Spurious Quasi-Resonances for Stabilized BIE-Volume Formulations for Helmholtz Transition Problem

Frederic Jørgensen. *June 3, 2022*

Abstract

tbd Lorem ipsum dolor sit amet, consetetur sadipscing elitr, sed diam nonumy eirmod tempor invidunt ut labore et dolore magna aliquyam erat, sed diam voluptua. At vero eos et accusam et justo duo dolores et ea rebum. Stet clita kasd gubergren, no sea takimata sanctus est Lorem ipsum dolor sit amet. Lorem ipsum dolor sit amet, consetetur sadipscing elitr, sed diam nonumy eirmod tempor invidunt ut labore et dolore magna aliquyam erat, sed diam voluptua. At vero eos et accusam et justo duo dolores et ea rebum. Stet clita kasd gubergren, no sea takimata sanctus est Lorem ipsum dolor sit amet.

1 Introduction

Let $\Omega^- \subset \mathbb{R}^d, d > 0$ be a bounded Lipschitz domain and define $\Omega^+ := \mathbb{R}^d \backslash \overline{\Omega^-}$, $\Gamma := \partial\Omega^-$. For any function f on $\mathbb{R}^d$ define $f^\pm := f|_{\Omega^\pm}$. Consider the Neumann and Dirichlet trace operators

$$\begin{aligned}\gamma_D^\pm :& H^1_{\text{loc}}\left(\Omega^\pm\right) \to H^{\frac{1}{2}}(\Gamma), \left(\gamma_D^\pm f\right)(x) := f(x)\\ \gamma_N^\pm :& H_{\text{loc}}\left(\Delta, \Omega^\pm\right) \to H^{-\frac{1}{2}}(\Gamma), \left(\gamma_N^\pm f\right)(x) := \nabla f(x) \cdot \mathbf{n}(x)\end{aligned}$$

where $H^1_{\text{loc}}(\Omega^\pm)$, $H_{\text{loc}}(\Delta, \Omega^\pm)$, $H^{\frac{1}{2}}(\Gamma)$, and $H^{-\frac{1}{2}}(\Gamma)$ are defined in chapter 3 of [1]. Now, the Cauchy trace is $\gamma_C^\pm : H^1_{\text{loc}}(\Omega^\pm, \Delta) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ with values given by $\gamma_C^\pm := \left(\gamma_D^\pm, \gamma_N^\pm\right)$. This concludes all required definitions to formulate the Helmholtz transmission problem.

Definition 1.1 (Helmholtz transmission problem)**.** *For $\tilde{\kappa}, c_i, c_o > 0$ and $\mathbf{f} \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ find $u \in H^1_{\text{loc}}\left(\mathbb{R}^d \backslash \Gamma\right)$ such that*

$$\begin{aligned}\left(\Delta + \tilde{\kappa}^2 c_i\right) U^- &= 0 \quad && \text{in } \Omega^-\\ \left(\Delta + \tilde{\kappa}^2 c_o\right) U^+ &= 0 && \text{in } \Omega^+\\ \gamma_C^+ U^+ - \gamma_C^- U^- &= \mathbf{f} && \text{on } \Gamma.\end{aligned} \tag{1}$$

Additionally, u must satisfy the Sommerfeld radiation condition $\lim\limits_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial U(x)}{\partial r} - i\sqrt{c_o}\tilde{\kappa}U(x)\right) = 0$ where r refers to the radial spherical coordinate.

This problem is well-posed and the solution is unique as shown in Lemma 2.2 in [2].
Hiptmair et al. considered the *single-trace formulations* (STF) of this problem [3]. This is a reformulation of this problem in terms of *boundary integral equations* (BIEs). When investigating the case $c_i < c_o$ they found that the involved *boundary integral operators* (BIOs) as a function of $\tilde{\kappa}$ exposed a nonphysical resonanace behavior. More specifically, the operator norm of the STF BIOs was found to have resonances (called *spurious quasi-resonances*) that the norm of the solution operator did not. Formally, Hiptmair et al. defined the solution operator as follows [3].

Definition 1.2. *Given positive real numbers k, c_i, and c_o, $S : H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ is the solution operator if $S(c_i, c_o)\mathbf{f} := \gamma_C^- u$ where u solves eq 1.*

Hiptmair et al. were able to remove these *spurious quasi-resonances* using an augmented formulation of the BIEs.
The goal of this paper is to investigate the occurence of *spurious quasi-resonance* in the *regularized variational formulation* of the Helmholtz transmission problem as proposed in P. Meury's doctoral thesis [4]. For easier notation let $\kappa = \tilde{\kappa}\sqrt{c_o}$ on the following pages.

Definition 1.3 (Regularized variational formulation). *Find $U \in H^1(\Omega), \theta \in H^{-1/2}(\Gamma)$ and $p \in H^1(\Gamma)$ such that for all $V \in H^1(\Omega), \varphi \in H^{-1/2}(\Gamma)$ and $q \in H^1(\Gamma)$ there holds*

$$\begin{aligned} \mathrm{q}_\kappa(U,V) + \left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), \gamma_D^- V\right)_\Gamma - \left((\frac{1}{2}\mathrm{ld} - \mathrm{K}'_\kappa)(\theta), \gamma_D^- V\right)_\Gamma &= g_1(V) \\ \left((\frac{1}{2}\mathrm{ld} - \mathrm{K}_\kappa)\left(\gamma_D^- U\right), \varphi\right)_\Gamma + (\mathrm{V}_\kappa(\theta),\varphi)_\Gamma + i\overline{\eta}(p,\varphi)_\Gamma &= \overline{g_2(\varphi)} \\ -\left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), q\right)_\Gamma - \left((\mathrm{K}'_\kappa + \frac{1}{2}\mathrm{Id})(\theta), q\right)_\Gamma + \mathrm{b}(p,q) &= g_3(q). \end{aligned} \tag{2}$$

where we have

$$\begin{aligned} g_1(V) &:= (f,V)_\Omega - \left(g_N, \gamma_D^- V\right)_\Gamma - \left(\mathrm{W}_\kappa\left(g_D\right), \gamma_D^- V\right)_\Gamma \\ g_2(V)(\varphi) &:= \left(\varphi, \left(\mathrm{K}_\kappa - \frac{1}{2}\mathrm{ld}\right)(g_D)\right)_\Gamma \\ g_3(V)(q) &:= \left(\mathrm{W}_\kappa\left(g_D\right), q\right)_\Gamma \\ \mathrm{q}_\kappa(U,V) &:= \int_\Omega \operatorname{grad} U \cdot \operatorname{grad} \bar{V} - \kappa^2 n(x) U \bar{V} \, \mathrm{d}x \\ \mathrm{b}(p,q) &:= (\operatorname{grad}_\Gamma p, \operatorname{grad}_\Gamma q)_\Gamma + (p,q)_\Gamma. \end{aligned}$$

