Notes for Introduction to Topology I

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DRAFT

ABSTRACT. This are very quick notes on some of the basic results and definitions of point set topology. These notes are a work in progress. Please note:

(1) **Draft status:** These notes are in their second main version, and are currently revised. There are various small errors left around, so be careful and email me any error you may find for correction in future versions.

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(3) **Thanks:** Several typos were found by D. Ash, R. Cardona Fuentes, G. Giron. Thanks!

Contents

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CHAPTER 1

Topological Spaces and continuous functions

1. The category of Topological spaces

DEFINITION 1.1. Let E be a set. A topology τ on E is a subset of 2^E such that:
  (1) $E \in \tau$ and $\varnothing \in \tau$.
  (2) For all $U, V \in \tau$ we have $U \cap V \in \tau$.
  (3) For all $\mathscr{U} \subseteq \tau$, we have $\bigcup \mathscr{U} \in \tau$.

A pair (E, τ) is a topological space when E is a set and τ is a topology on E. Moreover, a subset U of E is called *open* if $U \in \tau_E$ and *closed* if $\complement_E U \in \tau_E$.

The chapter on fundamental examples collect many examples of topologies. Note that $\{\varnothing, E\}$ and 2^E both are obvious topologies on any set E, but of course the examples in the next chapter are more substantial. For now, we will work the few basic properties of topologies.

PROPOSITION 1.2. *Let E be a (nonempty) set, (F, τ_F) be a topological space, and $f : E \to F$ a function. Define:*

$$\tau(f) = \{f^{-1}(V) : V \in \tau_F\}.$$

Then $(E, \tau(f))$ is a topological space. We call $\tau(f)$ the initial topology for f or the topology induced by f.

PROOF. By construction, $f^{-1}(F) = E$ and $f^{-1}(\varnothing) = \varnothing$ so $\varnothing, E \in \tau(f)$. Let $U_1, U_2 \in \tau(f)$. Then by definition, there exists $V_1, V_2 \in \tau_F$ such that $U_i = f^{-1}(V_i)$ for $i = 1, 2$. Thus:

$$U_1 \cap U_2 = f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2).$$

Hence by definition, $U_1 \cap U_2 \in \tau(f)$. Last, let $\mathscr{U} \subset \tau(f)$. By definition, each element of $\mathscr{U}$ is the preimage by f of some open set V_U in (F, τ_F). We set:

$$\mathscr{V} = \{V \in \tau_F : \exists U \in \mathscr{U} \ U = f^{-1}(V)\}.$$

Now $x \in \bigcup \mathscr{U}$ if and only if there exists $U \in \mathscr{U}$ such that $x \in U$. Now, $U \in \mathscr{U}$ if and only if there exists $V \in \mathscr{V}$ such that $U = f^{-1}(V)$. Hence $x \in \bigcup \mathscr{U}$ if and only if there exists $V \in \mathscr{V}$ such that $x \in f^{-1}(V)$, which in turn is equivalent to $x \in f^{-1}(\bigcup \mathscr{V})$. Hence:

$$\bigcup \mathscr{U} = f^{-1}(\bigcup \mathscr{V})$$

and therefore, we have shown that $\tau(f)$ is a topology on E. $\qquad\square$

DEFINITION 1.3. Let (E, τ_E) and (F, τ_F) be two topological spaces. Let $f : E \to F$ be a function. We say that f is continuous on E when:

$$\forall V \in \tau_F \ \ f^{-1}(V) \in \tau_E.$$

REMARK 1.4. It is immediate that $f : E \to F$ is continuous if and only if for all closed subset $C \subseteq F$, the set $f^{-1}(C)$ is closed as well.

REMARK 1.5. It is immediate by definition that any constant function is continuous. It is also obvious that the identity function on any topological space is continuous.

PROPOSITION 1.6. *Let (E, τ_E), (F, τ_F) and (G, τ_G) be three topological spaces. Let $f : E \to F$ and $g : F \to G$. If f and g are continuous, so is $g \circ f$.*

PROOF. Let $V \in \tau_G$. Then $g^{-1}(V) \in \tau_F$ by continuity of g. Hence $f^{-1}(g^{-1}(V)) \in \tau_E$ by continuity of f. $\qquad\square$

DEFINITION 1.7. The category whose objects are topological spaces and morphisms are continuous functions is the category of topological spaces.

DEFINITION 1.8. An isomorphism in the category of topological spaces is called an homeomorphism.

REMARK 1.9. An homeomorphism is by definition a continuous map which is also a bijection and whose inverse function is also continuous. There are continuous bijections which are not homeomorphisms (see the chapter Fundamental Examples).

2. Order on Topologies on a given set

The initial topology $\tau(f)$ induced by a function $f : E \to F$ is a subset of the topology on E if and only if f is continuous. This motivates the following definition.

DEFINITION 2.1. Let E be a set. Let τ_1 and τ_2 be two topologies on E. We say that τ_1 is finer than τ_2 and τ_2 is coarser than τ_1 when $\tau_2 \subseteq \tau_1$.

Of course, inclusion induces an order relation on topologies on a given set. A remarkable property is that any nonempty subset of the ordered set of topologies on a given set always admits a greatest lower bound.

THEOREM 2.2. *Let E be a set. Let $\mathscr{T}$ be a nonempty set of topologies on E. Then the set:*

$$\tau = \bigwedge \mathscr{T} = \{U \subseteq E : \forall \sigma \in \mathscr{T} \ \ U \in \sigma\}$$

is a topology on E and it is the greatest lower bound of $\mathscr{T}$ (where the order between topology is given by inclusion).

PROOF. We first show that τ is a topology. By definition, for all $\sigma \in \mathscr{T}$, since σ is a topology on E, we have $E, \varnothing \in \sigma$. Hence $\varnothing, E \in \tau$. Now, let $U, V \in \tau$. Let $\sigma \in \mathscr{T}$ be arbitrary. By definition of τ, we have $U \in \sigma$ and $V \in \sigma$. Therefore, $U \cap V \in \sigma$ since σ is a topology. Since σ was arbitrary in $\mathscr{T}$, we conclude that $U \cap V \in \tau$ by definition. The same proof can be made for unions. Let $\mathscr{U} \subseteq \tau$. Let $\sigma \in \mathscr{T}$ be arbitrary. By definition of τ, we have $\mathscr{U} \subseteq \sigma$. Since σ is a topology, $\bigcup \mathscr{U} \in \sigma$. Hence, as σ was arbitrary, $\bigcup \mathscr{U} \in \tau$. So τ is a topology on E. By construction, $\tau \subseteq \sigma$ for all $\sigma \in \mathscr{T}$, so τ is a lower bound for $\mathscr{T}$. Assume given a new topology ρ on E such that $\rho \subseteq \sigma$ for all σ in $\mathscr{T}$. Let $U \in \rho$. Then for all $\sigma \in \mathscr{T}$

we have $U \in \sigma$. Hence by definition $U \in \tau$. So $\rho \subseteq \tau$ and thus τ is the greatest lower bound of $\mathscr{T}$. □

COROLLARY 2.3. *Let E be a set and (F, τ_F) be a topological space. The smallest topology which makes a function $f : E \to F$ continuous is $\tau(f)$.*

PROOF. Let $\mathscr{T}$ be the set of all topologies on E such that f is continuous. By definition, $\tau(f)$ is a lower bound of $\mathscr{T}$. Moreover, $\tau(f) \in \mathscr{T}$. Hence it is the greatest lower bound. □

We can use Theorem (2.2) to define other interesting topologies. Note that trivially $\mathbf{2}^E$ is a topology, so given any $A \subseteq \mathbf{2}^E$ for some set E, there is at least one topology containing A. From this:

DEFINITION 2.4. Let E be a set. Let A be a subset of $\mathbf{2}^E$. The greatest lower bound of the set:
$$\mathscr{T}_A = \{\sigma \text{ is a topology on } E : A \subseteq \sigma\}$$
is the smallest topology on E containing A. We call it the topology induced by A on E and we denote it $\tau(A)$.

We note that since $\tau(A)$ belongs to $\mathscr{T}_A$ by construction, it is indeed the smallest topology containing A.

In general, we will want a better description of the topology induced by a particular set than the general intersection above. This is not always possible, but the concept of basis allows one to obtain useful descriptions of topologies.

3. Basis

When inducing a topology from a family $\mathscr{B}$ of subsets of some set E, the fact that $\mathscr{B}$ enjoys the following property greatly simplifies our understanding of $\tau(\mathscr{B})$.

DEFINITION 3.1. Let E be a set. A topological basis on E is a subset $\mathscr{B}$ of $\mathbf{2}^E$ such that:

(1) $E = \bigcup \mathscr{B}$,
(2) For all $B, C \in \mathscr{B}$ such that $B \cap C \neq \varnothing$, we have:
$$\forall x \in B \cap C \ \exists D \in \mathscr{B} \ x \in D \wedge D \subseteq B \cap C.$$

The main purpose for this definition stems from the following theorem:

THEOREM 3.2. *Let E be some set. Let $\mathscr{B}$ be a topological basis on E. Then the topology induced by $\mathscr{B}$ is:*
$$\tau(\mathscr{B}) = \{\bigcup \mathscr{U} : \mathscr{U} \subseteq \mathscr{B}\}.$$

PROOF. Denote, for this proof, the set $\{\bigcup \mathscr{U} : \mathscr{U} \subseteq \mathscr{B}\}$, by σ, and let us abbreviate $\tau(\mathscr{B})$ by τ. We wish to prove that $\tau = \sigma$. First, note that $\mathscr{B} \subseteq \sigma$ by construction. By definition, $\mathscr{B} \subseteq \tau$ and since τ is a topology, it is closed under arbitrary unions. Hence $\sigma \subseteq \tau$. To prove the converse, it is sufficient to show that σ is a topology. As it contains $\mathscr{B}$ and τ is the smallest such topology, this will provide us with the inverse inclusion. By definition, $\bigcup \varnothing = \varnothing$ and thus $\varnothing \in \sigma$. By assumption, since $\mathscr{B}$ is a basis, $E = \bigcup \mathscr{B}$ so $E \in \sigma$. As the union of unions

of elements in $\mathscr{B}$ is a union of elements in $\mathscr{B}$, σ is closed under arbitrary unions. Now, let U, V be elements of $\mathscr{B}$. If $U \cap V = \varnothing$ then $U \cap V \in \sigma$. Assume that U and V are not disjoints. Then by definition of a basis, for all $x \in U \cap V$ there exists $W_x \in \mathscr{B}$ such that $x \in W_x$ and $W_x \subseteq U \cap V$. So:

$$U \cap V = \bigcup_{x \in U \cap V} W_x$$

and therefore, by definition, $U \cap V \in \sigma$. We conclude that the intersection of two arbitary elements in σ is again in σ by using the distributivity of the union with respect to the intersection. □

The typical usage of this theorem is the following corollary. We shall say that a basis $\mathscr{B}$ on a set E is *a basis for a topology τ* on E when the smallest topology containing $\mathscr{B}$ is τ.

COROLLARY 3.3. *Let $\mathscr{B}$ be a topological basis for a topology τ on E. A subset U of E is in τ if and only if for any $x \in U$ there exists $B \in \mathscr{B}$ such that $x \in B$ and $B \subseteq U$.*

PROOF. We showed that any open set for the topology τ is a union of elements in $\mathscr{B}$: hence if $x \in U$ for $U \in \tau$ then there exists $B \in \mathscr{B}$ such that $x \in B$ and $B \subseteq U$. Conversely, if U is some subset of E such that for all $x \in U$ there exists $B_x \in \mathscr{B}$ such that $x \in B_x$ and $B_x \subseteq U$ then $U = \bigcup_{x \in U} B_x$ and thus $U \in \tau$. □

As a basic application, we show that:

COROLLARY 3.4. *Let (E, τ_E) and (F, τ_F) be two topological spaces. Let $\mathscr{B}$ be a basis for the topology τ_E. Let $f : E \to F$. Then f is continuous on E if and only if:*

$$\forall V \in \tau_F \ \ \forall x \in f^{-1}(V) \ \ \exists B \in \mathscr{B} \ \ x \in B \wedge B \subset f^{-1}(V).$$

COROLLARY 3.5. *Let (E, τ_E) and (F, τ_F) be two topological spaces. Let $\mathscr{B}$ be a basis for the topology τ_F. Let $f : E \to F$. Then f is continuous on E if and only if:*

$$\forall V \in \mathscr{B} \ \ f^{-1}(V) \in \tau_E.$$

PROOF. By definition, continuity of f implies 3.5. Conversely, assume 3.5 holds. Let $V \in \tau_F$. Then there exists $\mathscr{U} \subseteq \mathscr{B}$ such that $V = \bigcup \mathscr{U}$. Now by assumption, $f^{-1}(B) \in \tau_E$ for all $B \in \mathscr{U}$ and thus $f^{-1}(V) = \bigcup_{B \in \mathscr{U}} f^{-1}(B) \in \tau_E$ since τ_E is a topology. □

We leave to the reader to write the statement when both E and F have a basis.

4. Interior and Closure

Given a topological space (E, τ), an arbitrary subset A of E may be neither open nor closed. It is useful to find closest "approximations" for A which are either open or closed. The proper notions are as follows.

PROPOSITION 4.1. *Let (E, τ) be a topological space. Let $A \subseteq E$. There exists a (necessarily unique) largest open set in E contained in A and a (necessarily unique) smallest closed set in E containing A (where the order is given by inclusion).*

PROOF. Let $\mathscr{U} = \{U \in \tau : U \subseteq A\}$. Then since τ is a topology, $\bigcup \mathscr{U} \in \tau$. Moreover, by definition, if $x \in \bigcup \mathscr{U}$ then there exists $U \in \mathscr{U}$ such that $x \in U$ and since $U \subseteq A$ we conclude $x \in A$. Therefore $\bigcup \mathscr{U}$ is by construction the largest open subset of E contained in A.

The reasonning is similar for the smallest closed set. Namely, let $\mathscr{F} = \{F \subset E : \complement_E F \in \tau \wedge A \subset F\}$. Since closed sets are the complements of open sets, it follows that arbitrary intersections of closed sets is closed. Thus $\bigcap \mathscr{F}$ is a closed set, and it is the smallest containing A by construction. $\qquad\square$

DEFINITION 4.2. Let (E, τ) be a topological space. Let $A \subseteq E$. The interior of A is the largest open subset of E contained in A denoted by $\mathring{A}$. The closure of A is the smallest closed subset of E containing A and is denoted by $\overline{A}$.

Thus we always have $\mathring{A} \subseteq A \subseteq \overline{A}$. Moreover:

PROPOSITION 4.3. *Let (E, τ) be a topological space. Let $A, B \subseteq E$. Then:*
   (1) *If $A \subseteq B$ then $\mathring{A} \subseteq \mathring{B}$ and $\overline{A} \subseteq \overline{B}$,*
   (2) $\overset{\circ}{\overbrace{A \cap B}} = \mathring{A} \cap \mathring{B}$,
   (3) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

PROOF. If $A \subseteq B$ then $\mathring{A} \subseteq A \subseteq B$. Since $\mathring{A}$ is open and a subset of B, it is contained in $\mathring{B}$ by definition. Similarly, if $A \subseteq B$ then $A \subset \overline{B}$. Now, $\overline{B}$ is closed and contains A so it must contained the smallest closed subset containing A, namely $\overline{A}$.

Note that $A \cap B \subset A$ so $\overset{\circ}{\overbrace{A \cap B}} \subset \mathring{A}$. Similarly with B, so $\overset{\circ}{\overbrace{A \cap B}} \subseteq \mathring{A} \cap \mathring{B}$. On the other hand, $\mathring{A} \cap \mathring{B}$ is an open set (as the intersection of two open sets), and since $\mathring{A} \subset A$ and $\mathring{B} \subset B$, it is included in $A \cap B$. Hence it is included in the largest open subset contained in $A \cap B$, which completes this proof.

The same reasonning can be applied to the last assertion of this proposition. $\qquad\square$

The main theorem about closures is the following.

