THEOS: Complete Mathematical Formalization

Final Polished Version (Perplexity-Reviewed)

Preamble

This document presents the complete mathematical formalization of THEOS in two parts:

- **Part A (Sections 1–6):** Core definitions, assumptions, and theorems. These establish the fundamental mathematical structure of the THEOS reasoning system.
- **Part B (Sections 7–14):** Additional constructions, sufficient conditions, and illustrative parameter settings that demonstrate the assumptions are plausible and show how they can be met in practice.

Readers should interpret Part A as the rigorous mathematical foundation and Part B as supporting evidence and practical guidance.

PART A: RIGOROUS FOUNDATION

Section 1: State Spaces and Spiral Dynamics

Assumption 1.1: Metric Spaces and Completeness

Let $\mathcal{O}, \mathcal{I}, \mathcal{A}, \mathcal{D}, \mathcal{F}, \mathcal{W}$ be complete metric spaces with metrics $d_O, d_I, d_A, d_D, d_F, d_W$ respectively.

Define the combined state space: $$\mathcal{S} = \mathcal{I} \times \mathcal{A} \times \mathcal{D} \times \mathcal{F} \times \mathcal{W}$$

with product metric: $$d_S\big((I, A, D, \Phi, \gamma), (I', A', D', \Phi', \gamma')\big) = d_I(I, I') + d_A(A, A') + d_D(D, D') + d_F(\Phi, \Phi') + d_W(\gamma, \gamma')$$

We use $\|\cdot\|$ for norms on $\mathcal{D}$ and d_X for metrics on each space X.

Definition 1.2: Phase Operators

For induction, abduction, deduction, contradiction and wisdom update, define: $$\sigma_I : \mathcal{O} \times \mathcal{F} \to \mathcal{I}, \quad \sigma_A : \mathcal{I} \times \mathcal{W} \to \mathcal{A}, \quad \sigma_D : \mathcal{A} \to \mathcal{D} \mathrm{Contr} : \mathcal{D} \times \mathcal{D} \to \mathcal{F}, \quad \mathrm{Upd}_\gamma : \mathcal{W} \times \mathcal{Q} \times \mathcal{D} \times \mathcal{F} \to \mathcal{W}$$

Let $q \in \mathcal{Q}$ be a fixed query and $(O_n)_{n \geq 0}$ an observation process (possibly deterministic given q).

Definition 1.3: THEOS Cycle Map

Given dual deductions D_L, D_R (specified in Section 2 below), define the cycle map $T_q : \mathcal{S} \to \mathcal{S}$ by: $$T_q(S_n) = S_{n+1} = \big(I_{n+1}, A_{n+1}, D_{n+1}, \Phi_{n+1}, \gamma_{n+1}\big)$$

where, for the next observation O' (or sufficient statistic of it): $$I_{n+1} = \sigma_I(O', \Phi_n), \quad A_{n+1} = \sigma_A(I_{n+1}, \gamma_n), \quad D_{n+1} = \sigma_D(A_{n+1})\Phi_{n+1} = \mathrm{Contr}(D_L^{(n+1)}, D_R^{(n+1)}), \quad \gamma_{n+1} = \mathrm{Upd}_\gamma(\gamma_n, q, D_n, \Phi_n)$$

Assumption 1.4: Contractive Spiral

This is an assumption to be checked or approximated in concrete implementations, not automatically guaranteed.

For the fixed query q and observation process (O_n), there exists $\rho \in (0,1)$ such that: $$d_S\big(T_q(s), T_q(s')\big) \le \rho\, d_S(s,s') \quad \text{for all } s, s' \in \mathcal{S}$$

Theorem 1.5: Spiral Convergence

Under Assumption 1.4 (contractive spiral), for any initial state $S_0 \in \mathcal{S}$, the sequence: $$S_{n+1} = T_q(S_n), \quad n \ge 0$$

converges to a unique fixed point $S^\star \in \mathcal{S}$, and: $$d_S(S_n, S^\star) \le \rho^n d_S(S_0, S^\star)$$

Proof: This is the Banach fixed-point theorem applied to the complete metric space $(\mathcal{S}, d_S)$.

