

An Efficient Algorithm of Yau-Yau Method for Solving Nonlinear Filtering Problems

MEI-HENG YUEH , WEN-WEI LIN AND SHING-TUNG YAU

It is well known that the nonlinear filter has important applications in military, engineering and commercial industries. In this paper, we propose efficient and accurate numerical algorithms for the realization of the Yau-Yau method for solving nonlinear filtering problems by using finite difference schemes. The Yau-Yau method reduces the nonlinear filtering problem to the initial-value problem of Kolmogorov equations. We first solve this problem by the implicit Euler method, which is stable in most cases, but costly. Then, we propose a quasi-implicit Euler method which is feasible for acceleration by TFTs. Furthermore, we propose a superposition technique which enables us to deal with the nonlinear filtering problem in an off-time process and thus, save a large amount of computational cost. Next, we prove that the numerical solutions of Kolmogorov equations by our schemes are always nonnegative in each iteration. Consequently, our iterative process preserves the PDFs. In addition, we prove convergence of our schemes under some mild conditions. Numerical results show that the proposed algorithms are efficient and promising.

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1. Introduction

The nonlinear filtering problem has a variety of applications in military, engineering and commercial industries [15, 13, 14]. It involves the estimation of a stochastic process of the state $\mathbf{x} = \mathbf{x}(t)$ that cannot be directly measured during the process. Information containing \mathbf{x} is then obtained from observation process $\mathbf{y} = \mathbf{y}(t)$. The goal of the nonlinear filtering is to determine the conditional pdf $\rho(t, \mathbf{x})$ of the state $\mathbf{x}(t)$ via a given observation history of $\mathbf{y}(s)$, $s \in [0, t]$. In 1961, Kalman and Bucy [6] first proposed the finite-dimensional filter, known as Kalman-Bucy filter, for solving the linear filtering problem with Gaussian initial distribution, which is useful in natural science and engineering [1]. Although the Kalman-Bucy

filter is not feasible for the nonlinear filtering problem, it gives great inspiration to the posterity in solving the nonlinear filtering problem. The core issue of the nonlinear filtering problem is to solve the [Duncan-Mortensen-Zakai \[3, 12, 24\] \(DMZ\) equation](#) in real time. Yau and Yau [21, 22, 23] have proved that the real-time solution of the DMZ equation can be reduced to an off-time solution of the Kolmogorov equation. Luo/Yau [10, 11] have extended the algorithm in [23] to the most general setting of nonlinear filterings. Based on the work of Yau and Yau [22, 23], Liu/Dong/Ding [9] proposed a numerical method to solve the nonlinear filtering problem by using explicit finite difference schemes. In order to improve the reliability of the algorithm proposed in [9], in this paper, we propose an efficient and reliable quasi-implicit numerical scheme for solving the Kolmogorov equations and estimate approximate states of a given signal-observation model.

The nonlinear filtering problem considered here is to determine approximate states for a given observation history of the following signal-observation model [5, 2]

$$(1) \quad \begin{cases} d\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t)) dt + d\mathbf{v}(t) & \mathbf{x}(0) = \mathbf{x}_0, \\ d\mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t)) dt + d\mathbf{w}(t) & \mathbf{y}(0) = 0, \end{cases}$$

where $\mathbf{x}(t) = (x_1(t), \dots, x_D(t))^T \in \mathbb{R}^D$ and $\mathbf{y}(t) = (y_1(t), \dots, y_M(t))^T \in \mathbb{R}^M$ are the state and the measurement/observation vectors at time t , respectively, $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_D(\mathbf{x}))^T$ and $\mathbf{h}(\mathbf{x}) = (h_1(\mathbf{x}), \dots, h_M(\mathbf{x}))^T$ are given vector-valued functions, $\mathbf{v} \in \mathbb{R}^D$ and $\mathbf{w} \in \mathbb{R}^M$ are mutually independent standard Brownian processes. From the main results of Yau and Yau [20, 22, 23], the state vector $\mathbf{x}(t)$ can be estimated from the observation vectors $\{\mathbf{y}(s) | s \in [0, t]\}$ by solving the Kolmogorov equations.

Suppose that a set of observations $\{\mathbf{y}(\tau_0), \dots, \mathbf{y}(\tau_{N_\tau})\}$ is measured. For each time period $[\tau_{k-1}, \tau_k]$, $k = 1, \dots, N_\tau$, we solve the **Kolmogorov equations**:

$$(2) \quad \begin{cases} \frac{\partial \tilde{u}_k}{\partial t}(t, \mathbf{s}) = \frac{1}{2} \Delta \tilde{u}_k(t, \mathbf{s}) + \sum_{d=1}^D p_d(\mathbf{s}) \frac{\partial \tilde{u}_k}{\partial s_d}(t, \mathbf{s}) + q(\mathbf{s}) \tilde{u}_k(t, \mathbf{s}), & t \in [\tau_{k-1}, \tau_k], \\ \tilde{u}_k(\tau_{k-1}, \mathbf{s}) = \exp \left\{ \sum_{j=1}^M [y_j(\tau_{k-1}) - y_j(\tau_{k-2})] h_j(\mathbf{s}) \right\} \tilde{u}_{k-1}(\tau_{k-1}, \mathbf{s}), \\ \tilde{u}_1(0, \mathbf{s}) = \sigma_0(\mathbf{s}) \exp \left\{ \sum_{j=1}^M y_j(\tau_0) h_j(\mathbf{s}) \right\}, & \text{for } k = 2, \dots, N_\tau \text{ and } \tau_0 = 0 \end{cases}$$

u is the distribution of **x**

$$(3) \quad p_d(\mathbf{s}) = -f_d(\mathbf{s}), \quad d = 1, \dots, D,$$

$$(4) \quad q(\mathbf{s}) = - \left[\sum_{d=1}^D \frac{\partial f_d}{\partial s_d}(\mathbf{s}) + \frac{1}{2} \sum_{j=1}^M h_j^2(\mathbf{s}) \right].$$

For $t \in [\tau_{k-1}, \tau_k]$, $k = 2, \dots, N_\tau$, we compute the expectation

$$(5) \quad \hat{x}_d(t) = \int_{\mathbb{R}^D} s_d \tilde{u}_k(t, \mathbf{s}) \, d\mathbf{s},$$

for $d = 1, \dots, D$. $\hat{\mathbf{x}}(t) = (\hat{x}_1(t), \dots, \hat{x}_D(t))^\top$.

2. Numerical Algorithms

In order to solve the nonlinear filtering problem (1), we first generate states and observations $\{\mathbf{x}_k, \mathbf{y}_k\}_{k=0}^{N_\tau}$ by using Euler forward difference method with Gaussian noise. Based on the Yau-Yau's method, we first propose an [implicit Euler method \(IEM\)](#) for solving the Kolmogorov equations (2) which is stable and reliable, but costly. Furthermore, we develop the [quasi-implicit Euler method \(QIEM\)](#) for solving the Kolmogorov equations (2) which is also stable and reliable, but much more efficient because the FFT can be applied in the QIEM.

