1 Installing the VAR Toolbox

No installation is required. Simply extract the codes from the ZIP file and copy them to a specific folder, e.g. "`C:/UserFolder/VARToolbox`". Then, add the folder (with subfolders) to the Matlab path. To avoid clashes with other function it is recommendable to add and remove the Toolbox with the following commands at beginning and end of your scripts:

```
addpath(genpath('C:/AMPER/VARToolbox'))
...
rmpath(genpath('C:/AMPER/VARToolbox'))
```

To save Figures in high quality format, Ghostscript is needed (freely available at www.ghostscript.com). The VT 3.0 has been tested with Matlab R2016B on a Windows 10 machine.

2 VAR Toolbox: High level description

The VAR Toolbox is a collection of Matlab routines to perform VAR analysis. Vector autoregressive models (VARs) are one of the most successful, flexible, and easy to use models for the analysis of multivariate time series. It is a natural extension of the univariate autoregressive model to dynamic multivariate time series. In their well-known paper "Vector Autoregressions," [3] describe VAR models as especially useful (and successful) tools for i) describing the dynamic behavior of economic and financial time series and ii) for forecasting.

In addition to data description and forecasting, VAR models are also used for iii) structural inference and iv) policy analysis. In structural analysis, we generally need to impose certain assumptions about the causal structure of the data under investigation. The resulting "structural" VAR model can then be used to analyze the impact of unexpected shocks to specified variables on all the variables in the model. This is normally done by means of iimpulse responses, forecast error variance decompositions, and historical decompositions.

The VAR Toolbox allows for identification of structural shocks with zero short-run restrictions; zero long-run restrictions; sign restrictions; and with the external instrument approach (proxy SVAR). Impulse Response Functions (IR), Forecast Error Variance Decomposition (VD), and Historical Decompositions (HD) are computed according to the chosen identification. Error bands are obtained with bootstrapping. The VAR Toolbox makes use of few Matlab routines from the Econometrics Toolbox for Matlab by James P. LeSage (freely available at www.spatial-econometrics.com).

It also includes a collection of Matlab routines that allows the user to save and export high quality images from Matlab (using the Export_fig function by Oliver Woodford, freely available at https://www.mathworks.com/matlabcentral/fileexchange/23629-export_fig). To enable this option, the Toolbox requires Ghostscript installed on your computer (freely available at www.ghostscript.com).

The VAR Toolbox is not meant to be efficient, but rather to be transparent and allow the user to understand the econometrics of VARs step by step. The codes are grouped in six categories (and respective folders):

- **Auxiliary**: codes that I borrowed from other public sources. Each m-file has a reference to the original source.

- **ExportFig**: this is a toolbox available at Oliver Woodford website for exporting high quality figures. [add website]

- **Figure**: codes for plotting high quality figures, particularly thought for time series. For example, the functions in this folder allow to efficiently add dates to the x-axis, to control the size and font of Figures and the appearance of the legends, to plot charts with shaded error bands, etc.
- **Stats**: codes for the calculation of summary statistics, moving correlations, pairwise correlations, etc.
- **Utils**: codes that allow the smooth functioning if the Toolbox.
- **VAR**: the codes for VAR estimation, identification, computation of the impulse response functions, FEVD, HD.

The Matlab on examples page includes few examples on the functioning of the VAR Toolbox.

3 A Guided Example

The idea of this manual is to explain the functioning of the VT by means of a simple example – the idea being that is much easier to learn by doing rather than reading a technical manual and then go to the computer. As a result, many of the functions included in the VT are not covered in this manual, nor is a full description of the output of each function. Moreover, I will need to stop every now and then to introduce some concepts and/or notation. Sections that include technical details, derivations, etc will be labelled with [Note], while sections that include details on the practical example will be labelled with [Matlab].

Additional resources are available on my website:

- https://sites.google.com/site/ambropo/replications provide some lecture notes on the basics of VARs that are a good complement to this manual.
- https://sites.google.com/site/ambropo/matlab-examples provides a few examples on how to estimate VARs with different identification schemes (in a similar spirit to the example in this manual.
- https://sites.google.com/site/ambropo/replications) provide the replication codes for a few well-known VAR studies (e.g. [3], [1], [4], and [2]).

