

A Primer on Vector Autoregressions

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[DISCLAIMER]

These notes are meant to provide intuition on the basic mechanisms of VARs

As such, most of the material covered here is treated in a very informal way

If you crave a formal treatment of these topics, you should stop here and buy a copy of Hamilton's "Time Series Analysis"


The Matlab codes accompanying these notes are available at:

<https://github.com/ambropo/VAR-Toolbox>

The job of macro-econometricians

- ▶ In their 2001 Journal of Economic Perspectives' article "Vector Autoregressions" Stock and Watson describe the job of macroeconometricians as consisting of the following:
 - * Describe and summarize macroeconomic time series
 - * Make forecasts
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 - * Recover the structure of the macroeconomy from the data  Main focus of these notes
 - * Advise macroeconomic policy-makers
- ▶ Vector autoregressive models (VARs) are a statistical tool to perform these tasks

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- ▶ Consider a bivariate VAR with the following variables: real GDP growth (y_t) and the policy rate (r_t)

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- ▶ Consider a bivariate VAR with the following variables: real GDP growth (y_t) and the policy rate (r_t)
- ▶ A VAR can help us answering the following questions
 - [1] What is the dynamic behavior of these variables? How do these variables interact?
 - [2] What is the most likely behavior of GDP in the next few quarters?
 - [3] What is the effect of a monetary policy shock on GDP?
 - [4] What has been the historical contribution of monetary policy shocks to GDP fluctuations?

VAR Basics

What is a Vector Autoregression (VAR)?

- Consider a (2×1) vector of zero-mean time series x_t , composed of t observations and an initial condition x_0

$$x_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1t} \\ x_{21} & x_{22} & \dots & x_{2t} \end{bmatrix} \quad \text{and} \quad x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

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- Assume that the two time series in x_t are covariance stationary, which means (for $i = 1, 2$)
 - * Constant mean $\mathbb{E}[x_{it}] = \mu_i$
 - * Constant variance $\mathbb{V}[x_{it}] = \sigma_i$
 - * Constant autocovariance $\text{COV}[x_{it}, x_{it+\tau}] = \gamma_i(\tau)$

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- A **structural VAR** of order 1 is given by

$$x_t = \Phi x_{t-1} + B \varepsilon_t$$

where

- * Φ and B are (2×2) matrices of coefficients
- * ε_t is an (2×1) vector of unobservable zero-mean white noise processes

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- Or as a system of linear equations

$$\begin{cases} x_{1t} = \phi_{11}x_{1,t-1} + \phi_{12}x_{2,t-1} + b_{11}\varepsilon_{1t} + b_{12}\varepsilon_{2t} \\ x_{2t} = \phi_{21}x_{1,t-1} + \phi_{22}x_{2,t-1} + b_{21}\varepsilon_{1t} + b_{22}\varepsilon_{2t} \end{cases}$$

The structural shocks

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- ▶ We defined ε_t as a *vector of unobservable zero mean white noise processes*. **What does it mean?**
- ▶ The elements of ε_t are serially uncorrelated and independent of each other
- ▶ In other words we assumed

$$\varepsilon_t = (\varepsilon'_{1t}, \varepsilon'_{2t})' \sim \mathcal{N}(0, I_2)$$

where

$$\mathbb{V}(\varepsilon_t) = \Sigma_\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \text{CORR}(\varepsilon_t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Why is it called 'structural' VAR?

- Go back to our bivariate structural VAR(1)

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- The structural VAR can be thought of as a description of the true structure of the economy
 - * E.g.: an approximation of the structure of a DSGE model
- The structural shocks are shocks with a well-defined economic interpretation
 - * E.g.: TFP shocks or monetary policy shocks
 - * As $\varepsilon_t \sim \mathcal{N}(0, I_2)$ we can move one shock keeping the other shocks fixed
 - * That is: we can focus on the causal effect of one shock at the time

Structural VARs can answer many interesting questions...

- Go back to our bivariate structural VAR(1). To make a concrete example, assume that
 - * x_{1t} and x_{2t} are output growth (y_t) and the policy rate (r_t), both demeaned
 - * ε_{1t} and ε_{2t} are a demand shock (ε_t^{Demand}) and a monetary policy shock (ε_t^{MonPol})
 - * B is known (we'll get back to this in a second)

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 - * (We can add additional variables and look at other shocks: aggregate supply, oil price,...)

... but the estimation of structural VARs is tricky

- **Problem** The structural shocks ε_t are unobserved. How can we estimate B ?

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- Best we can do is to 'bundle' the ε_t into a single object:

$$u_t = B\varepsilon_t \Rightarrow \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix} \Rightarrow \begin{cases} u_{yt} = b_{11}\varepsilon_t^{Demand} + b_{12}\varepsilon_t^{MonPol} \\ u_{rt} = b_{21}\varepsilon_t^{Demand} + b_{22}\varepsilon_t^{MonPol} \end{cases}$$

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- Why is this useful? The VAR becomes

$$x_t = \Phi x_{t-1} + u_t$$

- Now we can estimate Φ and u_t with OLS (where u_t will be OLS residuals)

The reduced-form VAR

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- ▶ In matrix form

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- ▶ Or as a system of linear equations

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The reduced-form covariance matrix

- ▶ A key object of interest in VARs is the covariance matrix of the reduced-form residuals

$$\Sigma_u = \begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ \sigma_{yr}^2 & \sigma_r^2 \end{bmatrix}$$

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- ▶ Differently from the structural shocks (which are orthogonal), the reduced-form residuals are correlated among each other
- ▶ This is because the elements of u_t inherit all the contemporaneous relations among the endogenous variables x_t
 - * To see that, remember how the reduced form residuals are defined

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- ▶ To make causal statements (e.g. the effects on y_t of a shock to ε_t^{MonPol}) we need to find a way to recover B
- ▶ This is the essence of identification in VARs

The Wold representation

- ▶ Before turning to identification, let's introduce another representation of the VAR that will be useful later
- ▶ Start from the structural VAR representation

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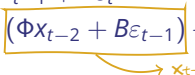
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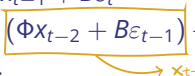
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$$\begin{aligned} x_t &= \Phi x_{t-1} + B\varepsilon_t \\ &= \Phi (\Phi x_{t-2} + B\varepsilon_{t-1}) + B\varepsilon_t = \Phi^2 x_{t-2} + \Phi B\varepsilon_{t-1} + B\varepsilon_t \\ &= \dots \\ &= \Phi^t x_0 + \sum_{j=0}^{t-1} \Phi^j B\varepsilon_{t-j} \end{aligned}$$


The Wold representation (cont'd)

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- Now let $t \rightarrow \infty$ to get

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$$x_t = \Phi^\infty x_{t-\infty} + \sum_{j=0}^{\infty} \Phi^j B \varepsilon_{t-j}$$

- ▶ But: we assumed that x_t is covariance stationary. How do these infinite sums relate to that assumption?
 - * Aren't the increasing powers of Φ exploding?

Stability of the VAR

- ▶ A VAR is stable if the effect of shocks progressively dissipate over time. For that to happen we need Φ^j to converge to zero

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- ▶ **Implication** In the absence of shocks, the VAR will converge to its equilibrium (i.e. its unconditional mean)

The unconditional mean of the VAR

- First note that if the eigenvalues of Φ are less than 1 in modulus we have

$$\Phi^\infty = 0 \quad \text{and} \quad \sum_{j=0}^{\infty} \Phi^j = (I_2 - \Phi)^{-1}$$

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- Because of white noise assumption of the ε_t , the unconditional mean is simply given by

$$\mathbb{E}[x_t] = \Phi^\infty x_{t-\infty} + \sum_{j=0}^{\infty} \Phi^j B \mathbb{E}[\varepsilon_{t-j}] = 0$$

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- First note that if the eigenvalues of Φ are less than 1 in modulus we have

$$\Phi^\infty = 0 \quad \text{and} \quad \sum_{j=0}^{\infty} \Phi^j = \boxed{(I_2 - \Phi)^{-1}} \quad \text{Geometric series}$$

- Because of white noise assumption of the ε_t , the unconditional mean is simply given by

$$\mathbb{E}[x_t] = \Phi^\infty x_{t-\infty} + \sum_{j=0}^{\infty} \Phi^j B \mathbb{E}[\varepsilon_{t-j}] = 0$$