Section 2: Dual Engines and Contradiction

Definition 2.1: Dual Engines

Let $f_L, f_R : \mathcal{I} \times \mathcal{A} \times \mathcal{W} \to \mathcal{D}$ be the left (constructive) and right (critical) deduction functions.

At cycle n, given $(I_n, A_n^{(L)}, A_n^{(R)}, \gamma_n)$, define: $$D_L^{(n)} = f_L(I_n, A_n^{(L)}, \gamma_n), \quad D_R^{(n)} = f_R(I_n, A_n^{(R)}, \gamma_n)$$

where $D_L^{(n)}$ and $D_R^{(n)}$ are the left and right deductions, distinct from the combined deduction D_n in the state vector.

Definition 2.2: Multi-Axis Contradiction

Let $\Delta_F, \Delta_N, \Delta_C, \Delta_D : \mathcal{D} \times \mathcal{D} \to [0, \infty)$ be four discrepancy functionals measuring factual, normative, constraint, and distributional differences.

For weights $\alpha_i \ge 0$, $\sum_{i=1}^4 \alpha_i = 1$, define the contradiction magnitude: $$\Phi_n = \Phi(D_L^{(n)}, D_R^{(n)}) := \sqrt{\alpha_1 \Delta_F^2 + \alpha_2 \Delta_N^2 + \alpha_3 \Delta_C^2 + \alpha_4 \Delta_D^2}$$

Definition 2.3: Productive Contradiction

Let $\mathrm{IG}(\Phi_n) \geq 0$ denote an information-gain functional (e.g., KL divergence between posteriors) and $\mathrm{Cost}_{\mathrm{resolve}}(\Phi_n) > 0$ the cost of resolving the contradiction.

Define: $$Q(\Phi_n) = \frac{\mathrm{IG}(\Phi_n)}{\mathrm{Cost}_{\mathrm{resolve}}(\Phi_n)}$$

A contradiction is called **productive** if $Q(\Phi_n) > \tau$ for some domain-dependent threshold $\tau > 0$.

Assumption 2.4: Error Dynamics with Productive Contradiction

Let $D^\star \in \mathcal{D}$ denote the (possibly implicit) ground-truth prediction and: $$e_n = d_D(D_n, D^\star)$$

the prediction error at cycle n. There exist constants $0 < \rho_1 < \rho_2 < 1$ such that:

- If $Q(\Phi_n) \leq \tau$ (non-productive contradiction), then $e_{n+1} \leq \rho_2 e_n$
- If $Q(\Phi_n) > \tau$ (productive contradiction), then $e_{n+1} \leq \rho_1 e_n$

Assumption 2.5: Single-Engine Baseline

A single-engine baseline process satisfies: $$e_{n+1}^{\mathrm{single}} \leq \bar{\rho}\, e_n^{\mathrm{single}}$$

for some $\bar{\rho} \in (\rho_1, \rho_2)$.

Assumption 2.6: Frequency of Productive Contradictions

The existence of a positive lower bound $p > 0$ on productive contradiction frequency is an empirical and design condition; Theorem 2.7 is conditional on this.

Along the dual-engine trajectory, productive contradictions occur with asymptotic frequency $p > 0$ in the sense that: $$\liminf_{N\to\infty} \frac{1}{N} \left|\{0 \leq n < N : Q(\Phi_n) > \tau\}\right| \geq p$$

Theorem 2.7: Asymptotic Dual-Engine Advantage

Under Assumptions 2.4–2.6, the dual-engine error sequence satisfies: $$\limsup_{n \to \infty} \bigl(e_n^{\mathrm{dual}}\bigr)^{1/n} \leq \rho_{\mathrm{eff}} \quad \text{with} \quad \rho_{\mathrm{eff}} = \rho_1^p \rho_2^{1-p} < \bar{\rho}$$

while the single-engine process satisfies: $\limsup_{n \to \infty} \bigl(e_n^{\mathrm{single}}\bigr)^{1/n} \leq \bar\rho$

Under these conditions, the dual engine achieves strictly faster asymptotic geometric convergence than the single engine.

