File: `ws2.pr`

Lemma 1 [*ax0*] $\texttt{a} = \texttt{a}$.

Lemma 2 [*ax1*] $gr(\texttt{a})$.

Lemma 3 [*ax2*] $\texttt{f}(\texttt{a}) \neq \texttt{f}(\texttt{b})$.

Lemma 4 [*ax3*] $\texttt{p} \vee \neg\texttt{p}$.

Lemma 5 [*ax4*] $\neg\texttt{a} = \texttt{b}$.

Lemma 6 [*ax5*] $\texttt{p} \leftrightarrow \texttt{p}$.

Lemma 7 [*p110:elim:egal*] $\forall x\, (x = \texttt{a} \wedge \texttt{p}(x)) \to \texttt{p}(\texttt{a})$.

Lemma 8 [*p111:egal:reflex*] $\forall x\, x = x$.

Lemma 9 [*p111:egal:sym*] $\forall x_1, x_2\, (x_1 = x_2 \to x_2 = x_1)$.

Lemma 10 [*p111:egal:trans*] $\forall x_1, x_2, x_3\, (x_1 = x_2 \wedge x_2 = x_3 \to x_1 = x_3)$.

Lemma 11 [*p112:elim:qqsoit0*] $\forall x\, \texttt{p}(x) \to \texttt{p}(\texttt{a})$.
Proof. $\forall x\, \texttt{p}(x) \to \texttt{p}(\texttt{a})$ by **GAP**. $\square$

Lemma 12 [*p112:elim:qqsoit1*] $\forall x\, \texttt{p}(x) \to \forall y\, \texttt{p}(y)$.
Proof. $\forall x\, \texttt{p}(x) \to \forall y\, \texttt{p}(y)$ by **GAP**. $\square$

Lemma 13 [*p112:elim:qqsoit2*] $\forall x\, \texttt{p}(x) \to \forall y\, \texttt{p}(\texttt{f}(y))$.
Proof. $\forall x\, \texttt{p}(x) \to \forall y\, \texttt{p}(\texttt{f}(y))$ by **GAP**. $\square$

Lemma 14 [*p113:intro:qqsoit0*] $\forall x\, \texttt{p}(x) \to \forall y\, \texttt{p}(y)$.
Proof.
Assumption$_0$: $\forall x\, \texttt{p}(x)$. $\texttt{p}(y)$.
Thus$_0$: $\forall x\, \texttt{p}(x) \to \forall y\, \texttt{p}(y)$. $\square$

Lemma 15 [*p114:0*] $\forall x\, (\texttt{p}(x) \to \texttt{q}(x)) \wedge \forall x\, \texttt{p}(x) \to \forall x\, \texttt{q}(x)$.
Proof.
Assumption$_0$: $\forall x\, (\texttt{p}(x) \to \texttt{q}(x)) \wedge \forall x\, \texttt{p}(x)$. $\texttt{p}(x_0) \to \texttt{q}(x_0)$. $\texttt{p}(x_0)$. $\texttt{q}(x_0)$.
Thus$_0$: $\forall x\, (\texttt{p}(x) \to \texttt{q}(x)) \wedge \forall x\, \texttt{p}(x) \to \forall x\, \texttt{q}(x)$. $\square$

Lemma 16 [*p114:1*] $\texttt{p}(\texttt{a}) \wedge \forall x\, (\texttt{p}(x) \to \neg\texttt{q}(x)) \to \neg\texttt{q}(\texttt{a})$.
Proof. $\texttt{p}(\texttt{a}) \wedge \forall x\, (\texttt{p}(x) \to \neg\texttt{q}(x)) \to \neg\texttt{q}(\texttt{a})$ by **GAP**. $\square$

Lemma 17 [*p117:0*] $\forall x\, \texttt{p}(x) \to \exists x\, \texttt{p}(x)$.
Proof.
Assumption$_0$: $\forall x\, \texttt{p}(x)$. $\texttt{p}(x)$.
Thus$_0$: $\forall x\, \texttt{p}(x) \to \exists x\, \texttt{p}(x)$. $\square$

Lemma 18 [*p117:1*] $\forall x\, (\mathtt{p}(x) \rightarrow \mathtt{q}(x)) \wedge \exists x\, \mathtt{p}(x) \rightarrow \exists x\, \mathtt{q}(x)$.

