

# Applying Spectral Methods To Mizer

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We are seeking to improve and implement the Mizer package:

Scott, F., Blanchard, J.L., Andersen, K.H., 2014a. Mizer: an R package for multispecies, trait-based and community size spectrum ecological modelling. *Methods Ecol. Evol.* 5, 1121-1125.

## A North Sea Model Runnig Via FFT

Before we go into details, let us show some code that uses sections modified from the Mizer package, upgraded by the inclusion of integration methods exploiting the fast fourier transform (fft).

```
setwd("C:/Users/me/Dropbox/Job/YorkJob/mizer")

source('./R/MizerParams-classNewSlots.R')
#source('./R/MizerParams-class.R')
source('./R/MizerSim-class.R')
#source('./R/MizerSim-classNewSlots.R')
#source('./R/project_methodsFFT.R')
  source('./R/project_methods.R')
source('./R/selectivity_funcs.R')
source('./R/summary_methods.R')
source('./R/wrapper_functions.R')
source('./R/plots.R')
source('./R/project.R')
library(ggplot2)
library(grid)
library(methods)
library(plyr)
library(reshape2)

params_data <- read.csv("./vignettes/NS_species_params.csv")
inter <- read.csv("./vignettes/inter.csv", row.names=1)
inter <- as(inter, "matrix")
params <- MizerParams(params_data, interaction = inter, no_w = 100)
```

## Note: No sel\_func column in species data frame. Setting selectivity to be 'knife\_edge' for all species.

## Note: No knife\_edge\_size column in species data frame. Setting knife edge selectivity equal to w\_inf.

## Note: No h column in species data frame so using f0 and k\_vb to calculate it.

## Note: No gamma column in species data frame so using f0, h, beta, sigma, lambda and kappa to calculate it.

## Note: No z0 column in species data frame so using  $z0 = z0pre * w\_inf \wedge z0exp$ .

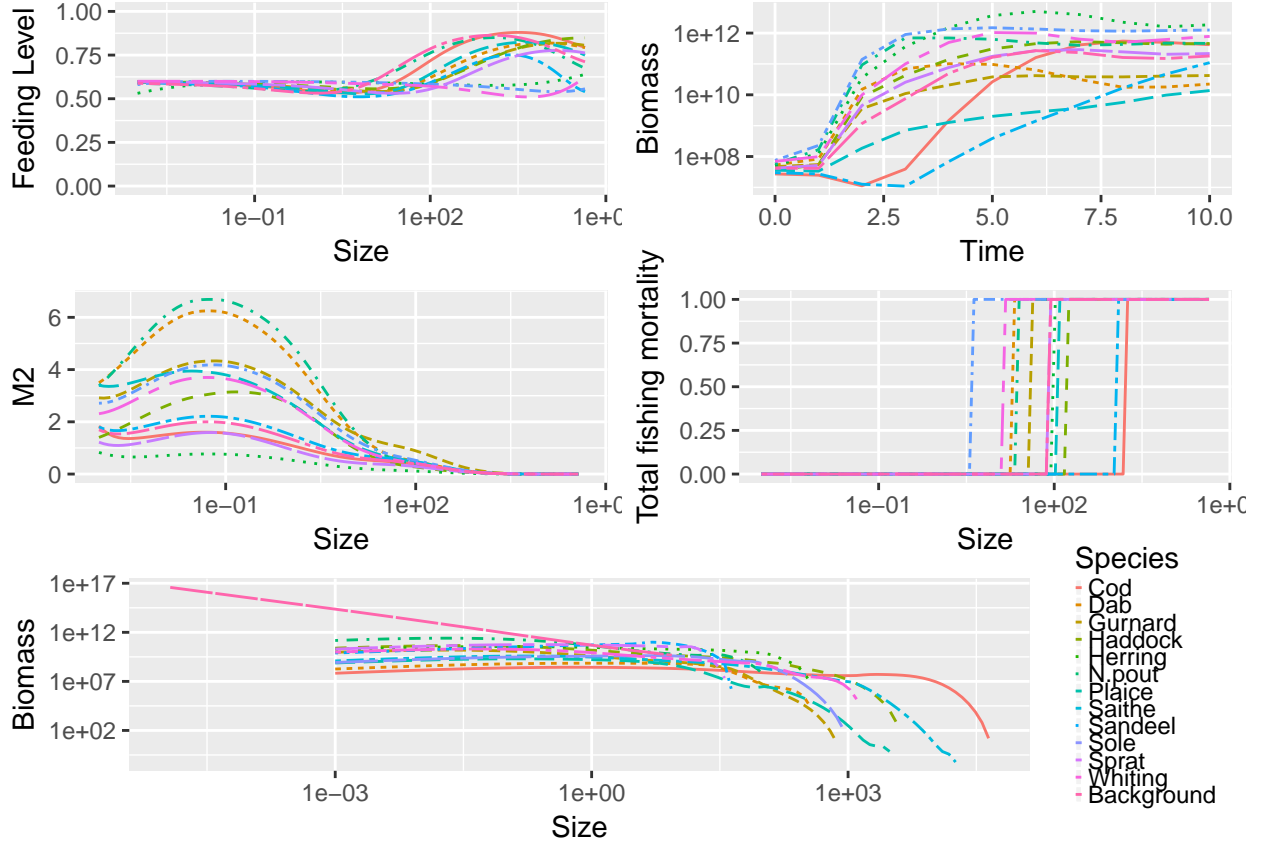
## Note: No ks column in species data frame so using  $ks = h * 0.2$ .

```
source('./R/project_methodsFFT2usingSlots.R')

sim <- project(params, effort = 1, t_max = 10, dt = 0.1, t_save = 1)

plot(sim)
```

