

A Gromov-Hausdorff distance for C^ -dynamical systems*

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- 2 The C^* -algebras will be endowed with *quantum metrics*.
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- 4 The actions will be by *Lipschitz morphisms*.

The Monge-Kantorovich metric

Let (X, \mathfrak{m}) be a compact metric space. The *Lipschitz seminorm* L induced by \mathfrak{m} is:

$$\mathsf{L}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{\mathfrak{m}(x, y)} : x, y \in X, x \neq y \right\}$$

for all $f \in \mathfrak{sa}(C(X)) = C(X, \mathbb{R})$ (allowing ∞).

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The *Monge-Kantorovich metric* on $\mathcal{S}(C(X))$ is given for all Borel-regular probability measures μ, ν by:

$$mk_L(\mu, \nu) = \sup \left\{ \left| \int_X f d\mu - \int_X f d\nu \right| : f \in \mathfrak{sa}(C(X)), L(f) \leq 1 \right\}.$$

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The Gelfand map $x \in (X, \mathfrak{m}) \mapsto \delta_x \in (\mathcal{S}(C(X)), \mathfrak{mk}_{\mathsf{L}})$ is an isometry.

Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; L., 13)

$(\mathfrak{A}, \mathbb{L})$ is a *quantum compact metric space* when:

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We call L an *L -seminorm*.

Quantum Tori as Compact Quantum Metric Spaces

Theorem (Rieffel, 98)

If α is a strongly continuous *action* of a *compact group* G , if ℓ is a continuous *length* function over G , and if for all $a \in \mathfrak{A}$ we set:

$$L(a) = \sup \left\{ \frac{\|\alpha^g(a) - a\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1\} \right\},$$

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Example

- *Quantum tori* are quantum compact metric spaces with $\mathfrak{A} = C^*(\mathbb{Z}^d, \sigma)$, $G = \mathbb{T}^d$ and ℓ any continuous length on \mathbb{T}^d ;
- *Noncommutative solenoids* $C^*\left(\mathbb{Z}\left[\frac{1}{p}\right]^2, \sigma\right)$ with $G = \mathcal{S}_p^2$ (L., Packer, 16).

The Dual Gromov-Hausdorff Propinquity

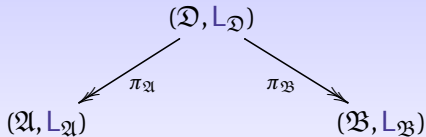


Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

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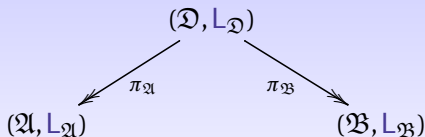


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Definition (The extent of a tunnel, L. 13,14)

The *extent* $\chi(\tau)$ of a tunnel $\tau = (\mathfrak{D}, L_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\max \left\{ \text{Haus}_{\text{mk}_{L_{\mathfrak{D}}}} \left(\mathcal{S}(\mathfrak{D}), \pi_{\mathfrak{E}}^*(\mathcal{S}(\mathfrak{E})) \right) : \mathfrak{E} \in \{\mathfrak{A}, \mathfrak{B}\} \right\}.$$

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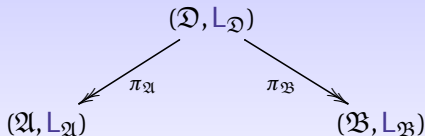


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The *dual propinquity* $\Lambda_F^*((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}))$ is given by:

$$\inf \left\{ \chi(\tau) : \tau \text{ any } F\text{-tunnel from } (\mathfrak{A}, L_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, L_{\mathfrak{B}}) \right\}.$$

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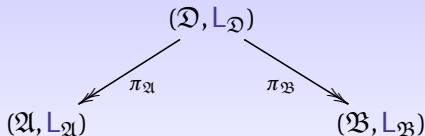


Figure: π_A, π_B are quantum isometries

Theorem (L., 13)

