Spurious Quasi-Resonances for Stabilized BIE-Volume Formulations for Helmholtz Transition Problem

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1 Theoretical Background

1.0.1 Lipschitz Domain

Remark 1.1. Henceforth we shall require that, roughly speaking that Ω is locally the set of points located above the graph of some Lipschitz function and the boundary is this graph.

Definition 1.2 (Lipschitz domain). Let $n \in \mathbb{N}$. Let Ω be a domain of $\mathbb{R}^n$ and let $\partial\Omega$ denote the boundary of Ω. Then Ω is called a Lipschitz domain if for every point $p \in \partial\Omega$ there exists a hyperplane H of dimension $n-1$ through p, a Lipschitz-continuous function $g : H \to \mathbb{R}$ over that hyperplane, and reals $r > 0$ and $h > 0$ such that

- $\Omega \cap C = \{x + y\vec{n} \mid x \in B_r(p) \cap H, -h < y < g(x)\}$
- $(\partial\Omega) \cap C = \{x + y\vec{n} \mid x \in B_r(p) \cap H, g(x) = y\}$

where $\vec{n}$ is a unit vector that is normal to H and $C := \{x + y\vec{n} \mid x \in B_r(p) \cap H, -h < y < h\}$.

1.0.2 Sobolev Space

Definition 1.3 (H^1). For a bounded domain $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, we define the Sobolev space $H^1(\Omega) := \left\{v \in L^2(\Omega) : \int_\Omega |\operatorname{grad} v(x)|^2 \,\mathrm{d}x < \infty\right\}$ as a Hilbert space with norm $\|v\|^2_{H^1(\Omega)} := \|v\|^2_{L^2(\Omega)} + |v|^2_{H^1(\Omega)}, \quad |v|^2_{H^1(\Omega)} := \int_\Omega |\operatorname{grad} v(x)|^2 \,\mathrm{d}x$

Definition 1.4 ($H^{1/2}$).

Definition 1.5 (H^k_{loc}).

1.0.3 Trace operators

Definition 1.6 (Trace operator). A trace operator is a linear mapping from a function space on the volume domain Ω to a function space on (parts of) the boundary $\partial\Omega$.

Definition 1.7 ((Layer) potential). A (layer) potential is a linear mapping from a function space on $\partial\Omega$ into a function space on the volume domain Ω.

Definition 1.8 (Dirichlet Trace). The Dirichlet trace (operator) T_D boils down to pointwise restriction for smooth functions:

$$(\mathrm{T}_D w)(\boldsymbol{x}) := w(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Gamma, \quad w \in C^\infty(\bar{\Omega}).$$

Definition 1.9 (Dirichlet trace space). The Dirichlet trace space $H^{\frac{1}{2}}(\Gamma)$ is the Hilbert space obtained by completion of $C^\infty(\bar{\Omega})\big|_\Gamma$ with respect to the energy norm

$$\|\mathfrak{u}\|_{H^{\frac{1}{2}}(\Gamma)} := \inf\left\{\|v\|_{H^1(\Omega)} : v \in C^\infty(\bar{\Omega}), \mathsf{T}_D v = \mathfrak{u}\right\}, \quad \mathfrak{u} \in C^\infty(\bar{\Omega})\big|_\Gamma.$$

theorem 1.10. The Dirichlet trace T_D according can be extended to a continuous and surjective linear operator $\mathrm{T}_D : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$

Definition 1.11 (Neumann Trace). For smooth functions the Neumann trace (operator) T_N is defined by

$$(\mathrm{T}_N w)(\boldsymbol{x}) := \operatorname{grad} w \cdot \boldsymbol{n}(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Gamma, w \in C^\infty(\bar{\Omega}).$$

