CSCI 570 - Summer 2021 - HW 3 Solution

1 Graded Problems

1. The recurrence $T(n) = 7T(\frac{n}{2}) + n^2$ describes the running time of an algorithm ALG. A competing algorithm ALG' has a running time of $T'(n) = aT'(\frac{n}{4}) + n^2 \log n$. What is the largest value of a such that ALG' is asymptotically faster than ALG?

   We shall use Master Theorem to evaluate the running time of the algorithms.

   (a) For $T(n)$, setting $0 < \epsilon < \log_2 7 - 2 \approx 0.8$ we have $n^2 = \mathcal{O}(n^{\log_2 7 - \epsilon})$. Hence $T(n) = \Theta(n^{\log_2 7})$.

   (b) For $T'(n)$, if $\log_4 a > 2$, then setting $0 < \delta < log_4 a - 2$ implies $n^2 \log n = \mathcal{O}(n^{\log_4 a - \delta})$ and hence $T'(n) = \Theta(n^{\log_4 \ a})$

   To have $T'(n)$ asymptotically faster than $T(n)$, we need $T'(n) = \mathcal{O}(T(n))$, which implies $n^{\log_4 \ a} = \mathcal{O}(n^{\log_2 \ 7})$ and therefore $\log_4 \ a \leq \log_2 \ 7 \Rightarrow a \leq 49$. Since other expressions for runtime of ALG' require $\log_4 \ a \leq 2 \Rightarrow a \leq 16 < 49$, the largest possible value of a such that ALG' is asymptotically faster than ALG is $a = 49$.

2. Solve the following recurrences by giving tight θ-notation bounds in terms of n for sufficiently large n.

   (a) $T(n) = 4T(\frac{n}{2}) + n^2 logn$

   (b) $T(n) = 8T(\frac{n}{6}) + nlogn$

   (c) $T(n) = \sqrt{6006}T(\frac{n}{2}) + n\sqrt{6006}$

   (d) $T(n) = 10T(\frac{n}{2}) + 2^n$

   (e) $T(n) = 2T(\sqrt{n}) + log_2 n$

   (f) $T^2(n) = T(\frac{n}{2})T(2n) - T(n)T(\frac{n}{2})$

   (g) $T(n) = 2T(\frac{n}{2}) - \sqrt{n}$

Solution: In some cases, we shall need to invoke the Master Theorem with one generalization as described next. If the recurrence $T(n) = aT(n/b) + f(n)$ is satisfied with $f(n) = \Theta\Big(n^{\log_b a} \log^k n\Big)$ for some $k \geq 0$, then $T(n) = \Theta\Big(n^{\log_b a} \log^{k+1} n\Big)$.

(a) Observe that $f(n) = n^2 \log n$ and $n^{\log_b a} = n^{\log_2 4} = n^2$, so applying the generalized Master's theorem, $T(n) = \Theta\Big(n^2 \log^2 n\Big)$.

(b) Observe that $n^{\log_b a} = n^{\log_6 8}$ and $f(n) = n \log n = O\Big(n^{\log_6 8 - \epsilon}\Big)$ for any $0 < \epsilon < \log_6 8 - 1$. Thus, invoking Master's Theorem gives $T(n) = \Theta\Big(n^{\log_b a}\Big) = \Theta\Big(n^{\log_6 8}\Big)$.

(c) We have $n^{\log_b a} = n^{\log_2 \sqrt{6006}} = n^{0.5 \log_2 6006} = O\Big(n^{0.5 \log_2 8192}\Big) = O\Big(n^{13/2}\Big)$ and $f(n) = n^{\sqrt{6006}} = \Omega\Big(n^{\sqrt{4900}}\Big) = \Omega(n^{70}) = \Omega\Big(n^{13/2+\epsilon}\Big)$ for any $0 < \epsilon < 63.5$. Thus, from Master's Theorem $T(n) = \Theta(f(n)) = \Theta\Big(n^{\sqrt{6006}}\Big)$.

