Math265 Real Analysis Class Notes

Based on lectures by Prof. Huang

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FOURTH EDITION

INTRODUCTION TO
REAL
ANALYSIS

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1 Preliminaries

1.1 Sets and Functions

1.2 Mathematical Induction

1.3 Finite and Infinite Sets

2 The Real Numbers

2.1 The Algebraic and Order Properties of $\mathbb{R}$

Axiom (**_Algebraic Property of_** $\mathbb{R}$)**.** On the set R of real numbers there are two binary operations, denoted by $+$ and $\cdot$ called **_addition_** and **_multiplication_**, respectively. These operations satisfy the following properties.

Axioms for Addition

(A1) For all $a, b \in \mathbb{R}$, $a + b = b + a$ (_commutative property of addition_)

(A2) For all $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$ (_associative property of addition_)

(A3) There exists an element $0 \in \mathbb{R}$ such that $0 + a = a + 0 = a$ for all $a \in \mathbb{R}$ (_existence of zero element (additive identity)_)

(A4) For all $a \in \mathbb{R}$, there exists an element $-a \in \mathbb{R}$ s.t. $a + (-a) = (-a) + a = 0$ (_existence of negative element (additive inverse)_)

Axioms for Multiplication

(M1) For all $a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$ (_commutative property of multiplication_)

(M2) For all $a, b, c \in \mathbb{R}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (_associative property of multiplication_)

(M3) There exists an element $1 \in \mathbb{R}$, $1 \neq 0$, s.t. $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{R}$ (_existence of unit element (multiplicative identity)_)

(M4) For all $a \in \mathbb{R}$, $a \neq 0$, there exists an element $\dfrac{1}{a} \in \mathbb{R}$ s.t. $a \cdot \dfrac{1}{a} = \dfrac{1}{a} \cdot a = 1$ (_existence of reciprocals (multiplicative inverse)_)

Distributive Law

(D) For all $a, b, c \in \mathbb{R}$, $a \cdot (b + c) = a \cdot b + a \cdot c$ (_distributive law of multiplication over addition_)

Theorem.

(a) If $z, a \in \mathbb{R}$ with $z + a = a$, then $z = 0$.

(b) If $u, b \in \mathbb{R}$, $u, b \neq 0$ with $u \cdot b = b$, then $u = 0$.

(c) If $a \in \mathbb{R}$, then $a \cdot 0 = 0$.

Proof. **(a)** Axiom (A3), (A4), (A2) gives

$$z = z + 0 = z + (a + (-a)) = (z + a) + (-a) = a + (-a) = 0$$

(b) Axiom (M3), (M4), (M2) gives

$$u = u \cdot 1 = u \cdot (b \cdot \frac{1}{b}) = (u \cdot b) \cdot \frac{1}{b} = b \cdot \frac{1}{b} = 1$$

(c) Axiom (M3), (D), (A3), theorem (a) gives

$$a + a \cdot 0 = a \cdot 1 + a \cdot 0 = a \cdot (1 + 0) = a \cdot 1 = a$$

$$a \cdot 0 = 0$$

$\blacksquare$

Theorem.

(a) If $a, b \in \mathbb{R}$, $a \neq 0$ with $a \cdot b = 1$, then $b = \dfrac{1}{a}$.

(b) If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

Proof. **(a)** Axiom (M3), (M4), (M2), $a \cdot b = 1$, (M3) gives

$$b = 1 \cdot b = (\frac{1}{a} \cdot a) \cdot b = \frac{1}{a} \cdot (a \cdot b) = \frac{1}{a} \cdot 1 = \frac{1}{a}$$

(b) It suffices to assume $a \neq 0$ and prove $b = 0$. (M2), (M4), (M3), and previous theorem(c) gives

$$b = 1 \cdot b = (\frac{1}{a} \cdot a) \cdot b = \frac{1}{a} \cdot (a \cdot b) = \frac{1}{a} \cdot 0 = 0$$

∎

Definition.

- We define $-$, called ***substraction***, as $a - b = a + (-b)$ for all $a, b \in \mathbb{R}$.

- We define $\div$, called ***division***, as $a \div b = a \cdot b = \dfrac{a}{b}$ for all $a, b \in \mathbb{R}$.

- We define a^n, called ***natural power***, as $a^n = (((a \cdot a) \cdot a) \cdot \dots) \cdot a$ (n-times) for all $a \in \mathbb{R}$, $n \in \mathbb{N}$.

Theorem ($\sqrt{2}$ ***Is Irrational Number***)**.** There is no rational number $r \in \mathbb{Q}$ such that $r^2 = 2$.

Proof. We will prove this by ***contradiction***.

Assume there are $p, q \in \mathbb{Z}$ satisfying $(\dfrac{p}{q})^2 = 2$. Without loss of generality, assume $p, q > 0$ and $\gcd(p, q) = 1$. Then

$$p^2 = 2q^2$$

So p^2 is even. This implies that p is also even(try to prove it!). Let $p = 2m$ for some $m \in \mathbb{Z}$. Then

$$p^2 = 4m^2 = 2q^2$$
$$2m^2 = q^2$$

By the same argument, if q^2 is even, then q is even. Since p, q are even, $\gcd(p, q)$ is at least 2, which contradicts our condition.

∎

Axiom (___Order of___ $\mathbb{R}$)**.** There is a non-empty set $\mathbb{P} \subset \mathbb{R}$ called the set of ___positive real numbers___ that satisfies the following properties:

1. If $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$.

2. If $a, b \in \mathbb{P}$, then $a \cdot b \in \mathbb{P}$.

3. If $a \in \mathbb{R}$, then one and only one of following holds:

$$a \in \mathbb{P}, \ a = 0, \ -a \in \mathbb{P}$$

___Note:___ for all $a \in \mathbb{R}$

- we write $a > 0$ and say a is ___positive___ if $a \in \mathbb{P}$,

- we write $a \geq 0$ and say a is ___non-negative___ if $a \in \mathbb{P} \cup \{0\}$.

- we write $a < 0$ and say a is ___negative___ if $a \in \mathbb{P}$,

- we write $a \leq 0$ and say a is ___non-positive___ if $-a \in \mathbb{P} \cup \{0\}$.

Definition. Let $a, b \in \mathbb{R}$.

1. If $a - b \in \mathbb{P}$, we denote that $a > b$ or $b < a$.

2. If $a - b \in \mathbb{P} \cup \{0\}$, we denote that $a \geq b$ or $b \leq a$.

___Note:___ For all $a, b \in \mathbb{R}$, there is one and only one of following holds:

$$a < b, \ a = b, \ a > b$$

And we call them a is ___greater than___, ___equal to___, or ___less than___ b respectively.

Theorem. Let $a, b, c \in \mathbb{R}$.

1. If $a > b$ and $b > c$, then $a > c$.

2. If $a > b$, then $a + c > b + c$.

3. If $a > b$ and $c > 0$, then $ca > cb$.
   If $a > b$ and $c < 0$, than $ca < cb$.

Proof.

1.By definition, $a - b \in \mathbb{P}$ and $b - c \in \mathbb{P}$. Then

$$(a - b) + (b - c) = a - c \in \mathbb{P} \Rightarrow a > c$$

2. By definition, $a - b \in \mathbb{P}$, then

$$(a + c) - (b + c) \in \mathbb{P} \Rightarrow a + c > b + c$$

3. By dfinition, $a - b \in \mathbb{P}$, then

$$\text{If } c > 0, \ c \in \mathbb{P}, \ ca - cb = c(a - b) \in \mathbb{P} \Rightarrow ca > cb$$

$$\text{If } c < 0, \ -c \in \mathbb{P}, \ cb - ca = (-c)(a - b) \in \mathbb{P} \Rightarrow cb > ca$$

$\blacksquare$

Theorem.

1. If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$.

2. $1 > 0$

3. If $n \in \mathbb{N}$, then $n > 0$.

Proof.

1. Since $a \neq 0$, there are only two cases:

   - If $a \in \mathbb{P}$, then $a^2 \in \mathbb{P}$.

   - If $-a \in \mathbb{P}$, then $(-a)(-a) = a^2 \in \mathbb{P}$. Proof of $(-a)(-a) = a^2$ is exercise

2. $1 \in \mathbb{P}$, so $1^2 \in \mathbb{P}$ by (1)

3. For all $n \in \mathbb{N}$, $n = (((1+1)+1) + \cdots + 1) + 1$. Thus, this is true by induction.

∎

Theorem. If $a \in \mathbb{R}$, $0 < a < \varepsilon$ for all $\varepsilon > 0$, then $a = 0$.

Proof. Assume that $a > 0$. Then if we take $\varepsilon_0 = \dfrac{1}{2}a$, we have $0 < \varepsilon_0 < a$ a contradiction.

Thus we conclude that $a = 0$. ∎

Theorem. If $ab > 0$, then either

- $a > 0$ and $b > 0$, or

- $a < 0$ and $b < 0$

Proof. Exercise. ∎

Corollary. If $ab < 0$, then either

- $a > 0$ and $b < 0$, or

- $a < 0$ and $b > 0$

Proof. Exercise. ∎

Proposition. Let $a \geq 0$, $b \geq 0$. Then

$$a < b \text{ iff } a^2 < b^2 \text{ iff } \sqrt{a} < \sqrt{b}$$

Proof.

- If $a = 0$, then it holds.

- If $a > 0$, then
  $$b^2 - a^2 = (b+a)(b-a) > 0 \Leftrightarrow b - a > 0$$
  And $\sqrt{a} < \sqrt{b} \Leftrightarrow a < b$ is a consequence of $a < b \Leftrightarrow a^2 < b^2$

∎

Theorem (***Arithmetic-geometric Mean Inequality***)**.** For all $a, b \in \mathbb{R}$, $a, b \geq 0$, $\sqrt{ab} \leq \dfrac{a+b}{2}$. Moreover, the inequality holds iff $a = b$.

Proof. If $a \neq b$, $\sqrt{a} \neq \sqrt{b}$, then

$$(\sqrt{a} - \sqrt{b})^2 > 0$$
$$a + b - 2\sqrt{ab} > 0$$
$$\sqrt{ab} < \frac{a+b}{2}$$

($\Leftarrow$) If $a = b$, then the equality holds.

($\Rightarrow$) If $\sqrt{ab} < \dfrac{a+b}{2}$, then we reverse the previous working,

$$(\sqrt{a} - \sqrt{b})^2 = 0$$

$$a = b$$

$\blacksquare$

Theorem (***Bernoulli's Inequality***)**.** If $x > -1$, then for all $n \in \mathbb{N}$,

$$(1 + x)^n \geq 1 + nx$$

Proof. We will prove this by mathematical induction.

- $n = 1$, then $1 + x \geq 1 + x$, so $P(1)$ holds.

- Assume $P(k)$ holds for all $k \in \mathbb{N}$. We want to show that $P(k+1)$ holds.

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)^k(1+x) \\
&\geq (1 + kx)(1 + x) = 1 + (k+1)x + kx^2 \\
&\geq 1 + (k+1)x
\end{aligned}$$

Thus $P(k+1)$ holds.

We conclude that $P(n)$ holds for all $n \in \mathbb{N}$. $\blacksquare$

2.2   Absolute Value and the Real Line

Definition. The ***absolute value*** of a real number a, denoted by $|a|$, is defined by

$$|a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } a < 0 \\ 0 & \text{if } a = 0 \end{cases}$$

Theorem.

1. $|ab| = |a| \cdot |b|$ for all $a, b \in \mathbb{R}$.

2. $|a|^2 = |a^2|$ for all $a \in \mathbb{R}$.

3. If $c \in \mathbb{R}$, $c \geq 0$, then
$$|a| \leq c \text{ iff } -c \leq a \leq c$$

4. $-|a| \leq a \leq |a|$.

Proof. Exercise.   ∎

Theorem (***Triangle Inequality***). Let $a, b \in \mathbb{R}$. Then

$$|a + b| \leq |a| + |b|$$

Proof. By previous theorem(4), we have

$$-|a| \leq a \leq |a| \text{ and } -|b| \leq b \leq |b|$$

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

Thus, by previuos theorem(3),
$$|a + b| \leq |a| + |b|$$

                                                                                              ∎

Remark. $ab \geq 0$ iff $|a + b| \leq |a| + |b|$

Proof. Exercise.   ∎

Corollary. Let $a, b \in \mathbb{R}$. Then

1. $||a| - |b|| \leq |a - b|$.

2. $|a - b| \leq |a| + |b|$.

Proof.

1. We write $a = a - b + b$. By triangle inequality,

$$|a| = |(a - b) + b| \leq |a - b| + |b|$$

$$|a| - |b| \leq |a - b|$$

   Similarly

$$|b| - |a| \leq |a - b|$$

   We thereby conclude that (1) holds.

2. We "put $-b$ in b" of triangle inequality:

$$|a - b| \leq |a| + |b|$$

∎

Corollary. If $a_1,\ a_2,\ \ldots,\ a_n \in \mathbb{R}$, then

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

Proof.

$$
\begin{aligned}
|a_1 + a_2 + \cdots + a_n| &\leq |a_1 + a_2 + \cdots + a_{n-1}| + |a_n| \\
&\leq |a_1 + a_2 + \cdots + a_{n-2}| + |a_{n-1}| + |a_n| \\
&\cdots \\
&\leq |a_1| + |a_2| + \cdots + |a_n|
\end{aligned}
$$

∎

Definition. Let $a \in \mathbb{R}$, $\varepsilon > 0$. Then the $\underline{\varepsilon\text{-}\textbf{\textit{neighborhood of }} a}$ is the set

$$V_\varepsilon(a) = \{x \in \mathbb{R}\,|\,|x - a| < \varepsilon\}$$

Theorem. Let $a \in \mathbb{R}$. If $\forall \varepsilon > 0,\ x \in V_\varepsilon(a)$, then $x = a$.

Proof. By definition of neighborhood, $\forall \epsilon > 0, |x - a| < \varepsilon$. Thus by previous theorem,

$$|x - a| = 0 \Rightarrow x - a = 0 \Rightarrow x = a$$

∎

Proposition. Let $a, b \in \mathbb{R}$ with $a < b$. Consider the $\underline{\textbf{\textit{open interval}}}$

$$(a, b) = \{x\,|\,a < x < b\}$$

Then for all $x \in (a, b)$, $\exists \varepsilon > 0$ s.t.

$$V_\varepsilon(x) \subseteq (a, b)$$

Proof. Choose $\varepsilon = \min\{|x - a|,\ |x - b|\}$. Suppose $y \in V_\varepsilon(x)$, then

$$|x - y| < \varepsilon$$

$$x - \varepsilon < y < x + \varepsilon$$

So it is either

- $-a < -b < y < b$ if $a > b$, or
- $-b < -a < y < a$ if $a < b$

We conclude that $y \in (a, b)$

∎

Proposition. Let $a, b \in \mathbb{R}$ with $a < b$. Consider the ___closed interval___

$$[a, b] = \{x | a \leq x \leq b\}$$

Then for all $\varepsilon > 0$,

$$V_\varepsilon(x) \nsubseteq (a, b)$$

Proof. Exercise.   ∎

Proposition. Let $x, y, a, b \in \mathbb{R}$. If $x \in V_\varepsilon(a)$, $y \in V_\varepsilon(b)$, then

$$x + y \in V_{2\varepsilon}(a + b)$$

Proof. By definintion of neighborhood,

$$
\begin{aligned}
|(x + y) - (a + b)| &= |(x - a) + (y - b)| \\
&\leq |x - a| + |y - b| \\
&< \varepsilon + \varepsilon = 2\varepsilon
\end{aligned}
$$

Thus $x + y \in V_{2\varepsilon}(a + b)$.   ∎

2.3   The Completeness Property of $\mathbb{R}$

Definition. Let $S \subseteq \mathbb{R}$ be a nonempty.

1. S is said to be ***boudned above*** if there exists a number $u \in \mathbb{R}$ such that $s \leq u$ for all $s \in S$. We say u is an ***upper bound*** of S.

2. S is said to be ***boudned below*** if there exists a number $u \in \mathbb{R}$ such that $u \geq s$ for all $s \in S$. We say u is an ***lower bound*** of S.

3. S is said to be ***boudned*** if it is both bounded above and below.

4. We say S is ***unbounded*** if it is not bounded above or bounded below.

Definition. Let $S \subseteq \mathbb{R}$ be nonempty.

1. If S is bounded above, then u is the ***supremum*** or ***least upper bound*** of S if

   - u is an upper bound, and

   - If v is an upper bound of S, then $u \leq v$.

2. If S is bounded below, then w is the ***infimum*** or ***greatest lower bound*** of S if

   - w is an lower bound, and

   - If t is an lower bound of S, then $t \leq w$.

Supremum and infimum are denoted by

$$\sup S, \ \inf S$$

Proposition. If S is bounded above, then $\sup S$ is ***unique***.

Proof. Suppose $u_1, \ u_2$ are supremum of S. Since u_1 is the supremum of S and u_2 is the upper bound of S,

$$u_1 \leq u_2$$

Similarly, we have

$$u_2 \leq u_1$$

Thus

$$u_1 = u_2$$

∎

Theorem. Let $S \subseteq \mathbb{R}$ be nonempty. If $\exists u = \sup S$, then the following statements are equivalent:

1. If v is an upper bound of S, then $u \leq v$.

2. If $z < u$, then z is not an upper bound of S.

3. If $z < u$, then there exists $s_z \in S$ such that $z < s_z$.

4. If $\varepsilon > 0$, then there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

Proof. Exercise.

∎

Lemma (***Alternative Definition***)**.** Let $S \subseteq \mathbb{R}$ be nonempty. $\exists u = \sup S$ iff

- $s \leq u$ for all $a \in S$.

- If $v < u$, then there exists $s' \in S$ such that $v < s'$.