I will start by introducing some (very light) notation.

3.1 VARs: Basics [Notes]

Given a $k \times 1$ vector of time series (x_t) a Structural Vector Autoregression (SVAR) of order p is given by:

$$x_t = \sum_{j=1}^{p} \Phi_j x_{t-j} + B\varepsilon_t, \tag{1}$$

where B is a $k \times k$ matrix and ε_t is a $k \times 1$ vector of serially uncorrelated error terms, generally called "*structural innovations*" or "*structural shocks*". All elements of ε_t are assumed to be mutually uncorrelated and $\varepsilon_{it} \sim i.i.d.(0, 1)$. Note that the fact that the variance of the structural shocks is equal to one is just a harmless normalization which does not involve a loss of generality (as long as the diagonal elements of A remain unrestricted.[1]

[1] An alternative (and equivalently valid) normalization would be to leave unrestricted the variance of the structural innovations, namely $\varepsilon_{it} \sim i.i.d.(0, \sigma_{it})$ and assume tha the diagonal elements of A to 1.

To keep the notation simple, consider a bivariate VAR(1), i.e. a VAR where the number of variables is $k = 2$ and the number of lags is $p = 1$. This simple bivariate VAR(1) can be written as a system of linear equations:

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}, \tag{2}$$

or:

$$\begin{aligned} x_{1t} &= \phi_{11} x_{1,t-1} + \phi_{12} x_{2,t-1} + b_{11}\varepsilon_{1t} + b_{12}\varepsilon_{2t}, \\ x_{2t} &= \phi_{21} x_{1,t-1} + \phi_{22} x_{2,t-1} + b_{21}\varepsilon_{1t} + b_{22}\varepsilon_{2t}, \end{aligned} \tag{3}$$

where $\varepsilon_t = (\varepsilon'_{1t}, \varepsilon'_{2t})'$ is a 2×1 vector of (unobservable) uncorrelated, zero mean, white noise processes. That is:

$$\mathbb{V}(\varepsilon_t) = \Sigma_\varepsilon = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] = I. \tag{4}$$

The assumption that the elements of ε_t are mutually uncorrelated is crucial. It implies that we can track the dynamic effects of a shock to, say, ε_{1t} to all variables in the VAR keeping the other shock to zero (and vice versa). The B matrix is also crucial. To see that, consider a unit surprise in ε_{1t}. What are the consequences for x_{1t} and x_{2t}? The answer to this question is in the first column of the B matrix: x_{1t} will increase by $b_{11}$and x_{2t} will increase by b_{21}. This is why the B matrix is also known as the structural impact matrix. The Φ matrix can then be used to track the dynamic effects in $t+1$, $t+2$, etc.

While all this sounds easy and great, there is a slight complication. The structural innovations are unobservable. An econometrician can only estimate:

$$\begin{aligned} x_{1t} &= \phi_{11} x_{1,t-1} + \phi_{12} x_{2,t-1} + u_{1t}, \\ x_{2t} &= \phi_{21} x_{1,t-1} + \phi_{22} x_{2,t-1} + u_{2t}, \end{aligned} \tag{5}$$

where the reduced-form innovations (u_t) are a linear combination of the structural innovations:

$$\begin{aligned} u_{1t} &= b_{11}\varepsilon_{1t} + b_{12}\varepsilon_{2t}, \\ u_{2t} &= b_{21}\varepsilon_{1t} + b_{22}\varepsilon_{2t}. \end{aligned} \tag{6}$$

The VAR in (**??**) is typically referred to as the *reduced-form representation* of the structural VAR, which can be written more compactly in matrix form as:

$$\left[\begin{array}{c} x_{1t} \\ x_{2t} \end{array} \right] = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \left[\begin{array}{c} x_{1t-1} \\ x_{2t-1} \end{array} \right] + \left[\begin{array}{c} u_{1t} \\ u_{2t} \end{array} \right] \tag{7}$$

where:

$$\left[\begin{array}{c} u_{1t} \\ u_{2t} \end{array} \right] = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}, \tag{8}$$

or:

