Math265 Real Analysis Class Notes

Based on lectures by Prof. Huang
Notes taken by Kaifeng Lu

Fall 2024

1 Preliminaries

1.1 Sets and Functions

1.2 Mathematical Induction

1.3 Finite and Infinite Sets

2 The Real Numbers

2.1 The Algebraic and Order Properties of $\mathbb{R}$

2.2 Absolute Value and the Real Line

2.3 The Completeness Property of $\mathbb{R}$

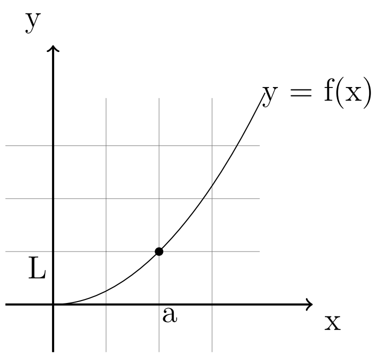
2.4 Applications of Supremum

2.5 Intervals

3 Sequences and Series

3.1 Sequences and their limits

Definition (**Seuqence**)**.** A ***sequence of real numbers*** is a *function* from $\mathbb{N}$ to $\mathbb{R}$.



We adopt the notation with a *sequence*:

$$a : \mathbb{N} \to \mathbb{R}$$

where instead of writing $a(1), a(2), \ldots$, we write it as $a_1, a_2, \ldots$ which we called them ***terms*** or ***elements*** of the sequence.

Notation.

$$(a_n)_{n=1}^{\infty} \text{ or } (a_n)_{n\in\mathbb{N}} \text{ or } (a_n) \text{ or } (a_n | n \in \mathbb{N})$$

Definition (**Converge to x**)**.** A sequence $(x_n) \in \mathbb{R}$ ***converges*** to $x \in \mathbb{R}$ if

$$\forall \epsilon > 0,\ \exists N_\epsilon \in \mathbb{N} \text{ such that } n \geq N_\epsilon \to |x_n - x| < \epsilon$$

We write

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (x_n) = x$$

Definition (**Convergent & Divergent**)**.** A sequence is ***convergent*** if it has a ***limit*** in $\mathbb{R}$, and is ***divergent*** if it has ***no limit*** in $\mathbb{R}$.

Theorem (Uniqueness of Limit). A sequence in $\mathbb{R}$ can have ***at most one*** limit. Or, the limit of a sequence is ***unique*** if the limit exists

Proof. Let (x_n) be a sequence of real numbers. Suppose x, x' are limits of (x_n). We want to prove $x = x'$ by controdiction.

Assume $|x - x'| > 0$. If we consider $\epsilon := \frac{1}{3}|x - x'| > 0$, then
The existence of $\lim_{x_n \to x}$ implies that $\exists N_1 \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ if $n \geq \mathbb{N}_1$.
Similarly, existence of $\lim_{x_n \to x'}$ implies that $\exists N_2 \in \mathbb{N}$ such that $|x_n - x'| < \epsilon$ if $n \geq \mathbb{N}_2$.
Thus,

$$\begin{aligned} |x - x'| &\leq |x - x_{N_1+N_2} + x_{N_1+N_2} - x'| \\ &\leq |x - x_{N_1+N_2}| + |x_{N_1+N_2} - x'| \text{ by triangle inequality} \\ &< \epsilon + \epsilon \\ &= \frac{2}{3}|x - x'| \end{aligned}$$

Then,

$$\frac{1}{3}|x - x'| < 0, \text{which is a controdiction}$$

we thereby prove by controdiction that

$$|x - x'| = 0, \text{ which is equvalent to } x = x'$$

∎

Example.

$$\lim_{n\to\infty}(\frac{1}{n}) = 0$$

Goal: $\forall \epsilon > 0$, want to find N_ϵ such that $\left|\frac{1}{n} - 0\right| < \epsilon$ for $n > N$, so it suffices to show that

$$\frac{1}{n} < \epsilon \Leftrightarrow \frac{1}{\epsilon} < n$$

Proof. Let $\epsilon > 0$. Apply Archemedian's property to $\frac{1}{\epsilon}$, then

$$\begin{aligned} &\exists N \in \mathbb{N} \text{ such that } \frac{1}{\epsilon} < N \\ &\Rightarrow \forall n \geq N, \left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon. \\ &\Rightarrow \lim_{n\to\infty}\frac{1}{n} = 0 \end{aligned}$$

