Spurious Quasi-Resonances for Stabilized BIE-Volume Formulations for Helmholtz Transition Problem

Frederic Jørgensen

1 Theoretical Background

1.0.1 Lipschitz Domain

Remark 1.1. *Henceforth we shall require that, roughly speaking that Ω is locally the set of points located above the graph of some Lipschitz function and the boundary is this graph.*

Definition 1.2 (Lipschitz domain)**.** *Let $n \in \mathbb{N}$. Let Ω be a domain of $\mathbb{R}^n$ and let $\partial\Omega$ denote the boundary of Ω. Then Ω is called a Lipschitz domain if for every point $p \in \partial\Omega$ there exists a hyperplane H of dimension $n-1$ through p, a Lipschitz-continuous function $g : H \to \mathbb{R}$ over that hyperplane, and reals $r > 0$ and $h > 0$ such that*

- $\Omega \cap C = \{x + y\vec{n} \mid x \in B_r(p) \cap H, -h < y < g(x)\}$
- $(\partial\Omega) \cap C = \{x + y\vec{n} \mid x \in B_r(p) \cap H, g(x) = y\}$

where $\vec{n}$ is a unit vector that is normal to H and $C := \{x + y\vec{n} \mid x \in B_r(p) \cap H, -h < y < h\}$.

1.0.2 Sobolev Space

Definition 1.3 (H^1)**.** *For a bounded domain $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, we define the Sobolev space $H^1(\Omega) := \left\{v \in L^2(\Omega) : \int_\Omega |\operatorname{grad} v(x)|^2 \, \mathrm{d}x < \infty\right\}$ as a Hilbert space with norm $\|v\|^2_{H^1(\Omega)} := \|v\|^2_{L^2(\Omega)} + |v|^2_{H^1(\Omega)}, \quad |v|^2_{H^1(\Omega)} := \int_\Omega |\operatorname{grad} v(x)|^2 \, \mathrm{d}x$*

Definition 1.4 ($H^{1/2}$)**.**

Definition 1.5 (H^k_{loc})**.**

1.0.3 Trace operators

Definition 1.6 (Trace operator)**.** *A trace operator is a linear mapping from a function space on the volume domain Ω to a function space on (parts of) the boundary $\partial\Omega$.*

Definition 1.7 ((Layer) potential)**.** *A (layer) potential is a linear mapping from a function space on $\partial\Omega$ into a function space on the volume domain Ω.*

Definition 1.8 (Dirichlet Trace)**.** *The Dirichlet trace (operator) T_D boils down to pointwise restriction for smooth functions:*

$$(\mathrm{T}_D w)(\boldsymbol{x}) := w(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Gamma, \quad w \in C^\infty(\bar{\Omega}).$$

Definition 1.9 (Dirichlet trace space)**.** *The Dirichlet trace space $H^{\frac{1}{2}}(\Gamma)$ is the Hilbert space obtained by completion of $C^\infty(\bar{\Omega})\big|_\Gamma$ with respect to the energy norm*

$$\|\mathfrak{u}\|_{H^{\frac{1}{2}}(\Gamma)} := \inf\left\{\|v\|_{H^1(\Omega)} : v \in C^\infty(\bar{\Omega}), \mathsf{T}_D v = \mathfrak{u}\right\}, \quad \mathfrak{u} \in C^\infty(\bar{\Omega})\big|_\Gamma .$$

Theorem 1.10. *The Dirichlet trace T_D according can be extended to a continuous and surjective linear operator $\mathrm{T}_D : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$*

Definition 1.11 (Neumann Trace)**.** *For smooth functions the Neumann trace (operator) T_N is defined by*

$$(\mathrm{T}_N w)(\boldsymbol{x}) := \operatorname{grad} w \cdot \boldsymbol{n}(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Gamma, w \in C^\infty(\bar{\Omega}).$$

Definition 1.12 (Neumann Trace Space)**.** *The Neumann trace space $H^{-\frac{1}{2}}(\Gamma)$ is the Hilbert space obtained by the completion of $C^0(\Gamma)$ with respect to the norm*

$$\|\phi\|_{H^{-\frac{1}{2}}(\Gamma)} := \|\widetilde{\phi}\|_{\widetilde{H}^{-1}(\Omega)}$$

where $\widetilde{\phi}$ is the "extension by zero to $\mathbb{R}^d$" of ϕ. We have the definition $\|\rho\|_{\widetilde{H}^{-1}(\Omega)} := |u|_{H^1(\mathbb{R}^3)}$ where u solves $\left\{\begin{array}{c} -\Delta u = \widetilde{\rho} \text{ in } \mathbb{R}^3 \\ u \text{ satisfies decay conditions} \end{array}\right.$.