This formulation is derived from the Helmholtz transmission problem by partially integrating the Helmholtz equation, applying Green's first formula, coupling the resulting variational problem to the BIEs using Dirichlet-to-Neumann maps, and transforming the Cauchy trace [4] [1].

Remark 1.4. *From here on, we will use the constant $\tilde{c} := \frac{c_i}{c_o}$. This reduces the amount of parameters to consider. Moreover, it makes more clear that the resonance behavior depends on the ratio and not the scaling of the refractive indices.*

To investigate the occurence of spurious quasi-resonances, we consider the simple example where $d = 2$ and $\Omega^- = B_1(0)$ in the following chapters if not mentioned otherwise.

2 Constructing the Galerkin Matrix

A numerical solution of the regularized variational formulation requires the construction of the Galerkin matrix of eq. 2. To proceed with this, we must first construct a basis of a subspace of the the solution space $H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma) \times H^1(\Gamma)$. Orthonormality is required since the representation matrix of an endomorphism is orthogonal if and only if the representation basis is orthogonal. As our goal is to approximate operator norms discretely, we must choose a orthonormal basis, allowing us to use an SVD for this[2].
Before we can derive the Galerkin matrix, we have to define an orthonormal basis of a finite subspace $H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma) \times H^1(\Gamma)$, so that we can formulate the problem in terms of finding the coefficients with respect to this basis.
To simplify the following derivations, we will consider the following space

Definition 2.1. *For $0 \leq s < \infty$ the space $\mathcal{H}^s_{\tilde{\kappa}}(X)$ is defined as the subspace of all functions $\varphi \in L^2(X)$ such that*

$$\sum_{n \in \mathbb{Z}} \left(\tilde{\kappa}^2 + n^2\right)^s |\varphi_n|^2 < \infty.$$

for the Fourier coefficients φ_n of φ. We define an inner product on this space by

$$(\varphi, \psi)_{\mathcal{H}^s(\mathbb{S})} := \sum_{n \in \mathbb{Z}} \left(\tilde{\kappa}^2 + n^2\right)^s \varphi_n \overline{\psi_n}.$$

A justification for this definition is given in Meury (lemma 3.22) and the norm is introduced in (2021 Spurious resonances).
The following proposition helps reducing our solution space.

Proposition 2.2. *There exists an operator $A : H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \times H^1(\Gamma) \to \mathbb{C}^3$ such that the left hand side operator of eq. 2 evaluated at $(U, \theta, p) \in H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \times H^1(\Gamma)$ is equal to $A(\gamma_D^- U, \theta, p)$ if U solves the Helmholtz equation.*

The following lemma is required for the proof.

Lemma 2.3. *Let $U, V \in H^1(\Omega^-)$ such that U solves the Helmholtz equation. Then $q_\kappa(U,V) = \int_{\partial\Omega^-} \overline{V} \nabla U \cdot n dS$ with normal vector n.*

1. The formulation here is equivalent to the case $U_i = 0, f = 0, n(x) = c_i/c_o$ in section 3 of [4].
2. How we will do this will be defined more specifically on the following pages.

Proof. Greens first formula $\int\limits_U(\psi\Delta\varphi+\nabla\psi\cdot\nabla\varphi)dx=\oint_{\partial U}\psi\nabla\varphi\cdot ndS$ with normal vector n implies

$$\begin{aligned}q_\kappa(U,V)&=\int_{\Omega^-}(\nabla U\nabla\overline{V}-\tilde{c}\kappa^2U\overline{V})dx\\&=\int_{\partial\Omega^-}\overline{V}\nabla U\cdot ndS-\int_{\Omega^-}(\tilde{c}\kappa^2U+\Delta U)\overline{V}\cdot ndS\\&=\int_{\partial\Omega^-}\overline{V}\nabla U\cdot ndS\end{aligned}$$

where we used the Helmholtz equation in the third equation. □

Proof of proposition 2.2. The previous lemma 2.3 already concluded this proof: Consider the operator A identical to the operator on the left hand side of eq. 2 except $q_\kappa(U,V)$ being replaced by $\int_{\partial\Omega^-}\overline{V}\nabla U\cdot ndS$. Then all entries of A only depend on γ_D^-U,θ,p □

3 Constructing the Galerkin Matrix

Now let's restrict ourselves to the space $\mathcal{S}_N^{\frac{1}{2}}\times\mathcal{S}_N^{-\frac{1}{2}}\times\mathcal{S}_N^1$ where $N\in\mathbb{N}$ and $\mathcal{S}_N^{\frac{1}{2}}$ is the restriction to $\operatorname{span}(U_{-N},-U_{-N+1},...,U_N)$ and similarly for $\mathcal{S}_N^{-\frac{1}{2}}$ and $\mathcal{S}_N^1$ and construct the Galerkin Matrix.

