Oscillatory Motion

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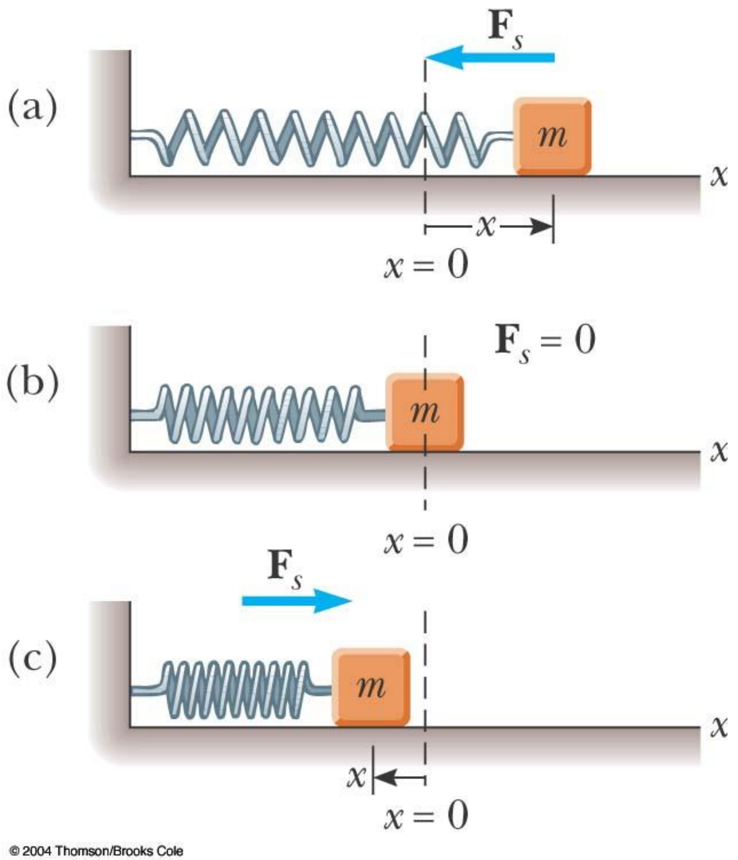
Periodic Motion

- ***Periodic motion*** is motion of an object that regularly repeats
  - The object returns to a given position after a fixed time interval
- A special kind of periodic motion occurs in mechanical systems when the force acting on the object is proportional to the displacement of the object relative to some equilibrium position
  - If the force is always directed toward the equilibrium position, the motion is called ***simple harmonic motion***

Motion of a Spring-Mass System

- A block of mass *m* is attached to a spring, the block is free to move on a frictionless horizontal surface
- When the spring is neither stretched nor compressed, the block is at the ***equilibrium position***
  - $x = 0$

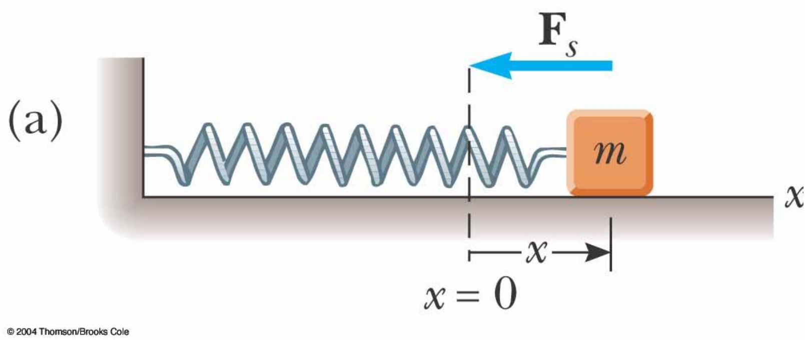


Hooke's Law

- Hooke's Law states $F_s = - kx$
  - F_s is the restoring force
    - It is always directed toward the equilibrium position
    - Therefore, it is always opposite the displacement from equilibrium
  - k is the force (spring) constant
  - x is the displacement

