A Primer On Vector Autoregressions

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[1]*Disclaimer*. These explanatory notes result from the elaboration of text and figures found in published papers, unpublished manuscripts, personal notes, and a lot of material found on the web that I collected over a long time. If you find any lack of attribution or error in quoting the source of the material in these notes —as well as if you find any errors or you would like to give me comments— please email me at ambrogio.cesabianchi@gmail.com.

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Chapter 1

INTRO

These notes summarize the basic concepts for understanding Vector Autoregressions. The notation used here is consistent with the notation used in the *VAR Toolbox,* a collection of MatLab codes for VAR analysis available on my website.[1]

The vector autoregression (VAR) model is one of the most successful, flexible, and easy to use models for the analysis of multivariate time series. It is a natural extension of the univariate autoregressive model to dynamic multivariate time series.

In their well-known paper "Vector Autoregressions," Stock and Watson (2001) describe VAR models as especially useful (and successful) tools for i) describing the dynamic behavior of economic and financial time series and ii) for forecasting.

In addition to data description and forecasting, VAR models are also used for iii) structural inference and iv) policy analysis. In structural analysis, we generally need to impose certain assumptions about the causal structure of the data under investigation. The resulting "structural" VAR model can then be used to analyze the impact of unexpected shocks to specified variables on all the variables in the model. This is normally done by means of impulse responses, forecast error variance decompositions, and historical decompositions.

[1] Go to https://sites.google.com/site/ambropo/MatlabCodes.

Chapter 2

VECTOR AUTOREGRESSIONS

2.1 Structural representation

Given a $k \times 1$ vector of time series (x_t) a **Structural Vector Autoregression** (SVAR) of order 1 is:

$$\mathbf{A}\mathbf{x}_t = \mathbf{B}\mathbf{x}_{t-1} + \varepsilon_t, \tag{2.1}$$

where $\boldsymbol{\varepsilon}_t$ is a $k \times 1$ vector of serially uncorrelated error terms, generally labelled "*structural innovations*" or "*structural shocks*". We assume that all elements of $\boldsymbol{\varepsilon}_t$ are mutually uncorrelated and $\varepsilon_{it} \sim i.i.d.(0,1)$. Note that the fact that the variance of the structural shocks is equal to one is just a harmless normalization which does not involve a loss of generality (as long as the diagonal elements of $\mathbf{A}$ remain unrestricted.[1]

A bivariate VAR(1) is a VAR model where the number of variables is $k = 2$ and the number of lags is $p = 1$. Note that a structural bivariate VAR(1) can be re-written as a system of linear equations:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}, \tag{2.2}$$

or:

$$\begin{aligned} a_{11}x_{1t} + a_{12}x_{2t} &= b_{11}x_{1,t-1} + b_{12}x_{2,t-1} + \varepsilon_{1t}, \\ a_{21}x_{1t} + a_{22}x_{2t} &= b_{21}x_{1,t-1} + b_{22}x_{2,t-1} + \varepsilon_{2t}. \end{aligned} \tag{2.3}$$

[1] An alternative (and equivalently valid) normalization would be to leave unrestricted the variance of the structural innovations, namely $\varepsilon_{it} \sim i.i.d.(0, \sigma_{it})$ and assume tha the diagonal elements of $\mathbf{A}$ to 1.

Finally, we defined ε_t as a $k \times 1$ vector of unobservable zero mean white noise processes. This implies that:

$$\mathsf{VCV}(\varepsilon_t) = \mathbf{\Sigma}_\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}.$$

NOTE

What is a variance-covariance matrix? The formula for the variance of a univariate time series $x = [x_1, x_2, ..., x_T]$ is:

$$\mathsf{VAR} = \sum_{t=0}^{T} \frac{(x_t - \bar{x})^2}{N} = \sum_{t=0}^{T} \frac{(x_t - \bar{x})(x_t - \bar{x})}{N}.$$

If we have a bivariate time series, such as:

$$\mathbf{x}_t = \begin{bmatrix} x_1 & x_2 & ... & x_T \\ x_1 & x_2 & ... & x_T \end{bmatrix},$$

the formula simply becomes:

$$\mathsf{VCV} = \begin{bmatrix} \sum_{t=0}^{T} \frac{(x_t-\bar{x})(x_t-\bar{x})}{N} & \sum_{t=0}^{T} \frac{(x_t-\bar{x})(x_t-x)}{N} \\ \sum_{t=0}^{T} \frac{(x_t-x)(x_t-\bar{x})}{N} & \sum_{t=0}^{T} \frac{(x_t-x)(x_t-x)}{N} \end{bmatrix} = \begin{bmatrix} \mathsf{VAR}(x) & \mathsf{COV}(x,x) \\ \mathsf{COV}(x,x) & \mathsf{VAR}(x) \end{bmatrix}.$$

Note that the basic VAR(1) model may be too poor to sufficiently summarize the main characteristics of the data. Deterministic terms (such as time trend or seasonal dummy variables) and exogenous variables (such as the price of oil) can be added to the basic specification. The general form of the **stationary structural VAR(p)** model with deterministic terms ($\mathbf{Z}_t$) and exogenous variables ($\mathbf{W}_t$) is given by:

$$\mathbf{A}\mathbf{x}_t = \mathbf{B}_1\mathbf{x}_{t-1} + \mathbf{B}_2\mathbf{x}_{t-2} + ... + \mathbf{B}_p\mathbf{x}_{t-p} + \mathbf{\Lambda}\mathbf{Z}_t + \mathbf{\Psi}\mathbf{W}_t + \varepsilon_t.$$

Two clarifications are in order here. First, why is it called structural VAR? You can think of the equations of a structural VAR define the true structure of the economy. The structural VAR is therefore a theoretical representation (as, for example, the simple real business cycle model). Second, why is it called stationary VAR? As we shall see, one of the main assumptions of standard VARs is stationarity of the data. Note that, loosely speaking, a stochastic process (x_t) is said covariance stationary if its first and second moments, $\mathsf{E}(\mathbf{x}_t)$ and $\mathsf{VCV}(\mathbf{x}_t)$ respectively, exist and are constant over time.

For example let's write down a trivariate VAR(1) in GDP growth (Δy_t), inflation

(π_t), and the policy rate (r_t):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \pi_t \\ r_t \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} \Delta y_{t-1} \\ \pi_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{\Delta yt} \\ \varepsilon_{\pi t} \\ \varepsilon_{rt} \end{bmatrix}.$$

Structural VARs potentially answers many interesting questions. The fact that $\boldsymbol{\varepsilon}_t = (\varepsilon'_{\Delta yt}, \varepsilon'_{\pi t}, \varepsilon'_{rt})' \sim \mathcal{N}(0, \mathbf{I})$ implies that we can interpret $\boldsymbol{\varepsilon}_t$ as structural shocks (for example, we could interpret $\varepsilon_{\Delta yt}$ as an aggregate shock, $\varepsilon_{\pi t}$ as a cost-push shock, and ε_{rt} as a monetary policy shock). If that is the case, a_{31} would be the impact multiplier of monetary policy shocks on GDP, while a_{32} would be the impact multiplier of monetary policy shocks on inflation. Moreover, by simulating the model, we could evaluate the time profile of a monetary policy shock on GDP.