Proof. $\forall x\, (\mathtt{p}(x) \rightarrow \mathtt{q}(x)) \wedge \exists x\, \mathtt{p}(x) \rightarrow \exists x\, \mathtt{q}(x)$ by **GAP**. □

Lemma 19 [*p118:1*] $\forall x\, (\mathtt{q}(x) \rightarrow \mathtt{r}(x)) \wedge \exists x\, (\mathtt{p}(x) \wedge \mathtt{q}(x)) \rightarrow \exists x\, (\mathtt{p}(x) \wedge \mathtt{r}(x))$.

Proof. $\forall x\, (\mathtt{q}(x) \rightarrow \mathtt{r}(x)) \wedge \exists x\, (\mathtt{p}(x) \wedge \mathtt{q}(x)) \rightarrow \exists x\, (\mathtt{p}(x) \wedge \mathtt{r}(x))$ by **GAP**. □

Lemma 20 [*p119:0*] $\exists x\, \mathtt{p}(x) \wedge \forall x, y\, (\mathtt{p}(x) \rightarrow \mathtt{q}(y)) \rightarrow \forall y\, \mathtt{q}(y)$.

Proof. $\exists x\, \mathtt{p}(x) \wedge \forall x, y\, (\mathtt{p}(x) \rightarrow \mathtt{q}(y)) \rightarrow \forall y\, \mathtt{q}(y)$ by **GAP**. □

Lemma 21 [*p122:1*] $\neg \forall x\, \mathtt{p}(x) \rightarrow \exists x\, \neg \mathtt{p}(x)$.

Proof. $\neg \forall x\, \mathtt{p}(x) \rightarrow \exists x\, \neg \mathtt{p}(x)$ by **GAP**. □

Lemma 22 [*p123:1*] $\exists x\, \neg \mathtt{p}(x) \rightarrow \neg \forall x\, \mathtt{p}(x)$.

Proof. $\exists x\, \neg \mathtt{p}(x) \rightarrow \neg \forall x\, \mathtt{p}(x)$ by **GAP**. □

Lemma 23 [*p124:0*] $\forall x\, \mathtt{p}(x) \wedge \mathtt{q} \rightarrow \forall x\, (\mathtt{p}(x) \wedge \mathtt{q})$.

Proof. $\forall x\, \mathtt{p}(x) \wedge \mathtt{q} \rightarrow \forall x\, (\mathtt{p}(x) \wedge \mathtt{q})$ by **GAP**. □

Lemma 24 [*p124:1*] $\exists x\, \mathtt{p}(x) \vee \exists x\, \mathtt{q}(x) \rightarrow \exists x\, (\mathtt{p}(x) \vee \mathtt{q}(x))$.

Proof. $\exists x\, \mathtt{p}(x) \vee \exists x\, \mathtt{q}(x) \rightarrow \exists x\, (\mathtt{p}(x) \vee \mathtt{q}(x))$ by **GAP**. □

Lemma 25 [*p125:0*] $\exists x\, (\mathtt{p}(x) \vee \mathtt{q}(x)) \rightarrow \exists x\, \mathtt{p}(x) \vee \exists x\, \mathtt{q}(x)$.

Proof. $\exists x\, (\mathtt{p}(x) \vee \mathtt{q}(x)) \rightarrow \exists x\, \mathtt{p}(x) \vee \exists x\, \mathtt{q}(x)$ by **GAP**. □

Lemma 26 [*p125:1*] $\exists x, y\, \mathtt{p}(x, y) \rightarrow \exists y, x\, \mathtt{p}(x, y)$.

Proof. $\exists x, y\, \mathtt{p}(x, y) \rightarrow \exists y, x\, \mathtt{p}(x, y)$ by **GAP**. □