Math265 Real Analysis Class Notes

Based on lectures by Prof. Huang

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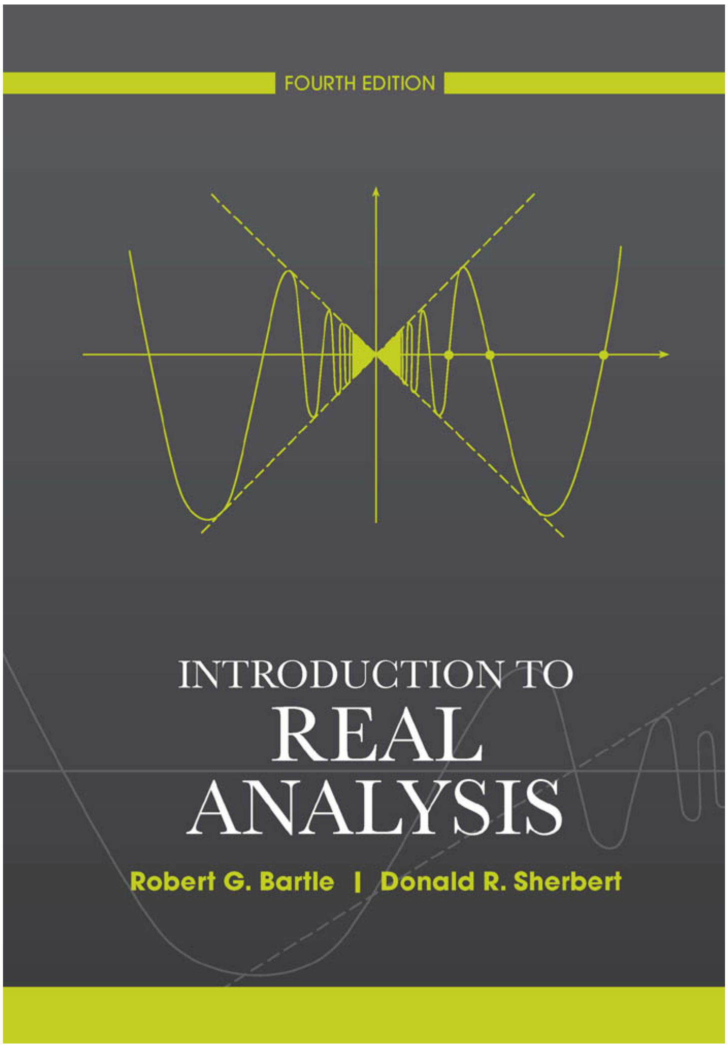


1 Preliminaries

1.1 Sets and Functions

Definition (*Image and Inverse Image*). Let $A, B \subseteq \mathbb{R},\ f : A \to B$.

If $E \subseteq A$, then the ***image*** of E is the set

$$f(E) = \{f(x) : x \in E\} \subseteq B$$

If $H \subseteq B$, the ***inverse image*** of H is the set

$$f^{-1}(H) = \{x \in A : f(x) \in H\} \subseteq A$$

Definition. Let $A, B \subseteq \mathbb{R},\ f : A \to B$.

- f is ***injective*** (or ***one-to-one***) if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

- f is ***surjective*** (or ***onto***) if

$$f(A) = B$$

- f is ***bijective*** if it is both injective and surjective

Definition (*Inverse Function*).

Let $A, B \subseteq \mathbb{R},\ f : A \to B$. If f is bijective, then we can define a function $g : B \Rightarrow A$ such that

$$g(y) = x \text{ if and only if(iff or } \Leftrightarrow \text{) } f(x) = y$$

g is called ***inverse function*** of f and denoted by $g = f^{-1}$

Definition. Let $A, B \subseteq \mathbb{R},\ f : A \to B,\ A_1 \subseteq A$. The ***restriction of f to A_1*** is the function $f|_{A_1} : A_1 \to B$ defined by

$$f|_{A_1}(x) = f(x) \text{ for all } x \in A_1$$

Example. Let $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$. Notice that f is not injective. However,

$$f|_{[0,\infty)} : [0, \infty) \to \mathbb{R}$$

is injective.

Proof. Suppose there are $x_1,\ x_2 \geq 0$ such that $f|_{[0,\infty)}(x_1) = f|_{[0,\infty)}(x_2)$. Then

$$x_1^2 = x_2^2$$

$$(x_1 + x_2)(x_1 - x_2) = 0$$

Thus $x_1 = -x_2$ or $x_1 = x_2$. Since x_1, x_2 are positive,

$$x_1 = x_2 = 0 \text{ or } x_1 = x_2$$

Hence, $x_1 = x_2$ and we conclude that $f|_{[0,\infty)}$ is injective. ■

Definition (***Relation***)**.** Let $A, B \subseteq \mathbb{R}$. A ***relation*** R from A to B is a subset of $R \subseteq A \times B$. We write aRb or $a \sim b$ to express $(a, b) \in R$

Example.

- Equality relation on $\mathbb{R}$, denoted by $=$, is defined by

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = y\} = \{(x, x) : x \in \mathbb{R}\}$$

Definition (***Equivalence Relation***)**.** Let $A \subseteq \mathbb{R}$. A relation R from A to A is an ***equivalent relation*** if it satisfies all of following:

1. $\forall a \in A,\ aRa$ (*Reflexivity*)
2. If for $a, b \in A,\ aRb$, then bRa. (*Symmetry*)
3. If for $a, b, c \in A,\ aRb$ and bRc, then aRc. (*Transitivity*)

Definition (***Equivalence Class***)**.** Let $A \subseteq \mathbb{R}$ with an equivalent relation $\sim$. An ***equivalence class*** under $\sim$ is a subset $B \subseteq A$ such that for some $a \in A$

$$\forall b \in B,\ a \sim b \Leftrightarrow a \in B$$

Example. Define relation R by $aRb \Leftrightarrow 3|a - b$. Then R is an equivalent relation(Try to prove it!)

Define $3\mathbb{Z} = \{n \in \mathbb{Z} : 3|n\} \subset \mathbb{Z}$. Then $3\mathbb{Z}$ is an equivalence class under R

Proof. Let $a \in 3\mathbb{Z}, b \in \mathbb{Z}$. Suppose aRb, then

$$3|a - b$$

$$a - b = 3n \text{ for some } n \in \mathbb{N}$$

$$b = a - 3n$$

Since $a \in 3\mathbb{Z}$, $a = 3m$ for some $m \in \mathbb{N}$. Thus,

$$b = 3m - 3n = 3(m - n) \in 3\mathbb{Z}$$

■

Proposition. Let S be a non-empty set, and $\sim$ be an equivalent relation on S. Then for any distinct equivalent classes A, B under R,

$$A \cap B = \phi$$

that is, equivalence classes of an equivalence relation are disjoints.

Proof. Let A, B be two distinct equivalence classes under $\sim$ in S

Assume $\exists c \in A \cap B$, then $\forall a \in A,\ a \sim c$ and $\forall b \in B,\ b \sim c$. Thus by transitivity,

$$\forall a \in A,\ b \in B,\ a \sim b$$

Thus

$$A \subseteq B \text{ and } B \subseteq A$$

$$A = B$$

which contradicts to the fact that $A \neq B$. ■

Proposition. Let S be a non-empty set, and $\sim$ be an equivalent relation on S. For all element $x \in A$, x is confined in a unique equivalence class denoted by $[x]_\sim$

Proof. Exercise. ■

Definition (*Quotient*). Let S be a non-empty set, and $\sim$ be an equivalent relation on S. The ***quotient*** of S under $\sim$ is the set $S/\sim$ defined by

$$S/\sim = \{[x]_\sim : x \in S\}$$

Example. $\mathbb{Z}/\sim = \{3\mathbb{Z},\ 1+3\mathbb{Z},\ 2+3\mathbb{Z}\} = \{\dots, [-1]_\sim,\ [0]_\sim,\ [1]_\sim,\ \dots\}$

1.2 Mathematical Induction

The reader should consult pages $12 - 15$ on the textbook.

1.3 Finite and Infinite Sets

Definition (***Cardinality***)**.** The number of elements a set S have is called the ***cardinality of*** S.

- The empty set ϕ has 0 elements, called ***cardinality 0***, denoted by

$$|\phi| = \#\phi = card(\phi) = 0$$

- Let $n \in \mathbb{N}$, a set $S \in \mathbb{R}$ has n elements, called ***cardinality n*** denoted by

$$|S| = \#S = card(S) = n$$

  if there exists a bijection

$$f : \mathbb{N}_n \to S$$

Definition. Let S be any set.

- S is said to be ***finite*** if it is empty or has cardinality n for some $n \in \mathbb{N}$.
- S is said to be ***infinite*** if it is not finite.

Theorem (***Uniqueness Theorem***)**.** If S is a finite set, then the number of elements in S is a unique number in $\mathbb{N}$.

Theorem. The set $\mathbb{N}$ of natural numbers is an infinite set.

Proof. Assume $\mathbb{N}$ is finite. Then there exists a natural number $n \in \mathbb{N}$ such that

$$|\mathbb{N}| = n$$

However, if we choose the first $n+1$ elements in $\mathbb{N}$, then

$$\{1,\ 2,\ \dots,\ n,\ n+1\} \subset \mathbb{N}$$

$$|\{1,\ 2,\ \dots,\ n,\ n+1\}| = n+1 < |\mathbb{N}| = n$$

which is a contradiction. ■

Theorem.

(a) Let A, B be sets and $\exists m, n \in \mathbb{N}$ s.t. $|A| = m,\ |B| = n,\ A \cap B = \phi$, then

$$|A \cup B| = m + n$$

(b) Let A be a set with $|A| = m$ for some $m \in \mathbb{N}$ and $C \subseteq A$ be a set with $|C| = 1$, then

$$|A \backslash C| = m - 1$$

(c) If C is an infinite set and B is a finite set, then

$$C \backslash B \text{ is an infinite set}$$

Proof. **(a)** By definition of cardinality, there exists two bijective function

$$f : \mathbb{N}_m \to A,\ g : \mathbb{N}_n \to B$$

We define $h : \mathbb{N}_{n+m} \to A \cup B$ such that

$$h(x) = \begin{cases} f(x) & \text{if } x \leq m \\ g(x-m) & \text{if } x > m \end{cases}$$

- ***Claim***: h is injective.

  Suppose $\exists x_1, x_2 \in \mathbb{N}_{n+m}$ s.t. $h(x_1) = h(x_2)$.

  Case 1: $x_1, x_2 \leq m$. Then $f(x_1) = f(x_1) \Rightarrow x_1 = x_2$ since f is injective.

  Case 2: $x_1 \leq m, x_2 > m$. Then $f(x_1) = g(x_2 - m)$ contradiction. Impossible.

  Case 3: $x_1, x_2 > m$. Then $g(x_1 - m) = g(x_2 - m) \Rightarrow x_1 = x_2$ since g is injective.

  We conclude that h is injective

- ***Claim***: h is surjective.

  Let $y \in A \cup B$.

  Case 1: $y \in A$. Then $\exists x \in \mathbb{N}_m \subseteq \mathbb{N}_{n+m}$ s.t. $f(x) = y$ since f is onto.

  Case 2: $y \in B$. Then $\exists x \in \mathbb{N}_n \subseteq \mathbb{N}_{n+m}$ s.t. $g(x) = y$ since f is onto.

  Since $x + m \in \mathbb{N}_{m+n}$ and $x + m > m$,

  $$h(x+m) = g(x+m-m) = g(x) = y$$

  We conclude that h is surjective.

(b)(c): *Exercise.* ■

Theorem. Let S, T be set such that $S \subseteq T$. Then

1. If T is finite, then S is finite.
2. If S is infinite, then T is infinite

Proof. **(a)** Prove by induction on cardinality of T.

For $|T| = 0$, $T = \phi \Rightarrow S = \phi$. Thus S is finite for $|T| = 0$

Suppose that the claim holds for all $S \subseteq T$ where $|T| = k$.

If $|T| = k + 1$, there are two cases:

Case 1: If $S = T$, then S is finite.

Case 2: $\exists t \in T$ s.t. $t \notin S$, then

$$|T \backslash \{t\}| = k \text{ and } S \subseteq T \backslash \{t\}$$

Then S is finite by induction hypothesis. Thus, we can prove for the rest of cases by induction.

(b) This assertion is the contrapositive of (a).

■

Definition.

- A set is said to be ***countably infinite*** if there exists a bijection of $\mathbb{N}$ onto S.
- A set is said to be ***countable*** if it is either finite or countably infinite.
- A set is said to be ***uncountable*** if it is not countable.

Example.

- $\mathbb{N}$ is countably infinite.
- $S = \{2, 3, 4, \dots\} \subseteq \mathbb{N}$ is countably infinite.
- The set of even natural numbers is countably infinite.
- $\mathbb{Z}$ is countably infinite.

Theorem. If A, B are both countably infinite, then $A \cup B$ is countably infinite.

Proof. By definition of countably infinity, $\exists f : \mathbb{N} \to A,\ g : \mathbb{N} \to B$ bijective.

Define $h : \mathbb{N} \to A \cup B$ by

$$h(n) = \begin{cases} f(\dfrac{n+1}{2}) & \text{if } n \text{ is odd} \\ g\left(\dfrac{n}{2}\right) & \text{if } n \text{ is even} \end{cases}$$

It suffices to show that h is surjective.

Let $x \in A \cup B$. If $x \in A$, then since f is surjective, $\exists n \in \mathbb{N}$ s.t. $f(n) = x$.

Consider $m = 2n - 1$. Then

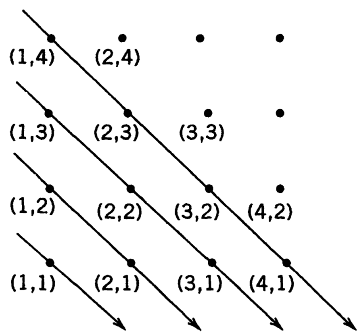
$$h(m) = f(\frac{m+1}{2}) = f(n) = x$$

We leave the rest of proof as exercise. ■

Theorem. The set $\mathbb{N} \times \mathbb{N}$ is countably infinite.

Proof. $\forall m, n \in \mathbb{N}$, we can enumerate these (m, n) pairs as

$$(1,1),\ (1,2),(2,1),\ (1,3),\ (2,2),\ (3,1),\ (1,4),\ldots$$



We see that the total number of points in diagonals 1 through k is given by$\psi : \mathbb{N} \to \mathbb{N}$

$$\psi(k) = \begin{cases} 0 & \text{if } k = 0 \\ \dfrac{1}{2}k(k+1) & \text{if } k \in \mathbb{N} \end{cases}$$

We define $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$\begin{aligned} h(m,n) &= \psi(m+n-2) + m \\ &= \frac{1}{2}(m+n-2)(m+n-1) + m \end{aligned}$$

Claim: h is a bijection. See Appendix B. ■

Theorem. The following statement are equivalent (Abbv.: TFAE):

(a) S is a countable set.

(b) There exists a surjection of $\mathbb{N}$ onto S.

(c) There exists an injection of S onto $\mathbb{N}$

Proof.

- (a)$\Rightarrow$(b): Suppose S is a countable set.
  - If S is finite, then $\exists f : \mathbb{N}_n \to S$ a bijection for some $n \in \mathbb{N}$, and $\exists g : \mathbb{N} \to \mathbb{N}_n$ s.t. $\forall m \in \mathbb{N},\ g(m) = m \mod n$. Thus g is onto. We have
  $$f \circ g : \mathbb{N} \to S \text{ is onto by exercise in homework}$$
  - If S is countably infinite, then $\exists f : \mathbb{N} \to S$ onto.
- (b)$\Rightarrow$(c): Suppose $\exists f : \mathbb{N} \to S$ onto. Define $g : S \to \mathbb{N}$ by
  $$g(s) = \min\{n \in \mathbb{N} : f(n) = s\}$$
  Suppose $\exists s, t \in S$ s.t. $g(s) = g(t) = n$ for some $n \in \mathbb{N}$. We have
  $$f(g(s)) = s \text{ and } f(g(t)) = t$$
  $$s = t$$
  Thus g is injective.
- (c)$\Rightarrow$(a): Suppose $\exists f : S \to \mathbb{N}$ injective. Then $\exists \hat{f} : S \to f(S)$ a bijection. Since $f(S) \subseteq \mathbb{N}$, $f(S)$ is countable by the previous theorem. Hence
  $$\exists h : \mathbb{N} \to f(s)$$
  It follows that
  $$\hat{f} \circ h : \mathbb{N} \to S$$
  is a bijection.

■

Theorem. $\mathbb{Q}$ is ***countable***.

Proof. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}_+$ where $\mathbb{Q}_+ = \{x \in \mathbb{Q} : x > 0\}$ by

$$f(m,n) = \frac{n}{m}$$

It is easy to see that f is surjective(Try to prove it!).

Since $\mathbb{N} \times \mathbb{N}$ is countable, $\exists g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ a bijection.

We compose f and g to get:

$$f \circ g : \mathbb{N} \to \mathbb{Q}_+$$

Thus $\mathbb{Q}_+$ is countable by the previous theorem.

Similarly, we can prove that $\mathbb{Q}_- = \{x \in \mathbb{Q} : x < 0\}$ is countable. Thus, $\mathbb{Q} = \mathbb{Q}_- \cup \mathbb{Q}_+ \cup \{0\}$ is countable by the previous theorem.

■

Theorem (*union of countable sets is countable*). If A_n is a countable set for $n \in \mathbb{N}$, then the union

$$A = \bigcup_{n=1}^{\infty} A_n$$

is countable. That is, the union of countable sets is countable.

Proof. $\forall n \in \mathbb{N}$, $\exists \psi_n : \mathbb{N} \to A_n$ a bijection.

Define $f : \mathbb{N} \times \mathbb{N} \to A$ by

$$f(m,n) = \psi_m(n)$$

Then f is surjective (Try to prove it!).

Since $\mathbb{N} \times \mathbb{N}$ is countable, by previous theorem, $\exists g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ a bijection. Thus

$$f \circ g : \mathbb{N} \to A$$

We conclude that by definition, A is countable.

■

Theorem (*Cantor's Theorem*). If A is a set, there is no surjection of A onto the power set of A.

Proof. Assume $\exists \psi : A \to P(A)$. For any element $a \in A$, $\psi(a) \subseteq A$. Thus either $a \in \psi(a)$ or a does not belong to this set. We define

$$D = \{a \in A : a \neq \psi(a)\}$$

Since $D \subseteq A$, if ψ is a surjection, then $\exists a_0 \in A$ s.t. $D = \psi(a_0)$.

- If $a_0 \in D$, then since $D = \psi(a_0)$, we must have $a_0 \in \psi(a_0)$ a contradiction to definition of D.
- Similarly, if $a_0 \notin \psi(a_0)$, then $a_0 \notin \psi(a_0)$ so that $a_0 \in D$, which is also a contradiction.

We thereby conclude that ψ cannot be a surjection. ■

Remark. This implies that the power set of natural number $\mathcal{P}(\mathbb{N})$ is uncountable.

2 The Real Numbers

2.1 The Algebraic and Order Properties of $\mathbb{R}$

Axiom (***Algebraic Property of*** $\mathbb{R}$)**.** On the set R of real numbers there are two binary operations, denoted by $+$ and $\cdot$ called ***addition*** and ***multiplication***, respectively. These operations satisfy the following properties.

Axioms for Addition

(A1) For all $a, b \in \mathbb{R},\ a + b = b + a$ (*commutative property of addition*)

(A2) For all $a, b, c \in \mathbb{R},\ (a + b) + c = a + (b + c)$ (*associative property of addition*)

(A3) There exists an element $0 \in \mathbb{R}$ such that $0 + a = a + 0 = a$ for all $a \in \mathbb{R}$ (*existence of zero element (additive identity)*)

(A4) For all $a \in \mathbb{R}$, there exists an element $-a \in \mathbb{R}$ s.t. $a + (-a) = (-a) + a = 0$ (*existence of negative element (additive inverse)*)

Axioms for Multiplication

(M1) For all $a, b \in \mathbb{R},\ a \cdot b = b \cdot a$ (*commutative property of multiplication*)

(M2) For all $a, b, c \in \mathbb{R},\ (a \cdot b) \cdot c = a \cdot (b \cdot c)$ (*associative property of multiplication*)

(M3) There exists an element $1 \in \mathbb{R},\ 1 \neq 0,$ s.t. $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{R}$ (*existence of unit element (multiplicative identity)*)

(M4) For all $a \in \mathbb{R},\ a \neq 0$, there exists an element $\frac{1}{a} \in \mathbb{R}$ s.t. $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$ (*existence of reciprocals (multiplicative inverse)*)

Distributive Law

(D) For all $a, b, c \in \mathbb{R},\ a \cdot (b + c) = a \cdot b + a \cdot c$ (*distributive law of multiplication over addition*)

Theorem.

(a) If $z, a \in \mathbb{R}$ with $z + a = a$, then $z = 0$.

(b) If $u, b \in \mathbb{R},\ u, b \neq 0$ with $u \cdot b = b$, then $u = 0$.

(c) If $a \in \mathbb{R}$, then $a \cdot 0 = 0$.

Proof. **(a)** Axiom (A3), (A4), (A2) gives

$$z = z + 0 = z + (a + (-a)) = (z + a) + (-a) = a + (-a) = 0$$

(b) Axiom (M3), (M4), (M2) gives

$$u = u \cdot 1 = u \cdot (b \cdot \frac{1}{b}) = (u \cdot b) \cdot \frac{1}{b} = b \cdot \frac{1}{b} = 1$$

(c) Axiom (M3), (D), (A3), theorem (a) gives

$$a + a \cdot 0 = a \cdot 1 + a \cdot 0 = a \cdot (1 + 0) = a \cdot 1 = a$$

$$a \cdot 0 = 0$$

∎

Theorem.

(a) If $a, b \in \mathbb{R},\ a \neq 0$ with $a \cdot b = 1$, then $b = \frac{1}{a}$.

