1 Installing the VAR Toolbox

No installation is required. Simply extract the codes from the ZIP file and copy them to a specific folder, e.g. "`C:/UserFolder/VARToolbox`". Then, add the folder (with subfolders) to the Matlab path. To avoid clashes with other function it is recommendable to add and remove the Toolbox with the following commands at beginning and end of your scripts:

```
addpath(genpath('C:/AMPER/VARToolbox'))
...
rmpath(genpath('C:/AMPER/VARToolbox'))
```

To save Figures in high quality format, Ghostscript is needed (freely available at `www.ghostscript.com`). The VT 3.0 has been tested with Matlab R2016B on a Windows 10 machine.

2 VAR Toolbox: High level description

The VAR Toolbox is a collection of Matlab routines to perform VAR analysis. Vector autoregressive models (VARs) are one of the most successful, flexible, and easy to use models for the analysis of multivariate time series. It is a natural extension of the univariate autoregressive model to dynamic multivariate time series. In their well-known paper "Vector Autoregressions," [5] describe VAR models as especially useful (and successful) tools for i) describing the dynamic behavior of economic and financial time series and ii) for forecasting.

In addition to data description and forecasting, VAR models are also used for iii) structural inference and iv) policy analysis. In structural analysis, we generally need to impose certain assumptions about the causal structure of the data under investigation. The resulting "structural" VAR model can then be used to analyze the impact of unexpected shocks to specified variables on all the variables in the model. This is normally done by means of iimpulse responses, forecast error variance decompositions, and historical decompositions.

The VAR Toolbox allows for identification of structural shocks with zero short-run restrictions; zero long-run restrictions; sign restrictions; and with the external instrument approach (proxy SVAR). Impulse Response Functions (IR), Forecast Error Variance Decomposition (VD), and Historical Decompositions (HD) are computed according to the chosen identification. Error bands are obtained with bootstrapping. The VAR Toolbox makes use of few Matlab routines from the Econometrics Toolbox for Matlab by James P. LeSage (freely available at `www.spatial-econometrics.com`).

It also includes a collection of Matlab routines that allows the user to save and export high quality images from Matlab (using the Export_fig function by Oliver Woodford, freely available at `https://www.mathworks.com/matlabcentral/fileexchange/23629-export_fig`). To enable this option, the Toolbox requires Ghostscript installed on your computer (freely available at `www.ghostscript.com`).

The VAR Toolbox is not meant to be efficient, but rather to be transparent and allow the user to understand the econometrics of VARs step by step. The codes are grouped in six categories (and respective folders):

- **Auxiliary**: codes that I borrowed from other public sources. Each m-file has a reference to the original source.

- **ExportFig**: this is a toolbox available at Oliver Woodford website for exporting high quality figures. [add website]

- **Figure**: codes for plotting high quality figures, particularly thought for time series. For example, the functions in this folder allow to efficiently add dates to the x-axis, to control the size and font of Figures and the appearance of the legends, to plot charts with shaded error bands, etc.

- **Stats**: codes for the calculation of summary statistics, moving correlations, pairwise correlations, etc.

- **Utils**: codes that allow the smooth functioning if the Toolbox.

- **VAR**: the codes for VAR estimation, identification, computation of the impulse response functions, FEVD, HD.

The idea of this manual is to explain the functioning of the VT by means of a simple example – the idea being that is much easier to learn by doing rather than reading a technical manual and then go to the computer. As a result, many of the functions included in the VT are not covered in this manual, nor is a full description of the output of each function. Moreover, I will need to stop every now and then to introduce some concepts and/or notation. Sections that include technical details, derivations, etc will be labelled with [Note], while sections that include details on the practical example will be labelled with [Matlab].

Additional resources are available on my website:

- `https://sites.google.com/site/ambropo/replications` provide some lecture notes on the basics of VARs that are a good complement to this manual.

- `https://sites.google.com/site/ambropo/matlab-examples` provides a few examples on how to estimate VARs with different identification schemes (in a similar spirit to the example in this manual).

- `https://sites.google.com/site/ambropo/replications` provide the replication codes for a few well-known VAR studies (e.g. [5], [1], [6], and [2]).

I will start by introducing some (very light) notation.

3 VAR basics [Notes]

Notation, VAR structural and reducd-form representation, impulse responses, variance decompositions, and historical decompositions

3.1 Vector Autoregressions

Given a $k \times 1$ vector of time series (x_t) a Structural Vector Autoregression (SVAR) of order p can be written as:

$$x_t = \sum_{j=1}^{p} \Phi_j x_{t-j} + B\varepsilon_t, \tag{1}$$

where B is a $k \times k$ matrix, generally referred to as **structrual impact matrix**; and ε_t is a $k \times 1$ vector of serially uncorrelated innovations, generally referred to as **structural shocks**, which are assumed to be mutually uncorrelated and $i.i.d.$ with zero mean and unit variance.[1]

[1]Note that the fact that the variance of the structural shocks is equal to one is just a harmless normalization which does not involve a loss of generality (as long as the diagonal elements of B remain unrestricted. An alternative (and equivalently valid) normalization would be to leave unrestricted the variance of the structural innovations, namely $\varepsilon_{it} \sim \mathcal{N}(0, \sigma_i^2)$ and assume tha the diagonal elements of B to 1.

You should think of the system of equations defined by the structural VAR () as approximating the true structure of the economy (for example, the structure of a DSGE model); and the structural shocks as having a well-defined economic interpretation (for example, TFP shocks or monetary policy shocks).

To keep the notation as simple as possible, consider a bivariate VAR(1), i.e. a VAR where the number of variables is $k = 2$ and the number of lags is $p = 1$. Also, to make a concrete example, let the two endogenous variables be output growth (y_t) and a short-term safe interest rate (r_t); and the two structural shocks be a demand shock ε_t^{Demand}) and a monetary policy shock ($\varepsilon_t^{Mon.\ Pol}$).[2] This simple bivariate VAR(1) can be written as a system of linear equations:

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{Mon.\ Pol} \end{bmatrix}, \tag{2}$$

or:

$$\begin{aligned} y_t &= \phi_{11} y_{t-1} + \phi_{12} r_{t-1} + b_{11}\varepsilon_t^{Demand} + b_{12}\varepsilon_t^{Mon.\ Pol}, \\ r_t &= \phi_{21} y_{t-1} + \phi_{22} r_{t-1} + b_{21}\varepsilon_t^{Demand} + b_{22}\varepsilon_t^{Mon.\ Pol}, \end{aligned} \tag{3}$$

where $\varepsilon_t = (\varepsilon_t^{Demand\prime}, \varepsilon_t^{Mon.\ Pol\prime})'$ is a 2×1 vector of (unobservable) uncorrelated, zero mean, white noise processes. That is:

$$\mathbb{V}(\varepsilon_t) \equiv \Sigma_\varepsilon = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] = I_2. \tag{4}$$

The assumption that the elements of ε_t are mutually uncorrelated is crucial. It implies that we can track the effects of a shock to, say, ε_t^{Demand} to all variables in the VAR keeping the other shock to zero (and vice versa). The B matrix is also crucial. To see that, consider a unit surprise in $\varepsilon_t^{Mon.\ Pol}$, i.e. a surprise tightening in monetary policy. What are the consequences for output grpwth y_t and r_t? The answer to this question is in the second column of the B matrix: y_t will move by b_{12} and r_t will move by b_{22}. This is why the B matrix is also known as the structural impact matrix. The Φ matrix can then be used to track the dynamic effects of the shocks in $t + 1$, $t + 2$, etc.