The *dual propinquity* Λ_F^* , defined for any two quantum compact metric spaces $(\mathcal{A}, L_{\mathcal{A}})$ by $(\mathcal{B}, L_{\mathcal{B}})$ by:

$$\inf \{ \chi(\tau) : \tau \text{ any } F\text{-tunnel from } (\mathcal{A}, L_{\mathcal{A}}) \text{ to } (\mathcal{B}, L_{\mathcal{B}}) \}$$

is a *complete metric* up to *full quantum isometry*.
 $\Lambda((\mathcal{A}, L_{\mathcal{A}}), (\mathcal{B}, L_{\mathcal{B}})) = 0$ iff there exists a $*$ -isomorphism $\pi : \mathcal{A} \rightarrow \mathcal{B}$ such that $L_{\mathcal{B}} \circ \pi = L_{\mathcal{A}}$.

The Dual Gromov-Hausdorff Propinquity

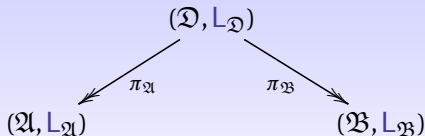


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is a *complete metric* up to *full quantum isometry*. Moreover Λ^* induces the topology of the *Gromov-Hausdorff distance* on compact metric spaces.

Quantum and Fuzzy Tori

Example:

Let ℓ be a *continuous length function* on \mathbb{T}^d . For any $G \subseteq \mathbb{T}^d$ a closed subgroup and σ a multiplier of \widehat{G} , for any $a \in C^*(\widehat{G}, \sigma)$, set:

$$\mathsf{L}_{G,\sigma}(a) = \sup \left\{ \frac{\|\alpha^g(a) - a\|_{C^*(\widehat{G}, \sigma)}}{\ell(g)} : g \in G \setminus \{1\} \right\}$$

where α is the dual action of G on $C^*(\widehat{G}, \sigma)$.

Rieffel showed in 1998 that $(C^*(\widehat{G}, \sigma), \mathsf{L}_{G,\sigma})$ is a Leibniz quantum compact metric space.

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Example: L., 13

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If $(G_n)_{n \in \mathbb{N}}$ is a *sequence of closed subgroups* of \mathbb{T}^d *converging* to \mathbb{T}^d for the *Hausdorff distance* Haus_ℓ , and if $(\sigma_n)_{n \in \mathbb{N}}$ is a sequence of multipliers of \mathbb{Z}^d converging pointwise to some σ , with $\sigma_n(g) = 1$ if g is the coset of 0 for \widehat{G}_n , then:

$$\lim_{n \rightarrow \infty} \Lambda^*((C^*(\widehat{G}_n, \sigma_n), \mathsf{L}_{\widehat{G}_n, \sigma_n}), (C^*(\mathbb{Z}^d, \sigma), \mathsf{L}_{\mathbb{Z}^d, \sigma})) = 0.$$

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How to build a tunnel from $(\mathfrak{A}, L_{\mathfrak{A}})$ to $(\mathfrak{B}, L_{\mathfrak{B}})$?

An idea is to embed \mathfrak{A} and \mathfrak{B} into a unital C^* -algebra \mathfrak{D} , pick ω and compute:

$$\zeta(\mathfrak{D}, \omega) = \max \left\{ \text{Haus}_{mk_{L_{\mathfrak{A}}}}(\mathcal{S}(\mathfrak{A}), \{\varphi \in \mathcal{S}(\mathfrak{A}) : \varphi(\omega \cdot) = \varphi(\cdot \omega) = \varphi\{$$

Covariance Property of Λ^*

Symmetries and metric convergence

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To study this question:

- 1 we define a distance on *proper monoids*,
- 2 we then define a *covariant* version of the propinquity,
- 3 we then prove a form of Arzéla-Ascoli theorem for the *dual propinquity*.

