Math265 Real Analysis Class Notes

Based on lectures by Prof. Huang

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Fall 2024

1 Preliminaries

Still working on it...(missed 1st month due to late enrollment)

2 The Real Numbers

Still working on it...

3 Sequences and Series

3.1 Sequences and their limits

Definition (Sequence). A ***sequence of real numbers*** is a *function* from $\mathbb{N}$ to $\mathbb{R}$.

We adopt the notation with a *sequence*:

$$a : \mathbb{N} \to \mathbb{R}$$

where instead of writing $a(1), a(2), \ldots$, we write it as $a_1, a_2, \ldots$ which we called them ***terms*** or ***elements*** of the sequence.

Notation.

$$(a_n)_{n=1}^{\infty} \text{ or } (a_n)_{n\in\mathbb{N}} \text{ or } (a_n) \text{ or } (a_n|n \in \mathbb{N})$$

Definition (Converge to x). A sequence $(x_n) \in \mathbb{R}$ ***converges*** to $x \in \mathbb{R}$ if

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} \text{ such that } n \geq N_\epsilon \to |x_n - x| < \epsilon$$

We write

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (x_n) = x.$$

Definition (Convergent & Divergent). A sequence is ***convergent*** if it has a ***limit*** in $\mathbb{R}$, and is ***divergent*** if it has ***no limit*** in $\mathbb{R}$.

Theorem (**Uniqueness of Limit**)**.** A sequence in $\mathbb{R}$ can have ***<u>at most one</u>*** limit. Or, the limit of a sequence is ***<u>unique</u>*** if the limit exists

Proof. Let (x_n) be a sequence of real numbers. Suppose x, x' are limits of (x_n). We want to prove $x = x'$ by contradiction.

Assume $|x - x'| > 0$. If we consider $\epsilon := \frac{1}{3}|x - x'| > 0$, then
The existence of $\lim_{x_n \to x}$ implies that $\exists N_1 \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ if $n \geq \mathbb{N}_1$.
Similarly, existence of $\lim_{x_n \to x'}$ implies that $\exists N_2 \in \mathbb{N}$ such that $|x_n - x'| < \epsilon$ if $n \geq \mathbb{N}_2$.
Thus,

$$\begin{aligned} |x - x'| &\leq |x - x_{N_1+N_2} + x_{N_1+N_2} - x'| \\ &\leq |x - x_{N_1+N_2}| + |x_{N_1+N_2} - x'| \text{ by triangle inequality} \\ &< \epsilon + \epsilon \\ &= \frac{2}{3}|x - x'| \end{aligned}$$

Then,

$$\frac{1}{3}|x - x'| < 0, \text{which is a contradiction}$$

we thereby prove by contradiction that

$$|x - x'| = 0, \text{ which is equivalent to } x = x'$$

■

Example.

$$\lim_{n \to \infty}(\frac{1}{n}) = 0$$

Goal: $\forall \epsilon > 0$, want to find N_ϵ such that $\left|\frac{1}{n} - 0\right| < \epsilon$ for $n > N$, so it suffices to show that

$$\frac{1}{n} < \epsilon \Leftrightarrow \frac{1}{\epsilon} < n$$

Proof. Let $\epsilon > 0$. Apply Archimedean's property to $\frac{1}{\epsilon}$, then

$$\begin{aligned} &\exists N \in \mathbb{N} \text{ such that } \frac{1}{\epsilon} < N \\ \Rightarrow &\forall n \geq N, \left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon. \\ \Rightarrow &\lim_{n \to \infty} \frac{1}{n} = 0 \end{aligned}$$

■

Theorem. Let (x_n) be a sequence of real numbers, and let $x \in \mathbb{R}$. The following are equivalent:

1. $x_n \to x$
2. $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|x_n - x| < \epsilon,$ for $n \geq \mathbb{N}$
3. $\ldots\ldots x - \epsilon < x_n < x + \epsilon \ldots\ldots$
4. $\forall\epsilon$-neighborhood $V_\epsilon(x), \exists N \in \mathbb{N}$ such that $x_n \in V_\epsilon(x)$ for $n \geq N$

Sketch of proof:

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$$

Proposition.

$$\lim_{n\to\infty} (2\sqrt{2n+1} - \sqrt{2n}) = 0$$

Proof. Let $\epsilon > 0$. Consider

$$N = \lceil \frac{1}{2}(\frac{1}{2\epsilon})^2 \rceil \in \mathbb{N}$$

$$n > N \Rightarrow n > \frac{1}{2}(\frac{1}{2\epsilon})^2 \Rightarrow \frac{1}{2\sqrt{2n}} < \epsilon \Rightarrow \left|\sqrt{2n+1} - \sqrt{2n}\right| = \cdots = \frac{1}{\sqrt{2n+1} + \sqrt{2n}} < \epsilon$$

■

Remark.

$$\lim_{n\to\infty} (-1)^n \text{ does not exist.}$$

Definition (m-tail). If (x_n) is a sequence of real numbers and $m \in \mathbb{N}$, then the ***m-tail*** of (x_n) is the sequence

$$\{x_{n+m} : n \in \mathbb{N}\} = \{x_{m+1}, x_{m+2}, \ldots\}$$

Theorem. Let (x_n) be a sequence and $m \in \mathbb{N}$. Then (x_n) is ***convergent*** iff (x_{n+m}) is ***convergent***. Moreover,

$$\lim_{n\to\mathbb{N}} (x_n) = \lim_{n\to\mathbb{N}} (x_{n+m})$$

Proof. $(\Rightarrow)$
Suppose $x_n \to x$. Let

$$\epsilon > 0, \exists N_\epsilon > 0, \text{ such that } |x_n - x| < \epsilon \text{ for } n \geq N_\epsilon$$

Consider $N_\epsilon{}' := N_\epsilon + m$ then

$$n + m \geq N_\epsilon{}' \Rightarrow n \geq N_\epsilon \Rightarrow n + m \geq N_\epsilon \Rightarrow |x_{n+m} - x| < \epsilon$$

It follows that

$$n \geq N_\epsilon \Rightarrow n + m \geq N_\epsilon \Rightarrow |x_{n+m} - x| < \epsilon$$

($\Leftarrow$)
Suppose $x_{n+m} \to x$.

$$\forall \epsilon > 0, \exists N_\epsilon > 0 \text{ such that } |x_{n+m} < \epsilon|, \forall n \geq N_\epsilon$$

Consider $N := N_\epsilon + m$. Then

$$\begin{aligned}
&n \geq N = N_\epsilon + m \\
&\Rightarrow n - m \geq N_\epsilon \\
&\Rightarrow \left|x_{(n-m)+m} - x\right| < \epsilon \\
&\Rightarrow |x_n - x| < \epsilon
\end{aligned}$$

■

Remark. We say that a sequence (x_n) ***ultimately*** has a property if that property holds for some tail of (x_n)

Theorem. Let x_n be a sequence of real numbers. Let a_n be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$. If $\exists c > 0, m \in \mathbb{N}, x \in \mathbb{R}$ such that

$$|x_n - x| \leq c \cdot a_n, \forall n \geq m$$

then

$$x_n \to x$$

Proof. We know that

$$\forall \epsilon > 0, \exists N \geq 0 \text{ s.t. } |a_n| < \frac{\epsilon}{c}, \forall n \geq N$$

Consider $N' = max\{N, m\}, \forall n \geq N'$. Then

$$|x_n - x| \leq Ca_n = c\,|a_n| < c \cdot \frac{\epsilon}{c} = \epsilon$$

$$\Rightarrow x_n \to x$$

■

Proposition.

$$\lim_{n\to\infty} \frac{17}{2+3n} = 0$$

Proof.

$$\left|\frac{17}{2+3n} - 0\right| = \frac{17}{2+3n} \leq \frac{13}{3n} = \frac{17}{3} \cdot \frac{1}{n}$$

