Math265 Real Analysis Class Notes

Based on lectures by Prof. Huang

Notes taken by Kaifeng Lu

Errata: klu15 at u dot rochester dot edu

Fall 2024

1 Preliminaries

Still working on it...(missed 1st month due to late enrollment)

2 The Real Numbers

Still working on it...

3 Sequences and Series

3.1 Sequences and their limits

Definition (**Sequence**). A ***sequence of real numbers*** is a *function* from $\mathbb{N}$ to $\mathbb{R}$.

We adopt the notation with a *sequence*:

$$a : \mathbb{N} \to \mathbb{R}$$

where instead of writing $a(1), a(2), \ldots$, we write it as $a_1, a_2, \ldots$ which we called them ***terms*** or ***elements*** of the sequence.

Notation.
$$(a_n)_{n=1}^{\infty} \text{ or } (a_n)_{n \in \mathbb{N}} \text{ or } (a_n) \text{ or } (a_n | n \in \mathbb{N})$$

Definition (**Converge to x**). A sequence $(x_n) \in \mathbb{R}$ ***converges*** to $x \in \mathbb{R}$ if

$$\forall \varepsilon > 0, \ \exists N_\varepsilon \in \mathbb{N} \text{ such that } n \geq N_\varepsilon \to |x_n - x| < \varepsilon$$

We write
$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} (x_n) = x.$$

Definition (**Convergent & Divergent**). A sequence is ***convergent*** if it has a ***limit*** in $\mathbb{R}$, and is ***divergent*** if it has ***no limit*** in $\mathbb{R}$.

Theorem (**Uniqueness of Limit**). A sequence in $\mathbb{R}$ can have **_at most one_** limit. Or, the limit of a sequence is **_unique_** if the limit exists

Proof. Let (x_n) be a sequence of real numbers. Suppose x, x' are limits of (x_n). We want to prove $x = x'$ by contradiction.

Assume $|x - x'| > 0$. If we consider $\varepsilon := \frac{1}{3}|x - x'| > 0$, then
The existence of $\lim_{x_n \to x}$ implies that $\exists N_1 \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ if $n \geq \mathbb{N}_1$.
Similarly, existence of $\lim_{x_n \to x'}$ implies that $\exists N_2 \in \mathbb{N}$ such that $|x_n - x'| < \varepsilon$ if $n \geq \mathbb{N}_2$.
Thus,

$$
\begin{aligned}
|x - x'| &\leq |x - x_{N_1+N_2} + x_{N_1+N_2} - x'| \\
&\leq |x - x_{N_1+N_2}| + |x_{N_1+N_2} - x'| \ \text{ by triangle inequality} \\
&< \varepsilon + \varepsilon \\
&= \frac{2}{3}|x - x'|
\end{aligned}
$$

Then,

$$
\frac{1}{3}|x - x'| < 0, \text{which is a contradiction}
$$

we thereby prove by contradiction that

$$
|x - x'| = 0, \text{ which is equivalent to } x = x'
$$

∎

Example.

$$
\lim_{n \to \infty} \left(\frac{1}{n}\right) = 0
$$

Goal: $\forall \varepsilon > 0$, want to find N_ε such that $\left|\frac{1}{n} - 0\right| < \varepsilon$ for $n > N$, so it suffices to show that

$$
\frac{1}{n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n
$$

Proof. Let $\varepsilon > 0$. Apply Archimedean's property to $\frac{1}{\varepsilon}$, then

$$
\exists N \in \mathbb{N} \text{ such that } \frac{1}{\varepsilon} < N
$$

$$
\Rightarrow \forall n \geq N, \left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.
$$

$$
\Rightarrow \lim_{n \to \infty} \frac{1}{n} = 0
$$

∎

Theorem. Let (x_n) be a sequence of real numbers, and let $x \in \mathbb{R}$. The following are equivalent:

1. $x_n \to x$

2. $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$, for $n \geq \mathbb{N}$

3. $\ldots \ldots x - \varepsilon < x_n < x + \varepsilon \ldots \ldots$

4. $\forall \varepsilon$-neighborhood $V_\varepsilon(x), \exists N \in \mathbb{N}$ such that $x_n \in V_\varepsilon(x)$ for $n \geq N$

Sketch of proof:
$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$$

Proposition.
$$\lim_{n \to \infty} (2\sqrt{2n+1} - \sqrt{2n}) = 0$$

Proof. Let $\varepsilon > 0$. Consider
$$N = \lceil \frac{1}{2}(\frac{1}{2\varepsilon})^2 \rceil \in \mathbb{N}$$

$$n > N \Rightarrow n > \frac{1}{2}(\frac{1}{2\varepsilon})^2 \Rightarrow \frac{1}{2\sqrt{2n}} < \varepsilon \Rightarrow \left| \sqrt{2n+1} - \sqrt{2n} \right| = \cdots = \frac{1}{\sqrt{2n+1} + \sqrt{2n}} < \varepsilon$$

$\blacksquare$

Remark.
$$\lim_{n \to \infty} (-1)^n \text{ does not exist.}$$

Definition (**m-tail**). If (x_n) is a sequence of real numbers and $m \in \mathbb{N}$, then the ***m-tail*** of (x_n) is the sequence

$$\{x_{n+m} : n \in \mathbb{N}\} = \{x_{m+1}, x_{m+2}, \ldots\}$$

Theorem. Let (x_n) be a sequence and $m \in \mathbb{N}$. Then (x_n) is ***convergent*** iff (x_{n+m}) is ***convergent***. Moreover,
$$\lim_{n \to \mathbb{N}} (x_n) = \lim_{n \to \mathbb{N}} (x_{n+m})$$

Proof. $(\Rightarrow)$
Suppose $x_n \to x$. Let

$$\varepsilon > 0, \exists N_\varepsilon > 0, \text{ such that } |x_n - x| < \varepsilon \text{ for } n \geq N_\varepsilon$$

Consider $N_\varepsilon' := N_\varepsilon + m$ then

$$n + m \geq N_\varepsilon' \Rightarrow n \geq N_\varepsilon \Rightarrow n + m \geq N_\varepsilon \Rightarrow |x_{n+m} - x| < \varepsilon$$

It follows that

$$n \geq N_\varepsilon \Rightarrow n + m \geq N_\varepsilon \Rightarrow |x_{n+m} - x| < \varepsilon$$

$(\Leftarrow)$

Suppose $x_{n+m} \to x$.

$$\forall \varepsilon > 0, \exists N_\varepsilon > 0 \text{ such that } |x_{n+m} < \varepsilon|, \forall n \geq N_\varepsilon$$

Consider $N := N_\varepsilon + m$. Then

$$
\begin{aligned}
& n \geq N = N_\varepsilon + m \\
\Rightarrow\ & n - m \geq N_\varepsilon \\
\Rightarrow\ & \left| x_{(n-m)+m} - x \right| < \varepsilon \\
\Rightarrow\ & |x_n - x| < \varepsilon
\end{aligned}
$$

∎

Remark. We say that a sequence (x_n) **_ultimately_** has a property if that property holds for some tail of (x_n)

Theorem. Let x_n be a sequence of real numbers. Let a_n be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$. If $\exists c > 0, m \in \mathbb{N}, x \in \mathbb{R}$ such that

$$|x_n - x| \leq c \cdot a_n, \forall n \geq m$$

then

$$x_n \to x$$

Proof. We know that

$$\forall \varepsilon > 0, \exists N \geq 0 \text{ s.t. } |a_n| < \frac{\varepsilon}{c}, \forall n \geq N$$

Consider $N' = max\{N, m\}, \forall n \geq N'$. Then

$$|x_n - x| \leq Ca_n = c\,|a_n| < c \cdot \frac{\varepsilon}{c} = \varepsilon$$

$$\Rightarrow x_n \to x$$

∎

Proposition.

$$\lim_{n\to\infty} \frac{17}{2 + 3n} = 0$$

Proof.

$$\left| \frac{17}{2 + 3n} - 0 \right| = \frac{17}{2 + 3n} \leq \frac{13}{3n} = \frac{17}{3} \cdot \frac{1}{n}$$

Apply the theorem above with

$$a_n = \frac{1}{n}, c = \frac{17}{3}, m = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{17}{2 + 3n} = 0, \text{ since } \lim_{n\to\infty} \frac{1}{n} = 0$$

∎

Proposition.
$$\forall c > 0, \lim_{n \to \infty} c^{\frac{1}{n}} = 1$$

Proof. **<u>Case 1: c = 1</u>**
$$\lim_{n \to \infty} c^{\frac{1}{n}} = 1$$

<u>Case 2: c > 1</u>

Let $d_n = c^{\frac{1}{n}} - 1$. Then $\forall n, d_n > 0$. It follows that

$$(d_n + 1) = c^{\frac{1}{n}} \Rightarrow c = (1 + d_n)^n \geq 1 + n \cdot d_n \text{ by Bernoulli's inequality}$$
$$\Rightarrow d_n \leq (c-1) \cdot \frac{1}{n}$$
$$\Rightarrow \left| c^{\frac{1}{n}} - 1 \right| = d_n \leq (c-1) \cdot \frac{1}{n}$$

Apply the theorem with

$$C = c - 1, a_n = \frac{1}{n}, m = 1, x = 1$$

$$\lim_{n \to \infty} c^{\frac{1}{n}} = 1$$

<u>Case 3: c < 1(Note that we cannot use Bernoulli inequality here)</u>

Define e_n to be a sequence that satisfies

$$c^{\frac{1}{n}} = \frac{1}{1 + e_n}$$

Then $e_n > 0 \forall n$.
$$c = \frac{1}{(1 + e_n)^n} \leq \frac{1}{1 + n \cdot e_n} < \frac{1}{n \cdot e_n}$$
$$\Rightarrow e_n < \frac{1}{c} \cdot \frac{1}{n}$$
$$1 - c^{\frac{1}{n}} = 1 - \frac{1}{1 + e_n} = \frac{e_n}{1 + e_n} < e_n < \frac{1}{c} \cdot \frac{1}{n}$$

Apply the theorem with

$$a_n = \frac{1}{n}, m = 1, C = \frac{1}{c}, x = 1$$
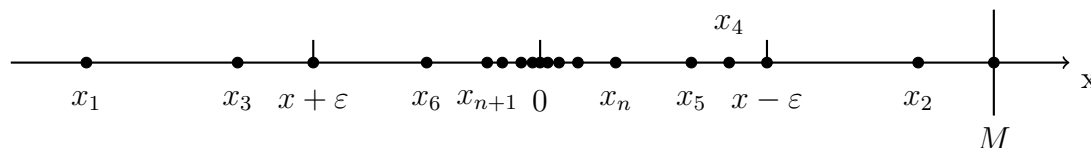$$\lim_{n \to \infty} c^{\frac{1}{n}} = 1$$

$\blacksquare$

3.2　Limit Theorems

Definition (Bounded sequence)**.** A sequence $(x_n) \in \mathbb{R}$ is **_bounded_** if

$$\exists M > 0 \text{ s.t. } \forall n \in \mathbb{N}, |x_n| \leq M$$

Theorem. A convergent sequence $(x_n) \in \mathbb{R}$ is bounded.

