

The Spectral Propinquity

Frédéric Latrémolière



*OdenSeaG
University of Southern Denmark
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Metric Spectral triples

Definition

A *metric spectral triple* $(\mathfrak{A}, \mathcal{H}, D)$ is given by:

- ① \mathfrak{A} is a unital C^* -algebra of operators acting on a Hilbert space \mathcal{H} ,
- ② D is a self-adjoint operator defined on a dense domain in \mathcal{H} with compact resolvent,
- ③ $\{a \in \mathfrak{A} : [D, a] \text{ is bounded}\}$ is a dense $*$ -subalgebra of \mathfrak{A} ,

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- ② D is a self-adjoint operator defined on a dense domain in \mathcal{H} with compact resolvent,
- ③ $\{a \in \mathfrak{A} : [D, a] \text{ is bounded}\}$ is a dense $*$ -subalgebra of \mathfrak{A} ,
- ④ if we set, for all states $\varphi, \psi \in \mathcal{S}(\mathfrak{A})$,

$$\text{mk}_L(\varphi, \psi) := \sup \{ |\varphi(a) - \psi(a)| : \| [D, a] \|_{\mathcal{H}} \leq 1 \}$$

then mk_L is a metric over $\mathcal{S}(\mathfrak{A})$ which induces the weak* topology.

1 A topology on the space of metric spectral triples

2 Examples

3 Spectral Properties

A topology for metric spectral triples

Our project

We wish to construct a metric, up to unitary equivalence, on the space of metric spectral triples, to formalize such heuristics as:

- ➊ the convergence of matrix models in physics,
- ➋ spectral triples on fractals or AF algebras as limits of simpler spectral triples,
- ➌ generalize ideas from Riemannian geometry.

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Gromov-Hausdorff distance

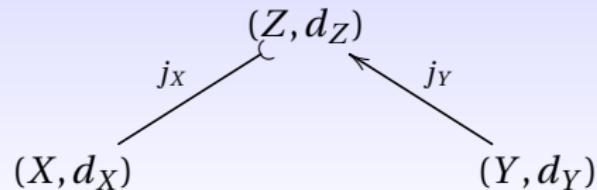


Figure: j_X, j_Y isometries

$$d_{\text{GH}}((X, d_X), (Y, d_Y)) := \inf \left\{ d_Z(j_X(X), j_Y(Y)) : \right.$$
$$\left. j_X : X \hookrightarrow Z, j_Y : Y \hookrightarrow Z \text{ isometries, } (Z, d_Z) \text{ compact m.s.} \right\}$$

Gromov-Hausdorff distance

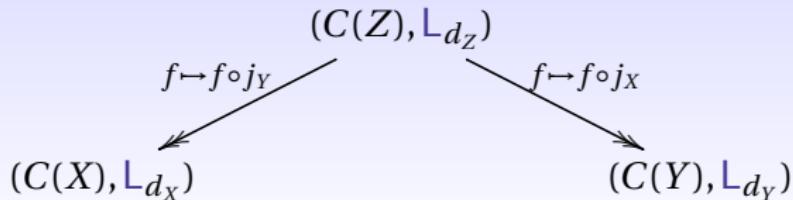


Figure:

$$\mathsf{L}_{d_Z} : f \in C(Z) \mapsto \sup \left\{ \frac{|f(x) - f(y)|}{d_Z(x, y)} : x, y \in Z, x \neq y \right\}.$$

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Gromov-type construction for metric spectral triples

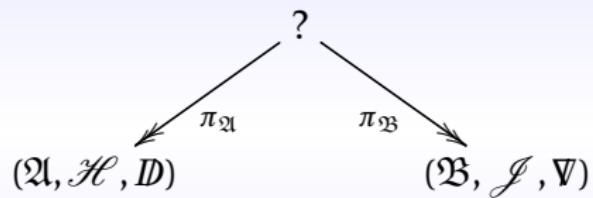


Figure:

We begin with the quantum metrics

$$\begin{array}{ccc} & (\mathfrak{D}, \textcolor{blue}{L}) & \\ \rho_{\mathfrak{A}} \swarrow & & \searrow \rho_{\mathfrak{B}} \\ (\mathfrak{A}, \|\cdot\|_{\mathcal{D}, \cdot}) & & (\mathfrak{B}, \|\cdot\|_{\mathfrak{D}, \cdot}) \end{array}$$

Figure: $\rho_{\mathfrak{A}}, \rho_{\mathfrak{B}}$ are quantum isometries

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Figure: $\rho_{\mathfrak{A}}, \rho_{\mathfrak{B}}$ are quantum isometries

$\varphi, \psi \in \mathcal{S}(\mathfrak{D}) \mapsto \text{mk}_{\textcolor{blue}{L}}(\varphi, \psi) := \sup \{ |\varphi(d) - \psi(d)| : \textcolor{blue}{L}(d) \leq 1 \}$ metrizes
 $\sigma(\mathfrak{D}^*, \mathfrak{D})$ on $\mathcal{S}(\mathfrak{D})$ and $\textcolor{blue}{L}$ satisfies a Leibniz property and $\textcolor{blue}{L}(1) = 0$.