(b) If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

Proof. **(a)** Axiom (M3), (M4), (M2), $a \cdot b = 1$, (M3) gives

$$b = 1 \cdot b = (\frac{1}{a} \cdot a) \cdot b = \frac{1}{a} \cdot (a \cdot b) = \frac{1}{a} \cdot 1 = \frac{1}{a}$$

(b) It suffices to assume $a \neq 0$ and prove $b = 0$. (M2), (M4), (M3), and previous theorem(c) gives

$$b = 1 \cdot b = (\frac{1}{a} \cdot a) \cdot b = \frac{1}{a} \cdot (a \cdot b) = \frac{1}{a} \cdot 0 = 0$$

■

Definition.

- We define $-$, called ***substraction***, as $a - b = a + (-b)$ for all $a, b \in \mathbb{R}$.
- We define $\div$, called ***division***, as $a \div b = a \cdot b = \frac{a}{b}$ for all $a, b \in \mathbb{R}$.
- We define a^n, called ***natural power***, as $a^n = (((a \cdot a) \cdot a) \cdot \ldots) \cdot a$ (n-times) for all $a \in \mathbb{R},\ n \in \mathbb{N}$.

Theorem ($\sqrt{2}$ ***Is Irrational Number***)**.** There is no rational number $r \in \mathbb{Q}$ such that $r^2 = 2$.

Proof. We will prove this by ***contradiction***.

Assume there are $p, q \in \mathbb{Z}$ satisfying $(\frac{p}{q})^2 = 2$. Without loss of generality, assume $p, q > 0$ and $\gcd(p, q) = 1$. Then

$$p^2 = 2q^2$$

So p^2 is even. This implies that p is also even(try to prove it!). Let $p = 2m$ for some $m \in \mathbb{Z}$. Then

$$p^2 = 4m^2 = 2q^2$$
$$2m^2 = q^2$$

By the same argument, if q^2 is even, then q is even. Since p, q are even, $\gcd(p, q)$ is at least 2, which contradicts our condition.

■

Axiom (***Order of*** $\mathbb{R}$)**.** There is a non-empty set $\mathbb{P} \subset \mathbb{R}$ called the set of ***positive real numbers*** that satisfies the following properties:

1. If $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$.
2. If $a, b \in \mathbb{P}$, then $a \cdot b \in \mathbb{P}$.
3. If $a \in \mathbb{R}$, then one and only one of following holds:

$$a \in \mathbb{P}, \ a = 0, \ -a \in \mathbb{P}$$

Note: for all $a \in \mathbb{R}$

- we write $a > 0$ and say a is ***positive*** if $a \in \mathbb{P}$,
- we write $a \geq 0$ and say a is ***non-negative*** if $a \in \mathbb{P} \cup \{0\}$.
- we write $a < 0$ and say a is ***negative*** if $a \in \mathbb{P}$,
- we write $a \leq 0$ and say a is ***non-positive*** if $-a \in \mathbb{P} \cup \{0\}$.

Definition. Let $a, b \in \mathbb{R}$.

1. If $a - b \in \mathbb{P}$, we denote that $a > b$ or $b < a$.
2. If $a - b \in \mathbb{P} \cup \{0\}$, we denote that $a \geq b$ or $b \leq a$.

Note: For all $a, b \in \mathbb{R}$, there is one and only one of following holds:

$$a < b, \ a = b, \ a > b$$

And we call them a is ***greater than***, ***equal to***, or ***less than*** b respectively.

Theorem. Let $a, b, c \in \mathbb{R}$.

1. If $a > b$ and $b > c$, then $a > c$.
2. If $a > b$, then $a + c > b + c$.
3. If $a > b$ and $c > 0$, then $ca > cb$.
   If $a > b$ and $c < 0$, than $ca < cb$.

Proof.

1.By definition, $a - b \in \mathbb{P}$ and $b - c \in \mathbb{P}$. Then

$$(a - b) + (b - c) = a - c \in \mathbb{P} \Rightarrow a > c$$

2. By definition, $a - b \in \mathbb{P}$, then

$$(a + c) - (b + c) \in \mathbb{P} \Rightarrow a + c > b + c$$

3. By dfinition, $a - b \in \mathbb{P}$, then

$$\text{If } c > 0, \ c \in \mathbb{P}, \ ca - cb = c(a - b) \in \mathbb{P} \Rightarrow ca > cb$$

$$\text{If } c < 0, \ -c \in \mathbb{P}, \ cb - ca = (-c)(a - b) \in \mathbb{P} \Rightarrow cb > ca$$

∎

Theorem.

1. If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$.
2. 1>0
3. If $n \in \mathbb{N}$, then $n > 0$.

Proof.

1. Since $a \neq 0$, there are only two cases:
   - If $a \in \mathbb{P}$, then $a^2 \in \mathbb{P}$.
   - If $-a \in \mathbb{P}$, then $(-a)(-a) = a^2 \in \mathbb{P}$. Proof of $(-a)(-a) = a^2$ is exercise
2. $1 \in \mathbb{P}$, so $1^2 \in \mathbb{P}$ by (1)
3. For all $n \in \mathbb{N}$, $n = (((1+1)+1)+\cdots+1)+1$. Thus, this is true by induction.

■

Theorem. If $a \in \mathbb{R},\ 0 < a < \varepsilon$ for all $\varepsilon > 0$, then $a = 0$.

Proof. Assume that $a > 0$. Then if we take $\varepsilon_0 = \dfrac{1}{2}a$, we have $0 < \varepsilon_0 < a$ a contradiction. Thus we conclude that $a = 0$. ■

Theorem. If $ab > 0$, then either

- $a > 0$ and $b > 0$, or
- $a < 0$ and $b < 0$

Proof. Exercise. ■

Corollary. If $ab < 0$, then either

- $a > 0$ and $b < 0$, or
- $a < 0$ and $b > 0$

Proof. Exercise. ■

Proposition. Let $a \geq 0,\ b \geq 0$. Then

$$a < b \text{ iff } a^2 < b^2 \text{ iff } \sqrt{a} < \sqrt{b}$$

Proof.

- If $a = 0$, then it holds.
- If $a > 0$, then
  $$b^2 - a^2 = (b+a)(b-a) > 0 \Leftrightarrow b - a > 0$$
  And $\sqrt{a} < \sqrt{b} \Leftrightarrow a < b$ is a consequence of $a < b \Leftrightarrow a^2 < b^2$

■

Theorem (***Arithmetic-geometric Mean Inequality***)**.** For all $a, b \in \mathbb{R},\ a, b \geq 0$, $\sqrt{ab} \leq \dfrac{a+b}{2}$. Moreover, the inequality holds iff $a = b$.

Proof. If $a \neq b$, $\sqrt{a} \neq \sqrt{b}$, then

$$\begin{aligned}(\sqrt{a} - \sqrt{b})^2 &> 0\\ a + b - 2\sqrt{ab} &> 0\\ \sqrt{ab} &< \frac{a+b}{2}\end{aligned}$$

($\Leftarrow$) If $a = b$, then the equality holds.

($\Rightarrow$) If $\sqrt{ab} < \dfrac{a+b}{2}$, then we reverse the previous working,

$$(\sqrt{a} - \sqrt{b})^2 = 0$$

$$a = b$$

■

Theorem (***Bernoulli's Inequality***)**.** If $x > -1$, then for all $n \in \mathbb{N}$,

$$(1 + x)^n \geq 1 + nx$$

Proof. We will prove this by mathematical induction.

- $n = 1$, then $1 + x \geq 1 + x$, so $P(1)$ holds.
- Assume $P(k)$ holds for all $k \in \mathbb{N}$. We want to show that $P(k + 1)$ holds.

$$\begin{aligned}(1 + x)^{k+1} &= (1 + x)^k(1 + x)\\ &\geq (1 + kx)(1 + x) = 1 + (k + 1)x + kx^2\\ &\geq 1 + (k + 1)x\end{aligned}$$

  Thus $P(k + 1)$ holds.

We conclude that $P(n)$ holds for all $n \in \mathbb{N}$. ■

2.2 Absolute Value and the Real Line

Definition. The ***absolute value*** of a real number a, denoted by $|a|$, is defined by

$$|a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } a < 0 \\ 0 & \text{if } a = 0 \end{cases}$$

Theorem.

1. $|ab| = |a| \cdot |b|$ for all $a, b \in \mathbb{R}$.
2. $|a|^2 = |a^2|$ for all $a \in \mathbb{R}$.
3. If $c \in \mathbb{R},\ c \geq 0$, then
$$|a| \leq c \text{ iff } -c \leq a \leq c$$
4. $-|a| \leq a \leq |a|$.

Proof. Exercise. ■

Theorem (*Triangle Inequality*). Let $a, b \in \mathbb{R}$. Then

$$|a + b| \leq |a| + |b|$$

Proof. By previous theorem(4), we have

$$-|a| \leq a \leq |a| \text{ and } -|b| \leq b \leq |b|$$

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

Thus, by previous theorem(3),

$$|a + b| \leq |a| + |b|$$

■

Remark. $ab \geq 0$ iff $|a + b| \leq |a| + |b|$

Proof. Exercise. ■

Corollary. Let $a, b \in \mathbb{R}$. Then

1. $||a| - |b|| \leq |a - b|$.
2. $|a - b| \leq |a| + |b|$.

Proof.

1. We write $a = a - b + b$. By triangle inequality,
$$|a| = |(a - b) + b| \leq |a - b| + |b|$$
$$|a| - |b| \leq |a - b|$$
Similarly
$$|b| - |a| \leq |a - b|$$
We thereby conclude that (1) holds.

2. We "put $-b$ in b" of triangle inequality:

$$|a-b| \leq |a| + |b|$$

$\blacksquare$

Corollary. If $a_1,\ a_2,\ \ldots,\ a_n \in \mathbb{R}$, then

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

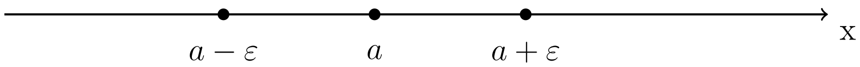
Proof.

$$\begin{aligned} |a_1 + a_2 + \cdots + a_n| &\leq |a_1 + a_2 + \cdots + a_{n-1}| + |a_n| \\ &\leq |a_1 + a_2 + \cdots + a_{n-2}| + |a_{n-1}| + |a_n| \\ &\cdots \\ &\leq |a_1| + |a_2| + \cdots + |a_n| \end{aligned}$$

$\blacksquare$

Definition. Let $a \in \mathbb{R}$, $\varepsilon > 0$. Then the ***ε-neighborhood of a*** is the set

$$V_\varepsilon(a) = \{x \in \mathbb{R} |\, |x-a| < \varepsilon\}$$



Theorem. Let $a \in \mathbb{R}$. If $\forall \varepsilon > 0,\ x \in V_\varepsilon(a)$, then $x = a$.

Proof. By definition of neighborhood, $\forall \epsilon > 0, |x-a| < \varepsilon$. Thus by previous theorem,

$$|x-a| = 0 \Rightarrow x - a = 0 \Rightarrow x = a$$

$\blacksquare$

Proposition. Let $a, b \in \mathbb{R}$ with $a < b$. Consider the ***open interval***

$$(a,b) = \{x | a < x < b\}$$

Then for all $x \in (a,b),\ \exists \varepsilon > 0$ s.t.

$$V_\varepsilon(x) \subseteq (a,b)$$

Proof. Choose $\varepsilon = \min\{|x-a|\,,\ |x-b|\}$. Suppose $y \in V_\varepsilon(x)$, then

$$|x-y| < \varepsilon$$

$$x - \varepsilon < y < x + \varepsilon$$

So it is either

- $-a < -b < y < b$ if $a > b$, or
- $-b < -a < y < a$ if $a < b$

We conclude that $y \in (a,b)$ $\blacksquare$

Proposition. Let $a, b \in \mathbb{R}$ with $a < b$. Consider the ***closed interval***

$$[a, b] = \{x | a \leq x \leq b\}$$

Then for all $\varepsilon > 0$,

$$V_\varepsilon(x) \nsubseteq (a, b)$$

Proof. Exercise. ■

Proposition. Let $x, y, a, b \in \mathbb{R}$. If $x \in V_\varepsilon(a),\ y \in V_\varepsilon(b)$, then

$$x + y \in V_{2\varepsilon}(a + b)$$

Proof. By definition of neighborhood,

$$\begin{aligned} |(x + y) - (a + b)| &= |(x - a) + (y - b)| \\ &\leq |x - a| + |y - b| \\ &< \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

Thus $x + y \in V_{2\varepsilon}(a + b)$. ■

2.3 The Completeness Property of $\mathbb{R}$

Definition. Let $S \subseteq \mathbb{R}$ be a nonempty.

1. S is said to be ***boudned above*** if there exists a number $u \in \mathbb{R}$ such that $s \leq u$ for all $s \in S$. We say u is an ***upper bound*** of S.
2. S is said to be ***boudned below*** if there exists a number $u \in \mathbb{R}$ such that $u \geq s$ for all $s \in S$. We say u is an ***lower bound*** of S.
3. S is said to be ***boudned*** if it is both bounded above and below.
4. We say S is ***unbounded*** if it is not bounded above or bounded below.

Definition. Let $S \subseteq \mathbb{R}$ be nonempty.

1. If S is bounded above, then u is the ***supremum*** or ***least upper bound*** of S if
   - u is an upper bound, and
   - If v is an upper bound of S, then $u \leq v$.
2. If S is bounded below, then w is the ***infimum*** or ***greatest lower bound*** of S if
   - w is an lower bound, and
   - If t is an lower bound of S, then $t \leq w$.

Supremum and infimum are denoted by

$$\sup S, \ \inf S$$

Proposition. If S is bounded above, then $\sup S$ is ***unique***.

Proof. Suppose $u_1,\ u_2$ are supremum of S. Since u_1 is the supremum of S and u_2 is the upper bound of S,

$$u_1 \leq u_2$$

Similarly, we have

$$u_2 \leq u_1$$

Thus

$$u_1 = u_2$$

■

Theorem. Let $S \subseteq \mathbb{R}$ be nonempty. If $\exists u = \sup S$, then the following statements are equivalent:

1. If v is an upper bound of S, then $u \leq v$.
2. If $z < u$, then z is not an upper bound of S.
3. If $z < u$, then there exists $s_z \in S$ such that $z < s_z$.
4. If $\varepsilon > 0$, then there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

Proof. Exercise. ■

Lemma (*Alternative Definition*). Let $S \subseteq \mathbb{R}$ be nonempty. $\exists u = \sup S$ iff

- $s \leq u$ for all $a \in S$.
- If $v < u$, then there exists $s' \in S$ such that $v < s'$.

Proof. Exercise. ■

Lemma (ε *Definition*). An upper bound u of nonempty set $S \subseteq \mathbb{R}$ is the supremum of S iff

$\forall \epsilon > 0,\ \exists s_\epsilon \in S$ s.t.

$$u - \varepsilon < s_\varepsilon$$

Proof. ($\Rightarrow$) Suppose $u = \sup S$. $\forall \varepsilon > 0, u - \varepsilon$ is NOT an upper bound of S. Thus

$$\exists s_\varepsilon \in S \text{ s.t. } u - \varepsilon < s_\varepsilon$$

($\Leftarrow$) Suppose u is an upper bound satisfying: $\forall \epsilon > 0,\ \exists s_\epsilon \in S$ s.t. $u - \varepsilon < s_\varepsilon$.

Let v be an upper bound of S.

Goal: $u \leq v$

Assume $u > v$. Let $\varepsilon = u - v > 0$. Then $\exists s_\varepsilon \in S$ s.t. $u - \varepsilon < s_\varepsilon$. Thus $u - \varepsilon$ is NOT an upper bound of S, which is a contradiction.

We thereby conclude that $u \leq v$.

■

Remark (*Maximum v.s. Supremum*). If a set is bounded above, then the ***maximum*** of a set may not exist, but the ***supremum*** of the set always exists.

Example. Let $S = (1, 2)$. Then there does NOT exist a maximum of S , but $\exists \sup S = 2$

This is also true for ***minimum*** and ***infimum***.

Remark. If a set S is bounded above, then $\sup S$ may NOT be an element of S.

Example. Let $S = (1, 2)$. $\exists \sup S = 2 \notin S$

Axiom (*Completeness Property of* $\mathbb{R}$).

Every nonempty set of real numbers has a supremum in $\mathbb{R} \cup \{\infty\}$. It is also called ***least-upper-bound-property*** or ***LUP*** or $\mathbb{R}$.

2.4 Applications of Supremum

Definition. Let $S \subseteq \mathbb{R},\ a \in \mathbb{R}$. We define

$$a + S = \{a + s : \forall s \in S\}$$

More generally,

$$A + B = \{a + b : \forall a \in A,\ b \in B\}$$

Proposition. $\sup(a + S) = a + \sup S$

Proof. Let $u = \sup S$. Then $\forall s \in S,\ u \geq s$. Thus

$$a + u \geq a + s$$

It follows that $a + u$ is an upper bound of $a + S$. Thus,

$$\sup(a + S) \leq a + u$$

Let v be an upper bound of $a + S$, so $\forall s \in S$

$$v \geq a_s$$
$$v - a \geq s$$

Thus $v - a$ is an upper bound of S.

$$v - a \geq \sup S = u$$
$$v \geq a + u$$
$$\sup(a + S) \geq a + u$$

Thus,

$$a + \sup(S) = \sup(a + S)$$

■

Proposition. Suppose $A, B \subseteq \mathbb{R}$. Then

$$\sup(A + B) = \sup A + \sup B$$

Proof. Exercise. ■

Proposition. Suppose $A, B \subseteq \mathbb{R}$ satisfying $\forall a \in A,\ b \in B,\ a \leq b$. Then

$$\sup A \leq \inf B$$

Proof. $\forall a \in A,\ b \in B,\ a \leq b \Rightarrow b$ is an upper bound for A.

$$\sup A \leq b$$

Thus $\sup A$ is a lower bound for B.

$$\sup A \leq \inf B$$

■

Exercise. Provide an example of $A, B \subseteq \mathbb{R}$ s.t. $\forall a \in A,\ b \in B,\ a < b$ but

$$\sup A = \inf B$$

Example.

$$A = (2, 3),\ B = [3, 4]$$

Proposition. Let $D \subseteq \mathbb{R},\ f, g : D \to \mathbb{R}$. If $\forall x \in D,\ f(x) \leq g(x)$, then

$$\sup_{x \in D} f(x) \leq \sup_{x \in D} g(x)$$

Proof. $\exists u = \sup g(D)$. Then $\forall x \in D$,

$$f(x) \leq g(x) \leq \sup g(D)$$

so $\sup g(D)$ is an upper bound of $f(D)$. Thus

$$\sup f(D) \leq \sup g(D)$$

■

Remark. $\forall x \in D,\ f(x) \leq g(x)$ does not imply any relation between $\sup f(D)$ and $\inf g(D)$.

Proposition. If $\forall x, y \in D,\ f(x) \leq g(y)$, then we may conclude that $\sup f(D) \leq \inf g(D)$.

Proof. It is a direct consequence of proposition. ■

Theorem (***Archimedean Property***)**.** $\forall x \in R,\ \exists n_x \in \mathbb{N}$ s.t. $x \leq n_x$.

Proof. Assume $\forall n \in \mathbb{N},\ n < x$. We have

$$\sup \mathbb{N} \leq x$$

. Consider $\sup \mathbb{N} - 1$. Then $\exists n \in \mathbb{N}$ s.t.

$$\sup \mathbb{N} - 1 < n$$

$$\sup \mathbb{N} < 1 + n \in \mathbb{N}$$

thus $\sup \mathbb{N}$ is NOT an upper bound of $\mathbb{N}$, which is a contradiction. ∎

Corollary. If $S = \{\frac{1}{n} : \forall n \in \mathbb{N}\}$, then

$$\inf S = 0$$

Proof. $\forall \varepsilon > 0$, by Archimedean Property, $\exists n \in \mathbb{N}$ s.t.

$$\frac{1}{\varepsilon} < n \Rightarrow \frac{1}{n} < \varepsilon$$

$$0 \leq \inf S \leq \frac{1}{n} < \varepsilon$$

By previous theorem of ε,

$$\inf S = 0$$

∎

Corollary. $\forall y > 0,\ \exists n_y \in \mathbb{N}$ s.t.

$$0 < \frac{1}{n_y} < y$$

Proof. Let $x = \frac{1}{y} \in \mathbb{R}$. By Archimedean Property, $\exists m_y \in \mathbb{N}$ s.t.

$$m_y \geq x = \frac{1}{y}$$

Let $n_y = m_y + 1 \in \mathbb{N}$. Then

$$n_y > \frac{1}{y} \Rightarrow 0 < \frac{1}{n_y} < y$$

∎

Corollary. $\forall y > 0, \exists n_y \in \mathbb{N}$ s.t.

$$n_y - 1 \leq y < n_y$$

Proof. Construct

$$E_y = \{m \in \mathbb{N} : y < m\}$$

By Archimedean Property, $E_y \neq \phi$. By Well-Ordering of $\mathbb{N}$, $\exists n_y \in \mathbb{N}$ s.t.

$$\exists n_y = \min E_y \Rightarrow y < n_y$$

Since we cannot have $y < n_y - 1$

$$n_y - 1 \leq y < n_y$$

■

Theorem. (***Existence of*** $\sqrt{2}$) $\exists x \in \mathbb{R},\ x > 0$ s.t. $x^2 = 2$

Proof. Let $S = \{s \in \mathbb{R} : s^2 < 2\}$. ***Claim:*** $\sup S$ satisfies $(\sup S)^2 = 2$.

Since $1^2 = 1 < 2$, $1 \in S$, so S is nonempty.

Assume 2 is not an upper bound of S. Then, $\exists s \in S$ s.t. $s > 2$. Thus

$$s^2 > 4 \ \wedge\ s^2 < 2$$

a contradiction. Thus, 2 is an upper bound of S. $\forall s \in S, s \leq 2$.

By Completeness Theorem Property, $\exists x = \sup S \in \mathbb{R}$.

Since we want to show $x^2 = 2$. it suffices to show that

$$x^2 \geq 2 \ \wedge x^2 \leq 2$$

We will prove those two statement by contradiction.