Lipschitz morphisms

Theorem-Definition (Lipschitz Morphisms)

Let $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *Lipschitz morphism* $\pi : (\mathfrak{A}, L_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, L_{\mathfrak{B}})$ is a unital $*$ -morphism from \mathfrak{A} to \mathfrak{B} such that any of the following equivalent statement holds:

- ① $(\exists k) \varphi \in \mathcal{S}(\mathfrak{B}) \mapsto \varphi \circ \pi$ is a *k-Lipschitz map from*
 $(\mathcal{S}(\mathfrak{B}), \text{mk}_{L_{\mathfrak{B}}})$ *to* $(\mathcal{S}(\mathfrak{A}), \text{mk}_{L_{\mathfrak{A}}})$,
- ② (Rieffel, 00) $(\exists k) L_{\mathfrak{B}} \circ \pi \leq k L_{\mathfrak{A}}$,
- ③ (L., 16) $\pi(\text{dom}(L_{\mathfrak{A}})) \subseteq \text{dom}(L_{\mathfrak{B}})$.

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Lipschitz morphisms are arrows for a category over quasi-Leibniz quantum compact metric spaces.

A distance on morphisms

Definition (L., 16)

Let $(\mathfrak{A}, \mathsf{L})$ be a quantum compact metric space and \mathfrak{B} be a unital C^* -algebra. For any two linear maps $\varphi, \psi : \mathfrak{A} \rightarrow \mathfrak{B}$ preserving the units and intertwining the adjoint operations, let:

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup \{ \|\varphi(a) - \psi(a)\|_{\mathfrak{B}} : a \in \mathfrak{A}, \mathsf{L}(a) \leq 1 \}.$$

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The topology induced by mk_{L} on the space of $*$ -morphisms from \mathfrak{A} to \mathfrak{B} is the topology of pointwise convergence, where a net $(\alpha_j)_{j \in J}$ converges to some α when:

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$$\forall a \in \mathfrak{A} \quad \lim_{j \in J} \|\alpha_j(a) - \alpha(a)\|_{\mathfrak{B}} = 0.$$

Moreover, a subset Ξ of $*$ -morphisms is totally bounded for $\mathbf{mk}_{\mathbb{L}}$ if there exists $C > 0$ such that $\mathbb{L} \leq C\mathbb{L} \circ \alpha$ for all $\alpha \in \Xi$.

A proper monoid Gromov-Hausdorff distance

Let (G_1, δ_1) and (G_2, δ_2) be two proper metric monoids. A *ε -isometric isomorphism* $(\varsigma_1, \varsigma_2)$ from (G_1, δ_1) to (G_2, δ_2) is a pair of functions $\varsigma_1 : G_1 \rightarrow G_2$ and $\varsigma_2 : G_2 \rightarrow G_1$ such that for all $\{j, k\} = \{1, 2\}$:

$$\forall g, g' \in G_j \left[\frac{1}{\varepsilon} \right], h \in G_k \left[\frac{1}{\varepsilon} \right]$$

$$|\delta_k(\varsigma_j(g)\varsigma_j(g'), h) - \delta_j(gg', \varsigma_k(h))| \leq \varepsilon,$$

and ς_1, ς_2 are unital.

We then set:

$$\Upsilon((G_1, \delta_1), (G_2, \delta_2)) = \min \left\{ \frac{\sqrt{2}}{2}, \inf \{ \varepsilon > 0 \mid \exists (\varsigma, \kappa) \text{ } \varepsilon\text{-iso-iso} \} \right\}.$$

Υ is a *metric up to isometric isomorphism of proper monoid*, and it dominates the pointed Gromov-Hausdorff distance.

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We can combine Λ^* and Υ . A *Lipschitz dynamical system* $(\mathfrak{A}, \mathbb{L}, G, \delta, \alpha)$ consists of:

- a quantum compact metric space $(\mathfrak{A}, \mathbb{L})$,
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- a strongly continuous action of G on \mathfrak{A} by Lipschitz morphisms.

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Definition (L., 18)

A *ε -covariant tunnel* $(\tau, \varsigma, \kappa)$ from $(\mathfrak{A}, \mathbb{L}, G, \delta, \alpha)$ to $(\mathfrak{B}, \mathbb{L}, H, \delta, \beta)$ is a tunnel τ from $(\mathfrak{A}, \mathbb{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathbb{L}_{\mathfrak{B}})$ and a ε -almost isometric isomorphism $(\varsigma_1, \varsigma_2)$ from (G, δ_G) to (H, δ_H) .