$$x_t = \Phi x_{t-1} + u_t \tag{9}$$

where $u_t = B\varepsilon_t$. Note that, in general, $\hat{\mathbf{\Sigma}}_u$, is a symmetric non-diagonal matrix:

$$\hat{\mathbf{\Sigma}}_u = \left[\begin{array}{cc} \sigma_1^2 & \sigma_{12}^2 \\ - & \sigma_2^2 \end{array} \right] \tag{10}$$

where its diagonal elements are the variances of the estimated reduced-form error terms, σ_1^2 and σ_2^2; and the off-diagonal elements are instead and the covariance between the estimated error terms ($\sigma_{12} = \sigma_{21}$).The covariance between the estimated reduced form resiuduals plays an important role VARs because it collects the information on the contemporaneous interaction of the variables in the structural system, which (as we have just seen) is summarized by by the

B matrix. Indeed, using (**??**) the covariance matrix of th ereduced for residulas can be written as:

$$\hat{\boldsymbol{\Sigma}}_u = \left[\begin{array}{cc} b_{11}^2 & b_{11}b_{21} + b_{12}b_{22} \\ b_{11}b_{21} + b_{12}b_{22} & b_{22}^2 \end{array} \right] \tag{11}$$

This shows that, differently from structural VARs, the reduced form innovations are not informative about how shocks propagate throught the system, as an innovation to u_{1t} could be driven by either ε_{1t} or ε_{2t} (and vice versa).

3.2 Load & Plot Data [Matlab]

The example uses data on US industrial production (IP_t), consumer prices (CPI_t), short-term interest rates (R_t), and the Excess Bond Premium (EBP_t) from 1979:M7 to 2015:M3 (this is the data used by [2]). The code below shows a general way of loading the data and managing it in a way that is consistent with the functioning of the VT.

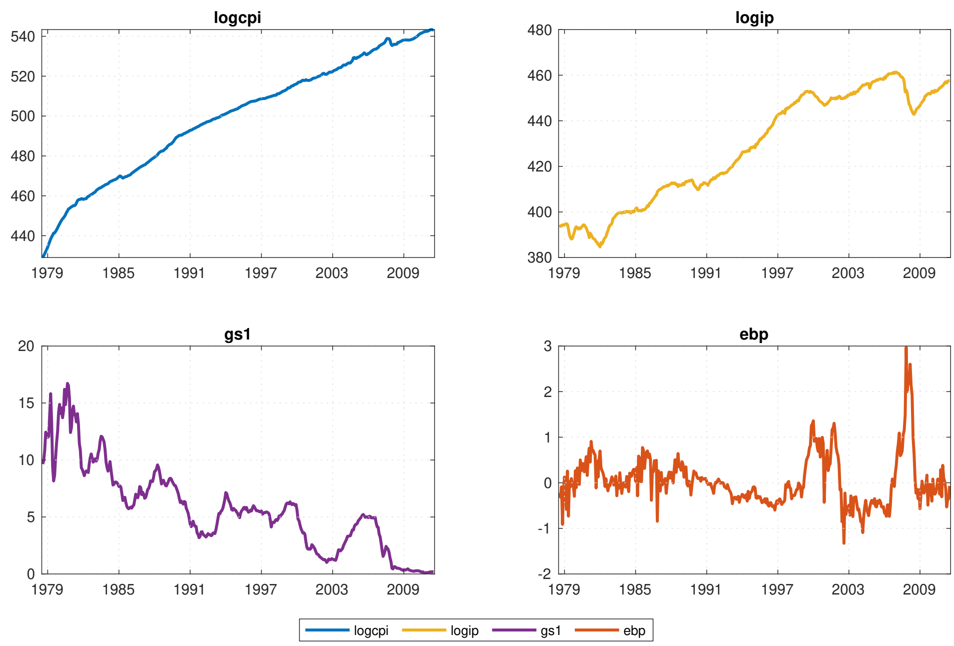
```
%% 1. LOAD & PLOT DATA
%-----------------------------------------------------------------------
% Load
[xlsdata, xlstext] = xlsread('GK2015_Data.xlsx','VAR_data');
data = Num2NaN(xlsdata(:,3:end));
vnames = xlstext(1,3:end);
for ii=1:length(vnames)
    DATA.(vnames{ii}) = data(:,ii);
end
year = xlsdata(1,1);
month = xlsdata(1,2);
% Observations
nobs = size(data,1);
% Set endogenous
VARvnames = {'gs1','logcpi','logip','ebp'};
VARvnames_long = {'Policy rate','CPI','Industrial Production','EBP'};
VARnvar = length(VARvnames);
% Create matrices of variables to be used in the VAR
ENDO = nan(nobs,VARnvar);
for ii=1:VARnvar
    ENDO(:,ii) = DATA.(VARvnames{ii});
end
```