- Note that if the VAR had a constant (α) an additional term would show up in the Wold representation

$$\mathbb{E}[x_t] = \Phi^\infty x_{t-\infty} + \sum_{j=0}^{\infty} \Phi^j \alpha + \sum_{j=0}^{\infty} \Phi^j B \mathbb{E}[\varepsilon_{t-j}] = 0$$

- The unconditional mean in this case would be

$$\mathbb{E}[x_t] = (I_2 - \Phi)^{-1} \alpha$$

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 - * Increase the number of lags (p)
 - * Add deterministic terms (e.g. time trend or seasonal dummy variables)
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- ▶ The general form of the VAR(p) model with deterministic terms (Z_t) and exogenous variables (W_t) is given by

$$x_t = \Phi_1 x_{t-1} + \Phi_2 x_{t-2} + \dots + \Phi_p x_{t-p} + \Lambda Z_t + \Psi W_t + B \varepsilon_t$$

The Identification Problem

Back to our reduced form VAR

- ▶ We have seen above that with OLS we can only estimate the reduced-form VAR (and not the structural VAR)
- ▶ Assume we already have an OLS estimate of $\hat{\Phi}$ and \hat{u}_t :

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} u_{yt} \\ u_{rt} \end{bmatrix}$$

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- ▶ **Question** What is the effect of a monetary policy shock on GDP growth?
- ▶ Unfortunately, the reduced form innovations (u_{yt} or u_{rt}) are not going to help us in answering the question

Reduced-form VARs do not tell us anything about causality

- To see that, assume that the ‘true’ (and unobserved) model of the economy is given by

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

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- ▶ How to know whether is [1] or [2]? This is the very nature of the **identification problem**!

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$$\Sigma_u = \mathbb{E} [u_t u_t'] = \mathbb{E} [B\varepsilon_t (B\varepsilon_t)'] = B\mathbb{E}(\varepsilon_t \varepsilon_t')B' = B\Sigma_\varepsilon B' = BB'$$

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- ▶ The identification problem simply boils down to finding a B matrix that satisfies $\Sigma_u = BB'$
- ▶ Unfortunately this is not as easy as it sounds. Why?
 - * Hint: There are infinite combinations of B that give you the same Σ_u

The identification problem (cont'd)

- Think of $\Sigma_u = BB'$ as a system of equations

$$\begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ - & \sigma_r^2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}$$

- Can be rewritten as

$$\begin{cases} \sigma_y^2 = b_{11}^2 + b_{12}^2 \\ \sigma_{yr}^2 = b_{11}b_{21} + b_{12}b_{22} \\ \sigma_{yr}^2 = b_{11}b_{21} + b_{12}b_{22} \\ \sigma_r^2 = b_{21}^2 + b_{22}^2 \end{cases}$$

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- **Problem** Because of the symmetry of the Σ_u matrix, the second and the third equation are identical
- We are left with 4 unknowns (the elements of B) but only 3 equations!

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- ▶ Economic theory can help in providing the 'missing' equation
 - * Example: If you believe that monetary policy works with a lag and has no effect on output growth on impact, you can *assume* $b_{12} = 0$
- ▶ The assumption buys us an additional equation

$$\begin{cases} \sigma_y^2 = b_{11}^2 + b_{12}^2 \\ \sigma_{yr}^2 = b_{11}b_{21} + b_{12}b_{22} \\ \sigma_r^2 = b_{21}^2 + b_{22}^2 \\ b_{12} = 0 \end{cases}$$

- ▶ The system now can be easily solved

Common Identification Schemes

Common identification schemes

- ▶ Zero (recursive) contemporaneous restrictions
- ▶ Zero (recursive) long-run restrictions
- ▶ Sign restrictions
- ▶ External instruments
- ▶ Combining sign restrictions and external instruments
- ▶ Other (narrative sign restrictions, maximization of forecast error variance,...)

Common Identification Schemes

Zero short-run restrictions

Zero contemporaneous restrictions

- ▶ **Intuition** Identification is achieved by assuming that some shocks have zero contemporaneous effect on some of the endogenous variables
- ▶ **References** Sims (1980), Christiano, Eichenbaum, Evans (1999)

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By assumption

- **Implication** We now have 3 structural parameters to estimate (instead of 4) and 3 restrictions implied by Σ_u

Zero contemporaneous restrictions

How to achieve identification?

- The system of equations implied by $\Sigma_u = BB'$ now becomes

$$\begin{bmatrix} \sigma_y^2 & \sigma_{yr} \\ - & \sigma_r^2 \end{bmatrix} = \begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} \\ 0 & b_{22} \end{bmatrix}$$

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- This yields

$$\begin{cases} \sigma_y^2 = b_{11}^2 \\ \sigma_{yr} = b_{11}b_{21} \\ \sigma_r^2 = b_{21}^2 + b_{22}^2 \end{cases}$$

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- ▶ And can be easily solved to get:

$$\begin{cases} b_{11} = \sigma_y \\ b_{21} = \sigma_{yr}/\sigma_y \\ b_{22} = \sqrt{\sigma_r^2 - \frac{\sigma_{yr}^2}{\sigma_y^2}} \end{cases}$$

Zero contemporaneous restrictions

Impact effects

- We can now derive the impact effects of shocks by simply re-writing the structural VAR as

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma_y^2 & 0 \\ \sigma_{yr}/\sigma_y^2 & \sqrt{\sigma_r^2 - \frac{\sigma_{yr}^2}{\sigma_y^2}} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

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- A one standard deviation shock to monetary policy ($\varepsilon_t^{MonPol} = 1$) in t leads to

$$\begin{cases} y_t = 0 \\ \pi_t = \sqrt{\sigma_r^2 - \frac{\sigma_{yr}^2}{\sigma_y^2}} \end{cases} \quad \begin{array}{l} \text{By assumption} \end{array}$$

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$$\begin{cases} y_t = \sigma_y^2 \\ \pi_t = \sigma_{yr}/\sigma_y^2 \end{cases}$$

Zero contemporaneous restrictions

Aka Cholesky identification

- ▶ This identification scheme is normally implemented via a Cholesky decomposition of Σ_u
- ▶ A Cholesky decomposition allows us to decompose Σ_u into the product of a lower triangular matrix P times its transpose

$$\Sigma_u = PP'$$

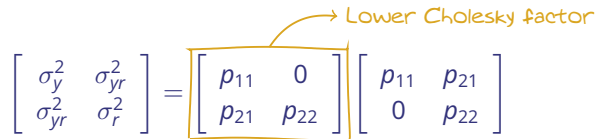
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- ▶ In matrix form we have


$$\begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ \sigma_{yr}^2 & \sigma_r^2 \end{bmatrix} = \begin{bmatrix} p_{11} & 0 \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} p_{11} & p_{21} \\ 0 & p_{22} \end{bmatrix}$$

Lower Cholesky factor

Cholesky decomposition of a matrix [\[Back to basics\]](#)

- ▶ The Cholesky decomposition is (roughly speaking) the square root of a matrix
 - * As for a square root, you can't compute a Cholesky decomposition for a non positive-definite matrix
- ▶ A symmetric and positive-definite matrix A can be decomposed as:

$$A = PP'$$

where P is a lower triangular matrix (and therefore P' is upper triangular)

- ▶ The formula for the decomposition of a 2×2 matrix is

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad P = \begin{bmatrix} \sqrt{a} & \frac{b}{\sqrt{a}} \\ 0 & \sqrt{c - \frac{b^2}{a}} \end{bmatrix}$$

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- ▶ As both P and B are lower triangular, it must follow that $P = B$

Common Identification Schemes

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- ▶ **References** Blanchard and Quah (1989), Gali (1999)

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- Re-write the VAR as

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Impact in t ← ← Impact in $t+1$ ← etc...

- We can rewrite

$$x_{t,t+\infty} = \sum_{j=0}^{\infty} \Phi^j B\varepsilon_t = (I - \Phi)^{-1} B\varepsilon_t = C\varepsilon_t$$

where $C \equiv (I - \Phi)^{-1}$ is the cumulative effect that ε_t has on output growth from time t to ∞ , i.e. the effect on output level

Zero long-run restrictions

How to compute the cumulative long-run effects of shocks?

- ▶ What is the intuition for C ?
- ▶ Go back to our output growth / policy rate example

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Zero long-run restrictions

How to compute the cumulative long-run effects of shocks?