Definition 1.13 (Space of function with square-integrable Laplacian)**.** *We define the Hilbert space*

$$H(\Delta, \Omega) := \left\{v \in H^1(\Omega) : \Delta v \in L^2(\Omega)\right\}$$

with norm

$$\|u\|^2_{H(\Delta,\Omega)} := \|u\|^2_{H^1(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}, \quad u \in H(\Delta,\Omega).$$

Theorem 1.14. *The Neumann trace* T_N *can be extended to a continuous mapping* $\mathrm{T}_N : H(\Delta,\Omega) \to H^{-\frac{1}{2}}(\Gamma)$.

Definition 1.15 **(**$C^\infty_{\mathrm{comp}}\left(\mathbb{R}^d\right)$**).**

1.0.4 Notations

- Consider a bounded Lipschitz open set $\Omega^- \subset \mathbb{R}^d$, $d = 2, 3$.
- $\Omega^+ := \mathbb{R}^d \backslash \overline{\Omega^-}$
- $\Gamma := \partial\Omega^- = \partial\Omega^+$
- $\mathbf{n}$ is the unit normal vector field on Γ pointing from Ω^-into Ω^+
- For any $\varphi \in L^2_{\mathrm{loc}}\left(\mathbb{R}^d\right)$, we let $\varphi^- := \varphi|_{\Omega^-}$and $\varphi^+ := \varphi|_{\Omega^+}$
- $H^1_{\mathrm{loc}}\left(\Omega^\pm, \Delta\right) := \left\{v : \chi v \in H^1\left(\Omega^\pm\right), \Delta(\chi v) \in L^2\left(\Omega^\pm\right) \text{for all } \chi \in C^\infty_{\mathrm{comp}}\left(\mathbb{R}^d\right)\right\}$
- Dirichlet and Neumann trace operators[1]: $\gamma^\pm_D : H^1_{\mathrm{loc}}\left(\Omega_\pm\right) \to H^{1/2}(\Gamma)$ and $\gamma^\pm_N : H^1_{\mathrm{loc}}\left(\Omega_\pm, \Delta\right) \to H^{-1/2}(\Gamma)$ with $\gamma^\pm_D v := v^\pm|_\Gamma$ and $\gamma^\pm_N$such that if $v \in H^2_{\mathrm{loc}}\left(\Omega_\pm\right)$then $\gamma^\pm_N v = \mathbf{n} \cdot \gamma^\pm_D(\nabla v)$
- Cauchy trace: $\gamma^\pm_C : H^1_{\mathrm{loc}}\left(\Omega^\pm, \Delta\right) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, $\gamma^\pm_C := \left(\gamma^\pm_D, \gamma^\pm_N\right)$
- Sommerfeld radiation condition: $\varphi \in C^1\left(\mathbb{R}^d \backslash B_R\right)$, for some ball $B_R := \{|\mathbf{x}| < R\}$, and $\kappa > 0$ satisfies this condition if $\lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial\varphi(\mathbf{x})}{\partial r} - \mathrm{i}\kappa\varphi(\mathbf{x})\right) = 0$ in all directions. We then write $\varphi \in \mathrm{SRC}(\kappa)$

Theorem 1.16 (Green's first formula)**.** *(From Wikipedia)* $\int_U \left(\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi\right) \mathrm{d}U = \int_{\partial U} \phi\frac{\partial\psi}{\partial n}\, \mathrm{d}S$

1. The Dirichlet trace operator boils down to pointwise restriction.

2 Problem

We start by formulating the problem. We want to investigate the occurence of spurious quasi-resonances in the variational formulation of the following Helmholtz transition problem.

