Dynamics of Complex Systems

Part 1:

Story so far: Assignments

Closer look at modelling growth

Part 2:

Multivariate Models

Potential models - Nonlinear Dynamics

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The right tools for the job?
Dynamic models…

• A model is (should be) a mathematical or logical implementation of a theory allowing for simulation of observed behaviour, prediction of measurement outcomes, interpretation... i.e evaluation of the theory

• If a model describes behaviour as a function of **time** in some way, it is *dynamic*

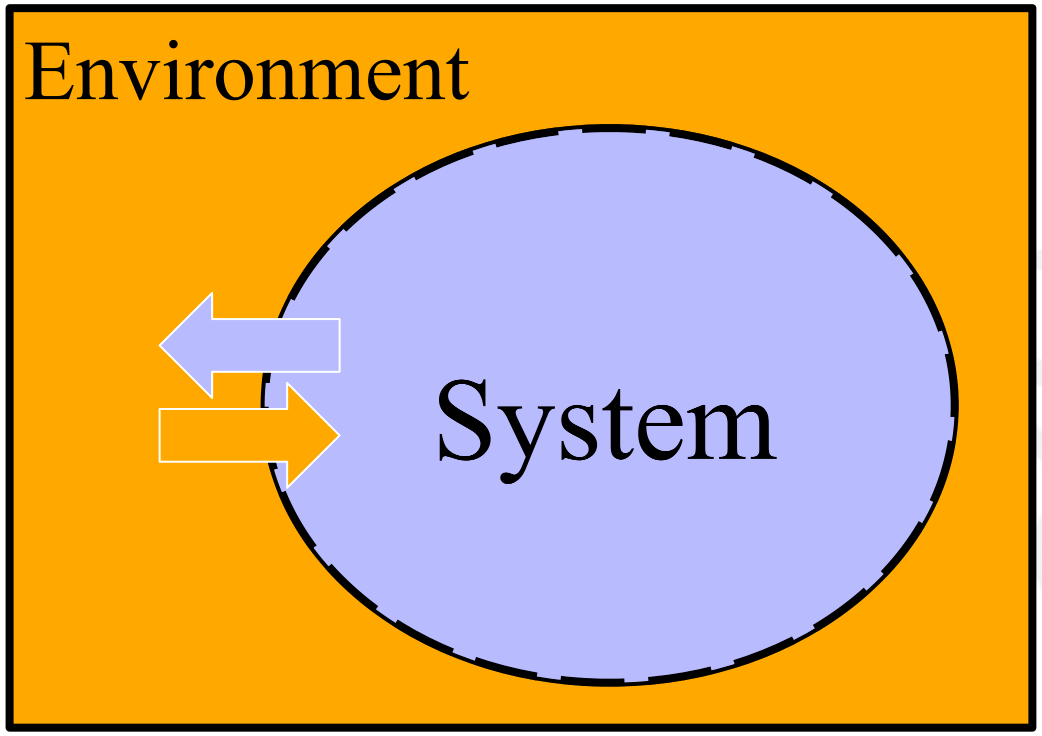
• If a dynamical model describes behaviour as an **interaction** of several processes or components it is “interacting on multiple scales”, maybe “multicausal” and perhaps even “self-organising” (i.e. *interaction dominant*)

• An interaction dominant dynamical model does not necessarily have to represent a “*dynamics of complex systems theory about emergence and self-organisation of behaviour*”. The tools used to build those models seem a good start!

• Many if not all theories in behavioural science deal with dynamics and interactions of variables, contexts... but are the right models used to evaluate those theories?

What is a system?

Closed and Open Systems



Continuous exchange of matter, energy, and information with the environment.

Story so far - Assignments session 1: Working with analytic solutions of dynamic systems

Solutions are rare!

Table 12-1. Summary of the Hierarchy of Dynamic Systems.

Type	Constraints	Description
Zero	Absolute	Constant state
I	Analytic integrals	Solvable dynamic system
II	Approximate analytic integrals	Amenable to perturbation theory
III	Quasi-deterministic; smooth but erratic trajectory	Chaotic dynamic system
IV	Rigorously defined only by averages over time or state space	Turbulent/stochastic

Table 12-2. A few examples of the types of dynamic systems.

Type	Examples
Zero	Images, gravity models, structures
I	Gear trains, 2-body problem, physical pendulum
II	Satellite orbits, lunar and planetary theories
III	Climatology, Lorenz equations, discrete logistic equation
IV	Quantum mechanics, turbulent flow, statistical mechanics

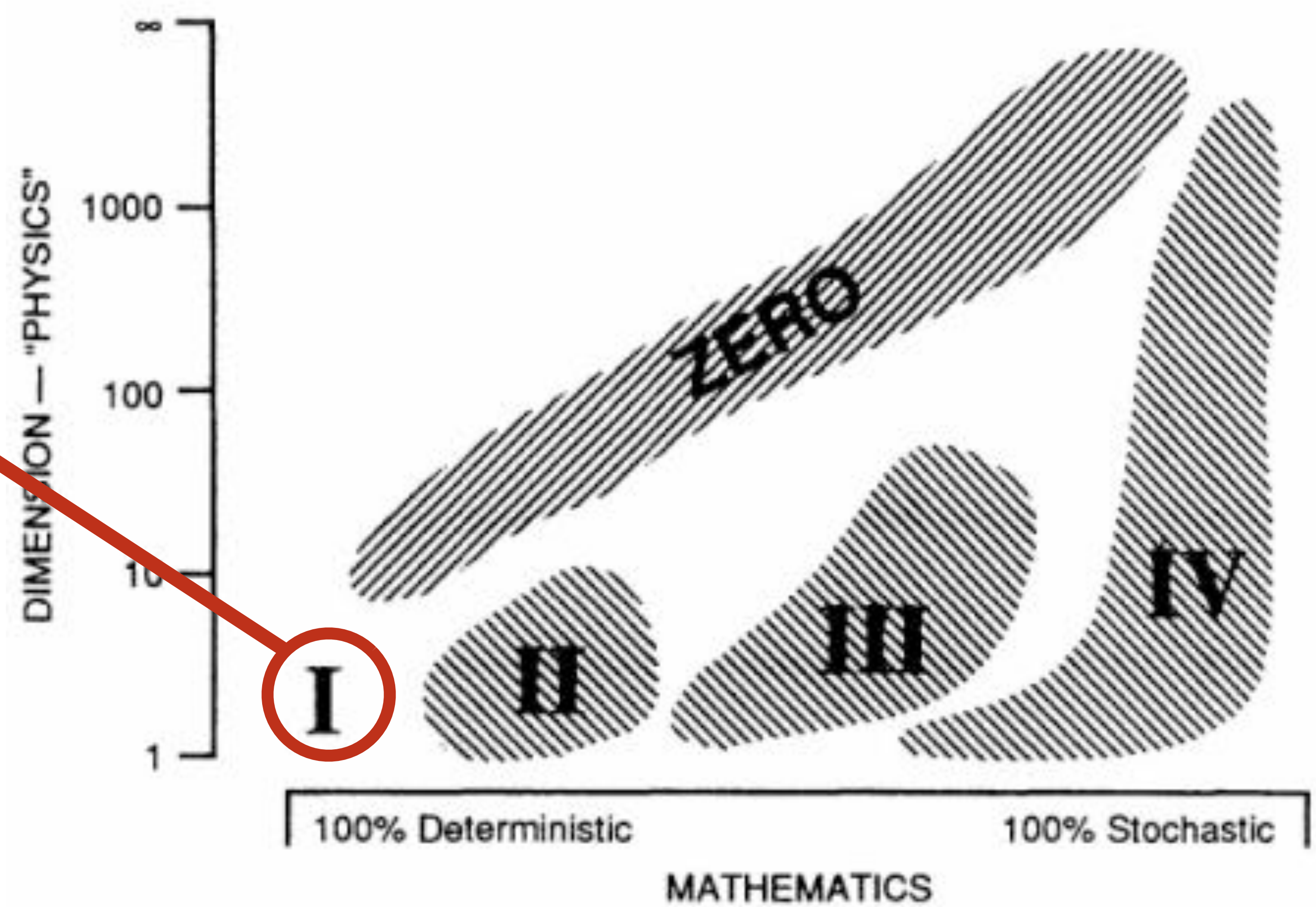


Figure 12-1. Schematic representation of the Hierarchy of Dynamic Systems.

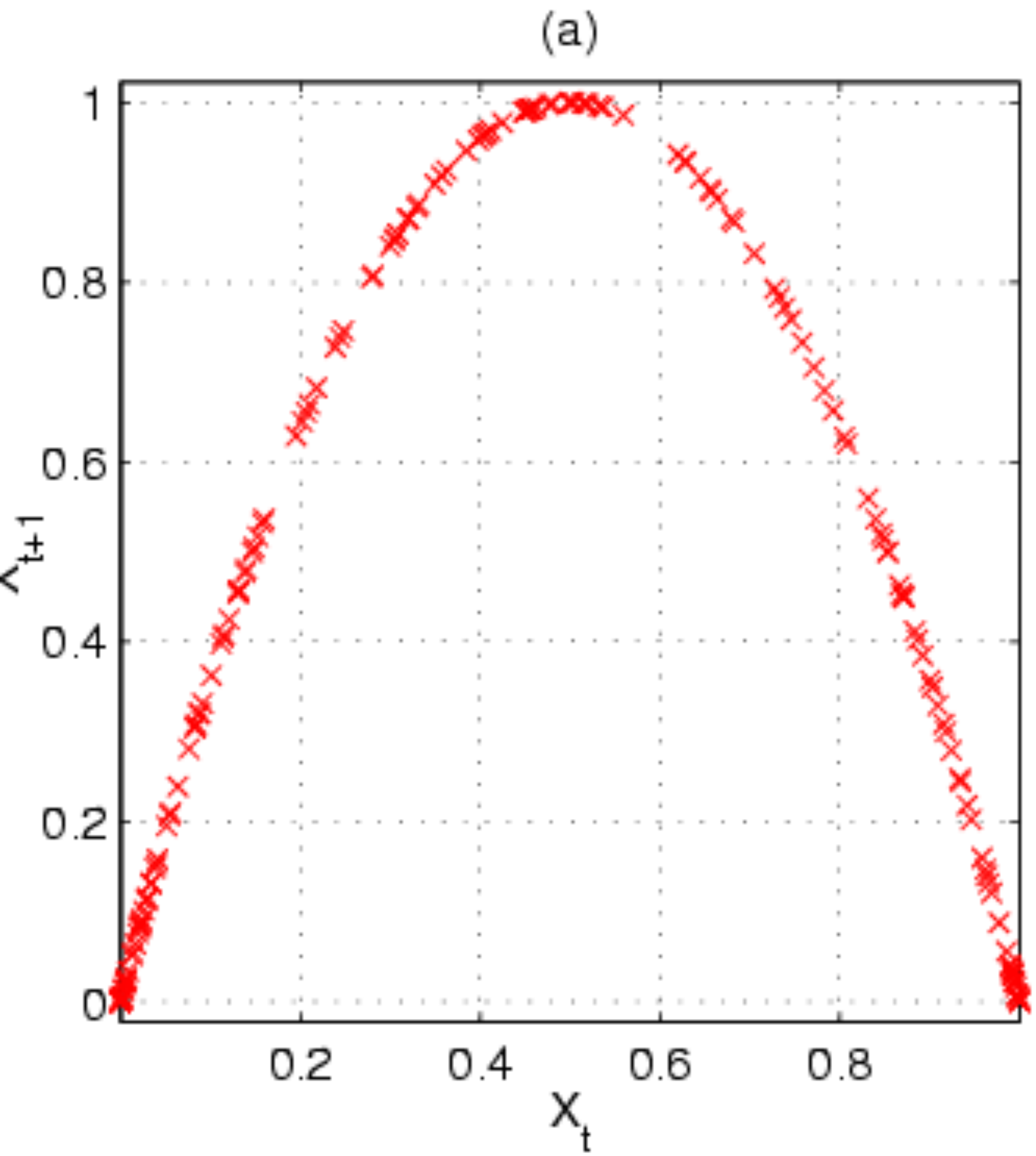
Story so far - Assignments session 1: Different ways to represent characteristics of change processes

- **Iterative processes -** (coupled) difference / differential equations that represent autocatalytic change processes, the time-evolution of a system observable
- **Timeseries -** a record of values generated by an iterative / change process
- **Solution -** if available, actual iterations of the function are not necessary. Only available for a very limited set of (coupled) equations.

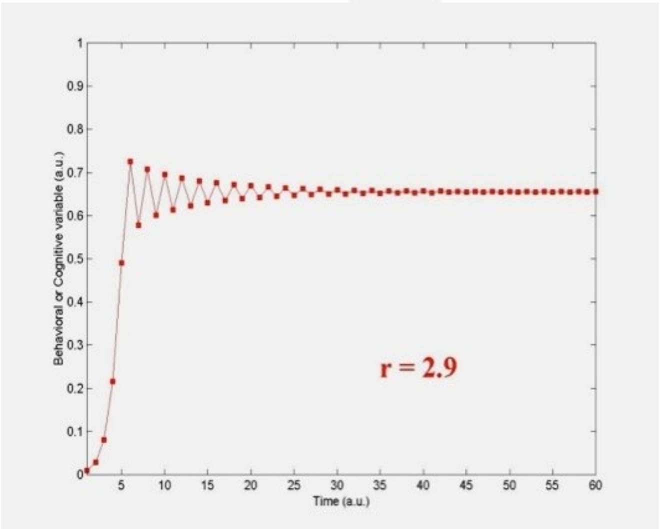
Story so far - Assignments session 1: Different ways to represent characteristics of change processes

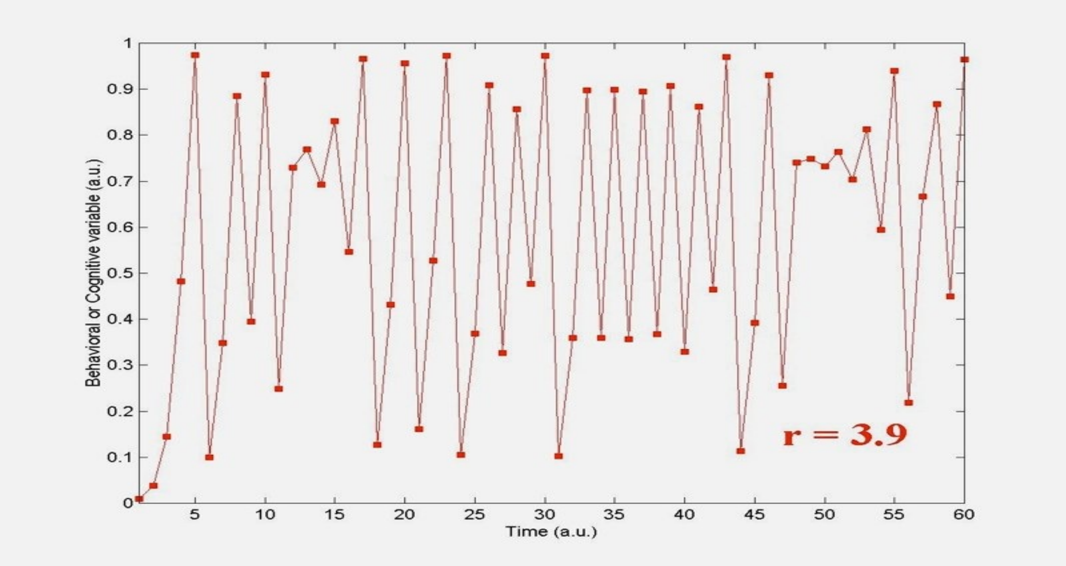
- **Iterative processes -** (coupled) difference / differential equations that represent autocatalytic change processes, the time-evolution of a system observable
- **Timeseries -** a record of values generated by an iterative / change process
- **Solution -** if available, actual iterations of the function are not necessary. Only available for a very limited set of (coupled) equations.
- **The return plot -** a scatterplot of Y_i vs. $Y_{i+1..n}$
- **The state / phase <u>space</u> -** A space spanned by **M** observable **dimensions** of the system.
  - Depending on parameter settings a system can be attracted to just a few states: *Attractors*
  - *Not discussed: The cobweb method*
- **The phase / bifurcation <u>diagram</u> -** diagram representing the parameter space of a system. Its dimensions represent the possible values of the control parameter(s) of the system. Stable regions are often labelled by an order parameter (solid, liquid, gas).
- Today: **Potential Functions -** A functions describing the relative stability of the ‘end-states’ of

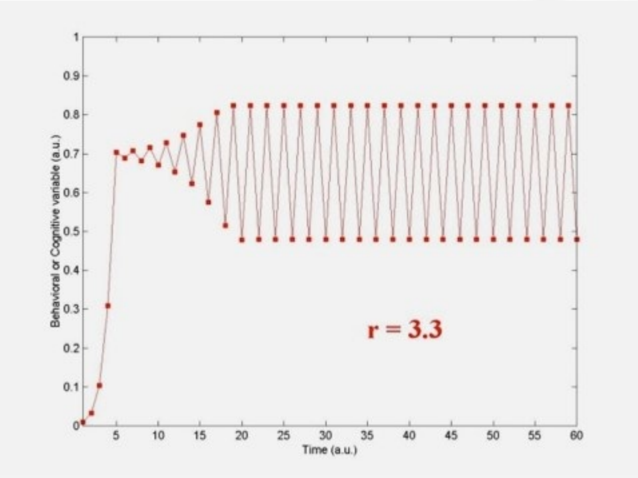
Story so far - Assignments session 1: Return plot of the logistic map



Why the same shape for all these different time series?

