File: `ws6_soln.pr`

Lemma 1 [*list:ex1*] $\exists x$ **S** `list`(x).

Proof. **S** `list`([]) by completion. $\exists x$ **S** `list`(x). □

Lemma 2 [*lista:ex2*] $\exists x$ **S** `lista`$([x, x])$.

Proof. **S** `lista`([a]) by completion. **S** `lista`([a, a]) by completion. $\exists x$ **S** `lista`$([x, x])$. □

Lemma 3 [*lista:ff*] **S** `lista`([b]) $\to \bot$.

Proof.
Assumption$_0$: **S** `lista`([b]). **F** `lista`([b]) by completion.
Thus$_0$: **S** `lista`([b]) $\to \bot$. □

Lemma 4 [*lista:f0*] $\neg$ **S** `lista`([b]).

Proof.
Contra$_0$: **S** `lista`([b]). **F** `lista`([b]) by completion. $\bot$.
Thus$_0$: $\neg$ **S** `lista`([b]). □

Lemma 5 [*lista:f1*] **F** `lista`([a, b]).

Proof. **F** `lista`([b]). **F** `lista`([a, b]). □

Lemma 6 [*lista:gr*] $\forall xs$ (**S** `lista`$(xs) \to gr(xs)$).

Proof.
Induction$_0$: $\forall xs$ (**S** `lista`$(xs) \to gr(xs)$).
  Hypothesis$_1$: none.
  Conclusion$_1$: gr([a]).
  Hypothesis$_1$: $gr(xs)$ and **S** `lista`(xs).
  Conclusion$_1$: $gr([\mathsf{a}|xs])$. □

Lemma 7 [*lista:list*] $\forall xs$ (**S** `lista`$(xs) \to$ **S** `list`(xs)).

Proof.
Induction$_0$: $\forall xs$ (**S** `lista`$(xs) \to$ **S** `list`(xs)).
  Hypothesis$_1$: none.
  Conclusion$_1$: **S** `list`([a]).
  Hypothesis$_1$: **S** `list`(xs) and **S** `lista`(xs).
  Conclusion$_1$: **S** `list`$([\mathsf{a}|xs])$. □

Lemma 8 [*lista:term*] $\forall xs$ (**S** `lista`$(xs) \to$ **T** `lista`(xs)).

Proof.
Induction$_0$: $\forall xs$ (**S** `lista`$(xs) \to$ **T** `lista`(xs)).
  Hypothesis$_1$: none.
  Conclusion$_1$: **T** `lista`([a]).
  Hypothesis$_1$: **T** `lista`(xs) and **S** `lista`(xs).
  Conclusion$_1$: **T** `lista`$([\mathsf{a}|xs])$. □

Lemma 9 [*lista:list:term*] $\forall xs\,(\mathbf{S}\,\texttt{lista}(xs) \to \mathbf{T}\,\texttt{list}(xs))$.

Proof.
Induction$_0$: $\forall xs\,(\mathbf{S}\,\texttt{lista}(xs) \to \mathbf{T}\,\texttt{list}(xs))$.
  Hypothesis$_1$: none.
  Conclusion$_1$: $\mathbf{T}\,\texttt{list}([\mathsf{a}])$.
  Hypothesis$_1$: $\mathbf{T}\,\texttt{list}(xs)$ and $\mathbf{S}\,\texttt{lista}(xs)$.
  Conclusion$_1$: $\mathbf{T}\,\texttt{list}([\mathsf{a}|xs])$. $\square$

Lemma 10 [*list:lista:term*] $\forall xs\,(\mathbf{S}\,\texttt{list}(xs) \to \mathbf{T}\,\texttt{lista}(xs))$.

Proof.
Induction$_0$: $\forall xs\,(\mathbf{S}\,\texttt{list}(xs) \to \mathbf{T}\,\texttt{lista}(xs))$.
  Hypothesis$_1$: none.
  Conclusion$_1$: $\mathbf{T}\,\texttt{lista}([])$.
  Hypothesis$_1$: $\mathbf{T}\,\texttt{lista}(xs)$ and $\mathbf{S}\,\texttt{list}(xs)$.
  Conclusion$_1$: $\mathbf{T}\,\texttt{lista}([x|xs])$. $\square$

Lemma 11 [*lista:lista*] $\forall xs\,(\mathbf{S}\,\texttt{lista}(xs) \to \mathbf{S}\,\texttt{lista}([\mathsf{a}|xs]))$.

Lemma 12 [*lista:member:term*] $\forall xs\,(\mathbf{S}\,\texttt{lista}(xs) \to \mathbf{T}\,\texttt{member}(\mathsf{a}, xs))$.