Proof. Exercise. ∎

Lemma (ε ***Definition***)**.** An upper bound u of nonempty set $S \subseteq \mathbb{R}$ is the supremum of S iff

$\forall \epsilon > 0, \ \exists s_\epsilon \in S$ s.t.

$$u - \varepsilon < s_\varepsilon$$

Proof. ($\Rightarrow$) Suppose $u = \sup S$. $\forall \varepsilon > 0, u - \varepsilon$ is NOT an upper bound of S. Thus

$$\exists s_\varepsilon \in S \text{ s.t. } u - \varepsilon < s_\varepsilon$$

($\Leftarrow$) Suppose u is an upper bound satisfying: $\forall \epsilon > 0, \ \exists s_\epsilon \in S$ s.t. $u - \varepsilon < s_\varepsilon$.

Let v be an upper bound of S.

Goal: $u \leq v$

Assume $u > v$. Let $\varepsilon = u - v > 0$. Then $\exists s_\varepsilon \in S$ s.t. $u - \varepsilon < s_\varepsilon$. Thus $u - \varepsilon$ is NOT an upper bound of S, which is a contradiction.

We thereby conclude that $u \leq v$.

∎

Remark (***Maximum v.s. Supremum***)**.** If a set is bounded above, then the ***maximum*** of a set may not exist, but the ***supremum*** of the set always exists.

Example. Let $S = (1, 2)$. Then there does NOT exist a maximum of S , but $\exists \sup S = 2$

This is also true for ***minimum*** and ***infimum***.

Remark. If a set S is bounded above, then $\sup S$ may NOT be an element of S.

Example. Let $S = (1, 2)$. $\exists \sup S = 2 \notin S$

Axiom (***Completeness Property of*** $\mathbb{R}$)**.**

Every nonempty set of real numbers has a supremum in $\mathbb{R} \cup \{\infty\}$.

3　Sequences and Series

3.1　Sequences and their limits

Definition (**Sequence**). A ***sequence of real numbers*** is a *function* from $\mathbb{N}$ to $\mathbb{R}$.

We adopt the notation with a *sequence*:

$$a : \mathbb{N} \to \mathbb{R}$$

where instead of writing $a(1), a(2), \ldots$, we write it as $a_1, a_2, \ldots$ which we called them ***terms*** or ***elements*** of the sequence.

Notation.

$$(a_n)_{n=1}^{\infty} \text{ or } (a_n)_{n\in\mathbb{N}} \text{ or } (a_n) \text{ or } (a_n | n \in \mathbb{N})$$

Definition (**Converge to x**). A sequence $(x_n) \in \mathbb{R}$ ***converges*** to $x \in \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ such that } n \geq N_\varepsilon \to |x_n - x| < \varepsilon$$

We write

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (x_n) = x.$$

Definition (**Convergent & Divergent**). A sequence is ***convergent*** if it has a ***limit*** in $\mathbb{R}$, and is ***divergent*** if it has ***no limit*** in $\mathbb{R}$.

Theorem (**Uniqueness of Limit**). A sequence in $\mathbb{R}$ can have **_at most one_** limit. Or, the limit of a sequence is **_unique_** if the limit exists

Proof. Let (x_n) be a sequence of real numbers. Suppose x, x' are limits of (x_n). We want to prove $x = x'$ by contradiction.

Assume $|x - x'| > 0$. If we consider $\varepsilon := \frac{1}{3}|x - x'| > 0$, then
The existence of $\lim_{x_n \to x}$ implies that $\exists N_1 \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ if $n \geq \mathbb{N}_1$.
Similarly, existence of $\lim_{x_n \to x'}$ implies that $\exists N_2 \in \mathbb{N}$ such that $|x_n - x'| < \varepsilon$ if $n \geq \mathbb{N}_2$.
Thus,

$$
\begin{aligned}
|x - x'| &\leq |x - x_{N_1+N_2} + x_{N_1+N_2} - x'| \\
&\leq |x - x_{N_1+N_2}| + |x_{N_1+N_2} - x'| \text{ by triangle inequality} \\
&< \varepsilon + \varepsilon \\
&= \frac{2}{3}|x - x'|
\end{aligned}
$$

Then,

$$\frac{1}{3}|x - x'| < 0, \text{which is a contradiction}$$

we thereby prove by contradiction that

$$|x - x'| = 0, \text{ which is equivalent to } x = x'$$

∎

Example.
$$\lim_{n \to \infty}\left(\frac{1}{n}\right) = 0$$

Goal: $\forall \varepsilon > 0$, want to find N_ε such that $\left|\frac{1}{n} - 0\right| < \varepsilon$ for $n > N$, so it suffices to show that

$$\frac{1}{n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n$$

Proof. Let $\varepsilon > 0$. Apply Archimedean's property to $\frac{1}{\varepsilon}$, then

$$\exists N \in \mathbb{N} \text{ such that } \frac{1}{\varepsilon} < N$$

$$\Rightarrow \forall n \geq N, \left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

$$\Rightarrow \lim_{n \to \infty}\frac{1}{n} = 0$$

∎

Theorem. Let (x_n) be a sequence of real numbers, and let $x \in \mathbb{R}$. The following are equivalent:

1. $x_n \to x$

2. $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$, for $n \geq \mathbb{N}$

3. $\ldots \ldots x - \varepsilon < x_n < x + \varepsilon \ldots \ldots$

4. $\forall \varepsilon$-neighborhood $V_\varepsilon(x), \exists N \in \mathbb{N}$ such that $x_n \in V_\varepsilon(x)$ for $n \geq N$

Sketch of proof:
$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$$

Proposition.
$$\lim_{n \to \infty} (2\sqrt{2n+1} - \sqrt{2n}) = 0$$

Proof. Let $\varepsilon > 0$. Consider
$$N = \lceil \frac{1}{2}(\frac{1}{2\varepsilon})^2 \rceil \in \mathbb{N}$$

$$n > N \Rightarrow n > \frac{1}{2}(\frac{1}{2\varepsilon})^2 \Rightarrow \frac{1}{2\sqrt{2n}} < \varepsilon \Rightarrow \left|\sqrt{2n+1} - \sqrt{2n}\right| = \cdots = \frac{1}{\sqrt{2n+1} + \sqrt{2n}} < \varepsilon$$

∎

Remark.
$$\lim_{n \to \infty} (-1)^n \text{ does not exist.}$$

Definition (**m-tail**). If (x_n) is a sequence of real numbers and $m \in \mathbb{N}$, then the ***m-tail*** of (x_n) is the sequence

$$\{x_{n+m} : n \in \mathbb{N}\} = \{x_{m+1}, x_{m+2}, \ldots\}$$

Theorem. Let (x_n) be a sequence and $m \in \mathbb{N}$. Then (x_n) is ***convergent*** iff (x_{n+m}) is ***convergent***. Moreover,
$$\lim_{n \to \mathbb{N}} (x_n) = \lim_{n \to \mathbb{N}} (x_{n+m})$$

Proof. $(\Rightarrow)$
Suppose $x_n \to x$. Let

$$\varepsilon > 0, \exists N_\varepsilon > 0, \text{ such that } |x_n - x| < \varepsilon \text{ for } n \geq N_\varepsilon$$

Consider $N_\varepsilon' := N_\varepsilon + m$ then

$$n + m \geq N_\varepsilon' \Rightarrow n \geq N_\varepsilon \Rightarrow n + m \geq N_\varepsilon \Rightarrow |x_{n+m} - x| < \varepsilon$$

It follows that

$$n \geq N_\varepsilon \Rightarrow n + m \geq N_\varepsilon \Rightarrow |x_{n+m} - x| < \varepsilon$$

($\Leftarrow$)

Suppose $x_{n+m} \to x$.

$$\forall \varepsilon > 0, \exists N_\varepsilon > 0 \text{ such that } |x_{n+m} < \varepsilon|, \forall n \geq N_\varepsilon$$

Consider $N := N_\varepsilon + m$. Then

$$n \geq N = N_\varepsilon + m$$
$$\Rightarrow n - m \geq N_\varepsilon$$
$$\Rightarrow \left| x_{(n-m)+m} - x \right| < \varepsilon$$
$$\Rightarrow |x_n - x| < \varepsilon$$

∎

Remark. We say that a sequence (x_n) **_ultimately_** has a property if that property holds for some tail of (x_n)

Theorem. Let x_n be a sequence of real numbers. Let a_n be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$. If $\exists c > 0, m \in \mathbb{N}, x \in \mathbb{R}$ such that

$$|x_n - x| \leq c \cdot a_n, \forall n \geq m$$

then

$$x_n \to x$$

Proof. We know that

$$\forall \varepsilon > 0, \exists N \geq 0 \text{ s.t. } |a_n| < \frac{\varepsilon}{c}, \forall n \geq N$$

Consider $N' = max\{N, m\}, \forall n \geq N'$. Then

$$|x_n - x| \leq Ca_n = c|a_n| < c \cdot \frac{\varepsilon}{c} = \varepsilon$$
$$\Rightarrow x_n \to x$$

∎

Proposition.
$$\lim_{n\to\infty} \frac{17}{2 + 3n} = 0$$

Proof.
$$\left| \frac{17}{2 + 3n} - 0 \right| = \frac{17}{2 + 3n} \leq \frac{13}{3n} = \frac{17}{3} \cdot \frac{1}{n}$$

Apply the theorem above with

$$a_n = \frac{1}{n}, c = \frac{17}{3}, m = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{17}{2 + 3n} = 0, \text{ since } \lim_{n\to\infty} \frac{1}{n} = 0$$

∎

Proposition.
$$\forall c > 0, \lim_{n \to \infty} c^{\frac{1}{n}} = 1$$

Proof. **<u>Case 1: c = 1</u>**
$$\lim_{n \to \infty} c^{\frac{1}{n}} = 1$$

<u>Case 2: c > 1</u>

Let $d_n = c^{\frac{1}{n}} - 1$. Then $\forall n, d_n > 0$. It follows that

$$(d_n + 1) = c^{\frac{1}{n}} \Rightarrow c = (1 + d_n)^n \geq 1 + n \cdot d_n \text{ by Bernoulli's inequality}$$
$$\Rightarrow d_n \leq (c - 1) \cdot \frac{1}{n}$$
$$\Rightarrow \left| c^{\frac{1}{n}} - 1 \right| = d_n \leq (c - 1) \cdot \frac{1}{n}$$

Apply the theorem with

$$C = c - 1, a_n = \frac{1}{n}, m = 1, x = 1$$

$$\lim_{n \to \infty} c^{\frac{1}{n}} = 1$$

<u>Case 3: c < 1(Note that we cannot use Bernoulli inequality here)</u>

Define e_n to be a sequence that satisfies

$$c^{\frac{1}{n}} = \frac{1}{1 + e_n}$$

Then $e_n > 0 \forall n$.

$$c = \frac{1}{(1 + e_n)^n} \leq \frac{1}{1 + n \cdot e_n} < \frac{1}{n \cdot e_n}$$
$$\Rightarrow e_n < \frac{1}{c} \cdot \frac{1}{n}$$
$$1 - c^{\frac{1}{n}} = 1 - \frac{1}{1 + e_n} = \frac{e_n}{1 + e_n} < e_n < \frac{1}{c} \cdot \frac{1}{n}$$

Apply the theorem with

$$a_n = \frac{1}{n}, m = 1, C = \frac{1}{c}, x = 1$$
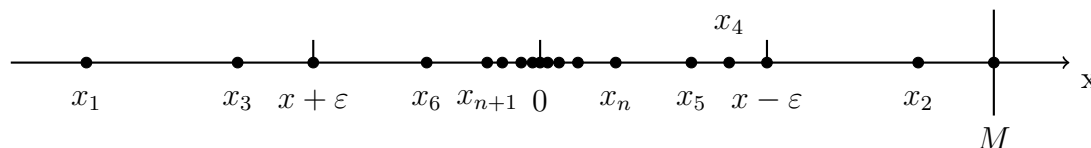$$\lim_{n \to \infty} c^{\frac{1}{n}} = 1$$

∎

3.2   Limit Theorems

Definition (Bounded sequence)**.** A sequence $(x_n) \in \mathbb{R}$ is **_bounded_** if

$$\exists M > 0 \text{ s.t. } \forall n \in \mathbb{N}, |x_n| \leq M$$

Theorem. A convergent sequence $(x_n) \in \mathbb{R}$ is bounded.

Proof. By definition of convergent sequence, let $\varepsilon = 1$:

$$\exists N > 0 \text{ s.t. } \forall n \geq N, |x_n - x| < 1$$

Thus we have

$$-1 < x_n - x < 1$$
$$\Rightarrow \qquad -1 + x < x_n < x + 1, \forall n \geq N$$

Then define

$$M := \max \{|x_1|, |x_2|, |x_3|, \ldots, |-1 + x|, |x + 1|\}$$
$$|x_n| \leq M$$

$\blacksquare$

Remark. By contrapositive, an unbounded sequence is divergent.

Definition. Given sequence $(x_n), (y_n) \in \mathbb{R}$, we define following operations of sequence:

- **_Sum_** $(x_n + y_n)$
- **_Difference_** $(x_n - y_n)$
- **_Product_** $(x_n \cdot y_n)$
- **_Quotient_** $\left(\frac{x_n}{y_n}\right)$ if $\forall n \in \mathbb{N}, y_n \neq 0$
- **_Multiple_** $(c \cdot x_n)$

Theorem (**Limit Laws**)**.** Let $(x_n), (y_n) \in \mathbb{R}$ be sequences of real numbers with $x_n \rightarrow x, y_n \rightarrow y$, and let $c \in \mathbb{R}$. Then

- $x_n + y_n \rightarrow x + y$
- $x_n - y_n \rightarrow x - y$
- $x_n \cdot y_n \rightarrow x \cdot y$
- $c \cdot x_n \rightarrow c \cdot x$
- If $\forall n \in \mathbb{N}, y_n \neq 0$ and $y \neq 0$, then $\frac{x_n}{y_n} \rightarrow \frac{x}{y}$

Proof of Sum. :

$\forall \varepsilon > 0$,

$$\exists N_1 \in \mathbb{N} \text{ s.t. } \forall n \geq N_1, |x_n - x| < \frac{\varepsilon}{2}$$

$$\exists N_2 \in \mathbb{N} \text{ s.t. } \forall n \geq N_2, |y_n - y| < \frac{\varepsilon}{2}$$

Consider

$$N := \max \{N_1, N_2\}$$

Then

$$\begin{aligned}
\forall n \geq N, |(x_n + y_n) - (x + y)| \\
= |x_n - x + y_n - y| \\
\leq |x_n - x| + |y_n - y| \\
= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \varepsilon
\end{aligned}$$

∎

Proof of Difference. :

Similarly,

$$\begin{aligned}
\forall n \geq N, |(x_n - y_n) - (x - y)| \\
\leq |x_n - x| + |y_n - y|
\end{aligned}$$

∎

Proof of Product. : Since (x_n) is convergent, it is also bounded. Thus,

$$\exists M \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, |x_n| \leq M$$

By definition of convergence, $\forall \varepsilon > 0$:

$$\exists N_1 > 0 \text{ s.t. } \forall n \geq N, |x_n - x| < \frac{\varepsilon}{2|y|}$$

$$\exists N_2 > 0 \text{ s.t. } \forall n \geq N, |y_n - y| < \frac{\varepsilon}{2M}$$

Then, $\exists N = \max \{N_1, N_2\}$ s.t. $\forall n \geq N$,

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\
&= |x_n(y_n - y) - y(x_n - x)| \\
&\leq |x_n| \, |y_n - y| + |y| \, |x_n - x| \\
&\leq M \cdot |y_n - y| + |y| \, |x_n - x| \\
&< M \cdot \frac{\varepsilon}{2M} + |y| \cdot \frac{\varepsilon}{2|y|} = \varepsilon
\end{aligned}$$

∎

Proof of Multiply. :

Exercise   ∎

Proof of Quotient.

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| = \left| \frac{x_n y - y_n x}{y_n y} \right|$$

$$= \left| \frac{x_n y - -x_n y_n + x_n y_n - y_n x}{y_n y} \right|$$

$$\leq \left| \frac{1}{y_n y} (|x_n| \, |y_n - y| + |y_n| \, |x_n - x|) \right|$$

Since $y_n \to y$

$$\exists N_1 \in \mathbb{N} \text{ s.t. } \forall n \geq N, ||y_n| - |y|| \leq |y_n - y| < \frac{|y|}{2}$$

Then

$$\Rightarrow \qquad -\frac{|y|}{2} < |y| - |y| < \frac{|y|}{2}$$

$$\Rightarrow \qquad \frac{|y|}{2} < |y_n|$$

$$\Rightarrow \qquad \frac{1}{|y_n|} < \frac{2}{|y|}$$

$$\Rightarrow \qquad \frac{1}{|y_n y|} < \frac{2}{|y^2|}$$

Define $F := \frac{2}{|y^2|}$.

Since (x_n) and (y_n) are bounded,

$\exists G \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$

$$|x_n| \leq G, |y_n| \leq G$$

Thus $\forall n > 0$

$$\exists N_2 > 0 \text{ s.t. } \forall n \geq N_2, |x_n - x| < \frac{\varepsilon}{2GF}$$

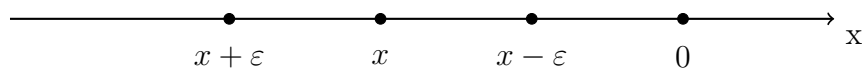$$\exists N_3 > 0 \text{ s.t. } \forall n \geq N_3, |y_n - y| < \frac{\varepsilon}{2GF}$$

Define $N := \max\{N_1, N_2, N_3\}$, then $\exists N \in \mathbb{N}$ s.t. $\forall n > N$,

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| < F(G \cdot \frac{\varepsilon}{2GF} + G \cdot \frac{\varepsilon}{2GF}) = \varepsilon$$

                                                         ∎

Theorem. If (x_n) is a convergent sequence in $\mathbb{R}$ of non-negative terms with $(x_n) \to x$, then $x \geq 0$.

