[plain]

A Primer on Vector Autoregressions

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[DISCLAIMER]

These notes are meant to provide intuition on the basic mechanisms of VARs

As such, most of the material covered here is treated in a very informal way

If you crave a formal treatment of these topics, you should stop here and buy a copy of Hamilton's "Time Series Analysis"

The Matlab codes accompanying these notes are available at:
`https://github.com/ambropo/VAR-Toolbox`

The job of macro-econometricians

- In their 2001 Journal of Economic Perspectives' article "Vector Autoregressions" Stock and Watson describe the job of macroeconometricians as consisting of the following:
  - Describe and summarize macroeconomic time series
  - Make forecasts
  - Recover the structure of the macroeconomy from the data
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  - Recover the structure of the macroeconomy from the data Main focus of these notes
  - Advise macroeconomic policy-makers
- Vector autoregressive models (VARs) are a statistical tool to perform these tasks

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- A VAR can help us answering the following questions

  [1] What is the dynamic behavior of these variables? How do these variables interact?

  [2] What is the most likely behavior of GDP in the next few quarters?

  [3] What is the effect of a monetary policy shock on GDP?

  [4] What has been the historical contribution of monetary policy shocks to GDP fluctuations?

What is a Vector Autoregression (VAR)?

- Consider a (2×1) vector of zero-mean time series x_t, composed of t observations and an initial condition x_0

$$x_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1t} \\ x_{21} & x_{22} & \dots & x_{2t} \end{bmatrix}$$

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- Assume that the two time series in x_t are covariance stationary, which means (for $i = 1, 2$)
  - Constant mean $E[x_{it}] = \mu_i$
  - Constant variance $V[x_{it}] = \sigma_i$
  - Constant autocovariance $COV[x_{it}, x_{it+\tau}] = \gamma_i(\tau)$

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- A **structural VAR** of order 1 is given by

$$x_t = \Phi x_{t-1} + B\varepsilon_t$$

where
  * Φ and B are (2×2) matrices of coefficients
  * ε_t is an (2×1) vector of unobservable zero-mean white noise processes

Three different ways of writing the same thing

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- Or as a system of linear equations

$$\begin{cases} x_{1t} = \phi_{11} x_{1,t-1} + \phi_{12} x_{1,t-1} + b_{11}\varepsilon_{1t} + b_{12}\varepsilon_{2t} \\ x_{2t} = \phi_{21} x_{2,t-1} + \phi_{22} x_{2,t-1} + b_{21}\varepsilon_{1t} + b_{22}\varepsilon_{2t} \end{cases}$$

The structural shocks

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The structural shocks

- We defined ε_t as a *vector of unobservable zero mean white noise processes*. **What does it mean?**
- The elements of ε_t are serially uncorrelated and independent of each other
- In other words

$$\varepsilon_t = (\varepsilon'_{1t}, \varepsilon'_{2t})' \sim \mathcal{N}(0, I_2)$$

where

$$\mathbb{V}(\varepsilon_t) = \Sigma_\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad CORR(\varepsilon_t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Why is it called 'structural' VAR?

- Go back to our bivariate structural VAR(1)

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- The structural VAR can be thought of as a description of the true structure of the economy
  - E.g.: an approximation of the structure of a DSGE model
- The structural shocks are shocks with a well-defined economic interpretation
  - E.g.: TFP shocks or monetary policy shocks
  - As $\varepsilon_t \sim \mathcal{N}(0, I_2)$ we can move one shock keeping the other shocks fixed
  - That is: we can focus on the causal effect of one shock at the time

Structural VARs can answer many interesting questions...

- Go back to our bivariate structural VAR(1). To make a concrete example, assume that
  - x_{1t} and x_{2t} are output growth (y_t) and the policy rate (r_t), both demeaned
  - ε_{1t} and ε_{2t} are a demand shock (ε_t^{Demand}) and a monetary policy shock (ε_t^{MonPol})
  - B is known (we'll get back to this in a second)

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Impact matrix

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  - The Φ matrix allows us to trace the 'dynamic effect' of the monetary policy shock over time
  - (We can add additional variables and look at other shocks: aggregate supply, oil price,...)

… but the estimation of structural VARs is tricky

- **Problem** The structural shocks ε_t are unobserved. How can we estimate B?

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- Best we can do is to 'bundle' the ε_t into a single object:

$$u_t = B\varepsilon_t \Rightarrow \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix} \Rightarrow \begin{cases} u_{yt} = b_{11}\varepsilon_t^{Demand} + b_{12}\varepsilon_t^{MonPol} \\ u_{rt} = b_{21}\varepsilon_t^{Demand} + b_{22}\varepsilon_t^{MonPol} \end{cases}$$

- Why is this useful? The VAR becomes

$$x_t = \Phi x_{t-1} + u_t$$

- Now we can estimate Φ and u_t with OLS (where u_t will be OLS residuals)

The reduced-form VAR

- This alternative formulation of the VAR is called the **reduced-form VAR** representation

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- In matrix form

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- Or as a system of linear equations

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The reduced-form covariance matrix

- A key object of interest in VARs is the covariance matrix of the **reduced-form residuals**

$$\Sigma_u = \begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ \sigma_{yr}^2 & \sigma_r^2 \end{bmatrix}$$

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- This is because the elements of u_t inherit all the contemporaneous relations among the endogenous variables x_t
  - To see that, remember how the reduced form residuals are defined

$$\begin{cases} u_{yt} = b_{11}\varepsilon_t^{Demand} + b_{12}\varepsilon_t^{MonPol} \\ u_{rt} = b_{21}\varepsilon_t^{Demand} + b_{22}\varepsilon_t^{MonPol} \end{cases}$$

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- This is the essence of **identification** in VARs

The Wold representation

- Before turning to identification, let's introduce another representation of the VAR that will be useful later

- Start from the structural VAR representation

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$$\begin{aligned} x_t &= \Phi x_{t-1} + B\varepsilon_t \\ &= \Phi \boxed{\left(\Phi x_{t-2} + B\varepsilon_{t-1}\right)} + B\varepsilon_t \end{aligned}$$

x_{t-1}

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$$\begin{aligned} x_t &= \Phi x_{t-1} + B\varepsilon_t \\ &= \Phi \underbrace{\left(\Phi x_{t-2} + B\varepsilon_{t-1}\right)}_{x_{t-1}} + B\varepsilon_t = \Phi^2 x_{t-2} + \Phi B\varepsilon_{t-1} + B\varepsilon_t \\ &= \dots \\ &= \Phi^t x_0 + \sum_{j=0}^{t-1} \Phi^j B\varepsilon_{t-j} \end{aligned}$$

The Wold representation (cont'd)

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  - The cumulative sum of current and past structural shocks

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Initial condition

Current & past shocks

  - The cumulative sum of current and past structural shocks
  - An initial condition

- If we let $t \to \infty$ we get

$$x_t = \Phi^{\infty} x_{t-\infty} + \sum_{j=0}^{\infty} \Phi^j B \varepsilon_{t-j}$$

- You may remember we assumed that x_t is covariance stationary
  - How do these infinite sums relate to that assumption? Aren't the increasing powers of Φ exploding?

Stability of the VAR

- A VAR is stable if the effect of shocks progressively dissipate over time. For that to happen we need Φ^j to converge to zero

$$x_t = \Phi^{\infty} x_{t-\infty} + \sum_{j=0}^{\infty} \Phi^j B \varepsilon_{t-j}$$

- Why does this matter? If shocks have permanent effects
  - The mean and the variance of x_t will depend on the history of shocks
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- **Definition** A VAR is called stable iff all the eigenvalues of Φ are less than 1 in modulus.More formally:

$$\det(\Phi - \lambda I_2) = 0 \quad |z| < 1$$

- **Implication** In the absence of shocks, the VAR will converge to its equilibrium (i.e. its unconditional mean)

The unconditional mean of the VAR

- First note that if the eigenvalues of Φ are less than 1 in modulus we have

$$\Phi^{\infty} = 0 \quad \text{and} \quad \sum_{j=0}^{\infty} \Phi^{j} = \boxed{(I_2 - \Phi)^{-1}}$$

Geometric series

- The unconditional mean therefore is simply given by

$$\mathbb{E}[x_t] = \Phi^{\infty} x_{t-\infty} + \sum_{j=0}^{\infty} \Phi^{j} \mathbb{E}[u_{t-j}] = 0$$

- Note that if the VAR had a constant (α) an additional term would show up in the Wold representation

$$\mathbb{E}[x_t] = \Phi^{\infty} x_{t-\infty} + \Phi^{\infty}\alpha + \sum_{j=0}^{\infty} \Phi^{j} \mathbb{E}[u_{t-j}] = 0$$

- The unconditional mean in this case would be

$$\mathbb{E}[x_t] = (I_2 - \Phi)^{-1}\alpha$$

The general form of the stationary structural VAR(p) model

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- Model can be enriched with along the following dimensions
  - Increase the number of endogenous variables (k)
  - Increase the number of lags (p)
  - Add deterministic terms (e.g. time trend or seasonal dummy variables)
  - Add exogenous variables (e.g. price of oil from the point of view of a small country)

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- The general form of the VAR(p) model with deterministic terms (Z_t) and exogenous variables (W_t) is given by

$$x_t = \Phi_1 x_{t-1} + \Phi_2 x_{t-2} + \ldots + \Phi_p x_{t-p} + \Lambda Z_t + \Psi W_t + B\varepsilon_t$$

The Identification Problem

Back to our reduced form VAR

- We have seen above that with OLS we can only estimate the reduced-form VAR (and not the structural VAR)

- Assume we already have an OLS estimate of $\hat{\Phi}$ and $\hat{u}_t$:

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} u_t^y \\ u_t^r \end{bmatrix}$$

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- **Question** What is the effect of a monetary policy shock on GDP growth?

