Dynamics of Complex Systems

Mathematics of Change

Simple (1D) Dynamical models

Notes on Deterministic Chaos

Time in the Social Sciences

All psychological processes are about *change*:

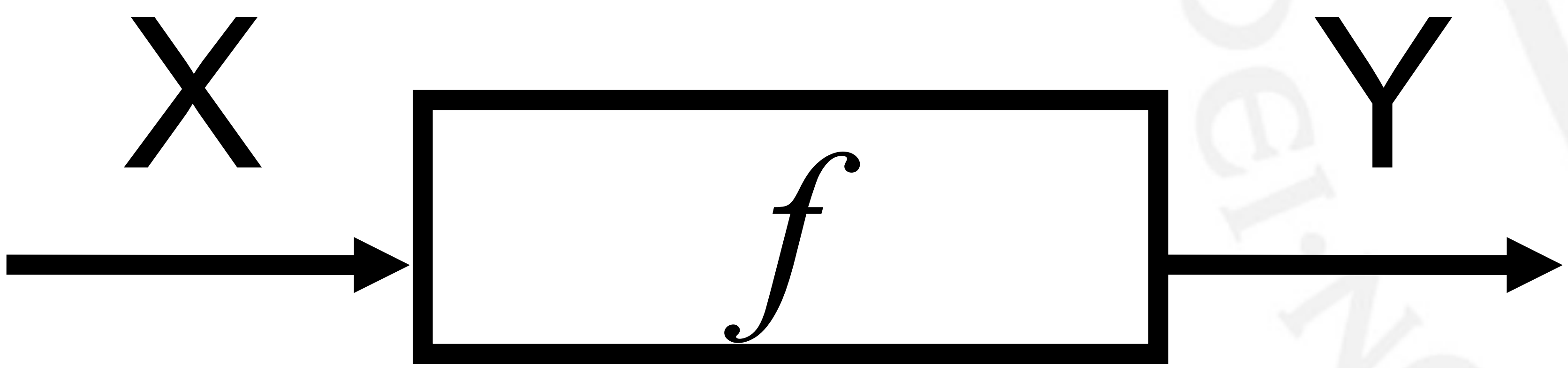
These processes involve the (in)stability of some behaviour, feature or variable over time:

- Sometimes gradual
- Sometimes abrupt/sudden
- Sometimes requiring a lot of effort or energy (resistance to change!)
- Sometimes seemingly without any resistance at all
- Think of common topics in psychological science:
  - *Developmental change*
  - *Change of behaviour between conditions in an experiment*
  - *Change after therapy or intervention*
  - *Stability of personality across the life span*

The mathematics of change

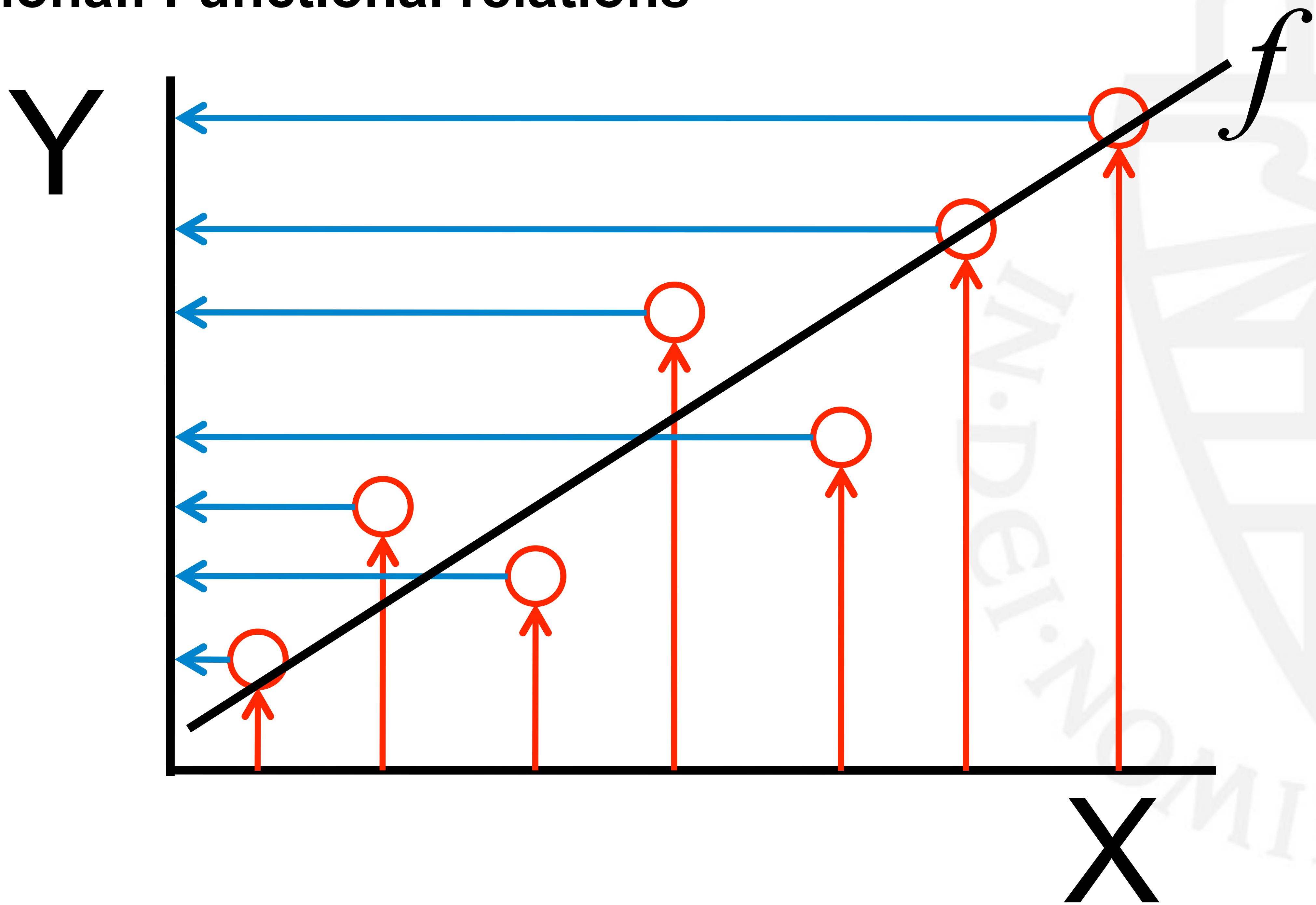
Traditional: Functional relations

$$Y = f(X)$$



The mathematics of change

Traditional: Functional relations



1refs

The mathematics of change

Complex systems however:

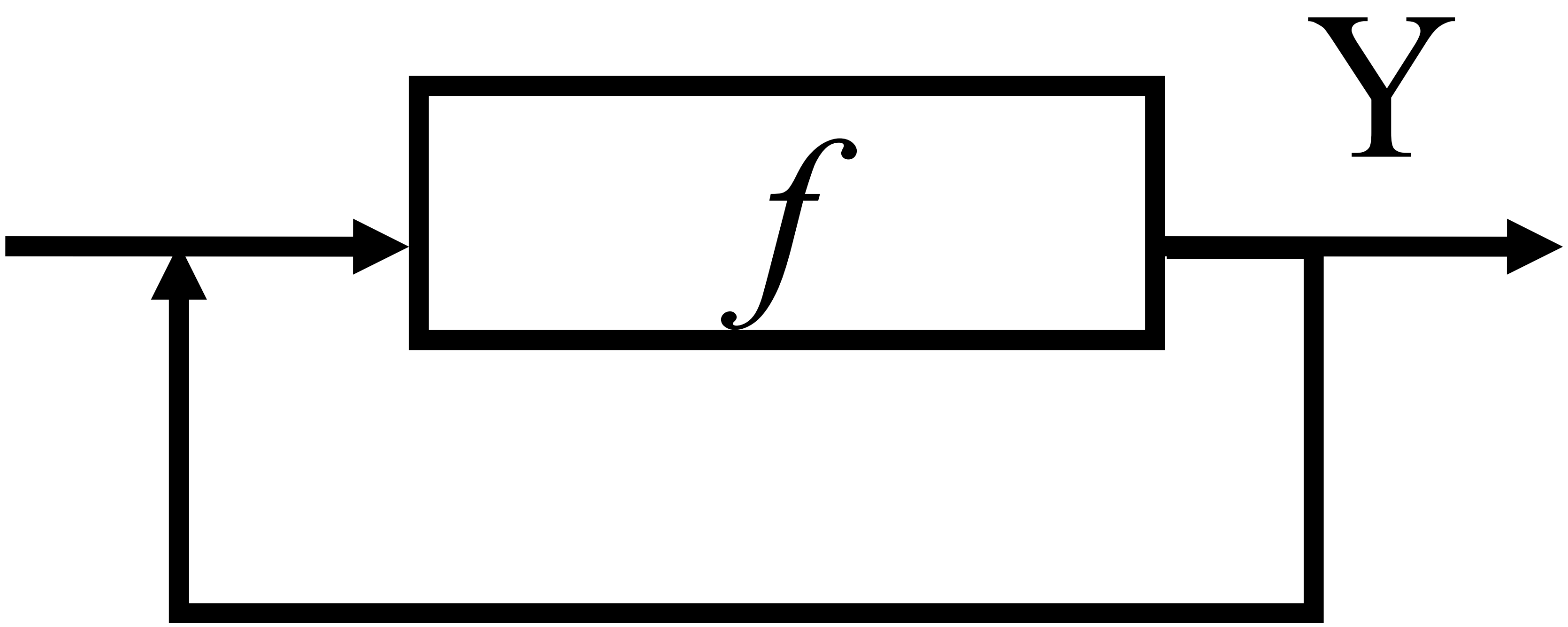
- Consist of feedback loops
- Are recurrent / recursive
- Have history
- Are characterised by multiplicative interactions between components

[1]refs

The mathematics of change

Complex systems: Recurrent processes / Feedback

$$\hat{Y} = f(Y)$$



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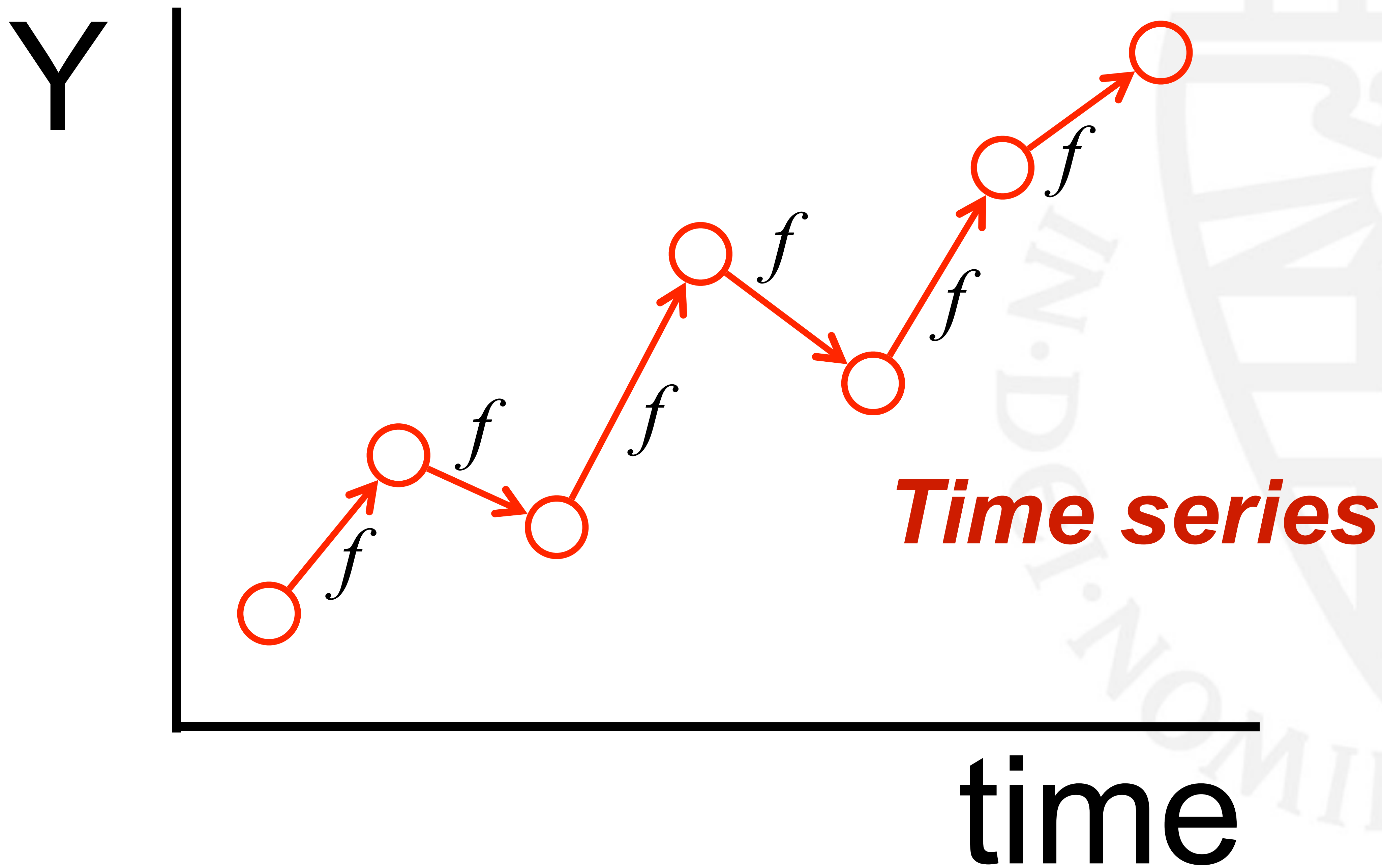
The mathematics of change