Apply the theorem above with

$$a_n = \frac{1}{n}, c = \frac{17}{3}, m = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{17}{2+3n} = 0, \text{ since } \lim_{n\to\infty} \frac{1}{n} = 0$$

■

Proposition.

$$\forall c > 0, \lim_{n\to\infty} c^{\frac{1}{n}} = 1$$

Proof. ***Case 1: c = 1***

$$\lim_{n\to\infty} c^{\frac{1}{n}} = 1$$

Case 2: c > 1

Let $d_n = c^{\frac{1}{n}} - 1$. Then $\forall n, d_n > 0$. It follows that

$$\begin{aligned}(d_n + 1) = c^{\frac{1}{n}} &\Rightarrow c = (1 + d_n)^n \geq 1 + n \cdot d_n \text{ by Bernoulli's inequality} \\ &\Rightarrow d_n \leq (c-1) \cdot \frac{1}{n} \\ &\Rightarrow \left| c^{\frac{1}{n}} - 1 \right| = d_n \leq (c-1) \cdot \frac{1}{n}\end{aligned}$$

Apply the theorem with

$$C = c - 1, a_n = \frac{1}{n}, m = 1, x = 1$$

$$\lim_{n\to\infty} c^{\frac{1}{n}} = 1$$

Case 3: c < 1(Note that we cannot use Bernoulli inequality here)

Define e_n to be a sequence that satisfies

$$c^{\frac{1}{n}} = \frac{1}{1 + e_n}$$

Then $e_n > 0 \forall n$.

$$c = \frac{1}{(1+e_n)^n} \leq \frac{1}{1 + n \cdot e_n} < \frac{1}{n \cdot e_n}$$

$$\Rightarrow e_n < \frac{1}{c} \cdot \frac{1}{n}$$

$$1 - c^{\frac{1}{n}} = 1 - \frac{1}{1+e_n} = \frac{e_n}{1+e_n} < e_n < \frac{1}{c} \cdot \frac{1}{n}$$

Apply the theorem with

$$a_n = \frac{1}{n}, m = 1, C = \frac{1}{c}, x = 1$$

$$\lim_{n\to\infty} c^{\frac{1}{n}} = 1$$

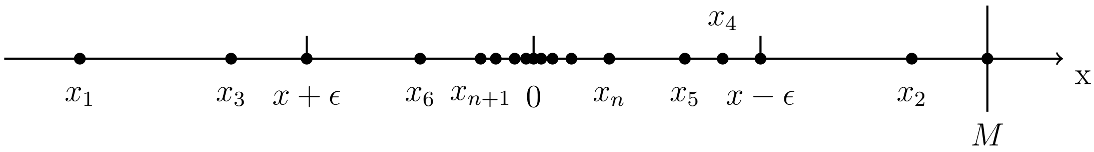
■

3.2 Limit Theorems

Definition (Bounded sequence)**.** A sequence $(x_n) \in \mathbb{R}$ is ***bounded*** if

$$\exists M > 0 \text{ s.t. } \forall n \in \mathbb{N}, |x_n| \leq M$$

Theorem. A convergent sequence $(x_n) \in \mathbb{R}$ is bounded.



Proof. By definition of convergent sequence, let $\epsilon = 1$:

$$\exists N > 0 \text{ s.t. } \forall n \geq N, |x_n - x| < 1$$

Thus we have

$$\begin{aligned} & -1 < x_n - x < 1 \\ \Rightarrow \quad & -1 + x < x_n < x + 1, \forall n \geq N \end{aligned}$$

Then define

$$M := \max\left\{|x_1|, |x_2|, |x_3|, \ldots, |-1 + x|, |x + 1|\right\}$$
$$|x_n| \leq M$$

■

Remark. By contrapositive, an unbounded sequence is divergent.

Definition. Given sequence $(x_n), (y_n) \in \mathbb{R}$, we define following operations of sequence:

- ***Sum*** $(x_n + y_n)$
- ***Difference*** $(x_n - y_n)$
- ***Product*** $(x_n \cdot y_n)$
- ***Quotient*** $(\frac{x_n}{y_n})$ if $\forall n \in \mathbb{N}, y_n \neq 0$
- ***Multiple*** $(c \cdot x_n)$

Theorem (**Limit Laws**)**.** Let $(x_n), (y_n) \in \mathbb{R}$ be sequences of real numbers with $x_n \to x, y_n \to y$, and let $c \in \mathbb{R}$. Then

- $x_n + y_n \to x + y$
- $x_n - y_n \to x - y$
- $x_n \cdot y_n \to x \cdot y$
- $c \cdot x_n \to c \cdot x$
- If $\forall n \in \mathbb{N}, y_n \neq 0$ and $y \neq 0$, then $\frac{x_n}{y_n} \to \frac{x}{y}$

Proof of Sum. :

$\forall \epsilon > 0,$

$$\exists N_1 \in \mathbb{N} \text{ s.t. } \forall n \geq N_1, |x_n - x| < \frac{\epsilon}{2}$$

$$\exists N_2 \in \mathbb{N} \text{ s.t. } \forall n \geq N_2, |y_n - y| < \frac{\epsilon}{2}$$

Consider

$$N := \max\{N_1, N_2\}$$

Then

$$\begin{aligned}\forall n \geq N, &|(x_n + y_n) - (x + y)| \\ &= |x_n - x + y_n - y| \\ &\leq |x_n - x| + |y_n - y| \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

■

Proof of Difference. :

Similarly,

$$\begin{aligned}\forall n \geq N, &|(x_n - y_n) - (x - y)| \\ &\leq |x_n - x| + |y_n - y|\end{aligned}$$

■

Proof of Product. : Since (x_n) is convergent, it is also bounded. Thus,

$$\exists M \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, |x_n| \leq M$$

By definition of convergence, $\forall \epsilon > 0$:

$$\exists N_1 > 0 \text{ s.t. } \forall n \geq N, |x_n - x| < \frac{\epsilon}{2|y|}$$

$$\exists N_2 > 0 \text{ s.t. } \forall n \geq N, |y_n - y| < \frac{\epsilon}{2M}$$

Then, $\exists N = \max\{N_1, N_2\}$ s.t. $\forall n \geq N$,

$$\begin{aligned}|x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &= |x_n(y_n - y) - y(x_n - x)| \\ &\leq |x_n| |y_n - y| + |y| |x_n - x| \\ &\leq M \cdot |y_n - y| + |y| |x_n - x| \\ &< M \cdot \frac{\epsilon}{2M} + |y| \cdot \frac{\epsilon}{2|y|} = \epsilon\end{aligned}$$

■

Proof of Multiply. :

Exercise ■

Proof of Quotient.

$$\begin{aligned}\left|\frac{x_n}{y_n}-\frac{x}{y}\right| &= \left|\frac{x_ny-y_nx}{y_ny}\right| \\ &= \left|\frac{x_ny--x_ny_n+x_ny_n-y_nx}{y_ny}\right| \\ &\leq \left|\frac{1}{y_ny}(|x_n|\,|y_n-y|+|y_n|\,|x_n-x|)\right|\end{aligned}$$

Since $y_n \to y$

$$\exists N_1 \in \mathbb{N} \text{ s.t. } \forall n \geq N, ||y_n|-|y|| \leq |y_n - y| < \frac{|y|}{2}$$

Then

$$\begin{aligned}\Rightarrow && -\frac{|y|}{2} < |y|-|y| < \frac{|y|}{2} \\ \Rightarrow && \frac{|y|}{2} < |y_n| \\ \Rightarrow && \frac{1}{|y_n|} < \frac{2}{|y|} \\ \Rightarrow && \frac{1}{|y_ny|} < \frac{2}{|y^2|}\end{aligned}$$

Define $F := \frac{2}{|y^2|}$.

