \in [0, t]),$$

then $w_s(x) \uparrow w_0(x)$, so $\{x : w_0(x) > 1 - \varepsilon\} = \cup_{s \in (0,t)}\{x : w_s(x) > 1 - \varepsilon\}$ is open.

Let $z \in \partial D$. Let $\varepsilon > 0$ and choose η such that $|f(w) - f(z)| < \varepsilon$ if $|w - z| < \eta$ and $w \in \partial D$. Pick t small so that $\mathbb{P}^0(\sup_{s \le t} |X_s| > \eta/2) < \varepsilon$; this is possible because Brownian motion has continuous paths. Because $z \in \partial D$ and every point of ∂D is regular for D^c, $\mathbb{P}^z(\tau_D \le t) = 1$. Finally choose $\delta < (\eta/2) \wedge \varepsilon$ so that if $|w - z| < \delta$ and $w \in D$, then $\mathbb{P}^w(\tau_D \le t) > 1 - \varepsilon$.

Now if $|w - z| < \delta$ and $w \in D$, then

$$\begin{aligned}\mathbb{P}^w(|X_{\tau_D} - z| < \eta) &\ge \mathbb{P}^w(\tau_D \le t, \sup_{s \le t} |X_s - w| \le \eta/2)\\ &\ge \mathbb{P}^w(\tau_D \le t) - \mathbb{P}^0(\sup_{s\le t} |X_s| > \eta/2)\\ &\ge (1-\varepsilon) - \varepsilon.\end{aligned}$$

The set ∂D is a bounded and closed subset of $\mathbb{R}^d$, hence compact, and since f is continuous on ∂D, there exists M such that $|f|$ is bounded by M. If $|w - z| < \delta$ and $w \in D$,

$$\begin{aligned}|h(w) - f(z)| &= |\mathbb{E}^w f(X_{\tau_D}) - f(z)|\\ &\le |\mathbb{E}^w[f(X_{\tau_D}); |X_{\tau_D} - z| < \eta] - f(z)\mathbb{P}^w(|X_{\tau_D} - z| < \eta)|\\ &\quad + 2M\mathbb{P}^w(|X_{\tau_D} - z| \ge \eta)\\ &\le \varepsilon\mathbb{P}^w(|X_{\tau_D} - z| < \eta) + 4M\varepsilon \le (1+4M)\varepsilon.\end{aligned}$$

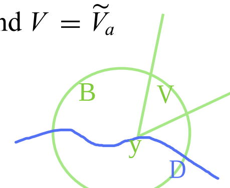
We used the fact that $|f(X_{\tau_D}) - f(z)| < \varepsilon$ if $|X_{\tau_D} - z| < \eta$. Since ε is arbitrary, this proves that $h(w) \to f(z)$ as $w \to z$ inside D. □

Let us give a sufficient condition for a point to be regular for a domain D. Let $\widetilde{V}_a = \{(x_1, \ldots, x_d) : x_1 > 0, (x_2^2 + \cdots + x_d^2) < a^2x_1^2\}$. The vertex of $\widetilde{V}_a$ is the origin. A cone V in $\mathbb{R}^d$ is a translation and rotation of $\widetilde{V}_a$ for some a.

Poincare cone condition

Proposition 21.8 *Suppose there exists a cone V with vertex $y \in \partial D$ such that $V \cap B(y, r) \subset D^c$ for some $r > 0$. Then y is regular for D^c.*

Proof By translation and rotation of the coordinates, we may suppose $y = 0$ and $V = \widetilde{V}_a$ for some a. Then for each t,



$$\begin{aligned}\mathbb{P}^0(\tau_D \le t) &\ge \mathbb{P}^0(X_t \in D^c) \ge \mathbb{P}^0(X_t \in V \cap B(0, r))\\ &\ge \mathbb{P}^0(X_t \in V) - \mathbb{P}^0(X_t \notin B(0, r)).\end{aligned}$$