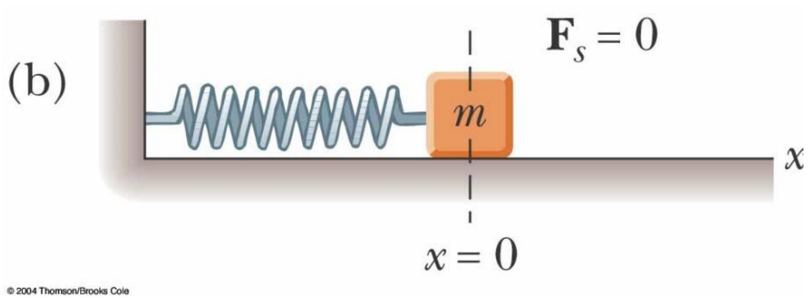
More About Restoring Force

- The block is displaced to the right of $x = 0$
  - The position is positive
- The restoring force is directed to the left



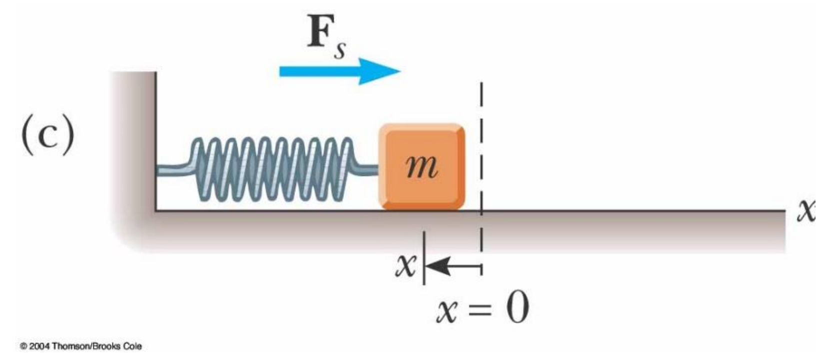
More About Restoring Force, 2

- The block is at the equilibrium position
  - $x = 0$
- The spring is neither stretched nor compressed
- The force is 0



More About Restoring Force, 3

- The block is displaced to the left of $x = 0$
  - The position is negative
- The restoring force is directed to the right



Acceleration

- The force described by Hooke's Law is the net force in Newton's Second Law

$$F_{\text{Hooke}} = F_{\text{Newton}}$$

$$-kx = ma_x$$

$$a_x = -\frac{k}{m}x$$

Acceleration, cont.

- The acceleration is proportional to the displacement of the block
- The direction of the acceleration is opposite the direction of the displacement from equilibrium
- An object moves with simple harmonic motion whenever its acceleration is proportional to its position and is oppositely directed to the displacement from equilibrium

Acceleration, final

- The acceleration is ***not*** constant
  - Therefore, the kinematic equations cannot be applied
  - If the block is released from some position $x = A$, then the initial acceleration is $-kA/m$
  - When the block passes through the equilibrium position, $a = 0$
  - The block continues to $x = -A$ where its acceleration is $+kA/m$

Motion of the Block

- The block continues to oscillate between $-A$ and $+A$
  - These are turning points of the motion
- The force is conservative
- In the absence of friction, the motion will continue forever
  - Real systems are generally subject to friction, so they do not actually oscillate forever

Orientation of the Spring

- When the block is hung from a vertical spring, its weight will cause the spring to stretch
- If the resting position of the spring is defined as $x = 0$, the same analysis as was done with the horizontal spring will apply to the vertical spring-mass system