Since (x_n) and (y_n) are bounded,

$\exists G \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$

$$|x_n| \leq G, |y_n| \leq G$$

Thus $\forall n > 0$

$$\exists N_2 > 0 \text{ s.t. } \forall n \geq N_2, |x_n - x| < \frac{\epsilon}{2GF}$$

$$\exists N_3 > 0 \text{ s.t. } \forall n \geq N_3, |y_n - y| < \frac{\epsilon}{2GF}$$

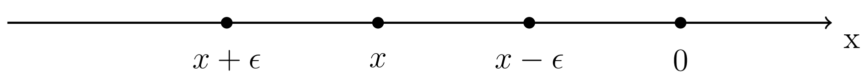
Define $N := \max\{N_1, N_2, N_3\}$, then $\exists N \in \mathbb{N}$ s.t. $\forall n > N$,

$$\left|\frac{x_n}{y_n}-\frac{x}{y}\right| < F(G\cdot\frac{\epsilon}{2GF}+G\cdot\frac{\epsilon}{2GF}) = \epsilon$$

■

Theorem. If (x_n) is a convergent sequence in $\mathbb{R}$ of non-negative terms with $(x_n) \to x$, then $x \geq 0$.

Sketch. : Assume $x < 0$, choose $\epsilon \leq |x|$, then $\forall n > N, x_n < 0$, which is a contradiction.



■

Theorem. If $x_n \to x, y_n \to y$ are sequences in $\mathbb{R}$ such that $\forall n \in \mathbb{N}, x_n \leq y_n$, then $x \leq y$.

Proof. Consider $z_n := y_n - x_n$. Then $z_n \geq 0$ and $z_n \to y - x$ by the limit law.

$$\Rightarrow y - x \geq 0$$

$$y \geq x$$

■

Theorem. If $x_n \to x$ is a sequence in $\mathbb{R}$, and let $a, b \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, a \leq x_n \leq b$, then $a \leq x \leq b$.

Proof. Consider constant sequence $a_n = a$ and $b_n = b$. Then this is true by the last theorem. ■

Theorem (Squeeze Theorem). Suppose $(x_n), (y_n), (z_n)$ are sequences of real numbers such that $\forall n \in \mathbb{N}, x_n \leq y_n \leq z_n$. If $\lim(x_n) = \lim(z_n)$, then (y_n) is convergent and

$$\lim(x_n) = \lim(y_n) = \lim(z_n)$$

Proof. Let $\forall \epsilon > 0$.

Write $L = \lim(x_n)$. Then by definition of limit,

$\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$|x_n - L| < \epsilon, |z_n - L| < \epsilon$$

$$\begin{aligned} \Rightarrow \quad & -\epsilon < x_n - L \leq y_n - L \leq z_n - L < \epsilon \\ \Rightarrow \quad & |y_n - L| < \epsilon \end{aligned}$$

Thus (y_n) converges to L by definition of limit.

■

Proposition. $\ln(n)$ is divergent.

Proof. Since $\ln(n)$ is unbounded, it is divergent.

■

Exercise. $\lim \frac{3n^2+2n+1}{5n^2-4} = \frac{3}{5}$

$$\lim \frac{3n^2 + 2n + 1}{5n^2 - 4} = \lim \frac{3 + 2\frac{1}{n} + \frac{1}{n^2}}{5 - 4\frac{1}{n^2}}$$

Notice that each term of $\frac{1}{n}$ and $\frac{1}{n^2}$ converges to 0. Thus by limit law,

$$\lim \frac{3 + 2\frac{1}{n} + \frac{1}{n^2}}{5 - 4\frac{1}{n^2}} = \frac{\lim\{3 + 2 \cdot 0 + 0\}}{\lim\{5 - 4 \cdot 0\}} = \frac{3}{5}$$

Proposition. $(-1)^n$ is divergent.

Proof:Exercise. ■

Theorem. If $x_n \to x$, then $|x_n| \to |x|$

Sketch of the proof. :

$$||x_n| - |x|| \leq |x_n - x|$$

■

Theorem. Suppose (x_n) is a sequence of non-negative real numbers, satisfiying $x_n \to x$. Then $\sqrt{x_n} \to \sqrt{x}$.

Proof. Let $\forall \epsilon > 0$

<u>Case1:</u> $x = 0$

$\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$|x_n - 0| < \epsilon^2$$
$$\Rightarrow \left|\sqrt{x} - 0\right| < \epsilon$$

<u>Case2:</u> $x > 0$

$\exists N$ s.t. $\forall n > N$,

$$|x_n - x| < \sqrt{x} \cdot \epsilon$$

Notice that

$$\begin{aligned} \left|\sqrt{x_n} - \sqrt{x}\right| &= \left|\frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}\right| \\ &\leq \frac{|x_n - x|}{\sqrt{x}} < \epsilon \end{aligned}$$

■

Theorem. Let (x_n) be a sequence of positive real numbers such that $L = \lim(\frac{x_{n+1}}{n})$ exists. If $L < 1$, then (x_n) converges to 0.

Proof. $\exists N_1 \in \mathbb{N}$ s.t. $\forall n > N_1$

$$\left|\left|\frac{x_{n+1}}{x_n}\right| - |L|\right| \leq \left|\frac{x_{n+1}}{x_n} - L\right| < \frac{1-L}{2}$$

Thus

$$\left|\frac{x_{n+1}}{x_n}\right| < \frac{1+L}{2}$$

Note that $\frac{1+L}{2} < 1$, write $r = \frac{1+L}{2}$. Then $\forall m \in \mathbb{N}$,

$$x_{N_1+m} < x_{N_1+(m-1)}r < x_{N_1+(m-2)}r^2 < \cdots < x_{N_1}r^m$$

Consider $(y_m) = (x_{N_1+m})$, $(z_m) = (x_{N_1}r^m)$. Then

$$0 \leq y_m \leq z_m$$

Since $z_m \to 0$

$$(y_m) \to 0 \text{ by squeeze theorem}$$

Thus we conclude that $(x_n) \to 0$ by m-th tail theorem. ■

3.3 Monotone sequence

Definition. Let (x_n) be a sequence of real. We say (x_n) is ...

- ***increasing*** if $\forall n \in \mathbb{N}, x_{n+1} \geq x_n$.
- ***strickly increasing*** if $\forall n \in \mathbb{N}, x_{n+1} > x_n$.
- ***decreasing*** if $\forall n \in \mathbb{N}, x_{n+1} \leq x_n$.
- ***strickly decreasing*** if $\forall n \in \mathbb{N}, x_{n+1} < x_n$.

Theorem (**Monotone Convergence Therorem**)**.** A monotone sequence of real numbers is ***convergent*** iff it is bounded. Moreover, if (x_n) is increaseing, then

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$$

If (x_n) is decreasing, then

$$\lim(x_n) = \inf\{x_n : n \in \mathbb{N}\}$$

Proof. $(\Rightarrow)$

A convergent sequence is always bounded.

$(\Leftarrow)$

Suppose (x_n) is a monotone and bounded sequence.

Case 1: (x_n) is increasing.

Write $x = \sup\{x_n : n \in \mathbb{N}\}$.

Let $\epsilon > 0$. Since $x = \sup(x_n)$:

$$x - \epsilon \text{ is NOT an upper boudn of } (x_n)$$

Then

$$\exists N \in \mathbb{N} \text{ s.t. } x_N > x - \epsilon$$

Since (x_n) is increasing,

$$\forall n \geq N,\ x_n > x - \epsilon$$

On the other hand,$x + \epsilon$ is an supper bound since x is an upper bound. Thus,

$$x_n < x + \epsilon$$

$$\Rightarrow \forall n \geq N, x - \epsilon < x_n < x + \epsilon$$

$$\Rightarrow |x_n - x| < \epsilon \Rightarrow (x_n) \to x$$

Case 2: (x_n) is decreasing.

