

Spurious Quasi-Resonances for Stabilized BIE-Volume Formulations for Helmholtz Transition Problem

Frederic Jørgensen



1 Theoretical Background

1.0.1 Lipschitz Domain

Remark 1.1. Henceforth we shall require that, roughly speaking that Ω is locally the set of points located above the graph of some Lipschitz function and the boundary is this graph.

Definition 1.2 (Lipschitz domain). Let $n \in \mathbb{N}$. Let Ω be a domain of \mathbb{R}^n and let $\partial\Omega$ denote the boundary of Ω . Then Ω is called a Lipschitz domain if for every point $p \in \partial\Omega$ there exists a hyperplane H of dimension $n - 1$ through p , a Lipschitz-continuous function $g : H \rightarrow \mathbb{R}$ over that hyperplane, and reals $r > 0$ and $h > 0$ such that

- $\Omega \cap C = \{x + y\vec{n} \mid x \in B_r(p) \cap H, -h < y < g(x)\}$
- $(\partial\Omega) \cap C = \{x + y\vec{n} \mid x \in B_r(p) \cap H, g(x) = y\}$

where \vec{n} is a unit vector that is normal to H and $C := \{x + y\vec{n} \mid x \in B_r(p) \cap H, -h < y < h\}$.

1.0.2 Sobolev Space

Definition 1.3 (H^1). For a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, we define the Sobolev space $H^1(\Omega) := \{v \in L^2(\Omega) : \int_{\Omega} |\text{grad } v(x)|^2 dx < \infty\}$ as a Hilbert space with norm $\|v\|_{H^1(\Omega)}^2 := \|v\|_{L^2(\Omega)}^2 + |v|_{H^1(\Omega)}^2$, $|v|_{H^1(\Omega)}^2 := \int_{\Omega} |\text{grad } v(x)|^2 dx$

Definition 1.4 ($H^{1/2}$).

Definition 1.5 (H_{loc}^k).

1.0.3 Trace operators

Definition 1.6 (Trace operator). A trace operator is a linear mapping from a function space on the volume domain Ω to a function space on (parts of) the boundary $\partial\Omega$.

Definition 1.7 ((Layer) potential). A (layer) potential is a linear mapping from a function space on $\partial\Omega$ into a function space on the volume domain Ω .

Definition 1.8 (Dirichlet Trace). The Dirichlet trace (operator) T_D boils down to pointwise restriction for smooth functions:

$$(T_D w)(\mathbf{x}) := w(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma, \quad w \in C^\infty(\bar{\Omega}).$$

Definition 1.9 (Dirichlet trace space). The Dirichlet trace space $H^{\frac{1}{2}}(\Gamma)$ is the Hilbert space obtained by completion of $C^\infty(\bar{\Omega})|_{\Gamma}$ with respect to the energy norm

$$\|u\|_{H^{\frac{1}{2}}(\Gamma)} := \inf \left\{ \|v\|_{H^1(\Omega)} : v \in C^\infty(\bar{\Omega}), T_D v = u \right\}, \quad u \in C^\infty(\bar{\Omega})|_{\Gamma}.$$

theorem 1.10. The Dirichlet trace T_D according can be extended to a continuous and surjective linear operator $T_D : H^1(\Omega) \rightarrow H^{\frac{1}{2}}(\Gamma)$

Definition 1.11 (Neumann Trace). For smooth functions the Neumann trace (operator) T_N is defined by

$$(T_N w)(\mathbf{x}) := \text{grad } w \cdot \mathbf{n}(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma, w \in C^\infty(\bar{\Omega}).$$

Definition 1.12 (Neumann Trace Space). The Neumann trace space $H^{-\frac{1}{2}}(\Gamma)$ is the Hilbert space obtained by the completion of $C^0(\Gamma)$ with respect to the norm

$$\|\phi\|_{H^{-\frac{1}{2}}(\Gamma)} := \|\tilde{\phi}\|_{\tilde{H}^{-1}(\Omega)}$$

where $\tilde{\phi}$ is the "extension by zero to \mathbb{R}^d " of ϕ . We have the definition $\|\rho\|_{\tilde{H}^{-1}(\Omega)} := |u|_{H^1(\mathbb{R}^3)}$ where u solves $\begin{cases} -\Delta u = \tilde{\rho} \text{ in } \mathbb{R}^3 \\ u \text{ satisfies decay condition} \end{cases}$

Definition 1.13 (Space of function with square-integrable Laplacian). We define the Hilbert space

$$H(\Delta, \Omega) := \{v \in H^1(\Omega) : \Delta v \in L^2(\Omega)\}$$

with norm

$$\|u\|_{H(\Delta, \Omega)}^2 := \|u\|_{H^1(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2, \quad u \in H(\Delta, \Omega).$$

theorem 1.14. The Neumann trace T_N can be extended to a continuous mapping $T_N : H(\Delta, \Omega) \rightarrow H^{-\frac{1}{2}}(\Gamma)$.

