

# *The Gromov-Hausdorff Propinquity*

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Noncommutative *metric* geometry is the study of noncommutative generalizations of algebras of *Lipschitz* functions over *metric* spaces.

- Pioneered by *Rieffel (1998–)*, inspired by *Connes (1989)*.
- Motivated by mathematical physics, addresses problems such as:
  - Can we approximate quantum spaces with finite dimensional algebras?
  - Are certain functions from a topological space to quantum spaces continuous? Lipschitz?
  - Are certain functions from a topological space to modules over quantum spaces continuous?

## 1 *The Gromov-Hausdorff Propinquity*

## 2 *Applications*

## 3 *The modular Propinquity*

## The Monge-Kantorovich metric

Let  $(X, \mathbf{m})$  be a compact metric space. The *Lipschitz seminorm*  $\mathbf{L}$  induced by  $\mathbf{m}$  is:

$$\mathbf{L}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{\mathbf{m}(x, y)} : x, y \in X, x \neq y \right\}$$

for all  $f \in \mathfrak{sa}(C(X)) = C(X, \mathbb{R})$  (allowing  $\infty$ ).

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The *Monge-Kantorovich metric* on  $\mathcal{S}(C(X))$  is given for all Borel-regular probability measures  $\mu, \nu$  by:

$$\text{mk}_L(\mu, \nu) = \sup \left\{ \left| \int_X f d\mu - \int_X f d\nu \right| : f \in \mathfrak{sa}(C(X)), \textcolor{blue}{L}(f) \leq 1 \right\}.$$

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The Gelfand map  $x \in (X, \textcolor{blue}{m}) \mapsto \delta_x \in (\mathcal{S}(C(X)), \text{mk}_{\textcolor{blue}{L}})$  is an isometry.

# *Compact Quantum Metric Spaces*

*Definition (Connes, 89; Rieffel, 98; L., 13)*

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We call  $\mathsf{L}$  an *L-seminorm*.

# Quantum Tori as Compact Quantum Metric Spaces

Theorem (Rieffel, 98)

If  $\alpha$  is a strongly continuous *action* of a *compact group*  $G$ , if  $\ell$  is a continuous *length* function over  $G$ , and if for all  $a \in \mathfrak{A}$  we set:

$$\mathsf{L}(a) = \sup \left\{ \frac{\|\alpha^g(a) - a\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1\} \right\},$$

then  $(\mathfrak{A}, \mathsf{L})$  is a *quantum compact metric space* if and only if  $\{a \in \mathfrak{A} : \forall g \in G \quad \alpha^g(a) = a\} = \mathbb{C}1_{\mathfrak{A}}$ .

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## Example

- *Quantum tori* are quantum compact metric spaces with  $\mathfrak{A} = C^*(\mathbb{Z}^d, \sigma)$ ,  $G = \mathbb{T}^d$  and  $\ell$  any continuous length on  $\mathbb{T}^d$ ;
- *Noncommutative solenoids*  $C^*\left(\mathbb{Z}\left[\frac{1}{p}\right]^2, \sigma\right)$  with  $G = \mathcal{S}_p^2$  (L., Packer, 16).

## *AF algebras*

If  $\mathfrak{A}$  is finite dimensional, then any seminorm  $L$  on  $\mathfrak{sa}(\mathfrak{A})$  with  $\ker L = \mathbb{R}1_{\mathfrak{A}}$  is an L-seminorm.

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*Theorem (Aguilar, L., 15)*

If  $\mathfrak{A} = \text{cl}(\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n)$  with  $(\mathfrak{A}_n)_{n \in \mathbb{N}}$  an increasing sequence of *finite dimensional subalgebras*, if  $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$  and if  $\tau$  is a faithful tracial state on  $\mathfrak{A}$ , and if for all  $a \in \mathfrak{A}$  we set:

$$L(a) = \sup \left\{ \frac{\|\mathbb{E}_n(a) - a\|_{\mathfrak{A}}}{(\dim \mathfrak{A}_n)^{-1}} : n \in \mathbb{N} \right\}$$

where  $\mathbb{E}_n : \mathfrak{A} \rightarrow \mathfrak{A}$  is the unique *conditional expectation from  $\mathfrak{A}$  onto  $\mathfrak{A}_n$*  such that  $\tau \circ \mathbb{E}_n = \tau$  for all  $n \in \mathbb{N}$ , then  $(\mathfrak{A}, L)$  is a quantum compact metric space.

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These examples include UHF algebras, Effros-Shen algebras, as well as commutative AF algebras such as  $C(\text{Cantor set})$ .

## *Certain Spectral Triples*

If  $(\mathfrak{A}, \mathcal{H}, D)$  is a spectral triple, then a natural candidate for an L-seminorm is given by:

$$\forall a \in \mathfrak{sa}(\mathfrak{A}) \quad \textcolor{blue}{L}_D(a) = \|[D, a]\|_{\mathcal{H}}$$

allowing for  $\infty$ .

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*Example: Connes, 89*

If  $M$  is a connected compact spin manifold and  $D$  is the Dirac operator of  $M$  acting on the Hilbert space  $\mathcal{H} = L^2(\Gamma S)$  of square integrable sections of the spinor bundle, then  $(C(M), \mathcal{H}, D)$  is a spectral triple such that  $\text{mk}_{\textcolor{blue}{L}_D}$  restricts to the geodesic distance on  $M$ .

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### *Problem*

Some spectral triples are known to produce a Monge-Kantorovich metric which does not even give finite radius to the state space!

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*Theorem (Ozawa, Rieffel, 02)*

Let  $G$  be a finitely generated group and  $l$  a word length function. On  $\ell^2(G)$ , we define the (unbounded) operator  $D$  as multiplication by  $l$ .

For all  $a \in C_{\text{red}}^*(G)$ , set:

$$\mathsf{L}_D(a) = \| [D, a] \|_{\ell^2(G)}.$$

If  $G$  is word-hyperbolic then  $(C_{\text{red}}^*(G), \mathsf{L})$  is a Leibniz quantum compact metric space.

