

# *The Spectral Propinquity*

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# Metric Spectral triples

## Definition

A *metric spectral triple*  $(\mathfrak{A}, \mathcal{H}, D)$  is given by:

- 1  $\mathfrak{A}$  is a unital  $C^*$ -algebra of operators acting on a Hilbert space  $\mathcal{H}$ ,
- 2  $D$  is a self-adjoint operator defined on a dense domain in  $\mathcal{H}$  with compact resolvent,
- 3  $\{a \in \mathfrak{A} : [D, a] \text{ is bounded} \}$  is a dense  $*$ -subalgebra of  $\mathfrak{A}$ ,

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- ❸  $\{a \in \mathfrak{A} : [D, a] \text{ is bounded}\}$  is a dense  $*$ -subalgebra of  $\mathfrak{A}$ ,
- ❹ if we set, for all states  $\varphi, \psi \in \mathcal{S}(\mathfrak{A})$ ,

$$\text{mk}_L(\varphi, \psi) := \sup \{ |\varphi(a) - \psi(a)| : \|[D, a]\|_{\mathcal{H}} \leq 1 \}$$

then  $\text{mk}_L$  is a metric over  $\mathcal{S}(\mathfrak{A})$  which induces the weak\* topology.

1 *A topology on the space of metric spectral triples*

2 *Examples*

3 *Spectral Properties*

# *A topology for metric spectral triples*

## *Our project*

We wish to construct a metric, up to unitary equivalence, on the space of metric spectral triples, to formalize such heuristics as:

- 1 the convergence of matrix models in physics,
- 2 spectral triples on fractals or AF algebras as limits of simpler spectral triples,
- 3 generalize ideas from Riemannian geometry.

# *A topology for metric spectral triples*

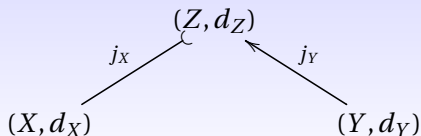
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## Gromov-Hausdorff distance



*Figure:*  $j_X, j_Y$  isometries

$$d_{\text{GH}}((X, d_X), (Y, d_Y)) := \inf \left\{ d_Z(j_X(X), j_Y(Y)) : \right. \\ \left. j_X : X \hookrightarrow Z, j_Y : Y \hookrightarrow Z \text{ isometries}, (Z, d_Z) \text{ compact m.s.} \right\}$$

## Gromov-Hausdorff distance

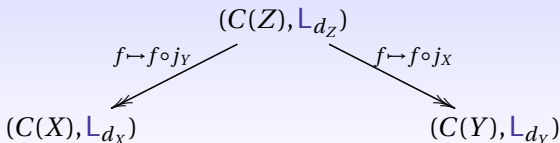


Figure:

$$\mathbb{L}_{d_Z} : f \in C(Z) \mapsto \sup \left\{ \frac{|f(x) - f(y)|}{d_Z(x, y)} : x, y \in Z, x \neq y \right\}.$$

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# Gromov-type construction for metric spectral triples

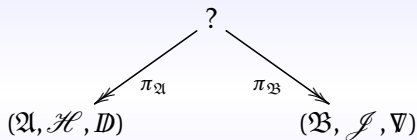
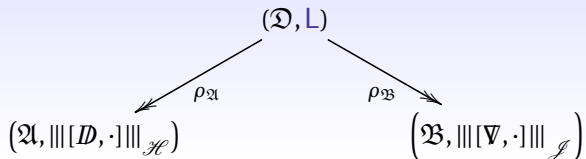


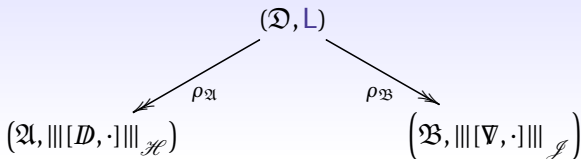
Figure:

We begin with the quantum metrics



*Figure:*  $\rho_{\mathfrak{A}}, \rho_{\mathfrak{B}}$  are quantum isometries

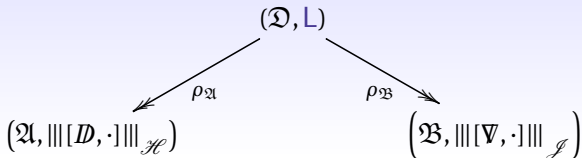
## We begin with the quantum metrics



*Figure:*  $\rho_{\mathfrak{A}}, \rho_{\mathfrak{B}}$  are quantum isometries

$\varphi, \psi \in \mathcal{S}(\mathfrak{D}) \mapsto \mathsf{mk}_{\mathsf{L}}(\varphi, \psi) := \sup \{ |\varphi(d) - \psi(d)| : \mathsf{L}(d) \leq 1 \}$  metrizes  $\sigma(\mathfrak{D}^*, \mathfrak{D})$  on  $\mathcal{S}(\mathfrak{D})$  and  $\mathsf{L}$  satisfies a Leibniz property and  $\mathsf{L}(1) = 0$ .