THEOREM 4.4. *Let (E, τ) be a topological space. Let $A \subseteq E$. Then $x \in \overline{A}$ if and only if for all $V \in \tau$ such that $x \in V$ we have $V \cap A \neq \varnothing$.*

PROOF. Assume first that $x \notin \overline{A}$. Then $x \in \complement_E \overline{A}$ and $\complement_E \overline{A}$ is open by definition; moreover by construction $\complement_E \overline{A} \cap A = \varnothing$ since $A \subseteq \overline{A}$ so $\complement_E \overline{A} \subseteq \complement_E A$. Conversely, let $x \in E$ and assume that there exists $V \in \tau$ such that $x \in V$ and yet $V \cap A = \varnothing$. Then $A \subseteq \complement_E V$ and $\complement_E V$ is closed, so by definition $\overline{A} \subseteq \complement_E V$, hence $x \notin \overline{A}$. $\qquad\square$

In general, one can thus write:

$$\overline{A} = \{x \in E : \forall V \in \tau \; x \in V \implies V \cap A \neq \varnothing\}.$$

The following is immediate, so we omit the proof.

COROLLARY 4.5. *Let (E, τ) be a topological space and $\mathscr{B}$ a basis for τ. Let $A \subseteq E$. Then $x \in \overline{A}$ if and only if for all $V \in \mathscr{B}$ such that $x \in V$ we have $V \cap A \neq \varnothing$.*

We can use the main theorem about closure to show that:

PROPOSITION 4.6. *Let (E, τ) be a topological space. Let $A \subseteq E$. Then $\overline{\complement_E A} = \complement_E \mathring{A}$ and $\widetilde{\mathring{\complement_E A}} = \complement_E \overline{A}$.*

PROOF. We shall only prove that $\overline{\complement_E A} = \complement_E \mathring{A}$ as both assertions are proved similarly. Note first that $\mathring{A} \subseteq A$ so $\complement_E A \subseteq \complement_E \mathring{A}$ and $\complement_E \mathring{A}$ is closed. Hence $\overline{\complement_E A} \subseteq \complement_E \mathring{A}$. Conversely, let $x \in \complement_E \mathring{A}$. Let $V \in \tau$ such that $x \in V$. Assume $V \cap \complement_E A = \varnothing$. Then $V \subseteq A$ so, as V open, $V \subseteq \mathring{A}$, which contradicts $x \in \complement_E \mathring{A}$. So $V \cap \complement_E A \neq \varnothing$, and thus $x \in \overline{\complement_E A}$ as desired. $\qquad\square$

5. Characterizations of Continuity

THEOREM 5.1. *Let (E, τ_E) and (F, τ_F) be topological spaces. Let $f : E \to F$ be given. Then f is continuous if and only if for all $A \subseteq E$, $f(\overline{A}) \subseteq \overline{f(A)}$.*

PROOF. Assume first that f is continuous. Let $A \subseteq E$. Since f is continuous, $f^{-1}(\overline{f(A)})$ is closed, and by definition of images and preimages,

$$A \subseteq f^{-1}(\overline{f(A)})$$

so $\overline{A} \subseteq f^{-1}(\overline{f(A)})$ as desired.

Let us now assume that for any $A \subseteq E$, we have $f(\overline{A}) \subseteq \overline{f(A)}$. Then let $C \subseteq F$ be a closed set. Let $D = f^{-1}(C)$. By assumption, $\overline{D} \subseteq f^{-1}(f(\overline{D})) \subseteq f^{-1}(\overline{f(D)}) \subseteq f^{-1}(\overline{C}) = f^{-1}(C) = D$. So $\overline{D} \subseteq D$, hence $D = \overline{D}$ so D is closed. Hence f is continous. $\qquad\square$

REMARK 5.2. In the next chapter on fundamental examples, one will encounter many topological spaces where singletons are not open. Note that in those cases, if $f : e \to F$ is constant, then $f(\mathring{E})$ would be empty, and thus we can not expect $f(\mathring{E}) \subseteq \widetilde{\mathring{f(E)}}$ in general for continuous functions.

THEOREM 5.3. *Let (E, τ_E) and (F, τ_F) be topological spaces. Let $f : E \to F$ be given. Then f is continuous if and only if for all $A \subseteq F$, we have $f^{-1}(\mathring{A}) \subseteq \widetilde{\mathring{f^{-1}(A)}}$.*

PROOF. Assume first that f is continuous. Let $A \subseteq F$. Then $f^{-1}(\mathring{A}) \in \tau_E$ since f is continuous. Since $f^{-1}(\mathring{A}) \subseteq f^{-1}(A)$, we conclude that $f^{-1}(\mathring{A}) \subseteq \widetilde{\mathring{f^{-1}(A)}}$ (by maximality of $\widetilde{\mathring{f^{-1}(A)}}$ among all open sets in E contained in A).

Conversely, assume that for all $A \subseteq F$, $f^{-1}(\mathring{A}) \subseteq \widetilde{\mathring{f^{-1}(A)}}$. Let $V \in \tau_F$. Then by assumption:

$$f^{-1}(V) = f^{-1}(\mathring{V}) \subseteq \widetilde{\mathring{f^{-1}(V)}} \subseteq f^{-1}(V)$$

so $f^{-1}(V) = \widetilde{\mathring{f^{-1}(V)}}$ i.e. $f^{-1}(V) \in \tau_E$. Since $V \in \tau_F$ was arbitrary, f is continuous. $\qquad\square$

THEOREM 5.4. *Let (E, τ_E) and (F, τ_F) be topological spaces. Let $f : E \to F$ be given. Then f is continuous if and only if for all $A \subseteq F$ we have $\overline{f^{-1}(A)} \subseteq f^{-1}(\overline{A})$.*

PROOF. Assume first that f is continuous. Let $A \subseteq F$. Then $A \subseteq \overline{A}$ so $f^{-1}(A) \subseteq f^{-1}(\overline{A})$. Since f is continuous, then $f^{-1}(\overline{A})$ is closed, and thus by minimality of $\overline{f^{-1}(A)}$, we have $\overline{f^{-1}(A)} \subseteq f^{-1}(\overline{A})$.

Conversely, assume that for all $A \subseteq F$ we have $\overline{f^{-1}(A)} \subseteq f^{-1}(\overline{A})$. Let $C \subseteq F$ be closed. Then:

$$\overline{f^{-1}(C)} \subseteq f^{-1}(\overline{C}) = f^{-1}(C)$$

so $f^{-1}(C) = \overline{f^{-1}(C)}$ i.e. $f^{-1}(C)$ is closed. Since C was an arbitrary closed subset of F, the function f is continuous. □

6. Topologies defined by functions

Continuity was phrased by stating the topology induced by the function is coarser than the topology on the domain. We can extend this idea to define topologies. Note that if E is endowed with the indiscrete topology 2^E then any function, to any topological space, is continuous. Hence the following definition can be made:

DEFINITION 6.1. Let E be a set. Let (F, τ_F) be a topological space. Let $\mathscr{F}$ be a nonempty set of functions from E to F. The smallest topology on E such that all the functions in $\mathscr{F}$ are continuous is the initial topology induced by $\mathscr{F}$ on E. We denote it by $\tau(\mathscr{F})$.

PROPOSITION 6.2. *Let E be a set. Let (F, τ) be a topological space. Let $\mathscr{F}$ be a nonempty set of functions from E to F. Let:*

$$\mathscr{B} = \{B_{(f_1,U_1),\ldots,(f_n,U_n)} : f_1,\ldots,f_n \in \mathscr{F}, U_1,\ldots,U_n \in \tau_F\}$$

where

$$B_{(f_1,U_1),\ldots,(f_n,U_n)} = f_1^{-1}(U_1) \cap \ldots \cap f_n^{-1}(U_n)$$

for all n-tuple $(f_1,\ldots,f_n)$ of functions in $\mathscr{F}$ and $U_1,\ldots,U_n \in \tau_F$. Then $\mathscr{B}$ is a basis for the initial topology induced by $\mathscr{F}$.

PROOF. Note that for all $f \in \mathscr{F}$, we must have $\tau(f) \subseteq \tau(\mathscr{F})$ by defintion of continuity. Hence, if τ is any topology on E such that all the functions in $\mathscr{F}$ are continuous, then $\mathscr{B} \subseteq \tau$. In particular, $\tau(\mathscr{B}) \subseteq \tau(\mathscr{F})$. On the other hand, by construction, every function in $\mathscr{F}$ is continuous for $\tau(\mathscr{B})$, so $\tau(\mathscr{F}) \subseteq \tau(\mathscr{B})$ by definition of the initial topology. Hence $\tau(\mathscr{F}) = \tau(\mathscr{B})$.

It remains to show that $\mathscr{B}$ is a basis on E. By definition, note that $E \in \mathscr{B}$ since $E = f^{-1}(F)$ for any $f \in \mathscr{F}$. Now, note that by definition, the intersection of two elements in $\mathscr{B}$ is still in $\mathscr{B}$, so $\mathscr{B}$ is a trivially a basis. □

The main two theorems regarding initial topologies describe its universal property:

THEOREM 6.3. *Let E be a set. Let (F, τ_F) and (D, τ_D) be topological spaces. Let $\mathscr{F}$ be a nonempty set of functions from E to F. A function $f : D \to E$ is continuous when E is endowed with the initial topology $\tau(\mathscr{F})$ if and only if $g \circ f$ is continuous for all $g \in \mathscr{F}$.*

PROOF. If f is continuous, then for all $g \in \mathscr{F}$ the function $g \circ f$ is continuous, as the composition of two continuous functions.

Conversely, assume that $g \circ f$ is continuous for all $g \in \mathscr{F}$. Let $g_1, \ldots, g_n \in \mathscr{F}$ and $V_1, \ldots, V_n \in \tau_F$ and set:

$$W = g_1^{-1}(V_1) \cap \ldots \cap g_n^{-1}(V_n).$$

Now, since $g_1 \circ f, \ldots, g_n \circ f$ are all continuous by assumption, $f^{-1}(g_1^{-1}(V_1)), \ldots, f^{-1}(g_n(V_n))$ are all in τ_D. Hence $f^{-1}(W) \in \tau_D$. Since W was an arbitrary set in a basis for the final topology on E, f is continuous.   □

THEOREM 6.4. *Let E be a set. Let (F, τ_F) and (D, τ_D) be topological spaces. Let $\mathscr{F}$ be a nonempty set of functions from E to F. The initial topology for $\mathscr{F}$ is the unique topology such that, given any topological space (D, τ_D) and given any function $f : D \to E$, then f is continuous if and only if $g \circ f$ is continuous for all $g \in \mathscr{F}$.*

PROOF. We showed that the initial topology for $\mathscr{F}$ has the desired property. Conversely, assume that the given property holds for some topology τ on E. In particular, let $f : E \to E$ be the identity, seen as a function from the topological space $(E, \tau(\mathscr{F}))$ into the space (E, τ). Using the specified universal property, we see that f is continuous, so $\tau(\mathscr{F})$ is finer than τ. Conversely, note that f^{-1} is also continuous for the same reason, so τ is finer than $\tau(\mathscr{F})$, so these two topologies agree. This completes the proof of this universal property.   □

Last, we can obtain a basis for initial topologies in a natural way:

THEOREM 6.5. *Let E be a set, (F, τ_F) be a topological space, $\mathscr{F}$ a nonempty set of functions from E to F. Let $\mathscr{B}$ be a basis for τ_F. Then the set:*

$$\mathscr{C} = \{f_1^{-1}(B_1) \cap \ldots \cap f_n^{-1}(B_n) : f_1, \ldots, f_n \in \mathscr{F}, B_1, \ldots, B_n \in \mathscr{B}\}$$

is a basis for the initial topology for $\mathscr{F}$.

PROOF. This is a quick computation.   □

The dual notion of initial topology is final topology:

DEFINITION 6.6. Let F be a set. Let $\mathscr{F}$ be a set of triplets (E, τ, f) where (E, τ) is a topological space, and $f : E \to F$ is a function. The final topology $\tau(\mathscr{F})$ on F is the largest topology such that all fuctions f such that $(E, \tau, f) \in \mathscr{F}$ are continuous.

In general, the union of topologies is not a topology, so the final topology definition deserves a quick comment. The set of all topologies on a fixed set form a complete lattice, whose meet operation is the intersection, and the join operation is defined by taking the smallest topology generated by the union. Using the notation of Definition (6.6), set:

$$\tau_F = \{U \subseteq F : f^{-1}(U) \in \tau\}.$$

It is easily check that τ_F is a topology and that for any topology strictly larger than τ, by definition, at least one set would not pull back by f to an open set in τ. One can check that this is the smallest topology generated by the union of all topologies which make f continuous (though this is a bit convoluted, as τ_F is the union of all these topologies and is a topology itself!).

In the following chapters, we will see that the product topology, the trace topology and the norm topology are examples of initial topologies. The quotient topology is an example of a final topology.

CHAPTER 2

Fundamental Examples

This chapter provides various examples of topological spaces which will be used all along these notes and are often at the core of the subject.

1. Trivial Topologies

DEFINITION 1.1. Let E be a set. The topology $\{\emptyset, E\}$ is the indiscrete topology on E. The topology $\mathbf{2}^E$ is the discrete topology on E.

PROPOSITION 1.2. *Let (F, τ) be a topological space. All functions from $(E, \mathbf{2}^E)$ into (F, τ) are continuous. The only continuous functions from $(E, \{\emptyset, E\})$ to F are constant if (F, τ) is T1.*

PROOF. When E is given the discrete topology, then for all open subsets V of F one has $f^{-1}(V)$ open in E. On the other hand, if E is given the indiscrete topology and (F, τ) is T1 then assume f takes two values, l and k. Then $\complement_F\{l\}$ is open, so $f^{-1}(\complement_F\{l\})$ must be open, and as it is nonempty (it contains $f^{-1}(\{k\})$), it is all of E, which is absurd. $\qquad\square$

2. Order topologies

DEFINITION 2.1. Let $(E, \leq)$ be a linearly ordered set. Let $\infty, -\infty$ be two symbols not in E. Define $\overline{E} = E \cup \{-\infty, \infty\}$ and extend $\leq$ to $\overline{E}$ by setting for all $x, y \in \overline{E}$:

$$x \leq y \iff (x \in E \wedge y \in E \wedge x \leq y) \vee (x = -\infty) \vee (y = \infty).$$

Let $<$ be the relation $\leq \cap \neq$ (i.e. $x < y \iff (x \leq y \wedge x \neq y)$) on $\overline{E}$. Let $a, b \in \overline{E}$. We define the set $(a, b) = \{x \in E : a < x \wedge x < b\}$.

DEFINITION 2.2. Let $(E, \leq)$ be a linearly ordered set. Then the topology induced by the set:

$$\mathscr{I}_E = \{(a, b) : a, b \in \overline{E}\}$$

is the order topology on E.

PROPOSITION 2.3. *The set $\mathscr{I}_E$ is a basis for the order topology on E.*

PROOF. Since E is linearly ordered, so is $\overline{E}$. It is immediate that $(a, b) \cap (c, d) = (a', d')$ if a' is the largest of a and c and d' is the smallest of b and d. $\qquad\square$

REMARK 2.4. The default topology on $\mathbb{R}$ is the order topology.

DEFINITION 2.5. Let $(E, \leq)$ be an ordered set, and $a, b \in E$. We define $[a, b] = \{x \in E : a \leq x \wedge x \leq b\}$.

3. Trace Topologies

PROPOSITION 3.1. *Let (E, τ_E) be a topological space. Let $A \subseteq E$. Then the trace topology on A induced by τ_E is the topology:*

$$\tau_A^E = \{U \cap A : U \in \tau_E\}.$$

PROOF. It is a trivial exercise to show that the above definition indeed gives a topology on A. □

Just as easy is the following observation:

PROPOSITION 3.2. *Let (E, τ) be a topological space. Let $\mathscr{B}$ be a basis for τ. Let $A \subseteq E$. Then the set $\{B \cap A : B \in \mathscr{B}\}$ is a basis for the trace topology on A induced by τ.*

PROOF. Trivial exercise. □

REMARK 3.3. The default topology on $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ is the trace topology induced by the order topology on $\mathbb{R}$. Since $\{n\} = (n - \frac{1}{2}, n + \frac{1}{2}) \cap \mathbb{Z}$ for all $n \in \mathbb{Z}$, we see that the natural topologies form $\mathbb{N}$ and $\mathbb{Z}$ are in fact the discrete topology.

REMARK 3.4. Let $A \subseteq E$, where (E, τ) is a topological space. Let $i : A \to E$ be the inclusion map. Then the trace topology is the initial topology for $\{i\}$, i.e. $\tau(i)$.

