Appendices

Appendix A: Appendix

This appendix records the conditional c.g.f. $\Lambda_n(\theta; z)$ and its derivatives, and provides the proof details for Lemmas 4.1–4.7 used in Section 4.

Proposition A.1. *$\Lambda_n(\theta; z)$ is the logarithmic moment generating function of $U^{(n)}X^{(n)}$ conditioned on $\mathcal{Z}_n = z$. That is,*

$$\Lambda_n(\theta; z) = \log\left[\lambda_{U^{(n)}}(\theta) F_\epsilon(\frac{v-z}{b^{(n)}}) + 1 - F_\epsilon(\frac{v-z}{b^{(n)}})\right].$$

The first and second partial derivatives are given by:

$$\frac{\partial}{\partial\theta}\Lambda_n(\theta; z) = \frac{F_\epsilon(\frac{v-z}{b^{(n)}})\lambda'_{U^{(n)}}(\theta)}{F_\epsilon(\frac{v-z}{b^{(n)}})\lambda_{U^{(n)}}(\theta) + 1 - F_\epsilon(\frac{v-z}{b^{(n)}})}$$

$$\frac{\partial}{\partial z}\Lambda_n(\theta; z) = \frac{\frac{1}{b^{(n)}} f_\epsilon(\frac{v-z}{b^{(n)}})(1 - \lambda_{U^{(n)}}(\theta))}{F_\epsilon(\frac{v-z}{b^{(n)}})\lambda_{U^{(n)}}(\theta) + 1 - F_\epsilon(\frac{v-z}{b^{(n)}})}$$

$$\frac{\partial^2}{\partial z\partial\theta}\Lambda_n(\theta; z) = \frac{-f_\epsilon\left(\frac{v-z}{b^{(n)}}\right)\frac{1}{b^{(n)}}\mathbf{E}[U^{(n)}e^{\theta U^{(n)}}]}{\left[F_\epsilon\left(\frac{v-z}{b^{(n)}}\right)\mathbf{E}[e^{\theta U^{(n)}}] + 1 - F_\epsilon\left(\frac{v-z}{b^{(n)}}\right)\right]^2}$$

$$\frac{\partial^2}{\partial\theta^2}\Lambda_n(\theta; z) = \frac{F_\epsilon\left(\frac{v-z}{b^{(n)}}\right)\left[\mathbf{E}[U^2e^{\theta U^{(n)}}]\left(F_\epsilon\left(\frac{v-z}{b^{(n)}}\right)\mathbf{E}[e^{\theta U^{(n)}}] + 1 - F_\epsilon\left(\frac{v-z}{b^{(n)}}\right)\right) - \left(\mathbf{E}[U^{(n)}e^{\theta U^{(n)}}]\right)^2 F_\epsilon\left(\frac{v-z}{b^{(n)}}\right)\right]}{\left[F_\epsilon\left(\frac{v-z}{b^{(n)}}\right)\mathbf{E}[e^{\theta U^{(n)}}] + 1 - F_\epsilon\left(\frac{v-z}{b^{(n)}}\right)\right]^2}$$

A.1. Proof of Lemma 4.1

Applying Proposition 3.3, it follows that

$$\begin{aligned}
h_n(t) =& \frac{1}{n}[-n\phi_n(-tM_n) + \log f_{\mathcal{Z}_n}(-tM_n)] \\
=& \frac{1}{n}\left[-nC_n(\theta_{x,n})\cdot\frac{\beta_\epsilon}{\gamma\xi_\epsilon}(\frac{v+tM_n}{b^{(n)}})^{1-\gamma}e^{-\xi_\epsilon(\frac{v+tM_n}{b^{(n)}})^\gamma}(1+o(1)) + (\log\beta_{\mathcal{Z}_n} - \xi_{\mathcal{Z}_n}(tM_n)^\gamma)(1+o(1))\right] \\
h_n'(t) =& \frac{1}{n}\left[-nC_n(\theta_{x,n})\cdot\frac{\beta_\epsilon}{\gamma\xi_\epsilon}e^{-\xi_\epsilon(\frac{v+tM_n}{b^{(n)}})^\gamma}\frac{M_n}{b^{(n)}}\left[(1-\gamma)(\frac{v+tM_n}{b^{(n)}})^{-\gamma} - \xi_\epsilon\gamma\right](1+o(1)) - \xi_{\mathcal{Z}_n}\gamma M_n^\gamma t^{\gamma-1}(1+o(1))\right] \\
=& \frac{1}{n}\left[nC_n(\theta_{x,n})\cdot\beta_\epsilon e^{-\xi_\epsilon(\frac{v+tM_n}{b^{(n)}})^\gamma}\frac{M_n}{b^{(n)}}(1+o(1)) - \xi_{\mathcal{Z}_n}\gamma M_n^\gamma t^{\gamma-1}(1+o(1))\right] \\
h_n''(t) =& \frac{1}{n}\left[nC_n(\theta_{x,n})\cdot\beta_\epsilon\frac{M_n}{b^{(n)}}e^{-\xi_\epsilon(\frac{v+tM_n}{b^{(n)}})^\gamma}\left(-\xi_\epsilon\gamma(\frac{v+tM_n}{b^{(n)}})^{\gamma-1}\frac{M_n}{b^{(n)}}\right) - \xi_{\mathcal{Z}_n}\gamma M_n^\gamma(\gamma-1)t^{\gamma-2}\right]
\end{aligned}$$

In the $h_n''(t)$ term, the first term behaves like $M_n^\gamma n M_n e^{-M_n^\gamma} \sim M_n^\gamma M_n$ since $M_n^\gamma \sim \log n$. The second term behaves like M_n^γ. Hence the first term is the dominant term and is negative when $t > 0$. As $n \to \infty$, $o(1)$ term can be dropped. For $\gamma \in (0, 2)$,

$$
\begin{aligned}
&h_n'(t_{0,n}) = 0\\
\Longleftrightarrow & nC_n(\theta_{x,n}) \cdot \beta_\epsilon e^{-\xi_\epsilon(\frac{v+t_{0,n}M_n}{b^{(n)}})^\gamma} \frac{M_n}{b^{(n)}}(1+o(1)) = \xi_{\mathcal{Z}_n}\gamma M_n^\gamma t_{0,n}^{\gamma-1}\\
\Longleftrightarrow & \log \frac{C_n(\theta_{x,n})\beta_\epsilon}{\xi_{\mathcal{Z}_n}\gamma b^{(n)}} + \log n + \log M_n - \xi_\epsilon(\frac{v+t_{0,n}M_n}{b^{(n)}})^\gamma = \gamma \log M_n + (\gamma-1)\log t_{0,n}\\
&\text{Notice that } (\frac{t_{0,n}M_n + v}{b^{(n)}})^\gamma = (\frac{t_{0,n}M_n}{b^{(n)}})^\gamma + \gamma\frac{v}{b^{(n)}}(\frac{t_{0,n}M_n}{b^{(n)}})^{\gamma-1}(1+O(M_n^{\gamma-2})) \text{ as } n \to \infty\\
\Longleftrightarrow & \log \frac{C_n(\theta_{x,n})\beta_\epsilon}{\xi_{\mathcal{Z}_n}\gamma b^{(n)}} + \log n + \log M_n - \xi_\epsilon(\frac{t_{0,n}M_n}{b^{(n)}})^\gamma - \frac{v\gamma\xi_\epsilon}{b^{(n)}}(\frac{t_{0,n}M_n}{b^{(n)}})^{\gamma-1}(1+O(M_n^{\gamma-2})) = \gamma\log M_n + (\gamma-1)\log t_{0,n}\\
\Longleftrightarrow & (t_{0,n}M_n)^\gamma = \frac{(b^{(n)})^\gamma}{\xi_\epsilon}\log n + (1-\gamma)\frac{(b^{(n)})^\gamma}{\xi_\epsilon}\log M_n - v\gamma(t_{0,n}M_n)^{\gamma-1}(1+O(M_n^{\gamma-2})) +\\
&\quad \frac{(b^{(n)})^\gamma}{\xi_\epsilon}\log\frac{C_n(\theta_{x,n})\beta_\epsilon}{\xi_{\mathcal{Z}_n}\gamma b^{(n)}} + (1-\gamma)\frac{(b^{(n)})^\gamma}{\xi_\epsilon}\log t_{0,n}\\
\Longleftrightarrow & t_{0,n}^\gamma = \frac{(b^{(n)})^\gamma}{\xi_\epsilon}\frac{\log n}{M_n^\gamma} + \frac{(1-\gamma)(b^{(n)})^\gamma}{\xi_\epsilon}\frac{\log M_n}{M_n^\gamma} - v\gamma t_{0,n}^{\gamma-1}\frac{1}{M_n}(1+O(M_n^{\gamma-2})) +\\
&\quad \frac{1}{M_n^\gamma}\left[\frac{(b^{(n)})^\gamma}{\xi_\epsilon}\log\frac{C_n(\theta_{x,n})\beta_\epsilon}{\xi_{\mathcal{Z}_n}\gamma b^{(n)}} + (1-\gamma)\frac{(b^{(n)})^\gamma}{\xi_\epsilon}\log t_{0,n}\right]\\
\Longleftrightarrow & t_{0,n}^\gamma = 1 + \frac{(1-\gamma)(b^{(n)})^\gamma}{\xi_\epsilon}\frac{\log M_n}{M_n^\gamma} - v\gamma t_{0,n}^{\gamma-1}\frac{1}{M_n}(1+O(M_n^{\gamma-2})) +\\
&\quad \frac{1}{M_n^\gamma}\left[\frac{(b^{(n)})^\gamma}{\xi_\epsilon}\log\frac{C_n(\theta_{x,n})\beta_\epsilon}{\xi_{\mathcal{Z}_n}\gamma b^{(n)}} + (1-\gamma)\frac{(b^{(n)})^\gamma}{\xi_\epsilon}\log t_{0,n}\right].
\end{aligned}
$$

And for $\gamma = 2$, it follows by similar methods that

$$
t_{0,n}^2 = 1 - \frac{2v}{M_n}t_{0,n} - \frac{(b^{(n)})^2\log M_n}{\xi_\epsilon M_n^2} + \frac{1}{M_n^2}\left(\frac{(b^{(n)})^2}{\xi_\epsilon}\log\frac{C_n(\theta_{x,n})\beta_\epsilon}{2\xi_{\mathcal{Z}_n}b^{(n)}} - \frac{(b^{(n)})^2}{\xi_\epsilon}\log t_{0,n} - v^2\right)(1+o(1))
$$

Notice that when $\gamma \in (0,1)$, $t_{0,n} = 1 + O\left(\frac{\log M_n}{M_n^\gamma}\right)$; when $\gamma = 1$, $t_{0,n} = 1 - \frac{v}{M_n} + O\left(\frac{1}{M_n^2}\right)$; when $\gamma \in (1,2)$, $t_{0,n} = 1 - \frac{v}{M_n} + O\left(\frac{\log M_n}{M_n^\gamma}\right)$; when $\gamma = 2$, $t_{0,n} = 1 - \frac{v}{M_n} + O(\frac{\log M_n}{M_n^2})$. And for all three cases, $\frac{t_{0,n}^{\gamma-1}}{M_n} = \frac{1}{M_n} + o\left(\frac{1}{M_n^\gamma}\right)$, hence

$$
t_{0,n}^\gamma = 1 + \frac{(1-\gamma)(b^{(n)})^\gamma}{\xi_\epsilon}\frac{\log M_n}{M_n^\gamma} - \frac{v\gamma}{M_n} + \frac{1}{M_n^\gamma}\left[\frac{(b^{(n)})^\gamma}{\xi_\epsilon}\log\frac{C_n(\theta_{x,n})\beta_\epsilon}{\xi_{\mathcal{Z}_n}\gamma b^{(n)}} - \frac{\log\eta_{\gamma,n}}{\xi_{\mathcal{Z}_n}}\right](1+o(1)),
$$

where $\eta_{2,n} = \exp\{-v^2\xi_{\mathcal{Z}_n}\}$ and $\eta_{\gamma,n} = 1$ for $\gamma \in (0,2)$. Since $t_{0,n}$ is the root of $h_n'(t) = 0$, that is

$$nC_n(\theta_{x,n}) \cdot \beta_\epsilon e^{-\xi_\epsilon(\frac{v+t_{0,n}M_n}{b^{(n)}})^\gamma} \frac{M_n}{b^{(n)}}(1+o(1)) = \xi_{\mathcal{Z}_n}\gamma M_n^\gamma t_{0,n}^{\gamma-1}.$$

$$\begin{aligned}
& nh_n(t_{0,n}) \\
=& -\xi_{\mathcal{Z}_n}\gamma M_n^\gamma t^{\gamma-1}\frac{b^{(n)}}{M_n}\frac{1}{\gamma\xi_\epsilon}\left(\frac{v+t_{0,n}M_n}{b^{(n)}}\right)^{1-\gamma} + \log\beta_{\mathcal{Z}_n} - \xi_{\mathcal{Z}_n}(t_{0,n}M_n)^\gamma \\
=& -\frac{\xi_{\mathcal{Z}_n}}{\xi_\epsilon}b^{(n)}(t_{0,n}M_n)^{\gamma-1}\left[\left(\frac{t_{0,n}M_n}{b^{(n)}}\right)^{1-\gamma} + (1-\gamma)\left(\frac{t_{0,n}M_n}{b^{(n)}}\right)^{-\gamma}\left(\frac{v}{b^{(n)}}\right)(1+o(1))\right] + \log\beta_{\mathcal{Z}_n} - \xi_{\mathcal{Z}_n}(t_{0,n}M_n)^\gamma \\
=& -\frac{\xi_{\mathcal{Z}_n}}{\xi_\epsilon}(b^{(n)})^\gamma(1+o(1)) + \log\beta_{\mathcal{Z}_n} - \xi_{\mathcal{Z}_n}(t_{0,n}M_n)^\gamma \\
=& -\xi_{\mathcal{Z}_n}M_n^\gamma - \frac{(1-\gamma)(b^{(n)})^\gamma\xi_{\mathcal{Z}_n}}{\xi_\epsilon}\log M_n + v\gamma\xi_{\mathcal{Z}_n}M_n^{\gamma-1} - \\
& \left(\underbrace{\frac{(b^{(n)})^\gamma\xi_{\mathcal{Z}_n}}{\xi_\epsilon}\log\frac{C_n(\theta_{x,n})\beta_\epsilon}{\xi_{\mathcal{Z}_n}\gamma b^{(n)}} + \frac{\xi_{\mathcal{Z}_n}}{\xi_\epsilon}(b^{(n)})^\gamma - \log\beta_{\mathcal{Z}_n}}_{\Delta_\gamma^{(n)}} - \log\eta_{\gamma,n}\right)(1+o(1)).
\end{aligned}$$

$$\begin{aligned}
& nh_n''(t_{0,n}) \\
=& -\gamma\xi_{\mathcal{Z}_n}M_n^\gamma t_{0,n}^{\gamma-1}\cdot\xi_\epsilon\gamma\left(\frac{v+t_{0,n}M_n}{b^{(n)}}\right)^{\gamma-1}\frac{M_n}{b^{(n)}}(1+o(1)) \\
=& -\xi_\epsilon\xi_{\mathcal{Z}_n}(\gamma M_n)^2(t_{0,n}M_n)^{\gamma-1}\frac{1}{b^{(n)}}\left[\left(\frac{t_{0,n}M_n}{b^{(n)}}\right)^{\gamma-1} + (\gamma-1)\left(\frac{t_{0,n}M_n}{b^{(n)}}\right)^{\gamma-2}\left(\frac{v}{b^{(n)}}\right)(1+o(1))\right] \\
=& -\xi_\epsilon\xi_{\mathcal{Z}_n}(\gamma M_n)^2 M_n^{2(\gamma-1)}(\frac{1}{b^{(n)}})^\gamma(1+o(1)).
\end{aligned}$$

Using the convergence of $b^{(n)}$ from Assumption (A2), convergence of $C_n(\cdot)$ from (L1), and convergence of $\theta_{x,n}$ from Proposition 3.2, we obtain $\Delta_\gamma^{(n)} \to \Delta_\gamma$, where $\Delta_\gamma = \left(\frac{b^\gamma\xi_{\mathcal{Z}}}{\xi_\epsilon}\log\frac{C_x\beta_\epsilon}{\xi_{\mathcal{Z}}\gamma b} + \frac{\xi_{\mathcal{Z}}}{\xi_\epsilon}b^\gamma - \log\beta_{\mathcal{Z}}\right)$, $\eta_{\gamma,n} \to \eta_\gamma$ where $\eta_2 = \exp\{-v^2\xi_{\mathcal{Z}}\}$ and $\eta_\gamma = 1$ for $\gamma \in (0,2)$, and $C_x = \lim_{n\to\infty} C_n(\theta_{x,n}) = \frac{\lambda_U(\theta_x)-1}{\lambda_U(\theta_x)}$. This completes the proof.