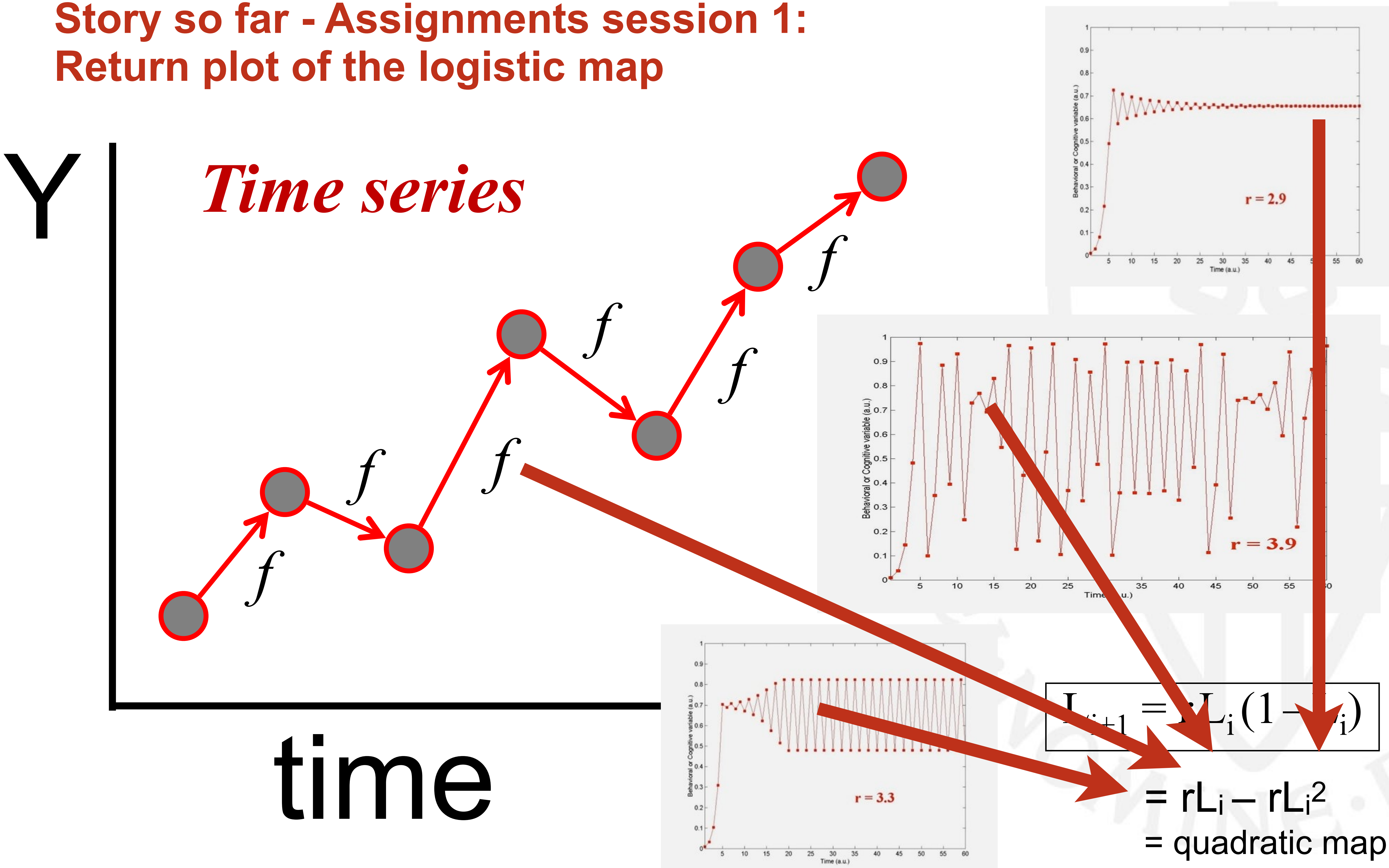






$$\boxed{L_{i+1} = r\,L_i\,(1 - L_i)}$$

Story so far - Assignments session 1:
Return plot of the logistic map



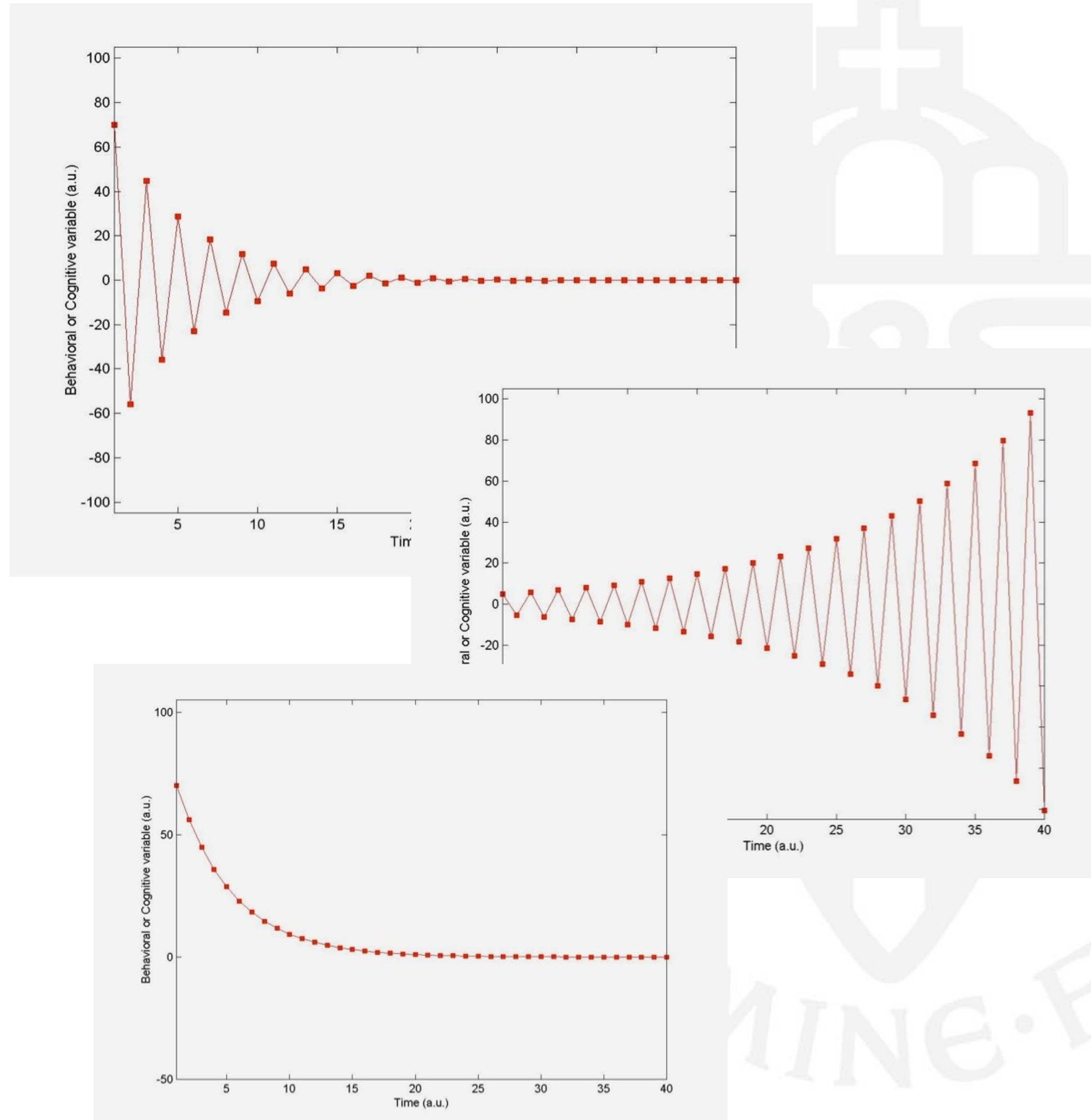
$$L_{i+1} = rL_i(1 - L_i)$$

$= rL_i – rL_i^2$

= quadratic map

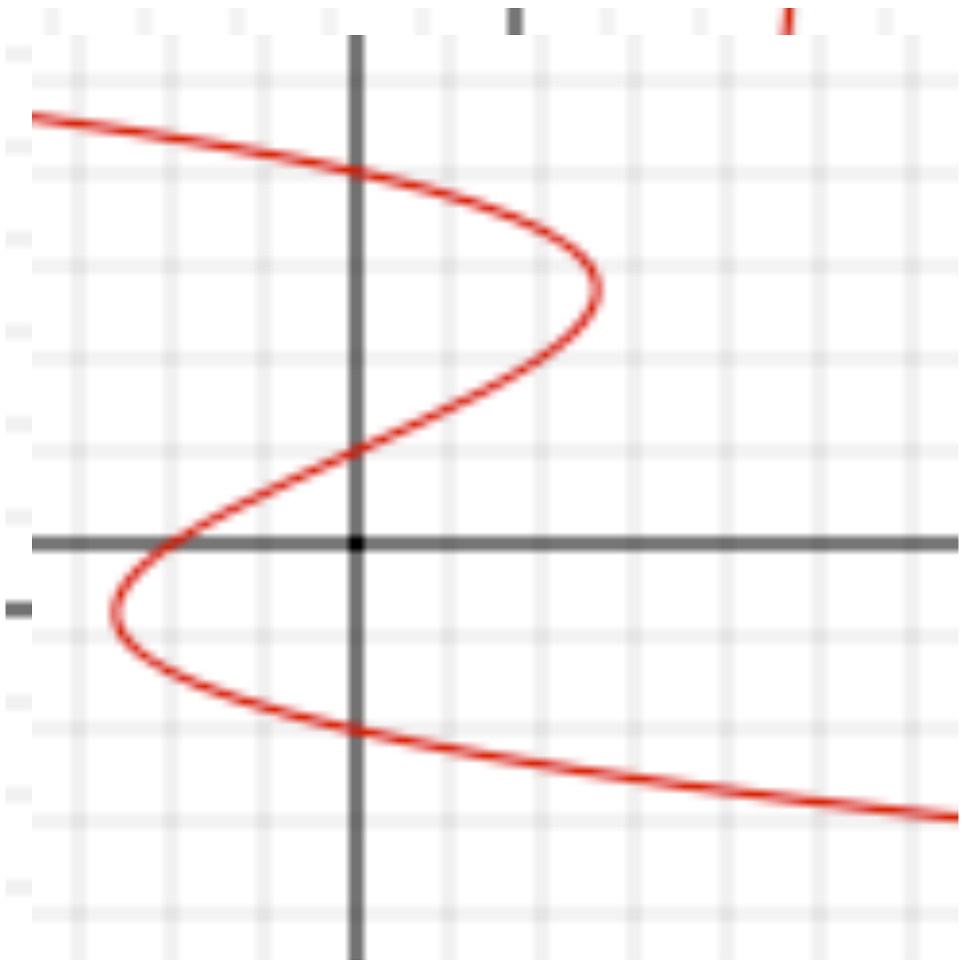
Return plot quiz

$$\boxed{Y_{i+1} = a \cdot Y_i}$$

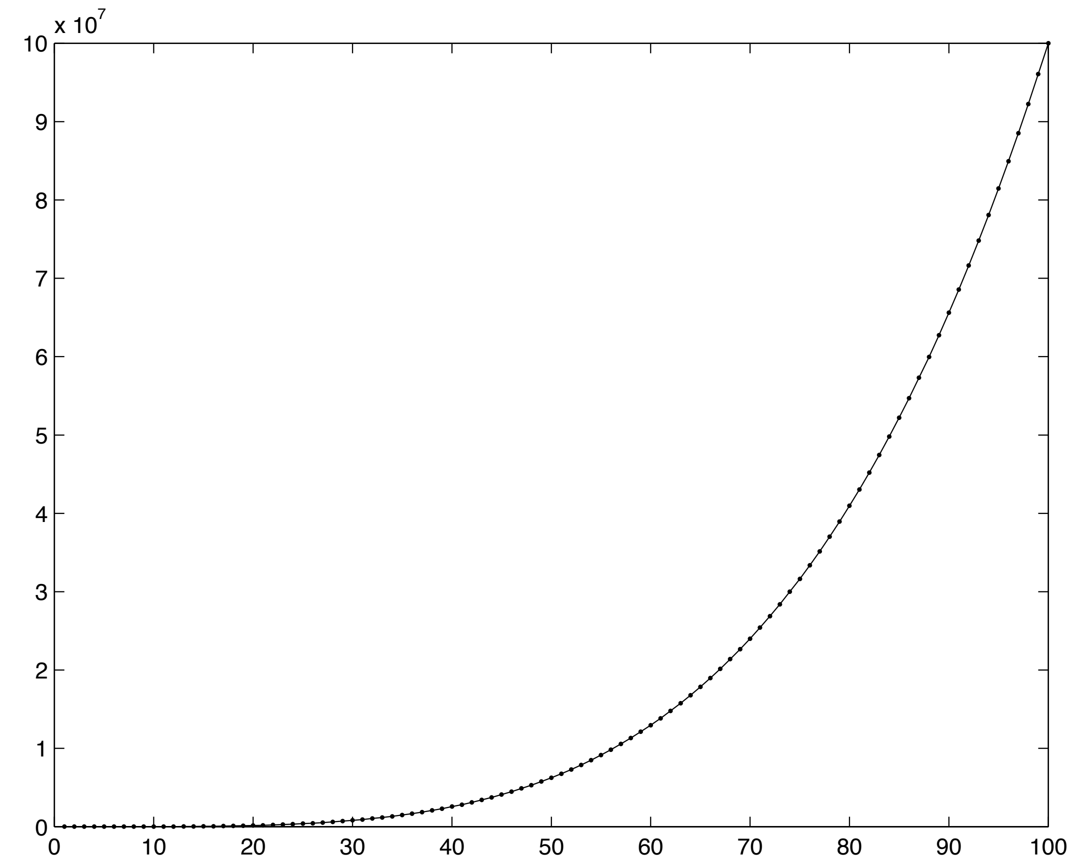


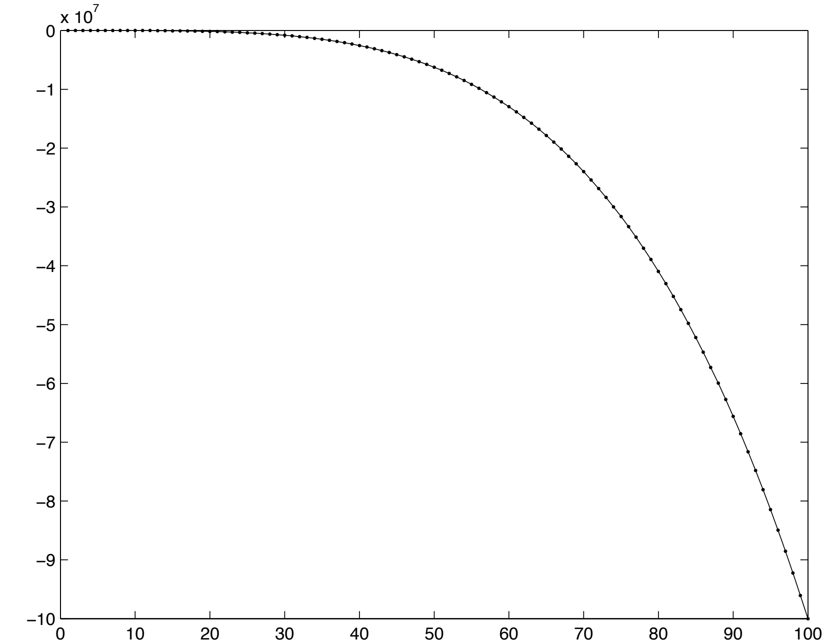
Return plot quiz

$$Y_{i+1} = a \cdot Y_i^3 + b \cdot Y_i^2 + \ldots$$



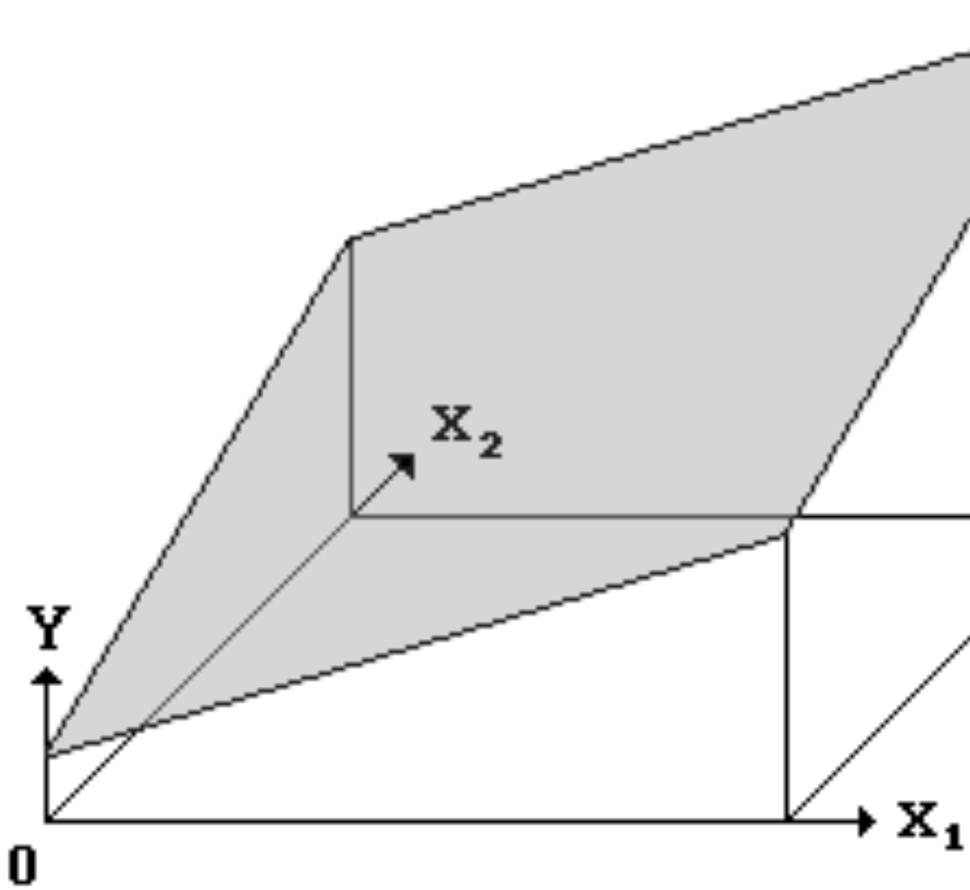
Remember this shape!



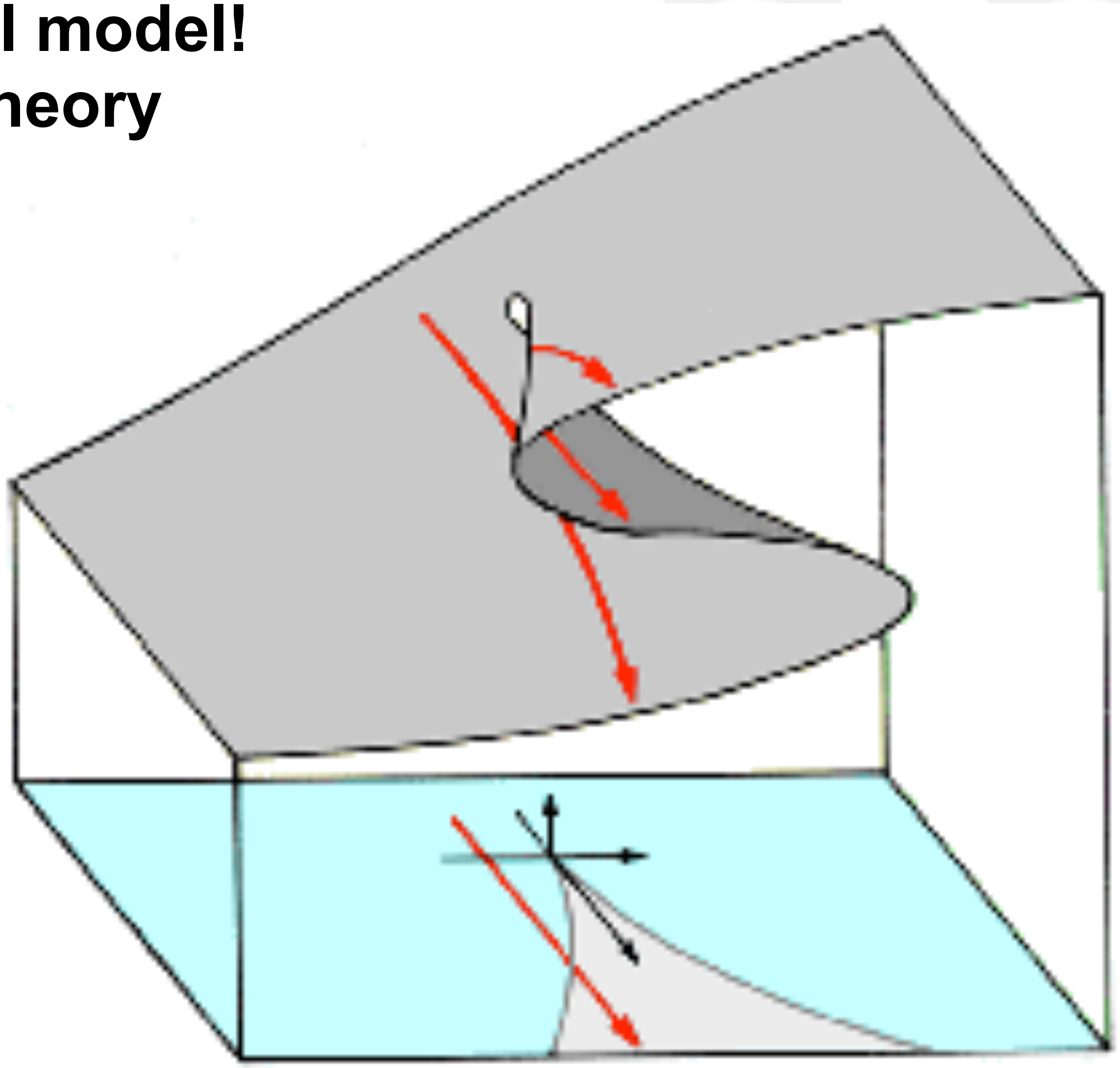


Linear vs. Dynamic models… fitting a response surface

same tools!
same general model!
different theory



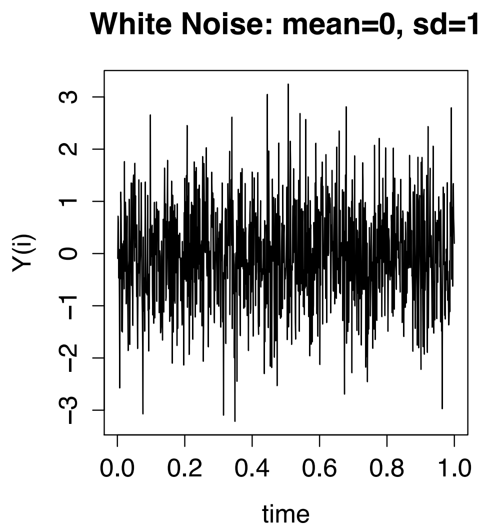
$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2$$