Simple Harmonic Motion – Mathematical Representation

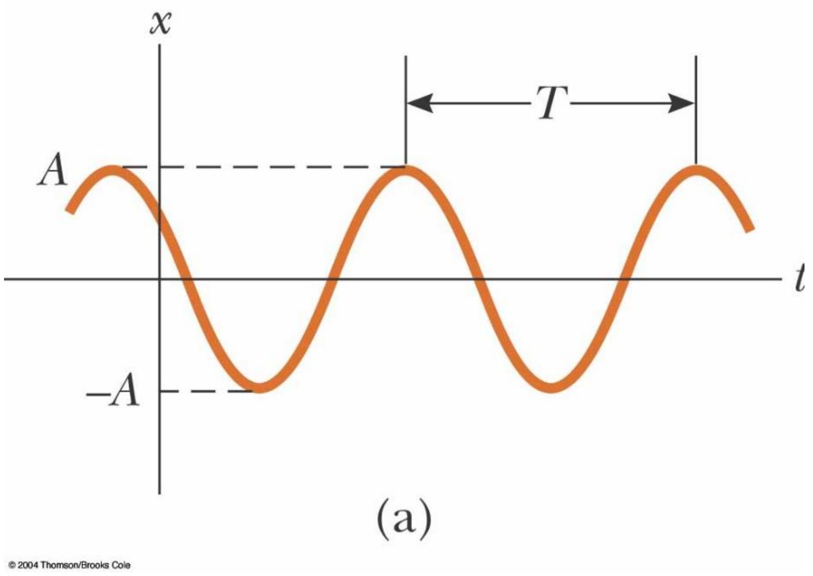
- Model the block as a particle
- Choose *x* as the axis along which the oscillation occurs
- Acceleration $a = \frac{d^2x}{dt^2} = -\frac{k}{m}x$
- We let $\omega^2 = \frac{k}{m}$
- Then $a = -\omega^2 x$

Simple Harmonic Motion – Mathematical Representation, 2

- A function that satisfies the equation is needed
  - Need a function $x(t)$ whose second derivative is the same as the original function with a negative sign and multiplied by ω^2
  - The sine and cosine functions meet these requirements

Simple Harmonic Motion – General Solution

- A solution is
- $x(t) = A\cos(\omega t + \phi)$
- A, ω, ϕ are all constants
- A cosine curve can be used to give physical significance to these constants



Simple Harmonic Motion – Definitions

- A is the amplitude of the motion
  - This is the maximum position of the particle in either the positive or negative direction
- ω is called the angular frequency
  - Units are rad/s
- ϕ is the phase constant or the initial phase angle

Simple Harmonic Motion, cont

- A and ϕ are determined uniquely by the position and velocity of the particle at $t = 0$
- If the particle is at $x = A$ at $t = 0$, then $\phi = 0$
- The ***phase*** of the motion is the quantity $(\omega t + \phi)$
- $x(t)$ is periodic and its value is the same each time ωt increases by 2π radians

Period

- The ***period***, *T*, is the time interval required for the particle to go through one full cycle of its motion
  - The values of *x* and *v* for the particle at time *t* equal the values of *x* and *v* at *t* + *T*

$$T = \frac{2\pi}{\omega}$$

Frequency

- The inverse of the period is called the ***frequency***
- The frequency represents the number of oscillations that the particle undergoes per unit time interval

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

- Units are cycles per second = hertz (Hz)

Summary Equations – Period and Frequency

- The frequency and period equations can be rewritten to solve for ω

$$\omega = 2\pi f = \frac{2\pi}{T}$$

- The period and frequency can also be expressed as:

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Period and Frequency, cont

- The frequency and the period depend only on the mass of the particle and the force constant of the spring
- They do not depend on the parameters of motion
- The frequency is larger for a stiffer spring (large values of k) and decreases with increasing mass of the particle