## Warning: Removed 12 rows containing missing values (geom\_path).



## Introduction

Now we shall discuss the details. We wish to evolve the McKendrik-von Foerster equation PDE

$$\frac{\partial N_i(w)}{\partial t} = -\frac{\partial [g_i(w)N_i(w)]}{\partial w} - \mu_i(w)N_i(w) \quad (1)$$

There is also include an extra boundary condition about new fish (eggs) of size  $w_0$  appearing over time due to reproduction, but we shall get into that later.

The point is that the above PDE (Equation 1) describes the time evolution of the individual-density function,  $N_i(w)$ , which is such that  $N_i(w)dw$  equals the number of individuals with a weight in the interval  $[w, w + dw]$ , when  $dw$  is small.

We can evolve this PDE numerically by repeatedly evaluating the right hand side of the PDE and then using this information into integrate-forward  $\frac{\partial N_i(w)}{\partial t}$ . This can be done by repeatedly evaluating the right hand side of the PDE, and then applying some method (e.g., Euler's method, or the method of lines), to approximate the time evolution of  $N_i(w)$ .

Since we have to evaluate the right hand side of Equation (1) many times in order to evolve our PDE, it is important to be able to do it quickly. The most time consuming parts of evaluating the right hand side of our PDE involve evaluating growth rate  $g_i(w)$  and mortality rate  $\mu_i(w)$ , which depend upon convolution integrals involving the entire size distribution.

The central point of this document is that we have determined/implemented ways of evaluating the integrals underlying  $g_i(w)$  and  $\mu_i(w)$  using spectral methods, which can evaluate the integrals many orders of magnitude faster than standard approaches (e.g., using Riemann sums). This is possible because the integrals in question are convolution integrals, and they can be evaluated quickly using spectral methods exploiting the fast Fourier transform (fft).

## The Integrals we want to evaluate faster

The growth rate  $g_i(w)$  involves many terms concerning search rates, feeding levels, partitions of energy into growth, reproduction and metabolism, but the computational bottleneck in evaluating the growth rate, at a particular time, is the determination of the available energy:

$$E_{a,i}(w) = \int_0^\infty \left( N_R(w_p) + \sum_{j=1}^s \theta_{i,j} N_j(w_p) \right) \Phi_i \left( \frac{w_p}{w} \right) w_p dw_p, \quad (2)$$

which is the amount of available food that predator  $w$  could get, over all species, resources, and prey masses.

The mortality rate  $\mu_i(w)$  of a fish of species  $i$  and mass  $w$  equals

$$\mu_i(w) = \mu_{p,i}(w) + \mu_{s,i}(w) + \mu_{b,i}(w) + \mu_{f,i}(w),$$

where  $\mu_{p,i}(w)$  is the predation mortality, and  $\mu_{s,i}(w)$  is the starvation mortality, and  $\mu_{b,i}(w)$  is the background mortality, and  $\mu_{f,i}(w)$  is the fishing mortality. The computational bottleneck to determining the predation mortality.

$$\mu_{p,i}(w) = \sum_{j=1}^s \int_0^\infty \Phi_j \left( \frac{w}{w_p} \right) (1 - f_j(w_p)) \gamma_j(w_p)^{p^*} \theta_{j,i} N_j(w_p) dw_p,$$

(given here for a prey of mass  $w$  of species  $i$ ). These parts of the predation mortality can be done quickly via spectral methods.