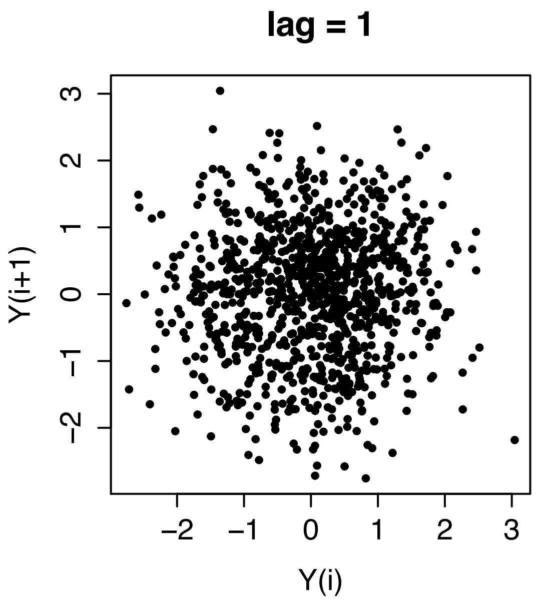
$$Y = \beta_0 + \beta_1 X_{control} + \beta_2 X_{bifur} * Y + \beta_3 Y^2 + \beta_4 Y^3$$

Y is entered as a predictor

Return plot quiz



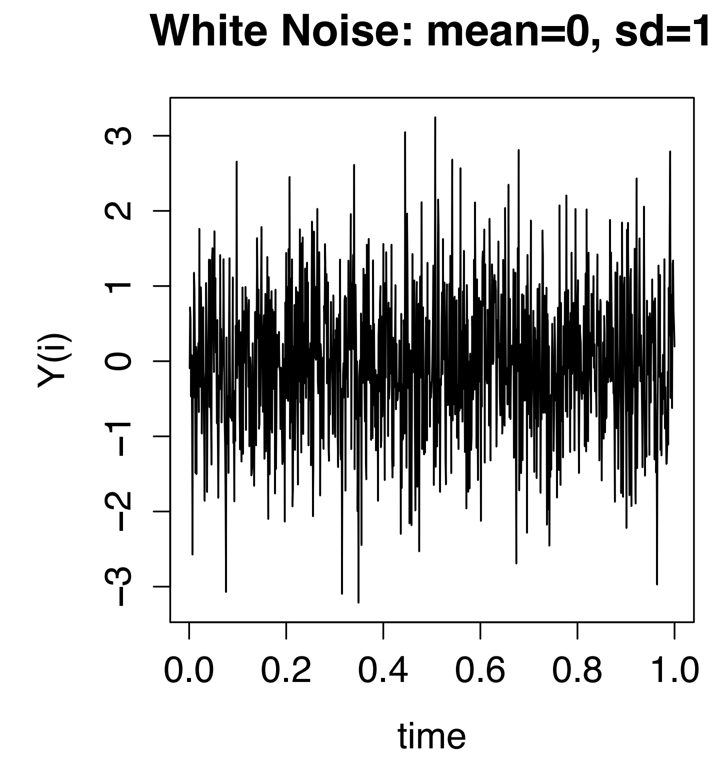
White noise
Completely random
(Gaussian distribution)

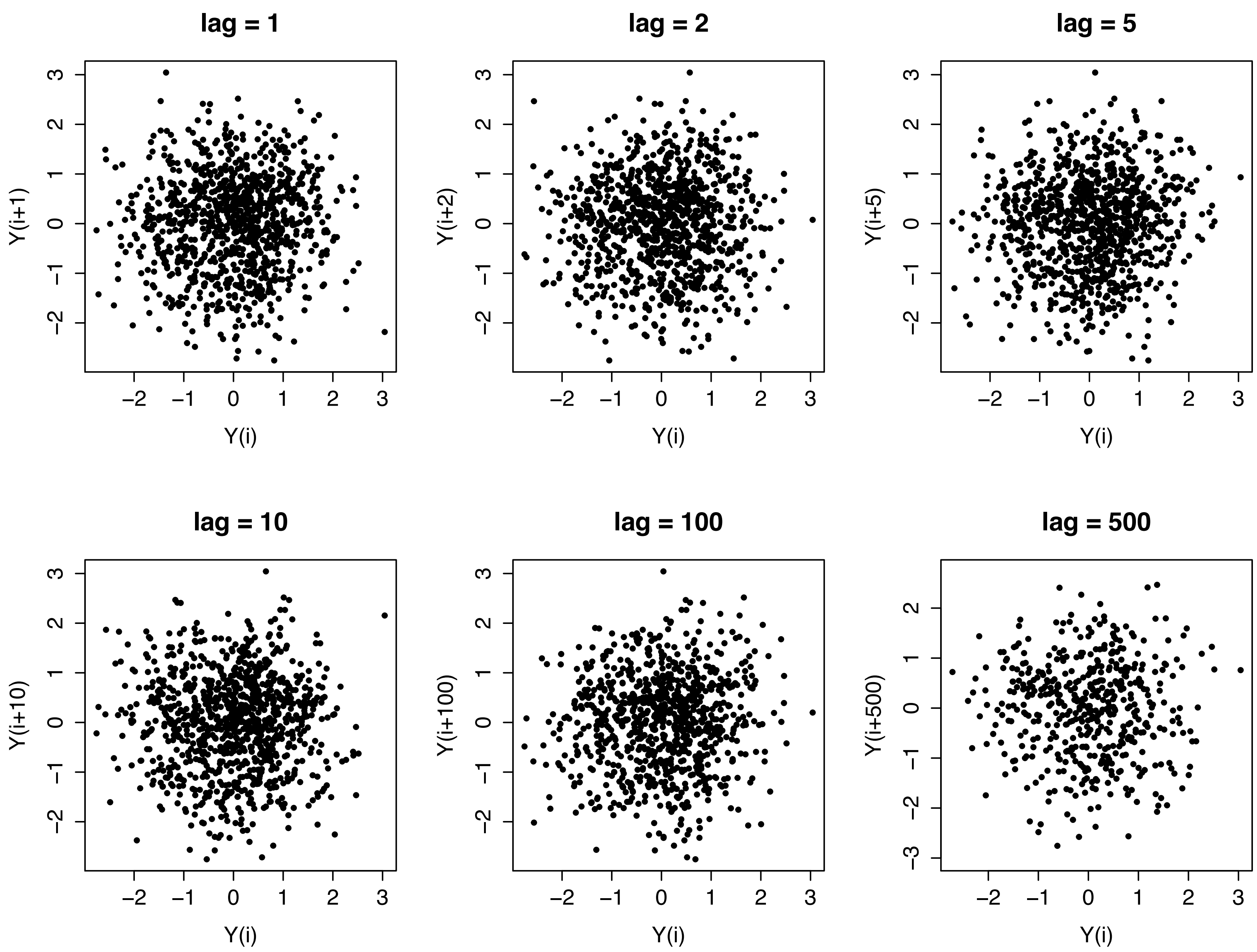


What happens at different lags?

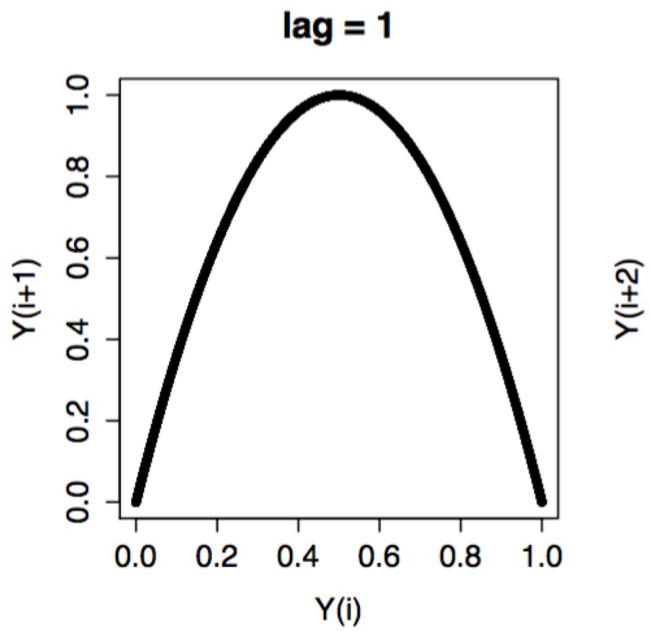
Return plot quiz

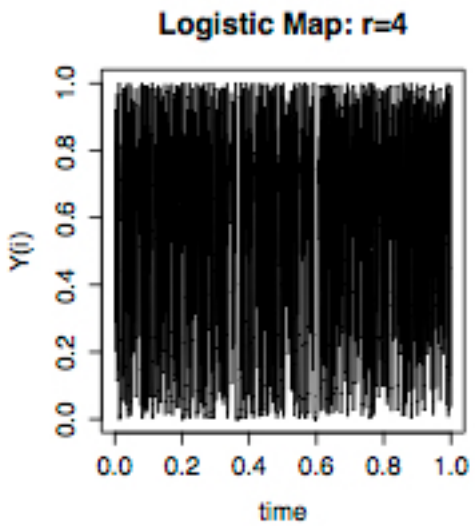
White noise
Completely random
(Gaussian distribution)





Return plot quiz



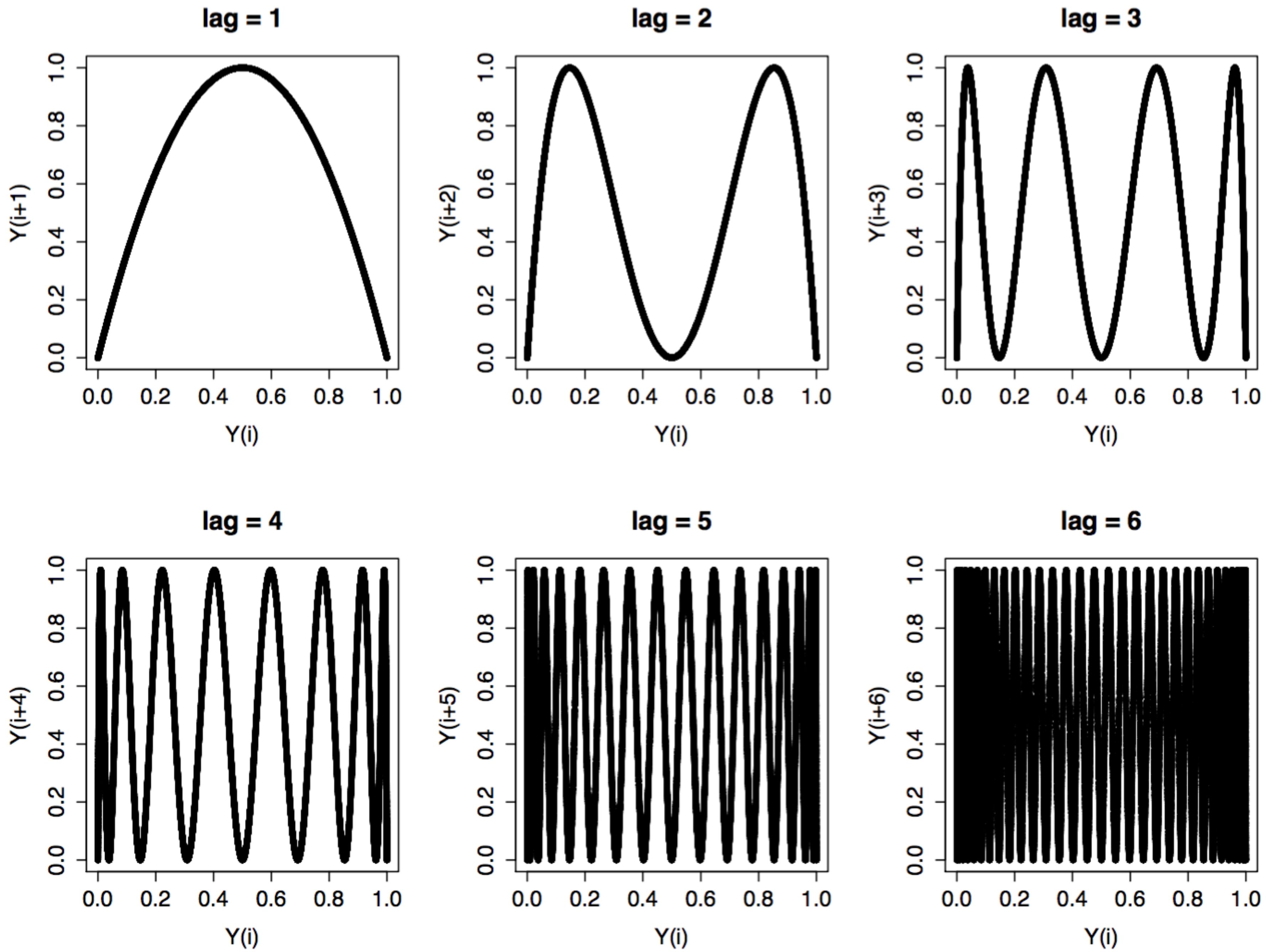


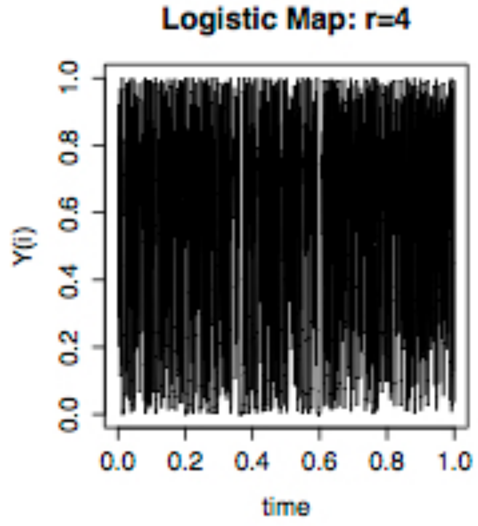
Iterative Process
Completely deterministic
(deterministic chaos)

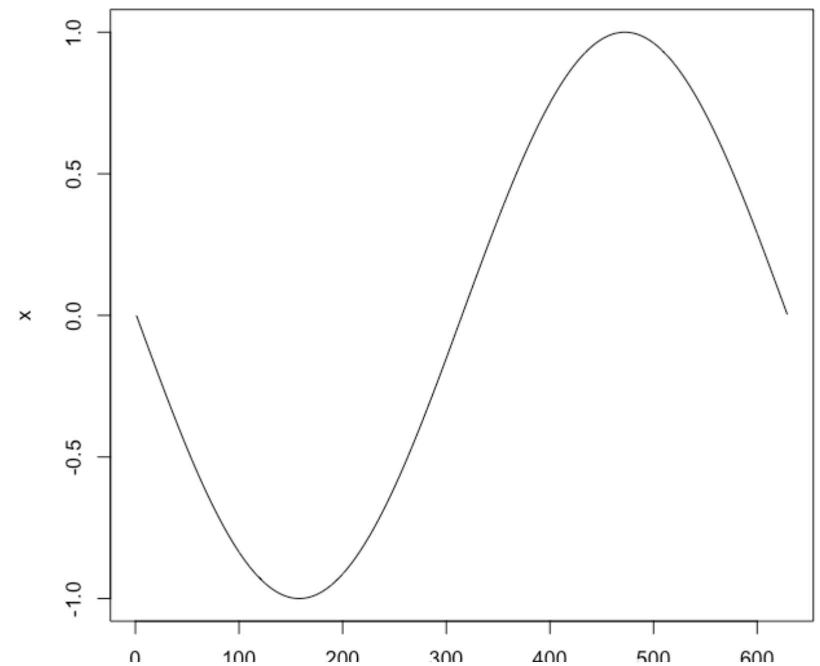
What happens at different lags?

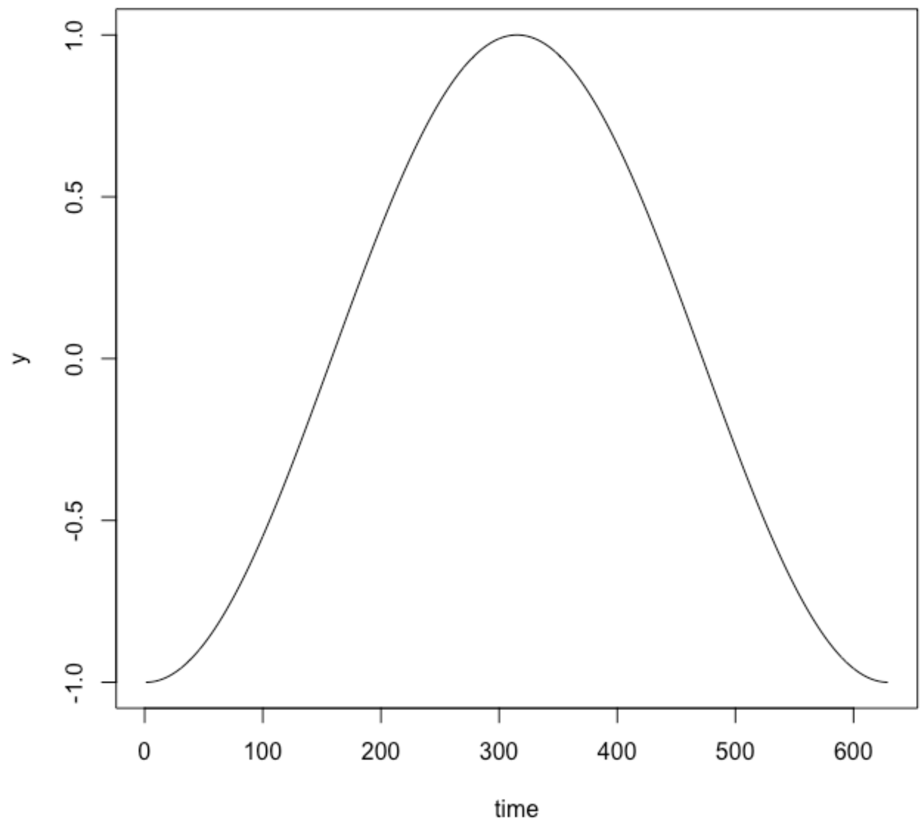
Return plot quiz

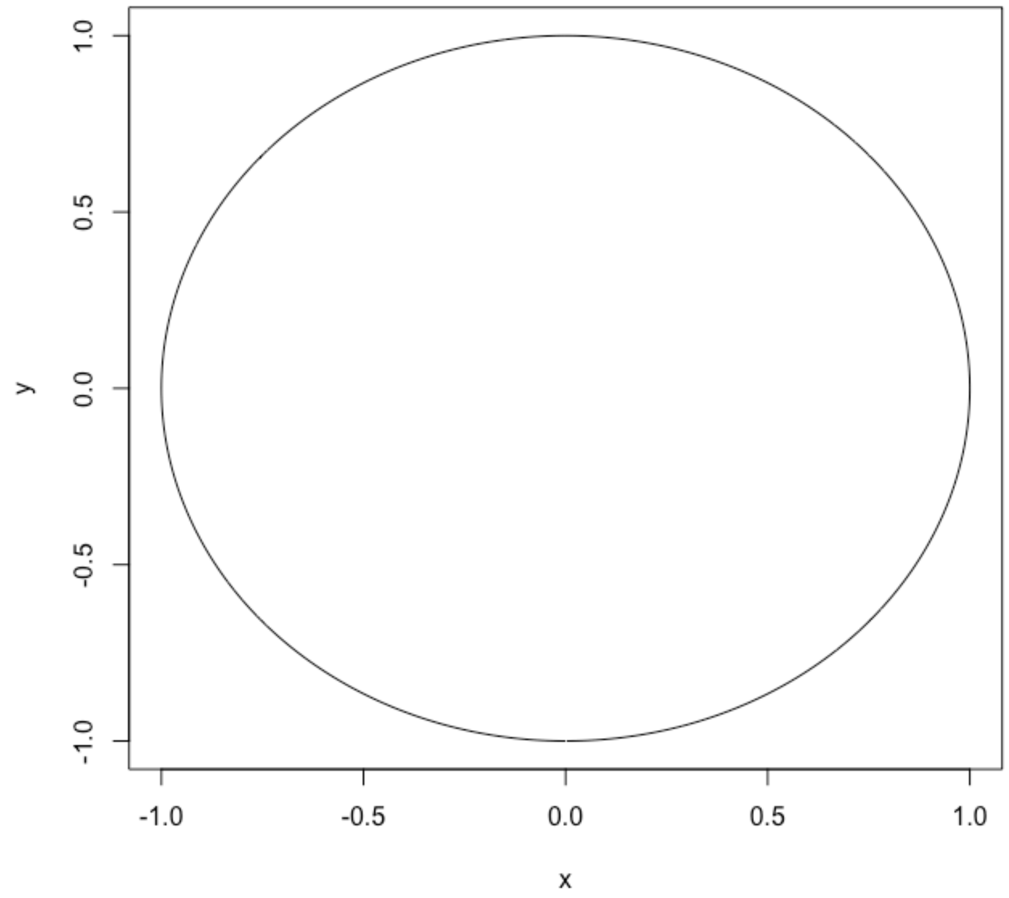
Iterative Process
Completely deterministic
(deterministic chaos)






x
1.0
0.5
0.0
-0.5
-1.0
0
100
200
300
400
500
600


y
1.0
0.5
0.0
-0.5
-1.0
0
100
200
300
400
500
600
time


y
1.0
0.5
0.0
-0.5
-1.0
-1.0
-0.5
0.0
0.5
1.0
x

Cobweb method

Cobweb Method

An insightful and easy way to visualize the dynamics of a one-dimensional map.

