Inside-Out Dissections of Polygons and Polyhedra

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Abstract

This work is motivated by a question posed at last year's CCCG open problems session. In particular, we investigate inside-out dissections of polygons and polyhedra. We present various examples of polygons and polyhedra that can be inside-out dissected. We prove that a regular polygon can be inside-out dissected with at most 6 pieces. Further, we establish the existence of inside-out dissections for every polyhedra in Euclidean 3-space.

1 Introduction

In 2014 Joseph O'Rourke [18] uploaded a post on Math Stack Exchange addressing the following question: Can a polygon be subdivided into finitely many pieces, meaning polygons, that can be rearranged into a congruent copy of the original polygon such that the perimeter of the new polygon consists entirely of interior cuts of the original? He called these rearrangements "inside-out dissections". When this question was originally posed, Joseph O'Rourke presented an inside-out dissection for an arbitrary triangle that uses 9 pieces. Aaron Meyerowitz responded to his post with an inside-out dissection of an arbitrary triangle that only uses 4 pieces [17] (see Figure 1).

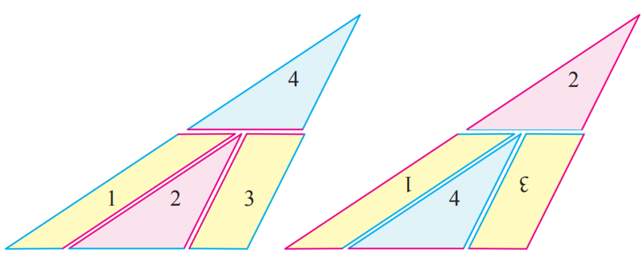

Figure 1: An inside-out-dissection of an obtuse-angled triangle using four pieces by Aaron Meyerowitz.[17]

Since every polygon with n vertices can be triangulated with $n-2$ triangles, it follows that every polygon can be inside-out dissected with at most $4(n-2)$ pieces. This raises the following question:

Question. *What is the minimum number of pieces required to inside-out dissect an arbitrary polygon with n edges?*

There was some initial discussion on the forum, but this and other questions remained unanswered. Therefore, Joseph O'Rourke posed this problem at the open problem session at CCCG 2024 [4]. The two main questions discussed at the conference were:

Question 1 *Let P be an arbitrary n-gon. Does there exist an inside-out dissection of P with k pieces $P_1, \ldots, P_k$, where $k < 4(n-2)$?*

Question 2 *Can every polyhedron be inside-out dissected?*

Question 1 was answered by A. Rivkin at CCCG 2024 by presenting a method that can inside-out dissect any n-gon with $2n+1$ pieces, see Theorem 1. Inspired by this result, we give an improved upper bound for some polygons by showing that all regular n-gons can be inside-out dissected with at most 6 pieces.

When Question 2 was presented at the open problems session, there were no known results on inside-out dissections of polyhedra. Figure 8 illustrates an inside-out dissection that was constructed by R. Akpanya. In this work, we answer Question 2 by showing that every polyhedron P with k faces can be inside-out dissected with at most $1+\sum_{i=1}^{k} 3\cdot 24\cdot(106n_i-216)$ pieces, where the i-th face of P has n_i edges.

The topic of polygonal dissections has been well-studied by the computational geometry (and the broader mathematical) community. A polygonal dissection problem gives two polygons P and Q as inputs. It then asks whether a set of polygons S can be produced via cuts of P, where the elements of S can then be transformed to create Q. Variations of dissection problems are obtained by:

1. Placing constraints on P or Q.
2. Restricting the types of cuts that are allowed.
3. Limitations on the transformations can be applied to elements of S.

The Wallace–Bolyai–Gerwien theorem [5, 14] from the early 1800s states that if P and Q are two polygons with the same area, then P can be dissected and turned into Q by using only translations and rotations.