## *Some other examples*

- *Li, 03*: Connes-Landi-du Bois-Voilette's spheres wrt spectral triple,
- *Hawkins, Skalski, White, Zacharias, 13*: C\*-crossed products for equicontinuous actions wrt spectral triple,
- *L., 16*: Curved quantum tori, with metric from spectral triple,
- *L., 16*: Perturbations (conformal and otherwise) of metric spectral triples,
- *Christ, Rieffel, 16*: Nilpotent group C\*-algebras with Connes' length spectral triple,
- *Aguilar, Kadd, 17*: Podlès spheres

# Lipschitz morphisms

## Theorem-Definition (Lipschitz Morphisms)

Let  $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$  and  $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$  be two quasi-Leibniz quantum compact metric spaces. A *Lipschitz morphism*  $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$  is a unital \*-morphism from  $\mathfrak{A}$  to  $\mathfrak{B}$  such that any of the following equivalent statement holds:

- ①  $(\exists k) \varphi \in \mathcal{S}(\mathfrak{B}) \mapsto \varphi \circ \pi$  is a *k-Lipschitz map from  $(\mathcal{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$  to  $(\mathcal{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$* ,
- ② (Rieffel, 00)  $(\exists k) \mathsf{L}_{\mathfrak{B}} \circ \pi \leq k \mathsf{L}_{\mathfrak{A}}$ ,
- ③ (L., 16)  $\pi(\text{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \text{dom}(\mathsf{L}_{\mathfrak{B}})$ .

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Lipschitz morphisms are arrows for a category over quasi-Leibniz quantum compact metric spaces.

# *Quantum Isometries*

If  $X \subset (Z, \text{m})$  and  $f \in \mathfrak{sa}(C(X))$ :

- if  $g \in C(Z)$  and  $g|_X = f$ , then  $\text{Lip}(f) \leq \text{Lip}(g)$ ,
- there exists  $g \in \mathfrak{sa}(C(Z))$  with  $g|_X = f$  and  $\text{Lip}(f) = \text{Lip}(g)$  (McShane, 34).

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*Definition (Rieffel (98), L. (13))*

A *quantum isometry*  $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$  is a \*-epimorphism such that:

$$\forall b \in \text{dom}(\mathsf{L}_{\mathfrak{B}}) \quad \mathsf{L}_{\mathfrak{B}}(b) = \inf \{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.$$

A *full quantum isometry*  $\pi$  is a \*-isomorphism such that  $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$ .

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The dual maps of quantum isometries are indeed isometries for the Monge-Kantorovich metric.

# The Gromov-Hausdorff Distance

## Definition

For any two compact metric spaces  $(X, \mathbf{m}_X)$  and  $(Y, \mathbf{m}_Y)$ , we define  $\text{Adm}(\mathbf{m}_X, \mathbf{m}_Y)$  as:

$$\left\{ (Z, \mathbf{m}_Z, \iota_X, \iota_Y) \middle| \begin{array}{l} (Z, \mathbf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right\}$$

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## Notation

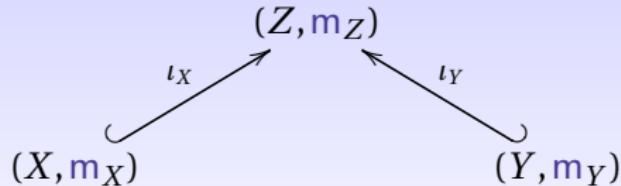
The *Hausdorff distance* on the compact subsets of a metric space  $(X, \mathbf{m})$  is denoted by  $\text{Haus}_{\mathbf{m}}$ .

## Definition (Edwards, 75; Gromov, 81)

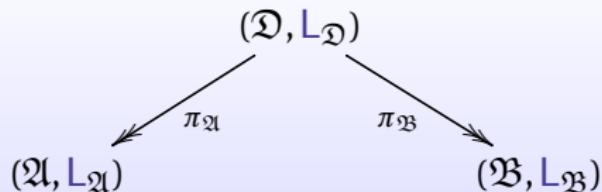
The *Gromov-Hausdorff distance* between two compact metric spaces  $(X, \mathbf{m}_X)$  and  $(Y, \mathbf{m}_Y)$  is:

$$\inf \{ \text{Haus}_{\mathbf{m}_Z} (\iota_X(X), \iota_Y(Y)) : (Z, \mathbf{m}_Z, \iota_X, \iota_Y) \in \text{Adm}(\mathbf{m}_X, \mathbf{m}_Y) \}.$$

# Tunnels



*Figure:* Gromov-Hausdorff Isometric Embeddings



*Figure:* A tunnel

*Legend:*  $\hookrightarrow$ : isometry ;  $\rightarrow\!\!\!\rightarrow$ : quantum isometry

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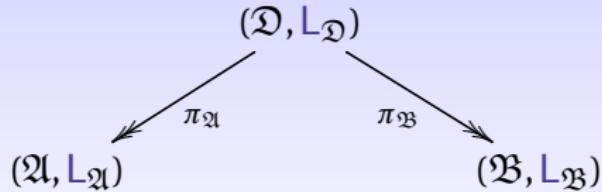


Figure: A tunnel

If  $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$  then set  $\pi^* : \varphi \in \mathcal{S}(\mathfrak{B}) \hookrightarrow \varphi \circ \pi \in \mathcal{S}(\mathfrak{A})$ .

