

# Math265 Real Analysis Class Notes

Based on lectures by Prof. Huang

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Fall 2024



# 1 Preliminaries

## 1.1 Sets and Functions

**Definition (Image and Inverse Image).** Let  $A, B \subseteq \mathbb{R}$ ,  $f : A \rightarrow B$ .

If  $E \subseteq A$ , then the image of  $E$  is the set

$$f(E) = \{f(x) : x \in E\} \subseteq B$$

If  $H \subseteq B$ , the inverse image of  $H$  is the set

$$f^{-1}(H) = \{x \in A : f(x) \in H\} \subseteq A$$

**Definition.** Let  $A, B \subseteq \mathbb{R}$ ,  $f : A \rightarrow B$ .

- $f$  is injective (or one-to-one) if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

- $f$  is surjective (or onto) if

$$f(A) = B$$

- $f$  is bijective if it is both injective and surjective

**Definition (Inverse Function).**

Let  $A, B \subseteq \mathbb{R}$ ,  $f : A \rightarrow B$ . If  $f$  is bijective, then we can define a function  $g : B \Rightarrow A$  such that

$$g(y) = x \text{ if and only if (iff or } \Leftrightarrow) f(x) = y$$

$g$  is called inverse function of  $f$  and denoted by  $g = f^{-1}$

**Definition.** Let  $A, B \subseteq \mathbb{R}$ ,  $f : A \rightarrow B$ ,  $A_1 \subseteq A$ . The restriction of  $f$  to  $A_1$  is the function  $f|_{A_1} : A_1 \rightarrow B$  defined by

$$f|_{A_1}(x) = f(x) \text{ for all } x \in A_1$$

**Example.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^2$ . Notice that  $f$  is not injective. However,

$$f|_{[0, \infty)} : [0, \infty) \rightarrow \mathbb{R}$$

is injective.

*Proof.* Suppose there are  $x_1, x_2 \geq 0$  such that  $f|_{[0, \infty)}(x_1) = f|_{[0, \infty)}(x_2)$ . Then

$$x_1^2 = x_2^2$$

$$(x_1 + x_2)(x_1 - x_2) = 0$$

Thus  $x_1 = -x_2$  or  $x_1 = x_2$ . Since  $x_1, x_2$  are positive,

$$x_1 = x_2 = 0 \text{ or } x_1 = x_2$$

Hence,  $x_1 = x_2$  and we conclude that  $f|_{[0, \infty)}$  is injective. ■

**Definition (Relation).** Let  $A, B \subseteq \mathbb{R}$ . A relation  $R$  from  $A$  to  $B$  is a subset of  $R \subseteq A \times B$ . We write  $aRb$  or  $a \sim b$  to express  $(a, b) \in R$

**Example.**

- Equality relation on  $\mathbb{R}$ , denoted by  $=$ , is defined by

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = y\} = \{(x, x) : x \in \mathbb{R}\}$$

**Definition (Equivalence Relation).** Let  $A \subseteq \mathbb{R}$ . A relation  $R$  from  $A$  to  $A$  is an equivalent relation if it satisfies all of following:

- $\forall a \in A, aRa$  (*Reflexivity*)
- If for  $a, b \in A, aRb$ , then  $bRa$ . (*Symmetry*)
- If for  $a, b, c \in A, aRb$  and  $bRc$ , then  $aRc$ . (*Transitivity*)

**Definition (Equivalence Class).** Let  $A \subseteq \mathbb{R}$  with an equivalent relation  $\sim$ . An equivalence class under  $\sim$  is a subset  $B \subseteq A$  such that for some  $a \in A$

$$\forall b \in B, a \sim b \Leftrightarrow a \in B$$

**Example.** Define relation  $R$  by  $aRb \Leftrightarrow 3|a - b$ . Then  $R$  is an equivalent relation (Try to prove it!)

Define  $3\mathbb{Z} = \{n \in \mathbb{Z} : 3|n\} \subset \mathbb{Z}$ . Then  $3\mathbb{Z}$  is an equivalence class under  $R$

*Proof.* Let  $a \in 3\mathbb{Z}, b \in \mathbb{Z}$ . Suppose  $aRb$ , then

$$3|a - b$$

$$a - b = 3n \text{ for some } n \in \mathbb{N}$$

$$b = a - 3n$$

Since  $a \in 3\mathbb{Z}, a = 3m$  for some  $m \in \mathbb{N}$ . Thus,

$$b = 3m - 3n = 3(m - n) \in 3\mathbb{Z}$$

■

**Proposition.** Let  $S$  be a non-empty set, and  $\sim$  be an equivalent relation on  $S$ . Then for any distinct equivalent classes  $A, B$  under  $R$ ,

$$A \cap B = \emptyset$$

that is, equivalence classes of an equivalence relation are disjoint.

*Proof.* Let  $A, B$  be two distinct equivalence classes under  $\sim$  in  $S$

Assume  $\exists c \in A \cap B$ , then  $\forall a \in A, a \sim c$  and  $\forall b \in B, b \sim c$ . Thus by transitivity,

$$\forall a \in A, b \in B, a \sim b$$

Thus

$$A \subseteq B \text{ and } B \subseteq A$$

$$A = B$$

which contradicts to the fact that  $A \neq B$ . ■

**Proposition.** Let  $S$  be a non-empty set, and  $\sim$  be an equivalent relation on  $S$ . For all element  $x \in A$ ,  $x$  is confined in a unique equivalence class denoted by  $[x]_{\sim}$

*Proof. Exercise.* ■

**Definition (Quotient).** Let  $S$  be a non-empty set, and  $\sim$  be an equivalent relation on  $S$ . The quotient of  $S$  under  $\sim$  is the set  $S/\sim$  defined by

$$S/\sim = \{[x]_{\sim} : x \in S\}$$

**Example.**  $\mathbb{Z}/\sim = \{3\mathbb{Z}, 1 + 3\mathbb{Z}, 2 + 3\mathbb{Z}\} = \{\dots, [-1]_{\sim}, [0]_{\sim}, [1]_{\sim}, \dots\}$

## 1.2 Mathematical Induction

The reader should consult pages 12 – 15 on the textbook.

### 1.3 Finite and Infinite Sets

**Definition (Cardinality).** The number of elements a set  $S$  have is called the cardinality of  $S$ .

- The empty set  $\phi$  has 0 elements, called cardinality 0, denoted by

$$|\phi| = \#\phi = \text{card}(\phi) = 0$$

- Let  $n \in \mathbb{N}$ , a set  $S \in \mathbb{R}$  has  $n$  elements, called cardinality  $n$  denoted by

$$|S| = \#S = \text{card}(S) = n$$

if there exists a bijection

$$f : \mathbb{N}_n \rightarrow S$$

**Definition.** Let  $S$  be any set.

- $S$  is said to be finite if it is empty or has cardinality  $n$  for some  $n \in \mathbb{N}$ .
- $S$  is said to be infinite if it is not finite.

**Theorem (Uniqueness Theorem).** If  $S$  is a finite set, then the number of elements in  $S$  is a unique number in  $\mathbb{N}$ .

**Theorem.** The set  $\mathbb{N}$  of natural numbers is an infinite set.

*Proof.* Assume  $\mathbb{N}$  is finite. Then there exists a natural number  $n \in \mathbb{N}$  such that

$$|\mathbb{N}| = n$$

However, if we choose the first  $n + 1$  elements in  $\mathbb{N}$ , then

$$\{1, 2, \dots, n, n + 1\} \subset \mathbb{N}$$

$$|\{1, 2, \dots, n, n + 1\}| = n + 1 < |\mathbb{N}| = n$$

which is a contradiction. ■

**Theorem.**

(a) Let  $A, B$  be sets and  $\exists m, n \in \mathbb{N}$  s.t.  $|A| = m$ ,  $|B| = n$ ,  $A \cap B = \phi$ , then

$$|A \cup B| = m + n$$

(b) Let  $A$  be a set with  $|A| = m$  for some  $m \in \mathbb{N}$  and  $C \subseteq A$  be a set with  $|C| = 1$ , then

$$|A \setminus C| = m - 1$$

(c) If  $C$  is an infinite set and  $B$  is a finite set, then

$$C \setminus B \text{ is an infinite set}$$

*Proof.* (a) By definition of cardinality, there exists two bijective function

$$f : \mathbb{N}_m \rightarrow A, g : \mathbb{N}_n \rightarrow B$$

We define  $h : \mathbb{N}_{n+m} \rightarrow A \cup B$  such that

$$h(x) = \begin{cases} f(x) & \text{if } x \leq m \\ g(x - m) & \text{if } x > m \end{cases}$$

- **Claim:**  $h$  is injective.

Suppose  $\exists x_1, x_2 \in \mathbb{N}_{n+m}$  s.t.  $h(x_1) = h(x_2)$ .

**Case 1:**  $x_1, x_2 \leq m$ . Then  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$  since  $f$  is injective.

**Case 2:**  $x_1 \leq m, x_2 > m$ . Then  $f(x_1) = g(x_2 - m)$  contradiction. Impossible.

**Case 3:**  $x_1, x_2 > m$ . Then  $g(x_1 - m) = g(x_2 - m) \Rightarrow x_1 = x_2$  since  $g$  is injective.

We conclude that  $h$  is injective

- **Claim:**  $h$  is surjective.

Let  $y \in A \cup B$ .

**Case 1:**  $y \in A$ . Then  $\exists x \in \mathbb{N}_m \subseteq \mathbb{N}_{n+m}$  s.t.  $f(x) = y$  since  $f$  is onto.

**Case 2:**  $y \in B$ . Then  $\exists x \in \mathbb{N}_n \subseteq \mathbb{N}_{n+m}$  s.t.  $g(x) = y$  since  $g$  is onto.

Since  $x + m \in \mathbb{N}_{m+n}$  and  $x + m > m$ ,

$$h(x + m) = g(x + m - m) = g(x) = y$$

We conclude that  $h$  is surjective.

(b)(c): *Exercise.* ■

**Theorem.** Let  $S, T$  be set such that  $S \subseteq T$ . Then

1. If  $T$  is finite, then  $S$  is finite.
2. If  $S$  is infinite, then  $T$  is infinite

*Proof.* (a) Prove by induction on cardinality of  $T$ .

For  $|T| = 0$ ,  $T = \emptyset \Rightarrow S = \emptyset$ . Thus  $S$  is finite for  $|T| = 0$

Suppose that the claim holds for all  $S \subseteq T$  where  $|T| = k$ .

If  $|T| = k + 1$ , there are two cases:

**Case 1:** If  $S = T$ , then  $S$  is finite.

**Case 2:**  $\exists t \in T$  s.t.  $t \notin S$ , then

$$|T \setminus \{t\}| = k \text{ and } S \subseteq T \setminus \{t\}$$

Then  $S$  is finite by induction hypothesis. Thus, we can prove for the rest of cases by induction.

(b) This assertion is the contrapositive of (a). ■

**Definition.**

- A set is said to be countably infinite if there exists a bijection of  $\mathbb{N}$  onto  $S$ .
- A set is said to be countable if it is either finite or countably infinite.
- A set is said to be uncountable if it is not countable.

**Example.**

- $\mathbb{N}$  is countably infinite.
- $S = \{2, 3, 4, \dots\} \subseteq \mathbb{N}$  is countably infinite.
- The set of even natural numbers is countably infinite.
- $\mathbb{Z}$  is countably infinite.



**Theorem.** If  $A, B$  are both countably infinite, then  $A \cup B$  is countably infinite.

*Proof.* By definition of countably infinity,  $\exists f : \mathbb{N} \rightarrow A, g : \mathbb{N} \rightarrow B$  bijective.

Define  $h : \mathbb{N} \rightarrow A \cup B$  by

$$h(n) = \begin{cases} f(\frac{n+1}{2}) & \text{if } n \text{ is odd} \\ g(\frac{n}{2}) & \text{if } n \text{ is even} \end{cases}$$

It suffices to show that  $h$  is surjective.

Let  $x \in A \cup B$ . If  $x \in A$ , then since  $f$  is surjective,  $\exists n \in \mathbb{N}$  s.t.  $f(n) = x$ .

Consider  $m = 2n - 1$ . Then

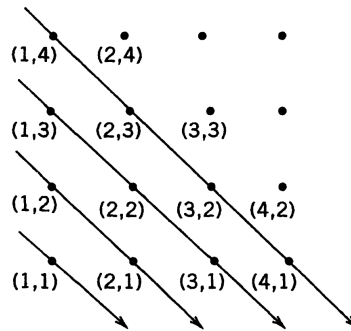
$$h(m) = f(\frac{m+1}{2}) = f(n) = x$$

We leave the rest of proof as exercise. ■

**Theorem.** The set  $\mathbb{N} \times \mathbb{N}$  is countably infinite.

*Proof.*  $\forall m, n \in \mathbb{N}$ , we can enumerate these  $(m, n)$  pairs as

$$(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), (1, 4), \dots$$



We see that the total number of points in diagonals 1 through  $k$  is given by  $\psi : \mathbb{N} \rightarrow \mathbb{N}$

$$\psi(k) = \begin{cases} 0 & \text{if } k = 0 \\ \frac{1}{2}k(k+1) & \text{if } k \in \mathbb{N} \end{cases}$$

We define  $h : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  by

$$\begin{aligned} h(m, n) &= \psi(m+n-2) + m \\ &= \frac{1}{2}(m+n-2)(m+n-1) + m \end{aligned}$$

**Claim:**  $h$  is a bijection. See Appendix B. ■

**Theorem.** The following statements are equivalent (Abb.: TFAE):

- (a)  $S$  is a countable set.
- (b) There exists a surjection of  $\mathbb{N}$  onto  $S$ .
- (c) There exists an injection of  $S$  onto  $\mathbb{N}$

*Proof.*

- (a) $\Rightarrow$ (b): Suppose  $S$  is a countable set.
  - If  $S$  is finite, then  $\exists f : \mathbb{N}_n \rightarrow S$  a bijection for some  $n \in \mathbb{N}$ , and  
 $\exists g : \mathbb{N} \rightarrow \mathbb{N}_n$  s.t.  $\forall m \in \mathbb{N}, g(m) = m \bmod n$ . Thus  $g$  is onto. We have

$f \circ g : \mathbb{N} \rightarrow S$  is onto by exercise in homework

- If  $S$  is countably infinite, then  $\exists f : \mathbb{N} \rightarrow S$  onto.
- (b) $\Rightarrow$ (c): Suppose  $\exists f : \mathbb{N} \rightarrow S$  onto. Define  $g : S \rightarrow \mathbb{N}$  by

$$g(s) = \min\{n \in \mathbb{N} : f(n) = s\}$$

Suppose  $\exists s, t \in S$  s.t.  $g(s) = g(t) = n$  for some  $n \in \mathbb{N}$ . We have

$$f(g(s)) = s \text{ and } f(g(t)) = t$$

$$s = t$$

Thus  $g$  is injective.

- (c) $\Rightarrow$ (a): Suppose  $\exists f : S \rightarrow \mathbb{N}$  injective. Then  $\exists \hat{f} : S \rightarrow f(S)$  a bijection.  
 Since  $f(S) \subseteq \mathbb{N}$ ,  $f(S)$  is countable by the previous theorem. Hence

$$\exists h : \mathbb{N} \rightarrow f(S)$$

It follows that

$$\hat{f} \circ h : \mathbb{N} \rightarrow S$$

is a bijection.

■

**Theorem.**  $\mathbb{Q}$  is countable.

*Proof.* Define  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q}_+$  where  $\mathbb{Q}_+ = \{x \in \mathbb{Q} : x > 0\}$  by

$$f(m, n) = \frac{n}{m}$$

It is easy to see that  $f$  is surjective (Try to prove it!).

Since  $\mathbb{N} \times \mathbb{N}$  is countable,  $\exists g : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$  a bijection.

We compose  $f$  and  $g$  to get:

$$f \circ g : \mathbb{N} \rightarrow \mathbb{Q}_+$$

Thus  $\mathbb{Q}_+$  is countable by the previous theorem.

Similarly, we can prove that  $\mathbb{Q}_- = \{x \in \mathbb{Q} : x < 0\}$  is countable. Thus,  $\mathbb{Q} = \mathbb{Q}_- \cup \mathbb{Q}_+ \cup \{0\}$  is countable by the previous theorem. ■

**Theorem (union of countable sets is countable).** If  $A_n$  is a countable set for  $n \in \mathbb{N}$ , then the union

$$A = \bigcup_{n=1}^{\infty} A_n$$

is countable. That is, the union of countable sets is countable.

*Proof.*  $\forall n \in \mathbb{N}$ ,  $\exists \psi_n : \mathbb{N} \rightarrow A_n$  a bijection.

Define  $f : \mathbb{N} \times \mathbb{N} \rightarrow A$  by

$$f(m, n) = \psi_m(n)$$

Then  $f$  is surjective (Try to prove it!).

Since  $\mathbb{N} \times \mathbb{N}$  is countable, by previous theorem,  $\exists g : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$  a bijection. Thus

$$f \circ g : \mathbb{N} \rightarrow A$$

We conclude that by definition,  $A$  is countable. ■

**Theorem (Cantor's Theorem).** If  $A$  is a set, there is no surjection of  $A$  onto the power set of  $A$ .

*Proof.* Assume  $\exists \psi : A \rightarrow P(A)$ . For any element  $a \in A$ ,  $\psi(a) \subseteq A$ . Thus either  $a \in \psi(a)$  or  $a$  does not belong to this set. We define

$$D = \{a \in A : a \notin \psi(a)\}$$

Since  $D \subseteq A$ , if  $\psi$  is a surjection, then  $\exists a_0 \in A$  s.t.  $D = \psi(a_0)$ .

- If  $a_0 \in D$ , then since  $D = \psi(a_0)$ , we must have  $a_0 \in \psi(a_0)$  a contradiction to definition of  $D$ .
- Similarly, if  $a_0 \notin \psi(a_0)$ , then  $a_0 \notin \psi(a_0)$  so that  $a_0 \in D$ , which is also a contradiction.

We thereby conclude that  $\psi$  cannot be a surjection. ■

**Remark.** This implies that the power set of natural number  $\mathcal{P}(\mathbb{N})$  is uncountable.

## 2 The Real Numbers

### 2.1 The Algebraic and Order Properties of $\mathbb{R}$

**Axiom (Algebraic Property of  $\mathbb{R}$ ).** On the set  $\mathbb{R}$  of real numbers there are two binary operations, denoted by  $+$  and  $\cdot$  called addition and multiplication, respectively. These operations satisfy the following properties.

#### Axioms for Addition

- (A1) For all  $a, b \in \mathbb{R}$ ,  $a + b = b + a$  (*commutative property of addition*)
- (A2) For all  $a, b, c \in \mathbb{R}$ ,  $(a + b) + c = a + (b + c)$  (*associative property of addition*)
- (A3) There exists an element  $0 \in \mathbb{R}$  such that  $0 + a = a + 0 = a$  for all  $a \in \mathbb{R}$  (*existence of zero element (additive identity)*)
- (A4) For all  $a \in \mathbb{R}$ , there exists an element  $-a \in \mathbb{R}$  s.t.  $a + (-a) = (-a) + a = 0$  (*existence of negative element (additive inverse)*)

#### Axioms for Multiplication

- (M1) For all  $a, b \in \mathbb{R}$ ,  $a \cdot b = b \cdot a$  (*commutative property of multiplication*)
- (M2) For all  $a, b, c \in \mathbb{R}$ ,  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  (*associative property of multiplication*)
- (M3) There exists an element  $1 \in \mathbb{R}$ ,  $1 \neq 0$ , s.t.  $1 \cdot a = a \cdot 1 = a$  for all  $a \in \mathbb{R}$  (*existence of unit element (multiplicative identity)*)
- (M4) For all  $a \in \mathbb{R}$ ,  $a \neq 0$ , there exists an element  $\frac{1}{a} \in \mathbb{R}$  s.t.  $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$  (*existence of reciprocals (multiplicative inverse)*)

#### Distributive Law

- (D) For all  $a, b, c \in \mathbb{R}$ ,  $a \cdot (b + c) = a \cdot b + a \cdot c$  (*distributive law of multiplication over addition*)

#### **Theorem.**

- (a) If  $z, a \in \mathbb{R}$  with  $z + a = a$ , then  $z = 0$ .
- (b) If  $u, b \in \mathbb{R}$ ,  $u, b \neq 0$  with  $u \cdot b = b$ , then  $u = 0$ .
- (c) If  $a \in \mathbb{R}$ , then  $a \cdot 0 = 0$ .

*Proof.* (a) Axiom (A3), (A4), (A2) gives

$$z = z + 0 = z + (a + (-a)) = (z + a) + (-a) = a + (-a) = 0$$

(b) Axiom (M3), (M4), (M2) gives

$$u = u \cdot 1 = u \cdot (b \cdot \frac{1}{b}) = (u \cdot b) \cdot \frac{1}{b} = b \cdot \frac{1}{b} = 1$$

(c) Axiom (M3), (D), (A3), theorem (a) gives

$$a + a \cdot 0 = a \cdot 1 + a \cdot 0 = a \cdot (1 + 0) = a \cdot 1 = a$$

$$a \cdot 0 = 0$$

■

**Theorem.**

(a) If  $a, b \in \mathbb{R}$ ,  $a \neq 0$  with  $a \cdot b = 1$ , then  $b = \frac{1}{a}$ .

(b) If  $a \cdot b = 0$ , then either  $a = 0$  or  $b = 0$ .

*Proof.* (a) Axiom (M3), (M4), (M2),  $a \cdot b = 1$ , (M3) gives

$$b = 1 \cdot b = \left(\frac{1}{a} \cdot a\right) \cdot b = \frac{1}{a} \cdot (a \cdot b) = \frac{1}{a} \cdot 1 = \frac{1}{a}$$

(b) It suffices to assume  $a \neq 0$  and prove  $b = 0$ . (M2), (M4), (M3), and previous theorem(c) gives

$$b = 1 \cdot b = \left(\frac{1}{a} \cdot a\right) \cdot b = \frac{1}{a} \cdot (a \cdot b) = \frac{1}{a} \cdot 0 = 0$$

■

**Definition.**

- We define  $-$ , called **subtraction**, as  $a - b = a + (-b)$  for all  $a, b \in \mathbb{R}$ .
- We define  $\div$ , called **division**, as  $a \div b = a \cdot b = \frac{a}{b}$  for all  $a, b \in \mathbb{R}$ .
- We define  $a^n$ , called **natural power**, as  $a^n = (((a \cdot a) \cdot a) \cdot \dots) \cdot a$  (n-times) for all  $a \in \mathbb{R}$ ,  $n \in \mathbb{N}$ .

**Theorem ( $\sqrt{2}$  Is Irrational Number).** There is no rational number  $r \in \mathbb{Q}$  such that  $r^2 = 2$ .

*Proof.* We will prove this by **contradiction**.

Assume there are  $p, q \in \mathbb{Z}$  satisfying  $\left(\frac{p}{q}\right)^2 = 2$ . Without loss of generality, assume  $p, q > 0$  and  $\gcd(p, q) = 1$ . Then

$$p^2 = 2q^2$$

So  $p^2$  is even. This implies that  $p$  is also even (try to prove it!). Let  $p = 2m$  for some  $m \in \mathbb{Z}$ . Then

$$p^2 = 4m^2 = 2q^2$$

$$2m^2 = q^2$$

By the same argument, if  $q^2$  is even, then  $q$  is even. Since  $p, q$  are even,  $\gcd(p, q)$  is at least 2, which contradicts our condition.