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A more exotic class of dissections is the class of *hinged dissections*. In a hinged dissection, pieces of a polygon P can be translated and rotated, however, each piece must be "hinged" to another (where hinges are points on the boundary of each polygon) and every transformation must be made subject to these hinges. Abbot et al. [1] proved that given starting and target polygons, P and Q, where P and Q have the same area, there is always a hinged dissection from P to Q. Hinged dissections on their own are a widely researched topic see, [2, 9, 11] to just name a few papers. The first published example of a hinged dissection is Dudeney's dissection [10], which shows a hinged dissection between a square and a triangle, using only 4 pieces. Even this simple construction of an early polygonal dissection is still of interest to the community. For instance, this construction was recently (late 2024) proved to be optimal (with respect to the number of pieces) [8].

In 2023, Eppstein [12] considered dissections of polygons into the minimum number of rectangles. Here, the starting polygon P is any axis-aligned polygon and the pieces of such a dissection can only be constructed by axis-aligned cuts of P. However, the target shape Q is no longer predefined, and we instead consider the optimization problem: What is the fewest number of rectangles that can be constructed? An exact algorithm is provided in the case that the only allowed transformations are translations, and Eppstein gives a 2-approximation when both translations and rotations are allowed.

One of the oldest examples of a polygonal dissection is the Tangram puzzle. In this puzzle the initial polygon P is always a square, and the set S is a predefined set of 7 pieces. Q is chosen by the player from a list of predefined shapes. Finally, pieces of S can be transformed by rotations, translations, and flips (see [19] for a formal analysis). For some other works on dissections we refer to [15, 16].

Regarding dissections of polyhedra, there is the notion of the Dehn Invariant [7]. Given two polyhedra, they can be dissected and rearranged into each other if and only if they have equal volume and Dehn Invariant. Crucially, this only addresses the existence of a dissection but not an inside-out-dissection, leaving Question 2 open.

2 Preliminaries

In this section, we present our notions of polygons and polyhedra. We further provide a formal definition of an inside-out dissection of a polygon (polyhedron). For this purpose, we denote the convex hull of a set $M \in \mathbb{R}^n$ as $\operatorname{conv}(M)$ in this paper.

Remark 1

1. *It is unclear how an inside-out dissection for a non-simple polygon would be defined. Hence, we assume that all polygons in this work are simple. Furthermore, we note that the polygons in this paper do not necessarily have to be convex.*
2. *In the literature, there are different notions of polyhedra, see [20] for instance. Here, a polyhedron is considered to be a subset of the real 3-space whose boundary solely consists of polygonal faces, straight line edges and vertices. Hence, the polyhedra in this work are allowed to be non-convex. Further, we will exploit the fact that a convex polyhedron can be constructed as the convex hull $\operatorname{conv}(M)$ of a finite subset $M \subset \mathbb{R}^3$. Moreover, δP denotes the boundary of P.*

Definition 1 *We say that polygons (polyhedra) $P_1, \dots, P_k$ form a decomposition of a polygon (polyhedron) P, if*

1. $\bigcup_{i=1}^{k} P_i = P$,
2. $P_i \cap P_j = \delta P_i \cap \delta P_j$ *for all* $1 \le i < j \le k$.

With these notions in place, we are able to formally define inside-out dissections.

Definition 2 *Let P be a polygon (polyhedron). An* **inside-out dissection** *of P is a decomposition of P into finitely many polygons (polyhedra) $P_1, \dots, P_k$ such that*

1. *the polygons $P_1, \dots, P_k$ can be rearranged by only applying rotations and translations to form a decomposition of a polygon (polyhedron) P' that is congruent to P,*
2. *the boundary of the polygon (polyhedron) P' is composed of internal cuts of P, i.e. no line segment (polygon) contained on the boundary of P appears as a line segment (polygon) on the boundary of P'.*

In the case that such a decomposition of P exists, we denote it by $(P_1, \dots, P_k)$ and say that P can be **inside-out dissected**. *We refer to the polygons (polyhedra) $P_1, \dots, P_k$ as* **pieces** *of P. Furthermore, we define $\mathcal{I}(n)$ as the smallest natural number such that every n-gon can be inside-out dissected with $k \le \mathcal{I}(n)$ pieces.*

As described in Section 1 every polygon can be inside-out dissected. In the following sections, we will provide various examples of inside-out dissections of polygons and polyhedra.

At last year's CCCG, A. Rivkin established a method to inside-out dissect an arbitrary polygon with n edges with $2n+1$ pieces, thereby improving the best previous upper bound of $4(n-2)$ pieces. This method has been illustrated in [4] without giving a formal proof. Thus, we give a formal proof of this method in this paper for completeness.