By scaling, the last term is $\mathbb{P}^0(X_1 \in V) - \mathbb{P}^0(X_1 \notin B(0, r/\sqrt{t}))$, which converges to

$$\mathbb{P}^0(X_1 \in V) = (2\pi)^{-d/2} \int_V e^{-|z|^2/2}\, dz > 0$$

as $t \to 0$. Observe $\mathbb{P}^0(\tau_D \le t)$ converges to $\mathbb{P}^0(\tau_D = 0)$. By the Blumenthal 0–1 law (Proposition 20.8), $\boxed{\mathbb{P}^0(\tau_D = 0) = 1}$. for all s, exsits t<s, Xt∈ Dc a.s. □

Continue to suppose $(X_t, \mathbb{P}^x)$ is a d-dimensional Brownian motion and D is a bounded domain, but now we suppose $d \ge 3$. Define

$$\boxed{U(x, A) = \mathbb{E}^x \int_0^\infty 1_A(X_s)\, ds, \qquad x \in D.}$$

This is the same as the λ-resolvent of 1_A with $\lambda = 0$. We write

$$\begin{aligned} U(x, A) &= \mathbb{E}^x \int_0^\infty 1_A(X_s)\, ds \\ &= \int_0^\infty \mathbb{P}^x(X_s \in A)\, ds \\ &= \int_0^\infty \int_A \frac{1}{(2\pi s)^{d/2}} e^{-|y-x|^2/2s}\, dy\, ds \\ &= \int_A \int_0^\infty \frac{1}{(2\pi s)^{d/2}} e^{-|y-x|^2/2s}\, ds\, dy. \end{aligned}$$

Some calculus shows that the inside integral = $c|x-y|^{2-d}$, denote by $u(x, y)$, we then have that

$$U(x, A) = \int_A u(x, y)\, dy. \tag{21.9}$$

The expression $u(x, y)$ is called the *Newtonian potential density*. Note that $u(x, y)$ is a function only of $|x-y|$, it blows up as $|x-y| \to 0$, and tends to 0 as $|x-y| \to \infty$.

If $x \in D\circ$, then $u(x, \cdot)$ will be bounded on ∂D. Define $h_x(z) = \mathbb{E}^{\,z} u(x, X_{\tau_D})$; we saw above that h_x is harmonic. Now define $g_D(x, y) = u(x, y) - h_x(y)$; this function of two variables is called the *Green's function* or *Green function* for D with pole at x. This is a well-known object in analysis – let us give a probabilistic interpretation. Since $u(x, y)$ is symmetric in x and y, if $A \subset D$ we have

$$\begin{aligned} \int_A g_D(x, y)\, dx &= \int_A u(x, y)\, dx - \int_A \mathbb{E}^y u(x, X_{\tau_D})\, dx \qquad (21.10) \\ &= \mathbb{E}^y \int_0^\infty 1_A(X_s)\, ds - \mathbb{E}^y \int_A u(x, X_{\tau_D})\, dx \\ &= \mathbb{E}^y \int_0^\infty 1_A(X_s)\, ds - \mathbb{E}^y \Big[\mathbb{E}^{X_{\tau_D}} \int_0^\infty 1_A(X_s)\, ds\Big]. \end{aligned}$$

Using the strong Markov property and then a change of variables,

$$\begin{aligned} \mathbb{E}^y\Big[\mathbb{E}^{X_{\tau_D}} \int_0^\infty 1_A(X_s)\, ds\Big] &= \mathbb{E}^y\Big[\mathbb{E}^y\Big[\int_0^\infty 1_A(X_s) \circ \theta_{\tau_D}\, ds \mid \mathcal{F}_{\tau_D}\Big]\Big] \\ &= \mathbb{E}^y \int_0^\infty 1_A(X_s) \circ \theta_{\tau_D}\, ds \\ &= \mathbb{E}^y \int_0^\infty 1_A(X_{\tau_D + s})\, ds \\ &= \mathbb{E}^y \int_{\tau_D}^\infty 1_A(X_s)\, ds. \end{aligned}$$

Substituting this in (21.10) we have

$$\int_A g_D(x, y)\, dx = \mathbb{E}^y \int_0^{\tau_D} 1_A(X_s)\, ds.$$

For this reason g_D is sometimes called the *occupation time density* for D.