Proof. By definition of convergent sequence, let $\varepsilon = 1$:

$$\exists N > 0 \text{ s.t. } \forall n \geq N, |x_n - x| < 1$$

Thus we have

$$-1 < x_n - x < 1$$
$$\Rightarrow \qquad -1 + x < x_n < x + 1, \forall n \geq N$$

Then define

$$M := \max \{|x_1|, |x_2|, |x_3|, \ldots, |-1+x|, |x+1|\}$$
$$|x_n| \leq M$$

$\blacksquare$

Remark. By contrapositive, an unbounded sequence is divergent.

Definition. Given sequence $(x_n), (y_n) \in \mathbb{R}$, we define following operations of sequence:

- **_Sum_** $(x_n + y_n)$
- **_Difference_** $(x_n - y_n)$
- **_Product_** $(x_n \cdot y_n)$
- **_Quotient_** $\left(\frac{x_n}{y_n}\right)$ if $\forall n \in \mathbb{N}, y_n \neq 0$
- **_Multiple_** $(c \cdot x_n)$

Theorem (**Limit Laws**)**.** Let $(x_n), (y_n) \in \mathbb{R}$ be sequences of real numbers with $x_n \to x, y_n \to y$, and let $c \in \mathbb{R}$. Then

- $x_n + y_n \to x + y$
- $x_n - y_n \to x - y$
- $x_n \cdot y_n \to x \cdot y$
- $c \cdot x_n \to c \cdot x$
- If $\forall n \in \mathbb{N}, y_n \neq 0$ and $y \neq 0$, then $\frac{x_n}{y_n} \to \frac{x}{y}$

Proof of Sum. :

$\forall \varepsilon > 0$,

$$\exists N_1 \in \mathbb{N} \text{ s.t. } \forall n \geq N_1, |x_n - x| < \frac{\varepsilon}{2}$$

$$\exists N_2 \in \mathbb{N} \text{ s.t. } \forall n \geq N_2, |y_n - y| < \frac{\varepsilon}{2}$$

Consider

$$N := \max\{N_1, N_2\}$$

Then

$$\begin{aligned}
\forall n \geq N, |(x_n + y_n) - (x + y)| \\
= |x_n - x + y_n - y| \\
\leq |x_n - x| + |y_n - y| \\
= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \varepsilon
\end{aligned}$$

■

Proof of Difference. :

Similarly,

$$\begin{aligned}
\forall n \geq N, |(x_n - y_n) - (x - y)| \\
\leq |x_n - x| + |y_n - y|
\end{aligned}$$

■

Proof of Product. : Since (x_n) is convergent, it is also bounded. Thus,

$$\exists M \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, |x_n| \leq M$$

By definition of convergence, $\forall \varepsilon > 0$:

$$\exists N_1 > 0 \text{ s.t. } \forall n \geq N, |x_n - x| < \frac{\varepsilon}{2|y|}$$

$$\exists N_2 > 0 \text{ s.t. } \forall n \geq N, |y_n - y| < \frac{\varepsilon}{2M}$$

Then, $\exists N = \max\{N_1, N_2\}$ s.t. $\forall n \geq N$,

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\
&= |x_n(y_n - y) - y(x_n - x)| \\
&\leq |x_n| |y_n - y| + |y| |x_n - x| \\
&\leq M \cdot |y_n - y| + |y| |x_n - x| \\
&< M \cdot \frac{\varepsilon}{2M} + |y| \cdot \frac{\varepsilon}{2|y|} = \varepsilon
\end{aligned}$$

■

Proof of Multiply. :

Exercise   ■

Proof of Quotient.

$$
\begin{aligned}
\left| \frac{x_n}{y_n} - \frac{x}{y} \right| &= \left| \frac{x_n y - y_n x}{y_n y} \right| \\
&= \left| \frac{x_n y - -x_n y_n + x_n y_n - y_n x}{y_n y} \right| \\
&\leq \left| \frac{1}{y_n y} (|x_n| \, |y_n - y| + |y_n| \, |x_n - x|) \right|
\end{aligned}
$$

Since $y_n \to y$

$$
\exists N_1 \in \mathbb{N} \text{ s.t. } \forall n \geq N, ||y_n| - |y|| \leq |y_n - y| < \frac{|y|}{2}
$$

Then

$$
\Rightarrow \qquad\qquad -\frac{|y|}{2} < |y| - |y| < \frac{|y|}{2}
$$

$$
\Rightarrow \qquad\qquad \frac{|y|}{2} < |y_n|
$$

$$
\Rightarrow \qquad\qquad \frac{1}{|y_n|} < \frac{2}{|y|}
$$

$$
\Rightarrow \qquad\qquad \frac{1}{|y_n y|} < \frac{2}{|y^2|}
$$

Define $F := \frac{2}{|y^2|}$.

Since (x_n) and (y_n) are bounded,

$\exists G \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$

$$
|x_n| \leq G, |y_n| \leq G
$$

Thus $\forall n > 0$

$$
\exists N_2 > 0 \text{ s.t. } \forall n \geq N_2, |x_n - x| < \frac{\varepsilon}{2GF}
$$

$$
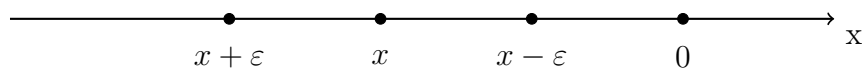\exists N_3 > 0 \text{ s.t. } \forall n \geq N_3, |y_n - y| < \frac{\varepsilon}{2GF}
$$

Define $N := \max \{N_1, N_2, N_3\}$, then $\exists N \in \mathbb{N}$ s.t. $\forall n > N$,

$$
\left| \frac{x_n}{y_n} - \frac{x}{y} \right| < F(G \cdot \frac{\varepsilon}{2GF} + G \cdot \frac{\varepsilon}{2GF}) = \varepsilon
$$

∎

Theorem. If (x_n) is a convergent sequence in $\mathbb{R}$ of non-negative terms with $(x_n) \to x$, then $x \geq 0$.

Sketch. : Assume $x < 0$, choose $\varepsilon \leq |x|$, then $\forall n > N, x_n < 0$, which is a contradiction.

∎

Theorem. If $x_n \to x, y_n \to y$ are sequences in $\mathbb{R}$ such that $\forall n \in \mathbb{N}, x_n \leq y_n$, then $x \leq y$.

Proof. Consider $z_n := y_n - x_n$. Then $z_n \geq 0$ and $z_n \to y - x$ by the limit law.

$$\Rightarrow y - x \geq 0$$

$$y \geq x$$

$\blacksquare$

Theorem. If $x_n \to x$ is a sequence in $\mathbb{R}$, and let $a, b \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, a \leq x_n \leq b$, then $a \leq x \leq b$.

Proof. Consider constant sequence $a_n = a$ and $b_n = b$. Then this is true by the last theorem. $\blacksquare$

Theorem (**Squeeze Theorem**). Suppose $(x_n), (y_n), (z_n)$ are sequences of real numbers such that $\forall n \in \mathbb{N}, x_n \leq y_n \leq z_n$. If $\lim (x_n) = \lim (z_n)$, then (y_n) is convergent and

$$\lim (x_n) = \lim (y_n) = \lim (z_n)$$

Proof. Let $\forall \varepsilon > 0$.

Write $L = \lim (x_n)$. Then by definition of limit,

$\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N$,
$$|x_n - L| < \varepsilon, |z_n - L| < \varepsilon$$

$$\Rightarrow \qquad -\varepsilon < x_n - L \leq y_n - L \leq z_n - L < \varepsilon$$
$$\Rightarrow \qquad |y_n - L| < \varepsilon$$

Thus (y_n) converges to L by definition of limit.

$\blacksquare$

Proposition. $\ln (n)$ is divergent.

Proof. Since $\ln(n)$ is unbounded, it is divergent.

$\blacksquare$

Exercise. $\lim \frac{3n^2+2n+1}{5n^2-4} = \frac{3}{5}$

$$\lim \frac{3n^2 + 2n + 1}{5n^2 - 4} = \lim \frac{3 + 2\frac{1}{n} + \frac{1}{n^2}}{5 - 4\frac{1}{n^2}}$$

Notice that each term of $\frac{1}{n}$ and $\frac{1}{n^2}$ converges to 0. Thus by limit law,

$$\lim \frac{3 + 2\frac{1}{n} + \frac{1}{n^2}}{5 - 4\frac{1}{n^2}} = \frac{\lim \{3 + 2 \cdot 0 + 0\}}{\lim\{5 - 4 \cdot 0\}} = \frac{3}{5}$$

Proposition. $(-1)^n$ is divergent.

Proof:Exercise. $\blacksquare$

Theorem. If $x_n \to x$, then $|x_n| \to |x|$

Sketch of the proof. :
$$||x_n| - |x|| \le |x_n - x|$$

■

Theorem. Suppose (x_n) is a sequence of non-negative real numbers, satisfiying $x_n \to x$. Then $\sqrt{x_n} \to \sqrt{x}$.

Proof. Let $\forall \varepsilon > 0$

<u>Case1:</u> $x = 0$

$\exists N \in \mathbb{N}$ s.t. $\forall n \ge N$,
$$|x_n - 0| < \varepsilon^2$$
$$\Rightarrow \left|\sqrt{x} - 0\right| < \varepsilon$$

<u>Case2:</u> $x > 0$

$\exists N$ s.t. $\forall n > N$,
$$|x_n - x| < \sqrt{x} \cdot \varepsilon$$

Notice that

$$\left|\sqrt{x_n} - \sqrt{x}\right| = \left|\frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}\right|$$
$$\le \frac{|x_n - x|}{\sqrt{x}} < \varepsilon$$

■

Theorem. Let (x_n) be a sequence of positive real numbers such that $L = \lim(\frac{x_{n+1}}{n})$ exists. If $L < 1$, then (x_n) converges to 0.