2.1. The State and Observation Generator

Given a terminal time Γ , we generate a set of states and observations $\{\mathbf{x}_k, \mathbf{y}_k\}_{k=0}^{N_\tau}$ by Euler forward difference method. We partition the time interval $[0, \Gamma]$:

$$(6) \quad \mathcal{P}_{[0, \Gamma]} = \{0 = \tau_0 < \tau_1 < \dots < \tau_{N_\tau} = \Gamma\},$$

where $\tau_k - \tau_{k-1} = \Delta\tau$, $k = 1, 2, \dots, N_\tau$. With the Euler forward discretization, the signal observation model in (1) can be simulated by

$$\begin{cases} \mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{f}(\mathbf{x}_k)\Delta\tau + \mathbf{v}\sqrt{\Delta\tau}, \\ \mathbf{y}_{k+1} = \mathbf{y}_k + \mathbf{h}(\mathbf{x}_k)\Delta\tau + \mathbf{w}\sqrt{\Delta\tau}, \end{cases}$$

where \mathbf{x}_0 is the initial vector and $\mathbf{y}_0 = 0$, \mathbf{v} and \mathbf{w} are mutually independent Brownian motion with $\mathbf{v}, \mathbf{w} \sim \mathcal{N}(0, 1)$.

Algorithm 1 State and Observation Generator

Input: a terminal time Γ , time step $\Delta\tau$, the initial state \mathbf{x}_0 , the vector-valued functions \mathbf{f}, \mathbf{h}

Output: the state $\mathbf{x}(t)$ and observation $\mathbf{y}(t)$ at $t = \tau_0, \dots, \tau_{N_\tau}$

- 1: $N_\tau = \frac{\Gamma}{\Delta\tau} + 1$.
 - 2: Set the initial state $\mathbf{x}(\tau_0) = \mathbf{x}_0$.
 - 3: Generate $\mathbf{v}, \mathbf{w} \sim \mathcal{N}(0, 1)$, $\mathbf{v} \perp \mathbf{w}$.
 - 4: **for** $k = 1$ to N_τ **do**
 - 5: $\mathbf{x}(\tau_k) = \mathbf{x}(\tau_{k-1}) + \mathbf{f}(\mathbf{x}(\tau_{k-1}))\Delta\tau + \mathbf{v}\sqrt{\Delta\tau}$.
 - 6: **end for**
 - 7: $\mathbf{y}(\tau_0) = 0$.
 - 8: **for** $k = 1$ to N_τ **do**
 - 9: $\mathbf{y}(\tau_k) = \mathbf{y}(\tau_{k-1}) + \mathbf{h}(\mathbf{x}(\tau_{k-1}))\Delta\tau + \mathbf{w}\sqrt{\Delta\tau}$.
 - 10: **end for**
-

2.2. Implicit Euler Method (IEM) for Kolmogorov Equations

We now propose the IEM for solving the Kolmogorov equations (2). From the equation (5), the state vector $\mathbf{x}(t)$ can be estimated by the solution of the equation (2). For each time interval, we partition $[\tau_{k-1}, \tau_k]$, $k = 1, \dots, N_\tau$, uniformly by

$$\mathcal{P}_{[\tau_{k-1}, \tau_k]} = \left\{ \tau_{k-1} = t_0^{(k)} < t_1^{(k)} < \dots < t_{N_t}^{(k)} = \tau_k \right\},$$

where $t_n^{(k)} - t_{n-1}^{(k)} = \Delta t$, $n = 1, \dots, N_t$. Then the partition

$$\mathcal{P}_{[0, \Gamma]}^* = \bigcup_{k=1}^{N_\tau} \mathcal{P}_{[\tau_{k-1}, \tau_k]}$$

$$= \left\{ 0 = \tau_0 = t_0^{(1)} < \cdots < t_{N_t}^{(1)} = \tau_1 = t_0^{(2)} < \cdots \right. \\ \left. < t_{N_t}^{(2)} = \tau_2 = t_0^{(3)} < \cdots < t_{N_t}^{(N_\tau)} = \tau_{N_\tau} = \Gamma \right\}$$

forms a refinement of the partition $\mathcal{P}_{[0,\Gamma]}$ in (6). On the other hand, the space interval $[-R, R]$ can be uniformly discretized by

$$\mathcal{P}_{[-R,R]} = \{-R = s_0 < s_2 < \cdots < s_{N_s} = R\},$$

where $s_j - s_{j-1} = \Delta s$, $j = 1, 2, \dots, N_s$ and R is a suitably large number so that the Gaussian distribution can be ignored outside $[-R, R]$. For the discretization of an D -cell $[-R, R]^D \subset \mathbb{R}^D$, we consider an ordered set of the power set of $\mathcal{P}_{[-R,R]}$,

$$\mathcal{P}_{[-R,R]}^D = \{\mathbf{s}_j\}_{j=1}^{(N_s)^D},$$

where $\mathbf{s}_j = (s_j^{(1)}, s_j^{(2)}, \dots, s_j^{(D)})^\top$, $s_j^{(d)} \in \mathcal{P}_{[-R,R]}$, $j = 1, \dots, (N_s)^D$, $d = 1, \dots, D$. In the discrete model of the equation (2), we set up the Dirichlet boundary condition of the domain $[-R, R]^D$ to be zero. For the d -th dimension of the space, $d = 1, \dots, D$, the second order PDO can be approximated by using the Euler central difference scheme

$$(7) \quad \frac{\partial^2 \tilde{u}}{\partial s^2}(t_n, s_j) \approx \left[\alpha \left(\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{(\Delta s)^2} \right) + \beta \left(\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta s)^2} \right) \right],$$

where $U_j^n \equiv \tilde{u}(t_n, s_j)$ and $\alpha + \beta = 1$, $\alpha, \beta \geq 0$. Similarly, the PDO can be approximated by

$$(8) \quad \frac{\partial \tilde{u}}{\partial s}(t_n, s_j) \approx \left[\alpha \left(\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta s} \right) + \beta \left(\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta s} \right) \right].$$

In other words, the discretized Laplacian operator in (2) can be represented by the matrix

$$(9) \quad \mathbb{T}_d \equiv \left[\left(\bigotimes_{k=1}^{D-d} I_{N_s} \right) \otimes \mathbb{T}_d \otimes \left(\bigotimes_{k=D-d+2}^D I_{N_s} \right) \right],$$

where \otimes denotes the Kronecker product (or tensor product), I_{N_s} is the identity matrix of size N_s and the matrix

$$\mathbb{T}_d = \frac{1}{(\Delta s)^2} \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}.$$

Similarly, the discretized PDO:

$$(10) \quad \mathbb{K}_d \equiv \left[\left(\bigotimes_{k=1}^{D-d} I_{N_s} \right) \otimes \mathbb{K}_d \otimes \left(\bigotimes_{k=D-d+2}^D I_{N_s} \right) \right],$$

where the matrix

$$\mathbb{K}_d = \frac{1}{2\Delta s} \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & -1 & 0 \end{bmatrix}.$$

For each time period $t_n^{(k)} \in [\tau_{k-1}, \tau_k]$, the partial differential of time $\frac{\partial \tilde{u}_k}{\partial t}(t_n, \mathbf{s})$ in (2) can be discretized by

$$(11) \quad \frac{\partial \tilde{u}_k}{\partial t}(t_n, \mathbf{s}) \approx \frac{U^{(k),n+1} - U^{(k),n}}{\Delta t},$$

where $U^{(k),n} \equiv (\tilde{u}_k(t_n, s_1), \tilde{u}_k(t_n, s_2), \dots, \tilde{u}_k(t_n, s_{(N_s)^D}))^\top$. Hence the numerical scheme can be written in the form

$$(12) \quad \frac{U^{(k),n} - U^{(k),n-1}}{\Delta t} = \alpha \mathbb{A} U^{(k),n} + \beta \mathbb{A} U^{(k),n-1},$$

where $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ and the matrix

$$(13) \quad \mathbb{A} = \frac{1}{2} \sum_{d=1}^D \mathbb{T}_d + \sum_{d=1}^D \mathbb{P}_d \mathbb{K}_d + \mathbb{Q},$$