First the code reads from an Excel file and stores all data into the structure DATA. The VAR Toolbox includes some functions that allow to plot time series quickly and export them as high-quality PDFs, so that they can be used directly in your papers. The code shows how to plot the four time series in ENDO.

```
% Open a figure of the desired size
FigSize(28,16)
for ii=1:VARnvar
    subplot(2,2,ii)
    H(ii) = plot(ENDO(:,ii),'LineWidth',2,'Color',cmap(ii));
    title(VARvnames_long(ii));
    DatesPlot(year+(month-1)/12,nobs,6,'m') % Set the x-axis label
    grid on;
end
% Legend
lopt = LegOption; lopt.handle = H; LegSubplot(VARvnames,lopt); FigFont(10);
% Save
SaveFigure('graphics/F1_PLOT',1)
```

Figure 1 reports the behavior of the interest rate on US 1-year Treasury bill, an index of industrial production, the CPI level and the Excess Bond Premium (GZ) over the 1979:M7 to 2015:M3 sample period.

Figure 1: Raw Data



Note. Raw data for interest rate on US 1-year Treasury bill, industrial production, CPI level and the Excess Bond Premium (GZ) from 1979;M7 to 2015:M3.

Some useful functions are:

- `FigSize.m`: allows the user to choose the proportions of the figure to plot. This is particularly useful when creating figures with many panels.

- `DatesPlot.m`: Adds dates to the horizontal axis of a chart (at monthly, quarterly, and annual frequency) using a specified number of ticks.

- `LegSubplot.m`: To be used in combination with the Matlab built-in function subplot. `Legsubplot.m` creates a single legend for the subplots, below the charts and centered.

- `FigFont.m`: Sets the font of axes, title, legends, etc to a specified font size.
- `FigFont.m`: SaveFigure saves the chart in the selected format (pdf, jpg, eps). The function allows the user to save the figure at high quality standard using the `export_fig.m` function created by Oliver Woodford. Note that you need Ghostscript to be able to use this function.

3.3 VAR estimation [Matlab]

A VAR model can be estimated with a simple line of code, using the `VARmodel.m` function. To do that you need to specify a matrix including the endogenous variables (ENDO), whether you want deterministic variables, like a constant or a trend for example (`det`), and the number of lags of the VAR (`nlags`). In the example, I specify a simple bivariate VAR(12) in industrial production and interest rates, with a constant.[2]

The code below shows how to create the matrix of endogenous variables.

```
%% VAR ESTIMATION
%--------------------------------------------------------------------------
% Set the deterministic variable in the VAR (1=constant, 2=trend)
det = 1;
% Set number of nlags
nlags = 12;
% Estimate VAR by OLS
[VAR, VARopt] = VARmodel(ENDO,nlags,det);
disp(VAR)
disp(VARopt)
% Add variable names to VARopt
VARopt.vnames = VARvnames;
VARopt.figname= 'graphics/';
% Print at screen and create table
[TABLE, beta] = VARprint(VAR,VARopt,2);
```

The cell array `VARvnames` defines the list of endogenous variables that will be used to estimate the VAR model (in this case, industrial production and interets rates, namely a subset of the data in DATA). The chosen data is then stored in the matrix ENDO. The convention in the VT is that each column is a variable and each row is an observation (with no missing observations allowed). That is:

$$\text{ENDO} = \begin{bmatrix} IP_1 & R_1 \\ IP_2 & R_2 \\ \dots & \dots \\ IP_T & R_T \end{bmatrix} = \left(IP_t', R_t'\right) = x_t'.$$

This convention implies that, using the notation defined in the previous section, $\text{ENDO} = x_t'$.