∎

Theorem. Let (x_n) be a sequence of real numbers, and let $x \in \mathbb{R}$. The following theorems are equvalent:

1. $x_n \to x$
2. $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$, for $n \geq \mathbb{N}$
3. $\ldots\ldots x - \epsilon < x_n < x + \epsilon \ldots\ldots$
4. $\forall \epsilon$-neighborhood $V_\epsilon(x), \exists N \in \mathbb{N}$ such that $x_n \in V_\epsilon(x)$ for $n \geq N$

Sketch of proof:

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$$

Proposition.

$$\lim_{n\to\infty}(2\sqrt{2n+1} - \sqrt{2n}) = 0$$

Proof. Let $\epsilon > 0$. Consider

$$N = \lceil \frac{1}{2}(\frac{1}{2\epsilon})^2 \rceil \in \mathbb{N}$$

$$n > N \Rightarrow n > \frac{1}{2}(\frac{1}{2\epsilon})^2 \Rightarrow \frac{1}{2\sqrt{2n}} < \epsilon \Rightarrow \left|\sqrt{2n+1} - \sqrt{2n}\right| = \cdots = \frac{1}{\sqrt{2n+1} + \sqrt{2n}} < \epsilon$$

■

Remark.

$$\lim_{n\to\infty}(-1)^n \text{ is undefined.}$$

Definition (m-tail). If (x_n) is a sequence of real numbers and $m \in \mathbb{N}$, then the ***m-tail*** of (x_n) is the sequence

$$\{x_{n+m} : n \in \mathbb{N}\} = \{x_{m+1}, x_{m+2}, \ldots\}$$

Theorem. Let (x_n) be a sequence and $m \in \mathbb{N}$. Then (x_n) is ***convergent*** iff (x_{n+m}) is ***convergent***. Moreover,

$$\lim_{n\to\mathbb{N}}(x_n) = \lim_{n\to\mathbb{N}}(x_{n+m})$$

Proof. $(\Rightarrow)$
Suppose $x_n \to x$. Let

$$\epsilon > 0, \exists N_\epsilon > 0, \text{ such that } |x_n - x| < \epsilon \text{ for } n \geq N_\epsilon$$

Consider $N_\epsilon{}' := N_\epsilon + m$ then

$$n + m \geq N_\epsilon{}' \Rightarrow n \geq N_\epsilon \Rightarrow n + m \geq N_\epsilon \Rightarrow |x_{n+m} - x| < \epsilon$$

It follows that

$$n \geq N_\epsilon \Rightarrow n + m \geq N_\epsilon \Rightarrow |x_{n+m} - x| < \epsilon$$

($\Leftarrow$)
Suppose $x_{n+m} \to x$.

$$\forall \epsilon > 0, \exists N_\epsilon > 0 \text{ such that } |x_{n+m} < \epsilon|, \forall n \geq N_\epsilon$$

Consider $N := N_\epsilon + m$. Then

$$\begin{aligned}
& n \geq N = N_\epsilon + m \\
& \Rightarrow n - m \geq N_\epsilon \\
& \Rightarrow \left|x_{(n-m)+m} - x\right| < \epsilon \\
& \Rightarrow |x_n - x| < \epsilon
\end{aligned}$$

■

Remark. We say that a sequence (x_n) ***ultimately*** has a property if that property holds for some tail of (x_n)

Theorem. Let x_n be a sequence of real numbers. Let a_n be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$. If $\exists c > 0, m \in \mathbb{N}, x \in \mathbb{R}$ such that

$$|x_n - x| \leq c \cdot a_n, \forall n \geq m$$

then

$$x_n \to x$$

Proof. We know that

$$\forall \epsilon > 0, \exists N \geq 0 \text{ s.t. } |a_n| < \frac{\epsilon}{c}, \forall n \geq N$$