Write $y = \inf(x_n)$. Let $\epsilon > 0$

Since $y = \inf(x_n)$,

$$y + \epsilon \text{is NOT and upper bound of } (x_n)$$

Thus

$$\exists N \in \mathbb{N} \text{ s.t. } x_n < y + \epsilon$$

Since (x_n) is decreasing,

$$\forall n > N, x_n < y + \epsilon$$

On the other hand, $y - \epsilon$ is a lower bound since y is a lower bound. Hence,

$$\forall n \in \mathbb{N}, y - \epsilon < x_n$$

$$\forall n \geq N, y - \epsilon < x_n < y + \epsilon \Rightarrow |x_n - y| < \epsilon$$
$$\Rightarrow (x_n) \to y$$

Remark. One can prove case 2 by following:

$(-x_n)$ is increasing and converges to $\sup(-x_n)$ by case 1. Also note that

$$(x_n) = (-(-x_n)) \to -\sup(-x_n)$$

by limit law. So it is easy to prove that

$$-\sup((-x_n)) = \inf(x_n)$$

■

Example. Consider the sequence (x_n) is given by

$$\begin{cases} x_0 = \frac{1}{2} \\ x_{n+1} = \frac{3}{2}x_n(1 - x_n) \end{cases}$$

(x_n) is decreasing and bounded.

Thoughts: Assume (x_n) converges, then by limit law,

$$x = \frac{3}{2}x(1 - x) \text{ where } x = \lim(x_n) \Rightarrow x = 0 \text{ or } 3$$

then, by proof of contradiction, it is not convergent.

Proof. ***Claim***: $\frac{1}{3} < x_{n+1} < x_n \leq \frac{1}{2}, \forall n \in \mathbb{N} \cup \{0\}$

Proof of the claim by induction:

When n = 0:

$$x_0 = \frac{1}{2}, x_1 = \frac{3}{2} \cdot \frac{1}{2}(1 - \frac{1}{2}) = \frac{3}{8}$$
$$\frac{1}{3} < \frac{3}{8} < \frac{1}{2} \leq \frac{1}{2}$$

Suppose this is true for n=k:

$$\frac{1}{3} < x_{k+1} < x_k \leq \frac{1}{2}$$

Goal:

$$\frac{1}{3} < x_{k+2} < x_{k+1} \leq \frac{1}{2}$$

$$x_{k+1} = \frac{3}{2}x_k(1 - x_k)$$

$$\frac{1}{3} < x_k \leq \frac{1}{2} \Rightarrow \frac{2}{3} > 1 - x_k \geq \frac{1}{2}$$

$$x_{k+1} < \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{2}$$

Complete the square:

$$\begin{aligned} x_{k+1} - \frac{1}{3} &= \frac{3}{2}x_k(1 - x_k) - \frac{1}{3} \\ &= -\frac{3}{2}[(x_k - \frac{1}{2})^2 - \frac{1}{36}] \end{aligned}$$

So

$$\begin{aligned} \frac{1}{3} < x_k \leq \frac{1}{2} &\Rightarrow \left|x_k - \frac{1}{2}\right| < \frac{1}{6} \\ &\Rightarrow (x_k - \frac{1}{2})^2 < \frac{1}{36} \\ &\Rightarrow x_{k+1} - \frac{1}{3} > 0 \\ &\Rightarrow \frac{1}{3} < x_{k+1} \leq \frac{1}{2} \end{aligned}$$

With the similar process, we can derive that

$$\frac{1}{3} < x_{k+2} \leq \frac{1}{2}$$

$$x_{k+2} = \frac{3}{2}x_{k+1}(1 - x_{k+1}) < \frac{3}{2}x_{k+1} \cdot \frac{3}{2} = x_{k+1}$$

Therefore, this claim is also true for n=k+1:

$$\frac{1}{3} < x_{k+2} < x_{k+1} \leq \frac{1}{2}$$

We thereby prove the theorem by induction:

$$\frac{1}{3} < x_{n+1} < x_n \leq \frac{1}{2}$$

∎

Exercise: Textbook p.75: A sequence that converges to $\sqrt{a}$ for $a > 0$.

Definition (Euler's Number).

$$e = \lim(1 + (\frac{1}{n})^n)$$

Goal: (x_n) is convergent where $x_n = (1 + \frac{1}{n})^n$

$$\begin{aligned} x_n = (1+\frac{1}{n})^n &= 1 + nC1 \cdot \frac{1}{n} + nC2 \cdot \frac{1}{n^2} + \cdots + nCn\frac{1}{n^n} \\ &= 1 + \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{2} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1) \cdot 3 \cdot 2 \cdot 1}{n!} \cdot \frac{1}{n^2} \\ &= 1 + 1 + \frac{1}{2}(1 - \frac{1}{n}) + \frac{1}{6}(1-n)(1-\frac{2}{n}) + \cdots + \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\ldots\frac{2}{n}\cdot\frac{1}{n} \end{aligned}$$

Write x_{n+1} in a similar way, we observe that

$$x_n < x_{n+1}$$

Facts $2^{m-1} \leq m!$ for $m \in \mathbb{N} \Rightarrow \frac{1}{m!} \leq \frac{1}{2^{m-1}}$

$$x_n < 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 1 + \frac{1(1-(\frac{1}{2})^n)}{1-\frac{1}{2}} < 3$$

$\Rightarrow (x_n)$ is increasing and bounded

3.4 Subsequence and the Bolzano-Weierstrass Theorem

Example.

$$(x_n) = ((-1)^n)$$
$$x_{2n} = (-1)^{2n}$$
$$x_{2n+1} = (-1)^{2n-1}$$

So $a_n = x_{2n}$ is a sequence, while $b_n = x_{2n+1}$ is a subsequence of x_n.

Definition (Subsequences). Let (x_n) be a real sequence and consider a strickly increasing seqence of natural numbers $n_1 < n_2 < n_3 < \dots$ The sequence

$$(x_{n_k} : k \in \mathbb{N})$$

is called a ***subsequence*** of (x_n)

Example. Any tails of a sequence is a subsequence: (x_n) n-th tail: (x_{m+k}), where $n = m + k, k \in \mathbb{N}$, is a subsequence.

Theorem. Suppose (x_n) converges to x. Then $x_{x_k} \to x$ for any subsequence of (x_n).

Proof. Let $\epsilon > 0$, then

$$\exists N_\epsilon > 0 \text{ s.t. } |x_n - x| < \epsilon \text{ for } n > N_\epsilon.$$

Note that

$$n_k \geq k, \forall k \in \mathbb{N}.$$

Exercise. By induction, $n_1 \geq 1, n_2 \geq n_1 \geq 1 \Rightarrow n_2 \geq 2$

When $k > N_\epsilon$, $n_k > N_\epsilon$, thus

$$|x_{n_k} - x| < \epsilon$$

Therefore

$$(x_{n_k}) \to x$$

∎

Theorem. Let (x_n) be a sequence of real numbers, and let $x \in \mathbb{R}$. Then the following are equivalent:

1. (x_n) does not converge to x.
2. $\exists \epsilon_0 > 0,$ s.t. $\forall k \in \mathbb{N}, \exists n_k \in \mathbb{N}$ s.t.

$$n_k \geq k \,\&\, |x_{n_k} - x| > \epsilon_0$$

3. $\exists \epsilon_0 > 0$ and a subsequence (x_{n_k}) s.t.

$$|x_{n_k} - x| > \epsilon_0, \forall k \in \mathbb{N}$$

Example.

$$x_n = (-1)^n \Rightarrow (x_n) \text{ does not converge to } 1$$

Proof.

$$\begin{aligned} 3 \to 1&: \text{ by the contrapositive statement of definition} \\ 3 \to 2&: \text{ 3 is a stronger statement of 2} \\ 1 \to 2&: \text{ left as exercise} \end{aligned}$$

■

Theorem. If (x_n) satisfies either of the following property, then it is ***divergent***:

1. There exists two subsequence (x_{n_k}) & (x_{m_k}) whose limits are NOT equal.
2. (x_n) is unbounded.

Example.