3.1 Redefining the problem

We reduce our problem to the space $\mathcal{S}^{\frac{1}{2}}\times\mathcal{S}^{-\frac{1}{2}}\times\mathcal{S}^1$:

Definition 3.1 (The restricted problem). *Find $(u,\theta,p)\in\mathcal{S}_N^{\frac{1}{2}}\times\mathcal{S}_N^{-\frac{1}{2}}\times\mathcal{S}_N^1$ such that for all $(V,\varphi,p)\in\mathcal{S}_N^{\frac{1}{2}}\times\mathcal{S}_N^{-\frac{1}{2}}\times\mathcal{S}_N^1$ there holds*

$$\begin{aligned}\mathrm{q}_\kappa(U,V)+\left(\mathrm{W}_\kappa\left(\gamma_D^-U\right),\gamma_D^-V\right)_\Gamma-\left((\frac{1}{2}\mathrm{ld}-\mathrm{K}'_\kappa)(\theta),\gamma_D^-V\right)_\Gamma&=g_1(V)\\\left((\frac{1}{2}\mathrm{ld}-\mathrm{K}_\kappa)\left(\gamma_D^-U\right),\varphi\right)_\Gamma+(\mathrm{V}_\kappa(\theta),\varphi)_\Gamma+i\overline{\eta}(p,\varphi)_\Gamma&=\overline{g_2(V)}(\varphi)\\-\left(\mathrm{W}_\kappa\left(\gamma_D^-U\right),q\right)_\Gamma-\left((\mathrm{K}'_\kappa+\frac{1}{2}\mathrm{Id})(\theta),q\right)_\Gamma+\mathrm{b}(p,q)&=g_3(V)(q).\end{aligned}$$

Now let's simplify the notation of this problem a bit. First, let's extend $(u,\theta,p)=\sum\limits_{n=-N}^{N}(C_n^UU_n,C_n^\theta\theta_n,C_n^pp_n)$, $f_1=\sum\limits_{j=-N}^{N}\frac{1}{2\pi}f_1^je^{ij\theta}$ and $f_2=\sum\limits_{j=-N}^{N}\frac{1}{2\pi}f_2^je^{ij\theta}$.
Define $\vec{C}_n=(C_n^U,C_n^\theta,C_n^p)$.
Let's introduce a few new constants before deriving the Galerkin Matrix.

Lemma 3.2. $q_\kappa(U_l,U_m)=2\pi\delta_{lm}|v_l|^2\kappa\frac{J_l'(\kappa)}{J_l(\kappa)}=:\alpha_l$.

Proof of Lemma. Greens first formula $\int\limits_U(\psi\Delta\varphi+\nabla\psi\cdot\nabla\varphi)dx=\oint_{\partial U}\psi\nabla\varphi\cdot ndS$ with normal vector n implies

$$\begin{aligned}q_\kappa(U_l,U_m)&=\int_{\Omega^-}(\nabla U_l\nabla\overline{U_m}-\tilde{c}\kappa^2U_l\overline{U_m})dx\\&=\int_{\partial\Omega^-}\overline{U_m}\nabla U_l\cdot ndS-\int_{\Omega^-}\underbrace{(\tilde{c}\kappa^2U_l+\Delta U_l)}_{=0}\overline{U_m}\cdot ndS\\&=\delta_{lm}\int_{\partial\Omega^-}\overline{U_l}\nabla U_l\cdot ndS\\&=\delta_{lm}|v_l|^2\sqrt{\tilde{c}}\kappa\frac{J_l'(\sqrt{\tilde{c}}\kappa)}{J_l(\sqrt{\tilde{c}}\kappa)}\int_{\partial\Omega^-}e^{in\phi}e^{-in\phi}dS\\&=\delta_{lm}2\pi|v_l|^2\kappa\sqrt{\tilde{c}}\frac{J_l'(\sqrt{\tilde{c}}\kappa)}{J_l(\sqrt{\tilde{c}}\kappa)}.\end{aligned}$$

□

Remark 3.3. *In the previous proof U_n is integrated over Ω^-, while U_n is technically only defined on Γ. However, note that, knowing the coefficients of U_n, it is straightforward to extend U to Ω^- using our basis vectors V_n. Also, the end result only depends on the restriction to Γ.* [3]

3. These two points is what justified the restriction of U_n to Γ in the first place.

Lemma 3.4. $b(p_n, p_m) = (p_n, p_m)_{H^1(\Gamma)} = \frac{(1+n^2)}{\tilde{\kappa}^2+n^2}$

Proof. $p_n \frac{\tilde{\kappa}^2+n^2}{1+n^2}$ is orthogonal w.r.t $(-,-)_{H^1(\Gamma)}$. $\square$

We note that, within the variational formulation, U is only evaluated on the boundary Γ[4], so that we can restrict U to $H^{\frac{1}{2}}(\Gamma)$, which simplifies our problem.
Now we can can define a complete orthogonal system $(H)^{\frac{1}{2}}(\Gamma) \times (H)^{-\frac{1}{2}}(\Gamma) \times (H)^{1}(\Gamma)$. We choose $U_n = v_n e^{in\phi}$, $\theta_n = w_n e^{in\phi}$, $p_n = l_n e^{in\phi}$ of $\mathcal{H}^{-\frac{1}{2}}(\Gamma)$ and $\mathcal{H}^1(\Gamma)$ respectively, where

$$
\begin{aligned}
v_n &= \frac{1}{\sqrt{2\pi\left(\tilde{\kappa}^2+n^2\right)}} \\
w_n &= \frac{\left(\tilde{\kappa}^2+n^2\right)^{\frac{1}{4}}}{\sqrt{2\pi}} \\
l_n &= \frac{1}{\sqrt{2\pi\left(\tilde{\kappa}^2+n^2\right)}}
\end{aligned}
\tag{3}
$$

Definition 3.5. *Define the notation $P_n^{ab} = (a_n, b_n)_\Gamma$ where $a_n, b_n \in L^2(\Gamma)$. In particular, the following values will be helpful:*

- $P_n^{UU} = 2\pi|v_n|^2$
- $P_n^{\theta U} = 2\pi w_n \overline{v_n}$
- $P_n^{U\theta} = \overline{P_n^{\theta U}}$
- $P_n^{\theta\theta} = 2\pi|w_n|^2$
- $P_n^{p\theta} = 2\pi l_n \overline{w_n}$
- $P_n^{Up} = 2\pi v_n \overline{l_n}$
- $P_n^{\theta p} = \overline{P_n^{p\theta}}$.