Theorem 1 *Every polygon with n edges can be inside-out dissected with $2n+1$ pieces, i.e. $\mathcal{I}(n) \leq 2n+1$.*

Proof. First, we show that a triangle T can be inside-out-dissected with 2 pieces per edge and one extra "scrap" piece. For every edge e of T, construct two congruent isosceles triangles t_1, t_2 such that their congruent edges are of length $|e|/2$, one of which lies on the perimeter of T. Thus, t_1 and t_2 can be rotated by π and then translated to replace each other. This rotation places the other (internal) edge of length $|e|/2$ of t_1 and t_2 on the perimeter of T, and the edge that was originally on the exterior of T is now internal to T. This can be done for each edge of T, inside-out dissecting T in 7 pieces (illustrated in Figure 2).

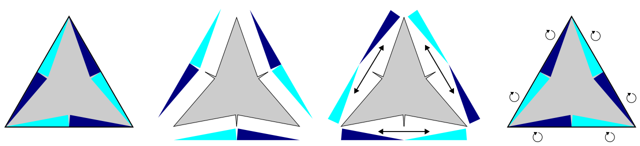


Figure 2: A triangle can be inside-out dissected with 7 pieces.

This method can be expended to any polygon P with n edges, yielding an inside-out dissection of P using $2n+1$ pieces. For every edge of P we construct two isosceles triangles just as we did in the triangle example, and perform the same procedure, rotating and exchanging each triangle (demonstrated in Figure 3).

□

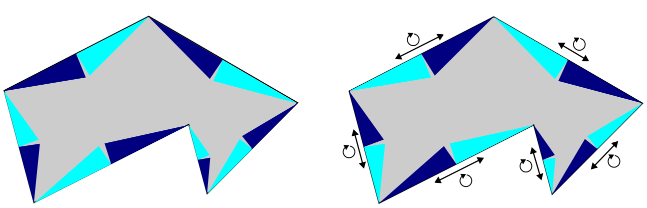


Figure 3: A polygon can be inside-out dissected with $2n+1$ pieces.

3 Inside-out Dissections of Polygons

Now, we consider inside-out dissections of regular polygons. We obtain the result that any regular polygon can be inside-out dissected using at most 6 pieces.

Proposition 1 *Any regular polygon P can be inside-out dissected with at most 6 pieces.*

Proof. For $n = 3, 4,$ and 5 we have ad hoc constructions (Figure 1 and Figure 4). For $n \geq 6$, the rough idea is to cut large rotationally symmetric pieces from P

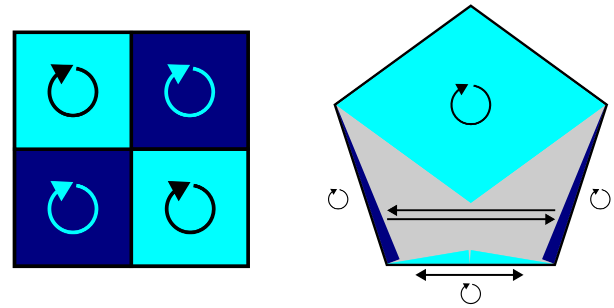


Figure 4: Regular squares and pentagons can be inside-out dissected with at most 6 pieces.

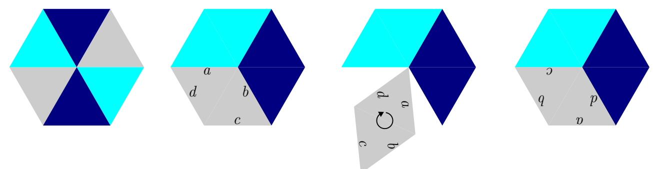


Figure 5: An inside-out dissection of a regular hexagon.

and rotate them so that perimeter edges become internal. This approach is inspired by the method presented in the proof of Theorem 1, where isosceles triangles are cut out and swapped. We start by demonstrating the technique for regular hexagons. For the regular polygon P, each internal angle is $\frac{\pi(n-2)}{n}$. Thus, for $n = 6$, every internal angle is $\frac{\pi(6-2)}{6} = \frac{2\pi}{3}$. We can then cut a hexagon into three congruent rhombi by bisecting every other angle. Each rhombus can now be rotated π radians, inside-out dissecting a regular hexagon with three pieces, demonstrated in Figure 5.