## FEEDING KERNELS

Let  $\{1, \dots, s\}$  denote the set of species.

The feeding Kernel  $\Phi_i$  measures amount of preference that a predator has weight  $W_p$  has for a prey of weight  $w$  is

$$\Phi_i \left( \frac{w_p}{w} \right) = \exp \left[ \frac{- \left( \ln \left( \frac{w}{w_p} \right) - \ln(\beta_i^*) \right)^2}{2\sigma_i^2} \right].$$

We shall normalize mass by dividing by the egg size  $w_0$ , and logs, to re-represent this information in the ‘x-space’ where  $x = \frac{\log(w)}{\log(w_0)}$ , and  $y = \frac{\log(w_p)}{\log(w_0)}$ . Before we consider the feeding kernel in these terms, let us note, that for any real number  $v$  we have:

$$\Phi_i(e^v) = \exp \left[ \frac{- (\ln(e^{-v}) - \ln(\beta_i^*))^2}{2\sigma_i^2} \right] = \exp \left[ \frac{-(v + \beta_i)^2}{2\sigma_i^2} \right],$$

where  $\beta_i = \ln(\beta_i^*)$ , and  $\Phi_i(e^v)$  is concentrated about  $-\beta_i$ .

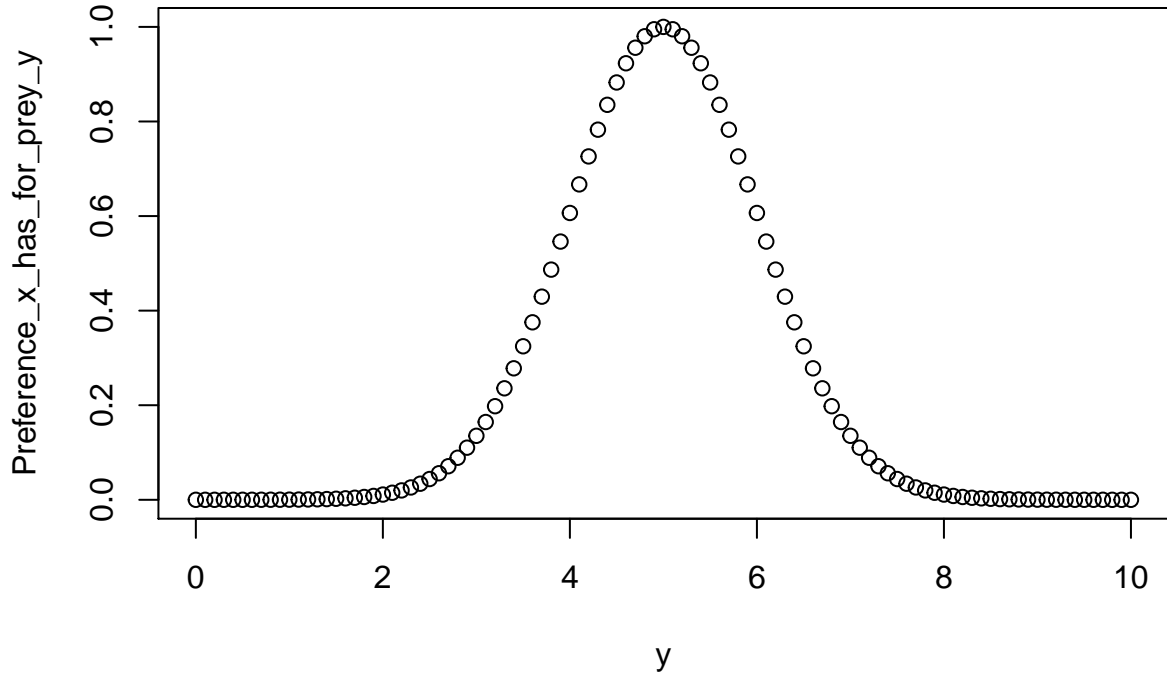
Let us return to consideration of  $\Phi_i \left( \frac{w_p}{w} \right)$ . By making our substitutions, we can rewrite  $\Phi_i$  in terms of  $x$  and  $y$  as

$$\Phi_i\left(\frac{w_p}{w}\right) = \Phi_i\left(\frac{w_0 e^y}{w_0 e^x}\right) = \Phi_i(e^{y-x}) = \exp\left[\frac{-(y-x+\beta_i)^2}{2\sigma_i^2}\right]$$

To illustrate, let us consider a practical example where the width of the feeding distribution is  $\sigma_i = 1$ , the (log-space) predator size is  $x = 8$ , and the log of the preferred predator-prey mass ratio is  $\beta_i = \ln(\beta_i^*) = 3$ .

