

# *A Gromov-Hausdorff distance for $C^*$ -dynamical systems*

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- ➍ The actions will be by *Lipschitz morphisms*.

# The Monge-Kantorovich metric

Let  $(X, \textcolor{blue}{m})$  be a compact metric space. The *Lipschitz seminorm*  $\textcolor{teal}{L}$  induced by  $\textcolor{blue}{m}$  is:

$$\textcolor{teal}{L}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{\textcolor{blue}{m}(x, y)} : x, y \in X, x \neq y \right\}$$

for all  $f \in \mathfrak{sa}(C(X)) = C(X, \mathbb{R})$  (allowing  $\infty$ ).

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The *Monge-Kantorovich metric* on  $\mathcal{S}(C(X))$  is given for all Borel-regular probability measures  $\mu, \nu$  by:

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The Gelfand map  $x \in (X, \textcolor{blue}{m}) \mapsto \delta_x \in (\mathcal{S}(C(X)), \text{mk}_{\textcolor{blue}{L}})$  is an isometry.

# Compact Quantum Metric Spaces

*Definition (Connes, 89; Rieffel, 98; L., 13)*

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We call  $\mathsf{L}$  an  *$L$ -seminorm*.

# Quantum Tori as Compact Quantum Metric Spaces

## Theorem (Rieffel, 98)

If  $\alpha$  is a strongly continuous *action* of a *compact group*  $G$ , if  $\ell$  is a continuous *length* function over  $G$ , and if for all  $a \in \mathfrak{A}$  we set:

$$\mathsf{L}(a) = \sup \left\{ \frac{\|\alpha^g(a) - a\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1\} \right\},$$

then  $(\mathfrak{A}, \mathsf{L})$  is a *quantum compact metric space* if and only if  $\{a \in \mathfrak{A} : \forall g \in G \quad \alpha^g(a) = a\} = \mathbb{C}1_{\mathfrak{A}}$ .

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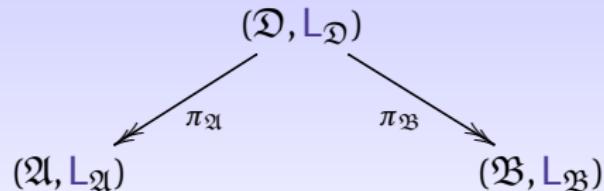
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## Example

- *Quantum tori* are quantum compact metric spaces with  $\mathfrak{A} = C^*(\mathbb{Z}^d, \sigma)$ ,  $G = \mathbb{T}^d$  and  $\ell$  any continuous length on  $\mathbb{T}^d$ ;
- *Noncommutative solenoids*  $C^*\left(\mathbb{Z}\left[\frac{1}{p}\right]^2, \sigma\right)$  with  $G = \mathcal{S}_p^2$  (L., Packer, 16).

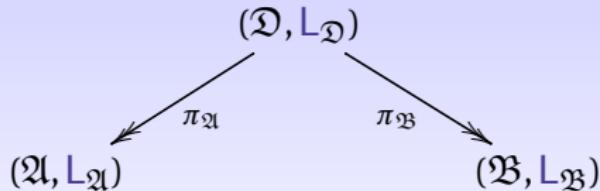


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*Definition (The extent of a tunnel, L. 13,14)*

The *extent*  $\chi(\tau)$  of a tunnel  $\tau = (\mathfrak{D}, \textcolor{violet}{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$  is:

$$\max \left\{ \text{Haus}_{\text{mk}_{\textcolor{violet}{L}_{\mathfrak{D}}}} \left( \mathcal{S}(\mathfrak{D}), \pi_{\mathfrak{E}}^*(\mathcal{S}(\mathfrak{E})) \right) : \mathfrak{E} \in \{\mathfrak{A}, \mathfrak{B}\} \right\}.$$

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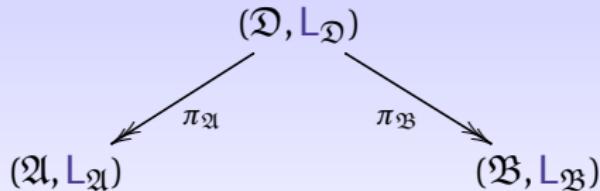


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## Definition (The Dual Propinquity, L. 13, 14)

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The *dual propinquity*  $\Lambda_F^*((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$  is given by:

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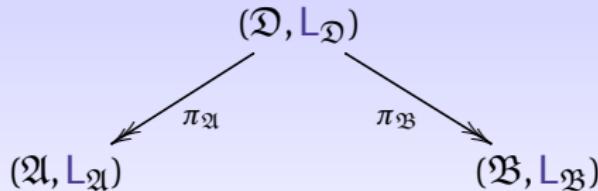


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## Theorem (L., 13)

The *dual propinquity*  $\Lambda_F^*$ , defined for any two quantum compact metric spaces  $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$  by  $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$  by:

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is a *complete metric* up to *full quantum isometry*:  
 $\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$  iff there exists a \*-isomorphism  $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$  such that  $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$ .