A.2. Proof of Lemma 4.2

Since $\tilde{M}_n = t_{0,n}M_n$, it follows that

$$\exp\{-n\phi_n(-\tilde{M}_n)\}[H_n(-\tilde{M}_n)]^{-1} = M_n e^{nh_n(t_{0,n})}\sqrt{\frac{1}{n|h_n''(t_{0,n})|}}.$$

Now, applying Lemma 4.1 and using algebraic manipulations, the RHS of the above expression reduces to $[M_n \tilde{R}_{1n} \tilde{R}_{2n} \tilde{R}_{3n}]^{-1} Q_n(x) + o(1)$, as $n \to \infty$, where for $j = 1, 2, 3$, $\tilde{R}_{jn}$ is same as R_{jn} where $b^{(n)}$ and $\mathcal{Z}_n$ are replaced by b and $\mathcal{Z}$, and $Q_n(x)$ is a constant. It follows from (L1) and $(\log n)|\xi_{\mathcal{Z}_n} - \xi_{\mathcal{Z}}| \to 0$ as $n \to \infty$ that $Q_n(x)$ converges to the RHS of (4.5).

A.3. Proof of Lemma 4.3

Since $0 < \inf_{n, z \in (-\infty, z_0)} \psi_n(z) \leq \sup_{n, z \in (-\infty, z_0)} \psi_n(z) < \infty$, there exist finite positive constants C_1, C_2 such that

$$\begin{aligned} J_{1n} = &\int_{-\infty}^{-M_n(1+\beta)} e^{-n\phi_n(z)} \psi_n(z) f_{\mathcal{Z}_n}(z) \mathrm{d}z \leq C_1 F_{\mathcal{Z}_n}(-M_n(1+\beta)) \\ \leq &C_2 (\log n)^{\frac{1-\gamma}{\gamma}} n^{-\frac{b^\gamma \xi_Z}{\xi_\epsilon}(1+\beta)^\gamma} = o(R_{1n} R_{2n} R_{3n}), \quad \text{as } n \to \infty. \end{aligned}$$

Turning to J_{3n}, notice that $\phi_n(z)$ is increasing in z. Hence, there exist finite positive constants C_1, C_2, C_3 such that

$$\begin{aligned} J_{3n} = &\int_{-M_n(1-\beta)}^{\infty} e^{-n\phi_n(z)} \psi_n(z) f_{\mathcal{Z}_n}(z) \mathrm{d}z \leq C_1 e^{-n\phi_n(-M_n(1-\beta))} \\ = &C_1 \exp\{-\frac{C_2}{[M_n(1-\beta)+v]^{\gamma-1}} n \exp\{-\frac{\xi_\epsilon}{b^\gamma}[v + M_n(1-\beta)]^\gamma\}\} \\ \leq &C_1 \exp\{-C_3 (\log n)^{(1-\gamma)\gamma^{-1}} \cdot n \cdot n^{-(1-\beta)^\gamma}\} \\ = &C_1 \exp\{-C_3 n^{1-(1-\beta)^\gamma} \cdot (\log n)^{(1-\gamma)\gamma^{-1}}\} = o(R_{1n} R_{2n} R_{3n}), \quad \text{as } n \to \infty, \end{aligned}$$

completing the proof of the Lemma.

A.4. Proof of Lemma 4.4

Applying Proposition 3.3, it follows that

$$\begin{aligned} h_n(t) = &\frac{1}{n}[-n\phi_n(-M_n^t) + \log f_{\mathcal{Z}_n}(-M_n^t) + t \log M_n] \\ = &\frac{1}{n}\left[-nC_n(\theta_{x,n})\left(\frac{v + M_n^t}{b^{(n)}}\right)^{-\alpha_\epsilon} L_\epsilon(M_n^t)(1+o(1)) + (\log L_{\mathcal{Z}_n}(M_n^t) - \alpha_{\mathcal{Z}_n} t \log M_n + \log \alpha_{\mathcal{Z}_n})(1+o(1))\right] \\ h_n'(t) = &\frac{1}{n}\left[nC_n(\theta_{x,n})\alpha_\epsilon\left(\frac{v + M_n^t}{b^{(n)}}\right)^{-\alpha_\epsilon - 1} \frac{1}{b^{(n)}} M_n^t (\log M_n) L_\epsilon(M_n^t)\right. \\ &\left. - nC_n(\theta_{x,n})\left(\frac{v + M_n^t}{b^{(n)}}\right)^{-\alpha_\epsilon} L_\epsilon'(M_n^t) M_n^t (\log M_n) + \frac{L_{\mathcal{Z}_n}'(M_n^t) M_n^t (\log M_n)}{L_{\mathcal{Z}_n}(M_n^t)} - \alpha_{\mathcal{Z}_n} \log M_n\right] \\ = &\frac{1}{n}\left\{nC_n(\theta_{x,n})\alpha_\epsilon\left(\frac{v + M_n^t}{b^{(n)}}\right)^{-\alpha_\epsilon - 1} \frac{1}{b^{(n)}} M_n^t (\log M_n) L_\epsilon(M_n^t)\left[1 - \frac{b^{(n)}}{\alpha_\epsilon}\left(\frac{v + M_n^t}{b^{(n)}}\right)\frac{L_\epsilon'(M_n^t)}{L_\epsilon(M_n^t)}\right]\right. \end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{L'_{\mathcal{Z}_n}(M_n^t)M_n^t(\log M_n)}{L_{\mathcal{Z}_n}(M_n^t)} - \alpha_{\mathcal{Z}_n}\log M_n \Bigg\} \\
&\quad \text{notice that } \frac{f'_{\mathcal{Z}_n}(z)}{f_{\mathcal{Z}_n}(z)} \approx (\alpha_{\mathcal{Z}_n}+1)|z|^{-1} \text{ and } \lim_{x\to\infty}\frac{xL'_\epsilon(x)}{L_\epsilon(x)} = 0, \\
&\approx \frac{1}{n}\left[nC_n(\theta_{x,n})\alpha_\epsilon \left(\frac{v+M_n^t}{b^{(n)}}\right)^{-\alpha_\epsilon-1}\frac{1}{b^{(n)}}M_n^t(\log M_n)L_\epsilon(M_n^t) + \frac{L'_{\mathcal{Z}_n}(M_n^t)M_n^t(\log M_n)}{L_{\mathcal{Z}_n}(M_n^t)} - \alpha_{\mathcal{Z}_n}\log M_n\right] \\
&= \frac{\log M_n}{n}\left[nC_n(\theta_{x,n})\alpha_\epsilon \left(\frac{v+M_n^t}{b^{(n)}}\right)^{-\alpha_\epsilon-1}\frac{1}{b^{(n)}}M_n^tL_\epsilon(M_n^t) + \frac{L'_{\mathcal{Z}_n}(M_n^t)M_n^t}{L_{\mathcal{Z}_n}(M_n^t)} - \alpha_{\mathcal{Z}_n}\right] \\
&\approx \frac{\log M_n}{n}\left[nC_n(\theta_{x,n})\alpha_\epsilon \left(\frac{v+M_n^t}{b^{(n)}}\right)^{-\alpha_\epsilon-1}\frac{1}{b^{(n)}}M_n^tL_\epsilon(M_n^t) - \alpha_{\mathcal{Z}_n}\right] \quad \left(\text{By } \lim_{x\to\infty}\frac{xL'_\epsilon(x)}{L_\epsilon(x)} = 0 \text{ and } t>0\right) \\
h''_n(t) &\approx \frac{\log M_n}{n}\frac{\mathrm{d}}{\mathrm{d}t}\left[nC_n(\theta_{x,n})\alpha_\epsilon \left(\frac{M_n^t}{b^{(n)}}\right)^{-\alpha_\epsilon} L_\epsilon(M_n^t) - \alpha_{\mathcal{Z}_n}\right] \quad \text{(approximation from equation (A.2))} \\
&\approx \frac{(\log M_n)^2M_n^t\alpha_\epsilon}{nb^{(n)}}\left[-nC_n(\theta_{x,n})\alpha_\epsilon \left(\frac{M_n^t}{b^{(n)}}\right)^{-\alpha_\epsilon-1} L_\epsilon(M_n^t)\right] \\
&= \frac{(\log M_n)^2\alpha_\epsilon}{n}\left[-nC_n(\theta_{x,n})\alpha_\epsilon \left(\frac{M_n^t}{b^{(n)}}\right)^{-\alpha_\epsilon} L_\epsilon(M_n^t)\right] < 0 \quad \left(\text{By } \lim_{x\to\infty}\frac{xL'_\epsilon(x)}{L_\epsilon(x)} = 0\right)
\end{aligned}
$$

$$
h'_n(t_{0,n}) = 0 \iff nC_n(\theta_{x,n})\alpha_\epsilon \left(\frac{v+M_n^{t_{0,n}}}{b^{(n)}}\right)^{-\alpha_\epsilon-1}\frac{1}{b^{(n)}}M_n^{t_{0,n}}L_\epsilon(M_n^{t_{0,n}}) = \alpha_{\mathcal{Z}_n} \tag{A.1}
$$

$$
\begin{aligned}
&\iff nC_n(\theta_{x,n})\alpha_\epsilon \left(\frac{M_n^{t_{0,n}}}{b^{(n)}}\right)^{-\alpha_\epsilon-1}\left(1+\frac{v}{M_n^{t_{0,n}}}\right)^{-\alpha_\epsilon-1}\frac{1}{b^{(n)}}M_n^{t_{0,n}}L_\epsilon(M_n^{t_{0,n}}) = \alpha_{\mathcal{Z}_n} \\
&\iff nC_n(\theta_{x,n})\alpha_\epsilon \left(\frac{M_n^{t_{0,n}}}{b^{(n)}}\right)^{-\alpha_\epsilon}\left(1+\frac{v}{M_n^{t_{0,n}}}\right)^{-\alpha_\epsilon-1}L_\epsilon(M_n^{t_{0,n}}) = \alpha_{\mathcal{Z}_n} \\
&\iff nC_n(\theta_{x,n})\alpha_\epsilon \left(\frac{M_n^{t_{0,n}}}{b^{(n)}}\right)^{-\alpha_\epsilon}\left(1-\frac{(\alpha_\epsilon+1)v}{M_n^{t_{0,n}}}+\frac{(\alpha_\epsilon+1)(\alpha_\epsilon+2)v^2}{2M_n^{2t_{0,n}}}+\cdots\right)L_\epsilon(M_n^{t_{0,n}}) = \alpha_{\mathcal{Z}_n}
\end{aligned}
$$

$$
\iff nC_n(\theta_{x,n})\alpha_\epsilon \left(\frac{M_n^{t_{0,n}}}{b^{(n)}}\right)^{-\alpha_\epsilon} L_\epsilon(M_n^{t_{0,n}}) = \alpha_{\mathcal{Z}_n} \quad \text{(for large } n\text{)} \tag{A.2}
$$

$$
\iff t_{0,n} = \frac{1}{\alpha_\epsilon\log M_n}\left[\log n + \log L_\epsilon(M_n^{t_{0,n}}) + \log\frac{C_n(\theta_{x,n})\alpha_\epsilon(b^{(n)})^{\alpha_\epsilon}}{\alpha_{\mathcal{Z}_n}}\right] \tag{A.3}
$$

$$
\approx 1 + \frac{\log L_\epsilon(M_n)}{\alpha_\epsilon\log M_n} + \frac{\log\frac{C_n(\theta_{x,n})\alpha_\epsilon(b^{(n)})^{\alpha_\epsilon}}{\alpha_{\mathcal{Z}_n}}}{\alpha_\epsilon\log M_n}
$$

Since $t_{0,n}$ is the root of $h'(t)=0$, and then by equation (A.1)

$$n\phi_n(-M_n^{t_{0,n}}) = nC_n(\theta_{x,n})\left(\frac{v+M_n^{t_{0,n}}}{b^{(n)}}\right)^{-\alpha_\epsilon} L_\epsilon(M_n^{t_{0,n}}) = \frac{\alpha_{\mathcal{Z}_n} b^{(n)}}{\alpha_\epsilon M_n^{t_{0,n}}}\left(\frac{v+M_n^{t_{0,n}}}{b^{(n)}}\right) = \frac{v\alpha_{\mathcal{Z}_n}}{\alpha_\epsilon M_n^{t_{0,n}}} + \frac{\alpha_{\mathcal{Z}_n}}{\alpha_\epsilon}.$$

$$\log M_n^{t_{0,n}} = \frac{\log n}{\alpha_\epsilon} + \frac{\log \frac{C_n(\theta_{x,n})\alpha_\epsilon (b^{(n)})^{\alpha_\epsilon}}{\alpha_{\mathcal{Z}_n}}}{\alpha_\epsilon} + \frac{\log L_\epsilon(M_n^{t_{0,n}})}{\alpha_\epsilon} \quad \text{(by (A.2))}$$

$$\begin{aligned}
h_n(t_{0,n}) =& \frac{1}{n}\left[-n\phi_n(-M_n^{t_{0,n}}) + \log L_{\mathcal{Z}_n}(M_n^t) - \alpha_{\mathcal{Z}_n} t \log M_n + \log \alpha_{\mathcal{Z}_n}\right] \\
=& \frac{1}{n}\left[-\frac{v\alpha_{\mathcal{Z}_n}}{\alpha_\epsilon M_n^{t_{0,n}}} - \frac{\alpha_{\mathcal{Z}_n}}{\alpha_\epsilon} + \log L_{\mathcal{Z}_n}(M_n^{t_{0,n}}) - \alpha_{\mathcal{Z}_n}\left(\frac{\log n}{\alpha_\epsilon} + \frac{\log \frac{C_n(\theta_{x,n})\alpha_\epsilon (b^{(n)})^{\alpha_\epsilon}}{\alpha_{\mathcal{Z}_n}}}{\alpha_\epsilon} + \frac{\log L_\epsilon(M_n^{t_{0,n}})}{\alpha_\epsilon}\right) + \log \alpha_{\mathcal{Z}_n}\right] \\
=& \frac{1}{n}\left[-\frac{\alpha_{\mathcal{Z}_n}}{\alpha_\epsilon} + \log L_{\mathcal{Z}_n}(M_n) - \frac{\alpha_{\mathcal{Z}_n}}{\alpha_\epsilon}\log n + \frac{\alpha_{\mathcal{Z}_n}}{\alpha_\epsilon}\log\frac{C_n(\theta_{x,n})\alpha_\epsilon (b^{(n)})^{\alpha_\epsilon}}{\alpha_{\mathcal{Z}_n}} + \frac{\alpha_{\mathcal{Z}_n}}{\alpha_\epsilon}\log L_\epsilon(M_n) + \log\alpha_{\mathcal{Z}_n} + o(1)\right] \\
=& \frac{1}{n}\left[-\frac{\alpha_{\mathcal{Z}_n}}{\alpha_\epsilon}\log n + \log L_{\mathcal{Z}_n}(M_n) + \frac{\alpha_{\mathcal{Z}_n}}{\alpha_\epsilon}\log L_\epsilon(M_n) + \underbrace{\frac{\alpha_{\mathcal{Z}_n}}{\alpha_\epsilon}\log\frac{C_n(\theta_{x,n})\alpha_\epsilon (b^{(n)})^{\alpha_\epsilon}}{\alpha_{\mathcal{Z}_n}} + \log\alpha_{\mathcal{Z}_n} - \frac{\alpha_{\mathcal{Z}_n}}{\alpha_\epsilon}}_{\Delta_n} + o(1)\right]
\end{aligned}$$

$$h_n''(t_{0,n}) = \frac{(\log M_n)^2\alpha_\epsilon}{n}\left[-nC_n(\theta_{x,n})\alpha_\epsilon\left(\frac{M_n^{t_{0,n}}}{b^{(n)}}\right)^{-\alpha_\epsilon} L_\epsilon(M_n^{t_{0,n}}) + o(1)\right] = -\frac{(\log M_n)^2\alpha_\epsilon\alpha_{\mathcal{Z}_n} + o(1)}{n} \quad \text{(by (A.2))}.$$

Using the convergence of $b^{(n)}$ from Assumption (A3), convergence of $C_n(\cdot)$ from (L1), convergence of $\alpha_{\mathcal{Z}_n}$ from assumption of the proposition, and convergence of $\theta_{x,n}$ from Proposition 3.2, we obtain $\Delta_n \to \Delta$, where $\Delta = \frac{\alpha_{\mathcal{Z}}}{\alpha_\epsilon}\log\frac{C_x\alpha_\epsilon b^{\alpha_\epsilon}}{\alpha_{\mathcal{Z}}} + \log\alpha_{\mathcal{Z}} - \frac{\alpha_{\mathcal{Z}}}{\alpha_\epsilon}$, and $C_x = \lim_{n\to\infty} C_n(\theta_{x,n}) = \frac{\lambda_U(\theta_x)-1}{\lambda_U(\theta_x)}$. This completes the proof.