Let's introduce the following matrix and vector:

Definition 3.6.

$$
A_n := \begin{pmatrix} (\alpha_n + \lambda_n^{(w)} P_n^{UU}) & -(\frac{1}{2} - \lambda_n^{(K')}) P_n^{\theta U} & 0 \\ (\frac{1}{2} - \lambda_n^{(K)}) P_n^{U\theta} & \lambda_n^{(V)} P^{\theta\theta} & i\eta P^{p\theta} \\ -\lambda_n^{(W)} P_n^{Up} & -(\lambda_n^{(K')} + \frac{1}{2}) P^{\theta p} & \beta_n \end{pmatrix}
$$

and

$$
\vec{b}_n = \begin{bmatrix} g_1(U_n) \\ \overline{g_2}(\theta_n) \\ g_3(p_n) \end{bmatrix} = \begin{bmatrix} -\overline{v_n} f_2^n - \lambda_n^{(W)} \overline{v_n} f_1^n \\ (\lambda_n^{(K)} - 0.5)\overline{w_n} f_1^n \\ \lambda_n^{(W)} \overline{l_n} f_1^n \end{bmatrix} = f_1^n \underbrace{\begin{bmatrix} -\lambda_n^{(W)} \overline{v_n} \\ (\lambda_n^{(K)} - 0.5)\overline{w_n} \\ \lambda_n^{(W)} \overline{l_n} \end{bmatrix}}_{\vec{x_1}} + f_2^n \underbrace{\begin{bmatrix} -\overline{v_n} \\ 0 \\ 0 \end{bmatrix}}_{\vec{x_1}}
$$

where α_n, β_n are constants that are defined below.

The following theorem summarises the Galerkin formulation of the problem:

Theorem 3.7. $(u, \theta, p) \in \mathcal{H}^{\frac{1}{2}} \times \mathcal{H}^{-\frac{1}{2}} \times \mathcal{H}^1$ *solves the problem in definition 3.1. if and only if $A_n \vec{C_n} = \vec{b}_n$ for all $n = -N, -N+1, ..., N$.*

Before we proof this we need the following Lemma from P. Meury doctoral thesis:

Lemma 3.8. *The following eigenvalue equations hold for $y_n = e^{in\phi}$:*

$$
\begin{aligned}
\mathrm{V}_\kappa(y_n) &= \lambda_n^{(\mathrm{V})} y_n, & \lambda_n^{(\mathrm{V})} &:= \frac{i\pi}{2} J_n(\kappa) H_n^{(1)}(\kappa) \\
\mathrm{K}_\kappa(y_n) &= \lambda_n^{(\mathrm{K})} y_n, & \lambda_n^{(\mathrm{K})} &:= \frac{i\pi\kappa}{2} J_n(\kappa) H_n^{(1)'}(\kappa) + \frac{1}{2} = \frac{i\pi\kappa}{2} J_n'(\kappa) H_n^{(1)}(\kappa) - \frac{1}{2} \\
\mathrm{K}'_\kappa(y_n) &= \lambda_n^{(\mathrm{K}')} y_n, & \lambda_n^{(\mathrm{K}')} &:= \frac{i\pi\kappa}{2} J_n(\kappa) H_n^{(1)'}(\kappa) + \frac{1}{2} = \frac{i\pi\kappa}{2} J_n'(\kappa) H_n^{(1)}(\kappa) - \frac{1}{2} \\
\mathcal{W}_\kappa(y_n) &= \lambda_n^{(\mathrm{W})} y_n, & \lambda_n^{(\mathrm{W})} &:= -\frac{i\pi\kappa^2}{2} J_n'(\kappa) H_n^{(1)'}(\kappa).
\end{aligned}
$$

in particular these eigenvalue equations are satisfied for U_n, θ_n, p_n.

Now we can proof theorem 3.7.

4. This is actually not clear for the term q_κ. However, we will prove this assumption later.

Proof of theorem 3.7. Both sides of the problem in definition 3.7 are linear in (V,φ,q). Therefore the condition in the problem is satisfied for all $(V,\varphi,q)\in\mathcal{S}_N^{\frac{1}{2}}\times\mathcal{S}_N^{-\frac{1}{2}}\times\mathcal{S}_N^{1}$ if and only it is satisfied for $(V,\varphi,q)\in(U_n,\theta_n,p_n)\forall n\in\{-N,-N+1,...,N\}$. So we end up with $(2N+1)$ systems of equations of the form

$$\begin{aligned}
\mathrm{q}_\kappa(U,U_n)+\left(\mathrm{W}_\kappa\left(\gamma_D^- U\right),U_n\right)_\Gamma-\left((\frac{1}{2}\mathrm{ld}-\mathrm{K}'_\kappa)(\theta),U_n\right)_\Gamma&=\mathrm{f}_2(U_n)\\
\left((\frac{1}{2}\mathrm{ld}-\mathrm{K}_\kappa)\left(\gamma_D^- U\right),\theta_n\right)_\Gamma+(\mathrm{V}_\kappa(\theta),\theta_n)_\Gamma+i\overline{\eta}(p,\theta_n)_\Gamma&=\overline{g_2(V)}(\theta_n)\ .\\
-\left(\mathrm{W}_\kappa\left(\gamma_D^- U\right),p_n\right)_\Gamma-\left((\mathrm{K}'_\kappa+\frac{1}{2}\mathrm{Id})(\theta),p_n\right)_\Gamma+\mathrm{b}(p,p_n)&=g_3(V)(p_n).
\end{aligned}$$

Now, before plugging in for (U,θ,p) note the following structural feature: Each of the summands of the left side of the problem has a factor that contains one of the following three bilinear forms:

- A scalar product $(a,b)_\Gamma$ of one of the entries of (U,θ,p) and a basis function.
- The bilinear form $b(a,b)$ of of the entries of (U,θ,p) and a basis function.
- The bilinear form $q_\kappa(a,b)$ of of the entries of (U,θ,p) and a basis function.

