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Linear Programming: an introduction

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CERMICS

Contents

CHAPTER 1

Overview

1.1 Introduction

1.1.1 Linear programming, what for?

The problem

Optimizing a linear criterion on a set of points defined by linear equalities and inequalities. Wlog:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c} \cdot \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \geqslant \boldsymbol{b}\,. \end{aligned} \tag{P}$$

A *polyhedron* is the intersection of finitely many closed half-spaces (bounded by hyperplanes). So linear programming is exactly the same thing as optimizing a linear criterion on a polyhedron.

Applications

- Direct industrial applications (transportation, production, energy, etc.).

  Example. m resources, n products. Producing one unit of j requires a_{ij} units of i, and generates a revenue of c_j. Maximize the revenue, under constraint resources.

- A bit more than linearity. maximum, absolute value, convex piecewise affine objective functions

- Integer polytopes, e.g., matchings in bipartite graph.

- Integer linear programming. Useful to solve "integer linear programming": bounds, TSP

  For instance, on an instance of more that 1.3 billions stars from the galaxy, a tour within 0.38% of optimality has been recently computed. Proving this optimality gap relies on linear programming.

1.1.2 History

Fourier (1827): a method for solving system of inequality constraints (not very efficient). This method is now known as the "Fourier–Motzkin" elimination (used for theory).

Kantorovich (1939) and Koopmans (1942): prix Nobel 1975. Pioneers of tranportation problems. Kantorovich was motivated by concrete applications to the production of plywood. Koopmans considered duality.

Dantzig (1947): simplex algorithm.

Dantzig (1948): duality (suggested by Von Neumann).

Khachyian (1979): ellipsoids, linear programming is polynomial.

Karmarkar (1984): interior-points methods, practically efficient polynomial algorithm for linear programming.

1.1.3 Algorithms

Several algorithms have been proposed to solve linear programs. Simplex, interior-point methods, ellipsoids are the most famous.

$10,000$ variables and constraints can be solved within seconds with best solvers on a standard laptop, and within one hour when dealing with integer variables. (Very rough numbers, and depend highly on the structure of the problem.)

1.2 Main results

1.2.1 The Minkowski–Weyl theorem and consequences

The *convex hull* of a finite set X of points is

$$\operatorname{conv}(X) := \left\{ \sum_{\boldsymbol{x} \in X} \lambda_{\boldsymbol{x}} \boldsymbol{x} \colon \sum_{\boldsymbol{x} \in X} \lambda_{\boldsymbol{x}} = 1,\ \lambda_{\boldsymbol{x}} \geqslant 0 \right\}.$$

The *positive hull* of a finite set X of points is

$$\operatorname{pos}(X) := \left\{ \sum_{\boldsymbol{x} \in X} \lambda_{\boldsymbol{x}} \boldsymbol{x} \colon \lambda_{\boldsymbol{x}} \geqslant 0 \right\}.$$

Theorem 1.1 (Minkowski–Weyl). *Let P be a polyhedron of $\mathbb{R}^n$. Then there exit two finite sets $V, R \subseteq \mathbb{R}^n$ such that $P = \operatorname{conv}(V) + \operatorname{pos}(R)$.*

The proof relies on the following lemma, which is actually a special case.

Lemma 1.2. *Given a matrix M, we have $\{\boldsymbol{x} \colon M\boldsymbol{x} \geqslant \boldsymbol{0}\} = \operatorname{pos}(S)$ for some finite set S of points.*

Proof. □

Proof of Theorem 1.1. Let $P = \{\boldsymbol{x}\colon A\boldsymbol{x} \geqslant \boldsymbol{b}\}$. Apply Lemma 1.2 to $M = \begin{pmatrix} A & -\boldsymbol{b} \\ \boldsymbol{0}^\top & 1 \end{pmatrix}$, we get the existence of S such that for every $\boldsymbol{x} \in P$ we have

$$\begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} = \sum_{\boldsymbol{s}\in S} \lambda_{\boldsymbol{s}} \boldsymbol{s}\,,$$

for some $\lambda_{\boldsymbol{s}} \geqslant 0$. Up to some scaling, we suppose w.l.o.g. that the last component of every $\boldsymbol{s} \in S$ is either 0, or 1. Let V' be the points in S with a last component equal to 1, and let R' be those with a last component equal to 0. Then setting V (resp. R) as the points of V' (resp. R') from which the last component has been removed, we get that $P \subseteq \operatorname{conv}(V) + \operatorname{pos}(R)$.

Conversely, since P is feasible, we have $V \neq \varnothing$, and let $\boldsymbol{x} \in \operatorname{conv}(V)$ and $\boldsymbol{z} \in \operatorname{pos}(R)$ (note that this latter set is always non-empty). Then, by construction, $A\boldsymbol{x} \geqslant \boldsymbol{b}$ and $A\boldsymbol{z} \geqslant \boldsymbol{0}$, which implies that $\boldsymbol{x} + \boldsymbol{z} \in P$. □

Moreover, the proof makes clear the following fact: when P contains no infinite line (we say it is *pointed*), the elements of V can be chosen to be of the form $(A')^{-1}\boldsymbol{b}$ for some non-singular submatrix A' of A. This has several important consequences. Let us mention one here. A *vertex* of P is a point of P that cannot be written as the convex combination of two distinct points of P. Suppose that P is pointed; then the elements in V can be chosen to be the vertices of P.

The Minkowski–Weyl theorem can be used to establish a result, which is sometimes called "the fundamental theorem of linear programming."

