Open Problems from CCCG 2024

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The following is a list of the problems presented during the open problem session at the 36th annual Candian Conference on Computational Geometry (CCCG), held at Brock University from July 17th to July 20th, 2024.

This year, the open problem session was held on the first day of the conference. While this is fairly typical for CCCG, it not always the case. This year a few attendees expressed that they thought it would be beneficial for the community if this schedule became the standard for future editions of CCCG. The authors of this document agree. Having the open problem session early in the program provides extra time for collaboration and progress to occur organically before attendees have to return home.

In this document, we include all progress (to the best of our knowledge) on the presented problems which was completed during CCCG. Further, courtesy of Erik Demaine a CoAuthor server has been created to track progress on these problems. Available (**TBD**)

Doming Polygons
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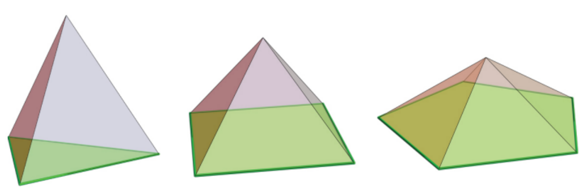

Figure 1: Examples of doming an equilateral triangle, a square, and a regular pentagon (left to right).[GAD+24]

> Given a convex polygon P, does there exists a convex polyhedron Q such that one face of Q is P and the remaining faces of Q admit a partition by congruent equilateral triangles. If such a polyhedron Q exists for the polygon P, we say that the polygon P can be *domed*, see Figure 1 for illustrations.

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For all $3 \leq n < 12$ with $n \neq 7$, we know that there exists a convex n-gon that can be domed. From the Gauss-Bonnet theorem, one can easily show there exists no convex n-gon with $n > 55$ that can be domed, hence we have an upper bound of $n < 55$. With a more detailed analysis of curvature, one can establish that there can be at most 11 vertices in the dome, which, when combined with an analysis of the degree of vertices in the base, a stronger bound of $n \leq 24$ can be achieved. The questions posed are the following.

1. Does there exist a convex 7-gon that can be domed?
2. Does there exist a convex n-gon with $n \geq 12$ that can be domed?
3. Does there exist a non-equilateral triangle that can be domed?

References

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Inside-Out-Dissections
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> Given a polygon (polyhedron) P_1, can we decompose P_1 into k pieces and then rearrange them by only applying rotations and translations to the different pieces such that (1) the rearranged pieces

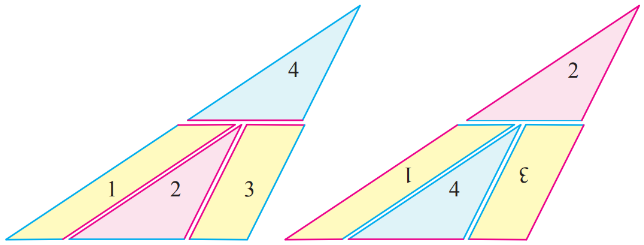
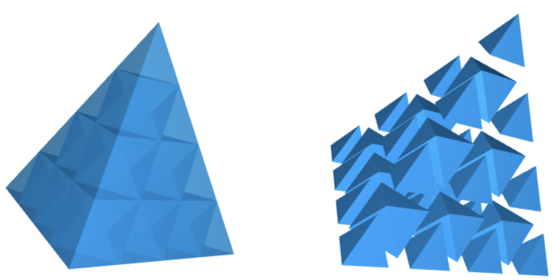


Figure 2: An inside-out-dissection of a triangle using four partitions by Aaron Meyerowitz.[Mey14]

form a polygon (polyhedron) P_2 that is congruent to P_1 and (2) the boundary of P_2 is composed of internal cuts of the dissection of P_1?

If we can, we say that P_1 can be *inside-out-dissected* (Figures 2, 3, 4). The goal is to find the minimum k for a class of polygon (polyhedron). For example, any triangle can be done with $k = 4$ (Figure 2). Therefore, any n-gon can be triangulated with $n - 2$ triangles and be naively inside-out-dissected with $k = 4 \cdot (n - 2)$. The questions posed are the following.

1. Can the bound of 4 times the number of triangles in the triangulation of P_1 be beaten?
2. Can every (or any) tetrahedron be inside-out-dissected?