Proof. $\exists N_1 \in \mathbb{N}$ s.t. $\forall n > N_1$
$$\left|\left|\frac{x_{n+1}}{x_n}\right| - |L|\right| \le \left|\frac{x_{n+1}}{x_n} - L\right| < \frac{1 - L}{2}$$

Thus
$$\left|\frac{x_{n+1}}{x_n}\right| < \frac{1 + L}{2}$$

Note that $\frac{1+L}{2} < 1$, write $r = \frac{1+L}{2}$. Then $\forall m \in \mathbb{N}$,

$$x_{N_1+m} < x_{N_1+(m-1)} r < x_{N_1+(m-2)} r^2 < \cdots < x_{N_1} r^m$$

Consider $(y_m) = (x_{N_1+m})$, $(z_m) = (x_{N_1} r^m)$. Then

$$0 \le y_m \le z_m$$

Since $z_m \to 0$
$$(y_m) \to 0 \text{ by squeeze theorem}$$

Thus we conclude that $(x_n) \to 0$ by m-th tail theorem.   ■

3.3   Monotone sequence

Definition. Let (x_n) be a sequence of real. We say (x_n) is ...

- **_increasing_** if $\forall n \in \mathbb{N}, x_{n+1} \geq x_n$.

- **_strictly increasing_** if $\forall n \in \mathbb{N}, x_{n+1} > x_n$.

- **_decreasing_** if $\forall n \in \mathbb{N}, x_{n+1} \leq x_n$.

- **_strictly decreasing_** if $\forall n \in \mathbb{N}, x_{n+1} < x_n$.

Theorem (**Monotone Convergence Therorem**). A monotone sequence of real numbers is **_convergent_** iff it is bounded. Moreover, if (x_n) is increaseing, then

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$$

If (x_n) is decreasing, then

$$\lim(x_n) = \inf\{x_n : n \in \mathbb{N}\}$$

Proof. $(\Rightarrow)$

A convergent sequence is always bounded.

$(\Leftarrow)$

Suppose (x_n) is a monotone and bounded sequence.

Case 1: (x_n) is increasing.

Write $x = \sup\{x_n : n \in \mathbb{N}\}$.

Let $\varepsilon > 0$. Since $x = \sup(x_n)$:

$$x - \varepsilon \text{ is NOT an upper boudn of } (x_n)$$

Then

$$\exists N \in \mathbb{N} \text{ s.t. } x_N > x - \varepsilon$$

Since (x_n) is increasing,

$$\forall n \geq N, x_n > x - \varepsilon$$

On the other hand,$x + \varepsilon$ is an supper bound since x is an upper bound. Thus,

$$x_n < x + \varepsilon$$

$$\Rightarrow \forall n \geq N, x - \varepsilon < x_n < x + \varepsilon$$

$$\Rightarrow |x_n - x| < \varepsilon \Rightarrow (x_n) \to x$$

Case 2: (x_n) is decreasing.

Write $y = \inf(x_n)$. Let $\varepsilon > 0$

Since $y = \inf(x_n)$,

$$y + \varepsilon \text{is NOT and upper bound of } (x_n)$$

Thus
$$\exists N \in \mathbb{N} \text{ s.t. } x_n < y + \varepsilon$$

Since (x_n) is decreasing,
$$\forall n > N, x_n < y + \varepsilon$$

On the other hand, $y - \varepsilon$ is a lower bound since y is a lower bound. Hence,

$$\forall n \in \mathbb{N}, y - \varepsilon < x_n$$

$$\forall n \geq N, y - \varepsilon < x_n < y + \varepsilon \Rightarrow |x_n - y| < \varepsilon$$
$$\Rightarrow (x_n) \to y$$

Remark. One can prove case 2 by following:

$(-x_n)$ is increasing and converges to $\sup(-x_n)$ by case 1. Also note that

$$(x_n) = (-(-x_n)) \to -\sup(-x_n)$$

by limit law. So it is easy to prove that

$$-\sup((-x_n)) = \inf(x_n)$$

∎

Example. Consider the sequence (x_n) is given by

$$\begin{cases} x_0 = \frac{1}{2} \\ x_{n+1} = \frac{3}{2}x_n(1 - x_n) \end{cases}$$

(x_n) is decreasing and bounded.

Thoughts: Assume (x_n) converges, then by limit law,

$$x = \frac{3}{2}x(1 - x) \text{ where } x = \lim(x_n) \Rightarrow x = 0 \text{ or } 3$$

then, by proof of contradiction, it is not convergent.

Proof. **_Claim_**: $\frac{1}{3} < x_{n+1} < x_n \leq \frac{1}{2}, \forall n \in \mathbb{N} \cup \{0\}$

Proof of the claim by induction:

When n = 0:
$$x_0 = \frac{1}{2}, \ x_1 = \frac{3}{2} \cdot \frac{1}{2}(1 - \frac{1}{2}) = \frac{3}{8}$$
$$\frac{1}{3} < \frac{3}{8} < \frac{1}{2} \leq \frac{1}{2}$$

Suppose this is true for n=k:
$$\frac{1}{3} < x_{k+1} < x_k \leq \frac{1}{2}$$

Goal:

$$\frac{1}{3} < x_{k+2} < x_{k+1} \leq \frac{1}{2}$$

$$x_{k+1} = \frac{3}{2}x_k(1 - x_k)$$

$$\frac{1}{3} < x_k \leq \frac{1}{2} \Rightarrow \frac{2}{3} > 1 - x_k \geq \frac{1}{2}$$

$$x_{k+1} < \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{2}$$

Complete the square:

$$x_{k+1} - \frac{1}{3} = \frac{3}{2}x_k(1 - x_k) - \frac{1}{3}$$
$$= -\frac{3}{2}[(x_k - \frac{1}{2})^2 - \frac{1}{36}]$$

So

$$\frac{1}{3} < x_k \leq \frac{1}{2} \Rightarrow \left| x_k - \frac{1}{2} \right| < \frac{1}{6}$$
$$\Rightarrow (x_k - \frac{1}{2})^2 < \frac{1}{36}$$
$$\Rightarrow x_{k+1} - \frac{1}{3} > 0$$
$$\Rightarrow \frac{1}{3} < x_{k+1} \leq \frac{1}{2}$$

With the similar process, we can derive that

$$\frac{1}{3} < x_{k+2} \leq \frac{1}{2}$$

$$x_{k+2} = \frac{3}{2}x_{k+1}(1 - x_{k+1}) < \frac{3}{2}x_{k+1} \cdot \frac{3}{2} = x_{k+1}$$

Therefore, this claim is also true for n=k+1:

$$\frac{1}{3} < x_{k+2} < x_{k+1} \leq \frac{1}{2}$$

We thereby prove the theorem by induction:

$$\frac{1}{3} < x_{n+1} < x_n \leq \frac{1}{2}$$

∎

Exercise: Textbook p.75: A sequence that converges to $\sqrt{a}$ for $a > 0$.

Definition (**Euler's Number**).

$$e = \lim(1 + (\frac{1}{n})^n)$$

Goal: (x_n) is convergent where $x_n = (1 + \frac{1}{n})^n$

$$
\begin{aligned}
x_n = (1 + \frac{1}{n})^n &= 1 + nC1 \cdot \frac{1}{n} + nC2 \cdot \frac{1}{n^2} + \cdots + nCn\frac{1}{n^n} \\
&= 1 + \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{2} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1) \cdot 3 \cdot 2 \cdot 1}{n!} \cdot \frac{1}{n^2} \\
&= 1 + 1 + \frac{1}{2}(1 - \frac{1}{n}) + \frac{1}{6}(1 - n)(1 - \frac{2}{n}) + \cdots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \ldots \frac{2}{n} \cdot \frac{1}{n}
\end{aligned}
$$

Write x_{n+1} in a similar way, we observe that

$$x_n < x_{n+1}$$

Facts $2^{m-1} \leq m!$ for $m \in \mathbb{N} \Rightarrow \frac{1}{m!} \leq \frac{1}{2^{m-1}}$

$$x_n < 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 1 + \frac{1(1 - (\frac{1}{2})^n)}{1 - \frac{1}{2}} < 3$$

$\Rightarrow (x_n)$ is increasing and bounded

3.4   Subsequence and the Bolzano-Weierstrass Theorem

Example.

$$(x_n) = ((-1)^n)$$
$$x_{2n} = (-1)^{2n}$$
$$x_{2n+1} = (-1)^{2n-1}$$

So $a_n = x_{2n}$ is a sequence, while $b_n = x_{2n+1}$ is a subsequence of x_n.

Definition (Subsequences). Let (x_n) be a real sequence and consider a strickly increasing seqence of natural numbers $n_1 < n_2 < n_3 < \ldots$ The sequence

$$(x_{n_k} \; : \; k \in \mathbb{N})$$

is called a ***subsequence*** of (x_n)

Example. Any tails of a sequence is a subsequence: (x_n) n-th tail: (x_{m+k}), where $n = m + k, k \in \mathbb{N}$, is a subsequence.

Theorem. Suppose (x_n) converges to x. Then $x_{x_k} \to x$ for any subsequence of (x_n).

Proof. Let $\varepsilon > 0$, then

$$\exists N_\varepsilon > 0 \text{ s.t. } |x_n - x| < \varepsilon \text{ for } n > N_\varepsilon.$$

Note that

$$n_k \geq k, \; \forall k \in \mathbb{N}.$$

Exercise. By induction, $n_1 \geq 1, n_2 \geq n_1 \geq 1 \Rightarrow n_2 \geq 2$

When $k > N_\varepsilon$, $n_k > N_\varepsilon$, thus

$$|x_{n_k} - x| < \varepsilon$$

Therefore

$$(x_{n_k}) \to x$$

$\blacksquare$

Theorem. Let (x_n) be a sequence of real numbers, and let $x \in \mathbb{R}$. Then the following are equivalent:

1. (x_n) does not converge to x.

2. $\exists \varepsilon_0 > 0$, s.t. $\forall k \in \mathbb{N}, \exists n_k \in \mathbb{N}$ s.t.

$$n_k \geq k \; \& \; |x_{n_k} - x| > \varepsilon_0$$

3. $\exists \varepsilon_0 > 0$ and a subsequence (x_{n_k}) s.t.

$$|x_{n_k} - x| > \varepsilon_0, \; \forall k \in \mathbb{N}$$

Example.

$$x_n = (-1)^n \Rightarrow (x_n) \text{ does not converge to } 1$$

Proof.