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 (\mathfrak{A}, \|\cdot\|_{[\mathcal{D}, \cdot]}|_{\mathcal{H}}) & & (\mathfrak{B}, \|\cdot\|_{[\mathcal{V}, \cdot]}|_{\mathcal{J}}) \\
 \end{array}$$

Figure: $\rho_{\mathfrak{A}}, \rho_{\mathfrak{B}}$ are quantum isometries

$$\begin{aligned}
 \chi(\tau) := \max \Big\{ & \text{Haus}_{\text{mk}_L} \left(\mathcal{S}(\mathfrak{D}), \{\varphi \circ \rho_{\mathfrak{A}} : \varphi \in \mathcal{S}(\mathfrak{A})\} \right), \\
 & \text{Haus}_{\text{mk}_L} \left(\mathcal{S}(\mathfrak{D}), \{\varphi \circ \rho_{\mathfrak{B}} : \varphi \in \mathcal{S}(\mathfrak{B})\} \right) \Big\}.
 \end{aligned}$$

What about the Hilbert spaces?

$$\begin{array}{ccc} & (\mathcal{E}, \textcolor{blue}{T}) & \\ \swarrow \Pi_{\mathfrak{A}} & & \searrow \Pi_{\mathfrak{B}} \\ (\mathcal{H}, \|\cdot\|_{\mathcal{H}} + \|D\cdot\|_{\mathcal{H}}) & & (\mathcal{J}, \|\cdot\|_{\mathcal{J}} + \|\nabla\cdot\|_{\mathcal{J}}) \end{array}$$

Figure:

What about the scalars?

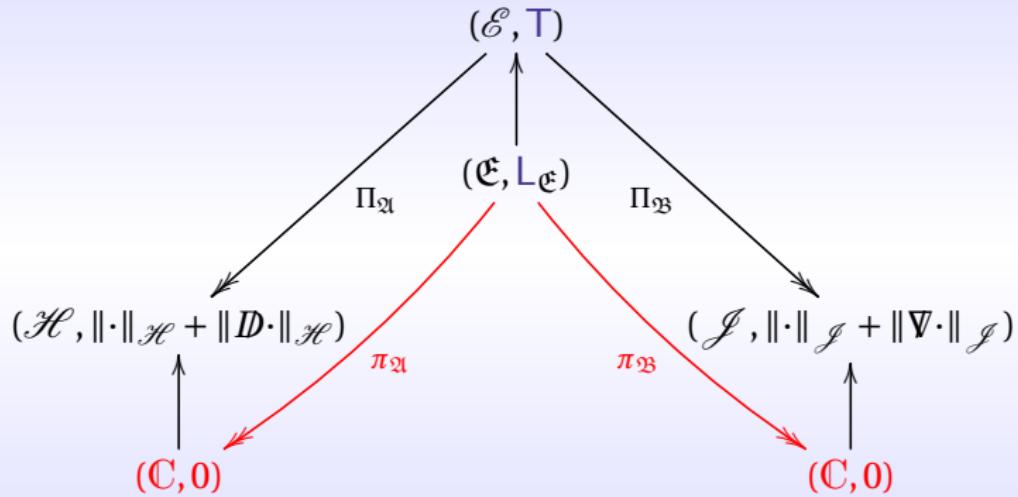


Figure:

Putting it together

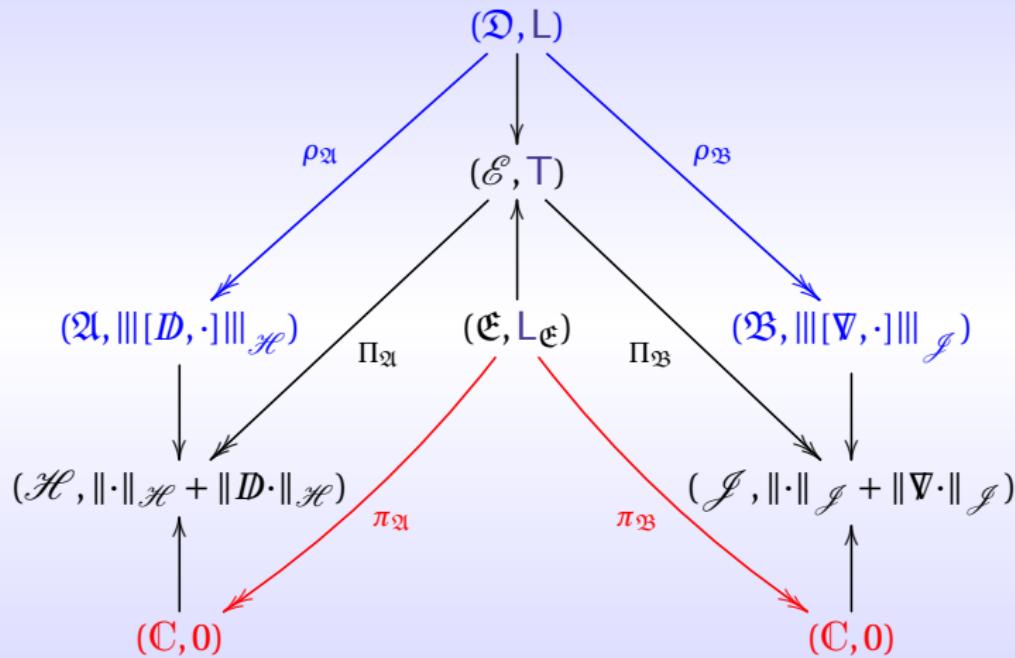


Figure:

Distorsion

Definition

$$K_\varepsilon(\xi, \eta) := \sup_{\substack{0 \leq t \leq \frac{1}{\varepsilon} \\ T(\omega) \leq 1}} \left\{ \left| \langle \exp(itD)\xi, \Pi_{\mathfrak{A}}(\omega) \rangle_{\mathcal{H}} - \langle \exp(itV)\eta, \Pi_{\mathfrak{B}}(\omega) \rangle_{\mathcal{J}} \right| \right\}$$

$$\text{sep}_\varepsilon(D, V|\tau) := \text{Haus}_{K_\varepsilon} \left(\left\{ \xi \in \text{dom}(D) : \|\xi\|_{\mathcal{H}} + \|D\xi\|_{\mathcal{H}} \leq 1 \right\}, \right. \\ \left. \left\{ \eta \in \text{dom}(V) : \|\eta\|_{\mathcal{J}} + \|V\eta\|_{\mathcal{J}} \leq 1 \right\} \right)$$

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A first collapse result

Theorem (L., 24)

If $(\mathfrak{A}, \mathcal{H}, \mathcal{D})$ is a metric spectral triple and $0 \in \text{Sp}(\mathcal{D})$, then

$$\lim_{\varepsilon \rightarrow 0} \Lambda^{\text{spec}} \left(\left(\mathfrak{A}, \mathcal{H}, \frac{1}{\varepsilon} \mathcal{D} \right), (\mathbb{C}, \ker \mathcal{D}, 0) \right) = 0.$$

If $0 \notin \text{Sp}(\mathcal{D})$, then $\left(\mathfrak{A}, \mathcal{H}, \frac{1}{\varepsilon} \mathcal{D} \right)_{\varepsilon > 0}$ does not converge for Λ^{spec} .

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Thank you!

- *The Quantum Gromov-Hausdorff Propinquity*, F. Latrémolière, *Transactions of the AMS* **368** (2016) 1, pp. 365–411, ArXiv: 1302.4058
- *The Dual Gromov-Hausdorff Propinquity*, F. Latrémolière, *Journal de Mathématiques Pures et Appliquées* **103** (2015) 2, pp. 303–351, ArXiv: 1311.0104
- *A compactness theorem for the dual Gromov-Hausdorff Propinquity*, F. Latrémolière, *Indiana Univ. Math. J.* **66** (2017) 5, 1707–1753, Arxiv: 1501.06121
- *The modular Gromov-Hausdorff propinquity*, F. Latrémolière, *Dissertationes Math.* 544 (2019), 70 pp. 46L89 (46L30 58B34)
- *The Gromov-Hausdorff propinquity for metric Spectral Triples*, F. Latrémolière, *Adv. Math.* 404 (2022), paper no. 108393, arXiv:1811.04534
- *Metric approximations of spectral triples on the Sierpiński gasket and other fractal curves*, T. Landry, M. Lapidus, F. Latrémolière, *Adv. Math.* 385 (2021), Paper No. 107771, 43 pp, arXiv:2010.06921.
- *Convergence of Spectral Triples on Fuzzy Tori to Spectral Triples on Quantum Tori*, F. Latrémolière, *Comm. Math. Phys.* **388** (2021) 2, 1049–1128, arXiv:2102.03729.