$\mathbb{P}_d \equiv \text{diag}\{p_d(\mathbf{s}_i)\}_{i=1}^{(N_s)^D}$, $\mathbb{Q} \equiv \text{diag}\{q(\mathbf{s}_i)\}_{i=1}^{(N_s)^D}$ are diagonal matrices. For each $t_j^{(k)} \in [\tau_{k-1}, \tau_k]$, $j = 1, \dots, N_t$, we solve the linear system

$$(14) \quad [I - \alpha(\Delta t)\mathbb{A}] U^{(k),n} = [I + \beta(\Delta t)\mathbb{A}] U^{(k),n-1},$$

for $k = 1, \dots, N_\tau$ with the initial vector $U^{(k),0} \equiv (U_1^{(k),0}, U_2^{(k),0}, \dots, U_{(N_s)^D}^{(k),0})^\top$, in which

$$(15) \quad U_j^{(k),0} = \exp \left\{ \sum_{d=1}^M [y(\tau_{k+1}) - y(\tau_k)] h_d(s_j) \right\} U^{(k-1),N_t},$$

$$j = 1, \dots, (N_s)^D.$$

Each vector $U^{(k),n}$ in (14) should be normalized such that $\sum_{j=1}^{(N_s)^D} U_j^{(k),n} = 1$. Then the vector $U^{(k),n}$ represents the probability distribution of the state at time $t_n^{(k)}$. Finally, we compute the expectation

$$(16) \quad \hat{\mathbf{x}}(t_n^{(k)}) = \sum_{j=1}^{(N_s)^D} s_j U_j^{(k),n}$$

as our estimation for the real state $\mathbf{x}(t_n^{(k)})$. In particular, we choose the parameter $\alpha = 1$ and $\beta = 0$, since the implicit scheme is stable in most of case while the explicit scheme ($\alpha = 0, \beta = 1$) is usually unstable. The algorithm in detail for solving the nonlinear filtering problem is stated in Algorithm 2.

2.3. Quasi-Implicit Euler Method (QIEM) for Kolmogorov Equations with Fast Fourier Transformations (FFTs)

The FFTs for the discretized Laplacian matrix is well-known. In the equation (13), we separate the matrix \mathbb{A} into 2 parts,

$$\mathbb{A} = \frac{1}{2} \Delta_D + \mathbb{A}_f,$$

where $\Delta_D = \sum_{d=1}^D \mathbb{T}_d$ and $\mathbb{A}_f = \sum_{d=1}^D \mathbb{P}_d \mathbb{K}_d + \mathbb{Q}$. Then we slightly modify the equation (12) into the form of the quasi-implicit scheme

$$(17) \quad \frac{U^{(k),n} - U^{(k),n-1}}{\Delta t} = \frac{1}{2} \Delta_D U^{(k),n} + \mathbb{A}_f U^{(k),n-1}.$$

Algorithm 2 IEM for Nonlinear Filter

Input: A terminal time Γ , the space interval $[-R, R]^D$, the time step size Δt , the space step size Δs , the vector-valued functions $\mathbf{f} = (f_1, \dots, f_D)^\top$, $\mathbf{h} = (h_1, \dots, h_M)^\top$, the observations $\{\mathbf{y}_k\}_{k=1}^{N_\tau}$ and the initial state σ_0

Output: the approximation of the state $\hat{\mathbf{x}}(t)$

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1: Set the values  $N_\tau = \frac{\Gamma}{\Delta\tau} + 1$ ,  $N_t = \frac{\Delta\tau}{\Delta t} + 1$  and  $N_s = \frac{2R}{\Delta s} + 1$ .
2: for  $j = 1$  to  $(N_s)^D$  do
3:    $U_j \leftarrow \sigma_0(s_j) \exp \left\{ \sum_{d=1}^M y_d(\tau_0) h_d(s_j) \right\}$ .
4: end for
5: for  $k = 1$  to  $N_\tau$  do
6:   for  $j = 1$  to  $(N_s)^D$  do
7:      $U_j \leftarrow \exp \left\{ \sum_{d=1}^M [y_d(\tau_{k+1}) - y_d(\tau_k)] h_d(s_j) \right\} U_j$  as in (15)
8:   end for
9:   for  $n = 1$  to  $N_t$  do
10:     $U_{tmp} = [I + \beta(\Delta t)\mathbb{A}] U$ .
11:    Solve the linear system  $[I - \alpha(\Delta t)\mathbb{A}] U = U_{tmp}$  as in (14)
12:    Normalize the solution  $U \leftarrow \frac{U}{\sum_{j=1}^{(N_s)^D} U_j}$ .
13:    Set the approximation of state  $\hat{\mathbf{x}}(t_n^{(k)}) = \sum_{j=1}^{(N_s)^D} s_j U_j$  as in (16)
14:   end for
15: end for

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Similarly, for each $t_j^{(k)} \in [\tau_{k-1}, \tau_k]$, $j = 1, \dots, N_t$, we solve the linear system

$$(18) \quad \left(I - \frac{\Delta t}{2} \Delta_D \right) U^{(k),n} = [I + \Delta t \mathbb{A}_f] U^{(k),n-1}.$$

Most important of all, the linear system (18) can be efficiently solved by FFTs. For the case of $D = 1$ (one-dimensional case), the Laplacian matrix in (18) satisfies $\Delta_1 \equiv T_1$. By Fourier sine transformation, we have the spectral decomposition

$$(19) \quad T_1 = \frac{1}{(\Delta s)^2} W S W^*,$$

where

$$W_{ij} = \sqrt{\frac{2}{N_s + 1}} \sin \left(\frac{ij\pi}{N_s + 1} \right),$$

and

$$S = \text{diag} \left\{ -4 \sin^2 \left(\frac{i\pi}{2(N_s + 1)} \right) \right\}_{i=1}^{N_s}.$$

Then the linear system of Δ_1 can be solved by calling the MATLAB function "FFT". The algorithm in detail for solving the D -dimensional nonlinear filtering problem by applying the FFTs is stated in Algorithm 3.

