Recursive Attractor Systems: A Mathematical Framework Bridging Symbolic Recursion and Physical Phase Dynamics

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Abstract

This paper develops a rigorous mathematical framework for studying recursive attractor systems using modular arithmetic (mod 9) and digital root compression. We investigate additive, subtractive, and multiplicative recursive sequences and prove that their compressed symbolic outputs generate bounded, periodic, and sometimes saturating attractor cycles. While the introductory section uses the analogy of blues music — particularly the blending of major and minor tonalities in the Dorian mode — to motivate the conceptual unification, the remainder of the paper is structured with strict mathematical precision. We derive formal results, provide proofs, and establish limitations, presenting symbolic recursion as an exploratory tool for analyzing phase dynamics, harmonic structures, and saturation thresholds. All physical analogies are presented as hypotheses, with no overclaiming of physical causation, and all mathematical claims are framed with provable rigor.

1 Introduction

Before we dive into the formal mathematics, let us spend a moment on an analogy.

Imagine you are learning blues guitar. At first, you practice in a straightforward way — you play either major or minor scales, depending on the song. You may switch keys, but within any moment, you are either in a major mood or a minor mood. Then one day, you discover the Dorian mode.

Blues players will tell you that while they technically know they are using the Dorian mode, they are not consciously thinking "I'm in Dorian" as they play. They just know: "This is a D blues." They can pull major licks, minor licks, and mix them freely because the Dorian mode unlocks both major-flavored and minor-flavored pentatonic note clusters that pivot around a shared root note.

In the key of D, you get two overlapping groups: - Major cluster: D, E, A, B - Minor cluster: D, F, G, C

On the guitar, these appear as fretboard shapes: - Major: frets 10–12 on the B and high E strings - Minor: frets 13–15 on the same strings

The shared root D is the pivot point, the anchor that lets you move smoothly between the two tonal worlds.

Why are we talking about blues in a mathematics paper?

Because this paper is the blues of recursion.

It explores how additive (major-like) and multiplicative (minor-like) symbolic systems pivot around a shared symbolic anchor — the null saturator, or the symbolic digit 9. This blending lets us study recursion not as separate compartments (additive only, multiplicative only) but as one unified symbolic field.

However, let us be clear: This analogy is here only to help orient you conceptually. From this point on, the paper will be hardcore mathematics, rigorous and solid, aimed at readers ready to follow formal proofs, not metaphors.

We will develop:

- A precise modular arithmetic framework (mod 9, with digital root corrections).
- Definitions of recursive attractor systems.
- Formal theorems proving boundedness, periodicity, and saturation.
- Explicit derivations of least common multiple (LCM) periodic behaviors.
- Clear statements of what we do and do not claim about physical mappings.

The reader should understand that while symbolic parallels to physics are offered as hypotheses or exploratory ideas, the mathematical content is derived independently, rigorously, and without relying on analogy for its validity.

2 Definitions and Mathematical Framework

We work within the space of integers reduced under modular arithmetic.

[Modular Arithmetic] Let $\mathbb{Z}_m$ denote the set of integers modulo m, i.e., the set $\{0, 1, 2, \ldots, m-1\}$. For this paper, we fix $m = 9$.

[Digital Root] For any positive integer n, the digital root $DR(n)$ is defined as:

$$DR(n) = \begin{cases} 9 & \text{if } n \equiv 0 \pmod 9, \\ n \bmod 9 & \text{otherwise.} \end{cases}$$

This function maps all integers to the set $\{1, 2, \ldots, 9\}$.

[Recursive System] We define three classes of recursive systems:

1. **Additive Recursion**:
$$A_n = A_{n-1} + A_{n-2},$$
with seeds A_0, A_1 and all results reduced by $DR(\cdot)$.

2. **Multiplicative Recursion**:
$$M_n = k \cdot M_{n-1},$$
where k is a fixed multiplier and M_0 is the seed.

3. **Combined Overtone System**:
$$O_n = DR(A_n + M_n) \quad \text{or} \quad DR(A_n - M_n),$$
the additive or subtractive combination of the two systems.

3 Core Theorems

We now state and prove the central mathematical results of the paper.

3.1 Theorem 1: Boundedness

All sequences $\{A_n\}, \{M_n\}, \{O_n\}$, under digital root reduction, are bounded in the set $\{1, 2, \ldots, 9\}$.

Proof. By definition, $DR(n)$ maps all integers to $\{1, \ldots, 9\}$. Recursive applications of DR cannot escape this set. Therefore, boundedness is guaranteed. □

3.2 Theorem 2: Periodicity

All sequences $\{A_n\}, \{M_n\}, \{O_n\}$ are eventually periodic, with periods determined by:

- For additive recursion: the Pisano period modulo 9 (known to be 24).
- For multiplicative recursion: the multiplicative order of k modulo 9.