Sketch. : Assume $x < 0$, choose $\varepsilon \leq |x|$, then $\forall n > N, x_n < 0$, which is a contradiction.

                                                         ∎

Theorem. If $x_n \to x, y_n \to y$ are sequences in $\mathbb{R}$ such that $\forall n \in \mathbb{N}, x_n \leq y_n$, then $x \leq y$.

Proof. Consider $z_n := y_n - x_n$. Then $z_n \geq 0$ and $z_n \to y - x$ by the limit law.

$$\Rightarrow y - x \geq 0$$

$$y \geq x$$

$\blacksquare$

Theorem. If $x_n \to x$ is a sequence in $\mathbb{R}$, and let $a, b \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, a \leq x_n \leq b$, then $a \leq x \leq b$.

Proof. Consider constant sequence $a_n = a$ and $b_n = b$. Then this is true by the last theorem. $\blacksquare$

Theorem (**Squeeze Theorem**). Suppose $(x_n), (y_n), (z_n)$ are sequences of real numbers such that $\forall n \in \mathbb{N}, x_n \leq y_n \leq z_n$. If $\lim (x_n) = \lim (z_n)$, then (y_n) is convergent and

$$\lim (x_n) = \lim (y_n) = \lim (z_n)$$

Proof. Let $\forall \varepsilon > 0$.

Write $L = \lim (x_n)$. Then by definition of limit,

$\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N$,
$$|x_n - L| < \varepsilon, |z_n - L| < \varepsilon$$

$$\Rightarrow \qquad -\varepsilon < x_n - L \leq y_n - L \leq z_n - L < \varepsilon$$
$$\Rightarrow \qquad |y_n - L| < \varepsilon$$

Thus (y_n) converges to L by definition of limit.

$\blacksquare$

Proposition. $\ln (n)$ is divergent.

Proof. Since $\ln(n)$ is unbounded, it is divergent.

$\blacksquare$

Exercise. $\lim \frac{3n^2 + 2n + 1}{5n^2 - 4} = \frac{3}{5}$

$$\lim \frac{3n^2 + 2n + 1}{5n^2 - 4} = \lim \frac{3 + 2\frac{1}{n} + \frac{1}{n^2}}{5 - 4\frac{1}{n^2}}$$

Notice that each term of $\frac{1}{n}$ and $\frac{1}{n^2}$ converges to 0. Thus by limit law,

$$\lim \frac{3 + 2\frac{1}{n} + \frac{1}{n^2}}{5 - 4\frac{1}{n^2}} = \frac{\lim \{3 + 2 \cdot 0 + 0\}}{\lim \{5 - 4 \cdot 0\}} = \frac{3}{5}$$

Proposition. $(-1)^n$ is divergent.

Proof:Exercise. $\blacksquare$

Theorem. If $x_n \to x$, then $|x_n| \to |x|$

Sketch of the proof. :
$$||x_n| - |x|| \le |x_n - x|$$

∎

Theorem. Suppose (x_n) is a sequence of non-negative real numbers, satisfiying $x_n \to x$. Then $\sqrt{x_n} \to \sqrt{x}$.

Proof. Let $\forall \varepsilon > 0$

Case1: $x = 0$

$\exists N \in \mathbb{N}$ s.t. $\forall n \ge N$,
$$|x_n - 0| < \varepsilon^2$$
$$\Rightarrow |\sqrt{x} - 0| < \varepsilon$$

Case2: $x > 0$

$\exists N$ s.t. $\forall n > N$,
$$|x_n - x| < \sqrt{x} \cdot \varepsilon$$

Notice that

$$\left|\sqrt{x_n} - \sqrt{x}\right| = \left|\frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}\right|$$
$$\le \frac{|x_n - x|}{\sqrt{x}} < \varepsilon$$

∎

Theorem. Let (x_n) be a sequence of positive real numbers such that $L = \lim\left(\frac{x_{n+1}}{n}\right)$ exists. If $L < 1$, then (x_n) converges to 0.

Proof. $\exists N_1 \in \mathbb{N}$ s.t. $\forall n > N_1$
$$\left|\left|\frac{x_{n+1}}{x_n}\right| - |L|\right| \le \left|\frac{x_{n+1}}{x_n} - L\right| < \frac{1 - L}{2}$$

Thus
$$\left|\frac{x_{n+1}}{x_n}\right| < \frac{1 + L}{2}$$

Note that $\frac{1+L}{2} < 1$, write $r = \frac{1+L}{2}$. Then $\forall m \in \mathbb{N}$,
$$x_{N_1+m} < x_{N_1+(m-1)}r < x_{N_1+(m-2)}r^2 < \cdots < x_{N_1}r^m$$

Consider $(y_m) = (x_{N_1+m})$, $(z_m) = (x_{N_1}r^m)$. Then
$$0 \le y_m \le z_m$$

Since $z_m \to 0$
$$(y_m) \to 0 \text{ by squeeze theorem}$$

Thus we conclude that $(x_n) \to 0$ by m-th tail theorem.   ∎

3.3   Monotone sequence

Definition. Let (x_n) be a sequence of real. We say (x_n) is ...

- **_increasing_** if $\forall n \in \mathbb{N}, x_{n+1} \geq x_n$.

- **_strictly increasing_** if $\forall n \in \mathbb{N}, x_{n+1} > x_n$.

- **_decreasing_** if $\forall n \in \mathbb{N}, x_{n+1} \leq x_n$.

- **_strictly decreasing_** if $\forall n \in \mathbb{N}, x_{n+1} < x_n$.

Theorem (**Monotone Convergence Therorem**). A monotone sequence of real numbers is **_convergent_** iff it is bounded. Moreover, if (x_n) is increaseing, then

$$\lim(x_n) = \sup \{x_n : n \in \mathbb{N}\}$$

If (x_n) is decreasing, then

$$\lim(x_n) = \inf \{x_n : n \in \mathbb{N}\}$$

Proof. $(\Rightarrow)$

A convergent sequence is always bounded.

$(\Leftarrow)$

Suppose (x_n) is a monotone and bounded sequence.

Case 1: (x_n) is increasing.

Write $x = \sup \{x_n : n \in \mathbb{N}\}$.

Let $\varepsilon > 0$. Since $x = \sup(x_n)$:

$$x - \varepsilon \text{ is NOT an upper boudn of } (x_n)$$

Then

$$\exists N \in \mathbb{N} \text{ s.t. } x_N > x - \varepsilon$$

Since (x_n) is increasing,

$$\forall n \geq N, \, x_n > x - \varepsilon$$

On the other hand,$x + \varepsilon$ is an supper bound since x is an upper bound. Thus,

$$x_n < x + \varepsilon$$

$$\Rightarrow \forall n \geq N, x - \varepsilon < x_n < x + \varepsilon$$

$$\Rightarrow |x_n - x| < \varepsilon \Rightarrow (x_n) \to x$$

Case 2: (x_n) is decreasing.

Write $y = \inf(x_n)$. Let $\varepsilon > 0$

Since $y = \inf(x_n)$,

$$y + \varepsilon \text{is NOT and upper bound of } (x_n)$$

Thus
$$\exists N \in \mathbb{N} \text{ s.t. } x_n < y + \varepsilon$$

Since (x_n) is decreasing,
$$\forall n > N, x_n < y + \varepsilon$$

On the other hand, $y - \varepsilon$ is a lower bound since y is a lower bound. Hence,

$$\forall n \in \mathbb{N}, y - \varepsilon < x_n$$

$$\forall n \geq N, y - \varepsilon < x_n < y + \varepsilon \Rightarrow |x_n - y| < \varepsilon$$
$$\Rightarrow (x_n) \to y$$

Remark. One can prove case 2 by following:

$(-x_n)$ is increasing and converges to $\sup(-x_n)$ by case 1. Also note that

$$(x_n) = (-(-x_n)) \to -\sup(-x_n)$$

by limit law. So it is easy to prove that

$$-\sup((-x_n)) = \inf(x_n)$$

$\blacksquare$

Example. Consider the sequence (x_n) is given by

$$\begin{cases} x_0 = \frac{1}{2} \\ x_{n+1} = \frac{3}{2}x_n(1 - x_n) \end{cases}$$

(x_n) is decreasing and bounded.

Thoughts: Assume (x_n) converges, then by limit law,

$$x = \frac{3}{2}x(1 - x) \text{ where } x = \lim(x_n) \Rightarrow x = 0 \text{ or } 3$$

then, by proof of contradiction, it is not convergent.

Proof. ***Claim***: $\frac{1}{3} < x_{n+1} < x_n \leq \frac{1}{2}, \forall n \in \mathbb{N} \cup \{0\}$

Proof of the claim by induction:

When n = 0:
$$x_0 = \frac{1}{2}, x_1 = \frac{3}{2} \cdot \frac{1}{2}(1 - \frac{1}{2}) = \frac{3}{8}$$
$$\frac{1}{3} < \frac{3}{8} < \frac{1}{2} \leq \frac{1}{2}$$

Suppose this is true for n=k:
$$\frac{1}{3} < x_{k+1} < x_k \leq \frac{1}{2}$$

Goal:

$$\frac{1}{3} < x_{k+2} < x_{k+1} \leq \frac{1}{2}$$

$$x_{k+1} = \frac{3}{2}x_k(1 - x_k)$$

$$\frac{1}{3} < x_k \leq \frac{1}{2} \Rightarrow \frac{2}{3} > 1 - x_k \geq \frac{1}{2}$$

$$x_{k+1} < \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{2}$$

Complete the square:

$$x_{k+1} - \frac{1}{3} = \frac{3}{2}x_k(1 - x_k) - \frac{1}{3}$$
$$= -\frac{3}{2}[(x_k - \frac{1}{2})^2 - \frac{1}{36}]$$

So

$$\frac{1}{3} < x_k \leq \frac{1}{2} \Rightarrow \left| x_k - \frac{1}{2} \right| < \frac{1}{6}$$
$$\Rightarrow (x_k - \frac{1}{2})^2 < \frac{1}{36}$$
$$\Rightarrow x_{k+1} - \frac{1}{3} > 0$$
$$\Rightarrow \frac{1}{3} < x_{k+1} \leq \frac{1}{2}$$

With the similar process, we can derive that

$$\frac{1}{3} < x_{k+2} \leq \frac{1}{2}$$

$$x_{k+2} = \frac{3}{2}x_{k+1}(1 - x_{k+1}) < \frac{3}{2}x_{k+1} \cdot \frac{3}{2} = x_{k+1}$$

Therefore, this claim is also true for n=k+1:

$$\frac{1}{3} < x_{k+2} < x_{k+1} \leq \frac{1}{2}$$

We thereby prove the theorem by induction:

$$\frac{1}{3} < x_{n+1} < x_n \leq \frac{1}{2}$$

■

Exercise: Textbook p.75: A sequence that converges to $\sqrt{a}$ for $a > 0$.

Definition (**Euler's Number**).

$$e = \lim(1 + (\frac{1}{n})^n)$$

Goal: (x_n) is convergent where $x_n = (1 + \frac{1}{n})^n$

$$
\begin{aligned}
x_n = (1 + \frac{1}{n})^n &= 1 + nC1 \cdot \frac{1}{n} + nC2 \cdot \frac{1}{n^2} + \cdots + nCn\frac{1}{n^n} \\
&= 1 + \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{2} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1) \cdot 3 \cdot 2 \cdot 1}{n!} \cdot \frac{1}{n^2} \\
&= 1 + 1 + \frac{1}{2}(1 - \frac{1}{n}) + \frac{1}{6}(1 - n)(1 - \frac{2}{n}) + \cdots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n})\ldots\frac{2}{n} \cdot \frac{1}{n}
\end{aligned}
$$

Write x_{n+1} in a similar way, we observe that

$$x_n < x_{n+1}$$

Facts $2^{m-1} \leq m!$ for $m \in \mathbb{N} \Rightarrow \frac{1}{m!} \leq \frac{1}{2^{m-1}}$

$$x_n < 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 1 + \frac{1(1 - (\frac{1}{2})^n)}{1 - \frac{1}{2}} < 3$$

$\Rightarrow (x_n)$ is increasing and bounded

3.4   Subsequence and the Bolzano-Weierstrass Theorem

Example.

$$(x_n) = ((-1)^n)$$
$$x_{2n} = (-1)^{2n}$$
$$x_{2n+1} = (-1)^{2n-1}$$

So $a_n = x_{2n}$ is a sequence, while $b_n = x_{2n+1}$ is a subsequence of x_n.

Definition (**Subsequences**). Let (x_n) be a real sequence and consider a strickly increasing seqence of natural numbers $n_1 < n_2 < n_3 < \ldots$ The sequence

$$(x_{n_k} \; : \; k \in \mathbb{N})$$

is called a ***subsequence*** of (x_n)

Example. Any tails of a sequence is a subsequence: (x_n) n-th tail: (x_{m+k}), where $n = m + k, k \in \mathbb{N}$, is a subsequence.

Theorem. Suppose (x_n) converges to x. Then $x_{x_k} \to x$ for any subsequence of (x_n).

Proof. Let $\varepsilon > 0$, then

$$\exists N_\varepsilon > 0 \text{ s.t. } |x_n - x| < \varepsilon \text{ for } n > N_\varepsilon.$$

Note that

$$n_k \geq k, \; \forall k \in \mathbb{N}.$$

Exercise. By induction, $n_1 \geq 1, n_2 \geq n_1 \geq 1 \Rightarrow n_2 \geq 2$

When $k > N_\varepsilon$, $n_k > N_\varepsilon$, thus
$$|x_{n_k} - x| < \varepsilon$$
Therefore
$$(x_{n_k}) \to x$$

∎

Theorem. Let (x_n) be a sequence of real numbers, and let $x \in \mathbb{R}$. Then the following are equivalent:

1. (x_n) does not converge to x.

2. $\exists \varepsilon_0 > 0$, s.t. $\forall k \in \mathbb{N}, \exists n_k \in \mathbb{N}$ s.t.

$$n_k \geq k \; \& \; |x_{n_k} - x| > \varepsilon_0$$

3. $\exists \varepsilon_0 > 0$ and a subsequence (x_{n_k}) s.t.

$$|x_{n_k} - x| > \varepsilon_0, \; \forall k \in \mathbb{N}$$

Example.
$$x_n = (-1)^n \Rightarrow (x_n) \text{ does not converge to } 1$$

Proof.

$$3 \to 1:\ \text{by the contrapositive statement of definition}$$
$$3 \to 2:\ 3 \text{ is a stronger statement of 2}$$
$$1 \to 2:\ \text{left as exercise}$$

∎

Theorem. If (x_n) satisfies either of the following property, then it is ***divergent***:

1. There exists two subsequence (x_{n_k}) & (x_{m_k}) whose limits are NOT equal.

2. (x_n) is unbounded.

Example.

1. $(-1)^n$

2. (n)

3. (x_n) such that

$$x_{2k} = k$$
$$x_{2k+1} = (-1)^k$$

Proof. exercise

∎

Theorem (**Bolzano-Weierstrass Theorem**)**.** A bounded sequence of real numbers has a ***convergent subsequence***.
Example.
$$x_n = (-1)^n$$

Proof.

Lemma. If (x_n) is a sequence of real numbers, there exists a subsequence of (x_n) which is monotone.

proof of lemma:
Call the m-th term x_m a *"peek"* if x_m is at least as large as any term after it in the sequence.
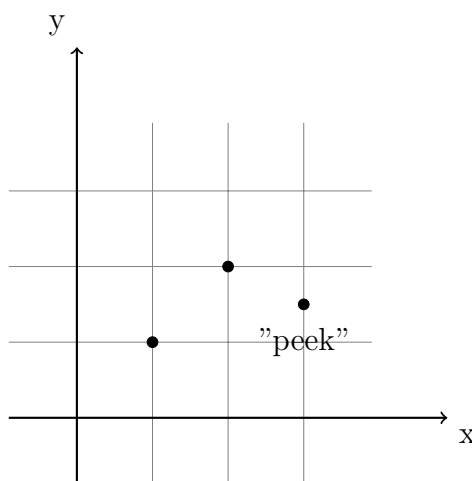
Case 1: (x_n) has infinitelyy many peaks

List the peaks of (x_n) in order of increasing index

$$x_{n_1}, x_{n_2}, \ldots$$

$\Rightarrow (x_{n_i})$ is a decreasing sequence.

Case 2: (x_n) has a finite number of peaks

Let s_1 be the first index after the last peak of (x_n). Then for every $n \geq s_1$, $\exists m \in \mathbb{N}$ such that $x_m > x_n$.

Choose

$$s_2 \geq s_1 \text{ such that } x_{s_2} > x_{s_1}, \ s_2 \geq s_1 \text{ such that } x_{s_2} > x_{s_1}, \ldots$$

$$\Rightarrow (x_{s_i}) \text{ is an increasing sequence.}$$

Remark. Lemma + monotone convergent theorem implies Bolzano-Weierstrass theorem.

Second proof

Suppose (x_n) is a bounded sequence.

$$\Rightarrow \exists I_1 = [a_1, b_1] \text{ such that } (x_n) \in I_1$$

Consider

$$I_2' = [a_1, \frac{a_1 + b_1}{2}], \ I_2'' = [\frac{a_1 + b_1}{2}, b_1]$$

Let $I_2 = [a_2, b_2]$ be one of I_2', I_2'' such that I_2 contains infinitely many terms of (x_n).

For $n \in \mathbb{N}$, define $I_n = [a_n, b_n]$ in a similar way.