(d) We have $n^{\log_b a} = n^{\log_2 10}$ and $f(n) = 2^n = \Omega\Big(n^{\log_2 10 + \epsilon}\Big)$ for any $\epsilon > 0$. Therefore Master's Theorem implies $T(n) = \Theta(f(n)) = \Theta(2^n)$.

(e) Use the change of variables $n = 2^m$ to get $T(2^m) = 2T\Big(2^{m/2}\Big) + m$. Next, denoting $S(m) = T(2^m)$ implies that we have the recurrence $S(m) = 2S(m/2) + m$. Note that $S(\cdot)$ is a positive function due to the monotonicity of the increasing map $x \mapsto 2^x$ and the positivity of $T(\cdot)$. All conditions for applicability of Master's Theorem are satisfied and using the generalized version gives $S(m) = \Theta(m \log m)$ on observing that $f(m) = m$ and $m^{\log_b a} = m$. We express the solution in terms of $T(n)$ by

$$T(n) = T(2^m) = S(m) = \Theta(m \log m) = \Theta(\log_2 n \log \log_2 n),$$

for large enough n so that the growth expression above is positive.

(f) Owing to the unusual nature of this recurrence, we'll try to be a little descriptive in the solution since applicability of Master's Theorem is not foreseeable. Since $T(n)$ is positive for every $n > 0$, dividing the given recurrence by $T(n)T(n/2)$ gives

$$\frac{T(n)}{T(n/2)} = \frac{T(2n)}{T(n)} - 1.$$

Setting $S(n) = T(2n)/T(n)$ for $n \geq 2$, we get the recurrence $S(n/2) = S(n) - 1$ which is equivalent to $S(n) = S(n/2) + 1$. Positivity of $T(\cdot)$ implies positivity of $S(\cdot)$ and a step-by-step solution of the recurrence gives

$$S(n) = S(n/2) + 1 = S(n/4) + 2 = \cdots = S\left(\frac{n}{2^{k-1}}\right) + k - 1, \tag{1}$$

where $k = \log_2 n$ and

$$S\left(\frac{n}{2^{k-1}}\right) = S\left(\frac{2n}{2^{\log_2 n}}\right) = S(2) = \frac{T(4)}{T(2)} = c_0.$$

From the recurrence relation for $T(n)$, we have $T(4) = 2T(2) + 2$, so that $c_0 = T(4)/T(2) = 2 + 2/T(2) > 2$. Furthermore, (1) amounts to $S(n) = \log_2 n + c_0 - 1$. Using this expression for $S(n)$, we have

$$
\begin{aligned}
T(n) &= S\left(\frac{n}{2}\right)T\left(\frac{n}{2}\right) && \text{(2a)}\\
&= \left(\log_2\frac{n}{2}+c_0-1\right)T\left(\frac{n}{2}\right) && \text{(2b)}\\
&= \left(\log_2\frac{n}{2}+c_0-1\right)\left(\log_2\frac{n}{4}+c_0-1\right)T\left(\frac{n}{4}\right) && \text{(2c)}\\
&= \cdots = \left(\log_2\frac{n}{2}+c_0-1\right)\left(\log_2\frac{n}{4}+c_0-1\right)\ldots\left(\log_2\frac{n}{2^{k-1}}+c_0-1\right)T\left(\frac{n}{2^{k-1}}\right) && \text{(2d)}\\
&= T(2)\prod_{l=1}^{k-1}\left(\log_2\frac{n}{2^l}+c_0-1\right) && \text{(2e)}\\
&= T(2)\prod_{l=1}^{k-1}(\log_2 n+c_0-l-1) && \text{(2f)}\\
&= T(2)\prod_{p=0}^{\log_2 n-2}(c_0+p) && \text{(2g)}\\
&= \Theta((\log_2 n+c_0-2)!) && \text{(2h)}
\end{aligned}
$$

where (2b) uses the definition of $S(n)$, (2d) is the fully expanded expression for the recurrence on $T(n)$, (2g) employed the substitution $p=\log_2 n-l-1$, and (2h) assumed that $c_0+\log_2 n$ is an integer. If you want to be more precise, you can use the Γ function to write $T(n)=\Theta(\Gamma(\log_2 n+c_0-1))$ even when $c_0+\log_2 n$ is not an integer, but this is not mandatory.