Definition 1.12 (Neumann Trace Space). The Neumann trace space $H^{-\frac{1}{2}}(\Gamma)$ is the Hilbert space obtained by the completion of $C^0(\Gamma)$ with respect to the norm

$$\|\phi\|_{H^{-\frac{1}{2}}(\Gamma)} := \|\widetilde{\phi}\|_{\widetilde{H}^{-1}(\Omega)}$$

where $\widetilde{\phi}$ is the "extension by zero to $\mathbb{R}^d$ " of ϕ. We have the definition $\|\rho\|_{\widetilde{H}^{-1}(\Omega)} := |u|_{H^1(\mathbb{R}^3)}$ where u solves $\left\{\begin{array}{c} -\Delta u = \widetilde{\rho} \text{ in } \mathbb{R}^3 \\ u \text{ satisfies decay condition} \end{array}\right.$

Definition 1.13 (Space of function with square-integrable Laplacian). We define the Hilbert space

$$H(\Delta,\Omega) := \left\{v \in H^1(\Omega) : \Delta v \in L^2(\Omega)\right\}$$

with norm

$$\|u\|^2_{H(\Delta,\Omega)} := \|u\|^2_{H^1(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}, \quad u \in H(\Delta,\Omega).$$

theorem 1.14. The Neumann trace T_N can be extended to a continuous mapping $\mathrm{T}_N : H(\Delta,\Omega) \to H^{-\frac{1}{2}}(\Gamma)$.

Definition 1.15 ($C^\infty_{\text{comp}}\left(\mathbb{R}^d\right)$).

1.0.4 Notations

- Consider a bounded Lipschitz open set $\Omega^- \subset \mathbb{R}^d$, $d = 2, 3$.
- $\Omega^+ := \mathbb{R}^d \backslash \overline{\Omega^-}$
- $\Gamma := \partial\Omega^- = \partial\Omega^+$
- $\mathbf{n}$ is the unit normal vector field on Γ pointing from Ω^-into Ω^+
- For any $\varphi \in L^2_{\text{loc}}\left(\mathbb{R}^d\right)$, we let $\varphi^- := \varphi|_{\Omega^-}$and $\varphi^+ := \varphi|_{\Omega^+}$
- $H^1_{\text{loc}}\left(\Omega^\pm, \Delta\right) := \left\{v : \chi v \in H^1\left(\Omega^\pm\right), \Delta(\chi v) \in L^2\left(\Omega^\pm\right) \text{for all } \chi \in C^\infty_{\text{comp}}\left(\mathbb{R}^d\right)\right\}$
- Dirichlet and Neumann trace operators[1]: $\gamma^\pm_D : H^1_{\text{loc}}\left(\Omega_\pm\right) \to H^{1/2}(\Gamma)$ and $\gamma^\pm_N : H^1_{\text{loc}}\left(\Omega_\pm, \Delta\right) \to H^{-1/2}(\Gamma)$ with $\gamma^\pm_D v := v^\pm|_\Gamma$ and $\gamma^\pm_N$such that if $v \in H^2_{\text{loc}}\left(\Omega_\pm\right)$then $\gamma^\pm_N v = \mathbf{n} \cdot \gamma^\pm_D(\nabla v)$
- Cauchy trace: $\gamma^\pm_C : H^1_{\text{loc}}\left(\Omega^\pm, \Delta\right) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, $\gamma^\pm_C := \left(\gamma^\pm_D, \gamma^\pm_N\right)$
- Sommerfeld radiation condition: $\varphi \in C^1\left(\mathbb{R}^d \backslash B_R\right)$, for some ball $B_R := \{|\mathbf{x}| < R\}$, and $\kappa > 0$ satisfies this condition if $\lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial\varphi(\mathbf{x})}{\partial r} - \mathrm{i}\kappa\varphi(\mathbf{x})\right) = 0$ in all directions. We then write $\varphi \in \mathrm{SRC}(\kappa)$

theorem 1.16 (Green's first formula). (From Wikipedia) $\int_U \left(\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi\right) \mathrm{d}U = \int_{\partial U} \phi\frac{\partial\psi}{\partial n}\, \mathrm{d}S$

1. The Dirichlet trace operator boils down to pointwise restriction.

2 Problem

We start by formulating the problem. We want to investigate the occurence of spurious quasi-resonances in the variational formulation of the following Helmholtz transition problem.