Consider $N' = max\{N, m\}, \forall n \geq N'$. Then

$$|x_n - x| \leq Ca_n = c\,|a_n| < c \cdot \frac{\epsilon}{c} = \epsilon$$

$$\Rightarrow x_n \to x$$

■

Proposition.

$$\lim_{n\to\infty} \frac{17}{2+3n} = 0$$

Proof.

$$\left|\frac{17}{2+3n} - 0\right| = \frac{17}{2+3n} \leq \frac{13}{3n} = \frac{17}{3} \cdot \frac{1}{n}$$

Apply the theorem above with

$$a_n = \frac{1}{n}, c = \frac{17}{3}, m = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{17}{2+3n} = 0, \text{ since } \lim_{n\to\infty} \frac{1}{n} = 0$$

■

Proposition.

$$\forall c > 0, \lim_{n\to\infty} c^{\frac{1}{n}} = 1$$

Proof. ***Case 1: c = 1***

$$\lim_{n\to\infty} c^{\frac{1}{n}} = 1$$

Case 2: c > 1

Let $d_n = c^{\frac{1}{n}} - 1$. Then $\forall n, d_n > 0$. It follows that

$$\begin{aligned}(d_n + 1) = c^{\frac{1}{n}} &\Rightarrow c = (1 + d_n)^n \geq 1 + n \cdot d_n \text{ by Bernoulli's inequality} \\ &\Rightarrow d_n \leq (c-1) \cdot \frac{1}{n} \\ &\Rightarrow \left|c^{\frac{1}{n}} - 1\right| = d_n \leq (c-1) \cdot \frac{1}{n}\end{aligned}$$

Apply the theorem with

$$C = c - 1, a_n = \frac{1}{n}, m = 1, x = 1$$

$$\lim_{n\to\infty} c^{\frac{1}{n}} = 1$$

Case 3: c < 1(Note that we cannot use Bernoulli inequality here)

Define e_n to be a sequence that satisfies

$$c^{\frac{1}{n}} = \frac{1}{1 + e_n}$$

Then $e_n > 0 \forall n$.

$$c = \frac{1}{(1 + e_n)^n} \leq \frac{1}{1 + n \cdot e_n} < \frac{1}{n \cdot e_n}$$

$$\Rightarrow e_n < \frac{1}{c} \cdot \frac{1}{n}$$

$$1 - c^{\frac{1}{n}} = 1 - \frac{1}{1 + e_n} = \frac{e_n}{1 + e_n} < e_n < \frac{1}{c} \cdot \frac{1}{n}$$

Apply the theorem with

$$a_n = \frac{1}{n}, m = 1, C = \frac{1}{c}, x = 1$$

$$\lim_{n\to\infty} c^{\frac{1}{n}} = 1$$

■

3.2 Limit Theorems

3.3 Monotone sequence

Definition. Let (x_n) be a sequence of real numbers
Theorem (Monotone Convergence Therorem). A monotone sequence of real numbers is ***convergent*** iff it is bounded. Moreover, if (x_n) is increaseing, then

$$\lim(x_n) = \sup x_n : n \in \mathbb{N}$$

If (x_n) is decreasing, then

$$\lim(x_n) = \inf x_n : n \in \mathbb{N}$$

Example. Consider the sequence (x_n) is given by

$$\begin{cases} x_0 = \frac{1}{2} \\ x_{n+1} = \frac{3}{2}x_n(1-x_n) \end{cases}$$

(x_n) is decreasing and bounded.

Thoughts: Assume (x_n) converges, then by limit law,

$$x = \frac{3}{2}x(1-x) \text{ where } x = \lim(x_n) \Rightarrow x = 0 \text{ or } 3$$

then, by proof of controdiction, it is not convergent.