1. $(-1)^n$
2. (n)
3. (x_n) such that

$$x_{2k} = k$$
$$x_{2k+1} = (-1)^k$$

Proof. exercise

■

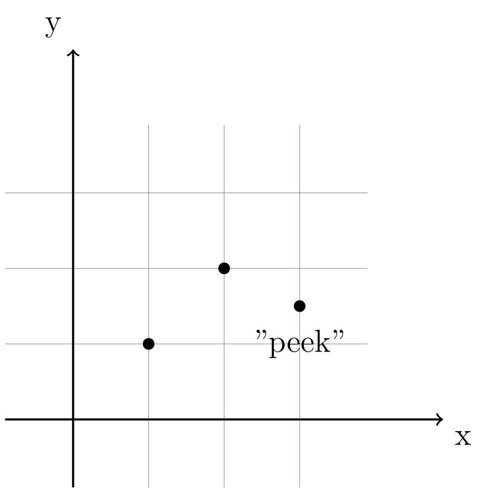
Theorem (Bolzano-Weierstrass Theorem). A bounded sequence of real numbers has a ***convergent subsequence***.
Example.

$$x_n = (-1)^n$$

Proof.

Lemma. If (x_n) is a sequence of real numbers, there exists a subsequence of (x_n) which is monotone.

proof of lemma:
Call the m-th term x_m a *"peek"* if x_m is at least as large as any term after it in the sequence.



Case 1: (x_n) has infinitelyy many peaks

List the peaks of (x_n) in order of increasing index

$$x_{n_1}, x_{n_2}, \ldots$$

$\Rightarrow (x_{n_i})$ is a decreasing sequence.

Case 2: (x_n) has a finite number of peaks

Let s_1 be the first index after the last peak of (x_n). Then for every $n \geq s_1$, $\exists m \in \mathbb{N}$ such that $x_m > x_n$.

Choose

$$s_2 \geq s_1 \text{ such that } x_{s_2} > x_{s_1},\ s_2 \geq s_1 \text{ such that } x_{s_2} > x_{s_1}, \ldots$$
$$\Rightarrow (x_{s_i}) \text{ is an increasing sequence.}$$

Remark. Lemma + monotone convergent theorem implies Bolzano-Weierstrass theorem.

Second proof

Suppose (x_n) is a bounded sequence.

$$\Rightarrow \exists I_1 = [a_1, b_1] \text{ such that } (x_n) \in I_1$$

Consider

$$I_2' = [a_1, \frac{a_1 + b_1}{2}],\ I_2'' = [\frac{a_1 + b_1}{2}, b_1]$$

Let $I_2 = [a_2, b_2]$ be one of I_2', I_2'' such that I_2 contains infinitely many terms of (x_n).

For $n \in \mathbb{N}$, define $I_n = [a_n, b_n]$ in a similar way.

For $i \in \mathbb{N}$, choose a term x_{n_i} such that $x_{n_i} \in I_i$ and $n_i > n_{i-1}$

Then

$$\begin{aligned}
&i = 1, && n_i = 1 \\
&i = 2, && \text{choose } n_2 \in \mathbb{N} \text{ such that } n_2 > n_1 \,\&\, x_{n_2} \in I_2 \\
&\ \vdots && \vdots
\end{aligned}$$

- $\forall i \in \mathbb{N}$, $a_i \leq x_{n_i} \leq b_i$
- (a_i) increases, bounded above by $b_1 \Rightarrow (a_i) \to \sup(a_i)$
- (b_i) decreases, bounded below by $a_1 \Rightarrow (b_i) \to \inf(b_i)$
- $\inf|a_i - b_i| = \inf \frac{b_1 - a_1}{2^n} = n \Rightarrow \sup(a_i) = \inf(b_i)$

Thus (x_{n_i}) is convergent by squeeze theorem.

■

Theorem. Let (x_n) be a bounded sequence, and $x \in \mathbb{R}$ has the property that every convergent subsequence of (x_n) converges to x. Then x_n converges to x

Proof. Let $\forall \epsilon > 0$

By Bolzano-Weierstrass theorem, $\exists$ a convergent subsequence (x_{n_i}) such that

$$\exists N_\epsilon \in \mathbb{N} \text{ s.t. for } i > N_\epsilon, |x_{n_i} - x| < \epsilon$$

Assume (x_n) does not converge to x. Then by previous theorem of subsequence,

$$\exists \epsilon_0 > 0 \text{ and a subsequence } (x_{n_k}) \text{ s.t. } |x_{n_k} - x| > \epsilon_0, \forall k \in \mathbb{N}$$

Since (x_n) is bounded, (x_{n_k}) is also bounded. Thus there exists a convergent subsequence of (x_{n_k}) as $(x_{n_{k_i}})$.

Note that $(x_{n_{k_i}})$ is a convergent subsequence of (x_n), thus

$$(x_{n_{k_i}}) \to x \text{ which is contradiction to previous assumption}$$

■

Definition. Let (x_n) be a sequence of real numbers. A point is called a ***subsequential limit*** of (x_n) if it is the limit of a subsequence of (x_n).

$$S = \{\alpha \in \mathbb{R} : x \text{ is a subsequential limit}\} \text{ NOTE: may be infinite set}$$

Definition. Let (x_n) be a sequence of real numbers.

- The ***limit superior*** of (x_n) is the infimum of the set of $v \in \mathbb{R}$ s.t. $v < x_n$ for at most a finite number of $n \in \mathbb{N}$. We write it as

$$\limsup(x_n) \text{ or } \limsup x_n \text{ or } \overline{\lim} x_n$$

- The ***limit inferior*** of (x_n) is the supremum of the set of $v \in \mathbb{R}$ s.t. $w > x_m$ for at most a finite number of $n \in \mathbb{N}$. We write it as

$$\liminf(x_n) \text{ or } \liminf x_n \text{ or } \overline{\lim} x_n$$

Intuition

- Suppose $v < x_n$ for at most finitely many $n \in \mathbb{N}$, then

Theorem. Let (x_n) be a bounded sequence, and $x \in \mathbb{R}$ has the property that every convergent subsequence of (x_n) converges to x. Then x_n converges to x

Proof. Let $\forall \epsilon > 0$

By Bolzano-Weierstrass theorem, $\exists$ a convergent subsequence (x_{n_i}) such that

$$\exists N_\epsilon \in \mathbb{N} \text{ s.t. for } i > N_\epsilon, |x_{n_i} - x| < \epsilon$$

Assume (x_n) does not converge to x. Then by previous theorem of subsequence,

$$\exists \epsilon_0 > 0 \text{ and a subsequence } (x_{n_k}) \text{ s.t. } |x_{n_k} - x| > \epsilon_0, \forall k \in \mathbb{N}$$

Since (x_n) is bounded, (x_{n_k}) is also bounded. Thus there exists a convergent subsequence of (x_{n_k}) as $(x_{n_{k_i}})$.

Note that $(x_{n_{k_i}})$ is a convergent subsequence of (x_n), thus

$$(x_{n_{k_i}}) \to x \text{ which is contradiction to previous assumption}$$

■

Definition. Let (x_n) be a sequence of real numbers. A point is called a ***subsequential limit*** of (x_n) if it is the limit of a subsequence of (x_n).

$$S = \{\alpha \in \mathbb{R} : x \text{ is a subsequential limit}\} \text{ (NOTE: may be infinite set)}$$

Example. Consider $(x_n) = \{(-1)^n | n \in \mathbb{N}\}$. Then

$$S \supseteq \{1, -1\}$$

Definition (lim sup and lim inf)**.** Let (x_n) be a sequence of real numbers.

