# Spurious Quasi-Resonances for Stabilized BIE-Volume Formulations for Helmholtz Transition Problem

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## 1 Problem

We start by formulating the problem. We want to investigate the occurence of spurious quasi-resonances in the variational formulation of the following Helmholtz transition problem.

**Definition 1.1** (Helmholtz Transmission Problem). *Find* $u \in H^1_{\text{loc}}\left(\mathbb{R}^d \backslash \Gamma\right) \cap \text{SRC}\left(k\sqrt{c_o}\right)$ *such that*

$$\begin{aligned} \left(\Delta + \tilde{\kappa}^2 c_i\right) u^- &= 0 && \text{in } \Omega^- \\ \left(\Delta + \tilde{\kappa}^2 c_o\right) u^+ &= 0 && \text{in } \Omega^+ \\ \gamma_C^+ u^+ &= \gamma_C^- u^- + \mathbf{f} && \text{on } \Gamma. \end{aligned} \tag{1}$$

### 1.1 Deriving the variational formulation

We will now state the variational formulation of the problem in $\Omega^-$.
Integration by parts in $\Omega^-$ using Green's first formula implies

$$a(U,V) - \left(\gamma_1^- U, \gamma_0^- V\right)_\Gamma = 0 \quad \forall v \in H^1\left(\Omega^-\right)$$

This variational equation can be coupled to the transmission conditions using the BIE. As the problem is equivalent to the problem in the doctoral thesis of P. Meury (section 3) with $U_i = 0, f = 0, n(x) = c_i/c_o, \kappa = \tilde{\kappa}\sqrt{c_o}$, we can use the same formulation.

**Remark 1.2.** *From here on, we will use the constant* $\tilde{c} := \frac{c_i}{c_o}$ *for two reasons. This reduces the amount of parameters we have to consider and makes it more clear that the resonance behavior depends only on the ratio and not the scaling of the refractive indices.*

The formulation is: Find $U \in H^1(\Omega), \theta \in H^{-1/2}(\Gamma)$ and $p \in H^1(\Gamma)$ such that for all $V \in H^1(\Omega), \varphi \in H^{-1/2}(\Gamma)$ and $q \in H^1(\Gamma)$ there holds

$$\begin{aligned} \mathrm{q}_\kappa(U,V) + \left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), \gamma_D^- V\right)_\Gamma - \left((\frac{1}{2}\mathrm{ld} - \mathrm{K}'_\kappa)(\theta), \gamma_D^- V\right)_\Gamma &= g_1(V) \\ \left((\frac{1}{2}\mathrm{ld} - \mathrm{K}_\kappa)\left(\gamma_D^- U\right), \varphi\right)_\Gamma + (\mathrm{V}_\kappa(\theta), \varphi)_\Gamma + i\overline{\eta}(p,\varphi)_\Gamma &= \overline{g_2(V)}(\varphi) \\ -\left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), q\right)_\Gamma - \left((\mathrm{K}'_\kappa + \frac{1}{2}\mathrm{Id})(\theta), q\right)_\Gamma + \mathrm{b}(p,q) &= g_3(V)(q). \end{aligned} \tag{2}$$

where we have

$$\begin{aligned} g_1(V) &:= (f,V)_\Omega - \left(g_N, \gamma_D^- V\right)_\Gamma - \left(\mathrm{W}_\kappa\left(g_D\right), \gamma_D^- V\right)_\Gamma \\ g_2(V)(\varphi) &:= \left(\varphi, \left(\mathrm{K}_\kappa - \frac{1}{2}\mathrm{ld}\right)\left(g_D\right)\right)_\Gamma \\ g_3(V)(q) &:= \left(\mathrm{W}_\kappa\left(g_D\right), q\right)_\Gamma, \end{aligned}$$

$\mathrm{q}_\kappa(U,V) := \int_\Omega \operatorname{grad} U \cdot \operatorname{grad} \bar{V} - \kappa^2 n(\mathbf{x}) U \bar{V} \, \mathrm{d}\mathbf{x}$ and $\mathrm{b}(p,q) := (\operatorname{grad}_\Gamma p, \operatorname{grad}_\Gamma q)_\Gamma + (p,q)_\Gamma$.

# 2 Galerkin Matrix

## 2.1 Constructing an orthonormal basis

Before we can derive the Galerkin matrix, we have to define an orthonormal basis of a finite subspace $H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma) \times H^1(\Gamma)$, so that we can formulate the problem in terms of finding the coefficients with respect to this basis.
To simplify the following derivations, we will consider the following space

**Definition 2.1.** *For $0 \leq s < \infty$ the space $\mathcal{H}^s_{\tilde{\kappa}}(X)$ is defined as the subspace of all functions $\varphi \in L^2(X)$ such that*

$$\sum_{n\in\mathbb{Z}} \left(\tilde{\kappa}^2 + n^2\right)^s |\varphi_n|^2 < \infty.$$

*for the Fourier coefficients $\varphi_n$ of $\varphi$. We define an inner product on this space by*

$$(\varphi, \psi)_{\mathcal{H}^s(\mathbb{S})} := \sum_{n\in\mathbb{Z}} \left(\tilde{\kappa}^2 + n^2\right)^s \varphi_n \overline{\psi_n}.$$