- Assume $x^2 < 2$. Then $\forall n \in \mathbb{N}$

$$(x + \frac{1}{n})^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} = x^2 + \frac{1}{n}(2x + \frac{1}{n}) \leq x^2 + \frac{1}{n}(2x+1)$$

Goal: Choose a $N \in \mathbb{N}$ large enough such that $(x + \frac{1}{n})^2 < 2$. Thus,

$$\begin{aligned} x^2 + \frac{1}{n}(2x+1) &< 2 \\ \frac{1}{n}(2x+1) &< 2 - x^2 \\ \frac{1}{n} &< \frac{2-x^2}{2x+1} \\ n &> \frac{2x+1}{2-x^2} \end{aligned}$$

Thus by Archimedean Property, $\exists n \in \mathbb{N},\ n = \left\lceil \frac{2x+1}{2-x^2} \right\rceil$ s.t.

$$(x + \frac{1}{n})^2 \leq x^2 + \frac{1}{n}(2x+1) < 2$$

$$x + \frac{1}{n} \in S$$

which is a contradiction to $x = \sup S$.

- Assume $x^2 > 2$. Then $\forall m \in \mathbb{N}$,

$$(x - \frac{1}{m})^2 = x^2 - \frac{2x}{n} + \frac{1}{m^2} > x^2 - \frac{2x}{m}$$

Goal: find $m \in \mathbb{N}$ large enough such that $(x - \frac{1}{m})^2 > 2$. It follows that

$$\begin{aligned} x^2 - \frac{2x}{m} &> 2 \\ \frac{2x}{m} &< x^2 - 2 \\ m &> \frac{2x}{x^2 - 2} \end{aligned}$$

By Archimedean Property, $\exists m \in \mathbb{N},\ m = \left\lceil \frac{2x}{x^2-2} \right\rceil$ s.t.

$$(x - \frac{1}{m})^2 > x^2 - \frac{2x}{x^2-2} > 2$$

thus $(x - \frac{1}{m})^2$ is an upper bound of S. However, since

$$x < x - \frac{1}{m}$$

so x is NOT an supremum of S, which is a contradiction.

Thus, we conclude that $\exists x \in \mathbb{R}$ s.t. $x^2 = 2$.

■

Remark. By Binomial Expansion, one may show the existence of positive n-th roots for all $n \in \mathbb{N}$.

Theorem (*Rationals Are Dense in* $\mathbb{R}$). If $x, y \in \mathbb{R}$ with $x < y$, then there exists a ***rational number*** $r \in \mathbb{Q}$ such that $x < r < y$.

Proof. WLOG, we may assume that $x, y > 0$.

Since $y - x > 0 \Rightarrow \frac{1}{y-x} > 0, \exists n \in \mathbb{N}$ s.t. $n > \frac{1}{y-x}$ by Archimedean Property. Thus

$$1 < ny - nx \Rightarrow nx + 1 < ny$$

By Archimedean Property, $\exists m \in \mathbb{N}$ s.t.

$$m - 1 \leq nx < m$$

$$\begin{aligned} nx < m \leq nx + 1 < ny \\ x < \frac{m}{n} < y \end{aligned}$$

■

Corollary (*__Irrationals Are Dense in__* $\mathbb{R}$). If $x, y \in \mathbb{R}$ with $x < y$, then there exists an ***__irrational number__*** $z \in \mathbb{Q}$ such that $x < z < y$.

Proof. Apply the density theorem of rationals to real number $\frac{x}{\sqrt{2}}$ and $\frac{y}{\sqrt{2}}$, we have

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$$

$$x < \sqrt{2}r < y$$

__Goal:__ Prove $\sqrt{2}r$ is irrational by contradiction.

Assume that $\sqrt{2}r$ is rational. Then $\exists p, q \in \mathbb{Z},\ q \neq 0$ s.t.

$$\sqrt{2}r = \frac{p}{q}$$

$$\sqrt{2} = \frac{p}{q}r \in \mathbb{Q}$$

which is a contradiction since $\sqrt{2} \in \mathbb{R}\backslash\mathbb{Q}$. Thus we conclude that $\sqrt{2}r \in \mathbb{R}\backslash\mathbb{Q}$. ■

2.5 Intervals

Definition (***Intervals***)**.** Let $a, b \in \mathbb{R}$. If

- $a < b$, then define ***open interval*** to be the set $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
- $a \leq b$, then define ***closed interval*** to be the set $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
- $a < b$, then define ***half-open interval*** to be the set $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$, or $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

Theorem (***Characterization Theorem***)**.** If $S \subseteq \mathbb{R}$ s.t. S contains at least 2 points and satisfies

$$\forall x, y \in S \ \wedge \ x < y \Rightarrow [x, y] \subseteq S$$

then S is an ***interval***.

Proof. Let $a = \inf S$, $b = \sup S$. Then $S \subseteq [a, b]$. If $a = -\infty$ or $b = \infty$, treat the boundary as open and this remains true.

Goal: Show that $S \supseteq (a, b)$

If $z \in (a, b)$, then z is NOT a lower bound of S. So $\exists x \in S$ with $a < x < z$.

Similarly, x is NOT an upper bound of S, so $\exists y \in S$ with $z < y < b$. Thus

$$z \in [x, y] \subseteq S$$

Therefore, the desire result then follows only on whether S includes a, b. ∎

Definition (***Nested Seqence of Intervals***)**.** A sequence of intervals $\{I_n\}_n \in \mathbb{N}$ is ***nested*** if

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots$$

Example. Consider $I_n = [0, \frac{1}{n}]$. Then $\forall n \in \mathbb{N}$,

$$I_n \supseteq I_{n+1}$$

Claim: $\bigcap_{n=1}^{\infty} I_n = \{0\}$.

Proof.

$(\supseteq)$:$\forall n \in \mathbb{N},\ 0 \in I_n$.

$(\subseteq)$: Assume $\exists x \in \mathbb{R} \backslash \{0\}$ s.t. $x \in \bigcap_{n=1}^{\infty} I_n$ Assume $x > 0$. Then by Archimedean Property, $\exists n \in \mathbb{N}$ s.t. $n > \frac{1}{x} \Rightarrow x > \frac{1}{n}$. Thus $x \notin I_n$, which is a contradiction.

Since x can not be negative, we conclude that $x = 0$ and $\bigcap_{n=1}^{\infty} I_n \subseteq \{0\}$ ∎

Remark. Not all nested sequence of intervals has single common point.

- $I_n = [0, 1 + \frac{1}{n}] \Rightarrow \bigcap_{n=1}^{\infty} I_n = [0, 1]$
- $I_n = [0, \frac{1}{n}] \Rightarrow \bigcap_{n=1}^{\infty} I_n = \phi$

Theorem (*Nested Interval Property*).

If $\forall n \in \mathbb{N},\ \exists a_n, b_n \in \mathbb{R}$ s.t. $I_n = [a_n, b_n]$ is a nested sequence of ***closed*** intervals, then there exists a real number ξ such that $\forall n \in \mathbb{N},\ \xi \in I_n$

Proof. Observe that $\forall n \in \mathbb{N},\ a_n \leq b_1$.

Hence $\{a_n : \forall n \in \mathbb{N}\}$ is bounded above.

Let $\xi = \sup\{a_n : \forall n \in \mathbb{N}\}$, then $\forall n \in \mathbb{N},\ \xi \geq a_n$.

It suffices to show $\forall n \in \mathbb{N},\ \xi \leq b_n$.

Observe that $\forall n \in \mathbb{N},\ b_n$ is an upper bound of $\{a_n : \forall n \in \mathbb{N}\}$. Thus by definition of supremum, $\forall n \in \mathbb{N}$

$$\xi \leq b_n$$

We thereby conclude that $\forall n \in \mathbb{N},\ \xi \in I_n$.

■

Theorem. If $\forall n \in \mathbb{N},\ \exists a_n, b_n \in \mathbb{R}$ s.t. $I_n = [a_n, b_n]$ is a nested sequence of ***closed*** intervals, and the length $b_n - a_n$ satisfies

$$\inf\{b_n - a_n : \forall n \in \mathbb{N}\} = 0$$

then $\xi \in \bigcap_{n=1}^{\infty} I_n$ is unique.

Proof. Assume $\exists \xi,\ \eta \in \mathbb{R}$ s.t.

$$\xi, \eta \in \bigcap_{n=1}^{\infty} I_n$$

Thus

$$0 \leq |\xi - \eta| \leq \inf\{b_N - a_N\} = 0$$

We thereby conclude that $\xi = \eta$.

■

Theorem. $\mathbb{R}$ is uncountable.

Proof. ***Goal:*** Show that $I = [0, 1]$ is uncountable.

Assume $I = [0, 1]$ is countable. Then we can enumerate the set as

$$I_0 = \{x_1,\ x_2,\ \dots,\ x_n, \dots\}$$

For all $n \in \mathbb{N}$, we choose a closed bounded interval $I_n \in I_{n-1}$ s.t. $x_n \notin I_n$. Then we have

$$I_1 \supseteq I_2 \supseteq \dots$$

By Nested Interval Property, $\exists \xi \in \bigcap_{n=1}^{\infty} I_n$. It follows that $\forall n \in \mathbb{N}, \xi \neq x_n$.

This means that I_0 is not a complete listing of elements of I_0, which contradicts the assumption that I_0 is countable. We thereby conclude that I is an uncountable set. ∎

Alternative Proof: Cantor's Diagonal Argument.

Assume $[0, 1]$ is countable. Then

$$[0, 1] = \{x_1,\ x_2,\ \dots,\ x_i,\ \dots\},\ \forall i \in \mathbb{N}$$

Consider the decimal representation of x_i:

$$x_1 = 0.b_{11}b_{12}b_{13}...$$

$$x_2 = 0.b_{21}b_{22}b_{23}...$$

$$x_i = 0.b_{i1}b_{i2}b_{i3}...$$

for all $j \in \mathbb{N}$ s.t. $b_{ij} \in \{0, 1, \dots, 9\}$. Here we define

$$y = 0.y_1y_2 \dots \in [0, 1] \text{ where } y_k = \begin{cases} 1 & \text{if } b_{kk} \neq 1 \\ 0 & \text{if } b_{kk} = 1 \end{cases},\ \forall k \in \mathbb{N}$$

Then $\forall i \in \mathbb{N}, y \neq x_i$ since $y_i \neq b_{ii}$. Thus $y \notin [0, 1]$, a contradiction. ∎

3 Sequences and Series

3.1 Sequences and their limits

Definition (Sequence). A ***sequence of real numbers*** is a *function* from $\mathbb{N}$ to $\mathbb{R}$.

We adopt the notation with a *sequence*:

$$a : \mathbb{N} \to \mathbb{R}$$

where instead of writing $a(1), a(2), \ldots$, we write it as $a_1, a_2, \ldots$ which we called them ***terms*** or ***elements*** of the sequence.

Notation.

$$(a_n)_{n=1}^{\infty} \text{ or } (a_n)_{n\in\mathbb{N}} \text{ or } (a_n) \text{ or } (a_n|n \in \mathbb{N})$$

Definition (Converge to x). A sequence $(x_n) \in \mathbb{R}$ ***converges*** to $x \in \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ such that } n \geq N_\varepsilon \to |x_n - x| < \varepsilon$$

We write

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (x_n) = x.$$

Definition (Convergent & Divergent). A sequence is ***convergent*** if it has a ***limit*** in $\mathbb{R}$, and is ***divergent*** if it has ***no limit*** in $\mathbb{R}$.

Theorem (Uniqueness of Limit). A sequence in $\mathbb{R}$ can have ***at most one*** limit. Or, the limit of a sequence is ***unique*** if the limit exists

Proof. Let (x_n) be a sequence of real numbers. Suppose x, x' are limits of (x_n). We want to prove $x = x'$ by contradiction.

Assume $|x - x'| > 0$. If we consider $\varepsilon := \frac{1}{3}|x - x'| > 0$, then
The existence of $\lim_{x_n \to x}$ implies that $\exists N_1 \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ if $n \geq \mathbb{N}_1$.
Similarly, existence of $\lim_{x_n \to x'}$ implies that $\exists N_2 \in \mathbb{N}$ such that $|x_n - x'| < \varepsilon$ if $n \geq \mathbb{N}_2$.
Thus,

$$\begin{aligned}
|x - x'| &\leq |x - x_{N_1+N_2} + x_{N_1+N_2} - x'| \\
&\leq |x - x_{N_1+N_2}| + |x_{N_1+N_2} - x'| \text{ by triangle inequality} \\
&< \varepsilon + \varepsilon \\
&= \frac{2}{3}|x - x'|
\end{aligned}$$

Then,

$$\frac{1}{3}|x - x'| < 0, \text{which is a contradiction}$$

we thereby prove by contradiction that

$$|x - x'| = 0, \text{ which is equivalent to } x = x'$$

■

Example.

$$\lim_{n\to\infty}(\frac{1}{n}) = 0$$

Goal: $\forall\varepsilon > 0$, want to find N_ε such that $\left|\frac{1}{n} - 0\right| < \varepsilon$ for $n > N$, so it suffices to show that

$$\frac{1}{n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n$$

Proof. Let $\varepsilon > 0$. Apply Archimedean's property to $\frac{1}{\varepsilon}$, then

$$\begin{aligned}
&\exists N \in \mathbb{N} \text{ such that } \frac{1}{\varepsilon} < N \\
\Rightarrow &\forall n \geq N, \left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon. \\
\Rightarrow &\lim_{n\to\infty} \frac{1}{n} = 0
\end{aligned}$$

■

Theorem. Let (x_n) be a sequence of real numbers, and let $x \in \mathbb{R}$. The following are equivalent:

1. $x_n \to x$
2. $\forall\varepsilon > 0, \exists N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$, for $n \geq \mathbb{N}$
3. $\ldots\ldots x - \varepsilon < x_n < x + \varepsilon \ldots\ldots$
4. $\forall\varepsilon$-neighborhood $V_\varepsilon(x), \exists N \in \mathbb{N}$ such that $x_n \in V_\varepsilon(x)$ for $n \geq N$

Sketch of proof:

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$$

Proposition.

$$\lim_{n\to\infty}(2\sqrt{2n+1} - \sqrt{2n}) = 0$$

Proof. Let $\varepsilon > 0$. Consider

$$N = \lceil \frac{1}{2}(\frac{1}{2\varepsilon})^2 \rceil \in \mathbb{N}$$

$$n > N \Rightarrow n > \frac{1}{2}(\frac{1}{2\varepsilon})^2 \Rightarrow \frac{1}{2\sqrt{2n}} < \varepsilon \Rightarrow \left|\sqrt{2n+1} - \sqrt{2n}\right| = \cdots = \frac{1}{\sqrt{2n+1} + \sqrt{2n}} < \varepsilon$$

■

Remark.

$$\lim_{n\to\infty}(-1)^n \text{ does not exist.}$$

Definition (m-tail). If (x_n) is a sequence of real numbers and $m \in \mathbb{N}$, then the ***m-tail*** of (x_n) is the sequence

$$\{x_{n+m} : n \in \mathbb{N}\} = \{x_{m+1}, x_{m+2}, \dots\}$$

Theorem. Let (x_n) be a sequence and $m \in \mathbb{N}$. Then (x_n) is ***convergent*** iff (x_{n+m}) is ***convergent***. Moreover,

$$\lim_{n \to \mathbb{N}} (x_n) = \lim_{n \to \mathbb{N}} (x_{n+m})$$

Proof. ($\Rightarrow$)
Suppose $x_n \to x$. Let

$$\varepsilon > 0, \exists N_\varepsilon > 0, \text{ such that } |x_n - x| < \varepsilon \text{ for } n \geq N_\varepsilon$$

Consider $N_\varepsilon{}' := N_\varepsilon + m$ then

$$n + m \geq N_\varepsilon{}' \Rightarrow n \geq N_\varepsilon \Rightarrow n + m \geq N_\varepsilon \Rightarrow |x_{n+m} - x| < \varepsilon$$

It follows that

$$n \geq N_\varepsilon \Rightarrow n + m \geq N_\varepsilon \Rightarrow |x_{n+m} - x| < \varepsilon$$

($\Leftarrow$)
Suppose $x_{n+m} \to x$.

$$\forall \varepsilon > 0, \exists N_\varepsilon > 0 \text{ such that } |x_{n+m} < \varepsilon|, \forall n \geq N_\varepsilon$$

Consider $N := N_\varepsilon + m$. Then

$$\begin{aligned} & n \geq N = N_\varepsilon + m \\ & \Rightarrow n - m \geq N_\varepsilon \\ & \Rightarrow \left| x_{(n-m)+m} - x \right| < \varepsilon \\ & \Rightarrow |x_n - x| < \varepsilon \end{aligned}$$

■

Remark. We say that a sequence (x_n) ***ultimately*** has a property if that property holds for some tail of (x_n)

Theorem. Let x_n be a sequence of real numbers. Let a_n be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$. If $\exists c > 0, m \in \mathbb{N}, x \in \mathbb{R}$ such that

$$|x_n - x| \leq c \cdot a_n, \forall n \geq m$$

then

$$x_n \to x$$

Proof. We know that

$$\forall \varepsilon > 0, \exists N \geq 0 \text{ s.t. } |a_n| < \frac{\varepsilon}{c}, \forall n \geq N$$

Consider $N' = max\{N, m\}, \forall n \geq N'$. Then

$$|x_n - x| \leq Ca_n = c\,|a_n| < c \cdot \frac{\varepsilon}{c} = \varepsilon$$

$$\Rightarrow x_n \to x$$

■

Proposition.

$$\lim_{n\to\infty} \frac{17}{2+3n} = 0$$

Proof.

$$\left|\frac{17}{2+3n} - 0\right| = \frac{17}{2+3n} \leq \frac{13}{3n} = \frac{17}{3} \cdot \frac{1}{n}$$

Apply the theorem above with

$$a_n = \frac{1}{n}, c = \frac{17}{3}, m = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{17}{2+3n} = 0, \text{ since } \lim_{n\to\infty} \frac{1}{n} = 0$$

■

Proposition.

$$\forall c > 0, \lim_{n\to\infty} c^{\frac{1}{n}} = 1$$

Proof. ***Case 1: c = 1***

$$\lim_{n\to\infty} c^{\frac{1}{n}} = 1$$

Case 2: c > 1

Let $d_n = c^{\frac{1}{n}} - 1$. Then $\forall n, d_n > 0$. It follows that

$$\begin{aligned}(d_n + 1) = c^{\frac{1}{n}} &\Rightarrow c = (1 + d_n)^n \geq 1 + n \cdot d_n \text{ by Bernoulli's inequality}\\ &\Rightarrow d_n \leq (c-1)\cdot\frac{1}{n}\\ &\Rightarrow \left|c^{\frac{1}{n}} - 1\right| = d_n \leq (c-1)\cdot\frac{1}{n}\end{aligned}$$

Apply the theorem with

$$C = c - 1, a_n = \frac{1}{n}, m = 1, x = 1$$

$$\lim_{n\to\infty} c^{\frac{1}{n}} = 1$$

Case 3: c < 1(Note that we cannot use Bernoulli inequality here)

Define e_n to be a sequence that satisfies

$$c^{\frac{1}{n}} = \frac{1}{1+e_n}$$

Then $e_n > 0 \forall n$.

$$c = \frac{1}{(1+e_n)^n} \leq \frac{1}{1+n\cdot e_n} < \frac{1}{n \cdot e_n}$$

$$\Rightarrow e_n < \frac{1}{c}\cdot\frac{1}{n}$$

$$1 - c^{\frac{1}{n}} = 1 - \frac{1}{1+e_n} = \frac{e_n}{1+e_n} < e_n < \frac{1}{c}\cdot\frac{1}{n}$$

Apply the theorem with

$$a_n = \frac{1}{n}, m = 1, C = \frac{1}{c}, x = 1$$

$$\lim_{n\to\infty} c^{\frac{1}{n}} = 1$$

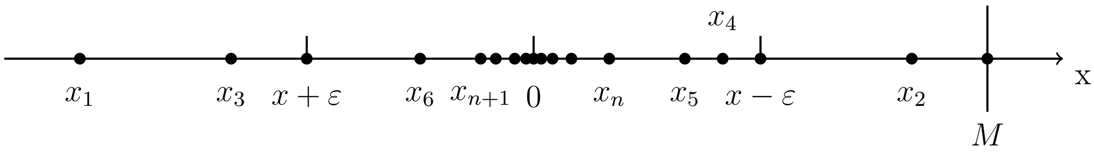
∎

3.2 Limit Theorems

Definition (Bounded sequence)**.** A sequence $(x_n) \in \mathbb{R}$ is ***bounded*** if

$$\exists M > 0 \text{ s.t. } \forall n \in \mathbb{N}, |x_n| \leq M$$

Theorem. A convergent sequence $(x_n) \in \mathbb{R}$ is bounded.



Proof. By definition of convergent sequence, let $\varepsilon = 1$:

$$\exists N > 0 \text{ s.t. } \forall n \geq N, |x_n - x| < 1$$

Thus we have

$$\begin{aligned} & -1 < x_n - x < 1 \\ \Rightarrow \quad & -1 + x < x_n < x + 1, \forall n \geq N \end{aligned}$$

Then define

$$M := \max \{|x_1|, |x_2|, |x_3|, \dots, |-1 + x|, |x + 1|\}$$
$$|x_n| \leq M$$

■

Remark. By contrapositive, an unbounded sequence is divergent.