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The *ε -reach* $\rho(\tau|\varepsilon)$ of a covariant tunnel $\tau = (\mathfrak{D}, \mathbb{L}, \pi_1, \pi_2, \varsigma_1, \varsigma_2)$ from $(\mathfrak{A}_1, \mathbb{L}_1, G_1, \delta_1, \alpha_1)$ to $(\mathfrak{A}_2, \mathbb{L}_2, G_2, \delta_2, \alpha_2)$ is:

$$\max_{\{j,k\}=\{1,2\}} \sup_{\varphi \in \mathcal{S}(\mathfrak{A}_j)} \inf_{\psi \in \mathcal{S}(\mathfrak{A}_k)} \sup_{g \in G_j[\frac{1}{\varepsilon}]}$$
$$\mathrm{mk}_{\mathbb{L}}(\varphi \circ \alpha_j^g \circ \pi_j, \psi \circ \alpha_k \circ \alpha_k^{\varsigma_k(g)} \circ \pi_k).$$

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Definition (L., 18)

The *covariant propinquity* $\Lambda^{\text{cov}}(\mathbb{A}, \mathbb{B})$ between two Lipschitz dynamical systems \mathbb{A} and \mathbb{B} is:

$$\inf \left\{ \frac{\sqrt{2}}{2}, \varepsilon > 0 \mid \right. \\ \left. \exists \tau \text{ a } \varepsilon\text{-covariant tunnel } \mathbb{A} \rightarrow \mathbb{B} \text{ with } \max\{\chi(\tau), \rho(\tau|\varepsilon)\} \leq \varepsilon \right\}$$

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Theorem (L., 18)

The *covariant propinquity* Λ^{cov} is a *metric* up to *equivariant* full quantum isometry, i.e it is symmetric, satisfies the triangle inequality, and:

$$\Lambda^{\text{cov}}((\mathfrak{A}, \mathbb{L}_{\mathfrak{A}}, G, \delta_G, \alpha), (\mathfrak{B}, \mathbb{L}_{\mathfrak{B}}, H, \delta_H, \beta)) = 0$$

if and only if there exists a full quantum isometry $\pi : (\mathfrak{A}, \mathbb{L}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \mathbb{L}_{\mathfrak{B}})$ and an isometric isomorphism $\varsigma : G \rightarrow H$ such that:

$$\forall a \in \text{dom}(\mathbb{L}_{\mathfrak{A}}) \quad \forall g \in G \quad \pi \circ \alpha^g = \beta^{\varsigma(g)} \circ \pi.$$

Theorem (L., 17, 18)

If $D : [0, \infty) \rightarrow [0, \infty)$ and:

- ❶ $\forall n \in \mathbb{N}$, $(\mathfrak{A}_n, \mathbb{L}_n, G_n, \delta_n, \alpha_n)$ is a Lipschitz dynamical F -system,
- ❷ $\lim_{n \rightarrow \infty} \Upsilon((G_n, \delta_n), (G, \delta)) = 0$,
- ❸ $\lim_{n \rightarrow \infty} \Lambda_F^*((\mathfrak{A}_n, \mathbb{L}_n), (\mathfrak{A}, \mathbb{L})) = 0$,
- ❹ for all $\varepsilon > 0$ there exists $\omega > 0, N \in \mathbb{N}$ such that:

$$\forall n \geq N \quad \forall g, h \in G_n \quad \delta_n(g, h) < \omega \implies \text{mk}_{\mathbb{L}_n}(\alpha_n^g, \alpha_n^h) < \varepsilon,$$

then there exists an action α of G on \mathfrak{A} and a $j : \mathbb{N} \rightarrow \mathbb{N}$ strictly increasing such that:

$$\lim_{n \rightarrow \infty} \Lambda_F^{\text{cov}}((\mathfrak{A}_{j(n)}, \mathbb{L}_{j(n)}, G_{j(n)}, \alpha_{j(n)}), (\mathfrak{A}, \mathbb{L}, G, \alpha)) = 0.$$

If G, G_n is a compact group and α_n *ergodic* $\forall_{\mathbb{N}} n$ then so is α .