Motion Equations for Simple Harmonic Motion

$$x(t) = A\cos(\omega t + \phi)$$

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$$

$$a = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi)$$

- Remember, simple harmonic motion is **not** uniformly accelerated motion

Maximum Values of v and a

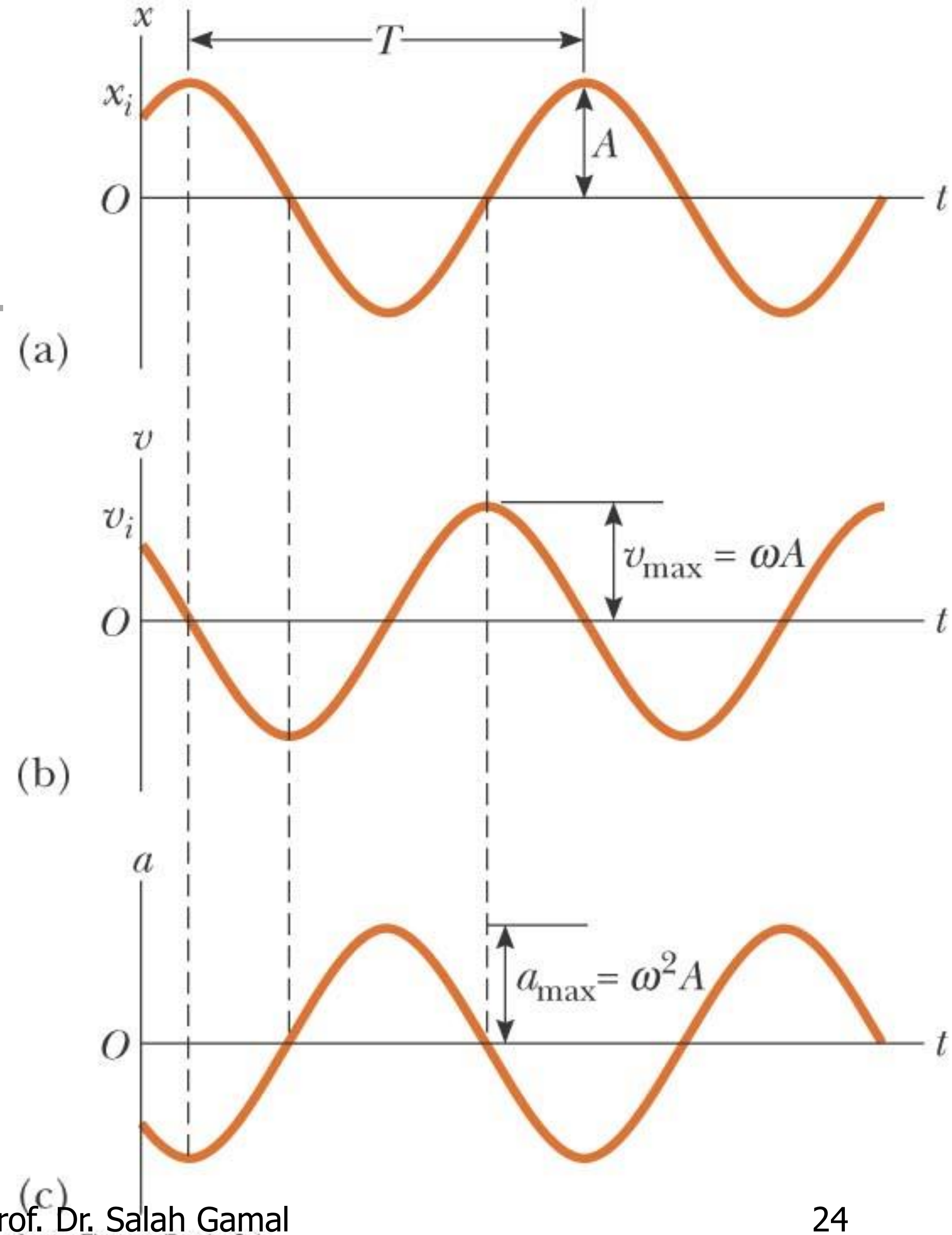
- Because the sine and cosine functions oscillate between ± 1, we can easily find the maximum values of velocity and acceleration for an object in SHM