A.5. Proof of Lemma 4.5

Notice that $\tilde{M}_n = M_n^{t_{0,n}}$, applying Lemma 4.4, we obtain

$$\begin{aligned}
\exp(-n\phi_n(-\tilde{M}_n)[H_n(-\tilde{M}_n)]^{-1} =& (\log M_n)e^{nh_n(t_{0,n})}\sqrt{\frac{1}{n|h_n''(t_{0,n})|}} \\
=& n^{-\frac{\alpha_{\mathcal{Z}_n}}{\alpha_\epsilon}}[L_\epsilon(n^{\frac{1}{\alpha_\epsilon}})]^{-\frac{\alpha_{\mathcal{Z}_n}}{\alpha_\epsilon}} L_{\mathcal{Z}_n}(n^{\frac{1}{\alpha_\epsilon}})e^{\Delta}\left(\frac{1}{\alpha_\epsilon\alpha_{\mathcal{Z}_n}}\right)^{\frac{1}{2}}(1+o(1)).
\end{aligned}$$

Using the assumptions that $(\log n)|\alpha_{\mathcal{Z}_n} - \alpha_{\mathcal{Z}}| \to 0$ and $L_{\mathcal{Z}_n}(n^{\frac{1}{\alpha_\epsilon}})\left[L_{\mathcal{Z}}(n^{\frac{1}{\alpha_\epsilon}})\right]^{-1} \to 1$ as $n\to\infty$, we can replace $\mathcal{Z}_n$ by $\mathcal{Z}$ and $L_{\mathcal{Z}_n}(n^{\frac{1}{\alpha_\epsilon}})$ by $L_{\mathcal{Z}}(n^{\frac{1}{\alpha_\epsilon}})$. We complete the proof by Setting $K_x = e^{\Delta}(\alpha_\epsilon\alpha_{\mathcal{Z}})^{-\frac{1}{2}}$.

A.6. Proof of Lemma 4.6

Since $0 < \inf_{n,z\in(-\infty,z_0)} \psi_n(z) \le \sup_{n,z\in(-\infty,z_0)} \psi_n(z) < \infty$, there exist finite positive constants C_1, C_2 such that as $n \to \infty$,

$$\begin{aligned} J'_{1n} =& \int_{-\infty}^{-M_n n^\beta} e^{-n\phi_n(z)} \psi_n(z) f_{Z_n}(z) \mathrm{d}z \le C_1 F_{Z_n}(-M_n n^\beta) \\ \le& C_2 n^{-\frac{\alpha_Z}{\alpha_\epsilon} - \beta\alpha_Z} = o\left((n^{-\frac{\alpha_Z}{\alpha_\epsilon}} [L_\epsilon(n^{\frac{1}{\alpha_\epsilon}})]^{-\frac{\alpha_Z}{\alpha_\epsilon}} L_Z(n^{\frac{1}{\alpha_\epsilon}}) \right). \end{aligned}$$

Turning to J_{3n}, notice that $\phi_n(z)$ is increasing in z. Hence, there exist finite positive constants C_1, C_2, C_3 such that as $n \to \infty$,

$$\begin{aligned} J'_{3n} =& \int_{-M_n n^{-\beta}}^{\infty} e^{-n\phi_n(z)} \psi_n(z) f_{Z_n}(z) \mathrm{d}z \le C_1 e^{-n\phi_n(-M_n n^{-\beta})} \\ \le& C_2 \exp\{-nC_x \left(\frac{v + n^{\frac{1}{\alpha_\epsilon} - \beta}}{b} \right)^{-\alpha_\epsilon} L_\epsilon(n^{\frac{1}{\alpha_\epsilon} - \beta})\} \\ \le& C_3 \exp\{-n^{\beta\alpha_\epsilon} L_\epsilon(n^{\frac{1}{\alpha_\epsilon} - \beta})\} = o\left(n^{-\frac{\alpha_Z}{\alpha_\epsilon}} [L_\epsilon(n^{\frac{1}{\alpha_\epsilon}})]^{-\frac{\alpha_Z}{\alpha_\epsilon}} L_Z(n^{\frac{1}{\alpha_\epsilon}}) \right), \end{aligned}$$

completing the proof of the Lemma.

A.7. Proof of Lemma 4.7

The proof is similar to the argument used in Theorem 2.1 (see Remark 4.1). We provide a brief sketch. By the envelope theorem,

$$\partial_z \Lambda^*(x,z) \; = \; -\partial_z \Lambda(\theta_x(z), z) \; = \; -\frac{\lambda_U(\theta_x(z)) - 1}{\lambda_U(\theta_x(z))\, p(z) + 1 - p(z)}\, p'(z),$$

with $p(z) = F_\epsilon((v-z)/b)$ and $p'(z) = -b^{-1} f_\epsilon((v-z)/b) < 0$. Since $\theta_x(z) > 0$ and $\lambda_U(\theta) > 1$ for $\theta > 0$, we get $\partial_z \Lambda^*(x,z) > 0$, i.e. strict increase in z. In the unbounded-left regime ($\kappa = -\infty$), $\Lambda^*(x,z) \downarrow \Lambda_U^*(x)$ as $z \to -\infty$.

A.8. Statement and proof of Lemma A.1

Lemma A.1 (Consistency of M_n and $\tilde{M}_n$). *Assume* (X1)*,* (X2)*,* (B1)*, and* (W1)-(W3)*. Let M_n be the balance scale from* (B1) *and $z_n^* = -\tilde{M}_n$ the maximizer of $\tilde{h}_n(z) := -\phi_n(z) + n^{-1} \log f_{Z_n}(z)$. Then*

$$\tilde{M}_n = M_n\{1 + o(1)\}, \qquad r_Z(M_n)(\tilde{M}_n - M_n) \to 0,$$

hence $z_n^* = -\tilde{M}_n = -M_n + o(1/r_Z(M_n))$.

Proof. Differentiate $\tilde{h}_n$:

$$\tilde{h}'_n(z) = -\phi'_n(z) + \frac{1}{n} \frac{f'_{Z_n}(z)}{f_{Z_n}(z)}.$$

Evaluating at $z = -M_n$ and using the balance ratio in (B1) gives $\tilde{h}_n'(-M_n) = o(r_{\mathcal{Z}}(M_n))$. By the log-smooth assumptions (X1)–(X2), and balance condition (B1), we also have $\tilde{h}_n''(-M_n) \sim -r_{\mathcal{Z}}(M_n)^2 < 0$. By the mean value theorem applied to the full exponent, there exists $\xi_n \in (-\tilde{M}_n, -M_n)$ such that

$$0 = (n\tilde{h}_n)'(-\tilde{M}_n) = (n\tilde{h}_n)'(-M_n) + (n\tilde{h}_n)''(\xi_n)\,(\tilde{M}_n - M_n).$$

Hence

$$\tilde{M}_n - M_n = -\frac{(n\tilde{h}_n)'(-M_n)}{(n\tilde{h}_n)''(\xi_n)}.$$

From (B1) we have $(n\tilde{h}_n)'(-M_n) = o(r_{\mathcal{Z}}(M_n))$, and by (X1)–(X2) the curvature satisfies $(n\tilde{h}_n)''(\xi_n) \sim -n\,r_{\mathcal{Z}}(M_n)^2$ uniformly along $(-\tilde{M}_n, -M_n)$, so

$$r_{\mathcal{Z}}(M_n)\,(\tilde{M}_n - M_n) \;\longrightarrow\; 0.$$

Finally, since $M_n \uparrow \infty$ by (B1) and $r_{\mathcal{Z}}(w) \uparrow \infty$ as $w \to \infty$ by (X2), we have $r_{\mathcal{Z}}(M_n)M_n \to \infty$; therefore

$$\frac{\tilde{M}_n - M_n}{M_n} = o\Big(\frac{1}{r_{\mathcal{Z}}(M_n)M_n}\Big) \;\longrightarrow\; 0,$$

i.e. $\tilde{M}_n = M_n\{1 + o(1)\}$. This completes the proof.

$\square$

Remark A.1. The proof repeats the displacement/window arguments already used for the GN and RV cases: see Lemma 5.1 and Lemma 5.4 (Newton step around the approximate maximizer; curvature of order $-r_Z^2$), now applied to $\tilde{h}_n$.

Appendix B: Appendix

B.1. Proof of Proposition 3.1

1. Let z be fixed. Then
   a) $\Lambda_n(\theta; z)$ is the cumulant generating function which is finite using the assumption (L1). The convexity follows from Hölder's inequality and differentiability in θ follows from the differentiability of $\lambda_{U^{(n)}}(\theta)$.
   b) Since $\Lambda_n(\theta; z)$ is strictly convex and differentiable, $\frac{\partial^2}{\partial\theta^2}\Lambda_n(\theta; z) > 0$; hence $\frac{\partial}{\partial\theta}\Lambda_n(\theta; z)$ is strictly increasing in θ.
   c) The formula for the derivative, $\frac{\partial}{\partial\theta}\Lambda_n(\theta; z)$, is in Appendix (Proposition A.1).
2. Let θ be fixed.
   a) By model assumption, ϵ is a random variable with continuous density, the differentiability follows by the differentiability of $F_\epsilon(\cdot)$.
   b) Since $U^{(n)} > 0$, we have $\lambda_{U^{(n)}}(\theta) > 0$; hence $\frac{\partial}{\partial z}\Lambda_n(\theta; z) < 0$
   c) Directly from the formula of $\Lambda_n(\theta; z)$.
3. Let $g_n(M_n) = 1 - F_\epsilon(\frac{v+M_n}{b^{(n)}})$ and $C_n(\theta) = \dfrac{\lambda_{U^{(n)}}(\theta) - 1}{\lambda_{U^{(n)}}(\theta)}$. First note that

$$\Lambda_n(\theta; -M_n) = \log\left[\lambda_{U^{(n)}}(\theta)F_\epsilon(\frac{v + M_n}{b^{(n)}}) + 1 - F_\epsilon(\frac{v + M_n}{b^{(n)}})\right]$$

$$= \log\left[\lambda_{U^{(n)}}(\theta)\right] - g_n(M_n)C_n(\theta) - \frac{1}{2}[g_n(M_n)]^2[C_n^*(\theta)]^2,$$

where $C_n^*(\theta)$ is a point between $\dfrac{\lambda_{U^{(n)}}(\theta)-1}{\lambda_{U^{(n)}}(\theta)}$ and $\dfrac{\lambda_{U^{(n)}}(\theta)-1}{\lambda_{U^{(n)}}(\theta)+g_n(M_n)(1-\lambda_{U^{(n)}}(\theta))}$. Hence it follows that

$$\Lambda_n(\theta;-M_n) = \Lambda_{U^{(n)}}(\theta) + \delta_n(\theta, M_n),$$

where $\delta_n(\theta, M_n) = -g_n(M_n)C_n(\theta) - \frac{1}{2}[g_n(M_n)]^2[C_n^*(\theta)]^2$. $|C_n^*(\theta)| < \infty$, $|C_n(\theta)| < \infty$ uniformly for any n, θ.

4. The proof follows from the assumption (A1)-(A3) and (L1).

B.2. Proof of Proposition 3.2

Let

$$G_n(\theta, z) = \frac{\partial}{\partial\theta}\Lambda_n(\theta; z) - x.$$

Since $\frac{\partial^2}{\partial\theta^2}\Lambda_n(\theta; z) > 0$, by the implicit function theorem, $\theta_{n,x}(z)$ is differentiable, and

$$\frac{d\theta_{n,x}(z)}{dz} = -\frac{\frac{\partial^2}{\partial z\partial\theta}\Lambda_n(\theta; z)}{\frac{\partial^2}{\partial\theta^2}\Lambda_n(\theta; z)}.$$

Using the calculations of Proposition A.1 in the Appendix, the numerator is positive since $f_\epsilon\left(\frac{v-z}{b_n}\right) > 0$, $\mathbf{E}[Ue^{\theta U}] > 0$, and $1 - F_\epsilon\left(\frac{v-z}{b_n}\right) > 0$. The denominator is positive due to the strict convexity of $\Lambda_n(\theta; z)$ in θ, guaranteed by the positivity of the variance term (variance of U under P_θ, where $\frac{dP_\theta}{dP} = e^{\theta U - \log\lambda_U(\theta)}$). Hence,

$$\frac{d\theta_{n,x}(z)}{dz} > 0,$$

proving strict monotonicity. Turning to the boundary limits, as $z \to -\infty$, we have $\Lambda_n(\theta; z) \to \Lambda_U(\theta)$, hence $\theta_{n,x}(z) \to \theta_x$ with $\Lambda_U'(\theta_x) = x$. As $z \to \infty$, we see $F_\epsilon((v-z)/b_n) \to 0$, forcing $\theta_{n,x}(z) \to \infty$ to maintain equality with x. Finally, again using the implicit function theorem and the continuous differentiability of $\Lambda_n(\theta; z)$, it follows that $\theta_{n,x}(z) \to \theta_x(z)$ as $n \to \infty$. This completes the proof.

B.3. Proof of Proposition 3.3

(i) For fixed θ, the function $F_\epsilon((v-z)/b_n)$ is strictly decreasing in z. Hence, $\Lambda_n(\theta; z)$ is strictly decreasing in z. Thus, $\theta x - \Lambda_n(\theta; z)$ is strictly increasing in z, and taking supremum preserves strict monotonicity.

(ii) For each fixed θ', we have

$$\lim_{z\to\infty}[\theta' x - \Lambda_n(\theta'; z)] = \theta' x.$$

Thus,

$$\liminf_{z\to\infty}\Lambda_n^*(x; z) = \liminf_{z\to\infty}\sup_\theta\{\theta x - \Lambda_n(\theta; z)\} \geq \theta' x.$$

Taking supremum over all θ', since $x > 0$, gives the result.

(iii) Define $g_n(z) = 1 - F_\epsilon((v - z)/b_n)$. Then

$$|\Lambda_n(\theta; z) - \Lambda_U(\theta)| = \left|\log\left[1 + g_n(z)\,\frac{1 - \mathbf{E}[e^{\theta U}]}{\mathbf{E}[e^{\theta U}]})\right]\right|.$$

Set $C_\theta = |(1 - \mathbf{E}[e^{\theta U}])/\mathbf{E}[e^{\theta U}]|$. Choose z_ϵ sufficiently negative so that for all $z \leq z_\epsilon$:

$$|g_n(z) C_\theta| < \delta = 1 - e^{-\epsilon},$$

uniformly for all n. Thus, $|\Lambda_n(\theta; z) - \Lambda_U(\theta)| < \epsilon$, uniformly in n, proving uniform convergence as $z \to -\infty$. Taking supremum over θ and using convergence of $\theta_n(x, z)$ completes the proof.