While all this sounds easy and great, there is a complication. The structural innovations ε_t are unobservable, which means that we can't directly estimate (3). The best we can do is to 'bundle' the ε_t into a single object, the **reduced-form innovations**:

$$\begin{aligned} u_{yt} &= b_{11}\varepsilon_t^{Demand} + b_{12}\varepsilon_t^{Mon.\ Pol}, \\ u_{rt} &= b_{21}\varepsilon_t^{Demand} + b_{22}\varepsilon_t^{Mon.\ Pol}. \end{aligned} \tag{5}$$

The reduced-form innovations $u_t = \left(u'_{yt}, u'_{rt}\right)'$ are a linear combination of the structural innovations. We can then rewrite our VAR as:

$$\begin{aligned} y_t &= \phi_{11} y_{t-1} + \phi_{12} r_{t-1} + u_{yt}, \\ r_t &= \phi_{21} y_{t-1} + \phi_{22} r_{t-1} + u_{rt}, \end{aligned} \tag{6}$$

The VAR in (6) is typically referred to as the **reduced-form representation** of the structural VAR, which can be written more compactly in matrix form as:

$$\left[\begin{array}{c} y_t \\ r_t \end{array} \right] = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \left[\begin{array}{c} y_{t-1} \\ r_{t-1} \end{array} \right] + \left[\begin{array}{c} u_{yt} \\ u_{rt} \end{array} \right] \tag{7}$$

where:

$$\left[\begin{array}{c} u_{yt} \\ u_{rt} \end{array} \right] = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{Mon.\ Pol} \end{bmatrix}, \tag{8}$$

[2] Note that this specification is too parsimoniuos to be realistic. But it is going to simplify the exposition.

or:

$$x_t = \Phi x_{t-1} + u_t \tag{9}$$

where $u_t = B\varepsilon_t$. Differently from the structural VAR (3), the parameters of the reduced form VAR (Φ) and its innovations (u_t) can be estimated with OLS.

The key difference between the structural and reduced for VARs lies in the covariance matrix of their innovations. While the covariance matrix of the structural VAR innovations is diagonal ($\Sigma_\varepsilon = I_2$), in general the covariance reduced form VAR innovations are correlated among themselves, which implies

$$\mathbb{V}(u_t) \equiv \Sigma_u = \left[\begin{array}{cc} \sigma_y^2 & \sigma_{yr}^2 \\ \sigma_{yr}^2 & \sigma_r^2 \end{array} \right]. \tag{10}$$

In other words, Σ_u is a symmetric non-diagonal matrix, where its diagonal elements are the variances of the estimated reduced-form error terms, σ_y^2 and σ_r^2; and the identical off-diagonal elements are instead equal to the covariance between the estimated reduced form residuals, σ_{yr}^2.

The covariance between the estimated reduced form residuals plays an important role VARs because it collects the information on the contemporaneous interaction of the variables in the structural system, which (as we have just seen) is summarized by the B matrix. Indeed, using (8) the covariance matrix of the reduced for residuals can be written as:

$$\Sigma_u = \mathbb{E}\left[u_t u_t'\right] = B\mathbb{E}\left[\varepsilon_t \varepsilon_t'\right] B' = BB' = \left[\begin{array}{cc} b_{11}^2 & b_{11}b_{21} + b_{12}b_{22} \\ b_{11}b_{21} + b_{12}b_{22} & b_{22}^2 \end{array} \right] \tag{11}$$

This shows that, differently from structural VARs, the reduced form innovations are not informative about how shocks (e.g. to demand or supply) propagate through the VAR. An innovation to u_{yt} could be driven by either ε_t^{Demand} or $\varepsilon_t^{Mon.\ Pol}$ (and vice versa). To be able to talk about the causal effects of a shock to the variables in the VAR we need to find a way to recover the B matrix from Σ_u. This will be the objective of the next Section, where we discuss the identification of structural shocks.

Before doing that, however, we introduce here another representation of the VAR that will be useful in the sections below, namely the **Wold representation**. To obtain the Wold representation, start from the structural VAR representation

$$x_t = \Phi x_{t-1} + B\varepsilon_t \tag{12}$$

and use it to substitute recursively the elements on the right hand side of the equal sign. Namely,

$$\begin{array}{rcl} x_t & = & \Phi x_{t-1} + B\varepsilon_t \\ & = & \Phi\left(\Phi x_{t-2} + B\varepsilon_{t-1}\right) + B\varepsilon_t = \Phi^2 x_{t-2} + \Phi B\varepsilon_{t-1} + B\varepsilon_t \\ & = & \ldots = \\ & = & \Phi^\infty x_{t-\infty} + \sum_{j=0}^{\infty} \Phi^j B\varepsilon_{t-j}. \end{array} \tag{13}$$

which show that each observation (x_t) can be re-written as the cumulative sum of the structural shocks plus an initial condition. Of course, in practical applications, we don't have time series of infinite length. For a finite time series of T observations (i.e. $x_1, x_2, ..., x_T$) plus an initial conditon x_0, the T observation can be written as

$$x_T = \Phi^T x_0 + \sum_{j=0}^{T-1} \Phi^j B\varepsilon_{T-j}. \tag{14}$$

We will show later that, if the eigenvalues of the matrix Φ are below 1 and number of observations is large enough, this sum converges to a finite number.

3.2 The Identification Problem

Assume that the true model of the economy is given by the structural VAR in (2)

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{Mon.\ Pol} \end{bmatrix}, \tag{15}$$

where the matrix B and the structural shocks ε_t are unobserved. In the previous section we have seen that we can't estimate (**??**), but we can estimate its reduced form representation:

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} u_{yt} \\ u_{rt} \end{bmatrix}. \tag{16}$$

Now, imagine that you are asked to estimate the effect of a shock to $\varepsilon_t^{Mon.\ Pol}$ on the endogenous variables y_t and r_t. Unfortunately, the reduced form innovation to the interest rate, u_{rt}, is not going to help us. The reason is that, as we discussed in Section 3, u_{rt} is a linear combination of the true structural shocks in the economy. So, it does not tell us anything about how ε_t^{Demand} affects y_t or r_t. To answer the question, we need to find out the values of b_{12} and b_{22}. But how can we go from the reduced form representation to the structural representation of the VAR? This is known as the identification problem.

We have seen above that $u_t = B\varepsilon_t$, so that we can write:

$$\Sigma_u = E\left[u_t u_t'\right] = E\left[B\varepsilon_t \left(B\varepsilon_t\right)'\right] = B\Sigma_\varepsilon B' = BB', \tag{17}$$

where remember that $\Sigma_\varepsilon = I_2$. This means that there is a mapping between the estimated covariance matrix of the reduced form residuals (Σ_u) and the unobserved matrix of structural impact coefficients (B). The identification problem simply boils down to finding a B matrix that satisfies $\Sigma_u = BB'$.

Unfortunately this is not as easy as it sounds. We can think of (43) as a system of nonlinear equations in the 4 unknown coefficients of the B matrix. The problem is that the Σ_u matrix, given its symmetric nature, leads to only 3 independent restrictions. In other words, we have

$$\begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ - & \sigma_r^2 \end{bmatrix} = \underbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}}_{B'}, \tag{18}$$

which can be rewritten as the following system of equations:

$$\begin{cases} \sigma_y^2 = b_{11}^2 + b_{12}^2 \\ \sigma_{yr}^2 = b_{11}b_{21} + b_{12}b_{22} \\ \sigma_{yr}^2 = b_{11}b_{21} + b_{12}b_{22} \\ \sigma_r^2 = b_{21}^2 + b_{22}^2 \end{cases} \tag{19}$$

Note that, because of the symmetry of the Σ_u matrix, the second and the third equation are identical. This means that we are left with 4 unknowns (the b's) but only 3 equations. The system is under-identified, meaning that there are infinite combination of the b's that solve the system of equations (45).

How to solve a system of 3 equations in 4 unknowns? The solution is (typically) to draw from economic theory an additional condition that allows us to recover a fourth equation – and therefore, solve the system of equations (45).

There are many ways of solving the identification problem described above. In the following section, we will cover a few popular identification schemes. We will show later how they can be implemented in the VT.

3.3 Common Identification schemes

Many solutions have been developed in the literature to address the identification problem described in the previous section. In this section, we go through the some popular ones namely, zero (recursive) contemporaneous restrictions, zero (recursive) long-run restrictions, sign restrictions, external instruments, combining sign restrictions and external instruments.

3.3.1 Identification by zero contemporaneous restrictions

Identification using zero contemporaneous restrictions (also known as Cholesky identification, for a reason that will be clear in a second) were developed by Sims1980, and are by far the most commonly used identification scheme used in the literature. In a recursive SVAR, identification is achieved by assuming that some shocks have zero contemporaneous effect on some of the endogenous variables. This amounts to setting some of the non-diagonal elements of the B matrix to zero – therefore reducing the number of unknown coefficients.