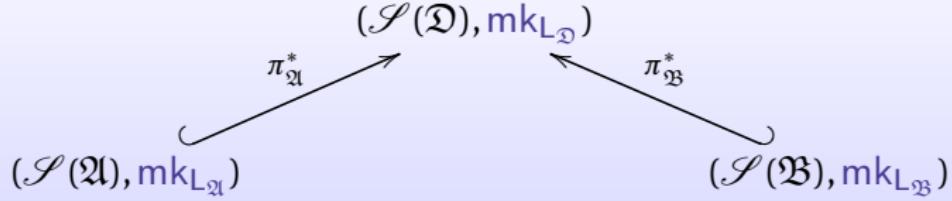
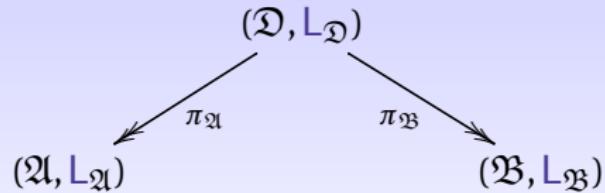


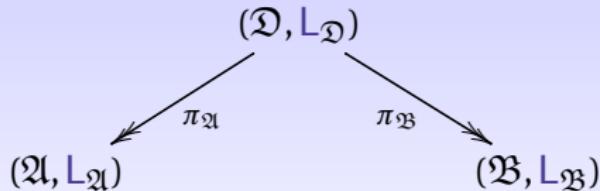
Figure: Gromov-Hausdorff Isometric Embeddings

# The Dual Gromov-Hausdorff Propinquity



*Figure:*  $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$  are quantum isometries

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*Definition (The extent of a tunnel, L. 13,14)*

The *extent*  $\chi(\tau)$  of a tunnel  $\tau = (\mathfrak{D}, \textcolor{violet}{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$  is:

$$\max \left\{ \text{Haus}_{\text{mk}_{\textcolor{violet}{L}_{\mathfrak{D}}}} \left( \mathcal{S}(\mathfrak{D}), \pi_{\mathfrak{E}}^*(\mathcal{S}(\mathfrak{E})) \right) : \mathfrak{E} \in \{\mathfrak{A}, \mathfrak{B}\} \right\}.$$

# The Dual Gromov-Hausdorff Propinquity

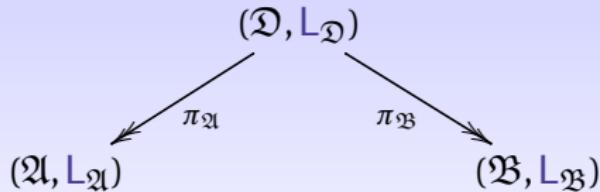


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## Definition (The Dual Propinquity, L. 13, 14)

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The *dual propinquity*  $\Lambda_F^*((\mathfrak{A}, \textcolor{violet}{L}_{\mathfrak{A}}), (\mathfrak{B}, \textcolor{violet}{L}_{\mathfrak{B}}))$  is given by:

$$\inf \{ \chi(\tau) : \tau \text{ any } F\text{-tunnel from } (\mathfrak{A}, \textcolor{violet}{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \textcolor{violet}{L}_{\mathfrak{B}}) \}.$$

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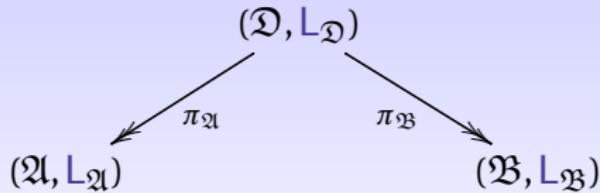


Figure:  $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$  are quantum isometries

## Theorem (L., 13)

The *dual propinquity*  $\Lambda_F^*$ , defined for any two quantum compact metric spaces  $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$  by  $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$  by:

$$\inf \{ \chi(\tau) : \tau \text{ any } F\text{-tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \}$$

is a *complete metric* up to *full quantum isometry*:  
 $\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$  iff there exists a \*-isomorphism  $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$  such that  $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$ .

# The Dual Gromov-Hausdorff Propinquity

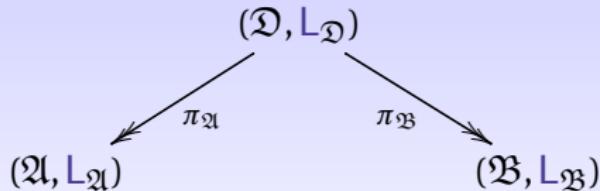


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is a *complete metric* up to *full quantum isometry*. Moreover  $\Lambda^*$  induces the topology of the *Gromov-Hausdorff distance* on compact metric spaces.

1 *The Gromov-Hausdorff Propinquity*

2 *Applications*

3 *The modular Propinquity*

## Examples: Quantum and Fuzzy Tori

Example:

Let  $\ell$  be a *continuous length function* on  $\mathbb{T}^d$ . For any  $G \subseteq \mathbb{T}^d$  a closed subgroup and  $\sigma$  a multiplier of  $\widehat{G}$ , for any  $a \in C^*(\widehat{G}, \sigma)$ , set:

$$\mathsf{L}_{G,\sigma}(a) = \sup \left\{ \frac{\|\alpha^g(a) - a\|_{C^*(\widehat{G}, \sigma)}}{\ell(g)} : g \in G \setminus \{1\} \right\}$$

where  $\alpha$  is the dual action of  $G$  on  $C^*(\widehat{G}, \sigma)$ .

Rieffel showed in 1998 that  $(C^*(\widehat{G}, \sigma), \mathsf{L}_{G,\sigma})$  is a Leibniz quantum compact metric space.

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where  $\alpha$  is the dual action of  $G$  on  $C^*(\widehat{G}, \sigma)$ .

If  $(G_n)_{n \in \mathbb{N}}$  is a *sequence of closed subgroups* of  $\mathbb{T}^d$  *converging* to  $\mathbb{T}^d$  for the *Hausdorff distance*  $\text{Haus}_\ell$ , and if  $(\sigma_n)_{n \in \mathbb{N}}$  is a sequence of multipliers of  $\mathbb{Z}^d$  converging pointwise to some  $\sigma$ , with  $\sigma_n(g) = 1$  if  $g$  is the coset of 0 for  $\widehat{G}_n$ , then:

$$\lim_{n \rightarrow \infty} \Lambda^*((C^*(\widehat{G}_n, \sigma_n), \mathsf{L}_{\widehat{G}_n, \sigma_n}), (C^*(\mathbb{Z}^d, \sigma), \mathsf{L}_{\mathbb{Z}^d, \sigma})) = 0.$$

# Finite Dimensional Approximations of quantum tori

Example: L. (13)