> $3 \to 1$: by the contrapositive statement of definition
>
> $3 \to 2$: 3 is a stronger statement of 2
>
> $1 \to 2$: left as exercise

∎

Theorem. If (x_n) satisfies either of the following property, then it is ***divergent***:

1. There exists two subsequence (x_{n_k}) & (x_{m_k}) whose limits are NOT equal.

2. (x_n) is unbounded.

Example.

1. $(-1)^n$

2. (n)

3. (x_n) such that

$$x_{2k} = k$$
$$x_{2k+1} = (-1)^k$$

Proof. exercise

∎

Theorem (**Bolzano-Weierstrass Theorem**). A bounded sequence of real numbers has a ***convergent subsequence***.
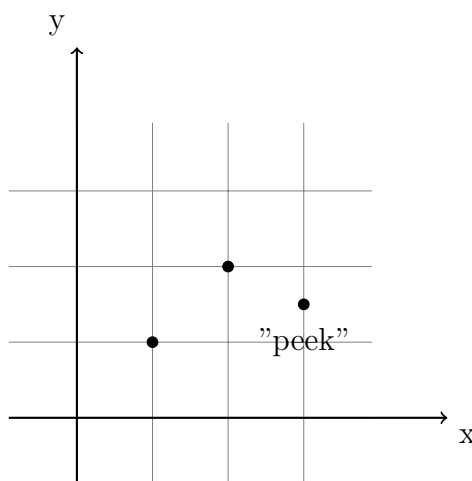
Example.

$$x_n = (-1)^n$$

Proof.

Lemma. If (x_n) is a sequence of real numbers, there exists a subsequence of (x_n) which is monotone.

proof of lemma:
Call the m-th term x_m a *"peek"* if x_m is at least as large as any term after it in the sequence.

Case 1: (x_n) has infinitelyy many peaks

List the peaks of (x_n) in order of increasing index

$$x_{n_1}, x_{n_2}, \ldots$$

$\Rightarrow (x_{n_i})$ is a decreasing sequence.

Case 2: (x_n) has a finite number of peaks

Let s_1 be the first index after the last peak of (x_n). Then for every $n \geq s_1$, $\exists m \in \mathbb{N}$ such that $x_m > x_n$.

Choose

$$s_2 \geq s_1 \text{ such that } x_{s_2} > x_{s_1}, \ s_2 \geq s_1 \text{ such that } x_{s_2} > x_{s_1}, \ldots$$

$$\Rightarrow (x_{s_i}) \text{ is an increasing sequence.}$$

Remark. Lemma + monotone convergent theorem implies Bolzano-Weierstrass theorem.

Second proof

Suppose (x_n) is a bounded sequence.

$$\Rightarrow \exists I_1 = [a_1, b_1] \text{ such that } (x_n) \in I_1$$

Consider

$$I_2' = [a_1, \frac{a_1 + b_1}{2}], \ I_2'' = [\frac{a_1 + b_1}{2}, b_1]$$

Let $I_2 = [a_2, b_2]$ be one of I_2', I_2'' such that I_2 contains infinitely many terms of (x_n).

For $n \in \mathbb{N}$, define $I_n = [a_n, b_n]$ in a similar way.

For $i \in \mathbb{N}$, choose a term x_{n_i} such that $x_{n_i} \in I_i$ and $n_i > n_{i-1}$

Then

$$
\begin{array}{ll}
i = 1, & n_i = 1 \\
i = 2, & \text{choose } n_2 \in \mathbb{N} \text{ such that } n_2 > n_1 \ \& \ x_{n_2} \in I_2 \\
\vdots & \vdots
\end{array}
$$

- $\forall i \in \mathbb{N}$, $a_i \leq x_{n_i} \leq b_i$
- (a_i) increases, bounded above by $b_1 \Rightarrow (a_i) \to \sup(a_i)$
- (b_i) decreases, bounded below by $a_1 \Rightarrow (b_i) \to \inf(b_i)$
- $\inf |a_i - b_i| = \inf \frac{b_1 - a_1}{2^n} = n \Rightarrow \sup(a_i) = \inf(b_i)$

Thus (x_{n_i}) is convergent by squeeze theorem. $\blacksquare$

Theorem. Let (x_n) be a bounded sequence, and $x \in \mathbb{R}$ has the property that every convergent subsequence of (x_n) converges to x. Then x_n converges to x

Proof. Let $\forall \varepsilon > 0$

By Bolzano-Weierstrass theorem, $\exists$ a convergent subsequence (x_{n_i}) such that

$$\exists N_\varepsilon \in \mathbb{N} \text{ s.t. for } i > N_\varepsilon, |x_{n_i} - x| < \varepsilon$$

Assume (x_n) does not converge to x. Then by previous theorem of subsequence,

$$\exists \varepsilon_0 > 0 \text{ and a subsequence } (x_{n_k}) \text{ s.t. } |x_{n_k} - x| > \varepsilon_0, \forall k \in \mathbb{N}$$

Since (x_n) is bounded, (x_{n_k}) is also bounded. Thus there exists a convergent subsequence of (x_{n_k}) as $(x_{n_{k_i}})$.

Note that $(x_{n_{k_i}})$ is a convergent subsequence of (x_n), thus

$$(x_{n_{k_i}}) \to x \text{ which is contradiction to previous assumption}$$

$$\blacksquare$$

Definition. Let (x_n) be a sequence of real numbers. A point is called a ***subsequential limit*** of (x_n) if it is the limit of a subsequence of (x_n).

$$S = \{\alpha \in \mathbb{R} : x \text{ is a subsequential limit}\} \text{ NOTE: may be infinite set}$$

Definition. Let (x_n) be a sequence of real numbers.

- The ***limit superior*** of (x_n) is the infimum of the set of $v \in \mathbb{R}$ s.t. $v < x_n$ for at most a finite number of $n \in \mathbb{N}$. We write it as

$$\limsup(x_n) \text{ or } \limsup x_n \text{ or } \overline{\lim} x_n$$

- The ***limit inferior*** of (x_n) is the supremum of the set of $v \in \mathbb{R}$ s.t. $w > x_m$ for at most a finite number of $n \in \mathbb{N}$. We write it as

$$\liminf(x_n) \text{ or } \liminf x_n \text{ or } \overline{\lim} x_n$$

Intuition

- Suppose $v < x_n$ for at most finitely many $n \in \mathbb{N}$, then

Theorem. Let (x_n) be a bounded sequence, and $x \in \mathbb{R}$ has the property that every convergent subsequence of (x_n) converges to x. Then x_n converges to x

Proof. Let $\forall \varepsilon > 0$

By Bolzano-Weierstrass theorem, $\exists$ a convergent subsequence (x_{n_i}) such that

$$\exists N_\varepsilon \in \mathbb{N} \text{ s.t. for } i > N_\varepsilon, |x_{n_i} - x| < \varepsilon$$

Assume (x_n) does not converge to x. Then by previous theorem of subsequence,

$$\exists \varepsilon_0 > 0 \text{ and a subsequence } (x_{n_k}) \text{ s.t. } |x_{n_k} - x| > \varepsilon_0, \forall k \in \mathbb{N}$$

Since (x_n) is bounded, (x_{n_k}) is also bounded. Thus there exists a convergent subsequence of (x_{n_k}) as $(x_{n_{k_i}})$.

Note that $(x_{n_{k_i}})$ is a convergent subsequence of (x_n), thus

$$(x_{n_{k_i}}) \to x \text{ which is contradiction to previous assumption}$$

$\blacksquare$

Definition. Let (x_n) be a sequence of real numbers. A point is called a **_subsequential limit_** of (x_n) if it is the limit of a subsequence of (x_n).

$$S = \{\alpha \in \mathbb{R} : x \text{ is a subsequential limit}\} \text{ (NOTE: may be infinite set)}$$

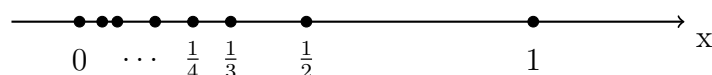**Example.** Consider $(x_n) = \{(-1)^n | n \in \mathbb{N}\}$. Then

$$S \supseteq \{1, -1\}$$

Definition (lim sup and lim inf). Let (x_n) be a sequence of real numbers.

- The **_limit superior_** of (x_n) is the infimum of the set of $v \in \mathbb{R}$ s.t. $v < x_n$ for at most a finite number of $n \in \mathbb{N}$. We write it as

  $$\limsup(x_n) = \limsup x_n = \overline{\lim} x_n = \inf \{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of n}\}$$

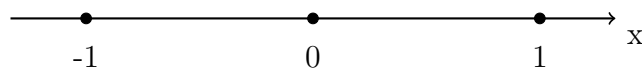 **Example.** Consider $(x_n) = \frac{1}{n}$

  Let

  $$X = \{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of n}\}$$

  - $-1 \notin X$ because there are infinitely many x_n such that $v < x_n$.

  - $\frac{1}{2} \in X$ because there are finitely many x_n such that $v < x_n$.

  - $2 \in X$ because there is no x_n such that $v < x_n$, which is smaller than finite and thereby satisfies the definition.

  We thus conclude that

  $$(0, \infty) \subset X \text{ and } \limsup x_n = \inf X$$

  Example. Consider $(x_n) = (-1)^n$:

  - $1 \in X$ because there is no x_n such that $v < x_n$.

  - $2 \in X$ because there is no x_n such that $v < x_n$.

  - $0, -1 \notin X$ because there are infinitely many x_n such that $v < x_n$.

  We thus conclude that

  $$[1, \infty) \subset X \text{ (in fact they are equal)}$$

- The **_limit inferior_** of (x_n) is the supremum of the set of $w \in \mathbb{R}$ s.t. $w > x_m$ for at most a finite number of $n \in \mathbb{N}$. We write it as

  $$\liminf(x_n) \text{ or } \liminf x_n \text{ or } \underline{\lim} x_n = \sup \{w \in \mathbb{R} | w > x_n \text{ for at most a finite number of n}\}$$

Intuition

- Suppose $v < x_n$ for at most finitely many $n \in \mathbb{N}$, then for all large n, $v \geq x_n$.
  $\Rightarrow$ No subsequential limit of (x_n) can possibly exceed v.