Definition 2.1 (Helmholtz Transmission Problem). Find $u \in H^1_{\text{loc}}\left(\mathbb{R}^d \backslash \Gamma\right) \cap \text{SRC}\left(k\sqrt{c_o}\right)$ such that

$$\begin{aligned}
\left(\Delta + \tilde{\kappa}^2 c_i\right) u^- &= 0 && \text{in } \Omega^- \\
\left(\Delta + \tilde{\kappa}^2 c_o\right) u^+ &= 0 && \text{in } \Omega^+ \\
\gamma_C^- u^- &= \gamma_C^+ u^+ + \mathbf{f} && \text{on } \Gamma.
\end{aligned}$$

2.1 Deriving the variational formulation

We will now state the variational formulation of the problem in Ω^-.
Integration by parts in Ω^- using Green's first formula implies

$$a(U,V) - \left(\gamma_1^- U, \gamma_0^- V\right)_\Gamma = 0 \quad \forall v \in H^1\left(\Omega^-\right)$$

This variational equation can be coupled to the transmission conditions using the BIE. As the problem is equivalent to the problem in the 2006 paper with $U_i = 0, f = 0, n(x) = c_i/c_o, \kappa = \tilde{\kappa}\sqrt{c_o}$, we can use the same formulation as in the 2006 paper.

The formulation is:
Find $U \in H^1(\Omega), \vartheta \in H^{-1/2}(\Gamma)$ and $p \in H^1(\Gamma)$ such that for all $V \in H^1(\Omega), \varphi \in H^{-1/2}(\Gamma)$ and $q \in H^1(\Gamma)$ there holds

$$\begin{aligned}
\mathrm{q}_\kappa(U,V) + \left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), \gamma_D^- V\right)_\Gamma - \left((\frac{1}{2}\mathrm{ld} - \mathrm{K}'_\kappa)(\vartheta), \gamma_D^- V\right)_\Gamma &= \mathrm{f}_2(V) \\
\left((\frac{1}{2}\mathrm{ld} - \mathrm{K}_\kappa)\left(\gamma_D^- U\right), \varphi\right)_\Gamma + (\mathrm{V}_\kappa(\vartheta), \varphi)_\Gamma + i\overline{\eta}(p,\varphi)_\Gamma &= \overline{\mathrm{g}_2}(V) \\
-\left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), q\right)_\Gamma - \left((\mathrm{K}'_\kappa + \frac{1}{2}\mathrm{Id})(\vartheta), q\right)_\Gamma + \mathrm{b}(p,q) &= \mathrm{h}_2(q).
\end{aligned}$$

where we have

$$\begin{aligned}
\mathrm{f}_2(V) &:= (f,V)_\Omega - \left(g_N, \gamma_D^- V\right)_\Gamma - \left(\mathrm{W}_\kappa\left(g_D\right), \gamma_D^- V\right)_\Gamma \\
\mathrm{g}_2(\varphi) &:= \left(\varphi, \left(\mathrm{K}_\kappa - \frac{1}{2}\mathrm{ld}\right)\left(g_D\right)\right)_\Gamma \\
\mathrm{h}_2(q) &:= \left(\mathrm{W}_\kappa\left(g_D\right), q\right)_\Gamma,
\end{aligned}$$

$\mathrm{q}_\kappa(U,V) := \int_\Omega \operatorname{grad} U \cdot \operatorname{grad} \bar{V} - \kappa^2 n(\mathbf{x}) U \bar{V}\, \mathrm{d}\mathbf{x}$ and $\mathrm{b}(p,q) := (\operatorname{grad}_\Gamma p, \operatorname{grad}_\Gamma q)_\Gamma + (p,q)_\Gamma$.