Typically, it is assumed that the first variable in the system is only affected by the first structural shock, the second is contemporaneously affected by the first and second structural shock, and so on. In our example, that means to assume that the structural VAR is

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{Mon.\ Pol} \end{bmatrix}, \tag{20}$$

where note that y_t is not contemporaneously affected by $\varepsilon_t^{Mon.\ Pol}$, while r_t is contemporaneously affected by both ε_t^{Demand} and $\varepsilon_t^{Mon.\ Pol}$ (via b_{21} and b_{22}). This assumption could be justified in the contect of a monetary VAR, by assuming that a monetary policy shock affects on impact the interest rate (r_t), but takes time to affect output (y_t).

What are the implications for the identification problem described above? The simple answer is that we now have 4 instead of 3 independent equations, and 4 parameters to estimate. That is, the system of equations (45) now becomes:

$$\begin{cases} \sigma_y^2 = b_{11}^2, \\ \sigma_{yr}^2 = b_{11} b_{21}, \\ \sigma_r^2 = b_{21}^2 + b_{22}^2. \end{cases} \tag{21}$$

which can be easily solved to get:

$$\begin{cases} b_{11} = \sigma_y^2, \\ b_{21} = \sigma_{yr}^2 / \sigma_y^2, \\ b_{22} = \sqrt{\sigma_r^2 - \frac{\sigma_{yr}^2}{\sigma_y^2}}. \end{cases}$$

The VAR is identified! This means that it is possible to compute the *impact* impulse response of all endogenous variables by simply looking at the estimated B matrix. For example, using the structural VAR representation

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma_y^2 & 0 \\ \sigma_{yr}^2/\sigma_y^2 & \sqrt{\sigma_r^2 - \frac{\sigma_{yr}^2}{\sigma_y^2}} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{Mon.\ Pol} \end{bmatrix}$$

we get that the response of x_t to ε_t^{Demand} is given by

$$\begin{aligned} y_t &= \sigma_y^2, \\ r_t &= \sigma_{yr}^2/\sigma_y^2, \end{aligned}$$

and the response of x_t to $\varepsilon_t^{Mon.\ Pol}$ is given by

$$\begin{aligned} y_t &= 0, \\ r_t &= \sqrt{\sigma_r^2 - \frac{\sigma_{yr}^2}{\sigma_y^2}}. \end{aligned}$$

This identification scheme is also known as Cholesky identification. The reason is the following. A symmetric and positive-definite matrix like Σ_u can always be decomposed as:

$$\Sigma_u = \begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ \sigma_{yr}^2 & \sigma_r^2 \end{bmatrix} = \begin{bmatrix} p_{11} & 0 \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ 0 & p_{22} \end{bmatrix} = PP'$$

where P is a lower triangular matrix that is known as the Cholesky factor of Σ_u.

Recalling that $\Sigma_u = BB'$ where remember that we assumed that B is also lower triangular ($b_{21} = 0$), it follows that $P = B$. The advantage is that the the Cholesky decomposition can be easily computed in Matlab without the need for explicitly solving the ssytem implied by BB', which can become quite complex as the dimensionality of the VAR increases.

3.3.2 Identification by zero long-run restrictions

BlanchardQuah1989 and Gali1992 proposed an alternative identification method using restrictions on the long-run effects of shocks. To fix ideas, consider a shock that hits in t. Its **cumulative effect** in the long-run (i.e. for t that goes to infinity) can be computed by recursively iterating forward the strucural VAR. The impact of a shock ε_t on the endogenous variables for each horizon is given by

$$\begin{aligned} x_t &= B\varepsilon_t, \\ x_{t+1} &= \Phi B\varepsilon_t, \\ &... \\ x_{t,t+\infty} &= \Phi^\infty B\varepsilon_t. \end{aligned} \tag{22}$$

The long run cumulative effect of the shock can be obtained by summing all the terms in (22)

$$x_{t,t+\infty} = B\varepsilon_t + \Phi B\varepsilon_t + \Phi^2 B\varepsilon_t + ... + \Phi^\infty B\varepsilon_t = \sum_{j=0}^{\infty} \Phi^j B\varepsilon_t, \tag{23}$$

where $x_{t,t+\infty}$ denotes the sum from t to $t+\infty$ of the elements in (22). Finally note that, if the VAR is stable (i.e. if the eigenvalues of Φ lie inside the unit circle), the infinite sum in equation (23) converges to

$$x_{t,t+\infty} = (I - \Phi)^{-1} B\varepsilon_t = C\varepsilon_t, \tag{24}$$

where

$$C \equiv (I - \Phi)^{-1} B \tag{25}$$

captures the cumulative effect of the shock ε_t on x_t from time t to $t+\infty$.

The identification through zero long-run restrictions imposes a zero restriction on the long-run impact matrix C. For example, if you believe in the long-run neutrality of monetary policy you would expect monetary policy to have zero effect on the level of output in the long run. Equation (24) allows us to impose exactly this restriction, as the element c_{12} captures the cumulative impact of $\varepsilon_t^{Mon.\ Pol}$ on the *level* of GDP in the long-run. That is, we can assume:

$$\begin{bmatrix} y_{t,t+\infty} \\ r_{t,t+\infty} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{Mon.\ Pol} \end{bmatrix}. \tag{26}$$

But how does this help with the identification of B? Clearly, $C \equiv (I - \Phi)^{-1} B$ is unknown as B is unobserved. But Equation (25) can be exploited to achieve identification. To see that, first define $\Omega \equiv CC'$ and note that

$$\Omega = \left((I - \Phi)^{-1}\right) BB' \left((I - \Phi)^{-1}\right)' = \left((I - \Phi)^{-1}\right) \Sigma_u \left((I - \Phi)^{-1}\right)', \tag{27}$$

which implies that Ω is known. Second, note that Ω is a positive-definite symmetric matrix, which implies it has a unique Cholesky decomposition:

$$\Omega = PP', \tag{28}$$

where the lower triangular matrix P is the Cholesky factor of Ω. Because of our assumption that C is lower triangular, it follows that $P = C$. Finally, as C is known, we can recover B from (25):

$$B = (I - \Phi)\, C. \tag{29}$$

3.3.3 Identification by sign restrictions

While the zero restrictions discussed in the previous sections can be justified by economic theory, in many applications these restrictions are implausible or hard to justify. The identification by sign restrictions provides an alternative approach that exploits prior beliefs (typically informed by theroetical models) about the sign that certain shocks should have on certain endogenous variables.

The idea is to impose restrictions on a set of orthogonalised impulse response functions. So, differently from the identification schemes described above (where there is a unique point estimate of the B matrix), sign restricted VARs are only set identified. In other words, the data are potentially consistent with a wide range of B matrices that are all admissible in that they satisfy the sign restrictions (Faust (1998), Canova and De Nicolo (2002), and Uhlig (2005))

To fix ideas, a demand shock (ε_t^{Demand}) should lead to an increase in output growth (y_t) and to an increase in the short term interest rate (r_t), as monetary policy responds to the shock to contain the boom. Differently, a monetary policy shock ($\varepsilon_t^{Mon.\ Pol}$) should lead to a fall in output growth for an increase in interest rates. That is:

	Demand (ε_t^{Demand})	Monetary Policy ($\varepsilon_t^{Mon.\ Pol}$)
Output growth (y_t)	+	-
Short-rate Int. Rate(r_t)	+	+

The key intuition is based on the following three steps. First, consider one of the infinite orthonormal matrices Q such that:

$$QQ' = I_2.$$

Second, consider the B matrix correspoding to the Cholesky factor of Σ_u, namely:

$$\Sigma_u = PP'.$$

We know from Section **??**) that B=P is a the unique structural impact matrix that would obtain under a zero contemporanesou restriction. Third, and finally, note that the following equality holds

$$\Sigma_u = PP' = PQQ'P' = \underbrace{(PQ)}_{B}\underbrace{(PQ)'}_{B'}$$

where the matrix $B = PQ$, which is not triangular anymore, is (i) known and (ii) a valid 'candidate' structural impact matrix that solves the identification problem.

But does $B = PQ$ represent a plausible solution? Identification is achieved by checking that the impulse responses associated with $B = PQ$ satisfy a set of a priori sign restrictions. Doing that is simple. Consider the structural representation of our VAR

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{Mon.\ Pol} \end{bmatrix}, \tag{30}$$

where the elements of B are known and equal to PQ. We can then easily check whether the impact response of the two shocks to prices and quantities satisfy the sign restrictions:

	Demand (ε_t^{Demand})	Monetary Policy ($\varepsilon_t^{Mon.\ Pol}$)
Output growth (y_t)	$b_{11} > 0$?	$b_{12} < 0$?
Short-rate Int. Rate(r_t)	$b_{21} > 0$?	$b_{22} > 0$?