Proof. ***Claim***: $\frac{1}{3} < x_{n+1} < x_n \leq \frac{1}{2}, \forall n \in \mathbb{N} \cup \{0\}$

Proof of the claim by induction:

When n = 0:

$$x_0 = \frac{1}{2}, x_1 = \frac{3}{2} \cdot \frac{1}{2}(1 - \frac{1}{2}) = \frac{3}{8}$$
$$\frac{1}{3} < \frac{3}{8} < \frac{1}{2} \leq \frac{1}{2}$$

Suppose this is true for n=k:

$$\frac{1}{3} < x_{k+1} < x_k \leq \frac{1}{2}$$

Goal:

$$\frac{1}{3} < x_{k+2} < x_{k+1} \leq \frac{1}{2}$$

$$x_{k+1} = \frac{3}{2}x_k(1-x_k)$$

$$\frac{1}{3} < x_k \leq \frac{1}{2} \Rightarrow \frac{2}{3} > 1 - x_k \geq \frac{1}{2}$$

$$x_{k+1} < \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{2}$$

Complete the square:

$$\begin{aligned} x_{k+1} - \frac{1}{3} &= \frac{3}{2}x_k(1-x_k) - \frac{1}{3} \\ &= -\frac{3}{2}[(x_k - \frac{1}{2})^2 - \frac{1}{36}] \end{aligned}$$

So

$$\begin{aligned} \frac{1}{3} < x_k \leq \frac{1}{2} &\Rightarrow \left|x_k - \frac{1}{2}\right| < \frac{1}{6} \\ &\Rightarrow (x_k - \frac{1}{2})^2 < \frac{1}{36} \\ &\Rightarrow x_{k+1} - \frac{1}{3} > 0 \\ &\Rightarrow \frac{1}{3} < x_{k+1} \leq \frac{1}{2} \end{aligned}$$

With the similar process, we can derive that

$$\frac{1}{3} < x_{k+2} \leq \frac{1}{2}$$

$$x_{k+2} = \frac{3}{2}x_{k+1}(1-x_{k+1}) < \frac{3}{2}x_{k+1} \cdot \frac{3}{2} = x_{k+1}$$

Therefore, this claim is also true for n=k+1:

$$\frac{1}{3} < x_{k+2} < x_{k+1} \leq \frac{1}{2}$$

We thereby prove the theorem by induction:

$$\frac{1}{3} < x_{n+1} < x_n \leq \frac{1}{2}$$

■

Exercise: Textbook p.75: A sequence that converges to $\sqrt{a}$ for $a > 0$.

Definition (Euler's Number).

$$e = \lim(1 + (\frac{1}{n})^n)$$

Goal: (x_n) is convergent where $x_n = (1 + \frac{1}{n})^n$

$$\begin{aligned} x_n = (1 + \frac{1}{n})^n &= 1 + nC1 \cdot \frac{1}{n} + nC2 \cdot \frac{1}{n^2} + \cdots + nCn\frac{1}{n^n} \\ &= 1 + \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{2} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1)\cdot 3 \cdot 2 \cdot 1}{n!} \cdot \frac{1}{n^2} \\ &= 1 + 1 + \frac{1}{2}(1 - \frac{1}{n}) + \frac{1}{6}(1-n)(1 - \frac{2}{n}) + \cdots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \ldots \frac{2}{n} \cdot \frac{1}{n} \end{aligned}$$

Write x_{n+1} in a similar way, we obserbe that

$$x_n < x_{n+1}$$

Facts $2^{m-1} \leq m!$ for $m \in \mathbb{N} \Rightarrow \frac{1}{m!} \leq \frac{1}{2^{m-1}}$

$$x_n < 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 1 + \frac{1(1 - (\frac{1}{2})^n)}{1 - \frac{1}{2}} < 3$$

$\Rightarrow (x_n)$ is increasing and bounded

3.4 Subsequence and the Bolzono-Weierstress Theorem

Example.

$$(x_n) = ((-1)^n)$$
$$x_{2n} = (-1)^{2n}$$
$$x_{2n+1} = (-1)^{2n-1}$$

So $a_n = x_{2n}$ is a sequence, while $b_n = x_{2n+1}$ is a subsequence of x_n.
Definition (Subsequences). Let (x_n) be a real sequence and consider a strickly increasing seuqence of natural numbers $n_1 < n_2 < n_3 < \dots$. The sequence

$$x_{n_k} : k \in \mathbb{N}$$

is call a ***subsequence*** of (x_n)
Example. Any tails of a sequence is a subsequence: (x_n) n-th dail: (x_{m+k}) is a subsequence
Theorem. Suppose (x_n) converges to x. Then $x_{x_k} \to x$ for any subsequence of (x_n).