4. Product Topologies

DEFINITION 4.1. Let I be some nonempty set. Let us assume given a family $(E_i, \tau_i)_{i \in I}$ of topological spaces. A basic open set of the cartesian product $\prod_{i \in I} E_i$ is a set of the form $\prod_{i \in I} U_i$ where $\{i \in I : U_i \neq E_i\}$ is finite and for all $i \in I$, we have $U_i \in \tau_i$.

DEFINITION 4.2. Let I be some nonempty set. Let us assume given a family $(E_i, \tau_i)_{i \in I}$ of topological spaces. The product topology on $\prod_{i \in I} E_i$ is the smallest topology containing all the basic open sets.

PROPOSITION 4.3. *Let I be some nonempty set. Let us assume given a family $(E_i, \tau_i)_{i \in I}$ of topological spaces. The collection of all basic open sets is a basis on the set $\prod_{i \in I} E_i$.*

PROOF. Trivial exercise. □

REMARK 4.4. The product topology is not just the basic open sets on the cartesian products: there are many more open sets!

PROPOSITION 4.5. *Let I be some nonempty set. Let us assume given a family $(E_i, \tau_i)_{i \in I}$ of topological spaces. The product topology on $\prod_{i \in I} E_i$ is the initial topology for the the set $\{p_i : i \in I\}$ where $p_i : \prod_{j \in I} E_j \to E_i$ is the canonical surjection for all $i \in I$.*

PROOF. Fix $i \in I$. Let $V \in \tau_{E_i}$. By definition, $p_i^{-1}(V) = \prod_{j \in I} U_j$ where $U_j = E_j$ for $j \in I \setminus \{i\}$, and $U_i = V$. Hence $p_i^{-1}(V)$ is open in the product topology. As V was an arbitrary open subset of E_i, the map p_i is continuous by definition. Hence, as i was arbitrary in I, the initial topology for $\{p_i : i \in I\}$ is coarser than the product topology.

Conversely, note that the product topology is generated by $\{p_i^{-1}(V) : i \in I, V \in \tau_{E_i}\}$, so it is coarser than the initial topology for $\{p_i : i \in I\}$. This concludes this proof. $\qquad\square$

COROLLARY 4.6. *Let I be some nonempty set. Let us assume given a family $(E_i, \tau_i)_{i \in I}$ of topological spaces. Let τ be the product topology on $F = \prod_{i \in I} E_i$. Let (D, τ_D) be a topological space. Then $f : D \to F$ is continuous if and only if $p_i \circ f$ is continuous from (D, τ_D) to (E_i, τ_{E_i}) for all $i \in I$, where p_i is the canonical surjection on E_i for all $i \in I$.*

PROOF. We simply applied the fundamental property of initial topologies. $\qquad\square$

REMARK 4.7. The box topology on the cartesian product is the smallest topology containing all possible cartesian products of open sets. It is finer than the product topology in general. Since the product topology is the coarsest topology which makes the canonical projections continuous, it is the preferred one on cartesian products. Of course, both agree on finite products.

REMARK 4.8. The product topology is the default topology on a cartesian product of topological spaces.

5. Metric spaces

DEFINITION 5.1. Let E be a set. A function $d : E \times E \to [0, \infty)$ is a distance on E when:

(1) For all $x, y \in E$, we have $d(x, y) = 0$ if and only if $x = y$,
(2) For all $x, y \in E$ we have $d(x, y) = d(y, x)$,
(3) For all $x, y, z \in E$ we have $d(x, y) \leq d(x, z) + d(z, y)$.

DEFINITION 5.2. A pair (E, d) is a metric space when E is a set and d a distance on E.

The following is often useful:

PROPOSITION 5.3. *Let (E,d) be a metric space. Let $x, y, z \in E$. Then:*

$$|d(x, y) - d(x, z)| \leq d(y, z).$$

PROOF. Since $d(x, y) \leq d(x, z) + d(z, y)$ we have $d(x, y) - d(x, z) \leq d(z, y) = d(y, z)$. Since $d(x, z) \leq d(x, y) + d(y, z)$ we have $d(x, z) - d(x, y) \leq d(y, z)$. Hence the proposition holds. $\qquad\square$

DEFINITION 5.4. Let (E, d) be a metric space. Let $x \in E$ and $r \in (0, \infty) \subseteq \mathbb{R}$. The open ball of center x and radius r in (E, d) is the set:

$$B(x, r) = \{y \in E : d(x, y) < r\}.$$

DEFINITION 5.5. Let (E, d) be a metric space. The metric topology on E induced by d is the smallest topology containing all the open balls of E.

THEOREM 5.6. *Let (E, d) be a metric space. The set of all open balls on E is a basis for the metric topology on E induced by d.*

PROOF. It is enough to show that the set of all open balls is a basis. By definition, $E = \bigcup_{x \in E} B(x, 1)$. Now, let us be given $B(x, r_x)$ and $B(y, r_y)$ for some $x, y \in E$ and $r_x, r_y > 0$. If the intersection of these two balls is empty, we are done; let us assume that there exists $z \in B(x, r_x) \cap B(y, r_y)$. Let ρ be the smallest of $r_x - d(x, z)$ and $r_y - d(y, z)$. Let $w \in B(z, \rho)$. Then:

$$d(x, w) \leq d(x, z) + d(z, w) < d(x, z) + r_x - d(x, z) = r_x$$

so $w \in B(x, r_x)$. Similarly, $w \in B(y, r_y)$. Hence, $B(z, \rho) \subseteq B(x, r_x) \cap B(y, r_y)$ as desired.   □

The following theorem shows that metric topologies are minimal in the sense of making the distance functions continuous.

THEOREM 5.7. *Let (E, d) be a metric space. For all $x \in E$, the function $y \in E \mapsto d(x, y)$ is continuous on E for the metric topology. Moreover, the metric topology is the smallest topology such that all the functions in the set $\{y \mapsto d(x, y) : x \in E\}$ are continuous.*

PROOF. Fix $x \in E$. It is sufficient to show that the preimage of $[0, r)$ and (r, ∞) by $d_x : y \in E \mapsto d(x, y)$ is open in the metric topology of E, where $r \geq 0$ is arbitrary. Indeed, these intervals form a basis for the topology of $[0, \infty)$. Let $r \geq 0$ be given. Then $d_x^{-1}([0, r)) = B(x, r)$ by definition, so it is open. Moreover, it shows that the minimal topology making all these maps continuous must indeed contain the metric topology. Now, let $y \in E$ such that $d(x, y) > r$. Let $\rho = d(x, y) - r > 0$. Then if $d(w, y) < \rho$ for some $w \in E$ then:

$$d(x, y) \leq d(x, w) + d(w, y) \text{ so } d(x, y) - d(w, y) \leq d(x, w)$$

so $d(x, w) > r$. Hence

$$B(y, \rho) \subset d_x^{-1}((r, \infty))$$

for all $y \in d_x^{-1}((r, \infty))$. Therefore, $d_x^{-1}((r, \infty))$ is open, as desired, and our proposition is proven.   □

REMARK 5.8. The topology on $[0, \infty)$ is the trace topology on $[0, \infty)$ induced by the usual, i.e. the order topology on $\mathbb{R}$.

REMARK 5.9. The metric topology is the default topology on a metric space.

There are more examples of continuous functions between metric spaces. More precisely, a natural category for metric spaces consists of metric spaces and Lipschitz maps as arrows, defined as follows:

DEFINITION 5.10. Let (E, d_E), (F, d_F) be metric spaces. A function $f : E \to F$ is k-Lipschitz for $k \in [0, \infty)$ if:

$$\forall x, y \in E \ \ d_F(f(x), f(y)) \leq k d_E(x, y).$$

DEFINITION 5.11. Let (E, d_E), (F, d_F) be metric spaces. Let $f : E \to F$ be a Lipschitz function. Then the Lipschitz constant of f is defined by:

$$\mathrm{Lip}(f) = \sup \left\{ \frac{d_F(f(x), f(y))}{d_E(x, y)} : x, y \in E, x \neq y \right\}.$$

REMARK 5.12. $\mathrm{Lip}(f) = 0$ if and only if f is constant.

PROPOSITION 5.13. *Let (E, d_E), (F, d_F) be metric spaces. If $f : E \to F$ is a Lipschitz function, then it is continuous.*

PROOF. Assume f is nonconstant (otherwise the result is trivial). Let k be the Lipschitz constant for f. Let $y \in F$ and $\epsilon > 0$. Let $x \in f^{-1}(B(y, \epsilon))$. Let $z \in E$ such that $d_E(x, z) < \delta_x = \frac{\epsilon - d(f(x), y)}{k}$ (note that the upper bound is nonzero).

$$
\begin{align}
d_F(f(z), y) &\leq d_F(f(z), f(x)) + d_F(f(x), y) \tag{1} \\
&\leq k d_E(x, z) + d_F(f(x), y) \tag{2} \\
&< \epsilon - d_F(f(x), y) + d_F(f(x), y) = \epsilon. \tag{3}
\end{align}
$$

Hence $f^{-1}(B(y, \epsilon)) = \bigcup_{x \in f^{-1}(B(y, \epsilon))} B(x, \delta_x)$. So f is continuous. $\square$

REMARK 5.14. The proof of continuity for Lipshitz maps can be simplified: it is a consequence of the squeeze theorem. We refer to the chapter on metric spaces for this.

REMARK 5.15. Using Lipshitz maps as morphisms for a category of metric spaces is natural. Another, more general type of morphisms, would be uniform continuous maps, which are discussed in the compact space chapter.

6. Co-Finite Topologies

A potential source for counter-examples, the family of cofinite topologies is easily defined:

PROPOSITION 6.1. *Let E be a set. Let:*

$$
\tau_{\mathrm{cof}}(E) = \{\emptyset\} \cup \{U \subset E : \complement_E U \text{ is finite }\}.
$$

Then $\tau_{\mathrm{cof}}(E)$ is a topology on E.

PROOF. By definition, $\emptyset \in \tau_{\mathrm{cof}}(E)$. Moreover, $\complement_E E = \emptyset$ which is finite, so $E \in \tau_{\mathrm{cof}}(E)$. Let $U, V \in \tau_{\mathrm{cof}}(E)$. If U or V is empty then $U \cap V = \emptyset$ so $U \cap V \in \tau_{\mathrm{cof}}(E)$. Otherwise, $\complement_E(U \cap V) = \complement_E U \cup \complement_E V$ which is finite, since by definition $\complement_E U$ and $\complement_E V$ are finite. Hence $U \cap V \in \tau_{\mathrm{cof}}(E)$. Last, let $\mathscr{U} \subseteq \tau_{\mathrm{cof}}(E)$. Again, if $\mathscr{U} = \{\emptyset\}$ then $\bigcup \mathscr{U} = \emptyset \in \tau_{\mathrm{cof}}(E)$. Let us now assume that $\mathscr{U}$ contains at least one nonempty set V. Then:

$$
\complement_E \bigcup \mathscr{U} = \bigcap \{\complement_E U : U \in \mathscr{U}\} \subseteq \complement_E V.
$$

Since $\complement_E V$ is finite by definition, so is $\complement_E \bigcup \mathscr{U}$, which is therefore in $\tau_{\mathrm{cof}}(E)$. This completes our proof. $\square$

7. The one-point compactification of $\mathbb{N}$

Limits of sequences is a central tool in topology and this section introduces the natural topology for this concept. The general notion of limit is the subject of the next chapter.

DEFINITION 7.1. Let ∞ be some symbol not found in $\mathbb{N}$. We define $\overline{\mathbb{N}}$ to be $\mathbb{N} \cup \{\infty\}$.

PROPOSITION 7.2. *The set:*

$$\tau_{\overline{\mathbb{N}}} = \{U \subseteq \overline{\mathbb{N}} : (U \subseteq \mathbb{N}) \vee (\infty \in U \wedge \complement_{\mathbb{N}} U \text{ is finite})\}$$

is a topology on $\overline{\mathbb{N}}$.

PROOF. By definition, $\varnothing \subseteq \mathbb{N}$ so $\varnothing \in \tau_{\overline{\mathbb{N}}}$. Moreover $\complement_{\overline{\mathbb{N}}} \overline{\mathbb{N}} = \varnothing$ which has cardinal 0 so $\overline{\mathbb{N}} \in \tau_{\overline{\mathbb{N}}}$. Let $U, V \in \tau_{\overline{\mathbb{N}}}$. If either U or V is a subset of $\mathbb{N}$ then $U \cap V$ is a subset of $\mathbb{N}$ so $U \cap V \in \tau_{\overline{\mathbb{N}}}$. Othwiwse, $\infty \in U \cap V$. Yet $\complement_{\overline{\mathbb{N}}}(U \cap V) = \complement_{\overline{\mathbb{N}}} U \cup \complement_{\overline{\mathbb{N}}} V$ which is finite as a finite union of finite sets. Hence $U \cap V \in \tau_{\overline{\mathbb{N}}}$ again.

Last, assume that $\mathscr{U} \subseteq \tau_{\overline{\mathbb{N}}}$. Of course, $\infty \in \bigcup \mathscr{U}$ if and only if $\infty \in U$ for some $U \in \mathscr{U}$. So, if $\infty \notin \bigcup \mathscr{U}$ then $\bigcup \mathscr{U} \in \tau_{\overline{\mathbb{N}}}$ by definition. If, on the other hand, $\infty \in \bigcup \mathscr{U}$, then there exists $U \in \mathscr{U}$ with $\complement_{\overline{\mathbb{N}}} U$ finite. Now, $\complement_{\overline{\mathbb{N}}} \bigcup \mathscr{U} = \bigcap \{\complement_{\overline{\mathbb{N}}} V : V \in \mathscr{U}\} \subseteq \complement_{\overline{\mathbb{N}}} U$ so it is finite, and thus again $\bigcup \mathscr{U} \in \tau_{\overline{\mathbb{N}}}$. □

CHAPTER 3

Limits

1. Topological Separation and Hausdorff spaces

The general definition of topology allows for examples where elements of a topological space, seen as a set, can not be distinguished from each others (for instance if the topology is indiscrete). When points can be topologically differentiated, a topology is in some sense separated. There are many axioms, or definitions, of separability, and we will use the most common and intuitive: namely, the notion of Hausdorff spaces. We do however present a few basic notions in this section which are weaker than Hausdorff separation, as such spaces are certainly common in mathematics.

DEFINITION 1.1. Let (E, τ) be a topological space. We say that τ is T0 when given any two points $x, y \in E$ with $x \neq y$, there exists an open set $U \in \tau$ such that either $x \in U, y \notin U$ or $y \in U, x \notin U$.

Thus a space is T0 when there are enough open sets to separate the points, i.e. when the set of all open sets containing one point is not the same as the set of all open sets containing a different point. However, this notion is not symmetric. The following definition add that the separation property should be symmetric:

DEFINITION 1.2. Let (E, τ) be a topological space. We say that τ is T1 when given any two points $x, y \in E$ with $x \neq y$, there exists two open sets $U, V \in \tau$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$.

A key advantage of T1 separation is:

PROPOSITION 1.3. Let (E, τ) be a topological space. Then τ is T1 if and only if for all $x \in E$ the set $\{x\}$ is closed.

PROOF. Assume τ is T1. Fix $x \in E$. Let $y \in E \setminus \{x\}$. Then there exists $U_y \in \tau$ such that $x \notin U_y$ and $y \in U_y$. We can thus write:

$$\complement_E \{x\} = \bigcup_{y \in E \setminus \{x\}} U_y$$

which shows that $\complement_E \{x\}$ is open as desired.

Assume now that all singletons are closed. Let $x, y \in E$ with $x \neq y$. Then $x \in \complement_E \{y\}$ and $y \in \complement_E \{x\}$, i.e. τ is T1. $\qquad\square$

Note that T0 is not enough for the above result.

COROLLARY 1.4. Let (E, τ) be a topological space. (E, τ) is T1 if and only if for all $x \in E$ we have $\{x\} = \bigcap \{U \in \tau : x \in U\}$.

PROOF. Assume (E, τ) is T1. Let $S = \bigcap \{U \in \tau : x \in U\}$. Let $y \in S$. If $y \neq x$ then there exists $U \in \tau$ such that $x \in U$ and $y \notin U$. This is a contradiction. So $S = \{x\}$.