2.2 From the structural to the reduced-form representation

If we premultiply the structural VAR in (2.1) by $\mathbf{A}^{-1}$ we get:

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{A}\mathbf{y}_t &= \mathbf{A}^{-1}\mathbf{B}\mathbf{y}_{t-1} + \mathbf{A}^{-1}\boldsymbol{\varepsilon}_t, \\ \mathbf{x}_t &= \mathbf{F}\mathbf{y}_{t-1} + \mathbf{u}_t, \end{aligned} \tag{2.4}$$

which is called the **reduced-form representation of the structural VAR** in (2.1). A reduced-form bivariate VAR(1) model is therefore a system of two linear equations in which each right hand side variable is in explained by its own lagged values plus 1 lagged value of the remaining variables:

$$\begin{aligned} x_{1t} &= f_{11}x_{1,t-1} + f_{12}x_{2,t-1} + u_{1t}, \\ x_{2t} &= f_{21}x_{1,t-1} + f_{22}x_{2,t-1} + u_{2t}, \end{aligned} \tag{2.5}$$

or, in matricial form:

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}. \tag{2.6}$$

NOTE

The inverse of a 2×2 matrix $\mathbf{X}$

$$\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

is given by following formula

$$\mathbf{X}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

It is now clear that the reduced-form innovations ($\mathbf{u}_t$) are in general a linear combination of the structural innovations:

$$\begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix},$$

which yields:

$$\begin{aligned} u_{1t} &= \frac{a_{22}\varepsilon_{1t} - a_{21}\varepsilon_{2t}}{\Delta}, \\ u_{2t} &= \frac{-a_{21}\varepsilon_{1t} + a_{11}\varepsilon_{2t}}{\Delta}, \end{aligned}$$

where $\Delta = a_{11}a_{22} - a_{12}a_{21}$. As a result, studying the response of the vector x_t to reduced-form shocks $\mathbf{u}_t$ will not tell us anything about the response of x_t to the structural shocks ε_t, which are our ultimate object of interest.

To summarize, in a standard reduced form VAR, each variable is expressed as a linear function of its own past values and past values of all other variables. The error terms ($\mathbf{u}_t$) are the surprise movements in the variables after taking past values into account. If the variables are contemporaneously correlated with each other (as often happens in macroeconomics), the error terms ($\mathbf{u}_t$) will also be correlated.

2.3 Estimation

The previous section showed that structural VARs are our ultimate object of interest. However, the estimation of a structural VAR with OLS violates one of the assumption of the classical regression models, i.e. the regressors and the error terms have to be orthogonal. $a_{11}x_{1t} + a_{12}x_{2t} = b_{11}x_{1,t-1} + b_{12}x_{2,t-1} + \varepsilon_{1t}$,

To see that, let's consider the second equation of the system in (2.3):

$$a_{22}x_{2t} = -a_{21}x_{1t} + b_{21}x_{1,t-1} + b_{22}x_{2,t-1} + \varepsilon_{2t}.$$

Below, I will compute the covariance between x_{1t} (one of the regressors) and ε_{2t} (the error term), and show that is not zero. Our object of interest is $\text{COV}[x_{1t}, \varepsilon_{2t}]$: note that we can substitute for x_{1t} using the first equation of the system in (2.3) and get:

$$\mathsf{COV}\left[\underbrace{-\frac{a_{12}}{a_{11}}x_{2t}+\frac{b_{11}}{a_{11}}x_{1,t-1}+\frac{b_{12}}{a_{11}}x_{2,t-1}+\frac{\varepsilon_{1t}}{a_{11}}}_{\text{from the first equation of (2.3)}},\varepsilon_{2t}\right].$$

But from the second equation of the system in (2.3) it follows that:

$$\mathsf{COV}\left[-\frac{a_{12}}{a_{11}}\left(\underbrace{-\frac{a_{21}}{a_{22}}x_{1t}+\frac{b_{21}}{a_{22}}x_{1,t-1}+\frac{b_{22}}{a_{22}}x_{2,t-1}+\frac{\varepsilon_{2t}}{a_{22}}}_{\text{from the second equation of (2.3)}}\right)+\frac{b_{11}}{a_{11}}x_{1,t-1}+\frac{b_{12}}{a_{11}}x_{2,t-1}+\frac{\varepsilon_{1t}}{a_{11}},\varepsilon_{2t}\right],$$

which implies that:

$$\mathsf{COV}\left[x_{1t},\varepsilon_{2t}\right]=\frac{a_{12}}{a_{11}}\frac{a_{21}}{a_{22}}\mathsf{COV}\left[x_{1t},\varepsilon_{2t}\right]-\frac{a_{12}}{a_{11}a_{22}}\mathsf{COV}\left[\varepsilon_{2t},\varepsilon_{2t}\right].$$

From the last equality it follows that

$$\mathsf{COV}[x_{1t},\varepsilon_{2t}]=\frac{a_{12}}{a_{12}a_{21}-a_{11}a_{22}}\neq 0$$

This is a violation of the assumptions of the classical regression model. Therefore, OLS estimates of equation for x_{2t} will yield inconsistent estimates of the parameters of the model unless we assume that $a_{12}=0$. The same happens if we want to estimate the structural equation for x_{1t}, unless we assume $a_{21}=0$.

The reduced form VAR does not suffer from the same problem and can be estimated by OLS. However, as we have seen above, the reduced-form errors terms will correlated and, therefore, it will not be possible to evaluate the effect of the structural shock on the system.

The OLS estimates of the parameters of a reduced form VAR(1) are:

$$\begin{aligned}\mathbf{F} &= (\mathbf{x}_{t-1}\mathbf{x}_{t-1}')^{-1}\mathbf{x}_{t-1}\mathbf{x}_t',\\ \hat{\mathbf{u}}_t &= \mathbf{x}_t-\hat{\mathbf{F}}\mathbf{y}_{t-1},\\ \hat{\mathbf{\Sigma}}_u &= \mathsf{E}\left[\hat{\mathbf{u}}_t\hat{\mathbf{u}}_t'\right]=\frac{\hat{\mathbf{u}}_t\hat{\mathbf{u}}_t'}{T-1}.\end{aligned}$$

Note that, in general, $\hat{\mathbf{\Sigma}}_u$, is a symmetric non-diagonal matrix:

$$\hat{\mathbf{\Sigma}}_u=\begin{bmatrix}\sigma_1^2 & \sigma_{12}^2\\ - & \sigma_2^2\end{bmatrix}.$$

Its diagonal elements are the variances of the estimated reduced-form error terms,

σ_1^2 and σ_2^2. The off-diagonal elements are instead and the covariance between the estimated error terms ($\sigma_{12} = \sigma_{21}$). The covariance between the estimated error terms plays an important role in estimated reduced form VARs because it collects the information on the contemporaneous interaction of the variables in the system.