Definition 2.1 (Helmholtz Transmission Problem)**.** *Find* $u \in H^1_{\text{loc}}\left(\mathbb{R}^d \backslash \Gamma\right) \cap \text{SRC}\left(k\sqrt{c_o}\right)$ *such that*

$$\begin{aligned} \left(\Delta + \tilde{\kappa}^2 c_i\right) u^- &= 0 && \textit{in } \Omega^- \\ \left(\Delta + \tilde{\kappa}^2 c_o\right) u^+ &= 0 && \textit{in } \Omega^+ \\ \gamma_C^+ u^+ &= \gamma_C^- u^- + \mathbf{f} && \textit{on } \Gamma. \end{aligned}$$

2.1 Deriving the variational formulation

We will now state the variational formulation of the problem in Ω^-.
Integration by parts in Ω^- using Green's first formula implies

$$a(U,V) - \left(\gamma_1^- U, \gamma_0^- V\right)_\Gamma = 0 \quad \forall v \in H^1\left(\Omega^-\right)$$

This variational equation can be coupled to the transmission conditions using the BIE. As the problem is equivalent to the problem in the 2006 paper with $U_i = 0, f = 0, n(x) = c_i/c_o, \kappa = \tilde{\kappa}\sqrt{c_o}$, we can use the same formulation as in the 2006 paper.

The formulation is:
Find $U \in H^1(\Omega), \theta \in H^{-1/2}(\Gamma)$ and $p \in H^1(\Gamma)$ such that for all $V \in H^1(\Omega), \varphi \in H^{-1/2}(\Gamma)$ and $q \in H^1(\Gamma)$ there holds

$$\begin{aligned} \mathrm{q}_\kappa(U,V) + \left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), \gamma_D^- V\right)_\Gamma - \left((\frac{1}{2}\mathrm{ld} - \mathrm{K}'_\kappa)(\theta), \gamma_D^- V\right)_\Gamma &= g_1(V) \\ \left((\frac{1}{2}\mathrm{ld} - \mathrm{K}_\kappa)\left(\gamma_D^- U\right), \varphi\right)_\Gamma + (\mathrm{V}_\kappa(\theta), \varphi)_\Gamma + i\overline{\eta}(p,\varphi)_\Gamma &= \overline{g_2(V)}(\varphi) \\ -\left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), q\right)_\Gamma - \left((\mathrm{K}'_\kappa + \frac{1}{2}\mathrm{Id})(\theta), q\right)_\Gamma + \mathrm{b}(p,q) &= g_3(V)(q). \end{aligned}$$

where we have

$$\begin{aligned} g_1(V) &:= (f,V)_\Omega - \left(g_N, \gamma_D^- V\right)_\Gamma - \left(\mathrm{W}_\kappa\left(g_D\right), \gamma_D^- V\right)_\Gamma \\ g_2(V)(\varphi) &:= \left(\varphi, \left(\mathrm{K}_\kappa - \frac{1}{2}\mathrm{ld}\right)(g_D)\right)_\Gamma \\ g_3(V)(q) &:= \left(\mathrm{W}_\kappa\left(g_D\right), q\right)_\Gamma, \end{aligned}$$

$\mathrm{q}_\kappa(U,V) := \int_\Omega \operatorname{grad} U \cdot \operatorname{grad} \bar{V} - \kappa^2 n(\mathbf{x}) U \bar{V} \, \mathrm{d}\mathbf{x}$ and $\mathrm{b}(p,q) := (\operatorname{grad}_\Gamma p, \operatorname{grad}_\Gamma q)_\Gamma + (p,q)_\Gamma$.

3 How to investigate Stability

To figure out how stable this formulation is, we use the inf-sup constant

Theorem 3.1. *Existence and uniqueness of discrete solutions $u_h \in V_h$ and the convergence $u_h \to u$ can only hold if the sesquilinear form a satisfies a discrete inf-sup condition, cf. [89, Eq. 2.3.6]: There exists a constant $\gamma > 0$ such that for all $v_h \in V_h$ and the whole family V_h with $h \to 0$, there holds*

$$\sup_{0 \neq w_h \in V_h} \frac{\left|\mathrm{a}\left(v_h, w_h\right)\right|}{\left\|w_h\right\|_V} \geq \gamma \left\|v_h\right\|_V$$

(from P.Meury thesis).