Proof: Over N steps, at least pN indices have contraction factor at most ρ_1, and at most $(1-p)N$ have at most ρ_2. Thus: $e_N^{\mathrm{dual}} \leq e_0^{\mathrm{dual}} \rho_1^{pN} \rho_2^{(1-p)N} = e_0^{\mathrm{dual}} \left(\rho_1^p \rho_2^{1-p}\right)^N$

Taking N-th roots and limsup yields the claim. The single-engine part is immediate from the assumed bound.

Section 3: Abduction as Optimization and Bracket

Definition 3.1: Abduction Quality

For fixed patterns $I \in \mathcal{I}$, define explanatory power $\mathrm{EP}(H, I) \geq 0$ and complexity $\mathrm{Complexity}(H) \geq 0$ (e.g., via Minimum Description Length), and let $\lambda > 0$.

The abduction quality is: $Q_A(H; I) = \frac{\mathrm{EP}(H,I)}{1+\lambda\,\mathrm{Complexity}(H)}$

Assumption 3.2: True Hypothesis and Nested Feasible Sets

There exists a (possibly idealized) ground-truth hypothesis $H^\star \in \mathcal{H}$ and a nested sequence $(\mathcal{H}_n)_{n\geq 0}$ with $\mathcal{H}_n \subseteq \mathcal{H}$ such that:

1. $H^\star \in \mathcal{H}_n$ for all n
2. $\mathcal{H}_{n+1} \subseteq \mathcal{H}_n$ for all n
3. The update $\mathcal{H}_n \to \mathcal{H}_{n+1}$ is induced by new evidence/contradictions and removes only hypotheses inconsistent with that evidence

Definition 3.3: Left/Right Abductions and Quality Bracket

On cycle n, define: $A_n^{(L)} \in \arg\max_{H\in\mathcal{H}_n} Q_A(H; I_n), \quad A_n^{(R)} \in \arg\min_{H\in\mathcal{H}_n} Q_A(H; I_n)$

The bracket $[A_n^{(R)}, A_n^{(L)}]$ is an interval in quality space (the scalar values Q_A), not necessarily an interval in hypothesis coordinates. Let: $q_n^{\max} = Q_A(A_n^{(L)}; I_n), \quad q_n^{\min} = Q_A(A_n^{(R)}; I_n), \quad \mathrm{Width}_n = q_n^{\max} - q_n^{\min}$

Assumption 3.4: Monotone Tightening of the Quality Bracket

There exists $\eta \in (0,1)$ such that for all n: $\mathrm{Width}_{n+1} \leq \eta\, \mathrm{Width}_n$

Theorem 3.5: Exponential Convergence in Quality

Under Assumptions 3.2–3.4: $\mathrm{Width}_n \leq \eta^n \mathrm{Width}_0 \to 0 \quad \text{as } n \to \infty$

Moreover, any accumulation point H_∞ of the maximizing (or minimizing) sequence satisfies: $Q_A(H_\infty; I_\infty) = \lim_{n\to\infty} q_n^{\max} = \lim_{n\to\infty} q_n^{\min}$

where I_∞ is the limit pattern state if it exists and is unique.

Proof: Immediate from the geometric bound on Width_n. The equality of the limits for accumulation points follows from continuity of Q_A under appropriate regularity.

Assumption 3.6: Local Regularity of Q_A

There exists a strictly increasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ and a neighborhood $\mathcal{U}$ of $H^\star$ such that for all $H, H' \in \mathcal{U}$: $d_A(H, H') \leq g\big(|Q_A(H; I_\infty) - Q_A(H'; I_\infty)|\big)$

Corollary 3.7: Abduction Convergence in Hypothesis Space

Under Assumptions 3.2–3.6, if $H^\star$ is the unique maximizer of $Q_A(\cdot; I_\infty)$ in $\mathcal{U}$, then: $d_A\big(A_n^{(L)}, A_n^{(R)}\big) \to 0 \quad \text{and} \quad A_n^{(L)}, A_n^{(R)} \to H^\star$

whenever the sequence eventually lies in $\mathcal{U}$.

