

# *A Gromov-Hausdorff Distance for Hilbert Modules*

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# Noncommutative metric geometry

## *Theorem (Gel'fand-Naimark duality)*

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## *Founding Allegory of Noncommutative **Metric** Geometry*

Noncommutative **metric** geometry is the study of noncommutative generalizations of algebras of **Lipschitz** functions on **metric spaces**.

Pioneered by Connes (1989: spectral triples), Rieffel (1998 onward: compact quantum metric spaces).

# *My program*

## *Geometry of classes of quantum metric spaces*

- Study *the geometry of entire classes* of quantum metric spaces using noncommutative analogues of the *Gromov-Hausdorff distance*: the *Gromov-Hausdorff propinquity* family.
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- Motivated by K-theory, KK-theory, Morita equivalence, as quantum vector bundles, ...
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Other directions / Other talks: Development of the theory of *locally compact quantum metric spaces*, Mathematical physics.



- 1 *Compact Quantum Metric Spaces*
- 2 *Convergence of quasi-Leibniz quantum compact metric space*
- 3 *Metrized quantum vector bundles*
- 4 *The modular Propinquity*

## The Monge-Kantorovich metric

Let  $(X, \mathfrak{m})$  be a compact metric space. The *Lipschitz seminorm*  $L$  induced by  $\mathfrak{m}$  is:

$$L(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\}$$

for all  $f \in C(X)$  (allowing  $\infty$ ).

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The *Monge-Kantorovich metric* on  $\mathcal{S}(C(X))$  is given for all Borel-regular probability measures  $\mu, \nu$  by:

$$\mathrm{mk}_L(\varphi, \psi) = \sup \left\{ \left| \int_X f d\mu - \int_X f d\nu \right| : f \in \mathfrak{sa}(C(X)), L(f) \leq 1 \right\}.$$

# Quasi-Leibniz Compact Quantum Metric Spaces

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We call  $L$  an *L-seminorm*.

## Ergodic actions of compact metric groups

### Theorem (Rieffel, 98)

Let  $G$  be a *compact group* endowed with a *continuous length function*  $\ell$ . Let  $\alpha$  be an *action* of  $G$  on some *unital  $C^*$ -algebra*  $\mathfrak{A}$ . Set:

$$\forall a \in \mathfrak{A} \quad L(a) = \sup \left\{ \frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1_G\} \right\}.$$

$(\mathfrak{A}, L)$  is a *Leibniz quantum compact metric space* if and only if  $\{a \in \mathfrak{A} : \forall g \in G \quad \alpha^g(a) = a\} = \mathbb{C}1_{\mathfrak{A}}$ .

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## Example: Quantum tori

- $G = \mathbb{T}^d$ ,
- $\mathfrak{A} = C^*(\mathbb{Z}^d, \sigma)$  (universal for  $U_j U_k = \sigma(j, k) U_{j+k}$ ),
- $\alpha$  : dual action ( $\alpha^z U_j = z^j U_j$ ).
- Associated with a differential calculus when  $\ell$  from invariant Riemannian metric.

## Spectral triples and quantum metrics

*Theorem (Rieffel, 02; Ozawa-Rieffel, 05; Christ-Rieffel, 15)*

Let  $G$  be a *discrete group*,  $l$  a *length function* on  $G$ , and  $\pi$  be the left regular representation of  $C_{\text{red}}^*(G)$  on  $\ell^2(G)$ . For all  $\xi \in \ell^2(G)$ , set  $D\xi : g \in G \mapsto l(g)\xi(g)$ . If  $G$  is hyperbolic or nilpotent and  $l$  is the word-length function for some finite generator set, then  $(C^*(G), \llbracket [D, \pi(\cdot)] \rrbracket_{\mathcal{H}})$  is a Leibniz quantum compact metric space.