A justification for this definition is given in Meury (lemma 3.22) and the norm is introduced in (2021 Spurious resonances).
We note that, within the variational formulation, $U$ is only evaluated on the boundary $\Gamma$[1], so that we can restrict $U$ to $H^{\frac{1}{2}}(\Gamma)$, which simplifies our problem.
Now we can can define a complete orthogonal system $(H)^{\frac{1}{2}}(\Gamma) \times (H)^{-\frac{1}{2}}(\Gamma) \times (H)^1(\Gamma)$. We choose $U_n = v_n V_n(1, \phi)$, $\theta_n = w_n e^{in\phi}$, $p_n = l_n e^{in\phi}$ of $\mathcal{H}^{-\frac{1}{2}}(\Gamma)$ and $\mathcal{H}^1(\Gamma)$ respectively, where

$$\begin{aligned} v_n = \frac{1}{\sqrt{2\pi\left(\tilde{\kappa}^2 + n^2\right)}\left|J_n\left(\sqrt{\frac{c_i}{c_o}}\kappa\right)\right|}, V_n = J_n\left(\sqrt{\frac{c_i}{c_o}}\kappa r\right) e^{in\phi} \\ w_n = \frac{\left(\tilde{\kappa}^2 + n^2\right)^{\frac{1}{4}}}{\sqrt{2\pi}} \\ l_n = \frac{1}{\sqrt{2\pi\left(\tilde{\kappa}^2 + n^2\right)}} \end{aligned} \tag{3}$$

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# 3 Constructing the Galerkin Matrix

Now let's restrict ourselves to the space $\mathcal{S}_N^{\frac{1}{2}} \times \mathcal{S}_N^{-\frac{1}{2}} \times \mathcal{S}_N^1$ where $N \in \mathbb{N}$ and $\mathcal{S}_N^{\frac{1}{2}}$ is the restriction to span$(U_{-N}, -U_{-N+1}, ..., U_N)$ and similarly for $\mathcal{S}_N^{-\frac{1}{2}}$ and $\mathcal{S}_N^1$ and construct the Galerkin Matrix.

## 3.1 Redefining the problem

We reduce our problem to the space $\mathcal{S}^{\frac{1}{2}} \times \mathcal{S}^{-\frac{1}{2}} \times \mathcal{S}^1$:

**Definition 3.1** (The restricted problem)**.** *Find $(u, \theta, p) \in \mathcal{S}_N^{\frac{1}{2}} \times \mathcal{S}_N^{-\frac{1}{2}} \times \mathcal{S}_N^1$ such that for all $(V, \varphi, p) \in \mathcal{S}_N^{\frac{1}{2}} \times \mathcal{S}_N^{-\frac{1}{2}} \times \mathcal{S}_N^1$ there holds*

$$\begin{aligned} \mathrm{q}_\kappa(U, V) + \left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), \gamma_D^- V\right)_\Gamma - \left((\frac{1}{2}\mathrm{ld} - \mathrm{K}'_\kappa)(\theta), \gamma_D^- V\right)_\Gamma = g_1(V) \\ \left((\frac{1}{2}\mathrm{ld} - \mathrm{K}_\kappa)\left(\gamma_D^- U\right), \varphi\right)_\Gamma + (\mathrm{V}_\kappa(\theta), \varphi)_\Gamma + i\overline{\eta}(p, \varphi)_\Gamma = \overline{g_2(V)}(\varphi) \\ -\left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), q\right)_\Gamma - \left((\mathrm{K}'_\kappa + \frac{1}{2}\mathrm{Id})(\theta), q\right)_\Gamma + \mathrm{b}(p, q) = g_3(V)(q). \end{aligned}$$

Now let's simplify the notation of this problem a bit. First, let's extend $(u, \theta, p) = \sum_{n=-N}^{N} (C_n^U U_n, C_n^\theta \theta_n, C_n^p p_n)$, $f_1 = \sum_{j=-N}^{N} \frac{1}{2\pi} f_1^j e^{ij\theta}$ and $f_2 = \sum_{j=-N}^{N} \frac{1}{2\pi} f_2^j e^{ij\theta}$.
Define $\vec{C}_n = (C_n^U, C_n^\theta, C_n^p)$.
Let's introduce a few new constants before deriving the Galerkin Matrix.

**Definition 3.2.** *Define the notation $P_n^{ab} = (a_n, b_n)_\Gamma$ where $a_n, b_n \in L^2(\Gamma)$. In particular, the following values will be helpful:*

- $P_n^{UU} = 2\pi|\tilde{v}_n|^2$
- $P_n^{\theta U} = 2\pi w_n \overline{\tilde{v}_n}$
- $P_n^{U\theta} = \overline{P_n^{\theta U}}$

1. This is actually not clear for the term $\mathrm{q}_\kappa$. However, we will prove this assumption later.

- $P_n^{\theta\theta} = 2\pi|w_n|^2$
- $P_n^{p\theta} = 2\pi l_n \overline{w_n}$
- $P_n^{Up} = \underline{2\pi\tilde{v}_n}\overline{l_n}$
- $P_n^{\theta p} = \overline{P_n^{p\theta}}$

*where* $\tilde{v}_n = J_n(\sqrt{\tilde{c}}\kappa)v_n$.