For $i \in \mathbb{N}$, choose a term x_{n_i} such that $x_{n_i} \in I_i$ and $n_i > n_{i-1}$

Then

$$\begin{aligned} i &= 1, & n_i &= 1 \\ i &= 2, & \text{choose } n_2 \in \mathbb{N} \text{ such that } n_2 > n_1 \ \& \ x_{n_2} &\in I_2 \\ &\vdots & &\vdots \end{aligned}$$

- $\forall i \in \mathbb{N}, \ a_i \leq x_{n_i} \leq b_i$

- (a_i) increases, bounded above by $b_1 \Rightarrow (a_i) \to \sup(a_i)$

- (b_i) decreases, bounded below by $a_1 \Rightarrow (b_i) \to \inf(b_i)$

- $\inf|a_i - b_i| = \inf \frac{b_1 - a_1}{2^n} = n \Rightarrow \sup(a_i) = \inf(b_i)$

Thus (x_{n_i}) is convergent by squeeze theorem. $\blacksquare$

Theorem. Let (x_n) be a bounded sequence, and $x \in \mathbb{R}$ has the property that every convergent subsequence of (x_n) converges to x. Then x_n converges to x

Proof. Let $\forall \varepsilon > 0$

By Bolzano-Weierstrass theorem, $\exists$ a convergent subsequence (x_{n_i}) such that

$$\exists N_\varepsilon \in \mathbb{N} \text{ s.t. for } i > N_\varepsilon, |x_{n_i} - x| < \varepsilon$$

Assume (x_n) does not converge to x. Then by previous theorem of subsequence,

$$\exists \varepsilon_0 > 0 \text{ and a subsequence } (x_{n_k}) \text{ s.t. } |x_{n_k} - x| > \varepsilon_0, \forall k \in \mathbb{N}$$

Since (x_n) is bounded, (x_{n_k}) is also bounded. Thus there exists a convergent subsequence of (x_{n_k}) as $(x_{n_{k_i}})$.

Note that $(x_{n_{k_i}})$ is a convergent subsequence of (x_n), thus

$$(x_{n_{k_i}}) \to x \text{ which is contradiction to previous assumption}$$

$$\blacksquare$$

Definition. Let (x_n) be a sequence of real numbers. A point is called a ***subsequential limit*** of (x_n) if it is the limit of a subsequence of (x_n).

$$S = \{\alpha \in \mathbb{R} : x \text{ is a subsequential limit}\} \text{ NOTE: may be infinite set}$$

Definition. Let (x_n) be a sequence of real numbers.

- The ***limit superior*** of (x_n) is the infimum of the set of $v \in \mathbb{R}$ s.t. $v < x_n$ for at most a finite number of $n \in \mathbb{N}$. We write it as

$$\limsup(x_n) \text{ or } \limsup x_n \text{ or } \overline{\lim} x_n$$

- The ***limit inferior*** of (x_n) is the supremum of the set of $v \in \mathbb{R}$ s.t. $w > x_m$ for at most a finite number of $n \in \mathbb{N}$. We write it as

$$\liminf(x_n) \text{ or } \liminf x_n \text{ or } \overline{\lim} x_n$$

Intuition

- Suppose $v < x_n$ for at most finitely many $n \in \mathbb{N}$, then

Theorem. Let (x_n) be a bounded sequence, and $x \in \mathbb{R}$ has the property that every convergent subsequence of (x_n) converges to x. Then x_n converges to x

Proof. Let $\forall \varepsilon > 0$

By Bolzano-Weierstrass theorem, $\exists$ a convergent subsequence (x_{n_i}) such that

$$\exists N_\varepsilon \in \mathbb{N} \text{ s.t. for } i > N_\varepsilon, |x_{n_i} - x| < \varepsilon$$

Assume (x_n) does not converge to x. Then by previous theorem of subsequence,

$$\exists \varepsilon_0 > 0 \text{ and a subsequence } (x_{n_k}) \text{ s.t. } |x_{n_k} - x| > \varepsilon_0, \forall k \in \mathbb{N}$$

Since (x_n) is bounded, (x_{n_k}) is also bounded. Thus there exists a convergent subsequence of (x_{n_k}) as $(x_{n_{k_i}})$.

Note that $(x_{n_{k_i}})$ is a convergent subsequence of (x_n), thus

$$(x_{n_{k_i}}) \to x \text{ which is contradiction to previous assumption}$$

■

Definition. Let (x_n) be a sequence of real numbers. A point is called a **_subsequential limit_** of (x_n) if it is the limit of a subsequence of (x_n).

$$S = \{\alpha \in \mathbb{R} : x \text{ is a subsequential limit}\} \text{ (NOTE: may be infinite set)}$$

Example. Consider $(x_n) = \{(-1)^n | n \in \mathbb{N}\}$. Then

$$S \supseteq \{1, -1\}$$

Definition (lim sup and lim inf)**.** Let (x_n) be a sequence of real numbers.

- The **_limit superior_** of (x_n) is the infimum of the set of $v \in \mathbb{R}$ s.t. $v < x_n$ for at most a finite number of $n \in \mathbb{N}$. We write it as

  $$\limsup(x_n) = \limsup x_n = \overline{\lim} x_n = \inf \{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of n}\}$$

  Example. Consider $(x_n) = \frac{1}{n}$
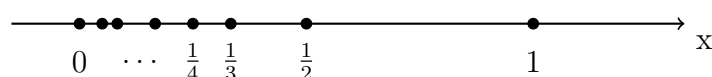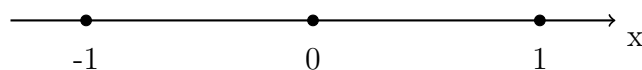

  Let

  $$X = \{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of n}\}$$

  - $-1 \notin X$ because there are infinitely many x_n such that $v < x_n$.

  - $\frac{1}{2} \in X$ because there are finitely many x_n such that $v < x_n$.

  - $2 \in X$ because there is no x_n such that $v < x_n$, which is smaller than finite and thereby satisfies the definition.

  We thus conclude that

  $$(0, \infty) \subset X \text{ and } \limsup x_n = \inf X$$

  Example. Consider $(x_n) = (-1)^n$:

  - $1 \in X$ because there is no x_n such that $v < x_n$.

  - $2 \in X$ because there is no x_n such that $v < x_n$.

  - $0, -1 \notin X$ because there are infinitely many x_n such that $v < x_n$.

  We thus conclude that

  $$[1, \infty) \subset X \text{ (in fact they are equal)}$$

- The **_limit inferior_** of (x_n) is the supremum of the set of $w \in \mathbb{R}$ s.t. $w > x_m$ for at most a finite number of $n \in \mathbb{N}$. We write it as

  $$\liminf(x_n) \text{ or } \liminf x_n \text{ or } \underline{\lim} x_n = \sup \{w \in \mathbb{R} | w > x_n \text{ for at most a finite number of n}\}$$

Intuition

- Suppose $v < x_n$ for at most finitely many $n \in \mathbb{N}$, then for all large n, $v \geq x_n$. $\Rightarrow$ No subsequential limit of (x_n) can possibly exceed v.

- Similar observation for $\underline{\lim} \, x_n$

Theorem. Let (x_n) be a bounded sequence of real numbers, and let $x^* \in \mathbb{R}$. Then TFAE:

1. $x^* = \lim \sup (x_n)$

2. If $\varepsilon > 0$, there are at most a finite number of $n \in \mathbb{N}$ s.t. $x^* + \varepsilon < x_n$, but infinitely many n for which $x^* - \varepsilon < x_n$

3. If $u_m = \sup \{x_n | n \geq m\}$ (sup of (m-1)-th tail), then $x* = \inf \{u_m | m \in \mathbb{N}\} = \lim u_m$

4. If S is the set of subsequential limits of x_n, then $x^* = \sup S$.

Remark. .

- u_m is decreasing.

- There is a similar such list of equivalent properties for $\lim \inf$.

Corollary. A bounded sequence (x_n) is convergent iff $\overline{\lim} x_n = \lim x_n$

Proof. A direct result of the theorem:

$$\overline{\lim} x_n = \sup S \text{ and } \underline{\lim} x_n = \inf S$$

∎

Proof of thm. (a) $\Rightarrow$ (b). Let $\varepsilon > 0$. Then

$$x^* + \varepsilon > x^* = X = \inf \{v \in \mathbb{R} | v < x_n \text{ for at most a finite number of n.}\}$$

$$\Rightarrow \exists v \in \mathbb{R} \text{ s.t. } x^* \leq v < x^* + \varepsilon$$

and there are only finitely mant n with $v < x_n$.

For any n for which $x^* + \varepsilon < x_n$, $v < x_n$. Thus there are only finitely many such n.

If $x^* - \varepsilon \notin X$, then there are infinityly many n such that $x^* - \varepsilon < x_n$ ∎

Proof of thm. (b) $\Rightarrow$ (c). Fix $\varepsilon > 0$.

By (b), there are only finitely many n with $x^* + \varepsilon < x$.

Take $N \in \mathbb{N}$ large enough such that

$$
\begin{aligned}
& x^* + \varepsilon \geq x_n && \forall n \geq N \\
\Rightarrow \quad & x^* + \varepsilon \geq u_N \\
\Rightarrow \quad & x^* + \varepsilon \geq \lim u_n \\
\Rightarrow \quad & x^* \geq \lim u_m && \forall n \geq N
\end{aligned}
$$

On the other hand, there are infinitely many n with $x^* - \varepsilon < x_n \leq u_n$.

Thus, there exists a subsequence of u_n, say u_{n_k}, satisfies

$$
\begin{aligned}
& x^* - \varepsilon \leq u_{n_k} \\
\Rightarrow \quad & x^* - \varepsilon \leq \lim u_{n_k} = \lim (u_n) \\
\Rightarrow \quad & x* \leq \lim (u_n)
\end{aligned}
$$

$\blacksquare$

Proof of thm. (c) $\Rightarrow$ (d). **_Goal_**:

$$x* = \lim (u_m), u_m = \sup x_n | n \geq m$$

$$\Rightarrow x^* = \sup S \text{ where S is the set of subsequential limits.}$$

Let (x_{n_k}) be a convergent subsequence of (x_n). Notice that $\lim (x_{n_k}) \in S$.

$$
\begin{aligned}
& && n \geq k \\
\Rightarrow \quad & x_{n_k} \leq \sup \{x_n | n \geq k\} = u_k \\
\Rightarrow \quad & \lim (x_{n_k}) \leq \lim (u_k) = x^* \\
\Rightarrow \quad & x^* \text{ is an upper bound of S..}
\end{aligned}
$$

For 1, $\exists n_1 \in \mathbb{N}$ s.t.
$$u_1 - 1 \leq x_{n_1} \leq u_1$$

For $\frac{1}{2}$, $\exists n_2 \in \mathbb{N}$ s.t.
$$u_2 - \frac{1}{2} \leq x_{n_2} \leq u_2$$
$$\dots$$

For $\frac{1}{k}$, $\exists n_k \in \mathbb{N}$ s.t.
$$u_k - \frac{1}{k} \leq x_{n_k} \leq u_k$$

When $k \to \infty$,
$$x^* - 0 \leq \lim (x_{n_k}) \leq x^*$$

By squeeze theorem,
$$\lim (x_{n_k}) = x^*$$

$\blacksquare$

Proof of thm. (d) $\Rightarrow$ (a). **Goal:**

$$x^* = \sup S$$

$$\Rightarrow x^* = \limsup x_n = \inf\left\{v \in \mathbb{R}\,|\,v < x_n \text{for at most finite many of n}\right\}$$

Fix $\varepsilon > 0$

There is no subsequence of x_n which has a limit exceeding $x^* + \varepsilon$.

$\Rightarrow$ There is only finitely many n with $x_n > x^* + \varepsilon$.

$\Rightarrow$ $$x^* + \varepsilon \in X$$

$\Rightarrow$ $$\inf X \leq x^* + \varepsilon$$

$\Rightarrow$ $$\limsup x_n \leq x^* + \varepsilon$$

$\Rightarrow$ $$\limsup x_n \leq x^*$$

Next, consider $x^* - \varepsilon$

Then, there exists a subsequential limit of x_n which is greater or equal to $x^* - \frac{1}{2}\varepsilon$.

There exists a convergent subsequence of (x_n), say (x_{n_k}), such that

$$\lim(x_{n_k}) \geq x^* - \frac{1}{2}\varepsilon$$

$\Rightarrow$ $$\text{There are infinitely many n with } x^* > x^* - \varepsilon.$$

$\Rightarrow$ $$\forall a \in X, x^* - \varepsilon \leq a$$

$\Rightarrow$ $$x^* - \varepsilon \leq \inf X$$

$\Rightarrow$ $$\limsup x_n \geq x^* - \varepsilon$$

$\Rightarrow$ $$\limsup x_n \geq x^*$$

In conclusion,

$$\limsup x_n = x^*$$

$\blacksquare$

3.5 Cauchy Criterion

Definition (**Cauchy Sequence**). A sequence (x_n) is ***Cauchy sequence*** if

$\forall \varepsilon > 0, \exists H \in \mathbb{N}$ s.t. $\forall n, m \in \mathbb{N}, n > 0, m > 0$,

$$|x_n - x_m| < \varepsilon$$

Example. $(\frac{1}{n})$ is a Cauchy sequence.

Proof. Observe that $\forall n, m \in \mathbb{N}, n \geq m$,

$$\left| \frac{1}{n} - \frac{1}{m} \right| = \left| \frac{m-n}{mn} \right| = \frac{n-m}{mn} \leq \frac{n}{mn} < \frac{1}{m}$$

Choose $H = \lceil \frac{1}{\varepsilon} \rceil + 1$, Then

$\forall n, m \geq H$,
$$\left| \frac{1}{n} - \frac{1}{m} \right| \leq \frac{1}{m} < \varepsilon$$

∎

Example. $(-1)^n$ is not a Cauchy sequence.

Proof. Choose $\varepsilon_0 = \frac{1}{2}, \forall H \in \mathbb{N}$, choose $n, m \geq H$ s.t. n is even, m is odd. Then

$$|(-1)^n - (-1)^m| = |1 - (-1)| = 2 > \frac{1}{2}$$

Thus $(-1)^n$ does not satisfies the definition of Cauchy sequence.

∎

Theorem. If (x_n) is convergent, then it is Cauchy sequence.

Proof. Let $\lim (x_n) = x$.

Goal:
$$|x_n - x| < \varepsilon \Rightarrow |x_n - x_m| < \varepsilon$$

$\forall n, m \geq N_\varepsilon$,

$$\begin{aligned}
|x_n - x_m| &= |(x_n - x) + (x - x_m)| \\
&\leq |x_n - x| + |x - x_m| \text{ by triangle inequality} \\
&< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon
\end{aligned}$$

∎

Lemma. If (x_n) is Cauchy, the it is bounded.

Proof. Choose $\varepsilon = 1, \exists H \geq 0$ s.t. $\forall n, m \geq H$,

$$|x_n - x_m| < 1$$

Since the choice of m satisfies $m \geq H$, we may choose $m = H$ s.t.

$$|x_n - x_H| < 1$$

It follows that

$$|x_n| - |x_H| \leq |x_n - x_H| < 1$$
$$|x_n| < |x_H| + 1, \forall n \geq H$$

Let

$$M := \max \{|x_1|, |x_2|, |x_3|, \ldots, , |x_{H-1}|, |x_H| + 1\}$$

Thus $\forall n \in \mathbb{N}, |x_n| \leq M$ ∎

Theorem (Cauchy Convergence Theorem). A sequence of real numbers is ***convergent*** if and only if it is ***Cauchy*** sequence.

(⇒). Done in the previous theorem. ∎

(⇐). Suppose (x_n) is Cauchy. By lemma, it is bounded.

By Bolzano-Weierstrass theorem, there exists a convergent subsequence (x_{n_k}).

Let $\lim(x_{n_k}) = x$.

Goal: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$

$$|x_n - x| < \varepsilon$$

We will use the trick of ***insert subsequence:***

By definition of convergence, $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall k \geq N_\varepsilon$

$$|x_{n_k} - x| < \frac{1}{2}\varepsilon$$

By definition of Cauchy sequence, $\forall \varepsilon > 0, \exists H \in \mathbb{N}, H \geq 0$ s.t. $\forall n, m \geq H$

$$|x_n - x_m| < \frac{1}{2}\varepsilon$$

Thus $\forall k \geq \max\{H, N_\varepsilon\}$,

$$
\begin{aligned}
|x_k - x| &= |(x_k - x_{n_k}) + (x_{n_k} - x)| \\
&\leq |x_k - x_{n_K}| + |x_{n_k} - x| \text{ since } n_k \geq k \geq \max\{H, N_\varepsilon\} \\
&< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon
\end{aligned}
$$

∎

Example. Let (h_n) be the sequence of harmonic series such that

$$h_n = \sum_{i=1}^{n} \frac{1}{i} \text{ , and } \lim(h_n) = \sum_{n=1}^{\infty} \frac{1}{n}$$

Claim: (h_n) is divergent

Goal: show that it is NOT Cauchy.

Proof. $\forall m, n \in \mathbb{N}$, WLOG suppose $m \geq n, h_m > h_n$, we have

$$
\begin{aligned}
|h_m - h_n| &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m-1} + \frac{1}{m} \\
&\geq \frac{m-n}{n}
\end{aligned}
$$

since there are (m-n)'s terms on the right side of the equation.

So we choose $\varepsilon_0 = \frac{1}{2}, \forall H \geq 0$, choose $n = H, m = 2H$, then

$$|h_m - h_n| \geq \frac{m-n}{m} = \frac{1}{2}$$

Thus, (h_n) is NOT Cauchy. $\qquad\blacksquare$

3.6 Property of Divergent Sequence

Example (divergent sequence)**.** :

- $(n) \to \infty$

- $(-n) \to -\infty$

- $(-1)^n \cdot n$ is divergent and unbounded.

- $(-1)^n$ is divergent and bounded.