(iv) In the following, the derivatives of $\Lambda_n(\theta; \kappa)$ and $\Lambda(\theta; \kappa)$ are taken with respect to θ. First notice that, $n|\Lambda_n(\theta; \kappa) - \Lambda(\theta; \kappa)|$ is bounded by

$$nF_\epsilon\left(\frac{v - \kappa}{b^{(n)}}\right)|\lambda_{U^{(n)}}(\theta) - \lambda_U(\theta)| + n\lambda_U(\theta)\left|F_\epsilon\left(\frac{v - \kappa}{b^{(n)}}\right) - F_\epsilon\left(\frac{v - \kappa}{b}\right)\right| + n\left|F_\epsilon\left(\frac{v - \kappa}{b}\right) - F_\epsilon\left(\frac{v - \kappa}{b^{(n)}}\right)\right|,$$

which is bounded above by

$$nF_\epsilon\left(\frac{v - \kappa}{b^{(n)}}\right)|\lambda_{U^{(n)}}(\theta) - \lambda_U(\theta)| + nC[b^{(n)} - b] \to 0,$$

using Assumption (L1). Next, turning to the derivatives, $n|\Lambda_n'(\theta; \kappa) - \Lambda'(\theta, \kappa)|$ is bounded by

$$nF_\epsilon\left(\frac{v - \kappa}{b^{(n)}}\right)|\lambda'_{U^{(n)}}(\theta) - \lambda'_U(\theta)| + n\lambda'_U(\theta)\left|F_\epsilon\left(\frac{v - \kappa}{b^{(n)}}\right) - F_\epsilon\left(\frac{v - \kappa}{b}\right)\right|,$$

which is bounded above by

$$C_1 n|\lambda'_{U^{(n)}}(\theta) - \lambda'_U(\theta)| + C_1 n|b^{(n)} - b| \to 0$$

where the convergence follows using (L1). Next, using

$$n\left|\frac{\Lambda_n'(\theta; \kappa)}{\Lambda_n(\theta; \kappa)} - \frac{\Lambda'(\theta; \kappa)}{\Lambda(\theta; \kappa)}\right| = \frac{n|\Lambda_n'(\theta; \kappa) - \Lambda'(\theta; \kappa)|}{\Lambda_n(\theta; \kappa)} - \frac{n\Lambda'(\theta; \kappa)|\Lambda_n(\theta; \kappa) - \Lambda(\theta; \kappa)|}{\Lambda_n(\theta; \kappa)\Lambda(\theta; \kappa)} \to 0,$$

uniformly in θ by (L1). Let $h_n(\theta) = \frac{\Lambda_n'(\theta;\kappa)}{\Lambda_n(\theta;\kappa)}$ and $h(\theta) = \frac{\Lambda'(\theta;\kappa)}{\Lambda(\theta;\kappa)}$, and $\theta_{x,n}$ be the root of $h_n(\theta) = x$ while θ_x is the root of $h(\theta) = x$. Using the mean value theorem, $h'(\theta) \in (0, \infty)$, and $\theta_{x,n} \to \theta_x$, it follows that

$$n|\theta_{x,n} - \theta_x| = \frac{n}{h'(\theta_n^*)}|h(\theta_{x,n}) - h(\theta_x)| \leq Cn|h(\theta_{x,n}) - h(\theta_x)| = Cn|h(\theta_{x,n}) - h_n(\theta_{x,n})| \to 0,$$

using the uniform convergence above. Therefore,

$$n|\Lambda_n^*(x, \kappa) - \Lambda^*(x, \kappa)| = xn|\theta_{x,n} - \theta_x| - n|h(\theta_{n,x}) - h(\theta_x)| \to 0.$$

(v) Let $\Delta_\theta = \theta_{x,n}(-M_n) - \theta_{x,n}$, and $\delta_n(\theta, M_n) = \Lambda_n(\theta; -M_n) - \Lambda_{U^{(n)}}(\theta) = -g_n(M_n)C_n(\theta) - \frac{1}{2}[g_n(M_n)]^2[C_n^*(\theta)]^2 = -g_n(M_n)C_n(\theta)(1 + o(1))$ using Proposition 3.1 (iii), where we have suppressed the dependence of Δ_θ on n and x. Notice that,

$$\Lambda_n^*(x; -M_n) = (\theta_{x,n} + \Delta_\theta)x - \Lambda_U(\theta_{x,n} + \Delta_\theta) - \delta_n(\theta_{x,n} + \Delta_\theta, M_n).$$

Now using a second-order Taylor expansion, the above can be expressed as

$$(\theta_{x,n}+\Delta_\theta)x-[\Lambda_U(\theta_{x,n})+\Delta_\theta\Lambda_U'(\theta_{x,n})+\frac{1}{2}\Lambda_U'(\theta_x^*)\Delta_\theta^2]-[\delta_n(\theta_{x,n},M_n)+\Delta_\theta\delta_n'(\theta_{x,n},M_n)+\frac{1}{2}\delta_n''(\theta_x^{**},M_n)\Delta_\theta^2],$$

which reduces to $x\theta_{x,n}-\Lambda_U(\theta_{x,n})-\delta_n(\theta_{x,n},M_n)-B_n$, where $B_n=\left[\frac{1}{2}\Lambda_U'(\theta_x^*)\Delta_\theta^2+\Delta_\theta\delta_n'(\theta_x,M_n)+\frac{1}{2}\delta_n''(\theta_x^{**},M_n)\Delta_\theta^2\right]$, $\theta_x^*\in[\theta_{x,n},\theta_{x,n}+\Delta_\theta]$, and $\theta_x^{**}\in[\theta_{x,n},\theta_{x,n}+\Delta_\theta]$. Next, since $[\theta x-\Lambda_U(\theta)-\delta_n(\theta,M_n)]$ attains its supremum at $\Delta_\theta+\theta_{x,n}$, it follows that Δ_θ can be solved by taking derivative of $\Lambda_n^*(x;-M_n)$ with respect to x and setting it to be zero; hence,

$$\Delta_\theta=-\frac{\delta_n'(\theta_x,M_n)}{\Lambda_U'(\theta_x^*)+\delta_n''(\theta_x^{**},M_n)}=O(g_n(M_n)),$$

since $\delta_n(\theta_x,M_n)=O(g_n(M_n))$, $\delta_n'(\theta_x,M_n)=O(g_n(M_n))$, and $\delta_n''(\theta_x,M_n)=O(g_n(M_n))$. This implies Δ_θ converges to 0 as $n\to\infty$. This implies that $B_n=o(\delta_n(\theta_x,M_n))\to 0$ and hence

$$\Lambda_n^*(x;-M_n)=\Lambda_U^*(x)-\delta_n(\theta_x,M_n)(1+o(1))=\Lambda_U^*(x)+g_n(M_n)C_n(\theta_x)(1+o(1)).$$

B.4. Prefactor stability lemma

Lemma B.1 (Prefactor stability under $O(1/n)$ level shifts). *Assume* (A1)-(A3) *and* (L1)-(L2). *Fix $x>\mu_U$ and let*

$$x_n(u)\ :=\ \frac{n}{n-1}x-\frac{u}{n-1}\,.$$

Let $A_n(\cdot)$ denote the Laplace prefactor from Theorems 2.1-2.3 , i.e.

$$A_n(x)=\frac{H_n(\tilde{M}_n(x))}{\sqrt{2\pi n}\theta_x(\tilde{M}_n(x))\sigma_x(\tilde{M}_n(x))}\exp\left\{n\phi_n(\tilde{M}_n(x);x)\right\},\quad \text{as } n\to\infty,$$

with $\phi_n(z;x):=\Lambda_n^*(x;z)-\Lambda_U^*(x)$ and $\partial_z\phi_n(\tilde{M}_n(x);x)=0$. Then there exist constants $C<\infty$ and n_0 not depending on Q and n) such that, for all $n\ge n_0$ and all $u\ge Q$,

$$\frac{A_n(x)}{A_{n-1}(x_n(u))}\ \le\ 1\ +\ \frac{C\,u}{n}\ +\ \frac{C\,u^2}{n^2}\,. \tag{B.1}$$

Moreover, the following uniform expansion holds:

$$\log\frac{A_n(x)}{A_{n-1}(x_n(u))}\ =\ \frac{\alpha_0+\alpha_1(u-x)}{n}\ +\ O\Big(\frac{1+u^2}{n^2}\Big),\qquad (u\ge Q), \tag{B.2}$$

where α_0,α_1 are bounded (not depending on n and Q), and the $O(\cdot)$ is uniform in $u\ge Q$.

Proof. Write $A_n(x)=\bar{A}_n(x)\exp\{n\phi_n(\tilde{M}_n(x);x)\}$ with

$$\bar{A}_n(x):=\frac{H_n(\tilde{M}_n(x))}{\sqrt{2\pi n}\,\theta_x(\tilde{M}_n(x))\,\sigma_x(\tilde{M}_n(x))}.$$

Set $\Delta_n(u) := x_n(u) - x = (x-u)/(n-1)$. By smoothness in x for all n (implicit function theorem for $\tilde{M}_n$ and C^1 dependence of H_n, θ_x, σ_x), a first-order expansion yields

$$\log \bar{A}_n(x) - \log \bar{A}_{n-1}(x_n(u)) = \frac{a_0}{n} + \frac{a_1}{n}(u-x) + O\Big(\frac{1+u^2}{n^2}\Big),$$

with bounded a_0, a_1 (uniform for $u \geq Q$). For the exponential part, use

$$\phi_n(\tilde{M}_n(x); x) = \Lambda_n^*(x; \tilde{M}_n(x)) - \Lambda_U^*(x)$$

and expand

$$\begin{aligned} &n\phi_n(\tilde{M}_n(x); x) - (n-1)\phi_{n-1}(\tilde{M}_{n-1}(x_n(u)); x_n(u)) \\ &\quad = \Big[n\Lambda_n^*(x; \tilde{M}_n(x)) - (n-1)\Lambda_{n-1}^*(x_n(u), \tilde{M}_{n-1}(x_n(u)))\Big] - \Big[n\Lambda_U^*(x) - (n-1)\Lambda_U^*(x_n(u))\Big]. \end{aligned}$$

The first bracket is $c_0 + \frac{b_1}{n}(u-x) + O((1+u^2)/n^2)$ by C^1 regularity in x (uniformly in n) together with the first-oder expansion in (4.10), while the second bracket is

$$\Lambda_U^*(x) + \theta_x u - \theta_x x + O\Big(\frac{1+u^2}{n}\Big)$$

by Taylor expansion and $(\Lambda^*)'(x) = \theta_x$. Combining both displays gives

$$\log \frac{A_n(x)}{A_{n-1}(x_n(u))} = \frac{\alpha_0 + \alpha_1(u-x)}{n} + O\Big(\frac{1+u^2}{n^2}\Big),$$

Exponentiating and using $e^y \leq 1 + y + Cy^2$ for small y yields (B.1). □

Lemma B.2 (Uniform compact TV for weighted weak limits). *Let $B_* = [-Q, Q]$. Assume $U^{(n)} \Rightarrow U$ and F_U has no atoms in B_*. Let $\phi_n, \phi : B_* \to [0, \infty)$ be bounded functions with $\|\phi_n - \phi\|_\infty \to 0$ and ϕ continuous on B_*. Define finite measures*

$$\mu_n(A) := \int_{A\cap B_*} \phi_n(u)\, \mathrm{d}F_{U^{(n)}}(u), \qquad \mu(A) := \int_{A\cap B_*} \phi(u)\, \mathrm{d}F_U(u).$$

Then

$$\sup_{B\subset B_*} |\mu_n(B) - \mu(B)| \longrightarrow 0.$$

Proof. Fix $\zeta > 0$ and set $M_* := \sup_{B_*} \phi$.

Step 1 (uniform Kolmogorov-type control for fixed weight). Define

$$H_n(x) \;:=\; \int_{(-\infty,x]\cap B_*} \phi(u)\, \mathrm{d}\,(F_{U^{(n)}} - F_U)(u), \qquad x \in \mathbb{R}.$$

Since ϕ is bounded and continuous on B_* and $U^{(n)} \Rightarrow U$, the finite measures $\nu_n(\cdot) := \int_{\cdot\cap B_*} \phi\, \mathrm{d}F_{U^{(n)}}$ converge weakly to $\nu(\cdot) := \int_{\cdot\cap B_*} \phi\, \mathrm{d}F_U$, by the Portmanteau Theorem (see Billingsley (2013), Theorem 2.1).

Moreover, ν has no atoms on $\mathbb{R}$ (because F_U is atomless on B_* and ϕ is continuous). Therefore, the *distribution functions* $x \mapsto \nu_n((-\infty, x])$ converge to $x \mapsto \nu((-\infty, x])$ *uniformly* on $\mathbb{R}$ (uniform convergence of cdfs to a continuous limit). Equivalently,

$$\|H_n\|_\infty := \sup_{x\in\mathbb{R}} |H_n(x)| \longrightarrow 0. \tag{B.2.1}$$

Step 2 (fixed finite grid on B_*). Choose a partition $-Q = x_0 < x_1 < \cdots < x_m = Q$ such that all gridpoints are F_U-continuity points (possible since F_U is atomless on B_*) and

$$\max_{1\le j\le m} \mu((x_{j-1}, x_j]) \le \frac{\zeta}{8}.$$

By weak convergence and $\|\phi_n - \phi\|_\infty \to 0$, there exists N_0 such that for all $n \ge N_0$,

$$\max_{1\le j\le m} |\mu_n((x_{j-1}, x_j]) - \mu((x_{j-1}, x_j])| \le \frac{\zeta}{8m}. \tag{B.2.2}$$

In particular, for $n \ge N_0$, $\max_j \mu_n((x_{j-1}, x_j]) \le \zeta/8 + \zeta/(8m) \le \zeta/4$.

Step 3 (uniform control on the algebra generated by the grid). Let $\mathcal{A}_m$ be the finite algebra generated by the cells $\{(x_{j-1}, x_j]\}_{j=1}^m$. For $S \in \mathcal{A}_m$, write it as a disjoint union of at most m cells, $S = \bigcup_{j\in J}(x_{j-1}, x_j]$. Using

$$\mu_n((x_{j-1}, x_j]) - \mu((x_{j-1}, x_j]) = H_n(x_j) - H_n(x_{j-1}),$$

we get for all n:

$$|\mu_n(S) - \mu(S)| \le \sum_{j\in J} (|H_n(x_j)| + |H_n(x_{j-1})|) \le 2m\,\|H_n\|_\infty.$$

Pick N_1 so large that $\|H_n\|_\infty \le \zeta/(8m)$ for all $n \ge N_1$ (by (B.2.1)). Then, for $n \ge N := \max\{N_0, N_1\}$,

$$\sup_{S\in\mathcal{A}_m} |\mu_n(S) - \mu(S)| \le \zeta/4.$$

Step 4 (Approximating B without an m–factor leak). Fix $\zeta > 0$. Because μ is a finite Borel measure on $\mathbb{R}$ and $B \subset B^*$, there exist a closed $F \subset B$ and an open $O \supset B$ with $\mu(O \setminus F) \le \zeta/4$. Moreover, by regularity on $\mathbb{R}$ we may take F and O to be finite unions of disjoint closed/open intervals with all endpoints chosen to be F_U–continuity points. Let $\{x_0 < \cdots < x_m\}$ be the ordered collection of these endpoints together with $\{-Q, Q\}$ and form the algebra $\mathcal{A}_m$ of finite unions of half–open cells $(x_{j-1}, x_j]$.