Assume now that for all $x \in E$ we have $\{x\} = \bigcap \{U \in \tau : x \in U\}$. Let $x, y \in E$ such that $x \neq y$. Then:

$$\bigcap \{U \in \tau : x \in U\} \nsubseteq \bigcap \{U \in \tau : y \in U\}$$

, so there exists $U \in \tau$ such that $x \in U$ and $y \notin U$. Similarly:

$$\bigcap \{U \in \tau : y \in U\} \nsubseteq \bigcap \{U \in \tau : x \in U\}$$

so there exists $U \in \tau$ such that $y \in U$ and $x \notin U$. Hence (E, τ) is T1. $\qquad \square$

EXAMPLE 1.5. The indiscrete topology is not T0.

EXAMPLE 1.6. Let E be an infinite set (for instance $E = \mathbb{R}$), endowed with the cofinite topology $\tau = \tau_{\mathrm{cof}}(E)$. By defintion, $\{x\}$ is closed for all $x \in E$ so (E, τ) is T1. We make a useful observation. Let $f : E \to \mathbb{R}$ be a continuous map, where the codomain is endowed with the usual order topology. Assume f is not constant: then there exists $x, y \in E$ such that $f(x) \neq f(y)$. Without loss of generality, we assume $f(x) < f(y)$. Let $r = \frac{1}{2}(f(y) - f(x))$. Let $U = (f(x) - r, f(x) + r)$ and $V = (f(y) - r, f(y) + r)$. Then $U \cap V = \varnothing$ and U, V are open sets in $\mathbb{R}$ such that $x \in U$ and $y \in V$. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open in E, i.e. are cofinite. Since U and V are disjoint, we conclude that $f^{-1}(U)$ is in the complement of $f^{-1}(V)$, which is finite. Since the complement of $f^{-1}(U)$ is finite as well, we conclude that E is finite, which is a contradiction. So f is a constant.

The cofinite topology on infinite set example shows that T1 still allows for counter intuitive situations. We also saw that we could find two disjoint open sets in $\mathbb{R}$ containing given distinct points: this stronger property is the separation axiom we will focus our attention to.

DEFINITION 1.7. Let (E, τ) be a topological space. We say that (E, τ) is a Hausdorff space (or T2) when for any $x, y \in E$ such that $x \neq y$ there exists $U, V \in \tau$ such that $U \cap V = \varnothing$, $x \in U$ and $y \in V$.

PROPOSITION 1.8. *If (E, τ) is Hausdorff, then it is T1.*

PROOF. This result holds by definition. $\qquad \square$

EXAMPLE 1.9. Let $(E, \leq)$ be a linearly ordered set and let τ be the associated order topology on E. Then (E, τ) is Hausdorff. Indeed, let $x, y \in E$ with $x \neq y$; without loss of generality we assume that $x < y$. Then if there exists $z \in E$ such that $x < z < y$ then $x \in (-\infty, z)$ and $y \in (z, \infty)$, where both intervals are disjoint and open. Otherwise, $(-\infty, y) \cap (x, \infty) = \varnothing$ and $x \in (-\infty, y), y \in (x, \infty)$.

EXAMPLE 1.10. In particular, $\mathbb{R}$ is Hausdorff.

EXAMPLE 1.11. Let (E, τ) be a Hausdorff space. Then by definition, it is immediate that the trace topology induced on any subset of E is also Hausdorff.

EXAMPLE 1.12. In particular, the topology on $\mathbb{Q}$ is Hausdorff.

EXAMPLE 1.13. The discrete topology on any set is always Hausdorff.

EXAMPLE 1.14. In particular, the topology on $\mathbb{Z}$ and $\mathbb{N}$ is Hausdorff.

EXAMPLE 1.15. Let (E, d) be a metric space. Then (E, τ_d) is Hausdorff. Indeed, let $x, y \in E$ such that $x \neq y$. Let $r = \frac{1}{2} d(x, y)$ and note that $r > 0$ by definition of a distance. Let $z \in B(x, r)$. Then:

$$d(x, y) \leq d(x, z) + d(z, y) \text{ so } r < d(z, y)$$

and by symmetry, if $d(w, y) < r$ then $d(w, x) > r$. Hence $B(x, r) \cap B(y, r) = \emptyset$.

EXAMPLE 1.16. Let $(E_i, \tau_i)_{i \in I}$ be some family of Hausdorff topological spaces. Then the cartesian product with the product topology is Hausdorff. Indeed, let $\mathbf{x} = (x_i)_{i \in I}$ and $\mathbf{y} = (y_i)_{i \in I}$ such that $\mathbf{x} \neq \mathbf{y}$. By definition, there exists $i_0 \in I$ such that $x_{i_0} \neq y_{i_0}$. Since E_{i_0} is Hausdorff, there exists U_{i_0}, V_{i_0} such that $x_{i_0} \in U_{i_0}$, $y_{i_0} \in V_{i_0}$ and $U_{i_0} \cap V_{i_0} = \emptyset$. Define the two families:

$$\mathbf{u} : i \in I \mapsto \begin{cases} E_i & \text{if } i \neq i_0, \\ U_{i_0} & \text{otherwise,} \end{cases}$$

and

$$\mathbf{v} : i \in I \mapsto \begin{cases} E_i & \text{if } i \neq i_0, \\ V_{i_0} & \text{otherwise,} \end{cases}$$

and set $U = \prod_{i \in I} \mathbf{u}_i$ and $V = \prod_{i \in I} \mathbf{v}_i$. Then by construction, $\mathbf{x} \in U$, $\mathbf{y} \in V$ and $U \cap V = \emptyset$.

The following result is a characterization of Hausdorff separation.

THEOREM 1.17. Let (E, τ) be a topological space. Then (E, τ) is Hausdorff if and only if $\Delta = \{(x, x) : x \in E\}$ is closed in E^2 for the product topology.

PROOF. Assume E is Hausdorff. Let $(x, y) \in \complement_{E^2} \Delta$ (so $x \neq y$!). Then there exists two disjoint open sets U and V in E such that $x \in U$ and $y \in V$. If $(z, z) \in U \times V$ then $z \in U$ and $z \in V$ which is impossible since U and V are disjoint. So $U \times V$, which is open in the product topology, contains (x, y) by definition, and is a subset of $\complement_{E^2} \Delta$. Hence $\complement_{E^2} \Delta$ is open.

Conversely, assume $\complement_{E^2} \Delta$ is open. Let $(x, y) \in \complement_{E^2} \Delta$. Since basic open sets form a basis for the product topology, there exists U, V open in E such that $(x, y) \in U \times V$ and $U \times V \subseteq \complement_{E^2} \Delta$. Now, by definition, $(U \times V) \cap \Delta = \emptyset$ so as above, $U \cap V = \emptyset$, as desired. So (E, τ) is Hausdorff. $\square$

Hausdorff spaces have a nice relation with continuous maps as well.

PROPOSITION 1.18. Let (E, τ_E) be a topological space. Let (F, τ_F) be a Hausdorff topological space. Let $f : E \to F$ and $g : E \to F$ be given.

(1) If f is continuous, then the kernel $\ker(f) = \{(x, y) \in E^2 : f(x) = f(y)\}$ of f is closed.
(2) If f is continuous, then the graph $\operatorname{graph}(f) = \{(x, f(x)) \in E \times F\}$ of f is closed.
(3) If f and g are continuous, then the equalizer $\operatorname{eq}(f, g) = \{x \in E : f(x) = g(x)\}$ of f, g is closed.

PROOF. Since (F, τ_F) is Hausdorff, the set $\Delta = \{(x, x) : x \in F\}$ is closed. We first prove that the kernel of f is closed when f is continuous. Let:

$$\kappa : E \times E \longrightarrow F \times F$$
$$(x, y) \longmapsto (f(x), f(y)).$$

The function $(x, y) \in E \times E \mapsto x$ is continuous since the product topology is the initial topology for the canonical surjections, and f is continuous by assumption, so $(x, y) \in E \times E \mapsto f(x)$ is continuous as the composition of two continuous functions. Similarly, $(x, y) \in E \times E \mapsto f(y)$ is continuous. Hence, by the universal property of the initial topology, κ is continuous.

Now, by definition, $\text{kernel}(f) = \kappa^{-1}(\Delta)$ and thus it is closed.

We now prove that the graph of f is closed when f is continuous. Let:

$$\delta : E \times F \longrightarrow F \times F$$
$$(x, y) \longmapsto (f(x), y).$$

By a similar argument as for κ, the map δ is continuous since $(x, y) \in E \times F \mapsto y$ and $(x, y) \in E \times F \mapsto x$ are continuous for the product topology, and f is continuous, so by composition of continuous functions, $(x, y) \in E \times F \mapsto f(x)$ is continuous. Hence by the universal property of the product topology (seen again as the initial topology for the canonical surjections), δ is continuous.

Now $\text{graph}(f) = \delta^{-1}(\Delta)$ so the graph of f is closed.

Last, we show the equalizer of f and g is closed when both f and g are continuous. Let:

$$\eta : E \longrightarrow F \times F$$
$$x \longmapsto (f(x), g(x)).$$

By assumption, f and g are continuous, so by the universal property of the product topology, η is continuous. So $\text{eq}(f, g) = \eta^{-1}(\Delta)$ is closed. $\qquad\square$

PROPOSITION 1.19. *Let (E, τ_E) and (F, τ_F) be two topological spaces. Let $f : E \to F$.*

(1) *If f open (i.e. $U \in \tau_E \implies f(U) \in \tau_F$) and a surjection, and if the kernel of f is closed then (F, τ_F) is Hausdorff.*
(2) *If f is a continuous, open surjection, then the kernel of f is closed if and only if (F, τ_F) is Hausdorff.*

PROOF. Let $x, y \in F$ with $x \neq y$. Since f is surjective, there exists $w, z \in E$ such that $f(w) = x$ and $f(z) = y$. Since $x \neq y$, (w, z) is in the complement of the kernel of f, which is open by assumption. Since $\{U \times V : U, V \in \tau_E\}$ is a basis for the product topology on E^2, there exists $U_w, U_z \in \tau_E$ such that $(w, z) \in U_w \times U_z$ and $U_w \times U_z$ is a subset of the complement of the kernel of f. Let $t \in f(U_w)$ and $r \in f(U_z)$. If $f(r) = f(t)$ then (t, r) lies in the kernel of f and in $U_w \times U_z$, which is a contradiction. Hence $V_x = f(U_w)$ and $V_y = f(U_z)$ are disjoint. Since f is open, they are open. By construction, $x \in V_x$ and $y \in V_y$. So (F, τ_F) is Hausdorff.

Assume now that f is a continuous, open surjection. We just proved that since f is an open surjection, if its kernel is closed then (F, τ_F) is closed. Conversely,

if f is continuous and (F, τ_F) is Hausdorff then its kernel is closed. Hence the equivalence stated. □

REMARK 1.20. We offer an alternative proof of the first assertion of the previous proposition when f is assumed to be open and *bijective*. We keep the notations used in that proposition.

Assume that f is an open bijection and $\ker(f)$ is closed. Since f is bijective, it has a right inverse $g : F \to E$. If U is open in E then $g^{-1}(U) = f(U)$ is open, so g is continuous. Moreover, letting $\Delta = \{(y, y) : y \in F\}$ and since f is surjective, we have:

$$\Delta = f(\text{kernel}(f)) = g^{-1}(\text{kernel}(f))$$

which is therefore closed by assumption, as the preimage of the closed set $\text{kernel}(f)$ by the continuous function g. Hence (F, τ_F) is Hausdorff.

REMARK 1.21. An open map may not be continuous. For instance, a non-constant map from $\{0, 1\}$ with the indiscrete topology to $\{0, 1\}$ with the discrete topology is always open but never continuous.

2. Limits along a set

Dealing with continuity can be complicated from the original definition. It is easier to introduce limits and the notion of continuity at a point. We first introduce a piece of vocabulary:

DEFINITION 2.1. Let (E, τ_E) be a topological space. Let $a \in E$. The set of open neighborhoods of a in τ_E is the set $\mathscr{V}_{\tau_E}(a)$ of all $U \in \tau_E$ such that $a \in U$.

DEFINITION 2.2. Let (E, τ_E) and (F, τ_F) be topological spaces. Assume (F, τ_F) is a Hausdorff space. Let $A \subseteq E$. Let $a \in \overline{A}$ and $l \in F$. Let $f : E \to F$. We say that f has limit l at a along A when:

$$\forall V \in \mathscr{V}_{\tau_F}(l) \; \exists U \in \mathscr{V}_{\tau_E}(a) \;\; f(U \cap A) \subseteq V.$$

REMARK 2.3. If $a \notin \overline{A}$ then the notion is silly, since we could find one open set U containing a and so that $U \cap A = \varnothing$ in the above definition.

PROPOSITION 2.4. *Let (E, τ_E) and (F, τ_F) be topological spaces. Assume (F, τ_F) is a Hausdorff space. Let $A \subseteq E$. Let $a \in \overline{A}$ and $l \in F$. Let $f : E \to F$ and $g : E \to F$. Assume that $f(x) = g(x)$ for all $x \in A$. Then f has limit l at a along A if and only if g has limit l at a along A.*

PROOF. Simply observe that the definition of limit involves only $f(\cdot \cap A)$ and $g(\cdot \cap A)$. □

We thus can define without ambiguity the limit of a partially defined function at a point in the closure of its domain:

DEFINITION 2.5. Let (E, τ_E) and (F, τ_F) be topological spaces. Assume $A \subseteq E$ is nonempty and $f : A \to F$. Then f has limit l at $a \in \overline{A}$ along A if any extension of f to E has limit l at a along A.

PROPOSITION 2.6. *Let (E, τ_E) and (F, τ_F) be topological spaces. Assume (F, τ_F) is a Hausdorff space. Let $B \subseteq A \subseteq E$. Let $a \in \overline{B}$ and $l \in F$. Let $f : E \to F$. Then if f has limit l at a along A then f has limit l at a along B.*

PROOF. First, since $B \subseteq A$, we have $\overline{B} \subseteq \overline{A}$, so $a \in \overline{B}$ implies that $a \in \overline{A}$, hence the notion of limits are well-defined. Let $V \in \mathscr{V}_{\tau_F}(l)$. Since $\lim_{x \to a, x \in A} f(x) = l$, there exists $U \in \mathscr{V}_{\tau_E}(a)$ such that $f(U \cap A) \subseteq V$. Since $B \subseteq A$ we have $f(U \cap B) \subseteq f(U \cap A) \subseteq V$ as desired. $\qquad\square$

PROPOSITION 2.7. *Let (E, τ_E) and (F, τ_F) be topological spaces. Assume (F, τ_F) is a Hausdorff space. Let $A \subseteq E$. Let $a \in \overline{A}$ and $l \in F$. Let $f : E \to F$. If f has a limit at a along A then this limit is unique and denoted by $\lim_{x \to a, x \in A} f(x)$.*

PROOF. Assume f has limit l and l' with $l \neq l'$ at a along A. Since F is Hausdorff, there exists V, V' open in F such that $l \in V$ and $l' \in V'$. Then by definition of limits, there exists U, U' open in E so that $a \in U$, $a \in U'$ and $f(U \cap A) \subseteq V$ and $f(U' \cap A) \subseteq V'$. So $f(A \cap U \cap U') \subseteq V \cap V' = \varnothing$. This is absurd since $a \in U \cap U'$ and $a \in \overline{A}$ so $A \cap U \cap U' \neq \varnothing$. $\qquad\square$

PROPOSITION 2.8. *Let (E, τ_E) and (F, τ_F) be topological spaces. Assume (F, τ_F) is a Hausdorff space. Let $A \subseteq E$. Let $a \in \overline{A}$ and $l \in F$. Let $f : E \to F$. If f has a limit l at a along A then $l \in \overline{f(A)}$.*

PROOF. Let $l = \lim_{x \to a, x \in A} f(x)$. Let $V \in \tau_F$ such that $l \in V$. Then by definition of limit, there exists $U \in \tau_E$ such that $a \in U$ and $f(U \cap A) \subseteq V$. By definition, $f(U \cap A) \subset f(A)$. Since $a \in \overline{A}$, we have $\varnothing \neq A \cap U$ so $\varnothing \neq f(A \cap U) \subseteq V \cap f(A)$. So $l \in \overline{f(A)}$. $\qquad\square$

REMARK 2.9. In general, it is difficult to compute $f(A)$, so the result is used as follows: if $f(A) \subseteq B$ then $\lim_{x \to a, x \in A} f(x) \in \overline{B}$.