We now move to the case that P is any regular polygon. In the same manner as we did for the hexagon, one can use the internal angle to compute the maximum rotationally symmetric $2k$-gon that can be cut out from P. These $2k$-gons consist of two identical convex polygonal chains, p_1 and p_2. One of the chains, say p_1, is simply a segment of the perimeter of P with k edges and its two endpoints vertices of P. The other chain, p_2, is the reflection of p_1. The polygon formed by $p_1 \cup p_2$ is a polygon with $2k$ edges, and $2k-2$ internal angles equal to exactly the internal angle of P. It then has equal angles θ that are not an internal angle of P.

We can therefore find a closed-form equation for θ as follows:

$$\begin{aligned}
& \pi(2k-2) = 2(k-1)\left(\frac{\pi(n-2)}{n}\right) + 2\theta \\
\Leftrightarrow \quad & \pi(k-1) - (k-1)\left(\frac{\pi(n-2)}{n}\right) = \theta \\
\Leftrightarrow \quad & \pi(k-1)\left(1 - \frac{n-2}{n}\right) = \theta.
\end{aligned}$$

With an expression for θ, we can compute the maximum k we can cut from the perimeter of P. This can be achieved by enforcing that θ should be half as large as an interior angle of P :

$$\begin{aligned}
& \theta = \pi(k-1)\left(1-\frac{n-2}{n}\right) \leq \pi\left(\frac{n-2}{n}\right)\left(\frac{1}{2}\right) \\
\Leftrightarrow \quad & (k-1)\left(1-\frac{n-2}{n}\right) \leq \frac{n-2}{2n} \\
\Leftrightarrow \quad & (k-1)\left(\frac{2}{n}\right) \leq \frac{n-2}{2n} \\
\Leftrightarrow \quad & (k-1) \leq \frac{n-2}{4} \\
\Leftrightarrow \quad & k = \left\lfloor \frac{n-2}{4} + 1 \right\rfloor .
\end{aligned}$$

So, we can inside-out dissect P with $\lceil \frac{n}{k} \rceil + 1 \leq 6$ pieces. □

For illustration, we present inside-out dissections of a regular 7-gon and a regular 10-gon using rotationally symmetric pieces as described in the proof of Proposition 1.

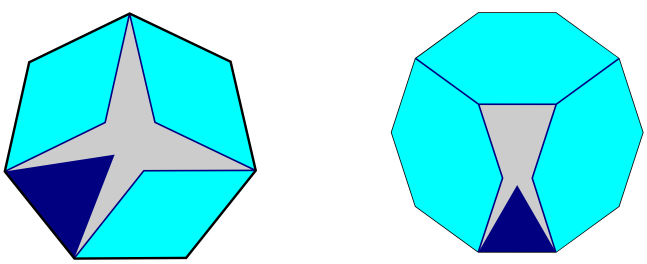


Figure 6: Two regular polygons with their pieces (Left) 7-gon, (right) 10-gon.

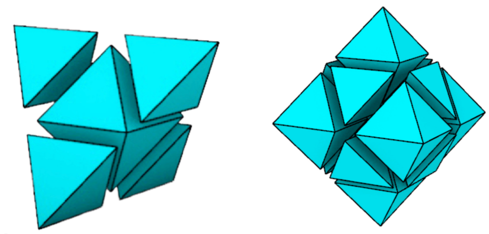
In [13] we provide a tool that allows us to visualise our constructed inside-out dissections for arbitrary regular polygons.

4 Inside-out Dissections of Polyhedra

Here, we study inside-out dissections of polyhedra in Euclidean 3-space.

As a first example of a polyhedron that can be inside-out dissected, we consider the regular tetrahedron, i.e. the Platonic solid consisting of four equilateral triangles as faces. In order to illustrate a corresponding inside-out dissection, we give the following remark.