Section 4: Wisdom and Cost

Definition 4.1: Wisdom Memory and Relevant Set

At cycle n, let: $W_n = \{(q_i, H_i, \mathrm{res}_i, \mathrm{conf}_i)\}_{i=1}^n$

be the wisdom memory, and $\mathrm{Sim} : \mathcal{Q} \times \mathcal{Q} \to [0,1]$ a similarity metric.

For a new query q, define the relevant subset: $W_{\mathrm{rel}}(q) = \{(q_i, H_i, \mathrm{res}_i, \mathrm{conf}_i) : \mathrm{Sim}(q, q_i) > \sigma\}$

for some threshold $\sigma \in (0, 1)$.

Definition 4.2: Per-Cycle Cost

The cost of cycle n is: $$\mathrm{Cost}_n = \mathrm{Cost}_{\text{base}} + \mathrm{Cost}_{\text{engines}} + \mathrm{Cost}_{\text{contr}} - \mathrm{Savings}_{\text{wisdom},n}$$

where $\mathrm{Savings}_{\text{wisdom},n} \geq 0$ is the computational savings attributable to using W_{rel}.

Assumption 4.3: Expected Marginal Savings

For a given domain distribution over queries, there exist constants $c > 0$ and $\kappa > 0$ such that: $$\mathbb{E}\left[\mathrm{Savings}_{\text{wisdom},n+1} - \mathrm{Savings}_{\text{wisdom},n}\right] \geq c\, e^{-\kappa n} \quad \text{for all } n \geq 0$$

Theorem 4.4: Upper Bound on Expected Cost

Under Assumption 4.3, there exist constants $C_1, C_2 > 0$ such that: $$\mathbb{E}[\mathrm{Cost}_n] \leq C_1 + C_2 e^{-\kappa n} \quad \text{for all } n \geq 0$$

Proof: Summing the marginal savings lower bound gives a convergent geometric series; rearranging yields a geometric upper bound on the expected cost term, up to additive constants that depend on the base and engine costs.

Section 5: Governor and Halting

Definition 5.1: Governor State

At cycle n, the governor state is: $$G_n = (\mathrm{cycles\_remaining}_n, \mathrm{budget\_remaining}_n, \mathrm{confidence}_n, \mathrm{contradiction\_trend}_n)$$

Definition 5.2: Objective for Halting

Let $\hat{E}(n)$ be an estimate of expected error cost if halting at cycle n, and: $$C(n) = \sum_{k=0}^{n-1} \mathrm{Cost}_k$$

the cumulative computation cost up to n. For a cost-sensitivity parameter $\lambda > 0$, define: $$J(n) = \hat{E}(n) + \lambda\, C(n)$$

Definition 5.3: Optimal Halting Cycle

An optimal halting cycle is: $n^\star \in \arg\min_{n\in\mathbb{N}} J(n)$

In practice, the governor maintains an online estimate $\hat{J}_n$ and halts when approximate local optimality and additional constraints are satisfied, e.g.:

- **Convergence:** $\|D_L^{(n)} - D_R^{(n)}\| < \epsilon_1$ and $Q(\Phi_n) < \tau$
- **Diminishing returns:** $\mathrm{IG}(\Phi_n)/\mathrm{IG}(\Phi_{n-1}) < \rho_{\min}$
- **Budget exhaustion:** $\mathrm{cycles\_remaining}_n = 0$ or $\mathrm{budget\_remaining}_n < \mathrm{Cost}_n$
- **Irreducible uncertainty:** $\mathrm{Entropy}(A_n) < \epsilon_2$ and $\Phi_n > \delta_{\min}$

These operational criteria approximate the minimizer $n^\star$ of $J(n)$ under uncertainty.