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- ▶ Take the first equation: $y_{t,t+\infty} = c_{11}\varepsilon_t^{Demand} + c_{12}\varepsilon_t^{MonPol}$
 - * The coefficient c_{12} represents the impact of a monetary policy shock (hitting in t) on the level of GDP in the long-run
 - * If you believe in the long-run neutrality of monetary policy you would expect $c_{12} = 0$

Zero long-run restrictions

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- ▶ Remember that $C \equiv (I - \Phi)^{-1} B$ and B is unobserved. So, how does this help with the identification of B ?

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- ▶ We achieved identification: $B = (I - \Phi) P$

Common Identification Schemes

Sign restrictions

Sign restrictions

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- ▶ For example
 - * Demand shocks should lead to an increase in output and interest rates
 - * Monetary policy shocks should lead to a fall in output for an increase in interest rates

	Demand (ε_t^{Demand})	Monetary Policy (ε_t^{MonPol})
Output growth (y_t)	$b_{11} > 0?$	$b_{12} < 0?$
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- ▶ But how can we impose restrictions on the signs of the effect of a structural shock?

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- The matrix $B = PQ$ is a valid 'candidate' impact matrix that solves the identification problem!

- * Differently from P , the matrix PQ is not lower triangular anymore

Orthonormal matrix [\[Back to basics\]](#)

- ▶ An orthonormal matrix Q is a real square matrix whose columns and rows are orthogonal unit vectors
- ▶ What does it mean? Take for example two 2×1 vectors q_1 and q_2 , then the matrix $Q = (q_1, q_2)$ is orthonormal if
 - * The vectors have unit norm: $\|q_i\| = 1$
 - * The vectors are mutually orthogonal: $q_1^T q_2 = 0$
- ▶ It follows that

$$QQ' = I$$

- ▶ And

$$Q' = Q^{-1}$$

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[2] Then check that the elements of B satisfy

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Sign restriction in steps

► Perform N replications of the following steps

[1] Draw a random orthonormal matrix Q

[2] Compute $B = PQ$ where Q is the Cholesky decomposition of the reduced form residuals Σ_u

[3] Compute the impact effects of shocks associated with B

[4] Are the sign restrictions satisfied?

[4.1] Yes. Store B and go back to [1]

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 - [4.1] Yes. Store B and go back to [1]
 - [4.2] No. Discard B and go back to [1]
- ▶ All matrices in the set $B^{(i)}$ (for $i = 1, 2, \dots, N$) represent admissible solutions to the identification problem
- ▶ In this sense, sign restricted VARs are only set identified

Common Identification Schemes

External Instruments (or Proxy SVARs)

External instruments

- ▶ **Intuition** Exploits information from a variable that is *external* to the VAR, but that is correlated with a particular shock of interest and uncorrelated with other shocks (the *instrument*)
- ▶ **References** Stock and Watson (2012), Mertens and Ravn (2013)

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- ▶ But how can this help in finding the B matrix?

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- ▶ Then, we can identify one column (in this example, the second one) of the B matrix:

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} - & b_{12} \\ - & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

External instruments

- **How does it work?** Recall that the reduced form residuals are a linear combination of the two structural shocks

$$\begin{cases} u_{yt} = b_{11}\varepsilon_t^{Demand} + b_{12}\varepsilon_t^{MonPol} \\ u_{rt} = b_{21}\varepsilon_t^{Demand} + b_{22}\varepsilon_t^{MonPol} \end{cases}$$

- First stage regression

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$$u_{rt} = \beta z_t + \xi_t$$

- ▶ The OLS estimate of β identifies b_{22} up to a scaling factor
- ▶ The OLS estimate of ξ_t collects everything else that is uncorrelated with ε_t^{MonPol}

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- ▶ If we normalize the effect of ε_t^{MonPol} on r_t to 1 (that is, we fix $b_{22} = 1$) we can easily recover b_{21} from the OLS estimates of γ

Structural Dynamic Analysis

Structural Dynamic Analysis

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- ▶ But what can we do with that?

Structural Dynamic Analysis

- ▶ We have seen how to 'identify' structural shocks by imposing some restrictions on the data
- ▶ But what can we do with that?
 - * Quantify the dynamic effect of a shock over time \Rightarrow Impulse responses
 - * Quantify how important a shock is in explaining the variation in the endogenous variables (on average) \Rightarrow Forecast error variance decomposition
 - * Quantify how important a shock was in driving the behavior of the endogenous variables in a specific time period in the past \Rightarrow Historical decompositions
- ▶ We'll turn to this structural dynamic analysis next

Structural Dynamic Analysis

Impulse responses

Impulse response functions

- ▶ Impulse response functions (*IR*) answer the following question:

What is the response over time of the VAR's endogenous variables to an innovation in the structural shocks, assuming that the other structural shocks are kept to zero?

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- ▶ *IR* allow to single out the effect of a shock (e.g. its impact and persistence) keeping all else equal
- ▶ **Example** What is the impact of a monetary policy shock to GDP?

How to compute impulse response functions

- Consider our simple bivariate VAR

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

- Define a 2×1 vector of impulse selection (s) that takes value of one for the structural shock that we want to consider.
- For example, to compute the IR to the demand shock, define s as:

$$s = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- The impulse responses to ε_t^{Demand} can be easily computed with the following equation

$$x_t = \Phi x_{t-1} + B s,$$

How to compute impulse response functions (cont'd)

- ▶ The IR can be computed recursively as follows

$$\begin{cases} IR_t = Bs, & \text{for } t = 0, \\ IR_t = \Phi \cdot IR_{t-1} & \text{for } t = 2, \dots, h. \end{cases}$$

- ▶ Note that the impact response is simply given by the elements of the impact matrix B selected by s ...

$$\begin{bmatrix} IR_0^y \\ IR_0^r \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$$

- ▶ ... while the responses at longer horizons are given the transition matrix

$$\begin{bmatrix} IR_h^y \\ IR_h^r \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} IR_h^y \\ IR_h^r \end{bmatrix}$$

The companion matrix [\[Back to basics\]](#)

- So far, we considered simple VAR(1) specifications. But what to do if the VAR has $p > 1$?
- Every VAR(p) can be written as a VAR(1) using the **companion representation**
 - * For example, take a VAR(2)

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11}^1 & \phi_{12}^1 \\ \phi_{21}^1 & \phi_{22}^1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} \phi_{11}^2 & \phi_{12}^2 \\ \phi_{21}^2 & \phi_{22}^2 \end{bmatrix} \begin{bmatrix} y_{t-2} \\ r_{t-2} \end{bmatrix} + \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

- * Re-write the VAR(2) as

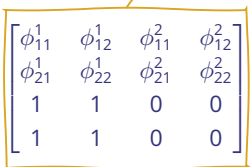
$$\begin{bmatrix} y_t \\ r_t \\ y_{t-1} \\ r_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_{11}^1 & \phi_{12}^1 & \phi_{11}^2 & \phi_{12}^2 \\ \phi_{21}^1 & \phi_{22}^1 & \phi_{21}^2 & \phi_{22}^2 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \\ y_{t-2} \\ r_{t-2} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \\ 0 \\ 0 \end{bmatrix}$$

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- * To get a VAR(1) where $\tilde{\Phi}$ is the **companion matrix**

$$\tilde{x}_t = \tilde{\Phi} \tilde{x}_{t-1} + \tilde{B} \varepsilon_t$$

Structural Dynamic Analysis

Forecast Error Variance Decompositions

Forecast error variance decompositions

- Forecast error variance decompositions (VD) answer the following question:

What portion of the variance of the VAR's forecast errors (at a given horizon h) is due to each structural shock?

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- ▶ Forecast error variance decompositions (VD) answer the following question:

What portion of the variance of the VAR's forecast errors (at a given horizon h) is due to each structural shock?

- ▶ VD provide information about the relative importance of each structural shock in affecting the forecast error variance of the VAR's endogenous variables
- ▶ **Example** What is the (average) importance of demand shocks in driving GDP forecast errors?