3 How to investigate Stability

To figure out how stable this formulation is, we use the inf-sup constant

theorem 3.1. Existence and uniqueness of discrete solutions $u_h \in V_h$ and the convergence $u_h \to u$ can only hold if the sesquilinear form a satisfies a discrete inf-sup condition, cf. [89, Eq. 2.3.6]: There exists a constant $\gamma > 0$ such that for all $v_h \in V_h$ and the whole family V_h with $h \to 0$, there holds

$$\sup_{0 \neq w_h \in V_h} \frac{\left|\mathrm{a}\left(v_h, w_h\right)\right|}{\left\|w_h\right\|_V} \geq \gamma \left\|v_h\right\|_V$$

(from P.Meury thesis).

We will now investgate the reliability of the variational formulation by investigating the inf-sup constant of the sesquilinear form on the left side of the variational problem.

Definition 3.2.

$$\gamma = \inf_{0 \neq v_h \in V_h} \sup_{0 \neq w_h \in V_h} \frac{\left|\mathrm{a}\left(v_h, w_h\right)\right|}{\left\|w_h\right\|_V \left\|v_h\right\|_V}$$

Remark 3.3. We can estimate the inf-sup constant from the minimum singular value of the Galerkin matrix (proven in P.Meury thesis).

So we now derive the Galerkin Matrix to find the inf-sup constant.

4 Derivation of Galerkin Matrix

First, we notice that we can restrict ourselves to finding functions on $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \times H^1(\Gamma)$, since the bilinear form only depends on the restriction $\gamma_D^- U$.

Now we will further restrict this space to finite subspaces and choose orthonormal bases for the finite subspaces.

4.1 Basis functions of $\mathcal{H}^{\frac{1}{2}}(\Gamma)$

We make the Fourier Ansatz $V_n = V_n^r e^{in\phi}$. Since U must satisfy $(\Delta+\kappa^2)U = 0$, we have

$$r^2\partial_r^2 V_n^r + r\partial_r V_n^r + (r^2\kappa^2 - n^2)V_n^r = 0.$$

This is Bessel's equation. Since we require convergence at the origin we have $V_n^r(r) = J_n(\kappa r)$.

We restrict our functions to $H^{\frac{1}{2}}$ and want to get an orthonormal basis $U_n(\phi) = v_n V_n(1,\phi)$ for this space:

$$(U_n, U_m)_{H^{\frac{1}{2}}(\Gamma)} = \delta_{nm}$$

This implies:

$$(U_n, U_m)_{H^{\frac{1}{2}}(\Gamma)} = (U_n, U_m)_\Gamma + (\boldsymbol{\nabla}\cdot U_n, \boldsymbol{\nabla}\cdot U_m)_\Gamma = \delta_{nm}$$

$$= |v_n|^2 2\pi(1+n^2)|J_n(\kappa)|^2\delta_{nm} \overbrace{=}^{!} \delta_{nm}$$

Therefore, we can choose real constants $v_n = \frac{1}{\sqrt{2\pi(1+n^2)}|J_n(\kappa)|}$.

Notes

In the calculation above we used $\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}}$ and eliminated the $r-$ dependency.

4.2 Other basis functions

We also pick orthonormal basis functions for $\mathcal{H}^{-\frac{1}{2}}(\Gamma)$ and $\mathcal{H}^1(\Gamma)$. We choose

$$\theta_n = w_n e^{in\phi} \text{ where } w_n = \frac{(1+n^2)^{\frac{1}{4}}}{\sqrt{2\pi}}$$

for $\mathcal{H}^{\frac{1}{2}}(\Gamma)$ and for $\mathcal{H}^{-\frac{1}{2}}(\Gamma)$ we choose

$$p_n = l_n e^{in\phi} \text{ where } l_n = \frac{1}{\sqrt{2\pi(1+n^2)}}.$$

Notes

See P. Meury 3.22 (Lemma)for explanation why we use these bases.

