Junhao Zhang “Freddie”

USC ID:

CSCI 570 – HW2

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**1. Graded Problems**

* **[Question 1]**

Sort set *A* and *B* in the decreasing order, and then let be the *i-th* element of set *A* so that be the greatest integer in the set *A*, and let be the *i-th* element of set *B* so that be the greatest integer in the set *B.* Then can receive the payoff on . This algorithm has runtime complexity since we need to sort the sets. However, if these two sets are sorted already, then then runtime complexity can be .

*Proof.*

Let’s prove its correctness by showing that if there is an optimal solution, then we can transform the optimal solution to our solution by making swaps and making such swaps won’t negatively impact the optimality.

Let and and > for some , , in the optimal solution:

\*

If we make inversion to transform the optimal solution to:

\*

Then we can observe:

, which is greater than 1, which means > .

Now we have proven that each time we make swap to transform the optimal solution to our solution will not have negative impact to the optimality. Thus, we have proven that our solution is also optimal.

* **[Question 2]**

We have a min-heap which has first values of all lists, and after each “pop”, if the list of the popped value is not empty then we push the next value in that list to our min-heap. We repeat *n* times, and we will finish merging these *k* sorted lists into one sorted list if the list saves the popped values in the order of being popped.

Since the heap has length *k*, thus each pop would cost *log* *k*, and we need to do that *n* times. So, the time complexity is *O* (*n log k*).

* **[Question 3]**

Let us denote the edge *e* has the minimal weight, and it is not in the minimum spanning tree. Then, let’s add *e* to the minimum spanning tree, would form a cycle. If we remove an edge in the cycle that is not *e*, then we generate a new tree with lighter total weight even than the minimum spanning tree, which is a contradiction. Thus, the edge *e* has to be in the minimum spanning tree.

* **[Question 4]**

First, all these spanning trees they must have the same number of edges, and let’s denote *N* be the number. The reason is, less than *N* edges are not enough to keep the graph connected, and more than *N* edges will cause cycles.

If we add one new edge to the minimal spanning tree, then there will be a cycle, but after we removed an edge of the cycle that is not the one that we just added, then a new spanning tree will be formed. Among all the spanning trees that can be formed this way, the one with the minimal total weight is the second-best minimal spanning tree.

* **[Question 5]**

Let the cow with the strongest strength take the most weight, which means the stronger strength a cow has, the lower position it takes in the cow tower.

*Proof.*

We are going to prove it by showing if there exists an optimal solution, then we can transform the optimal solution to our solution by swapping positions of cows without making negative impacts.

Let *i, j* ∈ {*1, 2, 3, … N*} and *i < j* (which means cow *i* is above the cow *j*). Moreover, let . Then, we can see that cow *i* has risk value and cow *j* has risk value and in this case cow *j* has the maximum risk value. But after swap, the cow *i* has risk value and cow *j* has risk value , and we can see both of these two values are less than the previous maximum risk value .

Thus, we have proven that making swap won’t negatively impact the solution, thus an optimal solution can be swapped to our solution without being negatively impacted, which makes our solution also an optimal solution.

**2. Practice Problems**

* **[Question 1]**

Let’s use an altered version of Kahn's algorithm to do it. The original algorithm needs to first calculate the indegree of all vertices by traversing over graph, and let’s alter the algorithm to make it keeps track of visited vertices at this stage, so if the current visiting vertex had been visited before, that means the graph has a cycle, and our altered version of algorithm will return that cycle. Otherwise, our algorithm will find topological order like what the original Kahn's algorithm does.

* **[Question 2]**

We will prove it by contradiction. First, we assume there is an edge *e* such that *e* exists in *G* but doesn’t exist in *T*. Since *T* has all nodes that *G* has, it must also have the two nodes that *e* connects, and because *T* is a DFS tree, thus one of the two nodes must be the ancestor of another. Let’s denote *x* be the ancestor, and *y* be the descendent. Moreover, since *e* connects *x* and *y*, thus the distance between these must be 1. However, there is a contradiction: since *x* is the ancestor of *y* and their distance is 1, thus when running BFS from *x*, *y* must be added to *T* as the direct child of *x*, which means *e* exists in T, but we already assumed *e* doesn’t exist in T. Contradiction! Therefore, we proved *G* cannot contain any edges that do not belong to *T*.

* **[Question 3]**

We first are going to use the union-find data structure to find cycles, then once we found a cycle, we delete the edge with the greatest weight from the graph. By deleting at most 9 edges (because a tree must have *n – 1* edge, and the problem states that the *near-tree* can have up to *n + 8* edges, that means we may need to remove up to 9 edges) we will have the minimum spanning tree. The time complexity is *O* (|*E*|).

* **[Question 4]**

Here is a counterexample:

A picture containing icon

Description automatically generated

A minimum-cost spanning tree

A picture containing schematic

Description automatically generated

The “Tree *T*”

* **[Question 5]**

1. We use the coin with the greatest value (i.e.: quarter) to add up until that is going to exceed the amount (*n* cents), then we use the coin with the second greatest value (i.e.: dime) to add up until that is going to exceed the rest value, then we use the coin with the third greatest value and do the same thing. Repeat until is equal to *n*.

*Proof.*

We are going to prove it by showing if there exists an optimal solution, then we can transform the optimal solution to our solution by substituting more quantity of less-value coins for less quantity of greater-value coins of the same amount without making negative impacts. For proving that, we have three cases, and we will go through them each.

Case 1: Pennies to nickels. Every 5 pennies the optimal solution has, we can substitute for 1 nickel. The quantity will be less so this will not make negative impact.

Case 2: Nickels to dimes. Every 2 nickels the optimal solution has, we can substitute for 1 dime. The quantity will be less so this will not make negative impact. Moreover, from Case 1 we can induct substituting pennies for dimes won’t make negative impact neither.

Case 3: Nickels to quarters. Every 5 nickels the optimal solution has, we can substitute for 1 quarter. The quantity will be less so this will not make negative impact. Moreover, from Case 2 we can induct replacing some of these five nickels with dimes of the same amount and getting quarters in that way won’t make negative impact neither.

Thus, we have shown that in all cases substituting less-value coins for greater-value coins of the same amount won’t have negative impacts to the optimal solution that we are trying to convert, which means the optimal solution can be converted to our solution, which makes our solution an optimal solution also.

1. {7 cents, 6 cents, 4 cents, 1 cent} and target amount is 10 cents.