■

**Axiom (Order of  $\mathbb{R}$ ).** There is a non-empty set  $\mathbb{P} \subset \mathbb{R}$  called the set of **positive real numbers** that satisfies the following properties:

1. If  $a, b \in \mathbb{P}$ , then  $a + b \in \mathbb{P}$ .
2. If  $a, b \in \mathbb{P}$ , then  $a \cdot b \in \mathbb{P}$ .
3. If  $a \in \mathbb{R}$ , then one and only one of following holds:

$$a \in \mathbb{P}, \quad a = 0, \quad -a \in \mathbb{P}$$

**Note:** for all  $a \in \mathbb{R}$

- we write  $a > 0$  and say  $a$  is **positive** if  $a \in \mathbb{P}$ ,
- we write  $a \geq 0$  and say  $a$  is **non-negative** if  $a \in \mathbb{P} \cup \{0\}$ .
- we write  $a < 0$  and say  $a$  is **negative** if  $a \in \mathbb{P}$ ,
- we write  $a \leq 0$  and say  $a$  is **non-positive** if  $-a \in \mathbb{P} \cup \{0\}$ .

**Definition.** Let  $a, b \in \mathbb{R}$ .

1. If  $a - b \in \mathbb{P}$ , we denote that  $a > b$  or  $b < a$ .
2. If  $a - b \in \mathbb{P} \cup \{0\}$ , we denote that  $a \geq b$  or  $b \leq a$ .

**Note:** For all  $a, b \in \mathbb{R}$ , there is one and only one of following holds:

$$a < b, \quad a = b, \quad a > b$$

And we call them  $a$  is **greater than**, **equal to**, or **less than**  $b$  respectively.

**Theorem.** Let  $a, b, c \in \mathbb{R}$ .

1. If  $a > b$  and  $b > c$ , then  $a > c$ .
2. If  $a > b$ , then  $a + c > b + c$ .
3. If  $a > b$  and  $c > 0$ , then  $ca > cb$ .  
If  $a > b$  and  $c < 0$ , then  $ca < cb$ .

*Proof.*

1. By definition,  $a - b \in \mathbb{P}$  and  $b - c \in \mathbb{P}$ . Then

$$(a - b) + (b - c) = a - c \in \mathbb{P} \Rightarrow a > c$$

2. By definition,  $a - b \in \mathbb{P}$ , then

$$(a + c) - (b + c) = a - b \in \mathbb{P} \Rightarrow a + c > b + c$$

3. By definition,  $a - b \in \mathbb{P}$ , then

$$\text{If } c > 0, \quad c \in \mathbb{P}, \quad ca - cb = c(a - b) \in \mathbb{P} \Rightarrow ca > cb$$

$$\text{If } c < 0, \quad -c \in \mathbb{P}, \quad cb - ca = (-c)(a - b) \in \mathbb{P} \Rightarrow cb > ca$$

■

**Theorem.**

1. If  $a \in \mathbb{R}$  and  $a \neq 0$ , then  $a^2 > 0$ .
2.  $1 > 0$
3. If  $n \in \mathbb{N}$ , then  $n > 0$ .

*Proof.*

1. Since  $a \neq 0$ , there are only two cases:
  - If  $a \in \mathbb{P}$ , then  $a^2 \in \mathbb{P}$ .
  - If  $-a \in \mathbb{P}$ , then  $(-a)(-a) = a^2 \in \mathbb{P}$ . Proof of  $(-a)(-a) = a^2$  is exercise
2.  $1 \in \mathbb{P}$ , so  $1^2 \in \mathbb{P}$  by (1)
3. For all  $n \in \mathbb{N}$ ,  $n = (((1 + 1) + 1) + \cdots + 1) + 1$ . Thus, this is true by induction. ■

**Theorem.** If  $a \in \mathbb{R}$ ,  $0 < a < \varepsilon$  for all  $\varepsilon > 0$ , then  $a = 0$ .*Proof.* Assume that  $a > 0$ . Then if we take  $\varepsilon_0 = \frac{1}{2}a$ , we have  $0 < \varepsilon_0 < a$  a contradiction.Thus we conclude that  $a = 0$ . ■**Theorem.** If  $ab > 0$ , then either

- $a > 0$  and  $b > 0$ , or
- $a < 0$  and  $b < 0$

*Proof. Exercise.* ■**Corollary.** If  $ab < 0$ , then either

- $a > 0$  and  $b < 0$ , or
- $a < 0$  and  $b > 0$

*Proof. Exercise.* ■**Proposition.** Let  $a \geq 0$ ,  $b \geq 0$ . Then

$$a < b \text{ iff } a^2 < b^2 \text{ iff } \sqrt{a} < \sqrt{b}$$

*Proof.*

- If  $a = 0$ , then it holds.
- If  $a > 0$ , then

$$b^2 - a^2 = (b + a)(b - a) > 0 \Leftrightarrow b - a > 0$$

And  $\sqrt{a} < \sqrt{b} \Leftrightarrow a < b$  is a consequence of  $a < b \Leftrightarrow a^2 < b^2$  ■

**Theorem (Arithmetic-geometric Mean Inequality).** For all  $a, b \in \mathbb{R}$ ,  $a, b \geq 0$ ,  $\sqrt{ab} \leq \frac{a+b}{2}$ . Moreover, the inequality holds iff  $a = b$ .

*Proof.* If  $a \neq b$ ,  $\sqrt{a} \neq \sqrt{b}$ , then

$$\begin{aligned}(\sqrt{a} - \sqrt{b})^2 &> 0 \\ a + b - 2\sqrt{ab} &> 0 \\ \sqrt{ab} &< \frac{a+b}{2}\end{aligned}$$

( $\Leftarrow$ ) If  $a = b$ , then the equality holds.

( $\Rightarrow$ ) If  $\sqrt{ab} < \frac{a+b}{2}$ , then we reverse the previous working,

$$\begin{aligned}(\sqrt{a} - \sqrt{b})^2 &= 0 \\ a &= b\end{aligned}$$

■

**Theorem (Bernoulli's Inequality).** If  $x > -1$ , then for all  $n \in \mathbb{N}$ ,

$$(1+x)^n \geq 1+nx$$

*Proof.* We will prove this by mathematical induction.

- $n = 1$ , then  $1+x \geq 1+x$ , so  $P(1)$  holds.
- Assume  $P(k)$  holds for all  $k \in \mathbb{N}$ . We want to show that  $P(k+1)$  holds.

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k(1+x) \\ &\geq (1+kx)(1+x) = 1 + (k+1)x + kx^2 \\ &\geq 1 + (k+1)x\end{aligned}$$

Thus  $P(k+1)$  holds.

We conclude that  $P(n)$  holds for all  $n \in \mathbb{N}$ .

■



## 2.2 Absolute Value and the Real Line

**Definition.** The absolute value of a real number  $a$ , denoted by  $|a|$ , is defined by

$$|a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } a < 0 \\ 0 & \text{if } a = 0 \end{cases}$$

**Theorem.**

$$1. |ab| = |a| \cdot |b| \text{ for all } a, b \in \mathbb{R}.$$

$$2. |a|^2 = |a^2| \text{ for all } a \in \mathbb{R}.$$

$$3. \text{ If } c \in \mathbb{R}, c \geq 0, \text{ then}$$

$$|a| \leq c \text{ iff } -c \leq a \leq c$$

$$4. -|a| \leq a \leq |a|.$$

*Proof. Exercise.* ■

**Theorem (Triangle Inequality).** Let  $a, b \in \mathbb{R}$ . Then

$$|a + b| \leq |a| + |b|$$

*Proof.* By previous theorem(4), we have

$$-|a| \leq a \leq |a| \text{ and } -|b| \leq b \leq |b|$$

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

Thus, by previous theorem(3),

$$|a + b| \leq |a| + |b|$$
■

**Remark.**  $ab \geq 0$  iff  $|a + b| \leq |a| + |b|$

*Proof. Exercise.* ■

**Corollary.** Let  $a, b \in \mathbb{R}$ . Then

$$1. ||a| - |b|| \leq |a - b|.$$

$$2. |a - b| \leq |a| + |b|.$$

*Proof.*

$$1. \text{ We write } a = (a - b) + b. \text{ By triangle inequality,}$$

$$|a| = |(a - b) + b| \leq |a - b| + |b|$$

$$|a| - |b| \leq |a - b|$$

Similarly

$$|b| - |a| \leq |a - b|$$

We thereby conclude that (1) holds.

2. We "put  $-b$  in  $b$ " of triangle inequality:

$$|a - b| \leq |a| + |b|$$

■

**Corollary.** If  $a_1, a_2, \dots, a_n \in \mathbb{R}$ , then

$$|a_1 + a_2 + \dots + a_n| \leq |a_1| + |a_2| + \dots + |a_n|$$

*Proof.*

$$\begin{aligned} |a_1 + a_2 + \dots + a_n| &\leq |a_1 + a_2 + \dots + a_{n-1}| + |a_n| \\ &\leq |a_1 + a_2 + \dots + a_{n-2}| + |a_{n-1}| + |a_n| \\ &\dots \\ &\leq |a_1| + |a_2| + \dots + |a_n| \end{aligned}$$

■

**Definition.** Let  $a \in \mathbb{R}$ ,  $\varepsilon > 0$ . Then the  $\varepsilon$ -neighborhood of  $a$  is the set

$$V_\varepsilon(a) = \{x \in \mathbb{R} \mid |x - a| < \varepsilon\}$$



**Theorem.** Let  $a \in \mathbb{R}$ . If  $\forall \varepsilon > 0$ ,  $x \in V_\varepsilon(a)$ , then  $x = a$ .

*Proof.* By definition of neighborhood,  $\forall \varepsilon > 0$ ,  $|x - a| < \varepsilon$ . Thus by previous theorem,

$$|x - a| = 0 \Rightarrow x - a = 0 \Rightarrow x = a$$

■

**Proposition.** Let  $a, b \in \mathbb{R}$  with  $a < b$ . Consider the open interval

$$(a, b) = \{x \mid a < x < b\}$$

Then for all  $x \in (a, b)$ ,  $\exists \varepsilon > 0$  s.t.

$$V_\varepsilon(x) \subseteq (a, b)$$

*Proof.* Choose  $\varepsilon = \min\{|x - a|, |x - b|\}$ . Suppose  $y \in V_\varepsilon(x)$ , then

$$|x - y| < \varepsilon$$

$$x - \varepsilon < y < x + \varepsilon$$

So it is either

- $-a < -b < y < b$  if  $a > b$ , or
- $-b < -a < y < a$  if  $a < b$

We conclude that  $y \in (a, b)$

■

**Proposition.** Let  $a, b \in \mathbb{R}$  with  $a < b$ . Consider the **closed interval**

$$[a, b] = \{x | a \leq x \leq b\}$$

Then for all  $\varepsilon > 0$ ,

$$V_\varepsilon(x) \not\subseteq (a, b)$$

*Proof. Exercise.* ■

**Proposition.** Let  $x, y, a, b \in \mathbb{R}$ . If  $x \in V_\varepsilon(a)$ ,  $y \in V_\varepsilon(b)$ , then

$$x + y \in V_{2\varepsilon}(a + b)$$

*Proof.* By definition of neighborhood,

$$\begin{aligned} |(x + y) - (a + b)| &= |(x - a) + (y - b)| \\ &\leq |x - a| + |y - b| \\ &< \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

Thus  $x + y \in V_{2\varepsilon}(a + b)$ . ■

## 2.3 The Completeness Property of $\mathbb{R}$

**Definition.** Let  $S \subseteq \mathbb{R}$  be a nonempty.

1.  $S$  is said to be **bounded above** if there exists a number  $u \in \mathbb{R}$  such that  $s \leq u$  for all  $s \in S$ . We say  $u$  is an **upper bound** of  $S$ .
2.  $S$  is said to be **bounded below** if there exists a number  $u \in \mathbb{R}$  such that  $u \leq s$  for all  $s \in S$ . We say  $u$  is an **lower bound** of  $S$ .
3.  $S$  is said to be **bounded** if it is both bounded above and below.
4. We say  $S$  is **unbounded** if it is not bounded above or bounded below.

**Definition.** Let  $S \subseteq \mathbb{R}$  be nonempty.

1. If  $S$  is bounded above, then  $u$  is the **supremum** or **least upper bound** of  $S$  if
  - $u$  is an upper bound, and
  - If  $v$  is an upper bound of  $S$ , then  $u \leq v$ .
2. If  $S$  is bounded below, then  $w$  is the **infimum** or **greatest lower bound** of  $S$  if
  - $w$  is a lower bound, and
  - If  $t$  is a lower bound of  $S$ , then  $t \leq w$ .

Supremum and infimum are denoted by

$$\sup S, \inf S$$

**Proposition.** If  $S$  is bounded above, then  $\sup S$  is **unique**.

*Proof.* Suppose  $u_1, u_2$  are supremum of  $S$ . Since  $u_1$  is the supremum of  $S$  and  $u_2$  is the upper bound of  $S$ ,

$$u_1 \leq u_2$$

Similarly, we have

$$u_2 \leq u_1$$

Thus

$$u_1 = u_2$$

■

**Theorem.** Let  $S \subseteq \mathbb{R}$  be nonempty. If  $\exists u = \sup S$ , then the following statements are equivalent:

1. If  $v$  is an upper bound of  $S$ , then  $u \leq v$ .
2. If  $z < u$ , then  $z$  is not an upper bound of  $S$ .
3. If  $z < u$ , then there exists  $s_z \in S$  such that  $z < s_z$ .
4. If  $\varepsilon > 0$ , then there exists  $s_\varepsilon \in S$  such that  $u - \varepsilon < s_\varepsilon$ .

*Proof. Exercise.*

■

**Lemma (Alternative Definition).** Let  $S \subseteq \mathbb{R}$  be nonempty.  $\exists u = \sup S$  iff

- $s \leq u$  for all  $s \in S$ .
- If  $v < u$ , then there exists  $s' \in S$  such that  $v < s'$ .

*Proof. Exercise.* ■

**Lemma ( $\varepsilon$  Definition).** An upper bound  $u$  of nonempty set  $S \subseteq \mathbb{R}$  is the supremum of  $S$  iff

$\forall \varepsilon > 0, \exists s_\varepsilon \in S$  s.t.

$$u - \varepsilon < s_\varepsilon$$

*Proof.* ( $\Rightarrow$ ) Suppose  $u = \sup S$ .  $\forall \varepsilon > 0, u - \varepsilon$  is NOT an upper bound of  $S$ . Thus

$$\exists s_\varepsilon \in S \text{ s.t. } u - \varepsilon < s_\varepsilon$$

( $\Leftarrow$ ) Suppose  $u$  is an upper bound satisfying:  $\forall \varepsilon > 0, \exists s_\varepsilon \in S$  s.t.  $u - \varepsilon < s_\varepsilon$ .

Let  $v$  be an upper bound of  $S$ .

**Goal:**  $u \leq v$

Assume  $u > v$ . Let  $\varepsilon = u - v > 0$ . Then  $\exists s_\varepsilon \in S$  s.t.  $u - \varepsilon < s_\varepsilon$ . Thus  $u - \varepsilon$  is NOT an upper bound of  $S$ , which is a contradiction.

We thereby conclude that  $u \leq v$ . ■

**Remark (Maximum v.s. Supremum).** If a set is bounded above, then the maximum of a set may not exist, but the supremum of the set always exists.

**Example.** Let  $S = (1, 2)$ . Then there does NOT exist a maximum of  $S$ , but  $\exists \sup S = 2$

This is also true for minimum and infimum.

**Remark.** If a set  $S$  is bounded above, then  $\sup S$  may NOT be an element of  $S$ .

**Example.** Let  $S = (1, 2)$ .  $\exists \sup S = 2 \notin S$

**Axiom (Completeness Property of  $\mathbb{R}$ ).**

Every nonempty set of real numbers has a supremum in  $\mathbb{R} \cup \{\infty\}$ . It is also called least-upper-bound-property or LUP or  $\mathbb{R}$ .

## 2.4 Applications of Supremum

**Definition.** Let  $S \subseteq \mathbb{R}$ ,  $a \in \mathbb{R}$ . We define

$$a + S = \{a + s : \forall s \in S\}$$

More generally,

$$A + B = \{a + b : \forall a \in A, b \in B\}$$

**Proposition.**  $\sup(a + S) = a + \sup S$

*Proof.* Let  $u = \sup S$ . Then  $\forall s \in S$ ,  $u \geq s$ . Thus

$$a + u \geq a + s$$

It follows that  $a + u$  is an upper bound of  $a + S$ . Thus,

$$\sup(a + S) \leq a + u$$

Let  $v$  be an upper bound of  $a + S$ , so  $\forall s \in S$

$$v \geq a_s$$

$$v - a \geq s$$

Thus  $v - a$  is an upper bound of  $S$ .

$$v - a \geq \sup S = u$$

$$v \geq a + u$$

$$\sup(a + S) \geq a + u$$

Thus,

$$a + \sup(S) = \sup(a + S)$$

■

**Proposition.** Suppose  $A, B \subseteq \mathbb{R}$ . Then

$$\sup(A + B) = \sup A + \sup B$$

*Proof.* Exercise.

■

**Proposition.** Suppose  $A, B \subseteq \mathbb{R}$  satisfying  $\forall a \in A, b \in B, a \leq b$ . Then

$$\sup A \leq \inf B$$

*Proof.*  $\forall a \in A, b \in B, a \leq b \Rightarrow b$  is an upper bound for  $A$ .

$$\sup A \leq b$$

Thus  $\sup A$  is a lower bound for  $B$ .

$$\sup A \leq \inf B$$

■

**Exercise.** Provide an example of  $A, B \subseteq \mathbb{R}$  s.t.  $\forall a \in A, b \in B, a < b$  but

$$\sup A = \inf B$$

**Example.**

$$A = (2, 3), B = [3, 4]$$

**Proposition.** Let  $D \subseteq \mathbb{R}$ ,  $f, g : D \rightarrow \mathbb{R}$ . If  $\forall x \in D$ ,  $f(x) \leq g(x)$ , then

$$\sup_{x \in D} f(x) \leq \sup_{x \in D} g(x)$$

*Proof.*  $\exists u = \sup g(D)$ . Then  $\forall x \in D$ ,

$$f(x) \leq g(x) \leq \sup g(D)$$

so  $\sup g(D)$  is an upper bound of  $f(D)$ . Thus

$$\sup f(D) \leq \sup g(D)$$

■

**Remark.**  $\forall x \in D$ ,  $f(x) \leq g(x)$  does not imply any relation between  $\sup f(D)$  and  $\inf g(D)$ .

**Proposition.** If  $\forall x, y \in D$ ,  $f(x) \leq g(y)$ , then we may conclude that  $\sup f(D) \leq \inf g(D)$ .

*Proof.* It is a direct consequence of proposition. ■

**Theorem (Archimedean Property).**  $\forall x \in \mathbb{R}, \exists n_x \in \mathbb{N}$  s.t.  $x \leq n_x$ .

*Proof.* Assume  $\forall n \in \mathbb{N}, n < x$ . We have

$$\sup \mathbb{N} \leq x$$

. Consider  $\sup \mathbb{N} - 1$ . Then  $\exists n \in \mathbb{N}$  s.t.

$$\sup \mathbb{N} - 1 < n$$

$$\sup \mathbb{N} < 1 + n \in \mathbb{N}$$

thus  $\sup \mathbb{N}$  is NOT an upper bound of  $\mathbb{N}$ , which is a contradiction. ■

**Corollary.** If  $S = \{\frac{1}{n} : \forall n \in \mathbb{N}\}$ , then

$$\inf S = 0$$

*Proof.*  $\forall \varepsilon > 0$ , by Archimedean Property,  $\exists n \in \mathbb{N}$  s.t.

$$\frac{1}{\varepsilon} < n \Rightarrow \frac{1}{n} < \varepsilon$$

$$0 \leq \inf S \leq \frac{1}{n} < \varepsilon$$

By previous theorem of  $\varepsilon$ ,

$$\inf S = 0$$
■

**Corollary.**  $\forall y > 0, \exists n_y \in \mathbb{N}$  s.t.

$$0 < \frac{1}{n_y} < y$$

*Proof.* Let  $x = \frac{1}{y} \in \mathbb{R}$ . By Archimedean Property,  $\exists m_y \in \mathbb{N}$  s.t.

$$m_y \geq x = \frac{1}{y}$$

Let  $n_y = m_y + 1 \in \mathbb{N}$ . Then

$$n_y > \frac{1}{y} \Rightarrow 0 < \frac{1}{n_y} < y$$
■

**Corollary.**  $\forall y > 0, \exists n_y \in \mathbb{N}$  s.t.

$$n_y - 1 \leq y < n_y$$



*Proof.* Construct

$$E_y = \{m \in \mathbb{N} : y < m\}$$

By Archimedean Property,  $E_y \neq \emptyset$ . By Well-Ordering of  $\mathbb{N}$ ,  $\exists n_y \in \mathbb{N}$  s.t.

$$\exists n_y = \min E_y \Rightarrow y < n_y$$

Since we cannot have  $y < n_y - 1$

$$n_y - 1 \leq y < n_y$$

■

**Theorem.** (*Existence of  $\sqrt{2}$* )  $\exists x \in \mathbb{R}, x > 0$  s.t.  $x^2 = 2$

*Proof.* Let  $S = \{s \in \mathbb{R} : s^2 < 2\}$ . **Claim:**  $\sup S$  satisfies  $(\sup S)^2 = 2$ .

Since  $1^2 = 1 < 2$ ,  $1 \in S$ , so  $S$  is nonempty.

Assume 2 is not an upper bound of  $S$ . Then,  $\exists s \in S$  s.t.  $s > 2$ . Thus

$$s^2 > 4 \wedge s^2 < 2$$

a contradiction. Thus, 2 is an upper bound of  $S$ .  $\forall s \in S, s \leq 2$ .

By Completeness Theorem Property,  $\exists x = \sup S \in \mathbb{R}$ .

Since we want to show  $x^2 = 2$ . it suffices to show that

$$x^2 \geq 2 \wedge x^2 \leq 2$$

We will prove those two statement by contradiction.

- Assume  $x^2 < 2$ . Then  $\forall n \in \mathbb{N}$

$$\left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} = x^2 + \frac{1}{n}(2x + \frac{1}{n}) \leq x^2 + \frac{1}{n}(2x + 1)$$

**Goal:** Choose a  $N \in \mathbb{N}$  large enough such that  $\left(x + \frac{1}{n}\right)^2 < 2$ . Thus,

$$\begin{aligned} x^2 + \frac{1}{n}(2x + 1) &< 2 \\ \frac{1}{n}(2x + 1) &< 2 - x^2 \\ \frac{1}{n} &< \frac{2 - x^2}{2x + 1} \\ n &> \frac{2x + 1}{2 - x^2} \end{aligned}$$

Thus by Archimedean Property,  $\exists n \in \mathbb{N}, n = \left\lceil \frac{2x + 1}{2 - x^2} \right\rceil$  s.t.

$$\left(x + \frac{1}{n}\right)^2 \leq x^2 + \frac{1}{n}(2x + 1) < 2$$

$$x + \frac{1}{n} \in S$$

which is a contradiction to  $x = \sup S$ .

- Assume  $x^2 > 2$ . Then  $\forall m \in \mathbb{N}$ ,

$$\left(x - \frac{1}{m}\right)^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m}$$

**Goal:** find  $m \in \mathbb{N}$  large enough such that  $\left(x - \frac{1}{m}\right)^2 > 2$ . It follows that

$$\begin{aligned} x^2 - \frac{2x}{m} &> 2 \\ \frac{2x}{m} &< x^2 - 2 \\ m &> \frac{2x}{x^2 - 2} \end{aligned}$$

By Archimedean Property,  $\exists m \in \mathbb{N}$ ,  $m = \left\lceil \frac{2x}{x^2 - 2} \right\rceil$  s.t.

$$\left(x - \frac{1}{m}\right)^2 > x^2 - \frac{2x}{x^2 - 2} > 2$$

thus  $\left(x - \frac{1}{m}\right)^2$  is an upper bound of  $S$ . However, since

$$x < x - \frac{1}{m}$$

so  $x$  is NOT an supremum of  $S$ , which is a contradiction.