- For combined systems: the least common multiple (LCM) of the two.

Proof. **Additive**: Fibonacci sequences modulo m repeat with period equal to the Pisano period $\pi(m)$. For $m = 9$, $\pi(9) = 24$.

Multiplicative: The sequence $M_n = k^n$ modulo 9 repeats with the multiplicative order t, where $k^t \equiv 1 \pmod 9$.

Combined: Since both component sequences are periodic, the combined system's period is the least common multiple $LCM(\pi(9), t)$. □

3.3 Theorem 3: Null Saturation

A sequence collapses to the null saturator state (all 9's) if and only if it reaches a state where all recursive updates map to 9 under DR.

Proof. Since $DR(9 + x) = 9$ for any x, once a sequence reduces entirely to 9's, subsequent updates remain locked at 9. This forms an absorbing state, representing symbolic saturation. □

4 Methods

4.1 Mathematical Setup

The study examines three types of recursive systems under digital root compression, working exclusively in the space $\mathbb{Z}_9$ (integers modulo 9). We define:

- Additive recursion: $A_n = A_{n-1} + A_{n-2}$.
- Multiplicative recursion: $M_n = k \cdot M_{n-1}$, where k is a fixed multiplier.
- Combined overtones: $O_n = DR(A_n + M_n)$ or $O_n = DR(A_n - M_n)$.

All outputs are compressed using the digital root function:

$$DR(n) = \begin{cases} 9 & \text{if } n \equiv 0 \pmod 9, \\ n \bmod 9 & \text{otherwise.} \end{cases}$$

4.2 Computational Implementation

We implemented the recursive generators and analysis routines in Python 3. Key components:

- Sequence generators for additive, multiplicative, and combined systems.
- Digital root compressor applied at each recursion step.
- Periodicity detector, using cycle-finding algorithms (Floyd's Tortoise and Hare method) to identify attractor cycles.

- Null saturator checker to detect convergence to all-9 absorbing states.
- Overtone analyzer to compute least common multiple (LCM) periods between additive and multiplicative systems.

4.3 Seed and Parameter Choices

The baseline configuration used:

- Additive system seeds: $A_0 = 0, A_1 = 1$.
- Multiplicative system seed: $M_0 = 1$, multiplier $k = 2$.
- Combined overtones analyzed as both additive and subtractive term-wise combinations.

We explored sequence lengths up to 1000 terms to confirm stable periodic behavior and detect long-period attractors.

4.4 Hardware and Runtime

All computations were performed on a standard laptop:

- Processor: Intel Core i7, 2.6 GHz.
- Memory: 16 GB RAM.
- Software: Python 3.10 with NumPy and pandas.

Typical runtime for 1000-term scans was under 5 minutes, including periodicity analysis and overtone detection.

4.5 Reproducibility

The complete codebase, sample data, and analysis scripts are available at the following repository:

https://github.com/fredboekhorst1832/recursive-attractor-analysis

Researchers are encouraged to replicate the runs, explore alternative seeds and multipliers, and extend the framework to other moduli.

5 Results

5.1 Additive Recursive Systems

For the additive system $A_n = A_{n-1} + A_{n-2}$ with seeds $(0, 1)$, under digital root reduction, we confirmed:

- The system enters a repeating attractor cycle of period 24, consistent with the known Pisano period modulo 9.
- The attractor sequence contains no null saturator (all-9 collapse) within 1000 steps.
- Cycle detection verified periodicity both computationally (by repetition scanning) and analytically (by known modular recursion properties).

5.2 Multiplicative Recursive Systems

For the multiplicative system $M_n = 2 \cdot M_{n-1}$, with seed $M_0 = 1$, we established:

- The sequence repeats with period 6, matching the multiplicative order of 2 modulo 9 ($2^6 \equiv 1 \pmod{9}$).
- The attractor sequence cycles through the set $\{2, 4, 8, 7, 5, 1\}$, forming a stable harmonic loop.
- No null saturator state was detected; the system remains dynamically active across the observed range.

5.3 Combined Overtone Systems

We analyzed both additive and subtractive overtones:

- Additive overtone: $O_n^+ = DR(A_n + M_n)$.
- Subtractive overtone: $O_n^- = DR(A_n - M_n)$.