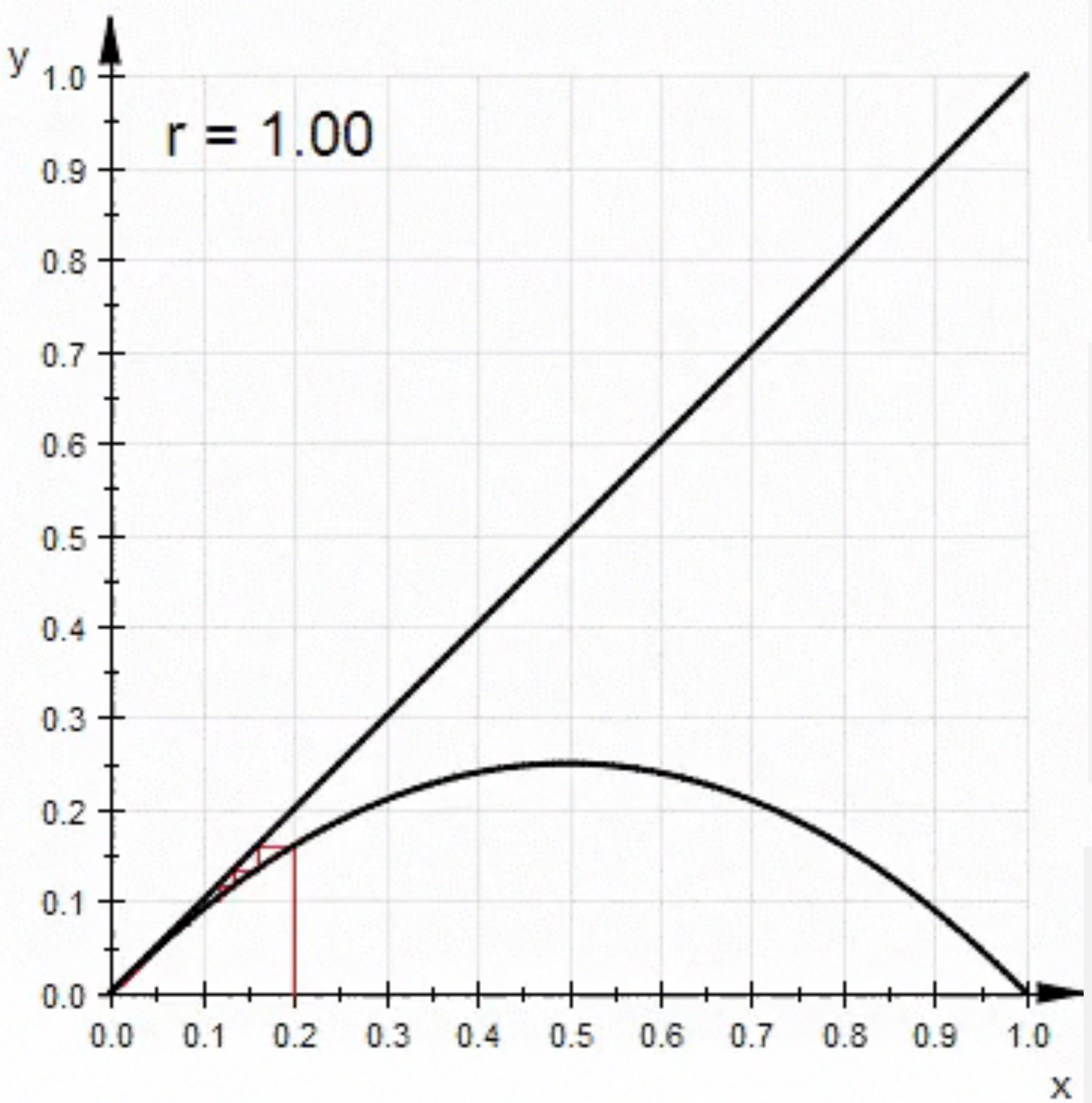
Useful when an algebraic solution can no longer be found.

Preparations (General):

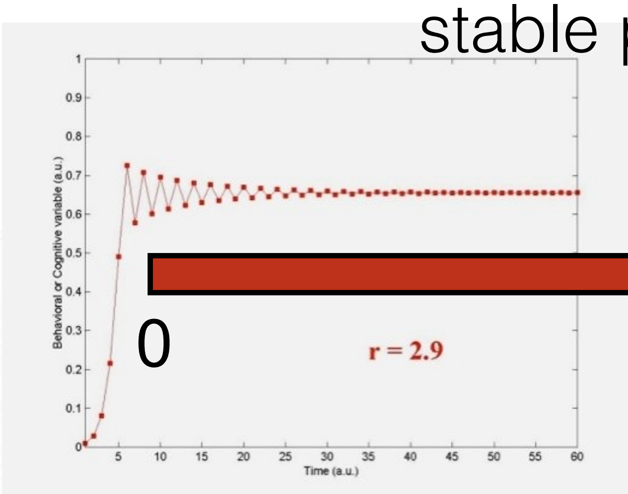
→ Draw the graph of $f(Y_i)$ (= Y_{i+1}) in the Y_{i+1} - Y_i (phase) plane;

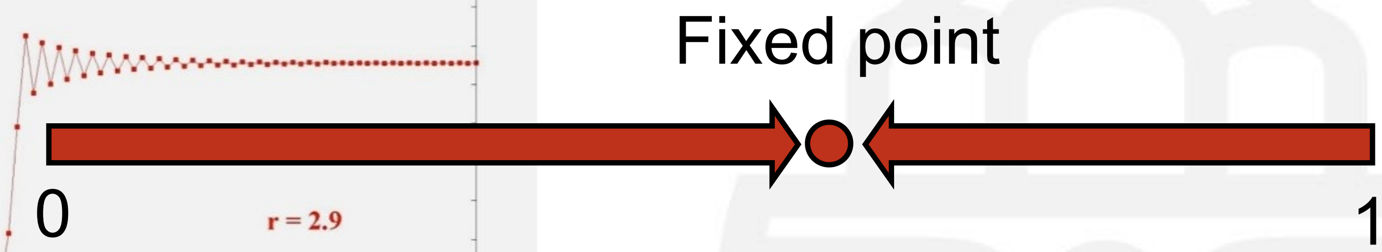
→ Also draw the line $Y_i = Y_{i+1}$ (45^o).

Cobweb method

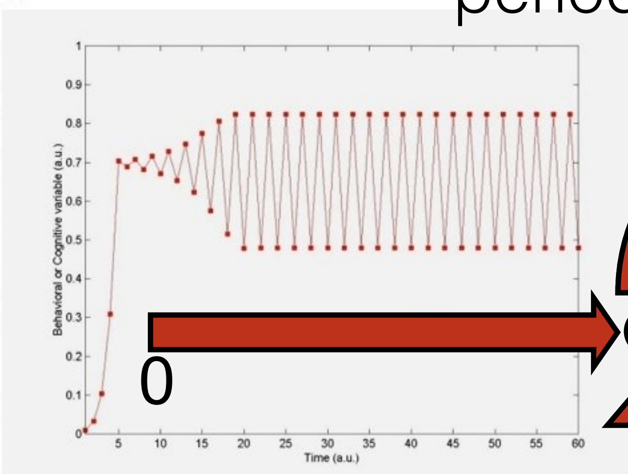


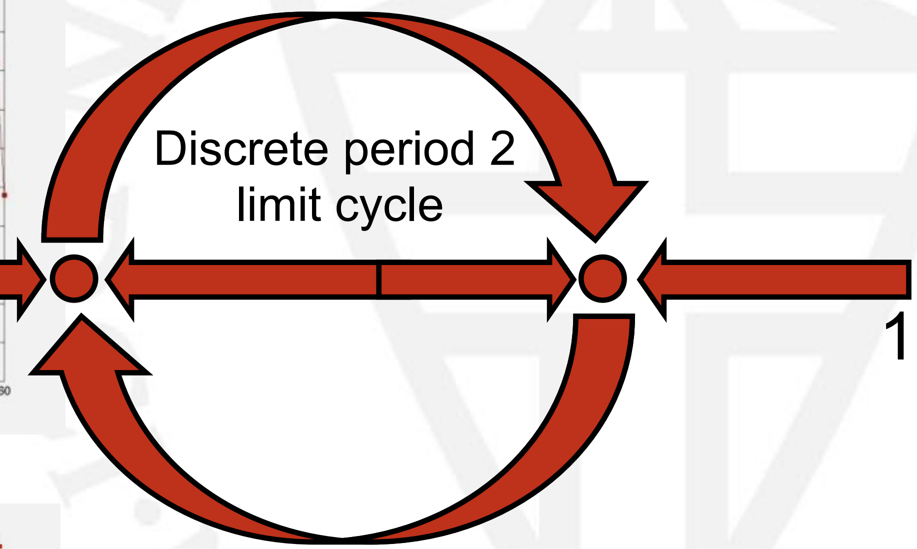
stable points (point attractors)

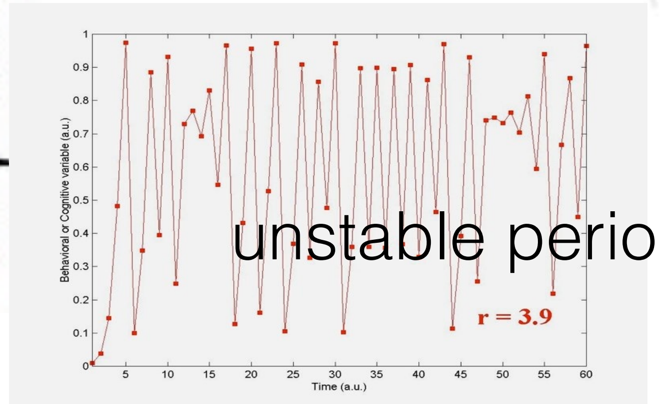




periodic orbits (limit cycles)







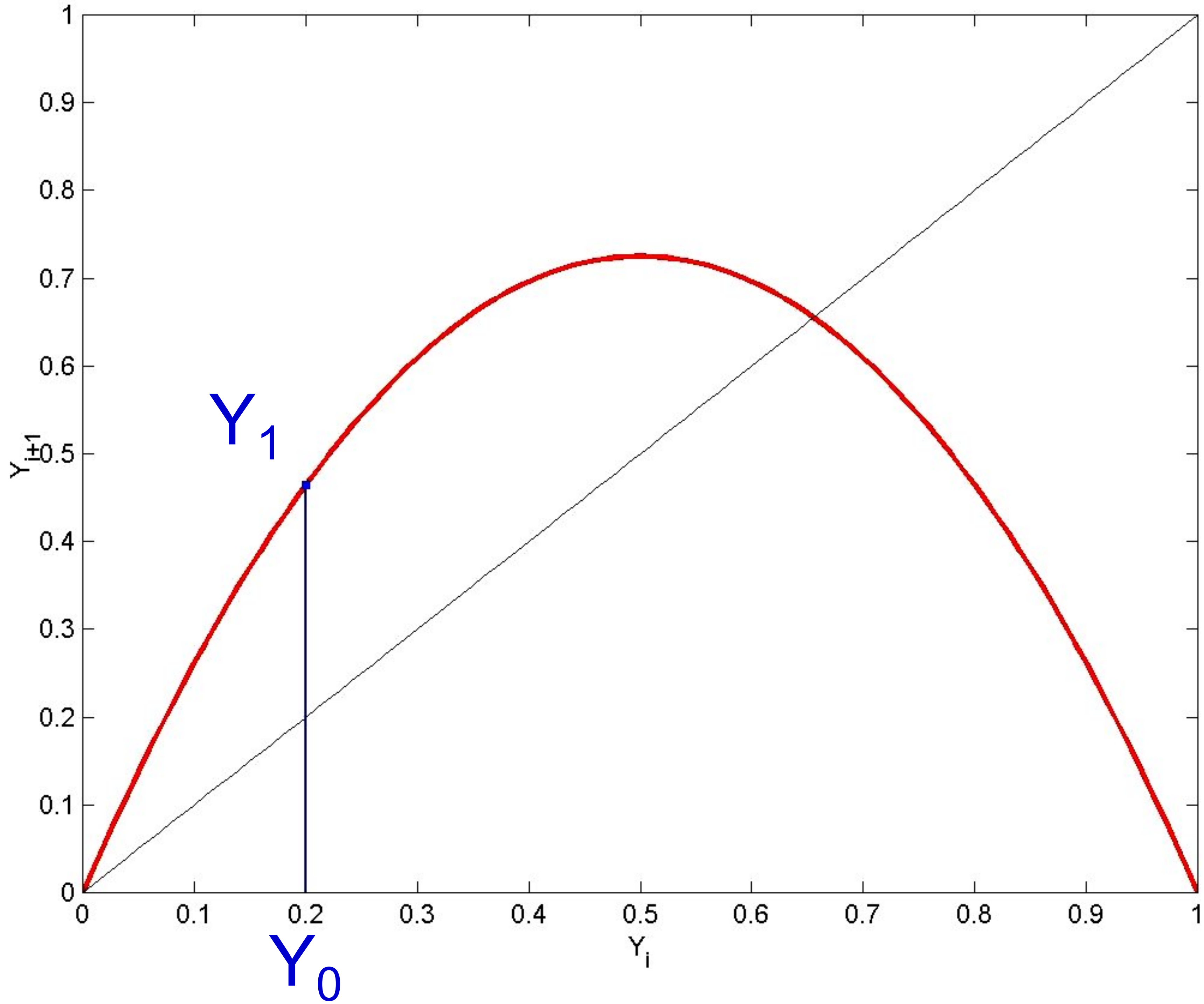
unstable periodic orbits (strange attractors)

Cobweb Method

Start with Y_0 on horizontal axis:

Go vertical to f: Gives you Y_1;

r=2.9



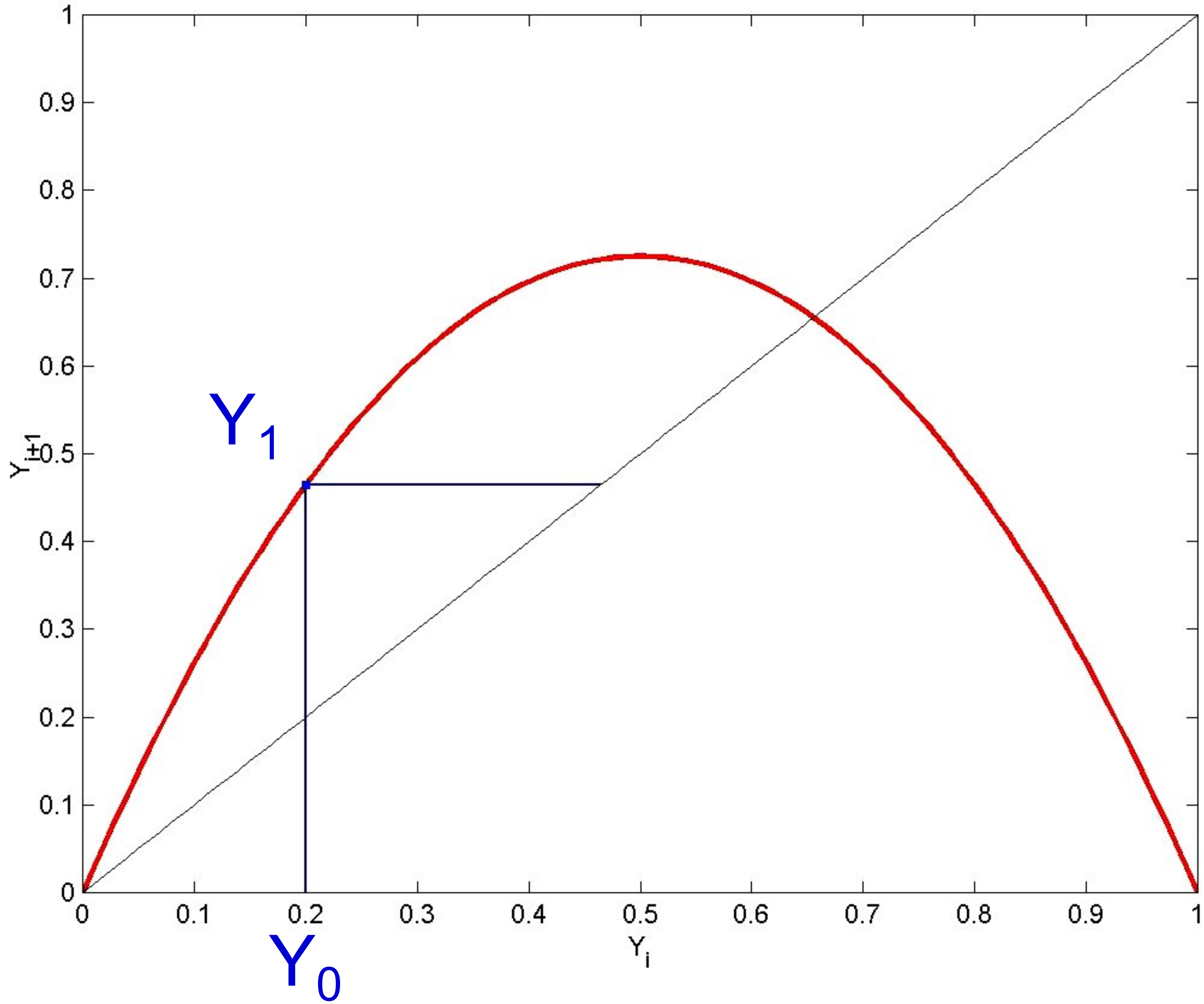
Cobweb Method

Start with Y_0 on horizontal axis:

Go vertical to *f*: Gives you Y_1;