4.3 Constructing the Galerkin Matrix

No we can construct the Galerkin Matrix. Because of orthogonality our matrix will have a blockdiagonal structure. We start by using Greens first formula $\int_U(\psi\Delta\varphi + \nabla\psi\cdot\nabla\varphi)dV = \oint_{\partial U}\psi\nabla\varphi\cdot d\mathbf{S}$ to see that

$$q_\kappa(U_n, U_m) = \int_{\Omega^-}(\boldsymbol{\nabla}\cdot U_n \boldsymbol{\nabla}\cdot \overline{U_m} - \kappa^2 U_n\overline{U_m})$$

$$= \int_{\partial\Omega^-}\overline{U_m}\boldsymbol{\nabla}\cdot U_n d\vec{S} - \int_{\Omega^-}\underbrace{(\kappa^2 U_n + \Delta U_n)}_{=0}\overline{U_m}d\vec{S}$$

$$= \delta_{nm}\int_{\partial\Omega^-}\overline{U_n}\boldsymbol{\nabla}\cdot U_n d\vec{S}$$

$$= \delta_{nm}|v_n|^2\kappa J_n(\kappa)J_n'(\kappa)\int_{\partial\Omega^-} e^{in\phi}e^{-in\phi}dS$$

$$= \delta_{nm}\frac{1}{1+n^2}\kappa\frac{J_n'(\kappa)}{J_n(\kappa)} =: \delta_{nm}\alpha_n.$$

Also clearly $b(p_n, p_m) = (p_n, p_m)_{H^1(\Gamma)} = \delta_{nm}$

PROVE ORTHOGONALITY OF MATRIX HERE, HERLEITUNG

4.4 Eigenvalue Equations

We are also using the following eigenvalue equations

$$\begin{aligned}
\mathrm{V}_\kappa\left(y_n\right) &= \lambda_n^{(\mathrm{V})} y_n, & \lambda_n^{(\mathrm{V})} &:= \frac{i\pi}{2} J_n(\kappa) H_n^{(1)}(\kappa) \\
\mathrm{K}_\kappa\left(y_n\right) &= \lambda_n^{(\mathrm{K})} y_n, & \lambda_n^{(\mathrm{K})} &:= \frac{i\pi\kappa}{2} J_n(\kappa) H_n^{(1)'}(\kappa) + \frac{1}{2} = \frac{i\pi\kappa}{2} J_n'(\kappa) H_n^{(1)}(\kappa) - \frac{1}{2} \\
\mathrm{K}_\kappa'\left(y_n\right) &= \lambda_n^{(\mathrm{K}')} y_n, & \lambda_n^{(\mathrm{K}')} &:= \frac{i\pi\kappa}{2} J_n(\kappa) H_n^{(1)'}(\kappa) + \frac{1}{2} = \frac{i\pi\kappa}{2} J_n'(\kappa) H_n^{(1)}(\kappa) - \frac{1}{2} \\
\mathcal{W}_\kappa\left(y_n\right) &= \lambda_n^{(\mathrm{W})} y_n, & \lambda_n^{(\mathrm{W})} &:= \frac{i\pi\kappa^2}{2} J_n'(\kappa) H_n^{(1)'}(\kappa)
\end{aligned}$$

(from P. Meury.)