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- **Question** What is the effect of a monetary policy shock on GDP growth?
- Unfortunately, the reduced form innovations (u_t^y or u_t^r) are not going to help us in answering the question

Reduced-form VARs do not tell us anything about causality

- To see that, assume that the 'true' (and unobserved) model of the economy is given by

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- It is obvious that the reduced form innovations are a linear combination of the two structural shocks

$$\begin{aligned} u_t^y &= b_{11}\varepsilon_t^{Demand} + b_{12}\varepsilon_t^{MonPol} \\ u_t^r &= b_{21}\varepsilon_t^{Demand} + b_{22}\varepsilon_t^{MonPol} \end{aligned}$$

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$$\begin{aligned} u_t^y &= b_{11}\varepsilon_t^{Demand} + b_{12}\varepsilon_t^{MonPol} \\ u_t^r &= b_{21}\varepsilon_t^{Demand} + b_{22}\varepsilon_t^{MonPol} \end{aligned}$$

- An increase in u_t^r is not a monetary policy shock!

Reduced-form VARs do not tell us anything about causality

- To see that, assume that the 'true' (and unobserved) model of the economy is given by

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

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- An increase in u_t^r could be due to
  - [1] A positive demand shock that increases both output growth and the policy rate ($b_{21} > 0$)
  - [2] Or a monetary policy shock that decreases output growth and increases the policy rate ($b_{22} > 0$)

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- How to know whether is [1] or [2]? This is the very nature of the **identification problem**

The identification problem

- The identification problem consists in finding a mapping from the reduced form VAR to its structural counterpart

$$u_t \Rightarrow B\varepsilon_t$$

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- To do that, we can exploit the relation between reduced form and structural innovations ($u_t = B\varepsilon_t$) to write

$$\Sigma_u = \mathbb{E}\left[u_t u_t'\right] = \mathbb{E}\left[B\varepsilon_t \left(B\varepsilon_t\right)'\right] = B\mathbb{E}(\varepsilon_t \varepsilon_t')B' = B\Sigma_\varepsilon B' = BB'$$

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- The identification problem simply boils down to finding a B matrix that satisfies $\Sigma_u = BB'$

- Unfortunately this is not as easy as it sounds. Why?

  * **Hint** There are infinite combinations of B that give you the same Σ_u

The identification problem (cont'd)

- Think of $\Sigma_u = BB'$ as a system of equations

$$\begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ - & \sigma_r^2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}$$

- Can be rewritten as

$$\begin{cases} \sigma_y^2 = b_{11}^2 + b_{12}^2 \\ \sigma_{yr}^2 = b_{11}b_{21} + b_{12}b_{22} \\ \sigma_{yr}^2 = b_{11}b_{21} + b_{12}b_{22} \\ \sigma_r^2 = b_{21}^2 + b_{22}^2 \end{cases}$$

- **Problem** Because of the symmetry of the Σ_u matrix, the second and the third equation are identical

- We are left with 4 unknowns (the elements of B) but only 3 equations!

How to solve the identification problem?

- How to solve a system of 3 equations in 4 unknowns?

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- **Example** If you believe that monetary policy works with a lag and has no effect on output growth on impact, you can *assume* $b_{12} = 0$

- The assumption buys us an additional equation

$$\begin{cases} \sigma_y^2 = b_{11}^2 + b_{12}^2 \\ \sigma_{yr}^2 = b_{11}b_{21} + b_{12}b_{22} \\ \sigma_r^2 = b_{21}^2 + b_{22}^2 \\ b_{12} = 0 \end{cases}$$

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- The system now can be easily solved

Common Identification Schemes

Common identification schemes

- Zero (recursive) contemporaneous restrictions
- Zero (recursive) long-run restrictions
- Sign restrictions
- External instruments
- Combining sign restrictions and external instruments
- Other (narrative sign restrictions, maximization of forecast error variance,...)

Common Identification Schemes

Zero short-run restrictions

Zero contemporaneous restrictions

- **Intuition** Identification is achieved by assuming that some shocks have zero contemporaneous effect on some of the endogenous variables

- **References** Sims (1980), Christiano, Eichenbaum, Evans (1999)

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Zero contemporaneous restrictions

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By assumption

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & \boxed{0} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

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- **Implication** We now have 3 structural parameters to estimate (instead of 4) and 3 restrictions implied by Σ_u

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How to achieve identification?

- The system of equations implied by $\Sigma_u = BB'$ now becomes

$$\begin{bmatrix} \sigma_y^2 & \sigma_{yr} \\ - & \sigma_r^2 \end{bmatrix} = \begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} \\ 0 & b_{22} \end{bmatrix}$$

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- And can be easily solved to get:

$$\begin{cases} b_{11} = \sigma_y^2 \\ b_{21} = \sigma_{yr} / \sigma_y^2 \\ b_{22} = \sqrt{\sigma_r^2 - \frac{\sigma_{yr}^2}{\sigma_y^2}} \end{cases}$$

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Impact effects

- We can now derive the impact effects of shocks by simply re-writing the structural VAR as

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma_y^2 & 0 \\ \sigma_{yr}/\sigma_y^2 & \sqrt{\sigma_r^2 - \frac{\sigma_{yr}^2}{\sigma_y^2}} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

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- A one standard deviation shock to monetary policy ($\varepsilon_t^{MonPol} = 1$) in t leads to

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- A one standard deviation shock to aggregate demand ($\varepsilon_t^{Demand} = 1$) in t leads to

$$\begin{cases} y_t = \sigma_y^2 \\ \pi_t = \sigma_{yr}/\sigma_y^2 \end{cases}$$

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Aka Cholesky identification

- This identification scheme is normally implemented via a Cholesky decomposition of Σ_u
- A Cholesky decomposition allows us to decompose Σ_u into the product of a lower triangular matrix P times its transpose

$$\Sigma_u = PP'$$

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- In matrix form we have

Lower Cholesky factor

$$\begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ \sigma_{yr}^2 & \sigma_r^2 \end{bmatrix} = \boxed{\begin{bmatrix} p_{11} & 0 \\ p_{21} & p_{22} \end{bmatrix}} \begin{bmatrix} p_{11} & p_{21} \\ 0 & p_{22} \end{bmatrix}$$

Cholesky decomposition of a matrix [Back to basics]

- The Cholesky decomposition is (roughly speaking) the square root of a matrix
  - As for a square root, you can't compute a Cholesky decomposition for a non positive-definite matrix

- A symmetric and positive-definite matrix A can be decomposed as:

$$A = PP'$$

  where P is a lower triangular matrix (and therefore P' is upper triangular)

- The formula for the decomposition of a 2×2 matrix is

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad P = \begin{bmatrix} \sqrt{a} & \frac{b}{\sqrt{a}} \\ 0 & \sqrt{c - \frac{b}{a}} \end{bmatrix}$$

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- Then we can use the Cholesky decomposition to write

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- As both P and B are lower triangular, it must follow that $P = B$

Common Identification Schemes

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- **Intuition** Identification is achieved by assuming that some shocks have zero cumulative effect on some of the endogenous variables in the long run

- **References** Blanchard and Quah (1989), Gali (1992)

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Impact in t

Impact in t+1

etc...

- We can rewrite

$$x_{t,t+\infty} = \sum_{j=0}^{\infty} \Phi^j B\varepsilon_t = (I - \Phi)^{-1} B\varepsilon_t = C\varepsilon_t$$

where $C \equiv (I - \Phi)^{-1}$ is the cumulative effect that ε_t has on output growth from time t to ∞, i.e. the effect on output level

Zero long-run restrictions

How to compute the cumulative long-run effects of shocks?