Section 6: Output Rule

When the governor halts at cycle N, define the output: $\mathrm{Output} = \begin{cases} D_N^{(L)} & \text{if } \Phi_N < \epsilon_1 \\ \mathrm{Blend}(D_N^{(L)}, D_N^{(R)}) & \text{if } \epsilon_1 \le \Phi_N < \epsilon_2 \\ (D_N^{(L)}, D_N^{(R)}, \Phi_N) & \text{if } \Phi_N \ge \epsilon_2 \end{cases}$

where the blending operator is, for example: $\mathrm{Blend}(D_L, D_R) = w_L D_L + w_R D_R, \quad w_L = \frac{1-\Phi_N/\epsilon_2}{2}, \quad w_R = \frac{1+\Phi_N/\epsilon_2}{2}$

PART B: SUPPORTING CONSTRUCTIONS AND ILLUSTRATIVE PARAMETERS

Section 7: Productive Contradiction Emergence

Proposition 7.1: Productive Contradiction Emergence from Finite Hypothesis Space

This is a sufficient condition for productive contradictions in finite settings.

Claim: Under the nested feasible set assumption, if new evidence eliminates at least one hypothesis per cycle, then productive contradictions must occur with positive frequency.

Formal Statement: Let $\mathcal{H}_n$ be the feasible set at cycle n with $|\mathcal{H}_n| = M_n$.

Assumption 7.1.1: The evidence from cycle n is informative: $M_{n+1} < M_n$ for all n (at least one hypothesis eliminated).

Proposition: If $M_0 < \infty$ (finite initial hypothesis space) and $M_{n+1} < M_n$ for all n, then:

1. The sequence terminates in finite time: $\exists N$ such that $M_N = 1$ (unique hypothesis)
2. Before termination, productive contradictions occur with frequency $p \geq \frac{1}{M_0}$

Proof Sketch:

- By pigeonhole principle, eliminating at least one hypothesis per cycle means we exhaust the finite space in at most M_0 cycles
- Each elimination corresponds to a contradiction that distinguished between hypotheses
- Such contradictions must be productive (they provide information that eliminates options)
- Therefore, productive contradictions occur at least once per M_0 cycles, giving $p \geq 1/M_0 > 0$

Implication: This shows that productive contradictions are plausible in finite hypothesis spaces and provides a constructive bound.

Section 8: Local Regularity Justification

Theorem 8.1: Sufficient Conditions for Local Regularity

These are sufficient, not necessary, conditions; in practice, we expect approximate regularity in many smooth Bayesian or neural abduction models.

Definition 8.1.1: Q_A satisfies **local regularity** near $H^\star$ if there exists a strictly increasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that: $$d_A(H, H') \leq g(|Q_A(H; I_\infty) - Q_A(H'; I_\infty)|)$$

Theorem: Local regularity holds if:

1. **Lipschitz Explanatory Power:** $\mathrm{EP}(H, I)$ is Lipschitz continuous in H with constant L_E
2. **Bounded Complexity:** $\mathrm{Complexity}(H)$ is bounded away from zero in a neighborhood of $H^\star$
3. **Non-degeneracy:** At $H^\star$, the gradient $\nabla_H Q_A(H^\star; I_\infty) \neq 0$

Proof Sketch:

- By Lipschitz continuity: $|Q_A(H) - Q_A(H')| \leq L_E \cdot d_A(H, H')$ (up to complexity term)
- By non-degeneracy: Q_A is strictly increasing near $H^\star$
- Therefore: $d_A(H, H') \leq L_E^{-1} |Q_A(H) - Q_A(H')|$ locally
- Set $g(x) = L_E^{-1} x$ to satisfy local regularity

Implication: This provides concrete, verifiable conditions for when Assumption 3.6 holds.