- Similar observation for $\underline{\lim} \, x_n$

Theorem. Let (x_n) be a bounded sequence of real numbers, and let $x^* \in \mathbb{R}$. Then TFAE:

1. $x^* = \lim\sup(x_n)$

2. If $\varepsilon > 0$, there are at most a finite number of $n \in \mathbb{N}$ s.t. $x^* + \varepsilon < x_n$, but infinitely many n for which $x^* - \varepsilon < x_n$

3. If $u_m = \sup\{x_n | n \geq m\}$(sup of (m-1)-th tail), then $x* = \inf\{u_m | m \in \mathbb{N}\} = \lim u_m$

4. If S is the set of subsequential limits of x_n, then $x^* = \sup S$.

Remark. .

- u_m is decreasing.

- There is a similar such list of equivalent properties for $\lim\inf$.

Corollary. A bounded sequence (x_n) is convergent iff $\overline{\lim} x_n = \underline{\lim} \, x_n$

Proof. A direct result of the theorem:

$$\overline{\lim} x_n = \sup S \text{ and } \underline{\lim} x_n = \inf S$$

$\blacksquare$

Proof of thm. (a) $\Rightarrow$ (b). Let $\varepsilon > 0$. Then

$$x^* + \varepsilon > x^* = X = \inf\{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of n.}\}$$

$$\Rightarrow \exists v \in \mathbb{R} \text{ s.t. } x^* \leq v < x^* + \varepsilon$$

and there are only finitely mant n with $v < x_n$.

For any n for which $x^* + \varepsilon < x_n$, $v < x_n$. Thus there are only finitely many such n.

If $x^* - \varepsilon \notin X$, then there are infinityly many n such that $x^* - \varepsilon < x_n$ $\blacksquare$

Proof of thm. (b) $\Rightarrow$ (c). Fix $\varepsilon > 0$.

By (b), there are only finitely many n with $x^* + \varepsilon < x$.

Take $N \in \mathbb{N}$ large enough such that

$$
\begin{aligned}
& x^* + \varepsilon \geq x_n && \forall n \geq N \\
\Rightarrow \quad & x^* + \varepsilon \geq u_N \\
\Rightarrow \quad & x^* + \varepsilon \geq \lim u_n \\
\Rightarrow \quad & x^* \geq \lim u_m && \forall n \geq N
\end{aligned}
$$

On the other hand, there are infinitely many n with $x^* - \varepsilon < x_n \leq u_n$.

Thus, there exists a subsequence of u_n, say u_{n_k}, satisfies

$$
\begin{aligned}
& x^* - \varepsilon \leq u_{n_k} \\
\Rightarrow \quad & x^* - \varepsilon \leq \lim u_{n_k} = \lim (u_n) \\
\Rightarrow \quad & x* \leq \lim (u_n)
\end{aligned}
$$

$\blacksquare$

Proof of thm. (c) $\Rightarrow$ (d). **_Goal_**:

$$
x* = \lim (u_m), u_m = \sup x_n | n \geq m
$$

$$
\Rightarrow x^* = \sup S \text{ where S is the set of subsequential limits.}
$$

Let (x_{n_k}) be a convergent subsequence of (x_n). Notice that $\lim (x_{n_k}) \in S$.

$$
\begin{aligned}
& && n \geq k \\
\Rightarrow \quad & x_{n_k} \leq \sup \{x_n | n \geq k\} = u_k \\
\Rightarrow \quad & \lim (x_{n_k}) \leq \lim (u_k) = x^* \\
\Rightarrow \quad & x^* \text{ is an upper bound of S..}
\end{aligned}
$$

For 1, $\exists n_1 \in \mathbb{N}$ s.t.
$$
u_1 - 1 \leq x_{n_1} \leq u_1
$$

For $\frac{1}{2}$, $\exists n_2 \in \mathbb{N}$ s.t.
$$
u_2 - \frac{1}{2} \leq x_{n_2} \leq u_2
$$
$$
\ldots
$$

For $\frac{1}{k}$, $\exists n_k \in \mathbb{N}$ s.t.
$$
u_k - \frac{1}{k} \leq x_{n_k} \leq u_k
$$

When $k \to \infty$,
$$
x^* - 0 \leq \lim (x_{n_k}) \leq x^*
$$

By squeeze theorem,
$$
\lim (x_{n_k}) = x^*
$$

$\blacksquare$

*Proof of thm. (d) $\Rightarrow$ (a). **Goal:***

$$x^* = \sup S$$

$$\Rightarrow x^* = \limsup x_n = \inf \{v \in \mathbb{R} | v < x_n \text{for at most finite many of n}\}$$

Fix $\varepsilon > 0$

There is no subsequence of x_n which has a limit exceeding $x^* + \varepsilon$.

$\Rightarrow$ There is only finitely many n with $x_n > x^* + \varepsilon$.

$\Rightarrow$ $$x^* + \varepsilon \in X$$

$\Rightarrow$ $$\inf X \leq x^* + \varepsilon$$

$\Rightarrow$ $$\limsup x_n \leq x^* + \varepsilon$$

$\Rightarrow$ $$\limsup x_n \leq x^*$$

Next, consider $x^* - \varepsilon$

Then, there exists a subsequential limit of x_n which is greater or equal to $x^* - \frac{1}{2}\varepsilon$.

There exists a convergent subsequence of (x_n), say (x_{n_k}), such that

$$\lim (x_{n_k}) \geq x^* - \frac{1}{2}\varepsilon$$

$\Rightarrow$ There are infinitely many n with $x^* > x^* - \varepsilon$.

$\Rightarrow$ $$\forall a \in X, x^* - \varepsilon \leq a$$

$\Rightarrow$ $$x^* - \varepsilon \leq \inf X$$

$\Rightarrow$ $$\limsup x_n \geq x^* - \varepsilon$$

$\Rightarrow$ $$\limsup x_n \geq x^*$$

In conclusion,

$$\limsup x_n = x^*$$

$\blacksquare$

3.5   Cauchy Criterion

Definition (**Cauchy Sequence**). A sequence (x_n) is ***Cauchy sequence*** if

$\forall \varepsilon > 0, \exists H \in \mathbb{N}$ s.t. $\forall n, m \in \mathbb{N}, n > 0, m > 0$,

$$|x_n - x_m| < \varepsilon$$

Example. $\left(\frac{1}{n}\right)$ is a Cauchy sequence.

Proof. Observe that $\forall n, m \in \mathbb{N}, n \geq m$,

$$\left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{m - n}{mn}\right| = \frac{n - m}{mn} \leq \frac{n}{mn} < \frac{1}{m}$$

Choose $H = \lceil \frac{1}{\varepsilon} \rceil + 1$, Then

$\forall n, m \geq H$,

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{m} < \varepsilon$$

∎

Example. $(-1)^n$ is not a Cauchy sequence.

Proof. Choose $\varepsilon_0 = \frac{1}{2}, \forall H \in \mathbb{N}$, choose $n, m \geq H$ s.t. n is even, m is odd. Then

$$|(-1)^n - (-1)^m| = |1 - (-1)| = 2 > \frac{1}{2}$$

Thus $(-1)^n$ does not satisfies the definition of Cauchy sequence.

∎

Theorem. If (x_n) is convergent, then it is Cauchy sequence.

Proof. Let $\lim (x_n) = x$.

Goal:
$$|x_n - x| < \varepsilon \Rightarrow |x_n - x_m| < \varepsilon$$

$\forall n, m \geq N_\varepsilon$,

$$
\begin{aligned}
|x_n - x_m| &= |(x_n - x) + (x - x_m)| \\
&\leq |x_n - x| + |x - x_m| \text{ by triangle inequality} \\
&< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon
\end{aligned}
$$

∎

Lemma. If (x_n) is Cauchy, the it is bounded.

Proof. Choose $\varepsilon = 1, \exists H \geq 0$ s.t. $\forall n, m \geq H$,

$$|x_n - x_m| < 1$$

Since the choice of m satisfies $m \geq H$, we may choose $m = H$ s.t.

$$|x_n - x_H| < 1$$

It follows that

$$|x_n| - |x_H| \leq |x_n - x_H| < 1$$
$$|x_n| < |x_H| + 1, \forall n \geq H$$

Let

$$M := \max\left\{|x_1|, |x_2|, |x_3|, \ldots, |x_{H-1}|, |x_H| + 1\right\}$$

Thus $\forall n \in \mathbb{N}, |x_n| \leq M$ ■

Theorem (**Cauchy Convergence Theorem**). A sequence of real numbers is **_convergent_** if and only if it is **_Cauchy_** sequence.

($\Rightarrow$). Done in the previous theorem.   ■

($\Leftarrow$). Suppose (x_n) is Cauchy. By lemma, it is bounded.

By Bolzano-Weierstrass theorem, there exists a convergent subsequence (x_{n_k}).

Let $\lim(x_{n_k}) = x$.

Goal: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$

$$|x_n - x| < \varepsilon$$

We will use the trick of **_insert subsequence:_**

By definition of convergence, $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall k \geq N_\varepsilon$

$$|x_{n_k} - x| < \frac{1}{2}\varepsilon$$

By definition of Cauchy sequence, $\forall \varepsilon > 0, \exists H \in \mathbb{N}, H \geq 0$ s.t. $\forall n, m \geq H$

$$|x_n - x_m| < \frac{1}{2}\varepsilon$$

Thus $\forall k \geq \max\{H, N_\varepsilon\}$,

$$
\begin{aligned}
|x_k - x| &= |(x_k - x_{n_k}) + (x_{n_k} - x)| \\
&\leq |x_k - x_{n_K}| + |x_{n_k} - x| \text{ since } n_k \geq k \geq \max\{H, N_\varepsilon\} \\
&< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon
\end{aligned}
$$

     ■

Example. Let (h_n) be the sequence of harmonic series such that

$$h_n = \sum_{i=1}^{n} \frac{1}{i} \text{ , and } \lim(h_n) = \sum_{n=1}^{\infty} \frac{1}{n}$$

Claim: (h_n) is divergent

Goal: show that it is NOT Cauchy.

Proof. $\forall m, n \in \mathbb{N}$, WLOG suppose $m \geq n, h_m > h_n$, we have

$$|h_m - h_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m-1} + \frac{1}{m}$$
$$\geq \frac{m-n}{n}$$

since there are (m-n)'s terms on the right side of the equation.

So we choose $\varepsilon_0 = \frac{1}{2}, \forall H \geq 0,$ choose $n = H, m = 2H$, then

$$|h_m - h_n| \geq \frac{m-n}{m} = \frac{1}{2}$$

Thus, (h_n) is NOT Cauchy.   ∎

3.6   Property of Divergent Sequence

Example (divergent sequence). :

- $(n) \to \infty$

- $(-n) \to -\infty$

- $(-1)^n \cdot n$ is divergent and unbounded.

- $(-1)^n$ is divergent and bounded.