2.3.1 Estimation with the VAR Toolbox

Before seeing how we can estimate a VAR in MatLab, note that In MatLab we generally use a different notation. Specifically, we generally write the two time series x_{1t} and x_{2t} in a matrix x of dimension $T \times 2$ (i.e., including 2 time series with T observations each). In other words, we can write:

$$\mathbf{x} = \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ \dots & \dots \\ x_{1T} & x_{2T} \end{bmatrix}.$$

Therefore we can write the system of equations in (2.5) as:

$$\begin{bmatrix} x_{1t} & x_{2t} \end{bmatrix} = \begin{bmatrix} x_{1t-1} & x_{2t-1} \end{bmatrix} \begin{bmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{bmatrix} + \begin{bmatrix} u_{1t} & u_{2t} \end{bmatrix}, \tag{2.7}$$

which clearly yields

$$\begin{cases} x_{1t} = f_{11}x_{1,t-1} + f_{12}x_{2,t-1} + u_{1t}, \\ x_{2t} = f_{21}x_{1,t-1} + f_{22}x_{2,t-1} + u_{2t}, \end{cases}$$

as in equation (2.5). Even though the matricial notation above is a different from the on used above, the reduced-form representation is of course the same. According to our above notation, equation 2.7 can be also expressed as:

$$\mathbf{x}_t = \mathbf{x}_{t-1}\mathbf{F}' + \mathbf{u}_t. \tag{2.8}$$

NOTE

In Matlab, to perform the estimation of a reduced form VAR you can use the procedure `VARmodel.m`, whose description is provided below.

```
function VARout = VARmodel(DATA,nlag,c_case,DATA_EX,nlag_ex)
% ==================================================================
% Performs vector autogressive estimation
% ==================================================================
% VARout = VARmodel(DATA,nlag,c_case,DATA_EX)
```

```
% -----------------------------------------------------------------
% INPUTS
% DATA = an (nobs x neqs) matrix of y-vectors
% nlag = the lag length
%
% OPTIONAL INPUTS
% c_case = 0 no constant; 1 with constant ; 2 constant +
%             time trend
% DATA_EX = optional matrix of variables (nobs x nvar_ex)
% nlag_ex = number of lags for exogeonus variables
%             (default = 0)
% =================================================================
```

The DATA matrix should be such that each column is a variable and each row is an observation (with no missing observations), as the following one:

$$\texttt{DATA} = \mathbf{x}' = \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ \ldots & \ldots \\ x_{1T} & x_{2T} \end{bmatrix}.$$

The other two inputs that are needed for the estimation are the number of lags and the deterministic variables (constant and trend). For example, to estimate a bivariate VAR(2) with constant and trend on the the matrix x you can type in Matlab:

```
VARout = VARmodel(Y,2,2);
```

The `VARmodel` code estimates the reduced form VAR and saves the results in the `VARout` structure. For example, a bivariate VAR(2) with trend and constant can be written as:

$$x_{1t} = \alpha_1 + \beta_1 tr + f_{11}x_{1t-1} + f_{12}x_{2t-1} + f_{11}^2 x_{1t-2} + f_{12}^2 x_{2t-2} + u_{1t},$$
$$x_{1t} = \alpha_2 + \beta_2 tr + f_{21}x_{1t-1} + f_{22}x_{2t-1} + f_{21}^2 x_{1t-2} + f_{22}^2 x_{2t-2} + u_{2t},$$

or as in equation (2.7):

$$\begin{bmatrix} x_{1t} & x_{2t} \end{bmatrix} = \begin{bmatrix} 1 & tr & x_{1t-1} & x_{2t-1} & x_{1t-2} & x_{2t-2} \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ f_{11} & f_{21} \\ f_{12} & f_{22} \\ f_{11}^2 & f_{21}^2 \\ f_{12}^2 & f_{22}^2 \end{bmatrix}}_{\mathbf{F}'}.$$

The matrix `VARout.beta` collects the estimated coefficients as in the equation above. Therefore, we have that:

$$\texttt{VARout.beta} = \mathbf{F}'.$$

The `VARout` structure includes many additional standard results of VAR analysis. I refer to the `VARmodel.m` file for a full description of those results.

2.3.2 A simple example

We can now estimate the VAR with some real data. We use UK quarterly data on GDP growth, inflation and, short term interest rates from 1985.I to 2013.III, plotted in the Figure below.

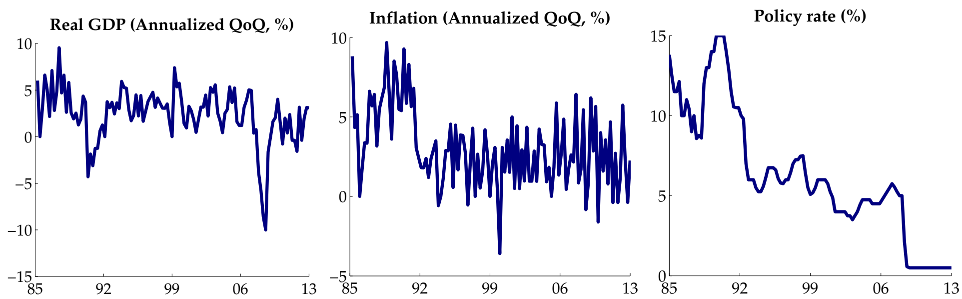

Figure 2.1 DATA: GDP GROWTH, INFLATION, SHORT-TERM INTEREST RATE

We estimate, as an example, a VAR(1) with a constant (namely, $\mathbf{x}_t = \mathbf{c} + \mathbf{F}\mathbf{x}_{t-1} + \mathbf{u}_t$). The typical VAR output is given by a matrix of coefficients ($\mathbf{F}'$):

	Real GDP	GDP Deflator	Policy Rate
c	1.14	1.19	-0.11
Real GDP(-1)	0.61	-0.07	0.06
GDP Deflator(-1)	-0.09	0.02	0.03
Policy Rate(-1)	0.01	0.30	0.96

and the correlation matrix of the reduced-form residuals:

	Real GDP	GDP Deflator	Policy Rate
Real GDP	1.000	-0.178	0.373
GDP Deflator	–	1.000	0.137
Policy Rate	–	–	1.000

After estimating the VAR we should check the goodness of our model. We do not cover this in detail but before interpreting the VAR results we should check a number of assumptions. Loosely speaking we need to check that the reduced-form residuals are, normally distributed, not autocorrelated, and not heteroskedastic (i.e., have constant variance). Finally, we need to check that the VAR is stationary (we'll see in a second what it means). But before moving to stationarity, why do we need to check the residuals? The reason is quite simple. The VAR believes that:

$$\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\sigma}) \quad \Longrightarrow \quad \begin{cases} \Delta y & \sim \quad \mathcal{N}(\mu^{\Delta y}, \sigma^{\Delta y}) \\ \pi & \sim \quad \mathcal{N}(\mu^{\pi}, \sigma^{\pi}) \\ i & \sim \quad \mathcal{N}(\mu^{i}, \sigma^{i}) \end{cases}$$

If the data that we feed into the VAR has not these features, the residuals will inherit them. Moreover note that:

- Mean (μ) and variance (σ) are constant $\Longrightarrow$ the data have to be stationary!
- The mean (μ) and variance (σ) are not known a priori