The VAR can then be estimated in a few lines of code.

[2]Note that this is a different specification from the original one used by [2]. This choice is purely pedagogical to make some of the derivations below easier.

```
%% VAR ESTIMATION
%--------------------------------------------------------------------------
% Set the deterministic variable in the VAR (1=constant, 2=trend)
det = 1;
% Set number of nlags
nlags = 12;
% Estimate VAR by OLS
[VAR, VARopt] = VARmodel(ENDO,nlags,det);
disp(VAR)
disp(VARopt)
% Add variable names to VARopt
VARopt.vnames = VARvnames;
VARopt.figname= 'graphics/';
% Print at screen and create table
[TABLE, beta] = VARprint(VAR,VARopt,2);
```

The results of the VAR estimation are stored in the structures `VAR` and `VARopt`. The structure `VAR` includes all the estimation results. These can be seen by executing the command `disp(VARprint(VAR)` which prints the following output in the command window:

```
>> disp(VAR)
         ENDO: [396x2 double]
         nlag: 12
        const: 1
         EXOG: []
         nobs: 384
         nvar: 4
      nvar_ex: 0
      nlag_ex: 0
       ncoeff: 24
    ntotcoeff: 25
          eq1: [1x1 struct]
          eq2: [1x1 struct]
          eq3: [1x1 struct]
          eq4: [1x1 struct]
           Ft: [49x4 double]
            F: [4x49 double]
        sigma: [4x4 double]
        resid: [384x4 double]
            X: [384x49 double]
            Y: [384x4 double]
        Fcomp: [48x48 double]
       maxEig: 0.9974
           Fp: [4x4x12 double]
            B: []
      BfromSR: []
          PSI: []
```

The structure `VAR` includes all the inputs to the `VARmodel.m` function, such as the matrix of endogenous variables (`VAR.ENDO`), the number of lags (`VAR.nlags`), and the number of endogenous variables (`VAR.nvar`). But also includes the estimation output. For example:

- The matrix `VAR.F` collects the estimated coefficients following the notation in (**??**), namely we have that `VAR.F`$= F$. For a VAR with 12 lags and 2 variables plus a constant, this means that `VAR.F` is a $2 \times (12 \times 2 + 1)$ matrix.

- The covariance matrix of the VAR residuals defined by (**??**) is instead stored in `VAR.sigma`$= \Sigma_u$, of size 2×2.
- Note that the structural impact matrix `VAR.B` $= B$, whihc we defined in equation (), is empty. This is because, for the moment we estimated only the reduced form VAR (1). The next sections will show how, with additional assumptions, the also the strutcural form of the VAR can be recovered.

Other outputs are the OLS equation-by-equation estimation results (structures `VAR.eq`), the VAR companion matrix (`VAR.Fcomp`), the maximum eigenvalue of the VAR (`VAR.maxEig`), etc.

The structure `VARopt` includes a few auxiliary variables that are created automatically by the `VARmodel.m` function and will be needed below for the calculation of impulse responses, variance decompositions, etc. The variables stored in `VARopt` can be seen by executing the command `disp(VARopt)` , which prints the following output in the Matlab command window:

```
>> disp(VARopt)
        vnames: []
     vnames_ex: []
        snames: []
        nsteps: 40
        impact: 0
          shut: 0
         ident: 'oir'
        recurs: 'wold'
        ndraws: 100
          pctg: 95
        method: 'bs'
          pick: 0
       quality: 0
      suptitle: 0
     firstdate: []
     frequency: 'q'
       figname: []
```

These variables include the number of steps for impulse response functions and variance decompositions (`nsteps`), the labels of the endogenous or exogenous variables for plots (`vnames` and `vnames_ex`), the confidence levels for the computation of error bands (`pctg`), etc. While some variables are automatically created by the VARmodel function, some other variables need to be inputted by the user. For example:

- `VARopt.vnames = VARvnames` stores in `VARopt` the endogenous variables' names.
- `VARopt.figname = 'graphics/'` stores in `VARopt` the name of the folder where all figures will be saved.