→ Go horizontal to the 45^o line;

r=2.9



Cobweb Method

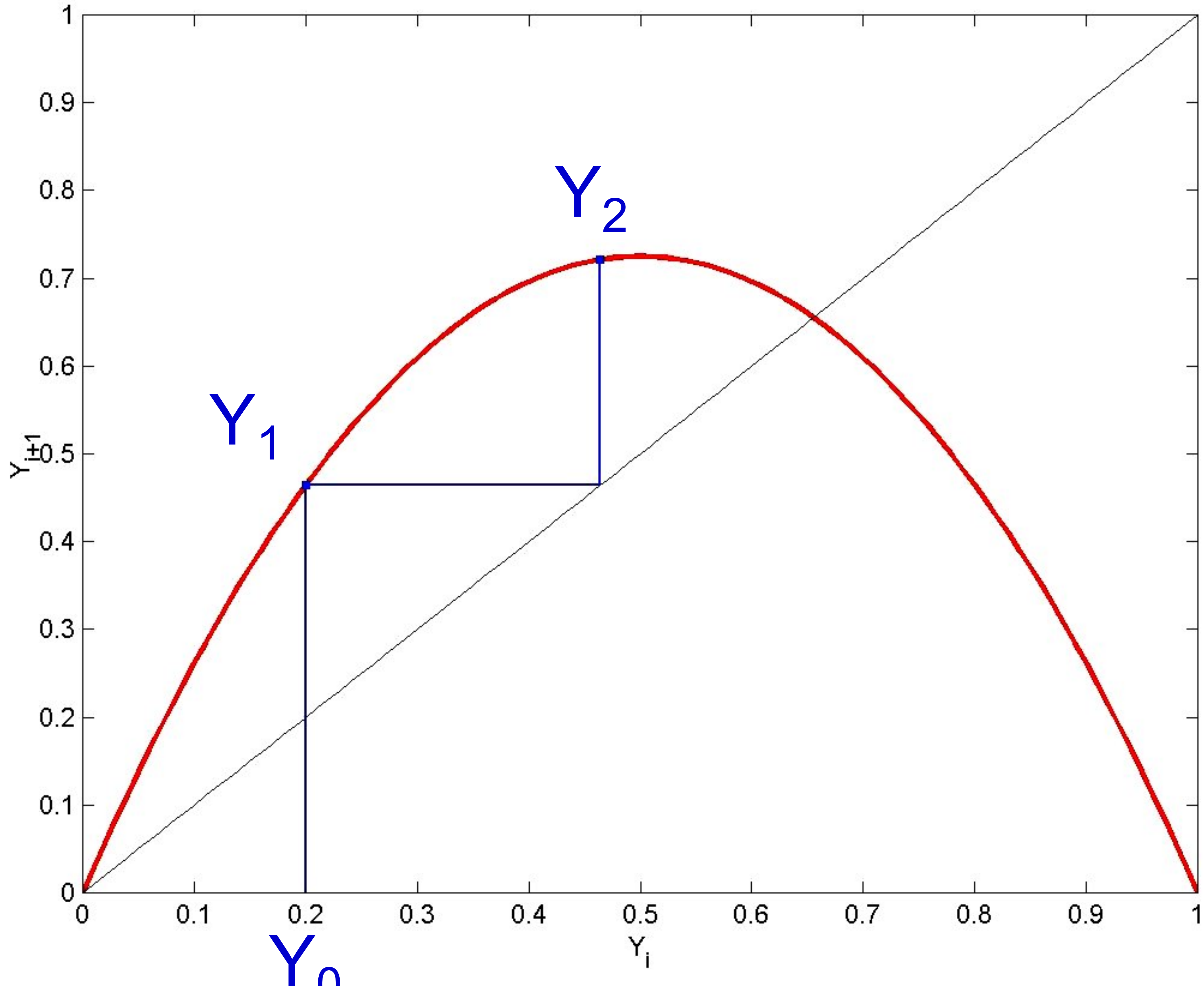
Start with Y_0 on horizontal axis:

Go vertical to *f*: Gives you Y_1;

→ Go horizontal to the 45° line;

→ Go vertical to *f*: Gives you Y_2;

r=2.9



Cobweb Method

Start with Y_0 on horizontal axis:

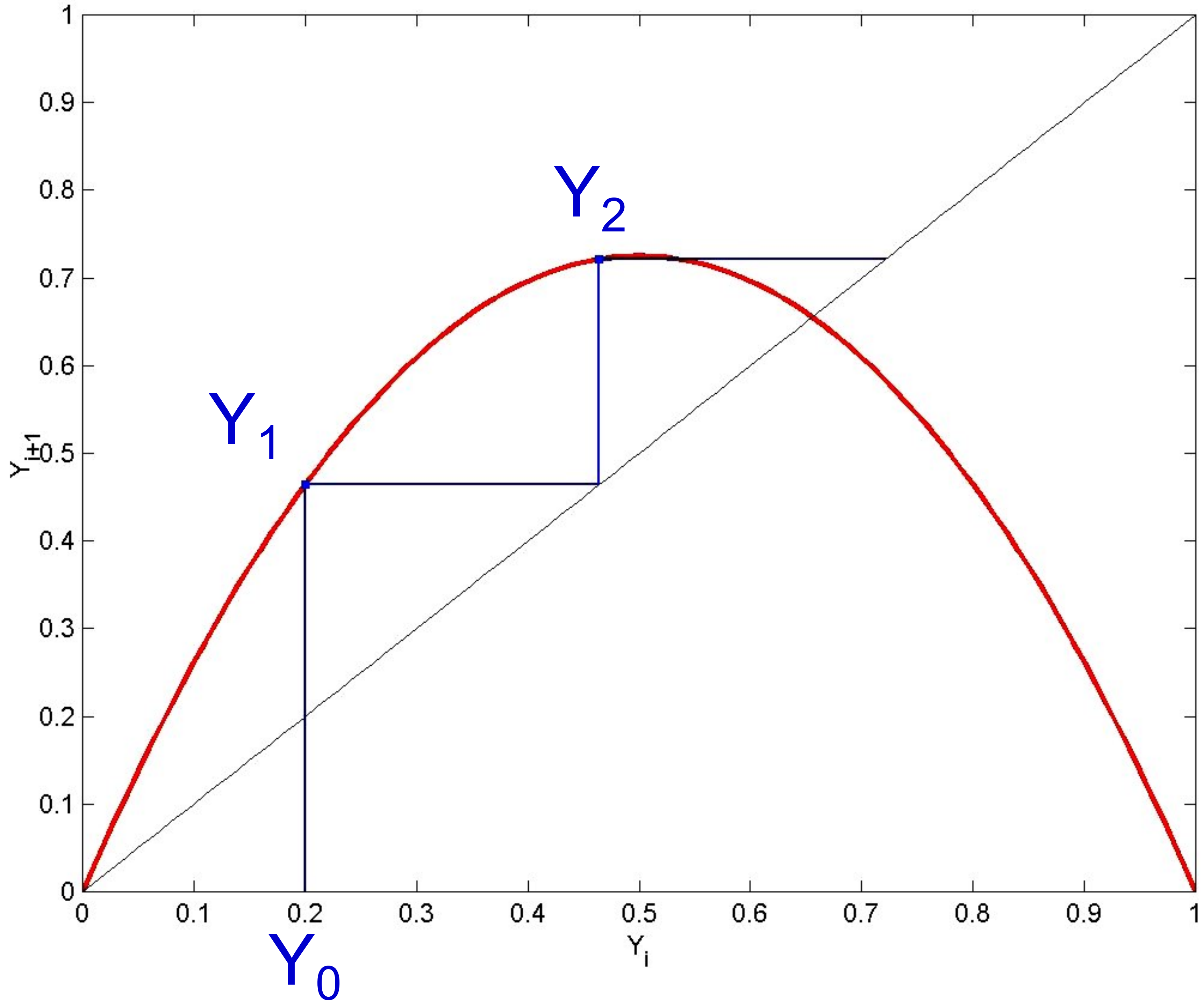
Go vertical to f: Gives you Y_1;

→ Go horizontal to the 45° line;

→ Go vertical to f: Gives you Y_2;

→ Go horizontal to the 45° line;

r=2.9



Cobweb Method

Start with Y_0 on horizontal axis:

Go vertical to *f*: Gives you Y_1;

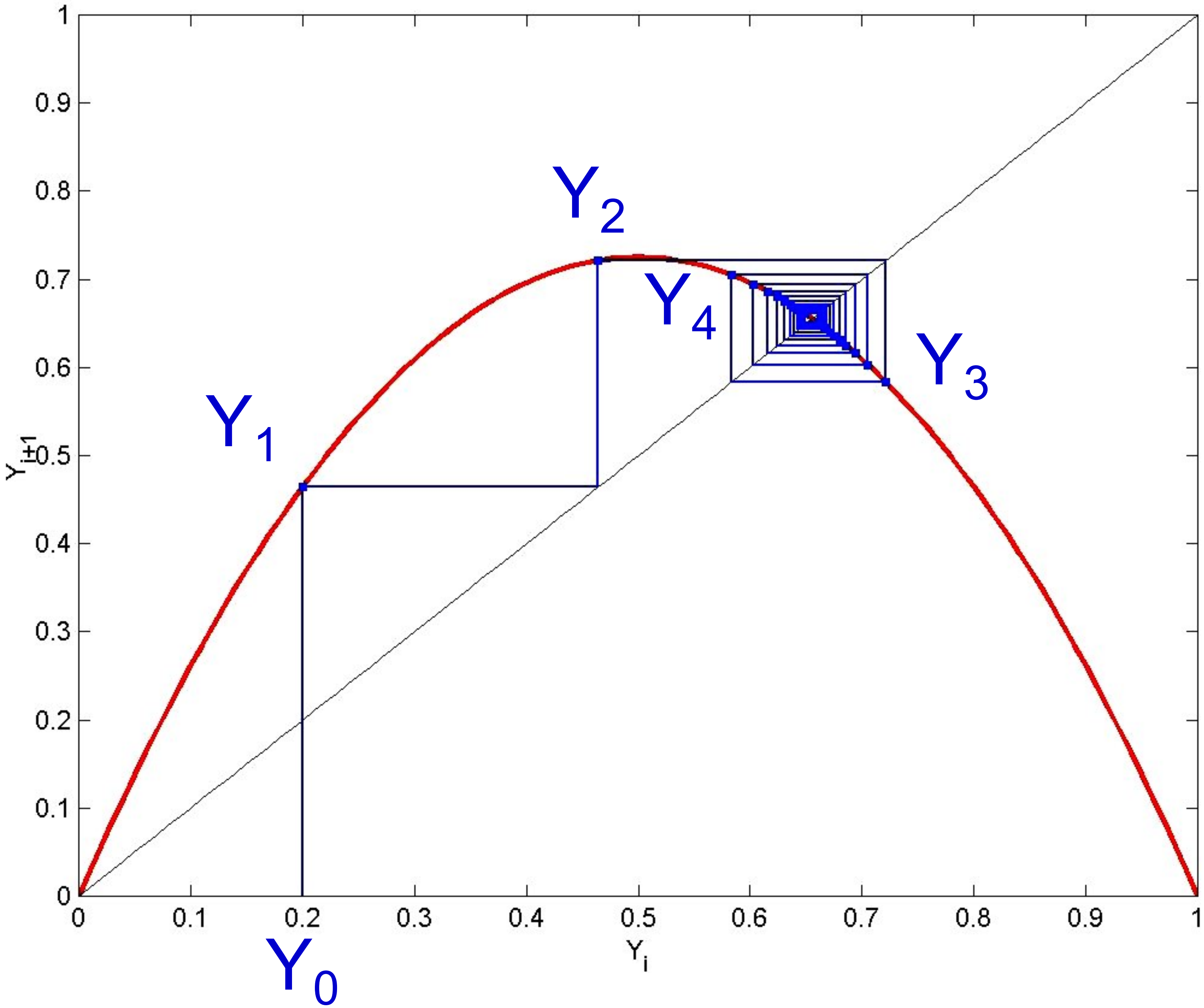
→ Go horizontal to the 45° line;

→ Go vertical to *f*: Gives you Y_2;

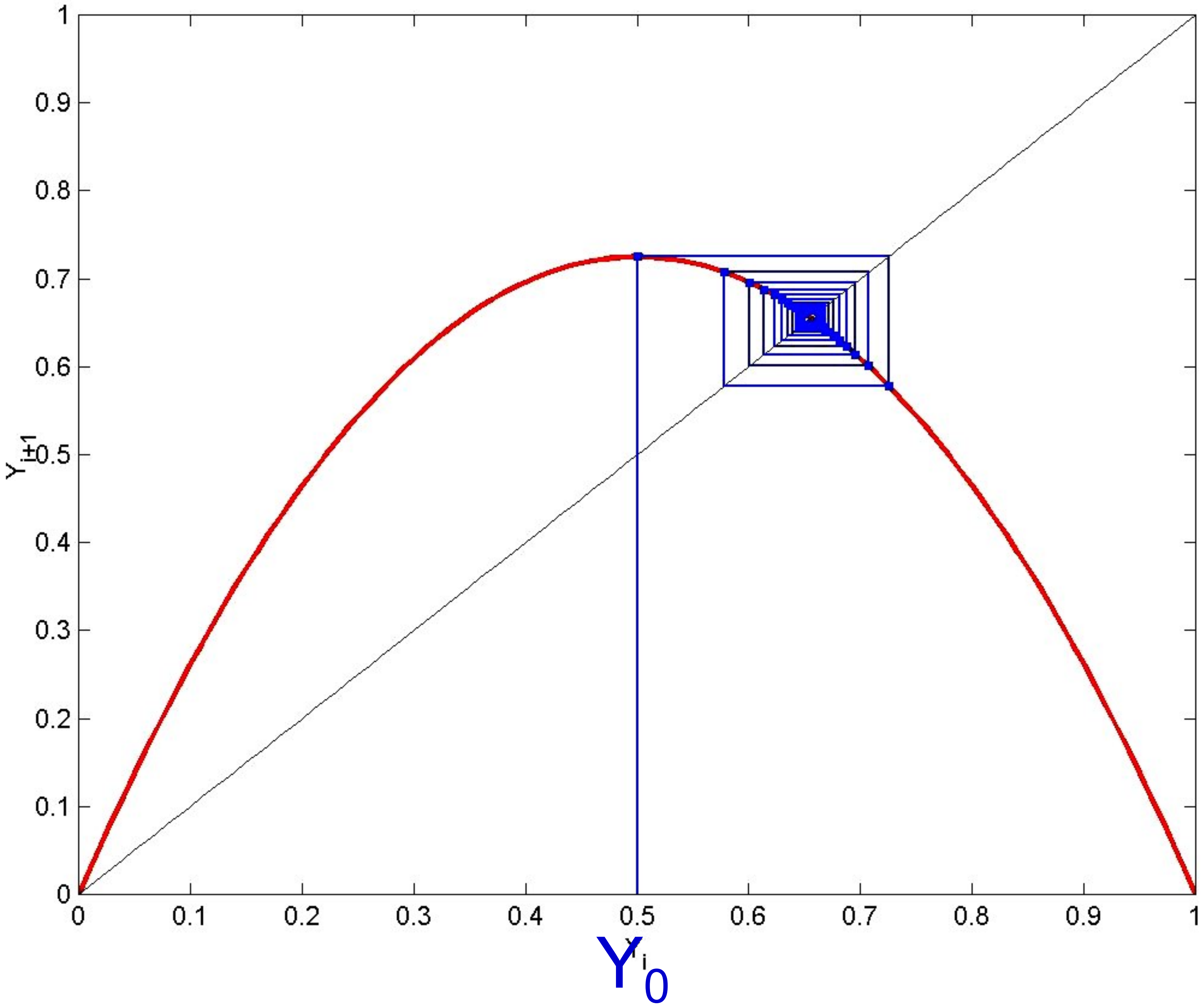
→ Go horizontal to the 45° line;

→ Etc. etc.

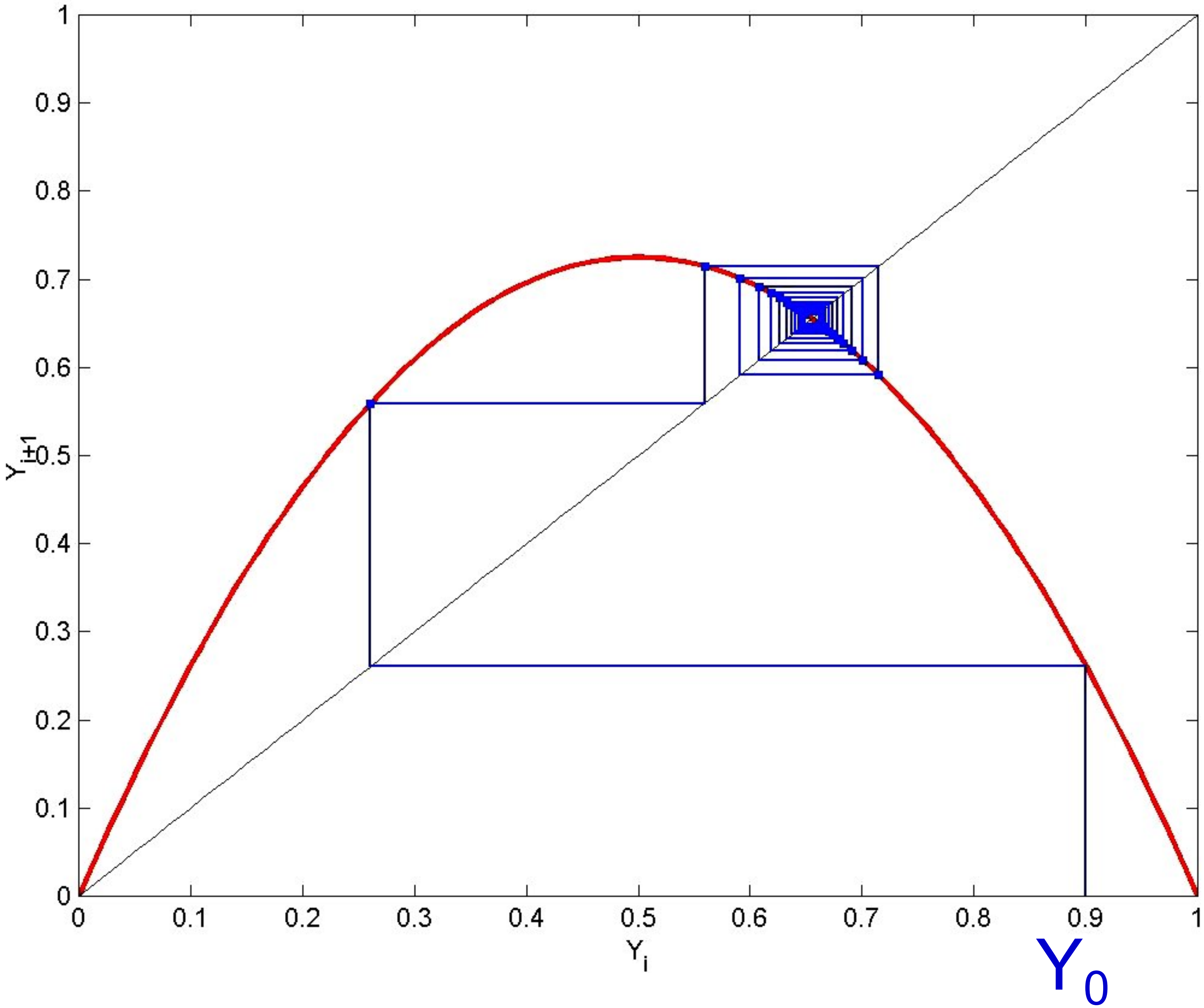
r=2.9


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Y_{i+1}
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Y_i
Y_0
Y_1
Y_2
Y_3
Y_4