- The ***limit superior*** of (x_n) is the infimum of the set of $v \in \mathbb{R}$ s.t. $v < x_n$ for at most a finite number of $n \in \mathbb{N}$. We write it as

  $\limsup(x_n) = \limsup x_n = \overline{\lim} x_n = \inf\{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of n}\}$

  Example. Consider $(x_n) = \frac{1}{n}$

  Let

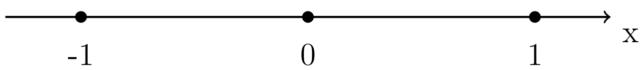
  $$X = \{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of n}\}$$

  - $-1 \notin X$ because there are infinitely many x_n such that $v < x_n$.
  - $\frac{1}{2} \in X$ because there are finitely many x_n such that $v < x_n$.
  - $2 \in X$ because there is no x_n such that $v < x_n$, which is smaller than finite and thereby satisfies the definition.

  We thus conclude that

  $$(0, \infty) \subset X \text{ and } \limsup x_n = \inf X$$

  Example. Consider $(x_n) = (-1)^n$:

  - $1 \in X$ because there is no x_n such that $v < x_n$.
  - $2 \in X$ because there is no x_n such that $v < x_n$.
  - $0, -1 \notin X$ because there are infinitely many x_n such that $v < x_n$.

  We thus conclude that

  $$[1, \infty) \subset X \text{ (in fact they are equal)}$$

- The ***limit inferior*** of (x_n) is the supremum of the set of $w \in \mathbb{R}$ s.t. $w > x_m$ for at most a finite number of $n \in \mathbb{N}$. We write it as

  $\liminf(x_n)$ or $\liminf x_n$ or $\overline{\lim} x_n = \sup\{w \in \mathbb{R} | w > x_n \text{ for at most a finite number of n}\}$

Intuition

- Suppose $v < x_n$ for at most finitely many $n \in \mathbb{N}$, then for all large n, $v \geq x_n$.
  $\Rightarrow$ No subsequential limit of (x_n) can possibly exceed v.
- Similar observation for $\underline{\lim}\, x_n$

Theorem. Let (x_n) be a bounded sequence of real numbers, and let $x^* \in \mathbb{R}$. Then TFAE:

1. $x^* = \limsup(x_n)$
2. If $\epsilon > 0$, there are at most a finite number of $n \in \mathbb{N}$ s.t. $x^* + \epsilon < x_n$, but infinitely many n for which $x^* - \epsilon < x_n$
3. If $u_m = \sup \{x_n | n \geq m\}$(sup of (m-1)-th tail), then $x* = \inf \{u_m | m \in \mathbb{N}\} = \lim u_m$
4. If S is the set of subsequential limits of x_n, then $x^* = \sup S$.

Remark. .

- u_m is decreasing.
- There is a similar such list of equivalent properties for lim inf.

Corollary. A bounded sequence (x_n) is convergent iff $\overline{\lim} x_n = \lim x_n$

Proof. A direct result of the theorem:

$$\overline{\lim} x_n = \sup S \text{ and } \underline{\lim} x_n = \inf S$$

$\blacksquare$

Proof of thm. (a) $\Rightarrow$ (b). Let $\epsilon > 0$. Then

$$x^* + \epsilon > x^* = X = \inf \{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of n.}\}$$

$$\Rightarrow \exists v \in \mathbb{R} \text{ s.t. } x^* \leq v < x^* + \epsilon$$

and there are only finitely mant n with $v < x_n$.

For any n for which $x^* + \epsilon < x_n$, $v < x_n$. Thus there are only finitely many such n.

If $x^* - \epsilon \notin X$, then there are infinityly many n such that $x^* - \epsilon < x_n$ $\blacksquare$

Proof of thm. (b) $\Rightarrow$ (c). Fix $\epsilon > 0$.

By (b), there are only finitely many n with $x^* + \epsilon < x$.

Take $N \in \mathbb{N}$ large enough such that

$$\begin{aligned}
& x^* + \epsilon \geq x_n && \forall n \geq N \\
\Rightarrow \quad & x^* + \epsilon \geq u_N && \\
\Rightarrow \quad & x^* + \epsilon \geq \lim u_n && \\
\Rightarrow \quad & x^* \geq \lim u_m && \forall n \geq N
\end{aligned}$$

On the other hand, there are infinitely many n with $x^* - \epsilon < x_n \leq u_n$.

Thus, there exists a subsequence of u_n, say u_{n_k}, satisfies

$$\begin{aligned}
& x^* - \epsilon \leq u_{n_k} \\
\Rightarrow \quad & x^* - \epsilon \leq \lim u_{n_k} = \lim (u_n) \\
\Rightarrow \quad & x* \leq \lim (u_n)
\end{aligned}$$

■

Proof of thm. (c) $\Rightarrow$ (d). ***Goal***:

$$x* = \lim (u_m), u_m = \sup x_n | n \geq m$$

$$\Rightarrow x^* = \sup S \text{ where S is the set of subsequential limits.}$$

Let (x_{n_k}) be a convergent subsequence of (x_n). Notice that $\lim (x_{n_k}) \in S$.

$$\begin{aligned}
& n \geq k \\
\Rightarrow \quad & x_{n_k} \leq \sup \{x_n | n \geq k\} = u_k \\
\Rightarrow \quad & \lim (x_{n_k}) \leq \lim (u_k) = x^* \\
\Rightarrow \quad & x^* \text{ is an upper bound of S..}
\end{aligned}$$

For 1, $\exists n_1 \in \mathbb{N}$ s.t.

$$u_1 - 1 \leq x_{n_1} \leq u_1$$

For $\frac{1}{2}$, $\exists n_2 \in \mathbb{N}$ s.t.

$$u_2 - \frac{1}{2} \leq x_{n_2} \leq u_2$$

$$\cdots$$

For $\frac{1}{k}$, $\exists n_k \in \mathbb{N}$ s.t.

$$u_k - \frac{1}{k} \leq x_{n_k} \leq u_k$$

When $k \to \infty$,

$$x^* - 0 \leq \lim (x_{n_k}) \leq x^*$$

By squeeze theorem,

$$\lim (x_{n_k}) = x^*$$

■

Proof of thm. (d) ⇒ (a). ***Goal:***

$$x^* = \sup S$$

$$\Rightarrow x^* = \limsup x_n = \inf \{v \in \mathbb{R} | v < x_n \text{for at most finite many of n}\}$$

Fix $\epsilon > 0$

There is no subsequence of x_n which has a limit exceeding $x^* + \epsilon$.

$$\begin{aligned}
\Rightarrow & \text{ There is only finitely many n with } x_n > x^* + \epsilon. \\
\Rightarrow & \quad x^* + \epsilon \in X \\
\Rightarrow & \quad \inf X \leq x^* + \epsilon \\
\Rightarrow & \quad \limsup x_n \leq x^* + \epsilon \\
\Rightarrow & \quad \limsup x_n \leq x^*
\end{aligned}$$

Next, consider $x^* - \epsilon$

Then, there exists a subsequential limit of x_n which is greater or equal to $x^* - \frac{1}{2}\epsilon$.

There exists a convergent subsequence of (x_n), say (x_{n_k}), such that

$$\begin{aligned}
& \lim (x_{n_k}) \geq x^* - \frac{1}{2}\epsilon \\
\Rightarrow & \quad \text{There are infinitely many n with } x^* > x^* - \epsilon. \\
\Rightarrow & \quad \forall a \in X, x^* - \epsilon \leq a \\
\Rightarrow & \quad x^* - \epsilon \leq \inf X \\
\Rightarrow & \quad \limsup x_n \geq x^* - \epsilon \\
\Rightarrow & \quad \limsup x_n \geq x^*
\end{aligned}$$

In conclusion,

$$\limsup x_n = x^*$$

■

3.5 Cauchy Criterion

Definition (Cauchy Sequence). A sequence (x_n) is ***Cauchy sequence*** if

$\forall \epsilon > 0, \exists H \in \mathbb{N}$ s.t. $\forall n, m \in \mathbb{N}, n > 0, m > 0,$

$$|x_n - x_m| < \epsilon$$

Example. $(\frac{1}{n})$ is a Cauchy sequence.