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*Theorem (Aguilar and L., 2015)*

Let  $\mathfrak{A} = \varinjlim \mathfrak{A}_n$  with  $\mathfrak{A}_n$  *f.d.* for all  $n \in \mathbb{N}$ . if  $\mathfrak{A}$  is unital and has a faithful tracial state  $\tau$ , and if for all  $a \in \mathfrak{A}$  we set:

$$L(a) = \sup \left\{ \frac{\|a - \mathbb{E}_n(a)\|_{\mathfrak{A}}}{\beta(n)} : n \in \mathbb{N} \right\}$$

where  $\beta \in (0, \infty)^{\mathbb{N}}$  with  $\lim_{\infty} \beta = 0$ , and  $\mathbb{E}_n : \mathfrak{A} \twoheadrightarrow \mathfrak{A}_n$  is the conditional expectation with  $\tau \circ \mathbb{E}_n = \tau$ , then  $(\mathfrak{A}, L)$  is a *quasi-Leibniz quantum compact metric space*.

## Other Examples

- 1 Quantum metrics from the standard spectral triple on quantum tori (Rieffel, 98 and 02)
- 2 Connes-Landi spheres (H. Li, 03)
- 3 Conformal deformations of quantum metric spaces from spectral triples (L., 15)
- 4 Curved quantum tori of Dabrowsky and Sitarz (L., 15)
- 5 Group  $C^*$ -algebras for groups with rapid decay
- 6 Certain  $C^*$ -crossed-products (J. Bellissard, M. Marcolli, Reihani, 10), (involves my work on locally compact quantum metric space).

# Lipschitz morphisms

## Definition

Let  $(\mathfrak{A}, L_{\mathfrak{A}})$  and  $(\mathfrak{B}, L_{\mathfrak{B}})$  be two quasi-Leibniz quantum compact metric spaces. A  *$k$ -Lipschitz morphism*  $\pi : (\mathfrak{A}, L_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, L_{\mathfrak{B}})$  is a unital  $*$ -morphism from  $\mathfrak{A}$  to  $\mathfrak{B}$  such that  $\varphi \in \mathcal{S}(\mathfrak{B}) \mapsto \varphi \circ \pi$  is a  *$k$ -Lipschitz map from  $(\mathcal{S}(\mathfrak{B}), \text{mk}_{L_{\mathfrak{B}}})$  to  $(\mathcal{S}(\mathfrak{A}), \text{mk}_{L_{\mathfrak{A}}})$ .*

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- ①  $\pi$  is a  $k$ -Lipschitz morphism,
- ② (Rieffel, 00)  $L_{\mathfrak{B}} \circ \pi \leq k L_{\mathfrak{A}}$ ,
- ③ (L., 16)  $\pi(\text{dom}(L_{\mathfrak{A}})) \subseteq \text{dom}(L_{\mathfrak{B}})$ .



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### *Theorem (McShane, 1934)*

Let  $(Z, \mathfrak{m})$  be a metric space and  $X \subseteq Z$  not empty. If  $f : X \rightarrow \mathbb{R}$  is a  $k$ -Lipschitz function on  $X$  then there exists a  $k$ -Lipschitz function  $g : Z \rightarrow \mathbb{R}$  such that  $g$  restricts to  $f$  on  $X$ .

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A *full quantum isometry*  $\pi$  is a  $*$ -isomorphism such that  $L_{\mathfrak{B}} \circ \pi = L_{\mathfrak{A}}$ .

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Quantum isometries are 1-Lipschitz morphisms.

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- 2 *Convergence of quasi-Leibniz quantum compact metric space*
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# The Gromov-Hausdorff Distance

## Definition

For any two compact metric spaces  $(X, \mathbf{m}_X)$  and  $(Y, \mathbf{m}_Y)$ , we define  $\text{Adm}(\mathbf{m}_X, \mathbf{m}_Y)$  as:

$$\left\{ (Z, \mathbf{m}_Z, \iota_X, \iota_Y) \left| \begin{array}{l} (Z, \mathbf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right. \right\}$$

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## Notation

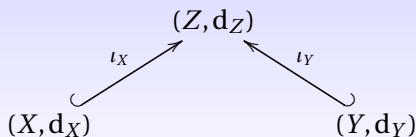
The *Hausdorff distance* on the compact subsets of a metric space  $(X, m)$  is denoted by  $\text{Haus}_m$ .