Definition (**Properly Divergent**)**.** Let (x_n) be a sequence of real numbers. We say that

1. (x_n) **_tends to_** ∞, or $\lim(x_n) = \infty$ if $\forall a \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$x_n > a$$

2. Similarly, (x_n) **_tends to_** $-\infty$, or $\lim(x_n) = -\infty$ if $\forall a \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$x_n < a$$

In either case, we say that (x_n) is **_properly divergent_**.

Example. Let $C > 0$. (C^n) is properly divergent. We write $\lim(C^n) = \infty$.

Proof. Notice that

$$C^n = [1 + (C - 1)]^n \geq 1 + n(C - 1) \text{ by Bernoulli's inequality}$$

Goal: $\forall a \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$1 + n(C - 1) > a_n \Leftrightarrow n > \frac{a_n - 1}{C - 1}$$

Then we choose $N = \lceil \frac{a-1}{c-1} \rceil + 1$. Thus $\forall n \geq N, C^n > a$. ∎

Theorem. A monotone sequence is **_divergent_** if and only if it is **_bounded_**.

Proof. Exercise. ∎

Theorem (**Comparison Test**)**.** Let (x_n) and (y_n) be two sequences. Suppose $\forall n \in \mathbb{N}, x_n \geq y_n$. Then,

1. If $(x_n) \to \infty$, then $(y_n) \to \infty$.

2. If $(y_n) \to -\infty$, then $(x_n) \to -\infty$.

Proof. Exercise. ∎

Theorem (**Limit Comparison Test**)**.** Let (x_n) and (y_n) be two sequences of positive real numbers. Suppose $\exists L \in \mathbb{R}, L > 0$ s.t.

$$\exists \lim(\frac{x_n}{y_n}) = L$$

then $\lim(x_n) = \infty$ if and only if $\lim(y_n) = \infty$

Proof. **_Claim:_** For N large enough,

$$\frac{1}{2}L \cdot y_n < x_n < 2L \cdot y_n$$

Goal: Claim+Comparison Theorem=Proof.

By definition of limit, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N$,

$$L - \varepsilon < \frac{x_n}{y_n} < L + \varepsilon$$

Choose $\varepsilon = \frac{1}{2}L$, we have

$$\frac{1}{2}L \cdot y_n < x_n < \frac{3}{2}L < 2L \cdot y_n$$

since $L > 0$ and these are positive sequences.

Thus by Comparison Theorem,

$$\lim(\frac{1}{2}L \cdot y_n) = \infty \Rightarrow \lim(x_n) = \infty$$

$\blacksquare$

3.7   Introduction to Infinite Series

Definition (**Infinite Series**). If (x_n) is a sequence of real numbers, then the ***infinite series*** generated by (x_n) is the sequence (s_k) defined by

$$s_k = \sum_{i=1}^{k} x_i$$

Terms s_k are called ***partial sums***.

Notation. $\sum x_i$ to mean this series or its limit at infinity $\lim(x_k)$.

Theorem (**Cauchy Criterion for Series**). The series $\sum x_i$ ***converges*** if and only if $\forall \varepsilon > 0, \exists M \in \mathbb{N}$ s.t. $\forall n, m \in \mathbb{N}, n > m \geq M$,

$$|x_{m+1} + x_{m+2} + \cdots + x_{n-1} + x_n| < \varepsilon$$

Or we write it as

$$|s_k - s_m| < \varepsilon$$

Proof: Exercise. ∎

Theorem (Montone Convergence for Series)**.** Let (x_n) be a sequence of non-negative real numbers. Then the series $\sum x_n$ ***converges*** if and only if (s_k) is bounded.

Proof. Exercise.

∎

Example. $\sum \frac{1}{n^2}$ is convergent.

Proof. **_Goal:_** find convergent subsequence (s_{k_j}).

Consider subsequence (s_{k_j}) where $k_j = 2^j - 1$.

Observe that:

$$s_{k_1} = 1$$
$$s_{k_2} = 1 + \frac{1}{2^2} + \frac{1}{3^2}$$
$$< s_{k_1} + 2 \cdot \frac{1}{2^2} = 1 + \frac{1}{2}$$
$$s_{k_3} = 1 + (\frac{1}{2^2} + \frac{1}{3^2}) + (\frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{9^2})$$
$$< s_{k_1} + 2 \cdot \frac{1}{2^2} = 1 + \frac{1}{2}$$

By induction(details are left as exercise), one can show that

$$s_{k_j} < \sum_{n=0}^{j-1} \frac{1}{2^n} < \sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \text{ (by limit of geometric series.)}$$

Thus (s_{k_j}) is bounded.

By theorem proved in homework, an increasing sequence with bounded(and thus convergent) subsequence implies that the sequence is convergent. ∎

4   Limits

4.1   Limits of Functions

Let $f : A \to B$ be a function where $A, B \subseteq \mathbb{R}$. Let $a \in A, L \in B$.

Goal: Define

$$\lim_{x \to a} f(x) = ?$$

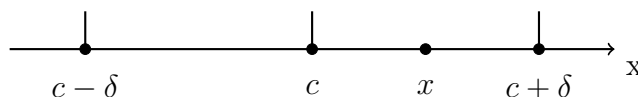**_Intuition:_** Define **closeness** on real line.

Definition (**Cluster Point**). Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a **_cluster point_** of A if

$\forall \delta > 0, \exists x \in A, x \neq c$ such that

$$|x - c| < \delta$$

Or

$$V_\delta(c) \cap (A \; \{c\}) \neq \phi$$

Theorem. $c \in \mathbb{R}$ is a cluster point if and only if

there exists a sequence $(a_n) \in A$ such that

$$\lim(a_n) = c \text{ and } \forall n \in \mathbb{N}, a_n \neq c$$

Sketch of Proof. $(\Rightarrow)$

$$\delta = 1, \exists a_1 \in A \backslash \{c\} \text{ s.t. } |a_1 - c| < 1$$
$$\delta = \frac{1}{2}, \exists a_2 \in A \backslash \{c\} \text{ s.t. } |a_2 - c| < \frac{1}{2}$$
$$\dots \text{ observe that:}$$
$$\forall \delta > 0, \exists a_n \in A \backslash \{c\} \text{ s.t. } |a_n - c| < \frac{1}{n} < \delta$$
$$\Rightarrow \lim(a_n) = c \text{ by squeeze theorem.}$$

$(\Leftarrow)$

$\forall \delta > 0, \exists N_\varepsilon > 0$ s.t.

$$|a_{N_\varepsilon} - c| < \delta$$

Note that $a_{N_\varepsilon} \in A \backslash \{c\}$ where c is a cluster point of A.   ■

Example. Let X be the set of cluster points of A.

- $A = (1,2) \cup (3,4) \Rightarrow X = [1,2] \cup [3,4]$. *Proof: Exercise.*

- $A = \{0\} \cup (1,2) \Rightarrow X = \phi$.

  Sketch of (2). :

  1. Let $x \in [1,2]$. Prove that $x \in X$. Thus $[1,2] \in X$.

  2. Prove that $0 \notin X$.

  3. Prove that $x \in X$ if $x \notin \{0\} \cup [1,2]$.

■

Remark. A may not be a subset of the set of cluster points of A.

Example. :

- $A = \mathbb{Z} \Rightarrow X = \phi$.

- $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} \Rightarrow X = \{0\}$ *Proof: Exercise.*

Definition (**Delta-Epsilon Definition of Limit**)**.** Let $A \subseteq \mathbb{R}$, c is a cluste point of A, $f : A \to \mathbb{R}$. A real number L is the ***limit of f at c*** if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$,

$$0 < |x - c| < \delta \to |f(x) - L| < \varepsilon$$

Theorem (**Uniqueness of Limit**)**.** If $f : A \to \mathbb{R}$ and c is a cluster of points of A, then f has at most 1 limit at c.

Proof. We will prove this by contradiction.

Let L_1 and L_2 be limits of f at c. Assume $L_1 \neq L_2$. Choose $\varepsilon = \frac{|L_1 - L_2|}{2} > 0$. Then,

$$\exists \delta_1 \text{ s.t. } 0 < |x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \frac{\varepsilon}{2}$$

$$\exists \delta_2 \text{ s.t. } 0 < |x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \frac{\varepsilon}{2}$$

Consider $\delta := \min\{\delta_1, \delta_2\}$.

Since c is a cluster point, $\exists x_0 \in A$ s.t.

$$0 < |x_0 - c| < \delta$$

Since

$$|f(x_0) - L_1| < \frac{\varepsilon}{2}, |f(x_0) - L_2| < \frac{\varepsilon}{2}$$

We have

$$|L_1 - L_2| \leq |L_1 - f(x_0)| + |f(x_0) - L_2|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
$$= \frac{|L_1 - L_2|}{2}$$

which is a contradiction.

■

Notation.
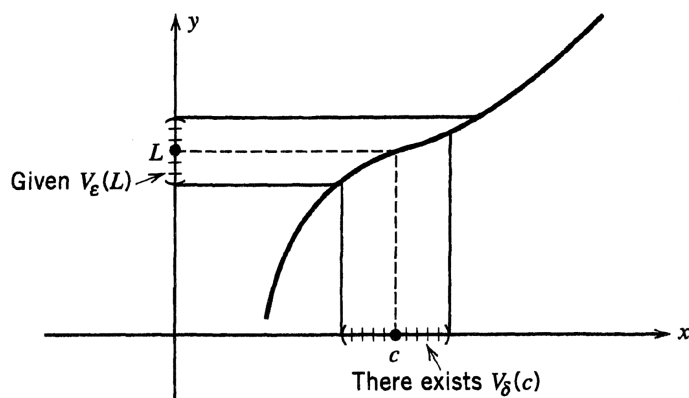$$L = \lim_{x \to c} f(x) \text{ or } L = \lim_{x \to c} f$$

And we say that *f(x) approaches to L as x approaches to c.*

Remark (**Divergence of function**). If the limit of $f(x)$ at c does not exists, we say that f **_diverges_** at c.

Theorem. Let $f : A \to \mathbb{R}$ and c be a cluster point of f. The following are equivalent:

1. $\lim_{x \to c} f(x) = L$

2. $\forall V_\varepsilon(L) \, \varepsilon$-neighborhood of L, $\exists V_\delta(c) \, \delta$-neighborhood of c s.t.

$$x \in V_\delta(c) \cap (A \backslash \{c\}) \Rightarrow f(x) \in V_\varepsilon(c)$$

Example. $\lim_{x \to c} f(x) = a$.

Proof. $\forall \varepsilon > 0, \exists \delta = 1$ s.t.

$$x \in V_\delta(c) \cap (\mathbb{R} \backslash \{c\}) \Rightarrow f(x) = a \in V_\varepsilon(L)$$

∎

Example. $\lim_{x \to c} f(x) = c$.

Proof. $\forall \varepsilon > 0, \exists \delta = \varepsilon$ s.t.

$$x \in V_\delta(c) \cap (A \backslash \{c\})$$
$$\Rightarrow f(x) = x \in V_\delta(c) \cap (A \backslash \{c\})$$
$$\Rightarrow f(x) \in V_\varepsilon(c) \cap (A \backslash \{c\}) \subseteq V_\varepsilon(c)$$

∎

Example.
$$\lim_{x \to c} x^2 = c^2$$

Proof. **_Goal_**: $\forall \varepsilon > 0$, find a $\delta(c, \varepsilon) > 0$ s.t.

$$\text{if } 0 < |x - c| < \delta, \text{ then } \left| x^2 - c^2 \right| < \varepsilon$$

which is equivalent to show

$$|x + c| \, |x - c| < \varepsilon$$

Since the choice of δ is dependent on ε and c only, $|x - c|$ can be easily confined with some constant. Let's assume that

$$|x - c| < 1$$

Now, the only task left is to find a way to confine $|x + c|$ with a constant by manipulating $|x - c| < 1$.

Here we will apply a common trick that estimates addition $|x + c|$ with subtraction $|x - c|$:

$$|x| - |c| \leq |x - c| < 1 \text{ by triangle inequality.}$$

Rearranging the inequality, we have:

$$|x| < |c| + 1$$

Adding the second $|c|$ to both side of inequality, then, we apply triangle inequality again:

$$|x + c| \leq |x| + |c| < 2\,|c| + 1$$

$$|x + c| < 2\,|c| + 1$$

Notice that this is equivalent to show

$$|x + c| \, |x - c| < (2\,|c| + \delta)\,|x - c| < \varepsilon$$

Rearrange the constant factor

$$|x - c| < \frac{\varepsilon}{2\,|c| + 1}$$

Thus, our choice of δ must satisfies two conditions at the same time:

$$\begin{cases} |x - c| < 1 \\ |x - c| < \frac{\varepsilon}{2|c|+1} \end{cases}$$

We achieve this by simply choosing

$$\delta = \min\left\{ \frac{\varepsilon}{2\,|c| + 1}, 1 \right\}$$

$\blacksquare$

Remark (**_The Toolbox of Proofs_**)**.** The readers should develop their "toolbox" of proof techniques. That is, the **_estimation against a constant_** + manipulation of **_triangle inequality_** + choice of δ that satisfies **_multiple conditions_**.

Example (Harder)**.**

$$\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$$

Proof. Observe that

$$\left| \frac{1}{x} - \frac{1}{c} \right| = \frac{|x - c|}{|cx|}$$

Here, we cannot choose $|x - c|$ smaller than some constant(why? try it on your own). Instead, we choose

$$|x - c| < \frac{|c|}{2}$$

By triangle inequality,

$$\frac{|c|}{2} < |x| < \frac{3|c|}{2}$$

Multiply $|c|$ to each term of the inequality,

$$\frac{c^2}{2} < |cx| < \frac{3c^2}{2}$$

Thus

$$\frac{1}{|cx|} < \frac{2}{c^2}$$

It follows that

$$\frac{|x - c|}{|cx|} < \frac{2}{c^2}|x - c|$$

In order to make the left term of the inequality less than ε, it suffices to confine

$$\frac{2}{c^2}|x - c| < \varepsilon$$

$$|x - c| < \frac{c^2}{2}\varepsilon$$

Thus, our choice of δ must satisfies two conditions at the same time:

$$\begin{cases} |x - c| < \frac{|c|}{2} \\ |x - c| < \frac{c^2}{2}\varepsilon \end{cases}$$

We achieve this by simply choosing

$$\delta = \min \left\{ \frac{|c|}{2}, \frac{c^2}{2}\varepsilon \right\}$$

■

Remark (***Factoring*** $|x - c|$)**.** Since we have the premice of $|x - c| < \delta$ for free, it would be much easier for us to confine $|f(x) - L|$ if we factor $|x - c|$ from the difference.

Example (Much Harder)**.**
$$\forall n \in \mathbb{N}, \lim_{x \to c} x^n = c^n$$

Proof. By difference of n-th powers factorization

$$\left|x^n - c^n\right| = |x - c| \left|\sum_{i=0}^{n-1} x^i c^{n-1-i}\right| \leq |x - c| \cdot \sum_{i=0}^{n-1} |x|^i |c|^{n-1-i}$$

If
$$|x - c| < 1$$

then by triangle inequality,
$$|x| < |c| + 1$$

It follows that

$$\left|x^n - c^n\right| < |x - c| \cdot \sum_{i=0}^{n-1} (|c| + 1)^i \left|c^{n-1-i}\right|$$

Similar to the previous example, it suffices to confine

$$|x - c| \cdot \sum_{i=0}^{n-1} (|c| + 1)^i \left|c^{n-1-i}\right| < \varepsilon$$

$$|x - c| < \frac{\varepsilon}{\sum_{i=0}^{n-1} (|c| + 1)^i |c^{n-1-i}|}$$

Thus, choose
$$\delta = \min\{1, \frac{\varepsilon}{\sum_{i=0}^{n-1} (|c| + 1)^i |c^{n-1-i}|}\}$$

■

Remark (***Estimate x against c***)**.** This is another trick by triangle inequality:

If $\exists k \in \mathbb{R}, k > 0$ s.t. $|x - c| < k$, then $|x| < |c| + k$

Remark (***difference of n-th powers factorization***)**.**

$$(x^n - c^n) = (x - c)(x^{n-1} + cx^{n-2} + \cdots + c^{n-2}x + c^{n-1}) = (x - c) \sum_{i=0}^{n-1} x^i c^{n-1-i}$$

Example (***Some Tedious Factorization***).

$$\lim_{x \to 2} \frac{x^3 + 2x - 1}{6x^2 - 5} = \frac{11}{19}$$

Proof.

$$\left| \frac{x^3 + 2x - 1}{6x^2 - 5} - \frac{11}{19} \right| = \left| \frac{19x^3 - 66x^2 + 38x + 36}{19(6x^2 - 5)} \right|$$

By some tedious factorization,

$$\ldots = |x - 2| \frac{|19x^3 - 28x - 18|}{19 |6x^2 - 5|}$$

We estimate $|x - 2|$ with constant 1

$$|x - 2| < 1$$

$$1 < x < 3 \text{ or } |x| < 3$$

It follows that

$$1 < x^2 < 9$$
$$1 < 6x^2 - 5 < 49$$
$$1 > \frac{1}{6x^2 - 5} > \frac{1}{49}$$

So

$$\frac{1}{19 |6x^2 - 5|} < \frac{1}{19}$$

Similarly,

$$\left| 19x^3 - 66x^2 + 38x + 36 \right| \le 19 |x|^2 + 28 |x| + 18 < 19 \cdot 3^2 + 28 \cdot 3 + 18 = 273$$

Thus

$$\ldots \le |x - 2| \cdot \frac{273}{19} < \varepsilon$$

$$|x - 2| < \frac{19}{273} \varepsilon$$

We conclude that it suffices to choose

$$\delta = \min \left\{ 1, \frac{19}{273} \varepsilon \right\}$$

■

Theorem (_ Sequential Criterion of Limits**_).** Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. $\lim_{x \to c} f = L$ if and only if for all sequence $(x_n) \in A$ that converge to c and $(x_n) \neq c, \forall n \in \mathbb{N}$, $(f(x_n))$ converges to L.