Section 9: Nested Feasible Sets Mechanism

Definition 9.1: Evidence-Driven Hypothesis Elimination

Mechanism: At cycle n, given contradiction Φ_n between $D_L^{(n)}$ and $D_R^{(n)}$:

1. **Contradiction Analysis:** Identify which hypotheses in $\mathcal{H}_n$ would predict $D_L^{(n)}$ vs. $D_R^{(n)}$
   - Let $\mathcal{H}_L = \{H \in \mathcal{H}_n : H \text{ predicts } D_L^{(n)}\}$
   - Let $\mathcal{H}_R = \{H \in \mathcal{H}_n : H \text{ predicts } D_R^{(n)}\}$
2. **Elimination Rule:** Eliminate hypotheses inconsistent with the resolved contradiction
   - If evidence favors $D_L^{(n)}$, eliminate $\mathcal{H}_R \setminus \mathcal{H}_L$
   - If evidence favors $D_R^{(n)}$, eliminate $\mathcal{H}_L \setminus \mathcal{H}_R$
   - If evidence supports both partially, eliminate only those inconsistent with both
3. **Update:** $\mathcal{H}_{n+1} = \mathcal{H}_n \setminus \{\text{eliminated hypotheses}\}$

Theorem 9.2: Nesting and Ground Truth Preservation

Theorem: The evidence-driven elimination mechanism maintains nesting and ensures $H^\star \in \mathcal{H}_n$ for all n.

Proof Sketch:

- By construction, $\mathcal{H}_{n+1} \subseteq \mathcal{H}_n$
- Since $H^\star$ generates ground truth, it predicts the actual outcome
- Therefore, $H^\star$ is never eliminated, so $H^\star \in \mathcal{H}_{n+1}$

Implication: This formalizes how contradictions drive hypothesis elimination, justifying Assumption 3.2.

Section 10: Wisdom Similarity Formalization

Definition 10.1: Semantic Similarity for Query Transfer

Assumption 10.1.1: Queries and hypotheses live in a semantic embedding space with distance metric d_Q.

Definition: Similarity between queries is: $$\text{Sim}(q, q_i) = \exp\left(-\frac{d_Q(q,q_i)^2}{2\sigma_q^2}\right)$$

where σ_q is a domain-dependent bandwidth parameter.

Theorem 10.2: Wisdom Transfer Benefit

Claim: If $\text{Sim}(q, q_i) > \sigma$ (above threshold), then the hypothesis H_i from query q_i provides useful information for query q.

Formal Statement: Let H_i be the resolved hypothesis for query q_i and $H_q^\star$ be the ground truth for query q.

Under the assumption that similar queries have similar ground truths (Lipschitz in query space): $$d_A(H_i, H_q^\star) \leq L_q \cdot d_Q(q, q_i)$$

where L_q is the Lipschitz constant.

Implication: If $\text{Sim}(q, q_i) > \sigma$, then $d_Q(q, q_i) < -2\sigma_q^2 \ln(\sigma)$, which means: $$d_A(H_i, H_q^\star) < L_q \cdot \sqrt{-2\sigma_q^2 \ln(\sigma)}$$

So H_i is close enough to $H_q^\star$ to provide useful initialization, reducing exploration cost.

Implication: This formalizes why wisdom transfer works and when it's beneficial.

Section 11: Information Gain Specification

Definition 11.1: Information Gain from Contradiction

Framework: Maintain posterior distributions over hypotheses:

- $P_n(H|\text{history}_n)$ = posterior after n cycles

Definition: Information gain from contradiction Φ_n: $\mathrm{IG}(\Phi_n) = \mathrm{KL}(P_{n+1}(H|\mathrm{history}_{n+1}) \| P_n(H|\mathrm{history}_n))$

This measures how much the posterior changes due to the contradiction.

Theorem 11.2: Information Gain Bounds Entropy Reduction

Claim: If $\mathrm{IG}(\Phi_n) > \tau_{\mathrm{IG}}$, then the entropy of the posterior decreases by at least τ_{IG}: $H(P_{n+1}) \leq H(P_n) - \tau_{\mathrm{IG}}$

Proof: By properties of KL divergence and entropy.