Define $S_B \in \mathcal{A}_m$ so that $F \subset S_B \subset O$ (e.g., take S_B to be the union of all cells contained in O). Then

$$\mu(B \triangle S_B) \le \mu(O \setminus F) \le \zeta/4.$$

By Step 3 (applied to this $\mathcal{A}_m$) there exists N_1 such that for all $n \ge N_1$,

$$\sup_{S\in\mathcal{A}_m} |\mu_n(S) - \mu(S)| \le \zeta/4.$$

Hence, for $n \geq N_1$,

$$\begin{aligned}|\mu_n(B)-\mu(B)| &\leq |\mu_n(S_B)-\mu(S_B)|+\mu_n(B\triangle S_B)+\mu(B\triangle S_B)\\ &\leq \zeta/4\ +\ \mu_n(B\triangle S_B)+\zeta/4.\end{aligned}$$

Finally, by (B.2.2) and the uniform control of cell masses (Step 2) we have $\mu_n(B\triangle S_B)\leq \mu(B\triangle S_B)+\zeta/8 \leq \zeta/4+\zeta/8$. Combining the displays, $|\mu_n(B)-\mu(B)|< 2\zeta$ for all $n \geq N_1$. As $\zeta > 0$ is arbitrary, the claim follows. ☐

Corollary B.1 (Uniform compact TV for weighted weak limits). *Let $B_* = [-Q, Q]$. Let $(U^{(n)})$ be real random variables with $U^{(n)} \Rightarrow U$ and F_U atomless on B_* (no point masses in B_*). Let $\phi_n, \phi : B_* \to [0,\infty)$ be bounded with $\|\phi_n-\phi\|_\infty \to 0$ and ϕ be continuous on B_*. Define finite measures on $\mathbb{B}(\mathbb{R})$ by*

$$\mu_n(A) := \int_{A\cap B_*} \phi_n(u)\,\mathrm{d}F_{U^{(n)}}(u), \qquad \mu(A) := \int_{A\cap B_*} \phi(u)\,\mathrm{d}F_U(u).$$

Then

$$\sup_{B\subset B_*} |\mu_n(B)-\mu(B)| \ \longrightarrow\ 0.$$

Proof. Apply Lemma B.2 with $B_* = [-Q, Q]$ and the weights $\tilde{\phi}_n = \phi_n \mathbf{1}_{B_*}$, $\tilde{\phi} = \phi\mathbf{1}_{B_*}$; the assumptions $\|\phi_n-\phi\|_\infty \to 0$, continuity of ϕ on B_*, and $U^{(n)} \Rightarrow U$ ensure its hypotheses. This gives $\sup_{B\subset B_*}|\mu_n(B)-\mu(B)| \to 0$. ☐

Remark B.1 (Direct application to Theorem 2.5). In the proof of Theorem 2.5, set

$$\phi_n(u) = e^{\theta_{x,n}(z)u-\Lambda_{U^{(n)}}(\theta_{x,n}(z))}$$

which converges uniformly to $\phi(u) = e^{\theta_x u-\Lambda_U(\theta_x)}$ on B_*. Apply Corollary B.1 with $B_* = [-Q, Q]$ to conclude

$$\sup_{B_1\subset B_*} |\nu_n(B_1)-\nu_{\theta_x}(B_1)| \ \to\ 0.$$

The same conclusion holds with n replaced by $(n-k)$ for any fixed k, since by (L1) and (L2) and strict convexity, we have $\sup_{z\leq z_0}|\theta_{x,n-k}(z)-\theta_{x,n}(z)| \to 0$ as $n \to \infty$ (implicit-function bound) and hence the weights differ by $o(1)$ uniformly on compact $u-$sets.

Lemma B.3 (Uniform tilted tail bound). *Under Assumption* (L1) *and* (L2) *there exist $t > 0$ and $C < \infty$ such that, for all $Q > 0$,*

$$\sup_n \int_{u>Q} \exp\{\theta_x u - \Lambda_U(\theta_x)\}\left(1+\frac{C\,u}{n}+\frac{C\,u^2}{n^2}\right) dF_{U^{(n)}}(u) \ \leq\ C\,e^{-tQ}.$$

Consequently*, $\sup_n \nu_n(U_1^{(n)} > Q) \ \leq\ C\,e^{-tQ} \to 0$ as $Q \to \infty$.*

Proof. By Assumption (L1), there exists $\theta_1 > \theta_x$ such that

$$\sup_{n\geq 1} \mathbf{E}\left[e^{\theta\,U^{(n)}}\right] < \infty \quad \text{and} \quad \sup_{n\geq 1} \mathbf{E}\left[|U^{(n)}|^k\, e^{\theta\,U^{(n)}}\right] < \infty \ \text{ for all } \ \theta < \theta_1, k = 1, 2. \tag{B.3}$$

Let $t > 0$ be such that $\theta_x + t < \theta_1$. Now, by Chernoff,

$$\int_{u>Q} u^k e^{\theta_x u}\, dF_{U^{(n)}}(u) \;\le\; e^{-tQ}\, \boldsymbol{E}[U^{(n)k} e^{(\theta_x+t)U^{(n)}}] \quad (k=0,1,2).$$

The upper bound in the Lemma now follows from (B.3). The consequently part in the lemma is the tail term in the main proof with the explicit factor from the $\Lambda^*_{n-1}(x_n(u))/\Lambda^*_n(x)$ ratio multiplied by Lemma B.1. □

Appendix C: Appendix

C.1. Conditional Central Limit Theorem

This appendix states the conditional central limit theorem and the conditional Bahadur-Rao estimate used in Section 4, together with auxiliary weak–limit arguments. Proposition C.1 (conditional CLT) provides the Gaussian control that underpins the conditional Bahadur-Rao estimate, Theorem C.1; the latter is the local prefactor input in the Laplace evaluations in Sections 4.1–4.2. Proposition C.2 records the marginal LLN/CLT baseline and is used to contrast the large–deviation regime with the near–mean regime (see Section 3 and Appendix E).

Proposition C.1. *$L_n = \sum_{j=1}^{n} T_j^{(n)}$, where $T_j^{(n)}|\mathcal{Z}_n$ are independent. $0 < (\sigma^{(n)}_{T_j|\mathcal{Z}_n})^2 = \mathbf{Var}[T_j^{(n)}|\mathcal{Z}_n] < \infty$ holds (a.e. w.r.t. the probability measure associated with $\mathcal{Z}_n$) uniformly for all j, n, and $\mathbf{E}[T_j^{(n)}|\mathcal{Z}_n] = 0$, $s_n^2 = \sum_{j=1}^{n} (\sigma^{(n)}_{T_j|\mathcal{Z}_n})^2$. If*

$$\lim_{n\to\infty} \frac{\sum_{j=1}^n \mathbf{E}\left[(T_j^{(n)})^2 \cdot \mathbf{1}(|T_j^{(n)}| > \delta s_n)|\mathcal{Z}_n\right]}{s_n^2} \overset{p}{=} 0$$

holds for all $\delta > 0$, then $\lim_{n\to\infty} \mathbf{P}\left(\frac{L_n}{s_n} \le x|\mathcal{Z}_n\right) = \Phi(x)$, where $\Phi(x)$ is the cdf of Gaussian distribution.

Proof. Following Sweeting (1989),to show $\frac{L_n}{s_n}|\mathcal{Z}_n \xrightarrow{d} \mathcal{N}(0,1)$, it is sufficient to show moment generating functions converge; that is,

$$\left|\mathbf{E}[e^{t\frac{L_n}{s_n}}|\mathcal{Z}_n] - e^{\frac{t^2}{2}}\right| \xrightarrow{p} 0.$$

We proceed in multiple steps. First, note that $T_j^{(n)} = T_j^{(n)} \cdot \mathbf{1}(|T_j^{(n)}| < \delta s_n) + T_j^{(n)} \cdot \mathbf{1}(|T_j^{(n)}| \ge \delta s_n)$. Since, $\mathbf{E}[T_j^{(n)}|\mathcal{Z}_n] = 0$, it follows that $\xi_j^{(n)}(\mathcal{Z}_n) = \mathbf{E}[T_j^{(n)} \cdot \mathbf{1}(|T_j^{(n)}| < \delta s_n)|\mathcal{Z}_n] = -\mathbf{E}[T_j^{(n)} \cdot \mathbf{1}(|T_j^{(n)}| \ge \delta s_n)|\mathcal{Z}_n]$. Now, define $V_{n,j} = T_j^{(n)} \cdot \mathbf{1}(|T_i^{(n)}| < \delta s_n) - \xi_j^{(n)}(\mathcal{Z}_n)$ and $W_{n,j} = T_j^{(n)} \cdot \mathbf{1}(|T_i^{(n)}| \ge \delta s_n) + \xi_j^{(n)}(\mathcal{Z}_n)$. Notice that $\mathbf{E}[V_{n,j}|\mathcal{Z}_n] = \mathbf{E}[W_{n,j}|\mathcal{Z}_n] = 0$, and $T_j^{(n)} = V_{n,j} + W_{n,j}$. We will now show that $\frac{\sum_{j=1}^n V_{n,j}}{s_n}|\mathcal{Z}_n \xrightarrow{d} \mathcal{N}(0,1)$ by verifying that

$$\left|\mathbf{E}[e^{t\frac{\sum_{j=1}^n V_{n,j}}{s_n}}|\mathcal{Z}_n] - e^{\frac{t^2}{2}}\right| \xrightarrow{p} 0.$$

Note that $\frac{V_{n,j}}{s_n}$ is a bounded random variable, since

$$\left|\frac{V_{n,j}}{s_n}\right| = \left|\frac{T_j^{(n)} \cdot \mathbf{1}(|T_i^{(n)}| < \delta s_n) - \xi_j^{(n)}(\mathcal{Z}_n)}{s_n}\right| = \left|\frac{T_j^{(n)} \cdot \mathbf{1}(|T_i^{(n)}| < \delta s_n) - \mathbf{E}[T_j^{(n)} \cdot \mathbf{1}(|T_j^{(n)}| < \delta s_n)|\mathcal{Z}_n]}{s_n}\right| \leq 2\delta.$$

Then conditional moment generating function of $\frac{V_{n,j}}{s_n}$ exists, and is given by

$$\begin{aligned}\mathbf{E}[e^{t\frac{V_{n,j}}{s_n}}|\mathcal{Z}_n] =& \mathbf{E}\left[1 + t \cdot V_{n,j} + \frac{t^2}{2}\left(\frac{V_{n,j}}{s_n}\right)^2 + \frac{t^3}{6}\left(\frac{V_{n,j}}{s_n}\right)^3 e^{\xi\frac{V_{n,j}}{s_n}}\middle|\mathcal{Z}_n\right] \\ =& 1 + \frac{t^2}{2}\frac{\mathbf{V}[V_{n,j}|\mathcal{Z}_n]}{s_n^2} + \underbrace{\frac{t^3}{6}\mathbf{E}\left[\left(\frac{V_{n,j}}{s_n}\right)^3 e^{\xi\frac{V_{n,j}}{s_n}}\middle|\mathcal{Z}_n\right]}_{R_{n,j}(t,\mathcal{Z}_n)}\end{aligned}$$

where $\xi \in (0, t)$ if $t > 0$ and $\xi \in (t, 0)$ if $t < 0$. Note that, since

$$\mathbf{Var}[V_{n,j}|\mathcal{Z}_n] \leq \mathbf{V}[T_j^{(n)}|\mathcal{Z}_n],$$

we have

$$\begin{aligned}\left|R_{n,j}(t,\mathcal{Z}_n)\right| =& \left|\frac{t^3}{6}\mathbf{E}\left[\left(\frac{V_{n,j}}{s_n}\right)^3 e^{\xi\frac{V_{n,j}}{s_n}}\middle|\mathcal{Z}_n\right]\right| \leq \frac{|t|^3}{6}\mathbf{E}\left[\left|\frac{V_{n,j}}{s_n}\right|^3 e^{|\xi|\left|\frac{V_{n,j}}{s_n}\right|}\middle|\mathcal{Z}_n\right] \\ =& \frac{|t|^3}{6}\mathbf{E}\left[\left|\frac{V_{n,j}}{s_n}\right| \cdot \left(\frac{V_{n,j}}{s_n}\right)^2 \cdot e^{|\xi|\left|\frac{V_{n,j}}{s_n}\right|}\middle|\mathcal{Z}_n\right] \\ & \left(\text{using } \left|\frac{V_{n,j}}{s_n}\right| \leq 2\delta\right) \\ \leq& \frac{|t|^3}{6} \cdot 2\delta \cdot e^{|t|\cdot 2\delta} \cdot \mathbf{E}\left[\left(\frac{V_{n,j}}{s_n}\right)^2\middle|\mathcal{Z}_n\right] \leq \frac{|t|^3}{6} \cdot 2\delta \cdot e^{|t|\cdot 2\delta} \cdot \frac{\mathbf{Var}[T_j^{(n)}|\mathcal{Z}_n]}{s_n^2}.\end{aligned}$$

Now returning to the conditional moment generating function of $\frac{\sum_{j=1}^n V_{n,j}}{s_n}$, since $V_{n,j}$ are independent given $\mathcal{Z}_n$,

$$\mathbf{E}[e^{t\frac{\sum_{j=1}^n V_{n,j}}{s_n}}|\mathcal{Z}_n] = \prod_{j=1}^n \mathbf{E}[e^{t\frac{V_{n,j}}{s_n}}|\mathcal{Z}_n] = \prod_{j=1}^n \left[1 + \frac{t^2}{2}\frac{\mathbf{V}[V_{n,j}|\mathcal{Z}_n]}{s_n^2} + R_{n,j}(t,\mathcal{Z}_n)\right].$$

To show

$$\mathbf{E}[e^{t\frac{\sum_{j=1}^n V_{n,j}}{s_n}}|\mathcal{Z}_n] \xrightarrow{p} e^{\frac{1}{2}t^2},$$

is equivalent to show

$$\prod_{j=1}^n \left[1 + \frac{t^2}{2}\frac{\mathbf{Var}[V_{n,j}|\mathcal{Z}_n]}{s_n^2} + R_{n,j}(t,\mathcal{Z}_n)\right] \xrightarrow{p} e^{\frac{1}{2}t^2},$$

only need to show

$$\sum_{j=1}^{n}\log\left[1+\underbrace{\frac{t^2}{2}\frac{\mathbf{Var}[V_{n,j}|\mathcal{Z}_n]}{s_n^2}+R_{n,j}(t,\mathcal{Z}_n)}_{Y_{n,j}(t,\mathcal{Z}_n)}\right]\xrightarrow{p}\frac{1}{2}t^2.$$

write $Y_{n,j}(t,\mathcal{Z}_n)=\frac{t^2}{2}\frac{\mathbf{Var}[V_{n,j}|\mathcal{Z}_n]}{s_n^2}+R_{n,j}(t,\mathcal{Z}_n)$. Then need to show

$$\sum_{j=1}^{n}\log\left[1+Y_{n,j}(t,\mathcal{Z}_n)\right]\xrightarrow{p}\frac{1}{2}t^2.$$

Note that for fixed t, $Y_{n,j}(t,\mathcal{Z}_n)\xrightarrow{a.s.}0$ for all $j,\mathcal{Z}_n$. Using Taylor expansion of $\log(1+x)=x+x^2K(x)$, where $K(x)=-\frac{1}{2}+\frac{x}{3}-\frac{x^2}{4}+\cdots$, and for $|x|\leq\frac{1}{2}$, $|K(x)|\leq\frac{1}{2}+\frac{1}{3}(\frac{1}{2})+\frac{1}{4}(\frac{1}{2})^2+\cdots\leq(\frac{1}{2})^0+(\frac{1}{2})^1+(\frac{1}{2})^2+\cdots\leq 2$.

$$\log\left[1+Y_{n,j}(t,\mathcal{Z}_n)\right]=Y_{n,j}(t,\mathcal{Z}_n)+Y_{n,j}^2(t,\mathcal{Z}_n)\cdot K(Y_{n,j}(t,\mathcal{Z}_n)).$$