Definition (**Properly Divergent**). Let (x_n) be a sequence of real numbers. We say that

1. (x_n) ***tends to*** ∞, or $\lim(x_n) = \infty$ if $\forall a \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$x_n > a$$

2. Similarly, (x_n) ***tends to*** $-\infty$, or $\lim(x_n) = -\infty$ if $\forall a \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$x_n < a$$

In either case, we say that (x_n) is ***properly divergent***.

Example. Let $C > 0$. (C^n) is properly divergent. We write $\lim(C^n) = \infty$.

Proof. Notice that

$$C^n = [1 + (C - 1)]^n \geq 1 + n(C - 1) \text{ by Bernoulli's inequality}$$

Goal: $\forall a \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$1 + n(C - 1) > a_n \Leftrightarrow n > \frac{a_n - 1}{C - 1}$$

Then we choose $N = \lceil \frac{a-1}{c-1} \rceil + 1$. Thus $\forall n \geq N, C^n > a$. ∎

Theorem. A monotone sequence is ***divergent*** if and only if it is ***bounded***.

Proof. Exercise. ∎

Theorem (**Comparison Test**). Let (x_n) and (y_n) be two sequences. Suppose $\forall n \in \mathbb{N}, x_n \geq y_n$. Then,

1. If $(x_n) \to \infty$, then $(y_n) \to \infty$.

2. If $(y_n) \to -\infty$, then $(x_n) \to -\infty$.

Proof. Exercise. ∎

Theorem (**Limit Comparison Test**). Let (x_n) and (y_n) be two sequences of positive real numbers. Suppose $\exists L \in \mathbb{R}, L > 0$ s.t.

$$\exists \lim(\frac{x_n}{y_n}) = L$$

then $\lim(x_n) = \infty$ if and only if $\lim(y_n) = \infty$

Proof. **<u>Claim:</u>** For N large enough,

$$\frac{1}{2}L \cdot y_n < x_n < 2L \cdot y_n$$

<u>Goal:</u> Claim+Comparison Theorem=Proof.

By definition of limit, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N$,

$$L - \varepsilon < \frac{x_n}{y_n} < L + \varepsilon$$

Choose $\varepsilon = \frac{1}{2}L$, we have

$$\frac{1}{2}L \cdot y_n < x_n < \frac{3}{2}L < 2L \cdot y_n$$

since $L > 0$ and these are positive sequences.

Thus by Comparison Theorem,

$$\lim(\frac{1}{2}L \cdot y_n) = \infty \Rightarrow \lim(x_n) = \infty$$

$\blacksquare$

3.7   Introduction to Infinite Series

Definition (**Infinite Series**). If (x_n) is a sequence of real numbers, then the ***infinite series*** generated by (x_n) is the sequence (s_k) defined by

$$s_k = \sum_{i=1}^{k} x_i$$

Terms s_k are called ***partial sums***.

Notation. $\sum x_i$ to mean this series or its limit at infinity $\lim(x_k)$.

Theorem (**Cauchy Criterion for Series**). The series $\sum x_i$ ***converges*** if and only if $\forall \varepsilon > 0, \exists M \in \mathbb{N}$ s.t. $\forall n, m \in \mathbb{N}, n > m \geq M$,

$$|x_{m+1} + x_{m+2} + \cdots + x_{n-1} + x_n| < \varepsilon$$

Or we write it as

$$|s_k - s_m| < \varepsilon$$

Proof: Exercise.   ∎

Theorem (Montone Convergence for Series). Let (x_n) be a sequence of non-negative real numbers. Then the series $\sum x_n$ ***converges*** if and only if (s_k) is bounded.

Proof. Exercise.

                                                                                                              ∎

Example. $\sum \frac{1}{n^2}$ is convergent.

Proof. ***Goal:*** find convergent subsequence (s_{k_j}).

Consider subsequence (s_{k_j}) where $k_j = 2^j - 1$.

Observe that:

$$s_{k_1} = 1$$
$$s_{k_2} = 1 + \frac{1}{2^2} + \frac{1}{3^2}$$
$$< s_{k_1} + 2 \cdot \frac{1}{2^2} = 1 + \frac{1}{2}$$
$$s_{k_3} = 1 + (\frac{1}{2^2} + \frac{1}{3^2}) + (\frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{9^2})$$
$$< s_{k_1} + 2 \cdot \frac{1}{2^2} = 1 + \frac{1}{2}$$

By induction(details are left as exercise), one can show that

$$s_{k_j} < \sum_{n=0}^{j-1} \frac{1}{2^n} < \sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \text{ (by limit of geometric series.)}$$

Thus (s_{k_j}) is bounded.

By theorem proved in homework, an increasing sequence with bounded(and thus convergent) subsequence implies that the sequence is convergent.   ∎

4 Limits

4.1 Limits of Functions

Let $f : A \to B$ be a function where $A, B \subseteq \mathbb{R}$. Let $a \in A, L \in B$.

Goal: Define

$$\lim_{x \to a} f(x) = ?$$

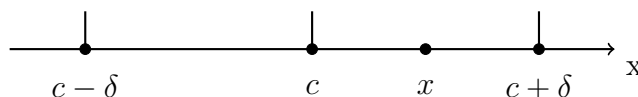***Intuition:*** Define **closeness** on real line.

Definition (**Cluster Point**). Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a ***cluster point*** of A if

$\forall \delta > 0, \exists x \in A, x \neq c$ such that

$$|x - c| < \delta$$

Or

$$V_\delta(c) \cap (A \, \{c\}) \neq \phi$$

Theorem. $c \in \mathbb{R}$ is a cluster point if and only if

there exists a sequence $(a_n) \in A$ such that

$$\lim(a_n) = c \text{ and } \forall n \in \mathbb{N}, a_n \neq c$$

Sketch of Proof. $(\Rightarrow)$

$$\delta = 1, \exists a_1 \in A \backslash \{c\} \text{ s.t. } |a_1 - c| < 1$$
$$\delta = \frac{1}{2}, \exists a_2 \in A \backslash \{c\} \text{ s.t. } |a_2 - c| < \frac{1}{2}$$
$$\dots \text{ observe that:}$$
$$\forall \delta > 0, \exists a_n \in A \backslash \{c\} \text{ s.t. } |a_n - c| < \frac{1}{n} < \delta$$
$$\Rightarrow \lim(a_n) = c \text{ by squeeze theorem.}$$

$(\Leftarrow)$

$\forall \delta > 0, \exists N_\varepsilon > 0$ s.t.

$$|a_{N_\varepsilon} - c| < \delta$$

Note that $a_{N_\varepsilon} \in A \backslash \{c\}$ where c is a cluster point of A. ∎

Example. Let X be the set of cluster points of A.

- $A = (1, 2) \cup (3, 4) \Rightarrow X = [1, 2] \cup [3, 4]$. *Proof: Exercise.*

- $A = \{0\} \cup (1, 2) \Rightarrow X = \phi$.

  Sketch of (2). :

    1. Let $x \in [1, 2]$. Prove that $x \in X$. Thus $[1, 2] \in X$.

    2. Prove that $0 \notin X$.

    3. Prove that $x \in X$ if $x \notin \{0\} \cup [1, 2]$.

$\blacksquare$

Remark. A may not be a subset of the set of cluster points of A.
Example. :

- $A = \mathbb{Z} \Rightarrow X = \phi$.

- $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} \Rightarrow X = \{0\}$ *Proof: Exercise.*

Definition (**Delta-Epsilon Definition of Limit**). Let $A \subseteq \mathbb{R}$, c is a cluste point of A, $f : A \to \mathbb{R}$. A real number L is the ***limit of f at c*** if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$,

$$0 < |x - c| < \delta \to |f(x) - L| < \varepsilon$$

Theorem (**Uniqueness of Limit**). If $f : A \to \mathbb{R}$ and c is a cluster of points of A, then f has at most 1 limit at c.

Proof. We will prove this by contradiction.

Let L_1 and L_2 be limits of f at c. Assume $L_1 \neq L_2$. Choose $\varepsilon = \frac{|L_1 - L_2|}{2} > 0$. Then,

$$\exists \delta_1 \text{ s.t. } 0 < |x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \frac{\varepsilon}{2}$$

$$\exists \delta_2 \text{ s.t. } 0 < |x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \frac{\varepsilon}{2}$$

Consider $\delta := \min\{\delta_1, \delta_2\}$.

Since c is a cluster point, $\exists x_0 \in A$ s.t.

$$0 < |x_0 - c| < \delta$$

Since

$$|f(x_0) - L_1| < \frac{\varepsilon}{2}, |f(x_0) - L_2| < \frac{\varepsilon}{2}$$

We have

$$|L_1 - L_2| \leq |L_1 - f(x_0)| + |f(x_0) - L_2|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
$$= \frac{|L_1 - L_2|}{2}$$

which is a contradiction.

$\blacksquare$

Notation.
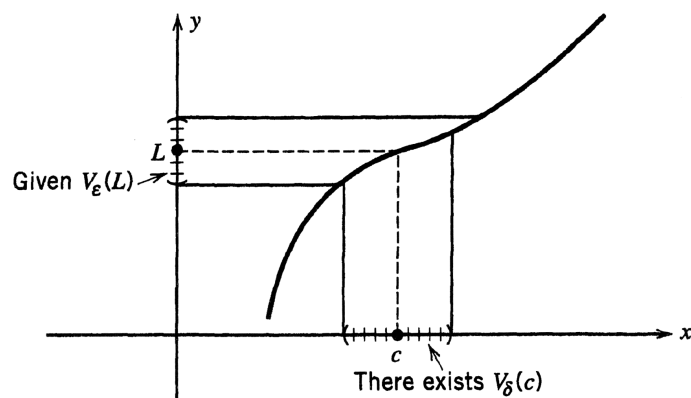$$L = \lim_{x \to c} f(x) \text{ or } L = \lim_{x \to c} f$$

And we say that *f(x) approaches to L as x approaches to c.*

Remark (**Divergence of function**). If the limit of $f(x)$ at c does not exists, we say that f **_diverges_** at c.

Theorem. Let $f : A \to \mathbb{R}$ and c be a cluster point of f. The following are equivalent:

1. $\lim_{x \to c} f(x) = L$

2. $\forall V_\varepsilon(L) \, \varepsilon$-neighborhood of L, $\exists V_\delta(c) \, \delta$-neighborhood of c s.t.

$$x \in V_\delta(c) \cap (A \backslash \{c\}) \Rightarrow f(x) \in V_\varepsilon(c)$$

Example. $\lim_{x \to c} f(x) = a$.