r=2.9

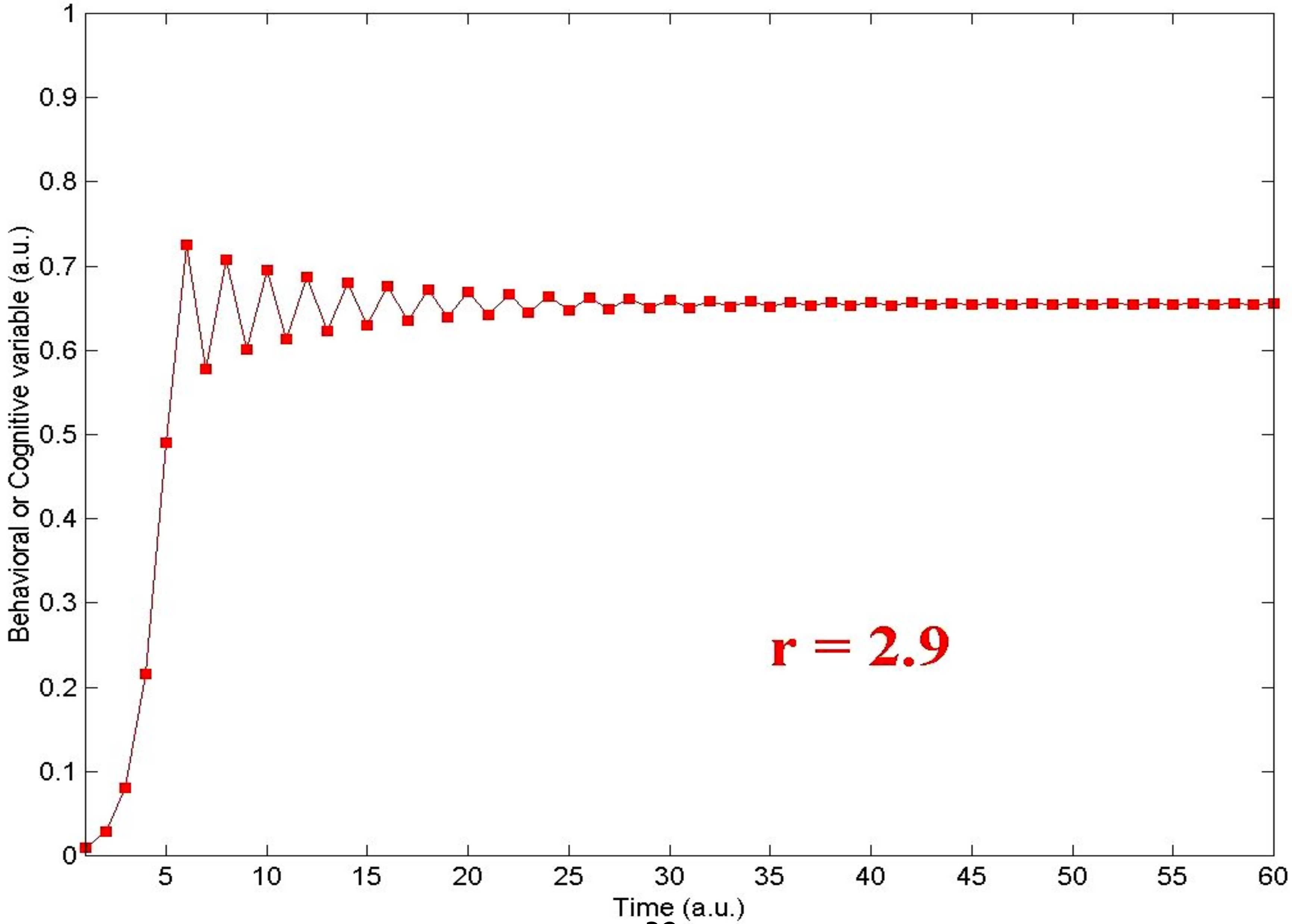


r=2.9



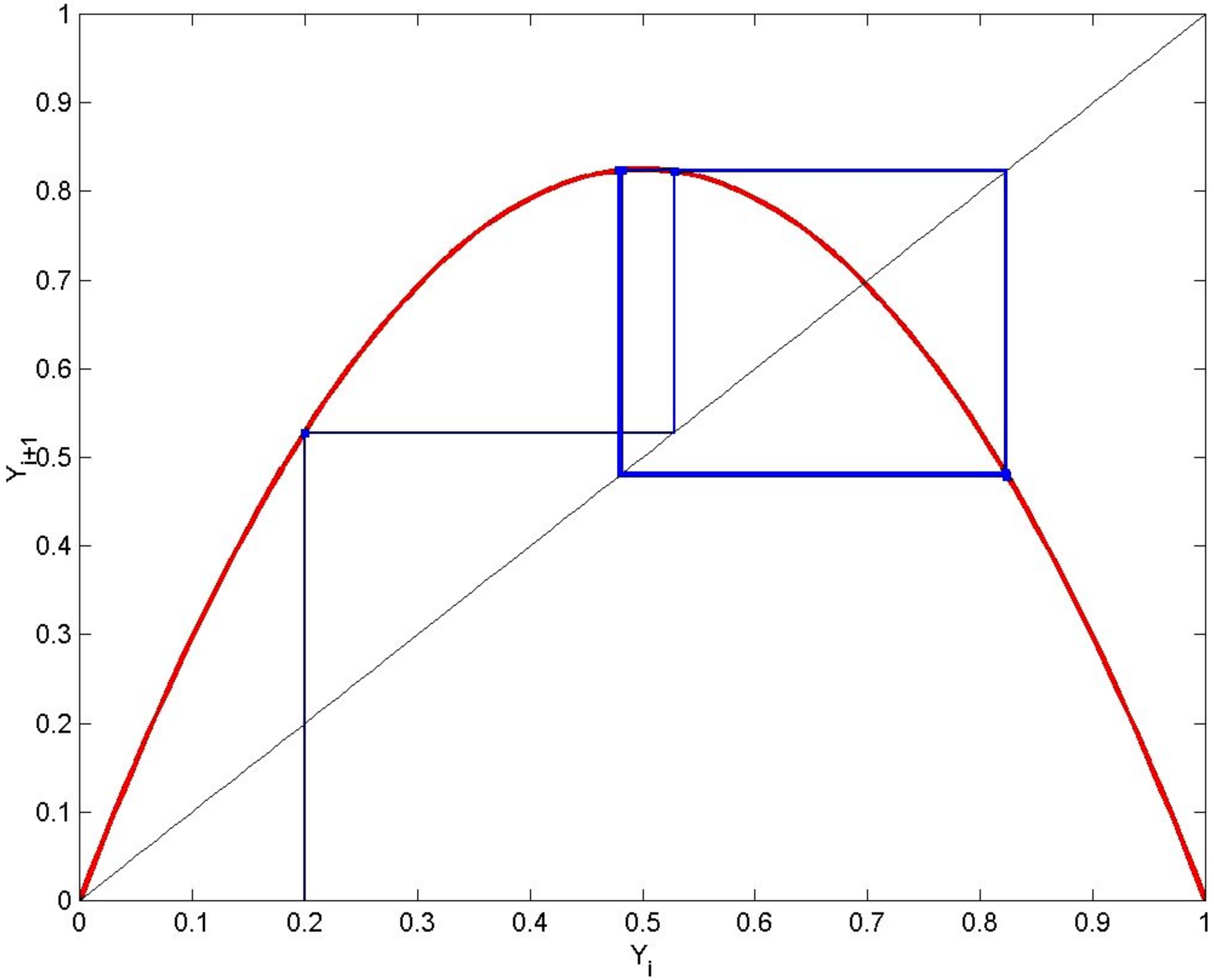
Logistic Map: Graphs

$$L_{i+1} = r L_i (1 - L_i)$$



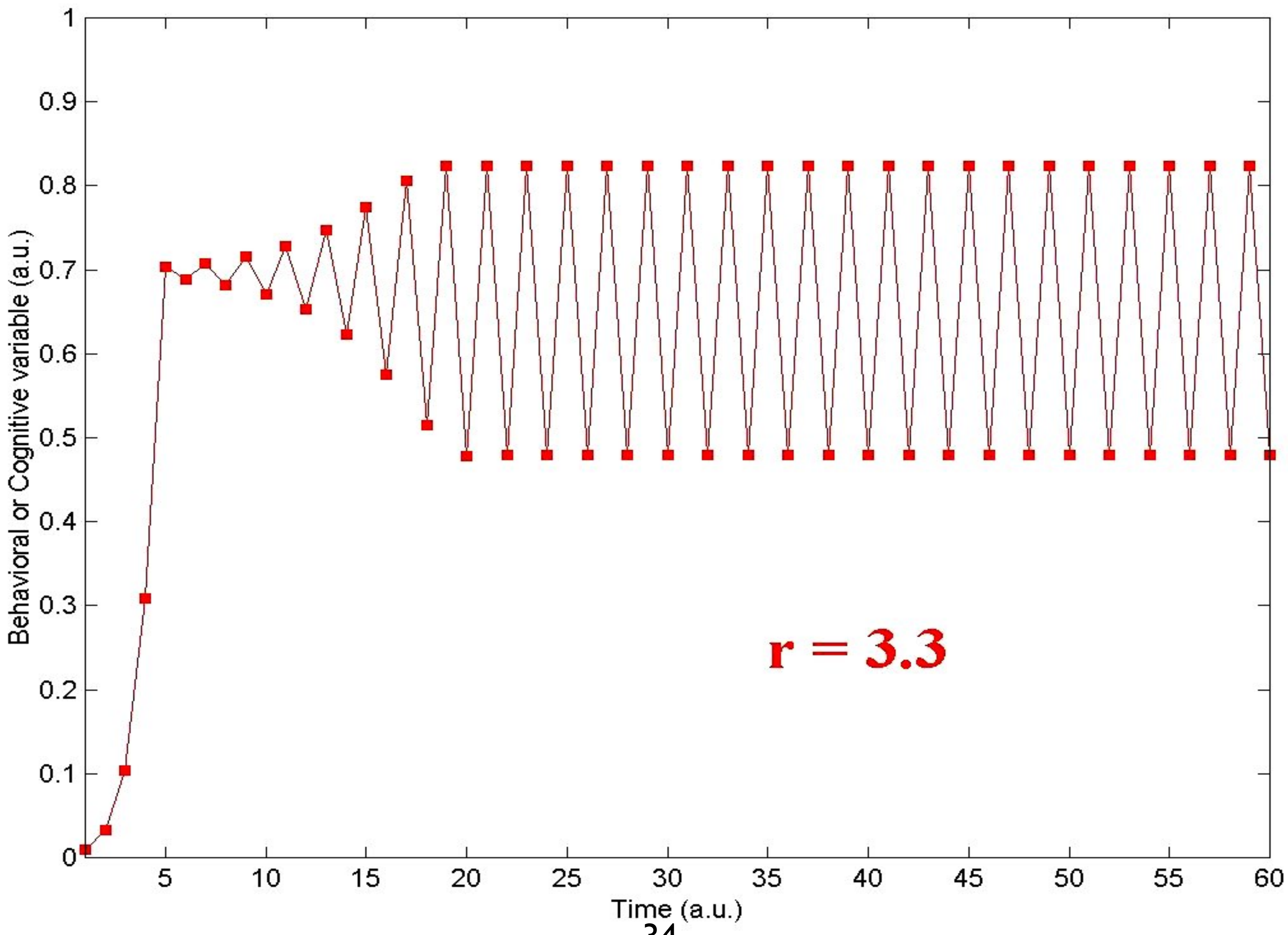
Teufelskreis (1)

r=3.3



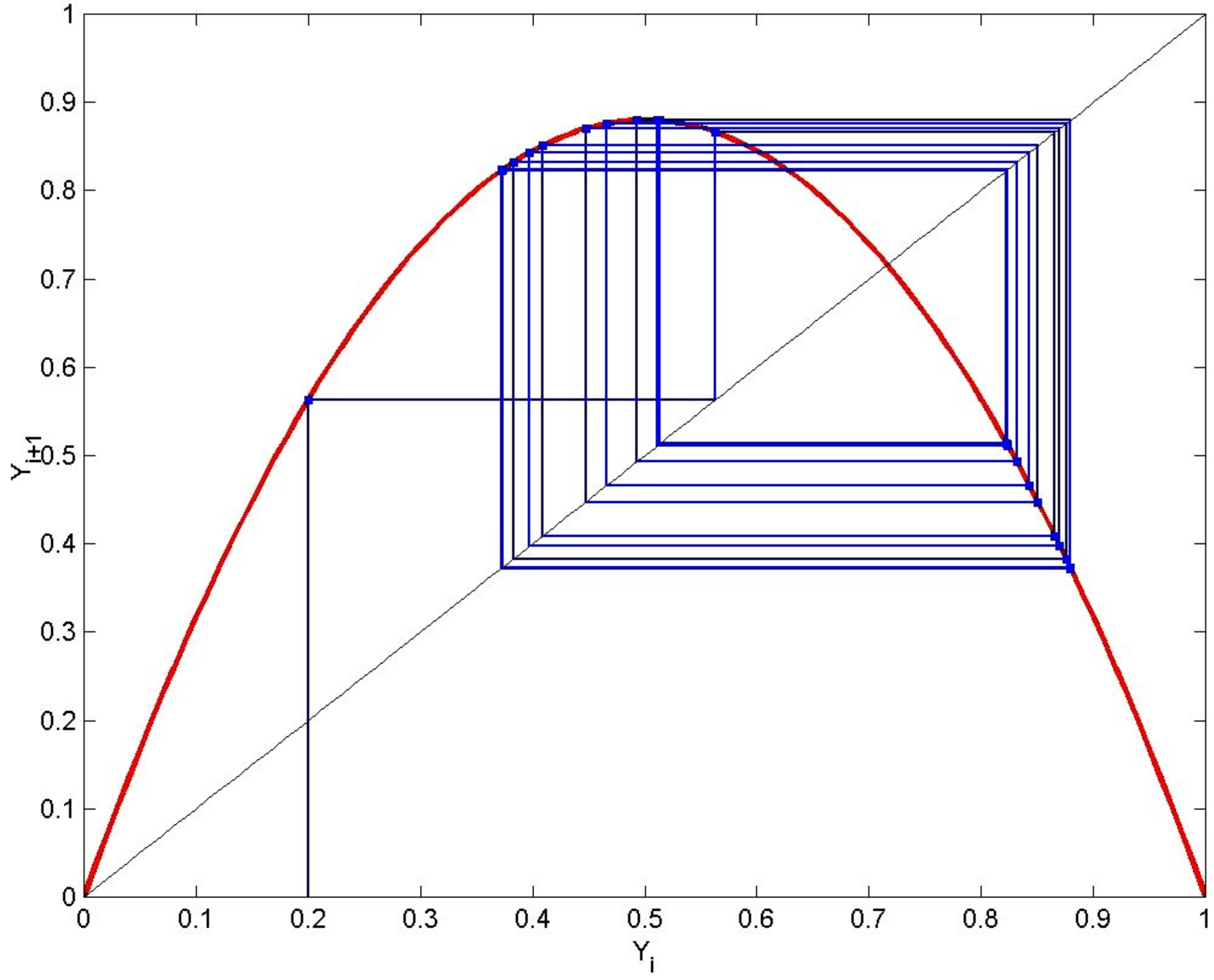
Logistic Map: Graphs

$$L_{i+1} = r L_i (1 - L_i)$$



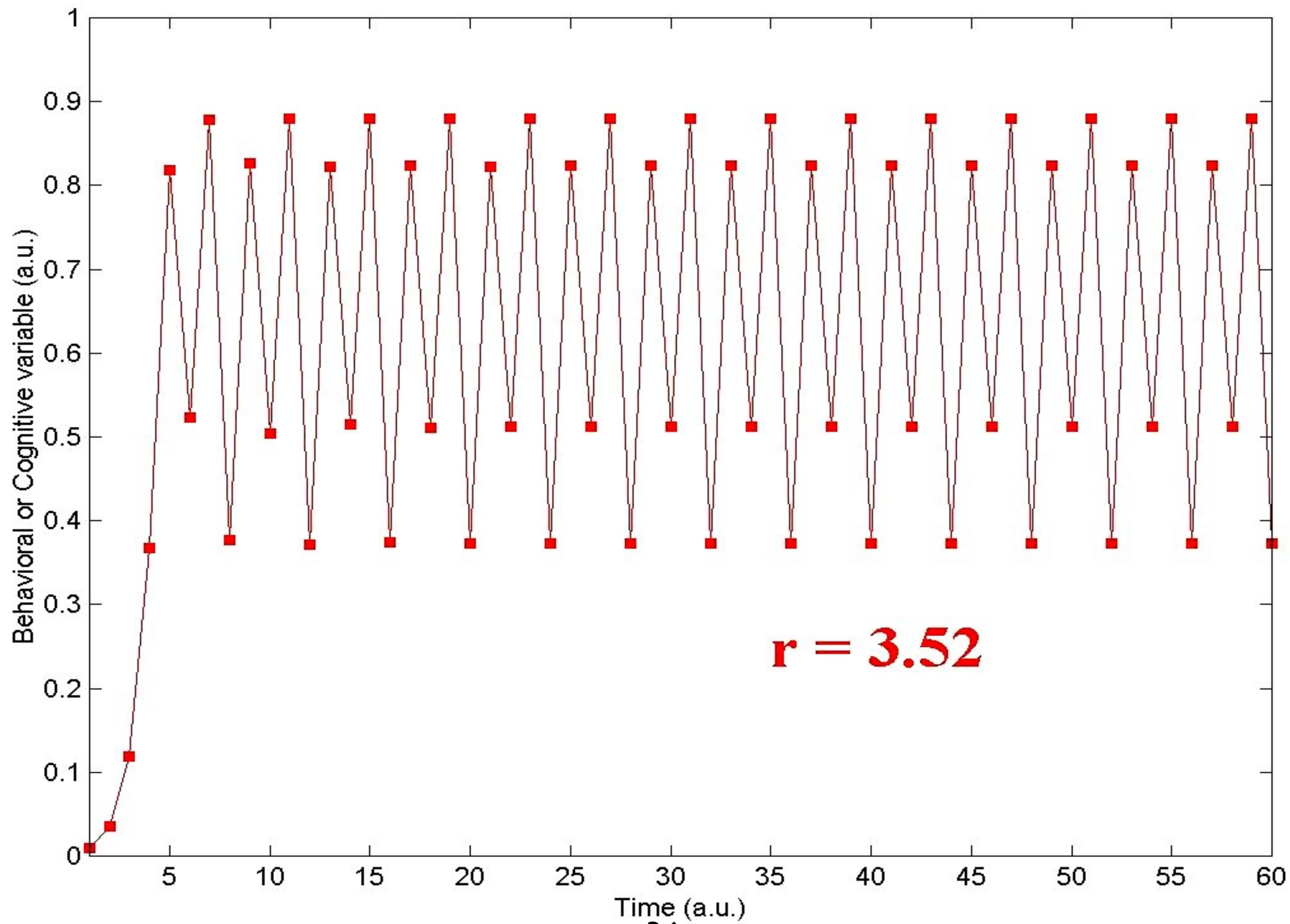
Teufelskreis (2)