Complex systems: Recurrent processes / Feedback



[1]refs

Two Flavors: Flows & Maps

Dynamical models of psychological processes can be formulated in:

'Clock' time	**'Metronome' time**
Continuous System	Discrete System
~ Flow ~	… Map …
(Differen*tial* equation)	(Differen*ce* equation)

1refs

Parameters & Bifurcations

Example 1:

The Linear Map

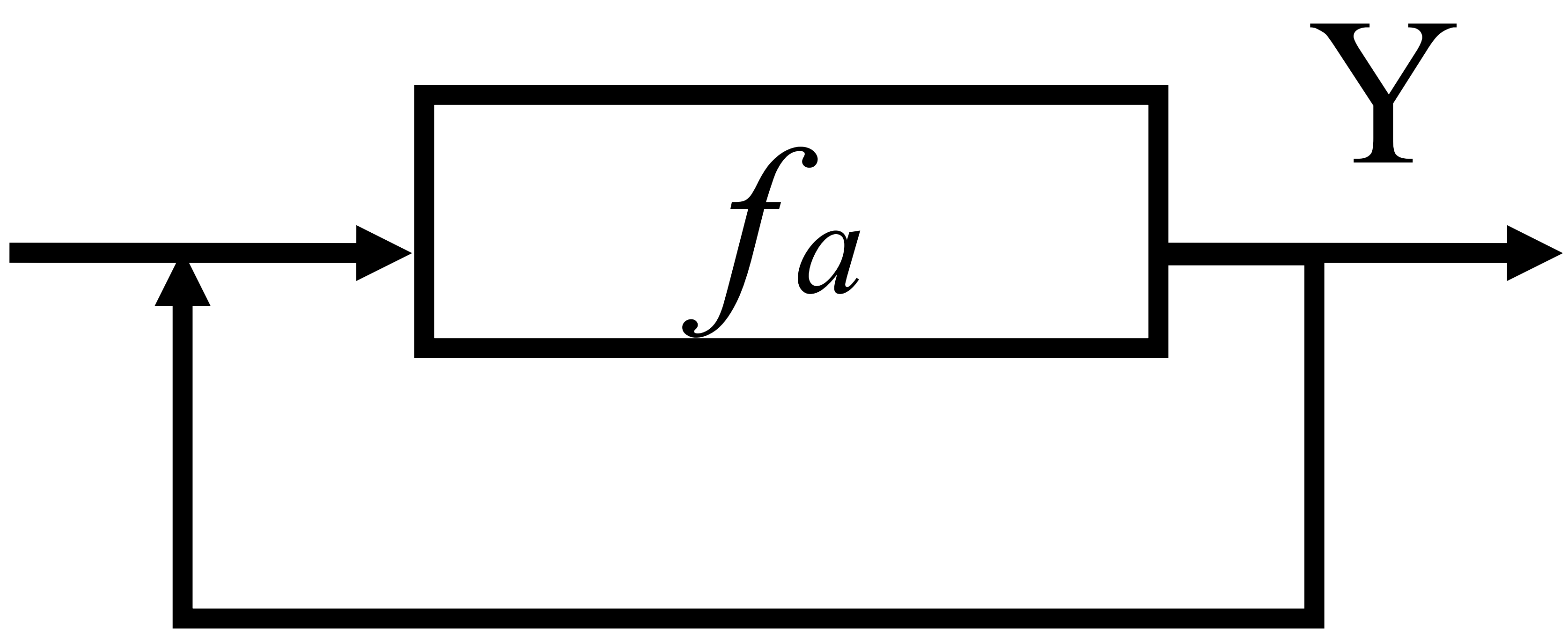
(Linear Growth)

1refs

The linear map

Dynamic Models: Parameter

$$\hat{Y} = f_a(Y)$$



1refs

The Linear Map …

The (rate of) change of the state of a system is proportional to its current state:

$$\boxed{Y_i+1 = a \cdot Y_i}$$

...Iteration...

[1]refs

The Linear Map

Initial value: Y_0

$$Y_1 = f(Y_0)$$

$$Y_2 = f(Y_1)$$

$$Y_3 = f(Y_2)$$

Iteration in general just means applying the function over and over again starting with an initial value

and subsequently to the result of the previous step

[1] refs

The Linear Map

$$\boxed{Y_{i+1} = f(Y_i)}$$

$$i = 0: \quad Y_0 \rightarrow Y_1 = f(Y_0)$$

$$i = 1: \quad Y_1 \rightarrow Y_2 = f(Y_1) = f(f(Y_0)) = f^2(Y_0)$$

$$i = 2: \quad Y_2 \rightarrow Y_3 = f(Y_2) = \ldots = f^3(Y_0)$$

$$\vdots \qquad \vdots$$

[1]refs

$$i = n: \quad Y_n \rightarrow Y_{n+1} = f(Y_n) = \ldots = f^n(Y_0)$$

Linear Map: Iteration with a parameter

$$\boxed{Y_{i+1} = a \cdot Y_i}$$

i = 0: $Y_0 \rightarrow Y_1 = a \cdot Y_0$

i = 1: $Y_1 \rightarrow Y_2 = a \cdot Y_1 = a \cdot a \cdot Y_0 = a^2 \cdot Y_0$

i = 2: $Y_2 \rightarrow Y_3 = a \cdot Y_2 = \ldots = a^3 \cdot Y_0$

$\vdots \qquad \vdots$

i = n: $Y_n \rightarrow Y_{n+1} = a \cdot Y_n = \ldots = a^{n+1} \cdot Y_0$

[1]refs

Linear Map: Iteration with a Parameter

$$Y_{i+1} = a \cdot Y_i$$

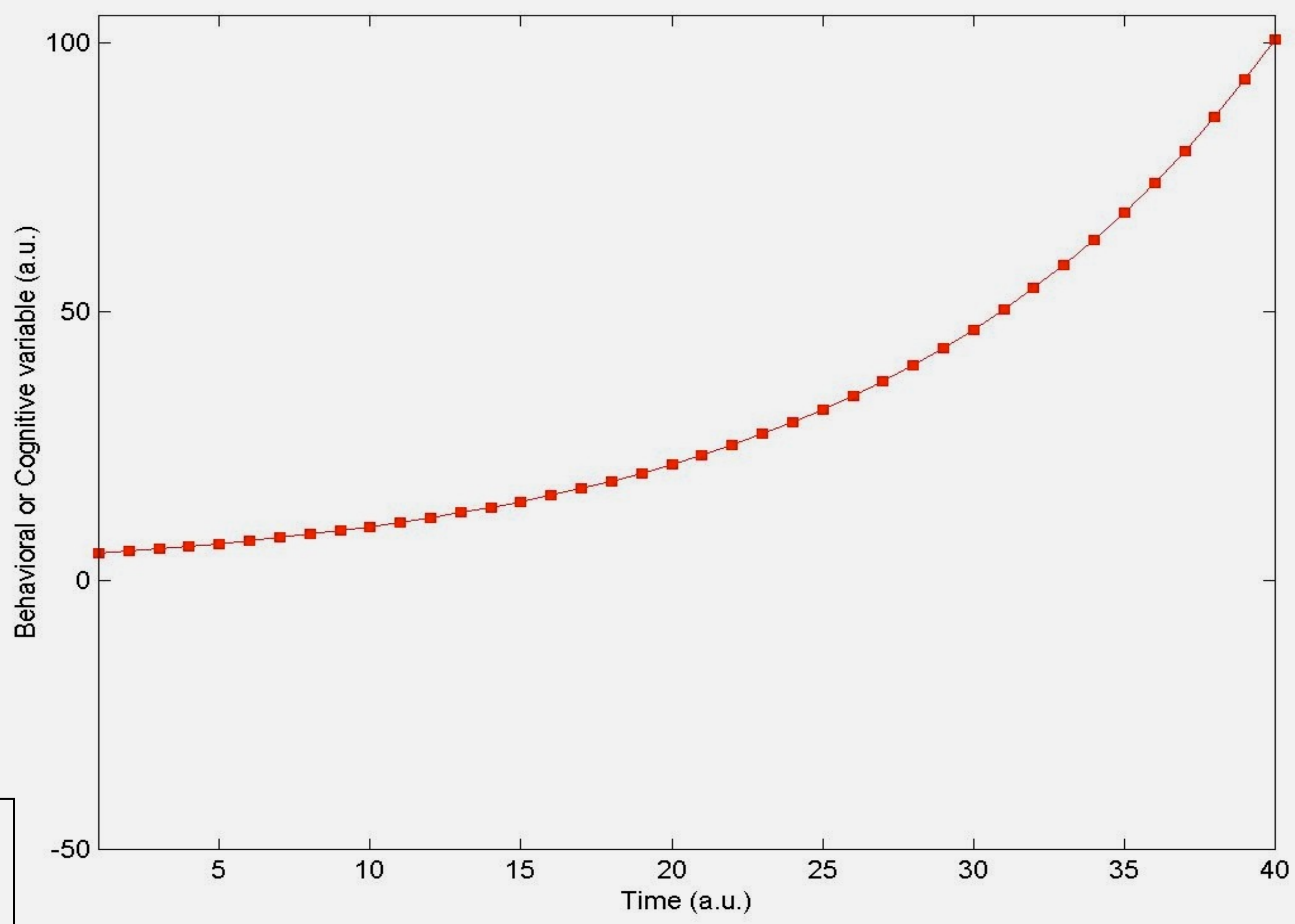
$0 < a < 1$	$-1 < a < 0$
$a > 1$	$a < -1$
$a = 1$	$a = -1$

