File: `ws2_soln.pr`

Lemma 1 [*ax0*] $\mathtt{a} = \mathtt{a}$.

Lemma 2 [*ax1*] $gr(\mathtt{a})$.

Lemma 3 [*ax2*] $\mathtt{f}(\mathtt{a}) \neq \mathtt{f}(\mathtt{b})$.

Lemma 4 [*ax3*] $\mathtt{p} \vee \neg\mathtt{p}$.

Lemma 5 [*ax4*] $\neg\mathtt{a} = \mathtt{b}$.

Lemma 6 [*ax5*] $\mathtt{p} \leftrightarrow \mathtt{p}$.

Lemma 7 [*p110:elim:egal*] $\forall x\,(x = \mathtt{a} \wedge \mathtt{p}(x)) \to \mathtt{p}(\mathtt{a})$.

Lemma 8 [*p111:egal:reflex*] $\forall x\, x = x$.

Lemma 9 [*p111:egal:sym*] $\forall x_1, x_2\,(x_1 = x_2 \to x_2 = x_1)$.

Lemma 10 [*p111:egal:trans*] $\forall x_1, x_2, x_3\,(x_1 = x_2 \wedge x_2 = x_3 \to x_1 = x_3)$.

Lemma 11 [*p112:elim:qqsoit0*] $\forall x\,\mathtt{p}(x) \to \mathtt{p}(\mathtt{a})$.
Proof.
Assumption$_0$: $\forall x\,\mathtt{p}(x)$.
Thus$_0$: $\forall x\,\mathtt{p}(x) \to \mathtt{p}(\mathtt{a})$. □

Lemma 12 [*p112:elim:qqsoit1*] $\forall x\,\mathtt{p}(x) \to \forall y\,\mathtt{p}(y)$.
Proof.
Assumption$_0$: $\forall x\,\mathtt{p}(x)$.
Thus$_0$: $\forall x\,\mathtt{p}(x) \to \forall y\,\mathtt{p}(y)$. □

Lemma 13 [*p112:elim:qqsoit2*] $\forall x\,\mathtt{p}(x) \to \forall y\,\mathtt{p}(\mathtt{f}(y))$.
Proof.
Assumption$_0$: $\forall x\,\mathtt{p}(x)$.
Thus$_0$: $\forall x\,\mathtt{p}(x) \to \forall y\,\mathtt{p}(\mathtt{f}(y))$. □

Lemma 14 [*p113:intro:qqsoit0*] $\forall x\,\mathtt{p}(x) \to \forall y\,\mathtt{p}(y)$.
Proof.
Assumption$_0$: $\forall x\,\mathtt{p}(x)$. $\mathtt{p}(y)$.
Thus$_0$: $\forall x\,\mathtt{p}(x) \to \forall y\,\mathtt{p}(y)$. □

Lemma 15 [*p114:0*] $\forall x\,(\mathtt{p}(x) \to \mathtt{q}(x)) \wedge \forall x\,\mathtt{p}(x) \to \forall x\,\mathtt{q}(x)$.
Proof.
Assumption$_0$: $\forall x\,(\mathtt{p}(x) \to \mathtt{q}(x)) \wedge \forall x\,\mathtt{p}(x)$. $\mathtt{p}(x_0) \to \mathtt{q}(x_0)$. $\mathtt{p}(x_0)$. $\mathtt{q}(x_0)$.
Thus$_0$: $\forall x\,(\mathtt{p}(x) \to \mathtt{q}(x)) \wedge \forall x\,\mathtt{p}(x) \to \forall x\,\mathtt{q}(x)$. □

Lemma 16 [*p114:1*] $\mathtt{p}(\mathtt{a}) \wedge \forall x\,(\mathtt{p}(x) \to \neg\mathtt{q}(x)) \to \neg\mathtt{q}(\mathtt{a})$.
Proof.
Assumption$_0$: $\mathtt{p}(\mathtt{a}) \wedge \forall x\,(\mathtt{p}(x) \to \neg\mathtt{q}(x))$. $\mathtt{p}(\mathtt{a}) \to \neg\mathtt{q}(\mathtt{a})$.
Thus$_0$: $\mathtt{p}(\mathtt{a}) \wedge \forall x\,(\mathtt{p}(x) \to \neg\mathtt{q}(x)) \to \neg\mathtt{q}(\mathtt{a})$. □

Lemma 17 [*p117:0*] $\forall x\, \mathtt{p}(x) \to \exists x\, \mathtt{p}(x)$.

Proof.
Assumption$_0$: $\forall x\, \mathtt{p}(x)$. $\mathtt{p}(x)$.
Thus$_0$: $\forall x\, \mathtt{p}(x) \to \exists x\, \mathtt{p}(x)$. □

Lemma 18 [*p117:1*] $\forall x\, (\mathtt{p}(x) \to \mathtt{q}(x)) \wedge \exists x\, \mathtt{p}(x) \to \exists x\, \mathtt{q}(x)$.