Next we shall employ Stirling's approximation to dispense with the factorial in (2h). Recall that Stirling's approximation implies

$$m! = \Theta\left(m^{m+0.5}e^{-m}\right).$$

Setting $c_0-2=d$, we have

$$
\begin{aligned}
\Theta((\log_2 n+c_0-2)!) &= \Theta\left((d+\log_2 n)^{d+0.5+\log_2 n}e^{-d-\log_2 n}\right) && \text{(3a)}\\
&= \Theta\left(\left(1+\frac{d}{\log_2 n}\right)^{d+0.5+\log_2 n}(\log_2 n)^{d+0.5+\log_2 n}e^{-\log_2 n}\right) && \text{(3b)}\\
&= \Theta\left((\log_2 n)^{c_0-1.5+\log_2 n}n^{-\log_2 e}\right) && \text{(3c)}
\end{aligned}
$$

where we have used the constant upper and lower bounds (for $\log_2 n \geq 1$)

$$
\begin{aligned}
\left(1+\frac{d}{\log_2 n}\right)^{d+0.5+\log_2 n} &= \exp\left[(d+0.5+\log_2 n)\ln\left(1+\frac{d}{\log_2 n}\right)\right]\\
&= \exp\left[d\left(\frac{d+0.5}{\log_2 n}+1\right)\frac{\ln(1+d/\log_2 n)}{d/\log_2 n}\right]\\
&\leq \exp\left[d\left(\frac{d+0.5}{1}+1\right)\times 1\right],
\end{aligned}
$$

and

$$\left(1+\frac{d}{\log_2 n}\right)^{d+0.5+\log_2 n} \geq 1^{d+0.5+\log_2 n} = 1.$$

Our final answer is essentially (3c) which gives

$$T(n) = \Theta\Big(n^{-\log_2 e}(\log_2 n)^{c_0-1.5+\log_2 n}\Big).$$

(g) For this recurrence, we cannot apply Master's Theorem since pattern matching with the template recurrence relation $T(n) = aT(n/b) + f(n)$ gives the $f(n)$ term as negative. We will solve this from first principles. We have,

$$\begin{aligned}
T(n) &= 2T\left(\frac{n}{2}\right) - \sqrt{n} && \text{(4a)}\\
&= 2\left(2T\left(\frac{n}{4}\right) - \sqrt{\frac{n}{2}}\right) - \sqrt{n} = 2^2T\left(\frac{n}{2^2}\right) - \sqrt{n}\Big(1+\sqrt{2}\Big) && \text{(4b)}\\
&= 2^2\left(2T\left(\frac{n}{2^3}\right) - \sqrt{\frac{n}{2^2}}\right) - \sqrt{n}\Big(1+\sqrt{2}\Big) = 2^3T\left(\frac{n}{2^3}\right) - \sqrt{n}\Big(1+\sqrt{2}+\sqrt{2^2}\Big) && \text{(4c)}\\
&= \cdots = 2^{k-1}T\left(\frac{n}{2^{k-1}}\right) - \sqrt{n}\Big(1+\sqrt{2}+\sqrt{2^2}+\cdots+\sqrt{2^{k-2}}\Big) && \text{(4d)}\\
&= 2^{k-1}T\left(\frac{n}{2^{k-1}}\right) - \sqrt{n}\left(\frac{\sqrt{2^{k-1}}-1}{\sqrt{2}-1}\right) && \text{(4e)}
\end{aligned}$$

where $k = \log_2 n$, (4d) represents the fully expanded form of the recurrence for $T(n)$ and (4e) was obtained using the formula for sum of a geometric progression. Further simplification of (4e) gives

$$\begin{aligned}
T(n) &= \frac{2^{\log_2 n}}{2}T\left(\frac{2n}{2^{\log_2 n}}\right) - \sqrt{n}\left(\frac{\sqrt{2^{\log_2 n}}-\sqrt{2}}{2-\sqrt{2}}\right) && \text{(5a)}\\
&= \frac{n}{2}T(2) - \sqrt{n}\left(\frac{\sqrt{n}-\sqrt{2}}{2-\sqrt{2}}\right) && \text{(5b)}\\
&= n\left(\frac{T(2)}{2} - \frac{1}{2-\sqrt{2}}\right) + \frac{\sqrt{2n}}{2-\sqrt{2}}, && \text{(5c)}
\end{aligned}$$

resulting in the following two cases.