r=3.52

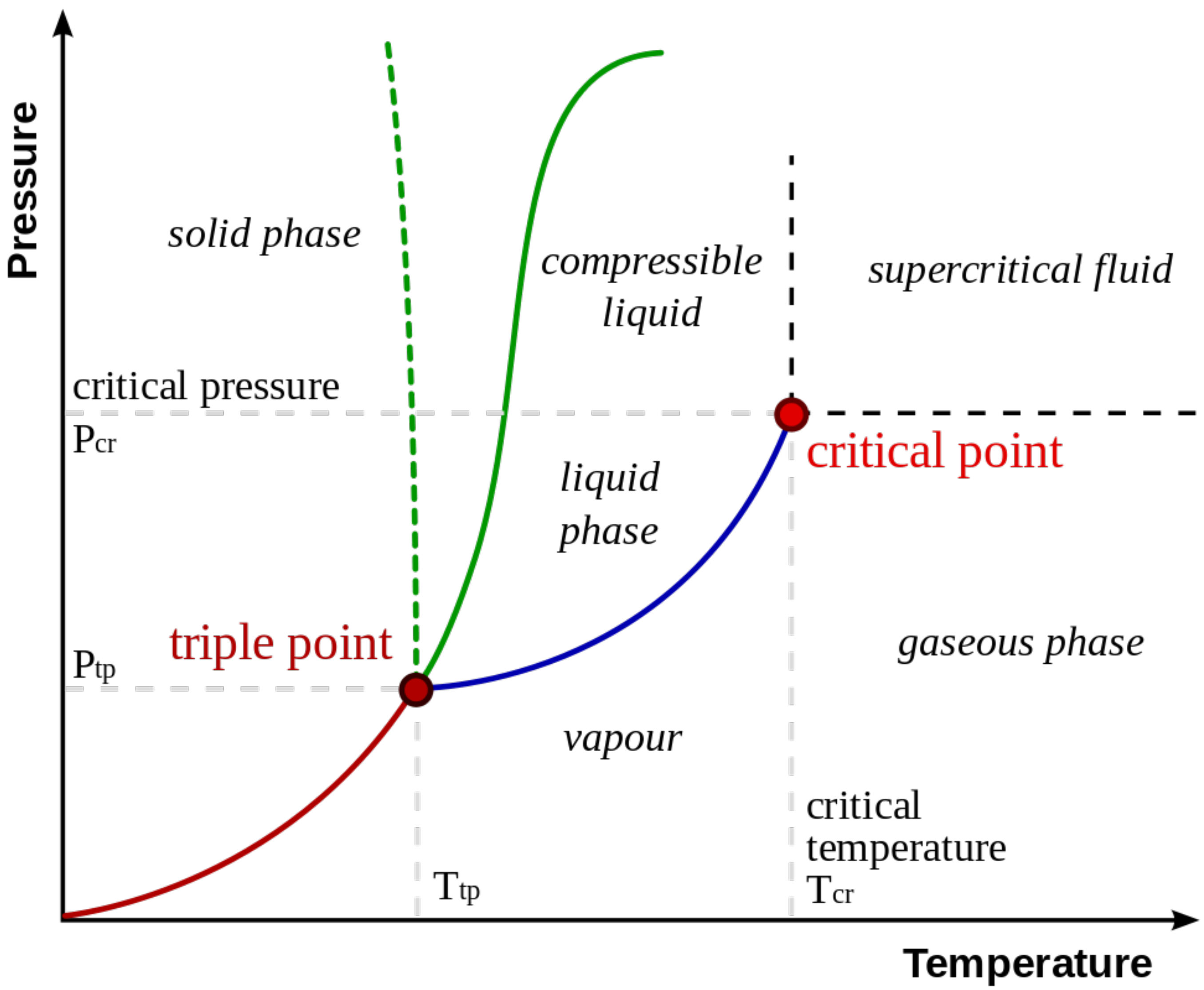


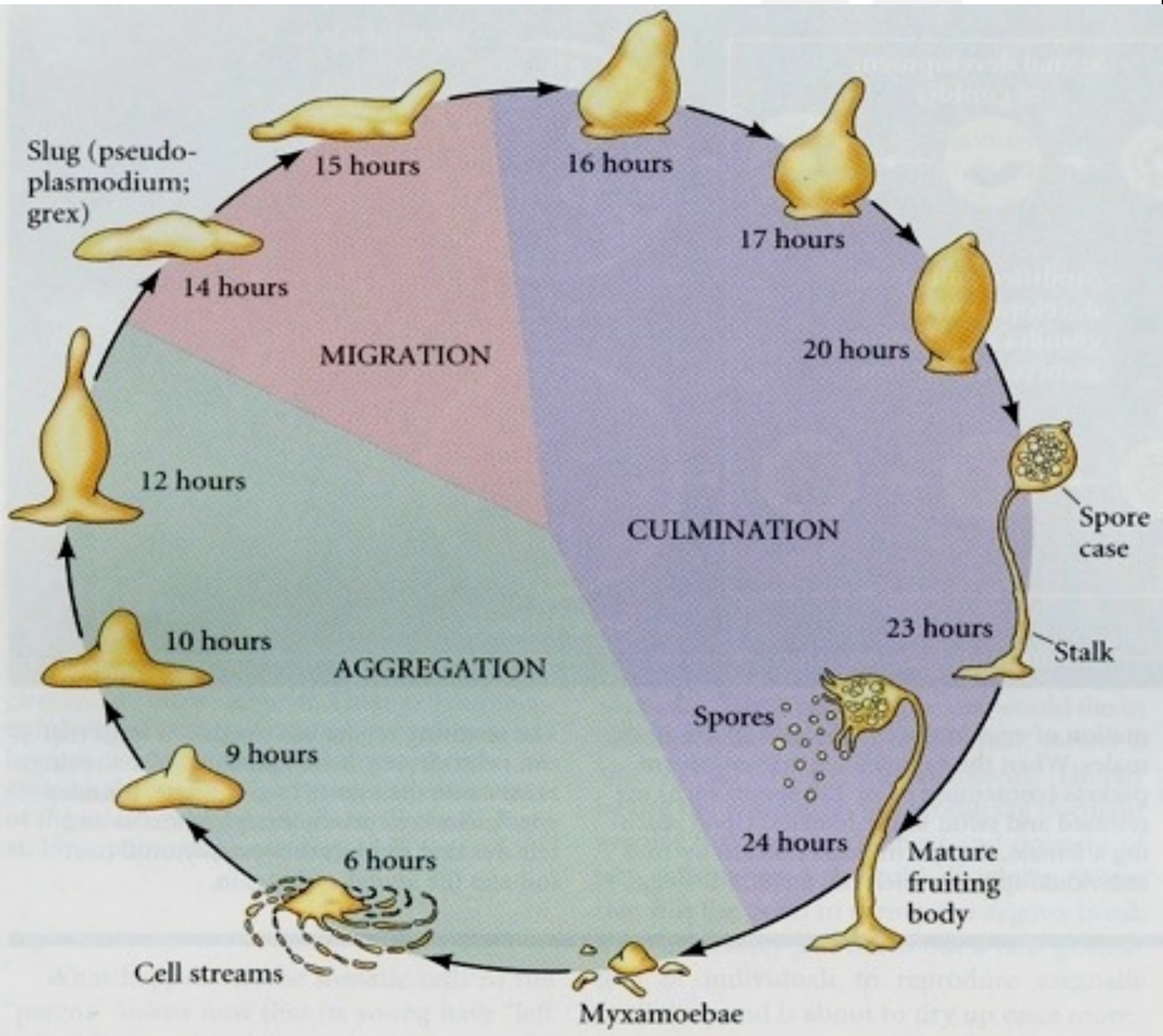
Logistic Map: Graphs

$$L_{i+1} = r L_i (1 - L_i)$$



Phase Diagram & Order parameter





The order parameter is often a qualitative description of a macro state / global organisation of the system, conditional on the control parameters:

H_2O: Ice (Solid), Water (Liquid), Steam (Vapour)

Disctyostelium: Aggregation (Mound), Migration (Slug), Culmination (Fruiting Body)

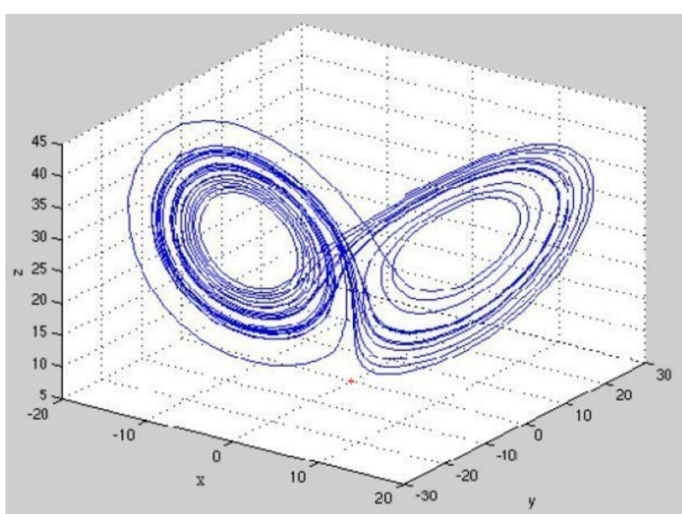
State Space / Phase Space

A state space is spanned by a system's state variables (dimensions), these are often iterative processes.

Every potential micro-scale configuration of a system is represented by a coordinate in state space. Each point is a state, a degree of freedom. *Change of states over time = Trajectory through state space, an orbit.*

Generally, state space is not completely filled by trajectories = not all d.o.f available = attracted to specific states / trajectories

$$\frac{dx}{dt} = a(y - x)$$
$$\frac{dy}{dt} = x(b - z) - y$$
$$\frac{dz}{dt} = xy - cz$$



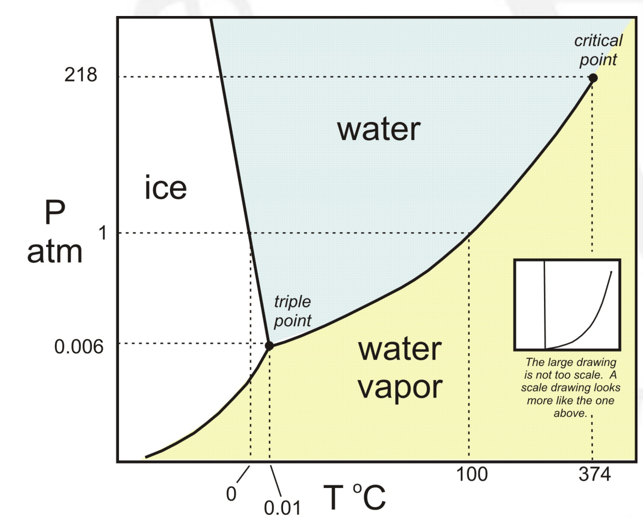
http://universe-review.ca/I01-18-LorenzEqs2.jpg

Phase Diagram / Phase Portrait

A phase diagram labels the order parameter of a system for different values of control parameter(s).

Regions represent qualitatively different states at the *global-scale*, the coordinates represent control parameter values associated with a global state. *Time does not have to be represented in the space.*

Generally, a phase diagram is completely filled with labelled regions of qualitatively different states, also called: phases / orders / regimes / modes of the system



http://cft.fis.uc.pt/eef/Fisica101/fluids/h2o_phase_diagram.jpg

Using Analytic Solutions

Van Geert's framework for cognitive growth

van Geert introduces an '*ecology of species of cognitive growers*'
(These terms are not accidentally chosen, these models also have applications in biology)

- Cognitive growth is an **autocatalytic** process (of discrete units)

- Cognitive growth is **limited** by available resources

- Cognitive growth may be a **delayed** process

- Cognitive growth rate may **vary** depending on certain events

- Cognitive growth may be in **competition** with, or receive **support** from other growth processes

Logistic growth

If we combine these 2 **linear** models we get **nonlinear** restricted (logistic) growth

$$\boxed{Y_{i+1} = rY_i(K - Y_i)}$$

$$\boxed{Y_{i+1} = rY_i(K - Y_i)}$$

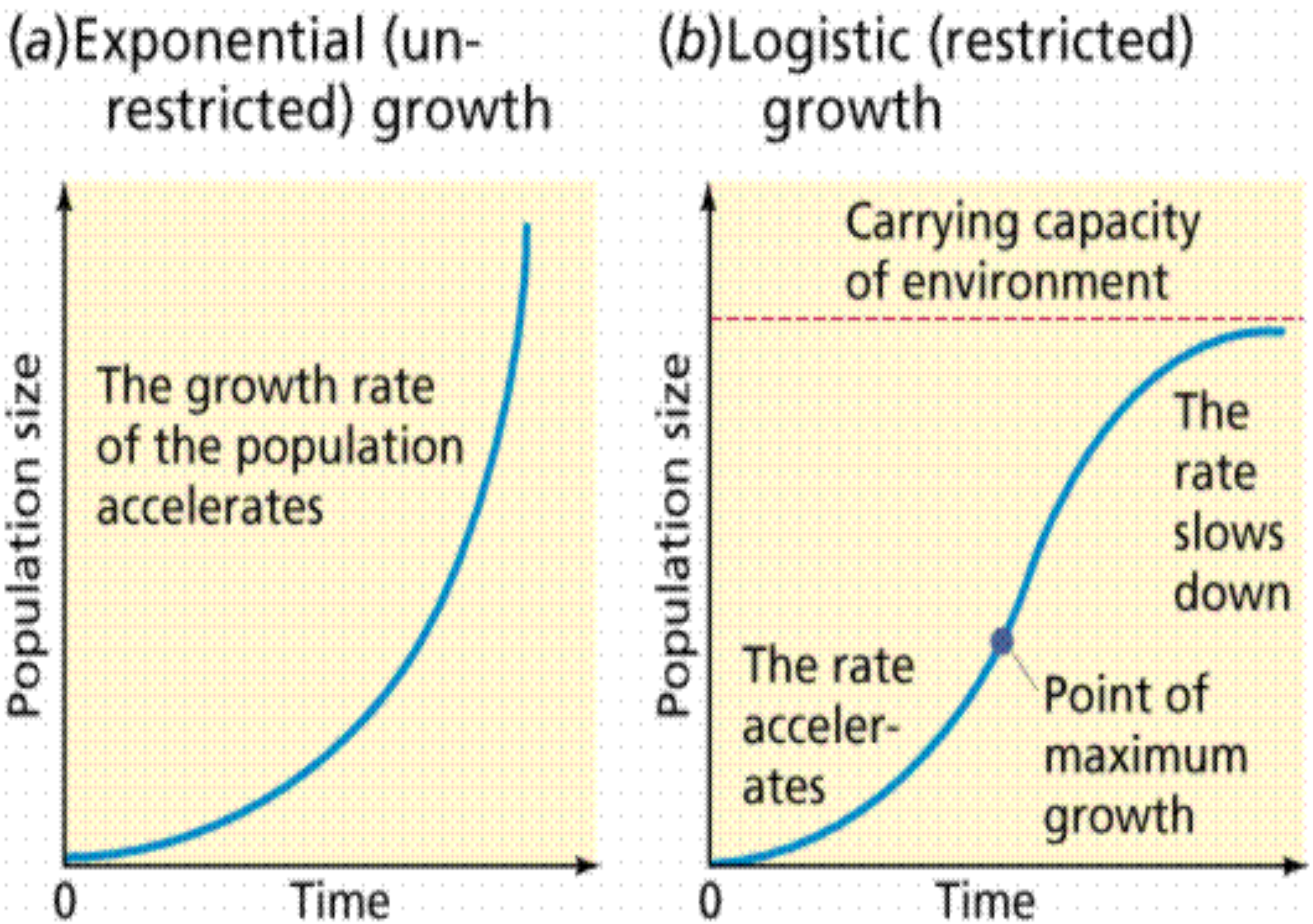
no analytic solution

$$\boxed{\frac{dY}{dt} = rY(K - Y)}$$

$$\downarrow$$

$$\boxed{Y(t) = \frac{KY_0}{Y_0 + (K - Y_0)e^{-Krt}}}$$

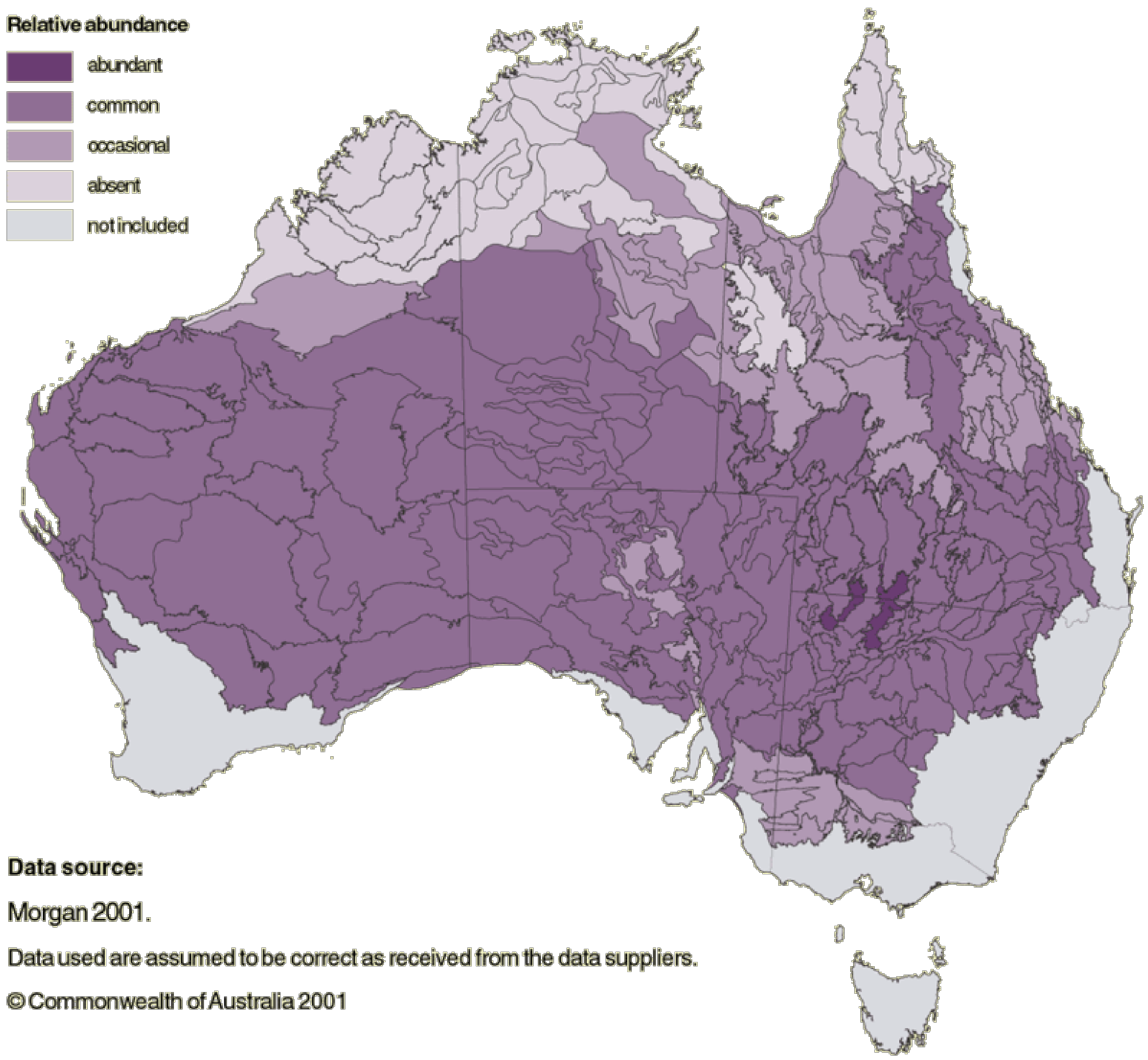
Logistic Growth (Flow ~)



Real world examples:

- Cell splitting
- Population growth

Logistic Growth (Flow ~)



Exponential population growth:
Start with 16 rabbits and
no restriction on growth
(or an extremely high
carrying capacity)

Van Geert's basic Growth Model

Same ingredients as logistic model:

$$L_{i+f} = (1 + r - r \cdot L_i/K) \cdot L_i$$

$$= \underbrace{(1 + r) \cdot L_i}_{\substack{\text{Driving} \\ (\textit{linear})}} - \underbrace{r \cdot L_i^2/K}_{\substack{\text{Damping} \\ (\textit{quadratic})}}$$

Important! – We still have one control parameter **r**
K just controls how much we can grow.