Proof.
Induction$_0$: $\forall xs\,(\mathbf{S}\,\texttt{lista}(xs) \to \mathbf{T}\,\texttt{member}(\mathsf{a}, xs))$.
  Hypothesis$_1$: none.
  Conclusion$_1$: $\mathbf{T}\,\texttt{member}(\mathsf{a}, [\mathsf{a}])$.
  Hypothesis$_1$: $\mathbf{T}\,\texttt{member}(\mathsf{a}, xs)$ and $\mathbf{S}\,\texttt{lista}(xs)$.
  Conclusion$_1$: $\mathbf{T}\,\texttt{member}(\mathsf{a}, [\mathsf{a}|xs])$. $\square$

Lemma 13 [*lista:member:succ*] $\forall xs\,(\mathbf{S}\,\texttt{lista}(xs) \to \mathbf{S}\,\texttt{member}(\mathsf{a}, xs))$.

Proof.
Induction$_0$: $\forall xs\,(\mathbf{S}\,\texttt{lista}(xs) \to \mathbf{S}\,\texttt{member}(\mathsf{a}, xs))$.
  Hypothesis$_1$: none.
  Conclusion$_1$: $\mathbf{S}\,\texttt{member}(\mathsf{a}, [\mathsf{a}])$.
  Hypothesis$_1$: $\mathbf{S}\,\texttt{member}(\mathsf{a}, xs)$ and $\mathbf{S}\,\texttt{lista}(xs)$.
  Conclusion$_1$: $\mathbf{S}\,\texttt{member}(\mathsf{a}, [\mathsf{a}|xs])$. $\square$

Lemma 14 [*mema*] $\forall x\,(\mathbf{S}\,\texttt{member}(x, [\mathsf{a}]) \to x = \mathsf{a})$.

Proof.
Assumption$_0$: $\mathbf{S}\,\texttt{member}(x, [\mathsf{a}])$. $\mathbf{D}\,\mathbf{S}\,\texttt{member}(x, [\mathsf{a}])$ by completion.
  Case$_1$: $x = \mathsf{a}$.
  Case$_1$: $\mathbf{S}\,\texttt{member}(x, [])$. $\mathbf{D}\,\mathbf{S}\,\texttt{member}(x, [])$ by completion. $\perp$.
  Hence$_1$, in all cases: $x = \mathsf{a}$.
Thus$_0$: $\mathbf{S}\,\texttt{member}(x, [\mathsf{a}]) \to x = \mathsf{a}$. $\square$

Lemma 15 [*lista:app*] $\forall x, xs\,(\mathbf{S}\,\texttt{lista}(xs) \wedge \mathbf{S}\,\texttt{member}(x, xs) \rightarrow x = \mathsf{a})$.

Proof.

Induction$_0$: $\forall xs\,(\mathbf{S}\,\texttt{lista}(xs) \rightarrow \forall x\,(\mathbf{S}\,\texttt{member}(x, xs) \rightarrow x = \mathsf{a}))$.

Hypothesis$_1$: none.

Assumption$_2$: $\mathbf{S}\,\texttt{member}(x, [\mathsf{a}])$. $x = \mathsf{a}$ by Lemma 14 [*mema*].

Thus$_2$: $\mathbf{S}\,\texttt{member}(x, [\mathsf{a}]) \rightarrow x = \mathsf{a}$.

Conclusion$_1$: $\forall x\,(\mathbf{S}\,\texttt{member}(x, [\mathsf{a}]) \rightarrow x = \mathsf{a})$.

Hypothesis$_1$: $\forall x\,(\mathbf{S}\,\texttt{member}(x, xs) \rightarrow x = \mathsf{a})$ and $\mathbf{S}\,\texttt{lista}(xs)$.

Assumption$_2$: $\mathbf{S}\,\texttt{member}(x, [\mathsf{a}|xs])$. $\mathbf{D}\,\mathbf{S}\,\texttt{member}(x, [\mathsf{a}|xs])$ by completion.

Case$_3$: $x = \mathsf{a}$.

Case$_3$: $\mathbf{S}\,\texttt{member}(x, xs)$.

Hence$_3$, in all cases: $x = \mathsf{a}$.

Thus$_2$: $\mathbf{S}\,\texttt{member}(x, [\mathsf{a}|xs]) \rightarrow x = \mathsf{a}$.

Conclusion$_1$: $\forall x\,(\mathbf{S}\,\texttt{member}(x, [\mathsf{a}|xs]) \rightarrow x = \mathsf{a})$. $\square$