Thus, we conclude that  $\exists x \in \mathbb{R}$  s.t.  $x^2 = 2$ . ■

**Remark.** By Binomial Expansion, one may show the existence of positive  $n$ -th roots for all  $n \in \mathbb{N}$ .

**Theorem (Rationals Are Dense in  $\mathbb{R}$ ).** If  $x, y \in \mathbb{R}$  with  $x < y$ , then there exists a rational number  $r \in \mathbb{Q}$  such that  $x < r < y$ .

*Proof.* WLOG, we may assume that  $x, y > 0$ .

Since  $y - x > 0 \Rightarrow \frac{1}{y - x} > 0$ ,  $\exists n \in \mathbb{N}$  s.t.  $n > \frac{1}{y - x}$  by Archimedean Property. Thus

$$1 < ny - nx \Rightarrow nx + 1 < ny$$

By Archimedean Property,  $\exists m \in \mathbb{N}$  s.t.

$$m - 1 \leq nx < m$$

$$nx < m \leq nx + 1 < ny$$

$$x < \frac{m}{n} < y$$
■

**Corollary (*Irrationals Are Dense in  $\mathbb{R}$* ).** If  $x, y \in \mathbb{R}$  with  $x < y$ , then there exists an ***irrational number***  $z \in \mathbb{Q}$  such that  $x < z < y$ .

*Proof.* Apply the density theorem of rationals to real number  $\frac{x}{\sqrt{2}}$  and  $\frac{y}{\sqrt{2}}$ , we have

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$$

$$x < \sqrt{2}r < y$$

**Goal:** Prove  $\sqrt{2}r$  is irrational by contradiction.

Assume that  $\sqrt{2}r$  is rational. Then  $\exists p, q \in \mathbb{Z}, q \neq 0$  s.t.

$$\sqrt{2}r = \frac{p}{q}$$

$$\sqrt{2} = \frac{p}{q}r \in \mathbb{Q}$$

which is a contradiction since  $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ . Thus we conclude that  $\sqrt{2}r \in \mathbb{R} \setminus \mathbb{Q}$ . ■

## 2.5 Intervals

**Definition (Intervals).** Let  $a, b \in \mathbb{R}$ . If

- $a < b$ , then define open interval to be the set  $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
- $a \leq b$ , then define closed interval to be the set  $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
- $a < b$ , then define half-open interval to be the set  $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ ,  
or  $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

**Theorem (Characterization Theorem).** If  $S \subseteq \mathbb{R}$  s.t.  $S$  contains at least 2 points and satisfies

$$\forall x, y \in S \wedge x < y \Rightarrow [x, y] \subseteq S$$

then  $S$  is an interval.

*Proof.* Let  $a = \inf S$ ,  $b = \sup S$ . Then  $S \subseteq [a, b]$ . If  $a = -\infty$  or  $b = \infty$ , treat the boundary as open and this remains true.

**Goal:** Show that  $S \supseteq (a, b)$

If  $z \in (a, b)$ , then  $z$  is NOT a lower bound of  $S$ . So  $\exists x \in S$  with  $a < x < z$ .

Similarly,  $x$  is NOT an upper bound of  $S$ , so  $\exists y \in S$  with  $z < y < b$ . Thus

$$z \in [x, y] \subseteq S$$

Therefore, the desired result then follows only on whether  $S$  includes  $a, b$ . ■

**Definition (Nested Sequence of Intervals).** A sequence of intervals  $\{I_n\}_n \in \mathbb{N}$  is nested if

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$$

**Example.** Consider  $I_n = [0, \frac{1}{n}]$ . Then  $\forall n \in \mathbb{N}$ ,

$$I_n \supseteq I_{n+1}$$

**Claim:**  $\bigcap_{n=1}^{\infty} I_n = \{0\}$ .

*Proof.*

$(\supseteq)$ :  $\forall n \in \mathbb{N}$ ,  $0 \in I_n$ .

$(\subseteq)$ : Assume  $\exists x \in \mathbb{R} \setminus \{0\}$  s.t.  $x \in \bigcap_{n=1}^{\infty} I_n$ . Assume  $x > 0$ . Then by Archimedean Property,  $\exists n \in \mathbb{N}$  s.t.  $n > \frac{1}{x} \Rightarrow x > \frac{1}{n}$ . Thus  $x \notin I_n$ , which is a contradiction.

Since  $x$  can not be negative, we conclude that  $x = 0$  and  $\bigcap_{n=1}^{\infty} I_n \subseteq \{0\}$  ■

**Remark.** Not all nested sequence of intervals has single common point.

$$\bullet \quad I_n = [0, 1 + \frac{1}{n}] \Rightarrow \bigcap_{n=1}^{\infty} I_n = [0, 1]$$

$$\bullet \quad I_n = [0, \frac{1}{n}] \Rightarrow \bigcap_{n=1}^{\infty} I_n = \phi$$

**Theorem** (*Nested Interval Property*).

If  $\forall n \in \mathbb{N}$ ,  $\exists a_n, b_n \in \mathbb{R}$  s.t.  $I_n = [a_n, b_n]$  is a nested sequence of **closed** intervals, then there exists a real number  $\xi$  such that  $\forall n \in \mathbb{N}$ ,  $\xi \in I_n$

*Proof.* Observe that  $\forall n \in \mathbb{N}$ ,  $a_n \leq b_1$ .

Hence  $\{a_n : \forall n \in \mathbb{N}\}$  is bounded above.

Let  $\xi = \sup\{a_n : \forall n \in \mathbb{N}\}$ , then  $\forall n \in \mathbb{N}$ ,  $\xi \geq a_n$ .

It suffices to show  $\forall n \in \mathbb{N}$ ,  $\xi \leq b_n$ .

Observe that  $\forall n \in \mathbb{N}$ ,  $b_n$  is an upper bound of  $\{a_n : \forall n \in \mathbb{N}\}$ . Thus by definition of supremum,  $\forall n \in \mathbb{N}$

$$\xi \leq b_n$$

We thereby conclude that  $\forall n \in \mathbb{N}$ ,  $\xi \in I_n$ . ■

**Theorem.** If  $\forall n \in \mathbb{N}$ ,  $\exists a_n, b_n \in \mathbb{R}$  s.t.  $I_n = [a_n, b_n]$  is a nested sequence of **closed** intervals, and the length  $b_n - a_n$  satisfies

$$\inf\{b_n - a_n : \forall n \in \mathbb{N}\} = 0$$

then  $\xi \in \bigcap_{n=1}^{\infty} I_n$  is unique.

*Proof.* Assume  $\exists \xi, \eta \in \mathbb{R}$  s.t.

$$\xi, \eta \in \bigcap_{n=1}^{\infty} I_n$$

Thus

$$0 \leq |\xi - \eta| \leq \inf\{b_n - a_n\} = 0$$

We thereby conclude that  $\xi = \eta$ . ■

**Theorem.**  $\mathbb{R}$  is uncountable.

*Proof.* **Goal:** Show that  $I = [0, 1]$  is uncountable.

Assume  $I = [0, 1]$  is countable. Then we can enumerate the set as

$$I_0 = \{x_1, x_2, \dots, x_n, \dots\}$$

For all  $n \in \mathbb{N}$ , we choose a closed bounded interval  $I_n \in I_{n-1}$  s.t.  $x_n \notin I_n$ . Then we have

$$I_1 \supseteq I_2 \supseteq \dots$$

By Nested Interval Property,  $\exists \xi \in \bigcap_{n=1}^{\infty} I_n$ . It follows that  $\forall n \in \mathbb{N}, \xi \neq x_n$ .

This means that  $I_0$  is not a complete listing of elements of  $I_0$ , which contradicts the assumption that  $I_0$  is countable. We thereby conclude that  $I$  is an uncountable set. ■

*Alternative Proof: Cantor's Diagonal Argument.*

Assume  $[0, 1]$  is countable. Then

$$[0, 1] = \{x_1, x_2, \dots, x_i, \dots\}, \forall i \in \mathbb{N}$$

Consider the decimal representation of  $x_i$ :

$$x_1 = 0.b_{11}b_{12}b_{13}\dots$$

$$x_2 = 0.b_{21}b_{22}b_{23}\dots$$

$$x_i = 0.b_{i1}b_{i2}b_{i3}\dots$$

for all  $j \in \mathbb{N}$  s.t.  $b_{ij} \in \{0, 1, \dots, 9\}$ . Here we define

$$y = 0.y_1y_2\dots \in [0, 1] \text{ where } y_k = \begin{cases} 1 & \text{if } b_{kk} \neq 1 \\ 0 & \text{if } b_{kk} = 1 \end{cases}, \forall k \in \mathbb{N}$$

Then  $\forall i \in \mathbb{N}, y \neq x_i$  since  $y_i \neq b_{ii}$ . Thus  $y \notin [0, 1]$ , a contradiction. ■

### 3 Sequences and Series

#### 3.1 Sequences and their limits

**Definition (Sequence).** A sequence of real numbers is a *function* from  $\mathbb{N}$  to  $\mathbb{R}$ .

We adopt the notation with a *sequence*:

$$a : \mathbb{N} \rightarrow \mathbb{R}$$

where instead of writing  $a(1), a(2), \dots$ , we write it as  $a_1, a_2, \dots$  which we called them terms or elements of the sequence.

**Notation.**

$$(a_n)_{n=1}^{\infty} \text{ or } (a_n)_{n \in \mathbb{N}} \text{ or } (a_n) \text{ or } (a_n | n \in \mathbb{N})$$

**Definition (Converge to x).** A sequence  $(x_n) \in \mathbb{R}$  converges to  $x \in \mathbb{R}$  if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ such that } n \geq N_\varepsilon \rightarrow |x_n - x| < \varepsilon$$

We write

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (x_n) = x.$$

**Definition (Convergent & Divergent).** A sequence is convergent if it has a limit in  $\mathbb{R}$ , and is divergent if it has no limit in  $\mathbb{R}$ .

**Theorem (Uniqueness of Limit).** A sequence in  $\mathbb{R}$  can have at most one limit. Or, the limit of a sequence is unique if the limit exists

*Proof.* Let  $(x_n)$  be a sequence of real numbers. Suppose  $x, x'$  are limits of  $(x_n)$ . We want to prove  $x = x'$  by contradiction.

Assume  $|x - x'| > 0$ . If we consider  $\varepsilon := \frac{1}{3} |x - x'| > 0$ , then

The existence of  $\lim_{x_n \rightarrow x}$  implies that  $\exists N_1 \in \mathbb{N}$  such that  $|x_n - x| < \varepsilon$  if  $n \geq N_1$ .

Similarly, existence of  $\lim_{x_n \rightarrow x'}$  implies that  $\exists N_2 \in \mathbb{N}$  such that  $|x_n - x'| < \varepsilon$  if  $n \geq N_2$ .

Thus,

$$\begin{aligned} |x - x'| &\leq |x - x_{N_1+N_2} + x_{N_1+N_2} - x'| \\ &\leq |x - x_{N_1+N_2}| + |x_{N_1+N_2} - x'| \text{ by triangle inequality} \\ &< \varepsilon + \varepsilon \\ &= \frac{2}{3} |x - x'| \end{aligned}$$

Then,

$$\frac{1}{3} |x - x'| < 0, \text{ which is a contradiction}$$

we thereby prove by contradiction that

$$|x - x'| = 0, \text{ which is equivalent to } x = x'$$

■

**Example.**

$$\lim_{n \rightarrow \infty} \left( \frac{1}{n} \right) = 0$$

Goal:  $\forall \varepsilon > 0$ , want to find  $N_\varepsilon$  such that  $\left| \frac{1}{n} - 0 \right| < \varepsilon$  for  $n > N$ , so it suffices to show that

$$\frac{1}{n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n$$

*Proof.* Let  $\varepsilon > 0$ . Apply Archimedean's property to  $\frac{1}{\varepsilon}$ , then

$$\begin{aligned} &\exists N \in \mathbb{N} \text{ such that } \frac{1}{\varepsilon} < N \\ \Rightarrow &\forall n \geq N, \left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon. \\ \Rightarrow &\lim_{n \rightarrow \infty} \frac{1}{n} = 0 \end{aligned}$$

■



**Theorem.** Let  $(x_n)$  be a sequence of real numbers, and let  $x \in \mathbb{R}$ . The following are equivalent:

1.  $x_n \rightarrow x$
2.  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  such that  $|x_n - x| < \varepsilon$ , for  $n \geq N$
3.  $\dots x - \varepsilon < x_n < x + \varepsilon \dots$
4.  $\forall \varepsilon$ -neighborhood  $V_\varepsilon(x), \exists N \in \mathbb{N}$  such that  $x_n \in V_\varepsilon(x)$  for  $n \geq N$

**Sketch of proof:**

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$$

**Proposition.**

$$\lim_{n \rightarrow \infty} (2\sqrt{2n+1} - \sqrt{2n}) = 0$$

*Proof.* Let  $\varepsilon > 0$ . Consider

$$N = \left\lceil \frac{1}{2} \left( \frac{1}{2\varepsilon} \right)^2 \right\rceil \in \mathbb{N}$$

$$n > N \Rightarrow n > \frac{1}{2} \left( \frac{1}{2\varepsilon} \right)^2 \Rightarrow \frac{1}{2\sqrt{2n}} < \varepsilon \Rightarrow \left| \sqrt{2n+1} - \sqrt{2n} \right| = \dots = \frac{1}{\sqrt{2n+1} + \sqrt{2n}} < \varepsilon$$

■

**Remark.**

$$\lim_{n \rightarrow \infty} (-1)^n \text{ does not exist.}$$

**Definition (m-tail).** If  $(x_n)$  is a sequence of real numbers and  $m \in \mathbb{N}$ , then the **m-tail** of  $(x_n)$  is the sequence

$$\{x_{n+m} : n \in \mathbb{N}\} = \{x_{m+1}, x_{m+2}, \dots\}$$

**Theorem.** Let  $(x_n)$  be a sequence and  $m \in \mathbb{N}$ . Then  $(x_n)$  is **convergent** iff  $(x_{n+m})$  is **convergent**. Moreover,

$$\lim_{n \rightarrow \mathbb{N}} (x_n) = \lim_{n \rightarrow \mathbb{N}} (x_{n+m})$$

*Proof.* ( $\Rightarrow$ )

Suppose  $x_n \rightarrow x$ . Let

$$\varepsilon > 0, \exists N_\varepsilon > 0, \text{ such that } |x_n - x| < \varepsilon \text{ for } n \geq N_\varepsilon$$

Consider  $N'_\varepsilon := N_\varepsilon + m$  then

$$n + m \geq N'_\varepsilon \Rightarrow n \geq N_\varepsilon \Rightarrow |x_{n+m} - x| < \varepsilon$$

It follows that

$$n \geq N_\varepsilon \Rightarrow n + m \geq N'_\varepsilon \Rightarrow |x_{n+m} - x| < \varepsilon$$

( $\Leftarrow$ )

Suppose  $x_{n+m} \rightarrow x$ .

$$\forall \varepsilon > 0, \exists N_\varepsilon > 0 \text{ such that } |x_{n+m} - x| < \varepsilon, \forall n \geq N_\varepsilon$$

Consider  $N := N_\varepsilon + m$ . Then

$$\begin{aligned} n &\geq N = N_\varepsilon + m \\ \Rightarrow n - m &\geq N_\varepsilon \\ \Rightarrow |x_{(n-m)+m} - x| &< \varepsilon \\ \Rightarrow |x_n - x| &< \varepsilon \end{aligned}$$

■

**Remark.** We say that a sequence  $(x_n)$  ***ultimately*** has a property if that property holds for some tail of  $(x_n)$

**Theorem.** Let  $x_n$  be a sequence of real numbers. Let  $a_n$  be a sequence of positive real numbers such that  $\lim_{n \rightarrow \infty} a_n = 0$ . If  $\exists c > 0, m \in \mathbb{N}, x \in \mathbb{R}$  such that

$$|x_n - x| \leq c \cdot a_n, \forall n \geq m$$

then

$$x_n \rightarrow x$$

*Proof.* We know that

$$\forall \varepsilon > 0, \exists N \geq 0 \text{ s.t. } |a_n| < \frac{\varepsilon}{c}, \forall n \geq N$$

Consider  $N' = \max\{N, m\}, \forall n \geq N'$ . Then

$$\begin{aligned} |x_n - x| &\leq C a_n = c |a_n| < c \cdot \frac{\varepsilon}{c} = \varepsilon \\ \Rightarrow x_n &\rightarrow x \end{aligned}$$

■

**Proposition.**

$$\lim_{n \rightarrow \infty} \frac{17}{2 + 3n} = 0$$

*Proof.*

$$\left| \frac{17}{2 + 3n} - 0 \right| = \frac{17}{2 + 3n} \leq \frac{13}{3n} = \frac{17}{3} \cdot \frac{1}{n}$$

Apply the theorem above with

$$a_n = \frac{1}{n}, c = \frac{17}{3}, m = 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{17}{2 + 3n} = 0, \text{ since } \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

■

**Proposition.**

$$\forall c > 0, \lim_{n \rightarrow \infty} c^{\frac{1}{n}} = 1$$

*Proof.* **Case 1:  $c = 1$**

$$\lim_{n \rightarrow \infty} c^{\frac{1}{n}} = 1$$

**Case 2:  $c > 1$**

Let  $d_n = c^{\frac{1}{n}} - 1$ . Then  $\forall n, d_n > 0$ . It follows that

$$\begin{aligned} (d_n + 1) &= c^{\frac{1}{n}} \Rightarrow c = (1 + d_n)^n \geq 1 + n \cdot d_n \text{ by Bernoulli's inequality} \\ \Rightarrow d_n &\leq (c - 1) \cdot \frac{1}{n} \\ \Rightarrow \left| c^{\frac{1}{n}} - 1 \right| &= d_n \leq (c - 1) \cdot \frac{1}{n} \end{aligned}$$

Apply the theorem with

$$C = c - 1, a_n = \frac{1}{n}, m = 1, x = 1$$

$$\lim_{n \rightarrow \infty} c^{\frac{1}{n}} = 1$$

**Case 3:  $c < 1$  (Note that we cannot use Bernoulli inequality here)**

Define  $e_n$  to be a sequence that satisfies

$$c^{\frac{1}{n}} = \frac{1}{1 + e_n}$$

Then  $e_n > 0 \forall n$ .

$$\begin{aligned} c &= \frac{1}{(1 + e_n)^n} \leq \frac{1}{1 + n \cdot e_n} < \frac{1}{n \cdot e_n} \\ \Rightarrow e_n &< \frac{1}{c} \cdot \frac{1}{n} \\ 1 - c^{\frac{1}{n}} &= 1 - \frac{1}{1 + e_n} = \frac{e_n}{1 + e_n} < e_n < \frac{1}{c} \cdot \frac{1}{n} \end{aligned}$$

Apply the theorem with

$$a_n = \frac{1}{n}, m = 1, C = \frac{1}{c}, x = 1$$

$$\lim_{n \rightarrow \infty} c^{\frac{1}{n}} = 1$$

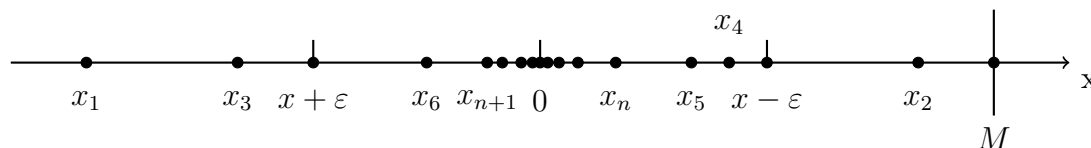
■

### 3.2 Limit Theorems

**Definition** (Bounded sequence). A sequence  $(x_n) \in \mathbb{R}$  is **bounded** if

$$\exists M > 0 \text{ s.t. } \forall n \in \mathbb{N}, |x_n| \leq M$$

**Theorem.** A convergent sequence  $(x_n) \in \mathbb{R}$  is bounded.



*Proof.* By definition of convergent sequence, let  $\varepsilon = 1$ :

$$\exists N > 0 \text{ s.t. } \forall n \geq N, |x_n - x| < 1$$

Thus we have

$$\begin{aligned} & -1 < x_n - x < 1 \\ \Rightarrow & -1 + x < x_n < x + 1, \forall n \geq N \end{aligned}$$

Then define

$$M := \max \{|x_1|, |x_2|, |x_3|, \dots, |-1 + x|, |x + 1|\}$$

$$|x_n| \leq M$$

■

**Remark.** By contrapositive, an unbounded sequence is divergent.

**Definition.** Given sequence  $(x_n), (y_n) \in \mathbb{R}$ , we define following operations of sequence:

- **Sum**  $(x_n + y_n)$
- **Difference**  $(x_n - y_n)$
- **Product**  $(x_n \cdot y_n)$
- **Quotient**  $(\frac{x_n}{y_n})$  if  $\forall n \in \mathbb{N}, y_n \neq 0$
- **Multiple**  $(c \cdot x_n)$

**Theorem (Limit Laws).** Let  $(x_n), (y_n) \in \mathbb{R}$  be sequences of real numbers with  $x_n \rightarrow x, y_n \rightarrow y$ , and let  $c \in \mathbb{R}$ . Then

- $x_n + y_n \rightarrow x + y$
- $x_n - y_n \rightarrow x - y$
- $x_n \cdot y_n \rightarrow x \cdot y$
- $c \cdot x_n \rightarrow c \cdot x$
- If  $\forall n \in \mathbb{N}, y_n \neq 0$  and  $y \neq 0$ , then  $\frac{x_n}{y_n} \rightarrow \frac{x}{y}$

*Proof of Sum. :*

$\forall \varepsilon > 0,$

$$\exists N_1 \in \mathbb{N} \text{ s.t. } \forall n \geq N_1, |x_n - x| < \frac{\varepsilon}{2}$$

$$\exists N_2 \in \mathbb{N} \text{ s.t. } \forall n \geq N_2, |y_n - y| < \frac{\varepsilon}{2}$$

Consider

$$N := \max \{N_1, N_2\}$$

Then

$$\begin{aligned} \forall n \geq N, |(x_n + y_n) - (x + y)| &= |x_n - x + y_n - y| \\ &\leq |x_n - x| + |y_n - y| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

■

*Proof of Difference. :*

Similarly,

$$\begin{aligned} \forall n \geq N, |(x_n - y_n) - (x - y)| &\leq |x_n - x| + |y_n - y| \end{aligned}$$

■

*Proof of Product. :* Since  $(x_n)$  is convergent, it is also bounded. Thus,

$$\exists M \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, |x_n| \leq M$$

By definition of convergence,  $\forall \varepsilon > 0$ :

$$\exists N_1 > 0 \text{ s.t. } \forall n \geq N, |x_n - x| < \frac{\varepsilon}{2|y|}$$

$$\exists N_2 > 0 \text{ s.t. } \forall n \geq N, |y_n - y| < \frac{\varepsilon}{2M}$$

Then,  $\exists N = \max \{N_1, N_2\}$  s.t.  $\forall n \geq N,$

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &= |x_n(y_n - y) - y(x_n - x)| \\ &\leq |x_n| |y_n - y| + |y| |x_n - x| \\ &\leq M \cdot |y_n - y| + |y| |x_n - x| \\ &< M \cdot \frac{\varepsilon}{2M} + |y| \cdot \frac{\varepsilon}{2|y|} = \varepsilon \end{aligned}$$

■

*Proof of Multiply. :*

*Exercise*

■

*Proof of Quotient.*

$$\begin{aligned} \left| \frac{x_n}{y_n} - \frac{x}{y} \right| &= \left| \frac{x_n y - y_n x}{y_n y} \right| \\ &= \left| \frac{x_n y - x_n y_n + x_n y_n - y_n x}{y_n y} \right| \\ &\leq \left| \frac{1}{y_n y} (|x_n| |y_n - y| + |y_n| |x_n - x|) \right| \end{aligned}$$

Since  $y_n \rightarrow y$

$$\exists N_1 \in \mathbb{N} \text{ s.t. } \forall n \geq N, ||y_n| - |y|| \leq |y_n - y| < \frac{|y|}{2}$$

Then

$$\begin{aligned} \Rightarrow & -\frac{|y|}{2} < |y| - |y_n| < \frac{|y|}{2} \\ \Rightarrow & \frac{|y|}{2} < |y_n| \\ \Rightarrow & \frac{1}{|y_n|} < \frac{2}{|y|} \\ \Rightarrow & \frac{1}{|y_n y|} < \frac{2}{|y^2|} \end{aligned}$$

Define  $F := \frac{2}{|y^2|}$ .