Let's introduce the following matrix and vector:

**Definition 3.3.**

$$A_n := \begin{pmatrix} (\alpha_n + \lambda_n^{(w)} P_n^{UU}) & -(\frac{1}{2} - \lambda_n^{(K')})P_n^{\theta U} & 0 \\ (\frac{1}{2} - \lambda_n^{(K)})P_n^{U\theta} & \lambda_n^{(V)} P^{\theta\theta} & i\eta P^{p\theta} \\ -\lambda_n^{(W)} P_n^{Up} & -(\lambda_n^{(K')} + \frac{1}{2})P^{\theta p} & \beta_n \end{pmatrix}$$

*and*

$$\vec{b}_n = \begin{bmatrix} g_1(U_n) \\ \overline{g_2}(\theta_n) \\ g_3(p_n) \end{bmatrix} = \begin{bmatrix} -\overline{\tilde{v}_n} f_2^n - \lambda_n^{(W)} \overline{\tilde{v}_n} f_1^n \\ (\lambda_n^{(K)} - 0.5)\overline{w_n} f_1^n \\ \lambda_n^{(W)} \overline{l_n} f_1^n \end{bmatrix} = f_1^n \underbrace{\begin{bmatrix} -\lambda_n^{(W)} \overline{\tilde{v}_n} \\ (\lambda_n^{(K)} - 0.5)\overline{w_n} \\ \lambda_n^{(W)} \overline{l_n} \end{bmatrix}}_{\vec{x_1}} + f_2^n \underbrace{\begin{bmatrix} -\overline{\tilde{v}_n} \\ 0 \\ 0 \end{bmatrix}}_{\vec{x_1}}$$

*where* $\alpha_n, \beta_n$ *are constants that are defined below.*

The following theorem summarises the Galerkin formulation of the problem:

**Theorem 3.4.** $(u, \theta, p) \in \mathcal{H}^{\frac{1}{2}} \times \mathcal{H}^{-\frac{1}{2}} \times \mathcal{H}^1$ *solves the problem in definition 3.1. if and only if* $A_n \vec{C_n} = \vec{b}_n$ *for all* $n = -N, -N+1, ..., N$.

Before we proof this we need the following Lemma from P. Meury doctoral thesis:

**Lemma 3.5.** *The following eigenvalue equations hold for* $y_n = e^{in\phi}$*:*

$$\begin{aligned} \mathrm{V}_\kappa(y_n) &= \lambda_n^{(\mathrm{V})} y_n, & \lambda_n^{(\mathrm{V})} &:= \frac{i\pi}{2} J_n(\kappa) H_n^{(1)}(\kappa) \\ \mathrm{K}_\kappa(y_n) &= \lambda_n^{(\mathrm{K})} y_n, & \lambda_n^{(\mathrm{K})} &:= \frac{i\pi\kappa}{2} J_n(\kappa) H_n^{(1)'}(\kappa) + \frac{1}{2} = \frac{i\pi\kappa}{2} J_n'(\kappa) H_n^{(1)}(\kappa) - \frac{1}{2} \\ \mathrm{K}'_\kappa(y_n) &= \lambda_n^{(\mathrm{K}')} y_n, & \lambda_n^{(\mathrm{K}')} &:= \frac{i\pi\kappa}{2} J_n(\kappa) H_n^{(1)'}(\kappa) + \frac{1}{2} = \frac{i\pi\kappa}{2} J_n'(\kappa) H_n^{(1)}(\kappa) - \frac{1}{2} \\ \mathcal{W}_\kappa(y_n) &= \lambda_n^{(\mathrm{W})} y_n, & \lambda_n^{(\mathrm{W})} &:= -\frac{i\pi\kappa^2}{2} J_n'(\kappa) H_n^{(1)'}(\kappa). \end{aligned}$$

*in particular these eigenvalue equations are satisfied for* $U_n, \theta_n, p_n$.

The following formulas will also be helpful for the proof.

**Lemma 3.6.** $q_\kappa(U_n, U_m) = 2\pi\delta_{nm}|v_n|^2 \kappa J_n(\kappa) J_n'(\kappa) =: \alpha_n$.

*Proof of Lemma.* We start by using Greens first formula $\int\limits_U (\psi\Delta\varphi + \nabla\psi \cdot \nabla\varphi) dV = \oint_{\partial U} \psi \nabla\varphi \cdot d\mathbf{S}$ to see that

$$\begin{aligned} q_\kappa(U_n, U_m) &= \int_{\Omega^-} (\boldsymbol{\nabla} \cdot U_n \boldsymbol{\nabla} \cdot \overline{U_m} - \tilde{c}\kappa^2 U_n \overline{U_m}) \\ &= \int_{\partial\Omega^-} \overline{U_m} \boldsymbol{\nabla} \cdot U_n d\vec{S} - \int_{\Omega^-} \underbrace{(\tilde{c}\kappa^2 U_n + \Delta U_n)}_{=0} \overline{U_m} d\vec{S} \\ &= \delta_{nm} \int_{\partial\Omega^-} \overline{U_n} \boldsymbol{\nabla} \cdot U_n d\vec{S} \\ &= \delta_{nm} |v_n|^2 \sqrt{\tilde{c}}\kappa J_n(\sqrt{\tilde{c}}\kappa) J_n'(\sqrt{\tilde{c}}\kappa) \int_{\partial\Omega^-} e^{in\phi} e^{-in\phi} dS \\ &= \delta_{nm} 2\pi |v_n|^2 \kappa \sqrt{\tilde{c}} J_n(\sqrt{\tilde{c}}\kappa) J_n'(\sqrt{\tilde{c}}\kappa) \end{aligned}$$