Exercises

21.1 Suppose $d = 2$, $(X_t, \mathbb{P}^x)$ is a two-dimensional Brownian motion, and $r > 0$. Imitate the argument of Proposition 21.1 but with $h(x) = \log(|x|)$ to show that $\mathbb{P}^x(X_t \text{ hits } B(0, r)) = 1$ when $|x| > r$. Then use the strong Markov property to show that there are times $T_i \to \infty$ with $X_{T_i} \in B(0, r)$. That is, two-dimensional Brownian motion is neighborhood recurrent.

21.2 In the proof of Lemma 21.4, justify each inequality in (21.6).

21.3 Let $(X_t, \mathbb{P}^x)$ be a Poisson process with parameter a and let F be defined by (21.5). Show $F(t, 1/2)/t$ does not converge to 0 as $t \to 0$.

21.4 Suppose $d \geq 3$, $(X_t, \mathbb{P}^x)$ is a d-dimensional Brownian motion, and

$$Uf(x) = \mathbb{E}^x \int_0^\infty f(X_s)\, ds.$$

Show that if f is bounded and measurable with compact support, then Uf is continuous and $|Uf(x)| \to 0$ as $|x| \to \infty$. Show that if $f \in C^2$ with compact support, then Uf is C^2. Show that $\frac{1}{2}\Delta Uf = -f$.

21.5 Let W_t be a Brownian motion and f a continuous function. Prove that if $f(W_t)$ is a submartingale, then f must be convex.

21.6 Prove the *maximum principle* for harmonic functions. This says that if h is harmonic in a bounded domain D, then

$$\sup_{x \in \overline{D}} |h(x)| \leq \sup_{x \in \partial D} |h(x)|.$$

21.7 If W is a d-dimensional Brownian motion started at 0, find $\mathbb{E}\, T$, where T is the first time W exits the ball of radius r centered at the origin.
Hint: Use the fact that $|W_t|^2 - dt$ is a martingale.

21.8 Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function with $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$. Let $D_\varepsilon = \{(x, y) \in \mathbb{R}^2 : f(x) < y < f(x) + \varepsilon\}$ for $\varepsilon \in (0, 1)$. Let $(X_t, \mathbb{P}^x)$ be a two-dimensional Brownian motion and let $\tau_\varepsilon = \inf\{t : X_t \notin D_\varepsilon\}$. Prove that there exists a constant c not depending on ε such that $\mathbb{E}^0 \tau_\varepsilon \leq c\varepsilon^2$.
Hint: By Exercise 21.7 the expected time for two-dimensional Brownian motion to leave a ball of radius 2ε is less than $c\varepsilon^2$. Then use the strong Markov property repeatedly at the times S_i, where S_i is the first time after time S_{i-1} that Brownian motion has moved at least 2ε from $X_{S_{i-1}}$.

Keys to prove thms:
1. Define stopping time ==> events
2. shift operator/notation
3. shift thm/ Strong markov property.

22

Transformations of Markov processes

There are a number of interesting transformations that make new Markov processes out of old: killing, conditioning, changing time, and stopping at a last exit time.

22.1 Killed processes

One sometimes wants to consider a Markov process up until a stopping time ζ, called the *lifetime* of the process. We affix to our state space $\mathcal{S}$ an isolated point Δ, called the *cemetery* state, and the topology on $\mathcal{S}_\Delta = \mathcal{S} \cup \{\Delta\}$ is the one generated by the collection of open sets of $\mathcal{S}$ together with the set $\{\Delta\}$. We define the *killed process* $\widehat{X}$ by

$$\widehat{X}_t = \begin{cases} X_t, & t < \zeta; \\ \Delta, & t \geq \zeta, \end{cases} \tag{22.1}$$

and we say we *kill* the process X at time ζ. Every function f on $\mathcal{S}$ is defined to be 0 at Δ.