Since  $(x_n)$  and  $(y_n)$  are bounded,

$\exists G \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}$

$$|x_n| \leq G, |y_n| \leq G$$

Thus  $\forall n > 0$

$$\exists N_2 > 0 \text{ s.t. } \forall n \geq N_2, |x_n - x| < \frac{\varepsilon}{2GF}$$

$$\exists N_3 > 0 \text{ s.t. } \forall n \geq N_3, |y_n - y| < \frac{\varepsilon}{2GF}$$

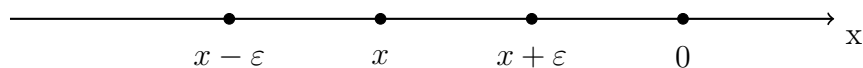
Define  $N := \max \{N_1, N_2, N_3\}$ , then  $\exists N \in \mathbb{N} \text{ s.t. } \forall n > N$ ,

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| < F(G \cdot \frac{\varepsilon}{2GF} + G \cdot \frac{\varepsilon}{2GF}) = \varepsilon$$

■

**Theorem.** If  $(x_n)$  is a convergent sequence in  $\mathbb{R}$  of non-negative terms with  $(x_n) \rightarrow x$ , then  $x \geq 0$ .

*Sketch.* : Assume  $x < 0$ , choose  $\varepsilon \leq |x|$ , then  $\forall n > N, x_n < 0$ , which is a contradiction.



■

**Theorem.** If  $x_n \rightarrow x, y_n \rightarrow y$  are sequences in  $\mathbb{R}$  such that  $\forall n \in \mathbb{N}, x_n \leq y_n$ , then  $x \leq y$ .

*Proof.* Consider  $z_n := y_n - x_n$ . Then  $z_n \geq 0$  and  $z_n \rightarrow y - x$  by the limit law.

$$\Rightarrow y - x \geq 0$$

$$y \geq x$$

■

**Theorem.** If  $x_n \rightarrow x$  is a sequence in  $\mathbb{R}$ , and let  $a, b \in \mathbb{R}$  such that  $\forall n \in \mathbb{N}, a \leq x_n \leq b$ , then  $a \leq x \leq b$ .

*Proof.* Consider constant sequence  $a_n = a$  and  $b_n = b$ . Then this is true by the last theorem. ■

**Theorem (Squeeze Theorem).** Suppose  $(x_n), (y_n), (z_n)$  are sequences of real numbers such that  $\forall n \in \mathbb{N}, x_n \leq y_n \leq z_n$ . If  $\lim(x_n) = \lim(z_n)$ , then  $(y_n)$  is convergent and

$$\lim(x_n) = \lim(y_n) = \lim(z_n)$$

*Proof.* Let  $\forall \varepsilon > 0$ .

Write  $L = \lim(x_n)$ . Then by definition of limit,

$\exists N_1 \in \mathbb{N}$  s.t.  $\forall n \geq N$ ,

$$|x_n - L| < \varepsilon, |z_n - L| < \varepsilon$$

$$\Rightarrow -\varepsilon < x_n - L \leq y_n - L \leq z_n - L < \varepsilon$$

$$\Rightarrow |y_n - L| < \varepsilon$$

Thus  $(y_n)$  converges to  $L$  by definition of limit. ■

**Proposition.**  $\ln(n)$  is divergent.

*Proof.* Since  $\ln(n)$  is unbounded, it is divergent. ■

**Exercise.**  $\lim_{n \rightarrow \infty} \frac{3n^2 + 2n + 1}{5n^2 - 4} = \frac{3}{5}$

$$\lim_{n \rightarrow \infty} \frac{3n^2 + 2n + 1}{5n^2 - 4} = \lim_{n \rightarrow \infty} \frac{3 + 2\frac{1}{n} + \frac{1}{n^2}}{5 - 4\frac{1}{n^2}}$$

Notice that each term of  $\frac{1}{n}$  and  $\frac{1}{n^2}$  converges to 0. Thus by limit law,

$$\lim_{n \rightarrow \infty} \frac{3 + 2\frac{1}{n} + \frac{1}{n^2}}{5 - 4\frac{1}{n^2}} = \frac{\lim\{3 + 2 \cdot 0 + 0\}}{\lim\{5 - 4 \cdot 0\}} = \frac{3}{5}$$

**Proposition.**  $(-1)^n$  is divergent.

*Proof: Exercise.* ■

**Theorem.** If  $x_n \rightarrow x$ , then  $|x_n| \rightarrow |x|$

*Sketch of the proof. :*

$$||x_n| - |x|| \leq |x_n - x|$$

■

**Theorem.** Suppose  $(x_n)$  is a sequence of non-negative real numbers, satisfying  $x_n \rightarrow x$ . Then  $\sqrt{x_n} \rightarrow \sqrt{x}$ .

*Proof.* Let  $\forall \varepsilon > 0$

**Case1:**  $x = 0$

$\exists N \in \mathbb{N}$  s.t.  $\forall n \geq N$ ,

$$\begin{aligned} |x_n - 0| &< \varepsilon^2 \\ \Rightarrow |\sqrt{x_n} - 0| &< \varepsilon \end{aligned}$$

**Case2:**  $x > 0$

$\exists N$  s.t.  $\forall n > N$ ,

$$|x_n - x| < \sqrt{x} \cdot \varepsilon$$

Notice that

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &= \left| \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}} \right| \\ &\leq \frac{|x_n - x|}{\sqrt{x}} < \varepsilon \end{aligned}$$

■

**Theorem.** Let  $(x_n)$  be a sequence of positive real numbers such that  $L = \lim(\frac{x_{n+1}}{x_n})$  exists. If  $L < 1$ , then  $(x_n)$  converges to 0.

*Proof.*  $\exists N_1 \in \mathbb{N}$  s.t.  $\forall n > N_1$

$$\left| \left| \frac{x_{n+1}}{x_n} \right| - |L| \right| \leq \left| \frac{x_{n+1}}{x_n} - L \right| < \frac{1-L}{2}$$

Thus

$$\left| \frac{x_{n+1}}{x_n} \right| < \frac{1+L}{2}$$

Note that  $\frac{1+L}{2} < 1$ , write  $r = \frac{1+L}{2}$ . Then  $\forall m \in \mathbb{N}$ ,

$$x_{N_1+m} < x_{N_1+(m-1)}r < x_{N_1+(m-2)}r^2 < \cdots < x_{N_1}r^m$$

Consider  $(y_m) = (x_{N_1+m})$ ,  $(z_m) = (x_{N_1}r^m)$ . Then

$$0 \leq y_m \leq z_m$$

Since  $z_m \rightarrow 0$

$(y_m) \rightarrow 0$  by squeeze theorem

Thus we conclude that  $(x_n) \rightarrow 0$  by m-th tail theorem.

■



### 3.3 Monotone sequence

**Definition.** Let  $(x_n)$  be a sequence of real. We say  $(x_n)$  is ...

- **increasing** if  $\forall n \in \mathbb{N}, x_{n+1} \geq x_n$ .
- **strictly increasing** if  $\forall n \in \mathbb{N}, x_{n+1} > x_n$ .
- **decreasing** if  $\forall n \in \mathbb{N}, x_{n+1} \leq x_n$ .
- **strictly decreasing** if  $\forall n \in \mathbb{N}, x_{n+1} < x_n$ .

**Theorem (Monotone Convergence Theorem).** A monotone sequence of real numbers is **convergent** iff it is bounded. Moreover, if  $(x_n)$  is increasing, then

$$\lim(x_n) = \sup \{x_n : n \in \mathbb{N}\}$$

If  $(x_n)$  is decreasing, then

$$\lim(x_n) = \inf \{x_n : n \in \mathbb{N}\}$$

*Proof.* ( $\Rightarrow$ )

A convergent sequence is always bounded.

( $\Leftarrow$ )

Suppose  $(x_n)$  is a monotone and bounded sequence.

**Case 1:**  $(x_n)$  is increasing.

Write  $x = \sup \{x_n : n \in \mathbb{N}\}$ .

Let  $\varepsilon > 0$ . Since  $x = \sup(x_n)$ :

$$x - \varepsilon \text{ is NOT an upper bound of } (x_n)$$

Then

$$\exists N \in \mathbb{N} \text{ s.t. } x_N > x - \varepsilon$$

Since  $(x_n)$  is increasing,

$$\forall n \geq N, x_n > x - \varepsilon$$

On the other hand,  $x + \varepsilon$  is an upper bound since  $x$  is an upper bound. Thus,

$$x_n < x + \varepsilon$$

$$\Rightarrow \forall n \geq N, x - \varepsilon < x_n < x + \varepsilon$$

$$\Rightarrow |x_n - x| < \varepsilon \Rightarrow (x_n) \rightarrow x$$

**Case 2:**  $(x_n)$  is decreasing.

Write  $y = \inf(x_n)$ . Let  $\varepsilon > 0$

Since  $y = \inf(x_n)$ ,

$$y + \varepsilon \text{ is NOT an upper bound of } (x_n)$$

Thus

$$\exists N \in \mathbb{N} \text{ s.t. } x_n < y + \varepsilon$$

Since  $(x_n)$  is decreasing,

$$\forall n > N, x_n < y + \varepsilon$$

On the other hand,  $y - \varepsilon$  is a lower bound since  $y$  is a lower bound. Hence,

$$\forall n \in \mathbb{N}, y - \varepsilon < x_n$$

$$\begin{aligned} \forall n \geq N, y - \varepsilon < x_n < y + \varepsilon &\Rightarrow |x_n - y| < \varepsilon \\ &\Rightarrow (x_n) \rightarrow y \end{aligned}$$

**Remark.** One can prove case 2 by following:

$(-x_n)$  is increasing and converges to  $\sup(-x_n)$  by case 1. Also note that

$$(x_n) = (-(-x_n)) \rightarrow -\sup(-x_n)$$

by limit law. So it is easy to prove that

$$-\sup(-x_n) = \inf(x_n)$$

■

**Example.** Consider the sequence  $(x_n)$  is given by

$$\begin{cases} x_0 = \frac{1}{2} \\ x_{n+1} = \frac{3}{2}x_n(1 - x_n) \end{cases}$$

$(x_n)$  is decreasing and bounded.

**Thoughts:** Assume  $(x_n)$  converges, then by limit law,

$$x = \frac{3}{2}x(1 - x) \text{ where } x = \lim(x_n) \Rightarrow x = 0 \text{ or } \frac{2}{3}$$

then, by proof of contradiction, it is not convergent.

*Proof.* **Claim:**  $\frac{1}{3} < x_{n+1} < x_n \leq \frac{1}{2}, \forall n \in \mathbb{N} \cup \{0\}$

**Proof of the claim by induction:**

When  $n = 0$ :

$$\begin{aligned} x_0 &= \frac{1}{2}, x_1 = \frac{3}{2} \cdot \frac{1}{2} \left(1 - \frac{1}{2}\right) = \frac{3}{8} \\ \frac{1}{3} &< \frac{3}{8} < \frac{1}{2} \leq \frac{1}{2} \end{aligned}$$

Suppose this is true for  $n=k$ :

$$\frac{1}{3} < x_{k+1} < x_k \leq \frac{1}{2}$$

Goal:

$$\begin{aligned}\frac{1}{3} < x_{k+2} < x_{k+1} &\leq \frac{1}{2} \\ x_{k+1} &= \frac{3}{2}x_k(1-x_k) \\ \frac{1}{3} < x_k \leq \frac{1}{2} &\Rightarrow \frac{2}{3} > 1-x_k \geq \frac{1}{2} \\ x_{k+1} &< \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{2}\end{aligned}$$

Complete the square:

$$\begin{aligned}x_{k+1} - \frac{1}{3} &= \frac{3}{2}x_k(1-x_k) - \frac{1}{3} \\ &= -\frac{3}{2}\left[\left(x_k - \frac{1}{2}\right)^2 - \frac{1}{36}\right]\end{aligned}$$

So

$$\begin{aligned}\frac{1}{3} < x_k \leq \frac{1}{2} &\Rightarrow \left|x_k - \frac{1}{2}\right| < \frac{1}{6} \\ &\Rightarrow \left(x_k - \frac{1}{2}\right)^2 < \frac{1}{36} \\ &\Rightarrow x_{k+1} - \frac{1}{3} > 0 \\ &\Rightarrow \frac{1}{3} < x_{k+1} \leq \frac{1}{2}\end{aligned}$$

With the similar process, we can derive that

$$\begin{aligned}\frac{1}{3} < x_{k+2} &\leq \frac{1}{2} \\ x_{k+2} &= \frac{3}{2}x_{k+1}(1-x_{k+1}) < \frac{3}{2}x_{k+1} \cdot \frac{3}{2} = x_{k+1}\end{aligned}$$

Therefore, this claim is also true for  $n=k+1$ :

$$\frac{1}{3} < x_{k+2} < x_{k+1} \leq \frac{1}{2}$$

We thereby prove the theorem by induction:

$$\frac{1}{3} < x_{n+1} < x_n \leq \frac{1}{2}$$

■

**Exercise:** Textbook p.75: A sequence that converges to  $\sqrt{a}$  for  $a > 0$ .

**Definition (Euler's Number).**

$$e = \lim(1 + (\frac{1}{n})^n)$$

**Goal:**  $(x_n)$  is convergent where  $x_n = (1 + \frac{1}{n})^n$

$$\begin{aligned} x_n &= (1 + \frac{1}{n})^n = 1 + nC1 \cdot \frac{1}{n} + nC2 \cdot \frac{1}{n^2} + \cdots + nCn \frac{1}{n^n} \\ &= 1 + \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{2} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1) \cdot 3 \cdot 2 \cdot 1}{n!} \cdot \frac{1}{n^2} \\ &= 1 + 1 + \frac{1}{2}(1 - \frac{1}{n}) + \frac{1}{6}(1 - n)(1 - \frac{2}{n}) + \cdots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots \frac{2}{n} \cdot \frac{1}{n} \end{aligned}$$

Write  $x_{n+1}$  in a similar way, we observe that

$$x_n < x_{n+1}$$

**Facts**  $2^{m-1} \leq m!$  for  $m \in \mathbb{N} \Rightarrow \frac{1}{m!} \leq \frac{1}{2^{m-1}}$

$$x_n < 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 1 + \frac{1(1 - (\frac{1}{2})^n)}{1 - \frac{1}{2}} < 3$$

$\Rightarrow (x_n)$  is increasing and bounded

### 3.4 Subsequence and the Bolzano-Weierstrass Theorem

**Example.**

$$\begin{aligned}(x_n) &= ((-1)^n) \\ x_{2n} &= (-1)^{2n} \\ x_{2n+1} &= (-1)^{2n-1}\end{aligned}$$

So  $a_n = x_{2n}$  is a sequence, while  $b_n = x_{2n+1}$  is a subsequence of  $x_n$ .

**Definition (Subsequences).** Let  $(x_n)$  be a real sequence and consider a strictly increasing sequence of natural numbers  $n_1 < n_2 < n_3 < \dots$ . The sequence

$$(x_{n_k} : k \in \mathbb{N})$$

is called a **subsequence** of  $(x_n)$

**Example.** Any tails of a sequence is a subsequence:  $(x_n)$   $n$ -th tail:  $(x_{m+k})$ , where  $n = m + k, k \in \mathbb{N}$ , is a subsequence.

**Theorem.** Suppose  $(x_n)$  converges to  $x$ . Then  $x_{n_k} \rightarrow x$  for any subsequence of  $(x_n)$ .

*Proof.* Let  $\varepsilon > 0$ , then

$$\exists N_\varepsilon > 0 \text{ s.t. } |x_n - x| < \varepsilon \text{ for } n > N_\varepsilon.$$

Note that

$$n_k \geq k, \forall k \in \mathbb{N}.$$

**Exercise.** By induction,  $n_1 \geq 1, n_2 \geq n_1 \geq 1 \Rightarrow n_2 \geq 2$

When  $k > N_\varepsilon$ ,  $n_k > N_\varepsilon$ , thus

$$|x_{n_k} - x| < \varepsilon$$

Therefore

$$(x_{n_k}) \rightarrow x$$

■

**Theorem.** Let  $(x_n)$  be a sequence of real numbers, and let  $x \in \mathbb{R}$ . Then the following are equivalent:

1.  $(x_n)$  does not converge to  $x$ .
2.  $\exists \varepsilon_0 > 0$ , s.t.  $\forall k \in \mathbb{N}, \exists n_k \in \mathbb{N}$  s.t.

$$n_k \geq k \text{ \& } |x_{n_k} - x| > \varepsilon_0$$

3.  $\exists \varepsilon_0 > 0$  and a subsequence  $(x_{n_k})$  s.t.

$$|x_{n_k} - x| > \varepsilon_0, \forall k \in \mathbb{N}$$

**Example.**

$$x_n = (-1)^n \Rightarrow (x_n) \text{ does not converge to } 1$$

*Proof.*

$3 \rightarrow 1$ : by the contrapositive statement of definition

$3 \rightarrow 2$ : 3 is a stronger statement of 2

$1 \rightarrow 2$ : left as exercise

■

**Theorem.** If  $(x_n)$  satisfies either of the following property, then it is **divergent**:

1. There exists two subsequence  $(x_{n_k})$  &  $(x_{m_k})$  whose limits are NOT equal.
2.  $(x_n)$  is unbounded.

**Example.**

1.  $(-1)^n$
2.  $(n)$
3.  $(x_n)$  such that

$$\begin{aligned} x_{2k} &= k \\ x_{2k+1} &= (-1)^k \end{aligned}$$

*Proof. exercise*

■

**Theorem (Bolzano-Weierstrass Theorem).** A bounded sequence of real numbers has a **convergent subsequence**.

**Example.**

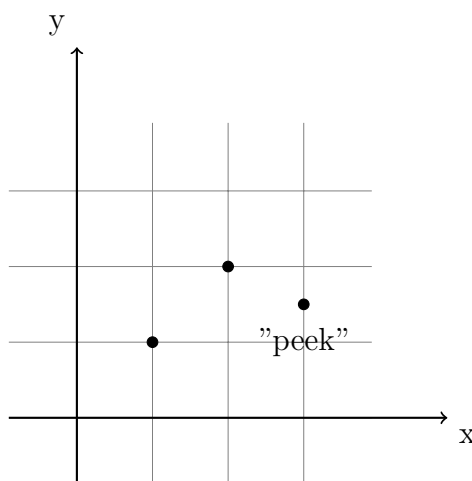
$$x_n = (-1)^n$$

*Proof.*

**Lemma.** If  $(x_n)$  is a sequence of real numbers, there exists a subsequence of  $(x_n)$  which is monotone.

**proof of lemma:**

Call the  $m$ -th term  $x_m$  a "*peek*" if  $x_m$  is at least as large as any term after it in the sequence.



**Case 1:**  $(x_n)$  has infinitely many peaks

List the peaks of  $(x_n)$  in order of increasing index

$$x_{n_1}, x_{n_2}, \dots$$

$\Rightarrow (x_{n_i})$  is a decreasing sequence.

**Case 2:**  $(x_n)$  has a finite number of peaks

Let  $s_1$  be the first index after the last peak of  $(x_n)$ . Then for every  $n \geq s_1$ ,  $\exists m \in \mathbb{N}$  such that  $x_m > x_n$ .

Choose

$$s_2 \geq s_1 \text{ such that } x_{s_2} > x_{s_1}, s_2 \geq s_1 \text{ such that } x_{s_2} > x_{s_1}, \dots$$

$$\Rightarrow (x_{s_i}) \text{ is an increasing sequence.}$$

**Remark.** Lemma + monotone convergent theorem implies Bolzano-Weierstrass theorem.

### Second proof

Suppose  $(x_n)$  is a bounded sequence.

$$\Rightarrow \exists I_1 = [a_1, b_1] \text{ such that } (x_n) \in I_1$$

Consider

$$I'_2 = [a_1, \frac{a_1 + b_1}{2}], I''_2 = [\frac{a_1 + b_1}{2}, b_1]$$

Let  $I_2 = [a_2, b_2]$  be one of  $I'_2, I''_2$  such that  $I_2$  contains infinitely many terms of  $(x_n)$ .

For  $n \in \mathbb{N}$ , define  $I_n = [a_n, b_n]$  in a similar way.

For  $i \in \mathbb{N}$ , choose a term  $x_{n_i}$  such that  $x_{n_i} \in I_i$  and  $n_i > n_{i-1}$

Then

$$\begin{array}{ll} i = 1, & n_i = 1 \\ i = 2, & \text{choose } n_2 \in \mathbb{N} \text{ such that } n_2 > n_1 \text{ \& } x_{n_2} \in I_2 \\ \vdots & \vdots \end{array}$$

- $\forall i \in \mathbb{N}, a_i \leq x_{n_i} \leq b_i$
- $(a_i)$  increases, bounded above by  $b_1 \Rightarrow (a_i) \rightarrow \sup(a_i)$
- $(b_i)$  decreases, bounded below by  $a_1 \Rightarrow (b_i) \rightarrow \inf(b_i)$
- $\inf |a_i - b_i| = \inf \frac{b_1 - a_1}{2^n} = n \Rightarrow \sup(a_i) = \inf(b_i)$

Thus  $(x_{n_i})$  is convergent by squeeze theorem. ■

**Theorem.** Let  $(x_n)$  be a bounded sequence, and  $x \in \mathbb{R}$  has the property that every convergent subsequence of  $(x_n)$  converges to  $x$ . Then  $x_n$  converges to  $x$

*Proof.* Let  $\forall \varepsilon > 0$

By Bolzano-Weierstrass theorem,  $\exists$  a convergent subsequence  $(x_{n_i})$  such that

$$\exists N_\varepsilon \in \mathbb{N} \text{ s.t. for } i > N_\varepsilon, |x_{n_i} - x| < \varepsilon$$

Assume  $(x_n)$  does not converge to  $x$ . Then by previous theorem of subsequence,

$$\exists \varepsilon_0 > 0 \text{ and a subsequence } (x_{n_k}) \text{ s.t. } |x_{n_k} - x| > \varepsilon_0, \forall k \in \mathbb{N}$$

Since  $(x_n)$  is bounded,  $(x_{n_k})$  is also bounded. Thus there exists a convergent subsequence of  $(x_{n_k})$  as  $(x_{n_{k_i}})$ .