Definition 1.15 ($C_{\text{comp}}^\infty(\mathbb{R}^d)$).

1.0.4 Notations

- Consider a bounded Lipschitz open set $\Omega^- \subset \mathbb{R}^d$, $d = 2, 3$.
- $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega^-}$
- $\Gamma := \partial\Omega^- = \partial\Omega^+$
- \mathbf{n} is the unit normal vector field on Γ pointing from Ω^- into Ω^+
- For any $\varphi \in L_{\text{loc}}^2(\mathbb{R}^d)$, we let $\varphi^- := \varphi|_{\Omega^-}$ and $\varphi^+ := \varphi|_{\Omega^+}$
- $H_{\text{loc}}^1(\Omega^\pm, \Delta) := \{v : \chi v \in H^1(\Omega^\pm), \Delta(\chi v) \in L^2(\Omega^\pm) \text{ for all } \chi \in C_{\text{comp}}^\infty(\mathbb{R}^d)\}$
- Dirichlet and Neumann trace operators¹: $\gamma_D^\pm : H_{\text{loc}}^1(\Omega_\pm) \rightarrow H^{1/2}(\Gamma)$ and $\gamma_N^\pm : H_{\text{loc}}^1(\Omega_\pm, \Delta) \rightarrow H^{-1/2}(\Gamma)$ with $\gamma_D^\pm v := v^\pm|_\Gamma$ and γ_N^\pm such that if $v \in H_{\text{loc}}^2(\Omega_\pm)$ then $\gamma_N^\pm v = \mathbf{n} \cdot \gamma_D^\pm(\nabla v)$
- Cauchy trace: $\gamma_C^\pm : H_{\text{loc}}^1(\Omega^\pm, \Delta) \rightarrow H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, $\gamma_C^\pm := (\gamma_D^\pm, \gamma_N^\pm)$
- Sommerfeld radiation condition: $\varphi \in C^1(\mathbb{R}^d \setminus B_R)$, for some ball $B_R := \{|\mathbf{x}| < R\}$, and $\kappa > 0$ satisfies this condition if $\lim_{r \rightarrow \infty} r^{\frac{d-1}{2}} \left(\frac{\partial \varphi(\mathbf{x})}{\partial r} - i\kappa \varphi(\mathbf{x}) \right) = 0$ in all directions. We then write $\varphi \in \text{SRC}(\kappa)$

theorem 1.16 (Green's first formula). (From Wikipedia) $\int_U (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) \, dU = \int_{\partial U} \phi \frac{\partial \psi}{\partial \mathbf{n}} \, dS$

1. The Dirichlet trace operator boils down to pointwise restriction.

2 Problem

We start by formulating the problem. We want to investigate the occurrence of spurious quasi-resonances in the variational formulation of the following Helmholtz transition problem.

Definition 2.1 (Helmholtz Transmission Problem). Find $u \in H_{\text{loc}}^1(\mathbb{R}^d \setminus \Gamma) \cap \text{SRC}(k\sqrt{c_o})$ such that

$$\begin{aligned} (\Delta + \kappa^2 n_i) u^- &= 0 & \text{in } \Omega^- \\ (\Delta + \kappa^2 n_o) u^+ &= 0 & \text{in } \Omega^+ \\ \gamma_C^- u^- &= \gamma_C^+ u^+ + \mathbf{f} & \text{on } \Gamma. \end{aligned}$$

2.1 Deriving the variational formulation

We will now state the variational formulation of the problem in Ω^- .

Integration by parts in Ω^- using Green's first formula implies

$$a(U, V) - (\gamma_1^- U, \gamma_0^- V)_\Gamma = 0 \quad \forall v \in H^1(\Omega^-)$$

This variational equation can be coupled to the transmission conditions using the BIE. As the problem is equivalent to the problem in the 2006 paper with $U_i = 0, f = 0$, we can use the same formulation as in the 2006 paper.

The formulation is:

Find $U \in H^1(\Omega), \vartheta \in H^{-1/2}(\Gamma)$ and $p \in H^1(\Gamma)$ such that for all $V \in H^1(\Omega), \varphi \in H^{-1/2}(\Gamma)$ and $q \in H^1(\Gamma)$ there holds

$$\begin{aligned} q_\kappa(U, V) + (W_\kappa(\gamma_D^- U), \gamma_D^- V)_\Gamma - \left(\left(\frac{1}{2} \text{Id} - K'_\kappa \right)(\vartheta), \gamma_D^- V \right)_\Gamma &= f_2(V) \\ \left(\left(\frac{1}{2} \text{Id} - K_\kappa \right)(\gamma_D^- U), \varphi \right)_\Gamma + (V_\kappa(\vartheta), \varphi)_\Gamma + i\bar{\eta}(p, \varphi)_\Gamma &= \bar{g}_2(V) \\ - (W_\kappa(\gamma_D^- U), q)_\Gamma - \left((K'_\kappa + \frac{1}{2} \text{Id})(\vartheta), q \right)_\Gamma + b(p, q) &= h_2(q). \end{aligned}$$