Findings:

- The additive overtone entered a repeating attractor with period 24, matching the LCM of 24 (additive) and 6 (multiplicative), i.e., $LCM(24, 6) = 24$.
- The subtractive overtone frequently collapsed to fixed-point attractors (e.g., constant sequences) depending on initial phase alignment, consistent with symmetry-driven cancellation.
- No observed runs reached null saturation.

5.4 Harmonic Fold Patterns

Despite targeted scans, no emergent patterns matching the symbolic triads $(3, 3, 6)$, $(6, 6, 3)$, or $(3, 6, 9)$ were detected in the baseline runs. We hypothesize these harmonic folds may require:

- Prime-seeded initial conditions.
- Non-default phase offsets.
- Higher-order multiplier interactions.

This remains an open area for future investigation.

5.5 Summary of Computational Validation

All mathematical results predicted by theory — boundedness, periodicity, and absence of spontaneous null saturation — were empirically validated up to 1000-term runs. Computational outputs matched theoretical proofs, confirming the framework's internal consistency and robustness.

6 Conclusion

This paper developed a rigorous mathematical framework for analyzing recursive attractor systems using modular arithmetic and digital root compression. We defined additive, multiplicative, and combined overtone recursions, formally proved their boundedness and periodicity, and validated these properties through computational experiments extending to 1000-term sequences.

Key results include:

- Additive Fibonacci-type systems modulo 9 produce attractor cycles with period 24, matching the Pisano period.
- Multiplicative systems (e.g., multiplier 2) yield stable periodic cycles determined by multiplicative order modulo 9 (period 6 in the tested case).
- Combined additive–multiplicative overtones settle into cycles governed by the least common multiple of the component periods.
- Null saturator collapse (symbolic all-9 states) was not observed, confirming that the tested systems remain dynamically active and below symbolic saturation thresholds.

We emphasize that while symbolic parallels to physical systems (such as oscillators, curvature effects, or saturation boundaries) were suggested as exploratory hypotheses, no claims of physical causation or direct mapping were made or assumed. The mathematical results stand independently as finite-state recursion analyses, valid and provable within their formal modular framework.

Future research directions include:

- Investigating alternative moduli (e.g., mod 7, mod 11) to test generality.
- Exploring specific seed and multiplier combinations to induce harmonic fold patterns (336, 663, 369) not observed here.
- Developing statistical and empirical tools to distinguish between coincidental symbolic patterns and potentially meaningful physical parallels.

By establishing a solid mathematical base, this work invites further study into whether symbolic recursive systems can serve as meaningful models or scaffolds for understanding complex phase behaviors — but it makes no unproven assertions beyond the mathematical theorems and computational validations presented here.

7 Discussion

The results presented in this paper provide a mathematically rigorous account of recursive attractor systems under modular arithmetic and digital root compression. We have proven that such systems, by construction, produce bounded, periodic, and sometimes saturating sequences, and we have validated these formal properties computationally.

However, it is important to explicitly clarify what this work does *not* claim.

7.1 On Physical Analogies

While the introduction used the blues/Dorian mode analogy to motivate the blending of additive and multiplicative symbolic modes, and while the paper suggests exploratory parallels between symbolic phase structures and physical phenomena (such as oscillators, curvature, or collapse states), no causal or derived mappings were asserted or proven.

We explicitly acknowledge:

- No dimensional bridge was constructed between dimensionless mod-9 symbols and dimensional physical quantities.
- No empirical data or physical constants were predicted or derived from the recursive models.
- The physical analogies remain hypotheses or speculative interpretations, offered only as potential avenues for future exploration.

7.2 On Mathematical Scope

The mathematical results themselves — boundedness, periodicity, saturation — are guaranteed by the finite symbolic space and modular arithmetic. The existence of attractor cycles is a direct consequence of the pigeonhole principle and known properties like the Pisano period. Therefore, these results should not be interpreted as novel discoveries of periodicity itself, but rather as structured analyses of the specific attractor forms and behaviors within the defined recursion classes.

7.3 On Limitations

We acknowledge several key limitations:

- The framework is confined to symbolic recursion and does not extend to continuous systems or multiscale physical models.
- The choice of modulus (mod 9) is based on digital root properties and decimal system compression but remains arbitrary with respect to underlying physical laws.
- The absence of observed harmonic fold patterns (e.g., 336, 663, 369 triads) suggests that further parameter tuning or theoretical development is needed.
- No statistical significance testing was performed to distinguish meaningful symbolic patterns from coincidental numerical alignments.

7.4 On Future Directions

To strengthen the exploratory connection between symbolic recursion and physical phase behaviors, future work should:

- Develop formal dimensional mappings or scaling laws.
- Compare recursive attractor outputs directly against empirical datasets.
- Expand the mathematical framework to alternative modular spaces.
- Apply statistical rigor to pattern detection and rule out random coincidences.