Note that  $(x_{n_{k_i}})$  is a convergent subsequence of  $(x_n)$ , thus

$$(x_{n_{k_i}}) \rightarrow x \text{ which is contradiction to previous assumption}$$

■

**Definition.** Let  $(x_n)$  be a sequence of real numbers. A point is called a **subsequential limit** of  $(x_n)$  if it is the limit of a subsequence of  $(x_n)$ .

$$S = \{\alpha \in \mathbb{R} : \alpha \text{ is a subsequential limit}\} \text{ NOTE: may be infinite set}$$

**Definition.** Let  $(x_n)$  be a sequence of real numbers.

- The **limit superior** of  $(x_n)$  is the infimum of the set of  $v \in \mathbb{R}$  s.t.  $v < x_n$  for at most a finite number of  $n \in \mathbb{N}$ . We write it as

$$\limsup(x_n) \text{ or } \limsup x_n \text{ or } \overline{\lim} x_n$$

- The **limit inferior** of  $(x_n)$  is the supremum of the set of  $v \in \mathbb{R}$  s.t.  $w > x_n$  for at most a finite number of  $n \in \mathbb{N}$ . We write it as

$$\liminf(x_n) \text{ or } \liminf x_n \text{ or } \underline{\lim} x_n$$

### **Intuition**

- Suppose  $v < x_n$  for at most finitely many  $n \in \mathbb{N}$ , then

**Theorem.** Let  $(x_n)$  be a bounded sequence, and  $x \in \mathbb{R}$  has the property that every convergent subsequence of  $(x_n)$  converges to  $x$ . Then  $x_n$  converges to  $x$

*Proof.* Let  $\forall \varepsilon > 0$

By Bolzano-Weierstrass theorem,  $\exists$  a convergent subsequence  $(x_{n_i})$  such that

$$\exists N_\varepsilon \in \mathbb{N} \text{ s.t. for } i > N_\varepsilon, |x_{n_i} - x| < \varepsilon$$

Assume  $(x_n)$  does not converge to  $x$ . Then by previous theorem of subsequence,

$$\exists \varepsilon_0 > 0 \text{ and a subsequence } (x_{n_k}) \text{ s.t. } |x_{n_k} - x| > \varepsilon_0, \forall k \in \mathbb{N}$$



Since  $(x_n)$  is bounded,  $(x_{n_k})$  is also bounded. Thus there exists a convergent subsequence of  $(x_{n_k})$  as  $(x_{n_{k_i}})$ .

Note that  $(x_{n_{k_i}})$  is a convergent subsequence of  $(x_n)$ , thus

$$(x_{n_{k_i}}) \rightarrow x \text{ which is contradiction to previous assumption}$$

■

**Definition.** Let  $(x_n)$  be a sequence of real numbers. A point is called a **subsequential limit** of  $(x_n)$  if it is the limit of a subsequence of  $(x_n)$ .

$$S = \{\alpha \in \mathbb{R} : \alpha \text{ is a subsequential limit}\} \text{ (NOTE: may be infinite set)}$$

**Example.** Consider  $(x_n) = \{(-1)^n | n \in \mathbb{N}\}$ . Then

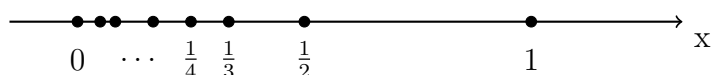
$$S \supseteq \{1, -1\}$$

**Definition** (lim sup and lim inf). Let  $(x_n)$  be a sequence of real numbers.

- The **limit superior** of  $(x_n)$  is the infimum of the set of  $v \in \mathbb{R}$  s.t.  $v < x_n$  for at most a finite number of  $n \in \mathbb{N}$ . We write it as

$$\limsup(x_n) = \limsup x_n = \overline{\lim} x_n = \inf \{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of } n\}$$

**Example.** Consider  $(x_n) = \frac{1}{n}$



Let

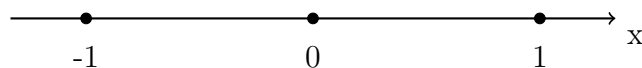
$$X = \{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of } n\}$$

- $-1 \notin X$  because there are infinitely many  $x_n$  such that  $v < x_n$ .
- $\frac{1}{2} \in X$  because there are finitely many  $x_n$  such that  $v < x_n$ .
- $2 \in X$  because there is no  $x_n$  such that  $v < x_n$ , which is smaller than finite and thereby satisfies the definition.

We thus conclude that

$$(0, \infty) \subset X \text{ and } \limsup x_n = \inf X$$

**Example.** Consider  $(x_n) = (-1)^n$ :



- $1 \in X$  because there is no  $x_n$  such that  $v < x_n$ .
- $2 \in X$  because there is no  $x_n$  such that  $v < x_n$ .
- $0, -1 \notin X$  because there are infinitely many  $x_n$  such that  $v < x_n$ .

We thus conclude that

$$[1, \infty) \subset X \text{ (in fact they are equal)}$$

- The **limit inferior** of  $(x_n)$  is the supremum of the set of  $w \in \mathbb{R}$  s.t.  $w > x_n$  for at most a finite number of  $n \in \mathbb{N}$ . We write it as

$$\liminf(x_n) \text{ or } \liminf x_n \text{ or } \underline{\lim} x_n = \sup \{w \in \mathbb{R} | w > x_n \text{ for at most a finite number of } n\}$$

**Intuition**

- Suppose  $v < x_n$  for at most finitely many  $n \in \mathbb{N}$ , then for all large  $n$ ,  $v \geq x_n$ .  
 $\Rightarrow$  No subsequential limit of  $(x_n)$  can possibly exceed  $v$ .
- Similar observation for  $\underline{\lim} x_n$

**Theorem.** Let  $(x_n)$  be a bounded sequence of real numbers, and let  $x^* \in \mathbb{R}$ . Then TFAE:

1.  $x^* = \limsup(x_n)$
2. If  $\varepsilon > 0$ , there are at most a finite number of  $n \in \mathbb{N}$  s.t.  $x^* + \varepsilon < x_n$ , but infinitely many  $n$  for which  $x^* - \varepsilon < x_n$
3. If  $u_m = \sup \{x_n | n \geq m\}$  (sup of  $(m-1)$ -th tail), then  $x^* = \inf \{u_m | m \in \mathbb{N}\} = \lim u_m$
4. If  $S$  is the set of subsequential limits of  $x_n$ , then  $x^* = \sup S$ .

**Remark.** .

- $u_m$  is decreasing.
- There is a similar such list of equivalent properties for  $\liminf$ .

**Corollary.** A bounded sequence  $(x_n)$  is convergent iff  $\overline{\lim} x_n = \lim x_n$

*Proof.* A direct result of the theorem:

$$\overline{\lim} x_n = \sup S \text{ and } \underline{\lim} x_n = \inf S$$

■

*Proof of thm. (a)  $\Rightarrow$  (b).* Let  $\varepsilon > 0$ . Then

$$x^* + \varepsilon > x^* = X = \inf \{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of } n.\}$$

$$\Rightarrow \exists v \in \mathbb{R} \text{ s.t. } x^* \leq v < x^* + \varepsilon$$

and there are only finitely many  $n$  with  $v < x_n$ .

For any  $n$  for which  $x^* + \varepsilon < x_n$ ,  $v < x_n$ . Thus there are only finitely many such  $n$ .

If  $x^* - \varepsilon \notin X$ , then there are infinitely many  $n$  such that  $x^* - \varepsilon < x_n$

■

*Proof of thm. (b)  $\Rightarrow$  (c).* Fix  $\varepsilon > 0$ .

By (b), there are only finitely many  $n$  with  $x^* + \varepsilon < x_n$ .

Take  $N \in \mathbb{N}$  large enough such that

$$\begin{aligned} & x^* + \varepsilon \geq x_n && \forall n \geq N \\ \Rightarrow & x^* + \varepsilon \geq u_N \\ \Rightarrow & x^* + \varepsilon \geq \lim u_n \\ \Rightarrow & x^* \geq \lim u_n && \forall n \geq N \end{aligned}$$

On the other hand, there are infinitely many  $n$  with  $x^* - \varepsilon < x_n \leq u_n$ .

Thus, there exists a subsequence of  $u_n$ , say  $u_{n_k}$ , satisfies

$$\begin{aligned} & x^* - \varepsilon \leq u_{n_k} \\ \Rightarrow & x^* - \varepsilon \leq \lim u_{n_k} = \lim (u_n) \\ \Rightarrow & x^* \leq \lim (u_n) \end{aligned}$$

■

*Proof of thm. (c)  $\Rightarrow$  (d).* **Goal:**

$$\begin{aligned} & x^* = \lim (u_m), u_m = \sup \{x_n | n \geq m\} \\ \Rightarrow & x^* = \sup S \text{ where } S \text{ is the set of subsequential limits.} \end{aligned}$$

Let  $(x_{n_k})$  be a convergent subsequence of  $(x_n)$ . Notice that  $\lim (x_{n_k}) \in S$ .

$$\begin{aligned} & n \geq k \\ \Rightarrow & x_{n_k} \leq \sup \{x_n | n \geq k\} = u_k \\ \Rightarrow & \lim (x_{n_k}) \leq \lim (u_k) = x^* \\ \Rightarrow & x^* \text{ is an upper bound of } S. \end{aligned}$$

For 1,  $\exists n_1 \in \mathbb{N}$  s.t.

$$u_1 - 1 \leq x_{n_1} \leq u_1$$

For  $\frac{1}{2}$ ,  $\exists n_2 \in \mathbb{N}$  s.t.

$$\begin{aligned} u_2 - \frac{1}{2} & \leq x_{n_2} \leq u_2 \\ & \dots \end{aligned}$$

For  $\frac{1}{k}$ ,  $\exists n_k \in \mathbb{N}$  s.t.

$$u_k - \frac{1}{k} \leq x_{n_k} \leq u_k$$

When  $k \rightarrow \infty$ ,

$$x^* - 0 \leq \lim (x_{n_k}) \leq x^*$$

By squeeze theorem,

$$\lim (x_{n_k}) = x^*$$

■

*Proof of thm. (d)  $\Rightarrow$  (a). **Goal:***

$$x^* = \sup S$$

$$\Rightarrow x^* = \limsup x_n = \inf \{v \in \mathbb{R} | v < x_n \text{ for at most finite many of } n\}$$

Fix  $\varepsilon > 0$

There is no subsequence of  $x_n$  which has a limit exceeding  $x^* + \varepsilon$ .

$\Rightarrow$  There is only finitely many  $n$  with  $x_n > x^* + \varepsilon$ .

$\Rightarrow$

$$x^* + \varepsilon \in X$$

$\Rightarrow$

$$\inf X \leq x^* + \varepsilon$$

$\Rightarrow$

$$\limsup x_n \leq x^* + \varepsilon$$

$\Rightarrow$

$$\limsup x_n \leq x^*$$

Next, consider  $x^* - \varepsilon$

Then, there exists a subsequential limit of  $x_n$  which is greater or equal to  $x^* - \frac{1}{2}\varepsilon$ .

There exists a convergent subsequence of  $(x_n)$ , say  $(x_{n_k})$ , such that

$$\lim (x_{n_k}) \geq x^* - \frac{1}{2}\varepsilon$$

$\Rightarrow$

There are infinitely many  $n$  with  $x_n > x^* - \varepsilon$ .

$\Rightarrow$

$$\forall a \in X, x^* - \varepsilon \leq a$$

$\Rightarrow$

$$x^* - \varepsilon \leq \inf X$$

$\Rightarrow$

$$\limsup x_n \geq x^* - \varepsilon$$

$\Rightarrow$

$$\limsup x_n \geq x^*$$

In conclusion,

$$\limsup x_n = x^*$$

■

### 3.5 Cauchy Criterion

**Definition (Cauchy Sequence).** A sequence  $(x_n)$  is Cauchy sequence if

$$\forall \varepsilon > 0, \exists H \in \mathbb{N} \text{ s.t. } \forall n, m \in \mathbb{N}, n > 0, m > 0,$$

$$|x_n - x_m| < \varepsilon$$

**Example.**  $(\frac{1}{n})$  is a Cauchy sequence.

*Proof.* Observe that  $\forall n, m \in \mathbb{N}, n \geq m$ ,

$$\left| \frac{1}{n} - \frac{1}{m} \right| = \left| \frac{m - n}{mn} \right| = \frac{n - m}{mn} \leq \frac{n}{mn} < \frac{1}{m}$$

Choose  $H = \lceil \frac{1}{\varepsilon} \rceil + 1$ , Then

$$\forall n, m \geq H,$$

$$\left| \frac{1}{n} - \frac{1}{m} \right| \leq \frac{1}{m} < \varepsilon$$

■

**Example.**  $(-1)^n$  is not a Cauchy sequence.

*Proof.* Choose  $\varepsilon_0 = \frac{1}{2}$ ,  $\forall H \in \mathbb{N}$ , choose  $n, m \geq H$  s.t.  $n$  is even,  $m$  is odd. Then

$$|(-1)^n - (-1)^m| = |1 - (-1)| = 2 > \frac{1}{2}$$

Thus  $(-1)^n$  does not satisfies the definition of Cauchy sequence.

■

**Theorem.** If  $(x_n)$  is convergent, then it is Cauchy sequence.

*Proof.* Let  $\lim (x_n) = x$ .

Goal:

$$|x_n - x| < \varepsilon \Rightarrow |x_n - x_m| < \varepsilon$$

$$\forall n, m \geq N_\varepsilon,$$

$$\begin{aligned} |x_n - x_m| &= |(x_n - x) + (x - x_m)| \\ &\leq |x_n - x| + |x - x_m| \text{ by triangle inequality} \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \end{aligned}$$

■

**Lemma.** If  $(x_n)$  is Cauchy, then it is bounded.

*Proof.* Choose  $\varepsilon = 1, \exists H \geq 0$  s.t.  $\forall n, m \geq H$ ,

$$|x_n - x_m| < 1$$

Since the choice of  $m$  satisfies  $m \geq H$ , we may choose  $m = H$  s.t.

$$|x_n - x_H| < 1$$

It follows that

$$|x_n| - |x_H| \leq |x_n - x_H| < 1$$

$$|x_n| < |x_H| + 1, \forall n \geq H$$

Let

$$M := \max \{|x_1|, |x_2|, |x_3|, \dots, |x_{H-1}|, |x_H| + 1\}$$

Thus  $\forall n \in \mathbb{N}, |x_n| \leq M$  ■

**Theorem (Cauchy Convergence Theorem).** A sequence of real numbers is convergent if and only if it is Cauchy sequence.

( $\Rightarrow$ ). Done in the previous theorem. ■

( $\Leftarrow$ ). Suppose  $(x_n)$  is Cauchy. By lemma, it is bounded.

By Bolzano-Weierstrass theorem, there exists a convergent subsequence  $(x_{n_k})$ .

Let  $\lim(x_{n_k}) = x$ .

**Goal:**  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  s.t.  $\forall n \geq N$

$$|x_n - x| < \varepsilon$$

We will use the trick of insert subsequence:

By definition of convergence,  $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$  s.t.  $\forall k \geq N_\varepsilon$

$$|x_{n_k} - x| < \frac{1}{2}\varepsilon$$

By definition of Cauchy sequence,  $\forall \varepsilon > 0, \exists H \in \mathbb{N}, H \geq 0$  s.t.  $\forall n, m \geq H$

$$|x_n - x_m| < \frac{1}{2}\varepsilon$$

Thus  $\forall k \geq \max\{H, N_\varepsilon\}$ ,

$$\begin{aligned} |x_k - x| &= |(x_k - x_{n_k}) + (x_{n_k} - x)| \\ &\leq |x_k - x_{n_k}| + |x_{n_k} - x| \quad \text{since } n_k \geq k \geq \max\{H, N_\varepsilon\} \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \end{aligned}$$

■

**Example.** Let  $(h_n)$  be the sequence of harmonic series such that

$$h_n = \sum_{i=1}^n \frac{1}{i}, \text{ and } \lim(h_n) = \sum_{n=1}^{\infty} \frac{1}{n}$$

**Claim:**  $(h_n)$  is divergent

**Goal:** show that it is NOT Cauchy.

*Proof.*  $\forall m, n \in \mathbb{N}$ , WLOG suppose  $m \geq n, h_m > h_n$ , we have

$$\begin{aligned} |h_m - h_n| &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m-1} + \frac{1}{m} \\ &\geq \frac{m-n}{n} \end{aligned}$$

since there are  $(m-n)$ 's terms on the right side of the equation.

So we choose  $\varepsilon_0 = \frac{1}{2}, \forall H \geq 0$ , choose  $n = H, m = 2H$ , then

$$|h_m - h_n| \geq \frac{m-n}{m} = \frac{1}{2}$$

Thus,  $(h_n)$  is NOT Cauchy. ■

### 3.6 Property of Divergent Sequence

**Example** (divergent sequence). :

- $(n) \rightarrow \infty$
- $(-n) \rightarrow -\infty$
- $(-1)^n \cdot n$  is divergent and unbounded.
- $(-1)^n$  is divergent and bounded.

**Definition (Properly Divergent).** Let  $(x_n)$  be a sequence of real numbers. We say that

1.  $(x_n)$  **tends to**  $\infty$ , or  $\lim(x_n) = \infty$  if  $\forall a \in \mathbb{R}, \exists N \in \mathbb{N}$  s.t.  $\forall n \geq N$ ,

$$x_n > a$$

2. Similarly,  $(x_n)$  **tends to**  $-\infty$ , or  $\lim(x_n) = -\infty$  if  $\forall a \in \mathbb{R}, \exists N \in \mathbb{N}$  s.t.  $\forall n \geq N$ ,

$$x_n < a$$

In either case, we say that  $(x_n)$  is **properly divergent**.

**Example.** Let  $C > 0$ .  $(C^n)$  is properly divergent. We write  $\lim(C^n) = \infty$ .

*Proof.* Notice that

$$C^n = [1 + (C - 1)]^n \geq 1 + n(C - 1) \text{ by Bernoulli's inequality}$$

**Goal:**  $\forall a \in \mathbb{R}, \exists N \in \mathbb{N}$  s.t.  $\forall n \geq N$ ,

$$1 + n(C - 1) > a_n \Leftrightarrow n > \frac{a_n - 1}{C - 1}$$

Then we choose  $N = \lceil \frac{a-1}{C-1} \rceil + 1$ . Thus  $\forall n \geq N, C^n > a$ . ■

**Theorem.** A monotone sequence is **divergent** if and only if it is **bounded**. ■

*Proof. Exercise.* ■

**Theorem (Comparison Test).** Let  $(x_n)$  and  $(y_n)$  be two sequences. Suppose  $\forall n \in \mathbb{N}, x_n \geq y_n$ . Then,

1. If  $(x_n) \rightarrow \infty$ , then  $(y_n) \rightarrow \infty$ .
2. If  $(y_n) \rightarrow -\infty$ , then  $(x_n) \rightarrow -\infty$ .

*Proof. Exercise.* ■



**Theorem (Limit Comparison Test).** Let  $(x_n)$  and  $(y_n)$  be two sequences of positive real numbers. Suppose  $\exists L \in \mathbb{R}, L > 0$  s.t.

$$\exists \lim\left(\frac{x_n}{y_n}\right) = L$$

then  $\lim(x_n) = \infty$  if and only if  $\lim(y_n) = \infty$

*Proof.* **Claim:** For  $N$  large enough,

$$\frac{1}{2}L \cdot y_n < x_n < 2L \cdot y_n$$

**Goal:** Claim+Comparison Theorem=Proof.

By definition of limit,  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  s.t.  $\forall n > N$ ,

$$L - \varepsilon < \frac{x_n}{y_n} < L + \varepsilon$$

Choose  $\varepsilon = \frac{1}{2}L$ , we have

$$\frac{1}{2}L \cdot y_n < x_n < \frac{3}{2}L < 2L \cdot y_n$$

since  $L > 0$  and these are positive sequences.

Thus by Comparison Theorem,

$$\lim\left(\frac{1}{2}L \cdot y_n\right) = \infty \Rightarrow \lim(x_n) = \infty$$

■

### 3.7 Introduction to Infinite Series

**Definition (Infinite Series).** If  $(x_n)$  is a sequence of real numbers, then the **infinite series** generated by  $(x_n)$  is the sequence  $(s_k)$  defined by

$$s_k = \sum_{i=1}^k x_i$$

Terms  $s_k$  are called **partial sums**.

**Notation.**  $\sum x_i$  to mean this series or its limit at infinity  $\lim(x_k)$ .

**Theorem (Cauchy Criterion for Series).** The series  $\sum x_i$  **converges** if and only if  $\forall \varepsilon > 0, \exists M \in \mathbb{N}$  s.t.  $\forall n, m \in \mathbb{N}, n > m \geq M$ ,

$$|x_{m+1} + x_{m+2} + \cdots + x_{n-1} + x_n| < \varepsilon$$

Or we write it as

$$|s_k - s_m| < \varepsilon$$

*Proof: Exercise.* ■

**Theorem (Montone Convergence for Series).** Let  $(x_n)$  be a sequence of non-negative real numbers. Then the series  $\sum x_n$  **converges** if and only if  $(s_k)$  is bounded.

*Proof. Exercise.* ■

**Example.**  $\sum \frac{1}{n^2}$  is convergent.

*Proof. Goal:* find convergent subsequence  $(s_{k_j})$ .

Consider subsequence  $(s_{k_j})$  where  $k_j = 2^j - 1$ .

Observe that:

$$\begin{aligned} s_{k_1} &= 1 \\ s_{k_2} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} \\ &< s_{k_1} + 2 \cdot \frac{1}{2^2} = 1 + \frac{1}{2} \\ s_{k_3} &= 1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{9^2}\right) \\ &< s_{k_1} + 2 \cdot \frac{1}{2^2} = 1 + \frac{1}{2} \end{aligned}$$

By induction(details are left as exercise), one can show that

$$s_{k_j} < \sum_{n=0}^{j-1} \frac{1}{2^n} < \sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \text{ (by limit of geometric series.)}$$

Thus  $(s_{k_j})$  is bounded.

By theorem proved in homework, an increasing sequence with bounded(and thus convergent) subsequence implies that the sequence is convergent. ■

## 4 Limits

### 4.1 Limits of Functions

Let  $f : A \rightarrow B$  be a function where  $A, B \subseteq \mathbb{R}$ . Let  $a \in A, L \in B$ .

**Goal:** Define

$$\lim_{x \rightarrow a} f(x) = ?$$

**Intuition:** Define *closeness* on real line.

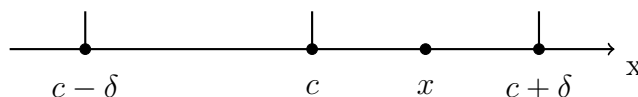
**Definition (Cluster Point).** Let  $A \subseteq \mathbb{R}$ . A point  $c \in \mathbb{R}$  is a **cluster point** of  $A$  if

$\forall \delta > 0, \exists x \in A, x \neq c$  such that

$$|x - c| < \delta$$

Or

$$V_\delta(c) \cap (A \setminus \{c\}) \neq \emptyset$$



**Theorem.**  $c \in \mathbb{R}$  is a cluster point if and only if

there exists a sequence  $(a_n) \in A$  such that

$$\lim(a_n) = c \text{ and } \forall n \in \mathbb{N}, a_n \neq c$$

*Sketch of Proof.* ( $\Rightarrow$ )

$$\delta = 1, \exists a_1 \in A \setminus \{c\} \text{ s.t. } |a_1 - c| < 1$$

$$\delta = \frac{1}{2}, \exists a_2 \in A \setminus \{c\} \text{ s.t. } |a_2 - c| < \frac{1}{2}$$

... observe that:

$$\forall \delta > 0, \exists a_n \in A \setminus \{c\} \text{ s.t. } |a_n - c| < \frac{1}{n} < \delta$$

$$\Rightarrow \lim(a_n) = c \text{ by squeeze theorem.}$$

( $\Leftarrow$ )

$\forall \delta > 0, \exists N_\varepsilon > 0$  s.t.

$$|a_{N_\varepsilon} - c| < \delta$$

Note that  $a_{N_\varepsilon} \in A \setminus \{c\}$  where  $c$  is a cluster point of  $A$ . ■

**Example.** Let  $X$  be the set of cluster points of  $A$ .

- $A = (1, 2) \cup (3, 4) \Rightarrow X = [1, 2] \cup [3, 4]$ . *Proof: Exercise.*
- $A = \{0\} \cup (1, 2) \Rightarrow X = \phi$ .