How to compute forecast error variance decompositions

- ▶ The forecast error of a variable at horizon $t + h$ is the change in the variable that couldn't have been forecast between $t - 1$ and $t + h$ due to the realization of the structural shocks.
- ▶ For example, at $h = 0$ we can compute the forecast error as

$$x_t - \mathbb{E}_{t-1}[x_t] = \Phi x_{t-1} + B\varepsilon_t - \Phi x_{t-1} = B\varepsilon_t$$

- ▶ At $h = 1$, we have

$$\begin{aligned} x_{t+1} - \mathbb{E}_{t-1}[x_{t+1}] &= \Phi x_t + B\varepsilon_{t+1} - \Phi^2 x_{t-1} = \\ &= \Phi(\Phi x_{t-1} + B\varepsilon_t) + B\varepsilon_{t+1} - \Phi^2 x_{t-1} = \Phi B\varepsilon_t + B\varepsilon_{t+1} \end{aligned}$$

- ▶ So, in general we have

$$FE_{t+h} = x_{t+h} - E_{t-1}[x_{t+h}] = \sum_{i=0}^h \Phi^{h-i} B\varepsilon_{t+i}$$

- ▶ What is the variance of FE_{t+h} ?

Basic properties of the variance [\[Back to basics\]](#)

- ▶ If X is a random variable x and a is a constant
 - * $\mathbb{V}(x + a) = \mathbb{V}(x)$
 - * $\mathbb{V}(ax) = a^2 \mathbb{V}(x)$
- ▶ If Y is a random variable and b is a constant
 - * $\mathbb{V}(aX + bY) = a^2 \mathbb{V}(x) + b^2 \mathbb{V}(Y) + 2ab \text{COV}(X, Y)$
- ▶ Since the structural errors are independent, it follows that $\text{COV}(\epsilon_{t+1}^{Demand}, \epsilon_{t+1}^{MonPol}) = 0$

How to compute forecast error variance decompositions (cont'd)

- For simplicity consider $h = 0$, namely

$$\mathbb{V}(x_t - E_{t-1}[x_t]) = \mathbb{V}(B\varepsilon_t)$$

- Recalling that $\mathbb{V}(\varepsilon_t) = I_2$ and the structural shocks are orthogonal to each other, the variance of the forecast error can be computed as

$$\mathbb{V}(y_t - E_{t-1}[y_t]) = b_{11}^2 \mathbb{V}(\varepsilon_t^{Demand}) + b_{12}^2 \mathbb{V}(\varepsilon_t^{MonPol}) = b_{11}^2 + b_{12}^2$$

$$\mathbb{V}(r_t - E_{t-1}[r_t]) = b_{21}^2 \mathbb{V}(\varepsilon_t^{Demand}) + b_{22}^2 \mathbb{V}(\varepsilon_t^{MonPol}) = b_{21}^2 + b_{22}^2$$

- What portion of the variance of the forecast error at $h = 0$ is due to each structural shock?

$$\underbrace{\begin{cases} VD_{y_0}^{\varepsilon^{Demand}} = \frac{b_{11}^2}{b_{11}^2 + b_{12}^2} \\ VD_{y_0}^{\varepsilon^{MonPol}} = \frac{b_{12}^2}{b_{11}^2 + b_{12}^2} \end{cases}}_{\text{This sums up to 1}} \quad \underbrace{\begin{cases} VD_{r_0}^{\varepsilon^{Demand}} = \frac{b_{21}^2}{b_{21}^2 + b_{22}^2} \\ VD_{r_0}^{\varepsilon^{MonPol}} = \frac{b_{22}^2}{b_{21}^2 + b_{22}^2} \end{cases}}_{\text{This sums up to 1}}$$

Structural Dynamic Analysis

Historical Decompositions

Historical decompositions

- ▶ Historical decompositions (*HD*) answer the following question:

What is the historical contribution of each structural shock in driving deviations of the VAR's the endogenous variables away from their equilibrium?

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- ▶ *HD* allow to track, at each point in time, the role of structural shocks in driving the VAR's endogenous variables away from their steady state
- ▶ **Example** What was the contribution of oil shocks in driving GDP growth in 1973?

How to compute historical decompositions

- ▶ As an example, let's compute the *HD* of the endogenous variables when $t = 2$ in our simple bivariate VAR
- ▶ Historical decompositions can be easily understood from the Wold representation of the VAR

$$x_t = \Phi^t x_0 + \sum_{j=0}^{t-1} \Phi^j B \varepsilon_{t-j}$$

How to compute historical decompositions

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- ▶ Historical decompositions can be easily understood from the Wold representation of the VAR

$$x_t = \Phi^t x_0 + \sum_{j=0}^{t-1} \Phi^j B \varepsilon_{t-j}$$

- ▶ Using the Wold representation, we can write x_2 as a function of present and past structural shocks (ε^{Demand} and ε^{MonPol}) plus the initial condition (x_0)

$$x_2 = \underbrace{\Phi^2 x_0}_{init} + \underbrace{\Phi B}_{\Theta_1} \varepsilon_1 + \underbrace{B}_{\Theta_0} \varepsilon_2$$

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- ▶ Re-write x_2 in matrix form

$$\begin{bmatrix} y_2 \\ r_2 \end{bmatrix} = \begin{bmatrix} init_y \\ init_r \end{bmatrix} + \begin{bmatrix} \theta_{11}^1 & \theta_{12}^1 \\ \theta_{21}^1 & \theta_{22}^1 \end{bmatrix} \begin{bmatrix} \varepsilon_2^{Demand} \\ \varepsilon_2^{MonPol} \end{bmatrix} + \begin{bmatrix} \theta_{11}^0 & \theta_{12}^0 \\ \theta_{21}^0 & \theta_{22}^0 \end{bmatrix} \begin{bmatrix} \varepsilon_2^{Demand} \\ \varepsilon_2^{MonPol} \end{bmatrix}$$

How to compute historical decompositions (cont'd)

► Then x_2 can be expressed as

$$\begin{cases} y_2 = init_y + \theta_{11}^1 \varepsilon_1^{Demand} + \theta_{12}^1 \varepsilon_1^{MonPol} + \theta_{11}^0 \varepsilon_2^{Demand} + \theta_{12}^0 \varepsilon_2^{MonPol} \\ r_2 = init_r + \theta_{21}^1 \varepsilon_1^{Demand} + \theta_{22}^1 \varepsilon_1^{MonPol} + \theta_{21}^0 \varepsilon_2^{Demand} + \theta_{22}^0 \varepsilon_2^{MonPol} \end{cases}$$

How to compute historical decompositions (cont'd)

- Then x_2 can be expressed as

$$\begin{cases} y_2 = \text{init}_y + \theta_{11}^1 \varepsilon_1^{\text{Demand}} + \theta_{12}^1 \varepsilon_1^{\text{MonPol}} + \theta_{11}^0 \varepsilon_2^{\text{Demand}} + \theta_{12}^0 \varepsilon_2^{\text{MonPol}} \\ r_2 = \text{init}_r + \theta_{21}^1 \varepsilon_1^{\text{Demand}} + \theta_{22}^1 \varepsilon_1^{\text{MonPol}} + \theta_{21}^0 \varepsilon_2^{\text{Demand}} + \theta_{22}^0 \varepsilon_2^{\text{MonPol}} \end{cases}$$

- The historical decomposition is given by

$$\underbrace{\begin{cases} HD_{y_2}^{\varepsilon^{\text{Demand}}} = \theta_{11}^1 \varepsilon_1^{\text{Demand}} + \theta_{11}^2 \varepsilon_2^{\text{Demand}} \\ HD_{y_2}^{\varepsilon^{\text{MonPol}}} = \theta_{12}^1 \varepsilon_1^{\text{MonPol}} + \theta_{12}^2 \varepsilon_2^{\text{MonPol}} \\ HD_{y_2}^{\text{init}} = \text{init}_y \end{cases}}_{\text{This sums up to } y_2}$$

$$\underbrace{\begin{cases} HD_{r_2}^{\varepsilon^{\text{Demand}}} = \theta_{21}^1 \varepsilon_1^{\text{Demand}} + \theta_{21}^0 \varepsilon_2^{\text{Demand}} \\ HD_{r_2}^{\varepsilon^{\text{MonPol}}} = \theta_{22}^1 \varepsilon_1^{\text{MonPol}} + \theta_{22}^0 \varepsilon_2^{\text{MonPol}} \\ HD_{r_2}^{\text{init}} = \text{init}_r \end{cases}}_{\text{This sums up to } r_2}$$

Practical Examples

The VAR Toolbox

- ▶ We'll see in practice how VARs work through a set of examples using the **VAR Toolbox 3.0**
- ▶ The VAR Toolbox is a collection of Matlab routines to perform VAR analysis
 - * Codes are available at <https://github.com/ambropo/VAR-Toolbox>
 - * No installation is required. Simply clone the folder from Github and add the folder (with subfolders) to your Matlab path
 - * To save figures in high quality format, you need to download and install Ghostscript (freely available at www.ghostscript.com).
 - ✦ The first time you'll be saving a figure using the Toolbox, you'll be asked to locate the Ghostscript on your local drive
- ▶ We'll start with a very simple example and then replicate the results from a few well-known papers