Can we recover the mean of our endogenous variables? Yes. It is given by $\mu = \mathbb{E}[\mathbf{x}_t]$

$$
\begin{aligned}
\mathsf{E}[\mathbf{x}_t] &= \mathsf{E}[\mathbf{c} + \mathbf{F}\mathbf{x}_{t-1} + \mathbf{u}_t] = \\
&= \mathsf{E}[\mathbf{c} + \mathbf{F}(\mathbf{c} + \mathbf{F}\mathbf{x}_{t-2} + \mathbf{u}_{t-1}) + \mathbf{u}_t] = \mathbf{E}[\mathbf{c} + \mathbf{F}\mathbf{c} + \mathbf{F}^2\mathbf{x}_{t-2} + \mathbf{F}\mathbf{u}_{t-1} + \mathbf{u}_t] = \\
&= \mathsf{E}[\mathbf{c} + \mathbf{F}\mathbf{c} + \mathbf{F}^2\mathbf{c} + ... + \mathbf{F}^{t-2}\mathbf{c} + \mathbf{F}^{t-1}\mathbf{x}_1 + \mathbf{F}^{t-2}\mathbf{u}_2 + ... + \mathbf{F}^2\mathbf{u}_{t-2} + \mathbf{F}\mathbf{u}_{t-1} + \mathbf{u}_t] = \\
&= ... = \\
&= \mathsf{E}\left[\sum_{j=0}^{t-2} \mathbf{F}^j\mathbf{c} + \mathbf{F}^{t-1}\mathbf{x}_1 + \sum_{j=0}^{t-2} \mathbf{F}^j\mathbf{u}_{t-j}\right]
\end{aligned}
$$

If VAR is stationary (wait one more minute to see what it means), from the last expression we get:

$$
\mathsf{E}[\mathbf{x}_t] = (\mathbf{I} - \mathbf{F})^{-1}\,\mathbf{c} = \mu
$$

NOTE

Geometric series. Consider the following sum $\sum_{j=0}^{T} Y^j$. When $T \to \infty$ we have:

$$
(\mathbf{1} + \mathbf{Y} + \mathbf{Y}^2 + ... + \mathbf{Y}^\infty) = (\mathbf{I} - \mathbf{Y})^{-1}
$$

if and only if:

- $|eig(\mathbf{Y})| < 1$ when Y is a matrix
- $Y < 1$ when Y is a number

Therefore, for T large enough we have

$$
\begin{aligned}
\mathsf{E}[\mathbf{x}_t] &= \mathsf{E}\left[\sum_{j=0}^{t-2} \mathbf{F}^j\mathbf{c} + \mathbf{F}^{t-1}\mathbf{x}_0 + \sum_{j=0}^{t-2} \mathbf{F}^j\mathbf{u}_{t-j}\right] = \\
&= \mathsf{E}\left[(\mathbf{I} - \mathbf{F})^{-1}\,\mathbf{c}\right] + \underbrace{\mathbf{F}^{t-1}\mathbf{x}_1}_{\approx 0} + \underbrace{\mathsf{E}\left[\sum_{j=0}^{t-2} \mathbf{F}^j\mathbf{u}_{t-j}\right]}_{\approx 0}
\end{aligned}
$$

A VAR is stationary (or stable) when:

$$
|eig(\mathbf{F})| < 1.
$$

In this case, the VAR thinks that $(\mathbf{I} - \mathbf{F})^{-1}\mathbf{c}$ is the unconditional mean of the stochastic processes governing our variables. Note that the unconditional mean is an interesting element of VAR analysis but it is often ignored. If we recover both $\mathbf{c}$ and

F from the estimated VAR, we can compute the unconditional mean as

$$(\mathbf{I}-\mathbf{F})^{-1}\mathbf{c} = \begin{bmatrix} 2.32 & -0.28 & -1.89 \\ 1.34 & 1.24 & 10.58 \\ 4.89 & 0.67 & 33.87 \end{bmatrix} \begin{bmatrix} 1.14 \\ 1.19 \\ -0.11 \end{bmatrix} = \begin{bmatrix} 2.51 \\ 1.80 \\ 2.49 \end{bmatrix} = \begin{bmatrix} \mu^{\Delta y} \\ \mu^{\pi} \\ \mu^{i} \end{bmatrix}$$

The stationarity of the data is important because in absence of shocks, each variable will converge to its unconditional mean. For example, if we start from an hypothetical point in time T where $x_T = 2\%$, $\pi_T = 2\%$, and $r_T = 4\%$, we would have:

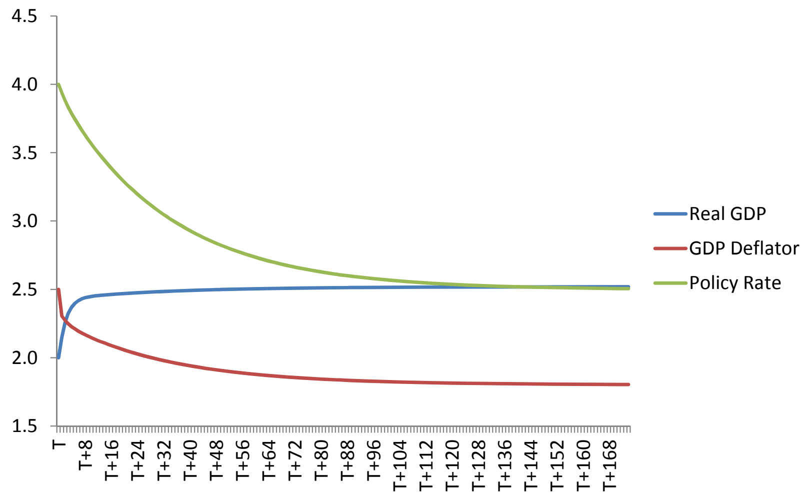

Figure 2.2 Convergence To The Unconditional Mean

Chapter 3

THE IDENTIFICATION PROBLEM

So far we have estimated our reduced form VAR. But what can we do with it?

- – What are the dynamic properties of these variables? [Look at lagged coefficients]
  – How do these variables interact? [Look at cross-variable coefficients]
  – What will be inflation tomorrow [Forecasting]
  – What is the effect of a monetary policy shock on GDP and inflation? [???]

The problem with reduced-form VARs is that they do not tell us anything about the structure of the economy. In particular, we cannot interpret the reduced-form error terms ($\mathbf{u}$) as structural shocks. Imagine that you have to interpret a movement in u_r? Since it is a linear combination of ε_r, $\varepsilon_{\Delta y}$, and ε_π it is hard to know what is the nature of the shock: is it a shock to aggregate demand that induces policy to move the interest rate? Or is it a monetary policy shock? This is the very nature of the identification problem

To answer this question we need to get back to the structural representation (where the error terms are uncorrelated). In practice, in a reduced form VAR we estimate $\mathbf{F}$ and $\mathbf{\Sigma}_u$, but we cannot easily revert to the $\mathbf{A}$, the $\mathbf{B}$, and the $\mathbf{\Sigma}_\varepsilon$ that are our ultimate object of interest. So how can we recover the structural parameters of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{\Sigma}_\varepsilon$?

From equation (2.4) we know that:

$$\mathbf{F} = \mathbf{A}^{-1}\mathbf{B}, \tag{3.1}$$
$$\hat{\mathbf{\Sigma}}_u = \mathsf{E}\left[\hat{\mathbf{u}}_t\hat{\mathbf{u}}_t'\right] = \mathsf{E}\left[\mathbf{A}^{-1}\varepsilon\left(\mathbf{A}^{-1}\varepsilon\right)'\right] = \mathbf{A}^{-1}\mathbf{\Sigma}_\varepsilon\left(\mathbf{A}^{-1}\right)' = \mathbf{A}^{-1}\mathbf{A}^{-1\prime}. \tag{3.2}$$

Therefore, if we recover $\mathbf{A}$ from equation (3.2), we will be able to recover $\mathbf{B}$ from equation (3.1).