Y_0 nonspecific

[1]refs

Linear Map: Iteration with a Parameter

a = 1.08
Y0 = 5

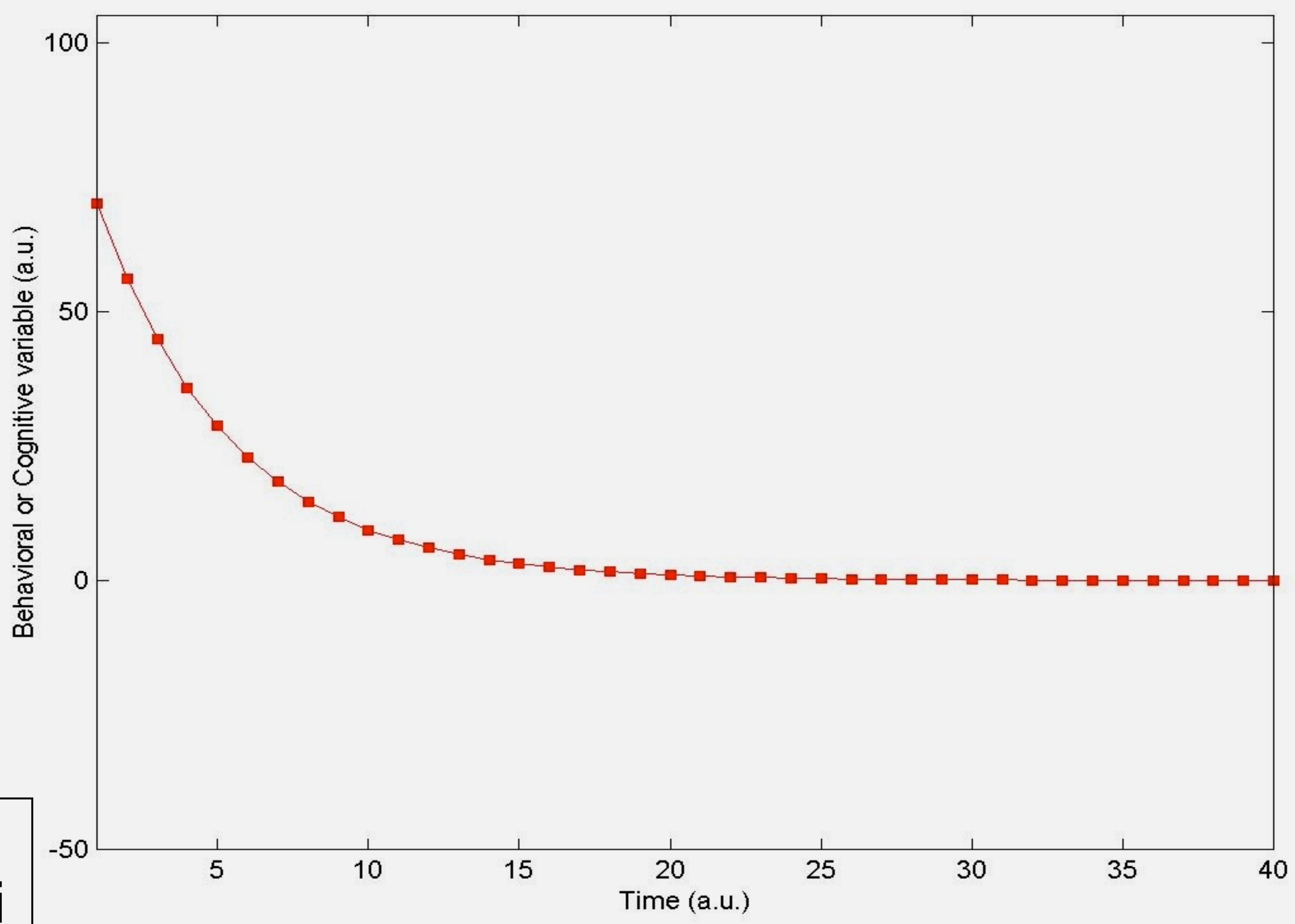


$$Y_{i+1} = a \cdot Y_i$$

[1]refs

Linear Map: Iteration with a Parameter

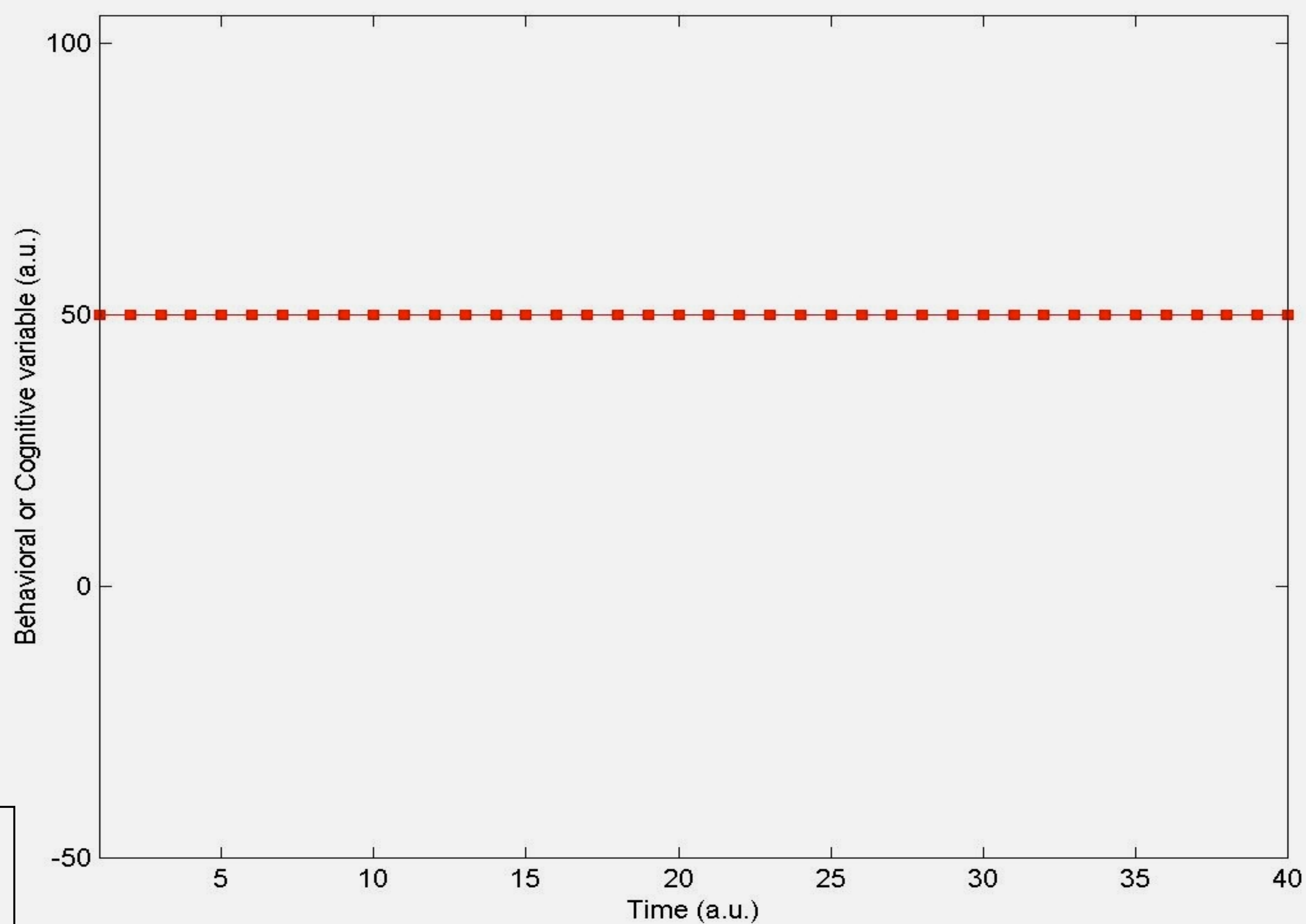
a = 0.8
Y0 = 70



$$Y_{i+1} = a \cdot Y_i$$

Linear Map: Iteration with a Parameter

a = 1.00
Y0 = 50

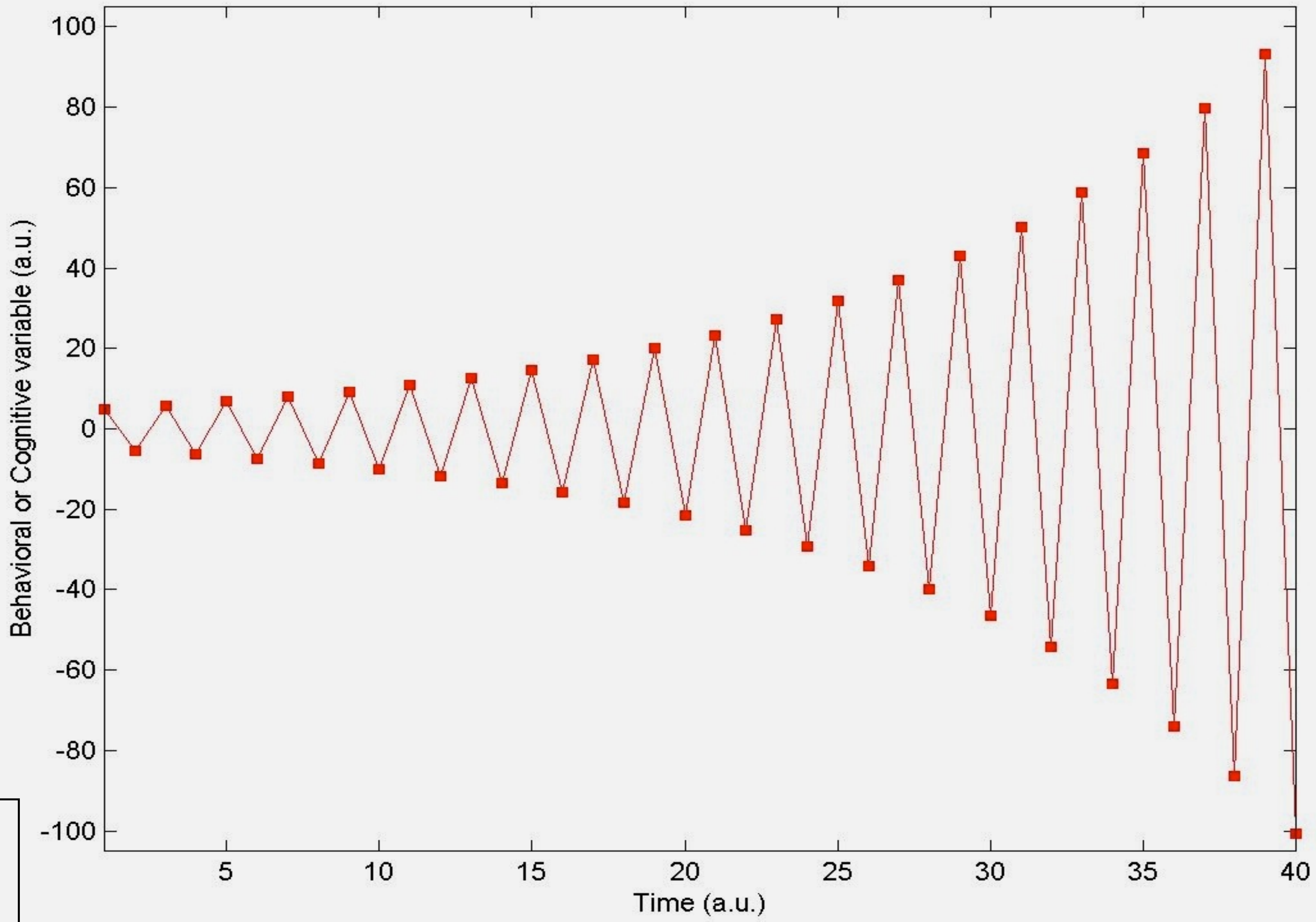


$$Y_{i+1} = a \cdot Y_i$$

[1]refs

Linear Map: Iteration with a Parameter

a = -1.08
Y0 = 5



$$Y_{i+1} = a \cdot Y_i$$

[1]refs

Linear Map: Iteration with a Parameter

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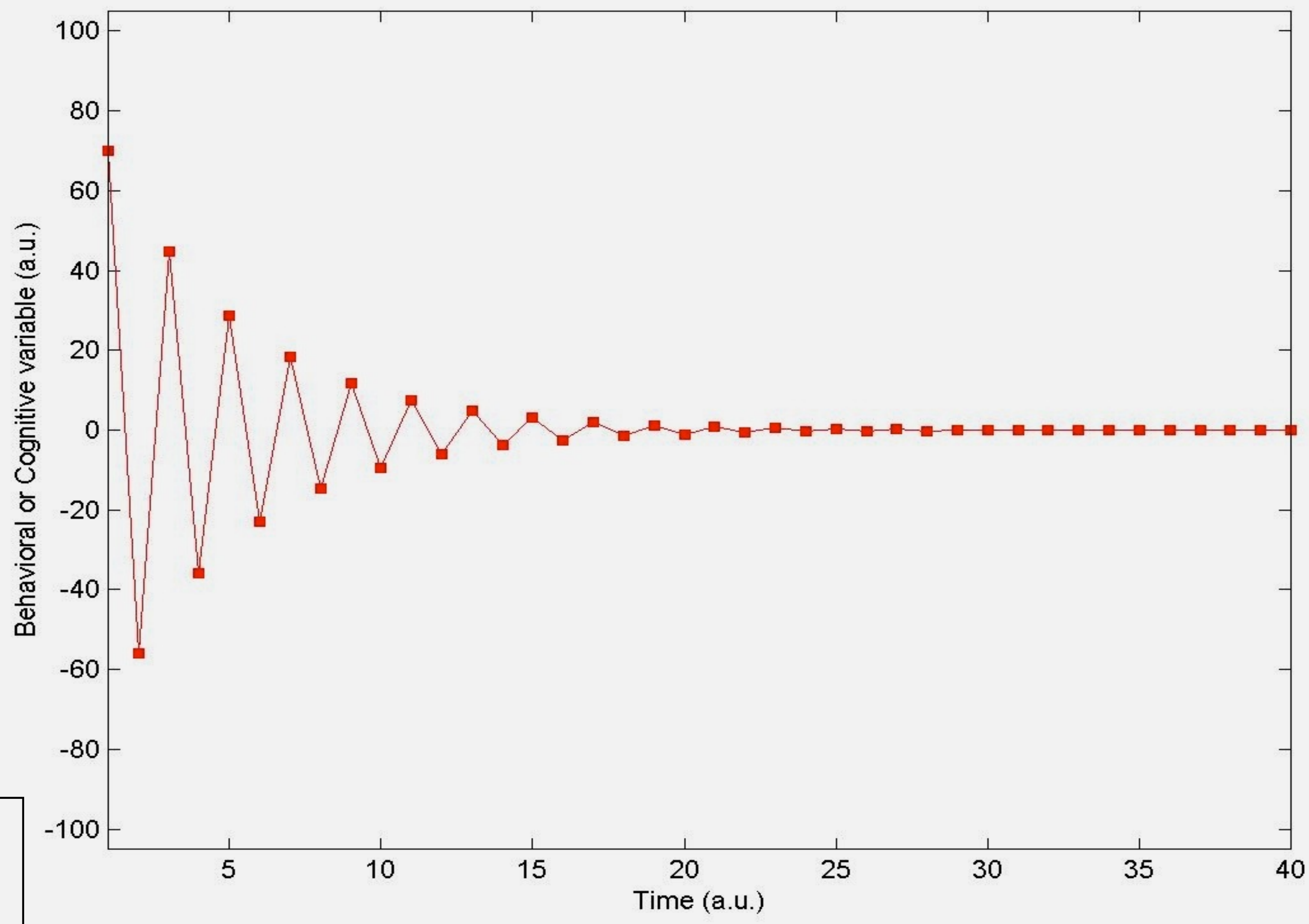
Y0 = 70

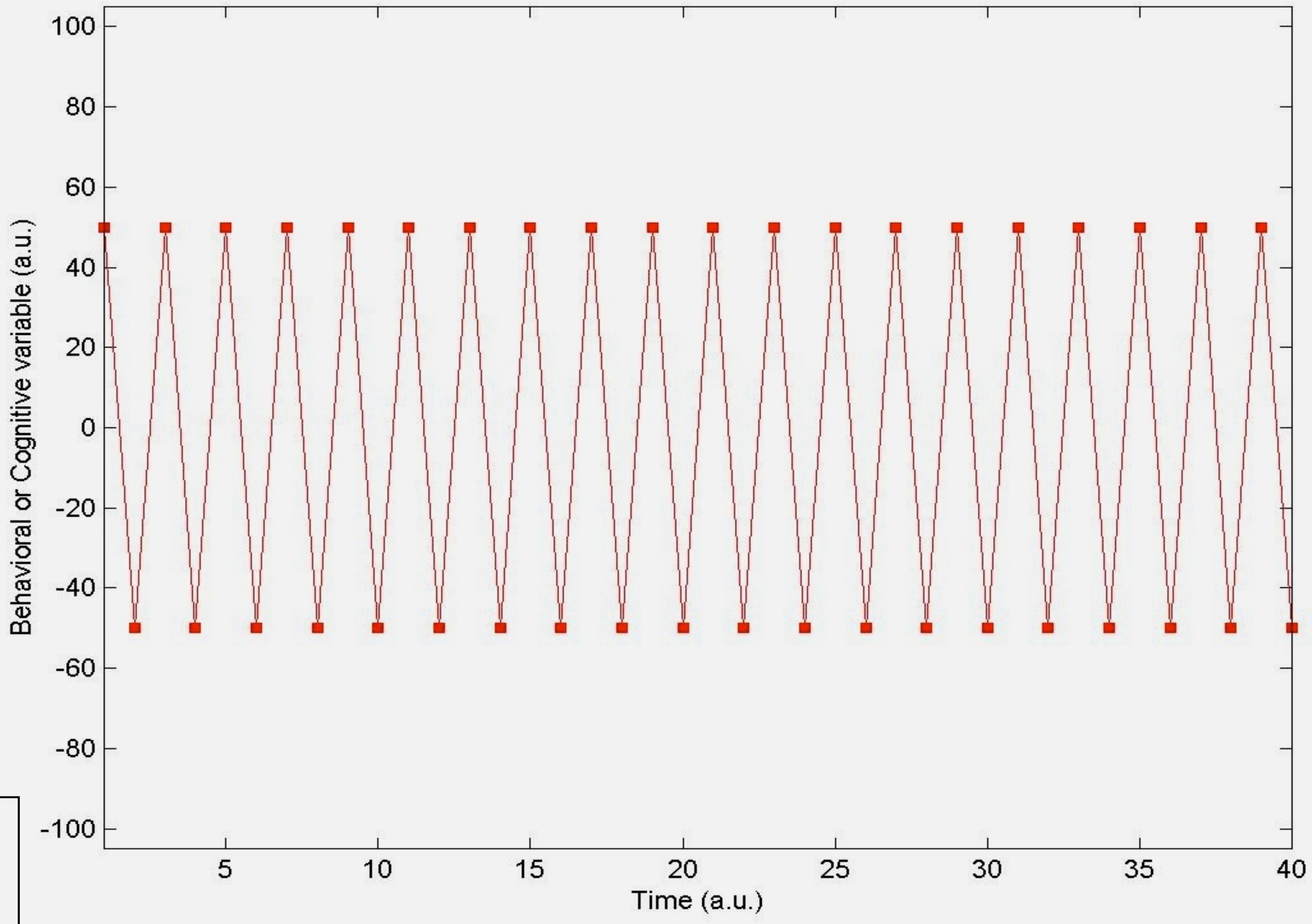


$$Y_{i+1} = a \cdot Y_i$$

[1]refs

Linear Map: Iteration with a Parameter

a = -1.00
Y0 = 50



$$Y_{i+1} = a \cdot Y_i$$

[1]refs

Linear Map: Iteration with a Parameter

Some interesting differences compared to a linear model:

- Change of behaviour over iterations
  - *Simple model vs. “time” or “occasion” as a predictor*
- Qualitatively different behaviour