We will now investgate the reliability of the variational formulation by investigating the inf-sup constant of the sesquilinear form on the left side of the variational problem.

Definition 3.2.

$$\gamma = \inf_{0 \neq v_h \in V_h} \sup_{0 \neq w_h \in V_h} \frac{\left|\mathrm{a}\left(v_h, w_h\right)\right|}{\left\|w_h\right\|_V \left\|v_h\right\|_V}$$

Remark 3.3. *We can estimate the inf-sup constant from the minimum singular value of the Galerkin matrix (proven in P.Meury thesis).*

So we now derive the Galerkin Matrix to find the inf-sup constant.

4 Derivation of Galerkin Matrix

First, we notice that we can restrict ourselves to finding functions on $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \times H^1(\Gamma)$, since the bilinear form only depends on the restriction $\gamma_D^- U$.

Now we will further restrict this space to finite subspaces and choose orthonormal bases for the finite subspaces.

4.1 Basis functions of $\mathcal{H}^{\frac{1}{2}}(\Gamma)$

We make the Fourier Ansatz $V_n = V_n^r e^{in\phi}$. Since U must satisfy $(\Delta + \frac{c_i}{c_o}\kappa^2)U = 0$, we have

$$r^2\partial_r^2 V_n^r + r\partial_r V_n^r + (r^2\frac{c_i}{c_o}\kappa^2 - n^2)V_n^r = 0.$$

This is Bessel's equation. Since we require convergence at the origin we have $V_n^r(r) = J_n(\sqrt{\frac{c_i}{c_o}}\kappa r)$.

We restrict our functions to $H^{\frac{1}{2}}$ and want to get an orthonormal basis $U_n(\phi) = v_n V_n(1,\phi)$ for this space:

$$(U_n, U_m)_{H^{\frac{1}{2}}(\Gamma)} = \delta_{nm}$$

This implies:

$$(U_n, U_m)_{H^{\frac{1}{2}}(\Gamma)} = (U_n, U_m)_\Gamma + (\boldsymbol{\nabla}_\Gamma U_n, \boldsymbol{\nabla}_\Gamma U_m)_\Gamma = \delta_{nm}$$

$$= |v_n|^2 2\pi(1+n^2)|J_n(\sqrt{\frac{c_i}{c_o}}\kappa)|^2\delta_{nm} \overbrace{=}^{!} \delta_{nm}$$

Therefore, we can choose real constants $v_n = \frac{1}{\sqrt{2\pi(1+n^2)}|J_n(\sqrt{\frac{c_i}{c_o}}\kappa)|}$.

Notes

In the calculation above we used $\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}}$ and eliminated the $r-$ dependency.

4.2 Other basis functions

We also pick orthonormal basis functions for $\mathcal{H}^{-\frac{1}{2}}(\Gamma)$ and $\mathcal{H}^1(\Gamma)$. We choose

$$\theta_n = w_n e^{in\phi} \text{ where } w_n = \frac{(1+n^2)^{\frac{1}{4}}}{\sqrt{2\pi}}$$

for $\mathcal{H}^{\frac{1}{2}}(\Gamma)$ and for $\mathcal{H}^{-\frac{1}{2}}(\Gamma)$ we choose

$$p_n = l_n e^{in\phi} \text{ where } l_n = \frac{1}{\sqrt{2\pi(1+n^2)}}.$$

Notes

See P. Meury 3.22 (Lemma)for explanation why we use these bases.

5 Constructing the Galerkin Matrix

Now let's restrict ourselves to the space $\mathcal{S}_N^{\frac{1}{2}} \times \mathcal{S}_N^{-\frac{1}{2}} \times \mathcal{S}_N^1$ where ... and $N \in \mathbb{N}$. Now we can construct the Galerkin Matrix.