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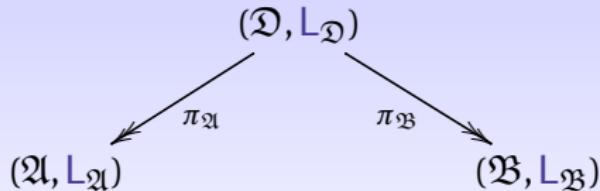


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is a *complete metric* up to *full quantum isometry*. Moreover  $\Lambda^*$  induces the topology of the *Gromov-Hausdorff distance* on compact metric spaces.



# Quantum and Fuzzy Tori

Example:

Let  $\ell$  be a *continuous length function* on  $\mathbb{T}^d$ . For any  $G \subseteq \mathbb{T}^d$  a closed subgroup and  $\sigma$  a multiplier of  $\widehat{G}$ , for any  $a \in C^*(\widehat{G}, \sigma)$ , set:

$$\mathsf{L}_{G,\sigma}(a) = \sup \left\{ \frac{\|\alpha^g(a) - a\|_{C^*(\widehat{G}, \sigma)}}{\ell(g)} : g \in G \setminus \{1\} \right\}$$

where  $\alpha$  is the dual action of  $G$  on  $C^*(\widehat{G}, \sigma)$ .

Rieffel showed in 1998 that  $(C^*(\widehat{G}, \sigma), \mathsf{L}_{G,\sigma})$  is a Leibniz quantum compact metric space.



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If  $(G_n)_{n \in \mathbb{N}}$  is a *sequence of closed subgroups* of  $\mathbb{T}^d$  *converging* to  $\mathbb{T}^d$  for the *Hausdorff distance*  $\text{Haus}_\ell$ , and if  $(\sigma_n)_{n \in \mathbb{N}}$  is a sequence of multipliers of  $\mathbb{Z}^d$  converging pointwise to some  $\sigma$ , with  $\sigma_n(g) = 1$  if  $g$  is the coset of 0 for  $\widehat{G}_n$ , then:

$$\lim_{n \rightarrow \infty} \Lambda^*((C^*(\widehat{G}_n, \sigma_n), \mathsf{L}_{\widehat{G}_n, \sigma_n}), (C^*(\mathbb{Z}^d, \sigma), \mathsf{L}_{\mathbb{Z}^d, \sigma})) = 0.$$



# *Bridges*

How to build a tunnel from  $(\mathfrak{A}, \mathcal{L}_{\mathfrak{A}})$  to  $(\mathfrak{B}, \mathcal{L}_{\mathfrak{B}})$ ?

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How to build a tunnel from  $(\mathfrak{A}, L_{\mathfrak{A}})$  to  $(\mathfrak{B}, L_{\mathfrak{B}})$ ?

An idea is to embedd  $\mathfrak{A}$  and  $\mathfrak{B}$  into a unital  $C^*$ -algebra  $\mathfrak{D}$ , pick  $\mathfrak{D}$  and compute:

$$\varsigma(\mathfrak{D}, \omega) = \max \left\{ \text{Haus}_{\text{mk}_{L_{\mathfrak{A}}}}(\mathcal{S}(\mathfrak{A}), \{\varphi \in \mathcal{S}(\mathfrak{A}) : \varphi(\omega \cdot) = \varphi(\cdot \omega) = \varphi\} \right\}$$

# Covariance Property of $\Lambda^*$

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Do symmetries or dynamics pass to the limit for the dual propinquity, under some *equicontinuity* condition ?

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To study this question:

- ➊ we define a distance on *proper monoids*,
- ➋ we then define a *covariant* version of the propinquity,
- ➌ we then prove a form of Arzéla-Ascoli theorem for the *dual propinquity*.