It is unpractical to check a statement for all open sets, as in general they are difficult to describe. The use of a basis makes things more amenable, and in fact it is the main role of basis.

PROPOSITION 2.10. *Let (E, τ_E) and (F, τ_F) be topological spaces, with (F, τ_F) Hausdorff. Assume that we are given a basis $\mathscr{B}_E$ for τ_E and a basis $\mathscr{B}_F$ for τ_F. Let $A \subseteq E$, $a \in \overline{A}$ and $l \in F$. Let $f : E \to F$. Let us denote by $\mathscr{B}_l$ the subset of all $B \in \mathscr{B}_F$ such that $l \in B$, and we define $\mathscr{B}_a$ similarly. Then f has limit l at a along A if and only if:*

$$\forall B \in \mathscr{B}_l \ \exists C \in \mathscr{B}_a \ f(C \cap A) \subset B.$$

PROOF. Assume 2.10. Let $V \in \tau_F$ such that $l \in V$. By definition, there exists $B \in \mathscr{B}_l$ such that $l \in B$ and $B \subseteq V$. By 2.10 there exists $C \in mathcalB_a$, i.e. an open set in E containing a, such that $f(C \cap A) \subseteq B \subseteq V$. This shows that f converges to l at a along A.

Conversely, assume that f converges to l at a along A. Let $B \in \mathscr{B}_l$. By definition, B is an open set in F containing l so there exists $U \in \tau_E$ with $a \in U$ such that $f(U \cap A) \subseteq B$ by definition of limit. There exists $C \in \mathscr{B}_E$ such that $a \in C$ and $C \subseteq U$ as $\mathscr{B}_E$ is a basis for τ_E. Hence, $f(C \cap A) \subset f(C \cap U) \subset C$ as desired. $\qquad\square$

PROPOSITION 2.11. *Let (E, τ_E) and (F, τ_F) be topological spaces, with (F, τ_F) T1. Let $f : E \to F$, $A \subseteq E$. If $a \in A$ and if f has a limit at a along A then this limit is $f(a)$.*

PROOF. Assume l is the limit of f at a along A and $l \neq f(a)$. Then since τ_F is T1, there exists $V \in \tau_F$ such that $l \in V, f(a) \notin V$. By definition of limit, there exists $U \in \tau_E$ with $a \in U$ and $f(U \cap A) \subseteq V$. Now, $a \in A \cap U$ so $f(a) \in f(A \cap U) \subseteq V$ which is absurd.   □

PROPOSITION 2.12. *Let (E, τ_E) be a topological space. Let (F, τ_F) be a Hausdorff topological space. Let $A, A' \subseteq E$. Let $a \in \overline{A} \cap \overline{A'}$. Then f has limit l at a along $A \cup A'$ if and only if f has limit l at a along A and along A'.*

PROOF. Note that $a \in \overline{A} \cap \overline{A'}$ implies that $a \in \overline{A \cup A'}$. The condition is necessary since $A \subseteq A \cup A'$ and $A' \subseteq A \cup A'$. Conversely, let $V \in \mathcal{V}_{\tau_F}(l)$. By definition of limits along A and A' there exist $U, U' \in \mathcal{V}_{\tau_E}(a)$ such that $f(A \cap U) \subseteq V$ and $f(A' \cap U') \subseteq V'$. Hence $f((U \cap U') \cap (A \cup A')) \subseteq V$. Since $U \cap U' \in \mathcal{V}_{\tau_E}(a)$, the proof is complete.   □

THEOREM 2.13. *Let (E, τ_E), (F, τ_F) and (G, τ_G) be three topological spaces, where τ_F and τ_G are Hausdorff. Let $f : E \rightarrow F$ and $g : F \rightarrow G$. Let $A \subseteq E$, $a \in \overline{A}$. Set $B = f(A)$. If b is the limit of f at a along A and l is the limit of g at b along B then l is the limit of $g \circ f$ at a along A.*

PROOF. Note first that $b \in \overline{B}$, as necessary to make sense of the statement of the theorem. Let $W \in \tau_G$ with $l \in W$. There exists $V \in \tau_F$ with $b \in V$ such that $g(V \cap B) \subseteq W$. Now, there exists $U \in \tau_E$ such that $a \in U$ and $f(U \cap A) \subseteq V$. By construction, $f(A \cap U) \subseteq B$ so $f(A \cap U) \subseteq B \cap V$. Hence, $g(f(U \cap A)) \subseteq W$ as desired.   □

REMARK 2.14. Beware of this theorem. Take $f(x) = x \sin(\frac{1}{x})$, $g(x) = 0$ for $x \neq 0$ and $g(0) = 1$, and $A = B = \mathbb{R} \setminus \{0\}$ and $a = b = 0$. Then:

$$0 = \lim_{x \to 0, x \neq 0} f(x) = \lim_{y \to 0, y \neq 0} g(y)$$

yet $g \circ f$ has no limit at 0 along the set of nonzero reals.

3. Continuity at a point

DEFINITION 3.1. Let (E, τ_E) and (F, τ_F) be topological spaces. Let $a \in E$. Then f is continuous at a if:

$$\forall V \in \mathcal{V}_{\tau_F}(f(a)) \exists U \in \mathcal{V}_{\tau_E}(a) \ f(U) \subseteq V.$$

The notion of continuity at a point looks more familiar when the codomain is Hausdorff.

PROPOSITION 3.2. *Let (E, τ_E) and (F, τ_F) be topological spaces. Assume (F, τ_F) is Hausdorff. Let $a \in E$. Then f is continuous at a if:*

$$\lim_{x \to a, x \in E} f(x) = f(a)$$

.

The main connection between continuity between spaces and continuity at a point is given in the following key theorem.

THEOREM 3.3. *Let (E, τ_E) and (F, τ_F) be two topological spaces. The function f is continuous at every $x \in E$ if and only if f is continuous on E.*

PROOF. Assume first that f is continuous at every $x \in E$. Let $V \in \tau_F$. Let $x \in f^{-1}(V)$. Then f is continuous at x so there exists $U_x \in \tau_E$ such that $x \in U_x$ and $f(U_x) \subseteq V$ (since $f(x) \in V$). Consequently, $U_x \subseteq f^{-1}(V)$. Hence:

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$$

and thus $f^{-1}(V)$ is open. So f is continuous on E.

Conversely, assume f is continuous on E. Let $x \in E$. Let $V \in \tau_F$ such that $f(x) \in V$. Then $f^{-1}(V)$ is open in E by continuity of f on E, and it contains x. Let $U = f^{-1}(V)$. Then $f(U) \subseteq V$. This concludes the proof.   □

4. Limit of Sequences

We refer to the chapter Fundamental Examples for the topology on the one point compactification $\overline{\mathbb{N}}$ of $\mathbb{N}$.

DEFINITION 4.1. Let E be a set. A sequence in E is a function from $\mathbb{N}$ to E.

NOTATION 4.2. Sequences are denoted as families. Namely, if x is sequence in E, then we write x_n for $x(n)$ for $n \in \mathbb{N}$, and we usually identify x with $(x_n)_{n \in \mathbb{N}}$.

REMARK 4.3. By abuse of language, we will also call sequences functions from a subset $\{n \in \mathbb{N} : n \geq N\}$ of $\mathbb{N}$, for some $N \in \mathbb{N}$. The obvious re-indexing is left implicit.

The definition of limit for functions apply to sequences.

DEFINITION 4.4. Let (E, τ) be a topological space. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in E. Then we say that $(x_n)_{n \in \mathbb{N}}$ converges to $l \in E$ when the limit of $(x_n)_{n \in \mathbb{N}}$ has limit l at ∞ along $\mathbb{N}$ in $\tau_{\overline{\mathbb{N}}}$.

REMARK 4.5. We learnt that if two functions agree on some set A inside of a topological space E, then their limits along a agree (including whether they exist!). So the previous definition is understood as follows: choose any extension of a sequence to $\overline{\mathbb{N}}$ (i.e. pick some element x_∞ in E). Then take its limit at ∞ in $\overline{\mathbb{N}}$ along $\mathbb{N}$. Then the existence and, if applicable, the value of this limit is independant of the choice the extension of our sequence. It is the limit of the sequence.

PROPOSITION 4.6. *Let (E, τ) be a Hausdorff topological space. A sequence $(x_n)_{n \in \mathbb{N}}$ converges to $l \in E$ if and only if:*

$$\forall V \in \mathscr{V}_{\tau_E}(l) \ \ \exists N \in \mathbb{N} \ \ \forall n \geq N \ \ x_n \in V.$$

PROOF. Assume that $(x_n)_{n \in \mathbb{N}}$ has limit l. Let $V \in \mathscr{V}_{\tau_E}(l)$. By definition of limits and of the topology $\tau_{\overline{\mathbb{N}}}$, there exists a finite set S in $\mathbb{N}$ such that $x_n \in V$ for all $n \in \mathbb{N} \setminus F$. Let N be the successor of the greatest element in F if F is nonempty, or 0 otherwise. Then for all $n \geq N$ we have $x_n \in V$ as desired.

The converse is obvious.   □

DEFINITION 4.7. A subsequence of a sequence $(x_n)_{n \in \mathbb{N}}$ in a set E is a sequence of the form $(x_{\phi(n)})_{n \in \mathbb{N}}$ for some strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$.

LEMMA 4.8. *Let $\phi : \mathbb{N} \to \mathbb{N}$ be strictly increasing. Then for all $n \in \mathbb{N}$ we have $\phi(n) \geq n$.*

PROOF. By definition, $\phi(0) \geq 0$. Assume that for some $n \in \mathbb{N}$ we have $\phi(n) \geq n$. Then by assumption, $\phi(n+1) > \phi(n) \geq n$ so $\phi(n+1) \geq n+1$. The lemma holds by the theorem of induction. $\square$

PROPOSITION 4.9. *Let (E, τ) be a topological space. If a sequence $(x_n)_{n \in \mathbb{N}}$ in E converges to l then all subsequences of $(x_n)_{n \in \mathbb{N}}$ converge to l.*

PROOF. Assume that $(x_n)_{n \in \mathbb{N}}$ converges to l and let $\phi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function. Let $V \in \mathscr{V}_\tau(l)$. By assumption, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n \in V$. Therefore, $x_{\phi(n)} \in V$ since $\phi(n) \geq n$. Hence the subsequence $(x_{\phi(n)})_{n \in \mathbb{N}}$ converges to l as well. $\square$

THEOREM 4.10. *Let (E, τ) be a topological space. A sequence $(x_n)_{n \in \mathbb{N}}$ in E converges to l if and only if every subsequence of $(x_n)_{n \in \mathbb{N}}$ has a subsequence which converges to l.*

PROOF. The condition is necessary by the previous result. Let us show it is sufficient. Assume that $(x_n)_{n \in \mathbb{N}}$ does not converge to l. Then there exists $V \in \mathscr{V}_\tau(l)$ such that for all $N \in \mathbb{N}$ there exists $n \geq N$ such that $x_n \notin V$. Set $\phi(0)$ to be the smallest $n \in \mathbb{N}$ such that $x_n \notin V$. Assume we have constructed $\phi(0) < \cdots < \phi(N)$ for some $N \in \mathbb{N}$ such that $x_{\phi(k)} \notin V$ for $k = 0, \ldots, N$. Then let $A_N = \{n \in \mathbb{N} : n \geq \phi(N) + 1\}$. Then $A_N \neq \varnothing$, so it has a smallest element $\phi(N+1)$. Note that by construction, $\phi(N+1) > \phi(N)$ and $x_{\phi(N+1)} \notin V$. Hence we have constructed a subsequence $(x_{\phi(n)})_{n \in \mathbb{N}}$ entirely contained in $\complement_E V$. By construction, it has no subsequence which converges to l. $\square$

CHAPTER 4

Limits and Continuity in Metric Spaces

In this section, the topology on any metric space is meant as the metric topology.

1. Convergence in metric spaces

PROPOSITION 1.1. *Let (E,d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ in E converges to l if and only if:*

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \ d(x_n, l) < \varepsilon.$$

PROOF. Since open balls form a basis for the metric topology, the definition of limit for a sequence at infinity (along $\mathbb{N}$) is equivalent to:

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \ x_n \in B(l, \varepsilon)$$

which is equivalent by definition to the statement in the proposition. $\square$

2. Squeeze Theorems

The following result is often useful.

THEOREM 2.1. *Let (E,d) be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E and $l \in E$. If there exists a sequence $(r_n)_{n\in\mathbb{N}}$ in $[0,\infty) \subseteq \mathbb{R}$ and $N \in \mathbb{N}$ such that $\lim_{n\to\infty} r_n = 0$ and $d(x_n, l) \leq r_n$ for all $n \geq N$, then $\lim_{n\to\infty} x_n = l$.*

PROOF. Let *varepsilon* > 0 be given. Since $(r_n)_{n\in\mathbb{N}}$ converges to 0 in $[0,\infty)$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $r_n \in [0,\varepsilon)$. Hence, for $n \geq N$ we have $d(x_n, l) \leq \varepsilon$, as desired. $\square$

This theorem shows that a decent supply of sequences converging to 0 in $\mathbb{R}$ is useful. The following results help in these constructions.

PROPOSITION 2.2. *Let $\varphi : \mathbb{N} \to (0,\infty)$ be increasing and not bounded. Let $M \in [0,\infty)$. Then $\left(\frac{M}{\varphi(n)}\right)_{n\in\mathbb{N}}$ converges to 0.*

PROOF. Let $\varepsilon > 0$ be given. By assumption, there exists $N \in \mathbb{N}$ such that $\varphi(n) > \frac{M}{\varepsilon}$. Since φ is increasing, for all $n \geq N$ we have $\varphi(n) \geq \frac{M}{\varepsilon}$. Hence for all $n \geq N$ we have $\frac{M}{\varphi(n)} \leq \frac{\varepsilon}{M} M = \varepsilon$, as desired. $\square$

The existence of unbounded functions φ is a consequence of the fact that $\mathbb{R}$ is Archimedean. Namely, the injection $i : \mathbb{N} \to \mathbb{R}$ is unbounded precisely because of the Archimedean principle (in fact, the two statements are equivalent). Therefore, $(\frac{1}{n})_{n\in\mathbb{N}^*}$ converges to 0, as desired.

3. Closures and Limits

THEOREM 3.1. *Let (E, d) be a metric space. Let $A \subseteq E$. Then $a \in \overline{A}$ if and only if there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in A converging to a.*

PROOF. If there exists a sequence in A converging to a then $a \in \overline{A}$.

Conversely, assume $a \in \overline{A}$. For $n \in \mathbb{N}^*$, there exists $x_n \in B(a.\frac{1}{n}) \cap A$. By construction, $d(x_n, a) \leq \frac{1}{n}$ for all $n \in \mathbb{N}^*$ so $(x_n)_{n \in \mathbb{N}^*}$ converges to a, which completes our proof. $\qquad\square$

Limits of subsequences can be characterized in metric spaces as follows:

THEOREM 3.2. *Let (E, d) be a metric space. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in E. Then $l \in E$ is a limit for a subsequence of $(x_n)_{n \in \mathbb{N}}$ if and only if:*

$$l \in \bigcap_{n \in \mathbb{N}} \overline{\{x_k : k \geq n\}}.$$

PROOF. Assume l is the limit of some subsequence $(x_{\phi(n)})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$. Let $n \in \mathbb{N}$. Then l is the limit of the truncated sequence $\{k \in \mathbb{N} : k \geq n\} \mapsto x_{\phi(k)}$ at ∞, so $l \in \overline{\{x_k : k \geq n\}}$ since $\phi(n) \geq n$. Hence the condition is necessary.

Assume now $l \in \bigcap_{n \in \mathbb{N}} \overline{\{x_k : k \geq n\}}$. To ease notations, write $X_n = \{x_k : k \geq n\}$ for all $n \in \mathbb{N}$. Since $l \in \overline{X_0}$, the set $\{k \in \mathbb{N} : d(x_k, l) < 1\}$ is not empty. Let $\phi(0)$ be its smallest element.