Proof.
Assumption$_0$: $\forall x\, (\mathtt{p}(x) \to \mathtt{q}(x)) \wedge \exists x\, \mathtt{p}(x)$.
  Let$_1$ x_0 with $\mathtt{p}(x_0)$. $\mathtt{p}(x_0) \to \mathtt{q}(x_0)$. $\mathtt{q}(x_0)$.
  Thus$_1$: $\exists x\, \mathtt{q}(x)$.
Thus$_0$: $\forall x\, (\mathtt{p}(x) \to \mathtt{q}(x)) \wedge \exists x\, \mathtt{p}(x) \to \exists x\, \mathtt{q}(x)$. □

Lemma 19 [*p118:0*] $\forall x\, (\mathtt{p}(x) \to \mathtt{q}(x)) \wedge \exists x\, \mathtt{p}(x) \to \exists x\, \mathtt{q}(x)$.

Proof.
Assumption$_0$: $\forall x\, (\mathtt{p}(x) \to \mathtt{q}(x)) \wedge \exists x\, \mathtt{p}(x)$.
  Let$_1$ x_0 with $\mathtt{p}(x_0)$. $\mathtt{p}(x_0) \to \mathtt{q}(x_0)$. $\mathtt{q}(x_0)$.
  Thus$_1$: $\exists x\, \mathtt{q}(x)$.
Thus$_0$: $\forall x\, (\mathtt{p}(x) \to \mathtt{q}(x)) \wedge \exists x\, \mathtt{p}(x) \to \exists x\, \mathtt{q}(x)$. □

Lemma 20 [*p118:1*] $\forall x\, (\mathtt{q}(x) \to \mathtt{r}(x)) \wedge \exists x\, (\mathtt{p}(x) \wedge \mathtt{q}(x)) \to \exists x\, (\mathtt{p}(x) \wedge \mathtt{r}(x))$.

Proof.
Assumption$_0$: $\forall x\, (\mathtt{q}(x) \to \mathtt{r}(x)) \wedge \exists x\, (\mathtt{p}(x) \wedge \mathtt{q}(x))$.
  Let$_1$ x_0 with $\mathtt{p}(x_0) \wedge \mathtt{q}(x_0)$. $\mathtt{q}(x_0) \to \mathtt{r}(x_0)$. $\mathtt{r}(x_0)$. $\mathtt{p}(x_0) \wedge \mathtt{r}(x_0)$.
  Thus$_1$: $\exists x\, (\mathtt{p}(x) \wedge \mathtt{r}(x))$.
Thus$_0$: $\forall x\, (\mathtt{q}(x) \to \mathtt{r}(x)) \wedge \exists x\, (\mathtt{p}(x) \wedge \mathtt{q}(x)) \to \exists x\, (\mathtt{p}(x) \wedge \mathtt{r}(x))$. □

Lemma 21 [*p119:0*] $\exists x\, \mathtt{p}(x) \wedge \forall x, y\, (\mathtt{p}(x) \to \mathtt{q}(y)) \to \forall y\, \mathtt{q}(y)$.

Proof.
Assumption$_0$: $\exists x\, \mathtt{p}(x) \wedge \forall x, y\, (\mathtt{p}(x) \to \mathtt{q}(y))$.
  Let$_1$ x_0 with $\mathtt{p}(x_0)$. $\mathtt{p}(x_0) \to \mathtt{q}(y_0)$. $\mathtt{q}(y_0)$.
  Thus$_1$: $\mathtt{q}(y_0)$.
Thus$_0$: $\exists x\, \mathtt{p}(x) \wedge \forall x, y\, (\mathtt{p}(x) \to \mathtt{q}(y)) \to \forall y\, \mathtt{q}(y)$. □

Lemma 22 [*p122:1*] $\neg\forall x\, \mathtt{p}(x) \to \exists x\, \neg\mathtt{p}(x)$.

Proof.
Assumption$_0$: $\neg\forall x\, \mathtt{p}(x)$.
  Indirect$_1$: $\neg\exists x\, \neg\mathtt{p}(x)$.
    Indirect$_2$: $\neg\mathtt{p}(x_0)$. $\exists x\, \neg\mathtt{p}(x)$. $\bot$.
    Thus$_2$: $\mathtt{p}(x_0)$. $\mathtt{p}(x_0)$. $\forall x\, \mathtt{p}(x)$. $\bot$.
  Thus$_1$: $\exists x\, \neg\mathtt{p}(x)$.
Thus$_0$: $\neg\forall x\, \mathtt{p}(x) \to \exists x\, \neg\mathtt{p}(x)$. □

Lemma 23 [*p123:1*] $\exists x\, \neg \mathtt{p}(x) \to \neg \forall x\, \mathtt{p}(x)$.