Algorithm 3 QIEM for Nonlinear Filter with FFTs

Input: A terminal time Γ , the space interval $[-R, R]^D$, the time step size Δt , the space step size Δs , the vector-valued functions $\mathbf{f} = (f_1, \dots, f_D)^\top$, $\mathbf{h} = (h_1, \dots, h_M)^\top$, the observations $\{\mathbf{y}_k\}_{k=1}^{N_\tau}$ and the initial state σ_0

Output: the approximation of the state $\hat{\mathbf{x}}(t)$

- 1: Set the values $N_\tau = \frac{\Gamma}{\Delta \tau} + 1$, $N_t = \frac{\Delta \tau}{\Delta t} + 1$ and $N_s = \frac{2R}{\Delta s} + 1$.
- 2: **for** $j = 1$ to $(N_s)^D$ **do**
- 3: $U_j \leftarrow \sigma_0(s_j) \exp \left\{ \sum_{d=1}^M y_d(\tau_0) h_d(s_j) \right\}$.
- 4: **end for**
- 5: **for** $k = 1$ to N_τ **do**
- 6: **for** $j = 1$ to $(N_s)^D$ **do**
- 7: $U_j \leftarrow \exp \left\{ \sum_{d=1}^M [y_d(\tau_{k+1}) - y_d(\tau_k)] h_d(s_j) \right\} U_j$ as in (15)
- 8: **end for**
- 9: **for** $n = 1$ to N_t **do**
- 10: $U_{tmp} \leftarrow \left[I + \Delta t \left(\sum_{d=1}^D \mathbb{P}_d \mathbb{K}_d + Q \right) \right] U$.
- 11: Call FFTs: $U \leftarrow \left(\bigotimes_{d=1}^D W^* \right) U_{tmp}$.
- 12: **for** $j = 1$ to N_s **do**
- 13: $U_j \leftarrow U_j / \left[1 + \frac{2\Delta t}{(\Delta s)^2} \sin^2 \left(\frac{j\pi}{2(N_s+1)} \right) \right]$.
- 14: **end for**
- 15: Call IFFTs: $U \leftarrow \left(\bigotimes_{d=1}^D W \right) U$
- 16: Normalize the solution $U \leftarrow \frac{U}{\sum_{j=1}^{(N_s)^D} U_j}$.
- 17: Set the approximation of state $\hat{\mathbf{x}} \left(t_n^{(k)} \right) = \sum_{j=1}^{(N_s)^D} s_j U_j$ as in (16)
- 18: **end for**
- 19: **end for**

2.4. Higher-Order QIEM for 2-D Kolmogorov Equations with FFTs

The Laplacian matrix in (18) is a second-order approximation of the Laplacian operator. Hereafter, we consider a fourth-order accurate scheme for Laplacian operator which reduces the size of the discretization matrix considerably, but preserves the same accuracy as the second-order approximation. Since there is no general form of the higher-order scheme for Laplacian operator, for convenience in practice, we consider the Kolmogorov equations (2) in two-dimensional case.

The 9-point scheme for the discretized Laplacian operator

$$(20) \quad \begin{aligned} \tilde{\Delta}_2 U_{i,j} = \frac{1}{6(\Delta s)^2} [& 4U_{i-1,j} + 4U_{i+1,j} + 4U_{i,j-1} + 4U_{i,j+1} \\ & + U_{i-1,j-1} + U_{i+1,j-1} + U_{i-1,j+1} + U_{i+1,j+1} - 20U_{i,j}] \end{aligned}$$

which is a fourth-order approximation of Laplacian operator.

$$(21) \quad \tilde{\Delta}_2 = \frac{1}{(\Delta s)^2} \begin{bmatrix} \Sigma & \Phi & & & \\ \Phi & \Sigma & \ddots & & \\ & \ddots & \ddots & \Phi & \\ & & \Phi & \Sigma & \end{bmatrix}$$

where

$$\Sigma = \frac{1}{3} \begin{bmatrix} -10 & 2 & & & \\ 2 & -10 & \ddots & & \\ & \ddots & \ddots & 2 & \\ & & 2 & -10 & \end{bmatrix}, \quad \Phi = \frac{1}{6} \begin{bmatrix} 4 & 1 & & & \\ 1 & 4 & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & 1 & 4 & \end{bmatrix}.$$

In the following, we derive the FFT for solving the linear system $\tilde{\Delta}_2 U = b$. Note that

$$\begin{aligned} \tilde{\Delta}_2 &= \left(\frac{1}{6} [(T_1 + 6I) \otimes (T_1 + 6I)] - 6I \right) \\ &= \frac{1}{6(\Delta s)^2} ((WSW^* + 6I) \otimes (WSW^* + 6I)) - 36I \\ &= \frac{1}{6(\Delta s)^2} ((W \otimes W) ((S + 6I) W^* \otimes (S + 6I) W^*) - 36I) \\ &= \frac{1}{6(\Delta s)^2} ((W \otimes W) (((S + 6I) \otimes (S + 6I)) - 36(I \otimes I)) (W^* \otimes W^*)), \end{aligned}$$

where $T_1 = \frac{1}{(\Delta s)^2} WSW^*$ as given in (19). Based on the QIEM Algorithm 3, the algorithm in detail for solving the D2 nonlinear filtering problem by applying the 4-order QIEM with FFTs is stated in Algorithm 4.

Algorithm 4 Fourth-order QIEM for 2-D Nonlinear Filter with FFTs

Input: A terminal time Γ , the space interval $[-R, R]$, the time step size Δt , the space step size Δs , the functions $\mathbf{f} = (f_1, f_2)^\top$, $\mathbf{h} = (h_1, h_2)^\top$, the observations $\{\mathbf{y}(\tau_k) = (y_1(\tau_k), y_2(\tau_k))^\top\}_{k=1}^{N_\tau}$ and the initial state σ_0

Output: the approximation of the state $\hat{\mathbf{x}}(t)$

```

1: Set the values  $N_\tau = \frac{\Gamma}{\Delta\tau} + 1$ ,  $N_t = \frac{\Delta\tau}{\Delta t} + 1$  and  $N_s = \frac{2R}{\Delta s} + 1$ .
2: for  $j = 1$  to  $(N_s)^2$  do
3:    $U_j = \sigma_0(s_j) \exp\{y_1(\tau_0)h_1(s_j) + y_2(\tau_0)h_2(s_j)\}$ .
4: end for
5: for  $k = 1$  to  $N_\tau$  do
6:   for  $j = 1$  to  $N_s$  do
7:      $U_j = \exp\{[y_1(\tau_{k+1}) - y_1(\tau_k)]h_1(s_j) + [y_2(\tau_{k+1}) - y_2(\tau_k)]h_2(s_j)\} U_j$ .
8:   end for
9:   for  $n = 1$  to  $N_t$  do
10:     $U_{tmp} \leftarrow [I + \Delta t (\mathbb{P}_1 \mathbb{D}_1 + \mathbb{P}_2 \mathbb{D}_2 + \mathbb{Q})] U$ .
11:    % Here the matrices  $W$  and  $S$  are defined in (19).
12:    Call FFT:  $U \leftarrow (W^* \otimes W^*) U_{tmp}$ .
13:    for  $j = 1$  to  $N_s$  do
14:       $U_j \leftarrow \left[ (I \otimes I) - \frac{\Delta t}{12(\Delta s)^2} ((S + 6I) \otimes (S + 6I) - 36(I \otimes I)) \right]^{-1} U_j$ .
15:    end for
16:    Call IFFT:  $U \leftarrow (W \otimes W) U$ .
17:    Normalize the solution  $U \leftarrow \frac{U}{\sum_{j=1}^{N_s} U_j}$ .
18:    Set the approximation of state  $\hat{\mathbf{x}}\left(t_n^{(k)}\right) = \sum_{j=1}^{(N_s)^2} s_j U_j$ .
19:   end for
20: end for

```

2.5. Superposition Technique

superposition technique.1,

$$\left\{ \delta_{\mathbf{c}_k}(\mathbf{s}) = e^{-\eta \|\mathbf{s} - \mathbf{c}_k\|^2} \middle| \mathbf{c}_k = (c_{k_1}, \dots, c_{k_D})^\top \in [-R, R]^D \right\}_{k=1}^{N_\delta}$$

as various initials to compute the approximate states for the nonlinear filtering problem, separately. Then we store all the fundamental solutions $\{v_k\}_{k=1}^{N_\delta}$ corresponding to $\{\mathbf{c}_k\}$.