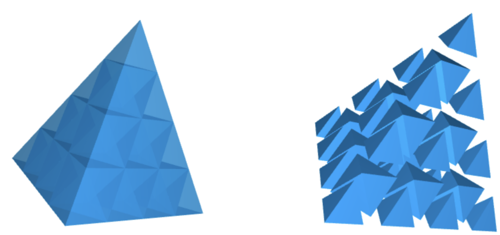
Remark 2 *Let $\mathcal{T}$ be a regular tetrahedron with all edge lengths being* 1*. Then T can be decomposed into four tetrahedra and one regular octahedron with all edge lengths being $\frac{1}{2}$. Additionally, if $\mathcal{O}$ is a regular octahedron which all edge lengths being* 1*, then $\mathcal{O}$ can be decomposed into eight regular tetrahedra and six regular octahedra with all edge lengths being $\frac{1}{2}$. These decompositions are illustrated in Figure 7.*



(a) (b)

Figure 7: Decomposition of (a) a regular tetrahedron into four regular tetrahedra and one regular octahedron and (b) a regular octahedron into six regular tetrahedra and eight regular octahedra. [3]

The above remark can be verified by examining the tetroctahedrille, i.e. the space-filling of the Euclidean 3-space consisting of regular tetrahedra and octahedra [6] and exploiting the fact that the vertices of the different tetrahedra and octahedra in this space-filling form a 3-dimensional lattice. By iterating the process of decomposing tetrahedra and octahedra, we observe that a regular tetrahedron can be decomposed into 24 regular tetrahedra and 10 regular octahedra, see Figure 8.



(a) (b)

Figure 8: (a) Decomposition of a regular tetrahedron into 24 regular tetrahedra and 10 regular octahedra (b) an exploded view of the described subdivision.

Hence, by applying suitable transformations to the tetrahedra and octahedra of the above decomposition, we obtain the desired inside-out dissection of a regular tetrahedron.

Proposition 2 *Let $\mathcal{T}$ be a regular tetrahedron. Then $\mathcal{T}$ can be inside-out dissected with* 34 *pieces forming regular tetrahedra and octahedra.*

Proof. We refer to a face of a polyhedron of the decomposition in Figure 8 as boundary face if it is contained in the boundary of $\mathcal{T}$. Thus, it can be observed that the above decomposition has exactly $2 \cdot 6 + 4 + 4 = 20$

tetrahedra and $6+4=10$ octahedra that have at least one boundary face. More precisely, there are exactly (1) 4 tetrahedra having exactly 3 boundary faces, (2) 12 tetrahedra having exactly 2 boundary faces, (3) 4 tetrahedra having exactly 1 boundary face, (4) 4 octahedra having exactly 3 boundary faces and (5) 6 octahedra having exactly 2 boundary faces. In order to construct the desired inside-out dissection of the tetrahedron $\mathcal{T}$ the tetrahedra described in (1) and (3) have to be swapped by applying a suitable transformation and the polyhedra in (2), (4) and (5) have to be rotated so that the boundary faces of these polyhedra do not form boundary faces of the rearranged tetrahedron. □

By observing the decomposition of the regular octahedron shown in Figure 7b and pursuing a similar strategy as presented in Proposition 2, we can show that a regular octahedron can be inside-out dissected.

Proposition 3 *Let $\mathcal{O}$ be a regular octahedron. Then $\mathcal{O}$ can be inside-out dissected with 14 pieces forming regular tetrahedra and octahedra.*

Since Proposition 1 establishes that any regular polygon can be inside-out dissected with at most 6 pieces, we are able to derive an infinite family of polyhedra, where every polyhedron of this family can be inside-out dissected with at most 12 pieces.