So that when executing `disp(VARopt)` we now get:

```
>> disp(VARopt)
        vnames: {'gs1'   'logcpi'   'logip'   'ebp'}
     vnames_ex: []
        snames: []
```

```
   nsteps: 40
   impact: 0
     shut: 0
    ident: 'oir'
   recurs: 'wold'
   ndraws: 100
     pctg: 95
   method: 'bs'
     pick: 0
  quality: 0
 suptitle: 0
firstdate: []
frequency: 'q'
  figname: 'graphics/'
```

3.4 The Identification Problem [Notes]

In the previous section we have seen how to estimate a reduced form VAR. Ignoring lagged variables beyond order 1 for ease of notation, we estimated the following:

$$\begin{bmatrix} IP_t \\ R_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} IP_{t-1} \\ R_{t-1} \end{bmatrix} + \text{other lags} + \begin{bmatrix} u_t^{IP} \\ u_t^{R} \end{bmatrix}, \tag{12}$$

Now, imagine that you are asked to estimate the effect of a moneatry policy shock to industrial production and consumer prices. Unfortunately, the reduced form innovation to the interets rate (u_t^R) is not going tpo help us. The reason is that, as we discussed in Section 3.1, u_t^R is a linear combination of the true structural shocks in the economy. So, it does not tell us anything about how monetary opoicy affects output and prices.

To see that more clearly, assume that the 'true' model of the economy is given by the following structural VAR:

$$\begin{bmatrix} IP_t \\ R_t \end{bmatrix} = \text{all lags} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{Mon.\ Pol} \end{bmatrix}, \tag{13}$$

where the matrix B and the structural shocks ε are unobserved. The SVAR in (XX) assumes that time series of industrial production and interest rates are driven by a combination of demand and moneatry policy shocks.[3] It is obvious that the reduced form innovation to the interets rate, u_t^R, is a linear combination of all shocks, and not just the monetary policy shock.

To answer the question of what are the effects of monetary policy on the economy, we need to find the values of the B matrix. This is known as the identification problem. For example, the coefficients b_{11} and b_{21} give us the impact effect of moneatry policy on all variables. The matrix of coefficient Φ, whihc we estimated in the reduced form VAR, can then be used to trace out the dynamic effects of moneatry policy on the economy beyond the impact effect.

So, how can we go from the reduced form representation to the structural representation of the VAR? We have seen abpove that we know that $u = B\varepsilon$, so that we can write:

$$\hat{\Sigma}_u = E\left[\hat{u}_t \hat{u}_t'\right] = E\left[B\varepsilon_t \left(B\varepsilon_t\right)'\right] = B\Sigma_\varepsilon B' = BB'. \tag{14}$$

where remember that $\Sigma_\varepsilon = I$. This means that there is a mapping between the estimated covariance matrix of the reduced form residuals ($\hat{\Sigma}_u$) and the unobserved matrix of structural

[3] Again this is not a very realistic assumption, but it simplifies the math that follows. a more realistic VAR woudl ave included more varibles and more shocks.

impact coefficients. We can think of (23) as a system of nonlinear equations in the 4 unknown coefficients of the B matrix. The problem the Σ_u matrix, given its symmetric nature, leads to only 3 independent restrictions. In other words, we have

$$\left[\begin{array}{cc} \sigma_1^2 & \sigma_{12}^2 \\ - & \sigma_2^2 \end{array}\right] = \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array}\right] \left[\begin{array}{cc} b_{11} & b_{21} \\ b_{12} & b_{22} \end{array}\right], \tag{15}$$

which can be rewritten as the following system of equations:

$$\begin{array}{l} \sigma_1^2 = b_{11}^2 + b_{12}^2 \\ \sigma_{12}^2 = b_{11}b_{21} + b_{12}b_{22} \\ \sigma_{12}^2 = b_{11}b_{21} + b_{12}b_{22} \\ \sigma_2^2 = b_{21}^2 + b_{22}^2 \end{array} \tag{16}$$

Note that, becasue of the symmetry of the Σ_u matrix, the second and the third equation are identical. This means that we are left with 4 unknowns (the b's) but only 3 equations. The system is clearly under-identified, meaning that we need additional conditions if we want to recover the structural parameters.

There are many ways of identifying solving the identificartion problem described above. In the following section, I will describe a few of the most popular ones, and how thy can be implemented in the VAR Tolbox.

