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Lemma 1 [*nat:ground*] $\forall x\,(\mathsf{S}\,\mathtt{nat}(x) \to gr(x))$.

Proof.
Induction$_0$: $\forall x\,(\mathsf{S}\,\mathtt{nat}(x) \to gr(x))$.
  Hypothesis$_1$: none.
  Conclusion$_1$: $gr(\mathtt{0})$.
  Hypothesis$_1$: $gr(x)$ and $\mathsf{S}\,\mathtt{nat}(x)$.
  Conclusion$_1$: $gr(\mathtt{s}(x))$. $\square$

Lemma 2 [*add:ground:1*] $\forall x, y, z\,(\mathsf{S}\,\mathtt{add}(x, y, z) \to gr(x))$.

Proof.
Induction$_0$: $\forall x, y, z\,(\mathsf{S}\,\mathtt{add}(x, y, z) \to gr(x))$.
  Hypothesis$_1$: none.
  Conclusion$_1$: $gr(\mathtt{0})$.
  Hypothesis$_1$: $gr(x)$ and $\mathsf{S}\,\mathtt{add}(x, y, z)$.
  Conclusion$_1$: $gr(\mathtt{s}(x))$. $\square$

Lemma 3 [*add:ground:2*] $\forall x, y, z\,(\mathsf{S}\,\mathtt{add}(x, y, z) \wedge gr(y) \to gr(z))$.

Proof.
Induction$_0$: $\forall x, y, z\,(\mathsf{S}\,\mathtt{add}(x, y, z) \to gr(y) \to gr(z))$.
  Hypothesis$_1$: none.
  Conclusion$_1$: $gr(y) \to gr(y)$.
  Hypothesis$_1$: $gr(y) \to gr(z)$ and $\mathsf{S}\,\mathtt{add}(x, y, z)$.
    Assumption$_2$: $gr(y)$. $gr(z)$. $gr(\mathtt{s}(z))$.
    Thus$_2$: $gr(y) \to gr(\mathtt{s}(z))$.
  Conclusion$_1$: $gr(y) \to gr(\mathtt{s}(z))$. $\square$

Lemma 4 [*add:ground:3*] $\forall x, y, z\,(\mathsf{S}\,\mathtt{add}(x, y, z) \wedge gr(z) \to gr(y))$.

Proof.
Induction$_0$: $\forall x, y, z\,(\mathsf{S}\,\mathtt{add}(x, y, z) \to gr(z) \to gr(y))$.
  Hypothesis$_1$: none.
  Conclusion$_1$: $gr(y) \to gr(y)$.
  Hypothesis$_1$: $gr(z) \to gr(y)$ and $\mathsf{S}\,\mathtt{add}(x, y, z)$.
    Assumption$_2$: $gr(\mathtt{s}(z))$. $gr(z)$.
    Thus$_2$: $gr(\mathtt{s}(z)) \to gr(y)$.
  Conclusion$_1$: $gr(\mathtt{s}(z)) \to gr(y)$. $\square$

Lemma 5 [*add:types:2*] $\forall x, y, z\,(\mathsf{S}\,\mathtt{add}(x, y, z) \wedge \mathsf{S}\,\mathtt{nat}(y) \to \mathsf{S}\,\mathtt{nat}(z))$.

Proof.
Induction$_0$: $\forall x, y, z\,(\mathsf{S}\,\mathtt{add}(x, y, z) \to \mathsf{S}\,\mathtt{nat}(y) \to \mathsf{S}\,\mathtt{nat}(z))$.
  Hypothesis$_1$: none.
  Conclusion$_1$: $\mathsf{S}\,\mathtt{nat}(y) \to \mathsf{S}\,\mathtt{nat}(y)$.
  Hypothesis$_1$: $\mathsf{S}\,\mathtt{nat}(y) \to \mathsf{S}\,\mathtt{nat}(z)$ and $\mathsf{S}\,\mathtt{add}(x, y, z)$.
  Conclusion$_1$: $\mathsf{S}\,\mathtt{nat}(y) \to \mathsf{S}\,\mathtt{nat}(\mathtt{s}(z))$. $\square$

Lemma 6 [*add:types:3*] $\forall x, y, z\, (\mathsf{S}\,\mathtt{add}(x, y, z) \land \mathsf{S}\,\mathtt{nat}(z) \to \mathsf{S}\,\mathtt{nat}(y))$.

Proof.