*Sketch of (2). :*

1. Let  $x \in [1, 2]$ . Prove that  $x \in X$ . Thus  $[1, 2] \in X$ .
2. Prove that  $0 \notin X$ .
3. Prove that  $x \in X$  if  $x \notin \{0\} \cup [1, 2]$ .

■

**Remark.**  $A$  may not be a subset of the set of cluster points of  $A$ .

**Example. :**

- $A = \mathbb{Z} \Rightarrow X = \phi$ .
- $A = \{\frac{1}{n} | n \in \mathbb{N}\} \Rightarrow X = \{0\}$  *Proof: Exercise.*

**Definition (Delta-Epsilon Definition of Limit).** Let  $A \subseteq \mathbb{R}$ ,  $c$  is a cluster point of  $A$ ,  $f : A \rightarrow \mathbb{R}$ . A real number  $L$  is the **limit of  $f$  at  $c$**  if

$\forall \varepsilon > 0, \exists \delta > 0$  s.t.  $\forall x \in A$ ,

$$0 < |x - c| < \delta \rightarrow |f(x) - L| < \varepsilon$$

**Theorem (Uniqueness of Limit).** If  $f : A \rightarrow \mathbb{R}$  and  $c$  is a cluster point of  $A$ , then  $f$  has at most 1 limit at  $c$ .

*Proof.* We will prove this by contradiction.

Let  $L_1$  and  $L_2$  be limits of  $f$  at  $c$ . Assume  $L_1 \neq L_2$ . Choose  $\varepsilon = \frac{|L_1 - L_2|}{2} > 0$ . Then,

$$\exists \delta_1 \text{ s.t. } 0 < |x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \frac{\varepsilon}{2}$$

$$\exists \delta_2 \text{ s.t. } 0 < |x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \frac{\varepsilon}{2}$$

Consider  $\delta := \min\{\delta_1, \delta_2\}$ .

Since  $c$  is a cluster point,  $\exists x_0 \in A$  s.t.

$$0 < |x_0 - c| < \delta$$

Since

$$|f(x_0) - L_1| < \frac{\varepsilon}{2}, |f(x_0) - L_2| < \frac{\varepsilon}{2}$$

We have

$$\begin{aligned} |L_1 - L_2| &\leq |L_1 - f(x_0)| + |f(x_0) - L_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \\ &= \frac{|L_1 - L_2|}{2} \end{aligned}$$

which is a contradiction.

■

**Notation.**

$$L = \lim_{x \rightarrow c} f(x) \text{ or } L = \lim_{x \rightarrow c} f$$

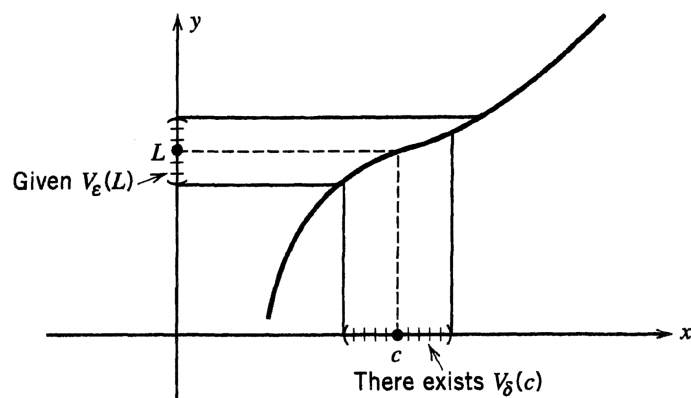
And we say that  $f(x)$  approaches to  $L$  as  $x$  approaches to  $c$ .

**Remark (Divergence of function).** If the limit of  $f(x)$  at  $c$  does not exist, we say that  $f$  diverges at  $c$ .

**Theorem.** Let  $f : A \rightarrow \mathbb{R}$  and  $c$  be a cluster point of  $f$ . The following are equivalent:

1.  $\lim_{x \rightarrow c} f(x) = L$
2.  $\forall V_\varepsilon(L)$   $\varepsilon$ -neighborhood of  $L$ ,  $\exists V_\delta(c)$   $\delta$ -neighborhood of  $c$  s.t.

$$x \in V_\delta(c) \cap (A \setminus \{c\}) \Rightarrow f(x) \in V_\varepsilon(L)$$



**Example.**  $\lim_{x \rightarrow c} f(x) = a$ .

*Proof.*  $\forall \varepsilon > 0, \exists \delta = 1$  s.t.

$$x \in V_\delta(c) \cap (\mathbb{R} \setminus \{c\}) \Rightarrow f(x) = a \in V_\varepsilon(L)$$

■

**Example.**  $\lim_{x \rightarrow c} f(x) = c$ .

*Proof.*  $\forall \varepsilon > 0, \exists \delta = \varepsilon$  s.t.

$$\begin{aligned} & x \in V_\delta(c) \cap (A \setminus \{c\}) \\ \Rightarrow & f(x) = x \in V_\delta(c) \cap (A \setminus \{c\}) \\ \Rightarrow & f(x) \in V_\varepsilon(c) \cap (A \setminus \{c\}) \subseteq V_\varepsilon(c) \end{aligned}$$

■

**Example.**

$$\lim_{x \rightarrow c} x^2 = c^2$$

*Proof.* **Goal:**  $\forall \varepsilon > 0$ , find a  $\delta(c, \varepsilon) > 0$  s.t.

$$\text{if } 0 < |x - c| < \delta, \text{ then } |x^2 - c^2| < \varepsilon$$

which is equivalent to show

$$|x + c| |x - c| < \varepsilon$$

Since the choice of  $\delta$  is dependent on  $\varepsilon$  and  $c$  only,  $|x - c|$  can be easily confined with some constant. Let's assume that

$$|x - c| < 1$$

Now, the only task left is to find a way to confine  $|x + c|$  with a constant by manipulating  $|x - c| < 1$ .

Here we will apply a common trick that estimates addition  $|x + c|$  with subtraction  $|x - c|$ :

$$|x| - |c| \leq |x - c| < 1 \text{ by triangle inequality.}$$

Rearranging the inequality, we have:

$$|x| < |c| + 1$$

Adding the second  $|c|$  to both side of inequality, then, we apply triangle inequality again:

$$|x + c| \leq |x| + |c| < 2|c| + 1$$

$$|x + c| < 2|c| + 1$$

Notice that this is equivalent to show

$$|x + c| |x - c| < (2|c| + \delta) |x - c| < \varepsilon$$

Rearrange the constant factor

$$|x - c| < \frac{\varepsilon}{2|c| + 1}$$

Thus, our choice of  $\delta$  must satisfies two conditions at the same time:

$$\begin{cases} |x - c| < 1 \\ |x - c| < \frac{\varepsilon}{2|c|+1} \end{cases}$$

We achieve this by simply choosing

$$\delta = \min \left\{ \frac{\varepsilon}{2|c| + 1}, 1 \right\}$$

■

**Remark (*The Toolbox of Proofs*).** The readers should develop their "toolbox" of proof techniques. That is, the ***estimation against a constant*** + manipulation of ***triangle inequality*** + choice of  $\delta$  that satisfies ***multiple conditions***.

**Example** (Harder).

$$\lim_{x \rightarrow c} \frac{1}{x} = \frac{1}{c}$$

*Proof.* Observe that

$$\left| \frac{1}{x} - \frac{1}{c} \right| = \frac{|x - c|}{|cx|}$$

Here, we cannot choose  $|x - c|$  smaller than some constant (why? try it on your own). Instead, we choose

$$|x - c| < \frac{|c|}{2}$$

By triangle inequality,

$$\frac{|c|}{2} < |x| < \frac{3|c|}{2}$$

Multiply  $|c|$  to each term of the inequality,

$$\frac{c^2}{2} < |cx| < \frac{3c^2}{2}$$

Thus

$$\frac{1}{|cx|} < \frac{2}{c^2}$$

It follows that

$$\frac{|x - c|}{|cx|} < \frac{2}{c^2} |x - c|$$

In order to make the left term of the inequality less than  $\varepsilon$ , it suffices to confine

$$\frac{2}{c^2} |x - c| < \varepsilon$$

$$|x - c| < \frac{c^2}{2} \varepsilon$$

Thus, our choice of  $\delta$  must satisfy two conditions at the same time:

$$\begin{cases} |x - c| < \frac{|c|}{2} \\ |x - c| < \frac{c^2}{2} \varepsilon \end{cases}$$

We achieve this by simply choosing

$$\delta = \min \left\{ \frac{|c|}{2}, \frac{c^2}{2} \varepsilon \right\}$$

■

**Remark** (**Factoring**  $|x - c|$ ). Since we have the premise of  $|x - c| < \delta$  for free, it would be much easier for us to confine  $|f(x) - L|$  if we factor  $|x - c|$  from the difference.

**Example** (Much Harder).

$$\forall n \in \mathbb{N}, \lim_{x \rightarrow c} x^n = c^n$$

*Proof.* By difference of  $n$ -th powers factorization

$$|x^n - c^n| = |x - c| \left| \sum_{i=0}^{n-1} x^i c^{n-1-i} \right| \leq |x - c| \cdot \sum_{i=0}^{n-1} |x|^i |c|^{n-1-i}$$

If

$$|x - c| < 1$$

then by triangle inequality,

$$|x| < |c| + 1$$

It follows that

$$|x^n - c^n| < |x - c| \cdot \sum_{i=0}^{n-1} (|c| + 1)^i |c|^{n-1-i}$$

Similar to the previous example, it suffices to confine

$$|x - c| \cdot \sum_{i=0}^{n-1} (|c| + 1)^i |c|^{n-1-i} < \varepsilon$$

$$|x - c| < \frac{\varepsilon}{\sum_{i=0}^{n-1} (|c| + 1)^i |c|^{n-1-i}}$$

Thus, choose

$$\delta = \min\left\{1, \frac{\varepsilon}{\sum_{i=0}^{n-1} (|c| + 1)^i |c|^{n-1-i}}\right\}$$

■

**Remark** (*Estimate  $x$  against  $c$* ). This is another trick by triangle inequality:

If  $\exists k \in \mathbb{R}, k > 0$  s.t.  $|x - c| < k$ , then  $|x| < |c| + k$

**Remark** (*difference of  $n$ -th powers factorization*).

$$(x^n - c^n) = (x - c)(x^{n-1} + cx^{n-2} + \cdots + c^{n-2}x + c^{n-1}) = (x - c) \sum_{i=0}^{n-1} x^i c^{n-1-i}$$



**Example (*Some Tedious Factorization*).**

$$\lim_{x \rightarrow 2} \frac{x^3 + 2x - 1}{6x^2 - 5} = \frac{11}{19}$$

*Proof.*

$$\left| \frac{x^3 + 2x - 1}{6x^2 - 5} - \frac{11}{19} \right| = \left| \frac{19x^3 - 66x^2 + 38x + 36}{19(6x^2 - 5)} \right|$$

By some tedious factorization,

$$\dots = |x - 2| \frac{|19x^3 - 28x - 18|}{19|6x^2 - 5|}$$

We estimate  $|x - 2|$  with constant 1

$$|x - 2| < 1$$

$$1 < x < 3 \text{ or } |x| < 3$$

It follows that

$$1 < x^2 < 9$$

$$1 < 6x^2 - 5 < 49$$

$$1 > \frac{1}{6x^2 - 5} > \frac{1}{49}$$

So

$$\frac{1}{19|6x^2 - 5|} < \frac{1}{19}$$

Similarly,

$$|19x^3 - 66x^2 + 38x + 36| \leq 19|x|^2 + 28|x| + 18 < 19 \cdot 3^2 + 28 \cdot 3 + 18 = 273$$

Thus

$$\dots \leq |x - 2| \cdot \frac{273}{19} < \varepsilon$$

$$|x - 2| < \frac{19}{273} \varepsilon$$

We conclude that it suffices to choose

$$\delta = \min \left\{ 1, \frac{19}{273} \varepsilon \right\}$$

■

**Theorem (Sequential Criterion of Limits).** Let  $f : A \rightarrow \mathbb{R}$  and let  $c$  be a cluster point of  $A$ .  $\lim_{x \rightarrow c} f = L$  if and only if for all sequence  $(x_n) \in A$  that converge to  $c$  and  $(x_n) \neq c, \forall n \in \mathbb{N}$ ,  $(f(x_n))$  converges to  $L$ .

*Proof.*  $(\Rightarrow)$ . By definition of convergent sequence,  $\forall \delta > 0, \exists K \in \mathbb{N}$  s.t.  $\forall n \in \mathbb{N}, n \geq K$ ,

$$|x_n - c| < \delta$$

By definition of limit,  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.  $\forall x \in A$ ,

$$|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Thus, choose  $\delta$  given  $\varepsilon$ , and choose  $K$  given  $\delta$ , we have:

$$|x_n - c| < \delta \Rightarrow |f(x_n) - L| < \varepsilon$$

■

$(\Leftarrow)$ . We will prove this by contrapositive.

Assume that there exists  $\varepsilon_0 > 0$  and a  $(x_n) \in A$  converges to  $c$  with  $(x_n) \neq c$  such that for all  $n \in \mathbb{N}$

$$0 < |x_n - c| < \frac{1}{n} \Rightarrow |f(x_n) - L| \geq \varepsilon_0$$

Thus the function does not have a limit at  $c$ . We thereby conclude the converse of the statement.

■

**Theorem (Sequential Criterion of Divergence).** Let  $A \subseteq \mathbb{R}, f : A \rightarrow \mathbb{R}$ ,  $c$  be a cluster point of  $A$ ,  $(x_n) \rightarrow c$  s.t.  $(x_n) \neq c$ , and  $L \in \mathbb{R}^1$ .

1.  $L$  is NOT the limit of  $f$  at  $c \iff f(x_n)$  does NOT converge to  $L$ .
2.  $f$  diverges  $\iff f(x_n)$  does NOT converge.

*Proof.* Exercise.

■

**Example.**

$$\lim_{x \rightarrow 0} \frac{1}{x} \text{ does NOT exist in } \mathbb{R}$$

*Proof.* Let  $(x_n) = \frac{1}{n} \rightarrow 0$ . Then

$$f(x_n) = \frac{1}{\frac{1}{n}} = n \rightarrow \infty$$

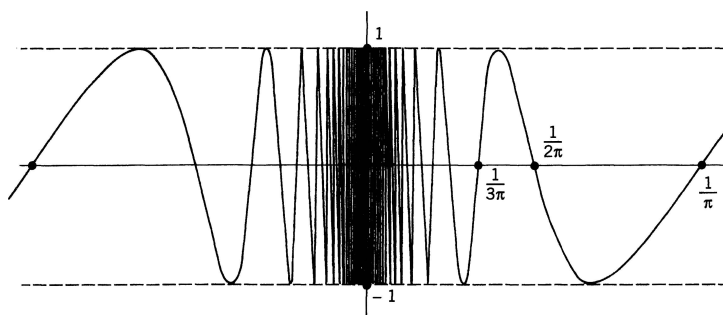
■

---

<sup>1</sup>The following theorems are NOT equivalent!!

**Example.**

$$\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right) \text{ DNE in } \mathbb{R}$$



*Proof.* Let  $(x_n) = \frac{1}{2n\pi} \rightarrow 0$  and  $(y_n) = \frac{1}{\frac{1}{2}\pi + 2n\pi} \rightarrow 0$ . Then

$$f(x_n) = \sin(2n\pi) = 0$$

$$f(y_n) = \sin\left(\frac{1}{2}\pi + 2n\pi\right) = 1$$

Thus limit of  $f$  at 0 does NOT exist in  $\mathbb{R}$ . ■

**Example.**

**Definition (Signum Function).** Let  $A \subseteq \mathbb{R}$  and  $\text{sgn}(x) : A \rightarrow \mathbb{R}$  s.t.

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$\lim_{x \rightarrow 0} \text{sgn}(x) \text{ DNE in } \mathbb{R}$$

*Proof.* Let  $(x_n) = \frac{1}{n} \rightarrow 0$  and  $(y_n) = \frac{1}{-n} \rightarrow 0$ . Then

$$f(x_n) = \text{sgn}(n) = 1$$

$$f(y_n) = \text{sgn}(-n) = -1$$

Thus limit of  $f$  at 0 does NOT exist in  $\mathbb{R}$ . ■

## 4.2 Limit Theorem

**Definition (Bounded Neighborhood of  $c$ ).** Let  $A \subseteq \mathbb{R}$ ,  $f : A \rightarrow \mathbb{R}$ ,  $c$  be a cluster point of  $A$ . Then we say  $f$  is a **bounded Neighborhood of  $c$**  if there exists  $\delta > 0$  and  $\exists M > 0$  s.t.  $\forall x \in A \cap V_\delta(c)$

$$|f(x)| \leq M$$

**Theorem (Existence of Bounded Neighborhood at Limit).** If  $f$  has a limit at  $c$ , then  $f$  is bounded on some neighborhood of  $c$ .

*Proof.* By definition of limit,  $\forall \epsilon > 0, \exists \delta > 0$  s.t.  $\forall x \in A \setminus \{c\}$

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$$

By triangle inequality,

$$|f(x)| - |L| \leq |f(x) - L| < \epsilon$$

$$|f(x)| < |L| + \epsilon$$

If  $f(c)$  is not defined on  $A$ , then let  $M = |L| + \epsilon$ . If  $f(c)$  is defined on  $A$ , then let  $M = \max\{f(c), |L| + \epsilon\}$ . Since the choice of  $\epsilon$  is arbitrary, we will choose  $\epsilon = 1$ . Thus,

$$f(x) \leq M$$

■

**Definition.** Let  $A \subseteq \mathbb{R}$ ;  $f, g : A \rightarrow \mathbb{R}$ ,  $c$  be a cluster point of  $A$ . Then  $\forall x \in A$

- **Sum** of function:  $(f + g)(x) = f(x) + g(x)$ .
- **Difference**:  $(f - g)(x) = f(x) - g(x)$ .
- **Multiple**:  $(bf)(x) = b \cdot f(x)$  for some  $b \in \mathbb{R}$ .
- **Product**:  $(f \cdot g)(x) = f(x) \cdot g(x)$ .
- **Quotient**:  $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$  if  $g(x) \neq 0$ .

**Theorem (Limit Theorem).** If  $\exists L, M \in \mathbb{R}$  s.t.  $\lim_{x \rightarrow c} f(x) = L$  and  $\lim_{x \rightarrow c} g(x) = M$ , then

- $\lim_{x \rightarrow c} (f + g)(x) = L + M$ .
- $\lim_{x \rightarrow c} (f - g)(x) = L - M$ .
- $\lim_{x \rightarrow c} (bf)(x) = b \cdot L$ .
- $\lim_{x \rightarrow c} (f \cdot g)(x) = L \cdot M$ .
- $\lim_{x \rightarrow c} (\frac{f}{g})(x) = \frac{L}{M}$  if  $\lim_{x \rightarrow c} g(x) \neq 0$ .

*Proof. Exercise.*

■

**Remark.** Always check conditions before applying: this is true only if both  $f$  and  $g$  has a limit at  $c$  !

**Example.**  $\lim_{x \rightarrow 4} \frac{(x-4)(x+3)}{4(x-4)(x-5)} = \lim_{x \rightarrow 4} \frac{x+3}{4(x-5)} = -\frac{7}{4}$

**Corollary** (***Polynomial Function***).

$$\lim_{x \rightarrow c} p(x) = \lim_{x \rightarrow c} \sum_{i=0}^n a_i \cdot x^i = \sum_{i=0}^n a_i \cdot \lim_{x \rightarrow c} x^i = \sum_{i=0}^n a_i \cdot c^i = p(c)$$

**Corollary** (***Rational Function***). For polynomial functions  $p(x), q(x)$  s.t.  
 $\lim_{x \rightarrow c} p(x) \rightarrow p(c), \lim_{x \rightarrow c} q(x) \rightarrow q(c) \neq 0$ ,

$$\lim_{x \rightarrow c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$

**Theorem.** Let  $A \subseteq \mathbb{R}$ ;  $f, g : A \rightarrow \mathbb{R}$ ,  $c$  be a cluster point of  $A$ .

If  $\exists a, b \in \mathbb{R}, \forall x \in A, x \neq c$  satisfying

$$a \leq f(x) \leq b \text{ and } \lim_{x \rightarrow c} f = L$$

then

$$a \leq \lim_{x \rightarrow c} f \leq b$$

*Proof.*  $\forall (x_n) \in A \setminus \{c\}, (x_n) \rightarrow c$ ,

$$a \leq f(x_n) \leq b \Rightarrow a \leq L \leq b$$

■

**Theorem** (***Squeeze Theorem***). Let  $A \subseteq \mathbb{R}$ ;  $f, g, h : A \rightarrow \mathbb{R}$ ,  $c$  be a cluster point of  $A$ .

If  $\forall x \in A, x \neq c$ ,

$$f(x) \leq g(x) \leq h(x) \text{ and } \lim_{x \rightarrow c} f = L = \lim_{x \rightarrow c} h$$

then

$$\lim_{x \rightarrow c} g = L$$

*Proof. Exercise. Try squeeze theorem of sequence.*

■

**Example.**  $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$

*Proof.* Notice that

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \Rightarrow -x \leq x \sin\left(\frac{1}{x}\right) \leq x$$

Since

$$\lim_{x \rightarrow 0} (-x) = 0 = \lim_{x \rightarrow 0} (x)$$

$$\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$$

■

**Example.** Let  $b \in \mathbb{R}, b > 0$ . Then  $\lim_{x \rightarrow 0} x^b = 0$ . Notice that when  $x \in [0, 1]$

$$x^{\lceil b \rceil} \leq x^b < x^{\lfloor b \rfloor}$$

The rest is left as exercise.

**Theorem.** Let  $A \subseteq \mathbb{R}$ ;  $f, g, h : A \rightarrow \mathbb{R}$ ,  $c$  be a cluster point of  $A$ . If  $\lim_{x \rightarrow c} f > 0$ , then  $\exists \delta > 0$  s.t.  $V_\delta(c)$  s.t.  $\forall x \in A \cap V_\delta(c) \setminus \{c\}$ ,

$$f(x) > 0$$

*Proof. Exercise.*

■

## 5 Continuous Functions

### 5.1 Continuity

**Definition (Counituuous).** Let  $A \subseteq \mathbb{R}$ ,  $f : A \rightarrow \mathbb{R}$ ,  $c \in A$ .

We say  $f$  is countinuous at c if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in A$$

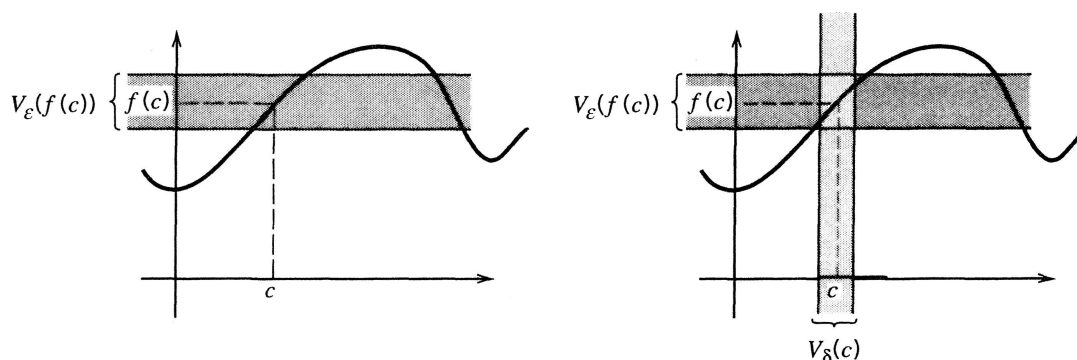
$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

We say  $f$  is discontinuous at c if  $f$  is NOT continuous at  $c$ .