Proof. Let $\epsilon > 0$

$$\exists N_\epsilon > 0 \text{ s.t. } |x_n - x| < \epsilon \text{ for } n > N_\epsilon.$$

Note that

$$n_k \geq k, \forall k \in \mathbb{N}.$$

The proof of the observation is exercise:

Exercise. By induction, $n_1 \geq 1, n_2 \geq n_1 \geq 1 \Rightarrow n_2 \geq 2$

When $k > N_k$, $n_k > N_\epsilon$,

$$\Rightarrow |x_{n_k} - x| < \epsilon$$

Therefore

$$(x_{n_k}) \to x$$

∎

Theorem. Let (x_n) be a sequence of real numbers, and let $x \in \mathbb{R}$. Then the following are equvalent:

1. (x_n) does not converge to x.
2. $\exists \epsilon_0 > 0,$ s.t. $\forall k \in \mathbb{N}, \exists n_k \in \mathbb{N}$ s.t.
$$n_k \geq k \,\&\, |x_{n_k} - x| > \epsilon_0$$
3. $\exists \epsilon_0 > 0$ & a subsequence x_{n_k} s.t.
$$|x_{n_k} - x| > \epsilon_0, \forall k \in \mathbb{N}$$

Proof.

$$3 \to 1\text{: by the contropositive statment of definition}$$
$$3 \to 2\text{: 3 is a stronger statement of 2}$$
$$1 \to 2\text{: left as exercise}$$

∎

Theorem. If x_n satisfies either of the following property if it is **divergent**:

- There exists two subsequence (x_{n_k}) & (x_{m_k}) whose limits are NOT equal.
- (x_n) is unbouded.

Example.

1. $(-1)^n$
2. (n)
3. (x_n) such that

$$x_{2k} = k$$
$$x_{2k+1} = (-1)^k$$

Proof. *exercise*

■

Theorem (Bolzano-Weierstrass Theorem). A boudned sequence of real numbers has a ***convergent subsequence***.

Example.

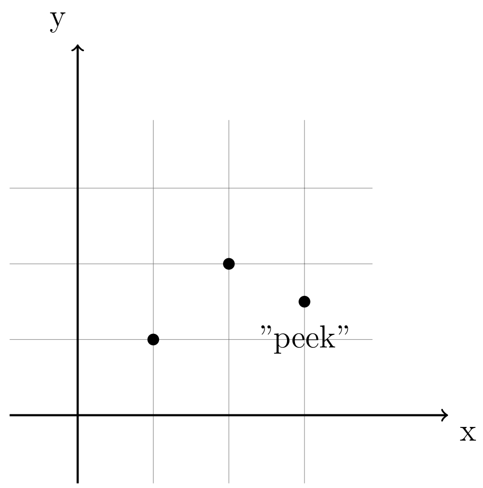
$$x_n = (-1)^n$$

Proof.

Lemma. If (x_n) is a sequence of real numbers, there exists a subsequence of (x_n) which is monotone.

proof of lemma:

Call the m-th term x_m a *"peek"* if x_m is at least as large as any term after it in the sequence.



Case 1: (x_n) has infinitelyy many peaks

List the peaks of (x_n) in order of increasing index

$$x_{n_1}, x_{n_2}, \ldots$$

$\Rightarrow (x_{n_i})$ is a decreasing sequence.

Case 2: (x_n) has a finite number of peaks

Let s_1 be the first index after the last peak of (x_n). Then for every $n \geq s_1$, $\exists m \in \mathbb{N}$ such that $x_m > x_n$.