Induction$_0$: $\forall x, y, z\, (\mathsf{S}\,\mathtt{add}(x, y, z) \to \mathsf{S}\,\mathtt{nat}(z) \to \mathsf{S}\,\mathtt{nat}(y))$.
  Hypothesis$_1$: none.
  Conclusion$_1$: $\mathsf{S}\,\mathtt{nat}(y) \to \mathsf{S}\,\mathtt{nat}(y)$.
  Hypothesis$_1$: $\mathsf{S}\,\mathtt{nat}(z) \to \mathsf{S}\,\mathtt{nat}(y)$ and $\mathsf{S}\,\mathtt{add}(x, y, z)$.
    Assumption$_2$: $\mathsf{S}\,\mathtt{nat}(\mathtt{s}(z))$. $\mathsf{D}\,\mathsf{S}\,\mathtt{nat}(\mathtt{s}(z))$ by completion.
    Thus$_2$: $\mathsf{S}\,\mathtt{nat}(\mathtt{s}(z)) \to \mathsf{S}\,\mathtt{nat}(y)$.
  Conclusion$_1$: $\mathsf{S}\,\mathtt{nat}(\mathtt{s}(z)) \to \mathsf{S}\,\mathtt{nat}(y)$. $\square$

Lemma 7 [*nat:termination*] $\forall x\, (\mathsf{S}\,\mathtt{nat}(x) \to \mathsf{T}\,\mathtt{nat}(x))$.

Proof.

Induction$_0$: $\forall x\, (\mathsf{S}\,\mathtt{nat}(x) \to \mathsf{T}\,\mathtt{nat}(x))$.
  Hypothesis$_1$: none.
  Conclusion$_1$: $\mathsf{T}\,\mathtt{nat}(\mathtt{0})$.
  Hypothesis$_1$: $\mathsf{T}\,\mathtt{nat}(x)$ and $\mathsf{S}\,\mathtt{nat}(x)$. $\mathsf{D}\,\mathsf{T}\,\mathtt{nat}(\mathtt{s}(x))$ by completion.
  Conclusion$_1$: $\mathsf{T}\,\mathtt{nat}(\mathtt{s}(x))$. $\square$

Lemma 8 [*add:termination:1*] $\forall x, y, z\, (\mathsf{S}\,\mathtt{nat}(x) \to \mathsf{T}\,\mathtt{add}(x, y, z))$.

Proof.

Induction$_0$: $\forall x\, (\mathsf{S}\,\mathtt{nat}(x) \to \forall y, z\ \mathsf{T}\,\mathtt{add}(x, y, z))$.
  Hypothesis$_1$: none.
  Conclusion$_1$: $\forall y, z\ \mathsf{T}\,\mathtt{add}(\mathtt{0}, y, z)$.
  Hypothesis$_1$: $\forall y, z\ \mathsf{T}\,\mathtt{add}(x, y, z)$ and $\mathsf{S}\,\mathtt{nat}(x)$.
  Conclusion$_1$: $\forall y, z\ \mathsf{T}\,\mathtt{add}(\mathtt{s}(x), y, z)$. $\square$

Lemma 9 [*add:termination:2*] $\forall x, y, z\, (\mathsf{S}\,\mathtt{nat}(z) \to \mathsf{T}\,\mathtt{add}(x, y, z))$.

Proof.

Induction$_0$: $\forall z\, (\mathsf{S}\,\mathtt{nat}(z) \to \forall x, y\ \mathsf{T}\,\mathtt{add}(x, y, z))$.
  Hypothesis$_1$: none.
  Conclusion$_1$: $\forall x, y\ \mathsf{T}\,\mathtt{add}(x, y, \mathtt{0})$.
  Hypothesis$_1$: $\forall v_0, y\ \mathsf{T}\,\mathtt{add}(v_0, y, x)$ and $\mathsf{S}\,\mathtt{nat}(x)$.
  Conclusion$_1$: $\forall v_0, y\ \mathsf{T}\,\mathtt{add}(v_0, y, \mathtt{s}(x))$. $\square$

Lemma 10 [*add:termination:3*] $\forall x, y, z\, (\mathsf{S}\,\mathtt{add}(x, y, z) \to \mathsf{T}\,\mathtt{add}(x, y, z))$.

Proof.

Induction$_0$: $\forall x, y, z\, (\mathsf{S}\,\mathtt{add}(x, y, z) \to \mathsf{T}\,\mathtt{add}(x, y, z))$.
  Hypothesis$_1$: none.
  Conclusion$_1$: $\mathsf{T}\,\mathtt{add}(\mathtt{0}, y, y)$.
  Hypothesis$_1$: $\mathsf{T}\,\mathtt{add}(x, y, z)$ and $\mathsf{S}\,\mathtt{add}(x, y, z)$.
  Conclusion$_1$: $\mathsf{T}\,\mathtt{add}(\mathtt{s}(x), y, \mathtt{s}(z))$. $\square$