## Definition (Gromov, 81)

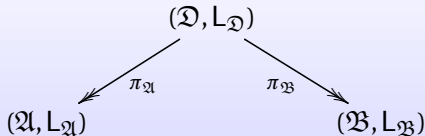
The *Gromov-Hausdorff distance* between two compact metric spaces  $(X, m_X)$  and  $(Y, m_Y)$  is:

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# A noncommutative Gromov-Hausdorff distance



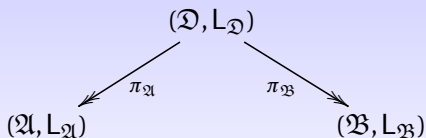
*Figure:* Gromov-Hausdorff Isometric Embeddings



*Figure:* A tunnel

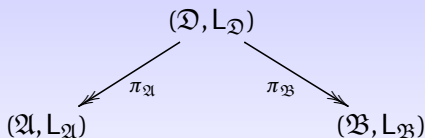


# The Dual Gromov-Hausdorff Propinquity



*Figure:* An  $F$ -tunnel: all spaces are  $F$ -quasi-Leibniz

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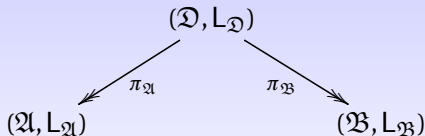
*Figure:* An  $F$ -tunnel: all spaces are  $F$ -quasi-Leibniz

## Definition (The extent of a tunnel)

The *extent* of a tunnel  $\tau = (\mathfrak{D}, L_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$  is:

$$\max \left\{ \text{Haus}_{\text{mk}_{L_{\mathfrak{D}}}} \left( \mathcal{S}(\mathfrak{D}), \pi_{\mathfrak{A}}^* (\mathcal{S}(\mathfrak{A})) \right), \text{Haus}_{\text{mk}_{L_{\mathfrak{D}}}} \left( \mathcal{S}(\mathfrak{D}), \pi_{\mathfrak{B}}^* (\mathcal{S}(\mathfrak{B})) \right) \right\}.$$

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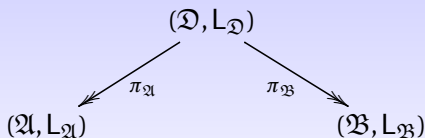
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## Definition (L. ,13, 14 / special case)

The *dual propinquity*  $\Lambda_F^*((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}))$  is given by:

$$\inf \left\{ \chi(\tau) : \tau \text{ any } F\text{-tunnel from } (\mathfrak{A}, L_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, L_{\mathfrak{B}}) \right\}.$$

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## Theorem (L., 13)

The dual propinquity is a *complete metric* up to *full quantum isometry*, which induces the same topology on classical compact metric spaces as the Gromov-Hausdorff distance.

## Target sets of tunnels

An  $F$ -tunnel  $(\mathfrak{A}, L_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, L_{\mathfrak{B}})$  gives rise to compact-set-valued function which has morphism-like properties.

- ① For  $a \in \text{dom}(L_{\mathfrak{A}})$  and  $l \geq L_{\mathfrak{A}}(a)$ , the  *$l$ -target set* of  $a$  is:

$$t_{\tau}(a|l) = \{\pi_{\mathfrak{B}}(d) \in \text{sa}(\mathfrak{B}) : L(d) \leq l, \pi_{\mathfrak{A}}(d) = a\}.$$

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- ② For all  $a \in \text{dom}(L_{\mathfrak{A}})$ , and  $l \geq L_{\mathfrak{A}}(a)$ :

$$\text{diam}(t_{\tau}(a|l), \|\cdot\|_{\mathfrak{B}}) \leq 4l\chi(\tau).$$

## Target sets of tunnels

An  $F$ -tunnel  $(\mathfrak{A}, L_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, L_{\mathfrak{B}})$  gives rise to compact-set-valued function which has morphism-like properties.