Below we plot the resulting feeding kernel, the horizontal axis gives the (log-space) prey size, and the vertical axis measures the amount of preference that our size  $x = 8$  predators have for this prey:

```
sigma <- 1
x <- 8
beta <- 3
dy <- 0.1
y <- seq(0, 10, by = dy)
Preference_x_has_for_pre_y <- exp(-((y-x+beta)^2)/(2*sigma^2))
plot(y, Preference_x_has_for_pre_y)
```



Because the size of the predator is  $x = 8$ , its most preferred prey size is

$$x - \beta_i = 8 - 3$$

, which is the  $y$  value at which the feeding preference function plotted above is concentrated at.

## Feeding Kernels in log space

For each real number  $v$  let us define  $\phi_i(v) = \Phi_i(e^v)$ . Essentially  $\phi_i$  is the feeding kernel, when the system has been transformed into log-space, and the convolution integrals we are concerned with involve them. From

the subsequent full line formula we have

$$\phi_i(v) = \Phi_i(e^v) = \exp \left[ \frac{-(\ln(e^{-v}) - \ln(\beta_i^*))^2}{2\sigma_i^2} \right] = \exp \left[ \frac{-(v + \beta_i)^2}{2\sigma_i^2} \right],$$

## How feeding kernels are handled in mizer and fft

The support of a function is the set of real inputs for which it has non-zero outputs. Our function  $\phi_i(v)$  is a Gaussian, and so technically it has an infinite support. However, in our practical use of this equation, in mizer, and also, for the sake of spectral methods, we have instead made small artificial changes to the type of feeding kernel we are using. To put it bluntly, in our code we assume  $\phi_i(v)$  returns the normal output for each real  $v$  in the interval  $[-\beta_i - 3\sigma_i, 0]$ , but we also suppose that our function returns a zero when  $v$  lies outside of this interval.

The assumption that  $v < -\beta_i - 3\sigma_i$  implies  $\phi_i(v) = 0$  is a realistic numerical approximation, since the overwhelming majority of the area under a Gaussian lies within three standard deviations  $\sigma_j$  of the mean  $-\beta_i$ , so for  $v < -\beta_i - 3\sigma_i$  the value of  $\phi_i(v)$  will be so small that it is negligible, with respect to the numerical methods we apply.

The assumption that  $v > 0$  implies  $\phi_i(v) = 0$  corresponds to the assumption that no predator would prefer to eat prey larger than themselves. This not a mild approximation, like in the previous assumption, this the padding part  $\phi_i$  with zeroes - adding zeros into parts of the function that would not even be referenced in ‘ecologically realistic’ scenarios where the chances of predators eating prey larger than themselves is negligible. So basically we take the definition of

$$\phi_i(v) = \Phi_i(e^v) = \exp \left[ \frac{-(v + \beta_i)^2}{2\sigma_i^2} \right],$$

and we suppose we are always dealing with ‘ecologically realistic’ scenarios where predators never predators never prefer to eat prey bigger than themselves, so  $v > 0 \Rightarrow \phi_i(v) = 0$ .

Under this **ecologically inspired' restriction, and our numerically convenient truncation decision'** we write

$$\text{sup}(\phi_i) = \{v \in \mathbb{R} : \phi_i(v) \neq 0\} = [-\beta_i - 3\sigma_i, 0],$$

Note also that  $\phi_i(v)$  is concentrated at  $-\beta_i$ .

Here the support of a function is the set of inputs that give non-zero outputs.

So essentially we are saying that, for the sake of our numerical calculations (fft's and such) we shall assume that  $\phi_i$  takes the form of a ‘truncated’ Gaussian feeding kernel  $\phi_j(v)$  such that  $\forall v \in \mathbb{R}$  we have

$$\Phi_j(e^v) = \phi_j(v) = \begin{cases} \exp \left( \frac{-(v + \beta_j)^2}{2\sigma_j^2} \right) & \text{if } v \in [-\beta_j - 3\sigma_j, 0] \\ 0 & \text{otherwise} \end{cases}$$

In ecologically realistic scenarios the effect of this truncation should be negligible.

We shall use feeding kernel  $\phi_i$  when we talk about the energy integral.