Implication: This connects information gain to hypothesis space reduction, justifying why productive contradictions drive progress.

Section 12: Finite-Time Error Bounds

Proposition 12.1: Illustrative Finite-Time Error Bound

This is an illustrative calculation with specific parameter choices, not a universal result.

Claim: Under the dual-engine setup with productive contradiction frequency p, the error at cycle n satisfies: $e_n^{\mathrm{dual}} \leq e_0 \cdot \rho_{\mathrm{eff}}^n + \epsilon_{\mathrm{residual}}$

where:

- $\rho_{\mathrm{eff}} = \rho_1^p \rho_2^{1-p}$ (effective contraction rate)
- $\epsilon_{\mathrm{residual}}$ is the irreducible error from unresolvable contradictions

Illustrative Example: For a representative choice of parameters:

- If $\rho_1 = 0.3$, $\rho_2 = 0.7$, $p = 0.5$: $\rho_{\mathrm{eff}} = 0.46$
- This means error reduces by ~54% per cycle
- To reach $e_n < 0.01 \cdot e_0$: need $n \geq \log(0.01)/\log(0.46) \approx 5$ cycles

Note: Other domains will yield different constants depending on their specific contraction factors and productive contradiction frequencies.

Section 13: Energy Efficiency Decomposition

Proposition 13.1: Illustrative Energy Efficiency Parameterization

This is an illustrative parameterization; other domains will yield different constants.

Qualitative Theorem: If (i) dual-engine cycle count is lower by factor $r > 1$, and (ii) per-cycle cost is reduced by factor $s \in (0, 1)$ via wisdom, then total energy is reduced by factor rs.

Proof: Total energy is proportional to (number of cycles) $\times$ (cost per cycle). If cycles are reduced by r and per-cycle cost by s, total energy is reduced by rs.

Illustrative Parameterization: For a representative choice of parameters:

1. Cycle Count Advantage: Dual engines converge faster

- Single engine: $N_{\text{single}} = \frac{\log(e_{\text{target}}/e_0)}{\log(\bar{\rho})}$
- Dual engine: $N_{\text{THEOS}} = \frac{\log(e_{\text{target}}/e_0)}{\log(\rho_{\text{eff}})}$
- Example ratio: $\frac{N_{\text{THEOS}}}{N_{\text{single}}} = \frac{\log(\bar{\rho})}{\log(\rho_{\text{eff}})} \approx 0.5$ (2x fewer cycles with $\bar{\rho} = 0.8$, $\rho_{\text{eff}} = 0.46$)

2. Per-Cycle Cost Advantage: Wisdom reduces per-cycle cost

- With wisdom: $C_{\text{THEOS}} = C_{\text{base}} + C_{\text{engines}} - \text{Savings}_{\text{wisdom}}$
- Example savings: $\text{Savings}_{\text{wisdom}} \approx 0.3 \cdot C_{\text{base}}$ after accumulation
- Example ratio: $\frac{C_{\text{THEOS}}}{C_{\text{single}}} \approx 0.7$ (30% cost reduction)

3. Combined Example: $\frac{E_{\text{THEOS}}}{E_{\text{single}}} \approx 0.5 \times 0.7 = 0.35$

Conclusion: For this representative parameterization, total energy is approximately 35% of single-engine cost, corresponding to ~65% energy savings. **Other domains will yield different percentages depending on their specific contraction factors, wisdom accumulation rates, and problem structure.**

Section 14: Irreducible Uncertainty Formalization

Definition 14.1: Irreducible Uncertainty

Claim: A contradiction Φ_n is irreducible if:

1. **Hypothesis Indistinguishability:** Both $D_L^{(n)}$ and $D_R^{(n)}$ are consistent with all available evidence
    - $\mathcal{H}_L \cap \mathcal{H}_R \neq \emptyset$ (overlapping predictions)
2. **Information Saturation:** Further cycles cannot distinguish between them
    - $\text{IG}(\Phi_{n+k}) < \tau_{\text{IG}}$ for all $k > 0$
3. **Fundamental Ambiguity:** The ground truth itself is ambiguous
    - Multiple valid hypotheses explain the data equally well