In practice, for any given initial pdf u_0 , we calculate a set of coefficients $\{\alpha_k\}$ of the linear combination of Dirac delta

$$u_0 \approx \sum_{k=1}^{N_\delta} \alpha_k \delta_{\mathbf{c}_k}.$$

Then the approximate pdf of the state v can be directly obtained by computing the linear combination of the fundamental solutions

$$v \approx \sum_{k=1}^{N_\delta} \alpha_k v_k.$$

This method significantly saves a large amount of computational cost.

3. Nonnegativity and Convergence of IEM/QIEM

In this section, first, we will show that the linear operator defined in (14) is a nonnegative operator. Then, we will prove the convergence of IEM and QIEM.

3.1. Sufficient Condition of Nonnegative Operator

In the linear system (14), the solution of each time step represents a probability distribution of the space. In order to guarantee the property that each solution is nonnegative, we find the sufficient condition such that the matrix $(I - \Delta t \mathbb{A})^{-1}$ is a nonnegative operator. First, let us introduce M -matrix.

Definition. A real matrix $B = [B_{ij}]$ is called an M -matrix if $B_{ij} \leq 0, i \neq j$ and B^{-1} exists with $B^{-1} \geq 0$.

Lemma 1 (Equivalence Condition of M -matrix). [4] *Let B be a real matrix with $B_{ij} \leq 0$ for $i \neq j$. Then B is an M -matrix iff there is a positive vector $v > 0$ such that $Bv > 0$.*

The following theorem shows that the vector U in each iteration of step 11 in Algo 2 preserves nonnegativity of the pdf.

Theorem 1 (Sufficient Condition of Nonnegative Operator). *Given real-valued functions $p_d, d = 1, \dots, D, q$, a time step Δt and a space step Δs . Let $B \equiv I - \Delta t \mathbb{A}$, where \mathbb{A} is defined as (13). If for each $\mathbf{s} \in [-R, R]^D$,*

$$(22) \quad |p_d(\mathbf{s})| < \frac{1}{\Delta s}, \quad |q(\mathbf{s})| < \frac{1}{\Delta t},$$

for $d = 1, \dots, D$, then B is an M -matrix. That is, $B^{-1} \geq 0$ is a nonnegative operator.

Proof. First we check $B_{ij} \leq 0$ for $i \neq j$. From the structure of the matrices in (9) and (10), we see that for $i \neq j$ either $B_{ij} = 0$ or

$$\begin{aligned}
 B_{ij} &= -\Delta t \left(\frac{1}{2(\Delta s)^2} + \frac{p_d(\mathbf{s})}{2\Delta s} \right) \\
 &= -\frac{\Delta t}{2(\Delta s)^2} (1 + \Delta s p_d(\mathbf{s})) \\
 &\leq -\frac{\Delta t}{2(\Delta s)^2} (1 - \Delta s |p_d(\mathbf{s})|) \\
 (23) \quad &< 0.
 \end{aligned}$$

The inequality (23) follows from the first equation of (22). Next, we check $B\mathbf{1} > 0$, where $\mathbf{1} \equiv (1, 1, \dots, 1)^\top > 0$. Hence

$$\begin{aligned}
 B\mathbf{1} &= 1 - \Delta t \left(\frac{-k}{2(\Delta s)^2} + \frac{\sum_{d=1}^k (-1)^{m_d} p_d(\mathbf{s})}{2\Delta s} + q(\mathbf{s}) \right) \\
 &\geq 1 - \Delta t \left(\frac{-k}{2(\Delta s)^2} + \frac{k \max_d |p_d(\mathbf{s})|}{2\Delta s} + q(\mathbf{s}) \right) \\
 &= (1 - \Delta t q(\mathbf{s})) + \frac{k\Delta t}{2(\Delta s)^2} \left(1 - \Delta s \max_d |p_d(\mathbf{s})| \right) \\
 &\geq (1 - \Delta t |q(\mathbf{s})|) + \frac{k\Delta t}{2(\Delta s)^2} \left(1 - \Delta s \max_d |p_d(\mathbf{s})| \right) \\
 (24) \quad &> 0,
 \end{aligned}$$

for some $k \in \{1, \dots, D\}$, $m_d \in \{0, 1\}$. The inequality (24) follows from (22). By Lemma 1, B is an M -matrix. That is, $B^{-1} \geq 0$. \square

Fact. the vector $U = [I - \Delta t \mathbb{A}]^{-1} U_{tmp}$ in Step 11 of Algo 2 is nonnegative.

3.2. Convergence of IEM/QIEM

In this section, we prove the convergence of the IEM and the QIEM by checking the consistency and the stability of the schemes.

Theorem 2 (Consistency of IEM/QIEM). *The local truncation errors of IEM (12) and QIEM (18) are $O(\Delta t + (\Delta s)^2)$. That is, IEM and QIEM are consistent.*

Proof. The first-order Taylor expansion of u at the point $(t + \Delta t, s)$ implies

$$(25) \quad \frac{\partial u}{\partial t}(t, s) = \frac{u(t + \Delta t, s) - u(t, s)}{\Delta t} + O(\Delta t).$$

The third-order Taylor expansions of u at the points $(t, s + \Delta s)$ and $(t, s - \Delta s)$, respectively, lead to

$$(26) \quad \begin{aligned} u(t, s + \Delta s) &= u(t, s) + \Delta s \frac{\partial u}{\partial s}(t, s) + \frac{(\Delta s)^2}{2} \frac{\partial^2 u}{\partial s^2}(t, s) \\ &\quad + \frac{(\Delta s)^3}{6} \frac{\partial^3 u}{\partial s^3}(t, s) + O((\Delta s)^4), \end{aligned}$$

and

$$(27) \quad \begin{aligned} u(t, s - \Delta s) &= u(t, s) - \Delta s \frac{\partial u}{\partial s}(t, s) + \frac{(\Delta s)^2}{2} \frac{\partial^2 u}{\partial s^2}(t, s) \\ &\quad - \frac{(\Delta s)^3}{6} \frac{\partial^3 u}{\partial s^3}(t, s) + O((\Delta s)^4). \end{aligned}$$

By adding the equations (26) and (27), we obtain

$$(28) \quad \frac{\partial^2 u}{\partial s^2}(t, s) = \frac{u(t, s + \Delta s) - 2u(t, s) + u(t, s - \Delta s)}{(\Delta s)^2} + O((\Delta s)^2).$$

Similarly, by subtracting the equation (27) from (26), we obtain

$$(29) \quad \frac{\partial u}{\partial s}(t, s) = \frac{u(t, s + \Delta s) - u(t, s - \Delta s)}{2\Delta s} + O((\Delta s)^2).$$

Hence, according to the equations (7), (8) and (11), respectively, the equations (28), (29) and (25) show the local truncation error of (12) is $O(\Delta t + (\Delta s)^2)$. \square