Definition. Given sequence $(x_n), (y_n) \in \mathbb{R}$, we define following operations of sequence:

- ***Sum*** $(x_n + y_n)$
- ***Difference*** $(x_n - y_n)$
- ***Product*** $(x_n \cdot y_n)$
- ***Quotient*** $(\frac{x_n}{y_n})$ if $\forall n \in \mathbb{N}, y_n \neq 0$
- ***Multiple*** $(c \cdot x_n)$

Theorem (**Limit Laws**)**.** Let $(x_n), (y_n) \in \mathbb{R}$ be sequences of real numbers with $x_n \to x, y_n \to y$, and let $c \in \mathbb{R}$. Then

- $x_n + y_n \to x + y$
- $x_n - y_n \to x - y$
- $x_n \cdot y_n \to x \cdot y$
- $c \cdot x_n \to c \cdot x$
- If $\forall n \in \mathbb{N}, y_n \neq 0$ and $y \neq 0$, then $\frac{x_n}{y_n} \to \frac{x}{y}$

Proof of Sum. :

$\forall \varepsilon > 0$,

$$\exists N_1 \in \mathbb{N} \text{ s.t. } \forall n \geq N_1, |x_n - x| < \frac{\varepsilon}{2}$$
$$\exists N_2 \in \mathbb{N} \text{ s.t. } \forall n \geq N_2, |y_n - y| < \frac{\varepsilon}{2}$$

Consider

$$N := \max\{N_1, N_2\}$$

Then

$$\begin{aligned}
\forall n \geq N, &|(x_n + y_n) - (x + y)| \\
&= |x_n - x + y_n - y| \\
&\leq |x_n - x| + |y_n - y| \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

■

Proof of Difference. :

Similarly,

$$\begin{aligned}
\forall n \geq N, &|(x_n - y_n) - (x - y)| \\
&\leq |x_n - x| + |y_n - y|
\end{aligned}$$

■

Proof of Product. : Since (x_n) is convergent, it is also bounded. Thus,

$$\exists M \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, |x_n| \leq M$$

By definition of convergence, $\forall \varepsilon > 0$:

$$\exists N_1 > 0 \text{ s.t. } \forall n \geq N, |x_n - x| < \frac{\varepsilon}{2|y|}$$
$$\exists N_2 > 0 \text{ s.t. } \forall n \geq N, |y_n - y| < \frac{\varepsilon}{2M}$$

Then, $\exists N = \max\{N_1, N_2\}$ s.t. $\forall n \geq N$,

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\
&= |x_n(y_n - y) - y(x_n - x)| \\
&\leq |x_n| |y_n - y| + |y| |x_n - x| \\
&\leq M \cdot |y_n - y| + |y| |x_n - x| \\
&< M \cdot \frac{\varepsilon}{2M} + |y| \cdot \frac{\varepsilon}{2|y|} = \varepsilon
\end{aligned}$$

■

Proof of Multiply. :

Exercise ■

Proof of Quotient.

$$\begin{aligned}
\left|\frac{x_n}{y_n} - \frac{x}{y}\right| &= \left|\frac{x_n y - y_n x}{y_n y}\right| \\
&= \left|\frac{x_n y - -x_n y_n + x_n y_n - y_n x}{y_n y}\right| \\
&\leq \left|\frac{1}{y_n y}(|x_n|\,|y_n - y| + |y_n|\,|x_n - x|)\right|
\end{aligned}$$

Since $y_n \to y$

$$\exists N_1 \in \mathbb{N} \text{ s.t. } \forall n \geq N, ||y_n| - |y|| \leq |y_n - y| < \frac{|y|}{2}$$

Then

$$\begin{aligned}
\Rightarrow &\quad -\frac{|y|}{2} < |y| - |y| < \frac{|y|}{2} \\
\Rightarrow &\quad \frac{|y|}{2} < |y_n| \\
\Rightarrow &\quad \frac{1}{|y_n|} < \frac{2}{|y|} \\
\Rightarrow &\quad \frac{1}{|y_n y|} < \frac{2}{|y^2|}
\end{aligned}$$

Define $F := \frac{2}{|y^2|}$.

Since (x_n) and (y_n) are bounded,

$\exists G \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$

$$|x_n| \leq G, |y_n| \leq G$$

Thus $\forall n > 0$

$$\exists N_2 > 0 \text{ s.t. } \forall n \geq N_2, |x_n - x| < \frac{\varepsilon}{2GF}$$

$$\exists N_3 > 0 \text{ s.t. } \forall n \geq N_3, |y_n - y| < \frac{\varepsilon}{2GF}$$

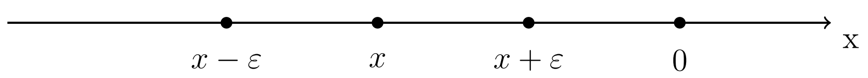
Define $N := \max\{N_1, N_2, N_3\}$, then $\exists N \in \mathbb{N}$ s.t. $\forall n > N$,

$$\left|\frac{x_n}{y_n} - \frac{x}{y}\right| < F(G \cdot \frac{\varepsilon}{2GF} + G \cdot \frac{\varepsilon}{2GF}) = \varepsilon$$

■

Theorem. If (x_n) is a convergent sequence in $\mathbb{R}$ of non-negative terms with $(x_n) \to x$, then $x \geq 0$.

Sketch. : Assume $x < 0$, choose $\varepsilon \leq |x|$, then $\forall n > N, x_n < 0$, which is a contradiction.



■

Theorem. If $x_n \to x, y_n \to y$ are sequences in $\mathbb{R}$ such that $\forall n \in \mathbb{N}, x_n \leq y_n$, then $x \leq y$.

Proof. Consider $z_n := y_n - x_n$. Then $z_n \geq 0$ and $z_n \to y - x$ by the limit law.

$$\Rightarrow y - x \geq 0$$

$$y \geq x$$

■

Theorem. If $x_n \to x$ is a sequence in $\mathbb{R}$, and let $a, b \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, a \leq x_n \leq b$, then $a \leq x \leq b$.

Proof. Consider constant sequence $a_n = a$ and $b_n = b$. Then this is true by the last theorem. ■

Theorem (**Squeeze Theorem**). Suppose $(x_n), (y_n), (z_n)$ are sequences of real numbers such that $\forall n \in \mathbb{N}, x_n \leq y_n \leq z_n$. If $\lim(x_n) = \lim(z_n)$, then (y_n) is convergent and

$$\lim(x_n) = \lim(y_n) = \lim(z_n)$$

Proof. Let $\forall \varepsilon > 0$.

Write $L = \lim(x_n)$. Then by definition of limit,

$\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$|x_n - L| < \varepsilon, |z_n - L| < \varepsilon$$

$$\begin{aligned} \Rightarrow \quad & -\varepsilon < x_n - L \leq y_n - L \leq z_n - L < \varepsilon \\ \Rightarrow \quad & |y_n - L| < \varepsilon \end{aligned}$$

Thus (y_n) converges to L by definition of limit.

■

Proposition. $\ln(n)$ is divergent.

Proof. Since $\ln(n)$ is unbounded, it is divergent.

■

Exercise. $\lim \frac{3n^2+2n+1}{5n^2-4} = \frac{3}{5}$

$$\lim \frac{3n^2+2n+1}{5n^2-4} = \lim \frac{3+2\frac{1}{n}+\frac{1}{n^2}}{5-4\frac{1}{n^2}}$$

Notice that each term of $\frac{1}{n}$ and $\frac{1}{n^2}$ converges to 0. Thus by limit law,

$$\lim \frac{3+2\frac{1}{n}+\frac{1}{n^2}}{5-4\frac{1}{n^2}} = \frac{\lim\{3+2\cdot 0+0\}}{\lim\{5-4\cdot 0\}} = \frac{3}{5}$$

Proposition. $(-1)^n$ is divergent.

Proof:Exercise. ■

Theorem. If $x_n \to x$, then $|x_n| \to |x|$

Sketch of the proof. :

$$||x_n| - |x|| \leq |x_n - x|$$

■

Theorem. Suppose (x_n) is a sequence of non-negative real numbers, satisfying $x_n \to x$. Then $\sqrt{x_n} \to \sqrt{x}$.

Proof. Let $\forall \varepsilon > 0$

Case1: $x = 0$

$\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$|x_n - 0| < \varepsilon^2$$
$$\Rightarrow \left|\sqrt{x} - 0\right| < \varepsilon$$

Case2: $x > 0$

$\exists N$ s.t. $\forall n > N$,

$$|x_n - x| < \sqrt{x} \cdot \varepsilon$$

Notice that

$$\begin{aligned} \left|\sqrt{x_n} - \sqrt{x}\right| &= \left|\frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}\right| \\ &\leq \frac{|x_n - x|}{\sqrt{x}} < \varepsilon \end{aligned}$$

■

Theorem. Let (x_n) be a sequence of positive real numbers such that $L = \lim(\frac{x_{n+1}}{n})$ exists. If $L < 1$, then (x_n) converges to 0.

Proof. $\exists N_1 \in \mathbb{N}$ s.t. $\forall n > N_1$

$$\left|\left|\frac{x_{n+1}}{x_n}\right| - |L|\right| \leq \left|\frac{x_{n+1}}{x_n} - L\right| < \frac{1 - L}{2}$$

Thus

$$\left|\frac{x_{n+1}}{x_n}\right| < \frac{1 + L}{2}$$

Note that $\frac{1+L}{2} < 1$, write $r = \frac{1+L}{2}$. Then $\forall m \in \mathbb{N}$,

$$x_{N_1+m} < x_{N_1+(m-1)}r < x_{N_1+(m-2)}r^2 < \cdots < x_{N_1}r^m$$

Consider $(y_m) = (x_{N_1+m})$, $(z_m) = (x_{N_1}r^m)$. Then

$$0 \leq y_m \leq z_m$$

Since $z_m \to 0$

$$(y_m) \to 0 \text{ by squeeze theorem}$$

Thus we conclude that $(x_n) \to 0$ by m-th tail theorem. ■

3.3 Monotone sequence

Definition. Let (x_n) be a sequence of real. We say (x_n) is ...

- ***increasing*** if $\forall n \in \mathbb{N}, x_{n+1} \geq x_n$.
- ***strickly increasing*** if $\forall n \in \mathbb{N}, x_{n+1} > x_n$.
- ***decreasing*** if $\forall n \in \mathbb{N}, x_{n+1} \leq x_n$.
- ***strickly decreasing*** if $\forall n \in \mathbb{N}, x_{n+1} < x_n$.

Theorem (Monotone Convergence Therorem). A monotone sequence of real numbers is ***convergent*** iff it is bounded. Moreover, if (x_n) is increasing, then

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$$

If (x_n) is decreasing, then

$$\lim(x_n) = \inf\{x_n : n \in \mathbb{N}\}$$

Proof. $(\Rightarrow)$

A convergent sequence is always bounded.

$(\Leftarrow)$

Suppose (x_n) is a monotone and bounded sequence.

Case 1: (x_n) is increasing.

Write $x = \sup\{x_n : n \in \mathbb{N}\}$.

Let $\varepsilon > 0$. Since $x = \sup(x_n)$:

$$x - \varepsilon \text{ is NOT an upper bound of } (x_n)$$

Then

$$\exists N \in \mathbb{N} \text{ s.t. } x_N > x - \varepsilon$$

Since (x_n) is increasing,

$$\forall n \geq N,\ x_n > x - \varepsilon$$

On the other hand,$x + \varepsilon$ is an supper bound since x is an upper bound. Thus,

$$x_n < x + \varepsilon$$

$$\Rightarrow \forall n \geq N, x - \varepsilon < x_n < x + \varepsilon$$

$$\Rightarrow |x_n - x| < \varepsilon \Rightarrow (x_n) \to x$$

Case 2: (x_n) is decreasing.

Write $y = \inf(x_n)$. Let $\varepsilon > 0$

Since $y = \inf(x_n)$,

$$y + \varepsilon \text{is NOT and upper bound of } (x_n)$$

Thus

$$\exists N \in \mathbb{N} \text{ s.t. } x_n < y + \varepsilon$$

Since (x_n) is decreasing,

$$\forall n > N, x_n < y + \varepsilon$$

On the other hand, $y - \varepsilon$ is a lower bound since y is a lower bound. Hence,

$$\forall n \in \mathbb{N}, y - \varepsilon < x_n$$

$$\forall n \geq N, y - \varepsilon < x_n < y + \varepsilon \Rightarrow |x_n - y| < \varepsilon$$

$$\Rightarrow (x_n) \to y$$

Remark. One can prove case 2 by following:

$(-x_n)$ is increasing and converges to $\sup(-x_n)$ by case 1. Also note that

$$(x_n) = (-(-x_n)) \to -\sup(-x_n)$$

by limit law. So it is easy to prove that

$$-\sup((-x_n)) = \inf(x_n)$$

■

Example. Consider the sequence (x_n) is given by

$$\begin{cases} x_0 = \frac{1}{2} \\ x_{n+1} = \frac{3}{2}x_n(1 - x_n) \end{cases}$$

(x_n) is decreasing and bounded.

Thoughts: Assume (x_n) converges, then by limit law,

$$x = \frac{3}{2}x(1-x) \text{ where } x = \lim(x_n) \Rightarrow x = 0 \text{ or } 3$$

then, by proof of contradiction, it is not convergent.

Proof. ***Claim***: $\frac{1}{3} < x_{n+1} < x_n \leq \frac{1}{2}$, $\forall n \in \mathbb{N} \cup \{0\}$

Proof of the claim by induction:

When n = 0:

$$x_0 = \frac{1}{2},\ x_1 = \frac{3}{2} \cdot \frac{1}{2}(1 - \frac{1}{2}) = \frac{3}{8}$$

$$\frac{1}{3} < \frac{3}{8} < \frac{1}{2} \leq \frac{1}{2}$$

Suppose this is true for n=k:

$$\frac{1}{3} < x_{k+1} < x_k \leq \frac{1}{2}$$

Goal:

$$\frac{1}{3} < x_{k+2} < x_{k+1} \leq \frac{1}{2}$$

$$x_{k+1} = \frac{3}{2}x_k(1-x_k)$$

$$\frac{1}{3} < x_k \leq \frac{1}{2} \Rightarrow \frac{2}{3} > 1 - x_k \geq \frac{1}{2}$$

$$x_{k+1} < \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{2}$$

Complete the square:

$$\begin{aligned} x_{k+1} - \frac{1}{3} &= \frac{3}{2}x_k(1-x_k) - \frac{1}{3} \\ &= -\frac{3}{2}[(x_k - \frac{1}{2})^2 - \frac{1}{36}] \end{aligned}$$

So

$$\begin{aligned} \frac{1}{3} < x_k \leq \frac{1}{2} &\Rightarrow \left| x_k - \frac{1}{2} \right| < \frac{1}{6} \\ &\Rightarrow (x_k - \frac{1}{2})^2 < \frac{1}{36} \\ &\Rightarrow x_{k+1} - \frac{1}{3} > 0 \\ &\Rightarrow \frac{1}{3} < x_{k+1} \leq \frac{1}{2} \end{aligned}$$

With the similar process, we can derive that

$$\frac{1}{3} < x_{k+2} \leq \frac{1}{2}$$

$$x_{k+2} = \frac{3}{2}x_{k+1}(1-x_{k+1}) < \frac{3}{2}x_{k+1} \cdot \frac{3}{2} = x_{k+1}$$

Therefore, this claim is also true for n=k+1:

$$\frac{1}{3} < x_{k+2} < x_{k+1} \leq \frac{1}{2}$$

We thereby prove the theorem by induction:

$$\frac{1}{3} < x_{n+1} < x_n \leq \frac{1}{2}$$

■

Exercise: Textbook p.75: A sequence that converges to $\sqrt{a}$ for $a > 0$.

Definition (Euler's Number).

$$e = \lim(1 + (\frac{1}{n})^n)$$

Goal: (x_n) is convergent where $x_n = (1 + \frac{1}{n})^n$

$$\begin{aligned} x_n = (1 + \frac{1}{n})^n &= 1 + nC1 \cdot \frac{1}{n} + nC2 \cdot \frac{1}{n^2} + \cdots + nCn\frac{1}{n^n} \\ &= 1 + \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{2} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1) \cdot 3 \cdot 2 \cdot 1}{n!} \cdot \frac{1}{n^2} \\ &= 1 + 1 + \frac{1}{2}(1 - \frac{1}{n}) + \frac{1}{6}(1-n)(1 - \frac{2}{n}) + \cdots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \dots \frac{2}{n} \cdot \frac{1}{n} \end{aligned}$$

Write x_{n+1} in a similar way, we observe that

$$x_n < x_{n+1}$$

Facts $2^{m-1} \leq m!$ for $m \in \mathbb{N} \Rightarrow \frac{1}{m!} \leq \frac{1}{2^{m-1}}$

$$x_n < 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 1 + \frac{1(1 - (\frac{1}{2})^n)}{1 - \frac{1}{2}} < 3$$

$\Rightarrow (x_n)$ is increasing and bounded

3.4 Subsequence and the Bolzano-Weierstrass Theorem

Example.

$$(x_n) = ((-1)^n)$$
$$x_{2n} = (-1)^{2n}$$
$$x_{2n+1} = (-1)^{2n-1}$$

So $a_n = x_{2n}$ is a sequence, while $b_n = x_{2n+1}$ is a subsequence of x_n.
Definition (**Subsequences**)**.** Let (x_n) be a real sequence and consider a strictly increasing sequence of natural numbers $n_1 < n_2 < n_3 < \dots$ The sequence

$$(x_{n_k} : k \in \mathbb{N})$$

is called a ***subsequence*** of (x_n)

Example. Any tails of a sequence is a subsequence: (x_n) n-th tail: (x_{m+k}), where $n = m + k, k \in \mathbb{N}$, is a subsequence.

Theorem. Suppose (x_n) converges to x. Then $x_{x_k} \to x$ for any subsequence of (x_n).

Proof. Let $\varepsilon > 0$, then

$$\exists N_\varepsilon > 0 \text{ s.t. } |x_n - x| < \varepsilon \text{ for } n > N_\varepsilon.$$

Note that

$$n_k \geq k, \forall k \in \mathbb{N}.$$

Exercise. By induction, $n_1 \geq 1, n_2 \geq n_1 \geq 1 \Rightarrow n_2 \geq 2$

When $k > N_\varepsilon$, $n_k > N_\varepsilon$, thus

$$|x_{n_k} - x| < \varepsilon$$

Therefore

$$(x_{n_k}) \to x$$

■

Theorem. Let (x_n) be a sequence of real numbers, and let $x \in \mathbb{R}$. Then the following are equivalent:

1. (x_n) does not converge to x.
2. $\exists \varepsilon_0 > 0,$ s.t. $\forall k \in \mathbb{N}, \exists n_k \in \mathbb{N}$ s.t.

$$n_k \geq k \,\&\, |x_{n_k} - x| > \varepsilon_0$$

3. $\exists \varepsilon_0 > 0$ and a subsequence (x_{n_k}) s.t.

$$|x_{n_k} - x| > \varepsilon_0, \forall k \in \mathbb{N}$$

Example.

$$x_n = (-1)^n \Rightarrow (x_n) \text{ does not converge to } 1$$

Proof.

$$3 \to 1: \text{ by the contrapositive statement of definition}$$
$$3 \to 2: \text{ 3 is a stronger statement of 2}$$
$$1 \to 2: \text{ left as exercise}$$

■

Theorem. If (x_n) satisfies either of the following property, then it is ***divergent***:

1. There exists two subsequence (x_{n_k}) & (x_{m_k}) whose limits are NOT equal.
2. (x_n) is unbounded.

Example.

1. $(-1)^n$
2. (n)
3. (x_n) such that

$$x_{2k} = k$$
$$x_{2k+1} = (-1)^k$$

Proof. exercise

■

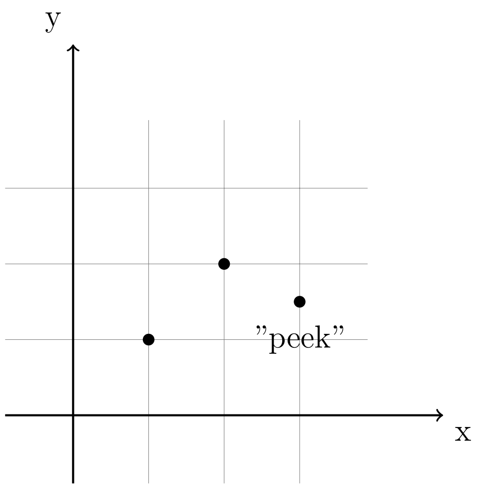
Theorem (Bolzano-Weierstrass Theorem). A bounded sequence of real numbers has a ***convergent subsequence***.
Example.

$$x_n = (-1)^n$$

Proof.

Lemma. If (x_n) is a sequence of real numbers, there exists a subsequence of (x_n) which is monotone.

proof of lemma:
Call the m-th term x_m a *"peek"* if x_m is at least as large as any term after it in the sequence.



Case 1: (x_n) has infinitely many peaks

List the peaks of (x_n) in order of increasing index

$$x_{n_1}, x_{n_2}, \dots$$

$\Rightarrow (x_{n_i})$ is a decreasing sequence.

Case 2: (x_n) has a finite number of peaks

Let s_1 be the first index after the last peak of (x_n). Then for every $n \geq s_1$, $\exists m \in \mathbb{N}$ such that $x_m > x_n$.

Choose

$$s_2 \geq s_1 \text{ such that } x_{s_2} > x_{s_1},\ s_2 \geq s_1 \text{ such that } x_{s_2} > x_{s_1}, \dots$$

$$\Rightarrow (x_{s_i}) \text{ is an increasing sequence.}$$

Remark. Lemma + monotone convergent theorem implies Bolzano-Weierstrass theorem.

Second proof

Suppose (x_n) is a bounded sequence.

$$\Rightarrow \exists I_1 = [a_1, b_1] \text{ such that } (x_n) \in I_1$$

Consider

$$I_2' = [a_1, \frac{a_1 + b_1}{2}],\ I_2'' = [\frac{a_1 + b_1}{2}, b_1]$$

Let $I_2 = [a_2, b_2]$ be one of I_2', I_2'' such that I_2 contains infinitely many terms of (x_n).

For $n \in \mathbb{N}$, define $I_n = [a_n, b_n]$ in a similar way.

For $i \in \mathbb{N}$, choose a term x_{n_i} such that $x_{n_i} \in I_i$ and $n_i > n_{i-1}$

Then

$$\begin{array}{lr} i = 1, & n_i = 1 \\ i = 2, & \text{choose } n_2 \in \mathbb{N} \text{ such that } n_2 > n_1 \,\&\, x_{n_2} \in I_2 \\ \vdots & \vdots \end{array}$$

- $\forall i \in \mathbb{N}$, $a_i \leq x_{n_i} \leq b_i$
- (a_i) increases, bounded above by $b_1 \Rightarrow (a_i) \to \sup(a_i)$
- (b_i) decreases, bounded below by $a_1 \Rightarrow (b_i) \to \inf(b_i)$
- $\inf|a_i - b_i| = \inf \frac{b_1 - a_1}{2^n} = n \Rightarrow \sup(a_i) = \inf(b_i)$

Thus (x_{n_i}) is convergent by squeeze theorem.