Note that for each of these bilinear forms s we have the orthogonality property $s(a_n,b_m)=0$ for $n\neq m$, $a,b\in\{U,\theta,p\}$. Also considering that our basis are also eigenvectors of the BIO, this implies that for each of the $(2N+1)$ problems all but one x_n is in the kernel of the left hand side. Or, said more simply, the problem can be written as a blockdiagonal problem with 3×3 blocks.
Plugging the relations derived in the former lemmata for $q_\kappa(U,U_n),b(p,p_n)$, the BIOs and the scalar products elementary calculation yields A_nx_n for the left hand side of the problem.
For the right, plugging in our extension for f_1,f_2 yields b_n. This concludes the proof. □

4 Validation

4.1 Solving a special case

To validate the correctness of our derived matrix we show that it yields the correct numerical solution for a simple example.
Consider the special case

$$\vec{f}=\begin{bmatrix}H_n^{(1)}(\kappa)-J_n(\sqrt{\tilde{c}}\kappa)\\ H_n'^{(1)}(\kappa)-J_n'(\sqrt{\tilde{c}}\kappa)\end{bmatrix}e^{in\phi}$$

where $n=-N,-N+1,...,N$
Then the solution is

$$u=J_n(\sqrt{\tilde{c}}\kappa r)e^{in\phi},x\in\Omega^-,u=H_n^{(1)}(\kappa r)e^{in\phi},x\in\Omega^+.$$

This can be seen directly by plugging in.

Remark 4.1. *Note that the discretized problem will yield the same solution as the original one as the restricted space contains the analytical solution.*

Proposition 4.2. *With regards to our chosen basis, the analytical solution can be written as* $(U,\theta,p)=(\frac{1}{v_n}U_n,\frac{1}{w_n}\kappa H_n'^{(1)}(\kappa)\theta_n,0)$.
So the solution vector should be

$$C_j^U=\delta_{nj}\frac{1}{v_n},C_j^\theta=\delta_{nj}\frac{1}{w_n}\kappa H_n'^{(1)}(\kappa),C_j^p=0,\forall j.$$

Proof. still to be written out, but the main point is: $\theta=\partial_r(u_{\Omega^+})_{|\Gamma}=-\frac{1}{w_0}\kappa_o H_1^{(1)}(\kappa_o)\theta_0$ and $p=0$, (from page 33 Thesis P. Meury). □

Let's validate whether we get the same numerical solution using our matrix A_n. We just use the $\vec{b}_n$ as computed above, with $f_1^j=\delta_{nj}2\pi(J_n(\sqrt{\tilde{c}}\kappa)-H_n^{(1)}(\kappa)),f_2^j=\delta_{nj}2\pi(\sqrt{\tilde{c}}\kappa J_1'(\sqrt{\tilde{c}}\kappa)-\kappa H_n'^{(1)}(\kappa))$.

To measure how good the solution is, let's introduce the ζ-number:

Definition 4.3 (ζ-number)**.** *For a fixed n and a fixed κ, let $\vec{C}_n^{num}(\kappa)$ be the numerical solution vector to the problem $diag(A_{-N},...,A_N)\vec{C}_n^{num}(\kappa)=(\vec{b}_{-N},...,\vec{b}_N)$. Let $\vec{C}_n^{ana}(\kappa)$ be the analytical solution vector for the same problem constructed in proposition 4.2. The ζ-number is defined as*

$$\zeta(\kappa)=\max_{n\in[-N,...,N]}\frac{\|\vec{C}_n^{num}(\kappa)-\vec{C}_n^{ana}(\kappa)\|}{\|\vec{C}_n^{ana}(\kappa)\|}$$

As seen in fig. **??**, the ζ-number is negligible across all values of κ and n which validates our derivation of A_n and b_n.

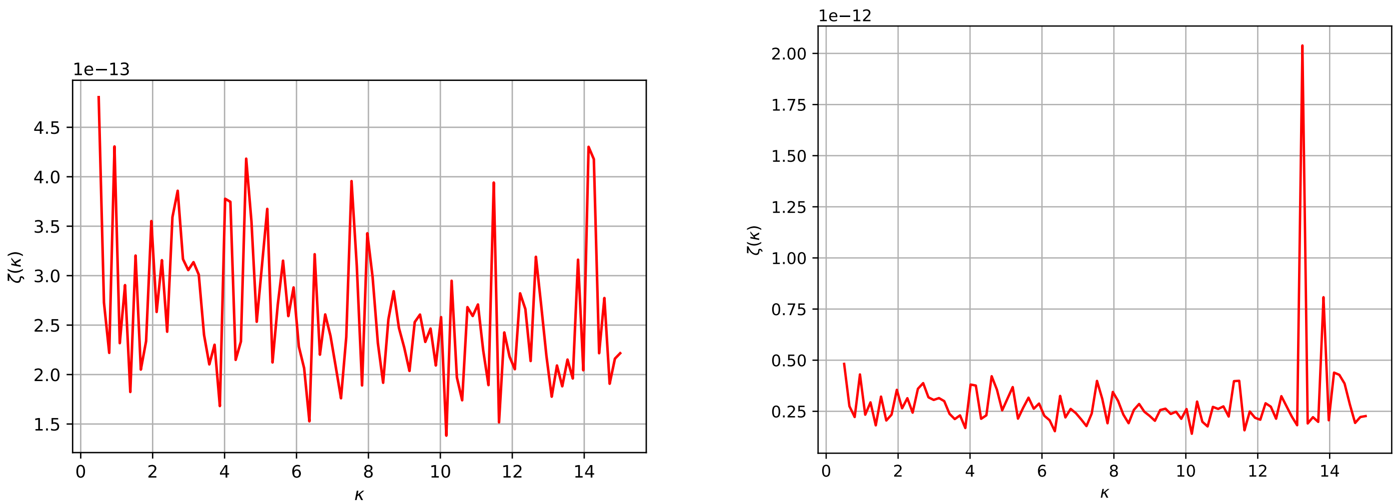

Fig. 1: Maximum relative residuum of the numerical solution by wave number. On the left plot we have $\tilde{c} = 1/3$ (f.e.$c_i = 1, c_o = 3$) and on the right plot we have $\tilde{c} = 3/1$. We always use $N = 100$ from here on.