3.5 Identification by zero contemporaneous resrictions

Identification using zero contemporaneous resrictions (also known as recursive identification, as it will be clear in a second) were developed by Sims1980, and are by far the most commonly used identification scheme used in the literature. In a recursive SVAR, identification is achieved by assuming that some shocks have zero contemporaneous effect on some endoenous variables. This amounts to setting some of the non-diagonal elements of the B matrix to zero – therefore reducing the number of unknown coefficients.

Typically, it is assumed that the first variable in the system is only affected by the first strcutral shock, the second is contemporaneously affected by the first and second structural shock, and so on. In our example, that means to assume that the structural VAR is

$$\begin{bmatrix} IP_t \\ R_t \end{bmatrix} = \text{all lags} + \left[\begin{array}{cc} b_{11} & 0 \\ b_{21} & b_{22} \end{array}\right] \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{Mon.\ Pol} \end{bmatrix}, \tag{17}$$

where note that the industrial production is not contemporaneously affected by the monetary policy shock (while interest rates are contemporaneously affected by both the demand and the moneatry policuy shock). This assumption could be justified by the fact that moneatry policy takes time to affect real variables like industrial production.

What are the implications for the identificariton problem descrobed above? The simple answer is that we now have 3 instead of 4 parameters to estimate, and 3 restrictions implied by the reduced form covariance matrix. That is, the system of equations (16) now becomes:

$$\begin{array}{l} \sigma_1^2 = b_{11}^2 \\ \sigma_{12}^2 = b_{11}b_{21} \\ \sigma_2^2 = b_{21}^2 + b_{22}^2 \end{array} \tag{18}$$

which can be easily solved to get:

$$\begin{array}{r} b_{11} = \sigma_1, \\ b_{21} = \sigma_{12}^2/\sigma_1, \\ b_{22} = \sqrt{\sigma_2^2 - \left(\sigma_{12}^2/\sigma_1^2\right)^2} \end{array}$$

The VAR is identified! This means that it is possible to compute the *impact* impulse response of all endogenous variables by simply looking at the estimated B matrix. For example, consider a one standard deviation shock to moneatry policy, i.e. $\varepsilon_t^{Mon.\ Pol} = 1$. Using the structural VAR representation we get:

$$\begin{aligned} \mathcal{IR}_{IP,0}^{Mon.\ Pol} &= 0 \\ \mathcal{IR}_{R,0}^{Mon.\ Pol} &= \sigma_{12}^2/\sigma_1 \end{aligned} \tag{19}$$

where $\mathcal{IR}_{i,0}^{j}$ denotes the impulse response at horizon 0 (i.e. on impact), of variable i to the structural shock j. Knowing the impact response it is then easy to compute the dynamic response of all endogenous variables (i.e. at horoizons > 0) with the reduced form Φ matrix.

In summary, we have:

$$\begin{aligned} \mathcal{IR}_0 &= B\varepsilon_t \\ \mathcal{IR}_h &= \Phi^h \mathcal{IR}_0^j \qquad h > 0 \end{aligned} \tag{20}$$

3.6 Variance decompositions

3.7 Historical decompositions

References

[1] Blanchard, O. J. and D. Quah (1989, September). The Dynamic Effects of Aggregate Demand and Supply Disturbances. *American Economic Review 79*(4), 655–73.

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[3] Stock, J. H. and M. W. Watson (2001, Fall). Vector Autoregressions. *Journal of Economic Perspectives 15*(4), 101–115.

[4] Uhlig, H. (2005, March). What are the Effects of Monetary Policy on Output? Results from an Agnostic Identification Procedure. *Journal of Monetary Economics 52*(2), 381–419.