■

Theorem. Let (x_n) be a bounded sequence, and $x \in \mathbb{R}$ has the property that every convergent subsequence of (x_n) converges to x. Then x_n converges to x

Proof. Let $\forall \varepsilon > 0$

By Bolzano-Weierstrass theorem, $\exists$ a convergent subsequence (x_{n_i}) such that

$$\exists N_\varepsilon \in \mathbb{N} \text{ s.t. for } i > N_\varepsilon, |x_{n_i} - x| < \varepsilon$$

Assume (x_n) does not converge to x. Then by previous theorem of subsequence,

$$\exists \varepsilon_0 > 0 \text{ and a subsequence } (x_{n_k}) \text{ s.t. } |x_{n_k} - x| > \varepsilon_0, \forall k \in \mathbb{N}$$

Since (x_n) is bounded, (x_{n_k}) is also bounded. Thus there exists a convergent subsequence of (x_{n_k}) as $(x_{n_{k_i}})$.

Note that $(x_{n_{k_i}})$ is a convergent subsequence of (x_n), thus

$$(x_{n_{k_i}}) \to x \text{ which is contradiction to previous assumption}$$

■

Definition. Let (x_n) be a sequence of real numbers. A point is called a ***subsequential limit*** of (x_n) if it is the limit of a subsequence of (x_n).

$$S = \{\alpha \in \mathbb{R} : x \text{ is a subsequential limit}\} \text{ NOTE: may be infinite set}$$

Definition. Let (x_n) be a sequence of real numbers.

- The ***limit superior*** of (x_n) is the infimum of the set of $v \in \mathbb{R}$ s.t. $v < x_n$ for at most a finite number of $n \in \mathbb{N}$. We write it as

$$\limsup(x_n) \text{ or } \limsup x_n \text{ or } \overline{\lim} x_n$$

- The ***limit inferior*** of (x_n) is the supremum of the set of $v \in \mathbb{R}$ s.t. $w > x_m$ for at most a finite number of $n \in \mathbb{N}$. We write it as

$$\liminf(x_n) \text{ or } \liminf x_n \text{ or } \overline{\lim} x_n$$

Intuition

- Suppose $v < x_n$ for at most finitely many $n \in \mathbb{N}$, then

Theorem. Let (x_n) be a bounded sequence, and $x \in \mathbb{R}$ has the property that every convergent subsequence of (x_n) converges to x. Then x_n converges to x

Proof. Let $\forall \varepsilon > 0$

By Bolzano-Weierstrass theorem, $\exists$ a convergent subsequence (x_{n_i}) such that

$$\exists N_\varepsilon \in \mathbb{N} \text{ s.t. for } i > N_\varepsilon, |x_{n_i} - x| < \varepsilon$$

Assume (x_n) does not converge to x. Then by previous theorem of subsequence,

$$\exists \varepsilon_0 > 0 \text{ and a subsequence } (x_{n_k}) \text{ s.t. } |x_{n_k} - x| > \varepsilon_0, \forall k \in \mathbb{N}$$

Since (x_n) is bounded, (x_{n_k}) is also bounded. Thus there exists a convergent subsequence of (x_{n_k}) as $(x_{n_{k_i}})$.

Note that $(x_{n_{k_i}})$ is a convergent subsequence of (x_n), thus

$$(x_{n_{k_i}}) \to x \text{ which is contradiction to previous assumption}$$

■

Definition. Let (x_n) be a sequence of real numbers. A point is called a ***subsequential limit*** of (x_n) if it is the limit of a subsequence of (x_n).

$$S = \{\alpha \in \mathbb{R} : x \text{ is a subsequential limit}\} \text{ (NOTE: may be infinite set)}$$

Example. Consider $(x_n) = \{(-1)^n | n \in \mathbb{N}\}$. Then

$$S \supseteq \{1, -1\}$$

Definition (lim sup and lim inf)**.** Let (x_n) be a sequence of real numbers.

- The ***limit superior*** of (x_n) is the infimum of the set of $v \in \mathbb{R}$ s.t. $v < x_n$ for at most a finite number of $n \in \mathbb{N}$. We write it as

  $\limsup(x_n) = \limsup x_n = \overline{\lim} x_n = \inf\{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of n}\}$

  Example. Consider $(x_n) = \frac{1}{n}$

  Let

  $$X = \{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of n}\}$$

  – $-1 \notin X$ because there are infinitely many x_n such that $v < x_n$.

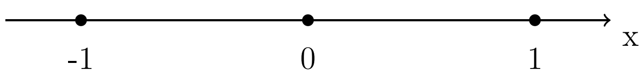
  – $\frac{1}{2} \in X$ because there are finitely many x_n such that $v < x_n$.

  – $2 \in X$ because there is no x_n such that $v < x_n$, which is smaller than finite and thereby satisfies the definition.

  We thus conclude that

  $$(0, \infty) \subset X \text{ and } \limsup x_n = \inf X$$

  Example. Consider $(x_n) = (-1)^n$:

  – $1 \in X$ because there is no x_n such that $v < x_n$.

  – $2 \in X$ because there is no x_n such that $v < x_n$.

  – $0, -1 \notin X$ because there are infinitely many x_n such that $v < x_n$.

  We thus conclude that

  $$[1, \infty) \subset X \text{ (in fact they are equal)}$$

- The ***limit inferior*** of (x_n) is the supremum of the set of $w \in \mathbb{R}$ s.t. $w > x_m$ for at most a finite number of $n \in \mathbb{N}$. We write it as

  $\liminf(x_n)$ or $\liminf x_n$ or $\overline{\lim} x_n = \sup\{w \in \mathbb{R} | w > x_n \text{ for at most a finite number of n}\}$

Intuition

- Suppose $v < x_n$ for at most finitely many $n \in \mathbb{N}$, then for all large n, $v \geq x_n$.
  $\Rightarrow$ No subsequential limit of (x_n) can possibly exceed v.
- Similar observation for $\underline{\lim}\, x_n$

Theorem. Let (x_n) be a bounded sequence of real numbers, and let $x^* \in \mathbb{R}$. Then TFAE:

1. $x^* = \limsup(x_n)$
2. If $\varepsilon > 0$, there are at most a finite number of $n \in \mathbb{N}$ s.t. $x^* + \varepsilon < x_n$, but infinitely many n for which $x^* - \varepsilon < x_n$
3. If $u_m = \sup\{x_n | n \geq m\}$(sup of (m-1)-th tail), then $x* = \inf\{u_m | m \in \mathbb{N}\} = \lim u_m$
4. If S is the set of subsequential limits of x_n, then $x^* = \sup S$.

Remark. .

- u_m is decreasing.
- There is a similar such list of equivalent properties for $\liminf$.

Corollary. A bounded sequence (x_n) is convergent iff $\overline{\lim} x_n = \lim x_n$

Proof. A direct result of the theorem:

$$\overline{\lim} x_n = \sup S \text{ and } \underline{\lim} x_n = \inf S$$

∎

Proof of thm. (a) $\Rightarrow$ (b). Let $\varepsilon > 0$. Then

$$x^* + \varepsilon > x^* = X = \inf\{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of n.}\}$$

$$\Rightarrow \exists v \in \mathbb{R} \text{ s.t. } x^* \leq v < x^* + \varepsilon$$

and there are only finitely mant n with $v < x_n$.

For any n for which $x^* + \varepsilon < x_n$, $v < x_n$. Thus there are only finitely many such n.

If $x^* - \varepsilon \notin X$, then there are infinitely many n such that $x^* - \varepsilon < x_n$ ∎

Proof of thm. (b) ⇒ (c). Fix $\varepsilon > 0$.

By (b), there are only finitely many n with $x^* + \varepsilon < x$.

Take $N \in \mathbb{N}$ large enough such that

$$\begin{aligned} & x^* + \varepsilon \geq x_n & \forall n \geq N \\ \Rightarrow \quad & x^* + \varepsilon \geq u_N & \\ \Rightarrow \quad & x^* + \varepsilon \geq \lim u_n & \\ \Rightarrow \quad & x^* \geq \lim u_m & \forall n \geq N \end{aligned}$$

On the other hand, there are infinitely many n with $x^* - \varepsilon < x_n \leq u_n$.

Thus, there exists a subsequence of u_n, say u_{n_k}, satisfies

$$\begin{aligned} & x^* - \varepsilon \leq u_{n_k} \\ \Rightarrow \quad & x^* - \varepsilon \leq \lim u_{n_k} = \lim (u_n) \\ \Rightarrow \quad & x* \leq \lim (u_n) \end{aligned}$$

■

Proof of thm. (c) ⇒ (d). ***<u>Goal</u>***:

$$x* = \lim (u_m), u_m = \sup x_n | n \geq m$$

$$\Rightarrow x^* = \sup S \text{ where S is the set of subsequential limits.}$$

Let (x_{n_k}) be a convergent subsequence of (x_n). Notice that $\lim (x_{n_k}) \in S$.

$$\begin{aligned} & n \geq k \\ \Rightarrow \quad & x_{n_k} \leq \sup \{x_n | n \geq k\} = u_k \\ \Rightarrow \quad & \lim (x_{n_k}) \leq \lim (u_k) = x^* \\ \Rightarrow \quad & x^* \text{ is an upper bound of S..} \end{aligned}$$

For 1, $\exists n_1 \in \mathbb{N}$ s.t.

$$u_1 - 1 \leq x_{n_1} \leq u_1$$

For $\frac{1}{2}$, $\exists n_2 \in \mathbb{N}$ s.t.

$$u_2 - \frac{1}{2} \leq x_{n_2} \leq u_2$$

$$\cdots$$

For $\frac{1}{k}$, $\exists n_k \in \mathbb{N}$ s.t.

$$u_k - \frac{1}{k} \leq x_{n_k} \leq u_k$$

When $k \to \infty$,

$$x^* - 0 \leq \lim (x_{n_k}) \leq x^*$$

By squeeze theorem,

$$\lim (x_{n_k}) = x^*$$

■

Proof of thm. (d) ⇒ (a). ***Goal:***

$$x^* = \sup S$$

$$\Rightarrow x^* = \limsup x_n = \inf \{v \in \mathbb{R} | v < x_n \text{for at most finite many of n}\}$$

Fix $\varepsilon > 0$

There is no subsequence of x_n which has a limit exceeding $x^* + \varepsilon$.

$$\begin{aligned}
&\Rightarrow \text{ There is only finitely many n with } x_n > x^* + \varepsilon. \\
&\Rightarrow \quad x^* + \varepsilon \in X \\
&\Rightarrow \quad \inf X \leq x^* + \varepsilon \\
&\Rightarrow \quad \limsup x_n \leq x^* + \varepsilon \\
&\Rightarrow \quad \limsup x_n \leq x^*
\end{aligned}$$

Next, consider $x^* - \varepsilon$

Then, there exists a subsequential limit of x_n which is greater or equal to $x^* - \frac{1}{2}\varepsilon$.

There exists a convergent subsequence of (x_n), say (x_{n_k}), such that

$$\begin{aligned}
&\lim (x_{n_k}) \geq x^* - \frac{1}{2}\varepsilon \\
\Rightarrow \quad &\text{There are infinitely many n with } x^* > x^* - \varepsilon. \\
\Rightarrow \quad &\forall a \in X, x^* - \varepsilon \leq a \\
\Rightarrow \quad &x^* - \varepsilon \leq \inf X \\
\Rightarrow \quad &\limsup x_n \geq x^* - \varepsilon \\
\Rightarrow \quad &\limsup x_n \geq x^*
\end{aligned}$$

In conclusion,

$$\limsup x_n = x^*$$

∎

3.5 Cauchy Criterion

Definition (Cauchy Sequence). A sequence (x_n) is ***Cauchy sequence*** if
$\forall \varepsilon > 0, \exists H \in \mathbb{N}$ s.t. $\forall n, m \in \mathbb{N}, n > 0, m > 0,$

$$|x_n - x_m| < \varepsilon$$

Example. $(\frac{1}{n})$ is a Cauchy sequence.

Proof. Observe that $\forall n, m \in \mathbb{N}, n \geq m,$

$$\left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{m-n}{mn}\right| = \frac{n-m}{mn} \leq \frac{n}{mn} < \frac{1}{m}$$

Choose $H = \lceil \frac{1}{\varepsilon} \rceil + 1$, Then
$\forall n, m \geq H,$

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{m} < \varepsilon$$

■

Example. $(-1)^n$ is not a Cauchy sequence.

Proof. Choose $\varepsilon_0 = \frac{1}{2}, \forall H \in \mathbb{N}$, choose $n, m \geq H$ s.t. n is even, m is odd. Then

$$|(-1)^n - (-1)^m| = |1 - (-1)| = 2 > \frac{1}{2}$$

Thus $(-1)^n$ does not satisfies the definition of Cauchy sequence.

■

Theorem. If (x_n) is convergent, then it is Cauchy sequence.

Proof. Let $\lim (x_n) = x$.
Goal:

$$|x_n - x| < \varepsilon \Rightarrow |x_n - x_m| < \varepsilon$$

$\forall n, m \geq N_\varepsilon,$

$$\begin{aligned} |x_n - x_m| &= |(x_n - x) + (x - x_m)| \\ &\leq |x_n - x| + |x - x_m| \text{ by triangle inequality} \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \end{aligned}$$

■

Lemma. If (x_n) is Cauchy, the it is bounded.

Proof. Choose $\varepsilon = 1, \exists H \geq 0$ s.t. $\forall n, m \geq H$,

$$|x_n - x_m| < 1$$

Since the choice of m satisfies $m \geq H$, we may choose $m = H$ s.t.

$$|x_n - x_H| < 1$$

It follows that

$$|x_n| - |x_H| \leq |x_n - x_H| < 1$$
$$|x_n| < |x_H| + 1, \forall n \geq H$$

Let

$$M := \max\{|x_1|, |x_2|, |x_3|, \ldots, , |x_{H-1}|, |x_H| + 1\}$$

Thus $\forall n \in \mathbb{N}, |x_n| \leq M$ ■

Theorem (Cauchy Convergence Theorem). A sequence of real numbers is ***convergent*** if and only if it is ***Cauchy*** sequence.

(⇒). Done in the previous theorem. ■

(⇐). Suppose (x_n) is Cauchy. By lemma, it is bounded.

By Bolzano-Weierstrass theorem, there exists a convergent subsequence (x_{n_k}).

Let $\lim(x_{n_k}) = x$.

Goal: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$

$$|x_n - x| < \varepsilon$$

We will use the trick of ***insert subsequence:***

By definition of convergence, $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall k \geq N_\varepsilon$

$$|x_{n_k} - x| < \frac{1}{2}\varepsilon$$

By definition of Cauchy sequence, $\forall \varepsilon > 0, \exists H \in \mathbb{N}, H \geq 0$ s.t. $\forall n, m \geq H$

$$|x_n - x_m| < \frac{1}{2}\varepsilon$$

Thus $\forall k \geq \max\{H, N_\varepsilon\}$,

$$\begin{aligned} |x_k - x| &= |(x_k - x_{n_k}) + (x_{n_k} - x)| \\ &\leq |x_k - x_{n_K}| + |x_{n_k} - x| \text{ since } n_k \geq k \geq \max\{H, N_\varepsilon\} \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \end{aligned}$$

■

Example. Let (h_n) be the sequence of harmonic series such that

$$h_n = \sum_{i=1}^{n} \frac{1}{i} \text{ , and } \lim(h_n) = \sum_{n=1}^{\infty} \frac{1}{n}$$

Claim: (h_n) is divergent

Goal: show that it is NOT Cauchy.

Proof. $\forall m, n \in \mathbb{N}$, WLOG suppose $m \geq n, h_m > h_n$, we have

$$\begin{aligned} |h_m - h_n| &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m-1} + \frac{1}{m} \\ &\geq \frac{m-n}{n} \end{aligned}$$

since there are (m-n)'s terms on the right side of the equation.

So we choose $\varepsilon_0 = \frac{1}{2}, \forall H \geq 0$, choose $n = H, m = 2H$, then

$$|h_m - h_n| \geq \frac{m-n}{m} = \frac{1}{2}$$

Thus, (h_n) is NOT Cauchy. ■

3.6 Property of Divergent Sequence

Example (divergent sequence)**.** :

- $(n) \to \infty$
- $(-n) \to -\infty$
- $(-1)^n \cdot n$ is divergent and unbounded.
- $(-1)^n$ is divergent and bounded.

Definition (Properly Divergent). Let (x_n) be a sequence of real numbers. We say that

1. (x_n) ***tends to*** ∞, or $\lim(x_n) = \infty$ if $\forall a \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,
$$x_n > a$$
2. Similarly, (x_n) ***tends to*** $-\infty$, or $\lim(x_n) = -\infty$ if $\forall a \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,
$$x_n < a$$

In either case, we say that (x_n) is ***properly divergent***.

Example. Let $C > 0$. (C^n) is properly divergent. We write $\lim(C^n) = \infty$.

Proof. Notice that

$$C^n = [1 + (C-1)]^n \geq 1 + n(C-1) \text{ by Bernoulli's inequality}$$

Goal: $\forall a \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$1 + n(C-1) > a_n \Leftrightarrow n > \frac{a_n - 1}{C - 1}$$

Then we choose $N = \lceil \frac{a-1}{c-1} \rceil + 1$. Thus $\forall n \geq N, C^n > a$. ∎

Theorem. A monotone sequence is ***divergent*** if and only if it is ***bounded***.

Proof. Exercise. ∎

Theorem (Comparison Test). Let (x_n) and (y_n) be two sequences. Suppose $\forall n \in \mathbb{N}, x_n \geq y_n$. Then,

1. If $(x_n) \to \infty$, then $(y_n) \to \infty$.
2. If $(y_n) \to -\infty$, then $(x_n) \to -\infty$.

Proof. Exercise. ∎

Theorem (Limit Comparison Test). Let (x_n) and (y_n) be two sequences of positive real numbers. Suppose $\exists L \in \mathbb{R}, L > 0$ s.t.

$$\exists \lim(\frac{x_n}{y_n}) = L$$

then $\lim(x_n) = \infty$ if and only if $\lim(y_n) = \infty$

Proof. ***Claim:*** For N large enough,

$$\frac{1}{2}L \cdot y_n < x_n < 2L \cdot y_n$$

Goal: Claim+Comparison Theorem=Proof.

By definition of limit, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N$,

$$L - \varepsilon < \frac{x_n}{y_n} < L + \varepsilon$$

Choose $\varepsilon = \frac{1}{2}L$, we have

$$\frac{1}{2}L \cdot y_n < x_n < \frac{3}{2}L < 2L \cdot y_n$$

since $L > 0$ and these are positive sequences.

Thus by Comparison Theorem,

$$\lim(\frac{1}{2}L \cdot y_n) = \infty \Rightarrow \lim(x_n) = \infty$$

■

3.7 Introduction to Infinite Series

Definition (Infinite Series). If (x_n) is a sequence of real numbers, then the ***infinite series*** generated by (x_n) is the sequence (s_k) defined by

$$s_k = \sum_{i=1}^{k} x_i$$

Terms s_k are called ***partial sums***.

Notation. $\sum x_i$ to mean this series or its limit at infinity $\lim(x_k)$.

Theorem (Cauchy Criterion for Series). The series $\sum x_i$ ***converges*** if and only if $\forall \varepsilon > 0, \exists M \in \mathbb{N}$ s.t. $\forall n, m \in \mathbb{N}, n > m \geq M$,

$$|x_{m+1} + x_{m+2} + \cdots + x_{n-1} + x_n| < \varepsilon$$

Or we write it as

$$|s_k - s_m| < \varepsilon$$

Proof: Exercise. ■

Theorem (Montone Convergence for Series)**.** Let (x_n) be a sequence of non-negative real numbers. Then the series $\sum x_n$ ***converges*** if and only if (s_k) is bounded.

Proof. Exercise.

■

Example. $\sum \frac{1}{n^2}$ is convergent.

Proof. ***Goal:*** find convergent subsequence (s_{k_j}).

Consider subsequence (s_{k_j}) where $k_j = 2^j - 1$.

Observe that:

$$\begin{aligned}
s_{k_1} &= 1 \\
s_{k_2} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} \\
&< s_{k_1} + 2 \cdot \frac{1}{2^2} = 1 + \frac{1}{2} \\
s_{k_3} &= 1 + (\frac{1}{2^2} + \frac{1}{3^2}) + (\frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{9^2}) \\
&< s_{k_1} + 2 \cdot \frac{1}{2^2} = 1 + \frac{1}{2}
\end{aligned}$$

By induction(details are left as exercise), one can show that

$$s_{k_j} < \sum_{n=0}^{j-1} \frac{1}{2^n} < \sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \text{ (by limit of geometric series.)}$$

Thus (s_{k_j}) is bounded.

By theorem proved in homework, an increasing sequence with bounded(and thus convergent) subsequence implies that the sequence is convergent. ■

4 Limits

4.1 Limits of Functions

Let $f : A \to B$ be a function where $A, B \subseteq \mathbb{R}$. Let $a \in A, L \in B$.

Goal: Define

$$\lim_{x \to a} f(x) = ?$$

Intuition: Define ***closeness*** on real line.