## Feeding kernel reversal for energy integral

The type of convolution integrals appearing in the mortality integrals are very similar to those appearing in the energy integrals we shall encounter. In the mortality integrals, the kernel  $\phi_i(v)$  appears naturally in the convolution integrals. However, for the available energy integrals, the kernel  $\psi_i$ , (which is the reflection of  $\phi_i$  about zero), appears in the convolution more naturally. Formally we give  $\psi_i(v)$  the definition that  $\psi_j(v) = \phi_j(-v)$ , for every real  $v$ . It follows that  $\psi_j(v)$  is concentrated at  $\beta_j$ , and has support

$$\text{supp}(\psi_i) = \{v \in \mathbb{R} : \psi_i(v) \neq 0\} = [0, \beta_i + 3\sigma_i],$$

and general form

$$\psi_i(v) = \phi_i(-v) = \exp \left[ \frac{-(v - \beta_i)^2}{2\sigma_i^2} \right].$$

## MORTALITY INTEGRALS

### Conversion of Mortality Equations

The predation mortality for a prey of mass  $w$  of species  $i$  is

$$\mu_{p,i}(w) = \sum_{j=1}^s \int_0^\infty \Phi_j \left( \frac{w}{w_p} \right) (1 - f_j(w_p)) \gamma_j(w_p)^{p^*} \theta_{j,i} N_j(w_p) dw_p$$

In this case, where we are discussing predation mortality, for convenience, we shall let  $w$  denote the mass of the prey, while  $w_p$  denotes the mass of the predator (note, we use the opposite convention when we discuss available energy, but at least our convention means integration always happens over  $w_p$ ).

Note that predation mortality, for an individual of species  $i$ , with mass  $w \in [w_0, W_i]$ , will be given by the sum of the predation rates that the various species  $j \in \{1, \dots, s\}$ , will have upon our individual, and so we will have

$$\mu_{p,i}(w) = \sum_{j=1}^s \mathbb{J}_{j,i}(w),$$

where  $\mathbb{J}_{j,i}(w)$  denotes the predation mortality of morality of a prey of mass  $w$  being predated upon by an individual of species  $j$ .

So now we seek to evaluate the ‘cross predation term’:

$$\mathbb{J}_{j,i}(w) = \int_0^\infty \Phi_j \left( \frac{w}{w_p} \right) (1 - F_j(w_p)) \gamma_j(w_p)^{p^*} \theta_{j,i} N_j(w_p) dw_p$$

Let

$$y' = \ln \left( \frac{w_p}{w_0} \right)$$

and let

$$x = \ln \left( \frac{w}{w_0} \right),$$

Under these substitutions it makes sense to let  $\mathbb{I}_{j,i}$  be such that  $\mathbb{I}_{j,i}(x) = \mathbb{J}_{j,i}(w_0 e^x)$ . Using substitutions we may rewrite our expression for  $\mathbb{I}_{j,i}(w)$  as

$$\mathbb{J}_{j,i}(w_0 e^x) = \mathbb{I}_{j,i}(x) = \int_{-\infty}^{\infty} \phi_j(x - y') f_{ji}(y') . dy'$$

where, for every real  $y'$ , we let

$$f_{ji}(y') = \left[ \left( 1 - F_j(w_0 e^{y'}) \right) \gamma_j(w_0 e^{y'})^{p^*} \theta_{j,i} N_j(w_0 e^{y'}) \right] w_0 e^{y'},$$

Under the substitution  $y' = x - y$  our expression for this log-space (part of the) predation mortality rate becomes

$$\mathbb{I}_{j,i}(x) = \int_{-\infty}^{\infty} \phi_j(y) f_{ji}(x - y) . dy$$

In order to determine the mortality rate  $\mu_{p,i}(x) = \sum_{j=1}^s \mathbb{I}_{j,i}(x)$ , we wish to evaluate each

$$\mathbb{I}_{j,i}(x) = \int_{-\infty}^{\infty} \phi_j(y) f_{ji}(x - y) . dy = \int_{-\beta_j - 3\sigma_j}^0 \phi_j(y) f_{ji}(x - y) . dy$$

$$\forall x \in [x_0, X_i] = [0, X_i]$$

$$\text{where } \sup(f_{j,i}) = [x_0, X_j] = [0, X_j]$$

We can do this directly, using spectral methods.