5.1 Redefining the problem

We reduce our problem to the space $\mathcal{H}^{\frac{1}{2}} \times \mathcal{H}^{-\frac{1}{2}} \times \mathcal{H}^1$:

Definition 5.1 (The restricted problem)**.** *Find $(u,\theta,p) \in \mathcal{S}_N^{\frac{1}{2}} \times \mathcal{S}_N^{-\frac{1}{2}} \times \mathcal{S}_N^1$ such that for all $(V,\varphi,p) \in \mathcal{S}_N^{\frac{1}{2}} \times \mathcal{S}_N^{-\frac{1}{2}} \times \mathcal{S}_N^1$ there holds*

$$\mathrm{q}_\kappa(U,V) + \left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), \gamma_D^- V\right)_\Gamma - \left((\frac{1}{2}\mathrm{ld} - \mathrm{K}'_\kappa)(\theta), \gamma_D^- V\right)_\Gamma = g_1(V)$$
$$\left((\frac{1}{2}\mathrm{ld} - \mathrm{K}_\kappa)\left(\gamma_D^- U\right), \varphi\right)_\Gamma + (\mathrm{V}_\kappa(\theta),\varphi)_\Gamma + i\overline{\eta}(p,\varphi)_\Gamma = \overline{g_2(V)}(\varphi)$$
$$-\left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), q\right)_\Gamma - \left((\mathrm{K}'_\kappa + \frac{1}{2}\mathrm{Id})(\theta), q\right)_\Gamma + \mathrm{b}(p,q) = g_3(V)(q).$$

Now let's simplify the notation of this problem a bit. First, let's extend $(u,\theta,p) = \sum_{n=-N}^{N}(C_n^U U_n, C_n^\theta \theta_n, C_n^p p_n)$, $f_1 = \sum_{j=-N}^{N}\frac{1}{2\pi}f_1^j e^{ij\theta}$ and $f_2 = \sum_{j=-N}^{N}\frac{1}{2\pi}f_2^j e^{ij\theta}$.

Define $\vec{C}_n = (C_n^U, C_n^\theta, C_n^p)$.

Let's introduce a few new constants before deriving the Galerkin Matrix.

Definition 5.2. *Define the notation $P_n^{ab} = (a_n, b_n)_\Gamma$ where $a_n, b_n \in L^2(\Gamma)$. In particular, the following values will be helpful:*

- $P_n^{UU} = 2\pi|\tilde{v}_n|^2$

- $P_n^{\theta U} = 2\pi w_n \overline{\tilde{v}_n}$
- $P_n^{U\theta} = \overline{P_n^{\theta U}}$
- $P_n^{\theta\theta} = 2\pi |w_n|^2$
- $P_n^{p\theta} = 2\pi l_n \overline{w_n}$
- $P_n^{Up} = 2\pi \tilde{v}_n \overline{l_n}$
- $P_n^{\theta p} = \overline{P_n^{p\theta}}$

where $\tilde{v}_n = J_n(\sqrt{\frac{c_i}{c_o}}\kappa) v_n$.

Let's introduce the following matrix and vector:

Definition 5.3.

$$A_n := \begin{pmatrix} (\alpha_n + \lambda_n^{(w)} P_n^{UU}) & -(\frac{1}{2} - \lambda_n^{(K')}) P_n^{\theta U} & 0 \\ (\frac{1}{2} - \lambda_n^{(K)}) P_n^{U\theta} & \lambda_n^{(V)} P^{\theta\theta} & i\eta P^{p\theta} \\ -\lambda_n^{(W)} P_n^{Up} & -(\lambda_n^{(K')} + \frac{1}{2}) P^{\theta p} & 1 \end{pmatrix}$$

and

$$\vec{b}_n = \begin{bmatrix} g_1(U_n) \\ \overline{g_2}(\theta_n) \\ g_3(p_n) \end{bmatrix} = \begin{bmatrix} -\overline{\tilde{v}_n} f_2^n - \lambda_n^{(W)} \overline{\tilde{v}_n} f_1^n \\ (\lambda_n^{(K)} - 0.5) \overline{w_n} f_1^n \\ \lambda_n^{(W)} \overline{l_n} f_1^n \end{bmatrix} = f_1^n \underbrace{\begin{bmatrix} -\lambda_n^{(W)} \overline{\tilde{v}_n} \\ (\lambda_n^{(K)} - 0.5) \overline{w_n} \\ \lambda_n^{(W)} \overline{l_n} \end{bmatrix}}_{\vec{x_1}} + f_2^n \underbrace{\begin{bmatrix} -\overline{\tilde{v}_n} \\ 0 \\ 0 \end{bmatrix}}_{\vec{x_1}}.$$

