Event-based MILP models for resource-constrained project scheduling problems

Problem Description

Resource-constrained project scheduling problem(RCPSP): NP-hard problem
A project involves activities, and resources (renewable or non-renewable), generally available in limited quantities. The processing of an activity requires throughout its duration, one or more units of one or more resources. The RCPSP deals with organizing in time the realization of activities, taking into account a number of precedence constraints, and constraints on the use and availability of the resources needed. A schedule is a solution that describes resource allocation over time, and aims at satisfying one or more objectives.

- A combinatorial optimization problem defined by a tuple (V, p, E, R, B, b)
  V: activities, p:vector of durations, E: precedence relations,
  R: resources, B: vector of resource availabilities, b: matrix of demands
- Activities constituting the project are identified by a set $\{0, \ldots, n+1\}$. Activity 0 represents by convention the start of the schedule, and activity $n+1$ represents symmetrically the end of the schedule. The set of non-dummy activities is identified by $\mathbf{A} = \{1, \ldots, n\}$.
- $p \in N^{n+2}$, p_i is the duration of activity A_i, with the special values: $p_0 = p_{n+1} = 0$
- $(i, j) \in E$ means that activity i must precede activity j
- $B \in N^m$, B_k is the availability of resource k, m is the number of available resources
- b is an $(n+2) \times m$ integer matrix, b_{ik} represents the amount of resource k used per time period during the execution of activity i, $b_{0k} = 0$, $b_{n+1,k} = 0$, for all $k \in R$.
- Schedule $S \in N^{n+2}$, S_i represents the start time of activity i, $S_0 = 0$, $\mathbf{H} = \{1, \ldots, T\}$ is the scheduling horizon, and T (the length of the scheduling horizon) is some upper bound for the makespan.

1 Basic Discrete-Time Formulation (DT)

1.1 Decision Variables

- $x_{it} = 1$ if activity i starts at time t, $x_{it} = 0$ otherwise

1.2 Objective Function

- $\min \sum_{t \in H} t x_{n+1,t}$

1.3 Constraints

- Precedence:

$$\forall (i, j) \in E, \quad \sum_{t \in H} t x_{jt} \geq \sum_{t \in H} t x_{it} + p_i \tag{1}$$

- Resource:

$$\forall t \in H, \forall k \in R, \quad \sum_{i=1}^{n} b_{ik} \sum_{\tau = t - p_i + 1}^{t} x_{i\tau} \leq B_k \tag{2}$$

- Non-preemptive:

$$\forall i \in A \cup \{0, n+1\}, \quad \sum_{t \in H} x_{it} = 1 \tag{3}$$

1.4 Problem Size

- Binary Variables: $(n+2)(T+1)$
- Constraints: $|E| + (T+1)m + n + 2$

2 Disaggregated discrete-time formulation (DDT)

- The only diffference between DT and DDT is how they formulate the precedence constraints.

$$\forall t \in H, \forall (i,j) \in E, \quad \sum_{\tau=t}^{T} x_{i\tau} + \sum_{\tau=0}^{t+p_i-1} x_{j\tau} \leq 1 \tag{4}$$

- Problem Size
  - Binary Variables: $(n+2)(T+1)$
  - Constraints: $(m + |E|)(T+1) + n + 2$

3 Flow-based continuous-time formulation (FCT)

3.1 Decision Variables

- Starting-time continuous variables S_i, for each activity i
- Sequential variables x_{ij} which are binary and indicate whether activity i is processed before activity j
- Continuous flow variables f_{ijk} to denote the quantity of resource k that is transferred from activity i (at the end of its processing) to activity j (at the start of its processing)

3.2 Objective Function

- $\min S_{n+1}$

3.3 Constraints

- Precedence:

$$\forall (i,j) \in (A \cup \{0, n+1\})^2, i{<}j, \quad x_{ij} + xji \leq 1 \tag{5}$$

$$\forall (i,j,k) \in (A \cup \{0, n+1\})^3, \quad x_{ik} \geq x_{ij} + x_{jk} - 1 \tag{6}$$

$$\forall (i,j) \in (A \cup \{0, n+1\})^2, \quad S_j - S_i \geq -M + (P_i + M)x_{ij} \tag{7}$$

  Constraints (5) state that for two distinct activities, either i precedes j, or j precedes i, or i and j are processed in parallel. Constraints (6) express the transitivity of the precedence relations. M is some large enough constant. M can be set to any valid upper bound of the makespan(e.g. $M = \sum_{i=1}^{n} p_i$).

- Resource:

$$\forall (i,j) \in (A \cup \{0\} \times A \cup \{n+1\}), \forall k \in R, \quad f_{ijk} \leq \min(b_{ik}, b_{jk})x_{ij} \tag{8}$$

$$\forall j \in A \cup \{0, n+1\}, \forall k \in R, \quad \sum_{j \in A \cup \{0,n+1\}} f_{ijk} = \overline{b_{ik}} \tag{9}$$

$$\forall i \in A \cup \{0, n+1\}, \forall k \in R, \quad \sum_{i \in A \cup \{0,n+1\}} f_{ijk} = \overline{b_{jk}} \tag{10}$$

$$\forall k \in R, \quad f_{(n+1)0k} = B_k \tag{11}$$

Constraint (8) indicates that if i precedes j, the maximum flow sent from i to j is set to $\min\{b_i, b_j\}$ while if i does not precede j the flow must be zero. Constraints (9) and (10) are resource flow conservation constraints. Constraint (11) ensures the conservation of the flow.