Theorem 14.2: Irreducible Contradictions Require Output Blending

Claim: When Φ_n is irreducible, the optimal output is a weighted blend: $$\text{Output} = w_L D_L^{(n)} + w_R D_R^{(n)}$$

where weights are proportional to posterior probabilities: $$w_L = \frac{P(D_L^{(n)}|\text{data})}{\sum_j P(D_j|\text{data})}$$

Implication: This formalizes when to output contradictions vs. blended predictions (see Definition 5.2).

PART C: SUMMARY AND DOMAIN APPLICABILITY

Section 15: Complete Framework Overview

The THEOS mathematical core consists of:

1. **Rigorous Foundation** (Sections 1–6): Complete metric space formalism, dual-engine dynamics, abduction optimization, wisdom accumulation, governor control, and output rules
2. **Supporting Constructions** (Sections 7–14): Justification of key assumptions, formalization of mechanisms, and illustrative parameterizations
3. **Key Properties:**
    - **Convergence:** Spiral converges to fixed point with exponential rate (Theorem 1.5)
    - **Dual-Engine Advantage:** Faster asymptotic convergence than single engine under explicit conditions (Theorem 2.7)
    - **Abduction Bracket:** Quality bracket tightens exponentially (Theorem 3.5)
    - **Wisdom Benefit:** Expected cost bounded with exponential decay (Theorem 4.4)
    - **Energy Efficiency:** Illustrative parameterization shows ~65% energy savings for representative parameters (Proposition 13.1)

Section 16: Domain Universality (Conditional)

THEOS defines a well-posed reasoning process in any domain where:

1. **State spaces and operators** can be instantiated (Definitions 1.2–1.3)
2. **The listed assumptions approximately hold:**
    - Contractivity of the cycle map (Assumption 1.4)
    - Productive contradictions occur with positive frequency (Assumption 2.6)
    - Local regularity of the abduction quality function (Assumption 3.6)
    - Wisdom transfer provides cost reduction (Assumption 4.3)

In domains where these conditions are met or approximately met, THEOS provides a mathematically well-defined framework for reasoning under uncertainty.

Section 17: Validation and Verification Checklist

To fully validate THEOS, verify:

Theoretical Validation

- [ ] Prove Theorem 1.5 for specific problem classes (e.g., linear hypothesis spaces)
- [ ] Verify Assumption 2.4 empirically on benchmark datasets
- [ ] Confirm Assumption 3.4 (bracket tightening) holds for real abduction tasks
- [ ] Validate Assumption 4.3 (marginal savings) with wisdom accumulation data

Empirical Validation

- [ ] Measure energy consumption: THEOS vs. single-engine on standard benchmarks
- [ ] Compare error rates on test sets (healthcare, finance, ethics domains)
- [ ] Track wisdom accumulation: cost reduction over time
- [ ] Test scalability: performance on problems of increasing size

Implementation Validation

- [ ] Implement cycle map T_q for a concrete domain
- [ ] Verify metric space properties hold in practice
- [ ] Test Governor halting criteria on real queries
- [ ] Validate output blending on irreducible contradictions

Section 18: Open Questions

1. **Productive Contradiction Frequency:** Is the bound $p \geq 1/M_0$ tight? Can we improve it for infinite hypothesis spaces?

2. **Local Regularity:** Are the sufficient conditions in Theorem 8.1 necessary? What if they fail?

3. **Wisdom Transfer:** How does the Gaussian kernel choice affect transfer quality? Are there better similarity metrics?
4. **Irreducible Uncertainty:** Can we characterize when contradictions are fundamentally unresolvable vs. just computationally hard?
5. **Finite-Time Bounds:** Can we tighten the $\epsilon_{\text{residual}}$ term in Proposition 12.1?
6. **Energy Efficiency:** How sensitive is the energy savings to parameter choices? What are typical ranges across domains?

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