- What is the intuition for C?
- Go back to our output growth / policy rate example

$$\begin{bmatrix} y_{t,t+\infty} \\ r_{t,t+\infty} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

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- Take the first equation: $y_{t,t+\infty} = c_{11}\varepsilon_t^{Demand} + c_{12}\varepsilon_t^{MonPol}$
  - The coefficient c_{12} represents the impact of a monetary policy shock (hitting in t) on the **level of GDP** in the long-run
  - If you believe in the long-run neutrality of monetary policy you would expect $c_{12} = 0$

Zero long-run restrictions

How to achieve identification?

- Remember that $C \equiv (I - \Phi)^{-1} B$ and B is unobserved. So, how does this help with the identification of B?

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- We achieved identification: $B = (I - \Phi) P$

Common Identification Schemes

Sign restrictions

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  - Demand shocks should lead to an increase in output and interest rates
  - Monetary policy shocks should lead to a fall in output for an increase in interest rates

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- The matrix $B = PQ$ is a valid 'candidate' impact matrix that solves the identification problem!
  - Differently from P, the matrix PQ is not lower triangular anymore

Orthonormal matrix [Back to basics]

- An orthonormal matrix Q is a real square matrix whose columns and rows are orthogonal unit vectors

- What does it mean? Take for example two 2×1 vectors q_1 and q_2, then the matrix $Q = (q_1, q_2)$ is orthonormal if
  - The vectors have unit norm: $\| q_i \| = 1$
  - The vectors are mutually orthogonal: $q_1^T q_2 = 0$

- It follows that

$$QQ' = I$$

- And

$$Q' = Q^{-1}$$

Sign restrictions

How to achieve identification?

- But Q is a random matrix... How can we check that $B = PQ$ represents a plausible solution?

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[2] Then check that the elements of B satisfy

	Demand (ε_t^{Demand})	Monetary Policy (ε_t^{MonPol})
Output growth (y_t)	$b_{11} > 0$?	$b_{12} < 0$?
Short-rate Int. Rate(r_t)	$b_{21} > 0$?	$b_{22} > 0$?

Sign restriction in steps

- Perform N replications of the following steps
  - [1] Draw a random orthonormal matrix Q
  - [2] Compute $B = PQ$ where Q is the Cholesky decomposition of the reduced form residuals Σ_u
  - [3] Compute the impact effects of shocks associated with B
  - [4] Are the sign restrictions satisfied?
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    - [4.1] Yes. Store B and go back to [1]
    - [4.2] No. Discard B and go back to [1]
- All matrices in the set $B^{(i)}$ (for $i = 1, 2, ..., N$) represent admissible solutions to the identification problem
- In this sense, sign restricted VARs are only **set identified**

Common Identification Schemes

External Instruments (or Proxy SVARs)

External instruments

- **Intuition** Exploits information from a variable that is *external* to the VAR, but that is correlated with a particular shock of interest and uncorrelated with other shocks (the *instrument*)

- **References** Stock and Watson (2012), Mertens and Ravn (2013)

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- For example, assume that you have some 'narrative' series of policy surprises (i.e. that are not just a response of policy to some development in the literature)
- But how can this help in finding the B matrix?

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\begin{aligned}
\mathbb{E}\left[\varepsilon_t^{Demand} z_t'\right] &= 0, \\
\mathbb{E}[\varepsilon_t^{MonPol} z_t'] &= c,
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$$

- Then, we can identify one column (in this example, the second one) of the B matrix:

$$
\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} - & b_{12} \\ - & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}
$$

External instruments

- **Intuition** Recall that the reduced form residuals are a linear combination of the two structural shocks

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- The OLS estimate of ξ_t collects everything else that is uncorrelated with ε_t^{MonPol}

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$$\hat{u}_{rt} = \hat{\beta} z_t$$

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- If we normalize the effect of ε_t^{MonPol} on r_t to 1 (that is, we fix $b_{22} = 1$) we can easily recover b_{21} from the OLS estimates of γ

Structural Dynamic Analysis

Structural Dynamic Analysis

- We have seen how to 'identify' structural shocks by imposing some restrictions on the data
- But what can we do with that?

Structural Dynamic Analysis

- We have seen how to ‘identify’ structural shocks by imposing some restrictions on the data
- But what can we do with that?
  - Quantify the dynamic effect of a shock over time $\Rightarrow$ **Impulse responses**
  - Quantify how important a shock is in explaining the variation in the endogenous variables (on average) $\Rightarrow$ **Forecast error variance decomposition**
  - Quantify how important a shock was in driving the behavior of the endogenous variables in a specific time period in the past $\Rightarrow$ **Historical decompositions**
- We'll turn to this structural dynamic analysis next

Structural Dynamic Analysis

Impulse responses

Impulse response functions

- Impulse response functions (*IR*) answer the following question:

  What is the response over time of the VAR's endogenous variables to an innovation in the structural shocks, assuming that the other structural shocks are kept to zero?

Impulse response functions

- Impulse response functions (*IR*) answer the following question:

  What is the response over time of the VAR's endogenous variables to an innovation in the structural shocks, assuming that the other structural shocks are kept to zero?

- *IR* allow to single out the effect of a shock (e.g. its impact an persistence) keeping all else equal

- **Example** What is the impact of a monetary policy shock to GDP?

How to compute impulse response functions

- Consider our simple bivariate VAR

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

- Define a 2×1 vector of impulse selection (s) that takes value of one for the structural shock that we want to consider.

- For example, to compute the *IR* to the demand shock, define s as:

$$s = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- The impulse responses to ε_t^{Demand} can be easily computed with the following equation

$$x_t = \Phi x_{t-1} + Bs,$$

How to compute impulse response functions (cont'd)

- The IR can be computed recursively as follows

$$\begin{cases} IR_t = Bs, & \text{for } t = 0, \\ IR_t = \Phi \cdot IR_{t-1} & \text{for } t = 2, ..., h. \end{cases}$$

- Note that the impact response is simply given by the elements of the impact matrix B selected by s...

$$\begin{bmatrix} IR_0^y \\ IR_0^r \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$$

- ... while the responses at longer horizons are given the transition matrix

$$\begin{bmatrix} IR_h^y \\ IR_h^r \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} IR_h^y \\ IR_h^r \end{bmatrix}$$

The companion matrix [Back to basics]

- So far, we considered simple VAR(1) specifications. But what to do if the VAR has $p > 1$?
- Every VAR(p) can be written as a VAR(1) using the **companion representation**
  - For example, take a VAR(2)

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi^1_{11} & \phi^1_{12} \\ \phi^1_{21} & \phi^1_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} \phi^2_{11} & \phi^2_{12} \\ \phi^2_{21} & \phi^2_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon^{Demand}_t \\ \varepsilon^{MonPol}_t \end{bmatrix}$$

  - Re-write the VAR(2) as

$$\begin{bmatrix} y_t \\ r_t \\ y_{t-1} \\ r_{t-1} \end{bmatrix} = \begin{bmatrix} \phi^1_{11} & \phi^1_{12} & \phi^2_{11} & \phi^2_{12} \\ \phi^1_{21} & \phi^1_{22} & \phi^2_{21} & \phi^2_{22} \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \\ y_{t-2} \\ r_{t-2} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon^{Demand}_t \\ \varepsilon^{MonPol}_t \\ 0 \\ 0 \end{bmatrix}$$

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Companion matrix

$$\begin{bmatrix} y_t \\ r_t \\ y_{t-1} \\ r_{t-1} \end{bmatrix} = \boxed{\begin{bmatrix} \phi_{11}^1 & \phi_{12}^1 & \phi_{11}^2 & \phi_{12}^2 \\ \phi_{21}^1 & \phi_{22}^1 & \phi_{21}^2 & \phi_{22}^2 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}} \begin{bmatrix} y_{t-1} \\ r_{t-1} \\ y_{t-2} \\ r_{t-2} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \\ 0 \\ 0 \end{bmatrix}$$

  - To get a VAR(1) where $\tilde{\Phi}$ is the **companion matrix**

$$\tilde{x}_t = \tilde{\Phi}\tilde{x}_{t-1} + \tilde{B}\varepsilon_t$$

Structural Dynamic Analysis

Forecast Error Variance Decompositions

Forecast error variance decompositions

- Forecast error variance decompositions (*VD*) answer the following question:

  What portion of the variance of the VAR's forecast errors (at a given horizon h) is due to each structural shock?

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  What portion of the variance of the VAR's forecast errors (at a given horizon h) is due to each structural shock?