Proof. $\forall \varepsilon > 0, \exists \delta = 1$ s.t.

$$x \in V_\delta(c) \cap (\mathbb{R} \backslash \{c\}) \Rightarrow f(x) = a \in V_\varepsilon(L)$$

∎

Example. $\lim_{x \to c} f(x) = c$.

Proof. $\forall \varepsilon > 0, \exists \delta = \varepsilon$ s.t.

$$x \in V_\delta(c) \cap (A \backslash \{c\})$$
$$\Rightarrow f(x) = x \in V_\delta(c) \cap (A \backslash \{c\})$$
$$\Rightarrow f(x) \in V_\varepsilon(c) \cap (A \backslash \{c\}) \subseteq V_\varepsilon(c)$$

∎

Example.
$$\lim_{x \to c} x^2 = c^2$$

Proof. **_Goal_**: $\forall \varepsilon > 0$, find a $\delta(c, \varepsilon) > 0$ s.t.

$$\text{if } 0 < |x - c| < \delta, \text{ then } \left|x^2 - c^2\right| < \varepsilon$$

which is equivalent to show
$$|x + c| \, |x - c| < \varepsilon$$

Since the choice of δ is dependent on ε and c only, $|x - c|$ can be easily confined with some constant. Let's assume that
$$|x - c| < 1$$

Now, the only task left is to find a way to confine $|x + c|$ with a constant by manipulating $|x - c| < 1$.

Here we will apply a common trick that estimates addition $|x + c|$ with subtraction $|x - c|$:

$$|x| - |c| \leq |x - c| < 1 \text{ by triangle inequality.}$$

Rearranging the inequality, we have:

$$|x| < |c| + 1$$

Adding the second $|c|$ to both side of inequality, then, we apply triangle inequality again:

$$|x + c| \leq |x| + |c| < 2\,|c| + 1$$

$$|x + c| < 2\,|c| + 1$$

Notice that this is equivalent to show

$$|x + c| \, |x - c| < (2\,|c| + \delta)\,|x - c| < \varepsilon$$

Rearrange the constant factor
$$|x - c| < \frac{\varepsilon}{2\,|c| + 1}$$

Thus, our choice of δ must satisfies two conditions at the same time:

$$\begin{cases} |x - c| < 1 \\ |x - c| < \frac{\varepsilon}{2|c|+1} \end{cases}$$

We achieve this by simply choosing

$$\delta = \min\left\{\frac{\varepsilon}{2\,|c| + 1}, 1\right\}$$

∎

Remark (**_The Toolbox of Proofs_**)**.** The readers should develop their "toolbox" of proof techniques. That is, the **_estimation against a constant_** + manipulation of **_triangle inequality_** + choice of δ that satisfies **_multiple conditions_**.

Example (Harder)**.**

$$\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$$

Proof. Observe that

$$\left| \frac{1}{x} - \frac{1}{c} \right| = \frac{|x - c|}{|cx|}$$

Here, we cannot choose $|x - c|$ smaller than some constant(why? try it on your own). Instead, we choose

$$|x - c| < \frac{|c|}{2}$$

By triangle inequality,

$$\frac{|c|}{2} < |x| < \frac{3|c|}{2}$$

Multiply $|c|$ to each term of the inequality,

$$\frac{c^2}{2} < |cx| < \frac{3c^2}{2}$$

Thus

$$\frac{1}{|cx|} < \frac{2}{c^2}$$

It follows that

$$\frac{|x - c|}{|cx|} < \frac{2}{c^2} |x - c|$$

In order to make the left term of the inequality less than ε, it suffices to confine

$$\frac{2}{c^2} |x - c| < \varepsilon$$

$$|x - c| < \frac{c^2}{2} \varepsilon$$

Thus, our choice of δ must satisfies two conditions at the same time:

$$\begin{cases} |x - c| < \frac{|c|}{2} \\ |x - c| < \frac{c^2}{2} \varepsilon \end{cases}$$

We achieve this by simply choosing

$$\delta = \min \left\{ \frac{|c|}{2}, \frac{c^2}{2} \varepsilon \right\}$$

■

Remark (***Factoring*** $|x - c|$)**.** Since we have the premice of $|x - c| < \delta$ for free, it would be much easier for us to confine $|f(x) - L|$ if we factor $|x - c|$ from the difference.

Example (Much Harder)**.**

$$\forall n \in \mathbb{N}, \lim_{x \to c} x^n = c^n$$

Proof. By difference of n-th powers factorization

$$\left| x^n - c^n \right| = |x - c| \left| \sum_{i=0}^{n-1} x^i c^{n-1-i} \right| \leq |x - c| \cdot \sum_{i=0}^{n-1} |x|^i \, |c|^{n-1-i}$$

If

$$|x - c| < 1$$

then by triangle inequality,

$$|x| < |c| + 1$$

It follows that

$$\left| x^n - c^n \right| < |x - c| \cdot \sum_{i=0}^{n-1} (|c| + 1)^i \left| c^{n-1-i} \right|$$

Similar to the previous example, it suffices to confine

$$|x - c| \cdot \sum_{i=0}^{n-1} (|c| + 1)^i \left| c^{n-1-i} \right| < \varepsilon$$

$$|x - c| < \frac{\varepsilon}{\sum_{i=0}^{n-1} (|c| + 1)^i \, |c^{n-1-i}|}$$

Thus, choose

$$\delta = \min\{1, \frac{\varepsilon}{\sum_{i=0}^{n-1} (|c| + 1)^i \, |c^{n-1-i}|}\}$$

$\blacksquare$

Remark (***Estimate x against c***)**.** This is another trick by triangle inequality:

If $\exists k \in \mathbb{R}, k > 0$ s.t. $|x - c| < k$, then $|x| < |c| + k$

Remark (***difference of n-th powers factorization***)**.**

$$(x^n - c^n) = (x - c)(x^{n-1} + cx^{n-2} + \cdots + c^{n-2}x + c^{n-1}) = (x - c) \sum_{i=0}^{n-1} x^i c^{n-1-i}$$

Example (***Some Tedious Factorization***).

$$\lim_{x \to 2} \frac{x^3 + 2x - 1}{6x^2 - 5} = \frac{11}{19}$$

Proof.

$$\left| \frac{x^3 + 2x - 1}{6x^2 - 5} - \frac{11}{19} \right| = \left| \frac{19x^3 - 66x^2 + 38x + 36}{19(6x^2 - 5)} \right|$$

By some tedious factorization,

$$... = |x - 2| \frac{|19x^3 - 28x - 18|}{19 |6x^2 - 5|}$$

We estimate $|x - 2|$ with constant 1

$$|x - 2| < 1$$

$$1 < x < 3 \text{ or } |x| < 3$$

It follows that

$$1 < x^2 < 9$$

$$1 < 6x^2 - 5 < 49$$

$$1 > \frac{1}{6x^2 - 5} > \frac{1}{49}$$

So

$$\frac{1}{19 |6x^2 - 5|} < \frac{1}{19}$$

Similarly,

$$\left| 19x^3 - 66x^2 + 38x + 36 \right| \le 19 |x|^2 + 28 |x| + 18 < 19 \cdot 3^2 + 28 \cdot 3 + 18 = 273$$

Thus

$$... \le |x - 2| \cdot \frac{273}{19} < \varepsilon$$

$$|x - 2| < \frac{19}{273} \varepsilon$$

We conclude that it suffices to choose

$$\delta = \min \left\{ 1, \frac{19}{273} \varepsilon \right\}$$

■

Theorem (_ Sequential Criterion of Limits**_).** Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. $\lim_{x \to c} f = L$ if and only if for all sequence $(x_n) \in A$ that converge to c and $(x_n) \neq c, \forall n \in \mathbb{N}$, $(f(x_n))$ converges to L.

Proof. ($\Rightarrow$). By definition of convergent sequence, $\forall \delta > 0, \exists K \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \geq K$,

$$|x_n - c| < \delta$$

By definition of limit, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$,

$$|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Thus, choose δ given ε, and choose K given δ, we have:

$$|x_n - c| < \delta \Rightarrow |f(x_n) - L| < \varepsilon$$

∎

($\Leftarrow$). We will prove this by contrapositive.

Assume that there exists $\varepsilon_0 > 0$ and a $(x_n) \in A$ converges to c with $(x_n) \neq c$ such that for all $n \in \mathbb{N}$

$$0 < |x_n - c| < \frac{1}{n} \Rightarrow |f(x_n) - L| \geq \varepsilon_0$$

Thus the function does not have a limit at c. We thereby conclude the converse of the statement.

∎

Theorem (_ Sequential Criterion of Divergence**_).** Let $A \subseteq \mathbb{R}, f : A \to \mathbb{R}, c$ be a cluster point of $A, (x_n) \to c$ s.t. $(x_n) \neq c$, and $L \in \mathbb{R}^1$.

1. L is NOT the limit of f at c $\iff$ $f(x_n)$ does NOT converge to L.

2. f diverges $\iff$ $f(x_n)$ does NOT converge.

Proof. Exercise.   ∎

Example.

$$\lim_{x \to 0} \frac{1}{x} \text{ does NOT exists in } \mathbb{R}$$
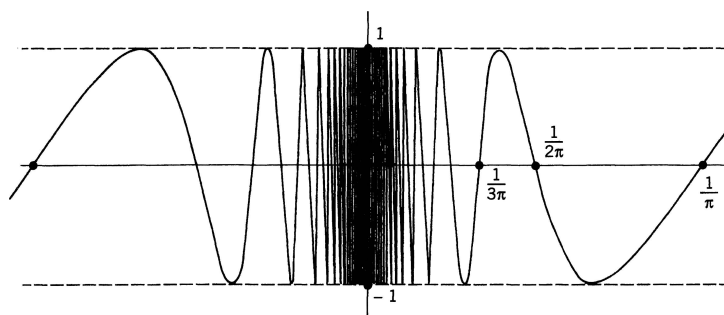
Proof. Let $(x_n) = \frac{1}{n} \to 0$. Then

$$f(x_n) = \frac{1}{\frac{1}{n}} = n \to \infty$$

∎

---

[1]The following theorems are NOT equivalent!!

Example.

$$\lim_{x \to 0} \sin(\frac{1}{x}) \text{ DNE in } \mathbb{R}$$

Proof. Let $(x_n) = \dfrac{1}{2n\pi} \to 0$ and $(y_n) = \dfrac{1}{\dfrac{1}{2}\pi + 2n\pi} \to 0$. Then

$$f(x_n) = \sin(2n\pi) = 0$$

$$f(y_n) = \sin(\frac{1}{2}\pi + 2n\pi) = 1$$

Thus limit of f at 0 does NOT exists in $\mathbb{R}$.