- ① For  $a \in \text{dom}(L_{\mathfrak{A}})$  and  $l \geq L_{\mathfrak{A}}(a)$ , the  $l$ -target set of  $a$  is:

$$\mathfrak{t}_{\tau}(a|l) = \{\pi_{\mathfrak{B}}(d) \in \mathfrak{sa}(\mathfrak{B}) : L(d) \leq l, \pi_{\mathfrak{A}}(d) = a\}.$$

- ② For all  $a \in \text{dom}(L_{\mathfrak{A}})$ , and  $l \geq L_{\mathfrak{A}}(a)$ :

$$\text{diam}(\mathfrak{t}_{\tau}(a|l), \|\cdot\|_{\mathfrak{B}}) \leq 4l\chi(\tau).$$

- ③ For all  $a, a' \in \text{dom}(L_{\mathfrak{A}})$ ,  $t \in \mathbb{R}$  and  $l \geq \max\{L_{\mathfrak{A}}(a), L_{\mathfrak{A}}(a')\}$ :

$$\mathfrak{t}_{\tau}(a|l) + t \cdot \mathfrak{t}_{\tau}(a'|l) \subseteq \mathfrak{t}_{\tau}(a + ta'|(1 + |t|)l)$$

$$\mathfrak{t}_{\tau}(a|l) \circ \mathfrak{t}_{\tau}(a'|l) \subseteq \mathfrak{t}_{\tau}(a \circ a' | F(\|a\|_{\mathfrak{A}} + 2l\chi(\tau), \|a'\|_{\mathfrak{A}} + 2l\chi(\tau), l, l))$$

with a similar expression for the Lie product.

# Bridges

## Definition (A bridge)

A *bridge*  $(\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$  from  $\mathfrak{A}$  to  $\mathfrak{B}$  is a unital  $C^*$ -algebra  $\mathfrak{D}$ , two unital  $*$ -monomorphisms  $\pi_{\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{D}$  and  $\pi_{\mathfrak{B}} : \mathfrak{B} \hookrightarrow \mathfrak{D}$  and  $\omega \in \mathfrak{D}$  such that  $\mathcal{S}(\mathfrak{D}|\omega) = \{\varphi \in \mathcal{S}(\mathfrak{D}) : \varphi(\cdot\omega) = \varphi(\omega\cdot) = \varphi\} \neq \emptyset$ .



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## Definition (Length of a bridge)

The length  $\lambda(\gamma)$  of a bridge  $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$  from  $(\mathfrak{A}, L_{\mathfrak{A}})$  to  $(\mathfrak{B}, L_{\mathfrak{B}})$  is the maximum of:

$$\max \left\{ \text{Haus}_{\text{mk}_{L_{\mathfrak{A}}}}(\mathcal{S}(\mathfrak{A}), \pi_{\mathfrak{A}}^*(\mathcal{S}(\mathfrak{D}|\omega))), \right. \\ \left. \text{Haus}_{\text{mk}_{L_{\mathfrak{B}}}}(\mathcal{S}(\mathfrak{B}), \pi_{\mathfrak{B}}^*(\mathcal{S}(\mathfrak{D}|\omega))) \right\},$$

and:

$$\text{Haus}_{\|\cdot\|_{\mathfrak{D}}}(\{a\omega \in \mathfrak{sa}(\mathfrak{A}) : L_{\mathfrak{A}}(a) \leq 1\}, \{\omega b \in \mathfrak{sa}(\mathfrak{A}) : L_{\mathfrak{B}}(b) \leq 1\}) ,$$

# The quantum Gromov-Hausdorff Propinquity

## Theorem (L. (13))

For any bridge  $\gamma$  from  $(\mathfrak{A}, L_{\mathfrak{A}})$  to  $(\mathfrak{B}, L_{\mathfrak{B}})$  and  $\lambda > 0$  with  $\lambda \geq \lambda(\gamma)$ , the following is a tunnel of extent at most  $2\lambda$ :

$$(\mathfrak{A} \oplus \mathfrak{B}, (a, b) \mapsto a, (a, b) \mapsto b,$$

$$(a, b) \mapsto \max \left\{ L_{\mathfrak{A}}(a), L_{\mathfrak{B}}(b), \frac{1}{\lambda} \|\pi_{\mathfrak{A}}(a)\omega - \omega\pi_{\mathfrak{B}}(b)\|_{\mathfrak{D}} \right\}.$$

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## Theorem (L, (13))

There exists a metric up to full quantum isometry on the class of  $F$ -quasi-Leibniz quantum compact metric spaces, called the *quantum propinquity*  $\Lambda$ , such that:

- ❶  $\Lambda^* \leq \Lambda$  ( $\leq$  GH in classical picture),
- ❷  $\Lambda((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) \leq \lambda(\gamma)$  for any bridge  $\gamma$  between any  $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})$ .