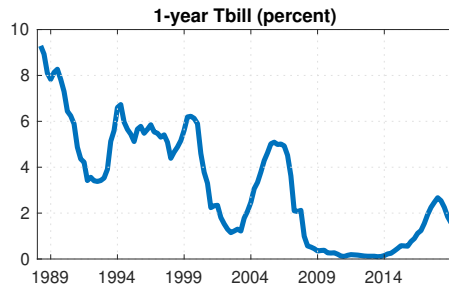
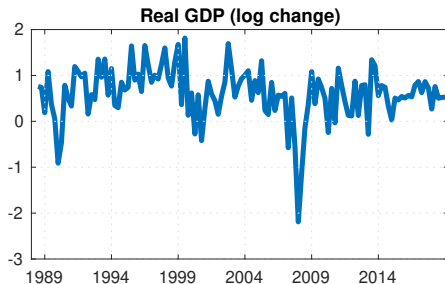
Adding the VAR Toolbox path to Matlab

- ▶ To avoid clashes with functions from other toolboxes, it is recommendable to add and remove the Toolbox at beginning and end of your scripts
- ▶ If you download the toolbox to C:/AMPER/VARToolbox, you can simply add the following lines at the beginning and end of your script

```
addpath(genpath('C:/VAR-Toolbox/'))  
...  
rmpath(genpath('C:/VARToolbox'))
```

A simple bivariate VAR model

- ▶ Our first example will be a simple bivariate VAR as the one considered above
- ▶ US quarterly data from 1989q1 to 2019q4 on output growth (y_t) and the 1-year T-bill (r_t)



A simple bivariate VAR model

- ▶ As both GDP growth and the 1-year rate are non-zero means, we fit the data with a VAR(1) with a constant

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \alpha_y \\ \alpha_\pi \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} u_{yt} \\ u_t^\pi \end{bmatrix}$$

- ▶ This means we will estimate the following parameters
 - * 2 + 4 coefficients, namely the elements of α and Φ
 - * 2 variances of the reduced-form residuals, namely σ_y^2 and σ_π^2
 - * 1 covariance of the reduced-form residuals, namely $\sigma_{y\pi}$
- ▶ We will store these coefficients in two Matlab matrices

$$\mathbf{F} = \begin{bmatrix} \alpha_1 & \phi_{11} & \phi_{12} \\ \alpha_2 & \phi_{21} & \phi_{22} \end{bmatrix} \quad \mathbf{sigma} = \begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ \sigma_{yr}^2 & \sigma_r^2 \end{bmatrix}$$

A simple bivariate VAR model

- In Matlab we store the data in the matrix `X`

$$X = \begin{bmatrix} y_1 & r_1 \\ y_2 & r_2 \\ \dots & \dots \\ y_T & r_T \end{bmatrix} = (y'_t, r'_t) = x'_t$$

- The VAR can then be easily estimated with a few lines of code

```
% Set the deterministic variable in the VAR (1=constant, 2=trend)
det = 1;
% Set number of nlags
nlags = 12;
% Estimate VAR by OLS
[VAR, VARopt] = VARmodel(ENDO,nlags,det);
```

OLS estimation: Typical VAR output (cont'd)


- ▶ The off-diagonal elements of Σ capture the average contemporaneous relation between the endogenous variables

	GDP growth (u_y)	1-year T-Bill(u_r)
Real GDP (u_y)	0.2891	0.0782
1-year T-Bill(u_r)	0.0782	0.1473

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- ▶ In our example output growth and inflation are contemporaneously positively correlated
 - * It means that, on average, when GDP growth increases inflation increases, too

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- ▶ In our example output growth and inflation are contemporaneously positively correlated
 - * It means that, on average, when GDP growth increases inflation increases, too
- ▶ Does it mean that a shock to output always increase inflation?
 - * No! Recall that reduced form residuals are not informative about structural shocks

Model checking & tuning

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 - * Not autocorrelated
 - * Not heteroskedastic (i.e., have constant variance)

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- ▶ Loosely speaking, you need to check that the reduced-form residuals are
 - * Normally distributed
 - * Not autocorrelated
 - * Not heteroskedastic (i.e., have constant variance)
- ▶ ... and that the VAR is stable

Stability and equilibrium (cont'd)

- ▶ As our VAR is stable, its Wold representation will converge to the (finite) unconditional mean of the data
- ▶ To see that, first consider the Wold representation in the presence of a constant

$$x_t = \Phi^t x_0 + \sum_{j=0}^{t-1} \Phi^j \alpha + \sum_{j=0}^{t-1} \Phi^j B \varepsilon_{t-j}$$

- ▶ For t large enough and taking expectations we get

$$\mathbb{E}[x_t] = \sum_{j=0}^{t-1} \Phi^j \alpha = (I_2 - \Phi)^{-1} \alpha$$

- ▶ In absence of shocks, the VAR's variable will converge to its equilibrium $(I_2 - \Phi)^{-1} \alpha$ at a rate that depends on Φ

Examples of different identification schemes

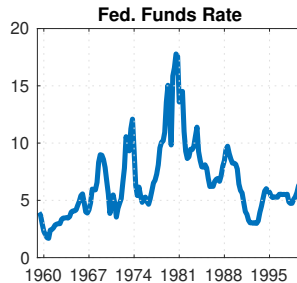
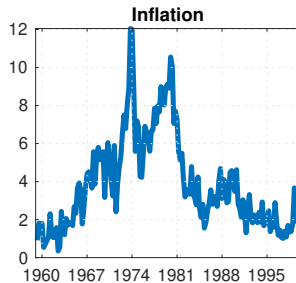
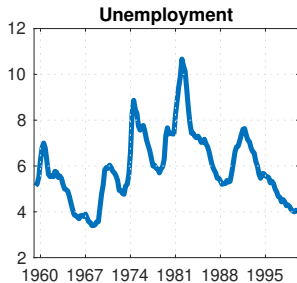
- ▶ Zero short-run restrictions
 - * Stock & Watson (2001). "Vector Autoregressions," *Journal of Economic Perspectives*
- ▶ Zero long-run restrictions
 - * Blanchard & Quah (1989). "The Dynamic Effects of Aggregate Demand and Supply Disturbances", *American Economic Review*
- ▶ Sign Restrictions
 - * Uhlig (2005). "What are the effects of monetary policy on output? Results from an agnostic identification procedure," *Journal of Monetary Economics*
- ▶ External instruments
 - * Gertler and Karadi (2015). "What are the effects of monetary policy on output? Results from an agnostic identification procedure," *American Economic Journal: Macroeconomics*
- ▶ External instruments & Sign restrictions
 - * Cesa-Bianchi and Sokol (2020). "Financial Shocks, Credit Spreads, and the International Credit Channel," *Unpublished manuscript*

Practical Examples

Stock & Watson (2001, JEP)

Stock & Watson (2001): Zero short-run restrictions

- ▶ Stock & Watson (2001). "Vector Autoregressions," *Journal of Economic Perspectives*
- ▶ US quarterly data from 1960Q1 to 2000Q4



Monetary policy shocks, inflation and unemployment

- ▶ **Objective** Infer the causal influence of monetary policy on unemployment and inflation

Monetary policy shocks, inflation and unemployment

- ▶ **Objective** Infer the causal influence of monetary policy on unemployment and inflation
- ▶ Assume a VAR with $p = 4$ with inflation (π_t), unemployment (u_t), and the fed funds rate (r_t)
- ▶ **Key identifying assumptions**
 - * MP (r_t) reacts contemporaneously to movements in inflation and in unemployment
 - * MP shocks (ε_t^{MonPol}) do not affect inflation and unemployment within the quarter of the shock

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$$\begin{bmatrix} \pi_t \\ u_t \\ r_t \end{bmatrix} = \sum_{p=1}^4 \Phi_p x_{t-p} + \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_t^1 \\ \varepsilon_t^2 \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

Monetary policy shocks, inflation and unemployment

- In Matlab, set lag length to 4 and estimate a VAR with a constant

```
% Set up and estimate VAR
det = 1;
nlags = 4;
[VAR, VARopt] = VARmodel(X,nlags,det);
```

- Then set the option for recursive identification `VARopt.ident = 'short'` and compute the *IR* with the `VARir` function. Note that the ordering of the variables matter!