- *VD* provide information about the relative importance of each structural shock in affecting the forecast error variance of the VAR's endogenous variables

- **Example** What is the (average) importance of demand shocks in driving GDP forecast errors?

How to compute forecast error variance decompositions

- The forecast error of a variable at horizon $t+h$ is the change in the variable that couldn't have been forecast between $t-1$ and $t+h$ due to the realization of the structural shocks.

- For example, at $h=0$ we can compute the forecast error as

$$x_t - \mathbb{E}_{t-1}[x_t] = \Phi x_{t-1} + B\varepsilon_t - \Phi x_{t-1} = B\varepsilon_t$$

- At $h=1$, we have

$$\begin{aligned} x_{t+1} - \mathbb{E}_{t-1}[x_{t+1}] &= \Phi x_t + B\varepsilon_{t+1} - \Phi^2 x_{t-1} = \\ &= \Phi(\Phi x_{t-1} + B\varepsilon_t) + B\varepsilon_{t+1} - \Phi^2 x_{t-1} = \Phi B\varepsilon_t + B\varepsilon_{t+1} \end{aligned}$$

- So, in general we have

$$FE_{t+h} = x_{t+h} - E_{t-1}[x_{t+h}] = \sum_{i=0}^{h} \Phi^{h-i} B\varepsilon_{t+h}$$

- What is the variance of FE_{t+h}?

Basic properties of the variance

- If X is a random variable x and a is a constant
  - $\mathbb{V}(x+a) = \mathbb{V}(x)$
  - $\mathbb{V}(ax) = a^2\mathbb{V}(x)$

- If Y is a random variable and b is a constant
  - $\mathbb{V}(aX+bY) = a^2\mathbb{V}(x) + b^2\mathbb{V}(Y) + 2ab\mathbb{COV}(X,Y)$

- Since the structural errors are independent, it follows that $\mathbb{COV}\left(\varepsilon_{t+1}^{Demand}, \varepsilon_{t+1}^{MonPol}\right) = 0$

How to compute forecast error variance decompositions (cont'd)

- For simplicity consider $h = 0$, namely

$$\mathbb{V}\left(x_t - E_{t-1}[x_t]\right) = \mathbb{V}\left(B\varepsilon_t\right)$$

- Recalling that $\mathbb{V}\left(\varepsilon_t\right) = I_2$ and the structural shocks are orthogonal to each other, the variance of the forecast error can be computed as

$$\begin{aligned} \mathbb{V}\left(y_t - E_{t-1}[y_t]\right) &= b_{11}^2 \mathbb{V}\left(\varepsilon_t^{Demand}\right) + b_{12}^2 \mathbb{V}\left(\varepsilon_t^{MonPol}\right) = b_{11}^2 + b_{12}^2 \\ \mathbb{V}\left(r_t - E_{t-1}[r_t]\right) &= b_{21}^2 \mathbb{V}\left(\varepsilon_t^{Demand}\right) + b_{22}^2 \mathbb{V}\left(\varepsilon_t^{MonPol}\right) = b_{21}^2 + b_{22}^2 \end{aligned}$$

- What portion of the variance of the forecast error at $h = 0$ is due to each structural shock?

$$\underbrace{\left\{ \begin{aligned} VD_{y_0}^{\varepsilon^{Demand}} &= \frac{b_{11}^2}{b_{11}^2 + b_{12}^2} \\ VD_{y_0}^{\varepsilon^{MonPol}} &= \frac{b_{12}^2}{b_{11}^2 + b_{12}^2} \end{aligned} \right.}_{\text{This sums up to 1}} \qquad \underbrace{\left\{ \begin{aligned} VD_{r_0}^{\varepsilon^{Demand}} &= \frac{b_{21}^2}{b_{21}^2 + b_{22}^2} \\ VD_{r_0}^{\varepsilon^{MonPol}} &= \frac{b_{22}^2}{b_{21}^2 + b_{22}^2} \end{aligned} \right.}_{\text{This sums up to 1}}$$

Structural Dynamic Analysis

Historical Decompositions

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- *HD* allow to track, at each point in time, the role of structural shocks in driving the VAR's endogenous variables away from their steady state

- **Example** What was the contribution of oil shocks in driving GDP growth in 1973?

How to compute historical decompositions

- As an example, let's compute the *HD* of of the endogenous variables when $t = 2$ in our simple bivariate VAR

- Historical decompositions can be easily understood from the Wold representation of the VAR

$$x_t = \Phi^t x_0 + \sum_{j=0}^{t-1} \Phi^j B \varepsilon_{t-j}$$

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- Using the Wold representation, we can write x_2 as a function of present and past structural shocks (ε^{Demand} and ε^{MonPol}) plus the initial condition (x_0)

$$x_2 = \underbrace{\Phi^2 x_0}_{init} + \underbrace{\Phi B}_{\boldsymbol{\Theta_1}} \varepsilon_1 + \underbrace{B}_{\boldsymbol{\Theta}_0} \varepsilon_2$$

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- Re-write x_2 in matrix form

$$\begin{bmatrix} y_2 \\ r_2 \end{bmatrix} = \begin{bmatrix} init_y \\ init_r \end{bmatrix} + \begin{bmatrix} \theta_{11}^1 & \theta_{12}^1 \\ \theta_{21}^1 & \theta_{22}^1 \end{bmatrix} \begin{bmatrix} \varepsilon_2^{Demand} \\ \varepsilon_2^{MonPol} \end{bmatrix} + \begin{bmatrix} \theta_{11}^0 & \theta_{12}^0 \\ \theta_{21}^0 & \theta_{22}^0 \end{bmatrix} \begin{bmatrix} \varepsilon_2^{Demand} \\ \varepsilon_2^{MonPol} \end{bmatrix}$$

How to compute historical decompositions (cont'd)

- Then x_2 can be expressed as

$$\begin{cases} y_2 = init_y + \theta_{11}^1 \varepsilon_1^{Demand} + \theta_{12}^1 \varepsilon_1^{MonPol} + \theta_{11}^0 \varepsilon_2^{Demand} + \theta_{12}^0 \varepsilon_2^{MonPol} \\ r_2 = init_r + \theta_{21}^1 \varepsilon_1^{Demand} + \theta_{22}^1 \varepsilon_1^{MonPol} + \theta_{21}^0 \varepsilon_2^{Demand} + \theta_{22}^0 \varepsilon_2^{MonPol} \end{cases}$$

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- Then x_2 can be expressed as

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- The historical decomposition is given by

$$\underbrace{\begin{cases} HD_{y_2}^{\varepsilon^{Demand}} = \theta_{11}^1 \varepsilon_1^{Demand} + \theta_{11}^2 \varepsilon_2^{Demand} \\ HD_{y_2}^{\varepsilon^{MonPol}} = \theta_{12}^1 \varepsilon_1^{MonPol} + \theta_{12}^2 \varepsilon_2^{MonPol} \\ HD_{y_2}^{init} = init_y \end{cases}}_{\text{This sums up to } y_2} \qquad \underbrace{\begin{cases} HD_{r_2}^{\varepsilon^{Demand}} = \theta_{21}^1 \varepsilon_1^{Demand} + \theta_{21}^0 \varepsilon_2^{Demand} \\ HD_{r_2}^{\varepsilon^{MonPol}} = \theta_{22}^1 \varepsilon_1^{MonPol} + \theta_{22}^0 \varepsilon_2^{MonPol} \\ HD_{r_2}^{init} = init_r \end{cases}}_{\text{This sums up to } r_2}$$

Practical Examples

The VAR Toolbox

- We'll see in practice how VARs work through a set of examples using the **VAR Toolbox 3.0**
- The VAR Toolbox is a collection of Matlab routines to perform VAR analysis
  - Codes are available at `https://github.com/ambropo/VAR-Toolbox`
  - No installation is required. Simply clone the folder from Github and add the folder (with subfolders) to your Matlab path
  - To save figures in high quality format, you need to download an install Ghostscript (freely available at `www.ghostscript.com`).
    - The first time you'll be saving a figure using the Toolbox, you'll be asked to locate the Ghostscript on your local drive
- We'll start with a very simple example and then replicate the results from a few well-known papers

Adding the VAR Toolbox path to Matlab [Matlab tip]

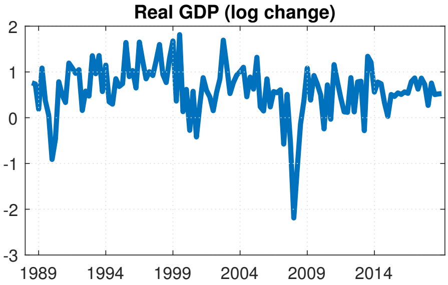
- To avoid clashes with functions from other toolboxes, it is recommendable to add and remove the Toolbox at beginning and end of your scripts
- If you download the toolbox to `C:/AMPER/VARToolbox`, you can simply add the following lines at the beginning and end of your script