Example of this situation would be to let $\zeta = \tau_D$, where D is a subset of $\mathcal{S}$ and $\tau_D = \inf\{t > 0 : X_t \notin D\}$, the first exit from the set D. Another common occurrence is to let $\zeta = S$, where S is a rv ~ exponential distribution with parameter λ, i.e., $\mathbb{P}(S > t) = e^{-\lambda t}$, such that S is independent of X. A third possibility would be to let $\zeta = \inf\{t : \int_0^t f(X_s)\, ds \geq 1\}$, where f is a non-negative function. The crucial property of ζ is that it be a *terminal time*:

$$\zeta = s + \zeta \circ \theta_s \qquad \text{if } s < \zeta. \tag{22.2}$$

Proposition 22.1 *If $(X_t, \mathbb{P}^x)$ is a strong Markov process and (22.2) holds, then $(\widehat{X}_t, \mathbb{P}^x)$ satisfies the Markov and strong Markov properties.*

Proof As in Section 20.2, we need to show

$$\mathbb{E}^x[f(\widehat{X}_t) \circ \theta_T | \mathcal{F}_T] = \mathbb{E}^{\widehat{X}_T} f(\widehat{X}_t), \qquad \mathbb{P}^x\text{-a.s.}$$

If $A \in \mathcal{F}_T$,

$$\mathbb{E}^x[f(\widehat{X}_t) \circ \theta_T; A] = \mathbb{E}^x[f(X_{t+T}); A, T + t < \zeta].$$

On the other hand,

$$\boxed{\mathbb{E}^{\widehat{X}_T} f(\widehat{X}_t) = \mathbb{E}^{X_T}[f(X_t); t < \zeta]1_{(T<\zeta)}}$$
$$= \mathbb{E}^x[f(X_t) \circ \theta_T; t \circ \theta_T < \zeta \circ \theta_T | \mathcal{F}_T]1_{(T<\zeta)}$$
$$= \mathbb{E}^x[f(X_{t+T}); T + t \circ \theta_T < T + \zeta \circ \theta_T, T < \zeta | \mathcal{F}_T]$$
$$= \mathbb{E}^x[f(X_{t+T}); T + t < \zeta | \mathcal{F}_T],$$

since $T + t \circ \theta_T = T + t$ and $T + \zeta \circ \theta_T = \zeta$ on $(T < \zeta)$.

□

22.2 Conditioned processes

Another type of transformation of a Markov process is by conditioning, also known as *Doob's h-path transform*. To motivate this, let D be a domain in $\mathbb{R}^d$ and let X_t be a Brownian motion killed on exiting the domain. One would like to give a precise meaning to the intuitive notion of Brownian motion conditioned to exit the domain at a certain point. Let h be a positive harmonic function in D (i.e., h is C^2 in D, and $\Delta h = 0$ there) and suppose that h is 0 everywhere on the bd$D$ except at one point z. The Poisson kernel for the ball or for the half-space gives examples of such harmonic functions. Then, heuristically, we have by the Markov property at time t,

$$\boxed{\begin{aligned}\mathbb{P}^x(X_t \in dy | X_{\tau_D} = z) &= \frac{\mathbb{P}^x(X_t \in dy, X_{\tau_D} = z)}{\mathbb{P}^x(X_{\tau_D} = z)} \\ &= \frac{\mathbb{P}^x(X_t \in dy)\mathbb{P}^y(X_{\tau_D} = z)}{\mathbb{P}^x(X_{\tau_D} = z)}.\end{aligned}}$$

If $p^0(t, x, dy)$ represents the probability that Brownian motion started at x and killed on leaving D is in dy at time t, we then expect that the analogous probability for Brownian motion conditioned to exit D at z ought to be $h(y)p^0(t, x, dy)/h(x)$.