If for all  $n \in \mathbb{N}$ , we set  $\mathcal{F}_n = C^*(U_n, V_n) = C^*(\mathbb{Z}_n^2, \rho_n)$  where:

$$U_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & \dots & 1 & 0 \end{pmatrix}, V_n = \begin{pmatrix} 1 & e^{\frac{2ip_n\pi}{n}} & & & \\ & e^{\frac{4ip_n\pi}{n}} & & & \\ & & \ddots & & \\ & & & & e^{\frac{2i(n-1)p_n\pi}{n}} \end{pmatrix}$$

with  $p_n \not\equiv 0 \pmod{n}$ , and if  $\lim_{n \rightarrow \infty} \frac{p_n}{n} = \theta$ , then:

$$\lim_{n \rightarrow \infty} \Lambda((\mathcal{F}_n, \mathsf{L}_n), (\mathcal{A}_\theta, \mathsf{L}_\theta)) = 0$$

where  $\mathcal{A}_\theta = C^*(U, V)$  and  $U, V$  are universal unitaries such that  $VU = e^{2i\pi\theta}UV$ , while  $\mathsf{L}_n$  and  $\mathsf{L}$  are L-seminorms from the dual actions, and for some *fixed* continuous length function on  $\mathbb{T}^2$ .

## Examples: AF algebras

### Example: Aguilar, L. 15

Let  $\mathfrak{A} = \text{cl}(\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n)$  with  $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$ ,  $\forall n \in \mathbb{N} \quad \mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$  and  $\dim_{\mathbb{C}} \mathfrak{A}_n < \infty$ , such that there exists a *faithful, tracial state*  $\tau$  on  $\mathfrak{A}$ . Let  $\mathbb{E}_n : \mathfrak{A} \rightarrow \mathfrak{A}_n$  for all  $n \in \mathbb{N}$  is the *conditional expectation* from  $\mathfrak{A}$  onto  $\mathfrak{A}_n$  with  $\tau \circ \mathbb{E}_n = \tau$ .

Set, for all  $a \in \mathfrak{A}$ :

$$\mathsf{L}(a) = \sup \left\{ \frac{\|a - \mathbb{E}_n(a)\|_{\mathfrak{A}}}{(\dim \mathfrak{A}_n)^{-1}} : n \in \mathbb{N} \right\}.$$

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Then:

- $(\mathfrak{A}, \mathbb{L}) = \Lambda^* - \lim_{n \rightarrow \infty} (\mathfrak{A}_n, \mathbb{L})$ ,
- the natural map from the Baire space to UHF algebras is Lipschitz.

# *Effros-Shen AF algebras*

*Example: Aguilar, L., 15*

For  $\theta \in (0, 1) \setminus \mathbb{Q}$ , let  $\theta = \lim_{n \rightarrow \infty} \frac{p_n^\theta}{q_n^\theta}$  with  $\frac{p_n^\theta}{q_n^\theta} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$  for  $a_1, \dots \in \mathbb{N}$ .

- Set  $\mathfrak{AF}_\theta = \varinjlim_{n \rightarrow \infty} (\mathfrak{M}_{q_n} \oplus \mathfrak{M}_{q_{n-1}}, \psi_{n,\theta})$  where  $\psi_{n,\theta}$  involves  $a_{n+1}$ .
- let  $\mathsf{L}_\theta$  the L-seminorm for this data.

For all  $\theta \in (0, 1) \setminus \mathbb{Q}$ , we have:

$$\lim_{\substack{\theta \rightarrow \theta \\ \theta \notin \mathbb{Q}}} \Lambda((\mathfrak{AF}_\theta, \mathsf{L}_\theta), (\mathfrak{AF}_\theta, \mathsf{L}_\theta)) = 0.$$

## Examples

### Example

- *Rieffel, 15* has constructed full matrix approximations for the propinquity of  $C(S^2)$  using metrics arising from actions of  $SU(2)$ .
- *L. and Packer, 16* proved continuity for the noncommutative solenoids as function of the multipliers of  $\mathbb{Z}[p^{-1}]^2$ .
- *L., 15* proved continuity for curved quantum tori.
- *L., 15* proved continuity for conformal deformations.
- *L., 15* proved that nuclear quasi-diagonal quantum compact metric spaces are limits of finite dimensional quantum compact metric spaces,
- and more ...

# Compactness

## Definition (L., 15)

If  $(\mathfrak{A}, \mathsf{L})$  is an  $F$ -quantum compact metric space and if  $\varepsilon > 0$  then

$$\text{cov}_F(\mathfrak{A}, \mathsf{L} | \varepsilon) = \min \left\{ \dim \mathfrak{B} : \Lambda_F^*((\mathfrak{A}, \mathsf{L}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leq \varepsilon \right\}.$$

## Theorem (L., 15)

A class  $\mathcal{C}$  of  $F$ -quantum compact metric spaces with finite covering numbers is *precompact* for  $\Lambda_F^*$  if and only if:

- ①  $\exists M > 0 \quad \forall (\mathfrak{A}, \mathsf{L}) \in \mathcal{C} \quad \text{diam}(\mathfrak{A}, \mathsf{L}) \leq M,$
- ②  $\exists G : (0, \infty) \rightarrow \mathbb{N} \quad \forall (\mathfrak{A}, \mathsf{L}) \in \mathcal{C} \quad \text{cov}_F(\mathfrak{A}, \mathsf{L} | \varepsilon) \leq G(\varepsilon).$

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## Example: L. 13 and L. 15

If  $d \in \mathbb{N} \setminus \{0, 1\}$ , then for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all multipliers  $\theta$  of  $\mathbb{Z}^d$ :

$$\text{cov}\left(\left(C^*\left(\mathbb{Z}^d, \theta\right), \mathsf{L}_{\theta}\right) \middle| \varepsilon\right) \leq N.$$

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## Example: Aguilar, L., 15

$$\left\{ (\mathfrak{A}, \mathsf{L}) \left| \begin{array}{l} \mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n} \text{ with } \forall n \in \mathbb{N} \quad \mathfrak{A}_n \subseteq \mathfrak{A}_{n+1} \\ \mathfrak{A} \text{ has a faithful tracial state} \\ \forall n \in \mathbb{N} \quad d(n) \leq \dim \mathfrak{A}_n \leq D(n) < \infty \\ \forall a \in \mathfrak{s}\mathfrak{a}(\mathfrak{A}) \quad \mathsf{L}(a) = \sup_{n \in \mathbb{N}} \dim \mathfrak{A}_n \|a - \mathbb{E}_n(a)\|_{\mathfrak{A}_n} \end{array} \right. \right\}$$

# *Covariance Property of $\Lambda^*$*

## *Symmetries and metric convergence*

Do symmetries or dynamics pass to the limit for the dual propinquity, under some *equicontinuity* condition ?

