Discrete Structures, CSCI 2824, Fall 2015

Problem Set 2. (5 problems) Due Thursday, October 15 (hard copy, in class!)

Problem 1. (5 points) In the last problem set, we looked at the Lucas numbers:

L(0) = 2
L(1) = 1
L(n) = L(n-1) + L(n-2) for all n>= 2

Again, the Lucas series begins as follows:
2, 1, 3, 4, 7, 11, 18, 29, etc.

Let L(n) be the nth Lucas number in the series (starting from L(0) = 2, L(1) = 1, and so forth), and let F(n) be the nth Fibonacci number (where F(0) = 0, F(1) = 1, and so forth). Show by induction that F(2n) = F(n)* L(n) for n>= 1.

Problem 2. (6 points) In class, we discussed the Chinese remainder theorem, and we showed an algorithm for "decoding" the remainders of two distinct primes. For example, suppose we want to know which number between 0 and 34 has remainder 1 when divided by 5 and 2 when divided by 7. We begin by finding the Bezout theorem parameters that would allow multiples of 5 and 7 to add up to 1:

5*3 + 7*(-2) = 1

Then, we multiply our two desired remainders–here, 1 and 2–by the two terms in our Bezout expression. The first remainder is multiplied into the second term, and the second remainder is multiplied into the first term:

5*3*2 + 7*(-2)*1 = 30-14 = 16

And now we can conclude (and check!) that 16 has remainder 1 when divided by 5 and remainder 2 when divided by 7.

2a (4 points). Your job is to extend this technique to 3 distinct primes. (We could continue on to more than 3 primes, but this problem provides the general idea of how to continue.) Suppose we choose 3 primes-let's say, 3, 5, and 7. Now, we want to find a number that has a remainder of 0 when divided by 3, 4 when divided by 5, and 5 when divided by 7. In other words, our desired number n should have the properties:

n mod 3 = 0

n mod 5 = 4

$n \bmod 7 = 5$

And we'll assume that n is in the range between 1 and 104 (since we are interested in numbers that are less than 3*5*7 = 105).

Show that you can extend the technique from class to find the desired number. Hint: you want to use the very same technique that you used for two primes, but now use that technique sequentially. First, find a number that has remainder 0 mod 3 and 4 mod 5; this will give you a number j with the right remainder mod 15. Now find a number between 1 and 104 that has remainder j mod 15 and remainder 5 mod 7. This will be your desired number.

2b (2 points). A Chinese general knows that his troops number somewhat above 1000, but otherwise isn't really sure of the number. He asks them to line up in rows of 11, and finds that the final "remainder" row has 9 soldiers. He asks them to line up in rows of 13, and finds that the final "remainder" row has 1 soldier. He asks them to line up in rows of 17, and finds 14 soldiers in the final "remainder" row. How many soldiers are in his army?

Problem 3. (4 points) *Divisors and the totient function.*

3a. Suppose we have a given positive natural number m that is greater than the nth prime but less than the n+1th prime. (For example, 12 is greater than 11, the 5th prime, but less than 13, the 6th prime.) Now we express our number m in the form:

$$2^{e_1}3^{e_2}5^{e_3}...p_n^{e_n}$$

where the exponents can include 0. Write an expression, in terms of the exponents e_1, ... e_n, that denotes the number of distinct divisors of our number including 1 and m itself. (For example, 12 has 6 divisors: 1, 2, 3, 4, 6, and 12.)

3b. Suppose a number m is the product of two distinct primes, p and q. Show that the Euler totient of m is equal to (p-1)(q-1). For instance, the Euler totient of 35 (which is 5 * 7) is equal to 4*6, or 24.

Problem 4. (6 points)

4a. Simplify the following three expressions using the Boolean algebra laws that you saw in class:

$$x'(x+y) + (y+x)(x+y')$$

$$xyz + xy'z + x'yz + x'yz' + x'y'z' + x'y'z$$

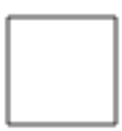
$$xyz+xy'z'+xy'z+x'yz+x'yz'+x'y'z$$

4b. At Boulder High School in the freshman class, the 37 students use a variety of forms of transportation to get to school every day. 20 of them sometimes arrive to school by car; 12 students bike to school at least occasionally; 16 students take the bus some days. If four students use all three of these options, two either bike or take the bus, six come by car or bike, and five arrive by car when they do not come on the bus, how many students always use some other type of transportation?

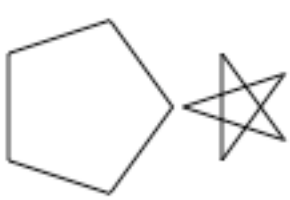
Problem 5. (5 points) Suppose we are interested in finding out how many distinct regular polygons there are with a given number of sides n. We allow sides to cross each other, but the n sides must make a complete closed polygon with equal-length sides and equal angles wherever two sides meet. Also, polygons that are congruent (by rotation, movement, or reflection) don't count as "distinct". To get us started (and in keeping with the Polya idea of "trying small values"), let's make a list of distinct regular polygons with 3, 4, 5, 6, 7, and 8 sides:

regular 3-gons:

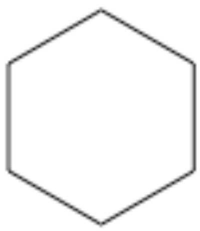
regular 4-gons:



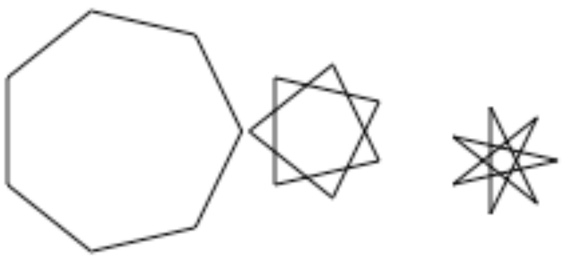
regular 5-gons:



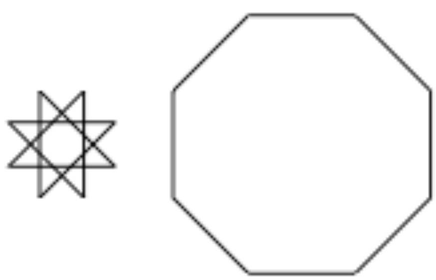
regular 6 gons:



regular 7-gons:



regular 8-gons:



To recap: we have found that there is 1 (unique) "species" of regular 3-gon, 1 4-gon, 2 5-gons, 1 6-gon, 3 7-gons, and 2 8-gons. Find a formula for the number of distinct n-gons, given n.