Let us look at a strong Markov process X. We say a function h is *invariant* wrt X if $P_t h(x) = h(x)$ for all t and x, where P_t is the semigroup associated with X. If h is invariant, by the Markov property,

$$\mathbb{E}^x[h(X_t) \mid \mathcal{F}_s] = P_{t-s}h(X_s) = h(X_s),$$

and so for each x, $h(X_t)$ is a martingale wrt $\mathbb{P}^x$. Conversely, if $h(X_t)$ is a martingale wrt $\mathbb{P}^x$ for all x,

$$P_t h(x) = \mathbb{E}^x h(X_t) = h(x)$$

by the definition of martingale, and so h is invariant. In the case of Brownian motion killed on leaving a domain, the invariant functions are thus the harmonic ones.

Now let h be a non-negative invariant function for a strong Markov process X. Letting $M_t = h(X_t)/h(X_0)$, M_t is a non-negative continuous martingale with $M_0 = 1$, $\mathbb{P}^x$-a.s., as long as $h(x) > 0$.

Define the h-path transform of the Markov process X

$$\mathbb{P}^x{}_h(A) := \mathbb{E}^x[M_t; A], \qquad A \in \mathcal{F}_t. \tag{22.3}$$

Since $M_0 = 1$, $\mathbb{P}^x_h(\Omega) = 1$. Observe that P^x_h gives more mass to paths where $h(X_t)$ is big and less to where it is small. Note the similarity to the Girsanov theorem.

Proposition 22.2 *Suppose $(X_t, \mathbb{P}^x)$ is a strong Markov process and that h is non-negative and invariant. Then $(X_t, \mathbb{P}^x_h)$ forms a strong Markov process.*

Proof Suppose $A \in \mathcal{F}_s$ and $h(x) \neq 0$. (We leave consideration of the case where $h(x) = 0$ to the reader.) Then

$$\begin{aligned}
\mathbb{E}^x_h[f(X_{t+s}); A] &= \frac{\mathbb{E}^x[f(X_{t+s})h(X_{t+s}); A]}{h(x)} \\
&= \frac{\mathbb{E}^x[\mathbb{E}^{X_s}[f(X_t)h(X_t)]; A]}{h(x)} \\
&= \mathbb{E}^x\Big[\frac{1}{h(X_s)}\mathbb{E}^{X_s}[f(X_t)h(X_t)]h(X_s); A\Big]
\end{aligned}$$

by the Markov property for X. This ==

$$\mathbb{E}^x\big[\mathbb{E}^{X_s}_h[f(X_t)]h(X_s); A\big]/h(x) = E^x_h[\mathbb{E}^{X_s}_h f(X_t); A].$$

The Markov property follows from this. The strong Markov property is proved in almost identical fashion. □

Example. Let $(X_t, \mathbb{P}^x)$ be a Brownian motion on the non-negative axis killed on first hitting 0. This is the same as a Brownian motion killed on exiting $(0, \infty)$. This will be a strong Markov process. Since the second derivative of the function $h(x) = x$ is 0, then h is harmonic on $(0, \infty)$, and so is invariant for the killed Brownian motion. Let us now condition using the function h to get Brownian motion conditioned to hit infinity before hitting zero.

To identify the resulting process, we argue as follows. Fix x and let $T_\varepsilon = \inf\{t > 0 : X_t < \varepsilon\}$. The Radon–Nikodym derivative of the law of $\mathbb{P}^x_h$ wrt $\mathbb{P}^x$ on $\mathcal{F}_{t\wedge T_\varepsilon}$ is $M_{t\wedge T_\varepsilon} = h(X_{t\wedge T_\varepsilon})/h(x)$. We can rewrite $M_{t\wedge T_\varepsilon}$ as