  - *One model produces at least four different types of behaviour*
  - *Not by adding predictors (components), by changing one parameter*

[1]refs

PARAMETERS & BIFURCATIONS

EXAMPLE 2:

The Logistic Map

(restricted growth)

Logistic Map …

$$\boxed{L_{i+1} = r\,L_i\,(1 - L_i)}$$

- Simplest nontrivial model often used as an introduction to DST and Chaos theory.
- Well-known model in ecology, physics, economics and social sciences.
- ‘Styled’ version of Van Geert’s model for language growth. (*Next meeting*)

[1]refs

Logistic Map: Iteration

$$\boxed{L_{i+1} = r L_i (1 - L_i)}$$

i = 0: $L_0 \rightarrow L_1 = r L_0 (1 - L_0)$

i = 1: $L_1 \rightarrow L_2 = r L_1 (1 - L_1)$

$$= r\, rL_0(1-L_0)\,(1- rL_0(1-L_0))$$

$$= -r^3L_0^{\,4} + 2r^3L_0^{\,3} - r^2(1+r)L_0^{\,2} + r^2L_0$$

[1]refs

Logistic Map: Parameter

$$L_{i+1} = r L_i (1 - L_i)$$

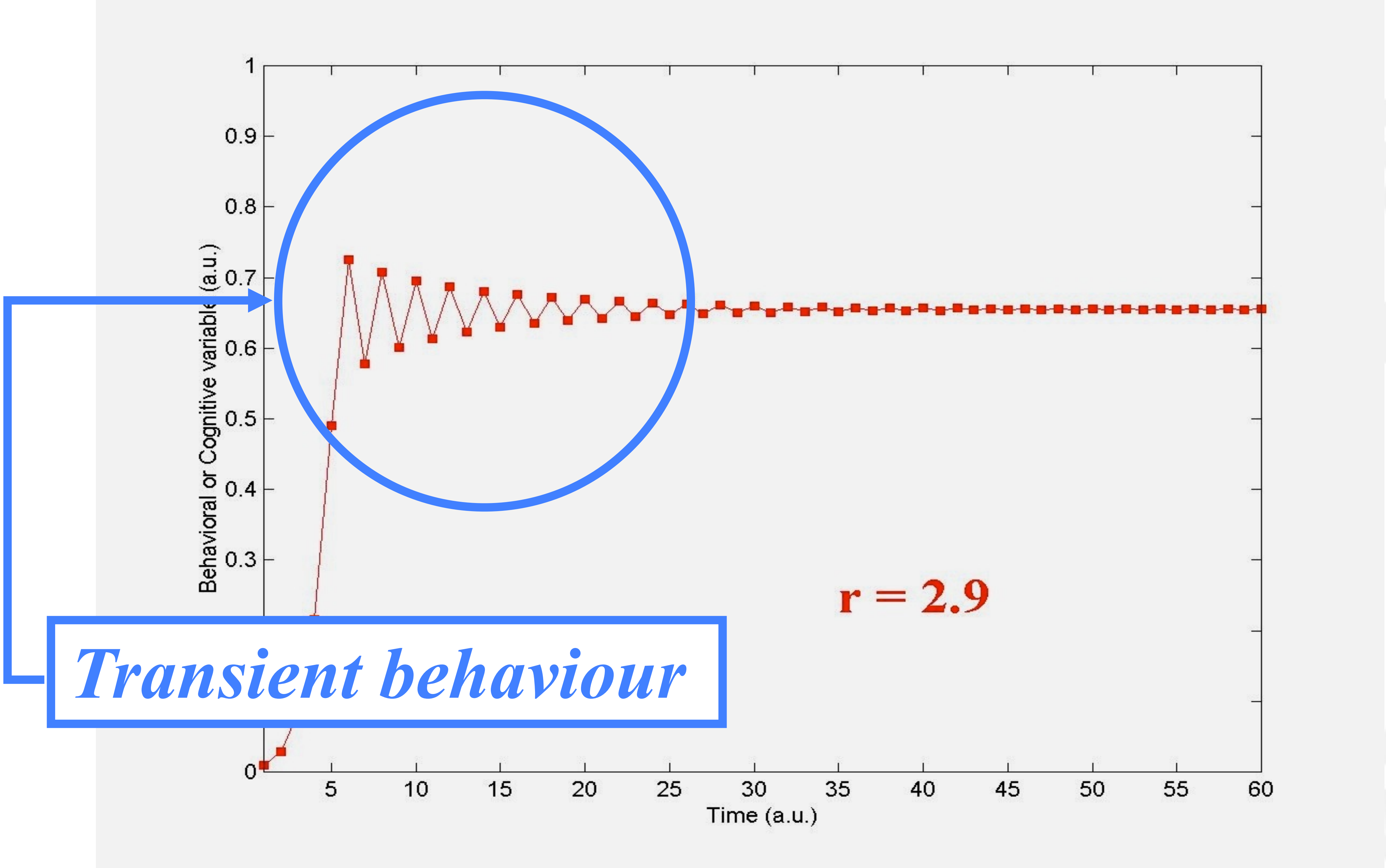
$r = 0.90$	$r = 3.30$
$r = 1.90$	$r = 3.52$
$r = 2.90$	$r = 3.90$

L_0 small

[1]refs

Logistic Map: Graphs

$$L_{i+1} = r L_i (1 - L_i)$$



[1]refs

An ecology of growth models?
Same principle!