## We begin with the quantum metrics



*Figure:*  $\rho_{\mathfrak{A}}, \rho_{\mathfrak{B}}$  are quantum isometries

$$\chi(\tau) := \max \left\{ \text{Haus}_{\text{mk}_L} \left( \mathcal{S}(\mathfrak{D}), \{ \varphi \circ \rho_{\mathfrak{A}} : \varphi \in \mathcal{S}(\mathfrak{A}) \} \right), \right. \\
 \left. \text{Haus}_{\text{mk}_L} \left( \mathcal{S}(\mathfrak{D}), \{ \varphi \circ \rho_{\mathfrak{B}} : \varphi \in \mathcal{S}(\mathfrak{B}) \} \right) \right\}.$$

## What about the Hilbert spaces?

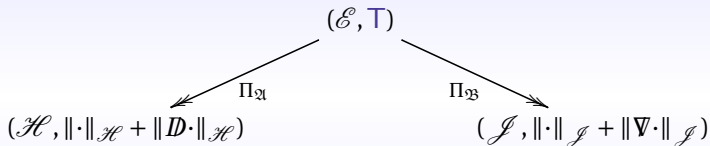


Figure:

## What about the scalars?

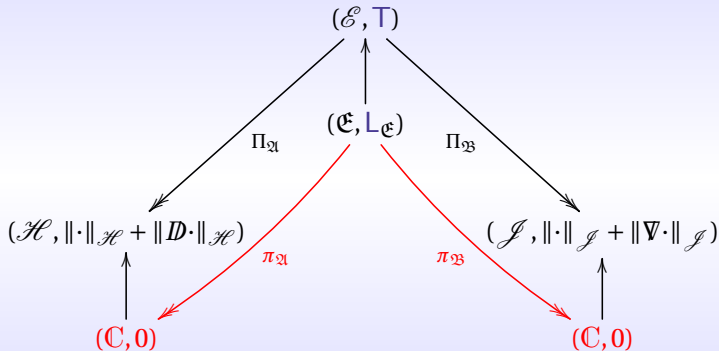


Figure:

# Putting it together

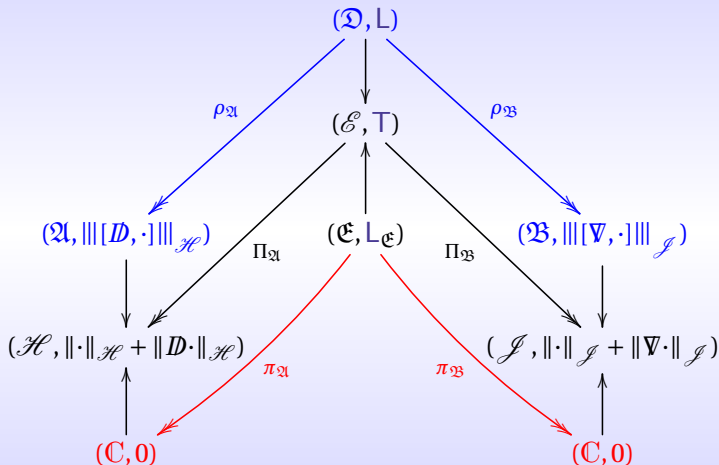


Figure:

# Distorsion

## Definition

$$K_{\varepsilon}(\xi, \eta) := \sup_{\substack{0 \leq t \leq \frac{1}{\varepsilon} \\ T(\omega) \leq 1}} \left\{ \left| \langle \exp(itD)\xi, \Pi_{\mathfrak{A}}(\omega) \rangle_{\mathcal{H}} - \langle \exp(it\nabla)\eta, \Pi_{\mathfrak{B}}(\omega) \rangle_{\mathcal{J}} \right| \right\}$$

$$\text{sep}_{\varepsilon}(D, \nabla | \tau) := \text{Haus}_{K_{\varepsilon}} \left( \left\{ \xi \in \text{dom}(D) : \|\xi\|_{\mathcal{H}} + \|D\xi\|_{\mathcal{H}} \leq 1 \right\}, \right. \\ \left. \left\{ \eta \in \text{dom}(\nabla) : \|\eta\|_{\mathcal{J}} + \|\nabla\eta\|_{\mathcal{J}} \leq 1 \right\} \right)$$



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## A first collapse result

### *Theorem (L., 24)*

If  $(\mathfrak{A}, \mathcal{H}, D)$  is a metric spectral triple and  $0 \in \operatorname{Sp}(D)$ , then

$$\lim_{\varepsilon \rightarrow 0} \Lambda^{\operatorname{spec}} \left( \left( \mathfrak{A}, \mathcal{H}, \frac{1}{\varepsilon} D \right), (\mathbb{C}, \ker D, 0) \right) = 0.$$

If  $0 \notin \operatorname{Sp}(D)$ , then  $(\mathfrak{A}, \mathcal{H}, \frac{1}{\varepsilon} D)_{\varepsilon > 0}$  does not converge for  $\Lambda^{\operatorname{spec}}$ .

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# Thank you!

- *The Quantum Gromov-Hausdorff Propinquity*, F. Latrémolière, *Transactions of the AMS* **368** (2016) 1, pp. 365–411, ArXiv: 1302.4058
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