$$M_{t\wedge T_\varepsilon} = \exp(\log X_{t\wedge T_\varepsilon} - \log x) = \exp\Big(\int_0^{t\wedge T_\varepsilon} \frac{1}{X_s}\, dX_s - \tfrac{1}{2}\int_0^{t\wedge T_\varepsilon}\Big(\frac{1}{X_s}\Big)^2 ds\Big),$$

using Itô's formula. By the Girsanov theorem, under $\mathbb{P}^x_h$,

$$W_{t\wedge T_\varepsilon} = X_{t\wedge T_\varepsilon} - \int_0^{t\wedge T_\varepsilon} \frac{1}{X_s}\, ds$$

is a martingale. By Exercise 13.2, its quadratic variation is $t \wedge T_\varepsilon$, and so by Exercise 12.3, $W_{t\wedge T_\varepsilon}$ is a Brownian motion stopped at time T_ε. We have

$$X_{t\wedge T_\varepsilon} = x + W_{t\wedge T_\varepsilon} + \int_0^{t\wedge T_\varepsilon} \frac{1}{X_s}\, ds,$$

or X satisfies the SDE

$$dX_t = dW_t + \frac{1}{X_t}\, dt$$

for $t \le T_\varepsilon$. We will see later (Section 24.3) that this is the stochastic differential equation defining the Bessel process of order 3. The same argument shows that Brownian motion killed on exiting $(0, a)$ and then conditioned to hit a before 0 is also a Bessel process of order 3 up until the time of first hitting a.

22.3 Time change

An *additive functional* is an increasing adapted process with $A_0 = 0$, a.s., such that

$$A_t = A_s + A_{t-s} \circ \theta_s \tag{22.4}$$

if $s < t$. The simplest examples are what are known as *classical additive functionals*: $A_t = \int_0^t f(X_r)\, dr$, where f is a non-negative measurable function. We have

$$A_t - A_s = \int_s^t f(X_r)\, dr = \int_0^{t-s} f(X_r)\, dr \circ \theta_s = A_{t-s} \circ \theta.$$

If we have the uniform limit of additive functionals, we again get an additive functional, and thus, for example, the local times L_t^x of a one-dimensional Brownian motion are also additive functionals.

Given a Markov process X and an additive functional A, let

$$B_t = \inf\{u : A_u > t\}$$

and

$$\boxed{X'_t = X_{B_t}.}$$

Let $\mathcal{F}'_t = \mathcal{F}_{B_t}$. Thus X' is a time change of X.

Proposition 22.3 *Let $(X_t, \mathbb{P}^x)$ be a strong Markov process and A_t an additive functional. With B defined as above, $(X'_t, \mathbb{P}^x)$ is also a strong Markov process.*

Proof We verify the strong Markov property. Let $\mathcal{F}'_t = \mathcal{F}_{B_t}$. Then if T is a stopping time for $\mathcal{F}'_t$, we have

$$\mathbb{E}^x[f(X'_{T+t}) \mid \mathcal{F}'_T] = \mathbb{E}^x[f(X(B_{T+t})) \mid \mathcal{F}_{B_T}].$$

B_T can be seen to be a stopping time wrt $\{\mathcal{F}_t\}$ and $B_{T+t} = B_t \circ \theta_{B_T}$ where the θ_t are the shift operators, so this is

$$\mathbb{E}^x \mathbb{E}^{X(B_T)} f(X_{B_t}) = \mathbb{E}^x \mathbb{E}^{X'_T} f(X'_t).$$

This suffices to show that X'_t is a strong Markov process. □

22.4 Last exit decompositions

let L be the last visit to A:

$$\boxed{L := \sup\{s : X_s \in A\}.}$$

$L = 0$ if X never hits A. The random time L is not a stopping time, but we can nevertheless kill the process X at time L. It turns out the resulting process Y is the process X conditioned by the function $h(x) = \mathbb{P}^x(T_A < \infty)$. The intuitive meaning of this is that Y is X conditioned to hit the set A.