Theorem 3 (Sufficient Condition for Stability of IEM). *The IEM (12) is stable if the function \mathbf{f} in (1) satisfies*

$$(30) \quad \boxed{\nabla \cdot \mathbf{f} \geq 0.}$$

Proof. We claim that IEM (12) is stable by applying von Neumann stability analysis. Let $U_j^n = \xi(k)^n e^{\imath kj(\Delta s)}$, where $\imath \equiv \sqrt{-1}$ and $\xi(k)$ is known as the amplification factor. Substituting U_j^n into the scheme (12), we obtain

$$\begin{aligned} \frac{\xi(k) - 1}{\Delta t} &= \frac{\xi(k)}{2(\Delta s)^2} \left(e^{\imath kj(\Delta s)} - 2 + e^{-\imath kj(\Delta s)} \right) \\ &\quad + \frac{\xi(k)}{2\Delta s} \left(e^{\imath kj(\Delta s)} - e^{-\imath kj(\Delta s)} \right) p(x) + \xi(k) q(s). \end{aligned}$$

That is,

$$\begin{aligned} \frac{1}{\xi(k)} &= 1 - \Delta t \left(\frac{e^{\imath kj(\Delta s)} - 2 + e^{-\imath kj(\Delta s)}}{2(\Delta s)^2} + \frac{e^{\imath kj(\Delta s)} - e^{-\imath kj(\Delta s)}}{2\Delta s} p(s) + q(s) \right) \\ &= 1 + \frac{\Delta t}{(\Delta s)^2} (1 - \cos(kj(\Delta s))) - (\Delta t) q(s) - \imath \frac{\Delta t}{\Delta s} \sin(kj(\Delta s)) p(s). \end{aligned}$$

If $\nabla \cdot \mathbf{f} \geq 0$, then $q(s) \equiv - \left[\nabla \cdot \mathbf{f} + \frac{1}{2} \sum_{j=1}^M h_j^2(\mathbf{s}) \right] \leq 0. \implies$

$$\begin{aligned} \frac{1}{|\xi(k)|^2} &= \left(1 + \frac{\Delta t}{(\Delta s)^2} (1 - \cos(kj(\Delta s))) - (\Delta t) q(s) \right)^2 \\ &\quad + \left(\frac{\Delta t}{\Delta s} \sin(kj(\Delta s)) p(s) \right)^2 \\ &\geq (1 - (\Delta t) q(s))^2 \geq 1. \end{aligned}$$

That is, $|\xi(k)|^2 \leq 1$. This implies that IEM (12) is stable under the assumption that $\nabla \cdot \mathbf{f} \geq 0$. \square

Theorem 4 (Sufficient Condition for Stability of QIEM). *The QIEM (18) is stable if both the step size of time Δt and the step size of space Δs are sufficient small. More precisely, Δt and Δs satisfy*

$$(31) \quad (\Delta s)^2 (2q(s) + q(s)^2 \Delta t) + \Delta t p(s)^2 \leq 2.$$

Proof. As in the proof of Theorem 3, we substitute $U_j^n = \xi(k)^n e^{\imath kj(\Delta s)}$ into the scheme (18) and obtain

$$\begin{aligned} \frac{\xi(k) - 1}{\Delta t} &= \frac{\xi(k)}{2(\Delta s)^2} \left(e^{\imath kj(\Delta s)} - 2 + e^{-\imath kj(\Delta s)} \right) \\ &\quad + \frac{1}{2\Delta s} \left(e^{\imath kj(\Delta s)} - e^{-\imath kj(\Delta s)} \right) p(s) + q(s). \end{aligned}$$

That is,

$$\xi(k) = \frac{1 + \Delta t q(s) + \frac{\Delta t}{\Delta s} \sin(kj(\Delta s)) p(s)}{1 - \frac{\Delta t}{(\Delta s)^2} (\cos(kj(\Delta s)) - 1)}.$$

If Δs and Δt satisfy $(\Delta s)^2 (2q(s) + q(s)^2 \Delta t) + \Delta t p(s)^2 \leq 2$, then

$$(\Delta s)^2 (2q(s) + q(s)^2 \Delta t) + \Delta t p(s)^2 \leq 2 + \frac{\Delta t}{(\Delta s)^2}.$$

Multiplying both side by $\frac{\Delta t}{(\Delta s)^2}$, we have

$$2q(s)\Delta t + q(s)^2(\Delta t)^2 + \frac{(\Delta t)^2}{(\Delta s)^2} p(s)^2 \leq \frac{2\Delta t}{(\Delta s)^2} + \frac{(\Delta t)^2}{(\Delta s)^4}.$$

Adding both side by 1, we obtain

$$(1 + \Delta t q(s))^2 + \left(\frac{\Delta t}{\Delta s} p(s) \right)^2 \leq \left(1 + \frac{\Delta t}{(\Delta s)^2} \right)^2.$$

\Rightarrow

$$\begin{aligned} |\xi(k)|^2 &= \frac{(1 + \Delta t q(s))^2 + \left(\frac{\Delta t}{\Delta s} \sin(kj(\Delta s)) p(s) \right)^2}{\left(1 - \frac{\Delta t}{(\Delta s)^2} (\cos(kj(\Delta s)) - 1) \right)^2} \\ &\leq \frac{(1 + \Delta t q(s))^2 + \left(\frac{\Delta t}{\Delta s} p(x) \right)^2}{\left(1 + \frac{\Delta t}{(\Delta s)^2} \right)^2} \leq 1. \end{aligned}$$

This implies QIEM (18) is stable under the given assumption. \square

Theorem 5 (Sufficient Conditions for Convergence of IEM/QIEM). *The IEM and QIEM converge if the conditions of (30) and (31) hold, respectively.*

Proof. From the consistency of IEM/QIEM in Theorem 2 as well as the stabilities of IEM and QIEM in Theorem 3 and Theorem 4, respectively, the convergence of IEM/QIEM follows by the Lax-Richtmyer equivalence theorem [8] immediately. \square

4. Numerical Experiments

In this section, we first show some numerical results by applying IEM/QIEM to 1-D and 2-D nonlinear filtering problems. Then we compare the accuracy of IEM/QIEM with the extended Kalman filtering method [16, 7, 17].

4.1. Numerical Results of IEM/QIEM

First, we consider the 1-D cubic sensor problem [19, 18] which is important and widely used in the nonlinear filtering control. The associated functions f and h in (1) for the 1-D cubic sensor problem are given by

$$(32) \quad f(s) = 0, \quad h(s) = s^3.$$

The parameters $\Delta s = 0.01$, $\Delta t = 0.0001$, $R = 10$ and $N_\delta = 201$. Figure 1(a)(b) show the numerical results by QIEM for the 1-D cubic sensor problem starting with two different random generators. The black and blue lines show the real states and the approximation for the real states, respectively. We see that two lines in Figure 1(a)(b) match very well.