∎

Example.
Definition (_Signum Function_). Let $A \subseteq \mathbb{R}$ and $sgn(x) : A \to \mathbb{R}$ s.t.

$$sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$\lim_{x \to 0} sgn(x) \text{ DNE in } \mathbb{R}$$

Proof. Let $(x_n) = \dfrac{1}{n} \to 0$ and $(y_n) = \dfrac{1}{-n} \to 0$. Then

$$f(x_n) = sgn(n) = 1$$

$$f(y_n) = sgn(-n) = -1$$

Thus limit of f at 0 does NOT exists in $\mathbb{R}$.

∎

4.2   Limit Theorem

Definition (***Bounded Neighborhood of c***)**.** Let $A \subseteq \mathbb{R}, f : A \to \mathbb{R}, c$ be a cluster point of A. Then we say f is a ***bounded Neighborhood of c*** if there exists $\delta > 0$ and $\exists M > 0$ s.t. $\forall x \in A \cap V_\delta(c)$

$$|f(x)| \le M$$

Theorem (***Existence of Bounded Neighborhood at Limit***)**.** If f has a limit at c, then f is bounded on some neighborhood of c.

Proof. By definition of limit, $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A \backslash \{c\}$

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$$

By triangle inequality,

$$|f(x)| - |L| \le |f(x) - L| < \epsilon$$

$$|f(x)| < |L| + \epsilon$$

If $f(c)$ is not defined on A, then let $M = |L| + \epsilon$. If $f(c)$ is defined on A, then let $M = \max\{f(c), |L| + \varepsilon\}$. Since the choice of ε is arbitrary, we will choose $\epsilon = 1$. Thus,

$$f(x) \le M$$

∎

Definition. Let $A \subseteq \mathbb{R}$; $f, g : A \to \mathbb{R}$, c be a cluster point of A. Then $\forall x \in A$

- ***Sum*** of function: $(f + g)(x) = f(x) + g(x)$.

- ***Difference:*** $(f - g)(x) = f(x) - g(x)$.

- ***Multiple:*** $(bf)(x) = b \cdot f(x)$ for some $b \in \mathbb{R}$.

- ***Product:*** $(f \cdot g)(x) = f(x) \cdot g(x)$.

- ***Quotient:*** $(\frac{f}{g})(x) = \dfrac{f(x)}{g(x)}$ if $g(x) \ne 0$.

Theorem (***Limit Theorem***)**.** If $\exists L, M \in R$ s.t. $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

- $\lim_{x \to c} (f + g)(x) = L + M$.

- $\lim_{x \to c} (f - g)(x) = L - M$.

- $\lim_{x \to c} (bf)(x) = b \cdot L$.

- $\lim_{x \to c} (f \cdot g)(x) = L \cdot M$.

- $\lim_{x \to c} (\frac{f}{g})(x) = \dfrac{L}{M}$ if $\lim_{x \to c} g(x) \ne 0$.

Proof. Exercise. ∎

Remark. Always check conditions befor applying: this is true only if both f and g has a limit at c !

Example. $\lim\limits_{x\to 4} \dfrac{(x-4)(x+3)}{4(x-4)(x-5)} = \lim\limits_{x\to 4} \dfrac{x+3}{4(x-5)} = -\dfrac{7}{4}$

Corollary (***Polynomial Function***).

$$\lim_{x\to c} p(x) = \lim_{x\to c} \sum_{i=0}^{n} a_i \cdot x^i = \sum_{i=0}^{n} a_i \cdot \lim_{x\to c} x^i = \sum_{i=0}^{n} a_i \cdot x^i = p(c)$$

Corollary (***Rational Function***). For polynomial functions $p(x), q(x)$ s.t. $\lim\limits_{x\to c} p(x) \to p(c), \lim\limits_{x\to c} q(x) \to q(c) \neq 0$,

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$

Theorem. Let $A \subseteq \mathbb{R}$; $f, g : A \to \mathbb{R}$, c be a cluster point of A.

If $\exists a, b \in \mathbb{R}, \forall x \in A, x \neq c$ satisfiying

$$a \leq f(x) \leq b \text{ and } \lim_{x\to c} f = L$$

then

$$a \leq \lim_{x\to c} f \leq b$$

Proof. $\forall (x_n) \in A\backslash\{c\}, (x_n) \to c$,

$$a \leq f(x_n) \leq b \Rightarrow a \leq L \leq b$$

∎

Theorem (***Squeeze Theorem***). Let $A \subseteq \mathbb{R}$; $f, g, h : A \to \mathbb{R}$, c be a cluster point of A.

If $\forall x \in A, x \neq c$,

$$f(x) \leq g(x) \leq h(x) \text{ and } \lim_{x\to c} f = L = \lim_{x\to c} h$$

then

$$\lim_{x\to c} g = L$$

Proof. Exercise. Try sequeeze theorem of sequence. ∎

Example. $\lim\limits_{x\to 0} x \sin(\dfrac{1}{x}) = 0$

Proof. Notice that

$$-1 \leq \sin(\frac{1}{x}) \leq 1 \Rightarrow -x \leq x \sin(\frac{1}{x}) \leq x$$

Since

$$\lim_{x\to 0} (-x) = 0 = \lim_{x\to 0} (x)$$

$$\lim_{x\to 0} x \sin(\frac{1}{x}) = 0$$

∎

Example. Let $b \in \mathbb{R}, b > 0$. Then $\lim_{x \to 0} x^b = 0$. Notice that when $x \in [0, 1]$

$$x^{\lceil b \rceil} \leq x^b < x^{\lfloor b \rfloor}$$

The rest is left as exercise.

Theorem. Let $A \subseteq \mathbb{R}$; $f, g, h : A \to \mathbb{R}$, c be a cluster point of A. If $\lim_{x \to c} f > 0$, then $\exists \delta > 0$ s.t. $V_\delta(c)$ s.t. $\forall x \in A \cap V_\delta(c) \backslash \{c\}$,

$$f(x) > 0$$

Proof. Excercise. ∎

5   Continuous Functions

Definition (***Counituous***)**.** Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$.

We say f is ***countinuous at c*** if

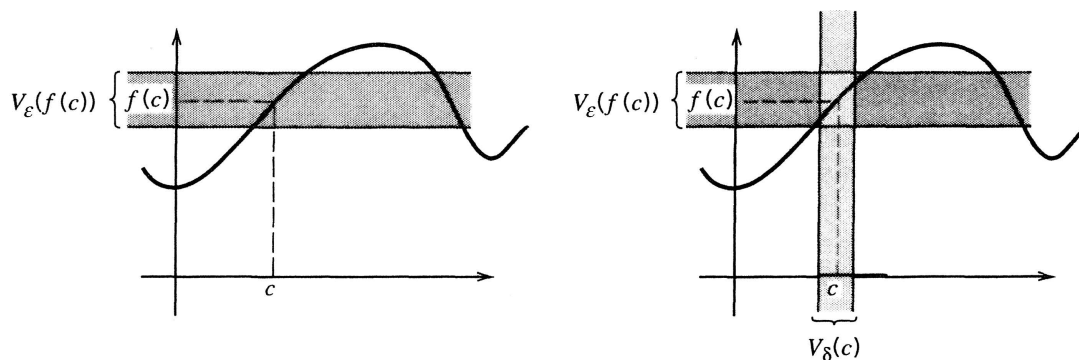$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

We say f is ***discontinuous at c*** if f is NOT continuous at c.

Definition (***Using Neighborhood***)**.** Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. Then f is continuous at c if and only if
$\forall \varepsilon > 0$ and its ε-neighborhood at $f(c)$, $V_{f(c)}(\varepsilon)$, $\exists \delta > 0$ and its δ-neighborhood at c , $V_c(\varepsilon)$, s.t.
$$f(V_\delta(c) \cap A) \subseteq V_\varepsilon(f(c))$$

Remark (***Comparison with Limit Definition***)**.** Continuity has 3 good properties that will be useful in the future study of Real Analysis:

- If c is a ***cluster point*** of A, then f is ***continuous*** if and only if

  1. $f(x)$ is defined at c.

     Counter-example: Any function $f : \mathbb{Q} \to \mathbb{R}$ is *undefined* at $\mathbb{R} \backslash \mathbb{Q}$ and thus *discontinuous* at $\mathbb{R} \backslash \mathbb{Q}$.

  2. $\lim\limits_{x \to c} f(x)$ exists.

     Counter-example: $\dfrac{1}{x}$ is *undefined* at 0 and thus *discontinuous* at 0.

  3. $f(c) = \lim\limits_{x \to c} f(x)$.

     Counter-example: $f(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is discontinuous at 0 as

     $f(c) = 0 \neq 1 = \lim\limits_{x \to c} f(x)$

- If c is ***NOT a cluster point*** of A, then c is an ***isolated point***, and

  $\exists V_\delta(c) \cap A = \{c\}$. Notice that an ***isolated point*** is ***automatically continuous*** as it *satisfies* the definition of continuity at c. This is because

  $$|x - c| = |c - c| = 0 < \delta \Rightarrow |f(x) - f(c)| = |f(c) - f(c)| = 0 < \varepsilon$$

Theorem (_Sequential Criterion for Continuity_).

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. Then f is continuous at c if and only if

$$\forall (x_n) \subseteq A \ \text{ s.t. } \ (x_n) \to c, \ f((x_n)) \to f(c).$$

Proof. ($\Rightarrow$). Continuity of f at c implies limit of f at c exists[2]. Thus we simply apply sequential criterion of limit.

($\Leftarrow$). This is exactly the statement of sequential criterion for limit at c. Thus,

$$\exists \lim_{x \to c} f(x) = f(c)$$

Notice that $c \in A$ implies that f is defined on c. Thus, we conclude that f is continuous on c by previous remark in p.43   ∎

Corollary (_Sequential Criterion for Discontinuity_).

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. Then f is discontinuous at c if and only if

$$\exists (x_n) \subseteq A \ \text{ s.t. } \ (x_n) \to c, \ f((x_n)) \not\to f(c).$$

Proof. Similar. Left to reader as exercise.   ∎

     Example.

- Polynomial functions are continuous in $\mathbb{R}$(try to prove it!).

- Rational functions are continuous in $\mathbb{R}$(try to prove it!).

---

[2]proof by definition