Let $T = \inf\{t : X_t \in A\}$, and set

$$Y_t = \begin{cases} X_t, & t < L, \\ \Delta, & t \geq L. \end{cases}$$

Let $\mathcal{H}_t = \sigma(Y_s; s \leq t)$.

Proposition 22.4 *If $(X_t, \mathbb{P}^x)$ is a strong Markov process, then $(Y_t, \mathbb{P}^x)$ is a Markov process wrt $\{\mathcal{H}_t\}$.*

Proof If $B \subset \mathcal{S}$ (so that $\Delta \notin B$), then

$$(Y_t \in B) = (X_t \in B, L > t) = (X_t \in B, T \circ \theta_t < \infty),$$

since L, the last time that X is in A, will be larger than t iff X hits A at some time after time t. We conclude that the function $x \to \mathbb{P}^x(Y_t \in B)$ is Borel measurable. Since

$$\mathbb{P}^x(Y_t = \Delta) = \mathbb{P}^x(L \leq t) = 1 - \mathbb{P}^x(L > t) = 1 - \mathbb{P}^x(T \circ \theta_t < \infty),$$

then the function $x \to \mathbb{P}^x(Y_t = \Delta)$ is also Borel measurable.

We need to show that if $C \in \mathcal{H}_s$,

$$\mathbb{E}^x[f(Y_t); C] = \mathbb{E}^x[Q_{t-s}f(Y_s); C], \tag{22.5}$$

where f is bounded and measurable, $h(x) = \mathbb{P}^x(L > 0)$, and

$$Q_t g(x) = \frac{1}{h(x)} P_t(gh)(x)$$

when $h(x) \neq 0$. (Set $Q_t g(x) = 0$ if $h(x) = 0$.)

It suffices to show (22.5) when $C = (Y_{r_1} \in B_1, \dots, Y_{r_n} \in B_n)$ for $r_1 \leq \dots \leq r_n \leq s$ and the $B_1, \dots, B_n$ are Borel sets. If we set

$$C_s = (X_{r_1} \in B_1, \dots, X_{r_n} \in B_n),$$

then $C_s \in \mathcal{F}_s$, $C \cap (L > s) = C_s \cap (L > s)$, and $C \cap (L > t) = C_s \cap (L > t)$.

We start with

$$\begin{aligned}\mathbb{E}^x[f(Y_t); C] &= \mathbb{E}^x[f(X_t); C, L > t] = \mathbb{E}^x[f(X_t); C_s, L > t] \\ &= \mathbb{E}^x[f(X_t); C_s, L \circ \theta_t > 0].\end{aligned}$$

Conditioning on $\mathcal{F}_t$, this ==

$$\mathbb{E}^x[f(X_t)\mathbb{P}^{X_t}(L > 0); C_s] = \mathbb{E}^x[f(X_t)h(X_t); C_s].$$

Conditioning on $\mathcal{F}_s$, this in turn ==

$$\begin{aligned} \mathbb{E}^x[P_{t-s}(fh)(X_{t-s}); C_s] &= \mathbb{E}^x[h(X_s)Q_{t-s}f(X_s); C_s] \\ &= \mathbb{E}^x[\mathbb{P}^{X_s}(L > 0)Q_{t-s}f(X_s); C_s] \\ &= \mathbb{E}^x[Q_{t-s}f(X_s); C_s, L \circ \theta_s > 0], \end{aligned} \tag{22.6}$$

where we used the Markov property for the last equality. Continuing, we have that the last line of (22.6) ==

$$\begin{aligned} \mathbb{E}^x[Q_{t-s}f(X_s); C_s, L > s] &= \mathbb{E}^x[Q_{t-s}f(X_s); C, L > s] \\ &= \mathbb{E}^x[Q_{t-s}f(Y_s); C], \end{aligned}$$

as desired. □

We can also look at X_{L+t}, where L is as above. This new process is again a strong Markov process, and this time is the process X conditioned by the function $h(x) = \mathbb{P}^x(T_A = \infty)$. The intuitive meaning of this is that X_{L+t} is X conditioned never to hit A. Since we are looking at the process after the last visit to A, this is entirely plausible. For a proof of the Markov property of X_{L+t}, see Meyer *et al.* (1972).