## Spectral Methods For Mortality Integrals

In this integral, we gave that  $f_{j,i}(v)$  is used  $\forall v = x - y \in [0, X_i + \beta_j + 3\sigma_j]$ . This means it makes treat  $\mathbb{I}_{j,i}(x)$  is a convolution integral, with period

$$P_{j,i} = \text{length}([- \beta_j - 3\sigma_j, 0]) + \text{length}([0, X_i]) = X_i + \beta_j + 3\sigma_j = P_{j,i}$$

Suppose  $\bar{\phi}_j(v)$  is the period  $P_{j,i}$  extension of  $\phi_j(v)$  that agrees with  $\phi_j(v)$ , for all  $v \in [-\beta_j - 3\sigma_j, -\beta_j - 3\sigma_j + P_{j,i}]$ .

Suppose  $\bar{f}_{j,i}(v)$  is the period  $P_{j,i}$  extension of  $f_{j,i}(v)$  that agrees with  $f_{j,i}(v)$ ,  $\forall v \in [0, X_i + \beta_j + 3\sigma_j] = [0, P]$ .

Now we have

$$\mathbb{I}_{j,i}(x) = \int_{-\beta_j - 3\sigma_j}^0 \phi_j(y) f_{ji}(x - y) . dy$$

can be rewritten (in terms more amenable to spectral methods) as

$$\mathbb{I}_{j,i}(x) = \int_{-\beta_j - 3\sigma_j}^{-\beta_j - 3\sigma_j + P_{j,i}} \bar{\phi}_j(y) \bar{f}_{j,i}(x - y) . dy$$

Also, since  $\bar{\phi}_j(y) \bar{f}_{j,i}(x - y)$  is a period  $P_{j,i}$  function, we can rewrite it as

$$\mathbb{I}_{j,i}(x) = \int_0^{P_{j,i}} \bar{\phi}_j(y) \bar{f}_{j,i}(x-y) dy,$$

we want

We can find the appropriate data to input into our spectral integration method by noting that  $v \in [0, P_{j,i}] \Rightarrow \bar{f}_{j,i}(v) = f_{j,i}(v)$  and

Note that  $\forall v \in [0, P_{j,i}] \Rightarrow \bar{\phi}_j(v) = \phi_j(v - P_{j,i})$ .

Now  $\forall k \in \{0, 1, \dots, N-1\}$ , let

$$x_k = \frac{kP_{j,i}}{N},$$

be the  $k$ th grid point in our discretization of  $[0, P_{j,i}]$ . In order to apply our spectral methods, we want to obtain samplings of  $\bar{f}_{j,i}(x_k)$  and  $\bar{\phi}_j(x_k)$  over these intervals, in particular, to apply our fft method we want to evaluate the sequences:

$$\mathcal{L}_f = (\bar{f}_{j,i}(x_k))_{k=0}^{N-1} = (f_{j,i}(x_k))_{k=0}^{N-1},$$

and

$$\mathcal{L}_\phi = (\bar{\phi}_j(x_k))_{k=0}^{N-1} = (\phi_j(x_k - P_{j,i}))_{k=0}^{N-1},$$

given this information we can apply our key transformation

$$(\mathbb{I}_{j,i}(x_k))_{k=0}^{N-1} \approx \left(\frac{P_{j,i}}{N}\right) \mathbb{F}_N^{-1} [\mathbb{F}_N [\mathcal{L}_f] \otimes \mathbb{F}_N [\mathcal{L}_\phi]]$$

Here applying the fast Fourier transform  $\mathbb{F}_N$  to a length  $N$  sequence  $\mathcal{L}_f$  results in another length  $N$  list. Here  $\mathbb{F}_N [\mathcal{L}_f] \otimes \mathbb{F}_N [\mathcal{L}_\phi]$  denotes the length  $N$  list you get by multiplying  $\mathbb{F}_N [\mathcal{L}_f]$  and  $\mathbb{F}_N [\mathcal{L}_\phi]$  component wise, and  $\mathbb{F}_N^{-1}$  is the inverse fast Fourier transform. After multiplying by the step size  $\left(\frac{P_{j,i}}{N}\right)$ , this gives us our output values (in  $N \times O(\log N)$  time), which are within the sequence  $(\mathbb{I}_{j,i}(x_k))_{k=0}^{N-1}$ . In particular, the sequence

$$(\mathbb{I}_{j,i}(x_k))_{k=0}^{\left\lfloor \frac{X_i}{\Delta x} \right\rfloor},$$

where  $\Delta x = \frac{P_{j,i}}{N}$ , gives the information about the values of  $E'_{a,i}(x)$  for discrete values of  $x$  in the interval  $[0, X_i]$ .