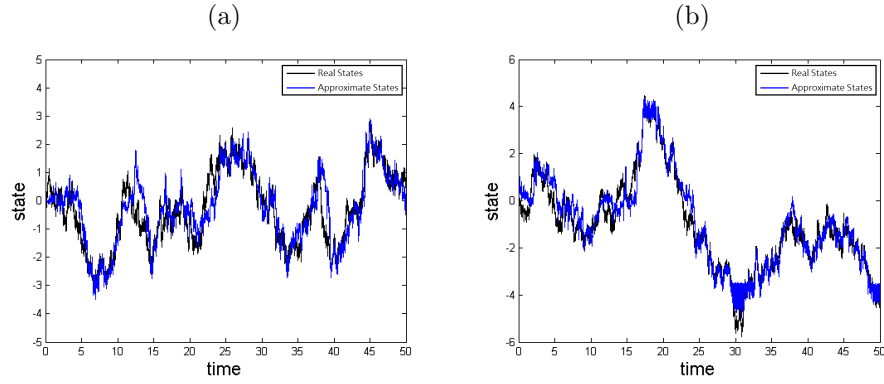


Figure 1: The numerical results by QIEM for 1-D cubic sensor problems

We now consider a 2-D nonlinear filtering problem with the associated functions \mathbf{f} and \mathbf{h} in (1) being given by

$$(33) \quad \begin{aligned} \mathbf{f}(s_1, s_2) &= (-\cos(s_1) - \sin(s_2), \cos(s_1) + \sin(s_2)), \\ \mathbf{h}(s_1, s_2) &= (s_1 - s_2, s_1 + s_2). \end{aligned}$$

Figure 3(a)(b) show the numerical results by QIEM for the 2-D nonlinear filtering problem starting with two different random generators. The black and blue lines show the real states and the approximation, respectively. We see that two lines in Figure 3(a)(b) still match well. In Figure 2, we show the density functions of the 2-D nonlinear filtering problem at $t = 0$ and $t = 1.2$, respectively.

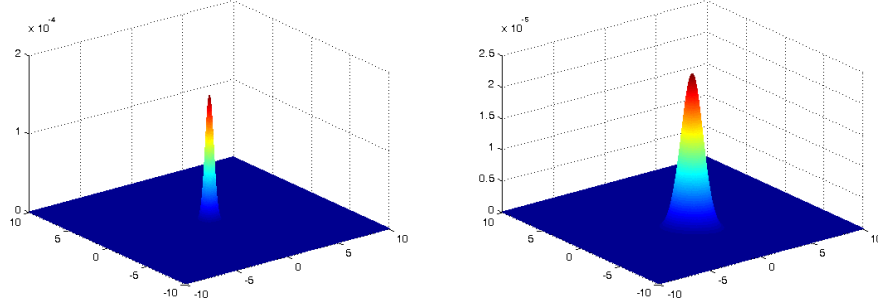
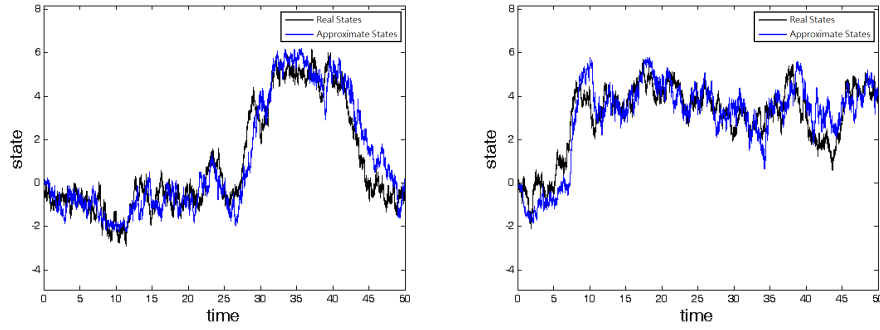


Figure 2: The density functions of the 2-D nonlinear filtering problem at $t = 0$ (left) and $t = 1.2$ (right), respectively.

(a) left: x_1 component, right: x_2 component



(b) left: x_1 component, right: x_2 component

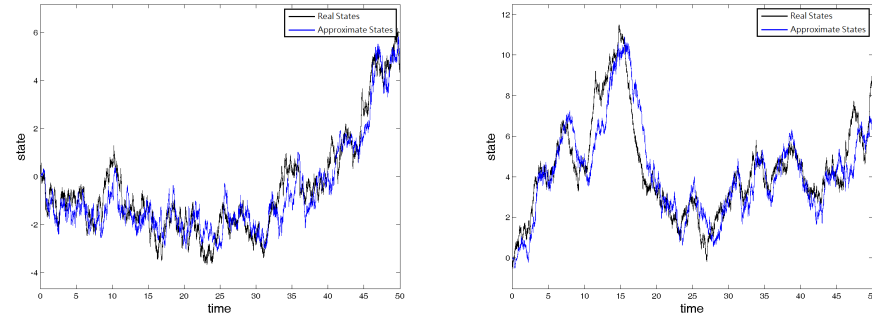


Figure 3: The numerical results of QIEM for the 2-D nonlinear filtering problems

According to our numerical experience, QIEM has the same accuracy as IEM, but QIEM has higher efficiency (less computational cost) than IEM for both 1-D and 2-D nonlinear filtering problems. Here, we ignore presenting the numerical results of IEM.

4.2. Comparison between Extended Kalman Filter and QIEM

The extended Kalman filter (EKF) [16, 7, 17] is a well-known method of dealing nonlinear filtering problem by using linear approach. We consider a 1-D nonlinear filtering problem in (1) with the associated functions $f(x) = \cos(x)$ and $h(x) = x^n$, for $n = 2, \dots, 6$.

In Figure 4, we show the comparison of numerical results between EKF and QIEM. The black, blue and red lines show the real states, the approximate states by QIEM and by EKF, respectively. We see that the approximate states by QIEM match much better than the approximate states by EKF.

In order to compare the accuracy of QIEM with EKF, we compute the root-mean-square error E_{RMS} between the approximate state $\hat{x}(\mathbf{t}) = (\hat{x}(t_1), \dots, \hat{x}(t_{N_\tau}))^\top$ and the real state $x(\mathbf{t}) = (x(t_1), \dots, x(t_{N_\tau}))^\top$ by

$$E_{RMS}(\hat{x}(\mathbf{t})) = \sqrt{\frac{\sum_{i=1}^{N_\tau} (\hat{x}(t_i) - x(t_i))^2}{N_\tau}}.$$

In Table 1, we show the root-mean-square error E_{RMS} of two methods. We see that the E_{RMS} of EKF increases from 0.64 to 1.33 when n of $h(x)$ grows from 2 to 6, while E_{RMS} of QIEM is between 0.43 to 0.56.

4.3. Numerical Results of Superposition Technique

In this subsection, we show some numerical results of the superposition technique (SPT) proposed in Section 2.5. We consider the 1-D cubic sensor problem and 2-D nonlinear filtering problem with \mathbf{f} and \mathbf{h} defined in (32) and (33), respectively. The initial probability density functions for 1-D and 2-D problems are chosen by $u_0(s) = e^s$ and $u(s_1, s_2) = e^{-5(s_1^2 + s_2^2)}$, respectively.

Figure 5(a)(b) and 6 show numerical results of SPT for 1-D cubic sensor problem and 2D nonlinear filtering problem, respectively. The black, blue and red lines show the real states, the approximate states by QIEM and by SPT, respectively. We see that SPT matches well as long as QIEM works well. Nevertheless, SPT saves a lot of computing time.