Basic Growth Models: Exponential + Restricted Growth

$$Population = rN \times (\frac{K - N}{K})$$

Additional Parameter: Carrying Capacity

$$CognitiveGrowth = L_i (1 + r \times \frac{K - L_i}{K})$$

$$StylizedLogistic = rY_i \times (\frac{1 - Y_i}{1})$$

Bifurcation Diagram

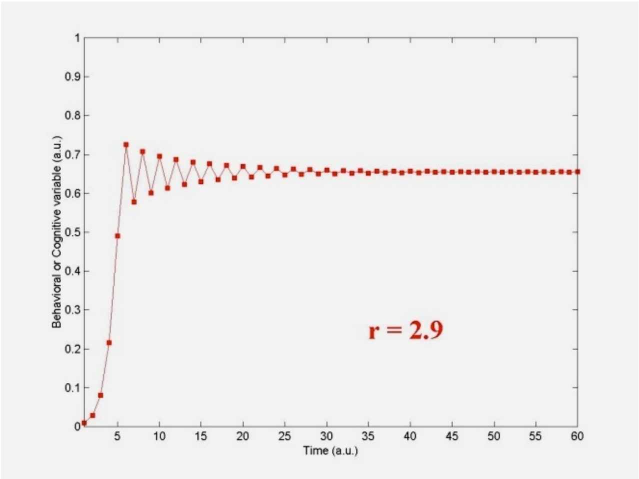
Bifurcation Diagram - Phase Diagram

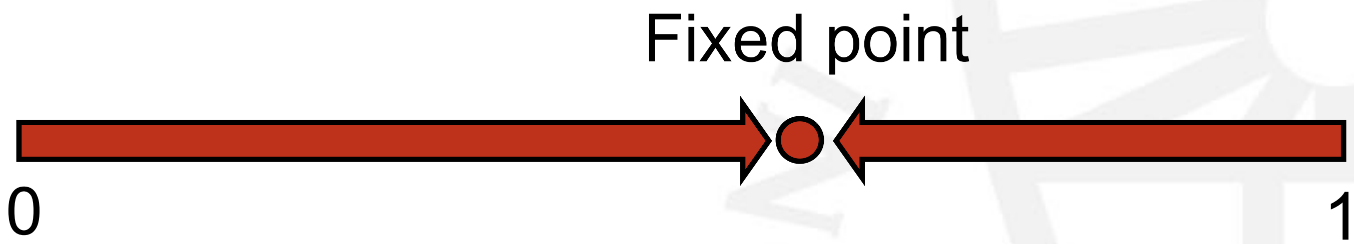
A graphical representation of the possible states a dynamical system can end up in for different values of one or more parameters.

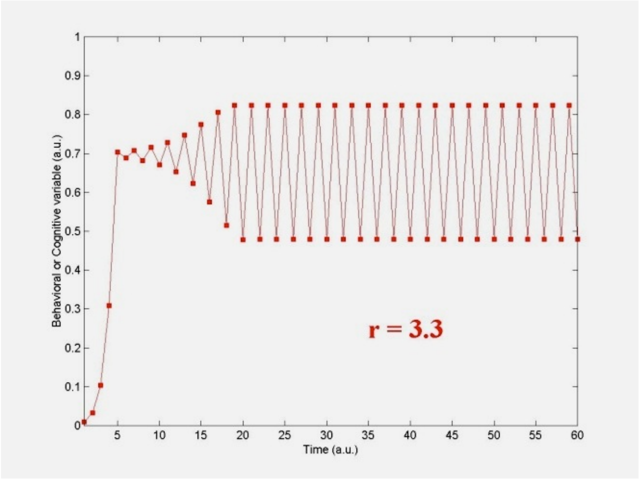
- The parameter is called the ***control parameter***.
- The end states are called ***attractors***.
- The change from one attractor (or set) to another is called a ***bifurcation***.

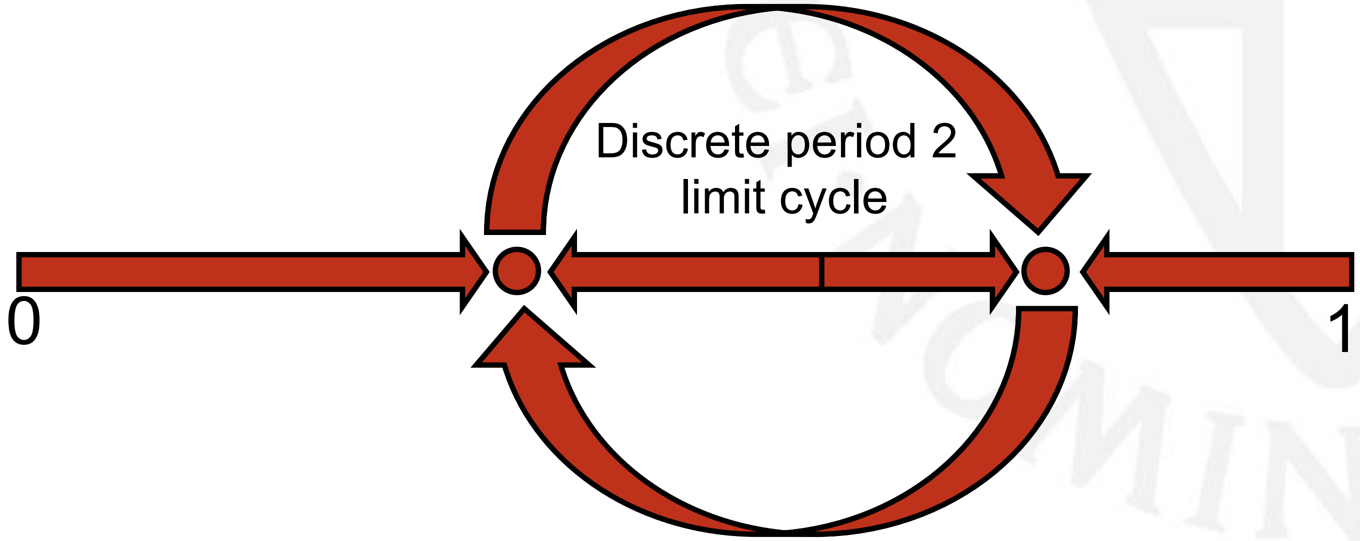
End states are attractors in state space: Attractor types

State Space is an abstract space used to represent the behaviour of a system. Its dimensions are the variables of the system. Thus a point in the phase space defines a potential state of the system. The points actually achieved by a system depend on its iterative function and initial condition (starting point).









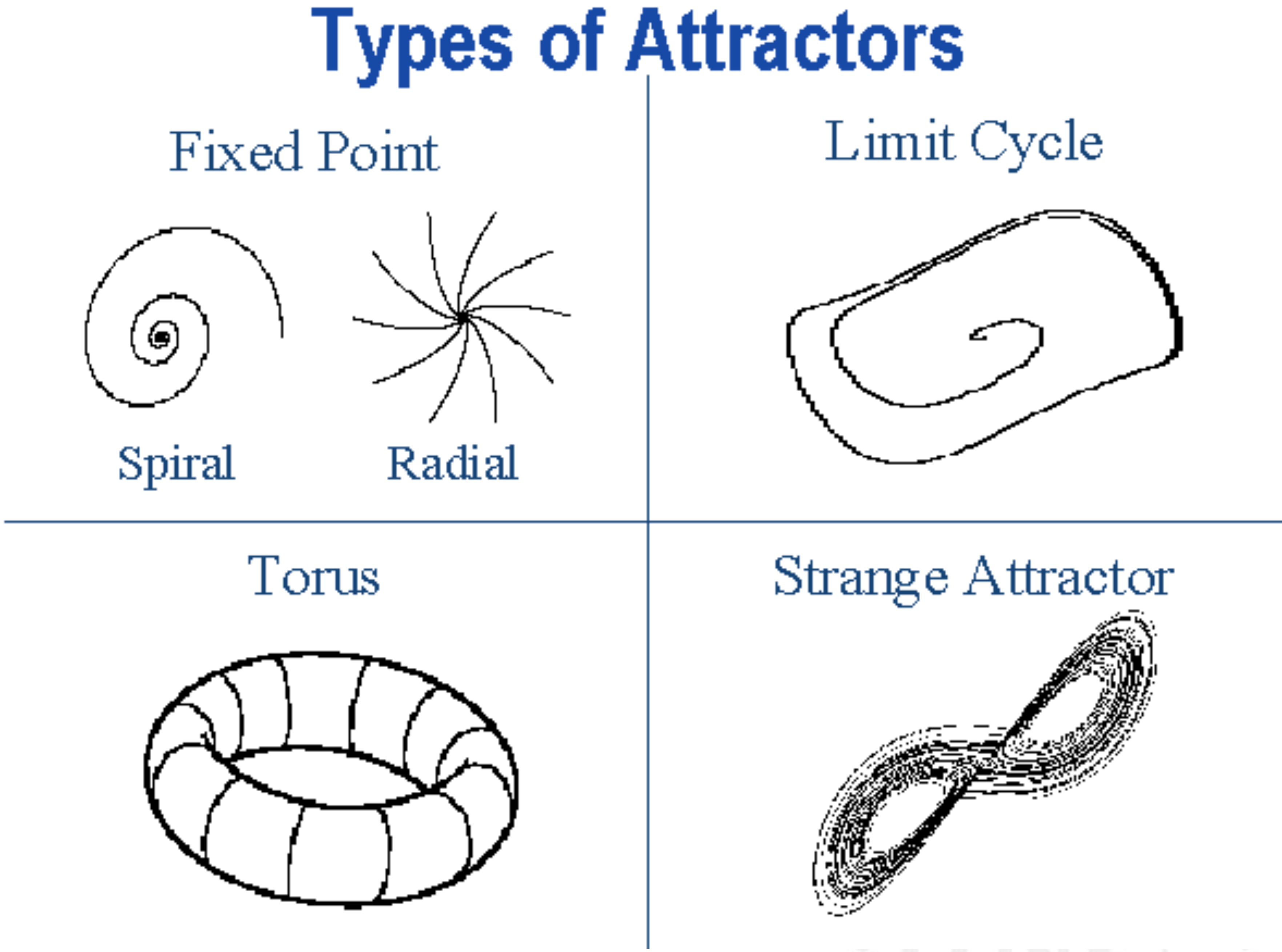
State space, Attractor types

Attractor: The state a dynamic system eventually "settles down to".

An attractor is a set of values in the phase space to which a system migrates over time, or iterations.

An attractor can be a single fixed point, a collection of points regularly visited, a loop, a complex orbit, or an infinite number of points.

It need not be one- or two-dimensional. Attractors can have as many dimensions as the number of variables that influence the system.