Application to Closures

Theorem (L., 17)

Let \mathcal{C} be the class of *finite dimensional quantum compact metric spaces* together with an ergodic action of $SU(2)$ by isometries. The *closure* of \mathcal{C} for the dual propinquity consists only of quantum compact metric spaces over *type I* C^* -algebras.

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Let \mathcal{F} be the class of *finite dimensional quantum compact metric spaces* together with an ergodic action of \mathbb{T}^2 by isometries. The *closure* of \mathcal{F} for the dual propinquity consists of fuzzy and quantum tori.

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Let \mathcal{F} be the class of *finite dimensional quantum compact metric spaces* together with an ergodic action of \mathbb{T}^2 by isometries. The *closure* of \mathcal{F} for the dual propinquity consists of fuzzy and quantum tori.

Theorem (L., 18)

The *covariant propinquity* is *complete* on the class of Lipschitz dynamical systems with a bi-invariant metric.

Convergence of dual actions of quantum tori

Let ℓ be a *continuous length function* on \mathbb{T}^d . For any $G \subseteq \mathbb{T}^d$ a closed subgroup and σ a multiplier of \widehat{G} , for any $a \in C^*(\widehat{G}, \sigma)$, set:

$$L_{G,\sigma}(a) = \sup \left\{ \frac{\|\alpha_{G,\sigma}^g(a) - a\|_{C^*(\widehat{G},\sigma)}}{\ell(g)} : g \in G \setminus \{1\} \right\}$$

where $\alpha_{G,\sigma}$ is the dual action of G on $C^*(\widehat{G}, \sigma)$.

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Theorem (L., 18)

If $(G_n)_{n \in \mathbb{N}}$ is a sequence of closed subgroups of \mathbb{T}^d converging to \mathbb{T}^d for the Hausdorff distance Haus_ℓ and if σ_n are (lift of) multipliers of \widehat{G}_n converging pointwise to some σ then:

$$\lim_{n \rightarrow \infty} \Lambda_F^{\text{cov}} \left((C^*(\widehat{G}_n, \eta), \mathsf{L}_{G_n, \eta}, G_n, \alpha_{G_n, \sigma_n}), \right. \\ \left. (C^*(\mathbb{Z}^d, \sigma), \mathsf{L}_{\mathbb{T}^d, \sigma}, \mathbb{T}^d, \alpha_{\mathbb{T}^d, \sigma}) \right) = 0.$$

Completeness of certain classes of systems

An interesting complication arises when trying to determine when the covariant propinquity is complete. We introduce a notion of uniform equicontinuity.

Definition (L., 18)

Let $S = (G_n, \delta_n)_{n \in \mathbb{N}}$ be a sequence of proper monoids. The set of *regular sequences* $\mathcal{R}(S)$ is:

$$\left\{ (g_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} G_n \left| \begin{array}{l} \forall \varepsilon > 0 \quad \exists \omega > 0 \quad \exists N \in \mathbb{N} \\ \forall n \geq N \quad \forall h, k \in G_n \\ \delta_n(h, k) < \omega \implies \delta_n(hg_n, kg_n) < \varepsilon \end{array} \right. \right\}.$$

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Theorem (L., 18)

$\mathcal{R}((G_n, \delta_n)_{n \in \mathbb{N}})$ is a monoid for the pointwise multiplication.

Complete classes for Υ

Theorem (L.,18)

Let $(G_n, \delta_n)_{n \in \mathbb{N}}$ be a equene of monoids, and $(\sum \varepsilon_n)_{n \in \mathbb{N}}$ be a convergent series of positive general term. If:

- ❶ for all $n \in \mathbb{N}$, there exists a ε_n almost iso-iso (ς_n, κ_n) from G_n to G_{n+1} ,
- ❷ for all $N \in \mathbb{N}$, with $r = \sum_{j=0}^N \varepsilon_j$, if $g \in G_N[r]$, then:

$$\left(\begin{array}{l} g_n = e_n \text{ if } n < N, \\ g_n = g \text{ if } n = N, \\ g_{n+1} = \varsigma_n(g_n) \text{ if } n \geq N \end{array} \right)_{n \in \mathbb{N}} \in \mathcal{R}((G_n, \delta_n)_{n \in \mathbb{N}}),$$

then there exists a proper monoid (G, δ) such that:

$$\lim_{n \rightarrow \infty} \Upsilon((G_n, \delta_n), (G, \delta)) = 0.$$

Examples

Corollary (L.,18)

The space of proper monoids equipped with bi-invariant metrics is complete for the metric Υ .