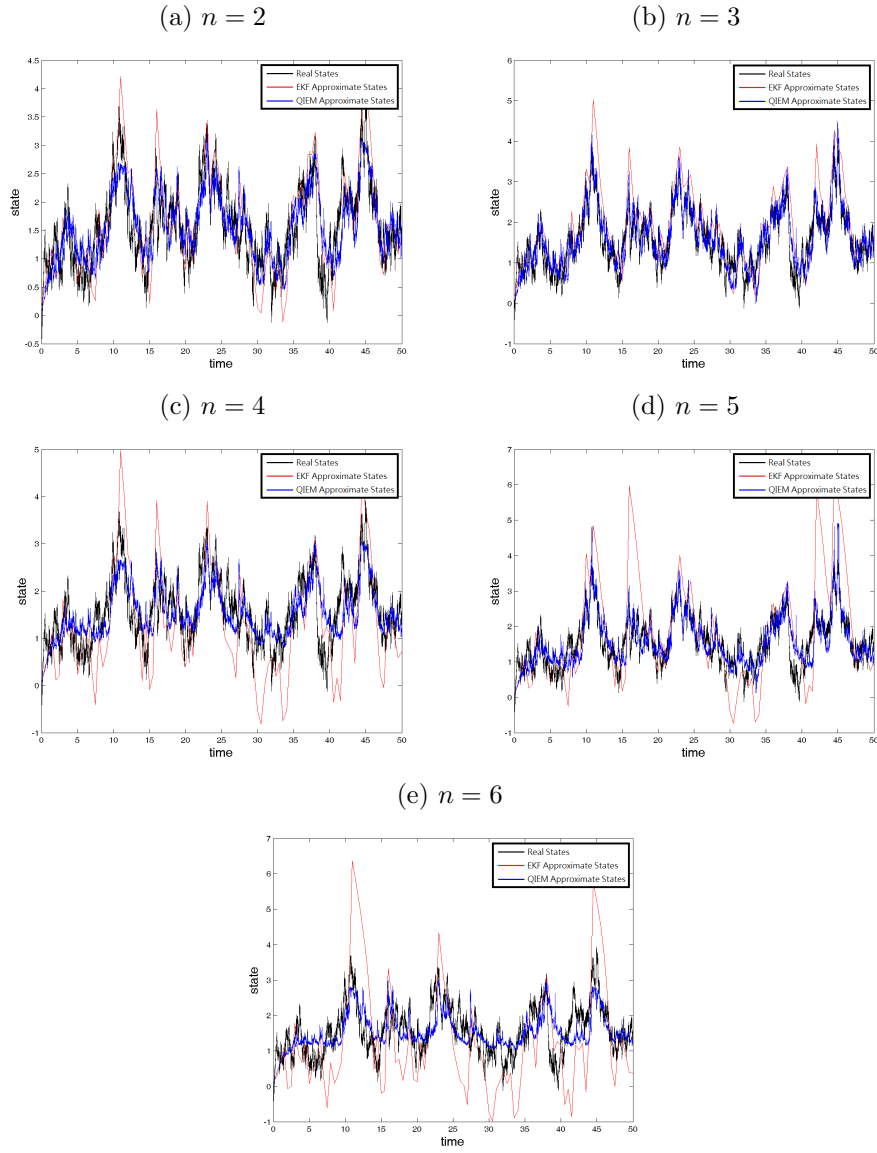
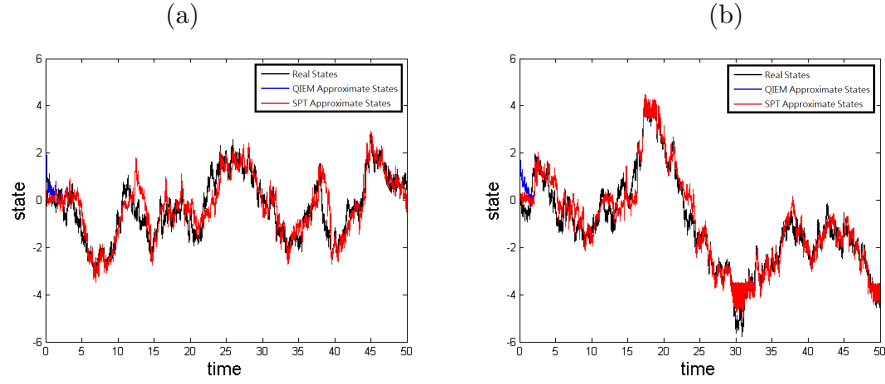


Figure 4: Comparison of real states and approximate states between EKF and QIEM with $f(x) = \cos x$, $h(x) = x^n$, $n = 2, \dots, 6$, respectively.

Table 1: The comparison of E_{RMS} between EKF and QIEM

Function Types \ Methods	EKF	QIEM
quadratic $h(x) = x^2$	0.6364	0.4630
cubic $h(x) = x^3$	0.6677	0.3600
quartic $h(x) = x^4$	0.9298	0.5089
quintic $h(x) = x^5$	1.2140	0.4279
sextic $h(x) = x^6$	1.3314	0.5597

Figure 5: The numerical results for 1-D cubic sensor problem with initial $u_0(s) = e^s$

5. Conclusions

In this paper, we propose efficient numerical algorithms, namely, IEM/QIEM/SPT for solving the nonlinear filtering problem by using the Yau-Yau method. QIEM is more efficient than IEM because the linear systems in QIEM is feasible by FFT-acceleration. SPT is an off-time process which saves a lot of computational cost. The solution of the initial-value problem of the Kolmogorov equations (2) forms a probability density function which must be nonnegative. We first show that the iterative matrix of the proposed schemes is an M -matrix whose inverse is a nonnegative operator and guarantees the nonnegativity of the solution in each iteration. Furthermore, we prove that the proposed numerical schemes converge to the initial-value problem (2) under some mild conditions. At the present stage, IEM/QIEM/SPT are capable for solving 1-D and 2-D nonlinear filtering problems efficiently and accurately. To make our algorithms feasible for solving the higher dimensional problems efficiently is our next goal and under investigation.

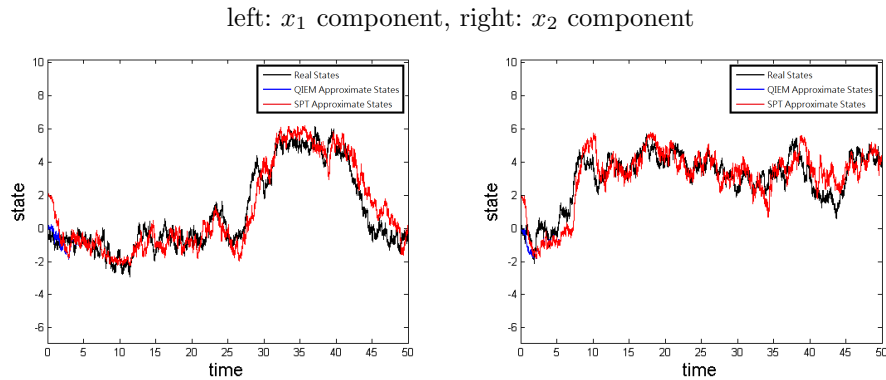


Figure 6: The numerical results for 2-D nonlinear filtering problem with initial $u(s_1, s_2) = e^{-5(s_1^2 + s_2^2)}$

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MEI-HENG YUEH
DEPARTMENT OF APPLIED MATHEMATICS
NATIONAL CHIAO TUNG UNIVERSITY
HSINCHU 300
TAIWAN
E-mail address: mhyueh@math.nctu.edu.tw

WEN-WEI LIN
DEPARTMENT OF APPLIED MATHEMATICS
NATIONAL CHIAO TUNG UNIVERSITY
HSINCHU 300
TAIWAN
E-mail address: wlin@math.nctu.edu.tw

SHING-TUNG YAU
DEPARTMENT OF MATHEMATICS
HARVARD UNIVERSITY
CAMBRIDGE, MA 02138
USA
E-mail address: yau@math.harvard.edu