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To study this question:

- ① we define a distance on *proper monoids*,
- ② we then define a *covariant* version of the propinquity,
- ③ we then prove a form of Arzéla-Ascoli theorem for the *dual propinquity*.

## *A proper monoid Gromov-Hausdorff distance*

Let  $(G_1, \delta_1)$  and  $(G_2, \delta_2)$  be two proper metric monoids. A

*$\varepsilon$ -isometric isomorphism*  $(\varsigma_1, \varsigma_2)$  from  $(G_1, \delta_1)$  to  $(G_2, \delta_2)$  is a pair of functions  $\varsigma_1 : G_1 \rightarrow G_2$  and  $\varsigma_2 : G_2 \rightarrow G_1$  such that for all  $\{j, k\} = \{1, 2\}$ :

$$\forall g, g' \in G_j \left[ \frac{1}{\varepsilon} \right], h \in G_k \left[ \frac{1}{\varepsilon} \right]$$

$$|\delta_k(\varsigma_j(g)\varsigma_j(g'), h) - \delta_j(gg', \varsigma_k(h))| \leq \varepsilon,$$

and  $\varsigma_1, \varsigma_2$  are unital.

We then set:

$$\Upsilon((G_1, \delta_1), (G_2, \delta_2)) = \inf \{\varepsilon > 0 | \exists (\varsigma, \kappa) \text{ } \varepsilon\text{-iso-iso}\}.$$

$\Upsilon$  is a *metric up to isometric isomorphism of proper monoid*, and it dominates the pointed Gromov-Hausdorff distance.

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## Definition (L., 18)

A  $\varepsilon$ -covariant tunnel  $(\tau, \varsigma, \kappa)$  from  $(\mathfrak{A}, \underline{\mathsf{L}}, G, \delta, \alpha)$  to  $(\mathfrak{B}, \underline{\mathsf{L}}, H, \delta, \beta)$  is a tunnel  $\tau$  from  $(\mathfrak{A}, \underline{\mathsf{L}}_{\mathfrak{A}})$  to  $(\mathfrak{B}, \underline{\mathsf{L}}_{\mathfrak{B}})$  and a  $\varepsilon$ -almost isometric isomorphism  $(\varsigma_1, \varsigma_2)$  from  $(G, \delta_G)$  to  $(H, \delta_H)$ .

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### Definition (L., 18)

The  *$\varepsilon$ -reach*  $\rho(\tau|\varepsilon)$  of a covariant tunnel  $\tau = (\mathfrak{D}, \mathsf{L}, \pi_1, \pi_2, \varsigma_1, \varsigma_2)$  from  $(\mathfrak{A}_1, \mathsf{L}_1, G_1, \delta_1, \alpha_1)$  to  $(\mathfrak{A}_2, \mathsf{L}_2, G_2, \delta_2, \alpha_2)$  is:

$$\max_{\{j,k\}=\{1,2\}} \sup_{\varphi \in \mathcal{S}(\mathfrak{A}_j)} \inf_{\psi \in \mathcal{S}(\mathfrak{A}_k)} \sup_{g \in G_j[\frac{1}{\varepsilon}]} \text{mk}_{\mathsf{L}}(\varphi \circ \alpha_j^g \circ \pi_j, \psi \circ \alpha_k \circ \alpha_k^{\varsigma_k(g)} \circ \pi_k).$$

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### Definition (L., 18)

The *covariant propinquity*  $\Lambda^{\text{cov}}(\mathbb{A}, \mathbb{B})$  between two Lipschitz dynamical systems  $\mathbb{A}$  and  $\mathbb{B}$  is:

$$\inf \{ \varepsilon > 0 \mid \exists \tau \text{ a } \varepsilon\text{-covariant tunnel } \mathbb{A} \rightarrow \mathbb{B} \text{ with } \max\{\chi(\tau), \rho(\tau|\varepsilon)\} \leq \varepsilon \}$$

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## Theorem (L., 18)

The *covariant propinquity*  $\Lambda^{\text{cov}}$  is a *metric* up to *equivariant* full quantum isometry, i.e it is symmetric, satisfies the triangle inequality, and:

$$\Lambda^{\text{cov}}((\mathfrak{A}, \underline{\mathsf{L}}_{\mathfrak{A}}, G, \delta_G, \alpha), (\mathfrak{B}, \underline{\mathsf{L}}_{\mathfrak{B}}, H, \delta_H, \beta)) = 0$$

if and only if there exists a full quantum isometry  $\pi : (\mathfrak{A}, \underline{\mathsf{L}}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \underline{\mathsf{L}}_{\mathfrak{B}})$  and an isometric isomorphism  $\varsigma : G \rightarrow H$  such that:

$$\forall a \in \text{dom}(\underline{\mathsf{L}}_{\mathfrak{A}}) \quad \forall g \in G \quad \pi \circ \alpha^g = \beta^{\varsigma(g)} \circ \pi.$$

## Theorem (L., 17, 18)

If  $D : [0, \infty) \rightarrow [0, \infty)$  and:

- ①  $\forall n \in \mathbb{N}, (\mathfrak{A}_n, \mathsf{L}_n, G_n, \delta_n, \alpha_n)$  is a Lipschitz dynamical  $F$ -system,
- ②  $\lim_{n \rightarrow \infty} Y((G_n, \delta_n), (G, \delta)) = 0,$
- ③  $\lim_{n \rightarrow \infty} \Lambda_F^*((\mathfrak{A}_n, \mathsf{L}_n), (\mathfrak{A}, \mathsf{L})) = 0,$
- ④ for all  $\varepsilon > 0$  there exists  $\omega > 0, N \in \mathbb{N}$  such that:

$$\forall n \geq N \quad \forall g, h \in G_n \quad \delta_n(g, h) < \omega \implies \sup \left\{ \left\| \alpha_n^g(a) - \alpha_n^h(a) \right\|_{\mathfrak{A}_n} \middle| \mathsf{L}_n(a) \leq 1 \right\} < \varepsilon,$$

then there exists an action  $\alpha$  of  $G$  on  $\mathfrak{A}$  and a  $j : \mathbb{N} \rightarrow \mathbb{N}$  strictly increasing such that:

$$\lim_{n \rightarrow \infty} \Lambda_F^{\text{cov}}((\mathfrak{A}_{j(n)}, \mathsf{L}_{j_n}, G_{j(n)}, \alpha_{j(n)}), (\mathfrak{A}, \mathsf{L}, G, \alpha)) = 0.$$

If  $G, G_n$  is a compact group and  $\alpha_n$  *ergodic*  $\forall_{\mathbb{N}} n$  then so is  $\alpha$ .