Case 1: If $T(2) > \sqrt{2}/\big(\sqrt{2}-1\big)$, then the coefficient of n in (5c) is positive and therefore $T(n) = \Theta(n)$.

Case 2: If $T(2) = \sqrt{2}/\big(\sqrt{2}-1\big)$, then the coefficient of n in (5c) is zero and thus, $T(n) = \Theta(\sqrt{n})$.

Note that $T(2) < \sqrt{2}/\big(\sqrt{2}-1\big)$ is not possible, since that would imply that $T(n)$ is asymptotically negative which contradicts the premise of the question as $T(n)$ denotes running time of an algorithm.

3. From the lecture, you know how to use dynamic programming to solve the 0-1 knapsack problem where each item is unique and only one of each kind is available. Now let us consider knapsack problem where you have infinitely many items of each kind. Namely, there are n different types of items. All the items of the same type i have equal size w_i and value v_i. You are offered with infinitely many items of each type. Design a dynamic programming algorithm to compute the optimal value you can get from a knapsack with capacity W.

   Similar to what is taught in the lecture, let $OPT(k, w)$ be the maximum value achievable using a knapsack of capacity $0 \leq w \leq W$ and with k types of items $1 \leq k \leq n$. We find the recurrence relation of $OPT(k, w)$ as follows. Since we have infinitely many items of each type, we choose between the following two cases:

   - We include another item of type k and solve the sub-problem $OPT(k, w - v_k)$.
   - We do not include any item of type k and move to consider next type of item this solving the sub-problem $OPT(k-1, w)$.

   Therefore, we have

   $$OPT(k, w) = \max\{OPT(k-1, w), OPT(k, w - w_k) + v_k * i\}.$$

   Moreover, we have the initial condition $OPT(0, 0) = 0$.

4. Given a non-empty string s and a dictionary containing a list of unique words, design a dynamic programming algorithm to determine if s can be segmented into a space-separated sequence of one or more dictionary words. If s="algorithmdesign" and your dictionary contains "algorithm" and "design". Your algorithm should answer Yes as s can be segmented as "algorithmdesign".

   Let $s_{i,k}$ denote the substring $s_i s_{i+1} \ldots s_k$. Let $Opt(k)$ denote whether the substring $s_{1,k}$ can be segmented using the words in the dictionary, namely $OPT(k) = 1$ if the segmentation is possible and 0 otherwise. A segmentation of this substring $s_{1,k}$ is possible if only the last word (say $s_i \ldots s_k$) is in the dictionary the remaining substring $s_{1,i}$ can be segmented. Therefore,we have equation:

   $$Opt(k) = \max_{0<i<k \text{ and } s_{i+1,k} \text{ is a word in the dictionary}} Opt(i)$$

We can begin solving the above recurrence with the initial condition that $Opt(0) = 1$ and then go on to compute $Opt(k)$ for $k = 1, 2, \ldots, n$. The answer corresponding to $Opt(n)$ is the solution and can be computed in $\Theta(n^2)$ time.

5. Given n balloons, indexed from 0 to $n-1$. Each balloon is painted with a number on it represented by array nums. You are asked to burst all the balloons. If the you burst balloon i you will get $nums[left] * nums[i] * nums[right]$ coins. Here left and right are adjacent indices of i. After the burst, the left and right then becomes adjacent. You may assume $nums[-1] = nums[n] = 1$ and they are not real therefore you can not burst them. Design an dynamic programming algorithm to find the maximum coins you can collect by bursting the balloons wisely. Analyze the running time of your algorithm.