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Corollary (L.,18)

If, for any proper monoids G and H , we set $\Upsilon'(G, H)$ as:

$$\left\{ \varepsilon > 0 \left| \begin{array}{l} \exists (\zeta, \kappa) \in \text{iso-iso } G \rightarrow H \\ \sup_{g \in G} |\text{dil}(h \mapsto hg) - \text{dil}(h \mapsto h\zeta(g))| < \varepsilon \\ \sup_{h \in H} |\text{dil}(g \mapsto gh) - \text{dil}(g \mapsto g\kappa(h))| < \varepsilon \end{array} \right. \right\}$$

then Υ' is a complete extended metric.

Completeness for the covariant propinquity

Theorem (L.,18)

Let $(\mathfrak{A}_n, \mathbf{L}_n, G_n, \delta_n, \alpha_n)_{n \in \mathbb{N}}$ be a sequence of Lipschitz dynamical system, $R : [0, \infty) \rightarrow [0, \infty)$ continuous, and $(\sum \varepsilon_n)_{n \in \mathbb{N}}$ be a convergent series of positive terms such that:

- ❶ $\Lambda((\mathfrak{A}_n, \mathbf{L}_n), (\mathfrak{A}_{n+1}, \mathbf{L}_{n+1})) \leq \varepsilon_n$,
- ❷ (in particular) there exists (ζ_n, κ_n) an ε_n -local almost isometry from G_n to G_{n+1} for all $n \in \mathbb{N}$,
- ❸ $\forall n \in \mathbb{N} \forall g \in G_n \quad \mathbf{L}_n \circ \alpha_n^g \leq R(\delta_n(e_n, g)) \mathbf{L}_n$,
- ❹ $\forall N \in \mathbb{N}, g \in G_N \quad (e_0, \dots, e_{N-1}, g, \zeta_N(g), \dots) \in \mathcal{R}((G_n, \delta_n)_{n \in \mathbb{N}})$,
- ❺ $\forall \varepsilon > 0 \quad \exists \omega > 0, N \in \mathbb{N} \quad \forall n \geq N, \forall g, h \in G_n \quad \delta_n(g, h) < \omega \implies \text{mk}_{\mathbf{L}}(\alpha_n^g, \alpha_n^h) < \varepsilon$

then there exists a Lipschitz dynamical system $(\mathfrak{A}, \mathbf{L}, G, \delta, \alpha)$ which is the limit of $(\mathfrak{A}_n, \mathbf{L}_n, G_n, \delta_n, \alpha_n)_{n \in \mathbb{N}}$ for the covariant propinquity.

Completeness

Corollary (L.,18)

Let $(\mathfrak{A}_n, \mathbb{L}_n, G_n, \delta_n, \alpha_n)_{n \in \mathbb{N}}$ be a Cauchy sequence of Lipschitz dynamical system for Λ^{cov} . Assume δ_n is bi-invariant for all $n \in \mathbb{N}$. Let $R : [0, \infty) \rightarrow [0, \infty)$ be continuous. If

- ❶ $\forall n \in \mathbb{N} \forall g \in G_n \quad \mathbb{L}_n \circ \alpha_n^g \leq R(\delta_n(e_n, g)) \mathbb{L}_n,$
- ❷ $\forall \varepsilon > 0 \quad \exists \omega > 0, N \in \mathbb{N} \quad \forall n \geq N, \forall g, h \in G_n \quad \delta_n(g, h) < \omega \implies \text{mk}_{\mathbb{L}}(\alpha_n^g, \alpha_n^h) < \varepsilon$

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Thank you!

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