Proof. ($\Rightarrow$). By definition of convergent sequence, $\forall \delta > 0, \exists K \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \geq K$,

$$|x_n - c| < \delta$$

By definition of limit, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$,

$$|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Thus, choose δ given ε, and choose K given δ, we have:

$$|x_n - c| < \delta \Rightarrow |f(x_n) - L| < \varepsilon$$

$\blacksquare$

($\Leftarrow$). We will prove this by contrapositive.

Assume that there exists $\varepsilon_0 > 0$ and a $(x_n) \in A$ converges to c with $(x_n) \neq c$ such that for all $n \in \mathbb{N}$

$$0 < |x_n - c| < \frac{1}{n} \Rightarrow |f(x_n) - L| \geq \varepsilon_0$$

Thus the function does not have a limit at c. We thereby conclude the converse of the statement.

$\blacksquare$

Theorem (_ Sequential Criterion of Divergence**_).** Let $A \subseteq \mathbb{R}, f : A \to \mathbb{R}, c$ be a cluster point of $A, (x_n) \to c$ s.t. $(x_n) \neq c$, and $L \in \mathbb{R}$[1].

1. L is NOT the limit of f at c $\iff$ $f(x_n)$ does NOT converge to L.

2. f diverges $\iff$ $f(x_n)$ does NOT converge.

Proof. Exercise. $\blacksquare$

Example.

$$\lim_{x \to 0} \frac{1}{x} \text{ does NOT exists in } \mathbb{R}$$

Proof. Let $(x_n) = \frac{1}{n} \to 0$. Then

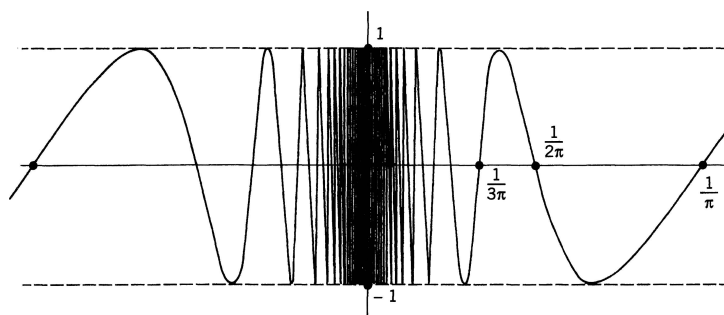$$f(x_n) = \frac{1}{\frac{1}{n}} = n \to \infty$$

$\blacksquare$

---

[1]The following theorems are NOT equivalent!!

Example.

$$\lim_{x \to 0} \sin(\frac{1}{x}) \text{ DNE in } \mathbb{R}$$

Proof. Let $(x_n) = \dfrac{1}{2n\pi} \to 0$ and $(y_n) = \dfrac{1}{\dfrac{1}{2}\pi + 2n\pi} \to 0$. Then

$$f(x_n) = \sin(2n\pi) = 0$$

$$f(y_n) = \sin(\frac{1}{2}\pi + 2n\pi) = 1$$

Thus limit of f at 0 does NOT exists in $\mathbb{R}$.

∎

Example.
Definition (_ Signum Function**_).** Let $A \subseteq \mathbb{R}$ and $sgn(x) : A \to \mathbb{R}$ s.t.

$$sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$\lim_{x \to 0} sgn(x) \text{ DNE in } \mathbb{R}$$

Proof. Let $(x_n) = \dfrac{1}{n} \to 0$ and $(y_n) = \dfrac{1}{-n} \to 0$. Then

$$f(x_n) = sgn(n) = 1$$

$$f(y_n) = sgn(-n) = -1$$

Thus limit of f at 0 does NOT exists in $\mathbb{R}$.

∎

4.2   Limit Theorem

Definition (***Bounded Neighborhood of c***)**.** Let $A \subseteq \mathbb{R}, f : A \to \mathbb{R}$, c be a cluster point of A. Then we say f is a ***bounded Neighborhood of c*** if there exists $\delta > 0$ and $\exists M > 0$ s.t. $\forall x \in A \cap V_\delta(c)$

$$|f(x)| \leq M$$

Theorem (***Existence of Bounded Neighborhood at Limit***)**.** If f has a limit at c, then f is bounded on some neighborhood of c.

Proof. By definition of limit, $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A \backslash \{c\}$

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$$

By triangle inequality,

$$|f(x)| - |L| \leq |f(x) - L| < \epsilon$$
$$|f(x)| < |L| + \epsilon$$

If $f(c)$ is not defined on A, then let $M = |L| + \epsilon$. If $f(c)$ is defined on A, then let $M = \max\{f(c), |L| + \varepsilon\}$. Since the choice of ε is arbitrary, we will choose $\epsilon = 1$. Thus,

$$f(x) \leq M$$

∎

Definition. Let $A \subseteq \mathbb{R}$; $f, g : A \to \mathbb{R}$, c be a cluster point of A. Then $\forall x \in A$

- ***Sum*** of function: $(f + g)(x) = f(x) + g(x)$.

- ***Difference:*** $(f - g)(x) = f(x) - g(x)$.

- ***Multiple:*** $(bf)(x) = b \cdot f(x)$ for some $b \in \mathbb{R}$.

- ***Product:*** $(f \cdot g)(x) = f(x) \cdot g(x)$.

- ***Quotient:*** $(\frac{f}{g})(x) = \dfrac{f(x)}{g(x)}$ if $g(x) \neq 0$.

Theorem (***Limit Theorem***)**.** If $\exists L, M \in R$ s.t. $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

- $\lim_{x \to c} (f + g)(x) = L + M$.

- $\lim_{x \to c} (f - g)(x) = L - M$.

- $\lim_{x \to c} (bf)(x) = b \cdot L$.

- $\lim_{x \to c} (f \cdot g)(x) = L \cdot M$.

- $\lim_{x \to c} (\frac{f}{g})(x) = \dfrac{L}{M}$ if $\lim_{x \to c} g(x) \neq 0$.

Proof. Exercise. ∎

Remark. Always check conditions befor applying: this is true only if both f and g has a limit at c !

Example. $\lim\limits_{x\to 4}\dfrac{(x-4)(x+3)}{4(x-4)(x-5)}=\lim\limits_{x\to 4}\dfrac{x+3}{4(x-5)}=-\dfrac{7}{4}$

Corollary (_ Polynomial Function**_).**

$$\lim_{x\to c}p(x)=\lim_{x\to c}\sum_{i=0}^{n}a_i\cdot x^i=\sum_{i=0}^{n}a_i\cdot\lim_{x\to c}x^i=\sum_{i=0}^{n}a_i\cdot x^i=p(c)$$

Corollary (_ Rational Function**_).** For polynomial functions $p(x),q(x)$ s.t. $\lim\limits_{x\to c}p(x)\to p(c),\lim\limits_{x\to c}q(x)\to q(c)\neq 0$,

$$\lim_{x\to c}\frac{p(x)}{q(x)}=\frac{p(c)}{q(c)}$$

Theorem. Let $A\subseteq\mathbb{R}$; $f,g:A\to\mathbb{R}$, c be a cluster point of A.

If $\exists a,b\in\mathbb{R},\forall x\in A,x\neq c$ satisfiying

$$a\leq f(x)\leq b\text{ and }\lim_{x\to c}f=L$$

then

$$a\leq\lim_{x\to c}f\leq b$$

Proof. $\forall(x_n)\in A\backslash\{c\},(x_n)\to c$,

$$a\leq f(x_n)\leq b\Rightarrow a\leq L\leq b$$

∎

Theorem (_ Squeeze Theorem**_).** Let $A\subseteq\mathbb{R}$; $f,g,h:A\to\mathbb{R}$, c be a cluster point of A.

If $\forall x\in A,x\neq c$,

$$f(x)\leq g(x)\leq h(x)\text{ and }\lim_{x\to c}f=L=\lim_{x\to c}h$$

then

$$\lim_{x\to c}g=L$$

Proof. Exercise. Try sequeeze theorem of sequence. ∎

Example. $\lim\limits_{x\to 0}x\sin(\dfrac{1}{x})=0$

Proof. Notice that

$$-1\leq\sin(\frac{1}{x})\leq 1\Rightarrow -x\leq x\sin(\frac{1}{x})\leq x$$

Since

$$\lim_{x\to 0}(-x)=0=\lim_{x\to 0}(x)$$

$$\lim_{x\to 0}x\sin(\frac{1}{x})=0$$

∎

Example. Let $b \in \mathbb{R}, b > 0$. Then $\lim\limits_{x \to 0} x^b = 0$. Notice that when $x \in [0, 1]$

$$x^{\lceil b \rceil} \le x^b < x^{\lfloor b \rfloor}$$

The rest is left as exercise.

Theorem. Let $A \subseteq \mathbb{R}$; $f, g, h : A \to \mathbb{R}$, c be a cluster point of A. If $\lim\limits_{x \to c} f > 0$, then $\exists \delta > 0$ s.t. $V_\delta(c)$ s.t. $\forall x \in A \cap V_\delta(c) \backslash \{c\}$,

$$f(x) > 0$$

Proof. Excercise.   ∎

5   Continuous Functions

5.1   Continuity

Definition (_Counituous_). Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$.

We say f is **_countinuous at c_** if

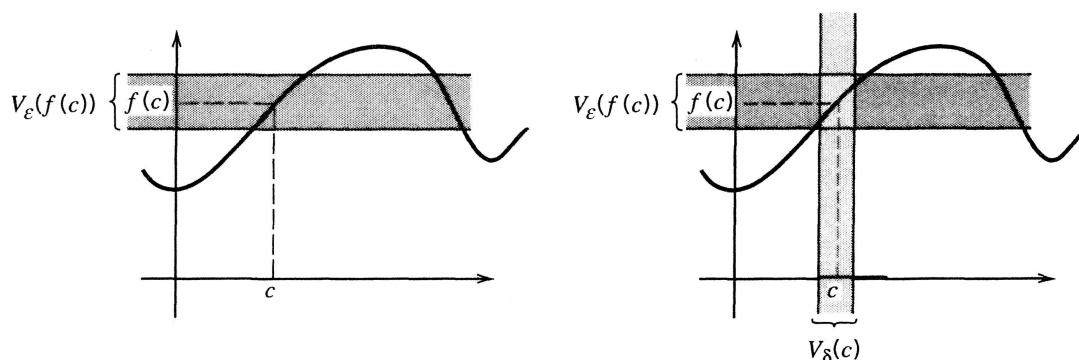$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

We say f is **_discontinuous at c_** if f is NOT continuous at c.

Definition (_Using Neighborhood_). Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. Then f is continuous at c if and only if
$\forall \varepsilon > 0$ and its ε-neighborhood at $f(c)$, $V_{f(c)}(\varepsilon)$, $\exists \delta > 0$ and its δ-neighborhood at c , $V_c(\varepsilon)$, s.t.

$$f(V_\delta(c) \cap A) \subseteq V_\varepsilon(f(c))$$

Remark (_Comparison with Limit Definition_).   Continuity has 3 good properties that will be useful in the future study of Real Analysis:

- If c is a **_cluster point_** of A, then f is **_continuous_** if and only if

    1. $f(x)$ is defined at c.

        Counter-example: Any function $f : \mathbb{Q} \to \mathbb{R}$ is *undefined* at $\mathbb{R}\backslash\mathbb{Q}$ and thus *discontinuous* at $\mathbb{R}\backslash\mathbb{Q}$.

    2. $\lim\limits_{x \to c} f(x)$ exists.
        Counter-example: $\dfrac{1}{x}$ is *undefined* at 0 and thus *discontinuous* at 0.

    3. $f(c) = \lim\limits_{x \to c} f(x)$.
        Counter-example: $f(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is discontinuous at 0 as
        $f(c) = 0 \neq 1 = \lim\limits_{x \to c} f(x)$

- If c is **_NOT a cluster point_** of A, then c is an **_isolated point_**, and

  $\exists V_\delta(c) \cap A = \{c\}$. Notice that an **_isolated point_** is **_automatically continuous_** as it *satisfies* the definition of continuity at c. This is because

$$|x - c| = |c - c| = 0 < \delta \Rightarrow |f(x) - f(c)| = |f(c) - f(c)| = 0 < \varepsilon$$

Theorem (_Sequential Criterion for Continuity_).

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. Then f is continuous at c if and only if

$\forall (x_n) \subseteq A$ s.t. $(x_n) \to c$, $f((x_n)) \to f(c)$.

Proof. ($\Rightarrow$). Continuity of f at c implies limit of f at c exists[2]. Thus we simply apply sequential criterion of limit.

($\Leftarrow$). This is exactly the statement of sequential criterion for limit at c. Thus,

$$\exists \lim_{x \to c} f(x) = f(c)$$

Notice that $c \in A$ implies that f is defined on c. Thus, we conclude that f is continuous on c by previous remark in p.43 ∎

Corollary (_Sequential Criterion for Discontinuity_).

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. Then f is discontinuous at c if and only if

$\exists (x_n) \subseteq A$ s.t. $(x_n) \to c$, $f((x_n)) \nrightarrow f(c)$.

Proof. Similar. Left to reader as exercise.

∎

Example. We begin with polynomial and rational functions:

- Constant function $f(x) = b$ is continuous in $\mathbb{R}$.

- Linear function $f(x) = ax + b$ is continuous in $\mathbb{R}$.

- Quadratic function $f(x) = x^2$ is continuous in $\mathbb{R}$.

- $f(x) = \dfrac{1}{x^2}$ is continuous in $\mathbb{R}$.

- Polynomial functions are continuous in $\mathbb{R}$(try to prove it!).

- Rational functions are continuous in $\mathbb{R}$(try to prove it!).

- $f(x) = \dfrac{1}{x}$ is NOT continuous at 0.

- $f(x) = sgn(x)$ is NOT continuous at 0.

---

[2]proof by definition

Example (*__Dirichlet's "discontinuous function"__*).

Let $A = \mathbb{R}$ and define Dirichlet's "discontinuous function" by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \text{ (rational)} \\ 0 & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \text{ (irrational)} \end{cases}$$

__Claim:__ this function is discontinuous on $\mathbb{R}$.

Proof. Let $c \in A$.

- If $c \in \mathbb{Q}$, then $\exists (x_n) \in \mathbb{R} \backslash \mathbb{Q}$ s.t. $\forall n \in \mathbb{N}$,

$$c < (x_n) < c + \frac{1}{n}$$

By squeeze theorem,

$$\lim c \leq \lim (x_n) \leq \lim (c + \frac{1}{n})$$

$$(x_n) \to c, \ (c + \frac{1}{n}) \to c$$

$$\lim f(x_n) = 1 \neq 0 = \lim f(c + \frac{1}{n})$$

Thus, by Sequential Criterion of Discontinuity, f is discontinuous for all $c \in \mathbb{Q}$.
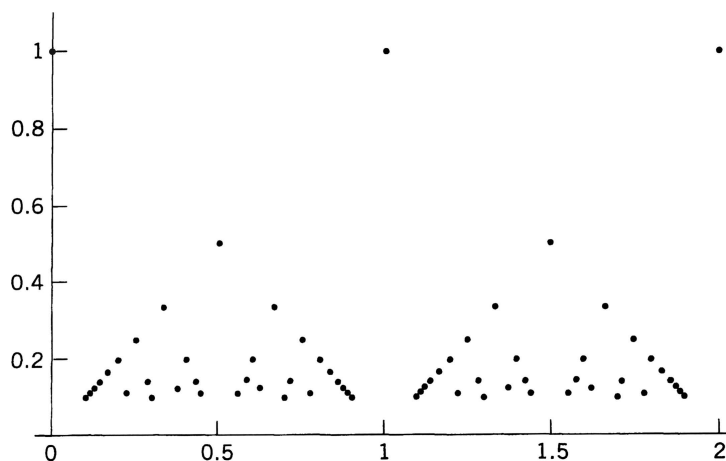
- If $c \in \mathbb{R} \backslash \mathbb{Q}$, then we adopt similar strategy. The rest of proofs is left as exercise.

■

Example (*__Thomae's function__*). *Notice that this example is harder. We skip it on class.

Let $f : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \text{ (irrational)} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ (rational)} \end{cases}$$

__Claim:__ f is discontinuous on $\mathbb{Q}$ and continuous on $\mathbb{R}^+ \backslash \mathbb{Q}$.

Proof.

- If c is rational, then $\exists (x_n) \in \mathbb{R}^+ \backslash \mathbb{Q} +$ s.t. $(x_n) \to c$. Then we have

$$\lim f(x_n) \to 0 \neq \frac{1}{q} = f(c)$$

  Since funtion value does not equal to limit value at c, f is discontinuous at $c \in \mathbb{Q}$

- If c is irrational, **_Goal:_** show that $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in R^+$,

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| = |f(x)| < \varepsilon$$

  - If $x \in \mathbb{R}^+ \backslash \mathbb{Q}$, then $|f(x)| = 0 < \varepsilon$ for some arbitrary choice of δ.

  - If $x \in \mathbb{Q}$, then by Archimedean Property, we could find a $\frac{1}{n_0} > \varepsilon$.