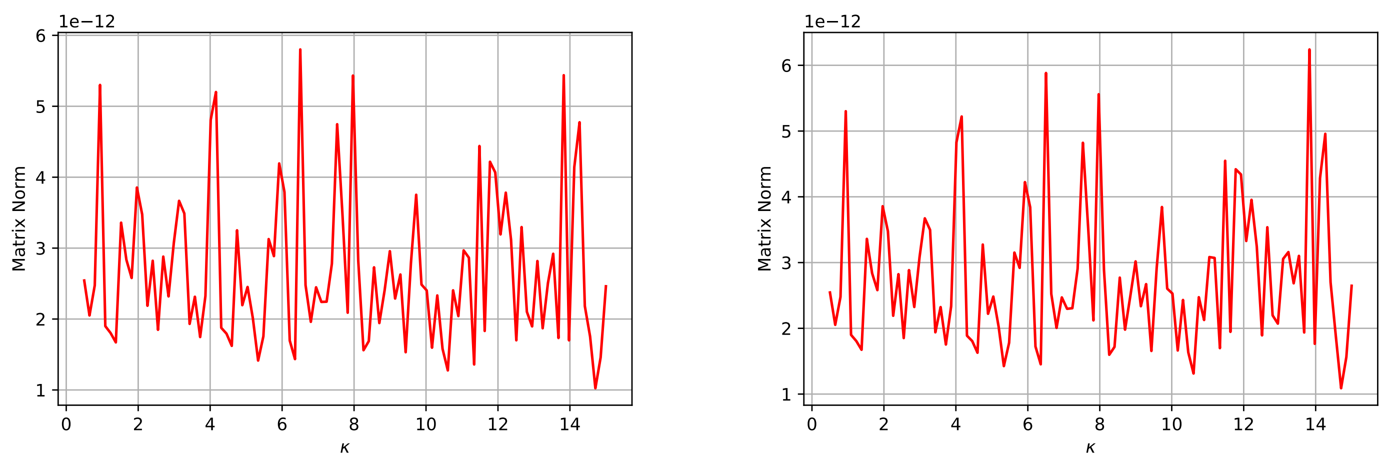

Fig. 2: Euclidian matrix norm of composed matrix $P_3 A_n^{-1} P_{V_{\vec{b}}}$. On the left plot we have $\tilde{c} = 1/3$ (f.e.$c_i = 1, c_o = 3$) and on the right plot we have $\tilde{c} = 3/1$.

4.2 p = 0

We validate our matrix A with another method, using the following remark from the doctoral thesis from P.Meury:

Remark 4.4. *At second glance, we realise that* $p = 0$*, if* (U, θ) *solve the problem.*

Let's assume A_n is invertible[5]. We can use the remark to validate that our derived matrix A_n is correct, using the following proposition:

Proposition 4.5. *Let* $V_{\vec{b}} := span(\vec{x}_1, \vec{x}_2)$ *and let* $P_{V_{\vec{b}}}$ *be the projector onto* $V_{\vec{b}}$*. Let* P_3 *be the projector onto* $(0, 0, 1)$*. Then* $P_3 A_n^{-1} P_{V_{\vec{b}}} = 0$.

Proof. Because of the remark 4.4, every solution of $A_n \vec{x} = \vec{b}_n$ satisfies $x_3 = 0$ where $\vec{b} \in V_{\vec{b}}$.
Therefore $V_{\vec{b}} \subset Ker(P_3 A_n^{-1})$. Since $\vec{x}_1, \vec{x}_2$ are linearly independent, $\dim V_{\vec{b}} = 2$, which implies $V_{\vec{b}} = Ker(P_3 A_n^{-1})$. So $P_3 A_n^{-1} P_{V_{\vec{b}}} = 0$. □

Let's check whether this is actually satisfied by calculating the euclidian matrix norm of A for a range of κ values.
As we see in fig 2, this is satisfied which is another validation of our matrix A.

5. This is fair to assume for most frequencies since the variational problem is supposed to have a unique solution.

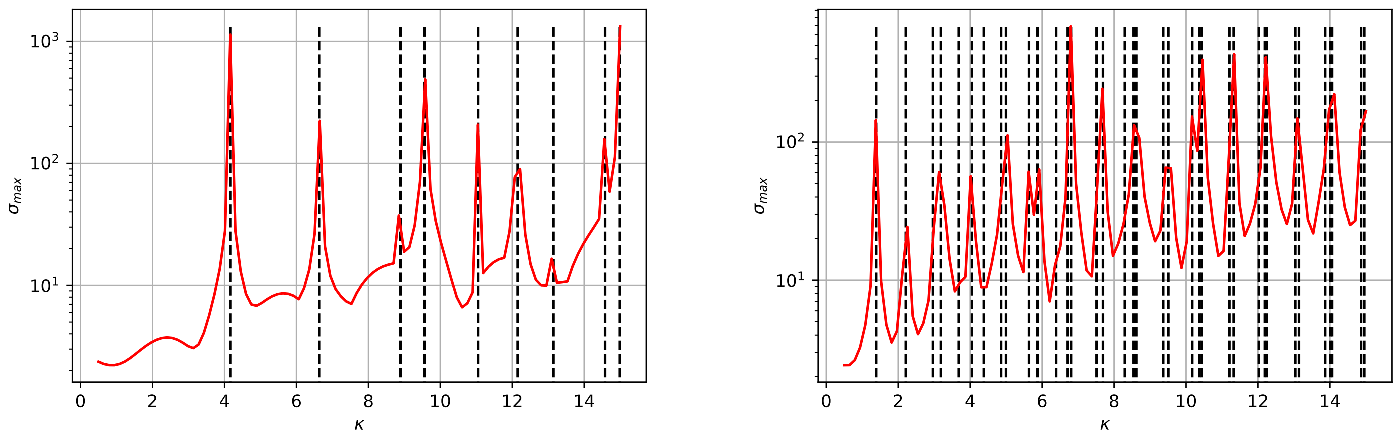

Fig. 3: Maximum singular value of the matrix $diag(A_n)$ by wave number κ. On the left plot we have $\tilde{c} = 1/3$ (f.e.$c_i = 1, c_o = 3$) and on the right plot we have $\tilde{c} = 3/1$.
The vertical dashed lines correspond to zeros of the Bessel functions.