If all the elements of B satisfy the sign restrinctions, then we retain the draw and store PQ. If at least one element of the B does not satisfy the restrictions, we discard the draw, compute a new Q matrix, and check again. After drawig a large number of Q matrices that satisfy the sign restrictions, we can then construct a distribution of the B matrix.

As hinited above, VARs identified with sign restrictions are only set-identified. In other words, the data are potentially consistent with a wide range of structural models that are all admissible in that they satisfy the identifying restrictions.

3.3.4 External instruments

The external instruments identification approach has been proposed by [4] and [3]. This identification strategy uses standard instrumental variable techniques to isolate the variation of the VAR reduced-form residuals that are due to the structural shock of interest.

The key element of this identification technique is the presence of an **instrument** that is correlated with a shock of interest and uncorrelated with all other shocks. For example, assume that such an instrument exists (z_t) and satifies the following properties:

$$\begin{aligned} \mathbb{E}\left[\varepsilon_t^{Demand} z_t'\right] &= c, \\ \mathbb{E}[\varepsilon_t^{Mon.\ Pol} z_t'] &= 0, \end{aligned}$$

namely is correlated with ε_t^{Demand} and uncorrelated with $\varepsilon_t^{Mon.\ Pol}$. If such an instrument exists, it is possible to identify the contemporaneous response of all endogenous variables in the VAR system to the shock of interest. That is, we can identify one coloumn (in this example, the first one) of the B matrix:

$$B = \begin{bmatrix} b_{11} & - \\ b_{21} & - \end{bmatrix}$$

The intuition is as follows. Recall that the reduced form residuals are a linear combination of two orthogonal shocks (see for example equation (5). Then, it is possible to isolate the variation in one of the two reduced-form residuals (say, u_{yt}) that is due only to the shock of interest with a regression of u_{yt} itself on the instrument z_t. By projecting the remaining residuals on the fitted values of the previous regression, it is possible to obtain an estimate of the contributions of the shock of interest to all variables in the system, thus providing an estimate of the first column of B (up to a scaling factor).

That is, in the 'first stage', we regress the reduced form residuals u_{yt} on z_t

$$u_{yt} = \beta z_t + \xi_t, \tag{31}$$

to construct the fitted values $\hat{u}_{1t}$. Then we regress the reduced form residuals u_{rt} on the fitted values ($\hat{u}_{1t}$) to get a consistent estimate of the ratio b_{21}/b_{11}:

$$u_{rt} = \underbrace{\gamma}_{b_{21}/b_{11}} \hat{u}_{1t} + \zeta_t, \tag{32}$$

where note that $\hat{u}_{1t}$ is orthogonal to ζ_t under the assumption that $\mathbb{E}[\varepsilon_t^{Demand} z_t'] = 0$. Now, if we normalize the effect of ε_t^{Demand} on y_t to 1 (that is, we fix $b_{11} = 1$) we can easily recover $b_{21} = \gamma$.

3.3.5 Combining sign restrictions and external instruments

The external instruments and sign restriction identification approaches can be combined as proposed by CesBianchiSokol2020. To have a meaningful example we need a trivariate VAR

$$\begin{bmatrix} y_t \\ r_t \\ x_{3t} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{12} & b_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{Mon.\ Pol} \\ \varepsilon_{3t} \end{bmatrix}. \tag{33}$$

For simplicity, and without loss of generality, we consider the case where we identify only the first structural shock (labelled ε_t^b) with the external instruments approach. Without loss of generality we choose this shock to be the shock associated with the first equation, so we set $\varepsilon_t^b \equiv \varepsilon_t^{Demand}$. We can then identify the remaining $n-1$ shocks, namely $\varepsilon_t^{\mathcal{B}} \equiv (\varepsilon_t^{Mon.\ Pol}, \varepsilon_{3t}')'$, with sign restrictions.

We start by partitioning the matrix B into a column vector b, which captures the impact of the shock in the first equation (e_t^b), and a matrix $\mathcal{B}$, whose columns capture the impact of the shocks in the remaining 2 equations ($e_t^{\mathcal{B}}$):

$$B = \left[\begin{array}{cc} b & \mathcal{B} \end{array}\right], \tag{34}$$

where b is an 3×1 vector and $\mathcal{B}$ is a 3×2 matrix. Assuming that a valid instrument exists, the first column of the B matrix (b) can be easily identified as explained in the previous section.

We now show how to combine the external instruments identification approach with a standard sign restriction approach to identify the remaining structural shocks ($e_t^{\mathcal{B}}$) 'conditional' on the shock identified with the external instrument (e_t^b). To identify $\mathcal{B}$ (i.e. the contemporaneous impact of the remaining shocks) we proceed as follows. First,

using (34), we re-write the covariance matrix of the reduced-form residuals as:

$$\Sigma_u = BB' = \left[\begin{array}{cc} b & \mathcal{B} \end{array}\right] \left[\begin{array}{cc} b & \mathcal{B} \end{array}\right]'. \tag{35}$$

As we have seen above, this decomposition of the covariance matrix is not unique. Let P be the Cholesky decomposition of the covariance matrix Σ_u, and let Q be an orthonormal matrix such that $QQ' = I$. Then we can write:

$$\Sigma_u = PP' = PQQ'C' = (PQ)(PQ)' \tag{36}$$

Our strategy consists precisely in constructing a large number of orthonormal matrices Q that satisfy the following condition:

$$CQ = \left[\begin{array}{cc} b & \mathcal{B} \end{array}\right],$$

where b is identified via the external instrument and $\mathcal{B}$ satisfies a set of sign restrictions. For example, assume we have an instrument for a monetary policy shock and we want to identify the effects of deamnd and supply shocks conditional on the monetary policy shock, namely:

	Monetary Policy	Demand (ε_t^{Demand})	Supply ($\varepsilon_t^{Mon.\ Pol}$)
Policy Rate (y_t)	Proxy	+	
Price (r_t)	Proxy	+	-
Quantity (x_{3t})	Proxy	+	+

We do that in three steps.

1. Find a normal vector q of dimension $n \times 1$ that rotates the first column of C, the Cholesky decomposition of Σ_u, into the vector b. That is, we find a $n \times 1$ normal vector q such that the following equality holds:
$$Cq = b \tag{37}$$
2. Given q, build the remaining $n-1$ columns of an orthonormal matrix Q following a standard Gram-Schmidt process.[3] That is, find an $(n \times n-1)$ matrix $\mathcal{Q}$ such that the following equality holds:
$$\begin{bmatrix} q & \mathcal{Q} \end{bmatrix} \begin{bmatrix} q & \mathcal{Q} \end{bmatrix}' = QQ' = I. \tag{38}$$
The matrix CQ then represents a candidate identification scheme because:
$$CQ = C \begin{bmatrix} q & \mathcal{Q} \end{bmatrix} = \begin{bmatrix} b & \mathcal{B} \end{bmatrix} = B. \tag{39}$$
3. Check that $\mathcal{B}$ satisfies our set of sign restrictions. If it does, we retain the matrix Q. If does not, we repeat steps (1) and (2) until we obtain a matrix $\mathcal{B}$ that satisfies the restrictions.

Finally we repeat steps (1)-(2)-(3) until we have M matrices B_i (with $i = 1, 2, ..., M$) consistent with our identification restrictions. This completes the (set) identification of structural matrix B.

3.4 Structural Dynamic Analysis

3.4.1 Impulse response functions

Impulse response functions (IR) allow us to answer the following question: 'What is the response over time of each of the variables in a VAR to an increase in the current value of one of the structural innovations, assuming that (i) the structural innovation returns to zero in subsequent periods and (ii) all other structrual innovations are equal to zero?

Of course, the implied thought experiment of shocking the innovations of one equation while holding the others constant makes sense only when the innovations are uncorrelated across equations – which can be done only once we know the structrual representation of the VAR, i.e. once we have identified the B matrix.