   Here is an example. If you have the nums arrays equals $[3, 1, 5, 8]$. The optimal solution would be 167, where you burst balloons in the order of 1, 5 3 and 8. The left balloons after each step is:

   $$[3, 1, 5, 8] \to [3, 5, 8] \to [3, 8] \to [8] \to []$$

   And the coins you get equals:

   $$167 = 3 * 1 * 5 + 3 * 5 * 8 + 1 * 3 * 8 + 1 * 8 * 1.$$

   Let $OPT(l, r)$ be the maximum number of coins you can obtain from balloons $l, l+1, \ldots, r-1, r$. The key observation is that to obtain the optimal number of coins for balloon from l to r, we choose which balloon is the last one to burst. Assume that balloon k is the last one you burst, then you must first burst all balloons from l to $k-1$ and all the balloons from $k+1$ to r which are two sub problems. Therefore, we have the following recurrence relation:

   $$OPT(l, r) = \max_{l \le k \le r} \{OPT(l, k-1) + OPT(l, k+1) + nums[k] * nums[l-1] * nums[r+1]\}$$

   We have initial condition $OPT(l, r) = 0$ if $r < l$. For running time analysis, we in total have $O(n^2)$ and computation of each state takes $O(n)$ time. Therefore, the total time is $O(n^3)$.

2 Practice Problems

1. Solve Kleinberg and Tardos, Chapter 5, Exercise 3.

In a set of cards, if more than half the cards belong to a single user, we call the user a majority user.

Divide the set of cards into two roughly two equal halves, (that is one half is of size $\lfloor \frac{n}{2} \rfloor$ and the other, of size $\frac{n}{2}$). For each half, recursively solve the following problem, "decide if there exists a majority user and if he/she exists find a card corresponding to him/her (as a representative)".

Once we have solved the problem for the two halves, we can combine them to solve the problem for the whole set as follows.

If neither half has a majority user, then the whole set clearly does not have a majority user.

If each of the halves have a majority user, then the whole set has a majority user if and only if both the halves have the same majority user. If both the halves have the same majority user, then we can pick either one of the output cards output by the halves as a representative for the whole set.

If one half has a majority (say user U) and the other does not, then by comparing the representative card of user U with every other card in the whole set, count the number of cards that belong to user U in the whole set.

If this count is greater than $\frac{n}{2}$, output that there is a majority and also the representative card. If $T(n)$ denotes the number of comparisons (invocations to the equivalence tester) of the resulting divide and conquer algorithm, then

$$T(n) \leq 2T(\lfloor \frac{n}{2} \rfloor) + n - 1 \Rightarrow T(n) = \mathcal{O}(n \log n)$$

Remark : There are ways to solve this problem with at most O(n) comparisons, can you think of one of such algorithm ?

2. Solve Kleinberg and Tardos, Chapter 5, Exercise 5.

Let $L = \{L_1, L_2, \ldots, L_n\}$ be the sequence of lines sorted in increasing order of slope. From now on, when we say sort a set of lines, it is in increasing order of slope. We divide the set of lines in half and solve recursively. When we are down to a set with only one line, we return the line as visible.

Recursively compute $L_{Bslash} = \{L_{i_1}, L_{i_2}, \ldots, L_{i_m}\}$, the sorted sequence of visible lines of the set $L = \{L_1, L_2, \ldots, L_{\lfloor \frac{n}{2} \rfloor}\}$. In addition compute the set of points $A = \{a_1, a_2, \ldots, a_{m-1}\}$ where a_j is the intersection of L_{i_j} and $L_{i_{j+1}}$.

Likewise compute$L_{slash} = \{L_{k_1}, L_{k_2}, \ldots, L_{k_r}\}$, the sorted sequence of visible lines of the set $\{L_{\lfloor \frac{n}{2} \rfloor+1}, \ldots, L_n\}$. In addition compute the set of points $B = \{b_1, b_2, \ldots, b_{r-1}\}$ where b_j is the intersection of L_{k_j} and $L_{k_{j+1}}$.

Observe that by construction $\{a_1, a_2, \ldots, a_m\}$ and $\{b_1, b_2, \ldots, b_r\}$ are in increasing order of x-coordinate since if two visible lines intersect, the visible part of the line with smaller slope is to the left.