We will now show that the convergence of

$$\sum_{j=1}^{n}\left[Y_{n,j}(t,\mathcal{Z}_n)+Y_{n,j}^2(t,\mathcal{Z}_n)\cdot K(Y_{n,j}(t,\mathcal{Z}_n))\right]\xrightarrow{p}\frac{1}{2}t^2.$$

This is equivalent to

$$\sum_{j=1}^{n}Y_{n,j}(t,\mathcal{Z}_n)+\sum_{j=1}^{n}Y_{n,j}^2(t,\mathcal{Z}_n)\cdot K(Y_{n,j}(t,\mathcal{Z}_n))\xrightarrow{p}\frac{1}{2}t^2$$

$$\underbrace{\sum_{j=1}^{n}\frac{t^2}{2}\frac{\mathbf{V}[V_{n,j}|\mathcal{Z}_n]}{s_n^2}}_{H_1(n,\mathcal{Z}_n)}+\underbrace{\sum_{j=1}^{n}R_{n,j}(t,\mathcal{Z}_n)}_{H_2(n,\mathcal{Z}_n)}+\underbrace{\sum_{j=1}^{n}Y_{n,j}^2(t,\mathcal{Z}_n)\cdot K(Y_{n,j}(t,\mathcal{Z}_n))}_{H_3(n,\mathcal{Z}_n)}\xrightarrow{p}\frac{1}{2}t^2.$$

First notice that $H_1(n,\mathcal{Z}_n)\xrightarrow{p}\frac{1}{2}t^2$ follows directly from the Lindeberg assumption; that is,

$$H_1(n,\mathcal{Z}_n)=\sum_{j=1}^{n}\frac{t^2}{2}\frac{\mathbf{Var}[V_{n,j}|\mathcal{Z}_n]}{s_n^2}=\frac{t^2}{2}\frac{\sum_{j=1}^{n}\mathbf{E}\left[(T_j^{(n)})^2\cdot\mathbf{1}(|T_j^{(n)}|\leq\delta s_n)|\mathcal{Z}_n\right]}{s_n^2}\xrightarrow{p}\frac{1}{2}t^2$$

We will now show that $H_2(n,\mathcal{Z}_n)\xrightarrow{p}0$. To this end, note that

$$\begin{aligned}|H_2(n,\mathcal{Z}_n)|=&\left|\sum_{j=1}^{n}R_{n,j}(t,\mathcal{Z}_n)\right|\leq\sum_{j=1}^{n}\left|R_{n,j}(t,\mathcal{Z}_n)\right|\leq\sum_{j=1}^{n}\frac{|t|^3}{6}\cdot2\delta\cdot e^{|t|\cdot2\delta}\cdot\frac{\mathbf{Var}[T_j^{(n)}|\mathcal{Z}_n]}{s_n^2}\\=&\frac{|t|^3}{6}\cdot2\delta\cdot e^{|t|\cdot2\delta}\cdot\frac{\sum_{j=1}^{n}\mathbf{Var}[T_j^{(n)}|\mathcal{Z}_n]}{s_n^2}.\end{aligned}$$

The result follows from the Lindeberg assumption, and δ being arbitrarily small. Thus, we have established that $\sum_{j=1}^{n} Y_{n,j}(t,\mathcal{Z}_n) \xrightarrow{p} \frac{1}{2}t^2$. We will now show that $H_3(n,\mathcal{Z}_n) \xrightarrow{p} 0$. Notice that

$$\begin{aligned}
|Y_{n,j}(t,\mathcal{Z}_n)| =& \left|\frac{t^2}{2}\frac{\mathbf{Var}[V_{n,j}|\mathcal{Z}_n]}{s_n^2} + R_{n,j}(t,\mathcal{Z}_n)\right| \le \frac{t^2}{2}\frac{\mathbf{Var}[V_{n,j}|\mathcal{Z}_n]}{s_n^2} + \left|R_{n,j}(t,\mathcal{Z}_n)\right| \\
\le& \frac{\mathbf{Var}[V_{n,j}|\mathcal{Z}_n]}{s_n^2}(\frac{1}{2}t^2 + \frac{|t|^3}{6}\cdot 2\delta \cdot e^{|t|\cdot 2\delta}) = (\frac{1}{2}t^2 + \frac{|t|^3}{6}\cdot 2\delta \cdot e^{|t|\cdot 2\delta}) \cdot \frac{\mathbf{Var}[V_{n,j}|\mathcal{Z}_n]}{s_n^2} \\
=& (\frac{1}{2}t^2 + \frac{|t|^3}{6}\cdot 2\delta \cdot e^{|t|\cdot 2\delta}) \frac{\mathbf{E}\left[(T_j^{(n)})^2 \cdot \mathbf{1}(|T_j^{(n)}| \le \delta s_n)|\mathcal{Z}_n\right]}{s_n^2} \le \delta^2(\frac{1}{2}t^2 + \frac{|t|^3}{6}\cdot 2\delta \cdot e^{|t|\cdot 2\delta})
\end{aligned}$$

For fixed t, we can find small enough δ such that $|Y_{n,j}(t,\mathcal{Z}_n)| \le \frac{1}{2}$, and then $|K\left(Y_{n,j}(t,\mathcal{Z}_n)\right)| \le 2$. Thus,

$$\begin{aligned}
|H_3(n,\mathcal{Z}_n)| =& \left|\sum_{j=1}^{n} Y_{n,j}^2(t,\mathcal{Z}_n)\cdot K(Y_{n,j}(t,\mathcal{Z}_n))\right| \le 2 \cdot \max_{1\le j\le n}\{|Y_{n,j}(t,\mathcal{Z}_n)|\} \cdot \sum_{j=1}^{n}|Y_{n,j}(t,\mathcal{Z}_n)| \\
\le& 2\delta^2(\frac{1}{2}t^2 + \frac{|t|^3}{6}\cdot 2\delta \cdot e^{|t|\cdot 2\delta}) \cdot \sum_{j=1}^{n}|Y_{n,j}(t,\mathcal{Z}_n)| \\
\le& 2\delta^2(\frac{1}{2}t^2 + \frac{|t|^3}{6}\cdot 2\delta \cdot e^{|t|\cdot 2\delta}) \cdot (|H_1(n,\mathcal{Z}_n)| + |H_2(n,\mathcal{Z}_n)|) \xrightarrow{p} 0.
\end{aligned}$$

Combining the above steps we have proved that

$$\sum_{j=1}^{n}\log\left[1 + Y_{n,j}(t,\mathcal{Z}_n)\right] \xrightarrow{p} \frac{1}{2}t^2.$$

Hence,

$$\mathbf{E}[e^{t\frac{\sum_{j=1}^{n} V_{n,j}}{s_n}}|\mathcal{Z}_n] \xrightarrow{p} e^{\frac{1}{2}t^2},$$

yielding $\frac{\sum_{j=1}^{n} V_{n,j}}{s_n}|\mathcal{Z}_n \xrightarrow{d} \mathcal{N}(0,1)$. The final step consists in showing that $\frac{\sum_{j=1}^{n} W_{n,j}}{s_n}|\mathcal{Z}_n \xrightarrow{p} 0$. By Chebyshev's inequality, and that $W_{n,j}$ are independent conditioned on $\mathcal{Z}_n$, and Lindeberg assumption,

$$\mathbf{P}\left(\left|\frac{\sum_{j=1}^{n} W_{n,j}}{s_n}\right| \ge \epsilon|\mathcal{Z}_n\right) \le \frac{\sum_{j=1}^{n}\mathbf{Var}[W_{n,j}|\mathcal{Z}_n]}{s_n^2\epsilon^2} = \frac{1}{\epsilon^2}\cdot\frac{\sum_{j=1}^{n}\mathbf{E}\left[(T_j^{(n)})^2\cdot\mathbf{1}(|T_j^{(n)}| > \delta s_n)|\mathcal{Z}_n\right]}{s_n^2} \to 0,$$

therefore $\frac{\sum_{j=1}^{n} W_{n,j}}{s_n}|\mathcal{Z}_n \xrightarrow{p} 0$. Combining the above steps we get $\lim_{n\to\infty}\mathbf{P}\left(\frac{L_n}{s_n} \le x|\mathcal{Z}_n\right) = \Phi(x)$, where $\Phi(x)$ is the cdf of Gaussian distribution. □

Corollary C.1 (Uniform Berry-Esseen under the tilt). *Fix a compact $K \subset (\mu_U, \infty)$ and $z_0 \in \mathbb{R}$. Let $\theta_{x,n}(z)$ solve $\partial_\theta\Lambda_n(\theta,z) = x$ and set $v_{x,n}^2(z) := \partial_\theta^2\Lambda_n(\theta_{x,n}(z),z)$. Define the standardized sum under the exponential tilt*

$$T_n(x,z) \;:=\; \frac{S_n(z) - nx}{v_{x,n}(z)\sqrt{n}}, \qquad S_n(z) = \sum_{i=1}^{n} U_i^{(n)} X_i^{(n)}(z).$$

Under (L1)–(L2)*, there exists $C < \infty$ such that for all large n,*

$$\sup_{x\in K}\ \sup_{z\le z_0}\ \sup_{t\in\mathbb{R}} |\mathbf{P}_{\theta_{x,n}(z)}(T_n(x,z)\le t) - \Phi(t)| \ \le\ \frac{C}{\sqrt{n}}.$$

The same bound holds uniformly for y in a shrinking window $|y-x| \le c/n$, with $\theta_{y,n}(z)$ and $v_{y,n}(z)$ in place of $\theta_{x,n}(z), v_{x,n}(z)$.

Remark C.1. The conditional CLT (Proposition 2.1) already provides (i) uniform bounds $0 < c \le v_{x,n}(z) \le C$ and $0 < c \le \theta_{x,n}(z) \le C$ on $K \times (-\infty, z_0]$ and (ii) a uniform Lindeberg control under the tilt. Keeping the next (cubic) term in the cumulant/Taylor expansion of $\log \mathbf{E}_\theta e^{it(Y-x)/v}$, and applying Esseen's smoothing inequality, yields

$$\sup_t |\mathbf{P}_\theta(T_n \le t) - \Phi(t)| \ \le\ \frac{C_{\mathrm{BE}}}{\sqrt{n}}\,\frac{\mathbf{E}_\theta|Y-x|^3}{(\mathrm{Var}_\theta(Y))^{3/2}} \ \le\ \frac{C}{\sqrt{n}},$$

with constants uniform in $(x,z) \in K \times (-\infty, z_0]$ by (L2).

C.2. Marginal Limit Distribution

Proposition C.2. *Assume that (A1)-(A3) hold. Then, $n^{-1}L_n$ converges in probability to $\mu_U F_\epsilon\left(\frac{v-\mathcal{Z}}{b}\right)$, as $n \to \infty$. Furthermore, under the additional assumption that $\mathcal{Z}_n \overset{a.s.}{\to} \mathcal{Z}$, and $\mathbf{E}[(U_1^{(n)} - \mu_{U^{(n)}})^{2+\delta}] \le C_1 < \infty$ for a fixed $\delta > 0$, then $n^{-1}L_n$ converges almost surely to $\mu F_\epsilon\left(\frac{v-\mathcal{Z}}{b}\right)$, as $n \to \infty$. Furthermore, as $n \to \infty$,*

$$\mathbf{P}\left(\frac{L_n - n\mu_{U^{(n)}}F_\epsilon\left(\frac{v-\mathcal{Z}_n}{b^{(n)}}\right)}{\sqrt{n}} \le x\right) \to \mathbf{E}\left[\Phi\left(\frac{x}{\sigma^{(\infty)}_{T_1|\mathcal{Z}}}\right)\right],$$

where $\Phi(x)$ is CDF of standard normal distribution and

$$\sigma^{(\infty)}_{T_1|\mathcal{Z}} = \sqrt{\sigma_U^2 F_\epsilon\left(\frac{v-\mathcal{Z}}{b}\right) + \mu_U^2 F_\epsilon\left(\frac{v-\mathcal{Z}}{b}\right)\cdot\left[1 - F_\epsilon\left(\frac{v-\mathcal{Z}}{b}\right)\right]}.$$

Proof. Let $\mathcal{F}_n$ denote the sigma-field generated by $\mathcal{Z}_n$. Let $\mu_U := \mathbf{E}[U]$ and set $\theta := \mu_U F_\epsilon\left(\frac{v-\mathcal{Z}}{b}\right)$. First notice that, $n^{-1}L_n - \theta = T_{n,1} + T_{n,2}$, where $T_{n,1} = \left(n^{-1}L_n - \mathbf{E}\left[n^{-1}L_n|\mathcal{F}_n\right]\right)$ and $T_{n,2} = \left(\mathbf{E}\left[n^{-1}L_n|\mathcal{F}_n\right] - \theta\right)$. We will first show that $T_{n,2}$ converges to zero in probability. To this end, observe that using the independence of $U_j^{(n)}$ and $X_j^{(n)}$ and noticing that $\mathcal{Z}_n$ is independent of $\{\epsilon_j\}$, we have that $\mathbf{E}[\mathbf{1}(Y_i^{(n)} \le v)|\mathcal{Z}_n] = F_\epsilon\left(\frac{v-\mathcal{Z}_n}{b^{(n)}}\right)$ with probability one (w.p.1). Hence, w.p.1,

$$\mathbf{E}[U_j^{(n)}X_j^{(n)}|\mathcal{Z}_n] = \mathbf{E}[U_j^{(n)}\mathbf{E}[X_j^{(n)}|\mathcal{Z}_n]] = \mu_{U^{(n)}}F_\epsilon\left(\frac{v-\mathcal{Z}_n}{b^{(n)}}\right).$$

Now, using (A1)-(A3), it follows using the continuity of $F_\epsilon(\cdot)$ that $\mathbf{E}\left[n^{-1}L_n|\mathcal{Z}_n\right]$ converges in probability to $\mu F_\epsilon\left(\frac{v-\mathcal{Z}}{b}\right)$ as $n \to \infty$. Thus, to complete the proof, it is sufficient to show that $T_{n,1}$ converges to zero in probability as $n \to \infty$. Towards this, note that

$$T_{n,1} = \frac{1}{n}\sum_{j=1}^{n} U_j^{(n)} X_j^{(n)} - \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}[U_j^{(n)} X_j^{(n)} | \mathcal{Z}_n] := T_{n,3} + T_{n,4},$$

where

$$T_{n,3} = \frac{1}{n}\sum_{j=1}^{n}(U_j^{(n)} - \mu_{U^{(n)}})X_j^{(n)} \quad \text{and} \quad T_{n,4} = \frac{\mu_{U^{(n)}}}{n}\sum_{j=1}^{n}(X_j^{(n)} - \mathbf{E}[X_j^{(n)}|\mathcal{Z}_n]).$$

Also, note that

$$\mathbf{Var}[T_{n,3}] = \frac{1}{n^2}\mathbf{Var}\left[\mathbf{E}\left[\sum_{j=1}^{n}(U_j^{(n)} - \mu_{U^{(n)}})X_j^{(n)}|\mathcal{Z}_n\right]\right] + \frac{1}{n^2}\mathbf{E}\left[\mathbf{Var}\left[\sum_{j=1}^{n}(U_j^{(n)} - \mu_{U^{(n)}})X_j^{(n)}|\mathcal{Z}_n\right]\right].$$

Now, using the independence of $\{U_j^{(n)}\}$ and $\{X_j^{(n)}\}$, the first term of the above expression is zero and the second term reduces to

$$\begin{aligned}
\mathbf{E}\left[\mathbf{Var}\left[\sum_{j=1}^{n}(U_j^{(n)} - \mu_{U^{(n)}})X_j^{(n)}|\mathcal{Z}_n\right]\right] &= \sum_{j=1}^{n}\mathbf{E}\left[\mathbf{Var}\left[(U_j^{(n)} - \mu_{U^{(n)}})X_j^{(n)}|\mathcal{Z}_n\right]\right] \\
&= n\mathbf{E}\left[(U_1^{(n)} - \mu_{U^{(n)}})^2\right] \cdot \mathbf{E}\left[(X_1^{(n)})^2|\mathcal{Z}_n\right] \\
&\leq n(\sigma^{(n)})^2. \qquad (C.1)
\end{aligned}$$

