Spurious Quasi-Resonances for Stabilized BIE-Volume Formulations for Helmholtz Transition Problem

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1 Problem

We start by formulating the problem ...

We want to find the minimal SV of the lhs of: Find $U \in H^1(\Omega), \vartheta \in H^{-1/2}(\Gamma)$ and $p \in H^1(\Gamma)$ such that for all $V \in H^1(\Omega), \varphi \in H^{-1/2}(\Gamma)$ and $q \in H^1(\Gamma)$ there holds

$$\mathrm{q}_\kappa(U,V) + \left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), \gamma_D^- V\right)_\Gamma - \left(\left(\frac{1}{2}\mathrm{ld} - \mathrm{K}'_\kappa\right)(\vartheta), \gamma_D^- V\right)_\Gamma = \mathrm{f}_2(V)$$

$$\left(\varphi, \left(\frac{1}{2}\mathrm{ld} - \mathrm{K}_\kappa\right)\left(\gamma_D^- U\right)\right)_\Gamma + (\varphi, \mathrm{V}_\kappa(\vartheta))_\Gamma - i\eta(\varphi, p)_\Gamma = \mathrm{g}_2(V)$$

$$-\left(\mathrm{W}_\kappa\left(\gamma_D^- U\right), q\right)_\Gamma - \left(\left(\mathrm{K}'_\kappa + \frac{1}{2}\mathrm{Id}\right)(\vartheta), q\right)_\Gamma + \mathrm{b}(p,q) = \mathrm{h}_2(q)$$

2 Derivation of Galerkin Matrix

First, we notice that we can restrict ourselves to finding functions on $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \times H^1(\Gamma)$, since the bilinear form only depends on the restriction $\gamma_D^- U$.

No we will further restrict this space to finite subspaces and choose orthonormal bases for the finite subspaces.

2.1 Condition for U

We make the Fourier Ansatz $V_n = V_n^r e^{in\phi}$. Since $U \in H^1(\Omega^-)$ must satisfy $(\Delta + \kappa^2 n_i)U = 0$, we have

$$r^2 \partial_r^2 V_n^r + r\partial_r V_n^r + (r^2\kappa^2 n_i - n^2)V_n^r = 0.$$

This is Bessel's equation. Since we require convergence at the origin we have $V_n^r(r) = J_n(\kappa\sqrt{n_i}r)$.

We restrict our functions to $H^{\frac{1}{2}}$ and want to get an orthonormal basis $U_n(\phi) = v_n V_n(1,\phi)$ for this space:

$$(U_n, U_m)_{H^1(\Gamma)} = \delta_{nm}$$

This implies:

$$(U_n, U_m)_{H^1(\Gamma)} = (U_n, U_m)_\Gamma + (\boldsymbol{\nabla}\cdot U_n, \boldsymbol{\nabla}\cdot U_m)_\Gamma$$

$$= |v_n|^2 2\pi\delta_{nm}\left((\int_0^1 drr(J_n(\kappa\sqrt{n_i}r)^2 + \kappa^2 n_i J'_n(\kappa\sqrt{n_i}r)^2) - n^2\right)$$

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where we used $\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}}$ and $\frac{d}{dx}J_n(x) = +\frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right)$ etc. check with Meury

Note (from NumPDE Advanced): 1. The Dirichlet trace space $H^{\frac{1}{2}}(\Gamma)$ is the Hilbert space obtained by completion of $C^\infty(\bar{\Omega})\big|_\Gamma$ with respect to the energy norm

$$\|\mathfrak{u}\|_{H^{\frac{1}{2}}(\Gamma)} := \inf\left\{\|v\|_{H^1(\Omega)} : v \in C^\infty(\bar{\Omega}), T_D v = \mathfrak{u}\right\}, \quad \mathfrak{u} \in C^\infty(\bar{\Omega})\big|_\Gamma$$

2. The Dirichlet trace T_D according to Def. 1.3.3 can be extended to a continuous and surjective linear operator $\mathrm{T}_D : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$

2.2 Other basis functions

Next to U_n we also pick orthonormal basis functions for $H^{-\frac{1}{2}}(\Gamma)$ and $H^1(\Gamma)$: $\theta_n = ...e^{in\phi}$ $p_n = ...e^{in\phi}$

Constructing the Galerkin Matrix

3 Validation

4 Numerical Results