    Notice that there are only finite number of $n \in \mathbb{N}$ such that $n < n_0$. Thus, there are only finite number of $\frac{p}{q} \in (c - 1, c + 1)$ with denominator $q < n_0$. Hence we could choose δ so small that the $V_\delta(c)$ contains no rational numbers with denominator less than n_0. It follows that

$$|x - b| < \delta \Rightarrow |f(x) - f(c)| = |f(x)| \leq \frac{1}{n_0} < \epsilon$$

  Thus, f is continuous at $c \in \mathbb{R} \backslash \mathbb{Q}$

  $\blacksquare$

5.2   Combinations of Continuous Functions

Theorem. Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$, $c \in A$, $b \in R$. Suppose f, g are ***continuous*** on c, then the following combination of functions are ***continuous***:

- ***Addition:*** $f + g$

- ***Subtraction:*** $f - g$

- ***Product:*** $f \cdot g$

- ***Multiplication:*** $b \cdot f$

- ***Quotient:*** $\dfrac{f}{g}$ if $g(x) \neq 0$

- ***Absolute:*** $|f|\,(x)$

- ***Square Root:*** $\sqrt{f}(x)$

Proof.

- If c is not a cluster point of A, then the theorem is automatically correct.

- If c is a cluster point of A, then we only need to show that the function value at c is equal to the limit of function at c. Detailed proof are left as exercise.

                         ∎

   Example.

- Polynomial functions are continuous on $\mathbb{R}$.

- Rational functions are continuous on $\mathbb{R}$.

Example.

- $\sin(x)$ is continuous on $\mathbb{R}$.

  Proof. Notice that $\forall x, y, z \in \mathbb{R}$, we have 3 inequalitis:

  $$|\sin(z) \leq |z|\,,\ \ |\cos(z)| \leq 1|\,,\ \ \sin(x) - \sin(y) = 2\sin[\tfrac{1}{2}(x - y)] \cdot \cos[\tfrac{1}{2}(x + y)]$$

  Hence if $c \in \mathbb{R}$, we have

  $$|\sin(x) - \sin(c)| \leq 2 \cdot \frac{1}{2}\,|x - c| \cdot 1 = |x - c|$$

  Thus $\sin(x)$ is continuous on $\mathbb{R}$.   ∎

- $\cos(x)$ is continuous on $\mathbb{R}$

  Proof. Exercise. Similar techniques as above.   ∎

- $\tan(x)$, cot, sec, csc are all continuous on *where they are defined.*

  Proof. Combinations of continuous functions on their domain are continuous.

                         ∎

Theorem. Let $A, B \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$, $c \in A$ s.t. $f(A) \subseteq B$. Suppose f is continuous at $c \in A$, g is ***continuous*** on $b = f(c) \in B$, then the composition $g \circ f$ is ***continuous*** at c.

Proof. Since f is continuous at c, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in A$,

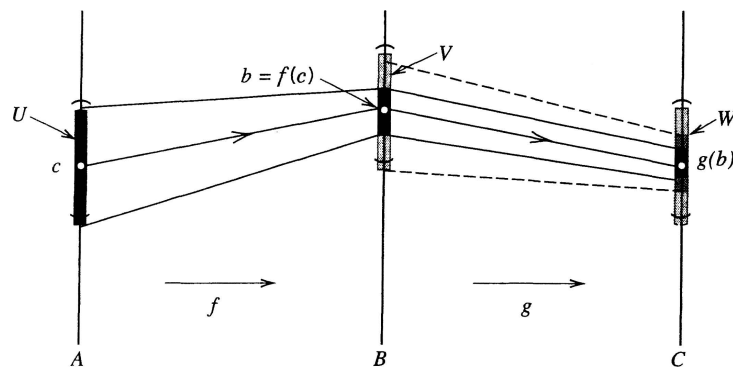$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

Since g is continuous at $b = f(c)$, $f(A) \subseteq B$, $\forall \zeta > 0$, $\exists \varepsilon > 0$ s.t. $\forall f(x) \in B$,

$$|f(x) - b| < \varepsilon \Rightarrow |g(f(x)) - g(b)| < \zeta$$

Thus,

$$|x - c| < \delta \Rightarrow |g(f(x)) - g(b)| = |g \circ f(x) - g \circ f(c)| < \zeta$$

<div align="right">■</div>

Example. $\sqrt{x^2 + 1}$ is continuous on $\mathbb{R}$.

Proof. Exercise.

<div align="right">■</div>

5.3   Continuous Function on Bounded Intervals

Continuous function on **_closed bounded interval_** has many good properties.

Definition (**_Bounded Function_**)**.** A function $f : A \to \mathbb{R}$ is said to be **_bounded on A_** if $\exists M > 0$ s.t. $\forall x \in A$

$$|f(x)| \leq M$$

A function is **_unbounded_** if $\forall M > 0,\ \exists x_M \in A$ s.t.

$$|f(x)| > M$$

Example. $f(x) = \dfrac{1}{x}$ is unbounded on $A = (0, \infty)$.

Proof. By Archimedean Property, $\forall M > 0, \exists x_M = \frac{1}{M+1} \in A$ s.t.

$$\left| f(\frac{1}{M+1}) \right| = \left| \frac{1}{\frac{1}{M+1}} \right| = |M+1| > M$$

We thereby conclude that $\dfrac{1}{x}$ is unbounded on A.   ∎

Theorem (**_Boundness Theorem_**)**.** Let $a, b \in \mathbb{R}$, $a < b$, $I = [a, b]$ a **_closed bounded interval_**. If $f : I \to \mathbb{R}$ is countinuous on I, then f is bounded on I.

Proof. Assume f is not bounded on I. Then, by definition, $\forall n \in \mathbb{N},\ \exists x_n \in I$ s.t.

$$|f(x_n)| > n$$

Since I is bounded, the sequence $(x_n) \in I$ is bounded. Therefore, by the Bolzano-Weierstrass Theorem, there exists a convergent subsequence $(x_{n_k}) \to x$ for some number $x \in I$. Since f is continuous on I,

$$f(x_{n_k}) \to f(x)$$

We thereby conclude that $f(x_{n_k})$ is a bounded sequence, which is a contradiction to unboundedness of $f(x_{n_k})$

$$\forall k \in \mathbb{N}, n_k \in \mathbb{N},\ |f(x_{n_k})| > n_k \geq k$$

∎

Remark. We will give 3 counter-examples to show that all of three conditions are necessary for Boundedness Theorem to be true.

- Interval must be **_closed_**
  Counter-example: $f(x) = \dfrac{1}{x}$ on $(0, 1]$ is countinous but unbounded.

- Interval must be **_bounded_**
  Counter-example: $g(x) = x$ is continuous but unbounded on $[0, \infty)$.

- The function must be **_continous_**
  Counter-example: Define $h : [0, 1] \to \mathbb{R}$, $h(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0, 1] \\ 1 & \text{if } x = 0 \end{cases}$ is discontinuous and unbounded.

Definition (***Maximum and Minimum***)**.** Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$.

We say that f has an ***absolute maximum*** on A if $\exists x^* \in A$ s.t. $\forall x \in A$

$$f(x^*) \geq f(x)$$

We say that f has an ***absolute minimum*** on A if $\exists x_* \in A$ s.t. $\forall x \in A$

$$f(x_*) \leq f(x)$$

Remark. Continuous function on a ***bounded set A*** does not necessarily have an maximum or minimum on *A*. For example:

- $f(x) = \dfrac{1}{x}$ has NO absolute maximum or absolute minimum on $A = (0, \infty)$

- $g(x) = x^2$ has only absolute minimum $x_* = 0$ on $\mathbb{R}$.

Theorem (***Maximum-Minimum Theorem***)**.** Let $I = [a, b] \in \mathbb{R}$ be a ***closed bounded interval***, $f : I \to \mathbb{R}$ be ***countinuous*** on I. Then f has an absolute maximum and absolute minimum on I.

Proof. By previous theorem, f is bounded on I. Thus,

$$\exists s^* = \sup f(I), \beta = \inf f(I)$$

We will first proof that f has an absolute maximum. It suffices to show that

$$\exists x^* \in I \text{ s.t. } f(x^*) = \sup f(I)$$

Since $s^* = \sup f(I)$, then $\forall n \in \mathbb{N}, s^* - \dfrac{1}{n}$ is not an upper bound of the set $f(I)$. Consequently, $\exists x_n \in I$ s.t. $\forall n \in \mathbb{N}$

$$s^* - \frac{1}{n} < f(x^*) \leq s^*$$

Since I is bounded, (x_n) is bounded, by the Bolzano-Weierstrass Theorem, there exists a subsequence $(x_{n_k}) \in I$ s.t. $(x_{n_k}) \to x^*$.
Since f is continuous on I,
$$\lim(f(x_{n_k})) = f(x*)$$

By squeeze theorem,
$$\lim(s^* - \frac{1}{n_k}) \leq^3 \lim f(x_{n_k}) \leq \lim s^*$$

$$s^* \leq f(x^*) \leq s^*$$

$$s^* = f(x^*)$$

Thus we conclude that x^* is the absolute maximum of f on I. The proof of absolute minimum is left as exercise using similar techniques. ∎

---

[3]The reader should find out why the strict inequality become weak inequality when limit applies.

The proof of next theorem provides an algorithm, know as **_Bisection Method_**, to calculate root to certain level of accuracy.

Theorem (**_Location of Roots_**). Let $I = [a, b]$, $f : I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$, then $\exists c \in (a, b)$ s.t. $f(c) = 0$.

Proof. WLOG, let's assume that $f(a) < 0 < f(b)$. Define a sequence of closed bounded nested interval I_n and midpoint p_n s.t. $\forall n \in \mathbb{N}$:

$$a_1 = a, \ b_1 = b, \ I_1 = [a_1, b_1], \ p_1 = \frac{1}{2}(a_1 + b_1)$$

Notice that if $f(p_n) = 0$, then $c = p_n$ and we are done. If not, then

$$I_n = \begin{cases} [a_{n-1}, p_{n-1}] & \text{if } f(p_{n-1}) > 0 \\ [p_{n-1}, a_{n-1}] & \text{if } f(p_{n-1}) < 0 \end{cases} \subset I_{n-1}$$

Observe that

- $\{I_n\}$ is a infinite sequence of **_nested intervals_** where $\forall n \in \mathbb{N}, I_n \subset I_{n-1}$

- $\lim\{b_n - a_n\} = \lim\{\dfrac{b - a}{2^{n-1}}\} = 0 \Rightarrow \lim(a_n) = \lim(b_n)$

Then by Nested Interval Property,

$$\exists c \in [a, b] \text{ s.t. } \forall n \in \mathbb{N}, \ c \in I_n, \ \bigcap_{n=1}^{\infty} I_n = \{c\}$$

Also notice that

$$a_n < c < b_n \Rightarrow \lim(a_n) \leq c \leq \lim(b_n)$$

By Sqeeze Theorem

$$\lim(a_n) = c = \lim(b_n)$$

Notice that

$$\begin{cases} f(a_n) < 0 \Rightarrow & \lim f(a_n) \leq 0 \\ 0 < f(b_n) \Rightarrow & 0 \leq \lim f(b_n) \end{cases}$$

Thus

$$c = \lim(a_n) = \lim(b_n) = 0$$

■

Theorem (***Bolzano's Intermediate Value Theorem***). Let $I \in \mathbb{R}$ be an interval[4], $f : I \to \mathbb{R}$ be continuous on I. If $\exists a, b \in I, k \in \mathbb{R}$ s.t.

$$f(a) < k < f(b)$$

then $\exists c \in I$ s.t.

$$f(a) < f(c) = k < f(b)$$

.

Proof. WLOG, assume $a < b$ and define $g(x) = f(x) - k$. Then

$$g(a) < 0 < g(b)$$

By previous theorem, $\exists c,\ a < c < b$ s.t. $0 = g(c) = f(c) - k$. Thus

$$f(c) = k$$

The similar proof also allpies for $b < a$. ∎

Corollary. Let $I = [a, b]$, $f : I \to \mathbb{R}$ be continuous on I. If $\exists k \in \mathbb{R}$ s.t.

$$\inf f(I) \leq k \leq \sup f(I)$$

then $\exists c \in I$ s.t.

$$f(c) = k$$

Proof. This is a direct result of previous theorem. ∎

Corollary. Let $I = [a, b]$, $f : I \to \mathbb{R}$ be continuous on I. Then

$$f(I) = [\inf f(I), \sup f(I)]$$

Proof. Let $m = \inf f(I)$, $M = \sup f(I)$. We know by the Maximum-Mininum Theorem that $m, M \in f(I)$. Thus

$$f(I) \subseteq [m, M]$$

Then by Bolzano's Intermediate Value Theorem, $\forall k \in [m, M], \exists c_k \in I$ s.t. $k = f(c_k)$. We thereby conclude that
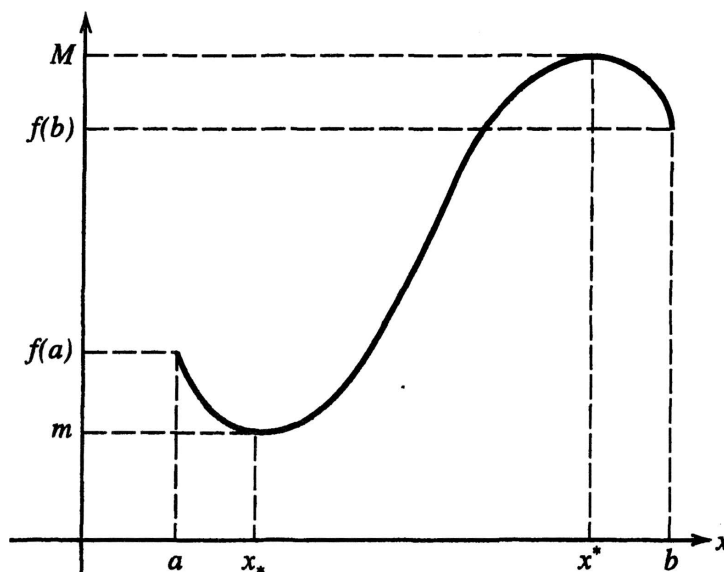
$$[m, M] \subseteq f(I)$$

It follows that

$$f(I) = [m, M]$$

∎

---

[4]Not necessarily closed or bounded. Thus this theorem is stronger

Remark. ***End points may not be extreme points***. Counter example:

$$f : [a,b] \to \mathbb{R}, \ f(I) \neq [f(a), f(b)]$$

Theorem (*Preservation of Interval Theorem*).

Let $I \in \mathbb{R}$ be an interval[5], $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is an interval.

Proof. Let $\alpha, \beta \in f(I)$ with $\alpha < \beta$. Then $\exists a, b \in I$ s.t.

$$\alpha = f(a), \ \beta = f(b)$$

By Bolzano's Intermediate Value Theorem, $\forall k \in (\alpha, \beta)$, $\exists c_k \in I$ s.t. $f(c) = k \in f(I)$. Thus

$$[\alpha, \beta] \subseteq f(I)$$

We conclude that $f(I)$ is an interval   ∎

---

[5]Not necessarily closed or bounded. Thus this theorem is stronger

5.4   Uniform Continuity

Recall the definition of continuity of f at $u \in A$:

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $\forall \varepsilon > 0, \exists \delta(\varepsilon, u)$ s.t. $\forall x \in A$

$$|x - u| < \delta(\varepsilon, u) \Rightarrow |f(x) - f(u)| < \varepsilon$$

Here we emphasize that the choice of *delta* depends on ***both ε and $u \in A$***. This implies that the change of function value $f(u)$ depends on choice of u. Consider $f(x) = \sin(\frac{1}{x})$. As x approaches 0, the function value changes more rapidly.

Example. In this example, δ depends on ε only:

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$. Then

$$|f(x) - f(u)| = 2\,|x - u|$$

So it suffice to choose $\delta = \dfrac{\varepsilon}{2}$.

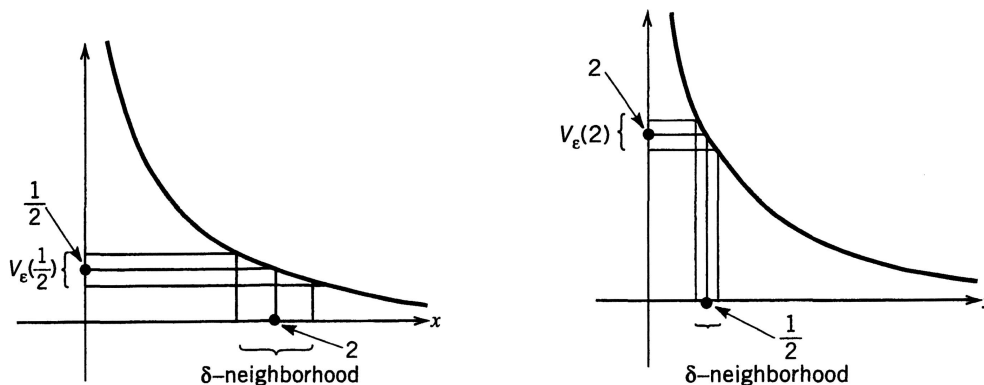**Example.** However, in certain cases, δ depends on both ε and u.

Let $g : (0, \infty) \to \mathbb{R}$, $g(x) = \dfrac{1}{x}$. Then

$$|f(x) - f(u)| = \left| \frac{u - x}{ux} \right|$$

It suffices to choose $\delta(\varepsilon, u) = \inf\{\frac{1}{2}u, \frac{1}{2}u^2\varepsilon\}$.

Notice that there is no way to choose a δ that will work for all $u > 0$. δ must depends on the position of u. As u tends to 0, the persissible value of δ tends to 0.

Definition (***Uniform Continuity***). Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$. We say that f is ***uniformly continuous on*** A if

$\forall \varepsilon > 0, \exists \delta(\varepsilon)$ s.t. $\forall x, y \in A$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Theorem (***Non-uniform Continuity Criterion***). Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$. The following statements are equivalent:

- f is NOT uniformly continuous.

- $\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0$, $\exists x_\delta$, $u_\delta \in A$ s.t.

$$|x_\delta - u_\delta| < \delta \Rightarrow |f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$$

- $\exists \varepsilon_0 > 0$, $\exists (x_n)$, $(u_n) \in A$ s.t. $\forall n \in \mathbb{N}$

$$\lim(x_n - u_n) = 0 \text{ and } |f(x_n) - f(u_n)| \geq \varepsilon_0$$

Proof. Exercise.   ∎

Theorem (***Uniform Continuity Theorem***). Let I be a ***closed bounded interval*** and let $f : I \to \mathbb{R}$ be ***continuous*** on I. Then f is ***uniformly continuous*** on I.