□

**Remark 3.7.** *In the previous proof* $U_n$ *is integrated over* $\Omega^-$*, while* $U_n$ *is technically only defined on* $\Gamma$*. However, note that, knowing the coefficients of* $U_n$*, it is straightforward to extend* $U$ *to* $\Omega^-$ *using our basis vectors* $V_n$*. Also, the end result only depends on the restriction to* $\Gamma$*.* [2]

2. These two points is what justified the restriction of $U_n$ to $\Gamma$ in the first place.

**Lemma 3.8.** $b(p_n, p_m) = (p_n, p_m)_{H^1(\Gamma)} = \frac{(1+n^2)}{\tilde{\kappa}^2+n^2}$

*Proof.* $p_n \frac{\tilde{\kappa}^2+n^2}{1+n^2}$ is orthogonal w.r.t $(-,-)_{H^1(\Gamma)}$. □

Now we can proof theorem 3.4.

*Proof of theorem 3.4.* Both sides of the problem in definition 3.4 are linear in $(V, \varphi, q)$. Therefore the condition in the problem is satisfied for all $(V, \varphi, q) \in \mathcal{S}_N^{\frac{1}{2}} \times \mathcal{S}_N^{-\frac{1}{2}} \times \mathcal{S}_N^1$ if and only it is satisfied for $(V, \varphi, q) \in (U_n, \theta_n, p_n) \forall n \in \{-N, -N+1, ..., N\}$. So we end up with $(2N+1)$ systems of equations of the form

$$\mathrm{q}_\kappa(U, U_n) + \left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), U_n\right)_\Gamma - \left((\frac{1}{2}\mathrm{ld} - \mathrm{K}'_\kappa)(\theta), U_n\right)_\Gamma = \mathrm{f}_2(U_n)$$
$$\left((\frac{1}{2}\mathrm{ld} - \mathrm{K}_\kappa)\left(\gamma_D^- U\right), \theta_n\right)_\Gamma + (\mathrm{V}_\kappa(\theta), \theta_n)_\Gamma + i\overline{\eta}(p, \theta_n)_\Gamma = \overline{g_2(V)}(\theta_n)\ .$$
$$-\left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), p_n\right)_\Gamma - \left((\mathrm{K}'_\kappa + \frac{1}{2}\mathrm{Id})(\theta), p_n\right)_\Gamma + \mathrm{b}(p, p_n) = g_3(V)(p_n).$$

Now, before plugging in for $(U, \theta, p)$ note the following structural feature: Each of the summands of the left side of the problem has a factor that contains one of the following three bilinear forms:

- A scalar product $(a, b)_\Gamma$ of one of the entries of $(U, \theta, p)$ and a basis function.
- The bilinear form $b(a, b)$ of of the entries of $(U, \theta, p)$ and a basis function.
- The bilinear form $q_\kappa(a, b)$ of of the entries of $(U, \theta, p)$ and a basis function.

Note that for each of these bilinear forms $s$ we have the orthogonality property $s(a_n, b_m) = 0$ for $n \neq m$, $a, b \in \{U, \theta, p\}$. Also considering that our basis are also eigenvectors of the BIO, this implies that for each of the $(2N+1)$ problems all but one $x_n$ is in the kernel of the left hand side. Or, said more simply, the problem can be written as a blockdiagonal problem with $3 \times 3$ blocks.
Plugging the relations derived in the former lemmata for $q_\kappa(U, U_n), b(p, p_n)$, the BIOs and the scalar products elementary calculation yields $A_n x_n$ for the left hand side of the problem.
For the right, plugging in our extension for $f_1, f_2$ yields $b_n$. This concludes the proof. □

# 4 Validation

## 4.1 Solving a special case

To validate the correctness of our derived matrix we show that it yields the correct numerical solution for a simple example.
Consider the special case

$$\vec{f} = \begin{bmatrix} H_n^{(1)}(\kappa) - J_n(\sqrt{\tilde{c}}\kappa) \\ H_n'^{(1)}(\kappa) - J_n'(\sqrt{\tilde{c}}\kappa) \end{bmatrix} e^{in\phi}$$

where $n = -N, -N+1, ..., N$
Then the solution is

$$u = J_n(\sqrt{\tilde{c}}\kappa r)e^{in\phi}, x \in \Omega^-, u = H_n^{(1)}(\kappa r)e^{in\phi}, x \in \Omega^+.$$

This can be seen directly by plugging in.