Corollary 1 *Let P be a regular prism. Then P can be inside-out dissected with at most 12 pieces.*

Proof. The polyhedron P has two types of faces, n square faces and two faces that are congruent regular n-gons, call this n-gon p. Without loss of generality, assume that P is oriented such that the two regular n-gon faces are the top and bottom faces of P, and the square faces are therefore oriented vertically between these two faces p_t and p_b (the top and bottom face respectively). By Proposition 1 the polygons p_t and p_b can both be inside-out dissected with at most 6 pieces such that the pieces of p_t are congruent to the pieces of p_b. Let the height of P (the distance between p_t and p_b) be h. Dissect both p_t and p_b and then extrude each piece to obtain prisms of height $h/2$. Now, every extruded prism having a face that forms a piece of p_t can be swapped with an extruded prism having a face that forms a piece of p_b congruent to the given piece of p_t. This can be achieved such that the rearranged prisms form an inside-out dissection of the polyhedron P. □

Next, we introduce Theorem 2 which establishes the existence of inside-out dissections of polyhedra. The main idea to prove this result is to subdivide the surface of a given polyhedron P into isosceles triangles and exploit this triangulation to derive the desired inside-out dissection of P. The key to constructing such a triangulation consisting of isosceles triangles is that every polygon admits a triangulation consisting of acute triangles, see [21]. Before proceeding with the proof of the desired result, we describe how acute triangles can be further subdivided into three isosceles triangles.

Remark 3 *Let t be an acute triangle with vertices $v_1, v_2, v_3 \in \mathbb{R}^2$. Since t is acute, the circumcenter c, i.e. the intersection point of all perpendicular bisectors of t, is contained within the interior of t, see Figure 9 for illustration. By construction the triangles $(v_1, v_2, c), (v_1, v_3, c), (v_2, v_3, c)$ all form isosceles triangles that subdivide t.*

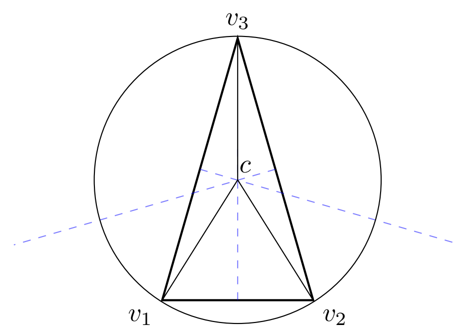


Figure 9: Exploiting the circumcenter of an acute triangulation to construct a subdivision of the given triangle into three isosceles triangles.

Inspired by Theorem 1, we aim to inside-out dissect a polyhedron by using rotationally symmetric pieces. We construct these pieces such that two faces of a given piece form congruent isosceles triangles as detailed below.

Remark 4 *Let t be an isosceles triangle with edge lengths (a, b, b), where a, b are positive real numbers. Because of t being isosceles, we know that the inequality $b > \frac{a}{2}$ holds. We construct a tetrahedron that is rotationally symmetric and has two faces that are congruent to t as follows: For $\alpha \in (0, \frac{\pi}{2})$ and $r := \sqrt{b^2 - \frac{a^2}{4}}$ we define the tetrahedron τ by its corresponding four vertices $v_1, \ldots, v_4 \in \mathbb{R}^3$ given as follows:*

$$v_1 := (\tfrac{a}{2}, 0, 0), v_2 := (-\tfrac{a}{2}, 0, 0),$$
$$v_3 := (0, r\cos(\alpha), r\sin(\alpha))$$
$$v_4 := (0, -r\cos(\alpha), r\sin(\alpha)).$$

Figure 10 illustrates the arising tetrahedron. By construction we see that the sets

$$t_1 := \mathrm{conv}(\{v_1, v_2, v_3\}), t_2 := \mathrm{conv}(\{v_1, v_2, v_4\})$$

form triangles with edge lengths (a, b, b). Moreover, we see that the symmetry group of τ admits a 180 degree rotation around the z-axis.

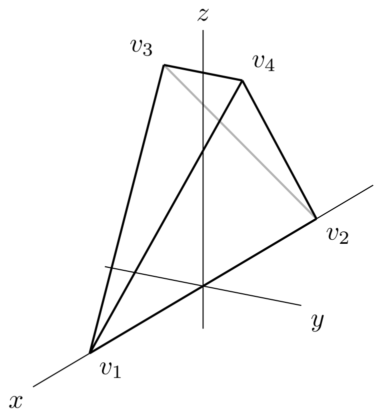


Figure 10: A rotationally symmetric tetrahedron with four isosceles faces

With the help of the above remarks, we are able to prove the main result of this section.