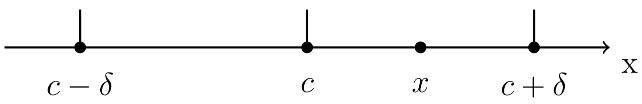
Definition (Cluster Point). Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a ***cluster point*** of A if

$\forall \delta > 0, \exists x \in A, x \neq c$ such that

$$|x - c| < \delta$$

Or

$$V_\delta(c) \cap (A\ \{c\}) \neq \phi$$



Theorem. $c \in \mathbb{R}$ is a cluster point if and only if

there exists a sequence $(a_n) \in A$ such that

$$\lim(a_n) = c \text{ and } \forall n \in \mathbb{N}, a_n \neq c$$

Sketch of Proof. $(\Rightarrow)$

$$\begin{aligned}
&\delta = 1, \exists a_1 \in A \backslash \{c\} \text{ s.t. } |a_1 - c| < 1 \\
&\delta = \frac{1}{2}, \exists a_2 \in A \backslash \{c\} \text{ s.t. } |a_2 - c| < \frac{1}{2} \\
&\ldots \text{ observe that:} \\
&\forall \delta > 0, \exists a_n \in A \backslash \{c\} \text{ s.t. } |a_n - c| < \frac{1}{n} < \delta \\
&\Rightarrow \lim(a_n) = c \text{ by squeeze theorem.}
\end{aligned}$$

$(\Leftarrow)$

$\forall \delta > 0, \exists N_\varepsilon > 0$ s.t.

$$|a_{N_\varepsilon} - c| < \delta$$

Note that $a_{N_\varepsilon} \in A \backslash \{c\}$ where c is a cluster point of A. ■

Example. Let X be the set of cluster points of A.

- $A = (1,2) \cup (3,4) \Rightarrow X = [1,2] \cup [3,4]$. *Proof: Exercise.*
- $A = \{0\} \cup (1,2) \Rightarrow X = \phi$.

  Sketch of (2). :

  1. Let $x \in [1,2]$. Prove that $x \in X$. Thus $[1,2] \in X$.
  2. Prove that $0 \notin X$.
  3. Prove that $x \in X$ if $x \notin \{0\} \cup [1,2]$.

■

Remark. A may not be a subset of the set of cluster points of A.
Example. :

- $A = \mathbb{Z} \Rightarrow X = \phi$.
- $A = \{\frac{1}{n} | \, n \in \mathbb{N}\} \Rightarrow X = \{0\}$ *Proof: Exercise.*

Definition (**Delta-Epsilon Definition of Limit**)**.** Let $A \subseteq \mathbb{R}$, c is a cluster point of A, $f : A \to \mathbb{R}$. A real number L is the ***limit of f at c*** if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$,

$$0 < |x - c| < \delta \to |f(x) - L| < \varepsilon$$

Theorem (**Uniqueness of Limit**)**.** If $f : A \to \mathbb{R}$ and c is a cluster of points of A, then f has at most 1 limit at c.

Proof. We will prove this by contradiction.

Let L_1 and L_2 be limits of f at c. Assume $L_1 \neq L_2$. Choose $\varepsilon = \frac{|L_1 - L_2|}{2} > 0$. Then,

$$\exists \delta_1 \text{ s.t. } 0 < |x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \frac{\varepsilon}{2}$$

$$\exists \delta_2 \text{ s.t. } 0 < |x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \frac{\varepsilon}{2}$$

Consider $\delta := \min\{\delta_1, \delta_2\}$.

Since c is a cluster point, $\exists x_0 \in A$ s.t.

$$0 < |x_0 - c| < \delta$$

Since

$$|f(x_0) - L_1| < \frac{\varepsilon}{2}, |f(x_0) - L_2| < \frac{\varepsilon}{2}$$

We have

$$\begin{aligned} |L_1 - L_2| &\leq |L_1 - f(x_0)| + |f(x_0) - L_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \\ &= \frac{|L_1 - L_2|}{2} \end{aligned}$$

which is a contradiction.

■

Notation.

$$L = \lim_{x \to c} f(x) \text{ or } L = \lim_{x \to c} f$$

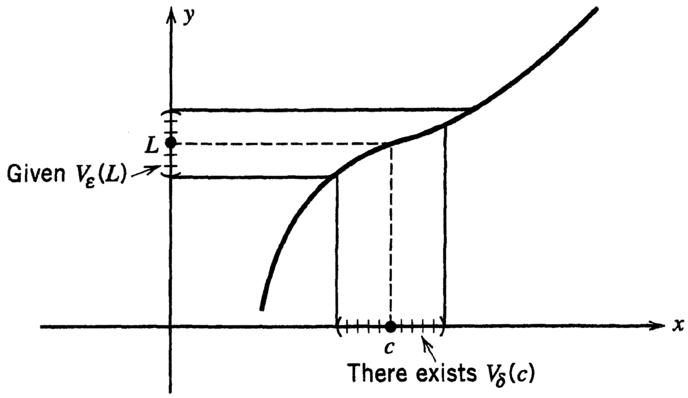
And we say that *f(x) approaches to L as x approaches to c.*

Remark (Divergence of function). If the limit of $f(x)$ at c does not exists, we say that f ***diverges*** at c.

Theorem. Let $f : A \to \mathbb{R}$ and c be a cluster point of f. The following are equivalent:

1. $\lim_{x \to c} f(x) = L$
2. $\forall\, V_\varepsilon(L)$ ε-neighborhood of L, $\exists\, V_\delta(c)$ δ-neighborhood of c s.t.

$$x \in V_\delta(c) \cap (A \backslash \{c\}) \Rightarrow f(x) \in V_\varepsilon(c)$$



Example. $\lim_{x \to c} f(x) = a$.

Proof. $\forall \varepsilon > 0, \exists \delta = 1$ s.t.

$$x \in V_\delta(c) \cap (\mathbb{R} \backslash \{c\}) \Rightarrow f(x) = a \in V_\varepsilon(L)$$

■

Example. $\lim_{x \to c} f(x) = c$.

Proof. $\forall \varepsilon > 0, \exists \delta = \varepsilon$ s.t.

$$\begin{aligned} & x \in V_\delta(c) \cap (A \backslash \{c\}) \\ \Rightarrow & f(x) = x \in V_\delta(c) \cap (A \backslash \{c\}) \\ \Rightarrow & f(x) \in V_\varepsilon(c) \cap (A \backslash \{c\}) \subseteq V_\varepsilon(c) \end{aligned}$$

■

Example.

$$\lim_{x \to c} x^2 = c^2$$

Proof. ***Goal***: $\forall \varepsilon > 0$, find a $\delta(c, \varepsilon) > 0$ s.t.

$$\text{if } 0 < |x - c| < \delta, \text{ then } \left|x^2 - c^2\right| < \varepsilon$$

which is equivalent to show

$$|x + c|\,|x - c| < \varepsilon$$

Since the choice of δ is dependent on ε and c only, $|x - c|$ can be easily confined with some constant. Let's assume that

$$|x - c| < 1$$

Now, the only task left is to find a way to confine $|x + c|$ with a constant by manipulating $|x - c| < 1$.

Here we will apply a common trick that estimates addition $|x + c|$ with subtraction $|x - c|$:

$$|x| - |c| \leq |x - c| < 1 \text{ by triangle inequality.}$$

Rearranging the inequality, we have:

$$|x| < |c| + 1$$

Adding the second $|c|$ to both side of inequality, then, we apply triangle inequality again:

$$|x + c| \leq |x| + |c| < 2\,|c| + 1$$

$$|x + c| < 2\,|c| + 1$$

Notice that this is equivalent to show

$$|x + c|\,|x - c| < (2\,|c| + \delta)\,|x - c| < \varepsilon$$

Rearrange the constant factor

$$|x - c| < \frac{\varepsilon}{2\,|c| + 1}$$

Thus, our choice of δ must satisfies two conditions at the same time:

$$\begin{cases} |x - c| < 1 \\ |x - c| < \frac{\varepsilon}{2|c|+1} \end{cases}$$

We achieve this by simply choosing

$$\delta = \min\left\{\frac{\varepsilon}{2\,|c| + 1}, 1\right\}$$

■

Remark (***The Toolbox of Proofs***)**.** The readers should develop their "toolbox" of proof techniques. That is, the ***estimation against a constant*** + manipulation of ***triangle inequality*** + choice of δ that satisfies ***multiple conditions***.

Example (Harder)**.**

$$\lim_{x\to c}\frac{1}{x}=\frac{1}{c}$$

Proof. Observe that

$$\left|\frac{1}{x}-\frac{1}{c}\right|=\frac{|x-c|}{|cx|}$$

Here, we cannot choose $|x-c|$ smaller than some constant(why? try it on your own). Instead, we choose

$$|x-c|<\frac{|c|}{2}$$

By triangle inequality,

$$\frac{|c|}{2}<|x|<\frac{3\,|c|}{2}$$

Multiply $|c|$ to each term of the inequality,

$$\frac{c^2}{2}<|cx|<\frac{3c^2}{2}$$

Thus

$$\frac{1}{|cx|}<\frac{2}{c^2}$$

It follows that

$$\frac{|x-c|}{|cx|}<\frac{2}{c^2}\,|x-c|$$

In order to make the left term of the inequality less than ε, it suffices to confine

$$\frac{2}{c^2}\,|x-c|<\varepsilon$$

$$|x-c|<\frac{c^2}{2}\varepsilon$$

Thus, our choice of δ must satisfies two conditions at the same time:

$$\begin{cases}|x-c|<\frac{|c|}{2}\\ |x-c|<\frac{c^2}{2}\varepsilon\end{cases}$$

We achieve this by simply choosing

$$\delta=\min\left\{\frac{|c|}{2},\frac{c^2}{2}\varepsilon\right\}$$

■

Remark (***Factoring*** $|x-c|$)**.** Since we have the premise of $|x-c|<\delta$ for free, it would be much easier for us to confine $|f(x)-L|$ if we factor $|x-c|$ from the difference.

Example (Much Harder)**.**

$$\forall n \in \mathbb{N}, \lim_{x \to c} x^n = c^n$$

Proof. By difference of n-th powers factorization

$$|x^n - c^n| = |x - c| \left| \sum_{i=0}^{n-1} x^i c^{n-1-i} \right| \leq |x - c| \cdot \sum_{i=0}^{n-1} |x|^i |c|^{n-1-i}$$

If

$$|x - c| < 1$$

then by triangle inequality,

$$|x| < |c| + 1$$

It follows that

$$|x^n - c^n| < |x - c| \cdot \sum_{i=0}^{n-1} (|c| + 1)^i \left| c^{n-1-i} \right|$$

Similar to the previous example, it suffices to confine

$$|x - c| \cdot \sum_{i=0}^{n-1} (|c| + 1)^i \left| c^{n-1-i} \right| < \varepsilon$$

$$|x - c| < \frac{\varepsilon}{\sum_{i=0}^{n-1} (|c| + 1)^i |c^{n-1-i}|}$$

Thus, choose

$$\delta = \min\{1, \frac{\varepsilon}{\sum_{i=0}^{n-1} (|c| + 1)^i |c^{n-1-i}|}\}$$

■

Remark (***Estimate*** x ***against*** c)**.** This is another trick by triangle inequality:

If $\exists k \in \mathbb{R}, k > 0$ s.t. $|x - c| < k$, then $|x| < |c| + k$

Remark (***difference of*** n***-th powers factorization***)**.**

$$(x^n - c^n) = (x - c)(x^{n-1} + cx^{n-2} + \cdots + c^{n-2}x + c^{n-1}) = (x - c) \sum_{i=0}^{n-1} x^i c^{n-1-i}$$

Example (*Some Tedious Factorization*).

$$\lim_{x \to 2} \frac{x^3 + 2x - 1}{6x^2 - 5} = \frac{11}{19}$$

Proof.

$$\left| \frac{x^3 + 2x - 1}{6x^2 - 5} - \frac{11}{19} \right| = \left| \frac{19x^3 - 66x^2 + 38x + 36}{19(6x^2 - 5)} \right|$$

By some tedious factorization,

$$... = |x - 2| \frac{|19x^3 - 28x - 18|}{19\,|6x^2 - 5|}$$

We estimate $|x - 2|$ with constant 1

$$|x - 2| < 1$$

$$1 < x < 3 \text{ or } |x| < 3$$

It follows that

$$1 < x^2 < 9$$

$$1 < 6x^2 - 5 < 49$$

$$1 > \frac{1}{6x^2 - 5} > \frac{1}{49}$$

So

$$\frac{1}{19\,|6x^2 - 5|} < \frac{1}{19}$$

Similarly,

$$\left|19x^3 - 66x^2 + 38x + 36\right| \leq 19\,|x|^2 + 28\,|x| + 18 < 19 \cdot 3^2 + 28 \cdot 3 + 18 = 273$$

Thus

$$... \leq |x - 2| \cdot \frac{273}{19} < \varepsilon$$

$$|x - 2| < \frac{19}{273}\varepsilon$$

We conclude that it suffices to choose

$$\delta = \min\left\{1, \frac{19}{273}\varepsilon\right\}$$

■

Theorem (***Sequential Criterion of Limits***)**.** Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. $\lim_{x\to c} f = L$ if and only if for all sequence $(x_n) \in A$ that converge to c and $(x_n) \neq c, \forall n \in \mathbb{N}$, $(f(x_n))$ converges to L.

Proof. $(\Rightarrow)$. By definition of convergent sequence, $\forall \delta > 0, \exists K \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \geq K$,

$$|x_n - c| < \delta$$

By definition of limit, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$,

$$|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Thus, choose δ given ε, and choose K given δ, we have:

$$|x_n - c| < \delta \Rightarrow |f(x_n) - L| < \varepsilon$$

■

$(\Leftarrow)$. We will prove this by contrapositive.

Assume that there exists $\varepsilon_0 > 0$ and a $(x_n) \in A$ converges to c with $(x_n) \neq c$ such that for all $n \in \mathbb{N}$

$$0 < |x_n - c| < \frac{1}{n} \Rightarrow |f(x_n) - L| \geq \varepsilon_0$$

Thus the function does not have a limit at c. We thereby conclude the converse of the statement.

■

Theorem (***Sequential Criterion of Divergence***)**.** Let $A \subseteq \mathbb{R}, f : A \to \mathbb{R}$, c be a cluster point of A, $(x_n) \to c$ s.t. $(x_n) \neq c$, and $L \in \mathbb{R}$[1].

1. L is NOT the limit of f at $c \iff f(x_n)$ does NOT converge to L.
2. f diverges $\iff f(x_n)$ does NOT converge.

Proof. Exercise. ■

Example.

$$\lim_{x\to 0} \frac{1}{x} \text{ does NOT exists in } \mathbb{R}$$

Proof. Let $(x_n) = \frac{1}{n} \to 0$. Then

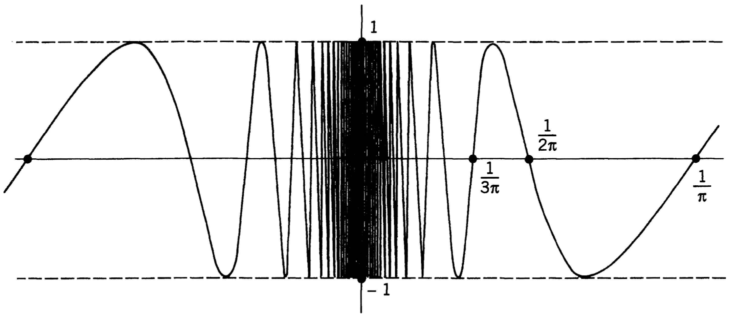
$$f(x_n) = \frac{1}{\frac{1}{n}} = n \to \infty$$

■

[1]The following theorems are NOT equivalent!!

Example.

$$\lim_{x\to 0} \sin(\frac{1}{x}) \text{ DNE in } \mathbb{R}$$



Proof. Let $(x_n) = \dfrac{1}{2n\pi} \to 0$ and $(y_n) = \dfrac{1}{\frac{1}{2}\pi + 2n\pi} \to 0$. Then

$$f(x_n) = \sin(2n\pi) = 0$$

$$f(y_n) = \sin(\frac{1}{2}\pi + 2n\pi) = 1$$

Thus limit of f at 0 does NOT exists in $\mathbb{R}$. ∎

Example.
Definition (***Signum Function***)**.** Let $A \subseteq \mathbb{R}$ and $sgn(x) : A \to \mathbb{R}$ s.t.

$$sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$\lim_{x\to 0} sgn(x) \text{ DNE in } \mathbb{R}$$

Proof. Let $(x_n) = \dfrac{1}{n} \to 0$ and $(y_n) = \dfrac{1}{-n} \to 0$. Then

$$f(x_n) = sgn(n) = 1$$

$$f(y_n) = sgn(-n) = -1$$

Thus limit of f at 0 does NOT exists in $\mathbb{R}$. ∎

4.2 Limit Theorem

Definition (***Bounded Neighborhood of c***)**.** Let $A \subseteq \mathbb{R}, f : A \to \mathbb{R}$, c be a cluster point of A. Then we say f is a ***bounded Neighborhood of c*** if there exists $\delta > 0$ and $\exists M > 0$ s.t. $\forall x \in A \cap V_\delta(c)$

$$|f(x)| \leq M$$

Theorem (***Existence of Bounded Neighborhood at Limit***)**.** If f has a limit at c, then f is bounded on some neighborhood of c.

Proof. By definition of limit, $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A \backslash \{c\}$

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$$

By triangle inequality,

$$|f(x)| - |L| \leq |f(x) - L| < \epsilon$$
$$|f(x)| < |L| + \epsilon$$

If $f(c)$ is not defined on A, then let $M = |L| + \epsilon$. If $f(c)$ is defined on A, then let $M = \max\{f(c), |L| + \varepsilon\}$. Since the choice of ε is arbitrary, we will choose $\epsilon = 1$. Thus,

$$f(x) \leq M$$

■

Definition. Let $A \subseteq \mathbb{R}$; $f, g : A \to \mathbb{R}$, c be a cluster point of A. Then $\forall x \in A$

- ***Sum*** of function: $(f + g)(x) = f(x) + g(x)$.
- ***Difference:*** $(f - g)(x) = f(x) - g(x)$.
- ***Multiple:*** $(bf)(x) = b \cdot f(x)$ for some $b \in \mathbb{R}$.
- ***Product:*** $(f \cdot g)(x) = f(x) \cdot g(x)$.
- ***Quotient:*** $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ if $g(x) \neq 0$.

Theorem (***Limit Theorem***)**.** If $\exists L, M \in R$ s.t. $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

- $\lim_{x\to c} (f + g)(x) = L + M$.
- $\lim_{x\to c} (f - g)(x) = L - M$.
- $\lim_{x\to c} (bf)(x) = b \cdot L$.
- $\lim_{x\to c} (f \cdot g)(x) = L \cdot M$.
- $\lim_{x\to c} (\frac{f}{g})(x) = \frac{L}{M}$ if $\lim_{x\to c} g(x) \neq 0$.

Proof. Exercise. ■

Remark. Always check conditions before applying: this is true only if both f and g has a limit at c !

Example. $\lim_{x\to 4} \frac{(x-4)(x+3)}{4(x-4)(x-5)} = \lim_{x\to 4} \frac{x+3}{4(x-5)} = -\frac{7}{4}$

Corollary (*Polynomial Function*).

$$\lim_{x\to c} p(x) = \lim_{x\to c} \sum_{i=0}^{n} a_i \cdot x^i = \sum_{i=0}^{n} a_i \cdot \lim_{x\to c} x^i = \sum_{i=0}^{n} a_i \cdot x^i = p(c)$$

Corollary (*Rational Function*). For polynomial functions $p(x), q(x)$ s.t. $\lim_{x\to c} p(x) \to p(c), \lim_{x\to c} q(x) \to q(c) \neq 0$,

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$

Theorem. Let $A \subseteq \mathbb{R}$; $f, g : A \to \mathbb{R}$, c be a cluster point of A.

If $\exists a, b \in \mathbb{R}, \forall x \in A, x \neq c$ satisfying

$$a \leq f(x) \leq b \text{ and } \lim_{x\to c} f = L$$

then

$$a \leq \lim_{x\to c} f \leq b$$

Proof. $\forall (x_n) \in A\backslash\{c\}, (x_n) \to c$,

$$a \leq f(x_n) \leq b \Rightarrow a \leq L \leq b$$

■

Theorem (*Squeeze Theorem*). Let $A \subseteq \mathbb{R}$; $f, g, h : A \to \mathbb{R}$, c be a cluster point of A.

If $\forall x \in A, x \neq c$,

$$f(x) \leq g(x) \leq h(x) \text{ and } \lim_{x\to c} f = L = \lim_{x\to c} h$$

then

$$\lim_{x\to c} g = L$$

Proof. Exercise. Try sequeeze theorem of sequence. ■

Example. $\lim_{x\to 0} x \sin(\frac{1}{x}) = 0$

Proof. Notice that

$$-1 \leq \sin(\frac{1}{x}) \leq 1 \Rightarrow -x \leq x \sin(\frac{1}{x}) \leq x$$

Since

$$\lim_{x\to 0} (-x) = 0 = \lim_{x\to 0} (x)$$

$$\lim_{x\to 0} x \sin(\frac{1}{x}) = 0$$

■

Example. Let $b \in \mathbb{R}, b > 0$. Then $\lim\limits_{x \to 0} x^b = 0$. Notice that when $x \in [0, 1]$

$$x^{\lceil b \rceil} \leq x^b < x^{\lfloor b \rfloor}$$

The rest is left as exercise.
Theorem. Let $A \subseteq \mathbb{R}$; $f, g, h : A \to \mathbb{R}$, c be a cluster point of A. If $\lim\limits_{x \to c} f > 0$, then $\exists \delta > 0$ s.t. $V_\delta(c)$ s.t. $\forall x \in A \cap V_\delta(c) \backslash \{c\}$,

$$f(x) > 0$$

Proof. Excercise. ■

5 Continuous Functions

5.1 Continuity

Definition (***Counituous***). Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$.

We say f is ***countinuous at c*** if

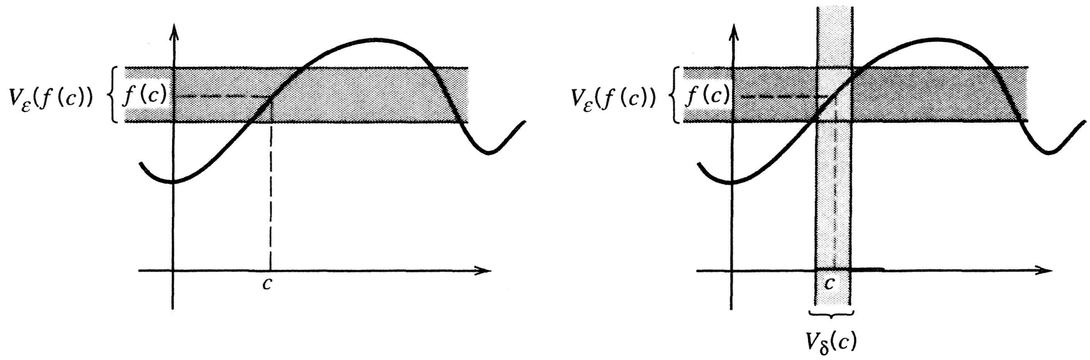
$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

We say f is ***discontinuous at c*** if f is NOT continuous at c.