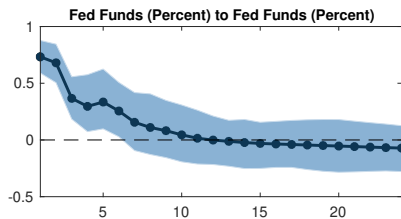
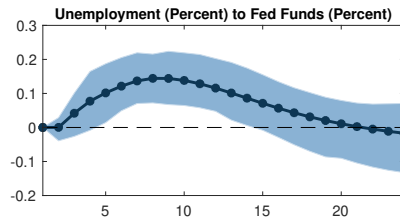
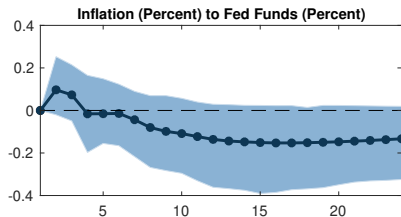
```
% For zero contemporaneous restrictions set:
VARopt.ident = 'short';
% Compute IR
[IR, VAR] = VARir(VAR,VARopt);
```

- The `VARirband` function allows to compute confidence intervals

```
% Compute IR
[IR, VAR] = VARir(VAR,VARopt);
```

The effect of a monetary policy shock

- Monetary policy shock raises inflation in the short run (price puzzle) and increases unemployment



The other two shocks are identified by definition... but how can we interpret them?

► How about ε_t^1 and ε_t^2 ?

- * The shock ε_t^1 affects all variables contemporaneously
- * The shock ε_t^2 affects r_t contemporaneously but not π_t

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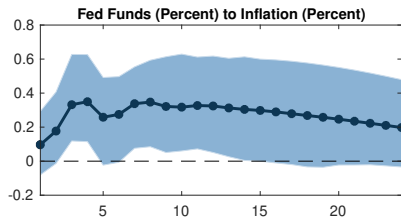
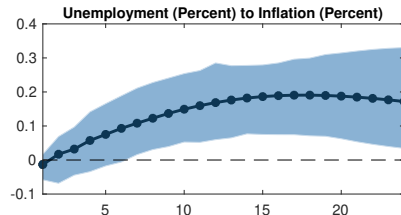
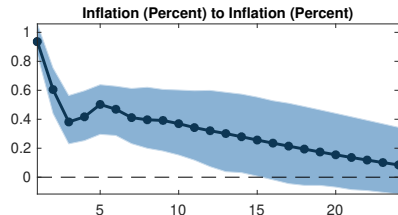
- ▶ How about ε_t^1 and ε_t^2 ?
 - * The shock ε_t^1 affects all variables contemporaneously
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- ▶ Can we interpret these shocks? Are the assumptions consistent with any theoretical mechanism?

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 - * The shock ε_t^1 affects all variables contemporaneously
 - * The shock ε_t^2 affects r_t contemporaneously but not π_t
- ▶ Can we interpret these shocks? Are the assumptions consistent with any theoretical mechanism?
- ▶ Some shocks may be better identified than others

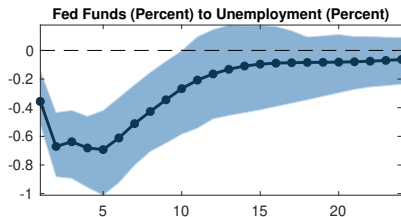
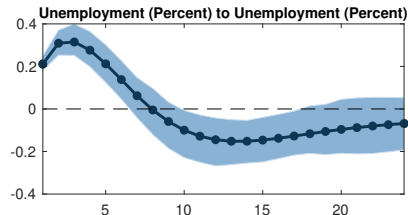
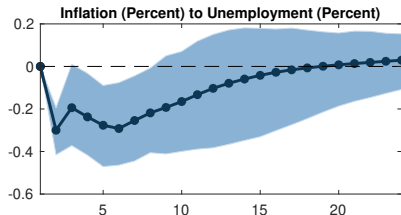
The other two shocks are identified by definition... but how can we interpret them?

- Shock to ε_t^1 behaves as a negative aggregate supply shock



The other two shocks are identified by definition... but how can we interpret them?

- Shock to ε_t^2 behaves as a negative aggregate demand shock



Forecast error variance & Historical decompositions

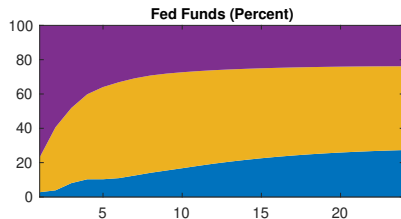
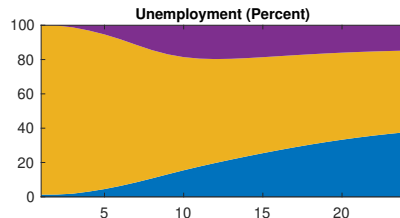
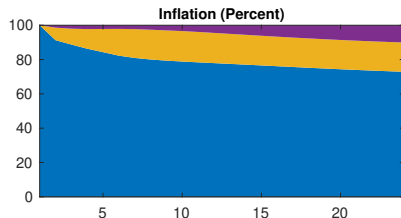
- In Matlab, set compute the VD with the `VARvd` function

```
% Compute VD  
[VD, VAR] = VARvd(VAR,VARopt);
```

- Similarly, the HD can be computed with the `VARhd` function

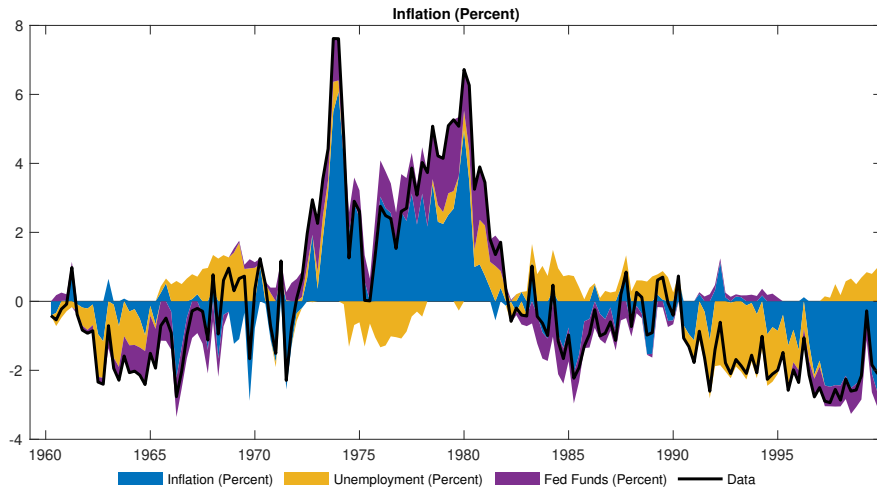
```
% Compute HD  
[HD, VAR] = VARhd(VAR,VARopt);
```

Forecast error variance decomposition

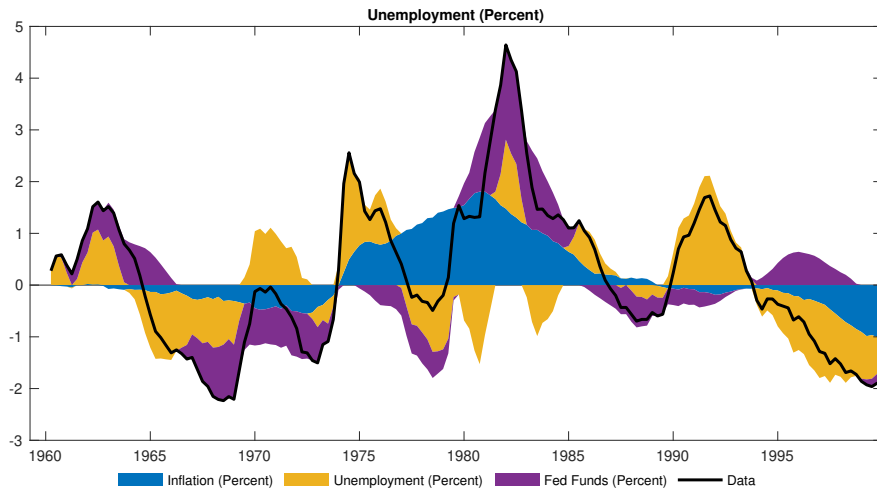


■ Inflation (Percent) ■ Unemployment (Percent) ■ Fed Funds (Percent)