Theorem 1.3. *Suppose* (P) *is feasible and has its objective function lower bounded on the set of its feasible solutions. Then* (P) *admits an optimal solution. If moreover the polyhedron formed by the feasible solutions is pointed, then there always exists a vertex of the polyhedron that is an optimal solution.*

The fact that the optimal value is attained is a real specificity of linear programming: consider for instance the minimization of $1/x$ over $[1, +\infty)$ to see that in general this might not hold.

Proof of Theorem 1.3. Denote by P the set of feasible solutions. Let V and R as in Theorem 1.1. If $\boldsymbol{c}\cdot\boldsymbol{r} < 0$ for some $\boldsymbol{r} \in R$, then clearly the objective function would be unbounded. So $\boldsymbol{c}\cdot\boldsymbol{r} \geqslant 0$ for all $\boldsymbol{r} \in R$. Consider $\boldsymbol{v}^\star \in V$ minimizing $\boldsymbol{c}\cdot\boldsymbol{v}$ on V. By convexity, $\boldsymbol{c}\cdot\boldsymbol{v}^\star \leqslant \boldsymbol{c}\cdot\boldsymbol{x}$ for all $\boldsymbol{x} \in \operatorname{conv}(V)$. Since $\boldsymbol{c}\cdot\boldsymbol{r} \geqslant 0$ for all $\boldsymbol{r} \in R$, we have further $\boldsymbol{c}\cdot\boldsymbol{v}^\star \leqslant \boldsymbol{c}\cdot\boldsymbol{x}$ for all $\boldsymbol{x} \in P$.

For the case when P is pointed, the statement results from the remark above: V can then be chosen as the set of vertices of P. □

1.2.2 Duality

Consider

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{b}\cdot\boldsymbol{y} \\ \text{s.c.} \quad & A^\top\boldsymbol{y} = \boldsymbol{c} \\ & \boldsymbol{y} \geqslant \boldsymbol{0}\,. \end{aligned} \tag{D}$$

Theorem 1.4. *If* (P) *or* (D) *is feasible then, their optimal values are equal.*

Strong duality is a consequence of Farkas's lemma, itself being a consequence of separation.

Lemma 1.5 (Farkas lemma)**.** *Exactly one of the following occurs:*

- *There exists $\boldsymbol{u}$ such that $M\boldsymbol{u} \geqslant \boldsymbol{p}$.*
- *There exists $\boldsymbol{v} \geqslant \boldsymbol{0}$ such that $M^\top \boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{p} \cdot \boldsymbol{v} = 1$.*

Proof. Suppose that there is no $\boldsymbol{u}$ such that $M\boldsymbol{u} \geqslant \boldsymbol{p}$, and consider $C = \{\boldsymbol{z} \colon \exists \boldsymbol{u} \text{ s.t. } \boldsymbol{z} \leqslant M\boldsymbol{u}\}$. By the separation theorem, there exist $\boldsymbol{v}$ and γ such that $\boldsymbol{v} \cdot \boldsymbol{z} \leqslant \gamma$ for all $\boldsymbol{z} \in C$ and $\boldsymbol{p} \cdot \boldsymbol{v} > \gamma$. Because C is unbounded, we can actually choose γ to be 0. Using again unboundedness, we see that necessarily $\boldsymbol{v} \geqslant \boldsymbol{0}$. By a simple scaling operation, we get $\boldsymbol{p} \cdot \boldsymbol{v} = 1$, as desired.

The reverse implication is immediate. □

Proof of Theorem 1.4. If the optimal value of (P) is $-\infty$ or the optimal value of (D) is $+\infty$, then the theorem results easily from weak duality. So it is sufficient to consider the case where (D) has a finite optimal value. (The case where (P) has a finite value will be a consequence, since the dual of the dual is the primal.) Denote by α the optimal value of (D). Let $M \coloneqq \left(\begin{smallmatrix} A \\ -\boldsymbol{c}^\top \end{smallmatrix}\right)$ and $\boldsymbol{p} \coloneqq \left(\begin{smallmatrix} \boldsymbol{b} \\ -\alpha \end{smallmatrix}\right)$.

Suppose for a contradiction that there exists $\boldsymbol{v} = \left(\begin{smallmatrix} \boldsymbol{y} \\ w \end{smallmatrix}\right) \geqslant \boldsymbol{0}$ such that $M^\top \boldsymbol{v} = 0$ and $\boldsymbol{p} \cdot \boldsymbol{v} = 1$. Denote by $\bar{\boldsymbol{y}}$ a feasible solution of (D). It exists since the optimal value of (D) is finite. If $w = 0$, then we would have $\boldsymbol{y} \geqslant \boldsymbol{0}$ such that $A^\top \boldsymbol{y} = \boldsymbol{0}$ and $\boldsymbol{b} \cdot \boldsymbol{y} = 1$, which would imply that $\bar{\boldsymbol{y}} + t\boldsymbol{y}$ is a feasible solution of (D) for all $t > 0$, which would in turn imply that the optimal value of (D) is $+\infty$; a contradiction. So $w > 0$, but this implies that there exists a feasible solution of (D) giving to the objective function a value larger than α; again a contradiction. Thus there is no such $\boldsymbol{v}$, and Farkas's lemma implies that there exists $\boldsymbol{u} = \boldsymbol{x}$ such that $A\boldsymbol{x} \geqslant \boldsymbol{b}$ and $\boldsymbol{c} \cdot x \leqslant \alpha$. Weak duality leads then to the desired conclusion. □

1.2.3 Polynomial solvability

1.3 Applications and special cases

1.3.1 Zero-sum games

1.3.2 Flows

1.3.3 TU matrices

1.4 Open questions

1.4.1 Strong polynomiality

1.4.2 Diameter of polytopes

Bibliography