Definition (***Using Neighborhood***). Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. Then f is continuous at c if and only if
$\forall \varepsilon > 0$ and its ε-neighborhood at $f(c)$, $V_{f(c)}(\varepsilon)$, $\exists \delta > 0$ and its δ-neighborhood at c , $V_c(\varepsilon)$, s.t.

$$f(V_\delta(c) \cap A) \subseteq V_\varepsilon(f(c))$$



Remark (***Comparison with Limit Definition***). Continuity has 3 good properties that will be useful in the future study of Real Analysis:

- If c is a ***cluster point*** of A, then f is ***continuous*** if and only if
  1. $f(x)$ is defined at c.

     Counter-example: Any function $f : \mathbb{Q} \to \mathbb{R}$ is *undefined* at $\mathbb{R}\backslash\mathbb{Q}$ and thus *discontinuous* at $\mathbb{R}\backslash\mathbb{Q}$.
  2. $\lim\limits_{x \to c} f(x)$ exists.

     Counter-example: $\dfrac{1}{x}$ is *undefined* at 0 and thus *discontinuous* at 0.
  3. $f(c) = \lim\limits_{x \to c} f(x)$.

     Counter-example: $f(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is discontinuous at 0 as

     $f(c) = 0 \neq 1 = \lim\limits_{x \to c} f(x)$

- If c is ***NOT a cluster point*** of A, then c is an ***isolated point***, and $\exists V_\delta(c) \cap A = \{c\}$. Notice that an ***isolated point*** is ***automatically continuous*** as it *satisfies* the definition of continuity at c. This is because

$$|x - c| = |c - c| = 0 < \delta \Rightarrow |f(x) - f(c)| = |f(c) - f(c)| = 0 < \varepsilon$$

Theorem (***Sequential Criterion for Continuity***)**.**

Let $A \subseteq \mathbb{R},\ f : A \to \mathbb{R},\ c \in A$. Then f is continuous at c if and only if

$\forall (x_n) \subseteq A$ s.t. $(x_n) \to c,\ f((x_n)) \to f(c)$.

Proof. $(\Rightarrow)$. Continuity of f at c implies limit of f at c exists[2]. Thus we simply apply sequential criterion of limit.

$(\Leftarrow)$. This is exactly the statement of sequential criterion for limit at c. Thus,

$$\exists \lim_{x \to c} f(x) = f(c)$$

Notice that $c \in A$ implies that f is defined on c. Thus, we conclude that f is continuous on c by previous remark in p.43 ■

Corollary (***Sequential Criterion for Discontinuity***)**.**

Let $A \subseteq \mathbb{R},\ f : A \to \mathbb{R},\ c \in A$. Then f is discontinuous at c if and only if

$\exists (x_n) \subseteq A$ s.t. $(x_n) \to c,\ f((x_n)) \not\to f(c)$.

Proof. Similar. Left to reader as exercise.

■

Example. We begin with polynomial and rational functions:

- Constant function $f(x) = b$ is continuous in $\mathbb{R}$.
- Linear function $f(x) = ax + b$ is continuous in $\mathbb{R}$.
- Quadratic function$f(x) = x^2$ is continuous in $\mathbb{R}$.
- $f(x) = \dfrac{1}{x^2}$ is continuous in $\mathbb{R}$.
- Polynomial functions are continuous in $\mathbb{R}$(try to prove it!).
- Rational functions are continuous in $\mathbb{R}$(try to prove it!).
- $f(x) = \dfrac{1}{x}$ is NOT continuous at 0.
- $f(x) = sgn(x)$ is NOT continuous at 0.

[2]proof by definition

Example (*Dirichlet's "discontinuous function"*).

Let $A = \mathbb{R}$ and define Dirichlet's "discontinuous function" by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \text{ (rational)} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \text{ (irrational)} \end{cases}$$

Claim: this function is discontinuous on $\mathbb{R}$.

Proof. Let $c \in A$.

- If $c \in \mathbb{Q}$, then $\exists (x_n) \in \mathbb{R}\backslash\mathbb{Q}$ s.t. $\forall n \in \mathbb{N}$,

$$c < (x_n) < c + \frac{1}{n}$$

By squeeze theorem,

$$\lim c \leq \lim (x_n) \leq \lim (c + \frac{1}{n})$$

$$(x_n) \to c, \ (c + \frac{1}{n}) \to c$$

$$\lim f(x_n) = 1 \neq 0 = \lim f(c + \frac{1}{n})$$

Thus, by Sequential Criterion of Discontinuity, f is discontinuous for all $c \in \mathbb{Q}$.

- If $c \in \mathbb{R}\backslash\mathbb{Q}$, then we adopt similar strategy. The rest of proofs is left as exercise.

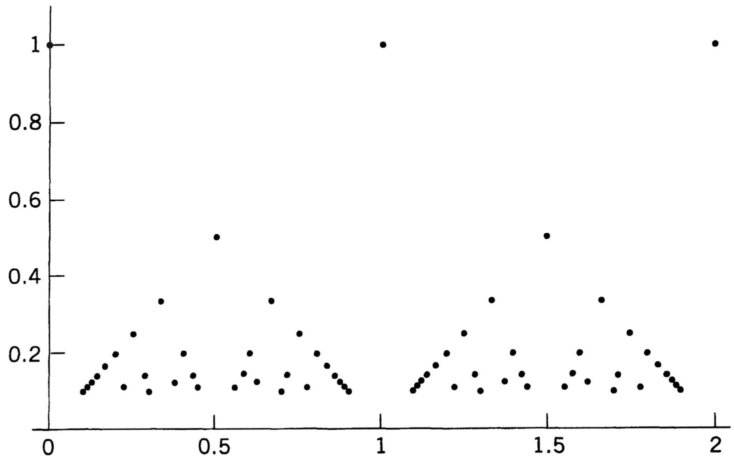
■

Example (*Thomae's function*). *Notice that this example is harder. We skip it on class.

Let $f : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \text{ (irrational)} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ (rational)} \end{cases}$$

Claim: f is discontinuous on $\mathbb{Q}$ and continuous on $\mathbb{R}^+\backslash\mathbb{Q}$.



Proof.

- If c is rational, then $\exists (x_n) \in \mathbb{R}^+ \backslash \mathbb{Q}+$ s.t. $(x_n) \to c$. Then we have

  $$\lim f(x_n) \to 0 \neq \frac{1}{q} = f(c)$$

  Since funtion value does not equal to limit value at c, f is discontinuous at $c \in \mathbb{Q}$

- If c is irrational, ***Goal:*** show that $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in R^+$,

  $$|x - c| < \delta \Rightarrow |f(x) - f(c)| = |f(x)| < \varepsilon$$

  - If $x \in \mathbb{R}^+ \backslash \mathbb{Q}$, then $|f(x)| = 0 < \varepsilon$ for some arbitrary choice of δ.
  - If $x \in \mathbb{Q}$, then by Archimedean Property, we could find a $\frac{1}{n_0} > \varepsilon$.

    Notice that there are only finite number of $n \in \mathbb{N}$ such that $n < n_0$. Thus, there are only finite number of $\dfrac{p}{q} \in (c-1, c+1)$ with denominator $q < n_0$. Hence we could choose δ so small that the $V_\delta(c)$ contains no rational numbers with denominator less than n_0. It follows that

    $$|x - b| < \delta \Rightarrow |f(x) - f(c)| = |f(x)| \leq \frac{1}{n_0} < \epsilon$$

    Thus, f is continuous at $c \in \mathbb{R} \backslash \mathbb{Q}$

■

5.2 Combinations of Continuous Functions

Theorem. Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$, $c \in A$, $b \in R$. Suppose f, g are ***continuous*** on c, then the following combination of functions are ***continuous***:

- ***Addition:*** $f + g$
- ***Subtraction:*** $f - g$
- ***Product:*** $f \cdot g$
- ***Multiplication:*** $b \cdot f$
- ***Quotient:*** $\dfrac{f}{g}$ if $g(x) \neq 0$
- ***Absolute:*** $|f|(x)$
- ***Square Root:*** $\sqrt{f}(x)$

Proof.

- If c is not a cluster point of A, then the theorem is automatically correct.
- If c is a cluster point of A, then we only need to show that the function value at c is equal to the limit of function at c. Detailed proof are left as exercise.

∎

Example.

- Polynomial functions are continuous on $\mathbb{R}$.
- Rational functions are continuous on $\mathbb{R}$.

Example.

- $\sin(x)$ is continuous on $\mathbb{R}$.

  Proof. Notice that $\forall x, y, z \in \mathbb{R}$, we have 3 inequalities:

  $$|\sin(z) \leq |z|\,,\ |\cos(z)| \leq 1|\,,\ \sin(x) - \sin(y) = 2\sin[\frac{1}{2}(x-y)] \cdot \cos[\frac{1}{2}(x+y)]$$

  Hence if $c \in \mathbb{R}$, we have

  $$|\sin(x) - \sin(c)| \leq 2 \cdot \frac{1}{2}\,|x - c| \cdot 1 = |x - c|$$

  Thus $\sin(x)$ is continuous on $\mathbb{R}$. ∎

- $\cos(x)$ is continuous on $\mathbb{R}$

  Proof. Exercise. Similar techniques as above. ∎

- $\tan(x)$, cot, sec, csc are all continuous on *where they are defined.*

  Proof. Combinations of continuous functions on their domain are continuous.

  ∎

Theorem. Let $A, B \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$, $c \in A$ s.t. $f(A) \subseteq B$. Suppose f is continuous at $c \in A$, g is ***continuous*** on $b = f(c) \in B$, then the composition $g \circ f$ is ***continuous*** at c.

Proof. Since f is continuous at c, $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $\forall x \in A$,

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

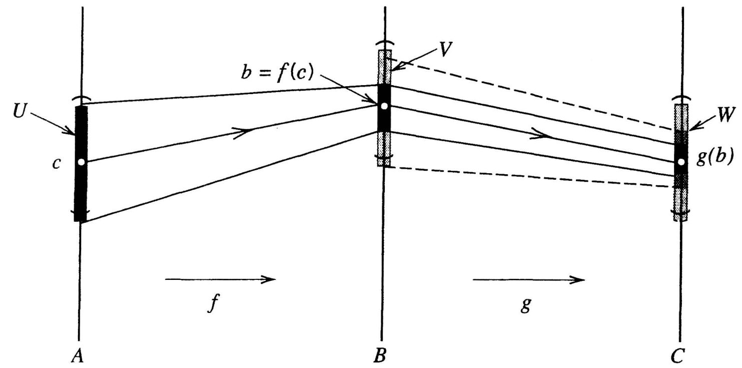
Since g is continuous at $b = f(c)$, $f(A) \subseteq B$, $\forall \zeta > 0,\ \exists \varepsilon > 0$ s.t. $\forall f(x) \in B$,

$$|f(x) - b| < \varepsilon \Rightarrow |g(f(x)) - g(b)| < \zeta$$

Thus,

$$|x - c| < \delta \Rightarrow |g(f(x)) - g(b)| = |g \circ f(x) - g \circ f(c)| < \zeta$$

■



Example. $\sqrt{x^2 + 1}$ is continuous on $\mathbb{R}$.

Proof. Exercise. ■

5.3 Continuous Function on Bounded Intervals

Continuous function on ***closed bounded interval*** has many good properties.

Definition (***Bounded Function***)**.** A function $f : A \to \mathbb{R}$ is said to be ***bounded on A*** if $\exists M > 0$ s.t. $\forall x \in A$

$$|f(x)| \leq M$$

A function is ***unbounded*** if $\forall M > 0,\ \exists x_M \in A$ s.t.

$$|f(x)| > M$$

Example. $f(x) = \dfrac{1}{x}$ is unbounded on $A = (0, \infty)$.

Proof. By Archimedean Property, $\forall M > 0, \exists x_M = \frac{1}{M+1} \in A$ s.t.

$$\left| f(\frac{1}{M+1}) \right| = \left| \frac{1}{\frac{1}{M+1}} \right| = |M+1| > M$$

We thereby conclude that $\dfrac{1}{x}$ is unbounded on A. ■

Theorem (***Boundness Theorem***)**.** Let $a, b \in \mathbb{R},\ a < b,\ I = [a, b]$ a ***closed bounded interval***. If $f : I \to \mathbb{R}$ is countinuous on I, then f is bounded on I.

Proof. Assume f is not bounded on I. Then, by definition, $\forall n \in \mathbb{N},\ \exists x_n \in I$ s.t.

$$|f(x_n)| > n$$

Since I is bounded, the sequence $(x_n) \in I$ is bounded. Therefore, by the Bolzano-Weierstrass Theorem, there exists a convergent subsequence $(x_{n_k}) \to x$ for some number $x \in I$. Since f is continuous on I,

$$f(x_{n_k}) \to f(x)$$

We thereby conclude that $f(x_{n_k})$ is a bounded sequence, which contradicts the fact that $f(x_{n_k})$ is unbounded.

$$\forall k \in \mathbb{N}, n_k \in \mathbb{N},\ |f(x_{n_k})| > n_k \geq k$$

■

Remark. We will give 3 counter-examples to show that all of three conditions are necessary for Boundedness Theorem to be true.

- Interval must be ***closed***
  Counter-example: $f(x) = \dfrac{1}{x}$ on $(0, 1]$ is continuos but unbounded.
- Interval must be ***bounded***
  Counter-example: $g(x) = x$ is continuous but unbounded on $[0, \infty)$.
- The function must be ***continous***
  Counter-example: Define $h : [0, 1] \to \mathbb{R},\ h(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0, 1] \\ 1 & \text{if } x = 0 \end{cases}$ is discontinuous and unbounded.

Definition (***Maximum and Minimum***)**.** Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$.

We say that f has an ***absolute maximum*** on A if $\exists x^* \in A$ s.t. $\forall x \in A$

$$f(x^*) \geq f(x)$$

We say that f has an ***absolute minimum*** on A if $\exists x_* \in A$ s.t. $\forall x \in A$

$$f(x_*) \leq f(x)$$

Remark. Continuous function on a ***bounded set A*** does not necessarily have an maximum or minimum on A. For example:

- $f(x) = \frac{1}{x}$ has NO absolute maximum or absolute minimum on $A = (0, \infty)$
- $g(x) = x^2$ has only absolute minimum $x_* = 0$ on $\mathbb{R}$.

Theorem (***Maximum-Minimum Theorem***)**.** Let $I = [a, b] \in \mathbb{R}$ be a ***closed bounded interval***, $f : I \to \mathbb{R}$ be ***countinuous*** on I. Then f has an absolute maximum and absolute minimum on I.

Proof. By previous theorem, f is bounded on I. Thus,

$$\exists s^* = \sup f(I), \beta = \inf f(I)$$

We will first proof that f has an absolute maximum. It suffices to show that

$$\exists x^* \in I \text{ s.t. } f(x^*) = \sup f(I)$$

Since $s^* = \sup f(I)$, then $\forall n \in \mathbb{N}, s^* - \frac{1}{n}$ is not an upper bound of the set $f(I)$.
Consequently, $\exists x_n \in I$ s.t. $\forall n \in \mathbb{N}$

$$s^* - \frac{1}{n} < f(x^*) \leq s^*$$

Since I is bounded, (x_n) is bounded, by the Bolzano-Weierstrass Theorem, there exists a subsequence $(x_{n_k}) \in I$ s.t. $(x_{n_k}) \to x^*$.
Since f is continuous on I,

$$\lim(f(x_{n_k})) = f(x*)$$

By squeeze theorem,

$$\lim(s^* - \frac{1}{n_k}) \leq {}^3 \lim f(x_{n_k}) \leq \lim s^*$$
$$s^* \leq f(x^*) \leq s^*$$
$$s^* = f(x^*)$$

Thus we conclude that x^* is the absolute maximum of f on I. The proof of absolute minimum is left as exercise using similar techniques. ■

[3] The reader should find out why the strict inequality become weak inequality when limit applies.

The proof of next theorem provides an algorithm, know as ***Bisection Method***, to calculate root to certain level of accuracy.

Theorem (***Location of Roots***)**.** Let $I = [a, b]$, $f : I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$, then $\exists c \in (a, b)$ s.t. $f(c) = 0$.

Proof. WLOG, let's assume that $f(a) < 0 < f(b)$. Define a sequence of closed bounded nested interval I_n and midpointp_n s.t. $\forall n \in \mathbb{N}$:

$$a_1 = a,\ b_1 = b,\ I_1 = [a_1, b_1],\ p_1 = \frac{1}{2}(a_1 + b_1)$$

Notice that if $f(p_n) = 0$, then$c = p_n$ and we are done. If not, then

$$I_n = \begin{cases} [a_{n-1}, p_{n-1}] & \text{if } f(p_{n-1}) > 0 \\ [p_{n-1}, a_{n-1}] & \text{if } f(p_{n-1}) < 0 \end{cases} \subset I_{n-1}$$

Observe that

- $\{I_n\}$ is a infinite sequence of ***nested intervals*** where $\forall n \in \mathbb{N}, I_n \subset I_{n-1}$
- $\lim\{b_n - a_n\} = \lim\{\frac{b-a}{2^{n-1}}\} = 0 \Rightarrow \lim(a_n) = \lim(b_n)$

Then by Nested Interval Property,

$$\exists c \in [a, b] \text{ s.t. } \forall n \in \mathbb{N},\ c \in I_n,\ \bigcap_{n=1}^{\infty} I_n = \{c\}$$

Also notice that

$$a_n < c < b_n \Rightarrow \lim(a_n) \leq c \leq \lim(b_n)$$

By Squeeze Theorem

$$\lim(a_n) = c = \lim(b_n)$$

Notice that

$$\begin{cases} f(a_n) < 0 \Rightarrow & \lim f(a_n) \leq 0 \\ 0 < f(b_n) \Rightarrow & 0 \leq \lim f(b_n) \end{cases}$$

Thus

$$c = \lim(a_n) = \lim(b_n) = 0$$

■

Theorem (*Bolzano's Intermediate Value Theorem*). Let $I \in \mathbb{R}$ be an interval[4], $f : I \to \mathbb{R}$ be continuous on I. If $\exists a, b \in I, k \in \mathbb{R}$ s.t.

$$f(a) < k < f(b)$$

then $\exists c \in I$ s.t.

$$f(a) < f(c) = k < f(b)$$

.

Proof. WLOG, assume $a < b$ and define $g(x) = f(x) - k$. Then

$$g(a) < 0 < g(b)$$

By previous theorem, $\exists c,\ a < c < b$ s.t. $0 = g(c) = f(c) - k$. Thus

$$f(c) = k$$

The similar proof also applies for $b < a$. ■

Corollary. Let $I = [a, b],\ f : I \to \mathbb{R}$ be continuous on I. If $\exists k \in \mathbb{R}$ s.t.

$$\inf f(I) \le k \le \sup f(I)$$

then $\exists c \in I$ s.t.

$$f(c) = k$$

Proof. This is a direct result of previous theorem. ■

Corollary. Let $I = [a, b],\ f : I \to \mathbb{R}$ be continuous on I. Then

$$f(I) = [\inf f(I), \sup f(I)]$$

Proof. Let $m = \inf f(I),\ M = \sup f(I)$. We know by the Maximum-Minimum Theorem that $m, M \in f(I)$. Thus

$$f(I) \subseteq [m, M]$$

Then by Bolzano's Intermediate Value Theorem, $\forall k \in [m, M], \exists c_k \in I$ s.t. $k = f(c_k)$. We thereby conclude that

$$[m, M] \subseteq f(I)$$

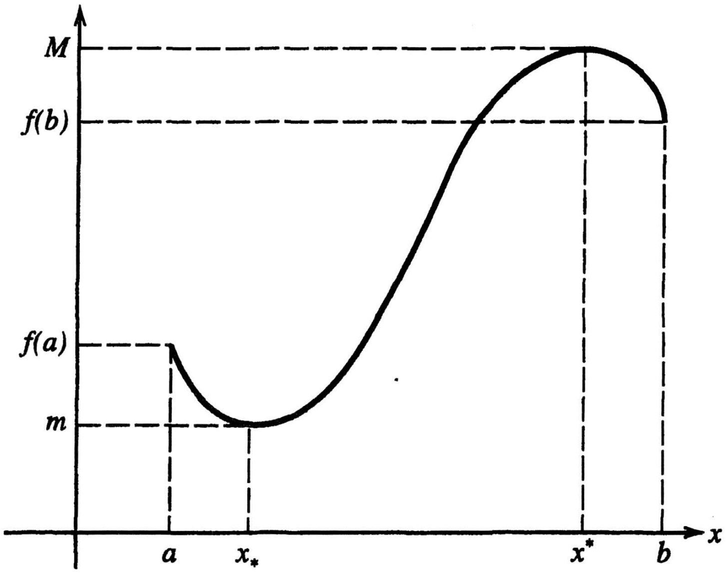
It follows that

$$f(I) = [m, M]$$

■

[4]Not necessarily closed or bounded. Thus this theorem is stronger

Remark. ***End points may not be extreme points***. Counter example:



$$f : [a, b] \to \mathbb{R},\ f(I) \neq [f(a), f(b)]$$

Theorem (*Preservation of Interval Theorem*).

Let $I \in \mathbb{R}$ be an interval[5], $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is an interval.