$$v_{\max} = \omega A = \sqrt{\frac{k}{m}}\, A$$

$$a_{\max} = \omega^2 A = \frac{k}{m}\, A$$

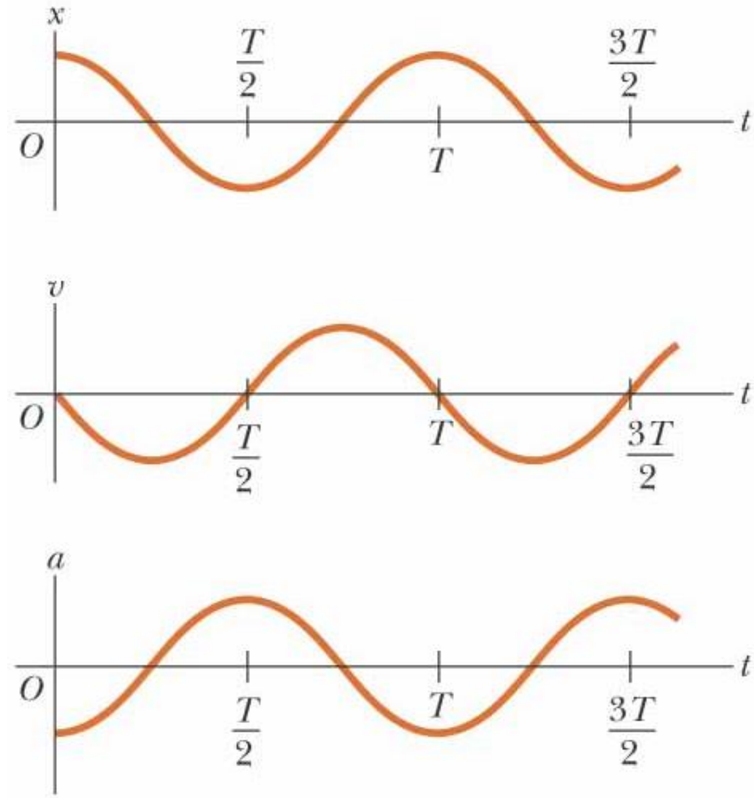
Graphs

- The graphs show:
  - (a) displacement as a function of time
  - (b) velocity as a function of time
  - (c) acceleration as a function of time
- The velocity is 90º out of phase with the displacement and the acceleration is 180º out of phase with the displacement



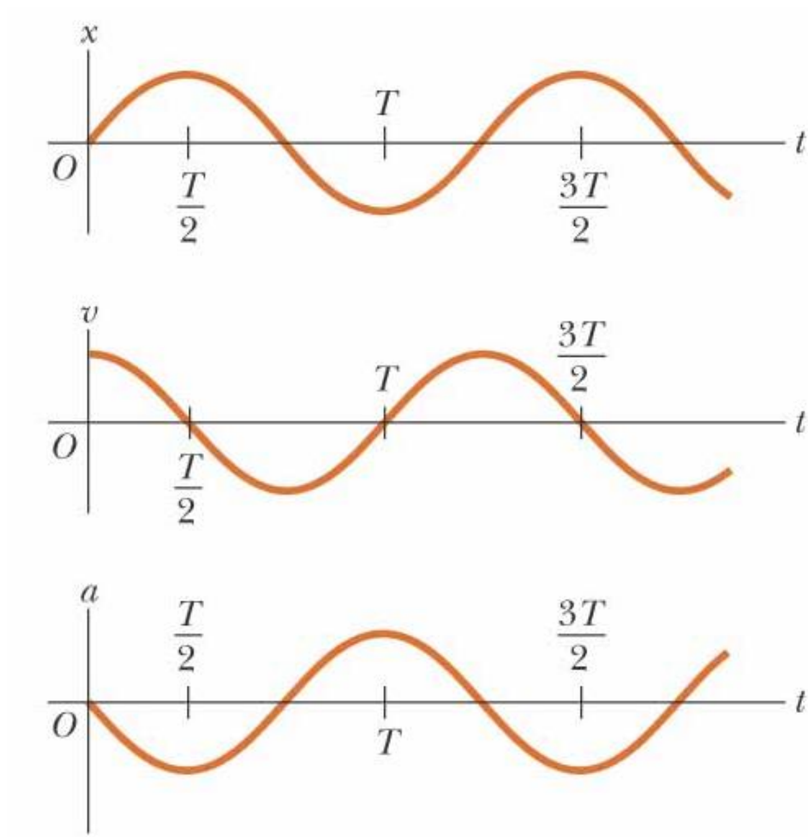
SHM Example 1

- Initial conditions at $t = 0$ are
  - $x(0) = A$
  - $v(0) = 0$
- This means $\phi = 0$
- The acceleration reaches extremes of $\pm \omega^2 A$
- The velocity reaches extremes of $\pm \omega A$



SHM Example 2

- Initial conditions at $t = 0$ are
  - $x(0)=0$
  - $v(0) = v_i$
- This means $\phi = -\pi/2$
- The graph is shifted one-quarter cycle to the right compared to the graph of $x(0) = A$