The following theorem summarises the Galerkin formulation of the problem:

Theorem 5.4. *$(u, \theta, p) \in \mathcal{H}^{\frac{1}{2}} \times \mathcal{H}^{-\frac{1}{2}} \times \mathcal{H}^1$ solves the problem in definition 5.1 if and only if $A_n \vec{C_n} = \vec{b}_n \forall n \in \{-N, -N+1, ..., N\}$.*

Before we proof this we need the following Lemma from (...):

Lemma 5.5. *We are also using the following eigenvalue equations*

$$\begin{aligned} \mathrm{V}_\kappa(y_n) &= \lambda_n^{(\mathrm{V})} y_n, \qquad \lambda_n^{(\mathrm{V})} := \frac{i\pi}{2} J_n(\kappa) H_n^{(1)}(\kappa) \\ \mathrm{K}_\kappa(y_n) &= \lambda_n^{(\mathrm{K})} y_n, \qquad \lambda_n^{(\mathrm{K})} := \frac{i\pi\kappa}{2} J_n(\kappa) H_n^{(1)'}(\kappa) + \frac{1}{2} = \frac{i\pi\kappa}{2} J_n'(\kappa) H_n^{(1)}(\kappa) - \frac{1}{2} \\ \mathrm{K}'_\kappa(y_n) &= \lambda_n^{(\mathrm{K}')} y_n, \quad \lambda_n^{(\mathrm{K}')} := \frac{i\pi\kappa}{2} J_n(\kappa) H_n^{(1)'}(\kappa) + \frac{1}{2} = \frac{i\pi\kappa}{2} J_n'(\kappa) H_n^{(1)}(\kappa) - \frac{1}{2} \\ \mathcal{W}_\kappa(y_n) &= \lambda_n^{(\mathrm{W})} y_n, \qquad \lambda_n^{(\mathrm{W})} := -\frac{i\pi\kappa^2}{2} J_n'(\kappa) H_n^{(1)'}(\kappa) \end{aligned}$$

in particular these eigenvalue equations are satisfied for U_n, θ_n, p_n.

The following lemmas will also come in handy for the proof.

Lemma 5.6. *$q_\kappa(U_n, U_m) = 2\pi \delta_{nm} |v_n|^2 \kappa J_n(\kappa) J_n'(\kappa) =: \alpha_n$.*

Proof of Lemma. We start by using Greens first formula $\int\limits_U (\psi \Delta\varphi + \nabla\psi \cdot \nabla\varphi) dV = \oint_{\partial U} \psi \nabla\varphi \cdot d\mathbf{S}$ to see that

$$\begin{aligned} q_\kappa(U_n, U_m) &= \int_{\Omega^-} (\boldsymbol{\nabla} \cdot U_n \boldsymbol{\nabla} \cdot \overline{U_m} - \frac{c_i}{c_o}\kappa^2 U_n \overline{U_m}) \\ &= \int_{\partial\Omega^-} \overline{U_m} \boldsymbol{\nabla} \cdot U_n d\vec{S} - \int_{\Omega^-} \underbrace{(\frac{c_i}{c_o}\kappa^2 U_n + \Delta U_n)}_{=0} \overline{U_m} d\vec{S} \\ &= \delta_{nm} \int_{\partial\Omega^-} \overline{U_n} \boldsymbol{\nabla} \cdot U_n d\vec{S} \\ &= \delta_{nm} |v_n|^2 \sqrt{\frac{c_i}{c_o}} \kappa J_n(\sqrt{\frac{c_i}{c_o}}\kappa) J_n'(\sqrt{\frac{c_i}{c_o}}\kappa) \int_{\partial\Omega^-} e^{in\phi} e^{-in\phi} dS \\ &= \delta_{nm} 2\pi |v_n|^2 \kappa \sqrt{\frac{c_i}{c_o}} J_n(\sqrt{\frac{c_i}{c_o}}\kappa) J_n'(\sqrt{\frac{c_i}{c_o}}\kappa) \end{aligned}$$

□

Remark 5.7. *In the previous proof U_n is integrated over Ω^-, while U_n is technically only defined on Γ. However, note that, knowing the coefficients of U_n, it is straightforward to extend U to Ω^- using our basis vectors V_n. Also, the end result only depends on the restriction to Γ.* [2]

2. These two points is what justified the restriction of U_n to Γ in the first place.

Lemma 5.8. $b(p_n, p_m) = (p_n, p_m)_{H^1(\Gamma)} = \delta_{nm}$.