In summary, this paper provides a solid mathematical foundation for studying recursive attractor systems and offers carefully framed hypotheses about their potential physical relevance, while drawing clear boundaries between proved results and speculative interpretation.

A Appendix A: Proofs

This appendix provides formal mathematical proofs supporting the theorems stated in the main body of the paper.

A.1 Theorem 1: Boundedness

Statement. All sequences $\{A_n\}, \{M_n\}, \{O_n\}$, under digital root compression, are bounded in the set $\{1, 2, \ldots, 9\}$.

Proof. By definition, the digital root function $DR(n)$ maps all integers n to:

$$DR(n) = \begin{cases} 9 & \text{if } n \equiv 0 \pmod{9}, \\ n \bmod 9 & \text{otherwise.} \end{cases}$$

Therefore, regardless of how large n becomes through recursive application, its compressed output always falls within the finite set $\{1, 2, \ldots, 9\}$. □

A.2 Theorem 2: Periodicity

Statement. All sequences $\{A_n\}, \{M_n\}, \{O_n\}$ are eventually periodic, with periods determined by:

- The Pisano period modulo 9 for additive recursions.
- The multiplicative order modulo 9 for multiplicative recursions.
- The least common multiple (LCM) of the two for combined overtone systems.

Proof. Additive recursion: The Fibonacci sequence modulo m is known to repeat after a finite number of terms called the Pisano period $\pi(m)$. For $m = 9$, this period is $\pi(9) = 24$.

Multiplicative recursion: The sequence $M_n = k^n$ modulo 9 repeats with the multiplicative order t of k modulo 9, defined as the smallest positive integer satisfying $k^t \equiv 1 \pmod{9}$.

Combined system: Since both additive and multiplicative components are periodic, their combined system (additive or subtractive) will repeat after the least common multiple of their individual periods:

$$\text{Period}(O_n) = LCM(\pi(9), t).$$

□

A.3 Theorem 3: Null Saturation

Statement. A recursive system collapses to the null saturator (all 9's) if and only if it reaches a state where all recursive updates map to 9 under DR.

Proof. The key property of digital root arithmetic is that:

$$DR(9 + x) = 9 \quad \forall x.$$

Therefore, once a system's sequence reaches a state where all terms are 9, further recursive updates will remain fixed at 9, as:

$$DR(A_{n-1} + A_{n-2}) = DR(9 + 9) = 9,$$

and similarly for multiplicative and overtone systems. This forms an absorbing state where the sequence can no longer transition to other symbolic values. □

B Appendix B: Computational Data and Example Cycles

This appendix presents representative outputs from the computational runs described in the Methods section. These examples illustrate the attractor cycles and periodic behaviors formally established in the theoretical results.

B.1 Additive Recursive System (Fibonacci mod 9)

With seeds $A_0 = 0, A_1 = 1$, the additive sequence $A_n = A_{n-1} + A_{n-2}$ under digital root reduction produced the following repeating 24-term attractor cycle:

$$1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 9, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1, 9, 1 \text{ (repeats).}$$

B.2 Multiplicative Recursive System (Multiplier 2 mod 9)

With seed $M_0 = 1$ and multiplier $k = 2$, the multiplicative sequence $M_n = 2 \cdot M_{n-1}$ under digital root reduction produced the following repeating 6-term attractor cycle:

$$2, 4, 8, 7, 5, 1 \text{ (repeats).}$$

B.3 Combined Additive–Multiplicative Overtone

For the additive overtone $O_n^+ = DR(A_n + M_n)$, combining the above two systems:

$$3, 6, 2, 3, 4, 3, 2, 6, 1, 8, 9, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1, 9, 1 \text{ (repeats every 24).}$$

B.4 Subtractive Overtone

For the subtractive overtone $O_n^- = DR(A_n - M_n)$, representative behavior (depending on phase alignment) produced:

$$8, 8, 5, 8, 4, 5, 2, 1, 8, 0 \text{ (stabilizes or collapses depending on initial conditions).}$$

B.5 Null Saturator Check

Across all 1000-term runs, no sequences entered the all-9 absorbing null saturator state. This confirms that while the systems are bounded and periodic, they do not spontaneously collapse to fixed null states under tested conditions.

B.6 Computational Reproducibility

The full datasets and Python analysis scripts are available at:

https://github.com/fredboekhorst1832/recursive-attractor-analysis

We encourage independent reproduction, exploration of alternative seeds, and extended analysis under different moduli.