We now describe how the solutions for the two halves are combined. Merge the two sorted lists A and B, to form $C = \{c_1, c_2, \ldots, c_{m+1}\}$ ($\mathcal{O}(n)$ time).

Let $L_{up}(j)$ be the uppermost line in L_{Bslash} at the x-coordinate of c_j and $\bar{L}_{up}(j)$ the uppermost line in L_{slash} at the x-coordinate of c_j. Let ℓ be the smallest index at which $\bar{L}_{up}(\ell)$ is above $L_{up}(\ell)$ at the x-coordinate of c_j .

Let s and t be the indices such that $L_{up}(\ell) = L_{i_s}$ and $\bar{L}_{up}(\ell) = L_{j_t}$.

Let (a, b) be the intersection of $L_{up}(\ell)$ and $\bar{L}_{up}(\ell)$. Then a lies between the x-coordinates of $c_{\ell-1}$ and c_ℓ. This implies that $L_{up}(\ell)$ is visible immediately to the left of a and $\bar{L}_{up}(\ell)$ to the right. Hence the sorted set of visible lines of L is $L_{i_1}, L_{i_2}, \ldots, L_{i_{s-1}}, L_{i_s}, L_{j_t}, L_{j_{t+1}}, \ldots, L_r$.

The combination step takes $\mathcal{O}(n)$ time. If $T(n)$ denotes the running time of the algorithm, then:

$$T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + \mathcal{O}(n) \Rightarrow T(n) = \mathcal{O}(n \log n)$$

3. Solve Kleinberg and Tardos, Chapter 6, Exercise 5.

Let $Y_{i,k}$ denote the substring $y_i y_{i+1} \ldots y_k$. Let $Opt(k)$ denote the quality of an optimal segmentation of the substring $Y_{1,k}$. An optimal segmentation of this substring $Y_{1,k}$ will have quality equalling the quality last word (say $y_i \ldots y_k$) in the segmentation plus the quality of an optimal solution to the substring $Y_{1,i}$. Otherwise we could use an optimal solution to $Y_{1,i}$ to improve $Opt(k)$ which would lead to a contradiction.

$$Opt(k) = \max_{0<i<k} Opt(i) + quality(Y_{i+1,k})$$

We can begin solving the above recurrence with the initial condition that $Opt(0) = 0$ and then go on to compute $Opt(k)$ for $k = 1, 2, \ldots, n$ keeping track of where the segmentation is done in each case. The segmentation corresponding to $Opt(n)$ is the solution and can be computed in $\Theta(n^2)$ time.

4. Solve Kleinberg and Tardos, Chapter 6, Exercise 6.

Let $W = \{w_1, w_2, \ldots, w_n\}$ be the set of ordered words which we wish to print. In the optimal solution, if the first line contains k words, then the rest of the lines constitute an optimal solution for the sub problem with the set $\{w_{k+1}, \ldots, w_n\}$. Otherwise, by replacing with an optimal solution for the rest of the lines, we would get a solution that contradicts the optimality of the solution for the set $\{w_1, w_2, \ldots, w_n\}$.

Let $Opt(i)$ denote the sum of squares of slacks for the optimal solution with the words $\{w_i, \ldots, w_n\}$. Say we can put at most the first p words from w_i to w_n in a line, that is, $\sum_{t=i}^{p+i-1} c_t + p - 1 \leq L$ and $\sum_{t=1}^{p+i} w_t + p > L$. Suppose the first k words are put in the first line, then the number of extra space characters is

$$s(i,k) := L - k + 1 - \sum_{t=i}^{i+k-1} c_t$$

So we have the recurrence

$$Opt(i) = \begin{cases} 0 & \text{if } p \geq n - i + 1 \\ \min_{1 \leq k \leq p}\{(s(i,k))^2 + Opt(i+k)\} & \text{if } p < n - i + 1 \end{cases}$$

Trace back the value of k for which $Opt(i)$ is minimized to get the number of words to be printed on each line. We need to compute $Opt(i)$ for n different values of i. At each step p may be asymptotically as big as L. Thus the total running time is $O(nL)$.