# Finite Dimensional Approximations of quantum tori

## Theorem (L. (13))

If for all  $n \in \mathbb{N}$ , we set  $\mathcal{F}_n = C^*(U_n, V_n)$  where:

$$U_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & \dots & 1 & 0 \end{pmatrix}, V_n = \begin{pmatrix} 1 & & & & \\ & \rho_n & & & \\ & & \rho_n^2 & & \\ & & & \ddots & \\ & & & & \rho_n^{n-1} \end{pmatrix}$$

with  $\rho_n = e^{2i\pi \frac{pn}{n}} \neq 1$ , and if  $\lim_{n \rightarrow \infty} \rho_n = \rho$ , then:

$$\lim_{n \rightarrow \infty} \Lambda((\mathcal{F}_n, L_n), (\mathcal{A}_\rho, L)) = 0$$

where  $\mathcal{A}_\rho = C^*(U, V)$  and  $U, V$  are universal unitaries such that  $VU = \exp(2i\pi\rho)UV$ , while  $L_n$  and  $L$  are  $L$ -seminorms from the dual actions, and for some *fixed* continuous length function on  $\mathbb{T}^2$ .

# Quantum Tori and the quantum propinquity

## Theorem (Latrémollière, 2013)

Let  $d \in \mathbb{N} \setminus \{0, 1\}$ ,  $\sigma$  a multiplier of  $\mathbb{Z}^d$ . For each  $n \in \mathbb{N}$ , let  $k_n \in \overline{\mathbb{N}}^d$  and  $\sigma_n$  be a multiplier of  $\mathbb{Z}_{k_n}^d = \mathbb{Z}^d / k_n \mathbb{Z}^d$  such that:

- 1  $\lim_{n \rightarrow \infty} k_n = (\infty, \dots, \infty)$ ,
- 2 the unique lifts of  $\sigma_n$  to  $\mathbb{Z}^d$  as multipliers converge pointwise to  $\sigma$ .

Then  $\lim_{n \rightarrow \infty} \Lambda \left( C^* (\mathbb{Z}^d, \sigma), C^* (\mathbb{Z}_{k_n}^d, \sigma_n) \right) = 0$ , where the Lip-norms are given by the dual actions for any *fixed* length function on  $\mathbb{T}^d$ .

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## Theorem (Aguilar and L. (15) / Informal)

The Effros-Shen AF algebras parametrized by the space of irrational real numbers (i.e. by the Baire space) are a continuous family of quasi-Leibniz quantum compact metric spaces for the quantum propinquity.

## Other examples

### Example: Other examples of convergence

- 1 Conformal perturbations of quantum metrics (L., 15)
- 2 Dabrowsky and Sitarz' Curved quantum tori (L., 15)
- 3 AF algebras as limits of their inductive sequence in a *metric* sense; UHF algebras and Effros-Shen algebras form continuous families (Aguilar and L., 15),
- 4 Spheres as limits of full matrix algebras (Rieffel, 15)
- 5 Nuclear quasi-diagonal quasi-Leibniz quantum compact metric spaces have finite dim approximations (L., 15),
- 6 There exists an analogue of Gromov's compactness theorem (L., 15)
- 7 Noncommutative solenoids form a continuous family and have approximations by quantum tori (L. and Packer, 16)
- 8 Closed balls for the noncommutative Lipschitz distance are totally bounded for  $\Lambda$  (L., 16)

- 1 *Compact Quantum Metric Spaces*
- 2 *Convergence of quasi-Leibniz quantum compact metric space*
- 3 ***Metrized quantum vector bundles***
- 4 *The modular Propinquity*



# *Metrics for Vector Bundles*

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Let  $V$  be a  $\mathbb{C}$ -vector bundles over a compact Riemannian manifold  $M$ . A *metric* on  $V$  is given as a (smooth) section of the associated bundle of sesquilinear products on the fibers of  $V$ .