Proof. ... $\square$

Proof of theorem .... Both sides of the problem in definition 5.1 are linear in (V, φ, q). Therefore the condition in the problem is satisfied for all $(V, \varphi, q) \in \mathcal{S}_N^{\frac{1}{2}} \times \mathcal{S}_N^{-\frac{1}{2}} \times \mathcal{S}_N^1$ if and only it is satisfied for $(V, \varphi, q) \in (U_n, \theta_n, p_n) \forall n \in \{-N, -N+1, ..., N\}$. So we end up with $(2N+1)$ systems of equations of the form

$$\begin{aligned} \mathrm{q}_\kappa(U, U_n) + \left(\mathrm{W}_\kappa \left(\gamma_D^- U\right), U_n\right)_\Gamma - \left((\frac{1}{2}\mathrm{Id} - \mathrm{K}'_\kappa)(\theta), U_n\right)_\Gamma &= \mathrm{f}_2(U_n) \\ \left((\frac{1}{2}\mathrm{Id} - \mathrm{K}_\kappa)\left(\gamma_D^- U\right), \theta_n\right)_\Gamma + (\mathrm{V}_\kappa(\theta), \theta_n)_\Gamma + i\overline{\eta}(p, \theta_n)_\Gamma &= \overline{g_2(V)}(\theta_n) \ . \\ -\left(\mathrm{W}_\kappa \left(\gamma_D^- U\right), p_n\right)_\Gamma - \left((\mathrm{K}'_\kappa + \frac{1}{2}\mathrm{Id})(\theta), p_n\right)_\Gamma + \mathrm{b}(p, p_n) &= g_3(V)(p_n). \end{aligned}$$

Now, before plugging in for (U, θ, p) note the following structural feature: Each of the summands of the left side of the problem has a factor that contains one of the following three bilinear forms:

- A scalar product $(a, b)_\Gamma$ of one of the entries of (U, θ, p) and a basis function.
- The bilinear form $b(a, b)$ of of the entries of (U, θ, p) and a basis function.
- The bilinear form $q_\kappa(a, b)$ of of the entries of (U, θ, p) and a basis function.

Note that for each of these bilinear forms s we have the orthogonality property $s(a_n, b_m) = 0$ for $n \neq m$, $a, b \in \{U, \theta, p\}$. Also considering that our basis are also eigenvectors of the BIO, this implies that for each of the $(2N+1)$ problems all but one x_n is in the kernel of the left hand side. Or, said more simply, the problem can be written as a blockdiagonal problem with 3×3 blocks.

Plugging the relations derived in the former lemmata for $q_(U, U_n), b(p, p_n)$, the BIOs and the scalar products elementary calculation yields $A_n x_n$ for the left hand side of the problem.
For the right, plugging in our extension for f_1, f_2 yields b_n. This concludes the proof. $\square$

5.2 Matrix

Now the first equation yields with coefficents:

$$(\alpha_n + \lambda_n^{(w)} P_n^{UU}) C_n^U - (\frac{1}{2} - \lambda_n^{(K')}) P_n^{\theta U} C_n^\theta = f_2(V)$$

second equation

$$(\frac{1}{2} - \lambda_n^{(K)}) P_n^{U\theta} C_n^U + \lambda_n^{(V)} P^{\theta\theta} C_n^\theta + i\eta P^{p\theta} C_n^p = \overline{g_2}(V)$$

third equation

$$-\lambda_n^{(W)} P_n^{Up} C_U^n - (\lambda_n^{(K')} + \frac{1}{2}) P^{\theta p} C_n^\theta + C_p^n = g_3(V)$$