Theorem 2 *Every polyhedron can be inside-out dissected.*

Proof. Let P be a polyhedron and f an arbitrary face of P. Since f is a face of P, it forms a polygon in $\mathbb{R}^2$. By [21] we know that f can be triangulated such that each arising triangle is acute, see Figure 11a for illustration. For simplicity, we denote these acute triangles by $t_1, \ldots, t_n$. Note that the circumcenter of t_i is contained within the interior of t_i for all $1 \leq i \leq m$. Thus, by connecting the circumcenter of each triangle t_i to its vertices as described in Remark 3, we obtain a more refined subdivision of f consisting of isosceles triangles, see Figure 11b for illustration.

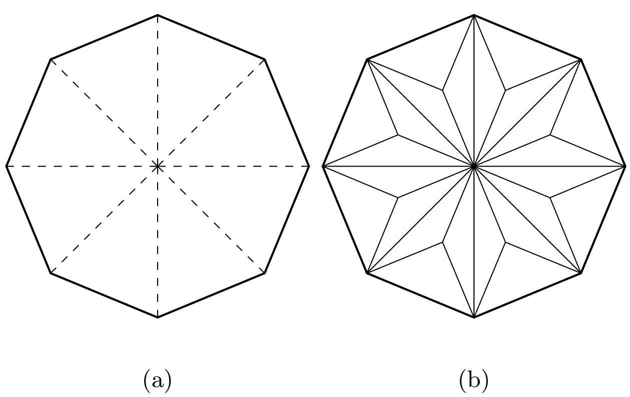


Figure 11: An acute triangulation triangulation of a regular octagon (a); a triangulation of a regular octagon consisting of isosceles triangles (b).

Since f is arbitrary, we know that the boundary of P can be triangulated with isosceles triangles $t'_1, \ldots, t'_m$. Note that if (a_i, b_i, b_i) are the edge lengths of the triangle t'_i, then $b_i > \frac{a_i}{2}$ holds and the edges of length b_i are not contained in any edge of P (by construction). Hence, we can exploit these triangles to construct polyhedra $p_1, \ldots, p_m$ such that

- the polyhedron p_i is a rotationally symmetric tetrahedron as described in Remark 4 for all $1 \leq i \leq m$,
- the triangle t'_i forms a face of the polyhedron p_i for all $1 \leq i \leq m$,
- the polyhedra p_i and p_j intersect at most in their boundary for all $1 \leq i < j \leq m$.

As each tetrahedron p_i is rotationally symmetric, it can be rotated to replace its external face f with its internal face that is congruent to f. Hence, the pieces $P \setminus \{p_1, \ldots, p_m\}, p_1, \ldots, p_m$ form the desired inside-out dissection of P. □

Note that [21] establishes that every n-gon can be triangulated with at most $24 \cdot (106n - 216)$ triangles forming acute triangles. We can exploit this fact to calculate an upper bound on the number of pieces of the inside-out dissection of a given polyhedron derived in the proof of Theorem 2. Let P be a polyhedron and $f_1, \ldots, f_k$ exactly the faces of P, where the face f_i forms a n_i-gon for $1 \leq i \leq k$. In our proof, every face of P is first triangulated with acute triangles. We then obtain an inside-out dissection of P by subdividing every arising acute triangle into three isosceles triangles. Thus, the derived inside-out dissection of P consists of at most

$$1 + \sum_{i=1}^{k} 3 \cdot 24 \cdot (106n_i - 216)$$

pieces.

5 Conclusion & Outlook

In this paper, we have studied inside-out dissections of polyhedra and polygons. We have established the existence of inside-out dissections for polyhedra and examined inside-out dissections of regular polygons. In future work, we want to address the following questions:

(1) Theorem 1 establishes that any n-gon can be inside-out dissected with $2n + 1$ pieces. We want to investigate whether this bound is sharp. Thus, we explore the existence of an n-gon that cannot be inside-out dissected with fewer than $2n + 1$ pieces.

(2) We want to improve the upper bound on the number of pieces of inside-out dissection of polyhedra established in the proof of Theorem 2.

(3) The inside-out dissections that have been produced for arbitrary polygons in Theorem 1 and arbitrary polyhedra in Theorem 2 contain one non-convex piece. Is it possible to inside-out dissect every polyhedron and every polygon with pieces that are all convex?

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