(a) (b)

Figure 3: Subdivision of a regular tetrahedron with edge length 1 into 24 regular tetrahedra and 10 regular octahedra with all edges having length $\frac{1}{4}$; This subdivision can be used to construct an inside-out dissection of the initial tetrahedron

Reversible hinged dissections [ADL20] are a similar concept. Here we are given two shapes and asked if we can dissect one of them and reassemble the pieces to form the other. There is an extra constraint requiring that the two shapes are connected by a continuous movement where pieces are connected at vertices (hinges) and they are allowed to rotate around such vertices. The "reversible" qualifier describes yet another constraint where the boundary

(a) (b)

Figure 4: Subdividing a square into a non-convex polygon and 8 congruent isosceles triangles; This subdivision can be exploited to create an inside out-dissection of the given square.

of one shape must lie in the interior of the other. Akiyama, Demaine and Langerman characterized such dissections showing that they exist if the two shapes are common nets of the same polyhedron.

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Contiguous Boundary Guarding of Art Galleries

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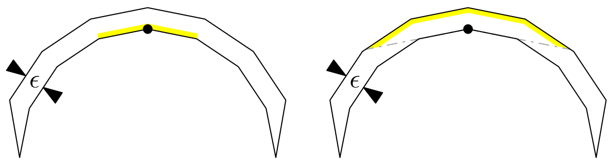
A paper presented at this year's CCCG [KCHA24] discussed the *Contiguous Watchtower Problem*. You are given a terrain T (a 2D polygonal chain), the goal is to find a placement for watchtowers to guard the terrain. Watchtowers are vertical segments with the bottom endpoint on the terrain. A watchtower w guards a point $p \in T$ if p is visible to the top endpoint of w; a straight line segment can be drawn from p to the top endpoint without crossing T. This problem is called the *Contiguous* Watchtower Problem as there is an additional constraint: the watchtowers can only guard a contiguous set of points along T. This inspired the following art gallery variant:

Given a simple polygon in the plane, we wish to guard its boundary (its interior is not necessarily covered) using the standard formulation of the art gallery problem, with one exception: A guard is only allowed to guard one *contiguous* region of the boundary (see Figure 5). So for example, Note



(a) (b)

Figure 5: A polygon P representing an art gallery to be guarded. A guard positioned at the small solid black disc either guards the portion of the boundary of P highlighted in (a), or the one highlighted in (b) (and not both).If the polygon is thin enough, i.e., if ϵ is small enough, then it is impossible to guard more than 2 full segments (except near the bottom vertices), yielding a linear lower bound on the number of guards. This example was presented by Ahmad Biniaz during a coffee break at CCCG.

that hardness proofs for typical art gallery variants require guards to guard several disconnected regions of a polygonal domain, so maybe this is in P? The questions posed are the following.

1. What are the combinatorial bounds on the minimum number of guards in the worst case?
2. Does there exists a polynomial time algorithm that computes the minimum number of guards and their positions?

References

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Max-Clique of Grounded Line Segments

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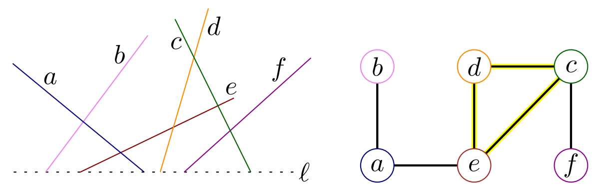
A grounded string graph is an intersection graph of a set of polygonal lines (or strings), where the polylines lie above a common horizontal line ℓ and have exactly one endpoint on ℓ. Given an intersection representation of such a graph, a maximum independent set can be computed in polynomial time [KMPV17] but finding a maximum clique is known to be APX-hard [KMMN22]. The hardness result holds even when each string is restricted to be a polyline with 1 bend. A natural problem in this context is to determine the time complexity of finding a maximum clique in grounded segment representations, i.e., when the strings are line segments [KMMN22]. Since a unit-segment graph is a proper subclass of the class of segment graphs [CJ17], it may be interesting to first tackle the problem for unit-segment graphs irrespective of whether the segments are grounded or not. We thus pose the following open question.