5 Constructing the Solution Operator

For better comparability, we construct the discretized solution operator for the considered problem.

Plugging the Fourier ansatz into eq. 1 and imposing convergence at the origin and the Sommerfeld radiation condition results in

$$u^- = \sum_{n=-\infty}^{\infty} u_n^- \frac{J_n(\sqrt{c_i}\tilde{\kappa}r)}{J_n(\sqrt{c_i}\tilde{\kappa})} e^{in\phi}$$

$$u^+ = \sum_{n=-\infty}^{\infty} u_n^+ \frac{H_n(\sqrt{c_o}\tilde{\kappa}r)}{H_n(\sqrt{c_o}\tilde{\kappa})} e^{in\phi}$$

where u_n^- and u_n^+ are the restrictions of u to Ω^- und Ω^+.
Extend $f_i = \sum_{n=-\infty}^{\infty} f_i^n e^{in\phi}$ for $i = 1, 2$. The transmission condition $\gamma_C^+ u^+ = \gamma_C^- u^- + \mathbf{f}$ implies

$$\begin{pmatrix} 1 & -1 \\ \sqrt{c_i}\tilde{\kappa}\frac{J_n'(\sqrt{c_i}\tilde{\kappa})}{J_n(\sqrt{c_i}\tilde{\kappa})} & -\sqrt{c_o}\tilde{\kappa}\frac{H_n'(\sqrt{c_o}\tilde{\kappa})}{H_n(\sqrt{c_o}\tilde{\kappa})} \end{pmatrix} \begin{pmatrix} u_n^- \\ u_n^+ \end{pmatrix} = \begin{pmatrix} f_n^1 \\ f_n^2 \end{pmatrix}. \tag{4}$$

By using the well-known inverse of a 2x2 matrix we obtain

$$u_n^- = \zeta((-\sqrt{c_o}\tilde{\kappa}J_n(\sqrt{c_i}\tilde{\kappa})H_n'(\sqrt{c_i}\tilde{\kappa}))f_n^1 + J_n(\sqrt{c_i}\tilde{\kappa})H_n(\sqrt{c_o}\tilde{\kappa})f_n^2). \tag{5}$$

where $\zeta := \frac{1}{\sqrt{c_i}\tilde{\kappa}J_n'(\sqrt{c_i}\tilde{\kappa})H_n(\sqrt{c_o}\tilde{\kappa}) - \sqrt{c_o}\tilde{\kappa}H_n'(\sqrt{c_o}\tilde{\kappa})J_n(\sqrt{c_i}\tilde{\kappa})}$. This presents the solution operator, as we get γ_D^- from restriction to the boundary. γ_N^- can be directly obtained by restriction of the normal derivative of u_n^- to the boundary which is equivalent to multiplying the Fourier coefficients by $\sqrt{c_i}\tilde{\kappa}\frac{J_n'(\sqrt{c_i}\tilde{\kappa})}{J_n(\sqrt{c_i}\tilde{\kappa})}$. After rescaling the u_n, f_i^n to the complete orthonormal system defined in eq. 3 we obtain the solution operator matrix:

$$S_{io}^n = \zeta \begin{pmatrix} -\sqrt{c_o}\tilde{\kappa}J_n(\sqrt{c_i}\tilde{\kappa})H_n'(\sqrt{c_i}\tilde{\kappa}) & \sqrt{n^2+\tilde{\kappa}^2}J_n(\sqrt{c_i}\tilde{\kappa})H_n(\sqrt{c_o}\tilde{\kappa}) \\ -\frac{1}{\sqrt{n^2+\tilde{\kappa}^2}}\sqrt{c_o}\sqrt{c_i}\tilde{\kappa}^2 J_n'(\sqrt{c_i}\tilde{\kappa})H_n'(\sqrt{c_i}\tilde{\kappa}) & \sqrt{c_i}\tilde{\kappa}J_n'(\sqrt{c_i}\tilde{\kappa})H_n(\sqrt{c_o}\tilde{\kappa}) \end{pmatrix} \tag{6}$$

that maps the Fourier coefficients of $\vec{f}$ in the complete orthonormal system for $\mathcal{H}^{\frac{1}{2}} \times \mathcal{H}^{-\frac{1}{2}}$ (as defined in eq. 3) to the Fouier coefficients of $\gamma_C^- u$ in the same basis.

6 Numerical Results

Now let's investigate maximal singular value of $A = diag(A_{-N}, A_{-N+1}, ..., A_N)$. Again, we look at the scenarios where $c_i = 1, c_o = 3$ and vice versa (fig. 3) We see that the peaks often coincide with the zeros of the Bessel function.

We can also consider the minimum singular value (fig. 4). Finally, let's consider the ratio of the minimum and maximum singular value (fig. 5). We see that in the case $c_i = 1, c_o = 3$, where the solution operator has weaker resonances resonances, the resonances from our matrix operator or also less strong than in the case $c_i = 3, c_o = 1$. [3]

Use word quasi-resonances (f.e. used in Hiptmair) for peaks of sol operator Remark 1.5. The physical reason for the existence of quasi-resonances when $n_i > n_o$ is that, in this case, geometric-optic rays can undergo total internal reflection when hitting Γ from Ω^-.