Proof. Let $\alpha, \beta \in f(I)$ with $\alpha < \beta$. Then $\exists a, b \in I$ s.t.

$$\alpha = f(a),\ \beta = f(b)$$

By Bolzano's Intermediate Value Theorem, $\forall k \in (\alpha, \beta),\ \exists c_k \in I$ s.t. $f(c) = k \in f(I)$. Thus

$$[\alpha, \beta] \subseteq f(I)$$

We conclude that $f(I)$ is an interval ■

[5]Not necessarily closed or bounded. Thus this theorem is stronger

5.4 Uniform Continuity

Recall the definition of continuity of f at $u \in A$:

Let $A \subseteq \mathbb{R},\ f : A \to \mathbb{R},\ \forall \varepsilon > 0, \exists \delta(\varepsilon, u)$ s.t. $\forall x \in A$

$$|x - u| < \delta(\varepsilon, u) \Rightarrow |f(x) - f(u)| < \varepsilon$$

Here we emphasize that the choice of *delta* depends on ***both*** ε ***and*** $u \in A$. This implies that the change of function value $f(u)$ depends on choice of u. Consider $f(x) = \sin(\frac{1}{x})$. As x approaches 0, the function value changes more rapidly.

Example. In this example, δ depends on ε only:

Let $f : \mathbb{R} \to \mathbb{R},\ f(x) = 2x$. Then

$$|f(x) - f(u)| = 2\,|x - u|$$

So it suffice to choose $\delta = \frac{\varepsilon}{2}$.

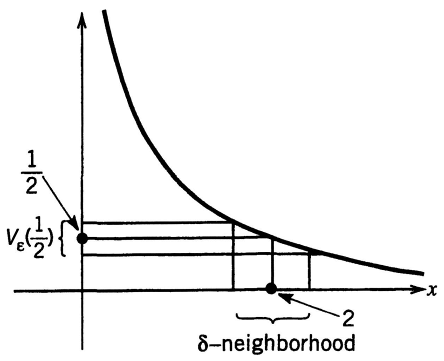
Example. However, in certain cases, δ depends on both ε and u.

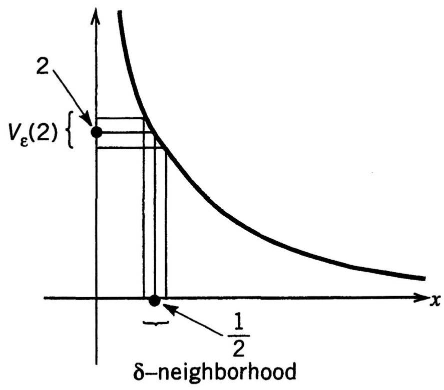
Let $g : (0, \infty) \to \mathbb{R},\ g(x) = \frac{1}{x}$. Then

$$|f(x) - f(u)| = \left|\frac{u - x}{ux}\right|$$

It suffices to choose $\delta(\varepsilon, u) = \inf\{\frac{1}{2}u, \frac{1}{2}u^2\varepsilon\}$.

Notice that there is no way to choose a δ that will work for all $u > 0$. δ must depends on the position of u. As u tends to 0, the permissible value of δ tends to 0.





Definition (***Uniform Continuity***)**.** Let $A \subseteq \mathbb{R},\ f : A \to \mathbb{R}$. We say that f is ***uniformly continuous on*** A if

$\forall \varepsilon > 0, \exists \delta(\varepsilon)$ s.t. $\forall x, y \in A$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Theorem (***Non-uniform Continuity Criterion***)**.** Let $A \subseteq \mathbb{R},\ f : A \to \mathbb{R}$. The following statements are equivalent:

- f is NOT uniformly continuous.
- $\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0,\ \exists x_\delta,\ u_\delta \in A$ s.t.

$$|x_\delta - u_\delta| < \delta \Rightarrow |f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$$

- $\exists \varepsilon_0 > 0,\ \exists (x_n),\ (u_n) \in A$ s.t. $\forall n \in \mathbb{N}$

$$\lim(x_n - u_n) = 0 \text{ and } |f(x_n) - f(u_n)| \geq \varepsilon_0$$

Proof. Exercise. ∎

Theorem (***Uniform Continuity Theorem***)**.** Let I be a ***closed bounded interval*** and let $f : I \to \mathbb{R}$ be ***continuous*** on I. Then f is ***uniformly continuous*** on I.

Proof. Assume f is not uniformly continuous on I. Then by precedence result, $\exists \varepsilon_0 > 0,\ \exists (x_n),\ (u_n),\ \in I$ s.t. $\forall n \in \mathbb{N}$

$$\lim(x_n - y_n) = 0 \text{ and } |f(x_n) - f(u_n)| \geq \varepsilon_0$$

Since I is bounded, the sequence (x_n) is bounded. By Bolzano-Weierstrass Theorem, there exists a convergent subsequence $(x_{n_k}),\ (u_{n_k})$ of $(x_n),\ (u_n)$ where

$$\lim(x_{n_k} - y_{n_k}) = 0 \text{ and } |f(x_{n_k}) - f(u_{n_k})| \geq \varepsilon_0$$

Since I is continuous on I

$$\lim f(x_{n_k}) = f(\lim(x_{n_k})),\ \lim f(u_{n_k}) = f(\lim(u_{n_k}))$$

Thus

$$\lim(f(x_{n_k})) = \lim(f(u_{n_k})) \Rightarrow \lim(f(x_{n_k}) - f(x_{n_k})) = 0$$

contradicts our assumption. We conclude that f must be uniformly continuous. ∎

Note: The next property conveniently ensures uniform continuity without requiring A to be a closed and bounded interval.

Definition (***Lipschitz Functions***)**.** Let $A \subseteq \mathbb{R},\ f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that $\forall x,\ u \in A$

$$|f(x) - f(u)| \leq K\,|x - u|$$

then f is said to be a ***Lipschitz function*** (***Lipschitz continuous***, or just ***Lipschitz***) on A.

Remark (***Lipschitze Function and Gradient***)**.** Rewrite the condition, we have

$$\left|\frac{f(x) - f(u)}{x - u}\right| \leq K$$

It follows that the absolute value is the gradient of a line segment joining the points $(x, f(x)),\ (u, f(u))$. Thus, f is Lipschitz if and only if the gradient of all line segments joining two points on the graph of $y = f(x)$ over I are bounded by some K.

Theorem. Let $A \subseteq \mathbb{R},\ f : A \to \mathbb{R}$. If f is ***Lipschitz***, then it is ***uniformly continuous*** on A.

Proof. For all $\varepsilon > 0$, we simply choose $\delta = \dfrac{\varepsilon}{K}$. Then

$$|x - u| < \delta \Rightarrow |f(x) - f(u)| < K \cdot \frac{\varepsilon}{K} = \varepsilon$$

Thus f is uniformly continuous on A ■

Remark. The ***converse*** may NOT true!

Counter-example: $f(x) = \sqrt{x}$ on $[0, 1]$ is uniformly continuous but NOT Lipschitz.

Example. If $f(x) = x^2$ on $A = [0, b]$ where $b > 0$, then $\forall x,\ y \in [0, b]$

$$|f(x) - f(y)| = |x + y|\,|x - y| \leq 2b\,|x - y|$$

It follows that f is Lipschitz on A given $K = 2b$, and thus f is uniformly continuous.

Theorem. Let $A \subseteq \mathbb{R},\ f : A \to \mathbb{R}$. If f is uniformly continuous on A, and $(x_n) \in A$ is a Cauchy sequence, then $(f(x_n))$ is a Cauchy sequence in R.

Proof. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in \delta$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Since (x_n) is Cauchy, for given $\delta > 0,\ \exists N_\delta \in \mathbb{N}$ s.t. $\forall n, m \in \mathbb{N},\ n, m > N_\delta$

$$|x_n - x_m| < \delta \Rightarrow |f(x_n) - f(x_m)| < \varepsilon$$

We conclude that $(f(x_n))$ is Cauchy. ■

Theorem (*Continuous Extension Theorem*). A function f is ***uniformly continuous*** on ***open*** interval $I = (a, b)$ if and only if it can be ***defined at the endpoints*** a and b such that the extended function is continuous on $[a, b]$.

Proof. ($\Leftarrow$). This direction is trivial.

($\Rightarrow$) Suppose $f : (a, b) \to \mathbb{R}$ is uniformly continuous.

Goal: Define f's extended function $F : [a, b] \to \mathbb{R}$ s.t.

$$F|_{(a,b)} = f,\ F \text{ is continuous}$$

Notice that this is equivalent to show $\lim_{x\to a} f(x)$ and $\lim_{x\to b} f(x)$ exist and define

$$F(x) = \begin{cases} f(x) & \text{if } x \in (a, b) \\ \lim_{x\to a} f(x) & \text{if } x = a \\ \lim_{x\to b} f(x) & \text{if } x = b \end{cases}$$

Choose $(x_n) \in (a, b)$ s.t. $(x_n) \to a$. Thus, (x_n) is Cauchy, and by preceding theorem, $(f(x_n))$ is Cauchy. Let $L = \lim(f(x_n))$. Then $\forall (y_n) \in (a, b)$ s.t. $(y_n) \to a$,

$$\lim(x_n - y_n) = a - a = 0$$

By uniform continuity, we have

$$\begin{aligned} \lim(f(y_n)) &= \lim(f(y_n) - f(x_n)) + \lim(f(x_n)) \\ &= 0 + L = L \end{aligned}$$

Thus, for all sequence (u_n) converging to a, $(f(x_n)) \to L$. By sequential criterion of limit, $L = \lim_{x\to a} f(x)$. If we define $f(a) = L$, then f is continuous at a.

The same argument applies to b. Thus, we conclude that f has a continuous extension to the interval $[a, b]$ ■

5.6 Monotone and Inverse Functions

Note: In this section, we will be focusing on ***monotone functions*** on an ***interval*** I. Specifically, we will discuss ***increasing functions***. It is easy to derive corresponding results for decreasing functions wit similar proof techniques.

Theorem. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on I. Suppose that $c \in I$ is NOT an endpoint of I. Then

1. $\lim_{x\to c-} f = \sup\{f(x) : x \in I,\ x < c\}$
2. $\lim_{x\to c+} f = \inf\{f(x) : x \in I,\ x < c\}$

Recall: $\lim_{x\to c-} f(x) = L$ if $\forall\varepsilon > 0, \exists\delta > 0$ s.t. $\forall x \in I$

$$0 < c - x < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Proof. (1). By monotonicity of f, $\forall x < c$,

$$f(x) \leq f(c)$$

Thus, the set is non-empty, and $f(c)$ is the upper bound of it. This indicates that

$$\exists L = \sup\{f(x) : x \in I, x < c\}$$

Then $\forall\varepsilon > 0,\ L - \varepsilon$ is not an upper bound of the set. Hence $\exists x_\varepsilon \in I, x_\varepsilon < c$ s.t.

$$L - \varepsilon < f(x_\varepsilon) \leq L$$

We choose $\delta = c - x_\varepsilon$. Then $\forall x \in I,\ 0 < c - x < \delta \to x_\epsilon < x$. Thus,

$$-\varepsilon < f(x_\delta) - L \leq f(x) - L \leq 0 < \varepsilon$$

$$|f(x) - L| < \varepsilon$$

Thus

$$L = \lim_{x\to c-} f(x)$$

The proof of (2) is similar. ■

Corollary (*One-sided Limits Criterion for Continuity*). Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on I. Suppose that $c \in I$ is NOT an endpoint of I. Then the following statements are equivalent:

- f is continuous at c.
- $\lim_{x\to c-} f(x) = f(c) = \lim_{x\to c+} f(x)$.
- $\sup\{f(x) : x \in,\ x < c\} = f(c) = \inf\{f(x) : x \in I,\ x > c\}$.

Proof. Exercise. ■

Recall. A function $f : I \to \mathbb{R}$ has ***inverse function*** if and only if f is ***injective***.

Note: It is not difficult to prove that a ***strictly monotone*** function is ***injective*** and thus has an inverse.

Theorem (*Continuous Inverse Theorem*).
Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be ***strictly monotone*** and ***continuous*** on I. Then the inverse function $g : f(I) \to I,\ g(y) = f^{-1}(y)$ is ***strictly monotone*** and ***continuous*** on $f(I)$.

Proof. WLOG, we assume that f is **strictly increasing**. We will first prove that g is strictly increasing.

Suppose $\exists y_1,\ y_2 \in f(I), y_1 < y_2$, then $\exists x_1,\ x_2 \in I$ s.t. $f(x_1) = y_1,\ f(x_2) = y_2$

1. Assume that $x_1 = x_2$.

   Then, $y_1 = f(x_1) = f(x_2) = y_2$ contradicts assumption. So $x_1 \neq x_2$.

2. Assume that $x_1 > x_2$.

   Then $y_1 = f(x_1) > f(x_2) = y_2$ contradicts assumption. So $x_1 \not> x_2$.

Thus, $x_1 < x_2$. It follows that

$$g(y_1) < g(y_2)$$

We conclude that g is strictly monotone.

Subsequently, we will prove that g is **continuous**.

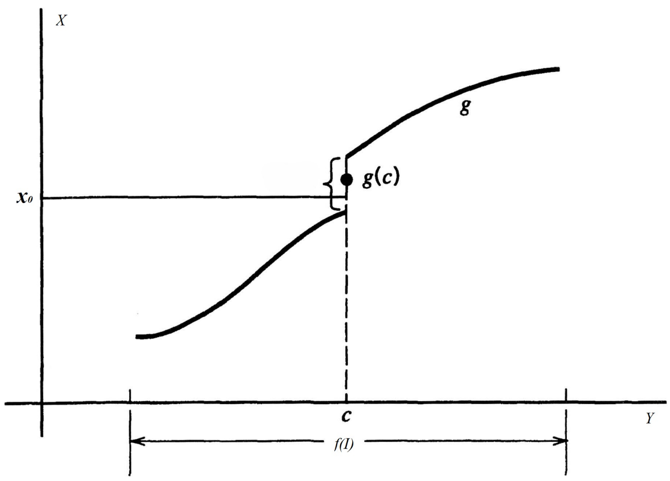
Assume g is discontinuous. Then, by inverse of previous criterion, $\exists c \in f(I)$, c is not endpoints of $f(I)$ s.t.

$$\lim_{y \to c-} g(y) < \lim_{y \to c+} g(y)$$

Let $x_0 \in (\lim_{y \to c-} g(y),\ \lim_{y \to c+} g(y)) \setminus g(c)$. Then

$$x_0 \notin g(f(I)) \subseteq I$$

which contradicts that fact that $x_0 \in I$ ■



8 Sequence of Functions

8.1 Pointwise and Uniform Convergence

Definition (***Sequence of Functions***)**.** Let $A \subseteq \mathbb{R}$.

We say that $(f_n(x))$ is a ***sequence of functions*** on A to $\mathbb{R}$ if

$$\forall n \in \mathbb{N},\ \exists f_n : A \to \mathbb{R}$$

Definition (***Convergence of Sequence of Functions***)**.** Let $(f_n(x))$ be a sequence of functions on A to $\mathbb{R}$. Let $A_0 \subseteq A$, and $f : A_0 \to \mathbb{R}$.

We say that (x_n) ***converges on A_0 to f*** if $\forall x \in A_0$,

$$\lim_{n\to\infty} (f_n(x)) = f(x) \text{ or } f_n \to f \text{ on } A_0$$

We call f the ***limit of f_n on A_0***. Or we say that ***(f_n) converges pointwise on A_0***.

Lemma (***$\varepsilon - \delta$ Definition***)**.** Let $A_0 \subseteq A \subseteq \mathbb{R}$, $f : A_0 \to \mathbb{R}$. A sequence of functions $(f_n(x)) : A \to \mathbb{R}$ converges pointwise to f if and only if

$\forall \varepsilon > 0, \exists N(\varepsilon,\ x) \in \mathbb{N}$ s.t. $\forall n \geq N(\varepsilon,\ x)$,

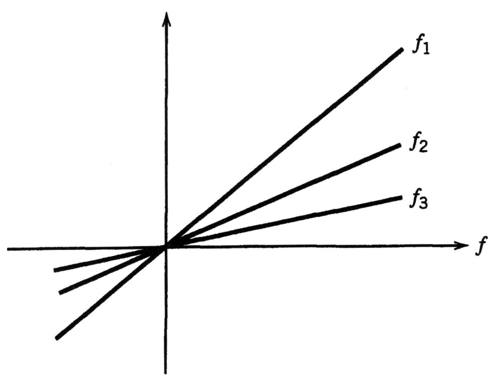
$$|f_n(x) - f(x)| < \varepsilon$$

Remark. We emphasize that the choice of $N(\varepsilon,\ x)$ depends on ***both*** ε and x.

Example. $\forall x \in \mathbb{R},\ \lim_{n\to\infty} \frac{x}{n} = 0$. Let $f_n(x) = \frac{x}{n},\ f(x) = 0$. We have

$$\lim_{n\to\infty} (f_n(x)) = \lim(\frac{x}{n}) = x \lim_{n\to\infty} \frac{1}{n} = x \cdot 0 = 0 = f(x)$$

Thus $(f_n(x)) \to 0$ pointwise on $\mathbb{R}$.



Or, $\forall \varepsilon > 0$,

$$\left|\frac{x}{n} - 0\right| = \frac{|x|}{n} < \varepsilon$$

So, it suffices to choose $K(\varepsilon,\ x) = \left\lceil \frac{|x|}{\varepsilon} \right\rceil$.

Example. Consider $\forall x \in \mathbb{R},\ n \in \mathbb{N},\ g_n(x) = x^n$.

By previous example, we know that

$$\lim(x^n) = \begin{cases} 0 & \text{if } -1 < x < 1, \\ 1 & \text{if } x = 1 \end{cases}$$

And if $x = -1,\ g_n(-1) = (-1)^n$ is divergent.

If $|x| > 1$, (x^n) is divergent as well.

Define $g : (-1, 1] \to \mathbb{R},\ g(x) = \lim(x^n)$. Then,

$$g_n \to g \text{ on } (-1, 1]$$

Definition (***Uniform Convergence***)**.** A sequence of $(f_n(x))$ functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ ***converges uniformly*** on $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if and only if $\forall \varepsilon > 0,\ \exists K(\varepsilon) \in \mathbb{N}$ s.t. $\forall n \geq K(\varepsilon),\ \forall x \in A_0$,

$$|f_n(x) - f(x)| < \varepsilon$$

In this case, we say that (f_n) is ***uniformly convergent*** on A_0.

Remark. The choice of $K(\varepsilon)$ depends on ***only*** ε.

Example. $f_n(x) = \dfrac{\sin(nx + n)}{n}$ converges uniformly to $f(x) = 0$.

Proof.

$$|f_n(x) - f(x)| = \left|\frac{\sin(nx + n)}{n}\right| < \frac{1}{n} < \varepsilon$$

So $N = \left\lceil \dfrac{1}{\varepsilon} \right\rceil$ will satisfies the condition.

■

Lemma. A sequence of $(f_n(x))$ functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ is ***NOT uniformly convergent*** on $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if and only if

- there exists $\varepsilon_0 > 0$ s.t. $\forall N \in \mathbb{N},\ \exists x \in A_0,\ \exists n \geq N$ s.t.

$$|f_n(x) - f(x)| \geq \varepsilon_0$$

- there exists $\varepsilon_0 > 0$, a subsequence (f_{n_k}) of (f_n), and a sequence $(x_k) \in A_0$ s.t.

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0,\ \forall k \in \mathbb{N}$$

Proof. Exercise. ■

Example. $f_n(x) = \dfrac{x}{n}$ does NOT converge uniformly to $f(x) = 0$ on $\mathbb{R}$.

Proof. Let $\varepsilon_0 = 1$

1. When $n_1 = 1,\ x_1 = 1$, we have $|f_1(x_1) - f(x_1)| = 1$
2. When $n_2 = 2,\ x_2 = 2$, we have $|f_2(x_2) - f(x_2)| = 1$
3. So choose $x_k = k,\ n_k = k$, we have

$$|f_{n_k}(x_k) - f(x_k)| = \left|\frac{x_k}{n_k} - 0\right| = 1 \geq \varepsilon_0$$

By preceding lemma, (f_n) does NOT converge to f uniformly on $\mathbb{R}$ ∎

Example. Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{x^2 + nx}{n}$

Claim: f_n does NOT uniformly converge to $f(x) = x$ on $\mathbb{R}$.

Proof. Choose $\varepsilon_0 = 1$. $\forall k \in \mathbb{N}$, if we choose $n_k = k,\ x_k = -k$

$$\begin{aligned}|f_{n_k}(x_k) - f(x_k)| &= \left|\frac{x_k^2 + n_k x_k}{n_k} - x_k\right| \\ &= \left|\frac{(-k)^2 + k \cdot (-k)}{k} - (-k)\right| \\ &= k \geq 1\end{aligned}$$

Thus, by previous lemma, f_n does NOT uniformly converge to f on $\mathbb{R}$. ∎

Claim: $(f_n) \to f$ uniformly on $[0, 1]$.

Proof. $\forall \varepsilon > 0$. Observe that $\forall x \in [0, 1],\ n \in \mathbb{N}$,

$$\left|\frac{x^2 + nx}{n} - x\right| = \left|x^2\right| \leq \frac{1}{n} < \varepsilon$$

Thus, it suffices to choose $N = \left\lceil \dfrac{1}{n} \right\rceil$ ∎

Example. Let $f_n : [0, 1] \to \mathbb{R},\ f_n(x) = x^n$.

Claim: f_n does NOT converge uniformly onn $[0, 1)$.

Choose $\varepsilon_0 = \dfrac{1}{2}$. $\forall k \in \mathbb{N},\ \exists n_k \in \mathbb{N}$ s.t.

$$n_k = k,\ x_k = (\frac{1}{2})^{\frac{1}{k}}$$

Then

$$\begin{aligned}|f_{n_k}(x_k) - f(x_k)| &= |x_k^{n_k} - 0| \\ &= (x_k)^k = \frac{1}{2} \geq \varepsilon_0\end{aligned}$$

We conclude that f_n is not uniformly convergent to f on $[0, 1]$.

NOTE: The lecture did NOT cover the sections starting from here. Begin if you have further interest in continuous study. That being said, in normal Advanced Calc/Intro Analysis course, this is a must-learn material. However, I suspect that removing chap 6 only happens in 2024 fall. Based on my survey, all of previous semesters have covered at least this chapter.

6 Differentiation

7 The Riemann Integral