# *Application to Closures*

## *Theorem (L., 17, new)*

Let  $\mathcal{C}$  be the class of *finite dimensional quantum compact metric spaces* together with an ergodic action of  $SU(2)$  by isometries. The *closure* of  $\mathcal{C}$  for the dual propinquity consists only of quantum compact metric spaces over *type I*  $C^*$ -algebras.

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## *Theorem (L., 17, alternate proof)*

Let  $\mathcal{F}$  be the class of *finite dimensional quantum compact metric spaces* together with an ergodic action of  $T^2$  by isometries. The *closure* of  $\mathcal{F}$  for the dual propinquity consists of fuzzy and quantum tori.

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## *Theorem (L., 18, very new)*

The *covariant propinquity* is *complete* on the class of Lipschitz dynamical systems with a bi-invariant metric.

## *Convergence of dual actions of quantum tori*

Let  $\ell$  be a *continuous length function* on  $\mathbb{T}^d$ . For any  $G \subseteq \mathbb{T}^d$  a closed subgroup and  $\sigma$  a multiplier of  $\widehat{G}$ , for any  $a \in C^*(\widehat{G}, \sigma)$ , set:

$$\mathsf{L}_{G,\sigma}(a) = \sup \left\{ \frac{\|\alpha_{G,\sigma}^g(a) - a\|_{C^*(\widehat{G}, \sigma)}}{\ell(g)} : g \in G \setminus \{1\} \right\}$$

where  $\alpha_{G,\sigma}$  is the dual action of  $G$  on  $C^*(\widehat{G}, \sigma)$ .

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### Theorem (L., 18)

If  $(G_n)_{n \in \mathbb{N}}$  is a sequence of closed subgroups of  $\mathbb{T}^d$  converging to  $\mathbb{T}^d$  for the Hausdorff distance  $\text{Haus}_\ell$  and if  $\sigma_n$  are (lift of) multipliers of  $\widehat{G}_n$  converging pointwise to some  $\sigma$  then:

$$\lim_{n \rightarrow \infty} \Lambda_F^{\text{cov}} \left( (C^*(\widehat{G}_n, \eta), \mathsf{L}_{G_n, \eta}, G_n, \alpha_{G_n, \sigma_n}), (C^*(\mathbb{Z}^d, \sigma), \mathsf{L}_{\mathbb{T}^d, \sigma}, \mathbb{T}^d, \alpha_{\mathbb{T}^d, \sigma}) \right) = 0.$$

## 1 *The Gromov-Hausdorff Propinquity*

## 2 *Applications*

## 3 *The modular Propinquity*

## *Metrics for Vector Bundles*

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$$\langle \omega, \xi \rangle_{C(M)} : x \in M \mapsto h_x(\omega_x, \xi_x) \in C(M).$$

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- Let  $\nabla$  be a metric connection on  $\Gamma V$ , i.e.  $\forall X \in \Gamma TM$ :

$$d_X \langle \omega, \xi \rangle = \langle \nabla_X \omega, \xi \rangle + \langle \omega, \nabla_X \xi \rangle.$$

$\nabla$  defines a norm on a dense subspace of  $\Gamma V$ :

$$\mathbb{D}(\omega) = \max \left\{ \sqrt{\langle \omega, \omega \rangle_{C(M)}}, \|\nabla \omega\|_{\Gamma V}^{\Gamma TM} \right\}.$$

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Our idea is to introduce a metric on objects of the form  $(\Gamma V, \langle \cdot, \cdot \rangle_{C(M)}, D, C(M), L)$ .

# Metrized quantum vector bundles

## Definition (metrized quantum vector bundle, L. (16))

A metrized quantum vector bundle  $\Omega = (\mathcal{M}, \langle \cdot, \cdot \rangle_{\mathcal{M}}, D, \mathfrak{A}, L)$  is given by:

- ①  $(\mathfrak{A}, L)$  is a *quantum compact metric space*,
- ②  $(\mathcal{M}, \langle \cdot, \cdot \rangle_{\mathcal{M}})$  is a *left Hilbert module* over  $\mathfrak{A}$ ,
- ③  $D$  is a *norm* on a dense subspace of  $\mathcal{M}$  such that:

- ①  $D \geq \| \cdot \|_{\mathcal{M}} := \sqrt{\langle \cdot, \cdot \rangle_{\mathcal{M}}}$
- ②  $\{\omega \in \mathcal{M} : D(\omega) \leq 1\}$  is compact in  $(\mathcal{M}, \| \cdot \|_{\mathcal{M}})$ ,
- ③  $D(a\omega) \leq G(\|a\|_{\mathfrak{A}}, L(a), D(\omega)),$
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## Full quantum isometries

$(\theta, \Theta)$  full quantum isometry when  $\theta$  full quantum isometry between bases and  $\Theta(a\xi) = \theta(a)\Theta(\xi)$ ,  $\Theta$  linear isomorphism preserving both the norms and the  $D$ -norms.

## *The Heisenberg Modules (Connes, 81; Rieffel)*

Fix  $\theta \in \mathbb{R}$ ,  $p \in \mathbb{Z}$ ,  $q \in \mathbb{N}$ ,  $d \in \mathbb{N} \setminus \{0\}$  such that  $\mathfrak{D} = \theta - \frac{p}{q} \neq 0$ . Let  $\mathcal{A}_\theta$  be the associated quantum torus.