Choose

$$s_2 \geq s_1 \text{ such that } x_{s_2} > x_{s_1},\ s_2 \geq s_1 \text{ such that } x_{s_2} > x_{s_1}, \ldots$$
$$\Rightarrow (x_{s_i}) \text{ is an increasing sequence.}$$

Remark. Lemma + monotone convergent theorem implies Bolzano-Weierstrass theorem.

Second proof

Suppose (x_n) is a bounded sequence.

$$\Rightarrow \exists I_1 = [a_1, b_1] \text{ such that } (x_n) \in I_1$$

Consider

$$I_2' = [a_1, \frac{a_1 + b_1}{2}],\ I_2'' = [\frac{a_1 + b_1}{2}, b_1]$$

Let $I_2 = [a_2, b_2]$ be one of I_2', I_2'' such that I_2 contains infinitely many terms of (x_n).

For $n \in \mathbb{N}$, define $I_n = [a_n, b_n]$ in a similar way.

For $i \in \mathbb{N}$, choose a term x_{n_i} such that $x_{n_i} \in I_i$ and $n_i > n_{i-1}$

Then

$$\begin{aligned}
&i = 1, && n_i = 1 \\
&i = 2, && \text{choose } n_2 \in \mathbb{N} \text{ such that } n_2 > n_1 \,\&\, x_{n_2} \in I_2 \\
&\vdots && \vdots
\end{aligned}$$

- $\forall i \in \mathbb{N}$, $a_i \leq x_{n_i} \leq b_i$
- (a_i) increases, bounded above by $b_1 \Rightarrow (a_i) \to \sup(a_i)$
- (b_i) decreases, bounded below by $a_1 \Rightarrow (b_i) \to \inf(b_i)$
- $\inf|a_i - b_i| = \inf \frac{b_1 - a_1}{2^n} = 0 \Rightarrow \sup(a_i) = \inf(b_i)$

Thus (x_{n_i}) is convergent by squeeze theorem.

■

Theorem. Let (x_n) be a bounded sequence, and $x \in \mathbb{R}$ has the property that every convergent subsequence of (x_n) converges to x. Then x_n converges to x

Proof. Let $\forall \epsilon > 0$

By Bolzano-Weierstrass theorem, $\exists$ a convergent subsequence (x_{n_i}) such that

$$\exists N_\epsilon \in \mathbb{N} \text{ s.t. for } i > N_\epsilon, |x_{n_i} - x| < \epsilon$$

Assume (x_n) does not converge to x. Then by previous theorem of subsequence,

$$\exists \epsilon_0 > 0 \text{ and a subsequence } (x_{n_k}) \text{ s.t. } |x_{n_k} - x| > \epsilon_0, \forall k \in \mathbb{N}$$

Since (x_n) is bounded, (x_{n_k}) is also bounded. Thus there exists a convergent subsequence of (x_{n_k}) as $(x_{n_{k_i}})$.

Note that $(x_{n_{k_i}})$ is a convergent subsequence of (x_n), thus

$$(x_{n_{k_i}}) \to x \text{ which is controdiction to previous assumption}$$

$\blacksquare$

Definition. Let (x_n) be a sequence of real numbers. A point is called a ***subsequential limit*** of (x_n) if it is the limit of a subsequence of (x_n).

$$S = \{\alpha \in \mathbb{R} : x \text{ is a subsequential limit}\} \text{ (NOTE: may be infinite set)}$$

Example. Consider $(x_n) = \{(-1)^n | n \in \mathbb{N}\}$. Then

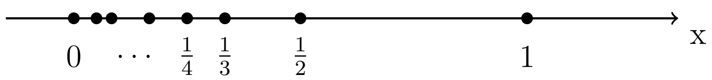
$$S \supseteq \{1, -1\}$$

Definition. Let (x_n) be a sequence of real numbers.