Assume now we have constructed $\phi(0) < \cdots < \phi(n)$ for some $n \in \mathbb{N}$ such that $d(x_{\phi(k)}, l) < \frac{1}{k+1}$ for $k = 0, \ldots, n$. Since $l \in \overline{X_{\phi(n)+1}}$, the set $\{k \in \mathbb{N} : k > \phi(n) \wedge d(x_{\phi(k)}, l) < \frac{1}{n+2}\}$ is not empty. Let $\phi(n+1)$ be its smallest element.

One checks that $l = \lim_{n \to \infty} x_{\phi(n)}$ as desired. $\qquad\square$

4. Limits at a point

THEOREM 4.1. *Let (E, d_E), (F, d_F) be metric spaces. Let $f : E \to F$ be a map. Let $x \in E$, $A \subseteq E$ with $x \in \overline{A}$, and $l \in F$. Then f is has limit l at x along A if and only if for any sequence $(x_n)_{n \in N}$ in A which converges to x, we have $\lim_{n \to \infty} f(x_n) = l$.*

PROOF. Note that since $x \in \overline{A}$ and (E, d) is metric, there is a sequence of elements of A converging to x. The theorem on composition of limits shows that the condition is necessary. Let us show that it is sufficient by contraposition. Assume f does not converge to l at x. Hence, there exists $\epsilon > 0$ such that for all $\delta > 0$, there exists $z \in E$ such that $d_E(x, z) < \delta$ yet $d_F(f(z), l) > \epsilon$. For each $n \in \mathbb{N}^*$, pick $x_n \in A$ such that $d_E(x_n, z) < \frac{1}{n}$ and $d_F(f(x_n), l) > \epsilon$. Note that since $x \in \overline{A}$, we can always choose such an x_n. By the squeeze theorem, we conclude that $lim_{n \to \infty} x_n = x$. On the other hand, $(f(x_n))_{n \in \mathbb{N}^*}$ does not converge to l. This proves our result. $\qquad\square$

5. Continuity at a Point

THEOREM 5.1. *Let (E, d_E), (F, d_F) be metric spaces. Let $f : E \to F$ be a map. Let $x \in E$. Then f is continuous at x if and only if for any sequence $(x_n)_{n \in N}$ which converges to x in E, we have $\lim_{n \in \mathbb{N}} f(x_n) = f(a)$.*

PROOF. This is a direct application of the result on limits in a metric space. $\quad\square$

Filters

This chapter introduces filters as a mean to analyze topological spaces. In a sense to be illustrated below, filters provide a generalization of sequences, albeit in a dual manner.

1. Filters on sets

DEFINITION 1.1. Let E be a set. A filter base on E is a subset $\mathscr{B}$ of 2^E such that:

(1) $\mathscr{B}$ is not empty and,
(2) $\varnothing \notin \mathscr{B}$
(3) If $A, B \in \mathscr{B}$ then there exists $C \in \mathscr{B}$ such that $C \subseteq A \cap B$.

DEFINITION 1.2. Let E be a set. A filter on E is a subset $\mathscr{F}$ of 2^E such that:

(1) $\varnothing \notin \mathscr{F}$,
(2) $E \in \mathscr{F}$,
(3) If $A \in \mathscr{F}$ and $C \subseteq E$ with $A \subseteq C$ then $C \in \mathscr{F}$,
(4) If $A, B \in \mathscr{F}$ then $A \cap B \in \mathscr{F}$.

Hence a filter is a proper nonempty upset closed under finite intersections. Note that the empty set is never in a filter. Note that a filter is a filter basis.

PROPOSITION 1.3. *Let $\mathscr{B}$ be a filter basis on a set E. Let:*

$$\mathscr{F}_{\mathscr{B}} = \{A \subseteq E : \exists B \in \mathscr{B} \ \ B \subseteq A\}.$$

Then $\mathscr{F}_{\mathscr{B}}$ is a filter on E, and it is the smallest filter on E containing $\mathscr{B}$. We call it the filter generated by $\mathscr{B}$.

PROOF. By definition, $\mathscr{B} \subseteq \mathscr{F}_{\mathscr{B}}$ so $\mathscr{F}$ is not empty. Moreover, if $A \in \mathscr{F}_{\mathscr{B}}$ and $C \subseteq E$ such that $A \subseteq C$ then, by construction, there exists $B \in \mathscr{B}$ such that $B \subseteq A$ and thus $B \subseteq C$ so $C \in \mathscr{F}_{\mathscr{B}}$. If $A_1, A_2 \in \mathscr{F}_{\mathscr{B}}$ then there exists $B_1, B_2 \in \mathscr{B}$ such that $B_i \subseteq A_i$ for $i = 1, 2$. Thus $B_1 \cap B_2 \subseteq A_1 \cap A_2$. Since $\mathscr{B}$ is a filter basis, there exists $C \in \mathscr{B}$ such that $C \subseteq B_1 \cap B_2$ so $C \subseteq A_1 \cap A_2$. Hence $A_1 \cap A_2 \in \mathscr{F}_{\mathscr{B}}$. Last, if $\varnothing \in \mathscr{F}_{\mathscr{B}}$ then there must be $A \in \mathscr{B}$ such that $A \subseteq \varnothing$, i.e. $\varnothing \in \mathscr{B}$ which is a contradiction. So $\varnothing \notin \mathscr{F}_{\mathscr{B}}$. So $\mathscr{F}_{\mathscr{B}}$ is a filter.

Let $\mathscr{G}$ be some filter containing $\mathscr{B}$. Let $A \in \mathscr{F}_{\mathscr{B}}$. Then by definition, there exists $B \in \mathscr{B}$ such that $B \subseteq A$. Sicne $B \in \mathscr{G}$ and $\mathscr{G}$ is a filter, we conclude $A \in \mathscr{G}$. Hence $\mathscr{F}_{\mathscr{B}} \subseteq \mathscr{G}$ as desired. □

PROPOSITION 1.4. *Let E, F be sets and $f : E \to F$ be a function. If $\mathscr{F}$ is a filter basis on E, then $f[\mathscr{F}]$, defined as:*

$$f[\mathscr{F}] = \{f(A) : A \in \mathscr{F}\}$$

is a filter basis.

PROOF. By definition, $f[\mathscr{F}]$ is not empty and does not contain the emptyset. Now, let $A, B \in f[\mathscr{F}]$. By definition, there exists $C, D \in \mathscr{F}$ such that $A = f(C)$ and $B = f(D)$. Now, since $\mathscr{F}$ is a filter basis, there exists $H \in \mathscr{F}$ such that $H \subseteq C \cap D$. Hence $f(H) \subseteq A \cap B$, and by definition, $f(H) \in f[\mathscr{F}]$. □

EXAMPLE 1.5. Let $f : E \to F$ be constant, equal to $t \in F$, while E, F have more than one element. Let $x \in E$. Then $\mathscr{F} = \{A \subseteq E : x \in A\}$ is a filter on E. Now, $f[\mathscr{F}] = \{\{t\}\}$. One checks easily that this is a filter basis, but not a filter (as it would contain F which is assumed to contain a set of cardinal 2). Hence, images of filters may not be filters. We could define $f[[\mathscr{F}]]$ to be the filter generated by the filter basis $f[\mathscr{F}]$, thus defining a map from filters to filters for each map from E to F.

2. Limits of Filters basis

DEFINITION 2.1. Let (E, τ) be a topological space. Let $x \in E$. Let $\mathscr{F}$ be a filter basis on E. We say that $\mathscr{F}$ converges to x when:

$$\forall U \in \tau \ (x \in U) \implies \exists B \in \mathscr{F} \ B \subseteq U.$$

REMARK 2.2. Equivalently, a filter basis $\mathscr{F}$ converges to x if the set $\mathscr{V}_\tau(x)$ of open neighborhoods of x is contained in the filter generated by $\mathscr{F}$.

EXAMPLE 2.3. The set $\mathscr{V}_\tau(x)$ is a filter basis which converges to x. The elements of the filter generated by $\mathscr{V}_\tau(x)$ are called neighborhoods of x.

The following theorem illustrates the connection between filters and sequences.

THEOREM 2.4. *Let (E, τ) be a topological space. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in E. For $n \in \mathbb{N}$ we define $X_n = \{x_k : k \geq n\}$. Then $\mathscr{X} = \{X_n : n \in \mathbb{N}\}$ is a filter basis. Moreover, $(x_n)_{n \in \mathbb{N}}$ converges to x if and only if $\mathscr{X}$ converges to x.*

PROOF. Since $X_{n+1} \subseteq X_n$ for all $n \in \mathbb{N}$, it is immediate that $\mathscr{X}$ is a filter basis. Assume that $(x_n)_{n \in \mathbb{N}}$ converges to x. Let $U \in \tau$ such that $x \in U$. Then by definition, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n \in U$. Hence $X_N \subseteq U$. Thus, $\mathscr{X}$ converges to x (since U was arbitrary). Conversely, assume

that $\mathscr{X}$ converges to x. Let $U \in \tau$ such that $x \in U$. By definition, there exists $X_N \in \mathscr{X}$ such that $X_N \subseteq U$. Hence for all $n \geq N$ we have $x_n \in U$. So $(x_n)_{n \in \mathbb{N}}$ converges to x (since U was arbitrary). □

REMARK 2.5. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence. Let $(x_{\varphi(n)})_{n \in \mathbb{N}}$ be a subsequence. The filter generated by the filter basis $\{\{x_k : k \geq n\} : n \in \mathbb{N}\}$ is a *subset* of the filter generated by the filter basis $\{\{x_{\phi(k)} : k \geq n\} : n \in \mathbb{N}\}$. So filters "grow" as we take subsequences. From this observation, we note that by definition, it is now immediate to see that if a sequence converge to some x then all of its subsequences do too. The following proposition generalizes this observation.

PROPOSITION 2.6. *Let (E, τ) be a topological space. Let $\mathscr{F}$ and $\mathscr{G}$ be two filter bases on E such that for all $F \in \mathscr{F}$ there exists $G \in \mathscr{G}$ such that $G \subseteq F$ (we say that $\mathscr{G}$ is finer than $\mathscr{F}$). If $\mathscr{F}$ converges to x, then $\mathscr{G}$ converges to x.*

PROOF. Let x be a limit of $\mathscr{F}$. Let $U \in \tau$ such that $x \in \tau$. There exists $F \in \mathscr{F}$ such that $F \subseteq U$. By assumption, there exists $G \in \mathscr{G}$ such that $G \subseteq F$. Hence $\mathscr{G}$ converges to x.   □

COROLLARY 2.7. *Let (E, τ) be a topological space. If $\mathscr{F} \subseteq \mathscr{G}$ are two filters on E and if $\mathscr{F}$ converges to x, then $\mathscr{G}$ converges to x.*

Limits of filters may not be unique. We have:

THEOREM 2.8. *Let (E, τ) be a topological space. The following are equivalent:*

  (1) *The space (E, τ) is Hausdorff,*
  (2) *Every convergent filter basis on E has a unique limit.*

PROOF. Assume that (E, τ) is Hausdorff. Let $\mathscr{F}$ be a filter basis on E. Assume it converges to x and y in E. If $x \neq y$ then, since τ is Hausdorff, there exists $U_x, U_y \in \tau$ such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$. By definition of convergence, there exist $B_x, B_y \in \mathscr{F}$ such that $B_x \subseteq U_x$ and $B_y \subseteq U_y$. By definition of filter basis, there exists $C \in \mathscr{F}$ such that $C \subseteq B_x \cap B_y = \emptyset$. This contradicts the fact $\mathscr{F}$, as a filter basis, does not contain the empty set. Hence limits are unique if they exist.

Assume now that all filter bases have unique limits in (E, τ). In particular, for any $x \in E$, the filter basis $\mathscr{V}_\tau(x)$ of open neighborhoods of x in τ, has x for unique limit. Let $y \neq x$. Suppose that for all $U \in \mathscr{V}_\tau(y)$, for all $V \in \mathscr{V}_\tau(x)$, we have $U \cap V \neq \emptyset$. Let:

$$\mathscr{F} = \{U \cap V : U \in \mathscr{V}_\tau(x), V \in \mathscr{V}_\tau(y)\}.$$

By assumption, $\mathscr{F}$ is a nonempty set of elements in τ not containing the emptyset. Moreover, it is closed by finite intersection (as τ is). Hence, it is a filter basis. Yet, for any $U \in \mathscr{V}_\tau(x)$ and any $V \in \mathscr{V}_\tau(y)$, we have $U, V \in \mathscr{F}$ so $\mathscr{F}$ converges to x and y by definition. This is a contradiction. Hence, for all $U \in \mathscr{V}_\tau(y)$ there exists $V \in \mathscr{V}_\tau(x)$ such that $U \cap V = \emptyset$, i.e. (E, τ) is Hausdorff.   □

NOTATION 2.9. When (E, τ) is Hausdorff and $\mathscr{F}$ is a filter basis converging to x then we write: $x = \lim \mathscr{F}$.

3. Some Applications of Filters

As a rule, filters can be used to replace sequences in general topology to obtain results limited to metric spaces. For instance:

LEMMA 3.1. *Let (E, τ) be a topological space. Let $A \subseteq E$. Let $\mathscr{F}$ be a filter basis in A which converges to x in τ. Then $x \in \overline{A}$.*

PROOF. Let $V \in \mathscr{V}_\tau(x)$. By definition of convergence, there exists $B \in \mathscr{F}$ such that $B \subseteq V$. Now, $B \subseteq A$ by assumption, so $emptyset \neq B \subseteq V \cap A$. So $x \in \overline{A}$ since V is arbitrary.   □

THEOREM 3.2. *Let (E, τ) be a topological space. Let $A \subseteq E$. The closure of A in τ is the set of all limits in E of filters of A.*

PROOF. All limits of filters of A are in $\overline{A}$ by the previous lemma. Assume $x \in \overline{A}$. If $U \in \mathscr{V}_\tau(x)$ then $A \cap U \neq \emptyset$. It is then obvious that:

$$\{U \cap A : U \in \mathscr{V}_\tau(x)\}$$

is a filter basis of A which converges to x. □

DEFINITION 3.3. Let E be a set, (F, τ) be a topological space, $\mathscr{F}$ a filter basis on E, and $f : E \to F$. Then we say that f converges to $x \in F$ along $\mathscr{F}$ if $f[\mathscr{F}]$ converges to x.

When (F, τ_F) is Hausdorff, we shall write: $\lim_{\mathscr{F}} f$ for this limit.

THEOREM 3.4. *Let (E, τ_E) and (F, τ_F) be two topological spaces, and let $f : E \to F$. Let $A \subseteq E$, $a \in \overline{A}$. Then f has limit l at a along A if and only if for all filter bases $\mathscr{F}$ of A converging to a, the filter basis $f[\mathscr{F}]$ converges to l in F.*

PROOF. Assume first that f converges. Let $\mathscr{F}$ be a filter basis of A converging to a. Note that we have shown such a filter basis exists. Now, let $U \in \tau_F$ such that $l \in U$. Since f converges to l at a along A, there exists $V \in \tau_E$ with $a \in V$ such that $f(A \cap V) \subseteq U$. Since $\mathscr{F}$ converges to a, there exists $C \in \mathscr{F}$ such that $C \subseteq V$. Since $C \subseteq A$ by assumption, $C = C \cap A \subseteq V \cap A$ so $f(C) \subseteq U$ as desired. Since U was arbitrary, $f[\mathscr{F}]$ converges to l as desired.

Conversely, assume that $f[\mathscr{F}]$ converges to l for all filter bases $\mathscr{F}$ of A converging to a. Let $U \in \tau_V$ such that $l \in U$. The filter basis $\mathscr{V} = \{V \cap A : V \in \tau, a \in V\}$ converges to a and is a subset of $\mathbf{2}^A$ so by assumption, $f[\mathscr{V}]$ converges to l. Thus, there exists $C \in \mathscr{V}$ such that $f(C) \subseteq U$. Hence, by definition, there exists $V \in \tau_E$ such that $f(V \cap A) \subseteq U$. As U is arbitrary in $\mathscr{V}_{\tau_F}(l)$, we conclude that f converges to l at a along A. □

COROLLARY 3.5. *Let (E, τ_E) and (F, τ_F) be topological spaces, $x \in E$ and $f : E \to F$. Then f is continuous at x if and only if the image by f of all filter bases of E converging to x converge to $f(x)$.*

4. Ultrafilters

DEFINITION 4.1. Let E be a set. An ultrafilter on E is a filter $\mathscr{F}$ such that for all $A \subseteq E$, we have either $A \in \mathscr{F}$ or $\complement_E A \in \mathscr{F}$.