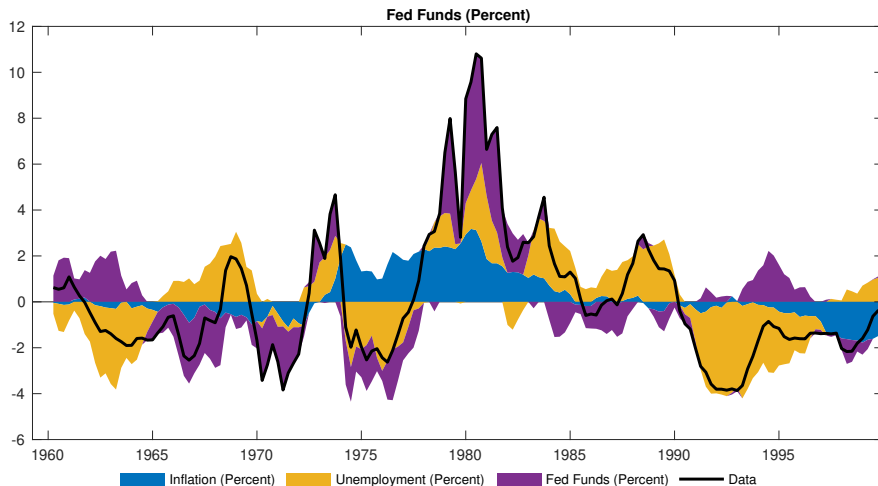
Historical decomposition



Historical decomposition



Historical decomposition

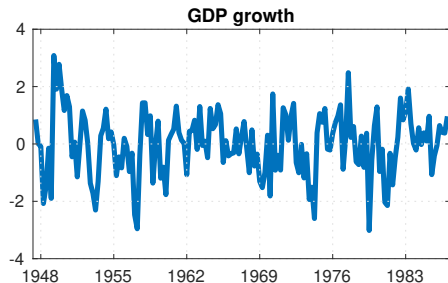


Practical Examples

Blanchard & Quah (1989, AER)

Blanchard & Quah (1989): Zero long-run restrictions

- ▶ Blanchard & Quah (1989). "The Dynamic Effects of Aggregate Demand and Supply Disturbances", *American Economic Review*
- ▶ US quarterly data from 1948Q1 to 1987Q4



What is the effect of demand and supply shocks?

- ▶ **Objective** Identify the effects of demand and supply shocks on output and unemployment

What is the effect of demand and supply shocks?

- ▶ **Objective** Identify the effects of demand and supply shocks on output and unemployment
- ▶ Assume a bivariate VAR with $p = 8$ with output growth (y_t) and unemployment (u_t)
- ▶ **Key identifying assumption** Demand-side shocks have no long-run effect on the level of output, while supply-side shocks do
 - * Blanchard & Quah impose zero long-run restrictions on the cumulative effect of demand shocks on output growth (i.e. on output level) to identify the shocks

$$\begin{bmatrix} y_{t,t+\infty} \\ u_{t,t+\infty} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Supply} \\ \varepsilon_t^{Demand} \end{bmatrix}$$

Monetary policy shocks, inflation and unemployment

- In Matlab, set lag length to 8 and estimate a VAR with a constant

```
% Set up and estimate VAR
det = 1;
nlags = 8;
[VAR, VARopt] = VARmodel(X,nlags,det);
```

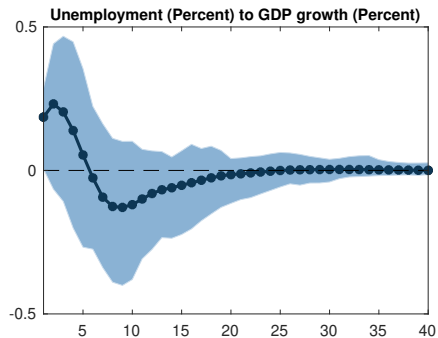
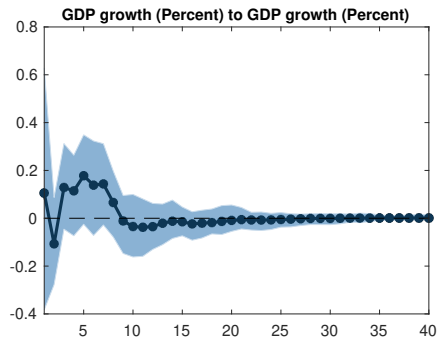
- Then set the option for zero long-run restrictions `VARopt.ident = 'long'` and compute the *IR* with the `VARir` function. Note that the ordering of the variables matter!

```
% For zero contemporaneous restrictions set:
VARopt.ident = 'long';
% Compute IR
[IR, VAR] = VARir(VAR,VARopt);
```

- The *B* matrix implied by the zero long-run restrictions is stored in `VAR.B`

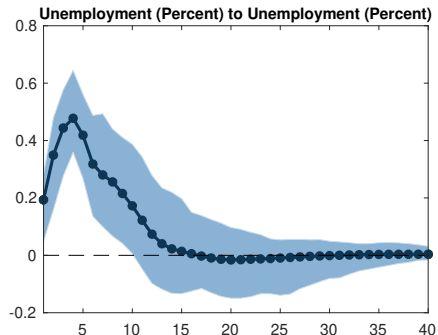
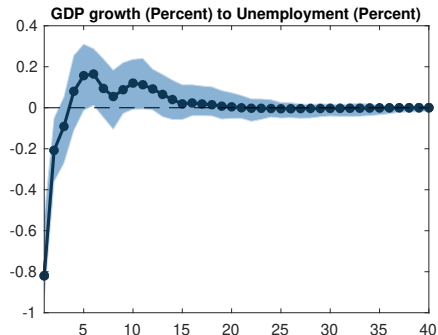
Aggregate supply shock

- Aggregate supply shock initially increases unemployment (puzzle of hours to productivity shocks)



Aggregate demand shock

- Aggregate demand shocks have a hump-shaped effect on output and unemployment

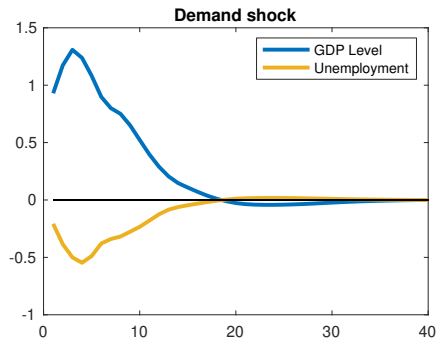
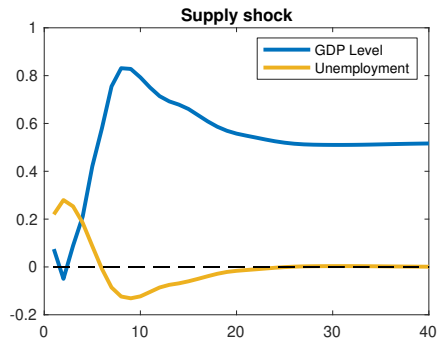


What is the long run effect of demand and supply shocks on output level?

- ▶ Blanchard & Quah report (Figure 1) the cumulative sum of the impulse responses of output growth (i.e. the response of output level)
- ▶ By assumption, it should be zero for demand shocks ✓

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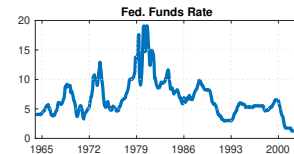
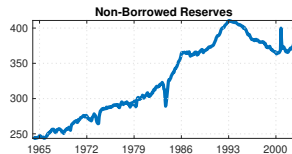
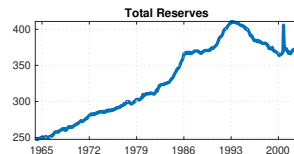
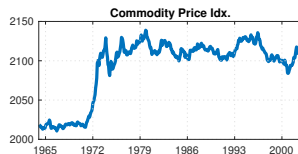
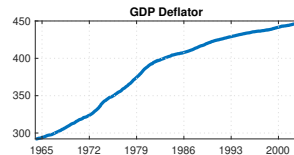


Practical Examples

Uhlig (2005, JME)

Uhlig (2005, JME): Sign restrictions

- ▶ Uhlig (2005). "What are the effects of monetary policy on output? Results from an agnostic identification procedure," *Journal of Monetary Economics*
- ▶ US monthly data from 1965M1 to 2003M12



What are the effects of monetary policy on output?