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Moreover, we always have a *metric connection*  $\nabla$ , such that:

$$d_X g(\omega, \eta) = g(\nabla_X \omega, \eta) + g(\omega, \nabla_X \eta).$$

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We propose that *both*  $g$  and  $|||\nabla \cdot |||$  contain metric information.

# Metrized quantum vector bundles

## Definition (metrized quantum vector bundle, L. (16))

A metrized quantum vector bundle  $\Omega = (\mathcal{M}, \langle \cdot, \cdot \rangle_{\mathcal{M}}, D, \mathfrak{A}, L)$  is given by:

- ①  $(\mathfrak{A}, L)$  is a quasi-Leibniz quantum compact metric space,
- ②  $(\mathcal{M}, \langle \cdot, \cdot \rangle_{\mathcal{M}})$  is a left Hilbert module over  $\mathfrak{A}$ ,
- ③  $D$  is a norm on a dense subspace of  $\mathcal{M}$  such that:
  - ①  $D \geq \| \cdot \|_{\mathcal{M}} := \sqrt{\langle \cdot, \cdot \rangle_{\mathcal{M}}}$
  - ②  $\{\omega \in \mathcal{M} : D(\omega) \leq 1\}$  is compact in  $(\mathcal{M}, \| \cdot \|_{\mathcal{M}})$ ,
  - ③  $D(a\omega) \leq G(\|a\|_{\mathfrak{A}}, L(a), D(\omega))$ ,
  - ④  $L(\langle \omega, \eta \rangle_{\mathcal{M}}) \leq H(D(\omega), D(\eta))$ .

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## Example: Classical picture

For a compact Riemannian manifold  $M$  and a  $\mathbb{C}$ -vector bundle  $V$  with a metric  $g$  and metric connection  $\nabla$ , we use the inner product  $\langle \omega, \eta \rangle_g = \int_M g_x(\omega_x, \eta_x) d\text{Vol}(x)$  and  $D(\omega) = \max\{\|\omega\|_g, \|\nabla \cdot \omega\|\}$ .

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## Example: Free modules

Given  $(\mathfrak{A}, L)$ , we set  $\langle (a_1, \dots, a_d), (b_1, \dots, b_d) \rangle_d = \sum_{j=1}^d a_j b_j^*$  and  $L_d(a_1, \dots, a_d) = \max \{L(\Re a_j), L(\Im a_j) : j \in \{1, \dots, d\}\}$ . Let  $D = \max \{\| \cdot \|_d, L_d\}$ . Then  $(\mathfrak{A}^d, \langle \cdot, \cdot \rangle_d, D, \mathfrak{A}, L)$  is a metrized quantum vector bundle.

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## Full quantum isometries

$(\theta, \Theta)$  full quantum isometry when  $\theta$  full quantum isometry between bases and  $\Theta(a\xi) = \theta(a)\Theta(\xi)$ ,  $\Theta$  linear isomorphism preserving both the norms and the  $D$ -norms.



## *The Heisenberg Modules (Connes, 81; Rieffel)*

Fix  $\theta \in \mathbb{R}$ ,  $p \in \mathbb{Z}$ ,  $q \in \mathbb{N}$ ,  $d \in \mathbb{N} \setminus \{0\}$  such that  $\tilde{\theta} = \theta - \frac{p}{q} \neq 0$ .

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- 1 Start with a representation of  $\left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R}^3 \right\}$  on

$L^2(\mathbb{R})$ :

$$\alpha_{\tilde{\theta}}^{x,y,t} \xi(s) = \exp(i\pi(t + 2xs)) \xi(s + \tilde{\theta}y).$$

Promote it to  $L^2(\mathbb{R}) \otimes \mathbb{C}^d$ .

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- ② Let  $W_1, W_2 \in U(d)$  with  $W_1 W_2 = e^{2i\pi p/q} W_2 W_1$  and  $W_1^n = W_2^n = 1$ . We get a  $\mathcal{A}_\theta = C^*(u_\theta, v_\theta)$ -module with:

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- ③ For Schwarz functions  $\xi, \omega$ , set:

$$\langle \xi, \omega \rangle_{\mathcal{H}_\theta^{p,q,d}} = \sum_{n,m \in \mathbb{Z}} \langle \varphi_{p,q,d,\vartheta}^{n,m} \xi, \omega \rangle_{L^2(\mathbb{R}, \mathbb{C}^d)} u_\theta^n v_\theta^m;$$

complete space of Schwarz functions to the *Heisenberg module*  
 $\mathcal{H}_\theta^{p,q,d}$ .