NOTE

The variance–covariance matrix of residuals. Why $\Sigma_{\mathbf{u}} = \mathsf{E}\left[\mathbf{u}_t\mathbf{u}_t'\right]$? The formula for the variance of a univariate time series is $x = [x_0, x_1, ..., x_T]$:

$$\mathsf{VAR} = \sum_{t=0}^{T}\frac{(x_t - \bar{x})^2}{N}.$$

But the residuals (both the $\mathbf{u}_t$ and the ε_t) have zero mean which implies that formula would be:

$$\mathsf{VAR} = \sum_{t=0}^{T}\frac{x_t^2}{N}.$$

In a bivariate VAR we would have:

$$\mathbf{u}_t\mathbf{u}_t' = \begin{bmatrix} u_1^1 & u_2^1 & \dots & u_T^1 \\ u_1^2 & u_2^2 & \dots & u_T^2 \end{bmatrix}\begin{bmatrix} u_1^1 & u_1^2 \\ u_2^1 & u_2^2 \\ \dots & \dots \\ u_T^1 & u_T^2 \end{bmatrix} = \begin{bmatrix} \Sigma_{t=0}^T\left(u_t^1\right)^2 & \Sigma_{t=0}^T\left(u_t^1u_t^2\right) \\ - & \Sigma_{t=0}^T\left(u_t^2\right)^2 \end{bmatrix}.$$

And therefore
$$\mathsf{E}\left[\mathbf{u}_t\mathbf{u}_t'\right] = \begin{bmatrix} \mathsf{VAR}\left[u^1\right] & \mathsf{COV}\left[u^1u^2\right] \\ - & \mathsf{VAR}\left[u^1\right] \end{bmatrix}$$

We can think of (3.2) as a system of nonlinear equations in the unknown parameters a_{11}, a_{22}, a_{12}, and a_{21}. More in general the matrix $\mathbf{A}$ would contain $k \times k$ unknown parameters. However, the $\mathbf{\Sigma}_u$ matrix —given its symmetric nature— would contain only $k(k \times 1)/2$ parameters. In other words, we have

$$\mathbf{\Sigma}_u = \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 \\ - & \sigma_2^2 \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix} = \mathbf{A}^{-1}\mathbf{A}^{-1\prime}$$

which shows that there are 4 unknowns $(a_{11}, a_{22}, a_{12}, a_{21})$ but only 3 equations (corresponding to $\sigma_1^2, \sigma_{12}^2, \sigma_2^2$). The system is clearly under-identified, meaning that we need at least one additional condition if we want to recover the structural parameters. Identification can be achieved in different ways:

1. ZERO SHORT-RUN RESTRICTIONS (or recursive VAR). In a recursive VAR identification is achieved by making the assumption on the contemporaneous relations among variables. Since non-diagonal elements of the **A** matrix represent the contemporaneous relation between the x_t, this is equivalent to making assumption about the non-diagonal elements of the **A** matrix. Specifically, we will assume that the first variable is not affected contemporaneously by the others, the second is contemporaneously affected by the first variable but not by the others, the third is contemporaneously affected by the first and the second and not by the others, and so on. In this way the error term in each regression is uncorrelated with the error in the preceding equations. The implementation of this method can be done by 1) estimating the equations of the VAR by carefully including in some of the equations the contemporaneous values of other variables as regressors or 2) applying a Cholesky decomposition to the error terms of a reduced form VAR (as we shall see below)
2. STRUCTURAL VAR. It uses economic theory to sort out the contemporaneous relationships between the variables (the elements of matrix A). For instance, we might know from economic theory that a_{12} is equal to 0.2.
3. ZERO LONG-RUN RESTRICTIONS. Similarly to the short-run restrictions, identification is achieved by making the assumption that some variables of the VAR cannot affect some other variables in the long-run. Specifically we will assume that the first variable is not affected in the long run by the others; the second is affected in the long run by the first variable but not by the others, and so on and so forth.
4. SIGN RESTRICTIONS. Identification is achieved by restricting the sign of the responses of selected model variables to structural shocks, using economic theory as a guidance

3.1 Zero short-run restrictions

This is the methodology proposed by Sims (1980). In the context of our bi-variate VAR(1), it consists in assuming that one variable has NO contemporaneous effect on the other one. For example, assume that x_{2t} is contemporaneously affected by x_{1t} but not vice-versa. Thus we assume that $a_{12} = 0$.

A different way to express the same idea is to say that the structural innovation ε_{1t} affects both x_{1t} and x_{2t}; while the structural innovation ε_{2t} affects only x_{2t}.[1] With

[1]Will see that this corresponds to a triangular decomposition of the error term, also called

this assumption, we now have 3 equations and 3 unknown structural parameters, and SVAR is exactly identified.

> **Assumption**. In what follows we assume that x_{2t} is contemporaneously affected by x_{1t} but not vice-versa. Thus we assume that $a_{12} = 0$.

This is equivalent to assume that A is lower triangular:

$$\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}.$$

Or in other words that:

-
  - $\varepsilon_{\Delta yt}$ affects contemporaneously all variables, namely Δy_t, π_t and r_t
  - $\varepsilon_{\pi t}$ affects contemporaneously only π_t and r_t, but no Δy_t
  - ε_{rt} affects contemporaneously only r_t

We now have 3 unknowns and 3 equations! To see what are the implications of our assumptions on $\mathbf{A}$, first remember that the inverse of a lower triangular matrix is also lower triangular

$$\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{a}_{11} & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} \end{bmatrix}$$

Now pre-multiply the VAR by $\mathbf{A}^{-1}$

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} 11 & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

Which implies that

$$\begin{cases} x_{1t} & = ... + \tilde{a}_{11}\varepsilon_1 \\ x_{2t} & = ... + \tilde{a}_{21}\varepsilon_1 + \tilde{a}_{22}\varepsilon_2 \end{cases}$$

We normally implement this identification scheme *via* a Cholesky decomposition of $\Sigma_{\mathbf{u}}$:

$$\Sigma_{\mathbf{u}} = \mathbf{P}'\mathbf{P},$$

where $\mathbf{P}'$ is lower triangular. Note that

$$\Sigma_{\mathbf{u}} = \mathbf{P}'\mathbf{P} \quad \text{but also} \quad \Sigma_{\mathbf{u}} = \mathbf{A}^{-1}\mathbf{A}^{-1\prime}$$

Cholesky decomposition.

and that $\mathbf{A}^{-1}$ is lower triangular as $\mathbf{P}'$. Then it musty follow that $\mathbf{P}' = \mathbf{A}^{-1} \implies$ Identification!