To show how to compute impulse response functions, consider our simple bivariate VAR in its structural representation

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{Mon.\ Pol} \end{bmatrix}, \tag{40}$$

[3] Let j index the columns of Q. Let Q_{j-1} denote the first $j-1$ columns of Q, such that $Q_{2-1} = Q_1 = q_1$. Let x_j be a draw from a Normal distribution on $\mathbb{R}^N$. Then the $j-$th column of Q can be constructed as:

$$q_j = \frac{\left(I_N - Q_{j-1}Q'_{j-1}\right) x_j}{\left\| \left(I_N - Q_{j-1}Q'_{j-1}\right) x_j \right\|}.$$

Then, define a 2×1 vector of impulse selection (s) that take value of one for the strcuctural shock that we wan to consider. For example, to compute the IR to the first structural shocks we define s as:

$$s = \left[\begin{array}{c} 1 \\ 0 \end{array} \right].$$

The impulse responses to the structural shock ε_t^{Demand} can be easily computed with the following equation

$$x_t = \Phi x_{t-1} + B s_t,$$

which can be computed recursively as follows

$$\begin{cases} IR_t = Bs, & \text{for } t = 0, \\ IR_t = \Phi \cdot IR_{t-1} & \text{for } t = 2, ..., h. \end{cases}$$

3.4.2 Forecast error variance decompositions

Forecast error variance decompositions (VD) answer the following question: What portion of the variance of the forecast error that is made when predicting $x_{i,t+h}$ is due to each structural shock in ε_t (on average, over the sample period)? As such, VD provide information about the relative importance of each structural shock in affecting the variables in the VAR

To show how to compute forecast error variance decompositions, consider the 1-step ahead forecast error in our simple bivariate VAR:

$$x_{t+1} - \mathbb{E}\left[x_{t+1}\right] = x_{t+1} - \mathbb{E}\left[\Phi x_t + u_{t+1}\right] = x_{t+1} - \Phi x_t = u_{t+1}$$

where note that the 1-step ahead forecast error is the 1-step ahead reduced form residual. As we know from the previous setion, the reduced form residual is related to the structural shocks throught the following equations

$$\begin{cases} u_{1t+1} = b_{11}\varepsilon_{1t+1} + b_{12}\varepsilon_{2t+1}, \\ u_{2t+1} = b_{21}\varepsilon_{1t+1} + b_{22}\varepsilon_{2t+1}. \end{cases}$$

So, what is the variance of the forecast error?

$$\begin{array}{c} \mathbb{V}\left(u_{yt}\right) = b_{11}^2 \mathbb{V}\left(\varepsilon_{1,t+1}\right) + b_{12}^2 \mathbb{V}\left(\varepsilon_{2,t+1}\right) = b_{11}^2 + b_{12}^2 \\ \mathbb{V}\left(u_{rt}\right) = b_{21}^2 \mathbb{V}\left(\varepsilon_{1,t+1}\right) + b_{22}^2 \mathbb{V}\left(\varepsilon_{2,t+1}\right) = b_{21}^2 + b_{22}^2 \end{array}$$

whihc follows from the fact that the variacne of ε_t is 1 and the structural shocks are orthogonal to each other. The final step to compute VD is to ask: what portion of the variance of the forecast error is due to each structural shock? The answer is given by teh following equation

$$\underbrace{\begin{cases} VD_y^{\varepsilon_1} = \frac{b_{11}^2}{b_{11}^2 + b_{12}^2} \\ VD_y^{\varepsilon_2} = \frac{b_{12}^2}{b_{11}^2 + b_{12}^2} \end{cases}}_{\text{This sums up to 1}} \qquad \underbrace{\begin{cases} VD_r^{\varepsilon_1} = \frac{b_{21}^2}{b_{21}^2 + b_{22}^2} \\ VD_r^{\varepsilon_2} = \frac{b_{22}^2}{b_{21}^2 + b_{22}^2} \end{cases}}_{\text{This sums up to 1}}$$

3.4.3 Historical decompositions

Historical decompositions (HD) answer the following question: What portion of the deviation of $x_{i,t}$ from its unconditional mean is due to each structural shock ε_t?

We showed before that, in the absence of shocks, the variables of a (stable) VAR will converge to their unconditional mean (i.e. their equilibrium values or steady state). As we ahev seen with impulse responses, when a structural shock hits, the endogenous variables move away from their equilibrium and then only slowly go back to it. If another shock hits in the next period, the variables will now be away from equilibrium because of (i) the effect of the new shock and (ii) the persistent effect of the old shock. Historical decompositions allow us to know, at each point in time, what shock is responsible for keeping the endogenous variables away from their steady state.

To show how to compute historical decompositions, we start from the Wold decomposition of a VAR in Equation (14), according to which each observation can be re-written as the cumulative sum of the structural shocks. In particular, we consider the Wold representation of our simple bivariate VAR for $t = 2$:

$$r = \underbrace{\Phi^2 x_0}_{init_2} + \underbrace{\Phi B}_{\mathbf{\Theta_1}} \varepsilon_1 + \underbrace{B}_{\Theta_2} \varepsilon_2,$$

which allows us to write r as a function of present and past structural shocks (ε_1 and ε_2) and the initial condition (x_0). In matrix form:

$$\begin{bmatrix} x_{1,2} \\ x_{2,2} \end{bmatrix} = \begin{bmatrix} init_{y,2} \\ init_{r,2} \end{bmatrix} + \begin{bmatrix} \theta^1_{11} & \theta^1_{12} \\ \theta^1_{21} & \theta^1_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,1} \\ \varepsilon_{2,1} \end{bmatrix} + \begin{bmatrix} \theta^2_{11} & \theta^2_{12} \\ \theta^2_{21} & \theta^2_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,2} \\ \varepsilon_{2,2} \end{bmatrix}$$

Therefore r can be expressed as

$$\begin{cases} x_{1,2} = init_{y,2} + \theta^1_{11}\varepsilon_{1,1} + \theta^1_{12}\varepsilon_{2,1} + \theta^2_{11}\varepsilon_{1,2} + \theta^2_{12}\varepsilon_{2,2} \\ x_{2,2} = init_{r,2} + \theta^1_{21}\varepsilon_{1,1} + \theta^1_{22}\varepsilon_{2,1} + \theta^2_{21}\varepsilon_{1,2} + \theta^2_{22}\varepsilon_{2,2} \end{cases}$$

The historical decomposition is given by

$$\underbrace{\begin{cases} HD^{\varepsilon_1}_{y,2} = \theta^1_{11}\varepsilon_{1,1} + \theta^2_{11}\varepsilon_{1,2} \\ HD^{\varepsilon_2}_{y,2} = \theta^1_{12}\varepsilon_{2,1} + \theta^2_{12}\varepsilon_{2,2} \\ HD^{init}_{y,2} = init_{y,2} \end{cases}}_{\text{This sums up to } x_{1,2}} \qquad \underbrace{\begin{cases} HD^{\varepsilon_1}_{r,2} = \theta^1_{21}\varepsilon_{1,2} + \theta^0_{21}\varepsilon_{1,2} \\ HD^{\varepsilon_2}_{r,2} = \theta^1_{22}\varepsilon_{2,2} + \theta^0_{22}\varepsilon_{2,2} \\ HD^{init}_{r,2} = init_{r,2} \end{cases}}_{\text{This sums up to } x_{2,2}}$$

where $HD^{\varepsilon_1}_{y,2}$ is the contribution of present and past shocks to ε_1 to y in period $t = 2$. Note that y, for example, is equal to the sum of the contribution of (i) present and past shocks to ε_1, (ii) present and past shocks to ε_2, and (iii) the initial condition. The first two elements are obvious: if shocks are persistent we would expect today's calue of y to be affectd by present and recent shocks. The third element depends on how far the first observation in our data (x_0) is from its unconditinoal mean. In this example, we are assuming that the unconditional mean of the data is zero. So if x_0 is very different from 0, the the initial condition will matter for many periods until it will become asymptotically small.