**Remark 4.1.** *Note that the discretized problem will yield the same solution as the original one as the restricted space contains the analytical solution.*

**Proposition 4.2.** *With regards to our chosen basis, the analytical solution can be written as* $(U, \theta, p) = (\frac{1}{v_n} U_n, \frac{1}{w_n} \kappa H_n'^{(1)}(\kappa)\theta_n, 0)$. *So the solution vector should be*

$$C_j^U = \delta_{nj}\frac{1}{v_n}, C_j^\theta = \delta_{nj}\frac{1}{w_n}\kappa H_n'^{(1)}(\kappa), C_j^p = 0, \forall j.$$

*Proof. still to be written out, but the main point is:* $\theta = \partial_r(u_{\Omega^+})_{|\Gamma} = -\frac{1}{w_0}\kappa_o H_1^{(1)}(\kappa_o)\theta_0$ and $p = 0$, (from page 33 Thesis P. Meury). □

Let's validate whether we get the same numerical solution using our matrix $A_n$. We just use the $\vec{b}_n$ as computed above, with $f_1^j = \delta_{nj} 2\pi(J_n(\sqrt{\tilde{c}}\kappa) - H_n^{(1)}(\kappa)), f_2^j = \delta_{nj} 2\pi(\sqrt{\tilde{c}}\kappa J_1'(\sqrt{\tilde{c}}\kappa) - \kappa H_n'^{(1)}(\kappa))$.

To measure how good the solution is, let's introduce the $\zeta$-number:

**Definition 4.3** ( $\zeta$-number)**.** *For a fixed n and a fixed* $\kappa$, *let* $\vec{C}_n^{num}(\kappa)$ *be the numerical solution vector to the problem* $diag(A_{-N}, ..., A_N)\vec{C}_n^{num}(\kappa) = (\vec{b}_{-N}, ..., \vec{b}_N)$. *Let* $\vec{C}_n^{ana}(\kappa)$ *be the analytical solution vector for the same problem constructed in proposition 4.2. The* $\zeta$*-number is defined as*

$$\zeta(\kappa) = \max_{n \in [-N, ..., N]} \frac{\|\vec{C}_n^{num}(\kappa) - \vec{C}_n^{ana}(\kappa)\|}{\|\vec{C}_n^{ana}(\kappa)\|}$$

As seen in fig. **??**, the $\zeta$-number is negligible across all values of $\kappa$ and $n$ which validates our derivation of $A_n$ and $b_n$.

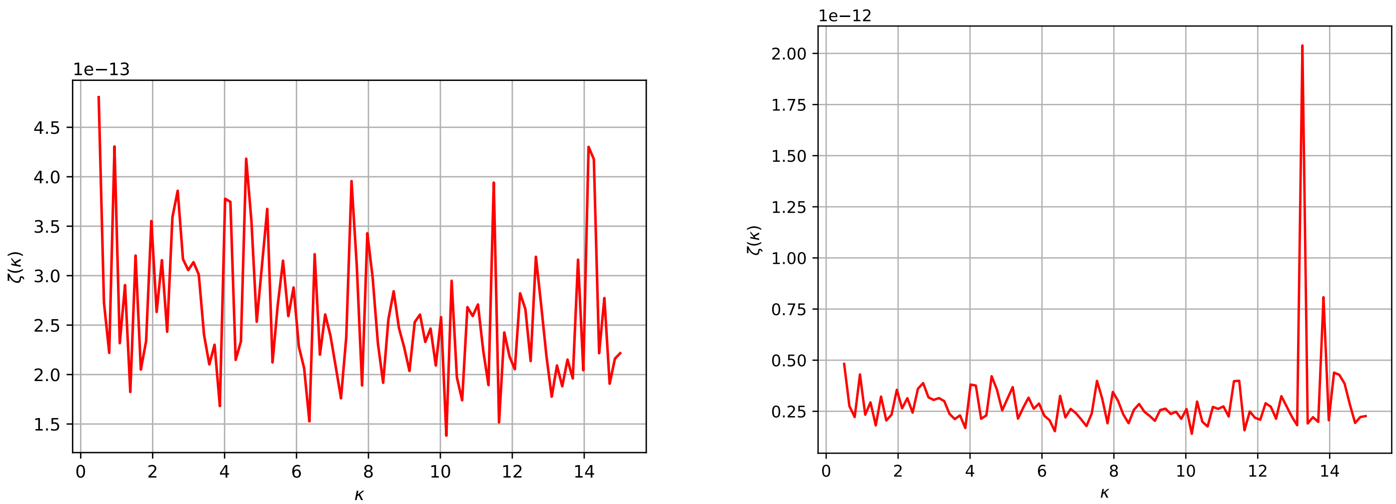


Fig. 1: Maximum relative residuum of the numerical solution by wave number. On the left plot we have $\tilde{c} = 1/3$ (f.e.$c_i = 1, c_o = 3$) and on the right plot we have $\tilde{c} = 3/1$. We always use $N = 100$ from here on.