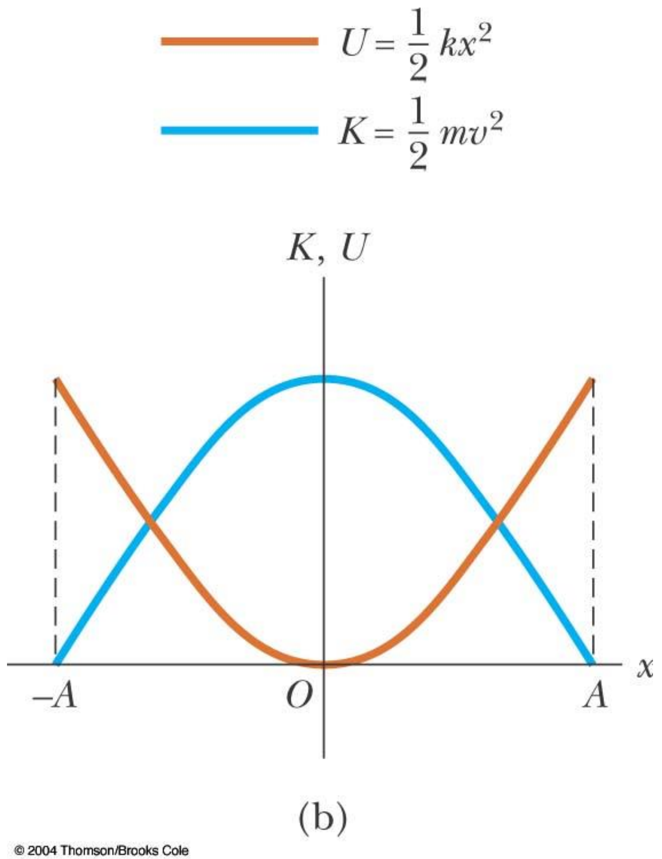


Energy of the SHM Oscillator

- Assume a spring-mass system is moving on a frictionless surface
- This tells us the total energy is constant
- The kinetic energy can be found by
  - $K = \frac{1}{2} mv^2 = \frac{1}{2} m\omega^2 A^2 \sin^2 (\omega t + \phi)$
- The elastic potential energy can be found by
  - $U = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \cos^2 (\omega t + \phi)$
- The total energy is $K + U = \frac{1}{2} kA^2$

Energy of the SHM Oscillator, cont

- The total mechanical energy is constant
- The total mechanical energy is proportional to the square of the amplitude
- Energy is continuously being transferred between potential energy stored in the spring and the kinetic energy of the block

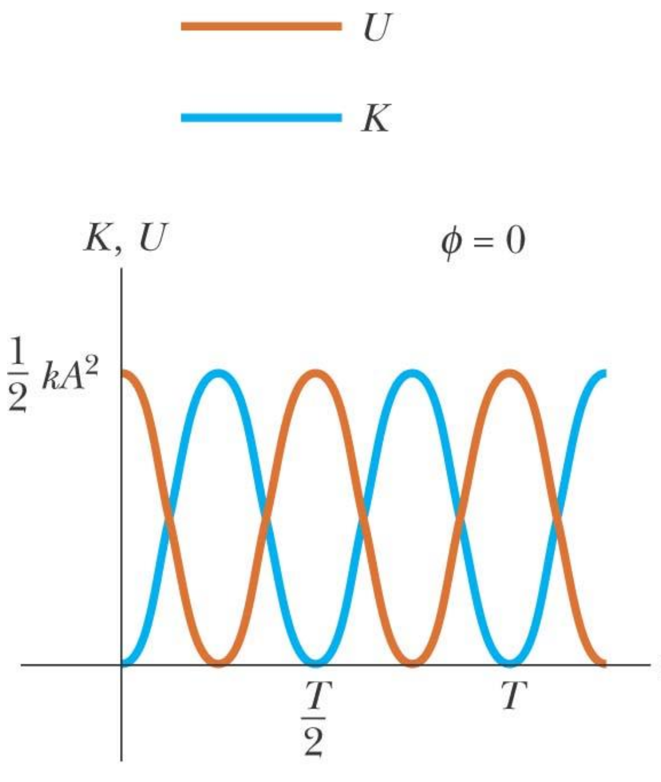


Energy of the SHM Oscillator, cont

- As the motion continues, the exchange of energy also continues
- Energy can be used to find the velocity