Proof. Assume f is not uniformly continuous on I. Then by precedence result, $\exists \varepsilon_0 > 0$, $\exists (x_n)$, (u_n), $\in I$ s.t. $\forall n \in \mathbb{N}$

$$\lim(x_n - y_n) = 0 \text{ and } |f(x_n) - f(u_n)| \geq \varepsilon_0$$

Since I is bounded, the sequence (x_n) is bounded. By Bolzano-Weierstrass Theorem, there exists a convergent subsequence (x_{n_k}), (u_{n_k}) of (x_n), (u_n) where

$$\lim(x_{n_k} - y_{n_k}) = 0 \text{ and } |f(x_{n_k}) - f(u_{n_k})| \geq \varepsilon_0$$

Since I is continuous on I

$$\lim f(x_{n_k}) = f(\lim(x_{n_k})), \ \lim f(u_{n_k}) = f(\lim(u_{n_k}))$$

Thus

$$\lim(f(x_{n_k})) = \lim(f(u_{n_k})) \Rightarrow \lim(f(x_{n_k}) - f(x_{n_k})) = 0$$

contradicts our assumption. We conclude that f must be uniformly continuous.   ∎

Note: The next property conveniently ensures uniform continuity without requiring A to be a closed and bounded interval.

Definition (***Lipschitz Functions***)**.** Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that $\forall x, \ u \in A$

$$|f(x) - f(u)| \leq K |x - u|$$

then f is said to be a ***Lipschitz function*** (***Lipschitz continuous***, or just ***Lipschitz***) on A.

Remark (***Lipschitze Function and Gradient***)**.** Rewrite the condition, we have

$$\left| \frac{f(x) - f(u)}{x - u} \right| \leq K$$

It follows that the absolute value is the gradient of a line segment joining the points $(x, f(x))$, $(u, f(u))$. Thus, f is Lipschitz if and only if the gradient of all line segments joining two points on the graph of $y = f(x)$ over I are bounded by some K.

Theorem. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$. If f is ***Lipschitz***, then it is ***uniformly continuous*** on A.

Proof. For all $\varepsilon > 0$, we simply choose $\delta = \dfrac{\varepsilon}{K}$. Then

$$|x - u| < \delta \Rightarrow |f(x) - f(u)| < K \cdot \frac{\varepsilon}{K} = \varepsilon$$

Thus f is uniformly continuous on A ∎

Remark. The ***converse*** may NOT true!

Counter-example: $f(x) = \sqrt{x}$ on $[0, 1]$ is uniformly continuous but NOT Lipschitz.

Example. If $f(x) = x^2$ on $A = [0, b]$ where $b > 0$, then $\forall x, \ y \in [0, b]$

$$|f(x) - f(y)| = |x + y| \, |x - y| \leq 2b \, |x - y|$$

It follows that f is Lipschitz on A given $K = 2b$, and thus f is uniformly continuous.

Theorem. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$. If f is uniformly continuous on A, and $(x_n) \in A$ is a Cauchy sequence, then $(f(x_n))$ is a Cauchy sequence in R.

Proof. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in \delta$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Since (x_n) is Cauchy, for given $\delta > 0$, $\exists N_\delta \in \mathbb{N}$ s.t. $\forall n, m \in \mathbb{N}, \ n, m > N_\delta$

$$|x_n - x_m| < \delta \Rightarrow |f(x_n) - f(x_m)| < \varepsilon$$

We conclude that $(f(x_n))$ is Cauchy.   ∎

Theorem (***Continuous Extension Theorem***). A function f is ***uniformly continuous*** on ***open*** interval $I = (a, b)$ if and only if it can be ***defined at the endpoints*** a and b such that the extended function is continuous on $[a, b]$.

Proof. ($\Leftarrow$). This direction is trivial.

($\Rightarrow$) Suppose $f : (a, b) \to \mathbb{R}$ is uniformly continuous.

Goal: Define f's extended function $F : [a, b] \to \mathbb{R}$ s.t.

$$F|_{(a,b)} = f, \ F \text{ is continuous}$$

Notice that this is equivalent to show $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to b} f(x)$ exist and define

$$F(x) = \begin{cases} f(x) & \text{if } x \in (a, b) \\ \lim\limits_{x \to a} f(x) & \text{if } x = a \\ \lim\limits_{x \to b} f(x) & \text{if } x = b \end{cases}$$

Choose $(x_n) \in (a, b)$ s.t. $(x_n) \to a$. Thus, (x_n) is Cauchy, and by preceding theorem, $(f(x_n))$ is Cauchy. Let $L = \lim(f(x_n))$. Then $\forall (y_n) \in (a, b)$ s.t. $(y_n) \to a$,

$$\lim(x_n - y_n) = a - a = 0$$

By uniform continuity, we have

$$\lim(f(y_n)) = \lim(f(y_n) - f(x_n)) + \lim(f(x_n))$$
$$= 0 + L = L$$

Thus, for all sequence (u_n) converging to a, $(f(x_n)) \to L$. By sequential criterion of limit, $L = \lim\limits_{x \to a} f(x)$. If we define $f(a) = L$, then f is continuous at a.

The same argument applies to b. Thus, we conclude that f has a continuous extension to the interval $[a, b]$ ∎

5.6 Monotone and Inverse Functions

Note: In this section, we will be focusing on ***monotone functions*** on an ***interval*** I. Specifically, we will discuss ***increasing functions***. It is easy to derive corresponding results for decreasing functions wit similar proof techniques.

Theorem. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on I. Suppose that $c \in I$ is NOT an endpoint of I. Then

1. $\lim\limits_{x \to c-} f = \sup\{f(x) : x \in I, \ x < c\}$

2. $\lim\limits_{x \to c+} f = \inf\{f(x) : x \in I, \ x < c\}$

Recall: $\lim\limits_{x \to c-} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in I$

$$0 < c - x < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Proof. (1). By monotonicity of f, $\forall x < c$,

$$f(x) \leq f(c)$$

Thus, the set is non-empty, and $f(c)$ is the upper bound of it. This indicates that

$$\exists L = \sup\{f(x) : x \in I, x < c\}$$

Then $\forall \varepsilon > 0$, $L - \varepsilon$ is not an upper bound of the set. Hence $\exists x_\varepsilon \in I, x_\varepsilon < c$ s.t.

$$L - \varepsilon < f(x_\varepsilon) \leq L$$

We choose $\delta = c - x_\varepsilon$. Then $\forall x \in I$, $0 < c - x < \delta \to x_\epsilon < x$. Thus,

$$-\varepsilon < f(x_\delta) - L \leq f(x) - L \leq 0 < \varepsilon$$

$$|f(x) - L| < \varepsilon$$

Thus

$$L = \lim\limits_{x \to c-} f(x)$$

The proof of (2) is similar. ∎

Corollary (***One-sided Limits Criterion for Continuity***). Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on I. Suppose that $c \in I$ is NOT an endpoint of I. Then the following statements are equivalent:

- f is continuous at c.

- $\lim\limits_{x \to c-} f(x) = f(c) = \lim\limits_{x \to c+} f(x)$.

- $\sup\{f(x) : x \in, \ x < c\} = f(c) = \inf\{f(x) : x \in I, \ x > c\}$.

Proof. Exercise. ∎

Recall.   A function $f : I \to \mathbb{R}$ has ***inverse function*** if and only if f is ***injective***.

Note: It is not difficult to prove that a ***strictly monotone*** function is ***injective*** and thus has an inverse.

Theorem (*Continuous Inverse Theorem*).
Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be ***strictly monotone*** and ***continuous*** on I. Then the inverse function $g : f(I) \to I$, $g(y) = f^{-1}(y)$ is ***strictly monotone*** and ***continuous*** on $f(I)$.

Proof. WLOG, we assume that f is **strictly increasing**. We will first prove that g is strictly increasing.

Suppose $\exists y_1, \ y_2 \in f(I), y_1 < y_2$, then $\exists x_1, \ x_2 \in I$ s.t. $\ f(x_1) = y_1, \ f(x_2) = y_2$

1. Assume that $x_1 = x_2$.

   Then, $y_1 = f(x_1) = f(x_2) = y_2$ contradicts assumption. So $x_1 \neq x_2$.

2. Assume that $x_1 > x_2$.

   Then $y_1 = f(x_1) > f(x_2) = y_2$ contradicts assumption. So $x_1 \not> x_2$.

Thus, $x_1 < x_2$. It follows that
$$g(y_1) < g(y_2)$$
We conclude that g is strictly monotone.

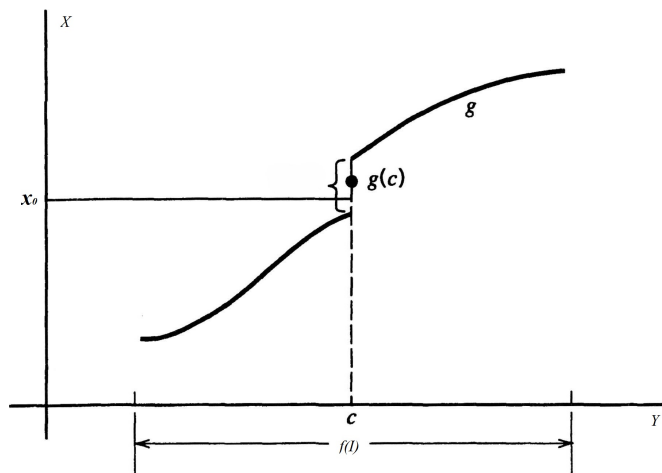Subsequently, we will prove that g is **continuous**.

Assume g is discontinuous. Then, by inverse of previous criterion, $\exists c \in f(I)$, c is not endpoints of $f(I)$ s.t.

$$\lim_{y \to c-} g(y) < \lim_{y \to c+} g(y)$$

Let $x_0 \in (\lim_{y \to c-} g(y), \ \lim_{y \to c+} g(y)) \setminus g(c)$. Then

$$x_0 \notin g(f(I)) \subseteq I$$

which contradicts that fact that $x_0 \in I$ ∎

8 Sequence of Functions

8.1 Pointwise and Uniform Convergence

Definition (_Sequence of Functions_). Let $A \subseteq \mathbb{R}$.

We say that $(f_n(x))$ is a **_sequence of functions_** on A to $\mathbb{R}$ if

$$\forall n \in \mathbb{N}, \ \exists f_n : A \to \mathbb{R}$$

Definition (_Convergence of Sequence of Functions_). Let $(f_n(x))$ be a sequence of functions on A to $\mathbb{R}$. Let $A_0 \subseteq A$, and $f : A_0 \to \mathbb{R}$.

We say that (x_n) **_converges on A_0 to f_** if $\forall x \in A_0$,

$$\lim_{n \to \infty} (f_n(x)) = f(x) \text{ or } f_n \to f \text{ on } A_0$$

We call f the **_limit of f_n on A_0_**. Or we say that **_(f_n) converges pointwise on A_0_**.

Lemma ($\varepsilon - \delta$ _Definition_). Let $A_0 \subseteq A \subseteq \mathbb{R}$, $f : A_0 \to \mathbb{R}$. A sequence of functions $(f_n(x)) : A \to \mathbb{R}$ converges pointwise to f if and only if

$\forall \varepsilon > 0, \exists N(\varepsilon, \ x) \in \mathbb{N}$ s.t. $\forall n \geq N(\varepsilon, \ x)$,
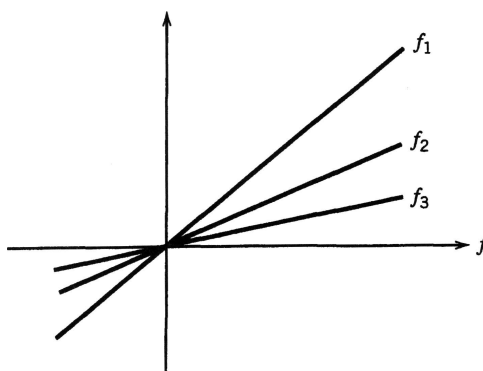
$$|f_n(x) - f(x)| < \varepsilon$$

Remark. We emphasize that the choice of $N(\varepsilon, \ x)$ depends on **_both_** ε and x.

Example. $\forall x \in \mathbb{R}, \ \lim_{n \to \infty} \dfrac{x}{n} = 0$. Let $f_n(x) = \dfrac{x}{n}, \ f(x) = 0$. We have

$$\lim_{n \to \infty} (f_n(x)) = \lim(\frac{x}{n}) = x \lim_{n \to \infty} \frac{1}{n} = x \cdot 0 = 0 = f(x)$$

Thus $(f_n(x)) \to 0$ pointwise on $\mathbb{R}$.

Or, $\forall \varepsilon > 0$,

$$\left| \frac{x}{n} - 0 \right| = \frac{|x|}{n} < \varepsilon$$

So, it surffices to choose $K(\varepsilon, \ x) = \left\lceil \dfrac{|x|}{\varepsilon} \right\rceil$.

Example. Consider $\forall x \in \mathbb{R}$, $n \in \mathbb{N}$, $g_n(x) = x^n$.

By previous example, we know that

$$\lim(x^n) = \begin{cases} 0 & \text{if } -1 < x < 1, \\ 1 & \text{if } x = 1 \end{cases}$$

And if $x = -1$, $g_n(-1) = (-1)^n$ is divergent.

If $|x| > 1$, (x^n) is divergent as well.

Define $g : (-1, 1] \to \mathbb{R}$, $g(x) = \lim(x^n)$. Then,

$$g_n \to g \text{ on } (-1, 1]$$

Definition (***Uniform Convergence***). A sequence of $(f_n(x))$ functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ ***converges uniformly*** on $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if and only if $\forall \varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ s.t. $\forall n \geq K(\varepsilon)$, $\forall x \in A_0$,

$$|f_n(x) - f(x)| < \varepsilon$$

In this case, we say that (f_n) is ***uniformly convergent*** on A_0.

Remark. The choice of $K(\varepsilon)$ depends on ***only*** ε.

Example. $f_n(x) = \dfrac{\sin(nx + n)}{n}$ converges uniformly to $f(x) = 0$.

Proof.
$$|f_n(x) - f(x)| = \left| \frac{\sin(nx + n)}{n} \right| < \frac{1}{n} < \varepsilon$$

So $N = \left\lceil \dfrac{1}{\varepsilon} \right\rceil$ will satisfies the condition.

∎

Lemma. A sequence of $(f_n(x))$ functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ is ***NOT uniformly convergent*** on $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if and only if

- there exists $\varepsilon_0 > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists x \in A_0$, $\exists n \geq N$ s.t.

$$|f_n(x) - f(x)| \geq \varepsilon_0$$

- there exists $\varepsilon_0 > 0$, a subsequence (f_{n_k}) of (f_n), and a sequence $(x_k) \in A_0$ s.t.

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0, \ \forall k \in \mathbb{N}$$

Proof. Exercise. ∎

Example. $f_n(x) = \dfrac{x}{n}$ does NOT converge unifromly to $f(x) = 0$ on $\mathbb{R}$.

Proof. Let $\varepsilon_0 = 1$

    1. When $n_1 = 1$, $x_1 = 1$, we have $|f_1(x_1) - f(x_1)| = 1$

    2. When $n_2 = 2$, $x_2 = 2$, we have $|f_2(x_2) - f(x_2)| = 1$

    3. So choose $x_k = k$, $n_k = k$, we have

$$|f_{n_k}(x_k) - f(x_k)| = \left| \frac{x_k}{n_k} - 0 \right| = 1 \geq \varepsilon_0$$

By preceding lemma, (f_n) does NOT congerge to f uniformly on $\mathbb{R}$ ■

Example. Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{x^2 + nx}{n}$

<u>Claim:</u> f_n does NOT uniformly converge to $f(x) = x$ on $\mathbb{R}$.

Proof. Choose $\varepsilon_0 = 1$. $\forall k \in \mathbb{N}$, if we choose $n_k = k$, $x_k = -k$

$$\begin{aligned}
|f_{n_k}(x_k) - f(x_k)| &= \left| \frac{x_k^2 + n_k x_k}{n_k} - x_k \right| \\
&= \left| \frac{(-k)^2 + k \cdot (-k)}{k} - (-k) \right| \\
&= k \geq 1
\end{aligned}$$

Thus, by previous lemma, f_n does NOT uniformly converge to f on $\mathbb{R}$. ■

<u>Claim:</u> $(f_n) \to f$ uniformly on $[0, 1]$.

Proof. $\forall \varepsilon > 0$. Observe that $\forall x \in [0, 1]$, $n \in \mathbb{N}$,

$$\left| \frac{x^2 + nx}{n} - x \right| = \left| x^2 \right| \leq \frac{1}{n} < \varepsilon$$

Thus, it suffices to choose $N = \left\lceil \dfrac{1}{n} \right\rceil$ ■

Example. Let $f_n : [0, 1] \to \mathbb{R}$, $f_n(x) = x^n$.

<u>Claim:</u> f_n does NOT converge uniformly onn $[0, 1)$.

Choose $\varepsilon_0 = \dfrac{1}{2}$. $\forall k \in \mathbb{N}$, $\exists n_k \in \mathbb{N}$ s.t.

$$n_k = k, \ x_k = (\frac{1}{2})^{\frac{1}{k}}$$

Then

$$\begin{aligned}
|f_{n_k}(x_k) - f(x_k)| &= |x_k^{n_k} - 0| \\
&= (x_k)^k = \frac{1}{2} \geq \varepsilon_0
\end{aligned}$$

We conclude that f_n is not uniformly convergent to f on $[0, 1]$.