4 Data: Load & Plot[Matlab]

In the following sections we are going to use the data set used by [2], which contains information on US industrial production (IP_t), consumer prices (CPI_t), the 1-year government bond interest rate (R_t) and the Excess Bond Premium (EBP_t) from 1979:M7 to 2012:M6.[4] The code below

[4] The data set also includes a series of high-frequency moneatry surprises around FOMC announcements that will be discussed in more detail below.

shows a general way of loading the data and managing it in a way that is consistent with the functioning of the VT.

```
%% 1. LOAD & PLOT DATA
%-------------------------------------------------
% Load
[xlsdata, xlstext] = xlsread('GK2015_Data.xlsx','VAR_data');
data = Num2NaN(xlsdata(:,3:end));
vnames = xlstext(1,3:end);
for ii=1:length(vnames)
    DATA.(vnames{ii}) = data(:,ii);
end
year = xlsdata(1,1);
month = xlsdata(1,2);
% Observations
nobs = size(data,1);
% Set endogenous
VARvnames = {'gs1','logcpi','logip','ebp'};
VARvnames_long = {'Policy rate','CPI','Industrial
Production','EBP'};
VARnvar = length(VARvnames);
% Create matrices of variables to be used in the VAR
ENDO = nan(nobs,VARnvar);
for ii=1:VARnvar
    ENDO(:,ii) = DATA.(VARvnames{ii});
end
```

First the code reads from an Excel file and stores all data into the structure `DATA`. The VAR Toolbox includes some functions that allow to plot time series quickly and export them as high-quality PDFs, so that they can be used directly in your papers. The code shows how to plot the four time series in `ENDO`.

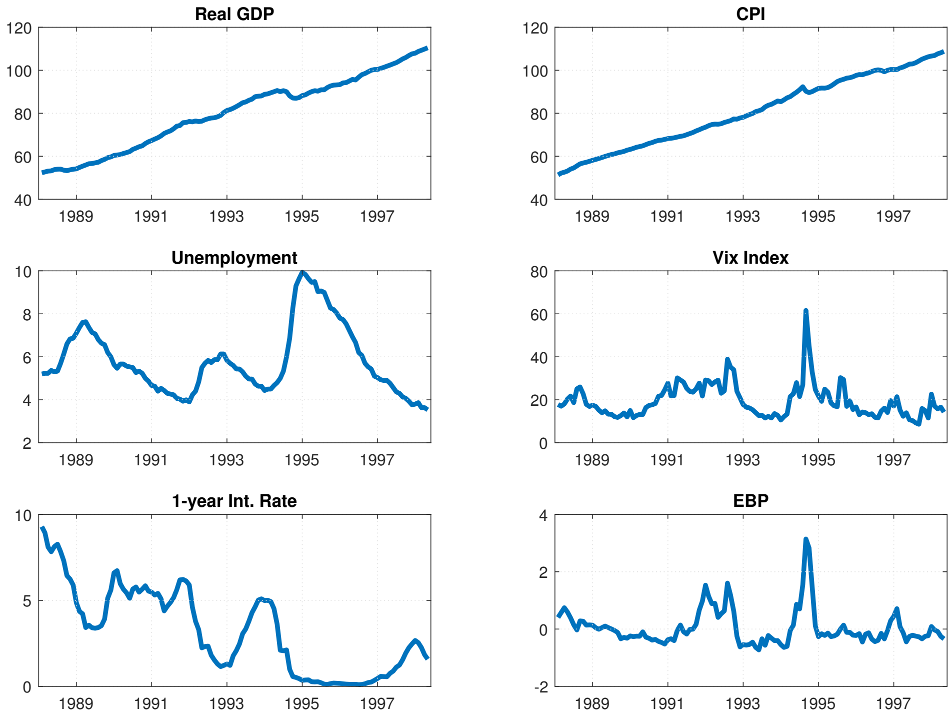
```
% Open a figure of the desired size
FigSize(28,16)
for ii=1:VARnvar
    subplot(2,2,ii)
    H(ii) = plot(ENDO(:,ii),'LineWidth',2,'Color',cmap(ii));
    title(VARvnames_long(ii));
    DatesPlot(year+(month-1)/12,nobs,6,'m') % Set the x-axis label
    grid on;
end
% Legend
lopt = LegOption; lopt.handle = H; LegSubplot(VARvnames,lopt);
FigFont(10);
% Save
SaveFigure('graphics/F1_PLOT',1)
```

Figure 1 reports the behavior of the interest rate on US 1-year Treasury bill, an index of industrial production, the CPI level and the Excess Bond Premium (GZ) over the 1979:M7 to 2015:M3 sample period.

Some useful functions are:

- `FigSize.m`: allows the user to choose the proportions of the figure to plot. This is particularly useful when creating figures with many panels.

Figure 1: RAW DATA



NOTE. Raw data for interest rate on US 1-year Treasury bill, industrial production, CPI level and the Excess Bond Premium (GZ) from 1979;M7 to 2015:M3.

- `DatesPlot.m`: Adds dates to the horizontal axis of a chart (at monthly, quarterly, and annual frequency) using a specified number of ticks.

- `SaveFigure.m`: saves the chart in the selected format (pdf, jpg, eps). The function allows the user to save the figure at high quality standard using the `export_fig.m` function created by Oliver Woodford. Note that you need Ghostscript to be able to use this function.

5 VAR estimation [Matlab]

To keep the exposition as simple as possible, we start from a vary stylized VAR(1) with a constant and only two endogenous variables, namely the monthly growth rate of industrial production and the monthly growth rate of the CPI (i.e. monthly inflation):

$$\begin{bmatrix} IP_t \\ R_t \end{bmatrix} = \begin{bmatrix} \alpha^{IP} \\ \alpha^{R} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} IP_{t-1} \\ R_{t-1} \end{bmatrix} + \begin{bmatrix} u_t^{IP} \\ u_t^{R} \end{bmatrix}.$$

While such a simple VAR cannot realistically describe the complex interactions of the US economy, it is a useful device to understand the functioning of the VT codes and the workings of VAR models more in general.

In the VT, a VAR model can be estimated with a simple line of code, using the `VARmodel.m` function. To do that you need to specify a matrix including the endogenous variables (`ENDO`), whether you want deterministic variables, like a constant or a trend for example (`det`), and the number of lags of the VAR (`nlags`). In the example, I specify a simple bivariate VAR(12) in industrial production and interest rates, with a constant.[5]

The code below shows how to create the matrix of endogenous variables.

```
%% VAR ESTIMATION
%--------------------------------------------------
% Set the deterministic variable in the VAR (1=constant, 2=trend)
det = 1;
% Set number of nlags
nlags = 12;
% Estimate VAR by OLS
[VAR, VARopt] = VARmodel(ENDO,nlags,det);
disp(VAR)
disp(VARopt)
% Add variable names to VARopt
VARopt.vnames = VARvnames;
VARopt.figname= 'graphics/';
% Print at screen and create table
[TABLE, beta] = VARprint(VAR,VARopt,2);
```

The cell array `VARvnames` defines the list of endogenous variables that will be used to estimate the VAR model (in this case, industrial production and interest rates, namely a subset of the data in `DATA`). The chosen data is then stored in the matrix `ENDO`. The convention in the VT is that each column is a variable and each row is an observation (with no missing observations allowed). That is:

$$\mathtt{ENDO} = \begin{bmatrix} IP_1 & R_1 \\ IP_2 & R_2 \\ \dots & \dots \\ IP_T & R_T \end{bmatrix} = \left(IP_t', R_t'\right) = x_t'.$$

This convention implies that, using the notation defined in the previous section, $\mathtt{ENDO} = x_t'$.

The VAR can then be estimated in a few lines of code.

```
%% VAR ESTIMATION
%--------------------------------------------------
% Set the deterministic variable in the VAR (1=constant, 2=trend)
det = 1;
% Set number of nlags
nlags = 12;
% Estimate VAR by OLS
[VAR, VARopt] = VARmodel(ENDO,nlags,det);
disp(VAR)
disp(VARopt)
% Add variable names to VARopt
VARopt.vnames = VARvnames;
VARopt.figname= 'graphics/';
% Print at screen and create table
[TABLE, beta] = VARprint(VAR,VARopt,2);
```

The results of the VAR estimation are stored in the structures `VAR` and `VARopt`. The structure `VAR` includes all the estimation results. These can be seen by executing the command

[5]Note that this is a different specification from the original one used by [2]. This choice is purely pedagogical to make some of the derivations below easier.