# Lipschitz morphisms

## Theorem-Definition (Lipschitz Morphisms)

Let  $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$  and  $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$  be two quasi-Leibniz quantum compact metric spaces. A *Lipschitz morphism*  $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$  is a unital \*-morphism from  $\mathfrak{A}$  to  $\mathfrak{B}$  such that any of the following equivalent statement holds:

- ①  $(\exists k) \varphi \in \mathcal{S}(\mathfrak{B}) \mapsto \varphi \circ \pi$  is a *k-Lipschitz map from  $(\mathcal{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$  to  $(\mathcal{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$* ,
- ② (Rieffel, 00)  $(\exists k) \mathsf{L}_{\mathfrak{B}} \circ \pi \leq k \mathsf{L}_{\mathfrak{A}}$ ,
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Lipschitz morphisms are arrows for a category over quasi-Leibniz quantum compact metric spaces.

# *A distance on morphisms*

## *Definition (L., 16)*

Let  $(\mathfrak{A}, \textcolor{blue}{L})$  be a quantum compact metric space and  $\mathfrak{B}$  be a unital  $C^*$ -algebra. For any two linear maps  $\varphi, \psi : \mathfrak{A} \rightarrow \mathfrak{B}$  preserving the units and intertwining the adjoint operations, let:

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The topology induced by  $\text{mk}_{\textcolor{blue}{L}}$  on the space of  $*$ -morphisms from  $\mathfrak{A}$  to  $\mathfrak{B}$  is the topology of pointwise convergence, where a net  $(\alpha_j)_{j \in J}$  converges to some  $\alpha$  when:

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Moreover, a subset  $\Xi$  of  $*$ -morphisms is totally bounded for  $\text{mk}_{\textcolor{blue}{L}}$  if there exists  $C > 0$  such that  $\textcolor{blue}{L} \leq C \textcolor{blue}{L} \circ \alpha$  for all  $\alpha \in \Xi$ .

## A proper monoid Gromov-Hausdorff distance

Let  $(G_1, \delta_1)$  and  $(G_2, \delta_2)$  be two proper metric monoids. A

**$\varepsilon$ -isometric isomorphism**  $(\varsigma_1, \varsigma_2)$  from  $(G_1, \delta_1)$  to  $(G_2, \delta_2)$  is a pair of functions  $\varsigma_1 : G_1 \rightarrow G_2$  and  $\varsigma_2 : G_2 \rightarrow G_1$  such that for all  $\{j, k\} = \{1, 2\}$ :

$$\forall g, g' \in G_j \left[ \frac{1}{\varepsilon} \right], h \in G_k \left[ \frac{1}{\varepsilon} \right]$$

$$|\delta_k(\varsigma_j(g)\varsigma_j(g'), h) - \delta_j(gg', \varsigma_k(h))| \leq \varepsilon,$$

and  $\varsigma_1, \varsigma_2$  are unital.

We then set:

$$\Upsilon((G_1, \delta_1), (G_2, \delta_2)) = \min \left\{ \frac{\sqrt{2}}{2}, \inf \{ \varepsilon > 0 | \exists (\varsigma, \kappa) \text{ } \varepsilon\text{-iso-iso} \} \right\}.$$

$\Upsilon$  is a *metric up to isometric isomorphism of proper monoid*, and it dominates the pointed Gromov-Hausdorff distance.



# *Covariant Propinquity*

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We can combine  $\Lambda^*$  and  $\Upsilon$ . A *Lipschitz dynamical system*  $(\mathfrak{A}, \underline{\mathcal{L}}, G, \delta, \alpha)$  consists of:

- a quantum compact metric space  $(\mathfrak{A}, \underline{\mathcal{L}})$ ,
- a proper monoid  $(G, \delta)$
- a strongly continuous action of  $G$  on  $\mathfrak{A}$  by Lipschitz morphisms.

# Covariant Propinquity

We can combine  $\Lambda^*$  and  $\Upsilon$ . A *Lipschitz dynamical system*  $(\mathfrak{A}, \underline{\mathcal{L}}, G, \delta, \alpha)$  consists of:

- a quantum compact metric space  $(\mathfrak{A}, \underline{\mathcal{L}})$ ,
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## Definition (L., 18)

A  *$\varepsilon$ -covariant tunnel*  $(\tau, \varsigma, \kappa)$  from  $(\mathfrak{A}, \underline{\mathcal{L}}, G, \delta, \alpha)$  to  $(\mathfrak{B}, \underline{\mathcal{L}}, H, \delta, \beta)$  is a tunnel  $\tau$  from  $(\mathfrak{A}, \underline{\mathcal{L}}_{\mathfrak{A}})$  to  $(\mathfrak{B}, \underline{\mathcal{L}}_{\mathfrak{B}})$  and a  $\varepsilon$ -almost isometric isomorphism  $(\varsigma_1, \varsigma_2)$  from  $(G, \delta_G)$  to  $(H, \delta_H)$ .