3.4 Problem Size

- Binary Variables: $(n+2)^2$
- Continous Variables: $m(n+2)^2 + n + 2$
- Constraints: $n^3 + (m + \frac{15}{2})n^2 + (4m + \frac{35}{2})n + 5m + 13$

4 Start/End Event-based formulation (SEE)

An event occurs when an activity starts or ends. $\varepsilon = \{0, 1, \ldots, n\}$ is the index set of the events.

4.1 Decision Variables

- $x_{ie} = 1$ if activity i starts at event e.
- $y_{ie} = 1$ if activity i ends at event e.
- Continous variable t_e represents the date of event e
- Continous variable r_ek represents the quantity of resource k required immediately after event e

4.2 Objective Function

- $\min t_n$

4.3 Constraints

- Activity i starts at event e and ends at event f, then $t_f \geq t_e + p_i$:

$$\forall (e,f) \in \varepsilon^2, f{>}e, \forall i \in A, \quad t_f \geq t_e + p_i x_{ie} - p_i(1 - y_{if}) \tag{12}$$

- A start event (respectively end event) has a single occurrence:

$$\forall i \in A, \quad \sum_{e \in \varepsilon} x_{ie} = 1, \sum_{e \in \varepsilon} y_{ie} = 1 \tag{13}$$

- The precedence relation between activities: If $i{<}j$ then i ends at event e or after, j cannot start before event e.

$$\forall (i,j) \in E, \forall e \in \varepsilon, \quad \sum_{e'=e}^{n} y_{ie'} + \sum_{e'=0}^{e-1} x_{je'} \leq 1 \tag{14}$$

- Resource conservation constraints that imply that for each resource k, its consumption immediately after event e is equal to its consumption immediately after the previous event $e-1$, plus the consumption required by the activities that start at event e, minus the consumption required by the activities that end at event e.

$$\forall e \in \varepsilon, e \geq 1, \forall k \in R, \quad r_{ek} = r_{(e-1)k} + \sum_{i \in A} b_{ik} x_{ie} - \sum_{i \in A} b_{ik} y_{ie} \tag{15}$$

- Limit the consumption of resources at each event to the availability of resources:

$$\forall e \in \varepsilon, \forall k \in R \quad r_{ek} \leq B_k \tag{16}$$

4.4 Problem Size

- Binary Variables: $2n(n+1)$
- Continous Variables: $n+1$
- Constraints: $\frac{1}{2}n^3 + n^2 + (3+|E|+m)n + |E| + m + 1$

5 On/Off Event-based formulation (OOE)

5.1 Decision Variables

- $z_{ie} = 1$ if activity i starts at event e or if it still being processed immediately after event e. Thus, z_{ie} remains equal to 1 for the duration of the process activity i.
- Continous variable t_e represents the date of event e

5.2 Objective Function

- $\min C_max$, C_{max} is the makespan.

5.3 Constraints

- $C_{max} \geq t_e + p_i$ if i is in process at event $e-1$ but not at event e

$$\forall e \in \varepsilon, \forall i \in A \quad C_{max} \geq t_e + (z_{ie} - z_{i(e-1)})p_i \tag{17}$$

- Link the binary optimization variables z_{ie} to the continuous optimization variables t_e, and ensures that the processing time of an activity is equal to the processing time of this activity.

$$\forall (e,f,i) \in \varepsilon^2 \times A, f{>}e \neq 0, \quad t_f \geq t_e + ((z_{ie} - z_{i(e-1)}) - (z_{if} - z_{i(f-1)}) - 1)p_i \tag{18}$$

- Ensure the adjacency of the events during which an activity being processed:

$$\forall e \neq 0 \in \varepsilon, \quad \sum_{e'=0}^{e-1} z_{ie'} \geq e(1 - (z_{ie} - z_{i(e-1)})) \tag{19}$$

$$\forall e \neq 0 \in \varepsilon, \quad \sum_{e'=e}^{n-1} z_{ie'} \geq e(1 + (z_{ie} - z_{i(e-1)})) \tag{20}$$

- Each activity is processed at least once during the project:

$$\forall i \in A, \quad \sum_{e \in \varepsilon} z_{ie} \geq 1 \tag{21}$$

- Precedence:

$$\forall e \in \varepsilon, \forall (i,j) \in E, \quad z_{ie} + \sum_{e'=0}^{e} z_{je'} \leq 1 + (1 - z_{ie})e \tag{22}$$

- Resource:

$$\forall e \in \varepsilon, \forall k \in R, \quad \sum_{i=0}^{n-1} b_{ik} z_{ie} \leq B_k \tag{23}$$

5.4 Problem Size

- Binary Variables: n^2
- Continous Variables: $n+1$
- Constraints: $\frac{1}{2}n^3 - \frac{1}{2}n^2 + (3+|E|+m)n - 2$