`disp(VARprint(VAR)` which prints the following output in the command window:

```
>> disp(VAR)
         ENDO: [396x2 double]
         nlag: 12
        const: 1
         EXOG: []
         nobs: 384
         nvar: 4
      nvar_ex: 0
      nlag_ex: 0
       ncoeff: 24
    ntotcoeff: 25
          eq1: [1x1 struct]
          eq2: [1x1 struct]
          eq3: [1x1 struct]
          eq4: [1x1 struct]
           Ft: [49x4 double]
            F: [4x49 double]
        sigma: [4x4 double]
        resid: [384x4 double]
            X: [384x49 double]
            Y: [384x4 double]
        Fcomp: [48x48 double]
       maxEig: 0.9974
           Fp: [4x4x12 double]
            B: []
     BfromSR: []
          PSI: []
```

The structure `VAR` includes all the inputs to the `VARmodel.m` function, such as the matrix of endogenous variables (`VAR.ENDO`), the number of lags (`VAR.nlags`), and the number of endogenous variables (`VAR.nvar`). But also includes the estimation output. For example:

- The matrix `VAR.F` collects the estimated coefficients following the notation in (**??**), namely we have that `VAR.F`$=\Phi$. For a VAR with 1 lags and 2 variables plus a constant, this means that `VAR.F` is a $2\times(1\times 2+1)$ matrix.
- The covariance matrix of the VAR residuals defined by (**??**) is instead stored in `VAR.sigma`$=\Sigma_u$, of size 2×2.
- Note that the structural impact matrix `VAR.B` $=B$, which we defined in equation (**??**), is empty. This is because, for the moment we estimated only the reduced form VAR (1). The next sections will show how, with additional assumptions, the also the structural form of the VAR can be recovered.

Other outputs are the OLS equation-by-equation estimation results (structures `VAR.eq`), the VAR companion matrix (`VAR.Fcomp`), the maximum eigenvalue of the VAR (`VAR.maxEig`), etc.

The structure `VARopt` includes a few auxiliary variables that are created automatically by the `VARmodel.m` function and will be needed below for the calculation of impulse responses, variance decompositions, etc. The variables stored in `VARopt` can be seen by executing the command `disp(VARopt)`, which prints the following output in the Matlab command window:

```
>> disp(VARopt)
         vnames: []
      vnames_ex: []
         snames: []
         nsteps: 40
         impact: 0
           shut: 0
          ident: 'oir'
         recurs: 'wold'
         ndraws: 100
           pctg: 95
         method: 'bs'
           pick: 0
        quality: 0
       suptitle: 0
      firstdate: []
      frequency: 'q'
        figname: []
```

These variables include the number of steps for impulse response functions and variance decompositions (`nsteps`), the labels of the endogenous or exogenous variables for plots (`vnames` and `vnames_ex`), the confidence levels for the computation of error bands (`pctg`), etc. While some variables are automatically created by the VARmodel function, some other variables need to be inputted by the user. For example:

- `VARopt.vnames = VARvnames` stores in `VARopt` the endogenous variables' names.
- `VARopt.figname = 'graphics/'` stores in `VARopt` the name of the folder where all figures will be saved.

So that when executing `disp(VARopt)` we now get:

```
>> disp(VARopt)
         vnames: {'gs1'   'logcpi'   'logip'   'ebp'}
      vnames_ex: []
         snames: []
         nsteps: 40
         impact: 0
           shut: 0
          ident: 'oir'
         recurs: 'wold'
         ndraws: 100
           pctg: 95
         method: 'bs'
           pick: 0
        quality: 0
       suptitle: 0
      firstdate: []
      frequency: 'q'
        figname: 'graphics/'
```

6 The Identification Problem [Notes]

In the previous section we have seen that we can estimate the following reduced form VAR with OLS:

$$\begin{bmatrix} IP_t \\ R_t \end{bmatrix} = \begin{bmatrix} \alpha^{IP} \\ \alpha^{R} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} IP_{t-1} \\ R_{t-1} \end{bmatrix} + \begin{bmatrix} u_t^{IP} \\ u_t^{R} \end{bmatrix}. \tag{41}$$

Now, imagine that you are asked to estimate the effect of a monetary policy shock on industrial production. Unfortunately, the reduced form innovation to the interest rate (u_t^R) is not going to help us. The reason is that, as we discussed in Section 3, u_t^R is a linear combination of the true structural shocks in the economy. So, it does not tell us anything about how monetary policy affects output.

To see that more clearly, assume that the 'true' model of the economy is given by the following structural VAR:

$$\begin{bmatrix} IP_t \\ R_t \end{bmatrix} = \begin{bmatrix} \alpha^{IP} \\ \alpha^{R} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} IP_{t-1} \\ R_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{Mon.\ Pol} \end{bmatrix}, \tag{42}$$

where the matrix B and the structural shocks ε are unobserved. The SVAR in (XX) assumes that time series of industrial production and interest rates are driven by a combination of demand and monetary policy shocks.[6] It is obvious that the reduced form innovation to the interest rate, u_t^R, is a linear combination of all shocks, and not just the monetary policy shock.

To answer the question of what are the effects of monetary policy on the economy, we need to find the values of the B matrix. This is known as the identification problem. For example, the coefficients b_{12} and b_{22} give us the impact effect of monetary policy on industrial production and interest rates. The matrix of coefficient Φ, which we estimated in the reduced form VAR, can then be used to trace out the dynamic effects of monetary policy on the economy beyond the impact effect.

So, how can we go from the reduced form representation to the structural representation of the VAR? We have seen above that $u_t = B\varepsilon_t$. so that we can write:

$$\Sigma_u = E\left[u_t u_t'\right] = E\left[B\varepsilon_t \left(B\varepsilon_t\right)'\right] = B\Sigma_\varepsilon B' = BB'. \tag{43}$$

where remember that $\Sigma_\varepsilon = I$. This means that there is a mapping between the estimated covariance matrix of the reduced form residuals (Σ_u) and the unobserved matrix of structural impact coefficients (B). The identification problem simply boils down to finding a B matrix that satisfies $\Sigma_u = BB'$.

Unfortunately this is not as easy as it sounds. We can think of (50) as a system of nonlinear equations in the 4 unknown coefficients of the B matrix. The problem the Σ_u matrix, given its symmetric nature, leads to only 3 independent restrictions. In other words, we have

$$\begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ - & \sigma_r^2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}, \tag{44}$$

which can be rewritten as the following system of equations:

$$\begin{aligned} \sigma_y^2 &= b_{11}^2 + b_{12}^2 \\ \sigma_{yr}^2 &= b_{11}b_{21} + b_{12}b_{22} \\ \sigma_{yr}^2 &= b_{11}b_{21} + b_{12}b_{22} \\ \sigma_r^2 &= b_{21}^2 + b_{22}^2 \end{aligned} \tag{45}$$

[6] Again this is not a very realistic assumption, but it simplifies the math that follows. A more realistic VAR would have included more varibles and more shocks.

Note that, because of the symmetry of the Σ_u matrix, the second and the third equation are identical. This means that we are left with 4 unknowns (the b's) but only 3 equations. The system is clearly under-identified, meaning that there are infinite combination of the b's that solve the system of equations (45).

How to solve a system of 3 equations in 4 unknowns? The solution typically is to use economic theory to derive an additional condition that allows us to recover a fourth equation – and therefore, solve the system of equations (45). For example, if you believe that monetary policy works with a lag and has no effect on ouput on impact, you can impose $b_{21} = 0$. If added to the system (45), this equation allows for a unique solution of the elements of B.

There are many ways of solving the identification problem described above. In the following section, we will cover a few popular identification schemes, and how thy can be implemented in the VT.

7 Common Identification schemes

Many solutions have been developed in the literature to address the identification problem described in the previous section. In this section, we go through the some popular ones namely, zero (recursive) contemporaneous restrictions, zero (recursive) long-run restrictions, sign restrictions, external instruments, combining sign restrictions and external instruments.

7.1 Identification by zero contemporaneous restrictions

Identification using zero contemporaneous restrictions (also known as Cholesky identification, for a reason that will be clear in a second) were developed by Sims1980, and are by far the most commonly used identification scheme used in the literature. In a recursive SVAR, identification is achieved by assuming that some shocks have zero contemporaneous effect on some of the endogenous variables. This amounts to setting some of the non-diagonal elements of the B matrix to zero – therefore reducing the number of unknown coefficients.