- The ***limit superior*** of (x_n) is the infimum of the set of $v \in \mathbb{R}$ s.t. $v < x_n$ for at most a finite number of $n \in \mathbb{N}$. We write it as

  $$\limsup(x_n) = \limsup x_n = \overline{\lim} x_n = \inf \{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of n}\}$$

  Example. Consider $(x_n) = \frac{1}{n}$

  Let

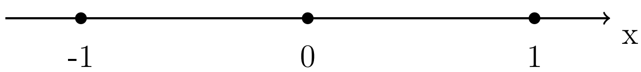
  $$X = \{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of n}\}$$

  - $-1 \notin X$ because there are infinitely many x_n such that $v < x_n$.
  - $\frac{1}{2} \in X$ becasue there are finitely many x_n such that $v < x_n$.
  - $2 \in X$ because there is no x_n such that $v < x_n$, which is smaller than finite and thereby satisfies the definition.

  We thus conclude that

  $$(0, \infty) \subset X \text{ and } \limsup x_n = \inf X$$

  Example. Consider $(x_n) = (-1)^n$:

  - $1 \in X$ because there is no x_n such that $v < x_n$.
  - $2 \in X$ because there is no x_n such that $v < x_n$.
  - $0, -1 \notin X$ becasue there are infinitely many x_n such that $v < x_n$.

  We thus conclude that

  $$[1, \infty) \subset X \text{ (in fact they are equal)}$$

- The ***limit inferior*** of (x_n) is the supremum of the set of $w \in \mathbb{R}$ s.t. $w > x_m$ for at most a finite number of $n \in \mathbb{N}$. We write it as

  $\liminf(x_n)$ or $\liminf x_n$ or $\overline{\lim} x_n = \sup \{w \in \mathbb{R} | w > x_n$ for at most a finite number of n$\}$

Intuition

- Suppose $v < x_n$ for at most finitely many $n \in \mathbb{N}$, then for all large n, $v \geq x_n$.
  $\Rightarrow$ No subsequential limit of (x_n) can possibly exceed v.
- Similar observation for $\underline{\lim}\, x_n$

Theorem. Let (x_n) be a bounded sequence of real numbers, and let $x^* \in \mathbb{R}$. Then TFAE:

1. $x^* = \limsup(x_n)$
2. If $\epsilon > 0$, there are at most a finite number of $n \in \mathbb{N}$ s.t. $x^* + \epsilon < x_n$, but infinitely many n for which $x^* - \epsilon < x_n$
3. If $u_m = \sup \{x_n | n \geq m\}$(sup of (m-1)-th tail), then $x* = \inf \{u_m | m \in \mathbb{N}\} = \lim u_m$
4. If S is the set of subsequential limits of x_n, then $x^* = \sup S$.

Remark. .

- u_m is decreasing.
- There is a similar such list of equivalent properties for $\liminf$.

Corollary. A bounded sequence (x_n) is convergent iff $\overline{\lim} x_n = \lim x_n$

Proof. A direct result of the theorem:

$$\overline{\lim} x_n = \sup S \text{ and } \underline{\lim} x_n = \inf S$$

■

Proof of thm. (a) $\Rightarrow$ (b). Let $\epsilon > 0$. Then

$$x^* + \epsilon > x^* = X = \inf \{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of n.}\}$$

$$\Rightarrow \exists v \in \mathbb{R} \text{ s.t. } x^* \leq v < x^* + \epsilon$$

and there are only finitely mant n with $v < x_n$.

For any n for which $x^* + \epsilon < x_n$, $v < x_n$. Thus there are only finitely many such n.

If $x^* - \epsilon \notin X$, then there are infinityly many n such that $x^* - \epsilon < x_n$ ■

Proof. ■

Proof of thm. (b) ⇒ (c). Fix $\epsilon > 0$.

By (b), there are only finitely many n with $x^* + \epsilon < x$.

Take $N \in \mathbb{N}$ large enough such that

$$\begin{aligned} & x^* + \epsilon \geq x_n && \forall n \geq N \\ \Rightarrow \quad & x^* + \epsilon \geq u_N \\ \Rightarrow \quad & x^* + \epsilon \geq \lim u_n \\ \Rightarrow \quad & x^* \geq \lim u_m && \forall n \geq N \end{aligned}$$

On the other hand, there are infinitely many n with $x^* - \epsilon < x_n \leq u_n$. Thus, there exists a subsequence o

$$x^* - \epsilon \leq u_{n_k}$$

■