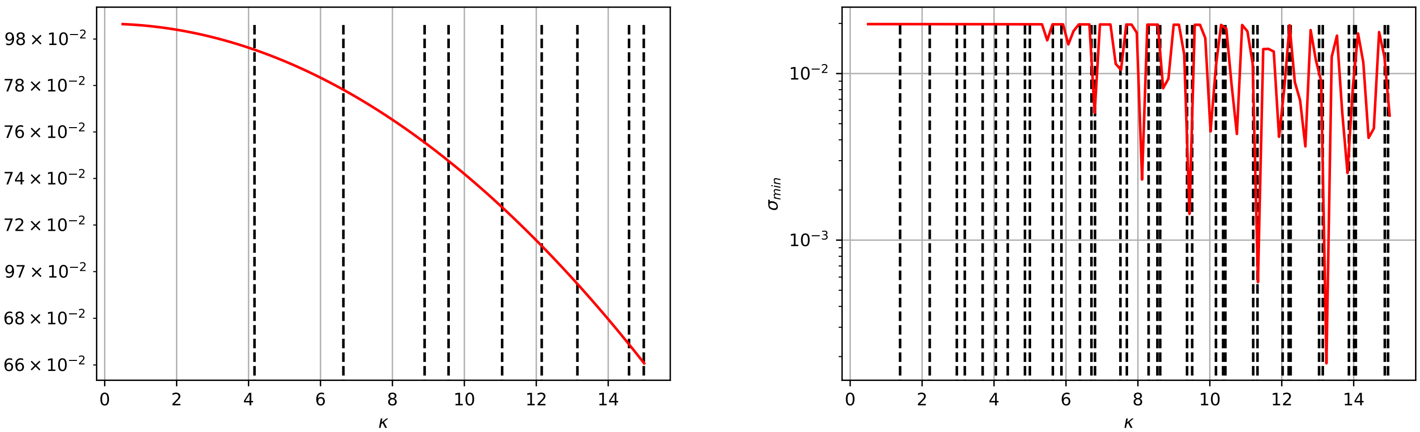

Fig. 4: Maximum singular value of the matrix $diag(A_n)$ by wave number κ. On the left plot we have $\tilde{c} = 1/3$ (f.e.$c_i = 1, c_o = 3$) and on the right plot we have $\tilde{c} = 3/1$. Again, the vertical dashed lines correspond to zeros of the Bessel functions.

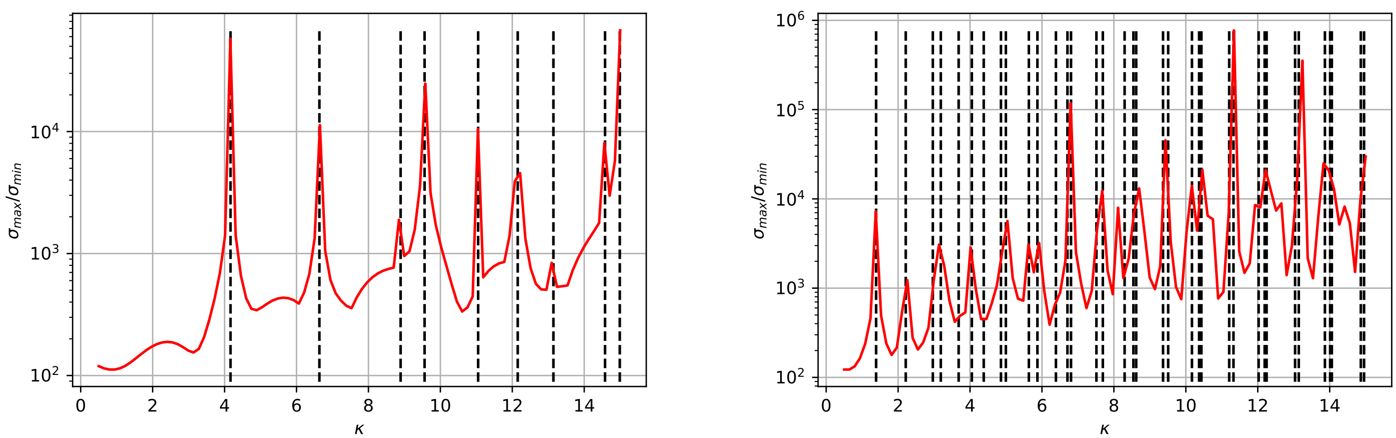

Fig. 5: Ratio of Maximum and Minimum singular value of the matrix $diag(A_n)$ by wave number κ. On the left plot we have $\tilde{c} = 1/3$ (f.e.$c_i = 1, c_o = 3$) and on the right plot we have $\tilde{c} = 3/1$. The vertical dashed lines correspond to zeros of the Bessel functions.

Rays "hugging" the boundary via a large number of bounces with total internal reflection correspond to solutions of the transmission problem localised near Γ; in the asymptotic-analysis literature these solutions are known as "whispering gallery" modes; see, e.g., [3, 4]. The existence of quasi- resonances of the transmission problem has only been rigorously established when $\Omega^-$is smooth and convex with strictly positive curvature. The understanding above via rays suggests that such quasi-resonances and quasimodes do not exist for polyhedral Ω^-(since sharp corners prevent rays from moving parallel to the boundary), although solutions with localisation qualitatively similar to that of quasimodes can be seen when $\Omega^-$is a pentagon [20, Figure 13] or a hexagon [6, Figure 23].

Interpretation: da $sigma_m in$ stabil ist Verfahren stabil

7 Appendix

resonances roots besselfunctions (Plots mit Roots in Anhang)

Weitere Werte von c_i, c_o, nu testen um zu sehen ob aehnliches Ergebnis

References

[1] W. McLean and W. C. H. McLean, *Strongly elliptic systems and boundary integral equations.* Cambridge university press, 2000.

[2] A. Moiola and E. A. Spence, "Acoustic transmission problems: wavenumber-explicit bounds and resonance-free regions," *Mathematical Models and Methods in Applied Sciences*, vol. 29, no. 02, pp. 317–354, 2019.

[3] R. Hiptmair, A. Moiola, and E. A. Spence, "Spurious quasi-resonances in boundary integral equations for the helmholtz transmission problem," *arXiv preprint arXiv:2109.08530*, 2021.

[4] P. E. Meury, *Stable finite element boundary element Galerkin schemes for acoustic and electromagnetic scattering.* PhD thesis, ETH Zurich, 2007.

Symbols

In this section we provide a definitions of all symbols used that are not defined in the text below.