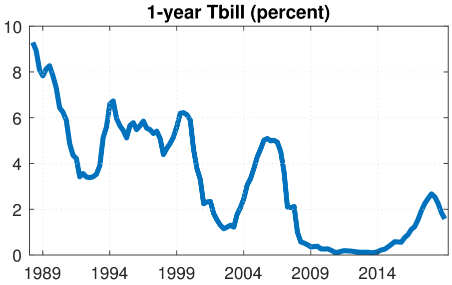
```
addpath(genpath('C:/VAR-Toolbox/'))
...
rmpath(genpath('C:/VARToolbox'))
```

A simple bivariate VAR model

- Our first example will be a simple bivariate VAR as the one considered above

- US quarterly data from 1989q1to 2019q4 on output growth (y_t) and the 1-year T-bill (r_t)





A simple bivariate VAR model

- As both GDP growth an the 1-year rate are non-zero means, we fit the data with a VAR(1) with a constant

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \alpha_y \\ \alpha_\pi \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} u_t^y \\ u_t^\pi \end{bmatrix}$$

- This means we will estimate the following parameters
  - $2 + 4$ coefficients, namely the elements of α and Φ
  - 2 variances of the reduced-form residuals, namely σ_y^2 and σ_π^2
  - 1 covariance of the reduced-form residuals, namely $\sigma_{y\pi}$

- We will store these coefficients in two Matlab matrices

$$\texttt{F} = \begin{bmatrix} \alpha_1 & \phi_{11} & \phi_{12} \\ \alpha_2 & \phi_{21} & \phi_{22} \end{bmatrix} \qquad \texttt{sigma} = \begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ \sigma_{yr}^2 & \sigma_r^2 \end{bmatrix}$$

A simple bivariate VAR model

- In Matlab we store the data in the matrix `X`

$$\texttt{X} = \begin{bmatrix} y_1 & r_1 \\ y_2 & r_2 \\ \dots & \dots \\ y_T & r_T \end{bmatrix} = (y_t', r_t') = x_t'$$

- The VAR can then be easily estimated with a few lines of code

```
% Set the deterministic variable in the VAR (1=constant, 2=trend)
det = 1;
% Set number of nlags
nlags = 12;
% Estimate VAR by OLS
[VAR, VARopt] = VARmodel(ENDO,nlags,det);
```

A simple bivariate VAR model: VAR output

- The code estimates the VAR equation by equation with OLS
- Results are stored in the `VAR` and `VARopt` structures
- The six estimated parameters (i.e. α and Φ) can be printed at screen by simply typing `disp(VAR.Ft)` to get

```
>> disp(VAR.F)
    0.3630    0.3788    0.0041
   -0.0729    0.2607    0.9541
```

- For the three elements of Σ_u type `disp(VAR.sigma)` to get

```
>> disp(VAR.sigma)
    0.2891    0.0782
    0.0782    0.1473
```

OLS estimation: Typical VAR output (cont'd)

- The off-diagonal elements of Σ capture the **average** contemporaneous relation between the endogenous variables

	GDP growth (u_y)	1-year T-Bill(u_r)
Real GDP (u_y)	0.2891	0.0782
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  - It means that, on average, when GDP growth increases inflation increases, too
- Does it mean that a shock to output always increase inflation?
  - No! Recall that reduced from residuals are not informative about structural shocks

Model checking & tuning

- These notes do not cover this aspect in detail but...
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- ... and that the VAR is stable

Stability and equilibrium

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```
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```

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    0.9559
```

- You can also check all of Φ's eigenvalues by executing Matlab's `eig` function on the VAR's companion matrix `Fcomp` (which, note, is built excluding the constant from `F`).

- In practice:

```
>> disp(eig(VAR.Fcomp))
    0.3769
    0.9559
```

Stability and equilibrium (cont'd)

- As our VAR is stable, its Wold representation will converge to the (finite) unconditional mean of the data
- To see that, first consider the Wold representation in the presence of a constant

$$x_t = \Phi^t x_0 + \sum_{j=0}^{t-1} \Phi^t \alpha + \sum_{j=0}^{t-1} \Phi^j B \varepsilon_{t-j}$$

- For t large enough and taking expectations we get

$$\mathbb{E}\left[x_t\right] = \sum_{j=0}^{t-1} \Phi^t \alpha = \left(I_2 - \Phi\right)^{-1} \alpha$$

- In absence of shocks, the VAR's variable will converge to its equilibrium $\left(I_2 - \Phi\right)^{-1} \alpha$ at a rate that depends on Φ

Examples of different identification schemes

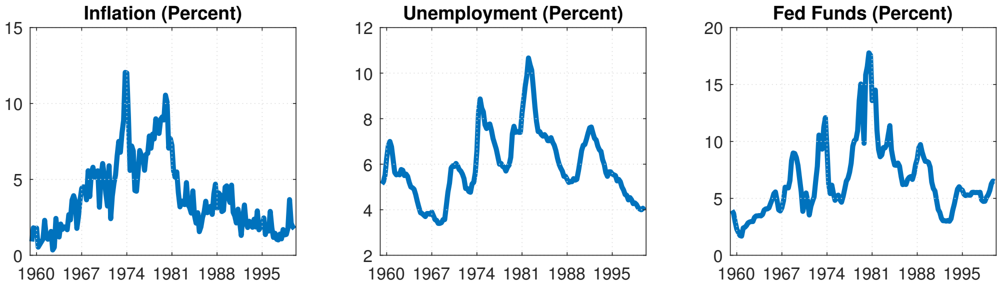
- Zero short-run restrictions
  - Stock & Watson (2001). "Vector Autoregressions," *Journal of Economic Perspectives*
- Zero long-run restrictions
  - Blanchard & Quah (1989). "The Dynamic Effects of Aggregate Demand and Supply Disturbances", *American Economic Review*
- Sign Restrictions
  - Uhlig (2005). "What are the effects of monetary policy on output? Results from an agnostic identification procedure," *Journal of Monetary Economics*
- External instruments
  - Gertler and Karadi (2015). "What are the effects of monetary policy on output? Results from an agnostic identification procedure," *American Economic Journal: Macroeconomics*
- External instruments & Sign restrictions
  - Cesa-Bianchi and Sokol (2020). "Financial Shocks, Credit Spreads, and the International Credit Channel," *Unpublished manuscript*

Practical Examples

Stock & Watson (2001, JEP)

Stock & Watson (2001): Zero short-run restrictions

- Stock & Watson (2001). "Vector Autoregressions," *Journal of Economic Perspectives*
- US quarterly data from 1960QI to 2000Q4



Monetary policy shocks, inflation and unemployment

- **Objective** Infer the causal influence of monetary policy on unemployment and inflation

Monetary policy shocks, inflation and unemployment

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- Assume a VAR with $p = 4$ with inflation (π_t), unemployment (u_t), and the fed funds rate (r_t)
- **Key identifying assumptions**
  - MP (r_t) reacts contemporaneously to movements in inflation and in unemployment
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$$\begin{bmatrix} \pi_t \\ u_t \\ r_t \end{bmatrix} = \sum_{p=1}^{4} \Phi_p x_{t-p} + \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_t^1 \\ \varepsilon_t^2 \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

Monetary policy shocks, inflation and unemployment

- In Matlab, set lag length to 4 and estimate a VAR with a constant