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- ➊ Start with a representation of  $\left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R}^3 \right\}$  on

$L^2(\mathbb{R})$ :

$$\alpha_{\tilde{\mathcal{D}}}^{x,y,t} \xi(s) = \exp(i\pi(t + 2xs)) \xi(s + \tilde{\mathcal{D}}y).$$

Promote it to  $L^2(\mathbb{R}) \otimes \mathbb{C}^d$ .

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- ➌ For Schwarz functions  $\xi, \omega$ , set:

$$\langle \xi, \omega \rangle_{\mathcal{H}_\theta^{p,q,d}} = \sum_{n,m \in \mathbb{Z}} \langle u_\theta^n v_\theta^m \xi, \omega \rangle_{L^2(\mathbb{R}, \mathbb{C}^d)} u_\theta^n v_\theta^m;$$

complete space of Schwarz functions to the *Heisenberg module*  
 $\mathcal{H}_\theta^{p,q,d}$ .

## *The Heisenberg Modules*

Fix  $\theta \in [0, 1)$ ,  $p \in \mathbb{Z}$ ,  $q \in \mathbb{N} \setminus \{0\}$ , and  $d \in q\mathbb{N}$ . The canonical generators of  $\mathcal{A}_\theta$  are  $U, V$  such that  $UV = \exp(2i\pi\theta)VU$ .

- Let  $\mathcal{S}^d$  be the space of  $\mathbb{C}^d$ -valued Schwarz functions.

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- The candidate *D-norm* is related to the norm of the operator:

$$(x, y) \in \mathbb{R}^2 \mapsto x\nabla_1 \xi + y\nabla_2 \xi,$$

for  $\xi \in \mathcal{S}^d$

# *D*-norms for Heisenberg Modules

*Theorem (L., 16)*

Fix some norm  $\|\cdot\|$  on  $\mathbb{R}^2$ . For all  $\xi \in \mathcal{H}_\theta^{p,q,d}$ , we set  $\textcolor{brown}{D}_\theta^{p,q,d}(\xi)$  as:

$$\sup \left\{ \|\xi\|_{\mathcal{H}_\theta^{p,q,d}}, \frac{\left\| \alpha_{\bar{\partial}}^{x,y,\frac{xy}{2}} \xi - \xi \right\|_{\mathcal{H}_\theta^{p,q,d}}}{2\pi |\bar{\partial}| \|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

$(\mathcal{H}_\theta^{p,q,d}, \langle \cdot, \cdot \rangle_{\mathcal{H}_\theta^{p,q,d}}, \textcolor{brown}{D}_\theta^{p,q,d}, \mathcal{A}_\theta, \textcolor{brown}{L}_\theta)$  is a metrized quantum vector bundle.

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As a note,  $\mathsf{D}_\theta^{p,q,d}(\xi)$  is actually the max of operator norm of  $\nabla \xi$  where  $\nabla$  is the Connes connection on  $\mathcal{H}_\theta^{p,q,d}$  and the usual Hilbert module norm.

# Tunnels between metrized quantum vector bundles

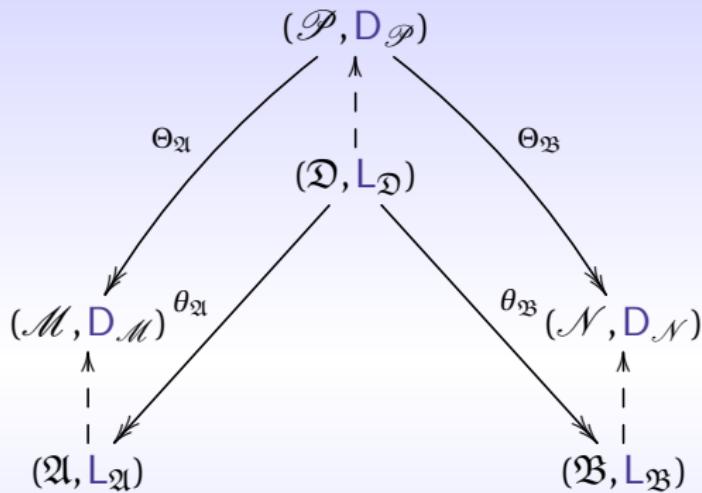


Figure: A modular tunnel

Definition (L., 18)

The *extent* of a modular tunnel is the extent of the underlying tunnel.

# *The modular propinquity*

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## Theorem (Heisenberg Modules (L. 17))

If  $(\theta_n)_{n \in \mathbb{N}}$  is a sequence in  $\mathbb{R} \setminus \mathbb{Q}$  converging to some irrational number  $\theta$ , and  $p, q \in \mathbb{Z} \setminus \{0\}, d \in \mathbb{N} \setminus \{0\}$  then:

$$\lim_{n \rightarrow \infty} \Lambda^{\text{mod}}((\mathcal{H}_{\theta_n}^{p,q,d}, \mathcal{H}_{\theta}^{p,q,d})) = 0.$$

### Theorem (L., 17)

Let  $\|\cdot\|$  be a norm on  $\mathbb{R}^2$  and  $p, q, d$  fixed. If for all  $\theta \in \mathbb{R}$ , and  $a \in \mathcal{A}_\theta$ :

$$\mathsf{L}_\theta(a) = \sup \left\{ \frac{\left\| \beta_\theta^{\exp(ix), \exp(iy)} a - a \right\|_{\mathcal{A}_\theta}}{\|(x, y)\|} : (x, y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