A Appendix

A.1 The Identification Problem [Notes]

In the previous section we have seen how to estimate a reduced form VAR. Ignoring lagged variables beyond order 1 for ease of notation, we estimated the following:

$$\begin{bmatrix} R_t \\ IP_t \\ CPI_t \\ EBP_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \\ \phi_{31} & \phi_{12} & \phi_{33} & \phi_{34} \\ \phi_{41} & \phi_{22} & \phi_{43} & \phi_{44} \end{bmatrix} \begin{bmatrix} R_{t-1} \\ IP_{t-1} \\ CPI_{t-1} \\ EBP_{t-1} \end{bmatrix} + \text{other lags} + \begin{bmatrix} u_t^R \\ u_t^{IP} \\ u_t^{CPI} \\ u_t^{EBP} \end{bmatrix}, \tag{21}$$

Now, imagine that you are asked to estimate the effect of a moneatry policy shock to industrial production and consumer prices. Unfortunately, the reduced form innovation to the interets rate (u_t^R) is not going tpo help us. The reason is that, as we discussed in Section 3.1, u_t^R is a linear combination of the true structural shocks in the economy. So, it does not tell us anything about how monetary opoicy affects output and prices.

To see that more clearly, assume that the 'true' model of the economy is given by the following structural VAR:

$$\begin{bmatrix} R_t \\ IP_t \\ CPI_t \\ EBP_t \end{bmatrix} = \text{all lags} + \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{12} & b_{33} & b_{34} \\ b_{41} & b_{22} & b_{43} & b_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Mon.\ Pol} \\ \varepsilon_t^{Demand} \\ \varepsilon_t^{Supply} \\ \varepsilon_t^{Financial} \end{bmatrix}, \tag{22}$$

where the matrix B and the structural shocks ε are unobserved. The SVAR in (XX) assumes that time series of interest rates, industrial production, consumer prices and the excess bond premium are driven by a combination of moneatry, demand, supply and financial shocks. It is obvious that the reduced form innovation to the interets rate, u_t^R, is a linear combination of all shocks, and not just the monetary policy shock.

To answer the question of what are the effects of monetary policy on the economy, we need to find the values of the B matrix. This is known as the identification problem. For example, the coefficients b_{11}, b_{21}, b_{31}, and b_{41} give us the impact effect of moneatry policy on all variables. The matrix of coefficient Φ, whihc we estimated in the reduced form VAR, can then be used to trace out the dynamic effects of moneatry policy on the economy beyond the impact effect.

So, how can we go from the reduced form representation to the structural representation of the VAR? From equation (**??**) we know that:

$$\hat{\Sigma}_u = E\left[\hat{u}_t \hat{u}_t'\right] = E\left[B\varepsilon\left(B\varepsilon\right)'\right] = B\Sigma_\varepsilon B' = BB'. \tag{23}$$

where remember that $\Sigma_\varepsilon = I$. This means that there is a mapping between the estimated covariance matrix of the reduced form residuals ($\hat{\Sigma}_u$) and the unobserved matrix of structural impact coefficients. We can think of (23) as a system of nonlinear equations in the 4×4 unknown coefficients of the B matrix. The problem the Σ_u matrix —given its symmetric nature— would contain only $4 + (4 \times 3)/2$ parameters. In other words, we have

$$\mathbf{\Sigma}_u = \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 & \sigma_{13}^2 & \sigma_{14}^2 \\ - & \sigma_2^2 & \sigma_{23}^2 & \sigma_{24}^2 \\ - & - & \sigma_3^2 & \sigma_{34}^2 \\ - & - & - & \sigma_4^2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{12} & b_{33} & b_{34} \\ b_{41} & b_{22} & b_{43} & b_{44} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{12} & b_{33} & b_{34} \\ b_{41} & b_{22} & b_{43} & b_{44} \end{bmatrix}'$$

which shows that there are 16 unknowns but only 10 independent equations. The system is clearly under-identified, meaning that we need additional conditions if we want to recover the structural parameters.

There are many ways of identifying solving the identificartion problem described above. In the following section, I will describe a few of the most popular ones, and how thy can be implemented in the VAR Tolbox.

A.2 Impulse responses

ZERO LONG-RUN RESTRICTIONS. Similarly to the short-run restrictions, identification is achieved by making the assumption that some variables of the VAR cannot affect some other variables in the long-run. Specifically we will assume that the first variable is not affected in the long run by the others; the second is affected in the long run by the first variable but not by the others, and so on and so forth.

SIGN RESTRICTIONS. Identification is achieved by restricting the sign of the responses of selected model variables to structural shocks, using economic theory as a guidance