```
% Set up and estimate VAR
det = 1;
nlags = 4;
[VAR, VARopt] = VARmodel(X,nlags,det);
```

- Then set the option for recursive identification `VARopt.ident ='short'` and compute the *IR* with the `VARir` function. Note that the ordering of the variables matter!

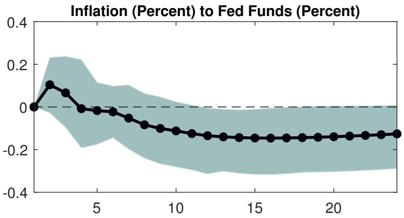
```
% For zero contemporaneous restrictions set:
VARopt.ident = 'short';
% Compute IR
[IR, VAR] = VARir(VAR,VARopt);
```

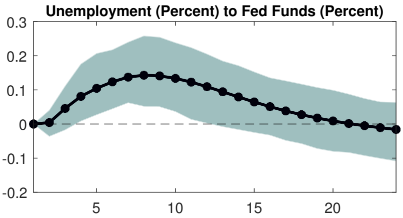
- The `VARirband` function allows tlo compute confidence intervals

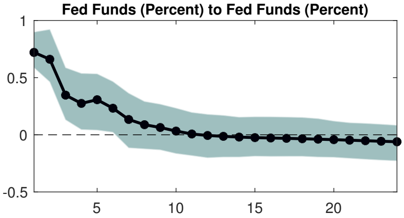
```
% Compute IR
[IR, VAR] = VARir(VAR,VARopt);
```

The effect of a monetary policy shock

- Monetary policy shock raises inflation in the short run (price puzzle) and increases unemployment







The other two shocks are identified by definition... but how can we interpret them?

- How about ε_t^1 and ε_t^2?
  - The shock ε_t^1 affects all variables contemporaneously
  - The shock ε_t^2 affects r_t contemporaneously but not π_t

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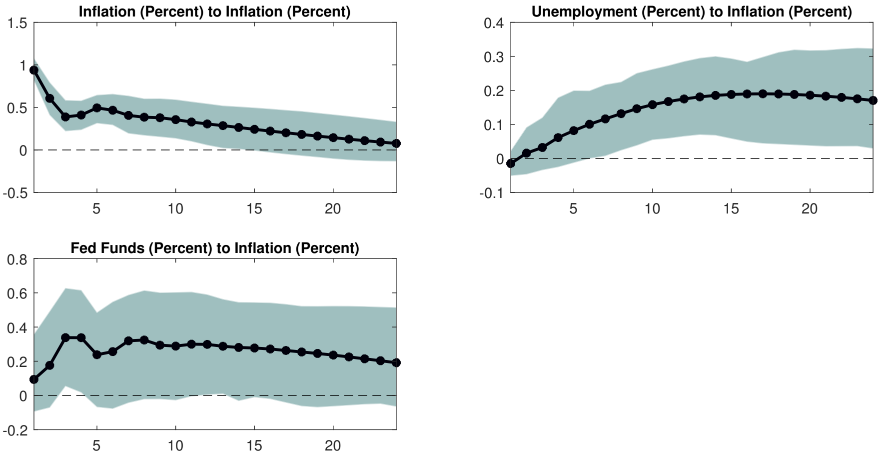
  - The shock ε_t^1 affects all variables contemporaneously
  - The shock ε_t^2 affects r_t contemporaneously but not π_t
- Can we interpret these shocks? Are the assumptions consistent with any theoretical mechanism?
- Some shocks may be better identified than others

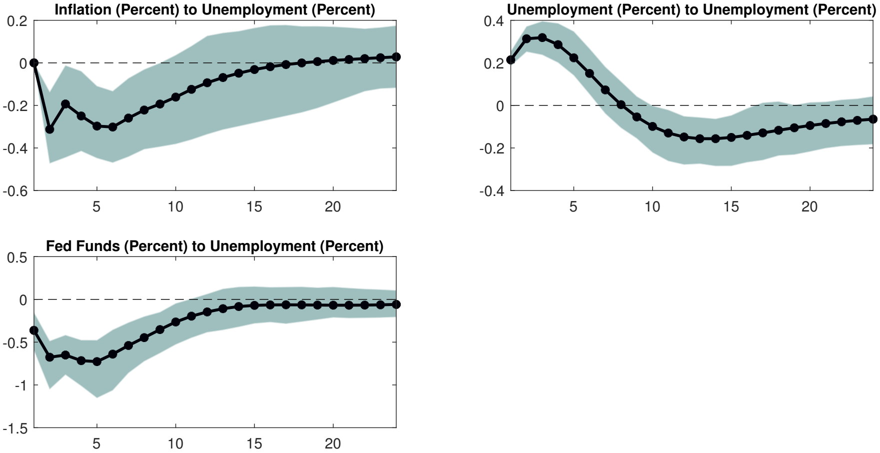
The other two shocks are identified by definition... but how can we interpret them?

- Shock to ε_t^1 behaves as a negative aggregate supply shock



The other two shocks are identified by definition... but how can we interpret them?

- Shock to ε_t^2 behaves as a negative aggregate demand shock



Forecast error variance & Historical decompositions

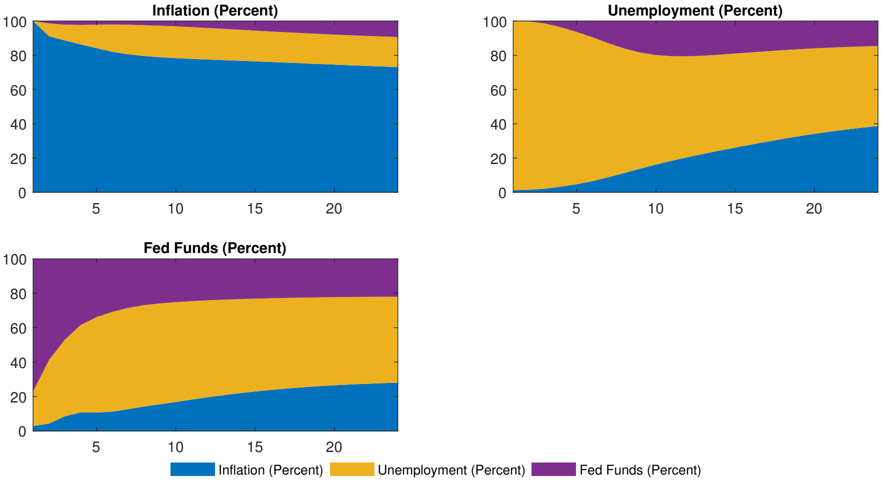
- In Matlab, set compute the *VD* with the `VARvd` function

```
% Compute VD
[VD, VAR] = VARvd(VAR,VARopt);
```

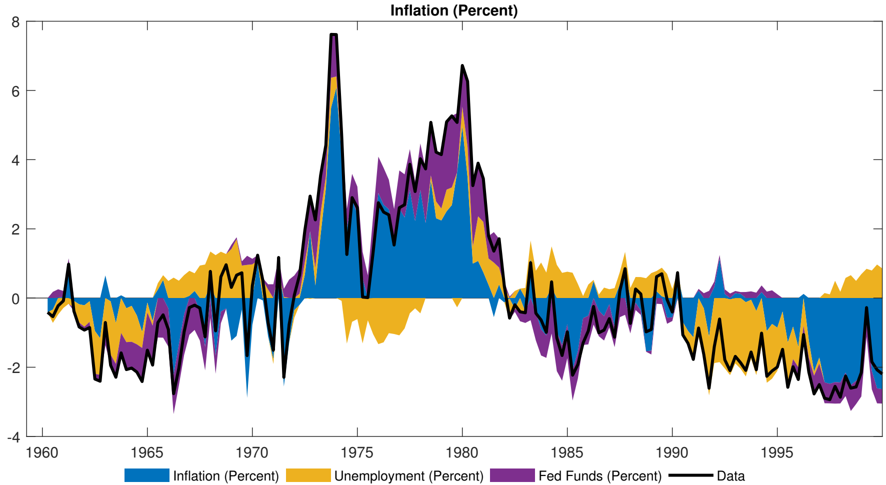
- Similarly, the *HD* can be computed with the `VARhd` function