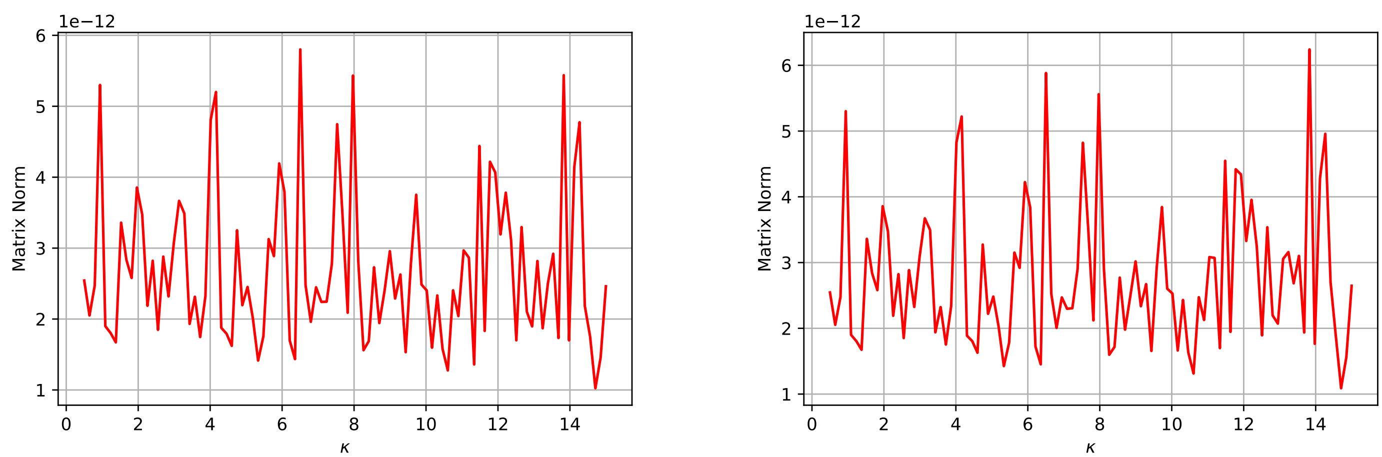


Fig. 2: Euclidian matrix norm of composed matrix $P_3 A_n^{-1} P_{V_{\vec{b}}}$. On the left plot we have $\tilde{c} = 1/3$ (f.e.$c_i = 1, c_o = 3$) and on the right plot we have $\tilde{c} = 3/1$.

### 4.2 p = 0

We validate our matrix $A$ with another method, using the following remark from the doctoral thesis from P.Meury:

**Remark 4.4.** *At second glance, we realise that* $p = 0$*, if* $(U, \theta)$ *solve the problem.*

Let's assume $A_n$ is invertible[3]. We can use the remark to validate that our derived matrix $A_n$ is correct, using the following proposition:

**Proposition 4.5.** *Let* $V_{\vec{b}} := span(\vec{x}_1, \vec{x}_2)$ *and let* $P_{V_{\vec{b}}}$ *be the projector onto* $V_{\vec{b}}$*. Let* $P_3$ *be the projector onto* $(0, 0, 1)$*. Then* $P_3 A_n^{-1} P_{V_{\vec{b}}} = 0$.

*Proof.* Because of the remark 4.4, every solution of $A_n \vec{x} = \vec{b}_n$ satisfies $x_3 = 0$ where $\vec{b} \in V_{\vec{b}}$.
Therefore $V_{\vec{b}} \subset Ker(P_3 A_n^{-1})$. Since $\vec{x}_1, \vec{x}_2$ are linearly independent, $\dim V_{\vec{b}} = 2$, which implies $V_{\vec{b}} = Ker(P_3 A_n^{-1})$. So $P_3 A_n^{-1} P_{V_{\vec{b}}} = 0$. ☐

Let's check whether this is actually satisfied by calculating the euclidian matrix norm of $A$ for a range of $\kappa$ values.
As we see in fig 2, this is satisfied which is another validation of our matrix $A$.

3. This is fair to assume for most frequencies since the variational problem is supposed to have a unique solution.

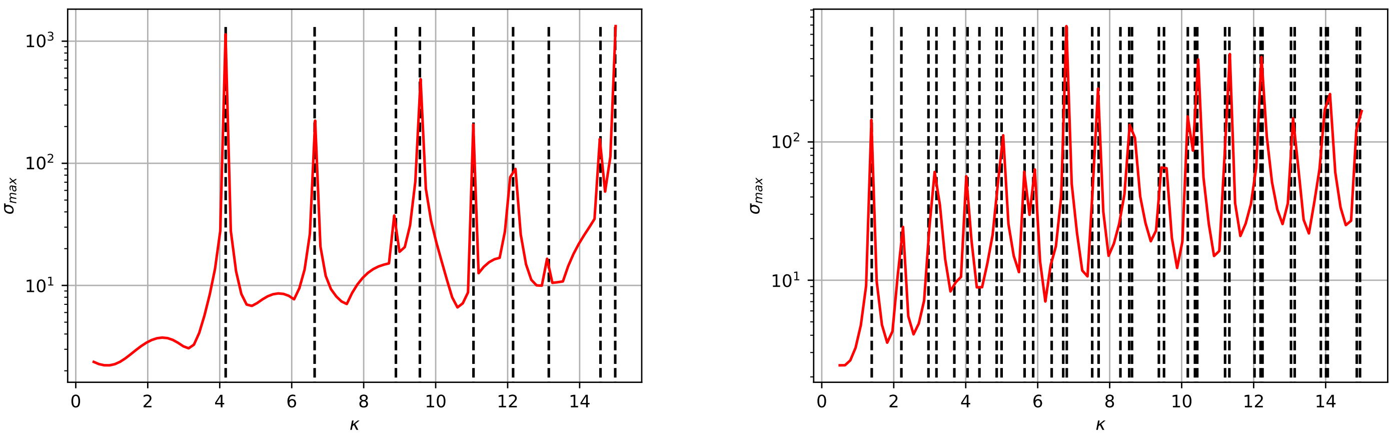


Fig. 3: Maximum singular value of the matrix $diag(A_n)$ by wave number $\kappa$. On the left plot we have $\tilde{c} = 1/3$ (f.e.$c_i = 1, c_o = 3$) and on the right plot we have $\tilde{c} = 3/1$.
The vertical dashed lines correspond to zeros of the Bessel functions.