EXAMPLE 4.2. Let $x \in E$. Then $\{A \subseteq E : x \in A\}$ is an ultrafilter.

Since for no set A can we have A and $\complement_E A$ in a filter of E, this definition suggests the following:

THEOREM 4.3. *A filter on a set E is an ultrafilter if and only if it is maximal among all filters on E (for the inclusion).*

PROOF. Assume first that $\mathscr{F}$ is an ultrafilter on E. Let $\mathscr{G}$ be a filter on E such that $\mathscr{F} \subseteq \mathscr{G}$. Let $A \in \mathscr{G}$. Assume $\complement_E A \in \mathscr{F}$. Then $\complement_E A \in \mathscr{G}$, and thus $\emptyset = A \cap \complement_E A$ is in $\mathscr{G}$, as filters are closed under finite intersections. This is a contradiction. Since $\mathscr{F}$ is an ultrafilter, $A \in \mathscr{F}$. As A was arbitrary, $\mathscr{G} = \mathscr{F}$.

Conversely, assume that $\mathscr{F}$ is a maximal filter for inclusion. Assume that it is not an ultrafilter. Let $Y \subseteq E$ such that $Y, \complement_E Y \notin \mathscr{F}$. Without loss of generality, assume $Y \neq \emptyset$. Let:

$$\mathscr{G} = \{C \subseteq E : \exists A \in \mathscr{F} \ \ A \cap Y \subseteq C\}.$$

Now, by construction, $\mathscr{G}$ is not empty. Assume $\emptyset \in \mathscr{G}$. Then there exists $A \in \mathscr{F}$ such that $A \cap Y = \emptyset$. Thus $A \subseteq \complement_E Y$. As $\mathscr{F}$ is a filter, this would imply $\complement_E Y \in \mathscr{F}$ which is a contradiction. So $\mathscr{G}$ does not contain the empty set. By construction, if $A \in \mathscr{G}$ and $A \subseteq C$ then $C \in \mathscr{G}$. Moreover, if $A_1, A_2 \in \mathscr{G}$ then there exist $B_1, B_2 \in \mathscr{F}$ such that $B_i \cap Y \subseteq A_i$ for $i = 1, 2$. Thus $B_1 \cap B_2 \cap Y \subseteq A_1 \cap A_2$. Since $B_1 \cap B_2 \in \mathscr{F}$ we conclude that $A_1 \cap A_2 \in \mathscr{G}$. Thus $\mathscr{G}$ is a filter. By definition, since $A \cap Y \subseteq A$ for all $A \in \mathscr{F}$, we have $\mathscr{F} \subseteq \mathscr{G}$. Since $E \cap Y = Y$ and $E \in \mathscr{F}$, we have $Y \in \mathscr{G}$, yet $Y \notin \mathscr{F}$. This contradicts the maximality of $\mathscr{F}$. So $\mathscr{F}$ is an ultrafilter. $\square$

Maximality lends itself to the application of Zorn's lemma to prove the existence of ultrafilters containing any given filer.

THEOREM 4.4. *Let E be a set, and let $\mathscr{F}$ be a filter basis on E. Then there exists an ultrafilter $\mathscr{G}$ containing $\mathscr{F}$.*

PROOF. Since every filter basis is contained in a filter, we will assume $\mathscr{F}$ is a filter. Let:

$$\mathscr{P} = \{\mathscr{H} \text{ filters on } E : \mathscr{F} \subseteq \mathscr{H}\}.$$

The set $\mathscr{P}$ is ordered by inclusion, and it is not empty (it contains $\mathscr{F}$. Note that $\mathscr{P}$ is an upper set, and thus a maximal element of $\mathscr{P}$ is a maximal element of the set of all filters on E. Thus, it is enough to show that $\mathscr{P}$ has a maximal element.

Let $\mathscr{U}$ be a chain in $\mathscr{P}$, i.e. a linearly ordered subset of $\mathscr{P}$. Let $\mathscr{Y} = \bigcup \mathscr{U}$. If $A, B \in \mathscr{Y}$ then there exists $U, V \in \mathscr{U}$ such that $A \in U$ and $B \in V$. Since $\mathscr{U}$ is linearly ordered, we may assume $U \subseteq V$. Thus $A, B \in V$, and V is a filter, so $A \cap B \in V \subseteq \bigcup \mathscr{Y}$. The set $\mathscr{Y}$ is not empty (filters never are), does not contain the empty set (filters never do), and is an upper set: if $A \in \mathscr{Y}$ and $A \subseteq C$ for some C, then there exists $U \in \mathscr{U}$ such that $A \in U$, and since U is a filter, $C \in U$ so $C \in \mathscr{Y}$. Therefore, $\mathscr{Y}$ is a filter. Trivially $\mathscr{Y} \in \mathscr{P}$. Hence, $\mathscr{U}$ admits an upper bound. Since $\mathscr{U}$ is an arbitrary chain, Zorn's lemma implies that $\mathscr{P}$ admits a maximal element $\mathscr{G}$. Thus $\mathscr{G}$ is a maximal filter, and hence an ultrafilter, containing $\mathscr{F}$. $\square$

Due to this result, it is possible to rephrase the theorems in the application section of this chapter in terms of ultrafilters only.

CHAPTER 6

Compactness

1. Compact space

DEFINITION 1.1. A topoligical space (E, τ) is compact if, given any $\mathscr{U} \subseteq \tau$ such that $E = \bigcup \mathscr{U}$, there exists a *finite* subset $\mathscr{V}$ of $\mathscr{U}$ such that $E = \bigcup \mathscr{V}$.

REMARK 1.2. A subset $\mathscr{U}$ of τ such that $E = \bigcup \mathscr{U}$ is called an open covering of E.

THEOREM 1.3. *Let (E, τ) be a topological space. Then the following are equivalent:*

(1) *(E, τ) is compact,*
(2) *For any set $\mathscr{F}$ of closed subsets of E whose intersection is empty, there exists a finite subset $\mathscr{G}$ of $\mathscr{F}$ whose intersection is empty.*

PROOF. The theorem follows by taking complements. $\square$

COROLLARY 1.4. *Let (E, τ) be a compact space. Let $(F_n)_{n \in \mathbb{N}}$ be a decreasing sequence of nonempty closed subsets of E (where the order on sets in inclusion). Then:*

$$\bigcap_{n \in \mathbb{N}} F_n \neq \varnothing.$$

PROOF. If $\bigcap_{n \in \mathbb{N}} F_n = \varnothing$ then since (E, τ) is compact, we can find $N \in \mathbb{N}$ such that $F_1 \cap F_2 \cap \cdots \cap F_N = \varnothing$. Yet since the family is decreasing, $F_1 \cap F_2 \cap \cdots \cap F_N = F_N$ which is assumed not empty. Our result follows by contraposition. $\square$

THEOREM 1.5. *Let (E, τ) be a topological space. Let $\mathscr{F}$ be a set of closed subsets of E such that given any finite $\mathscr{G} \subseteq \mathscr{F}$, we have $\bigcap \mathscr{G} \neq \varnothing$ (we say that $\mathscr{F}$ has the finite intersection property). Then $\bigcap \mathscr{F} \neq \varnothing$ if and only if (E, τ) is compact.*

PROOF. Assume first that (E, τ) is compact and consider a set $\mathscr{F}$ of closed subsets of E. Assume $\bigcap \mathscr{F} = \varnothing$. Since E is compact, there exists a finite set $\mathscr{G} \subseteq \mathscr{F}$ such that $\bigcap \mathscr{G} = \varnothing$. This proves the necessity of our theorem by contraposition.

Conversely, assume that for all sets $\mathscr{F}$ of closed subsets of E with the finite intersection property, $\bigcap \mathscr{F} \neq \varnothing$. Let $\mathscr{F}$ be a set of closed subsets of E such that $\bigcap \mathscr{F} = \varnothing$. Then by our assumption, there exists at least one finite subset $\mathscr{G} \subseteq \mathscr{F}$ such that $\bigcap \mathscr{G} = \varnothing$. $\square$

EXAMPLE 1.6. The space $(E, \mathbf{2}^E)$ is compact if and only if E is finite.

EXAMPLE 1.7. It is striaghtforward that if $U_n = (-n, \infty)$ in $\mathbb{R}$ for all $n \in \mathbb{N}$ then $\{U_n : n \in \mathbb{N}\}$ is an open covering of $\mathbb{R}$ which has no finite subcovering. So $\mathbb{R}$ is not compact.

2. Compact Subspaces

DEFINITION 2.1. *Let (E, τ) be a topological space. Let $A \subseteq E$. Then A is a compact subspace of (E, τ) if (A, τ_A) is a compact space (where τ_A is the trace topology on A).*

REMARK 2.2. We will say that A is compact in E when A is a compact subspace of E.

THEOREM 2.3. *Let (E, τ) be a topological space. Let $A \subseteq E$. Then A is compact if and only if for all $\mathscr{U} \subseteq \tau$ such that $A \subseteq \bigcup \mathscr{U}$, there exists $n \in \mathbb{N}$ and $U_1, \ldots, U_n \in \mathscr{U}$ such that:*

$$A \subseteq \bigcup_{k=1}^{n} U_k.$$

PROOF. Assume that (A, τ_A) is compact in (E, τ). Let $\mathscr{U} \subseteq \tau$ such that $A \subseteq \bigcup \mathscr{U}$. Then $A = \bigcup \{U \cap A : U \in \mathscr{U}\}$. By definition, $U \cap A \in \tau_A$ for all $U \in \tau$. Hence by compactness of A, there exists $\{U_1, \ldots, U_n\} \subseteq \mathscr{U}$ such that $A = \bigcup_{i=1}^{n} (A \cap U_i)$. Hence $A \subseteq \bigcup_{i=1}^{n} U_i$.

Assume now that any open covering of A in E admits a finite covering. Let $\mathscr{V} \subseteq \tau_A$ such that $A = \bigcup \mathscr{V}$. By definition of τ_A, for each $V \in \mathscr{V}$ there exists $U_V \in \tau$ such that $V = U_V \cap A$. Thus $A \subseteq \bigcup \{U_V : V \in \mathscr{V}\}$, so there exists $U_{V_1}, \ldots, U_{V_n} \in \tau$ such that $A \subseteq \bigcup_{i=1}^{n} U_{V_i}$. Hence $A = \bigcup_{i=1}^{n} V_i$. □

THEOREM 2.4. *Let (E, τ) be a topological space. Let $A \subseteq E$. Then A is compact if and only if for any family $(F_i)_{i \in I}$ of closed sets in E such that:*

$$A \cap \bigcap_{i \in I} F_i = \varnothing$$

there exists a finite subset $J \subseteq I$ such that:

$$A \cap \bigcap_{i \in J} F_j = \varnothing.$$

PROOF. Take complements in previous theorem. □

THEOREM 2.5. *Let (E, τ) be a Hausdorff topological space. Then if $A \subseteq E$ is compact, then A is closed.*

PROOF. Let $x \in \complement_E A$ and $y \in A$. Since τ is Hausdorff, there exists $U_{x,y}, V_{x,y} \in \tau$ such that $U_{x,y} \cap V_{x,y} = \varnothing$, with $y \in U_{x,y}$ and $x \in V_{x,y}$. Let $\mathscr{U}_x = \{U_{x,y} : y \in A\}$. By construction:

$$A \subseteq \bigcup \mathscr{U}_x$$

and thus, since A is compact, there exists $n \in \mathbb{N}$ and $y_1, ldots, y_n \in A$ such that:

$$A \subseteq \bigcup_{k=1}^{n} U_{x,y_k}.$$

Let:

$$W_x = \bigcap_{k=1}^{n} V_{x,y_k}.$$

As a finite intersection of open sets, W_x is open. Moreover:

$$
\begin{aligned}
A \quad &\subset \quad \bigcup_{k=1}^{n} U_{x,y_k} \\
&\subseteq \quad \bigcup_{k=1}^{n} \complement_E V_{x,y_k} \\
&= \quad \complement_E \left(\bigcap_{k=1}^{n} V_{x,y_k} \right) = \complement_E W_x.
\end{aligned}
$$

Hence, $W_x \subseteq \complement_E A$. Therefore:

$$
\complement_E A = \bigcup_{x \in A} W_x
$$

and thus $\complement_E A$ is open, so A is closed as desired. $\square$

THEOREM 2.6. *Let (E, τ) be a compact space. Then if $A \subseteq E$ is closed, then it is compact.*

PROOF. Let $\mathscr{U} \subseteq \tau$ such that $A \subseteq \bigcup \mathscr{U}$ and let A be closed. Then $\complement_E A \in tau$ so, if $\mathscr{V} = \mathscr{U} \cup \{\complement_E A\}$, then $E = \bigcup \mathscr{V}$. Since (E, τ) is compact, there exists a finite set $\mathscr{W} \subseteq \mathscr{V}$ such that $E = \bigcup \mathscr{W}$. Now, if $\mathscr{Z} = \mathscr{W} \setminus \{\complement_E A\}$, one checks readily that $A \subseteq \bigcup \mathscr{Z}$ and by construction, $\mathscr{Z} \subseteq \mathscr{U}$ with $\mathscr{Z}$ finite. $\square$

COROLLARY 2.7. *Let (E, τ) be a compact Hausdorff space. Then $A \subseteq E$ is compact if and only if A is closed.*

3. Continuous image of a compact space

THEOREM 3.1. *Let (E, τ_E) be a compact space, (F, τ_F) be a topological space, and $f : E \to F$ be a continuous function. Then $f(E)$ is a compact subspace of F.*

PROOF. Let $\mathscr{U} \subseteq \tau_F$ such that $f(E) \subseteq \bigcup \mathscr{U}$. Since f is continuous, for all $U \in \mathscr{U}$, we have $f^{-1}(U) \in \tau_E$. Let $\mathscr{V} = \{f^{-1}(U) : U \in \mathscr{U}\}$. Then $\mathscr{V} \subseteq \tau_E$ and by construction, $\bigcup \mathscr{V} = E$. Since (E, τ) is compact, there exists $n \in N$ and $U_1, \ldots, U_n \in \mathscr{U}$ such that $E = \bigcup_{k=1}^{n} f^{-1}(U_k)$. Hence $f(E) \subseteq \bigcup_{k=1}^{n} U_k$ as desired. $\square$

COROLLARY 3.2. *Let (E, τ_E) and (F, τ_F) be topological spaces. If $f : E \to F$ is continuous, and $A \subseteq E$ is compact, then $f(A)$ is compact.*

THEOREM 3.3. *Let (E, τ_E) be a compact space, and (F, τ_F) be a Hausdorff topological space. Let $f : E \to F$ be a continuous bijection. Then f is a homeomorphism.*

PROOF. It suffices to show that f^{-1} is continuous. Let $A \subseteq E$ be a closed set. Then A is compact since E is compact. So $f(A)$ is compact in F. Since F is Hausdorff, $f(A)$ is closed. Since f is a bijection, $(f^{-1})^{-1}(A) = f(A)$. So f^{-1} is continuous. $\square$

4. Compact Metric Spaces

THEOREM 4.1. *Let (E, d) be a metric space. Then the following are equivalent:*

(1) *E is compact in its metric topology.*
(2) Bolzanno-Weierstrass *Every sequence in E admits a convergent subsequence.*

PROOF. First, assume that E is compact. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E. Let $X_n = \overline{\{x_k : k \in \mathbb{N}, k \geq n\}}$ for all $n \in \mathbb{N}$. The sequence $(X_n)_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty closed subsets of E, which is compact, so $\bigcap_{n\in\mathbb{N}} X_n \neq \varnothing$. By Theorem (3.2), this implies that $(x_n)_{n\in\mathbb{N}}$ admits a convergent subsequence.

Second, assume that the Bolzanno-Weierstass axiom holds for E. We first prove a few lemmas.