**Definition (Using Neighborhood).** Let  $A \subseteq \mathbb{R}$ ,  $f : A \rightarrow \mathbb{R}$ ,  $c \in A$ . Then  $f$  is continuous at  $c$  if and only if

$\forall \varepsilon > 0$  and its  $\varepsilon$ -neighborhood at  $f(c)$ ,  $V_{f(c)}(\varepsilon)$ ,  $\exists \delta > 0$  and its  $\delta$ -neighborhood at  $c$ ,  $V_c(\delta)$ , s.t.

$$f(V_\delta(c) \cap A) \subseteq V_\varepsilon(f(c))$$



**Remark (Comparison with Limit Definition).** Continuity has 3 good properties that will be useful in the future study of Real Analysis:

- If  $c$  is a cluster point of  $A$ , then  $f$  is continuous if and only if

1.  $f(x)$  is defined at  $c$ .

**Counter-example:** Any function  $f : \mathbb{Q} \rightarrow \mathbb{R}$  is *undefined* at  $\mathbb{R} \setminus \mathbb{Q}$  and thus *discontinuous* at  $\mathbb{R} \setminus \mathbb{Q}$ .

2.  $\lim_{x \rightarrow c} f(x)$  exists.

**Counter-example:**  $\frac{1}{x}$  is *undefined* at 0 and thus *discontinuous* at 0.

3.  $f(c) = \lim_{x \rightarrow c} f(x)$ .

**Counter-example:**  $f(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$  is discontinuous at 0 as  $f(c) = 0 \neq 1 = \lim_{x \rightarrow c} f(x)$

- If  $c$  is **NOT a cluster point** of  $A$ , then  $c$  is an **isolated point**, and

$\exists V_\delta(c) \cap A = \{c\}$ . Notice that an **isolated point** is **automatically continuous** as it *satisfies* the definition of continuity at  $c$ . This is because

$$|x - c| = |c - c| = 0 < \delta \Rightarrow |f(x) - f(c)| = |f(c) - f(c)| = 0 < \varepsilon$$

**Theorem (Sequential Criterion for Continuity).**

Let  $A \subseteq \mathbb{R}$ ,  $f : A \rightarrow \mathbb{R}$ ,  $c \in A$ . Then  $f$  is continuous at  $c$  if and only if

$$\forall (x_n) \subseteq A \text{ s.t. } (x_n) \rightarrow c, f((x_n)) \rightarrow f(c).$$

*Proof.* ( $\Rightarrow$ ). Continuity of  $f$  at  $c$  implies limit of  $f$  at  $c$  exists<sup>2</sup>. Thus we simply apply sequential criterion of limit.

( $\Leftarrow$ ). This is exactly the statement of sequential criterion for limit at  $c$ . Thus,

$$\exists \lim_{x \rightarrow c} f(x) = f(c)$$

Notice that  $c \in A$  implies that  $f$  is defined on  $c$ . Thus, we conclude that  $f$  is continuous on  $c$  by previous remark in p.43 ■

**Corollary (Sequential Criterion for Discontinuity).**

Let  $A \subseteq \mathbb{R}$ ,  $f : A \rightarrow \mathbb{R}$ ,  $c \in A$ . Then  $f$  is discontinuous at  $c$  if and only if

$$\exists (x_n) \subseteq A \text{ s.t. } (x_n) \rightarrow c, f((x_n)) \not\rightarrow f(c).$$

*Proof.* Similar. Left to reader as exercise. ■

**Example.** We begin with polynomial and rational functions:

- Constant function  $f(x) = b$  is continuous in  $\mathbb{R}$ .
- Linear function  $f(x) = ax + b$  is continuous in  $\mathbb{R}$ .
- Quadratic function  $f(x) = x^2$  is continuous in  $\mathbb{R}$ .
- $f(x) = \frac{1}{x^2}$  is continuous in  $\mathbb{R}$ .
- Polynomial functions are continuous in  $\mathbb{R}$  (try to prove it!).
- Rational functions are continuous in  $\mathbb{R}$  (try to prove it!).
- $f(x) = \frac{1}{x}$  is NOT continuous at 0.
- $f(x) = \text{sgn}(x)$  is NOT continuous at 0.

---

<sup>2</sup>proof by definition



**Example (Dirichlet's “discontinuous function”).**

Let  $A = \mathbb{R}$  and define Dirichlet's “discontinuous function” by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \text{ (rational)} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \text{ (irrational)} \end{cases}$$

**Claim:** this function is discontinuous on  $\mathbb{R}$ .

*Proof.* Let  $c \in A$ .

- If  $c \in \mathbb{Q}$ , then  $\exists (x_n) \in \mathbb{R} \setminus \mathbb{Q}$  s.t.  $\forall n \in \mathbb{N}$ ,

$$c < (x_n) < c + \frac{1}{n}$$

By squeeze theorem,

$$\lim c \leq \lim (x_n) \leq \lim (c + \frac{1}{n})$$

$$(x_n) \rightarrow c, (c + \frac{1}{n}) \rightarrow c$$

$$\lim f(x_n) = 0 \neq 1 = \lim f(c + \frac{1}{n})$$

Thus, by Sequential Criterion of Discontinuity,  $f$  is discontinuous for all  $c \in \mathbb{Q}$ .

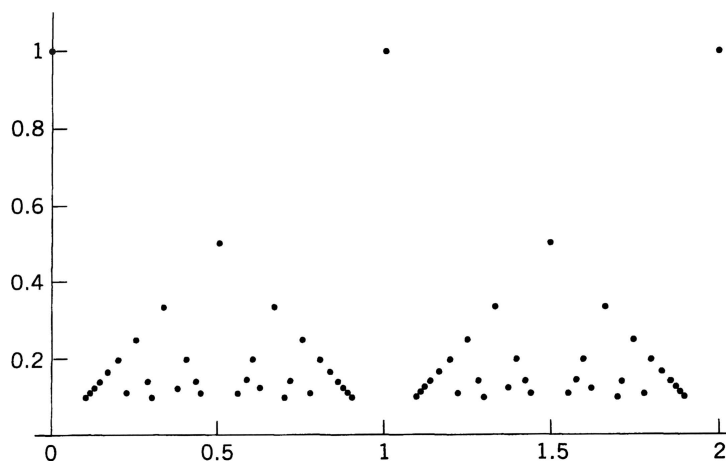
- If  $c \in \mathbb{R} \setminus \mathbb{Q}$ , then we adopt similar strategy. The rest of proofs is left as exercise. ■

**Example (Thomae's function).** \*Notice that this example is harder. We skip it on class.

Let  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \text{ (irrational)} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ (rational)} \end{cases}$$

**Claim:**  $f$  is discontinuous on  $\mathbb{Q}$  and continuous on  $\mathbb{R}^+ \setminus \mathbb{Q}$ .



*Proof.*

- If  $c$  is rational, then  $\exists (x_n) \in \mathbb{R}^+ \setminus \mathbb{Q}$  s.t.  $(x_n) \rightarrow c$ . Then we have

$$\lim f(x_n) \rightarrow 0 \neq \frac{1}{q} = f(c)$$

Since function value does not equal to limit value at  $c$ ,  $f$  is discontinuous at  $c \in \mathbb{Q}$

- If  $c$  is irrational, **Goal:** show that  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.  $\forall x \in \mathbb{R}^+$ ,

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| = |f(x)| < \varepsilon$$

- If  $x \in \mathbb{R}^+ \setminus \mathbb{Q}$ , then  $|f(x)| = 0 < \varepsilon$  for some arbitrary choice of  $\delta$ .
- If  $x \in \mathbb{Q}$ , then by Archimedean Property, we could find a  $\frac{1}{n_0} > \varepsilon$ .

Notice that there are only finite number of  $n \in \mathbb{N}$  such that  $n < n_0$ . Thus, there are only finite number of  $\frac{p}{q} \in (c - 1, c + 1)$  with denominator  $q < n_0$ .

Hence we could choose  $\delta$  so small that the  $V_\delta(c)$  contains no rational numbers with denominator less than  $n_0$ . It follows that

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| = |f(x)| \leq \frac{1}{n_0} < \varepsilon$$

Thus,  $f$  is continuous at  $c \in \mathbb{R} \setminus \mathbb{Q}$

■

## 5.2 Combinations of Continuous Functions

**Theorem.** Let  $A \subseteq \mathbb{R}$ ,  $f, g : A \rightarrow \mathbb{R}$ ,  $c \in A$ ,  $b \in \mathbb{R}$ . Suppose  $f, g$  are continuous on  $c$ , then the following combination of functions are continuous:

- **Addition:**  $f + g$
- **Subtraction:**  $f - g$
- **Product:**  $f \cdot g$
- **Multiplication:**  $b \cdot f$
- **Quotient:**  $\frac{f}{g}$  if  $g(x) \neq 0$
- **Absolute:**  $|f|(x)$
- **Square Root:**  $\sqrt{f}(x)$

*Proof.*

- If  $c$  is not a cluster point of  $A$ , then the theorem is automatically correct.
- If  $c$  is a cluster point of  $A$ , then we only need to show that the function value at  $c$  is equal to the limit of function at  $c$ . Detailed proof are left as exercise. ■

**Example.**

- Polynomial functions are continuous on  $\mathbb{R}$ .
- Rational functions are continuous on  $\mathbb{R}$ .

**Example.**

- $\sin(x)$  is continuous on  $\mathbb{R}$ .

*Proof.* Notice that  $\forall x, y, z \in \mathbb{R}$ , we have 3 inequalities:

$$|\sin(z)| \leq |z|, \quad |\cos(z)| \leq 1, \quad \sin(x) - \sin(y) = 2 \sin\left[\frac{1}{2}(x - y)\right] \cdot \cos\left[\frac{1}{2}(x + y)\right]$$

Hence if  $c \in \mathbb{R}$ , we have

$$|\sin(x) - \sin(c)| \leq 2 \cdot \frac{1}{2} |x - c| \cdot 1 = |x - c|$$

Thus  $\sin(x)$  is continuous on  $\mathbb{R}$ . ■

- $\cos(x)$  is continuous on  $\mathbb{R}$

*Proof. Exercise.* Similar techniques as above. ■

- $\tan(x)$ ,  $\cot$ ,  $\sec$ ,  $\csc$  are all continuous on *where they are defined*.

*Proof.* Combinations of continuous functions on their domain are continuous. ■

**Theorem.** Let  $A, B \subseteq \mathbb{R}$ ,  $f : A \rightarrow \mathbb{R}$ ,  $g : B \rightarrow \mathbb{R}$ ,  $c \in A$  s.t.  $f(A) \subseteq B$ . Suppose  $f$  is continuous at  $c \in A$ ,  $g$  is continuous on  $b = f(c) \in B$ , then the composition  $g \circ f$  is continuous at  $c$ .

*Proof.* Since  $f$  is continuous at  $c$ ,  $\forall \varepsilon > 0$ ,  $\exists \delta > 0$  s.t.  $\forall x \in A$ ,

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

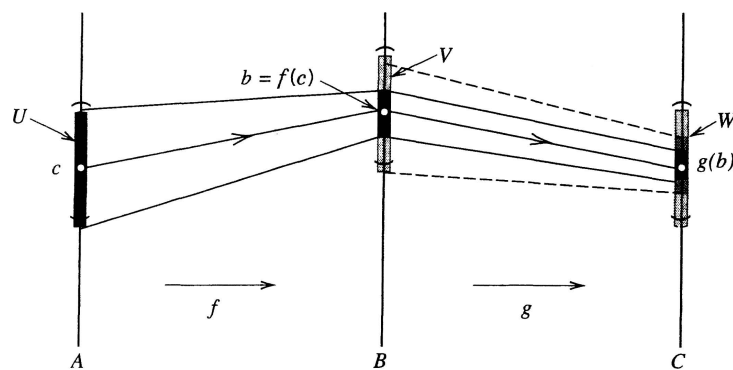
Since  $g$  is continuous at  $b = f(c)$ ,  $f(A) \subseteq B$ ,  $\forall \zeta > 0$ ,  $\exists \varepsilon > 0$  s.t.  $\forall f(x) \in B$ ,

$$|f(x) - b| < \varepsilon \Rightarrow |g(f(x)) - g(b)| < \zeta$$

Thus,

$$|x - c| < \delta \Rightarrow |g(f(x)) - g(b)| = |g \circ f(x) - g \circ f(c)| < \zeta$$

■



**Example.**  $\sqrt{x^2 + 1}$  is continuous on  $\mathbb{R}$ .

*Proof.* Exercise.

■

### 5.3 Continuous Function on Bounded Intervals

Continuous function on closed bounded interval has many good properties.

**Definition (Bounded Function).** A function  $f : A \rightarrow \mathbb{R}$  is said to be bounded on A if  $\exists M > 0$  s.t.  $\forall x \in A$

$$|f(x)| \leq M$$

A function is unbounded if  $\forall M > 0, \exists x_M \in A$  s.t.

$$|f(x)| > M$$

**Example.**  $f(x) = \frac{1}{x}$  is unbounded on  $A = (0, \infty)$ .

*Proof.* By Archimedean Property,  $\forall M > 0, \exists x_M = \frac{1}{M+1} \in A$  s.t.

$$\left| f\left(\frac{1}{M+1}\right) \right| = \left| \frac{1}{\frac{1}{M+1}} \right| = |M+1| > M$$

We thereby conclude that  $\frac{1}{x}$  is unbounded on  $A$ . ■

**Theorem (Boundness Theorem).** Let  $a, b \in \mathbb{R}, a < b, I = [a, b]$  a closed bounded interval. If  $f : I \rightarrow \mathbb{R}$  is continuous on  $I$ , then  $f$  is bounded on  $I$ .

*Proof.* Assume  $f$  is not bounded on  $I$ . Then, by definition,  $\forall n \in \mathbb{N}, \exists x_n \in I$  s.t.

$$|f(x_n)| > n$$

Since  $I$  is bounded, the sequence  $(x_n) \in I$  is bounded. Therefore, by the Bolzano-Weierstrass Theorem, there exists a convergent subsequence  $(x_{n_k}) \rightarrow x$  for some number  $x \in I$ . Since  $f$  is continuous on  $I$ ,

$$f(x_{n_k}) \rightarrow f(x)$$

We thereby conclude that  $f(x_{n_k})$  is a bounded sequence, which contradicts the fact that  $f(x_{n_k})$  is unbounded.

$$\forall k \in \mathbb{N}, n_k \in \mathbb{N}, |f(x_{n_k})| > n_k \geq k$$
■

**Remark.** We will give 3 counter-examples to show that all of three conditions are necessary for Boundedness Theorem to be true.

- Interval must be closed

**Counter-example:**  $f(x) = \frac{1}{x}$  on  $(0, 1]$  is continuous but unbounded.

- Interval must be bounded

**Counter-example:**  $g(x) = x$  is continuous but unbounded on  $[0, \infty)$ .

- The function must be continuous

**Counter-example:** Define  $h : [0, 1] \rightarrow \mathbb{R}, h(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0, 1] \\ 1 & \text{if } x = 0 \end{cases}$  is discontinuous and unbounded.

**Definition (Maximum and Minimum).** Let  $A \subseteq \mathbb{R}$ ,  $f : A \rightarrow \mathbb{R}$ .

We say that  $f$  has an absolute maximum on  $A$  if  $\exists x^* \in A$  s.t.  $\forall x \in A$

$$f(x^*) \geq f(x)$$

We say that  $f$  has an absolute minimum on  $A$  if  $\exists x_* \in A$  s.t.  $\forall x \in A$

$$f(x_*) \leq f(x)$$

**Remark.** Continuous function on a bounded set  $A$  does not necessarily have an maximum or minimum on  $A$ . For example:

- $f(x) = \frac{1}{x}$  has NO absolute maximum or absolute minimum on  $A = (0, \infty)$
- $g(x) = x^2$  has only absolute minimum  $x_* = 0$  on  $\mathbb{R}$ .

**Theorem (Maximum-Minimum Theorem).** Let  $I = [a, b] \in \mathbb{R}$  be a closed bounded interval,  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$ . Then  $f$  has an absolute maximum and absolute minimum on  $I$ .

*Proof.* By previous theorem,  $f$  is bounded on  $I$ . Thus,

$$\exists s^* = \sup f(I), \beta = \inf f(I)$$

We will first proof that  $f$  has an absolute maximum. It suffices to show that

$$\exists x^* \in I \text{ s.t. } f(x^*) = \sup f(I)$$

Since  $s^* = \sup f(I)$ , then  $\forall n \in \mathbb{N}$ ,  $s^* - \frac{1}{n}$  is not an upper bound of the set  $f(I)$ . Consequently,  $\exists x_n \in I$  s.t.  $\forall n \in \mathbb{N}$

$$s^* - \frac{1}{n} < f(x_n) \leq s^*$$

Since  $I$  is bounded,  $(x_n)$  is bounded, by the Bolzano-Weierstrass Theorem, there exists a subsequence  $(x_{n_k}) \in I$  s.t.  $(x_{n_k}) \rightarrow x^*$ .

Since  $f$  is continuous on  $I$ ,

$$\lim(f(x_{n_k})) = f(x^*)$$

By squeeze theorem,

$$\lim(s^* - \frac{1}{n_k}) \leq \lim f(x_{n_k}) \leq \lim s^*$$

$$s^* \leq f(x^*) \leq s^*$$

$$s^* = f(x^*)$$

Thus we conclude that  $x^*$  is the absolute maximum of  $f$  on  $I$ . The proof of absolute minimum is left as exercise using similar techniques. ■

---

<sup>3</sup>The reader should find out why the strict inequality become weak inequality when limit applies.

The proof of next theorem provides an algorithm, know as **Bisection Method**, to calculate root to certain level of accuracy.

**Theorem (Location of Roots)**. Let  $I = [a, b]$ ,  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$ . If  $f(a) < 0 < f(b)$  or  $f(b) < 0 < f(a)$ , then  $\exists c \in (a, b)$  s.t.  $f(c) = 0$ .

*Proof.* WLOG, let's assume that  $f(a) < 0 < f(b)$ . Define a sequence of closed bounded nested interval  $I_n$  and midpoint  $p_n$  s.t.  $\forall n \in \mathbb{N}$ :

$$a_1 = a, b_1 = b, I_1 = [a_1, b_1], p_1 = \frac{1}{2}(a_1 + b_1)$$

Notice that if  $f(p_n) = 0$ , then  $c = p_n$  and we are done. If not, then

$$I_n = \begin{cases} [a_{n-1}, p_{n-1}] & \text{if } f(p_{n-1}) > 0 \\ [p_{n-1}, a_{n-1}] & \text{if } f(p_{n-1}) < 0 \end{cases} \subset I_{n-1}$$

Observe that

- $\{I_n\}$  is a infinite sequence of **nested intervals** where  $\forall n \in \mathbb{N}, I_n \subset I_{n-1}$
- $\lim\{b_n - a_n\} = \lim\{\frac{b-a}{2^{n-1}}\} = 0 \Rightarrow \lim(a_n) = \lim(b_n)$

Then by Nested Interval Property,

$$\exists c \in [a, b] \text{ s.t. } \forall n \in \mathbb{N}, c \in I_n, \bigcap_{n=1}^{\infty} I_n = \{c\}$$

Also notice that

$$a_n < c < b_n \Rightarrow \lim(a_n) \leq c \leq \lim(b_n)$$

By Squeeze Theorem

$$\lim(a_n) = c = \lim(b_n)$$

Notice that

$$\begin{cases} f(a_n) < 0 \Rightarrow \lim f(a_n) \leq 0 \\ 0 < f(b_n) \Rightarrow 0 \leq \lim f(b_n) \end{cases}$$

Thus

$$c = \lim(a_n) = \lim(b_n) = 0$$

■

**Theorem (Bolzano's Intermediate Value Theorem).** Let  $I \in \mathbb{R}$  be an interval<sup>4</sup>,  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$ . If  $\exists a, b \in I, k \in \mathbb{R}$  s.t.

$$f(a) < k < f(b)$$

then  $\exists c \in I$  s.t.

$$f(a) < f(c) = k < f(b)$$

.

*Proof.* WLOG, assume  $a < b$  and define  $g(x) = f(x) - k$ . Then

$$g(a) < 0 < g(b)$$

By previous theorem,  $\exists c, a < c < b$  s.t.  $0 = g(c) = f(c) - k$ . Thus

$$f(c) = k$$

The similar proof also applies for  $b < a$ . ■

**Corollary.** Let  $I = [a, b]$ ,  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$ . If  $\exists k \in \mathbb{R}$  s.t.

$$\inf f(I) \leq k \leq \sup f(I)$$

then  $\exists c \in I$  s.t.

$$f(c) = k$$

*Proof.* This is a direct result of previous theorem. ■

**Corollary.** Let  $I = [a, b]$ ,  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$ . Then

$$f(I) = [\inf f(I), \sup f(I)]$$

*Proof.* Let  $m = \inf f(I)$ ,  $M = \sup f(I)$ . We know by the Maximum-Minimum Theorem that  $m, M \in f(I)$ . Thus

$$f(I) \subseteq [m, M]$$

Then by Bolzano's Intermediate Value Theorem,  $\forall k \in [m, M], \exists c_k \in I$  s.t.  $k = f(c_k)$ . We thereby conclude that

$$[m, M] \subseteq f(I)$$

It follows that

$$f(I) = [m, M]$$

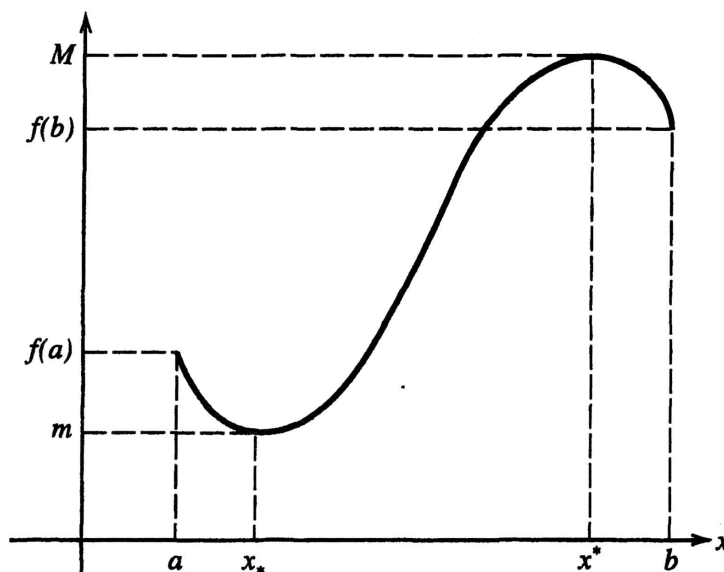
■

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<sup>4</sup>Not necessarily closed or bounded. Thus this theorem is stronger



**Remark.** End points may not be extreme points. Counter example:



$$f : [a, b] \rightarrow \mathbb{R}, f(I) \neq [f(a), f(b)]$$

**Theorem (Preservation of Interval Theorem).**

Let  $I \in \mathbb{R}$  be an interval<sup>5</sup>,  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$ . Then the set  $f(I)$  is an interval.

*Proof.* Let  $\alpha, \beta \in f(I)$  with  $\alpha < \beta$ . Then  $\exists a, b \in I$  s.t.

$$\alpha = f(a), \beta = f(b)$$

By Bolzano's Intermediate Value Theorem,  $\forall k \in (\alpha, \beta)$ ,  $\exists c_k \in I$  s.t.  $f(c_k) = k \in f(I)$ . Thus

$$[\alpha, \beta] \subseteq f(I)$$

We conclude that  $f(I)$  is an interval ■

---

<sup>5</sup>Not necessarily closed or bounded. Thus this theorem is stronger

## 5.4 Uniform Continuity

Recall the definition of continuity of  $f$  at  $u \in A$ :

Let  $A \subseteq \mathbb{R}$ ,  $f : A \rightarrow \mathbb{R}$ ,  $\forall \varepsilon > 0, \exists \delta(\varepsilon, u)$  s.t.  $\forall x \in A$

$$|x - u| < \delta(\varepsilon, u) \Rightarrow |f(x) - f(u)| < \varepsilon$$

Here we emphasize that the choice of *delta* depends on **both**  $\varepsilon$  **and**  $u \in A$ . This implies that the change of function value  $f(u)$  depends on choice of  $u$ . Consider  $f(x) = \sin(\frac{1}{x})$ . As  $x$  approaches 0, the function value changes more rapidly.