## AVAILABLE ENERGY INTEGRAL

The available (food) energy to a fish of mass  $w$  and species  $i \in \{1, \dots, 2\}$  is called the available energy, and described by the following equation.

$$E_{a,i}(w) = \int_0^\infty Q_i(w_p) \Phi_i\left(\frac{w_p}{w}\right) dw_p$$

where  $Q(w_p) = \left(N_R(w_p) + \sum_{j=1}^s \theta_{i,j} N_j(w_p)\right) w_p$  measures the energy that prey  $w$  gets from eating different fish and plankton of size  $w_p$ .



Let us transform the available energy integral into ‘x-space’, by making the substitutions  $x = \log\left(\frac{w}{w_0}\right)$  and  $y = \log\left(\frac{w_p}{w_0}\right)$ ,

so we have  $w = w_0 e^x$  and  $w_p = w_0 e^y$  and  $dw_p = w_p dy$ , we can rewrite our expression for available energy as

$$E'_{a.i}(x) := E_{a.i}(w) = E_{a.i}(w_0 e^x) = \int_{-\infty}^{\infty} Q_i(w_0 e^y) \Phi_i\left(\frac{w_0 e^y}{w_0 e^x}\right) w_0 e^y dy$$

$$E'_{a.i}(x) := E_{a.i}(w) = E_{a.i}(w_0 e^x) = \int_{-\infty}^{\infty} Q_i(w_0 e^y) \Phi_i\left(\frac{w_0 e^y}{w_0 e^x}\right) w_0 e^y dy$$

So  $E'_{a.i}(x)$  is the available energy  $E_{a.i}(w)$  to a prey of mass  $w = w_0 e^x$ ,

$$E'_{a.i}(x) = \int_{-\infty}^{\infty} Q_i(w_0 e^y) \Phi_i(e^{y-x}) w_0 e^y dy$$

If we define  $q_i(y) = Q_i(w_0 e^y) w_0 e^y$ , and we recall,  $\phi_i(y-x) = \Phi_i(e^{y-x})$ , as described in section  $\{\{*\}\}$ , we see that  $E'_{a.i}(x)$  can be rewritten as

$$E'_{a.i}(x) = \int_{-\infty}^{\infty} q_i(y) \phi_i(y-x) dy$$

This is a convolution integral, accessible to spectral methods, however to make things more clear, let us perform the substitution  $y' = x - y$ , so we rewrite  $E'_{a.i}(x)$  as

$$E'_{a.i}(x) = \int_{-\infty}^{\infty} q_i(x-y') \phi_i(-y') dy',$$

to see how spectral methods can be applied to this issue more easily, it makes sense for us to use the ‘reversed feeding kernel’  $\psi_i(v)$ , we defined previously. Recall that  $\psi_i(v) = \phi(-v)$ , for all reals  $v$ . So the function  $\psi_i(v)$  is such that for every real  $v$  we have

$$\psi_i(v) := \phi_i(-v) = \exp\left[\frac{-(v-\beta_i)^2}{2\sigma_i^2}\right].$$

Using this information, we can rewrite our expression for the available energy as  $E'_{a.i}(x)$  as

$$E'_{a.i}(x) = \int_{-\infty}^{\infty} q_i(x-y') \psi_i(y') dy',$$

We wish to evaluate this expression  $E'_{a.i}(x)$ ,  $\forall x \in [x_0, X_i] = [0, X_i]$ .

Recall  $X_i = \ln\left(\frac{W_i}{w_0}\right)$ , where  $W_i$  is the maximum weight of species  $i \in \{1, \dots, s\}$ .

Since

$$\text{sup}(\psi_i) = [0, \beta_i + 3\sigma_i],$$

we have

$$E'_{a.i}(x) = \int_0^{\beta_i + 3\sigma_i} q_i(x-y') \psi_i(y') dy',$$

Notice that, in the course of evaluating this integral, over those  $x$  values, we are only sampling the function  $q_i(x - y') = q_i(v)$  for  $v$  within the interval  $[0, X_i + \beta_i + 3\sigma_i]$ .

In our spectral method we shall choose period a period length  $\rho_i = X_i + \beta_i + 3\sigma_i$  with a length,  $\rho_i$  equal to this interval  $[0, X_i + \beta_i + 3\sigma_i]$ .