Proof. Observe that $\forall n, m \in \mathbb{N}, n \geq m,$

$$\left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{m-n}{mn}\right| = \frac{n-m}{mn} \leq \frac{n}{mn} < \frac{1}{m}$$

Choose $H = \lceil \frac{1}{\epsilon} \rceil + 1$, Then

$\forall n, m \geq H,$

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{m} < \epsilon$$

■

Example. $(-1)^n$ is not a Cauchy sequence.

Proof. Choose $\epsilon_0 = \frac{1}{2}, \forall H \in \mathbb{N}$, choose $n, m \geq H$ s.t. n is even, m is odd. Then

$$|(-1)^n - (-1)^m| = |1 - (-1)| = 2 > \frac{1}{2}$$

Thus $(-1)^n$ does not satisfies the definition of Cauchy sequence.

■

Theorem. If (x_n) is convergent, then it is Cauchy sequence.

Proof. Let $\lim (x_n) = x$.

Goal:

$$|x_n - x| < \epsilon \Rightarrow |x_n - x_m| < \epsilon$$

$\forall n, m \geq N_\epsilon,$

$$\begin{aligned} |x_n - x_m| &= |(x_n - x) + (x - x_m)| \\ &\leq |x_n - x| + |x - x_m| \text{ by triangle inequality} \\ &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon \end{aligned}$$

■

Lemma. If (x_n) is Cauchy, the it is bounded.

Proof. Choose $\epsilon = 1, \exists H \geq 0$ s.t. $\forall n, m \geq H$,

$$|x_n - x_m| < 1$$

Since the choice of m satisfies $m \geq H$, we may choose $m = H$ s.t.

$$|x_n - x_H| < 1$$

It follows that

$$|x_n| - |x_H| \leq |x_n - x_H| < 1$$
$$|x_n| < |x_H| + 1, \forall n \geq H$$

Let

$$M := \max \{|x_1|, |x_2|, |x_3|, \ldots,, |x_{H-1}|, |x_H| + 1\}$$

Thus $\forall n \in \mathbb{N}, |x_n| \leq M$ ■

Theorem (Cauchy Convergence Theorem). A sequence of real numbers is ***convergent*** if and only if it is ***Cauchy*** sequence.

(⇒). Done in the previous theorem. ■

(⇐). Suppose (x_n) is Cauchy. By lemma, it is bounded.

By Bolzano-Weierstrass theorem, there exists a convergent subsequence (x_{n_k}).

Let $\lim(x_{n_k}) = x$.

Goal: $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$

$$|x_n - x| < \epsilon$$

We will use the trick of ***insert subsequence:***

By definition of convergence, $\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}$ s.t. $\forall k \geq N_\epsilon$

$$|x_{n_k} - x| < \frac{1}{2}\epsilon$$

By definition of Cauchy sequence, $\forall \epsilon > 0, \exists H \in \mathbb{N}, H \geq 0$ s.t. $\forall n, m \geq H$

$$|x_n - x_m| < \frac{1}{2}\epsilon$$

Thus $\forall k \geq \max\{H, N_\epsilon\}$,

$$\begin{aligned} |x_k - x| &= |(x_k - x_{n_k}) + (x_{n_k} - x)| \\ &\leq |x_k - x_{n_K}| + |x_{n_k} - x| \text{ since } n_k \geq k \geq \max\{H, N_\epsilon\} \\ &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon \end{aligned}$$

■

Example. Let (h_n) be the sequence of harmonic series such that

$$h_n = \sum_{i=1}^{n} \frac{1}{i} \text{ , and } \lim(h_n) = \sum_{n=1}^{\infty} \frac{1}{n}$$

Claim: (h_n) is divergent

Goal: show that it is NOT Cauchy.

Proof. $\forall m, n \in \mathbb{N}$, WLOG suppose $m \geq n, h_m > h_n$, we have

$$\begin{aligned} |h_m - h_n| &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m-1} + \frac{1}{m} \\ &\geq \frac{m-n}{n} \end{aligned}$$

since there are (m-n)'s terms on the right side of the equation.

So we choose $\epsilon_0 = \frac{1}{2}, \forall H \geq 0$, choose $n = H, m = 2H$, then

$$|h_m - h_n| \geq \frac{m-n}{m} = \frac{1}{2}$$

Thus, (h_n) is NOT Cauchy. ■

3.6 Property of Divergent Sequence

Example (divergent sequence)**.** :

- $(n) \to \infty$
- $(-n) \to -\infty$
- $(-1)^n \cdot n$ is divergent and unbounded.
- $(-1)^n$ is divergent and bounded.

Definition (Properly Divergent). Let (x_n) be a sequence of real numbers. We say that

1. (x_n) ***tends to*** ∞, or $\lim(x_n) = \infty$ if $\forall a \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,
$$x_n > a$$
2. Similarly, (x_n) ***tends to*** $-\infty$, or $\lim(x_n) = -\infty$ if $\forall a \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,
$$x_n < a$$

In either case, we say that (x_n) is ***properly divergent***.

Example. Let $C > 0$. (C^n) is properly divergent. We write $\lim(C^n) = \infty$.

Proof. Notice that

$$C^n = [1 + (C-1)]^n \geq 1 + n(C-1) \text{ by Bernoulli's inequality}$$

Goal: $\forall a \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$1 + n(C-1) > a_n \Leftrightarrow n > \frac{a_n - 1}{C - 1}$$

Then we choose $N = \lceil \frac{a-1}{c-1} \rceil + 1$. Thus $\forall n \geq N, C^n > a$. ■

Theorem. A monotone sequence is ***divergent*** if and only if it is ***bounded***.

Proof. Exercise. ■

Theorem (Comparison Test). Let (x_n) and (y_n) be two sequences. Suppose $\forall n \in \mathbb{N}, x_n \geq y_n$. Then,

1. If $(x_n) \to \infty$, then $(y_n) \to \infty$.
2. If $(y_n) \to -\infty$, then $(x_n) \to -\infty$.

Proof. Exercise. ■

Theorem (Limit Comparison Test). Let (x_n) and (y_n) be two sequences of positive real numbers. Suppose $\exists L \in \mathbb{R}, L > 0$ s.t.

$$\exists \lim(\frac{x_n}{y_n}) = L$$

then $\lim(x_n) = \infty$ if and only if $\lim(y_n) = \infty$

Proof. ***Claim:*** For N large enough,

$$\frac{1}{2}L \cdot y_n < x_n < 2L \cdot y_n$$

Goal: Claim+Comparison Theorem=Proof.

By definition of limit, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N$,

$$L - \epsilon < \frac{x_n}{y_n} < L + \epsilon$$

Choose $\epsilon = \frac{1}{2}L$, we have

$$\frac{1}{2}L \cdot y_n < x_n < \frac{3}{2}L < 2L \cdot y_n$$

since $L > 0$ and these are positive sequences.

Thus by Comparison Theorem,

$$\lim(\frac{1}{2}L \cdot y_n) = \infty \Rightarrow \lim(x_n) = \infty$$

■

3.7 Introduction to Infinite Series

Definition (Infinite Series). If (x_n) is a sequence of real numbers, then the ***infinite series*** generated by (x_n) is the sequence (s_k) defined by

$$s_k = \sum_{i=1}^{k} x_i$$

Terms s_k are called ***partial sums***.

Notation. $\sum x_i$ to mean this series or its limit at infinity $\lim(x_k)$.

Theorem (Cauchy Criterion for Series). The series $\sum x_i$ ***converges*** if and only if $\forall \epsilon > 0, \exists M \in \mathbb{N}$ s.t. $\forall n, m \in \mathbb{N}, n > m \geq M$,

$$|x_{m+1} + x_{m+2} + \cdots + x_{n-1} + x_n| < \epsilon$$

Or we write it as

$$|s_k - s_m| < \epsilon$$

Proof: Exercise. ■

Theorem (Montone Convergence for Series)**.** Let (x_n) be a sequence of non-negative real numbers. Then the series $\sum x_n$ ***converges*** if and only if (s_k) is bounded.