Differences with time as a predictor in a linear model

Structural portion, *which embodies our hypothesis about the shape of each person's true trajectory of change over time*

Stochastic portion, *which allows for the effects of random error from the measurement of person i on occasion j. Usually assume* $\varepsilon_{ij} \sim N(0,\sigma_\varepsilon^2)$

Key assumption: In the population, COG_{ij} is a linear function of child i's AGE on occasion j

$$COG_{ij} = \left[\boxed{\pi_{0i}} + \boxed{\pi_{1i}}(AGE_{ij} - 1)\right] + \left[\varepsilon_{ij}\right]$$

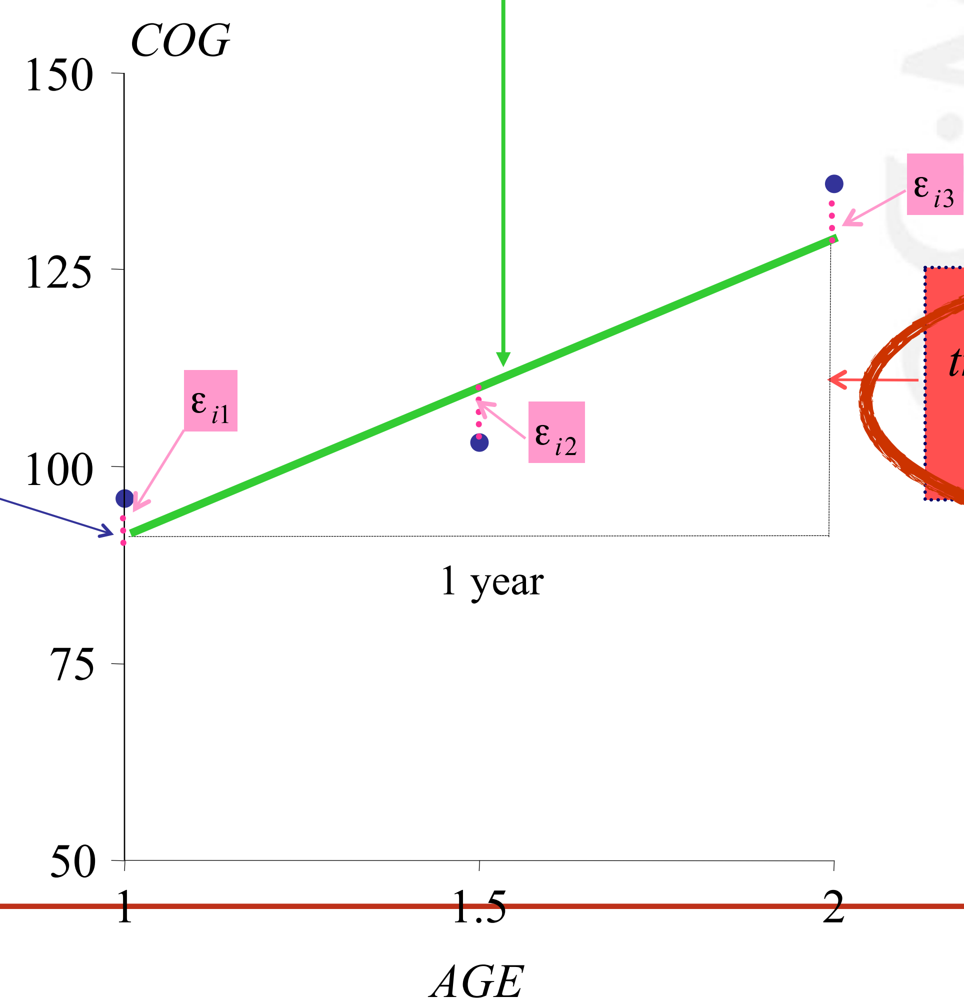
- ***i indexes persons (i=1 to 103)***
- ***j indexes occasions (j=1 to 3)***

*Individual i's hypothesized **true change trajectory***

*ε_{i1}, ε_{i2}, and ε_{i3} are deviations of i's true change trajectory **from linearity** on each occasion (measurement error)*

*π_{0i} is the **intercept** of i's true change trajectory, his true value of COG at AGE=1, his **"true initial status"***

*π_{1i} is the **slope** of i's true change trajectory, his yearly rate of change in true COG, his true "annual **rate of change**"*



(ALDA, Section 3.2, pp. 49-51)

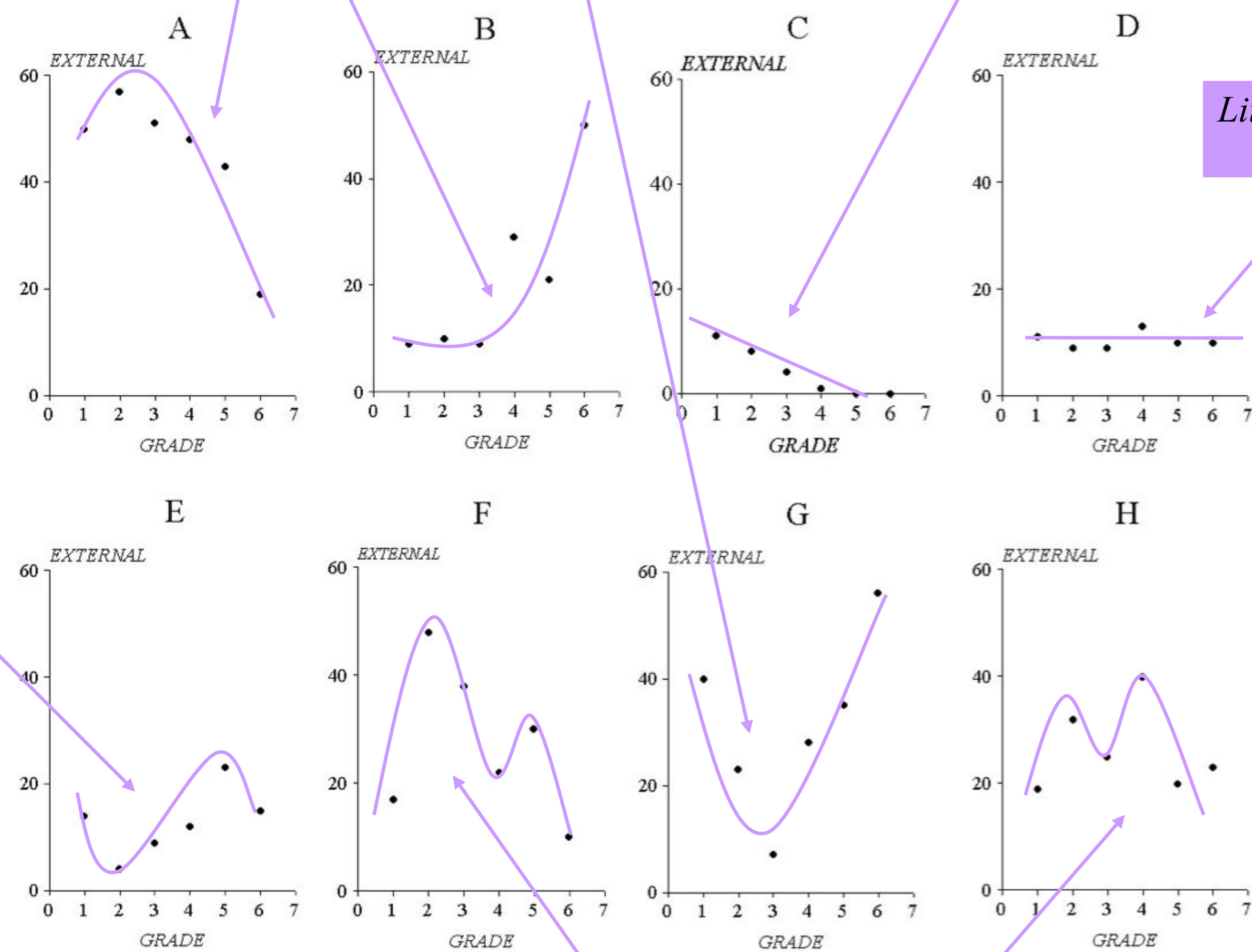
For a line:
2 parameters
(slope + intercept)

Other shapes?

Quadratic change (but with varying curvatures)

Linear decline (at least until 4th grade)

Little change over time (flat line?)



Two stationary points? (suggests a cubic)

Assumptions…

the true change trajectory is a polynomial function of time of unknown order in the population

Three stationary points? (suggests a quartic!!!)

When faced with so many different patterns, how do you select a common polynomial for analysis?

http://gseacademic.harvard.edu/~alda/Chapter%20presentations.htm

(ALDA, Section 6.3.2, pp 217-220)

Looking for explained variance of polynomial components, NOT based on a theoretical process

A logistic S-shape would require 4 fixed polynomial parameters

		Parameter	Model A No change	Model B Linear change	Model C Quadratic change	Model D Cubic change
Fixed Effects						
Composite model	Intercept (1st grade status)	γ_{00}	12.96***	13.29***	13.97***	13.79***
	TIME (linear term)	γ_{10}		−0.13	−1.15	−0.35
	$TIME^2$ (quadratic term)	γ_{20}			0.20	−0.23
	$TIME^3$ (cubic term)	γ_{30}				0.06
Variance Components						
Level-1:	Within-person	σ_ε^2	70.20***	53.72***	41.98***	40.10***
Level-2:	In 1st grade status	σ_0^2	87.42***	123.52***	107.08***	126.09***
	Linear term					
	variance	σ_1^2		4.69**	24.60*	88.71
	covar with 1st grade status	σ_{01}		−12.54*	−3.69	−51.73
	Quadratic term					
	variance	σ_2^2			1.22*	11.35
	covar with 1st grade status	σ_{02}			−1.36	22.83~
	covar with linear term	σ_{12}			−4.96*	−31.62
	Cubic term					
	variance	σ_3^2				0.08
	covar with 1st grade status	σ_{03}				−3.06~
	covar with linear term	σ_{13}				2.85
	covar with quadratic term	σ_{23}				−0.97
Goodness-of-fit						
	Deviance statistic		2010.3	1991.8	1975.8	1967.0
	AIC		2016.3	2003.8	1995.8	1997.0
	BIC		2021.9	2015.0	2014.5	2025.1

~$p < .10$; *$p < .05$; **$p < .01$; ***$p < .001$.

(ALDA, Section 6.3.3, pp 220-223)

Exponential growth (Flow ~)

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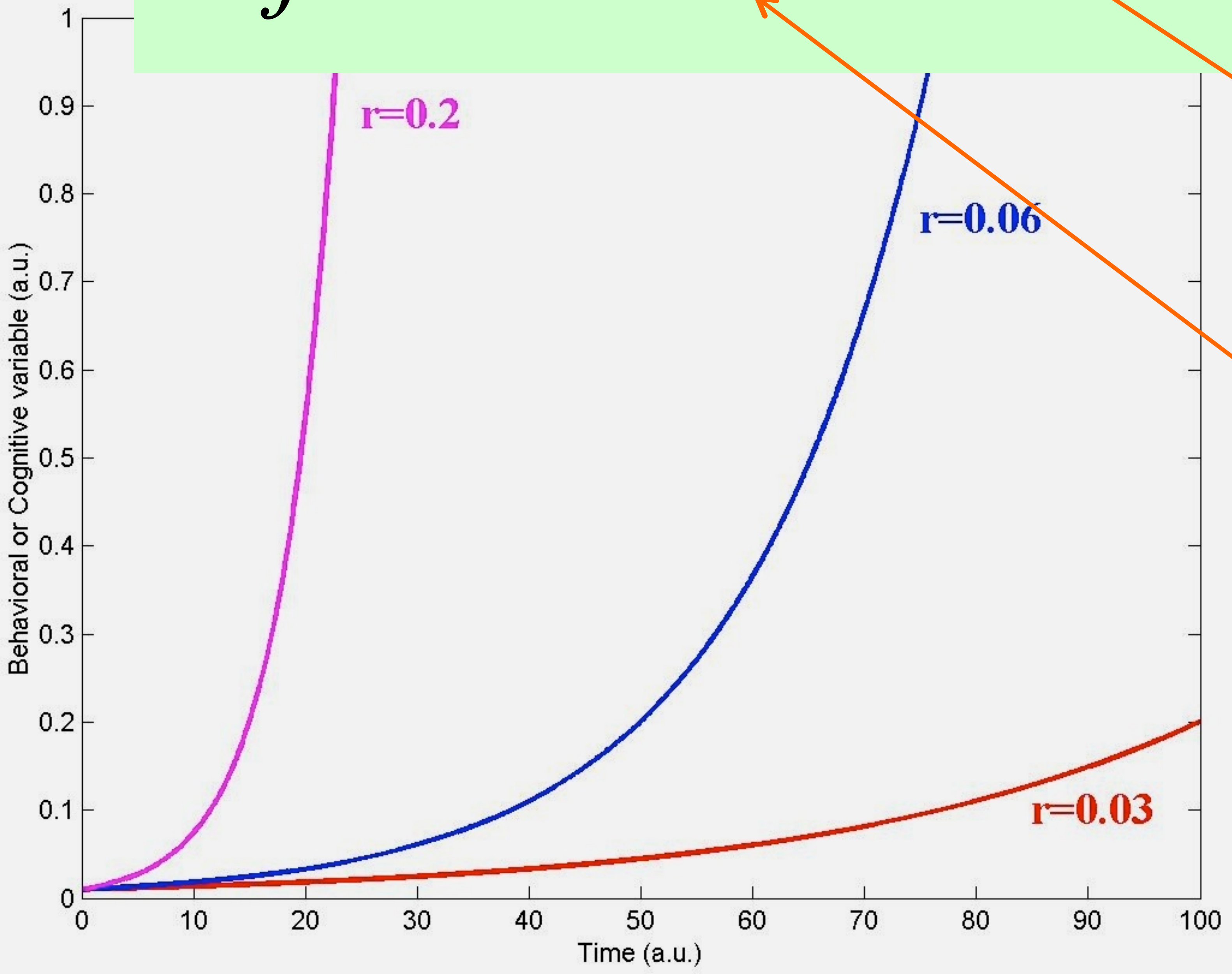
$$Y_{ij} = \pi_{0i} e^{\pi_{1i} TIME_{ij}} + \varepsilon_{ij}$$



$$\frac{dY}{dt} = r \cdot Y$$

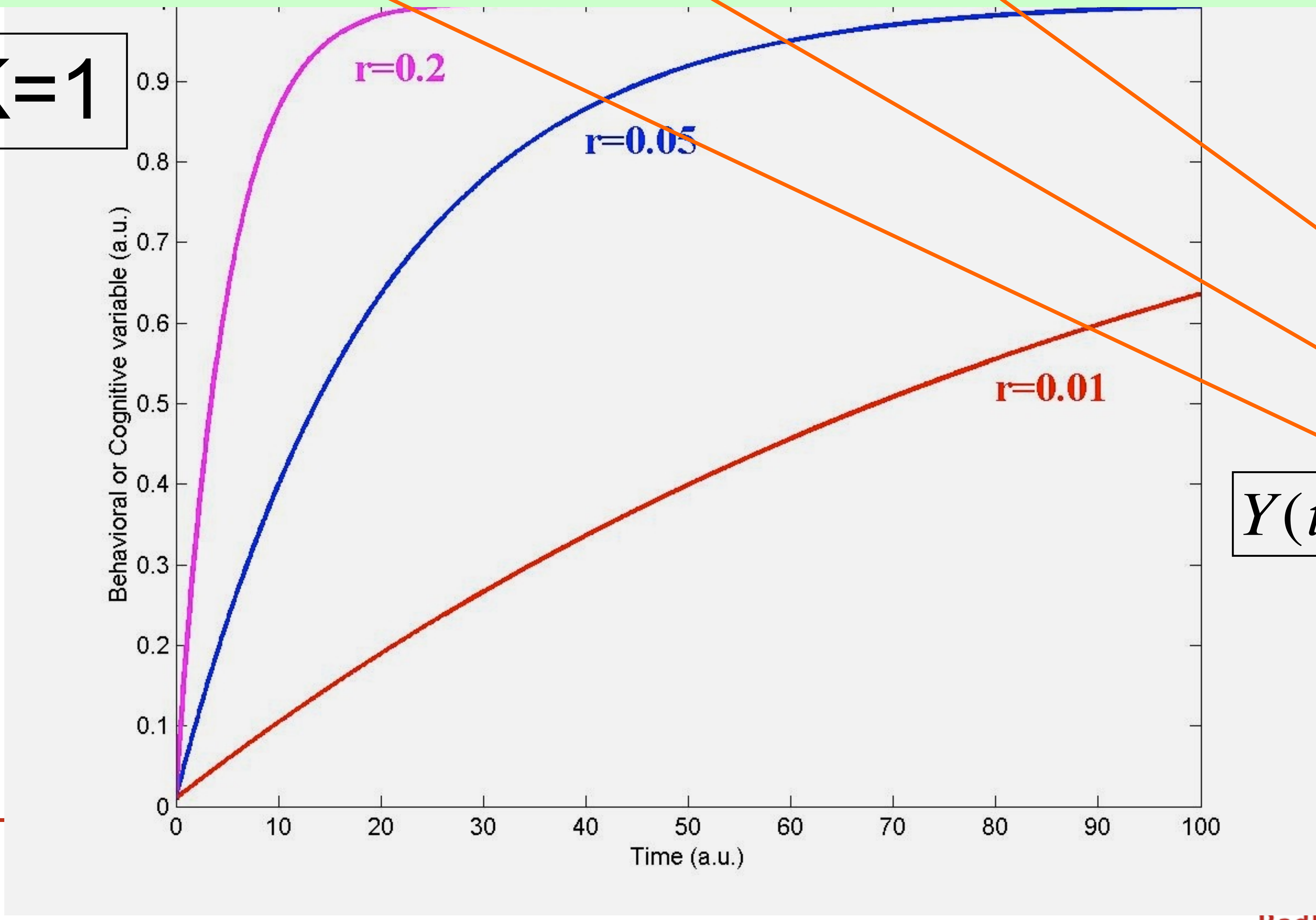
$$Y(t) = Y_0 \cdot e^{rt}$$

Y_0= Initial condition

Restricted growth

$$Y_{ij} = \alpha_i - (\alpha_i - \pi_{0i}) e^{-\pi_{1i} TIME_{ij}} + \varepsilon_{ij}$$

K=1

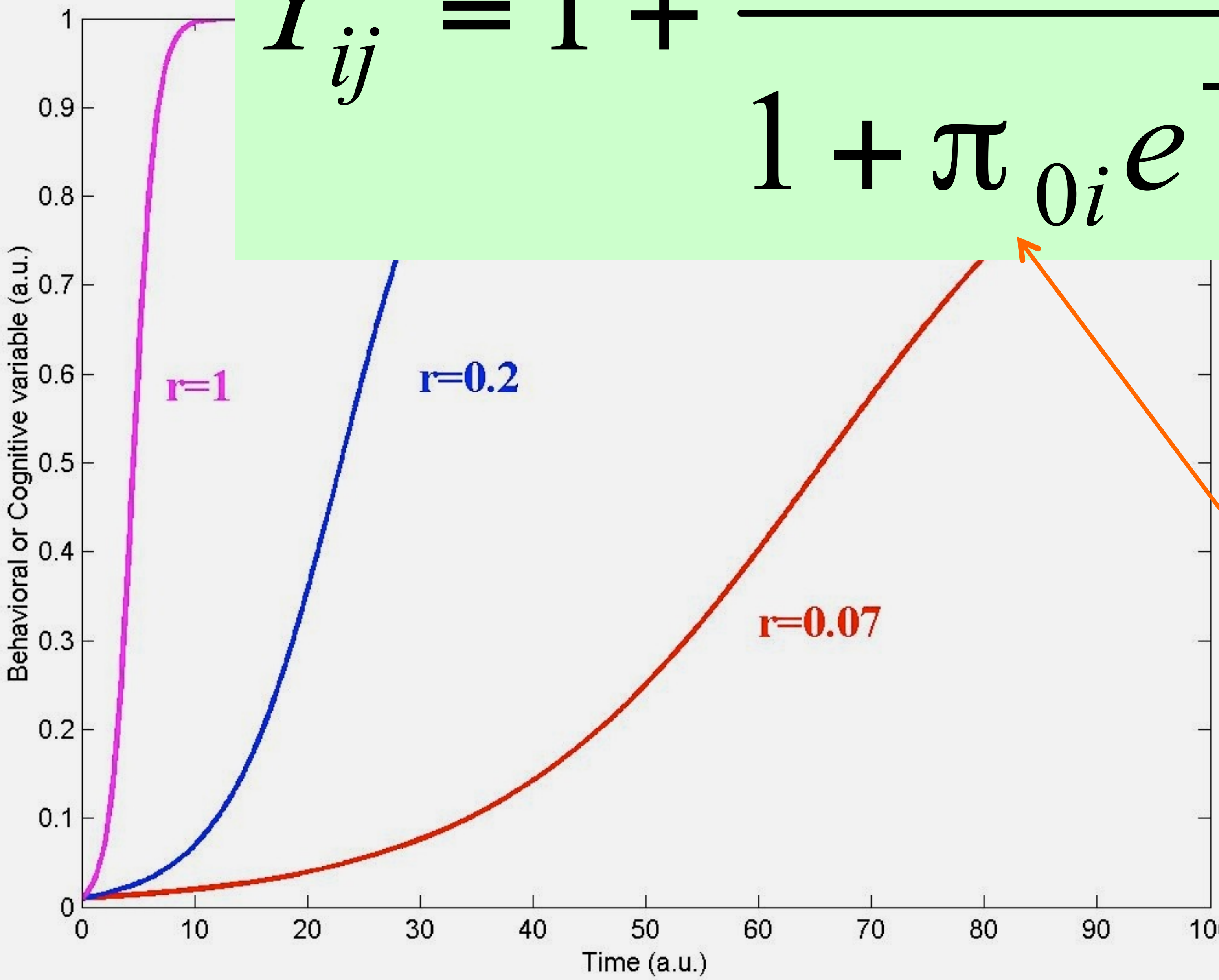


$$\frac{dY}{dt} = r \cdot (K - Y)$$

$$Y(t) = K - (K - Y_0) \cdot e^{-rt}$$

Logistic Growth (Flow ~)

$$Y_{ij} = 1 + \frac{19}{1 + \pi_{0i} e^{-(\pi_{1i} TIME_{ij})}} + \varepsilon_{ij}$$



$$\frac{dY}{dt} = rY(K - Y)$$

$$Y(t) = \frac{KY_0}{Y_0 + (K - Y_0)e^{-Krt}}$$