## Making periodic functions

To apply spectral methods to evaluate

$$E'_{a,i}(x) = \int_0^{\beta_i + 3\sigma_i} q_i(x - y') \psi_i(y') dy',$$

Let us define the period  $\rho_i$  function  $\bar{q}_i(v)$  which is a period  $\rho_i$  extension of  $q_i(v)$  which agrees with  $q_i$  on the interval  $[0, \rho_i] = [0, X_i + \beta_i + 3\sigma_i]$ . In other words, if  $v \in [0, \rho_i]$  then  $\bar{q}_i(v) = q_i(v)$ , otherwise  $\bar{q}_i(v) = q_i(v + n\rho_i)$ , where  $n$  is the integer such that  $v + n\rho_i \in [0, \rho_i]$ .

Let us also define the period  $\rho_i$  function  $\bar{\psi}_i(v)$ , which is a period  $\rho_i$  extension of  $\psi_i$ , which agrees with  $\psi_i$  on the interval  $[0, \rho_i]$ . In other words, if  $v \in [0, \rho_i]$  then  $\bar{\psi}_i(v) = \psi_i(v)$ , otherwise  $\bar{\psi}_i(v) = \psi_i(v + n\rho_i)$ , where  $n$  is the integer such that  $v + n\rho_i \in [0, \rho_i]$ .

We shall use now use these periodic functions to rewrite the integral in a way more amenable to spectral methods (i.e., fast evaluation of convolution integrals).

It should be checked that, when  $x \in [0, X_i]$  we can rewrite our expression of the available energy as

$$E'_{a,i}(x) = \int_0^{\rho_i} \bar{q}_i(x - y') \bar{\psi}_i(y') dy',$$

where  $\rho_i = X_i + \beta_i + 3\sigma_i$  is the period of the functions.

This problem can be solved quickly via spectral methods, as we discuss in the next section:

## Applying fft to energy convolution integral

We want to evaluate

$$E'_{a,i}(x) = \int_0^{\rho_i} \bar{q}_i(x - y') \bar{\psi}_i(y') dy',$$

$\forall x \in [0, X_i]$ .

Let us consider how to do this problem practically, using the fast Fourier transform (fft).

Suppose we are dealing with a discretised grid of  $N_i$  points. We index these points, so that  $\forall k \in \{0, 1, \dots, N - 1\}$ , we have

$$x_k = \frac{k\rho_i}{N},$$

denotes the  $k$ th point in interval  $[0, X_i]$ .

{[}DO WE NEED TO THINK ABOUT REMOVING REFERENCE TO RESOURCE, OR PICKING LOGER PERIOD HERE ?{]}

To do out fft we require the sequence

$$\mathcal{L}_q = (\bar{q}_i(x_k))_{k=0}^{N-1} = (q_i(x_k))_{k=0}^{N-1},$$

and we also require the sequence

$$\mathcal{L}_\psi = (\bar{\psi}_i(x_k))_{k=0}^{N-1} = (\psi_i(x_k))_{k=0}^{N-1},$$

In this case we have our key transformation

$$(E'_{a.i}(x_k))_{k=0}^{N-1} \approx \left(\frac{\rho_i}{N}\right) \mathbb{F}_N^{-1} [\mathbb{F}_N[\mathcal{L}_q] \otimes \mathbb{F}_N[\mathcal{L}_\psi]]$$

Here applying the fast Fourier transform  $\mathbb{F}_N$  to a length  $N$  sequence  $\mathcal{L}_q$  results in another length  $N$  list. Here  $\mathbb{F}_N[\mathcal{L}_q] \otimes \mathbb{F}_N[\mathcal{L}_\psi]$  denotes the length  $N$  list you get by multiplying  $\mathbb{F}_N[\mathcal{L}_q]$  and  $\mathbb{F}_N[\mathcal{L}_\psi]$  component wise, and  $\mathbb{F}_N^{-1}$  is the inverse fast Fourier transform. After multiplying by the step size  $\left(\frac{\rho_i}{N}\right)$ , this gives us our output values (in  $N \times O(\log N)$  time), which are within the sequence  $(E'_{a.i}(x_k))_{k=0}^{N-1}$ . In particular, the sequence

$$(E'_{a.i}(x_k))_{k=0}^{\left\lfloor \frac{X_i}{\Delta x} \right\rfloor},$$

where  $\Delta x = \frac{\rho_i}{N}$ , gives the information about the values of  $E'_{a.i}(x)$  for discrete values of  $x$  in the interval  $[0, X_i]$ .

## Differences in notation

In the mizer vignette they write  $\beta_i$ . I write  $\beta_i^*$  to mean the same thing.

In the mizer vignette they write  $\phi_i$ . I write  $\Phi_i$  to mean the same thing.

I used capital F for feeding function, inside mortality integral