4.5 Matrix

Now the first equation yields with coefficents:

$$(\alpha_n + \lambda_n^{(w)} P_n^{UU}) C_n^U - (\frac{1}{2} - \lambda_n^{(K')}) P_n^{\theta U} C_n^\theta = f_2(V)$$

second equation

$$(\frac{1}{2} - \lambda_n^{(K)}) P_n^{U\theta} C_n^U + \lambda_n^{(V)} P^{\theta\theta} C_n^\theta + i\eta P^{p\theta} C_n^p = \overline{g_2}(V)$$

third equation

$$-\lambda_n^{(W)} P_n^{Up} C_U^n - (\lambda_n^{(K')} + \frac{1}{2}) P^{\theta p} C_n^\theta + C_p^n = h_2(V)$$

Using all the previously derived equations we can construct the Galerkin matrix:

$$A_n = \begin{pmatrix} (\alpha_n + \lambda_n^{(w)} P_n^{UU}) & -(\frac{1}{2} - \lambda_n^{(K')}) P_n^{\theta U} & 0 \\ (\frac{1}{2} - \lambda_n^{(K)}) P_n^{U\theta} & \lambda_n^{(V)} P^{\theta\theta} & i\overline{\eta} P^{p\theta} \\ -\lambda_n^{(W)} P_n^{Up} & -(\lambda_n^{(K')} + \frac{1}{2}) P^{\theta p} & 1 \end{pmatrix}$$

4.6 Scalar Product Values

Introduce $\tilde{v}_n = J_n(\kappa) v_n$. The values of the scalar products are

- $P_n^{UU} = 2\pi |\tilde{v}_n|^2$
- $P_n^{\theta U} = 2\pi w_n \overline{\tilde{v}_n}$
- $P_n^{U\theta} = \overline{P_n^{\theta U}}$
- $P_n^{\theta\theta} = 2\pi |w_n|^2$
- $P_n^{p\theta} = 2\pi l_n \overline{w_n}$
- $P_n^{Up} = 2\pi \tilde{v}_n \overline{l_n}$
- $P_n^{\theta p} = \overline{P_n^{p\theta}}$

5 Validation

5.1 Setup

In preparation for the following steps to validate our matrix, we have to compute right side of the discretised variational problem. If we expand $f_1 = \sum_{j=-\infty}^{\infty} \frac{1}{2\pi} f_1^j e^{ij\theta}$ and $f_2 = \sum_{j=-\infty}^{\infty} \frac{1}{2\pi} f_2^j e^{ij\theta}$, we can write out the right side of the variational formulation

$$\vec{b}_n = \begin{bmatrix} f_1(U_n) \\ \overline{g_2}(\theta_n) \\ h_2(p_n) \end{bmatrix} = \begin{bmatrix} -\overline{\tilde{v}_n} f_2^n - \lambda_n^{(W)} \overline{\tilde{v}_n} f_1^n \\ (\lambda_n^{(K)} - 0.5) \overline{w_n} f_1^n \\ \lambda_n^{(W)} \overline{l_n} f_1^n \end{bmatrix} = f_1^n \underbrace{\begin{bmatrix} -\lambda_n^{(W)} \overline{\tilde{v}_n} \\ (\lambda_n^{(K)} - 0.5) \overline{w_n} \\ \lambda_n^{(W)} \overline{l_n} \end{bmatrix}}_{\vec{x_1}} + f_2^n \underbrace{\begin{bmatrix} -\overline{\tilde{v}_n} \\ 0 \\ 0 \end{bmatrix}}_{\vec{x_1}}.$$

5.2 p = 0

From P.Meury: " At second glance, we realise that $p = 0$, if (U, ϑ) solve the problems (3.23) and (3.24), respectively. This directly follows from corollary 3.12, theorem 1.8 and the definition of the exterior Calderón projector P_+. In short, p is a "dummy variable"."

Now to verify that our derived matrix A_n is correct we just have to show that

$$A_n \begin{bmatrix} C_n^U \\ C_n^\theta \\ C_n^p \end{bmatrix} = \vec{b}$$

for $\vec{b} \in span(\vec{x}_1, \vec{x}_2) =: V_{\vec{b}}$ implies $C_n^p = 0$.
Let's assume that A_n is invertible. Then we have to show that $V_{\vec{b}} \subset Ker(P_3 A_n^{-1} \vec{x_1})$. For dimensional reasons this is equivalent to $V_{\vec{b}} = Ker(P_3 A_n^{-1} \vec{x_1})$.
Let $P_{V_{\vec{b}}}$ be the projector on $V_{\vec{b}}$. Then we just have to show that $P_3 A_n^{-1} P_{V_{\vec{b}}}$ is approximately the zero map.