NOTE

Don't be scared of Cholesky decomposition! It's a kind of square root of a matrix. As in Excel you type `sqrt()` in Matlab you type `chol()`. A symmetric and positive definite matrix $\mathbf{X}$ can be decomposed as:

$$\mathbf{X} = \mathbf{P}'\mathbf{P},$$

where $\mathbf{P}$ is an upper triangular matrix (and therefore $\mathbf{P}'$ is lower triangular). The formula is

$$\mathbf{X} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} \sqrt{a} & \frac{b}{\sqrt{a}} \\ 0 & \sqrt{c - \frac{b^2}{a}} \end{bmatrix}$$

3.1.1 Zero short-run restrictions in Matlab

The computation of the $\mathbf{A}^{-1}$ matrix in Matlab is straightforward. Consider the following `DATA` matrix (as the one considered above):

$$\texttt{DATA} = \mathbf{x}' = \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ \dots & \dots \\ x_{1T} & x_{2T} \end{bmatrix}.$$

and a bivariate VAR(1) with no constant and no trend (for simplicity):

```
VARout = VARmodel(Y,0,1);
```

The results are stored in the `VARout` structure as follows:

$$\begin{aligned} \texttt{VARout.beta} &= \mathbf{F}' = \begin{bmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{bmatrix}, \\ \texttt{VARout.sigma} &= \mathbf{\Sigma}_u = \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 \\ - & \sigma_2^2 \end{bmatrix}. \end{aligned}$$

First, let's transform the matrix of estimated coefficients to be consistent with the notation above, namely:

```
F = transpose(VARout.beta);
```

where `F` $= \mathbf{F}$ and rename the VCV matrix of the reduced form residuals:

```
sigma = VARout.sigma;
```

where `sigma` $= \mathbf{\Sigma}_u$. The $\mathbf{A}^{-1}$ matrix can now be computed with the following line of code:

```
invA = chol(sigma)';
```

where `invA` $= \mathbf{A}^{-1}$ is a lower triangular matrix.

3.2 Structural VARs

Reduced form VAR representations, such as those introduced by Sims (1980), do not allow for instantaneous relationship among the dependent variables. As a result, there is a correlation structure among the error term that is left undetermined. The zero short-run restrictions identification scheme described above allows to decouple the correlation structure in the reduced-form residuals and to recover the structural errors.

The recursive structure implied by the zero short-run restrictions identification scheme, however, may not always be sensible. There have been many different types of restriction suggested in order to achieve the identification of the structural shocks. A summary of different types of restriction can be found in Christiano, Eichenbaum, and Evans (1999).

3.3 Zero long-run restrictions

This section will focus on identification based on long-run restrictions. As an example of long-run restrictions, let us consider the Blanchard and Quah (1989) identification of demand and supply shocks. Economic theory usually tells us a lot more about what will happen in the long-run, rather than exactly what will happen today. For instance, theory tells us that whatever positive aggregate demand shocks do in the short-run, in the long-run they have no effect on output and a positive effect on the price level. This suggests an alternative approach: to use these theoretically-inspired long-run restrictions to identify shocks and impulse responses.

Consider a reduced form bivariate VAR as in equation 2.4 and its structural representation:

$$\begin{aligned}\mathbf{x}_t &= \mathbf{F}\mathbf{x}_{t-1} + \mathbf{u}_t \\ \mathbf{A}\mathbf{x}_t &= \mathbf{B}\mathbf{x}_{t-1} + \boldsymbol{\varepsilon}_t.\end{aligned}$$

To simplify the notation, as we have done above we define $\mathbf{A}^{-1} \equiv \tilde{\mathbf{A}}$. We can therefore re-write the VAR as:

$$\mathbf{x}_t = \boldsymbol{\Phi}\mathbf{x}_{t-1} + \tilde{\mathbf{A}}\boldsymbol{\varepsilon}_t,$$

and express the VCV matrix of the residuals of the reduced form VAR as:

$$\boldsymbol{\Sigma}_u = \mathsf{E}\left[\mathbf{u}_t\mathbf{u}_t'\right] = \mathsf{E}\left[\tilde{\mathbf{A}}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}'\tilde{\mathbf{A}}'\right] = \tilde{\mathbf{A}}\boldsymbol{\Sigma}_\varepsilon\tilde{\mathbf{A}}' = \tilde{\mathbf{A}}\tilde{\mathbf{A}}'. \tag{3.3}$$

If a structural shock hits in time t, its long run impact on the level of $\mathbf{x}_t$ —given that the VAR is stable, i.e. that is if eigenvalues of $\mathbf{F}$ are are inside unit circle— is given by:

$$\begin{aligned}&\mathbf{x}_{t+\infty} = \tilde{\mathbf{A}}\boldsymbol{\varepsilon}_t + \mathbf{F}\tilde{\mathbf{A}} + \mathbf{F}^2\tilde{\mathbf{A}}\boldsymbol{\varepsilon}_t + ... + \mathbf{F}^\infty\tilde{\mathbf{A}}\boldsymbol{\varepsilon}_t, \\ &\mathbf{x}_t = (\mathbf{I} - \mathbf{F})^{-1}\tilde{\mathbf{A}}\boldsymbol{\varepsilon}_t,\end{aligned} \tag{3.4}$$

where remember that the series $\mathbf{I} + \mathbf{F} + \mathbf{F}^2 + ...\mathbf{F}^\infty = (\mathbf{I} - \mathbf{F})^{-1}$ if the eigenvalues of $\mathbf{F}$ are smaller than one. Moreover we can simplify the above expression by defining:

$$\mathbf{x}_t = \mathbf{D}\boldsymbol{\varepsilon}_t, \tag{3.5}$$

where $\mathbf{D} = (\mathbf{I} - \mathbf{F})^{-1}\tilde{\mathbf{A}}$. The matrix $\mathbf{D}$ represents the cumulative effect (from t to $t + \infty$) of a shock hitting in period t.

The idea of Blanchard and Quah is the following. According to the natural rate hypothesis, demand-side shocks will have no long-run effect on the level of output, while supply-side shocks (e.g., productivity shocks) will have a permanent effect on it. Assume that ε_{1t} is the supply side shock and that ε_{2t} is the demand side shock: we can rewrite () such that the cumulated effect of ε_{2t} on x_{1t} is equal to zero by assuming:

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} d_{11} & 0 \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix},$$

by imposing $d_{12} = 0$.

How can we compute the value of $\mathbf{D}$? First let's compute $\mathbf{D}\mathbf{D}'$:

$$\mathbf{D}\mathbf{D}' = (\mathbf{I} - \mathbf{F})^{-1}\tilde{\mathbf{A}}\tilde{\mathbf{A}}'\left((\mathbf{I} - \mathbf{F})^{-1}\right)',$$

and then remember from equation (3.3) that $\tilde{\mathbf{A}}\tilde{\mathbf{A}}' = \mathbf{\Sigma}_u$. Therefore, all elements of that the right hand side of the previous equation are known and $\mathbf{DD}'$ is pinned down:

$$\mathbf{DD}' = (\mathbf{I}-\mathbf{F})^{-1}\mathbf{\Sigma}_u\left((\mathbf{I}-\mathbf{F})^{-1}\right)'.$$

Now that we know $\mathbf{DD}'$, how can we pin down $\mathbf{D}$? To solve for $\mathbf{D}$, remember that both $\mathbf{DD}'$ and $(\mathbf{I}-\mathbf{F})^{-1}\mathbf{\Sigma}_u\left((\mathbf{I}-\mathbf{F})^{-1}\right)'$ are symmetric matrices.