Proof.
Assumption$_0$: $\exists x\, \neg \mathtt{p}(x)$.
  Contra$_1$: $\forall x\, \mathtt{p}(x)$.
    Let$_2$ x_0 with $\neg \mathtt{p}(x_0)$. $\mathtt{p}(x_0)$. $\bot$.
    Thus$_2$: $\bot$. $\bot$.
  Thus$_1$: $\neg \forall x\, \mathtt{p}(x)$.
Thus$_0$: $\exists x\, \neg \mathtt{p}(x) \to \neg \forall x\, \mathtt{p}(x)$. $\square$

Lemma 24 [*p124:0*] $\forall x\, \mathtt{p}(x) \wedge \mathtt{q} \to \forall x\, (\mathtt{p}(x) \wedge \mathtt{q})$.

Proof.
Assumption$_0$: $\forall x\, \mathtt{p}(x) \wedge \mathtt{q}$. $\mathtt{p}(x_0)$. $\mathtt{p}(x_0) \wedge \mathtt{q}$.
Thus$_0$: $\forall x\, \mathtt{p}(x) \wedge \mathtt{q} \to \forall x\, (\mathtt{p}(x) \wedge \mathtt{q})$. $\square$

Lemma 25 [*p124:1*] $\exists x\, \mathtt{p}(x) \vee \exists x\, \mathtt{q}(x) \to \exists x\, (\mathtt{p}(x) \vee \mathtt{q}(x))$.

Proof.
Assumption$_0$: $\exists x\, \mathtt{p}(x) \vee \exists x\, \mathtt{q}(x)$.
  Case$_1$: $\exists x\, \mathtt{p}(x)$.
    Let$_2$ x_0 with $\mathtt{p}(x_0)$. $\mathtt{p}(x_0) \vee \mathtt{q}(x_0)$.
    Thus$_2$: $\exists x\, (\mathtt{p}(x) \vee \mathtt{q}(x))$.
  Case$_1$: $\exists x\, \mathtt{q}(x)$.
    Let$_2$ x_0 with $\mathtt{q}(x_0)$. $\mathtt{p}(x_0) \vee \mathtt{q}(x_0)$.
    Thus$_2$: $\exists x\, (\mathtt{p}(x) \vee \mathtt{q}(x))$.
  Hence$_1$, in all cases: $\exists x\, (\mathtt{p}(x) \vee \mathtt{q}(x))$.
Thus$_0$: $\exists x\, \mathtt{p}(x) \vee \exists x\, \mathtt{q}(x) \to \exists x\, (\mathtt{p}(x) \vee \mathtt{q}(x))$. $\square$

Lemma 26 [*p125:0*] $\exists x\, (\mathtt{p}(x) \vee \mathtt{q}(x)) \to \exists x\, \mathtt{p}(x) \vee \exists x\, \mathtt{q}(x)$.

Proof.
Assumption$_0$: $\exists x\, (\mathtt{p}(x) \vee \mathtt{q}(x))$.
  Let$_1$ x_0 with $\mathtt{p}(x_0) \vee \mathtt{q}(x_0)$.
    Case$_2$: $\mathtt{p}(x_0)$. $\exists x\, \mathtt{p}(x) \vee \exists x\, \mathtt{q}(x)$.
    Case$_2$: $\mathtt{q}(x_0)$. $\exists x\, \mathtt{p}(x) \vee \exists x\, \mathtt{q}(x)$.
    Hence$_2$, in all cases: $\exists x\, \mathtt{p}(x) \vee \exists x\, \mathtt{q}(x)$.
  Thus$_1$: $\exists x\, \mathtt{p}(x) \vee \exists x\, \mathtt{q}(x)$.
Thus$_0$: $\exists x\, (\mathtt{p}(x) \vee \mathtt{q}(x)) \to \exists x\, \mathtt{p}(x) \vee \exists x\, \mathtt{q}(x)$. $\square$

Lemma 27 [*p125:1*] $\exists x, y\, \mathtt{p}(x, y) \to \exists y, x\, \mathtt{p}(x, y)$.

Proof.
Assumption$_0$: $\exists x, y\, \mathtt{p}(x, y)$.
  Let$_1$ x_0, y_0 with $\mathtt{p}(x_0, y_0)$.
  Thus$_1$: $\exists y, x\, \mathtt{p}(x, y)$.
Thus$_0$: $\exists x, y\, \mathtt{p}(x, y) \to \exists y, x\, \mathtt{p}(x, y)$. $\square$