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The  *$\varepsilon$ -reach*  $\rho(\tau|\varepsilon)$  of a covariant tunnel  $\tau = (\mathfrak{D}, \underline{\mathsf{L}}, \pi_1, \pi_2, \varsigma_1, \varsigma_2)$  from  $(\mathfrak{A}_1, \underline{\mathsf{L}}_1, G_1, \delta_1, \alpha_1)$  to  $(\mathfrak{A}_2, \underline{\mathsf{L}}_2, G_2, \delta_2, \alpha_2)$  is:

$$\max_{\{j,k\}=\{1,2\}} \sup_{\varphi \in \mathcal{S}(\mathfrak{A}_j)} \inf_{\psi \in \mathcal{S}(\mathfrak{A}_k)} \sup_{g \in G_j[\frac{1}{\varepsilon}]} \text{mk}_{\underline{\mathsf{L}}}(\varphi \circ \alpha_j^g \circ \pi_j, \psi \circ \alpha_k \circ \alpha_k^{\varsigma_k(g)} \circ \pi_k).$$

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## Definition (L., 18)

The *covariant propinquity*  $\Lambda^{\text{cov}}(\mathbb{A}, \mathbb{B})$  between two Lipschitz dynamical systems  $\mathbb{A}$  and  $\mathbb{B}$  is:

$$\inf \left\{ \frac{\sqrt{2}}{2}, \varepsilon > 0 \mid \right.$$

$\exists \tau \text{ a } \varepsilon\text{-covariant tunnel } \mathbb{A} \rightarrow \mathbb{B} \text{ with } \max\{\chi(\tau), \rho(\tau|\varepsilon)\} \leq \varepsilon \}$

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## Theorem (L., 18)

The *covariant propinquity*  $\Lambda^{\text{cov}}$  is a *metric* up to *equivariant* full quantum isometry, i.e it is symmetric, satisfies the triangle inequality, and:

$$\Lambda^{\text{cov}}((\mathfrak{A}, \underline{\mathsf{L}}_{\mathfrak{A}}, G, \delta_G, \alpha), (\mathfrak{B}, \underline{\mathsf{L}}_{\mathfrak{B}}, H, \delta_H, \beta)) = 0$$

if and only if there exists a full quantum isometry  $\pi : (\mathfrak{A}, \underline{\mathsf{L}}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \underline{\mathsf{L}}_{\mathfrak{B}})$  and an isometric isomorphism  $\varsigma : G \rightarrow H$  such that:

$$\forall a \in \text{dom}(\underline{\mathsf{L}}_{\mathfrak{A}}) \quad \forall g \in G \quad \pi \circ \alpha^g = \beta^{\varsigma(g)} \circ \pi.$$



## Theorem (L., 17, 18)

If  $D : [0, \infty) \rightarrow [0, \infty)$  and:

- ①  $\forall n \in \mathbb{N}, (\mathfrak{A}_n, \mathsf{L}_n, G_n, \delta_n, \alpha_n)$  is a Lipschitz dynamical  $F$ -system,
- ②  $\lim_{n \rightarrow \infty} Y((G_n, \delta_n), (G, \delta)) = 0,$
- ③  $\lim_{n \rightarrow \infty} \Lambda_F^*((\mathfrak{A}_n, \mathsf{L}_n), (\mathfrak{A}, \mathsf{L})) = 0,$
- ④ for all  $\varepsilon > 0$  there exists  $\omega > 0, N \in \mathbb{N}$  such that:

$$\forall n \geq N \quad \forall g, h \in G_n \quad \delta_n(g, h) < \omega \implies \text{mk}_{\mathsf{L}_n}(\alpha_n^g, \alpha_n^h) < \varepsilon,$$

then there exists an action  $\alpha$  of  $G$  on  $\mathfrak{A}$  and a  $j : \mathbb{N} \rightarrow \mathbb{N}$  strictly increasing such that:

$$\lim_{n \rightarrow \infty} \Lambda_F^{\text{cov}}((\mathfrak{A}_{j(n)}, \mathsf{L}_{j_n}, G_{j(n)}, \alpha_{j(n)}), (\mathfrak{A}, \mathsf{L}, G, \alpha)) = 0.$$

If  $G, G_n$  is a compact group and  $\alpha_n$  ergodic  $\forall_{\mathbb{N}} n$  then so is  $\alpha$ .