(a) (b)

Figure 6: (a) Six unit segments grounded on ℓ and labeled $a, \ldots, f$. (b) The corresponding intersection graph, called the unit-segment graph. In this example, the only maximum clique is $\{c, d, e\}$ (highlighted in yellow).

Open Question: Does there exist a polynomial time algorithm to find a maximum clique in a (grounded) unit-segment graph (see Figure 6)?

References

[CJ17] Sergio Cabello and Miha Jejčič. Refining the hierarchies of classes of geometric intersection graphs. *The Electronic Journal of Combinatorics*, pages 1–33, 2017. URL: `https://www.combinatorics.org/ojs/index.php/eljc/article/view/v24i1p33/6724`.

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[KMPV17] J Mark Keil, Joseph SB Mitchell, Dinabandhu Pradhan, and Martin Vatshelle. An algorithm for the maximum weight independent set problem on outerstring graphs. *Computational Geometry*, 60:19–25, 2017. Also appears in Canadian Conference on Computational Geometry 2015. `doi:10.1016/j.comgeo.2016.05.001`.

Independent Set for Outer-String Graphs in a Polygon With Holes

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Recall that an outerstring graph G is a geometric intersection graph of Jordan curves, $J_1, \ldots, J_n$, where one point on each curve falls along the boundary of the disk and all other points are internal to or fall along the boundary of the disk. Here, in the special case where G is an intersection graph of n chords in a circle, assuming the real RAM model, and provided an intersection model (or diagram) for the chords, we know that an $O(n^2)$ algorithm exists for the problem of finding a Maximum Independent Set (MIS) of vertices. Specifically, an $O(n^3)$ algorithm was originally given by [Gav73], and this was subsequently improved to have an $O(n^2)$ time complexity [AIM91],[GT94],[Sup87].

Much more recently, letting G be an outerstring graph admitting an intersection model where all Jordan curves are chains of straight-line segments connected at their endpoints, and where there are a total of r straight line segments for all curves (i.e., where r is the total number of straight-line segments in the intersection model), [KMPV17] showed that an $O(r^3)$ algorithm exists for finding a maximum independent set of vertices for G.

Provided this context, let's now consider a variation on the notion of outerstring graphs in which we require the existence of an intersection model where: (i) all points along the intersecting Jordan curves are internal to or fall along the boundary of a simple polygon with holes P (note that the curves are not permitted to enter the holes); (ii) all Jordan curves are segmented curves corresponding to chains of straight-line segments connected at their endpoints; (iii) all Jordan curves have a point along *either* the outer boundary of P *or* the boundary of a hole in P. Letting h be the number of holes in P, perhaps we can call such outerstring graphs *h-weak-outerstring* graphs (Figuer 7).

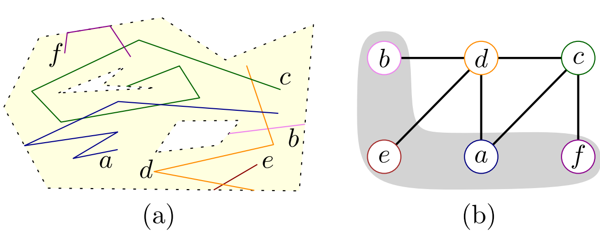

Figure 7: (a) A model of a 2-weak-outerstring graph G (with $r = 19$). (b) The corresponding G; the only maximum independent set is $\{a, b, e, f\}$ (shaded in grey).

Question: Assuming the real RAM model and letting G be an instance of an h-weak-outerstring graph, is there a Fixed-Parameter Tractable (FPT) algorithm for finding a maximum independent set of vertices for G where the one or more parameters depend on the number of holes h in an interesting way? Briefly, what I mean here by "interesting" is that we don't simply count the number of Jordan curves we need to "cut through" in the diagram to draw connecting lines between the holes and the outer boundary, reducing to the current problem to the one [KMPV17] originally considered. We can also pose the same question for the problem of recognizing G as an h-weak-outerstring graph.

References

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[Sup87] Kenneth J Supowit. Finding a maximum planar subset of a set of nets in a channel. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 6(1):93–94, 1987. `doi:10.1109/TCAD.1987.1270250`.