```
% Compute HD
[HD, VAR] = VARhd(VAR,VARopt);
```

Forecast error variance decomposition



Historical decomposition



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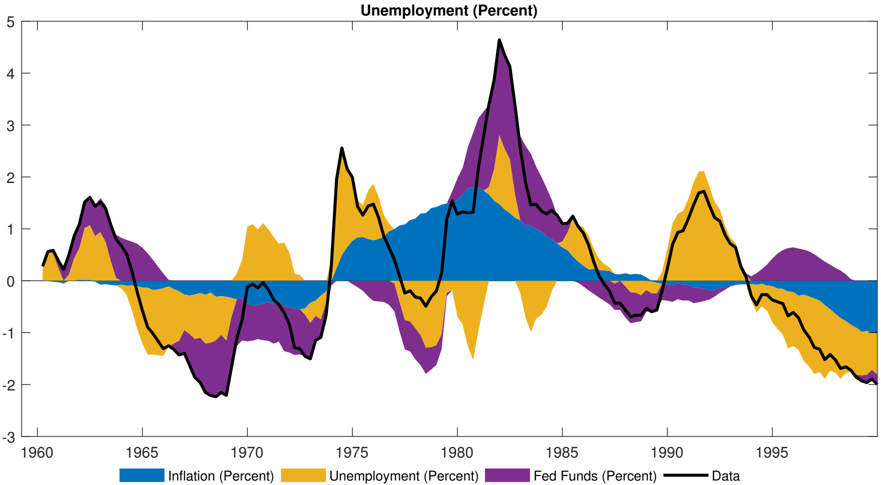

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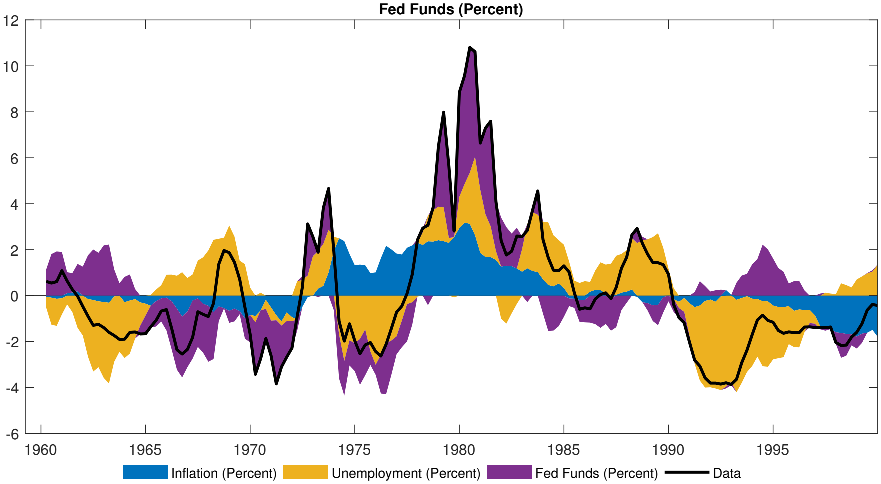


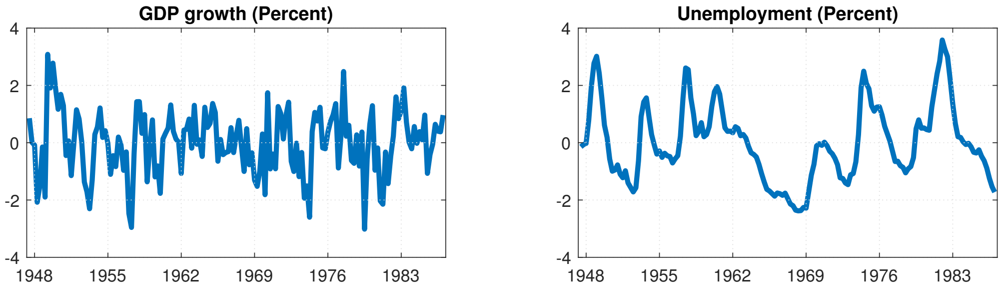
Practical Examples

Blanchard & Quah (1989, AER)

Blanchard & Quah (1989): Zero long-run restrictions

- Blanchard & Quah (1989). "The Dynamic Effects of Aggregate Demand and Supply Disturbances", *American Economic Review*

- US quarterly data from 1948Q1 to 1987Q4



What is the effect of demand and supply shocks?

- **Objective** Identify the effects of demand and supply shocks on output and unemployment

What is the effect of demand and supply shocks?

- **Objective** Identify the effects of demand and supply shocks on output and unemployment
- Assume a bivariate VAR with $p = 8$ with output growth (y_t) and unemployment (u_t)
- **Key identifying assumption** Demand-side shocks have no long-run effect on the level of output, while supply-side shocks do
  - Blanchard & Quah impose zero long-run restrictions on the cumulative effect of demand shocks on output growth (i.e. on output level) to identify the shocks

$$\begin{bmatrix} y_{t,t+\infty} \\ u_{t,t+\infty} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Supply} \\ \varepsilon_t^{Demand} \end{bmatrix}$$

Monetary policy shocks, inflation and unemployment

- In Matlab, set lag length to 8 and estimate a VAR with a constant

```
% Set up and estimate VAR
det = 1;
nlags = 8;
[VAR, VARopt] = VARmodel(X,nlags,det);
```

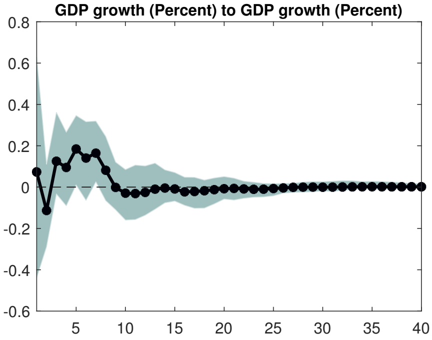
- Then set the option for zero long-run restrictions `VARopt.ident ='long'` and compute the *IR* with the `VARir` function. Note that the ordering of the variables matter!

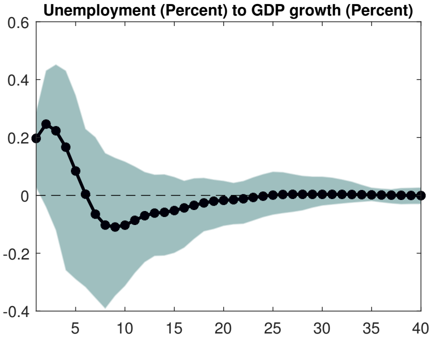
```
% For zero contemporaneous restrictions set:
VARopt.ident = 'long';
% Compute IR
[IR, VAR] = VARir(VAR,VARopt);
```

- The *B* matrix implied by the zero long-run restrictions is stored in `VAR.B`

Aggregate supply shock

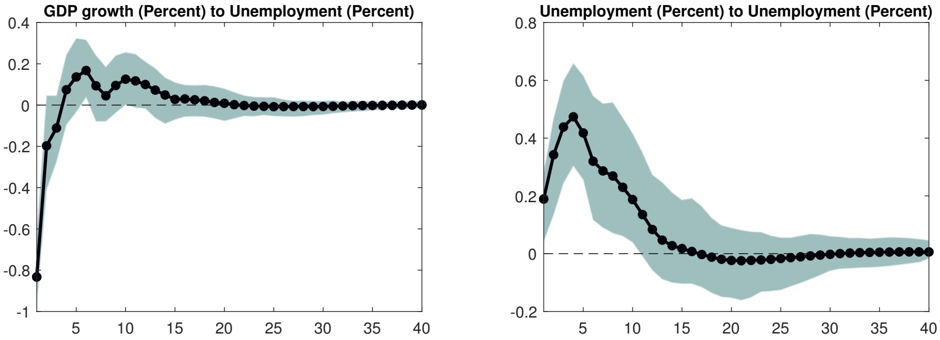
- Aggregate supply shock initially increases unemployment (puzzle of hours to productivity shocks)





Aggregate demand shock

- Aggregate demand shocks have a hump-shaped effect on output and unemployment



What is the long run effect of demand and supply shocks on output level?

- Blanchard & Quah report (Figure 1) the cumulative sum of the impulse responses of output growth (i.e. the response of output level)
- By assumption, it should be zero for demand shocks ✔

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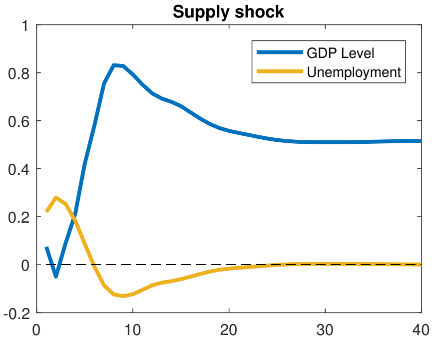
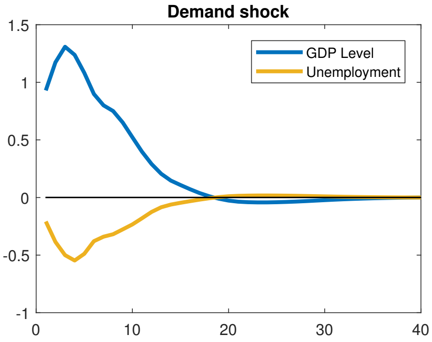


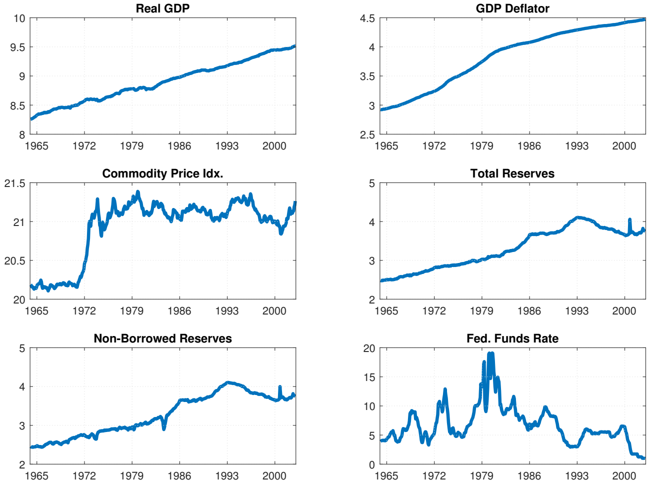


Practical Examples

Uhlig (2005, JME)

Uhlig (2005, JME): Sign restrictions

- Uhlig (2005). "What are the effects of monetary policy on output? Results from an agnostic identification procedure," *Journal of Monetary Economics*
- US monthly data from 1965M1 to 2003M12



What are the effects of monetary policy on output?

- **Objective** Infer the causal influence of monetary policy on real GDP

What are the effects of monetary policy on output?

- **Objective** Infer the causal influence of monetary policy on real GDP
- Assume a VAR with $p = 12$ with real GDP, real GDP deflator, a commodity price index, total reserves, non-borrowed reserves, and the fed. funds rate
- **Key identifying assumptions** According to conventional wisdom, monetary contractions should
  - Raise the federal funds rate
  - Lower prices
  - Decrease non-borrowed reserves
- Real GDP is left unrestricted

Monetary policy shock: The sign restrictions

- Uhlig imposes the following sign restrictions on the impulse responses of the VAR

	Monetary Policy Shock
Real GDP (y_t)	?
Real GDP deflator (p_t)	< 0
commodity price index,	?
total reserves,	?
non-borrowed reserves,	< 0
Fed. Funds Rate	> 0

- Restrictions are imposed for 6 periods

Monetary policy shock: The sign restrictions

- In Matlab, the sign restrictions can be set as follows

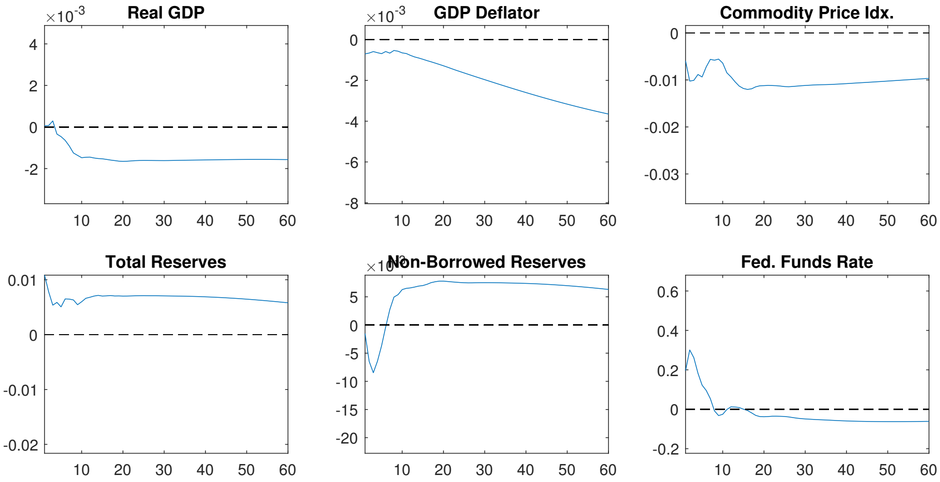
```
% Define the shock names
VARopt.snames = {'Mon. Policy Shock'};
% Define sign restrictions : positive 1, negative -1, unrestricted 0
SIGN = [ 0,0,0,0,0,0; % Real GDP
-1,0,0,0,0,0; % Deflator
-1,0,0,0,0,0; % Commodity Price
0,0,0,0,0,0; % Total Reserves
-1,0,0,0,0,0; % NonBorr. Reserves
1,0,0,0,0,0]; % Fed Fund
% Define the number of steps the restrictions are imposed for:
VARopt.sr_hor = 6;
```

- The routine is then implemented with the `SR` function