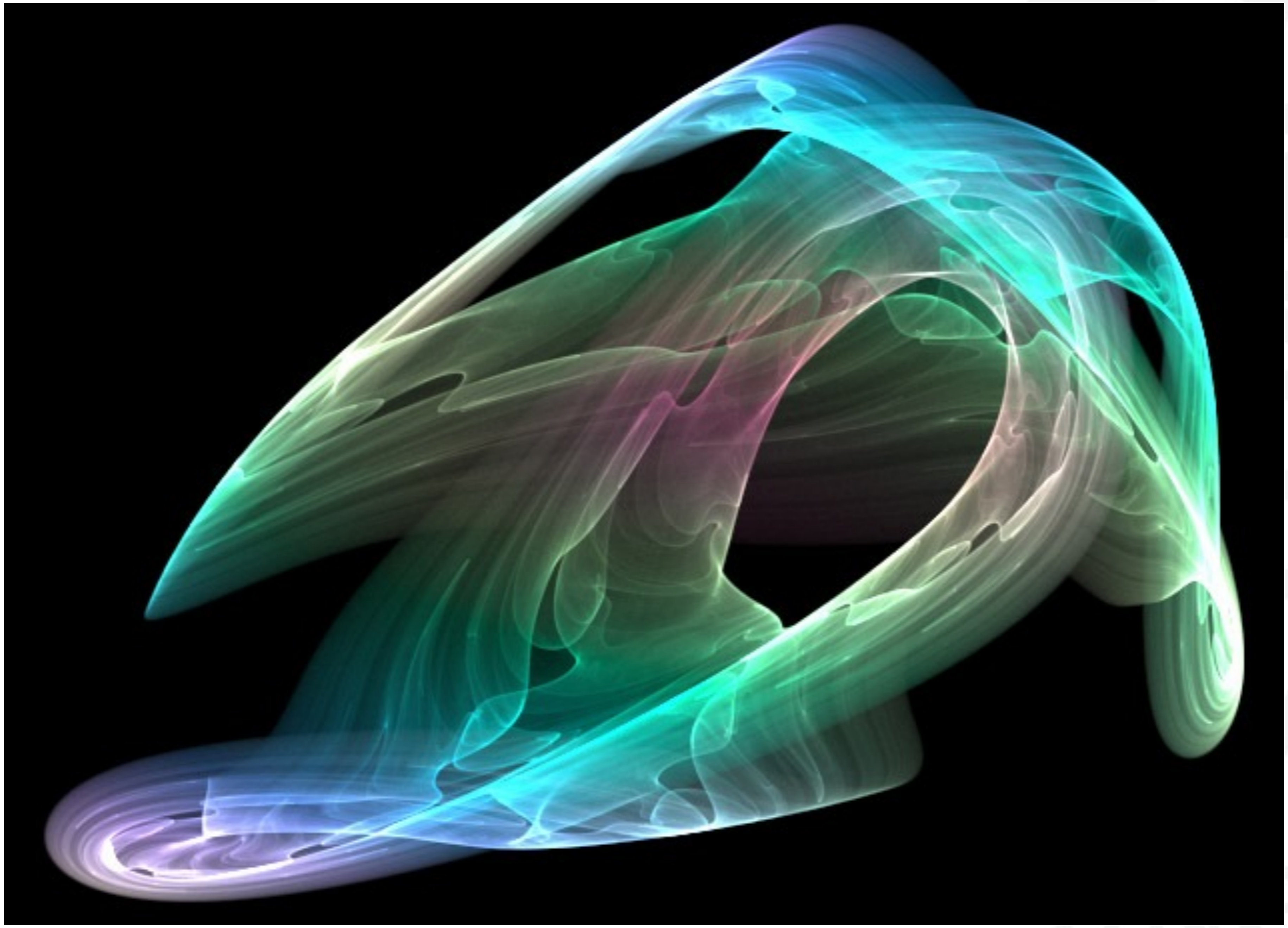
State space, Attractor types

“Saturn”
attractor

Strange attractors
are quasi periodic
and bounded

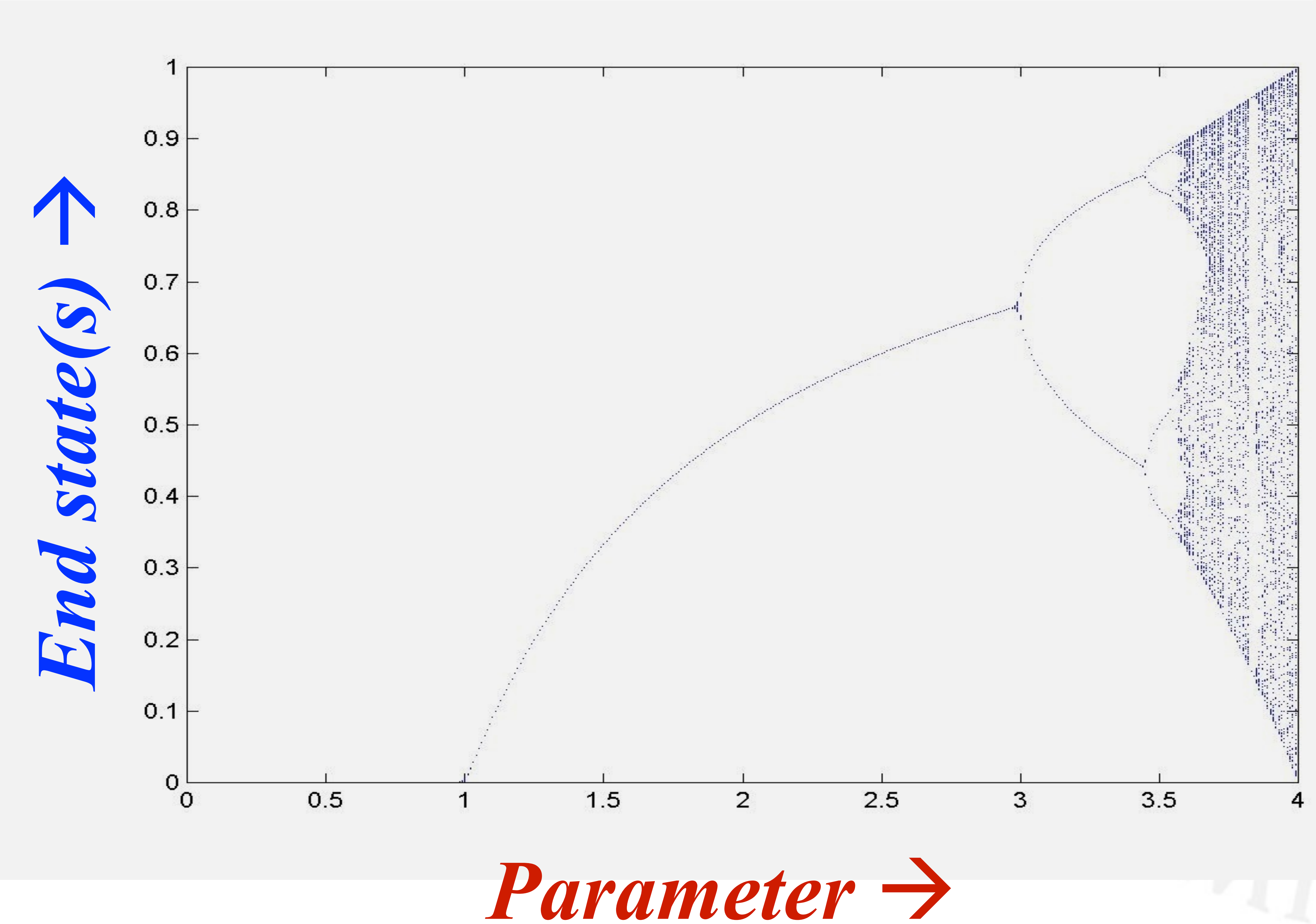
Bottom line:

An attractor means
a limited region
of state space
is visited.
Not all DF actually
available
to the system
are used.



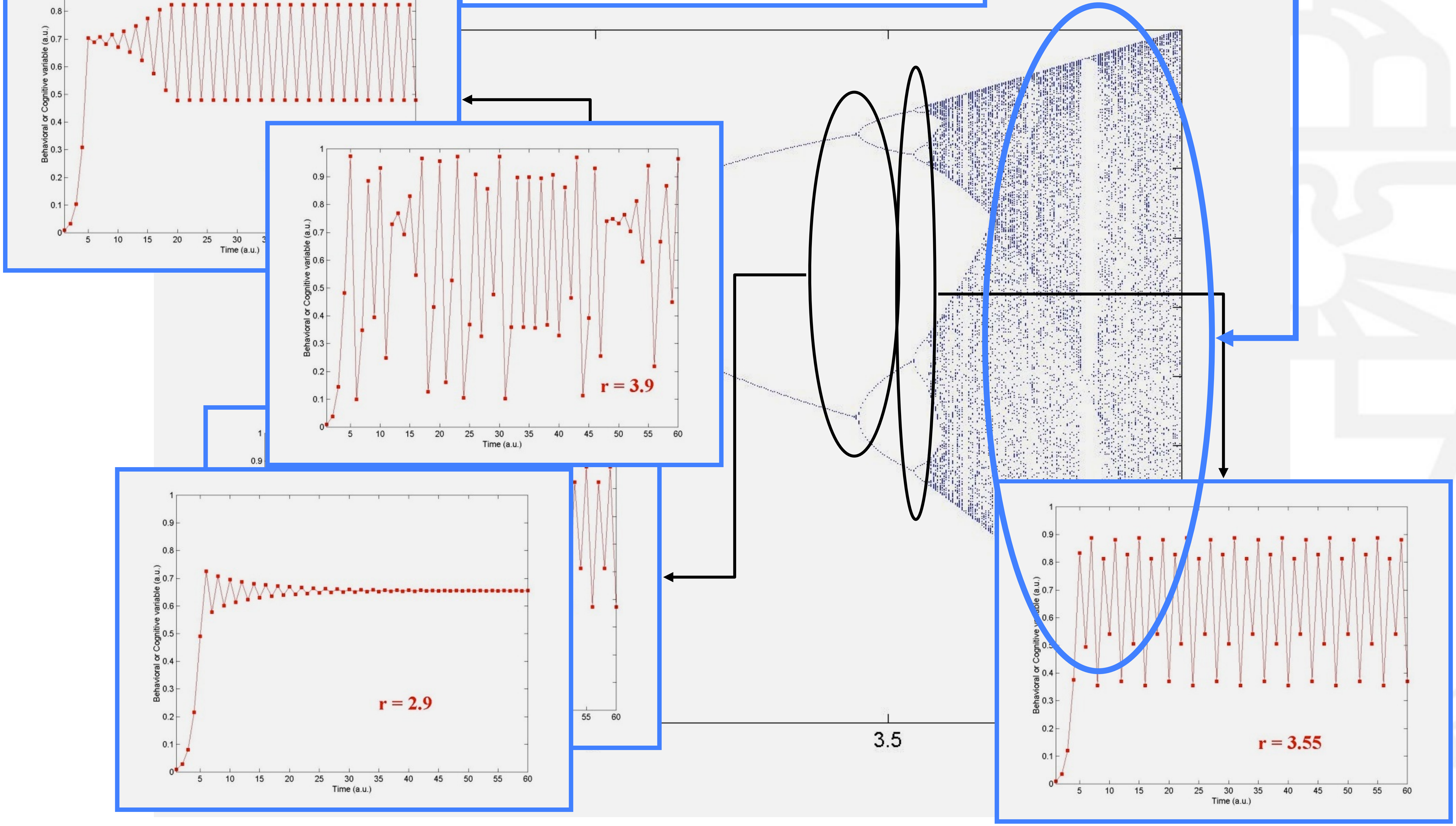
http://www.da4ga.nl/wp-content/uploads/2012/03/PastedGraphic-2-1.jpg