All symmetric matrices have a unique upper diagonal matrix $\mathbf{P}$ such that $\mathbf{P}'\mathbf{P}$ equals the original symmetric matrix (the Cholesky factor of the symmetric matrix). If we take the Cholesky decomposition of $(\mathbf{I}-\mathbf{F})^{-1}\mathbf{\Sigma}_u\left((\mathbf{I}-\mathbf{F})^{-1}\right)'$ such that:

$$\mathbf{P}'\mathbf{P} = (\mathbf{I}-\mathbf{F})^{-1}\mathbf{\Sigma}_u\left((\mathbf{I}-\mathbf{F})^{-1}\right)',$$

and we assume that $d_{12} = 0$ (as we did above), $\mathbf{D}$ is uniquely pinned down by:

$$\mathbf{D} = \begin{bmatrix} d_{11} & 0 \\ d_{21} & d_{22} \end{bmatrix} = \mathbf{P}'$$

Now that we know $\mathbf{D}$, we can easily recover the structural matrix $\mathbf{A}^{-1} = \tilde{\mathbf{A}} = (\mathbf{I}-\mathbf{F})\,\mathbf{D}$.

NOTE

The computation of the $\mathbf{D}$ matrix in Matlab is straightforward. Consider a bivariate VAR(1) with no constant and no trend (as the one considered above):

```
VARout = VARmodel(Y,0,1);
F = transpose(VARout.beta);
sigma = VARout.sigma;
```

where `F` $= \mathbf{F}$ and `sigma` $= \mathbf{\Sigma}_u$. The $\mathbf{D}$ matrix can now be computed with the following lines of code:

```
Finf = inv(eye(length(F))-F);
D = chol(Finf*sigma*Finf')';
```

where `Finf` $= (\mathbf{I}-\mathbf{F})^{-1}$ and `D` $= \mathbf{D}$. Finally, the structural matrix $\mathbf{A}^{-1}$ can also be computed as

```
invA = Finf\D;
```

where `invA` $= \tilde{\mathbf{A}} = \mathbf{A}^{-1}$.

Therefore, we just computed a matrix $\mathbf{D}$, such that ε_{2t} has no long-run effect on x_{1t}. In their original paper, Blanchard and Quah used a bivariate VAR in the log-difference of GDP (Δy_t) and the level of unemployment rate (u_t). The lower-diagonal assumption thus meant that of the two structural shocks only ε_{1t} —what they label a supply shock— could have a long-run effect on the level of output (i.e., the cumulated response of Δy_t).

3.4 Sign restrictions

In the zero short-run restriction identification we used the fact that

$$\hat{\mathbf{\Sigma}}_u = \mathbf{A}^{-1}\mathbf{\Sigma}_\varepsilon \left(\mathbf{A}^{-1}\right)' \quad \text{and} \quad \hat{\mathbf{\Sigma}}_u = \mathbf{P}'\mathbf{P},$$

where the lower triangular $\mathbf{P}'$ matrix is the Cholesky decomposition of the variance-covariance matrix of the reduced-form error terms. Remember that Identification was achieved by assuming that $\mathbf{A}^{-1}$ was also lower triangular.

Note however that such decomposition is not unique. Specifically, for a given orthonormal matrix $\mathbf{S}$ such that:

$$\mathbf{S}'\mathbf{S} = \mathbf{I},$$

we have that:

$$\hat{\mathbf{\Sigma}}_u = \mathbf{A}^{-1}\mathbf{\Sigma}_\varepsilon \left(\mathbf{A}^{-1}\right)' = \mathbf{P}'\mathbf{S}'\mathbf{S}\mathbf{P} = \mathcal{P}'\mathcal{P}.$$

where $\mathcal{P}'$ is generally not lower triangular anymore. Therefore, the system of equations implied by $\mathbf{A}^{-1}\left(\mathbf{A}^{-1}\right)' = \mathcal{P}'\mathcal{P}$ can be solved (there are as many equations as the number of unknowns!): but is the solution plausible? Identification is achieved by checking whether the impulse response function (formally defined in the section below) implied by this solution satisfy a set of a priori (and possibly theory-driven) sign restrictions.

Specifically, the sign restriction identification procedure can be summarized by the following steps:

1. Draw a random orthonormal matrix $\mathbf{S}'$
2. Compute the implied $\mathbf{A}^{-1} = \mathbf{P}'\mathbf{S}'$ where $\mathbf{P}'$ is the Cholesky decomposition of $\hat{\mathbf{\Sigma}}_u$
3. Compute the impulse response associated with $\mathbf{A}^{-1}$. Are the sign restrictions satisfied?
   (a) Yes. Store the impulse response

(b) No. Discard the impulse response

4. Perform N replications and report the median impulse response (and its confidence intervals)

NOTE

In Matlab we can easily implement the sign restriction identification scheme. Consider a bivariate VAR(1) with no constant and no trend (as the one considered above):

```
VARout = VARmodel(Y,0,1);
sigma = VARout.sigma;
```

where `sigma` $= \mathbf{\Sigma}_u$. For each draw we can compute $\mathbf{A}^{-1} = \mathbf{P}'\mathbf{S}'$ as follows. First we draw a random orthonormal matrix with the `OrthNorm.m` function

```
S = OrthNorm(length(sigma));
```

where `S` $= \mathbf{S}$. Then we compute

```
invA = chol(sigma)'*S';
```

where `invA` $= \mathbf{A}^{-1} = \mathbf{P}'\mathbf{S}'$.

Chapter 4

IMPULSE RESPONSE FUNCTIONS

In a VAR we are often interested in obtaining the impulse response functions. Impulse responses trace out the response of current and future values of each of the variables to a one-unit increase (or to a one-standard deviation increase, when the scale matters) in the current value of one of the VAR errors, assuming that this error returns to zero in subsequent periods and that all other errors are equal to zero.

The implied thought experiment of changing one error while holding the others constant makes most sense when the errors are uncorrelated across equations, so impulse responses are typically calculated for recursive and structural VARs. On the contrary, the estimated error terms u_{1t} and u_{2t} are correlated: it doesn't make much sense to ask "what if u_{1t} has a unit impulse with no change in u_{2t}", because when I am moving u_{1t}, I am moving the structural shock of both equations (remember that the u's are a linear combination of the structural shocks).

4.1 IRF and the moving average representation

Consider the general structural form of a bivariate VAR(1), $\mathbf{A}\mathbf{x}_t = \mathbf{B}\mathbf{x}_{t-1} + \varepsilon_t$, and its reduced form $x_t = \mathbf{F}\mathbf{x}_{t-1} + \mathbf{u}_t$. Using the lag operator, we can re-write the same model as:

$$(\mathbf{I} - \mathbf{F}L)\,\mathbf{x}_t = \mathbf{u}_t.$$

If the roots of $\mathbf{I} - \mathbf{F}L$ lie outside the unit circle, than the time series is stationary and we can write:

$$\mathbf{x}_t = \sum_{i=0}^{\infty} \mathbf{F}^i \mathbf{u}_{t-i} = \sum_{i=0}^{\infty} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}^i \begin{bmatrix} u_{1t-i} \\ u_{2t-i} \end{bmatrix}.$$

Moreover, since we know that $\tilde{\mathbf{A}}\boldsymbol{\varepsilon}$, we can write the above expression as a function of the structural errors:

$$\mathbf{x}_t = \sum_{i=0}^{\infty} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}^i \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix},$$

or, by defining $\mathbf{F}^i \tilde{\mathbf{A}} = \mathbf{\Gamma}^i$:

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \sum_{i=0}^{\infty} \begin{bmatrix} \gamma_{11} & \gamma_{22} \\ \gamma_{12} & \gamma_{22} \end{bmatrix}^i \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix},$$

which in terms of a system of linear equations is given by:

$$\begin{aligned} x_{1t} &= \varepsilon_{1t} + [\gamma_{11}^1 \varepsilon_{1,t-1} + \gamma_{12}^1 \varepsilon_{2,t-1}] + [\gamma_{11}^2 \varepsilon_{1,t-2} + \gamma_{12}^2 \varepsilon_{2,t-2}] + ... \\ x_{2t} &= \varepsilon_{2t} + [\gamma_{21}^1 \varepsilon_{1,t-1} + \gamma_{22}^1 \varepsilon_{2,t-1}] + [\gamma_{21}^2 \varepsilon_{1,t-2} + \gamma_{22}^2 \varepsilon_{2,t-2}] + ... \end{aligned}$$

Now suppose we are interested in the response of x_{1t} to a shock in the structural error of equation 1, ε_{1t}, from period t to period $t+h$. First, let's forward the system by h periods:

$$\begin{aligned} x_{1t+h} &= \varepsilon_{1t+h} + [\gamma_{11}^1 \varepsilon_{1,t+h-1} + \gamma_{12}^1 \varepsilon_{2,t+h-1}] + ... + [\gamma_{11}^h \varepsilon_{1,t} + \gamma_{12}^h \varepsilon_{2,t}] + ... \\ x_{2t+h} &= \varepsilon_{2t+h} + [\gamma_{21}^1 \varepsilon_{1,t+h-1} + \gamma_{22}^1 \varepsilon_{2,t+h-1}] + ... + [\gamma_{21}^h \varepsilon_{1,t} + \gamma_{22}^h \varepsilon_{2,t}] + ... \end{aligned}$$

Then, we can compute the effect we are interested in as:

$$\frac{\partial x_{1,t+h}}{\partial \varepsilon_{1,t}} = \gamma_{11}^h.$$

The Impulse Response Function is defined as $IRF = \Gamma^h$ for $n = 0, 1, ...H$. The elements of the matrix Γ^h are called **impact multipliers** while the **long-run cumulated effect** is given by $\sum_{i=1}^{\infty} \Gamma^i$.

4.2 How to compute IRFs in practice?

Let's define $\mathbf{s}_t = (s'_{1t}, s'_{2t})'$ as a vector of exogenous impulses that we want to impose to the structural errors of the system. Impulse responses allow us to simulate the effect of $\mathbf{s}_t$ on the dynamics of our VAR model.

For example, suppose we are interested in a one standard deviation impulse to the structural error of equation of x_1 that returns to zero in subsequent periods. Therefore, the time profile of the impulse will be:

Time (τ)	1	2	h
Impulse to ε_{1t} (s_1)	$s_{1,1} = 1$	$s_{1,2} = 0$	$s_{1,h} = 0$
Impulse to ε_{2t} (s_2)	$s_{2,1} = 0$	$s_{2,2} = 0$	$s_{2,h} = 0$

.

After identifying the VAR with one of the methodologies described above, we can use the reduced-form representation of our VAR to compute the IRFs. Specifically, ewe can write

$$\mathbf{x}_t = \mathbf{F}\mathbf{x}_{t-1} + \mathbf{A}^{-1}\mathbf{s}_t,$$

which implies that the impulse response $IRF\tau$ in time $\tau = 1$ is:

$$\begin{aligned} \mathbf{IRF}_1 &= \mathbf{A}^{-1}\mathbf{s}_1, \\ \mathbf{IRF}_\tau &= \mathbf{FIRF}_{\tau-1}, \quad \text{for } \tau = 2, ...h. \end{aligned}$$

Chapter 5

HISTORICAL DECOMPOSITIONS

Take a VAR with trend and constant. Each observation can be written as

$$
\begin{aligned}
\mathbf{x}_1 &= \mathbf{c} + 1 + \mathbf{F}\mathbf{y}_0 + \mathbf{u}_1 \\
\mathbf{x}_2 &= \mathbf{c} + 2 + \mathbf{F}\mathbf{y}_1 + \mathbf{u}_2 = \mathbf{c} + 2 + \mathbf{F}(\mathbf{c} + \mathbf{1} + \mathbf{F}\mathbf{y_0} + \mathbf{u_1}) + \mathbf{u}_2 = \mathbf{F}^2\mathbf{x}_0 + \underbrace{(\mathbf{F}\mathbf{c} + \mathbf{c})}_{cc} + (\mathbf{F} + \mathbf{2}) + \underbrace{(\mathbf{F}\mathbf{u}_1 + \mathbf{u}_2)}_{uu} \\
\mathbf{x}_3 &= \mathbf{c} + 3 + \mathbf{F}\mathbf{y}_2 + \mathbf{u}_3 = \mathbf{F}^3\mathbf{x}_0 + (\mathbf{F}^2\mathbf{c} + \mathbf{F}\mathbf{c} + \mathbf{c}) + (\mathbf{F}^2 + 2\mathbf{F} + \mathbf{3}) + (\mathbf{F}^2\mathbf{u}_1 + \mathbf{F}\mathbf{u}_2 + \mathbf{u}_3) \\
&= \ldots = \\
\mathbf{x}_T &= \mathbf{c} + T + \mathbf{F}\mathbf{y}_{T-1} + \mathbf{u}_T = \mathbf{F}^T\mathbf{x}_0 + (\mathbf{F}^T\mathbf{c} + \ldots + \mathbf{F}\mathbf{c} + \mathbf{c}) + (\mathbf{F}^T + \ldots + \mathbf{F}(T-1) + T) + (\mathbf{F}^T\mathbf{u}_0 + \ldots + \mathbf{F}
\end{aligned}
$$

Now take structural errors

$$
\begin{aligned}
\mathbf{x}_1 &= \mathbf{F}\mathbf{y}_0 + \mathbf{A}^{-1}\varepsilon_1 \\
\mathbf{x}_2 &= \mathbf{F}\mathbf{y}_1 + \mathbf{A}^{-1}\varepsilon_2 = \mathbf{F}(\mathbf{F}\mathbf{y_0} + \mathbf{A}^{-1}\varepsilon_1) + \mathbf{A}^{-1}\varepsilon_2 = \mathbf{F}^2\mathbf{x}_0 + \underbrace{(\mathbf{F}\mathbf{A}^{-1}\varepsilon_1 + \mathbf{A}^{-1}\varepsilon_2)}_{uu} \\
\mathbf{x}_3 &= \mathbf{c} + 3 + \mathbf{F}\mathbf{y}_2 + \mathbf{u}_3 = \mathbf{F}^3\mathbf{x}_0 + (\mathbf{F}^2\mathbf{c} + \mathbf{F}\mathbf{c} + \mathbf{c}) + (\mathbf{F}^2 + 2\mathbf{F} + \mathbf{3}) + (\mathbf{F}^2\mathbf{u}_1 + \mathbf{F}\mathbf{u}_2 + \mathbf{u}_3) \\
&= \ldots = \\
\mathbf{x}_T &= \mathbf{c} + T + \mathbf{F}\mathbf{y}_{T-1} + \mathbf{u}_T = \mathbf{F}^T\mathbf{x}_0 + (\mathbf{F}^T\mathbf{c} + \ldots + \mathbf{F}\mathbf{c} + \mathbf{c}) + (\mathbf{F}^T + \ldots + \mathbf{F}(T-1) + T) + (\mathbf{F}^T\mathbf{u}_0 + \ldots + \mathbf{F}
\end{aligned}
$$

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