Typically, it is assumed that the first variable in the system is only affected by the first structural shock, the second is contemporaneously affected by the first and second structural shock, and so on. In our example, that means to assume that the structural VAR is

$$\begin{bmatrix} IP_t \\ R_t \end{bmatrix} = \begin{bmatrix} \alpha^{IP} \\ \alpha^{R} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} IP_{t-1} \\ R_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{Mon.\ Pol} \end{bmatrix}, \tag{46}$$

where note that the industrial production is not contemporaneously affected by the monetary policy shock (while interest rates are contemporaneously affected by both the demand and the monetary policy shock). This assumption could be justified by the fact that monetary policy takes time to affect real variables like industrial production.

What are the implications for the identification problem described above? The simple answer is that we now have 3 instead of 4 structural parameters to estimate, and 3 restrictions implied by the reduced form covariance matrix. That is, the system of equations (45) now becomes:

$$\begin{cases} \sigma_y^2 = b_{11}^2, \\ \sigma_{yr}^2 = b_{11} b_{21}, \\ \sigma_r^2 = b_{21}^2 + b_{22}^2. \end{cases} \tag{47}$$

which can be easily solved to get:

$$\begin{cases} b_{11} = \sigma_y^2, \\ b_{21} = \sigma_{yr}^2 / \sigma_y^2, \\ b_{22} = \sqrt{\sigma_r^2 - \frac{\sigma_{yr}^2}{\sigma_y^2}}. \end{cases}$$

The VAR is identified! This means that it is possible to compute the *impact* impulse response of all endogenous variables by simply looking at the estimated B matrix. For example, consider a one standard deviation shock to monetary policy, i.e. $\varepsilon_t^{Mon.\ Pol} = 1$. Using the structural VAR representation

$$\begin{bmatrix} IP_t \\ R_t \end{bmatrix} = \begin{bmatrix} \alpha^{IP} \\ \alpha^R \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} IP_{t-1} \\ R_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma_y^2 & 0 \\ \sigma_{yr}^2 / \sigma_y^2 & \sqrt{\sigma_r^2 - \frac{\sigma_{yr}^2}{\sigma_y^2}} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{Mon.\ Pol} \end{bmatrix}.$$

we get:

$$\begin{aligned} IP_t &= 0, \\ R_t &= \sqrt{\sigma_r^2 - \frac{\sigma_{yr}^2}{\sigma_y^2}}. \end{aligned}$$

A one standard deviation shock to aggregate demand ($\varepsilon_t^{Demand} = 1$) in t leads to

$$\begin{aligned} IP_t &= \sigma_y^2, \\ R_t &= \sigma_{yr}^2 / \sigma_y^2. \end{aligned}$$

7.2 Identification by zero long-run restrictions

Identification using zero contemporaneous restrictions (also known as recursive identification, for a reason that will be

7.3 Variance decompositions

7.4 Historical decompositions

References

[1] Blanchard, O. J. and D. Quah (1989, September). The Dynamic Effects of Aggregate Demand and Supply Disturbances. *American Economic Review 79*(4), 655–73.

[2] Gertler, M. and P. Karadi (2015, January). Monetary Policy Surprises, Credit Costs, and Economic Activity. *American Economic Journal: Macroeconomics 7*(1), 44–76.

[3] Mertens, K. and M. O. Ravn (2013, June). The Dynamic Effects of Personal and Corporate Income Tax Changes in the United States. *American Economic Review 103*(4), 1212–47.

[4] Stock, J. and M. Watson (2012). Disentangling the Channels of the 2007-2009 Recession. *Brookings Papers on Economic Activity Spring*, 81–135.

[5] Stock, J. H. and M. W. Watson (2001, Fall). Vector Autoregressions. *Journal of Economic Perspectives 15*(4), 101–115.

[6] Uhlig, H. (2005, March). What are the Effects of Monetary Policy on Output? Results from an Agnostic Identification Procedure. *Journal of Monetary Economics 52*(2), 381–419.

A Appendix

A.1 The Identification Problem [Notes]

In the previous section we have seen how to estimate a reduced form VAR. Ignoring lagged variables beyond order 1 for ease of notation, we estimated the following:

$$\begin{bmatrix} R_t \\ IP_t \\ CPI_t \\ EBP_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \\ \phi_{31} & \phi_{12} & \phi_{33} & \phi_{34} \\ \phi_{41} & \phi_{22} & \phi_{43} & \phi_{44} \end{bmatrix} \begin{bmatrix} R_{t-1} \\ IP_{t-1} \\ CPI_{t-1} \\ EBP_{t-1} \end{bmatrix} + \text{other lags} + \begin{bmatrix} u_t^R \\ u_t^{IP} \\ u_t^{CPI} \\ u_t^{EBP} \end{bmatrix}, \tag{48}$$

Now, imagine that you are asked to estimate the effect of a monetary policy shock to industrial production and consumer prices. Unfortunately, the reduced form innovation to the interest rate (u_t^R) is not going to help us. The reason is that, as we discussed in Section 3, u_t^R is a linear combination of the true structural shocks in the economy. So, it does not tell us anything about how monetary policy affects output and prices.

To see that more clearly, assume that the 'true' model of the economy is given by the following structural VAR:

$$\begin{bmatrix} R_t \\ IP_t \\ CPI_t \\ EBP_t \end{bmatrix} = \text{all lags} + \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{12} & b_{33} & b_{34} \\ b_{41} & b_{22} & b_{43} & b_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Mon.\ Pol} \\ \varepsilon_t^{Demand} \\ \varepsilon_t^{Supply} \\ \varepsilon_t^{Financial} \end{bmatrix}, \tag{49}$$

where the matrix B and the structural shocks ε are unobserved. The SVAR in (XX) assumes that time series of interest rates, industrial production, consumer prices and the excess bond premium are driven by a combination of monetary, demand, supply and financial shocks. It is obvious that the reduced form innovation to the interest rate, u_t^R, is a linear combination of all shocks, and not just the monetary policy shock.

To answer the question of what are the effects of monetary policy on the economy, we need to find the values of the B matrix. This is known as the identification problem. For example, the coefficients b_{11}, b_{21}, b_{31}, and b_{41} give us the impact effect of monetary policy on all variables. The matrix of coefficient Φ, which we estimated in the reduced form VAR, can then be used to trace out the dynamic effects of monetary policy on the economy beyond the impact effect.

So, how can we go from the reduced form representation to the structural representation of the VAR? From equation (**??**) we know that:

$$\hat{\Sigma}_u = E\left[\hat{u}_t \hat{u}_t'\right] = E\left[B\varepsilon\left(B\varepsilon\right)'\right] = B\Sigma_\varepsilon B' = BB'. \tag{50}$$

where remember that $\Sigma_\varepsilon = I$. This means that there is a mapping between the estimated covariance matrix of the reduced form residuals ($\hat{\Sigma}_u$) and the unobserved matrix of structural impact coefficients. We can think of (50) as a system of nonlinear equations in the 4×4 unknown coefficients of the B matrix. The problem the Σ_u matrix —given its symmetric nature—

would contain only $4 + (4 \times 3)/2$ parameters. In other words, we have

$$\boldsymbol{\Sigma}_u = \begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 & \sigma_{13}^2 & \sigma_{14}^2 \\ - & \sigma_r^2 & \sigma_{23}^2 & \sigma_{24}^2 \\ - & - & \sigma_3^2 & \sigma_{34}^2 \\ - & - & - & \sigma_4^2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{12} & b_{33} & b_{34} \\ b_{41} & b_{22} & b_{43} & b_{44} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{12} & b_{33} & b_{34} \\ b_{41} & b_{22} & b_{43} & b_{44} \end{bmatrix}'$$

which shows that there are 16 unknowns but only 10 independent equations. The system is clearly under-identified, meaning that we need additional conditions if we want to recover the structural parameters.

There are many ways of identifying solving the identification problem described above. In the following section, I will describe a few of the most popular ones, and how thy can be implemented in the VAR Tolbox.

A.2 Impulse responses

ZERO LONG-RUN RESTRICTIONS. Similarly to the short-run restrictions, identification is achieved by making the assumption that some variables of the VAR cannot affect some other variables in the long-run. Specifically we will assume that the first variable is not affected in the long run by the others; the second is affected in the long run by the first variable but not by the others, and so on and so forth.

SIGN RESTRICTIONS. Identification is achieved by restricting the sign of the responses of selected model variables to structural shocks, using economic theory as a guidance