Logistic Map: Bifurcation Diagram

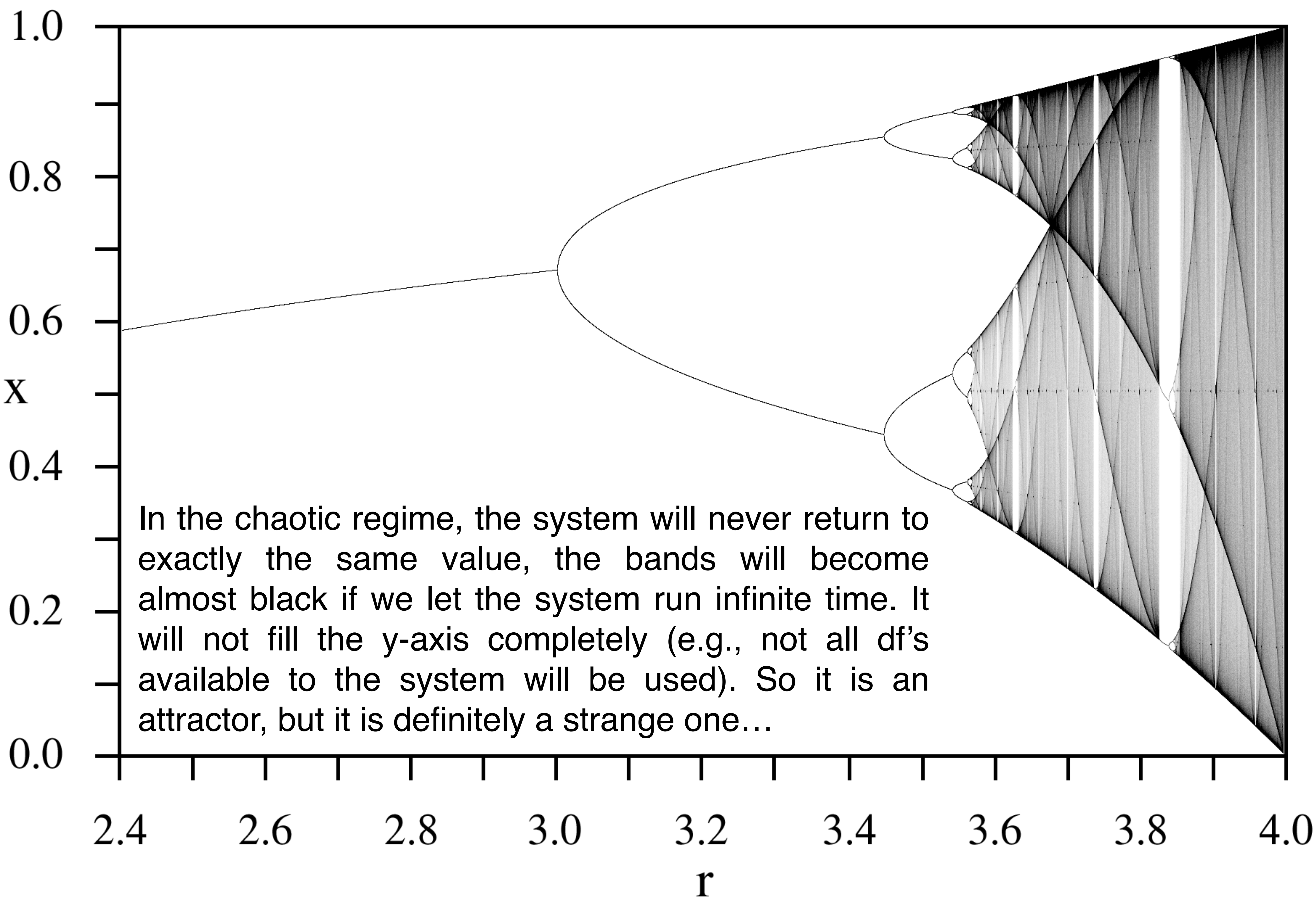


[1]refs

Chaotic regime



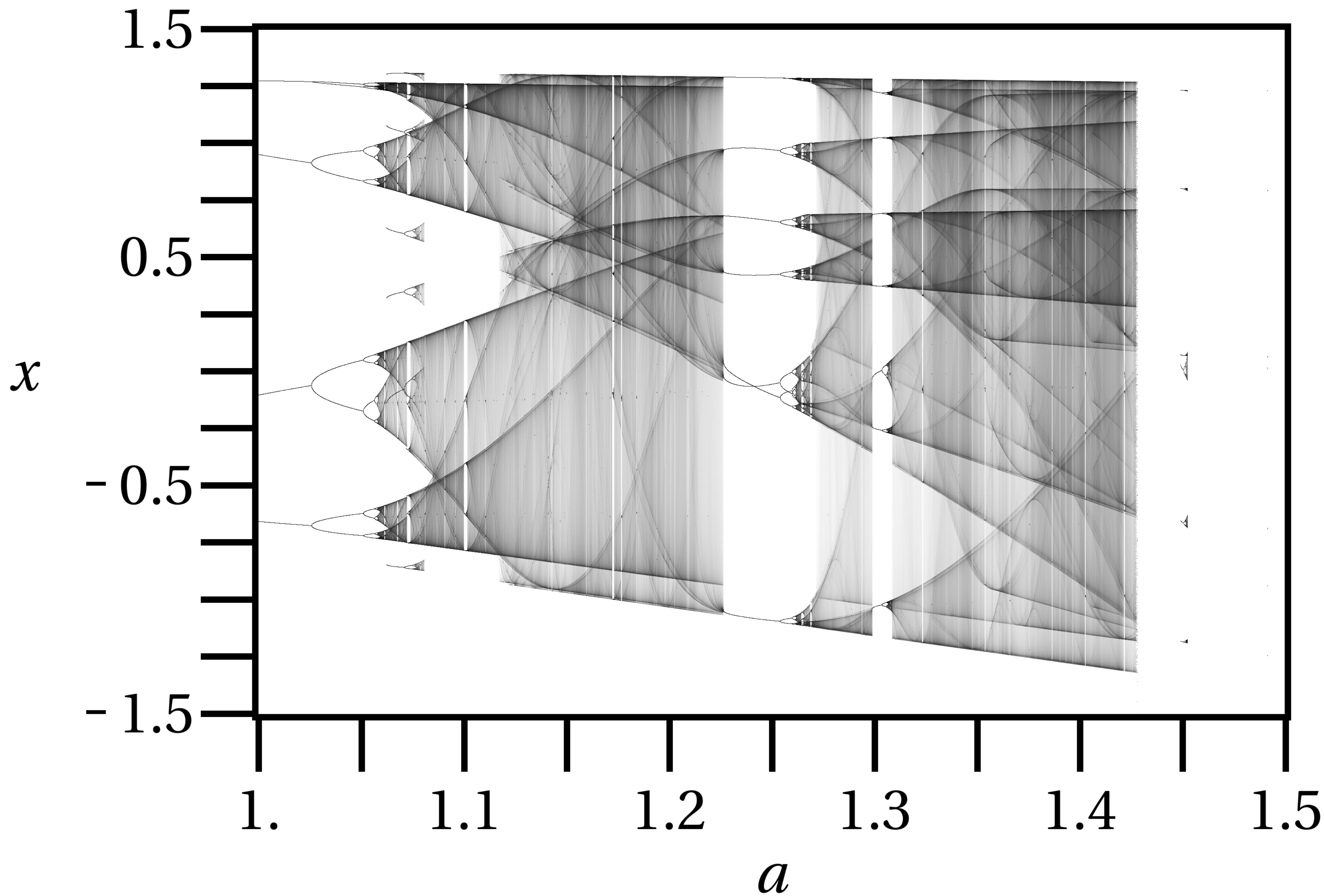
Logistic Map: Bifurcation Diagram



In the chaotic regime, the system will never return to exactly the same value, the bands will become almost black if we let the system run infinite time. It will not fill the y-axis completely (e.g., not all df's available to the system will be used). So it is an attractor, but it is definitely a strange one…

http://upload.wikimedia.org/wikipedia/commons/7/7d/LogisticMap_BifurcationDiagram.png

Henon Map: Bifurcation Diagram



http://upload.wikimedia.org/wikipedia/commons/c/cd/Henon_bifurcation_map_b%3D0.3.png

Deterministic Chaos

1refs

Chaos

There is no real definition of chaos, but there are at least four ingredients:

*The dynamics is **a-periodic** and **bounded**, and the system is **deterministic** and **sensitively depends on initial conditions**.*

[1]refs

Deterministic Chaos… Paradox?

Something that is ***deterministic***, is:

- *Mathematically exact;*
- *Predictable*.

Something that is ***’chaotic’***, shows:

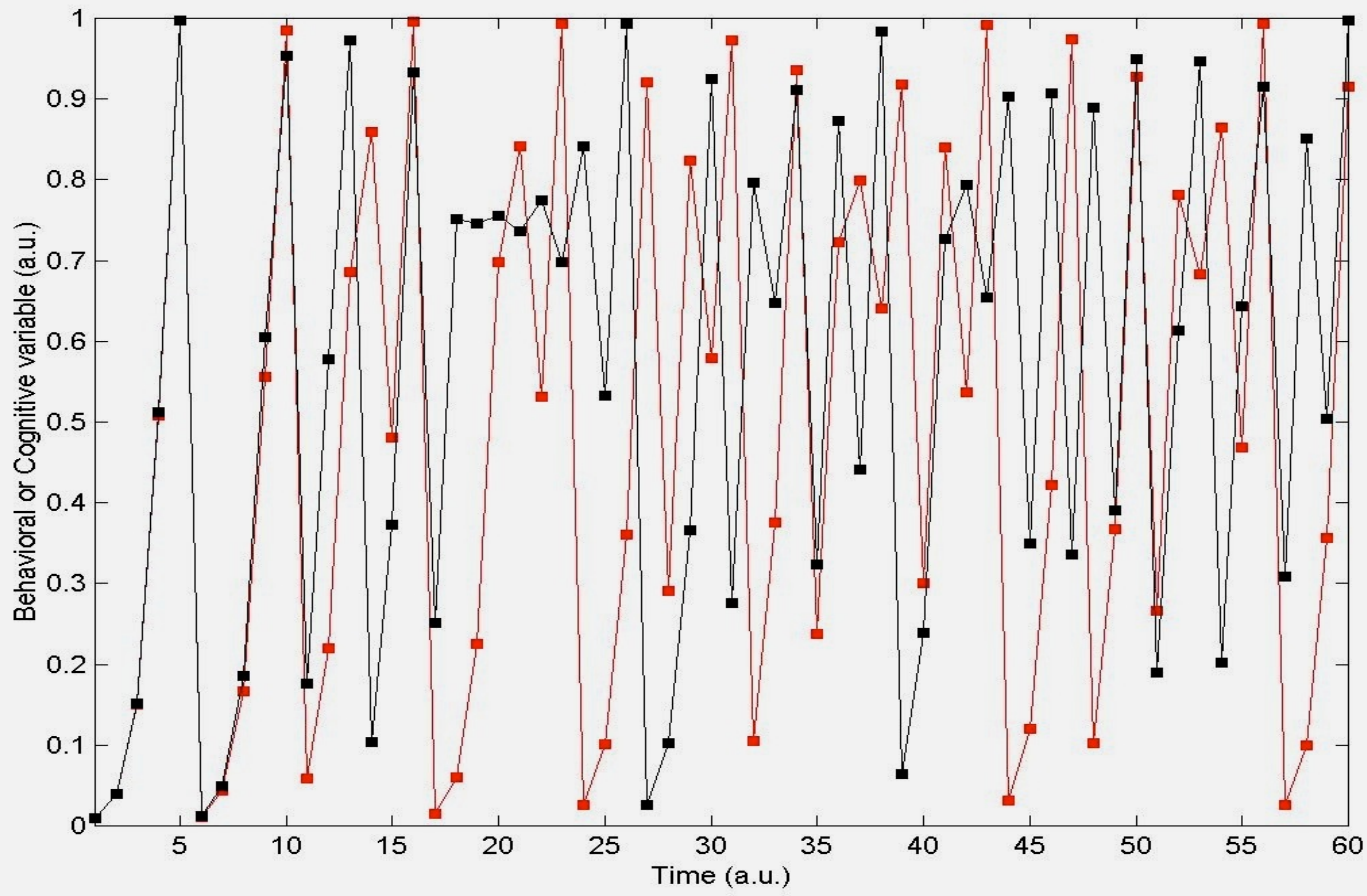
- *Disorderly behaviour;*
- *Extreme sensitivity*.