## 5 Constructing the Solution Operator

For better comparability, we construct the discretized solution operator for the considered problem.

**Definition 5.1.** *Given positive real numbers $k, c_i$, and $c_o$, let $S\left(c_i, c_o\right)\mathbf{f} := \gamma_C^- u$ be the solution operator where $u$ solves eq 1.*

Plugging the Fourier ansatz into eq. 1 and imposing convergence at the origin and the Sommerfeld radiation condition results in

$$u^- = \sum_{n=-\infty}^{\infty} u_n^- \frac{J_n(\sqrt{c_i}\tilde{\kappa} r)}{J_n(\sqrt{c_i}\tilde{\kappa})} e^{in\phi}$$

$$u^+ = \sum_{n=-\infty}^{\infty} u_n^+ \frac{H_n(\sqrt{c_o}\tilde{\kappa} r)}{H_n(\sqrt{c_o}\tilde{\kappa})} e^{in\phi}$$

where $u_n^-$ and $u_n^+$ are the restrictions of $u$ to $\Omega^-$ und $\Omega^+$.
Extend $f_i = \sum\limits_{n=-\infty}^{\infty} f_i^n e^{in\phi}$ for $i = 1, 2$. The transmission condition $\gamma_C^+ u^+ = \gamma_C^- u^- + \mathbf{f}$ implies

$$\begin{pmatrix} 1 & -1 \\ \sqrt{c_i}\tilde{\kappa} \frac{J_n'(\sqrt{c_i}\tilde{\kappa})}{J_n(\sqrt{c_i}\tilde{\kappa})} & -\sqrt{c_o}\tilde{\kappa} \frac{H_n'(\sqrt{c_o}\tilde{\kappa})}{H_n(\sqrt{c_o}\tilde{\kappa})} \end{pmatrix} \begin{pmatrix} u_n^- \\ u_n^+ \end{pmatrix} = \begin{pmatrix} f_n^1 \\ f_n^2 \end{pmatrix}. \tag{4}$$

By using the well-known inverse of a 2x2 matrix we obtain

$$u_n^- = \zeta((-\sqrt{c_o}\tilde{\kappa} J_n(\sqrt{c_i}\tilde{\kappa}) H_n'(\sqrt{c_i}\tilde{\kappa})) f_n^1 + J_n(\sqrt{c_i}\tilde{\kappa}) H_n(\sqrt{c_o}\tilde{\kappa}) f_n^2). \tag{5}$$

where $\zeta := \frac{1}{\sqrt{c_i}\tilde{\kappa} J_n'(\sqrt{c_i}\tilde{\kappa}) H_n(\sqrt{c_o}\tilde{\kappa}) - \sqrt{c_o}\tilde{\kappa} H_n'(\sqrt{c_o}\tilde{\kappa}) J_n(\sqrt{c_i}\tilde{\kappa})}$. This presents the solution operator, as we get $\gamma_D^-$ from restriction to the boundary. $\gamma_N^-$ can be directly obtained by restriction of the normal derivative of $u_n^-$ to the boundary which is equivalent to multiplying the Fourier coefficients by $\sqrt{c_i}\tilde{\kappa}\frac{J_n'(\sqrt{c_i}\tilde{\kappa})}{J_n(\sqrt{c_i}\tilde{\kappa})}$. After rescaling the $u_n$, $f_i^n$ to the complete orthonormal system defined in eq. 3 we obtain the solution operator matrix:

$$S_{io}^n = \zeta \begin{pmatrix} -\sqrt{c_o}\tilde{\kappa} J_n(\sqrt{c_i}\tilde{\kappa}) H_n'(\sqrt{c_i}\tilde{\kappa}) & \sqrt{n^2+\tilde{\kappa}^2} J_n(\sqrt{c_i}\tilde{\kappa}) H_n(\sqrt{c_o}\tilde{\kappa}) \\ -\frac{1}{\sqrt{n^2+\tilde{\kappa}^2}} \sqrt{c_o}\sqrt{c_i}\tilde{\kappa}^2 J_n'(\sqrt{c_i}\tilde{\kappa}) H_n'(\sqrt{c_i}\tilde{\kappa}) & \sqrt{c_i}\tilde{\kappa} J_n'(\sqrt{c_i}\tilde{\kappa}) H_n(\sqrt{c_o}\tilde{\kappa}) \end{pmatrix} \tag{6}$$

that maps the Fourier coefficients of $\vec{f}$ in the complete orthonormal system for $\mathcal{H}^{\frac{1}{2}} \times \mathcal{H}^{-\frac{1}{2}}$ (as defined in eq. 3) to the Fouier coefficients of $\gamma_C^- u$ in the same basis.