Exercises

22.1 Let $(X_t, \mathbb{P}^x)$ be a one-dimensional Brownian motion, L_t^x the local time of Brownian motion at x, and m a positive finite measure on $\mathbb{R}$. Show that $A_t = \int L_t^x \, m(dx)$ is an additive functional.

22.2 We consider the *space-time process*. Let $V_t = V_0 + t$. The process V_t is simply the process that increases deterministically at unit speed. Thus V_t can represent time. If $(X_t, \mathbb{P}^x)$ is a Markov process, show that $((X_t, V_t), \mathbb{P}^{(x,v)})$ is also a Markov process. Is $((X_t, V_t), \mathbb{P}^{(x,v)})$ necessarily a strong Markov process if $(X_t, \mathbb{P}^x)$ is a strong Markov process?

For some applications, one lets $V_t = V_0 - t$, and one thinks of time running backwards. Space-time processes are useful when considering parabolic PDEs.

22.3 Suppose $(X_t,\mathbb{P}^x)$ is a strong Markov process and f is a non-negative invariant function for $(X_t, \mathbb{P}^x)$. Write $\mathbb{Q}^x$ for $\mathbb{P}^x_f$. Suppose g is a non-negative invariant function for $(X_t, \mathbb{Q}^x)$. Show that fg is a non-negative invariant function for $(X_t, \mathbb{P}^x)$ and that $\mathbb{Q}^x_g = \mathbb{P}^x_{fg}$.

22.4 Suppose A and B are additive functionals for a Markov process and A and B have continuous paths. Prove that if $\mathbb{E}^x A_t = \mathbb{E}^x B_t$ for all x and t, then

$$\mathbb{P}^x(A_t \neq B_t \text{ for some } t \geq 0) = 0$$

for all x.

Hint: Show $A_t - B_t$ is a martingale.

22.5 Suppose A and B are additive functionals with continuous paths and suppose $\mathbb{E}^x A_\infty = \mathbb{E}^x B_\infty < \infty$ for each x. Show

$$\mathbb{P}^x(A_t \neq B_t \text{ for some } t \geq 0) = 0$$

for each x.

Hint: If $f(x) = \mathbb{E}^x A_\infty$, then

$$\mathbb{E}^x[A_\infty \mid \mathcal{F}_t] - A_t = \mathbb{E}^{X_t} A_\infty = f(X_t),$$

and similarly with B in place of A. Then $A - B$ is a $\mathbb{P}^x$ martingale for each x.

22.6 Let A be an additive functional with continuous paths. Suppose there exists $K > 0$ such that $\mathbb{E}^x A_\infty \leq K$ for each x. Prove that there exists a constant c depending only on K such that

$$\mathbb{E}\, e^{cA_\infty} < \infty, \qquad x \in \mathcal{S}.$$

22.7 Here is an argument that the law of a Brownian motion conditioned to have a maximum at a certain level is a Bessel process of order 3.

Let W be a one-dimensional Brownian motion killed on hitting 0. Let $S_t = \sup_{s \leq t} W_s$ be the maximum. By Exercise 19.1, $X = (W, S)$ is a Markov process. Determine the law of X for $t \leq L$, where L is the last time X hits the diagonal. To define L more precisely, let $D = \{(w, s) : w = s, w > 0\}$ and $L = \sup\{t \geq 0 : X_t \in D\}$. L is finite, a.s., because W will hit 0 in finite time with probability one.

Notes

Markov processes are in some sense supposed to have the property that the past and the future are independent given the present. From this point of view, one might hope that a Markov process run backwards is again a Markov process. This is, more or less, the case; see Chung and Walsh (1969) or Rogers and Williams (2000a).