```
% The functin SR performs the sign restrictions identification and computes
% IRs, VDs, and HDs. All the results are stored in SRout
SRout = SR(VAR,SIGN,VARopt);
```

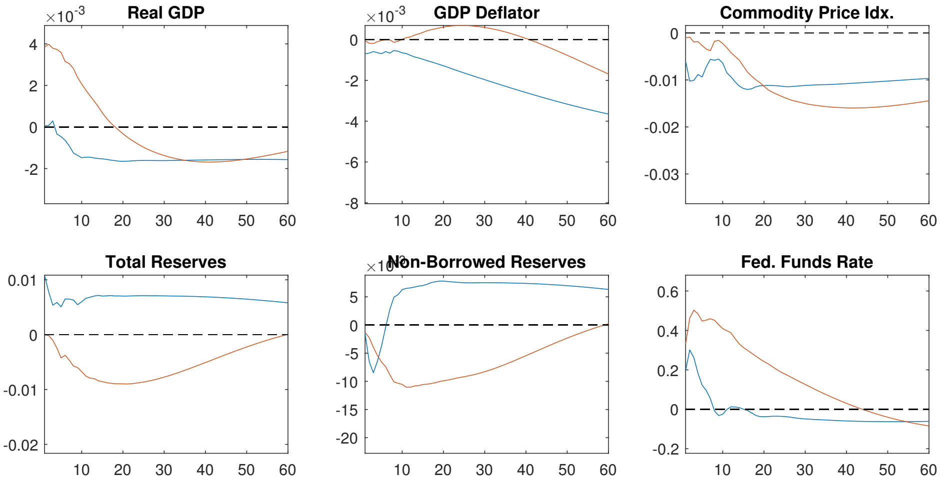
What happens when you do sign restrictions

- Start drawing orthonormal matrices Q until you find one that satisfies the restrictions...



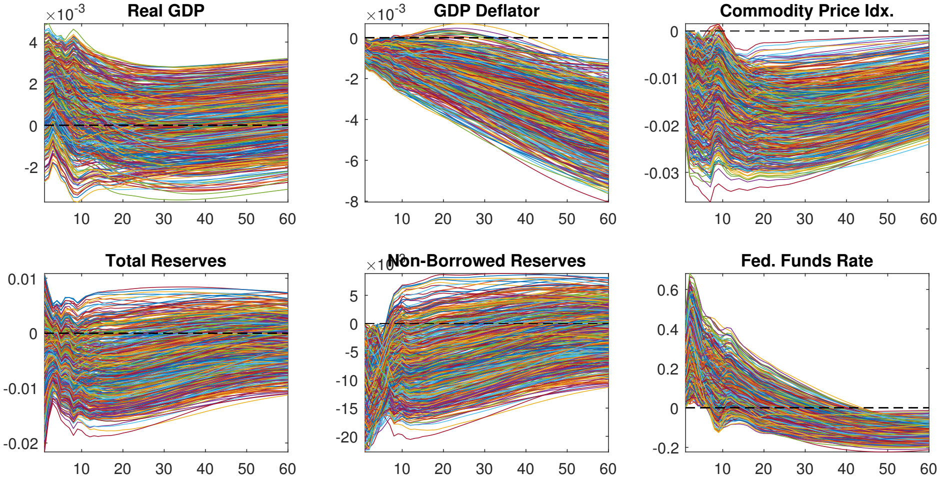
What happens when you do sign restrictions

- Start drawing Q again until you find another one...



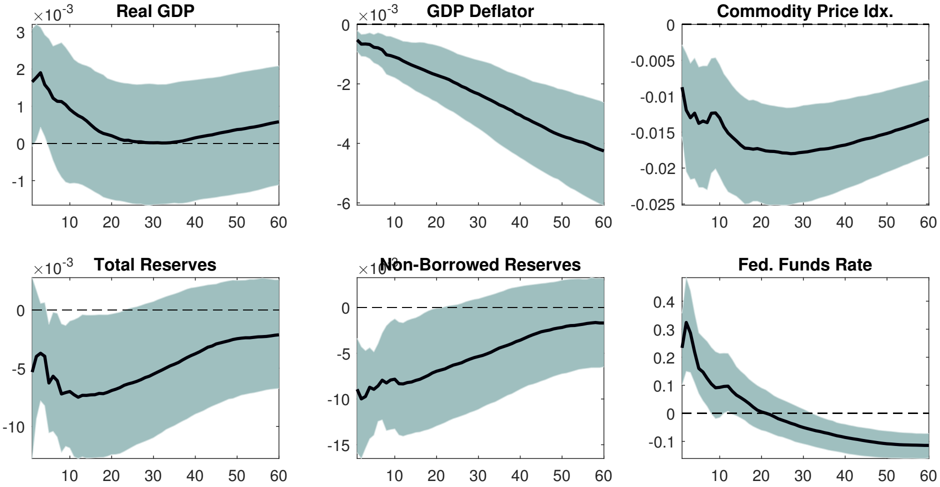
What happens when you do sign restrictions

- After a while...



What are the effects of monetary policy on output?

- Ambiguous effect on real GDP $\Longrightarrow$ Long-run monetary neutrality



Practical Examples

Gertler and Karadi (2015, AEJ:M)

Practical Examples

Cesa-Bianchi and Sokol (2020)