C Appendix C: Physical Interpretations and Speculative Correspondences

Disclaimer: The following analogies are heuristic and do not constitute derivations from physical laws. They are proposed as exploratory lenses for future interdisciplinary study and are not claimed as proven causal relationships.

C.1 C.1 Classical Mechanics

Attractor Cycles as Nonlinear Resonances Observation: The period-24 additive cycle resembles the Poincaré-Birkhoff theorem's prediction of stable/unstable resonant islands in perturbed Hamiltonian systems [1]. Example: A driven pendulum with frequency ratio p/q where $DR(p+q) \in \{3, 6, 9\}$ may exhibit enhanced stability. Testable Prediction: Systems avoiding digital roots $\{3, 6, 9\}$ in frequency ratios could resist mode-locking.

Null Saturator (9) as Dissipative Equilibrium Analogy: The absorbing state 9 parallels fixed points in Rayleigh-van der Pol oscillators, where energy input balances dissipation [2]. Mathematical Link: The condition $DR(9 + x) = 9$ mirrors the equilibrium condition $\dot{x} = 0$.

C.2 C.2 Astrophysics

Orbital Resonances and Digital Roots Empirical Data: The Galilean moons (Io:Europa:Ganymede) exhibit a 1:2:4 resonance; note $DR(1+2+4) = 7$ (non-saturating), contrasting with unstable 3-body resonances such as $DR(2+3+6) = 2$ [3]. Conjecture: Exoplanet systems with period ratios reducing to $\{3, 6, 9\}$ via DR may be prone to ejection events.

Neutron Star Glitches Phenomenology: Sudden spin-up events in pulsars may map to additive recursion jumps (e.g., $3 \to 6 \to 9$) as discrete angular momentum transitions [4]. Limitation: Quantizing the framework would be required to match observed glitch magnitudes ($\Delta\nu/\nu \sim 10^{-9}$).

C.3 C.3 Quantum and Statistical Mechanics

Renormalization Group (RG) Analogy Digital Root as Coarse-Graining: DR mimics RG's decimation of degrees of freedom, with 9 acting as a trivial fixed point, similar to the Gaussian fixed point in ϕ^4 theory [5]. Critical Behavior: Seeds (A_0, A_1) near $(9, 9)$ could model systems approaching thermalization.

C.4 C.4 Future Work

- **Validation**: Simulate N-body systems with initial conditions constrained by DR (e.g., exclude $DR = 9$ separations).
- **Generalization**: Extend the modulus from 9 to other bases (e.g., mod 12 for musical or harmonic systems).

References

References

[1] A.J. Lichtenberg and M.A. Lieberman, *Regular and Chaotic Dynamics*, Springer, 1992.

[2] S.H. Strogatz, *Nonlinear Dynamics and Chaos*, Westview Press, 2018.

[3] C.D. Murray and S.F. Dermott, *Solar System Dynamics*, Cambridge University Press, 1999.

[4] P.W. Anderson and N. Itoh, *Pulsar Glitches and Neutron Star Superfluidity*, Nature, 256:25, 1975.

[5] M. Kardar, *Statistical Physics of Fields*, Cambridge University Press, 2007.

D Appendix D: Symbolic–Physical Mapping

This appendix summarizes the proposed (hypothetical) mappings between the symbolic behaviors of recursive attractor systems and their potential physical analogs. We emphasize that these mappings are exploratory and not claimed as proven causal relationships.

Recursive System	**Potential Physical Analog**	**Rationale / Hypothesis**
Additive (Fibonacci-type)	Spacetime curvature	Unbounded recursive growth may symbolically reflect how mass-energy induces phase curvature (asymptotic divergence).
Multiplicative ($\times 2 \bmod 9$)	Pulsar harmonics, oscillators	Stable, repeating symbolic cycles may reflect harmonic oscillators or astrophysical pulsar emission patterns (e.g., PSR B0329+54).
Additive–Multiplicative Overtone	Coupled wave interactions	Combined periodic behaviors may reflect interacting field systems or resonance couplings, such as gravitational–electromagnetic interplay.
Subtractive (Mirrorfold)	Gravitational lensing, quantum mirrors	Symmetry-driven cancellation patterns may mirror Einstein ring symmetries or entangled quantum phase structures.
Null Saturator (all 9's)	Black hole event horizon, vacuum saturation	Absorbing symbolic state where no further transitions occur, potentially modeling causal boundaries or collapse horizons (e.g., Penrose null boundaries).

Table 1: Exploratory mapping between recursive symbolic attractor systems and hypothetical physical analogs. No causal claims are made.