Now, using Chebyshev's inequality and (C.1), it follows that $T_{n,3}$ converges in probability to zero. Turning to $T_{n,4}$, using similar calculation it follows that $\mathbf{Var}[T_{n,4}]$ is bounded above by $n^{-1}(\mu_{U^{(n)}})^2$, verifying that $T_{n,4}$ converges in probability to zero. Turning to the almost sure convergence, we follow the same notation and methods as above to obtain $T_{n,2} \stackrel{a.s.}{\to} 0$. Turning to $T_{n,1}$, notice that it is enough to show that for any $\epsilon > 0$,

$$P\left(\lim_{n\to\infty}|T_{n,1}| > \epsilon\right) = \mathbf{E}\left[P\left(\lim_{n\to\infty}|T_{n,1}| > \epsilon|\mathcal{Z}_n\right)\right] = 0. \qquad (C.2)$$

Decomposing $T_{n,1} = T_{n,3} + T_{n,4}$, where

$$T_{n,3} = \frac{1}{n}\sum_{j=1}^{n}(U_j^{(n)} - \mu_{U^{(n)}})X_j^{(n)} \quad \text{and} \quad T_{n,4} = \frac{\mu_{U^{(n)}}}{n}\sum_{j=1}^{n}(X_j^{(n)} - \mathbf{E}[X_j^{(n)}|\mathcal{Z}_n].$$

we will show each of the terms converges to zero almost surely. To this end, first note that by Markov's inequality

$$\mathbf{P}(|T_{n,3}| > \epsilon) \leq \frac{\mathbf{E}[|T_{n,3}|^{2+\delta}]}{\epsilon^{2+\delta}}, \quad \mathbf{P}(|T_{n,4}| > \epsilon) \leq \frac{\mathbf{E}[|T_{n,4}|^{2+\delta}]}{\epsilon^{2+\delta}}.$$

Let $A_j^{(n)} = (U_j^{(n)} - \mu_{U^{(n)}})X_j^{(n)}$. Then $T_{n,3} = \frac{1}{n}\sum_{j=1}^{n} A_j^{(n)}$. We first calculate the conditional expectation and then take the expectation on both sides. Applying the martingale version of the Marcinkiewicz–Zygmund inequality (see Chow and Teicher (2012)), by conditioning on $\mathcal{Z}_n$ and taking expectations and using Minkowski's inequality, we obtain

$$\mathbf{E}[|T_{n,3}|^{2+\delta}] \leq \left(\frac{1}{n}\right)^{2+\delta} B_{2+\delta}\mathbf{E}\left[\left(\sum_{j=1}^{n}(A_j^{(n)})^2\right)^{\frac{2+\delta}{2}}\right] = \left(\frac{1}{n}\right)^{2+\delta} B_{2+\delta}\left(\left\|\sum_{j=1}^{n}(A_j^{(n)})^2\right\|_{\frac{2+\delta}{2}}\right)^{\frac{2+\delta}{2}}$$
$$\leq \left(\frac{1}{n}\right)^{2+\delta} B_{2+\delta}\left(\sum_{j=1}^{n}\left\|(A_j^{(n)})^2\right\|_{\frac{2+\delta}{2}}\right)^{\frac{2+\delta}{2}} = \left(\frac{1}{n}\right)^{2+\delta} B_{2+\delta}\left(n\left(\mathbf{E}\left[(A_j^{(n)})^{2+\delta}\right]\right)^{\frac{2}{2+\delta}}\right)^{\frac{2+\delta}{\delta}}.$$

The RHS is $n^{-(1+\delta/2)}B_{2+\delta}\mathbf{E}\left[(A_j^{(n)})^{2+\delta}\right]$, where $B_{2+\delta}\in(0,\infty)$ only depends on δ. Next, using the independence of $U_1^{(n)}$ and $X_1^{(n)}$, we obtain

$$\begin{aligned}\mathbf{E}[(A_1^{(n)})^{2+\delta}|\mathcal{Z}_n] =& \mathbf{E}[(U_1^{(n)}-\mu_{U^{(n)}})^{2+\delta}]\cdot\mathbf{E}[(X_1^{(n)})^{2+\delta}|\mathcal{Z}_n]\\ \leq& \mathbf{E}[(U_1^{(n)}-\mu_{U^{(n)}})^{2+\delta}] \leq C_1.\end{aligned}$$

Now, plugging into $\mathbf{E}[|T_{n,3}|^{2+\delta}|\mathcal{Z}_n]$, we get

$$\mathbf{E}[|T_{n,3}|^4|\mathcal{Z}_n] \leq \left(\frac{1}{n}\right)^{1+\delta/2} B_{2+\delta}C_1.$$

Noticing that $\mathbf{E}[|T_{n,3}|^{2+\delta}|\mathcal{Z}_n]>0$ and taking expectation on both side, we obtain

$$\mathbf{E}[|T_{n,3}|^{2+\delta}] \leq \left(\frac{1}{n}\right)^{1+\delta/2} B_{2+\delta}C_1.$$

Hence

$$\sum_{n=1}^{\infty}\mathbf{P}(|T_{n,3}|>\epsilon) \leq B_{2+\delta}C_1\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{1+\delta/2} < \infty.$$

By Borel–Cantelli lemma, it follows that $T_{n,3}\overset{a.s.}{\to}0$. Similar calculation also yields that $T_{n,4}\overset{a.s.}{\to}0$. Combining we conclude that $T_{n,1}\overset{a.s.}{\to}0$. Turning to the CLT part of the proposition, define

$$S_n=\sum_{j=1}^{n}\left[U_j^{(n)}X_j^{(n)}-\mu_{U^{(n)}}F_\epsilon\left(\frac{v-\mathcal{Z}_n}{b^{(n)}}\right)\right] := \sum_{j=1}^{n}T_j^{(n)}.$$

Notice that

$$\begin{aligned}\mathbf{Var}[T_j^{(n)}|\mathcal{Z}_n] =& \mathbf{Var}[U_j^{(n)}X_j^{(n)}|\mathcal{Z}_n]\\ =& \mathbf{E}\left[\mathbf{Var}[U_j^{(n)}X_j^{(n)}|\mathcal{Z}_n,\epsilon_j]|\mathcal{Z}_n\right]+\mathbf{Var}\left[\mathbf{E}[U_j^{(n)}X_j^{(n)}|\mathcal{Z}_n,\epsilon_i]|\mathcal{Z}_n\right]\\ =& \mathbf{E}\left[X_j^{(n)}\mathbf{Var}(U_j^{(n)})|\mathcal{Z}_n\right]+\mathbf{Var}\left[X_j^{(n)}\mathbf{E}[U_j^{(n)}]|\mathcal{Z}_n\right]\\ =& (\sigma^{(n)})^2\mathbf{E}\left[X_j^{(n)}|\mathcal{Z}_n\right]+(\mu_{U^{(n)}})^2\mathbf{Var}\left[X_j^{(n)}|\mathcal{Z}_n\right]\end{aligned}$$

$$=(\sigma^{(n)})^2 F_\epsilon\left(\frac{v-\mathcal{Z}_n}{b^{(n)}}\right)+(\mu_{U^{(n)}})^2(\sigma^{(n)}_{X^{(n)}_j|\mathcal{Z}_n})^2,$$

where $(\sigma^{(n)}_{X_j|\mathcal{Z}_n})^2 = \mathbf{V}[X^{(n)}_j|\mathcal{Z}_n] = F_\epsilon\left(\frac{v-\mathcal{Z}_n}{b^{(n)}}\right)\cdot\left(1-F_\epsilon\left(\frac{v-\mathcal{Z}_n}{b^{(n)}}\right)\right)$. Denote $\mathbf{Var}[T^{(n)}_j|\mathcal{Z}_n]$ by $(\sigma^{(n)}_{T_i|\mathcal{Z}_n})^2$, which is finite since $X^{(n)}_j$ is an indicator. Then $\mathbf{Var}[L_n|\mathcal{Z}_n]$ is given by s_n^2, where

$$s_n^2=\sum_{j=1}^n(\sigma^{(n)}_{T_j|\mathcal{Z}_n})^2=n\cdot(\sigma^{(n)}_{T_1|\mathcal{Z}_n})^2.$$

Now we show that, conditionally on $\mathcal{Z}_n$, $\frac{S_n}{\sqrt{n}}$ satisfies the Lindeberg conditions. Notice that $\mathbf{E}[S_n|\mathcal{Z}_n]=0$, and for any $\delta>0$,

$$\frac{\sum_{j=1}^n\mathbf{E}\left[(T^{(n)}_j)^2\cdot\mathbf{1}(|T^{(n)}_j|>\delta s_n)|\mathcal{Z}_n\right]}{s_n^2}=\frac{\mathbf{E}\left[(T^{(n)}_1)^2\cdot\mathbf{1}(|T^{(n)}_1|>\delta s_n)|\mathcal{Z}_n\right]}{(\sigma^{(n)}_{T_1|\mathcal{Z}_n})^2}.$$

Next we show that the numerator on the RHS converges to 0. To this end, note that

$$\begin{aligned}\mathbf{E}\left[(T^{(n)}_1)^2\cdot\mathbf{1}(|T^{(n)}_1|>\delta s_n)|\mathcal{Z}_n\right]=&\mathbf{E}\left[\left(U^{(n)}_1X^{(n)}_1-\mu_{U^{(n)}}F_\epsilon\left(\frac{v-\mathcal{Z}_n}{b^{(n)}}\right)\right)^2\cdot\mathbf{1}(|T^{(n)}_1|>\delta s_n)|\mathcal{Z}_n\right]\\ \le&2\mathbf{E}\left[\left(U^{(n)}_1X^{(n)}_1\right)^2\cdot\mathbf{1}(|T^{(n)}_1|>\delta s_n)|\mathcal{Z}_n\right]\\ &+2\mathbf{E}\left[\left(\mu_{U^{(n)}}F_\epsilon\left(\frac{v-\mathcal{Z}_n}{b^{(n)}}\right)\right)^2\cdot\mathbf{1}(|T^{(n)}_1|>\delta s_n)|\mathcal{Z}_n\right]\\ =&2(\mu_{U^{(n)}})^2\mathbf{E}\left[\left(X^{(n)}_1\right)^2\cdot\mathbf{1}(|T^{(n)}_1|>\delta s_n)|\mathcal{Z}_n\right]\\ &+2\left(\mu_{U^{(n)}}F_\epsilon\left(\frac{v-\mathcal{Z}_n}{b^{(n)}}\right)\right)^2\mathbf{E}\left[\mathbf{1}(|T^{(n)}_1|>\delta s_n)|\mathcal{Z}_n\right]\end{aligned}$$

Now using $|X^{(n)}_1|\le 1$ and $F_\epsilon\left(\frac{v-\mathcal{Z}_n}{b^{(n)}}\right)\le 1$, it follows that the above is bounded by

$$2(\mu_{U^{(n)}})^2\mathbf{E}\left[\mathbf{1}(|T^{(n)}_1|>\delta s_n)|\mathcal{Z}_n\right]+2\left(\mu_{U^{(n)}}\right)^2\mathbf{E}\left[\mathbf{1}(|T^{(n)}_1|>\delta s_n)|\mathcal{Z}_n\right]\to 0.$$

since $\mathbf{E}\left[\mathbf{1}(|T^{(n)}_1|>\delta s_n)|\mathcal{Z}_n\right]\to 0$ as $n\to\infty$. Thus,

$$\lim_{n\to\infty}\frac{\sum_{j=1}^n\mathbf{E}\left[(T^{(n)}_j)^2\cdot\mathbf{1}(|T^{(n)}_j|>\delta s_n)|\mathcal{Z}_n\right]}{s_n^2}=\frac{\lim_{n\to\infty}\mathbf{E}\left[(T^{(n)}_1)^2\cdot\mathbf{1}(|T^{(n)}_1|>\delta s_n)|\mathcal{Z}_n\right]}{\lim_{n\to\infty}(\sigma^{(n)}_{T_1|\mathcal{Z}_n})^2}=0.$$

And also $\frac{s_n}{\sqrt{n}} = \sigma^{(n)}_{T_1|\mathcal{Z}_n} \xrightarrow{P} \sigma^{(\infty)}_{T_1|\mathcal{Z}}$ by (A3). Hence, using the CLT in Appendix C (Proposition C.1) we obtain

$$\mathbf{P}\left(\frac{L_n - n\mu_{U^{(n)}} F_\epsilon\left(\frac{v-\mathcal{Z}_n}{b^{(n)}}\right)}{\sqrt{n}} \le x\right) = \mathbf{E}\left[\mathbf{P}\left(\frac{L_n - n\mu_{U^{(n)}} F_\epsilon\left(\frac{v-\mathcal{Z}_n}{b^{(n)}}\right)}{\sqrt{n}} \le x|\mathcal{Z}_n\right)\right]$$

$$= \mathbf{E}\left[\mathbf{P}\left(\frac{S_n}{\sqrt{n}} \le x|\mathcal{Z}_n\right)\right] \to \mathbf{E}\left[\Phi\left(\frac{x}{\sigma^{(\infty)}_{T_1|\mathcal{Z}}}\right)\right],$$

where $\Phi(x)$ is CDF of standard normal distribution and

$$\sigma^{(\infty)}_{T_1|\mathcal{Z}} = \sqrt{\sigma_U^2 F_\epsilon\left(\frac{v-\mathcal{Z}}{b}\right) + \mu_U^2 F_\epsilon\left(\frac{v-\mathcal{Z}}{b}\right)\cdot\left[1 - F_\epsilon\left(\frac{v-\mathcal{Z}}{b}\right)\right]}.$$

This completes the proof. □

C.3. Sharp Conditional Large Deviations

The Key ingredient to the proofs of Theorem 2.1 and Theorem 2.2 is the conditional sharp large deviation Theorem and an evaluation of the behavior of the conditional rate function in the tails of the distribution of the common factors. This involves a careful decomposition of the integral and requires identification of an optimal point similar to the Laplace method for exponential integrals. We start with conditional sharp large deviations, whose proof is standard and hence omitted. Below, $\Lambda_n^{*'}(\cdot, z)$ and $\Lambda_n^{*''}(\cdot, z)$ are derivatives for a fixed z. The proof follows from Proposition C.1 via Berry-Esseen bounds for the conditional triangular array.

Theorem C.1 (Conditional Bahadur-Rao LDP). *Assume that conditions (A1)-(A3), (L1)-(L2), (and additional conditions in Theorem 2.2 for degenerate case) hold. Then for any $x > \mu_U$ ($x > q_\kappa$ for the degenerate case)*

$$\mathbf{P}(L_n > nx|\mathcal{Z}_n = z) = n^{-\frac{1}{2}} e^{-n\Lambda_n^*(x;z)} \frac{1}{\sqrt{2\pi}\theta_{x,n}(z)\sigma_{x,n}(z)}(1 + r_n(z)),$$

where $\Lambda_n^*(x; z) = \sup_\theta\{\theta x - \Lambda_n(\theta; z)\}$ and $\Lambda_n(\theta; z)$ is the logarithmic moment generating function of $U^{(n)}X^{(n)}$ conditioned on $\mathcal{Z}_n = z$; that is

$$\Lambda_n(\theta; z) = \log \mathbf{E}\left[\exp(\theta U^{(n)} X^{(n)})|\mathcal{Z}_n = z\right] = \log\left[\lambda_{U^{(n)}}(\theta) F_\epsilon\left(\frac{v-z}{b^{(n)}}\right) + 1 - F_\epsilon\left(\frac{v-z}{b^{(n)}}\right)\right].$$

Also, $\sigma_{x,n}(z) = [\Lambda_n^{*''}(x; z)]^{-\frac{1}{2}}$ and $\theta_{x,n}(z) = \Lambda_n^{*'}(x; z)$. Furthermore, for any z_0 positive, $\sup_{z\le z_0} r_n(z) \to 0$ as $n \to \infty$ and $\theta_{x,n}(z)$ and $\sigma_{x,n}(z)$ are continuous in z and x and converge to $\theta_x(z)$ and $\sigma_x(z)$ respectively, as $n \to \infty$.