5.3 Solving a special case

(Would expect exact solution because in same space).

Another thing we can to to validate the correctness of our derived matrix is to show that it yields the correct numerical solution. Define $\kappa_j = \sqrt{n_j}\kappa$ for $j = i, o$ and consider the special case

$$\vec{f} = \begin{bmatrix} H_0^{(1)}(\kappa_o) - J_0(\kappa_i) \\ \kappa_i J_1(\kappa_i) - \kappa_o H_1^{(1)}(\kappa_o) \end{bmatrix}.$$

Then the solution is

$$u = J_0(\kappa_i r), x \in \Omega^-, u = H_0^{(1)}(\kappa_o r), x \in \Omega^+.$$

This can be seen directly by plugging in, using $J_0'(x) = -J_1(x)$ and $H_0'(x) = -H_1'(x)$.
Let's validate if we get the same numerical solution using our matrix A. We just use the $\vec{b}_n$ as computed above, with $f_1^j = \delta_{0j} 2\pi(H_0^{(1)}(\kappa_o) - J_0(\kappa_i))$, $f_2^j = \delta_{0j} 2\pi(\kappa_i J_1(\kappa_i) - \kappa_o H_1^{(1)}(\kappa_o))$.
The analytical solution can be written as $u = \frac{1}{v_0} U_0$. Also, we expect $\vartheta = \partial_r (u_{\Omega^+})_{|\Gamma} = -\frac{1}{w_0} \kappa_o H_1^{(1)}(\kappa_o) \theta_0$ and $p = 0$.

So the solution vector should be

$$C_j^U = \delta_{0j} \frac{1}{v_0}, C_j^\theta = -\delta_{0j} \frac{1}{w_0} \kappa_o H_1^{(1)}(\kappa_o), C_j^p = 0, \forall j.$$

5.4 Even more special case

In addition to the assumptions above, pick $c_o = 1$, $\eta = 1$, and $\kappa = 5.502$ (then $J_0(\kappa) = 0$).
Then we have $\lambda_0^{(V)} = 0$, $\lambda_n^{(K)} = \lambda_n^{(K')} = 1/2$. For easier notation write $\lambda_n^{(W)} =: \nu$. Now we have

$$A_0 = \begin{pmatrix} \alpha_0 + \nu P_0^{UU} & 0 & 0 \\ 0 & 0 & iP^{p\theta} \\ -\nu P_0^{Up} & -P_0^{\theta p} & 1 \end{pmatrix}.$$

Moreover we have

$$b_0 = 2\pi H_0^{(1)}(\kappa) \begin{bmatrix} -\nu \overline{v_0} \\ 0 \\ \nu \overline{l_0} \end{bmatrix} + 2\pi(H_1^{(1)}(\kappa) - J_1^{(1)}(\kappa)) \begin{bmatrix} -\overline{v_0} \\ 0 \\ 0 \end{bmatrix}.$$

We have

$$A_0^{-1} = \begin{pmatrix} (\alpha_n + \nu P_n^{UU})^{-1} & 0 & 0 \\ -\frac{\nu P_0^{(Up)}}{(\alpha_0 + \nu P_0^{UU}) P_0^{\theta p}} & -\frac{i}{(P_0^{\theta p})^2} & -\frac{1}{P_0^{\theta p}} \\ 0 & -\frac{i}{P_0^{p\theta}} & 0 \end{pmatrix}.$$

So $x = A_0^{-1} b_0 = ..).$ (

6 Numerical Results