# The D-norm

## Definition (L., 16)

Fix some norm  $\|\cdot\|$  on  $\mathbb{R}^2$ . For all  $\xi \in \mathcal{H}_\theta^{p,q,d}$ , we set:

$$D_\theta^{p,q,d}(\xi) = \sup \left\{ \frac{\left\| \alpha_{\tilde{\partial}}^{x,y,\frac{xy}{2}} \xi - \xi \right\|_{\mathcal{H}_\theta^{p,q,d}}}{2\pi|\tilde{\partial}|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

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$(\mathcal{H}_\theta^{p,q,d}, \langle \cdot, \cdot \rangle_{\mathcal{H}_\theta^{p,q,d}}, D_\theta^{p,q,d}, \mathcal{A}_\theta, L_\theta)$  is a metrized quantum vector bundle.

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The proof rests on the compactness, for  $f$  radial, of:

$$\xi \mapsto \alpha_{\tilde{\partial}}^f \xi = \iint_{\mathbb{R}^2} f(x,y) \alpha_{\tilde{\partial}}^{x,y, \frac{xy}{2}} \xi \, dx dy$$

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For  $f$  a linear combination of certain *Laguerre functions*,  $\alpha_{\tilde{\partial}}^f$  is finite rank.

$$r \longmapsto$$



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We can fabricate an  $L^1$ -approximate unit of linear combinations of Laguerre functions using *Thangavelu* results on *Césaro sums of Laguerre expansions*.

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We then check  $\alpha_{\tilde{\partial}}^f(\xi) \leq D_\theta^{p,q,d}(\xi) \iint f(x,y)(2\pi|\tilde{\partial}|\|(x,y)\|) dx dy$ .

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$$D_\theta^{p,q,d}(\xi) = \sup \left\{ \frac{\left\| \alpha_{\tilde{\partial}}^{x,y, \frac{xy}{2}} \xi - \xi \right\|_{\mathcal{H}_\theta^{p,q,d}}}{2\pi|\tilde{\partial}|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

## Theorem (L., 16)

$(\mathcal{H}_\theta^{p,q,d}, \langle \cdot, \cdot \rangle_{\mathcal{H}_\theta^{p,q,d}}, D_\theta^{p,q,d}, \mathcal{A}_\theta, L_\theta)$  is a metrized quantum vector bundle.

Many details involved which are different from Rieffel's ergodic action result (which relied on finite dimension of spectral subspaces).

# The D-norm

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The D-norm, when restricted to Schwarz functions, is the *norm of a connection studied by Connes, Rieffel for the Yang Mills problem on the quantum 2-torus*. The connection arises from the infinitesimal rep of the Heisenberg Lie algebra form  $\alpha_{\tilde{\partial}}$ .

- 1 *Compact Quantum Metric Spaces*
- 2 *Convergence of quasi-Leibniz quantum compact metric space*
- 3 *Mettrized quantum vector bundles*
- 4 *The modular Propinquity*

## Bridges for modules

Fix  $\Omega_{\mathfrak{A}} = (\mathcal{M}_{\mathfrak{A}}, \langle \cdot, \cdot \rangle_{\mathfrak{A}}, D_{\mathfrak{A}}, \mathfrak{A}, L_{\mathfrak{A}})$  and  $\Omega_{\mathfrak{B}} = (\mathcal{M}_{\mathfrak{B}}, \langle \cdot, \cdot \rangle_{\mathfrak{B}}, D_{\mathfrak{B}}, \mathfrak{B}, L_{\mathfrak{B}})$  be two metrized quantum vector bundles.

### Definition (L., 16)

A *modular bridge*  $(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, (\omega_j)_{j \in J}, (\eta_j)_{j \in J})$  is a bridge  $(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$  and two families  $(\omega_j)_{j \in J} \in \mathcal{M}_{\mathfrak{A}}$ ,  $(\eta_j)_{j \in J} \in \mathcal{M}_{\mathfrak{B}}$  with  $D_{\mathfrak{A}}(\omega_j), D_{\mathfrak{B}}(\eta_j) \leq 1$  for all  $j \in J$ .