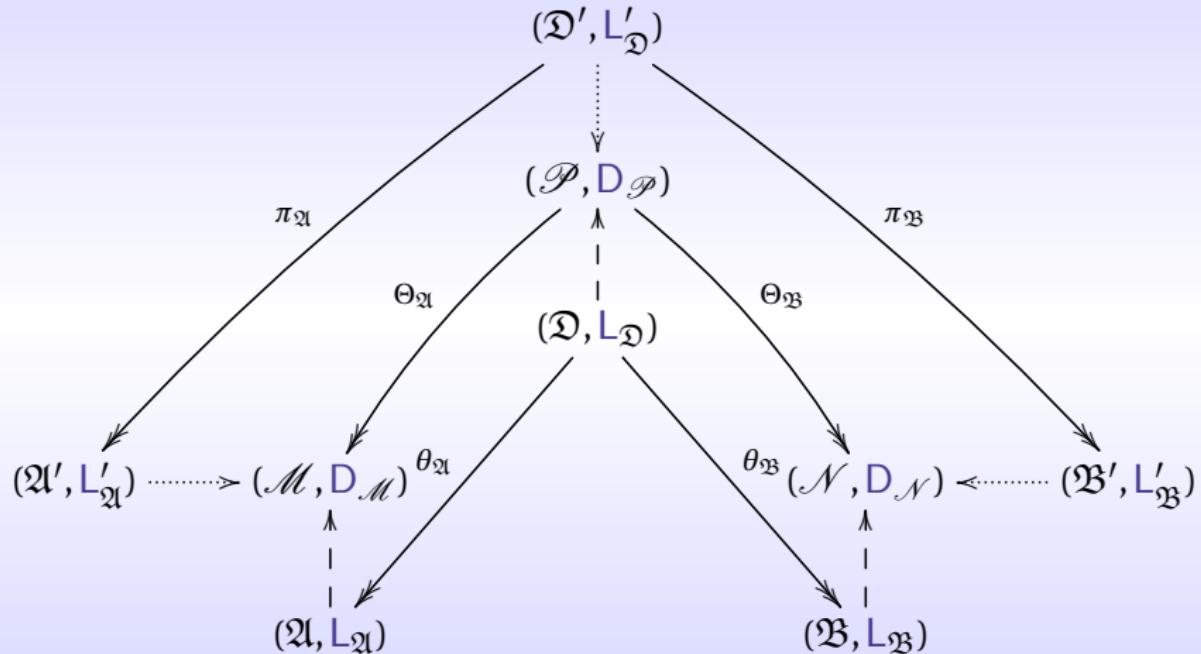
where  $\beta_\theta$  is the dual action, and for all  $\xi \in \mathcal{H}_\theta^{p, q, d}$  we set:

$$\mathsf{D}_\theta^{p, q, d}(\xi) = \sup \left\{ \frac{\left\| \alpha_{\bar{\partial}}^{x, y, \frac{xy}{2}} \xi - \xi \right\|_{\mathcal{H}_\theta^{p, q, d}}}{2\pi |\bar{\partial}| \|(x, y)\|} : (x, y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where  $\bar{\partial} = \theta - p/q$ , then:

$$\lim_{\theta \rightarrow 0} \Lambda^{\text{mod}} \left( \left( \mathcal{H}_\theta^{p, q, d}, \langle \cdot, \cdot \rangle_{\mathcal{H}_\theta^{p, q, d}}, \mathsf{D}_\theta^{p, q, d}, \mathcal{A}_\theta, \mathsf{L}_\theta \right), \right. \\ \left. \left( \mathcal{H}_\theta^{p, q, d}, \langle \cdot, \cdot \rangle_{\mathcal{H}_\theta^{p, q, d}}, \mathsf{D}_\theta^{p, q, d}, \mathcal{A}_\theta, \mathsf{L}_\theta \right) \right) = 0.$$

# Metrical Quantum Vector Bundles



*Figure:* A metrical modular tunnel

# Spectral Triples

If  $(\mathfrak{A}, \mathcal{H}, D)$  is a *metric* spectral triple, and if we set:

$$\text{qvb}(\mathfrak{A}, \mathcal{H}, D) = (\mathcal{H}, \mathbf{D}_D, \mathbb{C}, 0, \mathfrak{A}, \mathbf{L})$$

where  $\mathbf{D}_D(\xi) = \|\xi\|_{\mathcal{H}} + \|D\xi\|_{\mathcal{H}}$  and  $\mathbf{L}(a) = \|[D, a]\|_{\mathcal{H}}$  for all  $\xi \in \mathcal{H}$  and  $a \in \mathfrak{A}$ , then:

## Theorem (L., 18)

$\Lambda^{\text{mod}}(\text{qvb}(,,), \text{qvb}(,,))$  defines a pseudo-metric on metric spectral triples such that  $(\mathfrak{A}, \mathcal{H}_{\mathfrak{A}}, D_{\mathfrak{A}})$  and  $(\mathfrak{B}, \mathcal{H}_{\mathfrak{B}}, D_{\mathfrak{B}})$  are at distance zero if, and only if there exists  $U : \mathcal{H}_{\mathfrak{A}} \rightarrow \mathcal{H}_{\mathfrak{B}}$  unitary such that  $UD_{\mathfrak{A}}^2 U^* = D_{\mathfrak{B}}^2$ .

# Spectral Triples

If  $(\mathfrak{A}, \mathcal{H}, D)$  is a *metric* spectral triple, and if we set:

$$\text{qvb}(\mathfrak{A}, \mathcal{H}, D) = (\mathcal{H}, \mathsf{D}_D, \mathbb{C}, 0, \mathfrak{A}, \mathsf{L})$$

where  $\mathsf{D}_D(\xi) = \|\xi\|_{\mathcal{H}} + \|D\xi\|_{\mathcal{H}}$  and  $\mathsf{L}(a) = \|[D, a]\|_{\mathcal{H}}$  for all  $\xi \in \mathcal{H}$  and  $a \in \mathfrak{A}$ , then:

## Theorem (L., 18)

$\Lambda^{\text{mod}}(\text{qvb}(,,), \text{qvb}(,,))$  defines a pseudo-metric on metric spectral triples such that  $(\mathfrak{A}, \mathcal{H}_{\mathfrak{A}}, D_{\mathfrak{A}})$  and  $(\mathfrak{B}, \mathcal{H}_{\mathfrak{B}}, D_{\mathfrak{B}})$  are at distance zero if, and only if there exists  $U : \mathcal{H}_{\mathfrak{A}} \rightarrow \mathcal{H}_{\mathfrak{B}}$  unitary such that  $UD_{\mathfrak{A}}^2 U^* = D_{\mathfrak{B}}^2$ .

What about conserving the entire information in the spectral triple?

# *The covariant modular propinquity*

*Thank you!*

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