Proof. Exercise.

■

Example. $\sum \frac{1}{n^2}$ is convergent.

Proof. ***Goal:*** find convergent subsequence (s_{k_j}).

Consider subsequence (s_{k_j}) where $k_j = 2^j - 1$.

Observe that:

$$\begin{aligned}
s_{k_1} &= 1 \\
s_{k_2} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} \\
&< s_{k_1} + 2 \cdot \frac{1}{2^2} = 1 + \frac{1}{2} \\
s_{k_3} &= 1 + (\frac{1}{2^2} + \frac{1}{3^2}) + (\frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{9^2}) \\
&< s_{k_1} + 2 \cdot \frac{1}{2^2} = 1 + \frac{1}{2}
\end{aligned}$$

By induction(details are left as exercise), one can show that

$$s_{k_j} < \sum_{n=0}^{j-1} \frac{1}{2^n} < \sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \text{ (by limit of geometric series.)}$$

Thus (s_{k_j}) is bounded.

By theorem proved in homework, an increasing sequence with bounded(and thus convergent) subsequence implies that the sequence is convergent. ■

4 Limits

4.1 Limits of Functions

Let $f : A \to B$ be a function where $A, B \subseteq \mathbb{R}$. Let $a \in A, L \in B$.

Goal: Define

$$\lim_{x \to a} f(x) = ?$$

Intuition: Define ***closeness*** on real line.

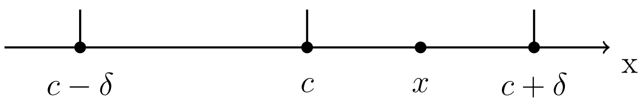
Definition (Cluster Point). Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a ***cluster point*** of A if

$\forall \delta > 0, \exists x \in A, x \neq c$ such that

$$|x - c| < \delta$$

Or

$$V_\delta(c) \cap (A \ \{c\}) \neq \phi$$



Theorem. $c \in \mathbb{R}$ is a cluster point if and only if

there exists a sequence $(a_n) \in A$ such that

$$\lim(a_n) = c \text{ and } \forall n \in \mathbb{N}, a_n \neq c$$

Sketch of Proof. $(\Rightarrow)$

$$\begin{aligned}
&\delta = 1, \exists a_1 \in A \backslash \{c\} \text{ s.t. } |a_1 - c| < 1 \\
&\delta = \frac{1}{2}, \exists a_2 \in A \backslash \{c\} \text{ s.t. } |a_2 - c| < \frac{1}{2} \\
&\ldots \text{ observe that:} \\
&\forall \delta > 0, \exists a_n \in A \backslash \{c\} \text{ s.t. } |a_n - c| < \frac{1}{n} < \delta \\
&\Rightarrow \lim(a_n) = c \text{ by squeeze theorem.}
\end{aligned}$$

$(\Leftarrow)$

$\forall \delta > 0, \exists N_\epsilon > 0$ s.t.

$$|a_{N_\epsilon} - c| < \delta$$

Note that $a_{N_\epsilon} \in A \backslash \{c\}$ where c is a cluster point of A. ■

Example. Let X be the set of cluster points of A.

- $A = (1,2) \cup (3,4) \Rightarrow X = [1,2] \cup [3,4]$. *Proof: Exercise.*
- $A = \{0\} \cup (1,2) \Rightarrow X = \phi$.

  Sketch of (2). :

  1. Let $x \in [1,2]$. Prove that $x \in X$. Thus $[1,2] \in X$.
  2. Prove that $0 \notin X$.
  3. Prove that $x \in X$ if $x \notin \{0\} \cup [1,2]$.

■

Remark. A may not be a subset of the set of cluster points of A.
Example. :

- $A = \mathbb{Z} \Rightarrow X = \phi$.
- $A = \{\frac{1}{n} | \, n \in \mathbb{N}\} \Rightarrow X = \{0\}$ *Proof: Exercise.*

Definition (Delta-Epsilon Definition of Limit). Let $A \in \mathbb{R}$, c is a cluste point of A, $f : A \to \mathbb{R}$. A real number L is the ***limit of f at c*** if

$\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$,

$$0 < |x - c| < \delta \to |f(x) - L| < \epsilon$$

Theorem (Uniqueness of Limit). If $f : A \to \mathbb{R}$ and c is a cluster of points of A, then f has at most 1 limit at c.

Proof. We will prove this by contradiction.

Let L_1 and L_2 be limits of f at c. Assume $L_1 \neq L_2$. Choose $\epsilon = \frac{|L_1 - L_2|}{2} > 0$. Then,

$$\exists \delta_1 \text{ s.t. } 0 < |x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \frac{\epsilon}{2}$$

$$\exists \delta_2 \text{ s.t. } 0 < |x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \frac{\epsilon}{2}$$

Consider $\delta := \min\{\delta_1, \delta_2\}$.

Since c is a cluster point, $\exists x_0 \in A$ s.t.

$$0 < |x_0 - c| < \delta$$

Since

$$|f(x_0) - L_1| < \frac{\epsilon}{2}, |f(x_0) - L_2| < \frac{\epsilon}{2}$$

We have

$$\begin{aligned} |L_1 - L_2| &\leq |L_1 - f(x_0)| + |f(x_0) - L_2| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\ &= \frac{|L_1 - L_2|}{2} \end{aligned}$$

which is a contradiction.

■

Notation.

$$L = \lim_{x \to c} f(x) \text{ or } L = \lim_{x \to c} f$$

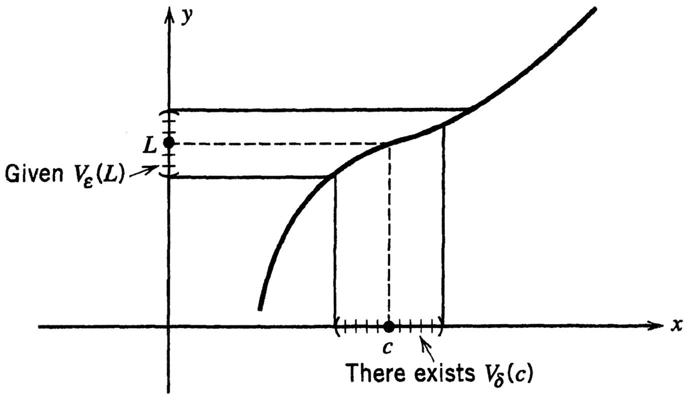
And we say that *f(x) approaches to L as x approaches to c.*

Remark (Divergence of function). If the limit of $f(x)$ at c does not exists, we say that f ***diverges*** at c.

Theorem. Let $f : A \to \mathbb{R}$ and c be a cluster point of f. The following are equivalent:

1. $\lim_{x \to c} f(x) = L$
2. $\forall\, V_\epsilon(L)$ ϵ-neighborhood of L, $\exists\, V_\delta(c)$ δ-neighborhood of c s.t.

$$x \in V_\delta(c) \cap (A \backslash \{c\}) \Rightarrow f(x) \in V_\epsilon(c)$$



Example. $\lim_{x \to c} f(x) = a$.

Proof. $\forall \epsilon > 0, \exists \delta = 1$ s.t.

$$x \in V_\delta(c) \cap (\mathbb{R} \backslash \{c\}) \Rightarrow f(x) = a \in V_\epsilon(L)$$

■

Example. $\lim_{x \to c} f(x) = c$.

Proof. $\forall \epsilon > 0, \exists \delta = \epsilon$ s.t.

$$\begin{aligned} & x \in V_\delta(c) \cap (A \backslash \{c\}) \\ \Rightarrow & f(x) = x \in V_\delta(c) \cap (A \backslash \{c\}) \\ \Rightarrow & f(x) \in V_\epsilon(c) \cap (A \backslash \{c\}) \subseteq V_\epsilon(c) \end{aligned}$$

■