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The **length** of a modular bridge is the maximum of the length of its basic bridge, and the sum of:

- 1 the maximum of  $\text{Haus}_k(\{\omega_j : j \in J\}, \{\omega : D_{\mathfrak{A}}(\omega) \leq 1\})$  and its counterpart in  $\Omega_{\mathfrak{B}}$ , where:

$$k(\omega, \xi) = \sup \{ \|\langle \omega, \eta \rangle_{\mathfrak{A}} - \langle \xi, \eta \rangle_{\mathfrak{A}}\|_{\mathfrak{A}} : D_{\mathfrak{A}}(\eta) \leq 1 \},$$

- 2  $\max \{ \|\pi_{\mathfrak{A}}(\langle \omega_j, \omega_k \rangle_{\mathfrak{A}}) \omega - \omega \pi_{\mathfrak{B}}(\langle \eta_j, \eta_k \rangle_{\mathfrak{B}})\|_{\mathfrak{D}} : j \in J \}.$

# The modular propinquity

## Definition (L., 16)

The *modular propinquity* is the largest pseudo-metric  $\Lambda^{\text{mod}}$  such that  $\Lambda^{\text{mod}}(\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}) \leq \lambda(\gamma)$  for any modular  $\gamma$  from  $\Omega_{\mathfrak{A}}$  to  $\Omega_{\mathfrak{B}}$ .



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## Theorem (Free modules; L., 16)

If  $(\mathfrak{A}, L_{\mathfrak{A}})$ ,  $(\mathfrak{B}, L_{\mathfrak{B}})$  are quasi-Leibniz quantum compact metric space then:

$$\Lambda((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) \leq$$

$$\Lambda^{\text{mod}}((\mathfrak{A}^n, D_{\mathfrak{A}}^n, \mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}^n, D_{\mathfrak{B}}^n, \mathfrak{B}, L_{\mathfrak{B}})) \leq 2n\Lambda((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}))$$

where  $D_{\mathfrak{A}}^n(a_1, \dots, a_n) = \max_{j=1, \dots, n} \{\|a_j\|_{\mathfrak{A}}, L_{\mathfrak{A}}(\Re(a_j)), L_{\mathfrak{A}}(\Im(a_j))\}$ .

### Theorem (L, 16)

Let  $\|\cdot\|$  be a norm on  $\mathbb{R}^2$  and  $p, q, d$  fixed. If for all  $\theta \in \mathbb{R}$ , and  $a \in \mathcal{A}_\theta$ :

$$L_\theta(a) = \sup \left\{ \frac{\left\| \beta_\theta^{\exp(ix), \exp(iy)} a - a \right\|_{\mathcal{A}_\theta}}{\|(x, y)\|} : (x, y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where  $\beta_\theta$  is the dual action, and for all  $\xi \in \mathcal{H}_\theta^{p,q,d}$  we set:

$$D_\theta^{p,q,d}(\xi) = \sup \left\{ \frac{\left\| \alpha_{\tilde{\theta}}^{x,y, \frac{xy}{2}} \xi - \xi \right\|_{\mathcal{H}_\theta^{p,q,d}}}{2\pi|\tilde{\theta}|\|(x, y)\|} : (x, y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where  $\tilde{\theta} = \theta - p/q$ , then:

$$\lim_{\vartheta \rightarrow \theta} \Lambda^{\text{mod}} \left( \left( \mathcal{H}_\vartheta^{p,q,d}, \langle \cdot, \cdot \rangle_{\mathcal{H}_\vartheta^{p,q,d}}, D_\vartheta^{p,q,d}, \mathcal{A}_\vartheta, L_\vartheta \right), \right. \\ \left. \left( \mathcal{H}_\theta^{p,q,d}, \langle \cdot, \cdot \rangle_{\mathcal{H}_\theta^{p,q,d}}, D_\theta^{p,q,d}, \mathcal{A}_\theta, L_\theta \right) \right) = 0.$$

# *Convergence of Heisenberg modules II*