$$v = \pm\sqrt{\frac{k}{m}\left(A^2 - x^2\right)}$$

$$= \pm\omega\sqrt{A^2 - x^2}$$



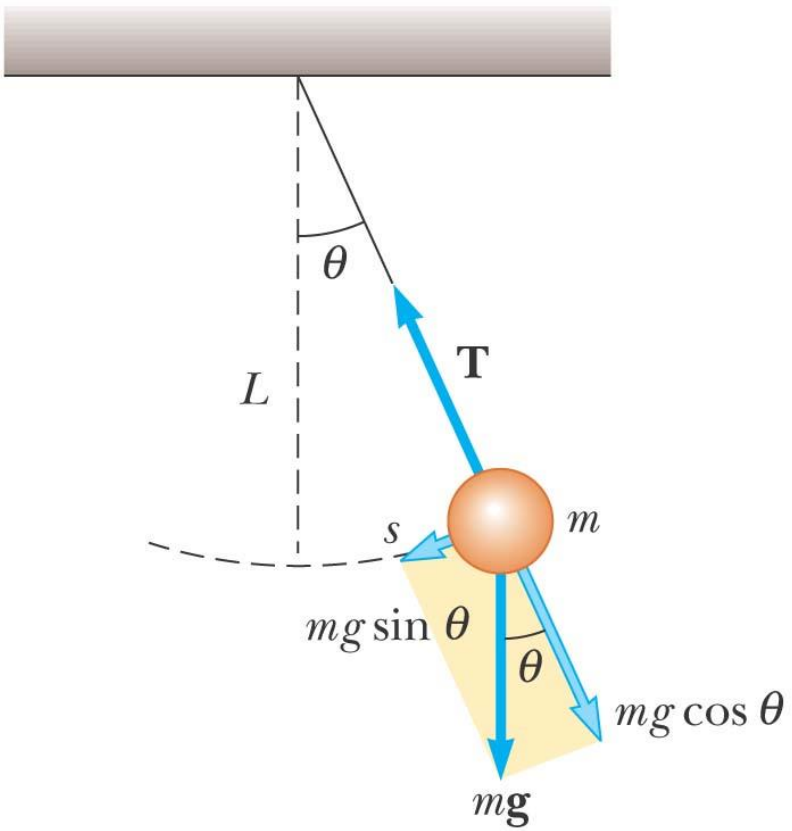
(a)

Simple Pendulum

- A simple pendulum also exhibits periodic motion
- The motion occurs in the vertical plane and is driven by gravitational force
- The motion is very close to that of the SHM oscillator
  - If the angle is $<10^o$

Simple Pendulum, 2

- The forces acting on the bob are **T** and *m***g**
  - **T** is the force exerted on the bob by the string
  - *m***g** is the gravitational force
- The tangential component of the gravitational force is a restoring force



Simple Pendulum, 3

- In the tangential direction, the equation of motion reads,

$F_t = m\,a_t$, with $F_t = -mg\sin\theta$

$$F_t = -mg\sin\theta = m\frac{d^2 s}{dt^2}$$

- But $s = L\,\theta$, and for small values of θ, $\sin\theta = \theta$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta = -\frac{g}{L}\theta$$

Simple Pendulum, 4

- Thus,

$$\frac{d^2\theta}{dt^2} = -\,\omega^2\theta$$

- where,

$$\omega^2 = \frac{g}{L}$$

which conforms to SHM

Simple Pendulum, 5

- The function θ can be written as

$$\theta = \theta_{\text{max}} \cos(\omega t + \phi)$$

The angular frequency is $\omega = \sqrt{\dfrac{g}{L}}$

The period is $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{L}{g}}$

Simple Pendulum, Summary

- The period and frequency of a simple pendulum depend only on the length of the string and the acceleration due to gravity
- The period is independent of the mass
- All simple pendula that are of equal length and are at the same location oscillate with the same period