3 How to investigate Stability

To figure out how stable this formulation is, we use the inf-sup constant

theorem 3.1. Existence and uniqueness of discrete solutions $u_h \in V_h$ and the convergence $u_h \rightarrow u$ can only hold if the sesquilinear form satisfies a discrete inf-sup condition, cf. [89, Eq. 2.3.6]: There exists a constant $\gamma > 0$ such that for all $v_h \in V_h$ and the whole family V_h with $h \rightarrow 0$, there holds

$$\sup_{0 \neq w_h \in V_h} \frac{|\mathbf{a}(v_h, w_h)|}{\|w_h\|_V} \geq \gamma \|v_h\|_V$$

(from P.Meury thesis).

We will now investigate the reliability of the variational formulation by investigating the inf-sup constant of the sesquilinear form on the left side of the variational problem.

Definition 3.2.

$$\gamma = \inf_{0 \neq v_h \in V_h} \sup_{0 \neq w_h \in V_h} \frac{|\mathbf{a}(v_h, w_h)|}{\|w_h\|_V \|v_h\|_V}$$

Remark 3.3. We can estimate the inf-sup constant from the minimum singular value of the Galerkin matrix (proven in P.Meury thesis).

So we now derive the Galerkin Matrix to find the inf-sup constant.

4 Derivation of Galerkin Matrix

First, we notice that we can restrict ourselves to finding functions on $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \times H^1(\Gamma)$, since the bilinear form only depends on the restriction $\gamma_D^- U$.

Now we will further restrict this space to finite subspaces and choose orthonormal bases for the finite subspaces.

4.1 Basis functions of $\mathcal{H}^{\frac{1}{2}}(\Gamma)$

We make the Fourier Ansatz $V_n = V_n^r e^{in\phi}$. Since U must satisfy $(\Delta + \kappa^2 n_i)U = 0$, we have

$$r^2 \partial_r^2 V_n^r + r \partial_r V_n^r + (r^2 \kappa^2 n_i - n^2) V_n^r = 0.$$

This is Bessel's equation. Since we require convergence at the origin we have $V_n^r(r) = J_n(\kappa \sqrt{n_i} r)$.

We restrict our functions to $H^{\frac{1}{2}}$ and want to get an orthonormal basis $U_n(\phi) = v_n V_n(1, \phi)$ for this space:

$$(U_n, U_m)_{H^{\frac{1}{2}}(\Gamma)} = \delta_{nm}$$

This implies:

$$\begin{aligned} (U_n, U_m)_{H^{\frac{1}{2}}(\Gamma)} &= (U_n, U_m)_{\Gamma} + (\nabla \cdot U_n, \nabla \cdot U_m)_{\Gamma} = \delta_{nm} \\ &= |v_n|^2 2\pi(1+n^2) |J_n(\kappa \sqrt{n_i})|^2 \delta_{nm} \stackrel{!}{=} \delta_{nm} \end{aligned}$$

Therefore, we can choose real constants $v_n = \frac{1}{\sqrt{2\pi(1+n^2)|J_n(\kappa \sqrt{n_i})|}}$.

Notes

In the calculation above we used $\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}}$ and eliminated the r -dependency.

4.2 Other basis functions

We also pick orthonormal basis functions for $\mathcal{H}^{-\frac{1}{2}}(\Gamma)$ and $\mathcal{H}^1(\Gamma)$. We choose

$$\theta_n = w_n e^{in\phi} \text{ where } w_n = \frac{(1+n^2)^{\frac{1}{4}}}{\sqrt{2\pi}}$$

for $\mathcal{H}^{\frac{1}{2}}(\Gamma)$ and for $\mathcal{H}^{-\frac{1}{2}}(\Gamma)$ we choose

$$p_n = l_n e^{in\phi} \text{ where } l_n = \frac{1}{\sqrt{2\pi(1+n^2)}}.$$

Notes

See P. Meury 3.22 (Lemma) for explanation why we use these bases.