[1]refs

Sensitive Dependence on Initial Values: The "Butterfly Effect"

Y_0=0.0100

Y_0=0.0101



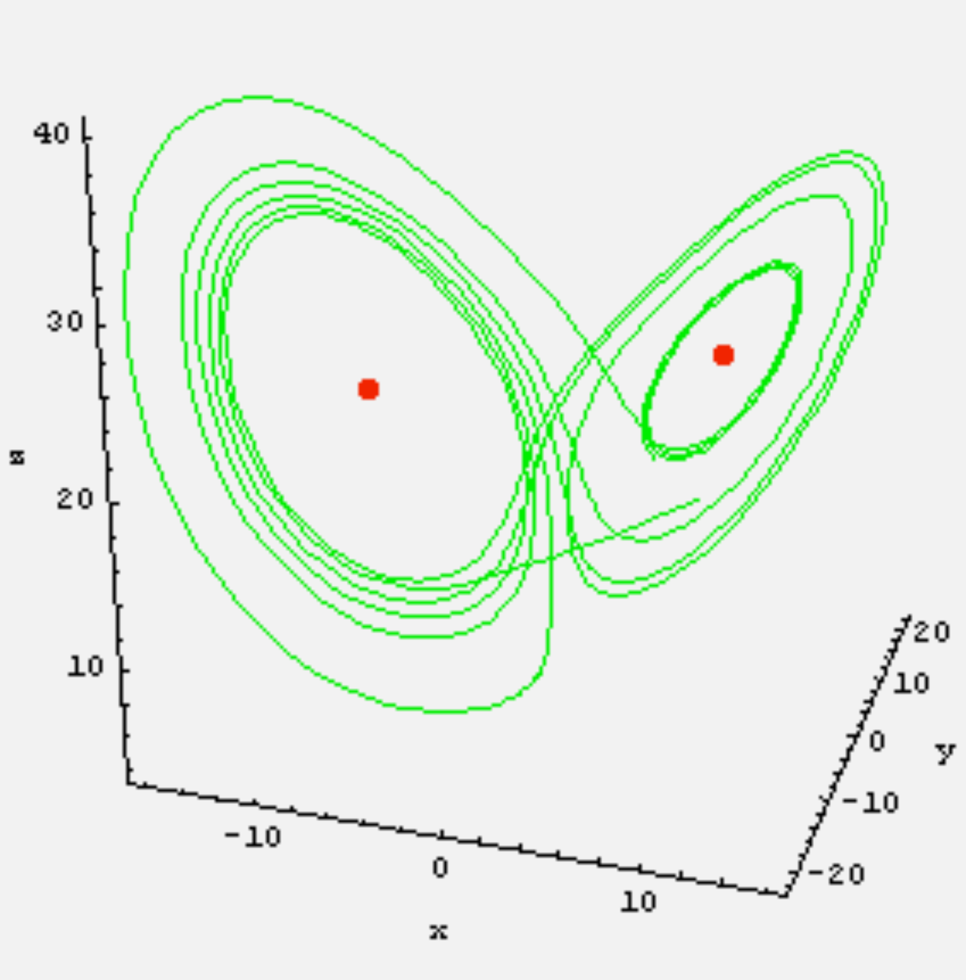
Lorenz Attractor

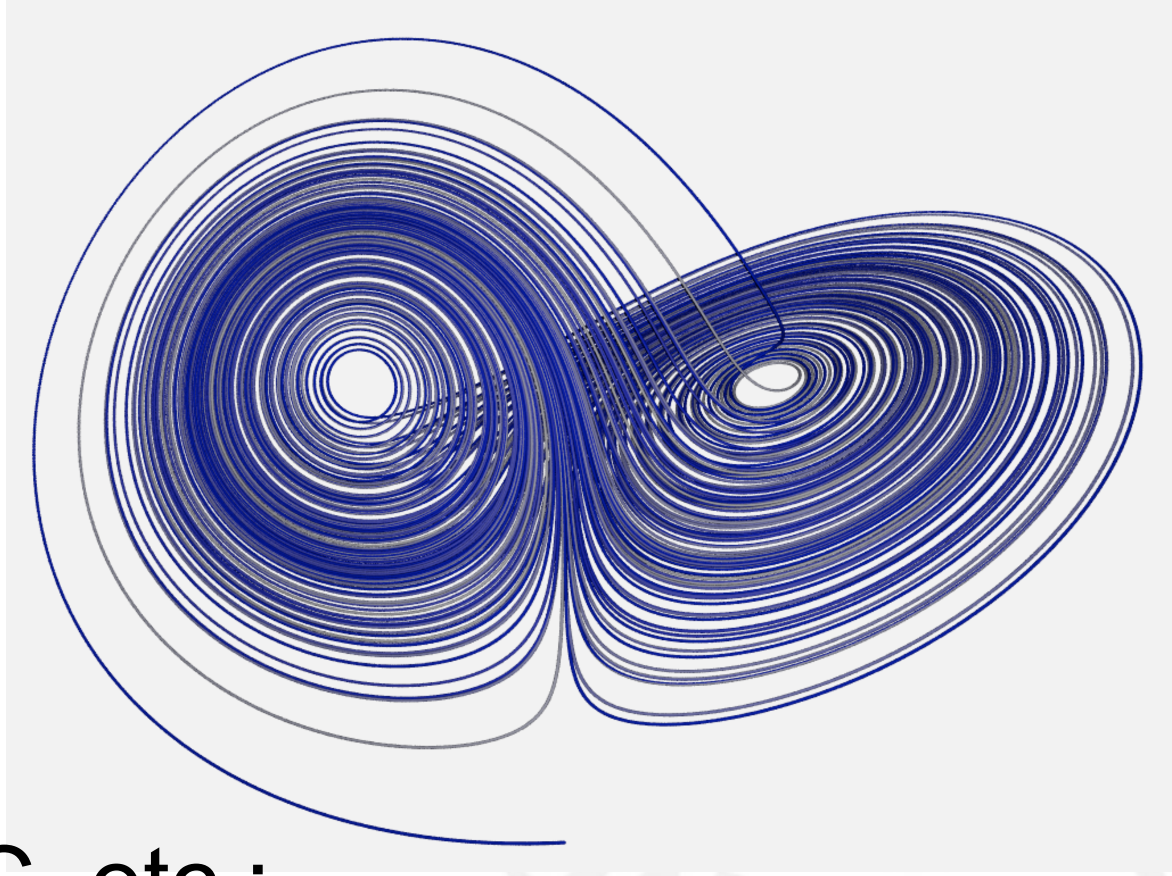
$$\frac{dx}{dt} = a(y - x),$$
$$\frac{dy}{dt} = x(b - z) - y,$$
$$\frac{dz}{dt} = xy - cz.$$

Deterministic Chaos

Maps: linear map, 1D state space

Flows: Need 3 coupled ODEs (ordinary differential equations) Minimum is 3D state space





Lorenz about chaos, fractals, SOC, etc.:
"*Study of things that look random -but are not*"

Deterministic Chaos

Table 12-1. Summary of the Hierarchy of Dynamic Systems.

Type	Constraints	Description
Zero	Absolute	Constant state
I	Analytic integrals	Solvable dynamic system
II	Approximate analytic integrals	Amenable to perturbation theory
III	Quasi-deterministic; smooth but erratic trajectory	Chaotic dynamic system
IV	Rigorously defined only by averages over time or state space	Turbulent/stochastic

Table 12-2. A few examples of the types of dynamic systems.

Type	Examples
Zero	Images, gravity models, structures
I	Gear trains, 2-body problem, physical pendulum
II	Satellite orbits, lunar and planetary theories
III	Climatology, Lorenz equations, discrete logistic equation
IV	Quantum mechanics, turbulent flow, statistical mechanics

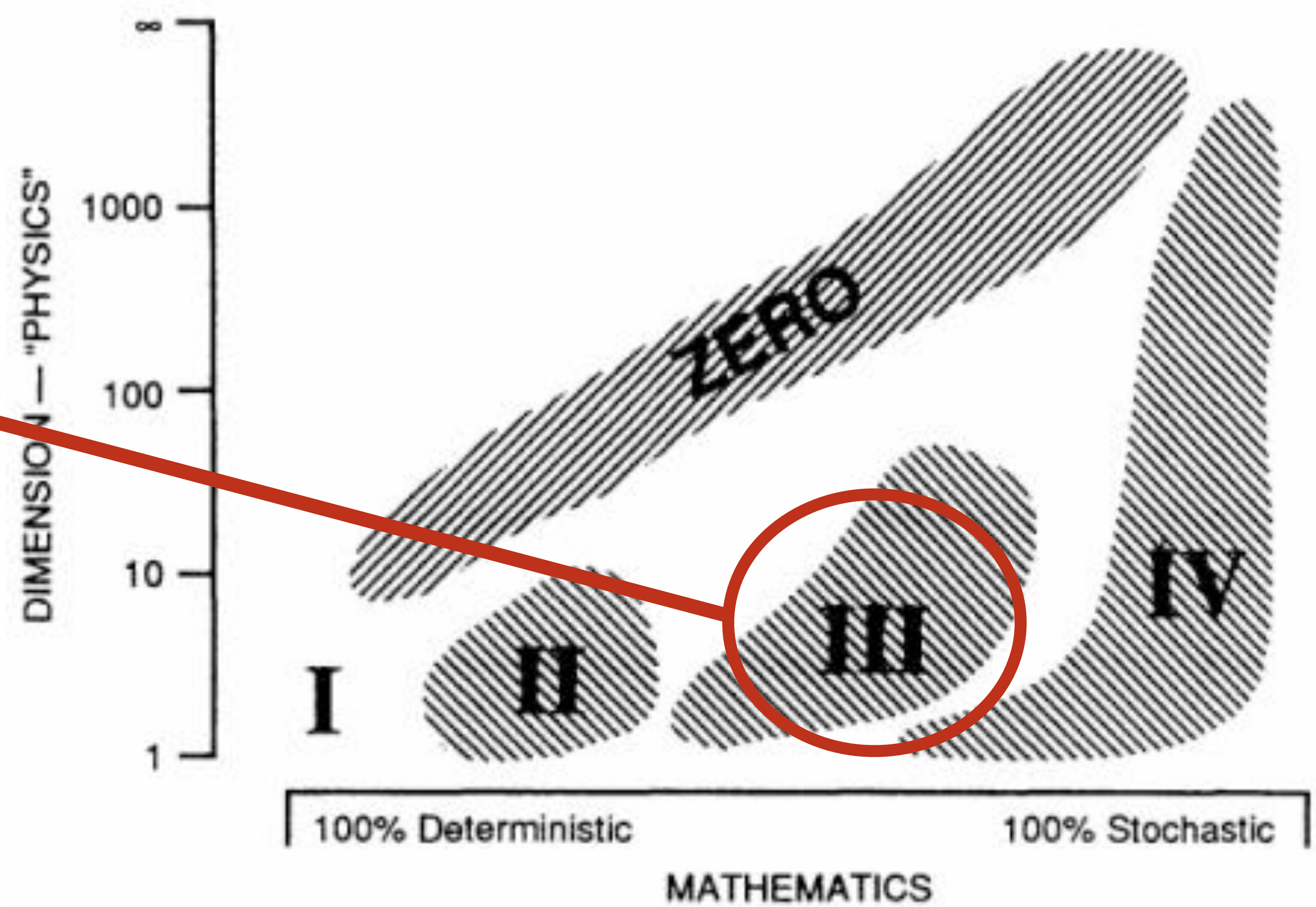


Figure 12-1. Schematic representation of the Hierarchy of Dynamic Systems.

Next session

Model for Cognitive Growth - van Geert
(read chapter on dynamic system models)

Multivariate models = Coupled equations
(Predator-Prey dynamics)

[1]refs