**Example.** In this example,  $\delta$  depends on  $\varepsilon$  only:

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = 2x$ . Then

$$|f(x) - f(u)| = 2|x - u|$$

So it suffice to choose  $\delta = \frac{\varepsilon}{2}$ .

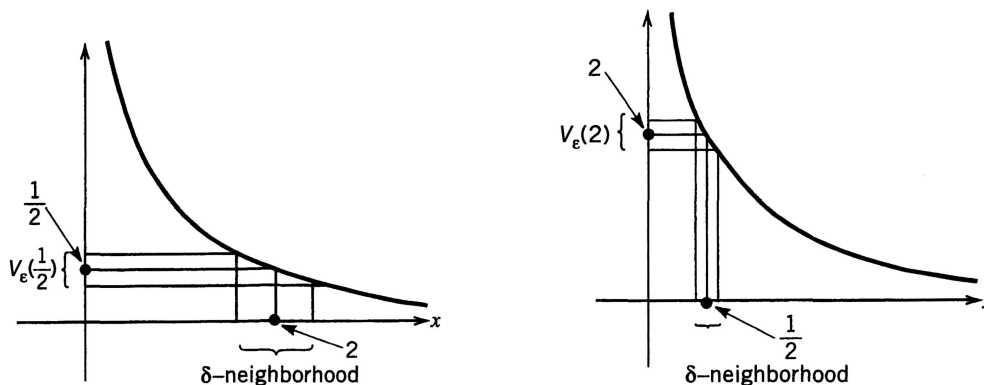
**Example.** However, in certain cases,  $\delta$  depends on both  $\varepsilon$  and  $u$ .

Let  $g : (0, \infty) \rightarrow \mathbb{R}$ ,  $g(x) = \frac{1}{x}$ . Then

$$|f(x) - f(u)| = \left| \frac{u - x}{ux} \right|$$

It suffices to choose  $\delta(\varepsilon, u) = \inf\{\frac{1}{2}u, \frac{1}{2}u^2\varepsilon\}$ .

Notice that there is no way to choose a  $\delta$  that will work for all  $u > 0$ .  $\delta$  must depends on the position of  $u$ . As  $u$  tends to 0, the permissible value of  $\delta$  tends to 0.



**Definition (Uniform Continuity).** Let  $A \subseteq \mathbb{R}$ ,  $f : A \rightarrow \mathbb{R}$ . We say that  $f$  is uniformly continuous on  $A$  if

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) \text{ s.t. } \forall x, y \in A$$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

**Theorem (Non-uniform Continuity Criterion).** Let  $A \subseteq \mathbb{R}$ ,  $f : A \rightarrow \mathbb{R}$ . The following statements are equivalent:

- $f$  is NOT uniformly continuous.
- $\exists \varepsilon_0 > 0$  s.t.  $\forall \delta > 0, \exists x_\delta, u_\delta \in A$  s.t.

$$|x_\delta - u_\delta| < \delta \Rightarrow |f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$$

- $\exists \varepsilon_0 > 0, \exists (x_n), (u_n) \in A$  s.t.  $\forall n \in \mathbb{N}$

$$\lim(x_n - u_n) = 0 \text{ and } |f(x_n) - f(u_n)| \geq \varepsilon_0$$

*Proof. Exercise.* ■

**Theorem (Uniform Continuity Theorem).** Let  $I$  be a closed bounded interval and let  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$ . Then  $f$  is uniformly continuous on  $I$ .

*Proof.* Assume  $f$  is not uniformly continuous on  $I$ . Then by precedence result,  $\exists \varepsilon_0 > 0, \exists (x_n), (u_n) \in I$  s.t.  $\forall n \in \mathbb{N}$

$$\lim(x_n - y_n) = 0 \text{ and } |f(x_n) - f(u_n)| \geq \varepsilon_0$$

Since  $I$  is bounded, the sequence  $(x_n)$  is bounded. By Bolzano-Weierstrass Theorem, there exists a convergent subsequence  $(x_{n_k}), (u_{n_k})$  of  $(x_n), (u_n)$  where

$$\lim(x_{n_k} - y_{n_k}) = 0 \text{ and } |f(x_{n_k}) - f(u_{n_k})| \geq \varepsilon_0$$

Since  $I$  is continuous on  $I$

$$\lim f(x_{n_k}) = f(\lim(x_{n_k})), \lim f(u_{n_k}) = f(\lim(u_{n_k}))$$

Thus

$$\lim(f(x_{n_k})) = \lim(f(u_{n_k})) \Rightarrow \lim(f(x_{n_k}) - f(u_{n_k})) = 0$$

contradicts our assumption. We conclude that  $f$  must be uniformly continuous. ■

**Note:** The next property conveniently ensures uniform continuity without requiring  $A$  to be a closed and bounded interval.

**Definition (Lipschitz Functions).** Let  $A \subseteq \mathbb{R}$ ,  $f : A \rightarrow \mathbb{R}$ . If there exists a constant  $K > 0$  such that  $\forall x, u \in A$

$$|f(x) - f(u)| \leq K |x - u|$$

then  $f$  is said to be a **Lipschitz function** (**Lipschitz continuous**, or just **Lipschitz**) on  $A$ .

**Remark (Lipschitz Function and Gradient).** Rewrite the condition, we have

$$\left| \frac{f(x) - f(u)}{x - u} \right| \leq K$$

It follows that the absolute value is the gradient of a line segment joining the points  $(x, f(x))$ ,  $(u, f(u))$ . Thus,  $f$  is Lipschitz if and only if the gradient of all line segments joining two points on the graph of  $y = f(x)$  over  $I$  are bounded by some  $K$ .

**Theorem.** Let  $A \subseteq \mathbb{R}$ ,  $f : A \rightarrow \mathbb{R}$ . If  $f$  is **Lipschitz**, then it is **uniformly continuous** on  $A$ .

*Proof.* For all  $\varepsilon > 0$ , we simply choose  $\delta = \frac{\varepsilon}{K}$ . Then

$$|x - u| < \delta \Rightarrow |f(x) - f(u)| < K \cdot \frac{\varepsilon}{K} = \varepsilon$$

Thus  $f$  is uniformly continuous on  $A$  ■

**Remark.** The **converse** may NOT true!

**Counter-example:**  $f(x) = \sqrt{x}$  on  $[0, 1]$  is uniformly continuous but NOT Lipschitz.

**Example.** If  $f(x) = x^2$  on  $A = [0, b]$  where  $b > 0$ , then  $\forall x, y \in [0, b]$

$$|f(x) - f(y)| = |x + y| |x - y| \leq 2b |x - y|$$

It follows that  $f$  is Lipschitz on  $A$  given  $K = 2b$ , and thus  $f$  is uniformly continuous.

**Theorem.** Let  $A \subseteq \mathbb{R}$ ,  $f : A \rightarrow \mathbb{R}$ . If  $f$  is uniformly continuous on  $A$ , and  $(x_n) \in A$  is a Cauchy sequence, then  $(f(x_n))$  is a Cauchy sequence in  $\mathbb{R}$ .

*Proof.*  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.  $\forall x, y \in \delta$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Since  $(x_n)$  is Cauchy, for given  $\delta > 0$ ,  $\exists N_\delta \in \mathbb{N}$  s.t.  $\forall n, m \in \mathbb{N}$ ,  $n, m > N_\delta$

$$|x_n - x_m| < \delta \Rightarrow |f(x_n) - f(x_m)| < \varepsilon$$

We conclude that  $(f(x_n))$  is Cauchy. ■

**Theorem** (*Continuous Extension Theorem*). A function  $f$  is uniformly continuous on open interval  $I = (a, b)$  if and only if it can be defined at the endpoints  $a$  and  $b$  such that the extended function is continuous on  $[a, b]$ .

*Proof.* ( $\Leftarrow$ ). This direction is trivial.

( $\Rightarrow$ ) Suppose  $f : (a, b) \rightarrow \mathbb{R}$  is uniformly continuous.

**Goal:** Define  $f$ 's extended function  $F : [a, b] \rightarrow \mathbb{R}$  s.t.

$$F|_{(a,b)} = f, \quad F \text{ is continuous}$$

Notice that this is equivalent to show  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow b} f(x)$  exist and define

$$F(x) = \begin{cases} f(x) & \text{if } x \in (a, b) \\ \lim_{x \rightarrow a} f(x) & \text{if } x = a \\ \lim_{x \rightarrow b} f(x) & \text{if } x = b \end{cases}$$

Choose  $(x_n) \in (a, b)$  s.t.  $(x_n) \rightarrow a$ . Thus,  $(x_n)$  is Cauchy, and by preceding theorem,  $(f(x_n))$  is Cauchy. Let  $L = \lim(f(x_n))$ . Then  $\forall (y_n) \in (a, b)$  s.t.  $(y_n) \rightarrow a$ ,

$$\lim(x_n - y_n) = a - a = 0$$

By uniform continuity, we have

$$\begin{aligned} \lim(f(y_n)) &= \lim(f(y_n) - f(x_n)) + \lim(f(x_n)) \\ &= 0 + L = L \end{aligned}$$

Thus, for all sequence  $(u_n)$  converging to  $a$ ,  $(f(x_n)) \rightarrow L$ . By sequential criterion of limit,  $L = \lim_{x \rightarrow a} f(x)$ . If we define  $f(a) = L$ , then  $f$  is continuous at  $a$ .

The same argument applies to  $b$ . Thus, we conclude that  $f$  has a continuous extension to the interval  $[a, b]$  ■

## 5.6 Monotone and Inverse Functions

**Note:** In this section, we will be focusing on **monotone functions** on an **interval**  $I$ . Specifically, we will discuss **increasing functions**. It is easy to derive corresponding results for decreasing functions with similar proof techniques.

**Theorem.** Let  $I \subseteq \mathbb{R}$  be an interval and let  $f : I \rightarrow \mathbb{R}$  be increasing on  $I$ . Suppose that  $c \in I$  is NOT an endpoint of  $I$ . Then

1.  $\lim_{x \rightarrow c-} f = \sup\{f(x) : x \in I, x < c\}$
2.  $\lim_{x \rightarrow c+} f = \inf\{f(x) : x \in I, x > c\}$

**Recall:**  $\lim_{x \rightarrow c-} f(x) = L$  if  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.  $\forall x \in I$

$$0 < c - x < \delta \Rightarrow |f(x) - L| < \varepsilon$$

*Proof.* (1). By monotonicity of  $f$ ,  $\forall x < c$ ,

$$f(x) \leq f(c)$$

Thus, the set is non-empty, and  $f(c)$  is the upper bound of it. This indicates that

$$\exists L = \sup\{f(x) : x \in I, x < c\}$$

Then  $\forall \varepsilon > 0$ ,  $L - \varepsilon$  is not an upper bound of the set. Hence  $\exists x_\varepsilon \in I, x_\varepsilon < c$  s.t.

$$L - \varepsilon < f(x_\varepsilon) \leq L$$

We choose  $\delta = c - x_\varepsilon$ . Then  $\forall x \in I$ ,  $0 < c - x < \delta \rightarrow x_\varepsilon < x$ . Thus,

$$-\varepsilon < f(x_\varepsilon) - L \leq f(x) - L \leq 0 < \varepsilon$$

$$|f(x) - L| < \varepsilon$$

Thus

$$L = \lim_{x \rightarrow c-} f(x)$$

The proof of (2) is similar. ■

**Corollary (One-sided Limits Criterion for Continuity).** Let  $I \subseteq \mathbb{R}$  be an interval and let  $f : I \rightarrow \mathbb{R}$  be increasing on  $I$ . Suppose that  $c \in I$  is NOT an endpoint of  $I$ . Then the following statements are equivalent:

- $f$  is continuous at  $c$ .
- $\lim_{x \rightarrow c-} f(x) = f(c) = \lim_{x \rightarrow c+} f(x)$ .
- $\sup\{f(x) : x \in I, x < c\} = f(c) = \inf\{f(x) : x \in I, x > c\}$ .

*Proof. Exercise.* ■

**Recall.** A function  $f : I \rightarrow \mathbb{R}$  has inverse function if and only if  $f$  is injective.

**Note:** It is not difficult to prove that a strictly monotone function is injective and thus has an inverse.

**Theorem (Continuous Inverse Theorem).**

Let  $I \subseteq \mathbb{R}$  be an interval and let  $f : I \rightarrow \mathbb{R}$  be strictly monotone and continuous on  $I$ . Then the inverse function  $g : f(I) \rightarrow I$ ,  $g(y) = f^{-1}(y)$  is strictly monotone and continuous on  $f(I)$ .

*Proof.* WLOG, we assume that  $f$  is **strictly increasing**. We will first prove that  $g$  is strictly increasing.

Suppose  $\exists y_1, y_2 \in f(I), y_1 < y_2$ , then  $\exists x_1, x_2 \in I$  s.t.  $f(x_1) = y_1, f(x_2) = y_2$

1. Assume that  $x_1 = x_2$ .

Then,  $y_1 = f(x_1) = f(x_2) = y_2$  contradicts assumption. So  $x_1 \neq x_2$ .

2. Assume that  $x_1 > x_2$ .

Then  $y_1 = f(x_1) > f(x_2) = y_2$  contradicts assumption. So  $x_1 \not> x_2$ .

Thus,  $x_1 < x_2$ . It follows that

$$g(y_1) < g(y_2)$$

We conclude that  $g$  is strictly monotone.

Subsequently, we will prove that  $g$  is **continuous**.

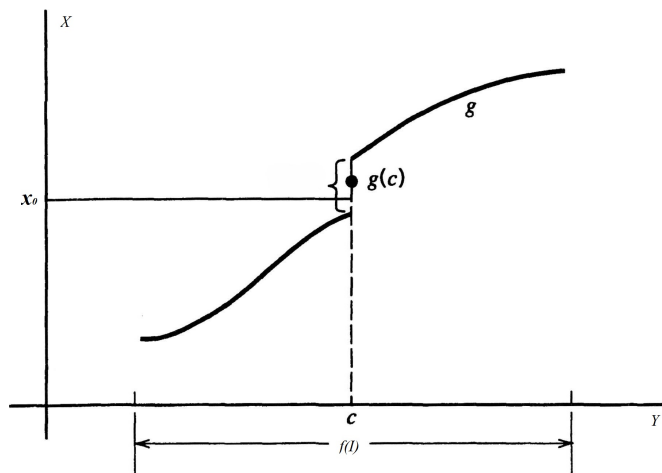
Assume  $g$  is discontinuous. Then, by inverse of previous criterion,  $\exists c \in f(I)$ ,  $c$  is not endpoints of  $f(I)$  s.t.

$$\lim_{y \rightarrow c^-} g(y) < \lim_{y \rightarrow c^+} g(y)$$

Let  $x_0 \in (\lim_{y \rightarrow c^-} g(y), \lim_{y \rightarrow c^+} g(y)) \setminus g(c)$ . Then

$$x_0 \notin g(f(I)) \subseteq I$$

which contradicts that fact that  $x_0 \in I$  ■



## 8 Sequence of Functions

### 8.1 Pointwise and Uniform Convergence

**Definition (Sequence of Functions).** Let  $A \subseteq \mathbb{R}$ .

We say that  $(f_n(x))$  is a sequence of functions on  $A$  to  $\mathbb{R}$  if

$$\forall n \in \mathbb{N}, \exists f_n : A \rightarrow \mathbb{R}$$

**Definition (Convergence of Sequence of Functions).** Let  $(f_n(x))$  be a sequence of functions on  $A$  to  $\mathbb{R}$ . Let  $A_0 \subseteq A$ , and  $f : A_0 \rightarrow \mathbb{R}$ .

We say that  $(x_n)$  converges on  $A_0$  to  $f$  if  $\forall x \in A_0$ ,

$$\lim_{n \rightarrow \infty} (f_n(x)) = f(x) \text{ or } f_n \rightarrow f \text{ on } A_0$$

We call  $f$  the limit of  $f_n$  on  $A_0$ . Or we say that  $(f_n)$  converges pointwise on  $A_0$ .

**Lemma ( $\varepsilon - \delta$  Definition).** Let  $A_0 \subseteq A \subseteq \mathbb{R}$ ,  $f : A_0 \rightarrow \mathbb{R}$ . A sequence of functions  $(f_n(x)) : A \rightarrow \mathbb{R}$  converges pointwise to  $f$  if and only if

$$\forall \varepsilon > 0, \exists N(\varepsilon, x) \in \mathbb{N} \text{ s.t. } \forall n \geq N(\varepsilon, x),$$

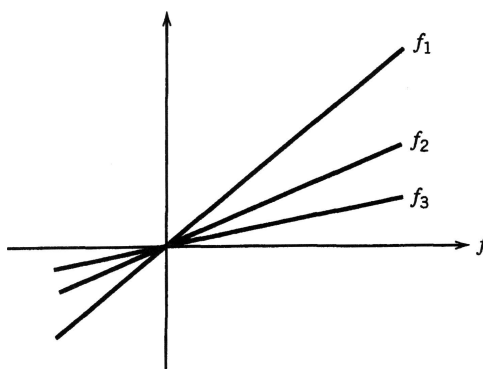
$$|f_n(x) - f(x)| < \varepsilon$$

**Remark.** We emphasize that the choice of  $N(\varepsilon, x)$  depends on both  $\varepsilon$  and  $x$ .

**Example.**  $\forall x \in \mathbb{R}$ ,  $\lim_{n \rightarrow \infty} \frac{x}{n} = 0$ . Let  $f_n(x) = \frac{x}{n}$ ,  $f(x) = 0$ . We have

$$\lim_{n \rightarrow \infty} (f_n(x)) = \lim_{n \rightarrow \infty} \left(\frac{x}{n}\right) = x \lim_{n \rightarrow \infty} \frac{1}{n} = x \cdot 0 = 0 = f(x)$$

Thus  $(f_n(x)) \rightarrow 0$  pointwise on  $\mathbb{R}$ .



Or,  $\forall \varepsilon > 0$ ,

$$\left| \frac{x}{n} - 0 \right| = \frac{|x|}{n} < \varepsilon$$

So, it suffices to choose  $K(\varepsilon, x) = \left\lceil \frac{|x|}{\varepsilon} \right\rceil$ .



**Example.** Consider  $\forall x \in \mathbb{R}, n \in \mathbb{N}, g_n(x) = x^n$ .

By previous example, we know that

$$\lim(x^n) = \begin{cases} 0 & \text{if } -1 < x < 1, \\ 1 & \text{if } x = 1 \end{cases}$$

And if  $x = -1$ ,  $g_n(-1) = (-1)^n$  is divergent.

If  $|x| > 1$ ,  $(x^n)$  is divergent as well.

Define  $g : (-1, 1] \rightarrow \mathbb{R}, g(x) = \lim(x^n)$ . Then,

$$g_n \rightarrow g \text{ on } (-1, 1]$$

**Definition (*Uniform Convergence*).** A sequence of  $(f_n(x))$  functions on  $A \subseteq \mathbb{R}$  to  $\mathbb{R}$  converges uniformly on  $A_0 \subseteq A$  to a function  $f : A_0 \rightarrow \mathbb{R}$  if and only if  $\forall \varepsilon > 0, \exists K(\varepsilon) \in \mathbb{N}$  s.t.  $\forall n \geq K(\varepsilon), \forall x \in A_0$ ,

$$|f_n(x) - f(x)| < \varepsilon$$

In this case, we say that  $(f_n)$  is uniformly convergent on  $A_0$ .

**Remark.** The choice of  $K(\varepsilon)$  depends on only  $\varepsilon$ .

**Example.**  $f_n(x) = \frac{\sin(nx + n)}{n}$  converges uniformly to  $f(x) = 0$ .

*Proof.*

$$|f_n(x) - f(x)| = \left| \frac{\sin(nx + n)}{n} \right| < \frac{1}{n} < \varepsilon$$

So  $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$  will satisfies the condition. ■

**Lemma.** A sequence of  $(f_n(x))$  functions on  $A \subseteq \mathbb{R}$  to  $\mathbb{R}$  is NOT uniformly convergent on  $A_0 \subseteq A$  to a function  $f : A_0 \rightarrow \mathbb{R}$  if and only if

- there exists  $\varepsilon_0 > 0$  s.t.  $\forall N \in \mathbb{N}, \exists x \in A_0, \exists n \geq N$  s.t.

$$|f_n(x) - f(x)| \geq \varepsilon_0$$

- there exists  $\varepsilon_0 > 0$ , a subsequence  $(f_{n_k})$  of  $(f_n)$ , and a sequence  $(x_k) \in A_0$  s.t.

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0, \forall k \in \mathbb{N}$$

*Proof. Exercise.* ■

**Example.**  $f_n(x) = \frac{x}{n}$  does NOT converge uniformly to  $f(x) = 0$  on  $\mathbb{R}$ .

*Proof.* Let  $\varepsilon_0 = 1$

1. When  $n_1 = 1$ ,  $x_1 = 1$ , we have  $|f_1(x_1) - f(x_1)| = 1$
2. When  $n_2 = 2$ ,  $x_2 = 2$ , we have  $|f_2(x_2) - f(x_2)| = 1$
3. So choose  $x_k = k$ ,  $n_k = k$ , we have

$$|f_{n_k}(x_k) - f(x_k)| = \left| \frac{x_k}{n_k} - 0 \right| = 1 \geq \varepsilon_0$$

By preceding lemma,  $(f_n)$  does NOT converge to  $f$  uniformly on  $\mathbb{R}$  ■

**Example.** Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f_n(x) = \frac{x^2 + nx}{n}$

**Claim:**  $f_n$  does NOT uniformly converge to  $f(x) = x$  on  $\mathbb{R}$ .

*Proof.* Choose  $\varepsilon_0 = 1$ .  $\forall k \in \mathbb{N}$ , if we choose  $n_k = k$ ,  $x_k = -k$

$$\begin{aligned} |f_{n_k}(x_k) - f(x_k)| &= \left| \frac{x_k^2 + n_k x_k}{n_k} - x_k \right| \\ &= \left| \frac{(-k)^2 + k \cdot (-k)}{k} - (-k) \right| \\ &= k \geq 1 \end{aligned}$$

Thus, by previous lemma,  $f_n$  does NOT uniformly converge to  $f$  on  $\mathbb{R}$ . ■

**Claim:**  $(f_n) \rightarrow f$  uniformly on  $[0, 1]$ .

*Proof.*  $\forall \varepsilon > 0$ . Observe that  $\forall x \in [0, 1]$ ,  $n \in \mathbb{N}$ ,

$$\left| \frac{x^2 + nx}{n} - x \right| = |x^2| \leq \frac{1}{n} < \varepsilon$$

Thus, it suffices to choose  $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$  ■

**Example.** Let  $f_n : [0, 1] \rightarrow \mathbb{R}$ ,  $f_n(x) = x^n$ .

**Claim:**  $f_n$  does NOT converge uniformly on  $[0, 1]$ .

Choose  $\varepsilon_0 = \frac{1}{2}$ .  $\forall k \in \mathbb{N}$ ,  $\exists n_k \in \mathbb{N}$  s.t.

$$n_k = k, \quad x_k = \left(\frac{1}{2}\right)^{\frac{1}{k}}$$

Then

$$\begin{aligned} |f_{n_k}(x_k) - f(x_k)| &= |x_k^{n_k} - 0| \\ &= (x_k)^k = \frac{1}{2} \geq \varepsilon_0 \end{aligned}$$

We conclude that  $f_n$  is not uniformly convergent to  $f$  on  $[0, 1]$ .