Using all the previously derived equations we can construct the Galerkin matrix:

$$A_n = \begin{pmatrix} (\alpha_n + \lambda_n^{(w)} P_n^{UU}) & -(\frac{1}{2} - \lambda_n^{(K')}) P_n^{\theta U} & 0 \\ (\frac{1}{2} - \lambda_n^{(K)}) P_n^{U\theta} & \lambda_n^{(V)} P^{\theta\theta} & i\overline{\eta} P^{p\theta} \\ -\lambda_n^{(W)} P_n^{Up} & -(\lambda_n^{(K')} + \frac{1}{2}) P^{\theta p} & 1 \end{pmatrix}$$

6 Validation

6.1 Solving a special case

To validate the correctness of our derived matrix is to show that it yields the correct numerical solution for a simple example.
Consider the special case

$$\vec{f} = \begin{bmatrix} H_n^{(1)}(\kappa) - J_n(\tilde{c}\kappa) \\ H_n'^{(1)}(\kappa) - J_n'(\tilde{c}\kappa) \end{bmatrix} e^{in\phi}$$

where $\tilde{c} = \sqrt{\frac{c_i}{c_o}}$.
Then the solution is

$$u = J_n(\tilde{c}\kappa r) e^{in\phi}, x \in \Omega^-, u = H_n^{(1)}(\kappa r) e^{in\phi}, x \in \Omega^+.$$

This can be seen directly by plugging in.

Remark 6.1. *Note that the discretized problem will yield the same solution as the original one as the restricted space contains the analytical solution.*

Fig. 1: Validation of the numerical solution for the considered case case.

Proposition 6.2. *With regards to our chosen basis, the analytical solution can be written as* $(U, \theta, p) = (\frac{1}{v_n} U_n, \frac{1}{w_n} \kappa H_n'^{(1)}(\kappa)\theta_n, 0)$. *So the solution vector should be*

$$C_j^U = \delta_{nj}\frac{1}{v_n}, C_j^\theta = \delta_{nj}\frac{1}{w_n}\kappa H_n'^{(1)}(\kappa), C_j^p = 0, \forall j.$$

Proof. .. Also, we expect $\theta = \partial_r(u_{\Omega^+})_{|\Gamma} = -\frac{1}{w_0}\kappa_o H_1^{(1)}(\kappa_o)\theta_0$ and $p = 0$, from ... □

Let's validate if we get the same numerical solution using our matrix A_n. We just use the $\vec{b}_n$ as computed above, with $f_1^j = \delta_{nj} 2\pi(J_n(\tilde{c}\kappa) - H_n^{(1)}(\kappa)), f_2^j = \delta_{nj} 2\pi(\tilde{c}\kappa J_1'(\tilde{c}\kappa) - \kappa H_n'^{(1)}(\kappa))$.

To measure how good the solution is, let's introduce the ζ-number:

$$\zeta(\kappa) = \max_{n \in [-N, \dots, N]} \frac{\|\vec{C}_{num}^n - \vec{C}_{ana}^n\|}{\|\vec{C}_{ana}^n\|}$$

As seen in fig. 1, the ζ-number is negligible across all values of κ and n which validates our derivation of A_n and b_n.

6.2 p = 0

From P.Meury:

Remark 6.3. *At second glance, we realise that* $p = 0$, *if* (U, θ) *solve the problem.*

We can use this to validate that our derived matrix A_n is correct we just have to show that every solution of $A_n\vec{x} = \vec{b}_n$ satisfies $x_3 = 0$, where $\vec{b} \in span(\vec{x}_1, \vec{x}_2) =: V_{\vec{b}}$.
Let's assume that A_n is invertible. Then we have to show that $V_{\vec{b}} \subset Ker(P_3 A_n^{-1})$. For dimensional reasons this is equivalent to $V_{\vec{b}} = Ker(P_3 A_n^{-1})$.
Let $P_{V_{\vec{b}}}$ be the projector on $V_{\vec{b}}$. Then we just have to show that $P_3 A_n^{-1} P_{V_{\vec{b}}}$ is approximately the zero map. For that, we just calculate the euclidian matrix norm of A for a range of κ values.

7 Numerical Results