# *Application to Closures*

## *Theorem (L., 17)*

Let  $\mathcal{C}$  be the class of *finite dimensional quantum compact metric spaces* together with an ergodic action of  $SU(2)$  by isometries. The *closure* of  $\mathcal{C}$  for the dual propinquity consists only of quantum compact metric spaces over *type I*  $C^*$ -algebras.

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## *Theorem (L., 17, alternate proof)*

Let  $\mathcal{F}$  be the class of *finite dimensional quantum compact metric spaces* together with an ergodic action of  $T^2$  by isometries. The *closure* of  $\mathcal{F}$  for the dual propinquity consists of fuzzy and quantum tori.

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## *Theorem (L., 18)*

The *covariant propinquity* is *complete* on the class of Lipschitz dynamical systems with a bi-invariant metric.

## *Convergence of dual actions of quantum tori*

Let  $\ell$  be a *continuous length function* on  $\mathbb{T}^d$ . For any  $G \subseteq \mathbb{T}^d$  a closed subgroup and  $\sigma$  a multiplier of  $\widehat{G}$ , for any  $a \in C^*(\widehat{G}, \sigma)$ , set:

$$\mathsf{L}_{G,\sigma}(a) = \sup \left\{ \frac{\|\alpha_{G,\sigma}^g(a) - a\|_{C^*(\widehat{G}, \sigma)}}{\ell(g)} : g \in G \setminus \{1\} \right\}$$

where  $\alpha_{G,\sigma}$  is the dual action of  $G$  on  $C^*(\widehat{G}, \sigma)$ .

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### Theorem (L., 18)

If  $(G_n)_{n \in \mathbb{N}}$  is a sequence of closed subgroups of  $\mathbb{T}^d$  converging to  $\mathbb{T}^d$  for the Hausdorff distance  $\text{Haus}_\ell$  and if  $\sigma_n$  are (lift of) multipliers of  $\widehat{G}_n$  converging pointwise to some  $\sigma$  then:

$$\lim_{n \rightarrow \infty} \Lambda_F^{\text{cov}} \left( (C^*(\widehat{G}_n, \eta), \mathsf{L}_{G_n, \eta}, G_n, \alpha_{G_n, \sigma_n}), (C^*(\mathbb{Z}^d, \sigma), \mathsf{L}_{\mathbb{T}^d, \sigma}, \mathbb{T}^d, \alpha_{\mathbb{T}^d, \sigma}) \right) = 0.$$

# *Completeness of certain classes of systems*

An interesting complication arises when trying to determine when the covariant propinquity is complete. We introduce a notion of uniform equicontinuity.

## *Definition (L., 18)*

Let  $S = (G_n, \delta_n)_{n \in \mathbb{N}}$  be a sequence of proper monoids. The set of *regular sequences*  $\mathcal{R}(S)$  is:

$$\left\{ (g_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} G_n \middle| \begin{array}{l} \forall \varepsilon > 0 \quad \exists \omega > 0 \quad \exists N \in \mathbb{N} \\ \forall n \geq N \quad \forall h, k \in G_n \\ \delta_n(h, k) < \omega \implies \delta_n(hg_n, kg_n) < \varepsilon \end{array} \right\}.$$

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## *Theorem (L., 18)*

$\mathcal{R}((G_n, \delta_n)_{n \in \mathbb{N}})$  is a monoid for the pointwise multiplication.

# Complete classes for $\Upsilon$

## Theorem (L, 18)

Let  $(G_n, \delta_n)_{n \in \mathbb{N}}$  be a sequence of monoids, and  $(\sum \varepsilon_n)_{n \in \mathbb{N}}$  be a convergent series of positive general term. If:

- ① for all  $n \in \mathbb{N}$ , there exists a  $\varepsilon_n$  almost iso-iso  $(\varsigma_n, \kappa_n)$  from  $G_n$  to  $G_{n+1}$ ,
- ② for all  $N \in \mathbb{N}$ , with  $r = \sum_{j=0}^N \varepsilon_j$ , if  $g \in G_N[r]$ , then:

$$\left( \begin{array}{l} g_n = e_n \text{ if } n < N, \\ g_n = g \text{ if } n = N, \\ g_{n+1} = \varsigma_n(g_n) \text{ if } n \geq N \end{array} \right)_{n \in \mathbb{N}} \in \mathcal{R}((G_n, \delta_n)_{n \in \mathbb{N}}),$$

then there exists a proper monoid  $(G, \delta)$  such that:

$$\lim_{n \rightarrow \infty} \Upsilon((G_n, \delta_n), (G, \delta)) = 0.$$

# Examples

## Corollary (L., 18)

The space of proper monoids equipped with bi-invariant metrics is complete for the metric  $\Upsilon$ .

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### Corollary (L., 18)

If, for any proper monoids  $G$  and  $H$ , we set  $\Upsilon'(G, H)$  as:

$$\left\{ \varepsilon > 0 \middle| \begin{array}{l} \exists(\zeta, \kappa) \text{ } \varepsilon \text{ iso-iso } G \rightarrow H \\ \sup_{g \in G} |\text{dil}(h \mapsto hg) - \text{dil}(h \mapsto h\zeta(g))| < \varepsilon \\ \sup_{h \in H} |\text{dil}(g \mapsto gh) - \text{dil}(g \mapsto g\kappa(h))| < \varepsilon \end{array} \right\}$$

then  $\Upsilon'$  is a complete extended metric.

# Completeness for the covariant propinquity

## Theorem (L.18)

Let  $(\mathfrak{A}_n, \mathsf{L}_n, G_m, \delta_n, \alpha_n)_{n \in \mathbb{N}}$  be a sequence of Lipschitz dynamical system,  $R : [0, \infty) \rightarrow [0, \infty)$  continuous, and  $(\sum \varepsilon_n)_{n \in \mathbb{N}}$  be a convergent series of positive terms such that:

- ①  $\Lambda((\mathfrak{A}_n, \mathsf{L}_n), (\mathfrak{A}_{n+1}, \mathsf{L}_{n+1})) \leq \varepsilon_n,$
- ② (in particular) there exists  $(\varsigma_n, \kappa_n)$  an  $\varepsilon_n$ -local almost isometry from  $G_n$  to  $G_{n+1}$  for all  $n \in \mathbb{N}$ ,
- ③  $\forall n \in \mathbb{N} \forall g \in G_n \quad \mathsf{L}_n \circ \alpha_n^g \leq R(\delta_n(e_n, g)) \mathsf{L}_n,$
- ④  $\forall N \in \mathbb{N}, g \in G_N \quad (e_0, \dots, e_{N-1}, g, \varsigma_N(g), \dots) \in \mathcal{R}((G_n, \delta_n)_{n \in \mathbb{N}}),$
- ⑤  $\forall \varepsilon > 0 \quad \exists \omega > 0, N \in \mathbb{N} \quad \forall n \geq N, \forall g, h \in G_n \quad \delta_n(g, h) < \omega \implies \text{mk}_{\mathsf{L}}(\alpha_n^g, \alpha_n^h) < \varepsilon$

then there exists a Lipschitz dynamical system  $(\mathfrak{A}, \mathsf{L}, G, \delta, \alpha)$  which is the limit of  $(\mathfrak{A}_n, \mathsf{L}_n, G_n, \delta_n, \alpha_n)_{n \in \mathbb{N}}$  for the covariant propinquity.

# Completeness

## Corollary (L., 18)

Let  $(\mathfrak{A}_n, \mathsf{L}_n, G_m, \delta_n, \alpha_n)_{n \in \mathbb{N}}$  be a Cauchy sequence of Lipschitz dynamical system for  $\Lambda^{\text{cov}}$ . Assume  $\delta_n$  is bi-invariant for all  $n \in \mathbb{N}$ . Let  $R : [0, \infty) \rightarrow [0, \infty)$  be continuous. If

- ①  $\forall n \in \mathbb{N} \ \forall g \in G_n \quad \mathsf{L}_n \circ \alpha_n^g \leq R(\delta_n(e_n, g)) \mathsf{L}_n,$
- ②  $\forall \varepsilon > 0 \quad \exists \omega > 0, N \in \mathbb{N} \quad \forall n \geq N, \forall g, h \in G_n \quad \delta_n(g, h) < \omega \implies \text{mk}_{\mathsf{L}}(\alpha_n^g, \alpha_n^h) < \varepsilon$

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# Thank you!

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