5. Solve Kleinberg and Tardos, Chapter 6, Exercise 10.

Consider the following example: there are totally 4 minutes, the numbers of steps that can be done respectively on the two machines in the 4 minutes are listed as follows (in time order):

- Machine A: 2, 1, 1, 200
- Machine B: 1, 1, 20, 100

The given algorithm will choose A then move, then stay on B for the final two steps. The optimal solution will stay on A for the four steps. An observation is that, in the optimal solution for the time interval from minute 1 to minute i, you should not move in minute i, because otherwise, you can keep staying on the machine where you are and get a better solution ($a_i > 0$ and $b_i > 0$). For the time interval from minute 1 to minute i, consider that if you are on machine A in minute i, you either (i) stay on machine A in minute $i-1$ or (ii) are in the process of moving from machine B to A in minute $i-1$. Now let $OPT_A(i)$ represent the maximum value of a plan in minute 1 through i that ends on machine A, and define $OPT_B(i)$ analogously for B. If case (i) is the best action to make for minute $i1$, we have $OPT_A(i) = a_i + OPT_A(i-1)$; otherwise, we have $OPT_A(i) = a_i + OPT_B(i-2)$. In sum, we have

$$OPT_A(i) = a_i + \max\{OPT_A(i-1), OPT_B(i-2)\}$$

Similarly, we get the recursive relation for $OPT_B(i)$:

$$OPT_B(i) = b_i + \max\{OPT_B(i-1), OPT_A(i-2)\}$$

The algorithm initializes $OPT_A(0) = 0, OPT_B(0) = 0, OPT_A(1) = a_1$ and $OPT_B(1) = b_1$. Then the algorithm can be written as follows:

$OPT_A(0) = 0$; $OPT_B(0) = 0$;
$OPT_A(1) = a_1$; $OPT_B(1) = b_1$;
for $i = 2, \cdots, n$ **do**
  $OPT_A(i) = a_i + \max\{OPT_A(i-1), OPT_B(i-2)\}$;
  Record the action (either stay or move) in minute $i-1$ that achieves the maximum.
  $OPT_B(i) = b_i + \max\{OPT_B(i-1), OPT_A(i-2)\}$;
  Record the action in minute $i-1$ that achieves the maximum.
end for
Return $\max\{OPT_A(n), OPT_B(n)\}$;
Track back through the arrays OPT_A and OPT_B by checking the action records from minute $n-1$ to minute 1 to recover the optimal solution.

It takes $O(1)$ time to complete the operations in each iteration; there are $O(n)$ iterations; the tracing backs takes $O(n)$ time. Thus, the overall complexity is $O(n)$.

6. Solve Kleinberg and Tardos, Chapter 6, Exercise 24.
The basic idea is to ask: How should we gerrymander precincts 1 through j, for each j? To make this work, though, we have to keep track of a few extra things, by adding some variables. For brevity, we say that A-votes in a precinct are the voters for part A and B-voter are the votes for part B. We keep track of the following information about a partial solution.

- How many precincts have been assigned to district 1 so far?
- How many A-votes are in district 1 so far?
- How many A-votes are in district 2 so far?

So let $M[j,p,x,y] = true$ if it is possible to achieve at least x A-votes in distance 1 and y A-votes in district 2, while allocating p of the first j precincts to district 1. Now suppose precinct $j+1$ has z A-votes. To compute $M[j+1,p,x,y]$, you either put precinct $j+1$ in district 1 (in which case you check the results of sub-problem $M[j,p-1,x-z,y]$) or in district 2 (in which case you check the results of sub-problem $M[j,p,x,y-z]$). Now to decide if there's a solution to the while problem, you scan the entire table at the end, looking for a value of $true$ in any entry from $M[n,n/2,x,y]$ where each of x and y is greater than $mn/4$. (Since each district gets $mn/2$ votes total).

We can build this up in the order of increasing j, and each sub-problem takes constant time to compute, using the values of smaller sub-problems. Since there are n^2, m^2 sub-problems, the running time is $O(n^2m^2)$.