4.3 Constructing the Galerkin Matrix

No we can construct the Galerkin Matrix. Because of orthogonality our matrix will have a blockdiagonal structure. We start by using Greens first formula $\int_U (\psi \Delta \varphi + \nabla \psi \cdot \nabla \varphi) dV = \oint_{\partial U} \psi \nabla \varphi \cdot d\vec{S}$ to see that

$$\begin{aligned} q_{\kappa}(U_n, U_m) &= \int_{\Omega^-} (\nabla \cdot U_n \nabla \cdot \overline{U_m} - \kappa^2 n_i U_n \overline{U_m}) \\ &= \int_{\partial \Omega^-} \overline{U_m} \nabla \cdot U_n d\vec{S} - \int_{\Omega^-} \underbrace{(\kappa^2 n_i U_n + \Delta U_n)}_{=0} \overline{U_m} d\vec{S} \\ &= \delta_{nm} \int_{\partial \Omega^-} \overline{U_n} \nabla \cdot U_n d\vec{S} \\ &= \delta_{nm} |v_n|^2 \kappa \sqrt{n_i} J_n(\kappa \sqrt{n_i}) J'_n(\kappa \sqrt{n_i}) \int_{\partial \Omega^-} e^{in\phi} e^{-in\phi} dS \\ &= \delta_{nm} \frac{1}{1+n^2} \kappa \sqrt{n_i} \frac{J'_n(\kappa \sqrt{n_i})}{J_n(\kappa \sqrt{n_i})} =: \delta_{nm} \alpha_n. \end{aligned}$$

Also clearly $b(p_n, p_m) = (p_n, p_m)_{H^1(\Gamma)} = \delta_{nm}$

PROVE ORTHOGONALITY OF MATRIX HERE, HERLEITUNG

4.4 Eigenvalue Equations

$$V_\kappa(y_n) = \lambda_n^{(V)} y_n, \quad \lambda_n^{(V)} := \frac{i\pi}{2} J_n(\kappa) H_n^{(1)}(\kappa)$$

$$K_\kappa(y_n) = \lambda_n^{(K)} y_n, \quad \lambda_n^{(K)} := \frac{i\pi\kappa}{2} J_n(\kappa) H_n^{(1)'}(\kappa) + \frac{1}{2} = \frac{i\pi\kappa}{2} J_n'(\kappa) H_n^{(1)}(\kappa) - \frac{1}{2}$$

$$K'_\kappa(y_n) = \lambda_n^{(K')} y_n, \quad \lambda_n^{(K')} := \frac{i\pi\kappa}{2} J_n(\kappa) H_n^{(1)'}(\kappa) + \frac{1}{2} = \frac{i\pi\kappa}{2} J_n'(\kappa) H_n^{(1)}(\kappa) - \frac{1}{2}$$

$$W_\kappa(y_n) = \lambda_n^{(W)} y_n, \quad \lambda_n^{(W)} := \frac{i\pi\kappa^2}{2} J_n'(\kappa) H_n^{(1)'}(\kappa)$$

(from P. Meury.)

4.5 Matrix

Now the first equation yields with coefficients:

$$(\alpha_n + \lambda_n^{(w)} P_n^{UU}) C_n^U - (\frac{1}{2} - \lambda_n^{(K')}) P_n^{\theta U} C_n^\theta = f_2(V)$$

second equation

$$(\frac{1}{2} - \lambda_n^{(K)}) P_n^{U\theta} C_n^U + \lambda_n^{(V)} P_n^{\theta\theta} C_n^\theta + i\eta P^{p\theta} C_n^p = \overline{g_2}(V)$$

third equation

$$-\lambda_n^{(W)} P_n^{Up} C_n^U - (\lambda_n^{(K')} + \frac{1}{2}) P_n^{\theta p} C_n^\theta + C_n^p = h_2(V)$$

Using all the previously derived equations we can construct the Galerkin matrix:

$$A_n = \begin{pmatrix} (\alpha_n + \lambda_n^{(w)} P_n^{UU}) & -(\frac{1}{2} - \lambda_n^{(K')}) P_n^{\theta U} & 0 \\ (\frac{1}{2} - \lambda_n^{(K)}) P_n^{U\theta} & \lambda_n^{(V)} P_n^{\theta\theta} & i\eta P^{p\theta} \\ -\lambda_n^{(W)} P_n^{Up} & -(\lambda_n^{(K')} + \frac{1}{2}) P_n^{\theta p} & 1 \end{pmatrix}$$

4.6 Scalar Product Values

Introduce $\tilde{v}_n = J_n(\kappa\sqrt{n_i})v_n$. The values of the scalar products are

- $P_n^{UU} = 2\pi|\tilde{v}_n|^2$
- $P_n^{\theta U} = 2\pi w_n \overline{\tilde{v}_n}$
- $P_n^{U\theta} = \overline{P_n^{\theta U}}$
- $P_n^{\theta\theta} = 2\pi|w_n|^2$
- $P_n^{p\theta} = 2\pi l_n \overline{w_n}$
- $P_n^{Up} = 2\pi \tilde{v}_n \overline{l_n}$
- $P_n^{\theta p} = \overline{P_n^{p\theta}}$

5 Validation

6 Numerical Results