- ▶ **Objective** Infer the causal influence of monetary policy on real GDP

What are the effects of monetary policy on output?

- ▶ **Objective** Infer the causal influence of monetary policy on real GDP
- ▶ Assume a VAR with $p = 12$ with real GDP, real GDP deflator, a commodity price index, total reserves, non-borrowed reserves, and the fed. funds rate
- ▶ **Key identifying assumptions** According to conventional wisdom, monetary contractions should
 - * Raise the federal funds rate
 - * Lower prices
 - * Decrease non-borrowed reserves
- ▶ Real GDP is left unrestricted

Monetary policy shock: The sign restrictions

- Uhlig imposes the following sign restrictions on the impulse responses of the VAR

	Monetary Policy Shock
Real GDP)	?
Real GDP deflator)	< 0
Commodity price index	?
Total reserves	?
Non-borrowed reserves	< 0
Fed. Funds Rate	> 0

- Restrictions are imposed for 6 periods

Monetary policy shock: The sign restrictions

- In Matlab, the sign restrictions can be set as follows

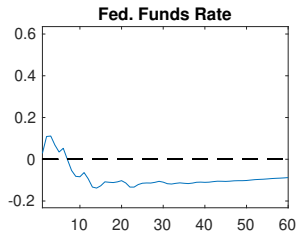
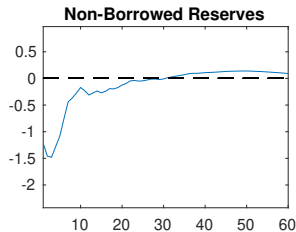
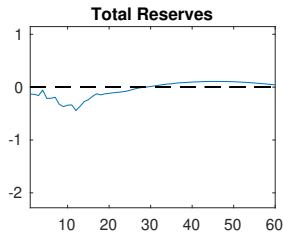
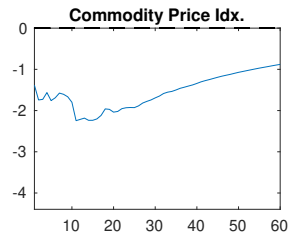
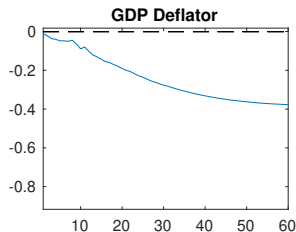
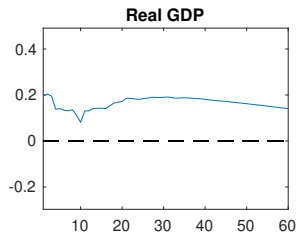
```
% Define the shock names
VARopt.snames = {'Mon. Policy Shock'};
% Define sign restrictions : positive 1, negative -1, unrestricted 0
SIGN = [ 0,0,0,0,0,0,0; % Real GDP
        -1,0,0,0,0,0,0; % Deflator
        -1,0,0,0,0,0,0; % Commodity Price
         0,0,0,0,0,0,0; % Total Reserves
        -1,0,0,0,0,0,0; % NonBorr. Reserves
         1,0,0,0,0,0,0]; % Fed Fund
% Define the number of steps the restrictions are imposed for:
VARopt.sr_hor = 6;
```

- The routine is then implemented with the `SR` function

```
% The function SR performs the sign restrictions identification and computes
% IRs, VDs, and HDs. All the results are stored in SRout
SRout = SR(VAR,SIGN,VARopt);
```

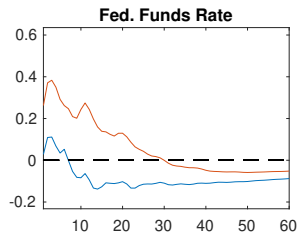
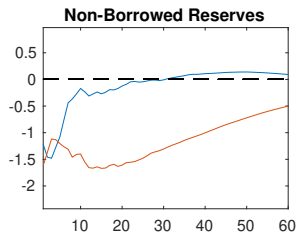
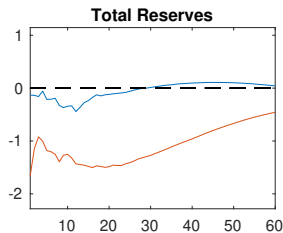
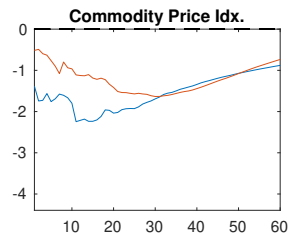
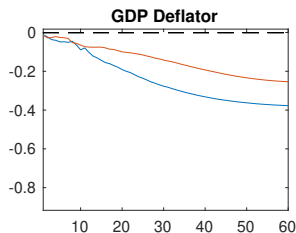
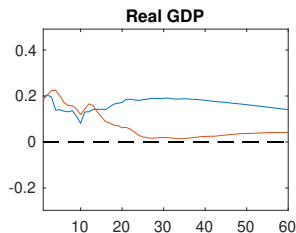
What happens when you do sign restrictions

- Start drawing orthonormal matrices Q until you find one that satisfies the restrictions...



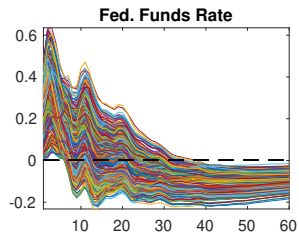
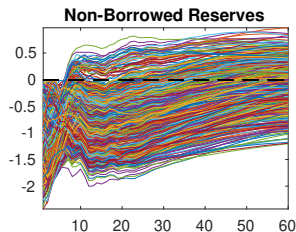
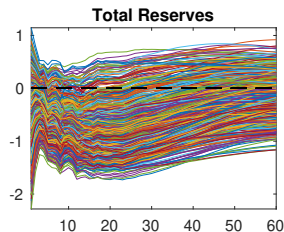
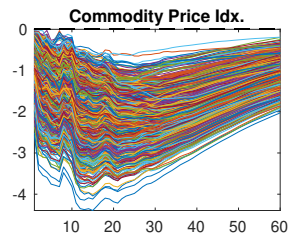
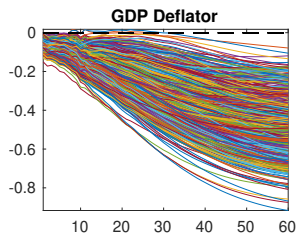
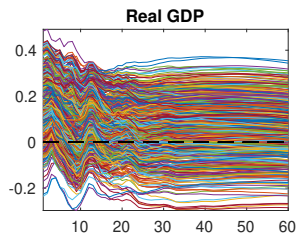
What happens when you do sign restrictions

- Start drawing Q again until you find another one...



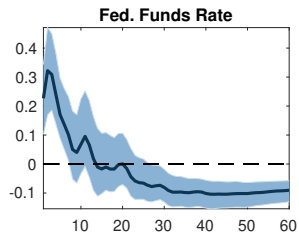
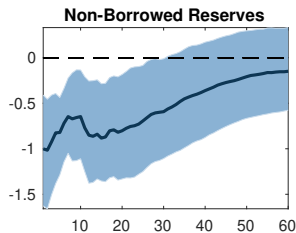
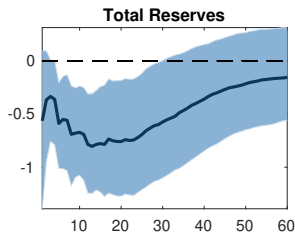
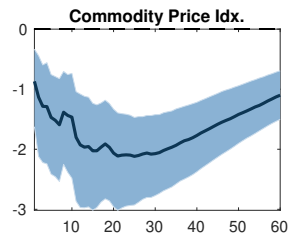
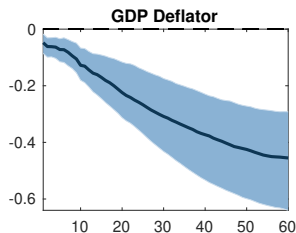
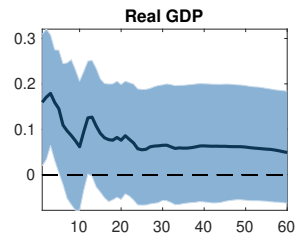
What happens when you do sign restrictions

► After a while...



What are the effects of monetary policy on output?

- Ambiguous effect on real GDP \Rightarrow Long-run monetary neutrality



Practical Examples

Gertler and Karadi (2015, AEJ:M)

Practical Examples

Cesa-Bianchi and Sokol (2020)