## 6 Numerical Results

Now let's investigate maximal singular value of $A = diag(A_{-N}, A_{-N+1}, ..., A_N)$. Again, we look at the scenarios where $c_i = 1, c_o = 3$ and vice versa (fig. 3) We see that the peaks often coincide with the zeros of the Bessel function.

We can also consider the minimum singular value (fig. 4). Finally, let's consider the ratio of the minimum and maximum singular value (fig. 5). We see that in the case $c_i = 1, c_o = 3$, where the solution operator has weaker resonances resonances, the resonances from our matrix operator or also less strong than in the case $c_i = 3, c_o = 1$.

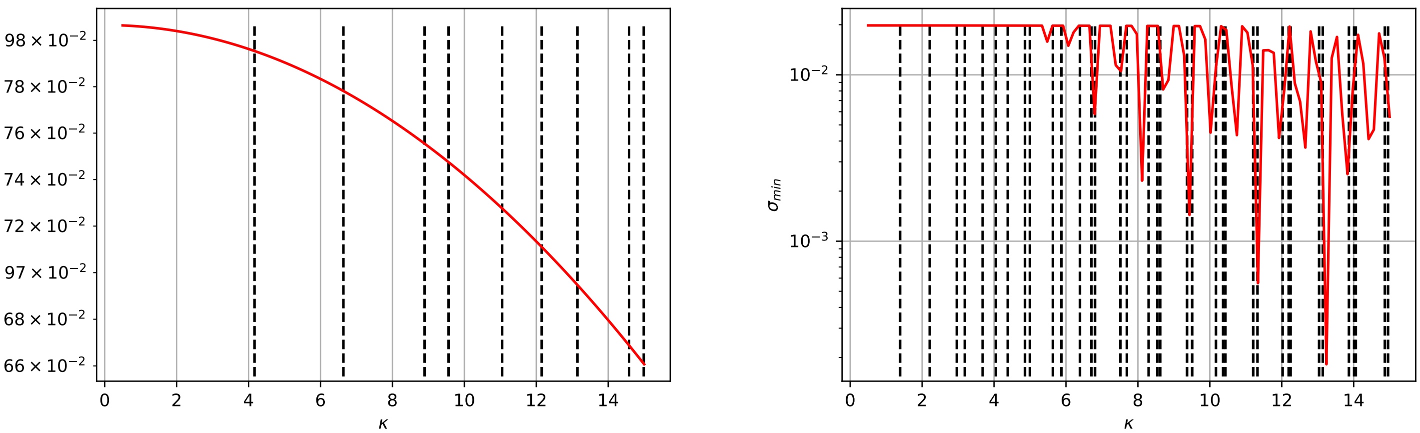


Fig. 4: Maximum singular value of the matrix $diag(A_n)$ by wave number $\kappa$. On the left plot we have $\tilde{c} = 1/3$ (f.e.$c_i = 1, c_o = 3$) and on the right plot we have $\tilde{c} = 3/1$. Again, the vertical dashed lines correspond to zeros of the Bessel functions.

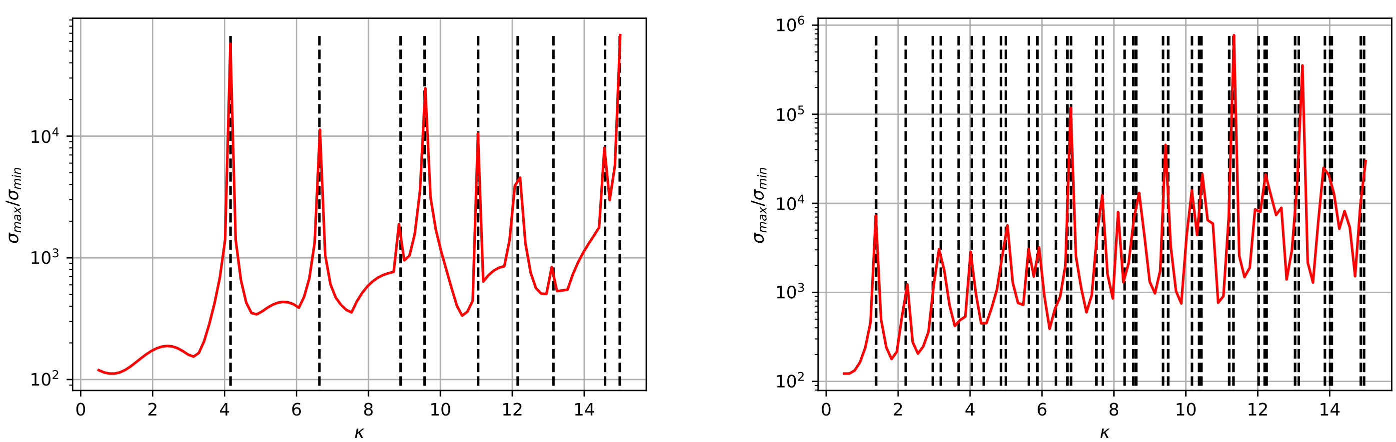


Fig. 5: Ratio of Maximum and Minimum singular value of the matrix $diag(A_n)$ by wave number $\kappa$. On the left plot we have $\tilde{c} = 1/3$ (f.e.$c_i = 1, c_o = 3$) and on the right plot we have $\tilde{c} = 3/1$. The vertical dashed lines correspond to zeros of the Bessel functions.