LEMMA 4.2 (Lebesgues number). *Let $\mathscr{U} \subseteq \tau_d$ be an open covering of E. There exists $\epsilon > 0$ such that for all $x \in E$, there exists $U \in \mathscr{U}$ such that the open ball $B(x, \epsilon)$ of center x and radius ϵ is contained in U.*

PROOF. Assume that for all $\epsilon > 0$, there exists $x \in E$ such that for all $U \in \mathscr{U}$, we have $B(x, \epsilon) \not\subseteq U$. Let $n \in \mathbb{N}$. There exists $x_n \in E$ such that $B(x_n, \frac{1}{n+1}) \not\subseteq U$ for all $U \in \mathscr{U}$. The sequence $(x_n)_{n\in\mathbb{N}}$ admits a convergent subsequence $(x_{\phi(n)})_{n\in\mathbb{N}}$ by the Bolzanno-Weierstrass axiom. Let $l \in E$ be its limit. Since $\mathscr{U}$ covers E, there exists $U \in \mathscr{U}$ such that $l \in U$. Since U is open, there exists $\delta > 0$ such that $B(l, \delta) \subseteq U$. Since $(x_{\phi(n)})_{n\in\mathbb{N}}$ converges to l, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $x_{\phi(n)} \in B(l, \frac{\delta}{2})$. Let $N \in \mathbb{N}$ such that $\frac{1}{N+1} < \frac{\delta}{2}$ and $N \geq N_2$. Let $z \in B(x_{\phi(N)}, \frac{1}{\phi(N)+1})$. Then:

$$d(z, l) \leq d(z, x_{\phi(N)}) + d(x_{\phi(N)}, l) < \frac{1}{\phi(N)+1} + \frac{\delta}{2} < \delta$$

since $\phi(N) \geq N$. Hence, $z \in B(l, \delta)$ so $B(x_{\phi(N)}, \frac{1}{\phi(N)+1}) \subseteq B(l, \delta) \subseteq U$. This is a contradiction. $\square$

LEMMA 4.3 (Precompact). *For any $\epsilon > 0$ there exists $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$ such that:*

$$E = \bigcup_{j=1}^{n} B(x_j, \epsilon).$$

.

PROOF. Assume that there exists $\epsilon > 0$ such E is not the union of finitely many open balls of radius ϵ. Let $x_0 \in E$: since $E \subsetneq B(x_0, \epsilon)$, there exists $x_1 \in E$ such that $d(x_0, x_1) \geq \epsilon$.

Assume we have constructed $x_0, \ldots, x_n \in E$ such that $d(x_i, x_j) \geq \epsilon$ for $i \neq j \in \{0, \ldots, n\}$. Since $E \subsetneq \bigcup_{j=0}^{n} B(x_j, \epsilon)$, there exists $x_{n+1} \in E \setminus (\bigcup_{j=0}^{n} B(x_j, \epsilon))$, i.e. $d(x_{n+1}, x_i) \geq \epsilon$.

By induction, we have constructued a sequence $(x_n)_{n\in\mathbb{N}}$ which admits no convergent subseqence. This contradicts the Bolzanno-Weierstrass axiom. $\square$

We can now conclude our proof. Assume $\mathscr{U}$ is an open covering of E. There exists $\epsilon > 0$ such that any open ball of radius ϵ or less is contained in some $U \in \mathscr{U}$.

Now, there exists $x_1, \ldots, x_n \in E$ such that $E = \bigcup_{j=1}^{n} B(x_j, \epsilon)$. Let $U_j \in \mathscr{U}$ such that $B(x_j, \epsilon) \subseteq U_j$. Then $E = \bigcup_{j=1}^{n} U_j$. So E is compact, as desired. $\qquad\square$

COROLLARY 4.4. *A compact metric space is bounbed.*

PROOF. The precompact lemma shows that a compact metric space can be covered with finitely many balls of radius 1. $\qquad\square$

REMARK 4.5. The definition of compacity by means of open covering is referred to the Heine-Borel axioms of compactity. Thus we have shown that in metric spaces, Heine-Borel and Bolzanno-Weierstrass are equivalent.

5. Compacts of ℝ

We provide two proofs of the following result, both based on your real analysis class.

THEOREM 5.1. *A subset A of ℝ is compact if and only if it is closed and bounded.*

NECESSITY. Since ℝ is Hausdorff, compact subsets are closed. Since ℝ is metric, compact subsets are bounded. $\qquad\square$

FIRST PROOF OF SUFFICIENCY: MONOTONICITY. Assume that $A \subseteq ℝ$ is closed and bounded. Let $(x_n)_{n \in ℕ}$ be a sequence in A. By the monotone subsequence theorem, there exists a monotone subsequence of $(x_n)_{n \in ℕ}$. Since A is bounded, this subsequence must converge. Hence A satisfies Bolzanno-Weierstrass axiom in a metric space and is thus compact. $\qquad\square$

SECOND PROOF OF SUFFICIENCY: DICHOTOMY. Let $a_0 < b_0$ be real numbers. Let $(x_n)_{n \in ℕ}$ be a sequence in $[a_0, b_0]$. Set $\phi(0) = 0$.

Assume that for some $n \in ℕ$ we have constructed $\phi(0) < \ldots < \phi(n)$, $a_0 \leq a_1 \leq \ldots \leq a_n \leq b_0$ and $a_0 \leq b_n \leq \ldots \leq b_0$ such that $x_{\phi(k)} \in [a_k, b_k]$ and $b_k - a_k = \frac{1}{2^k}(b_0 - a_0)$. Let $m_n = \frac{a_n + b_n}{2}$. If $\{n \in ℕ : n \geq \phi(n) \wedge x_n \in [a_n, m_n]\}$ is infinite, then let $\phi(n+1)$ be its smallest element. We also let $a_{n+1} = a_n$, $b_{n+1} = m_n$. Otherwise, we set $\phi(n+1)$ to be the smallest element of the (necessarily infinite) set $\{n \in ℕ : n \geq \phi(n) \wedge x_n \in [m_n, b_n]\}$, and we let $a_{n+1} = m_n$, $b_{n+1} = b_n$. One checks that by induction, we have constructed an subsequence $(x_{\phi(n)})_{n \in ℕ}$ and monotone sequences $(a_n)_{n \in ℕ}$ and $(b_n)_{n \in ℕ}$ such that:

(1) $(a_n)_{n \in ℕ}$ is increasing and bounded above by b_0 so it converges.
(2) $(b_n)_{n \in ℕ}$ is decreasing and bounded above by a_0 so it converges.
(3) We have $\lim_{n \to \infty} a_n - b_n = 0$ so $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$.
(4) Thus by the squeeze theorem, $\lim_{n \to \infty} x_{\phi(n)} = \lim_{n \to \infty} a_n$.

Hence again we have proven the Bolzanno-Weierstrass axiom for all closed intervals. Now, if $A \subseteq ℝ$ is closed and bounded, then it is a closed subset of some closed interval, i.e. of a compact space. Hence it is compact, as needed. $\qquad\square$

REMARK 5.2. The second proof is longer but can be applied as is to prove that *compact subsets of $ℝ^n$ are closed bounded subsets of $ℝ^n$.* Indeed, rather than cut an interval in two, one can repeat the argument, cutting a square in four, a cube in eight, and so forth. This process works well as long as we divide an hypercube into

finitely many hypercubes at each stage. This proofs fails for infinite dimensional normed vector spaces, and in fact so does the result: one can show that locally compact normed vector spaces are all finite dimensional.

6. Uniform Continuity and Heine Theorem

DEFINITION 6.1. Let (E, d_E) and (F, d_F) be metric spaces. A function $f : E \to F$ is uniformly continuous when:

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x, y \in E \ \ d_E(x, y) < \delta \implies d_F(f(x), f(y)) < \epsilon.$$

PROPOSITION 6.2. *A uniformly continuous function is continuous.*

PROOF. Obvious. □

EXAMPLE 6.3. A L-Lipshitz function is uniformly continuous: take $\delta = \frac{\epsilon}{L}$ if $L > 0$ (i.e. nonconstant functions), and δ arbitrary for constants.

EXAMPLE 6.4. The function $x \in \mathbb{R} \mapsto x^2$ is not uniformly continuous in $\mathbb{R}$. Indeed basic precalculus shows that $x^2 < kx$ if and only if $x \in (-k, k)$.

The following shows that continuity can be strengthened to uniform continuity when working on a compact metric space.

THEOREM 6.5. *Let (E, d_E) be a compact metric space. Let (F, d_F) be a metric space. Let $f : E \to F$ be given. Then f is continuous if and only if f is uniformly continuous on E.*

PROOF. If f is uniformly continuous then it is continuous. Let us prove the converse. Assume f is continuous on E. Let $\epsilon > 0$. Then for all $x \in E$ there exists $\delta_x > 0$ such that if $y \in B(x, \delta_x)$ then $d_F(f(y), f(x)) < \frac{\epsilon}{2}$. Now, since $E = \bigcup_{x \in E} B(x, \frac{\delta_x}{2})$, and E is compact, there exists $x_1, \ldots, x_n$ such that $E = \bigcup_{j=1}^{n} B(x_j, \frac{\delta_{x_j}}{2})$. Let $\delta = \frac{1}{2} \min\{\delta_{x_j} : j = 1, \ldots, n\} > 0$. Let $x, y \in E$ with $d_E(x, y) < \delta$. There exists $j \in \{1, \ldots, n\}$ such that $x \in B(x_j, \delta_{x_j})$. Then

$$d_E(y, x_j) \leq d_E(y, x) + d_E(x, x_j) < \delta + \frac{1}{2}\delta_{x_j} \leq \delta_{x_j}.$$

Thus:

$$d_F(f(x), f(y)) \leq d_F(f(x), f(x_j)) + d(f(x_j), f(y)) < \epsilon$$

as desired.

□

ALTERNATE, USE REAL ANALYSIS. If f is uniformly continuous then it is continuous. Let us prove the converse. Assume f is continuous but not uniformly continuous. There exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $x, y \in E$ such that $d_E(x, y) < \delta$ and $d_F(f(x), f(y)) \geq \epsilon$. For $n \in \mathbb{N}$ let $x_n, y_n \in E$ such that $d_E(x_n, y_n) < \frac{1}{n+1}$ and $d_F(f(x_n), f(y_n)) \geq \epsilon$. Since E is compact, there exists a convergent subsequence $(x_{\phi(n)})_{n \in \mathbb{N}}$. Then there exists a convergent subsequence $(y_{\phi(\psi(n))})_{n \in \mathbb{N}}$ of $(y_{\phi(n)})_{n \in \mathbb{N}}$. Let $l_x = \lim_{n \to \infty} x_{\phi(\psi(n))}$ and $l_y = \lim_{n \to \infty} y_{\phi(\psi(n))}$. Now $l_x - l_y = \lim_{n \to \infty} y_{\phi(\psi(n)))} - x_{\phi(\psi(n))} = 0$. On the other hand, using the continuity of f, we have $\epsilon \leq d_F(f(l_x), f(l_y))$ which is a contradiction. □

REMARK 6.6. The notion of completeness of metric spaces is not topological: there exist two homeomorphic metric spaces, one complete and one not. A possible choice of morphisms for the category of metric spaces are the uniformly continuous functions, in which case isomorphic metric spaces are either all complete or all not complete.

Now, if E and F are *compact* metric spaces, and if they are homeomorphic, then E is complete if and only if F is complete. This is a very remarkable fact: a purely topological notion (compactness) imposes a metric notion (completeness). This is seen by observing that a Cauchy sequence has at most one accumulation point, and we have seen that every sequence of a compact space has at least one. With the Heine theorem, we see a different reason: an homeomorphism between compact metric spaces is always bi-uniformly continuous, and hence preserve Cauchy sequences.

7. Filter characterization of compactness

THEOREM 7.1. *Let (E, τ) be a topological space. (E, τ) is compact if and only if every ultrafilter in E converge.*

PROOF. Assume that (E, τ) is compact. Let $\mathscr{F}$ be an ultrafilter in E. Assume it does not converge. Then for all $x \in E$ there exists $U_x \in \tau$ with $x \in U_x$ such that $U_x \notin \mathscr{F}$. As $\mathscr{F}$ is an ultrafilter, $\complement_E U_x \in \mathscr{F}$. Now, $E = \bigcup_{x \in E} U_x$, and as τ is compact, there exists a finite subset F of E such that $E = \bigcup_{x \in F} U_x$. Now, $\varnothing = \bigcap_{x \in F} \complement_E U_x \in \mathscr{F}$ which is a contradiction. So $\mathscr{F}$ converges.

Assume now that all ultrafilters in E converge in τ. Assume τ is not compact. Let $\mathscr{W} \subseteq \tau$ such that $E = \bigcup \mathscr{W}$, with $\mathscr{W}$ infinite, and such that no finite subset of $\mathscr{W}$ covers E. Let:

$$\mathscr{F} = \{\complement_E \bigcup_{U \in F} U : F \subseteq \mathscr{W}, \text{finite}\}.$$

Since no finite subset of $\mathscr{W}$ covers E, the emptyset is not in $\mathscr{F}$. By construction, $\mathscr{F}$ is closed by finite intersections, and thus it is a filter basis. Let $\mathscr{U}$ be an ultafilter containing $\mathcal{F}$. There exists $x \in E$ such that $\mathscr{U}$ converges to x by assumption. Now, there exists $W \in \mathscr{W}$ such that $x \in W$ and thus $W \in \mathscr{U}$. Yet $E \setminus W \in \mathscr{F}$, so $\varnothing = W \cap E \setminus W \in \mathscr{U}$ which is a contradiction. □

8. Tychonoff theorem

THEOREM 8.1. *Let $(E_i, \tau_i)_{i \in I}$ be a family of compact topological spaces. Let τ be the product topology of $(\tau_i)_{i \in I}$. Then $(\prod_{i \in I} E_i, \tau)$ is compact.*

PROOF. Let $\mathscr{F}$ be an ultrafilter on $\prod_{i \in I} E_i$. Let $i \in I$. Let $p_i : \prod_{j \in I} E_j \twoheadrightarrow E_i$ be the canonical surjection. Let $\mathcal{F}_i = p_i[\mathcal{F}]$. Now, let $A \in \mathcal{F}_i$ and $C \subseteq E_i$ such that $A \subseteq C$. By definition of $\mathcal{F}_i$, there exists $B \in \mathcal{F}$ such that $A = p_i(B)$. Now $B \subseteq p_i^{-1}(p_i(B)) = p_i^{-1}(A) \subseteq p_i^{-1}(C)$; as $\mathcal{F}$ is a filter, $p_i^{-1}(A) \in \mathcal{F}$, hence $p_i^{-1}(C) \in \mathcal{F}$. So $p_i(p_i^{-1}(C)) = C$ is in $\mathcal{F}_i$ (note: p_i is surjective).

If $A, B \in \mathcal{F}_i$ then a similar reasoning proves that $A \cap B \in \mathcal{F}_i$. Moreover, $\varnothing \notin \mathcal{F}_i$ since $\varnothing \notin \mathcal{F}$. So $\mathcal{F}$ is a filter.

Let now $A \subseteq E_i$ for some $i \in I$. Since $\mathcal{F}$ is an ultrafilter, then either $p_i^{-1}(A)$ or $p_i^{-1}(E_i \setminus A)$ lies in $\mathcal{F}$. As p_i is a surjection, either $A = p_i(p_i^{-1}(A))$ or its complement lies in $\mathcal{F}_i$. Hence $\mathcal{F}_i$ converges to some x_i as τ_i is compact.

Let us now show that $\mathcal{F}$ converges to $(x_i)_{i \in I}$ in the product topology. Let $U \in \tau$ containing $(x_i)_{i \in I}$. Let $W \in \tau$ and $J \subseteq I$ finite such that $p_i(W) = E_i$ for $i \in I \setminus J$, and $W \subseteq U$, $(x_i)_{i \in I} \in W$. Write $W_i = p_i(W)$ for all $i \in I$, and note that $x_i \in W_i$, with $W_i \in \tau_i$ for all $i \in I$. Also note that $W = \bigcap_{i \in J} p_i^{-1}(W_i)$. Now, since $\mathcal{F}_i$ converges to x_i, we have $W_i \in \mathcal{F}_i$. Hence, as above, $p_i^{-1}(W_i) \in \mathcal{F}$. As $\mathcal{F}$ is closed under finite intersection, $W \in \mathcal{F}$. As $\mathcal{F}$ is a filter, $U \in \mathcal{F}$. So $\mathcal{F}$ converges to $(x_i)_{i \in I}$ as desired. $\qquad\square$