Proof. Fix x in a compact $K \subset (\mu_U, \infty)$ and $z \le z_0$. Let $S_n(z) = \sum_{i=1}^n Y_i^{(n)}(z)$ with $Y_i^{(n)}(z) = U_i^{(n)} X_i^{(n)}(z)$, and let $\theta = \theta_{x,n}(z)$ solve $\partial_\theta \Lambda_n(\theta, z) = x$. Write $v^2 = \sigma^2_{x,n}(z) = \partial^2_\theta \Lambda_n(\theta, z)$. Perform the Cramér tilt at θ:

under $\mathbf{P}_\theta$, $\mathbf{E}_\theta Y_i^{(n)}(z) = x$ and $\mathrm{Var}_\theta(Y_i^{(n)}(z)) = v^2$. Then

$$\mathbf{P}(S_n(z) \geq nx) = e^{-n(\theta x - \Lambda_n(\theta;z))}\,\mathbf{E}_\theta\left[e^{-\theta(S_n(z)-nx)}\,\mathbf{1}\{S_n(z) \geq nx\}\right].$$

Set $T_n := \frac{S_n(z)-nx}{v\sqrt{n}}$ and $a_n := \theta v\sqrt{n}$. The bracket equals $\mathbf{E}_\theta[e^{-a_n T_n}\mathbf{1}\{T_n \geq 0\}]$. Next, we invoke the *uniform CLT* from Proposition C.1. By the uniform tilted Berry-Esseen bound (Prop. C.1) and (L2),

$$\sup_{t\in\mathbb{R}} |\mathbf{P}_\theta(T_n \leq t) - \Phi(t)| \leq \frac{C_1}{\sqrt{n}},$$

with C_1 independent of $(x,z) \in K \times (-\infty, z_0]$. By (L1)–(L2) and strict convexity, there exist $0 < c \leq C < \infty$ such that $c \leq \theta \leq C$ and $c \leq v \leq C$ uniformly on $K \times (-\infty, z_0]$; hence $a_n = \theta v\sqrt{n} \asymp \sqrt{n}$ uniformly. The last step is the evaluation of the *Laplace transform of the positive part*. Let F_n be the cdf of T_n under $\mathbf{P}_\theta$ and write

$$\mathbf{E}_\theta[e^{-a_n T_n}\mathbf{1}\{T_n \geq 0\}] = \int_0^\infty e^{-a_n t}\,dF_n(t) = \int_0^\infty e^{-a_n t}\,d\Phi(t) \; + \; \int_0^\infty e^{-a_n t}\,d(F_n - \Phi)(t).$$

For the Gaussian term, a direct computation gives

$$\int_0^\infty e^{-at}\,d\Phi(t) = \frac{1}{a\sqrt{2\pi}}\Big(1 + O(a^{-2})\Big), \qquad a \to \infty.$$

For the error term, integrate by parts and use the Berry-Esseen bound:

$$\left|\int_0^\infty e^{-a_n t}\,d(F_n - \Phi)(t)\right| \;\leq\; a_n \int_0^\infty e^{-a_n t}\,\sup_s |F_n(s) - \Phi(s)|\,dt \;\leq\; \frac{C_1}{\sqrt{n}}.$$

Since $a_n \asymp \sqrt{n}$, the $O(a_n^{-2})$ from the Gaussian term is $O(n^{-1})$ and thus dominated by $C_1/\sqrt{n}$. Therefore,

$$\mathbf{E}_\theta[e^{-a_n T_n}\mathbf{1}\{T_n \geq 0\}] = \frac{1}{\theta v\sqrt{2\pi n}}\Big(1 + O(n^{-1/2})\Big),$$

with constants uniform on $(x,z) \in K \times (-\infty, z_0]$. Finally, combining with the tilt identity,

$$\mathbf{P}(S_n(z) \geq nx) = \frac{e^{-n\Lambda_n^*(x,z)}}{\sqrt{2\pi n}\,\theta\,v}\Big(1 + O(n^{-1/2})\Big),$$

which is the claimed formula with $v = \sigma_{x,n}(z)$ and uniform remainder. The nonlattice assumption in (L2) guarantees the tilted Berry-Esseen bound; if a lattice correction is needed, the same argument holds with the usual continuity correction, and the remainder remains $O(n^{-1/2})$. □

Appendix D: Appendix

In this appendix, we provide a brief description of self-neglecting functions.

Definition D.1 (Self-neglecting at infinity). A measurable function $f : [x_0, \infty) \to (0, \infty)$ is called *self-neglecting* (abbreviated $f \in \mathrm{SN}$) if

$$\lim_{x\to\infty} \sup_{|t|\leq T} \left|\frac{f(x + t\,f(x))}{f(x)} - 1\right| = 0 \qquad \text{for every } T < \infty,$$

and, in addition, $f(x) = o(x)$ as $x \to \infty$. We will also use the equivalent "little-o shift" notation

$$\frac{f(x + o(f(x)))}{f(x)} \longrightarrow 1 \qquad (x \to \infty),$$

which follows from the uniform formulation above.

Proposition D.1 (Equivalent characterizations of SN). *Let $f(\cdot)$ be eventually positive and measurable. The following are equivalent:*

1. $f \in \mathrm{SN}$ *in the sense of Definition D.1.*
2. $f(x + o(f(x)))/f(x) \to 1$ *as* $x \to \infty$.
3. *For any sequence $x_n \to \infty$ and any bounded sequence (t_n), one has $f(x_n + t_n a(x_n))/f(x_n) \to 1$.*

Proof(Sketch). *(1)⇒(2) is immediate; (2)⇒(3) by choosing $o(f(x_n)) = t_n f(x_n)$; (3)⇒(1) follows by a standard diagonal/compactness argument in t.*

Lemma D.1 (A convenient sufficient condition). *If $f(\cdot)$ is eventually C^1 with $f(x) = o(x)$ and $f'(x) \to 0$ as $x \to \infty$, then $f(\cdot) \in \mathrm{SN}$. In particular, if $Q \in C^2(\mathbb{R})$ with $Q'(x) \to \infty$ and $Q''(x) = o(Q'(x)^2)$, then*

$$f(x) := \frac{1}{Q'(x)} \in \mathrm{SN}.$$

Proof sketch. *By the mean value theorem, $f(x + t f(x)) = f(x) + t\, f(x)\, f'(\xi_x)$ for some ξ_x between x and $x + t f(x)$; since $f'(\xi_x) \to 0$ uniformly on $|t| \le T$, the ratio tends to 1. For the particular case, $f'(x) = -Q''(x)/Q'(x)^2 \to 0$ and $f(x) = o(x)$ as $Q' \to \infty$.*

Remark D.1 (Time change and window scaling). If $f(\cdot) \in \mathrm{SN}$ and $S(x) := \int_{x_0}^{x} \frac{dt}{f(t)}$, then $S(x) \to \infty$ and

$$S(x + t\, f(x)) - S(x) \longrightarrow t \quad \text{as } x \to \infty, \text{ uniformly for } |t| \le T.$$

This identity underlies the width of the Laplace window (of order $1/\sqrt{n|\tilde{h}_n''(z_*)|}$) in our sharp Large Deviation analysis.

Remark D.2 (Relation to slow variation). (a) If L is slowly varying in the Karamata sense, then for any $f(\cdot) \in \mathrm{SN}$, $L(x + t\, f(x))/f(x) \to 1$ uniformly on compact t ("Beurling slow variation" relative to $f(\cdot)$).
(b) If $f(x) = x^{\alpha} L_0(x)$ with $0 \le \alpha < 1$ and L_0 slowly varying, then $f(\cdot) \in \mathrm{SN}$.
(c) The functions $f(x) = (\log x)^{\beta}$ ($\beta > 0$); $f(x) = x^{\alpha}$ ($0 < \alpha < 1$); $f(x) = x/\log x$ are standard Beurling slow variation examples.

Remark D.3 (How we use SN here). In assumption (X1) we set $a_\epsilon(t) := 1/Q'_\epsilon(t)$. The condition $Q''_\epsilon = o((Q'_\epsilon)^2)$ implies $a'_\epsilon \to 0$, hence $a_\epsilon \in \mathrm{SN}$ by Lemma D.1. This justifies the notation $a_\epsilon(t + o(a_\epsilon(t))) \sim a_\epsilon(t)$ used in the Laplace localization.

Next, we provide some examples in the log-smooth class.

(a) Generalized gamma and Weibull-type (right tail for ϵ). If $g_\epsilon(t) = 1 - F_\epsilon(t) \sim c_\epsilon(t)\exp\{-Q_\epsilon(t)\}$ with

$$Q_\epsilon(t) = \xi_\epsilon\, t^{m_\epsilon} \quad (m_\epsilon > 1,\ \xi_\epsilon > 0),$$

then $Q'_\epsilon(t) = \xi_\epsilon m_\epsilon t^{m_\epsilon - 1} \uparrow \infty$ and $Q''_\epsilon(t)/\{Q'_\epsilon(t)\}^2 = O(t^{-m_\epsilon}) \to 0$. Thus (X1) holds with $a_\epsilon(t) = 1/Q'_\epsilon(t)$ self-neglecting. This covers the classical Weibull tail and, more generally, the *generalized gamma* law on $\mathbb{R}_+$ (density $\propto t^{k_\epsilon - 1}\exp\{-(t/\theta_\epsilon)^{m_\epsilon}\}$) by absorbing polynomial factors into $c_\epsilon(t)$.

(b) Generalized normal. For GN(β, ξ, γ), the right tail has $Q_\epsilon(t) = \xi\, t^\gamma$. When $\gamma > 1$, $Q'_\epsilon(t) = \xi\gamma t^{\gamma-1} \uparrow \infty$ and $Q''_\epsilon/Q'^2 \to 0$, so (X1) holds. The borderline case $\gamma = 1$ (Laplace/exponential) does *not* satisfy $Q'_\epsilon \uparrow \infty$, but it is already covered by the class $\mathcal{C}$ results in Theorem 2.1 and Proposition 2.1.

(c) Log-smooth left tails for the common factor $\mathcal{Z}_n$. If $f_{\mathcal{Z}_n}(-w) \sim c_{\mathcal{Z}_n}(w)\exp\{-R_{\mathcal{Z}_n}(w)\}$ with

$$R_{\mathcal{Z}_n}(w) = \xi_{\mathcal{Z}_n}\, w^{m_{\mathcal{Z}_n}} \quad (m_{\mathcal{Z}_n} > 1,\ \xi_{\mathcal{Z}_n} > 0),$$

then $r_{\mathcal{Z}_n}(w) := R'_{\mathcal{Z}_n}(w) = \xi_{\mathcal{Z}_n} m_{\mathcal{Z}_n} w^{m_{\mathcal{Z}_n} - 1} \uparrow \infty$ and $R''_{\mathcal{Z}_n}(w)/\{r_{\mathcal{Z}_n}(w)\}^2 = O(w^{-m_{\mathcal{Z}_n}}) \to 0$, so (X2) holds. This includes Weibull-type left tails and two-sided exponential-power and GN models with $\gamma > 1$.

(d) Mixed log-smooth pairs. (X1)–(X2) do not require matching shapes: e.g., ϵ Weibull ($m_\epsilon > 1$) and $\mathcal{Z}_n$ generalized gamma ($m_{\mathcal{Z}_n} > 1$) are admissible; all arguments go through with the balance condition (B1) selecting M_n.

(e) Borderline and other cases. Exponential ($m = 1$) and regularly varying tails fall outside (X1)–(X2) but are already treated by the class $\mathcal{C}$ theorems (Theorem 2.1 and Proposition 2.1 for generalized normal and regularly varying distributions), ensuring that our results cover both Weibull and generalized-gamma–type light tails and heavy tails under a unified presentation.

Remark D.4 (Connection to extreme-value theory). The log-smooth assumptions (X1)-(X2) are of von Mises type and guarantee that the tails of ϵ and $\mathcal{Z}$ lie in the *maximum domain of attraction (MDA)* of the Gumbel law. In extreme-value theory (EVT), the MDA of a limiting distribution refers to the class of distributions whose suitably normalized maxima converge to that limit law. In particular, distributions with

$$\bar{F}(x) = c(x)\, e^{-Q(x)}, \quad Q'(x) \uparrow \infty, \quad Q''(x)/Q'(x)^2 \to 0,$$

are in the Gumbel MDA. Thus, the generalized gamma (Weibull-type with shape $m > 1$), generalized normal/exponential-power with $\gamma > 1$, and many other light-tailed models fall into our framework automatically. This situates our log-smooth setting within the well-established Gumbel domain of attraction in EVT.

Appendix E: Appendix

We now illustrate the consequences of the sharp large deviation results by deriving asymptotic expansions for two standard portfolio risk measures: Value-at-Risk (VaR) and Expected Shortfall (ES). These results demonstrate how our prefactor refinements translate into practical risk quantification, and clarify when portfolios operate in the large-deviation regime rather than in a central-limit or near-critical regime. For orientation, Proposition C.2 provides the near-mean LLN/CLT baseline; the expansions below quantify the correction when the portfolio operates in the large-deviation regime of Theorems 2.1–2.3. Recall that assumptions (L1) and (L2) hold.

E.1. Value-at-Risk

For $\alpha \in (0,1)$, the Value-at-Risk at level α is the quantile $x_{\alpha,n} := \mathrm{VaR}_\alpha(L_n/n)$ defined by

$$\mathbf{P}\left(\frac{L_n}{n} \geq x_{\alpha,n}\right) = 1-\alpha,$$

where the quantile is understood in the usual sense (e.g., via the infimum definition, or under continuity of the law of L_n/n). Applying Theorem 2.1 yields

$$n^{-1/2} e^{-n\Lambda_U^*(x_{\alpha,n})} [H_n(-\tilde{M}_n)]^{-1} e^{-n\phi_n(-\tilde{M}_n)} C = 1-\alpha+o(1), \qquad n\to\infty,$$

where $C=\psi_\infty$, from which a second-order expansion of $x_{\alpha,n}$ follows. Tables E.1 and E.2 in Appendix E contain numerical values of VaR_α for $\alpha=0.95$, $\alpha=0.99$, and $\alpha=0.999$ under Gaussian and Pareto tails.

Let $\mu_U = \mathbf{E}[U]$ and $\sigma_U^2 = \mathrm{Var}(U)$. To extract explicit expansions, we expand $\Lambda_U^*(x)$ around μ_U. Setting $x_{\alpha,n} = \mu_U + y_n$ and applying a Taylor expansion up to third order around μ_U, we obtain

$$\Lambda_U^*(x_{\alpha,n}) = \Lambda_U^*(\mu_U+y_n) = \Lambda_U^*(\mu_U) + (\Lambda_U^*)'(\mu_U)y_n + \frac{1}{2}(\Lambda_U^*)''(\mu_U)y_n^2 + O(y_n^3), \tag{E.1}$$

Noticing that $\Lambda_U^*(\mu_U)=0$, $(\Lambda_U^*)'(\mu_U)=0$, and $(\Lambda_U^*)''(\mu_U)=1/\sigma_U^2$ we obtain, by substituting into (E.1), that

$$\frac{1}{n}\left(-\log(1-\alpha) - \log H_n(-\tilde{M}_n) - \frac{1}{2}\log n - n\phi_n(-\tilde{M}_n) + \log C\right) = \frac{1}{2}(\Lambda_U^*)''(\mu_U)y_n^2 + o(1), \quad \text{as } n\to\infty$$

which yields

$$y_n = \left(\frac{2\Lambda_U^*(x_{\alpha,n})}{(\Lambda_U^*)''(\mu_U)}\right)^{\frac{1}{2}} + o(1) \quad \text{as } n\to\infty.$$

Now using the definition of y_n, we obtain that as $n\to\infty$

$$x_{\alpha,n} = \mu_U + \left(\frac{2\,\sigma_U^2[-\log(1-\alpha) - \log H_n(-\tilde{M}_n) - \frac{1}{2}\log n - n\phi_n(-\tilde{M}_n) + \log C]}{n}\right)^{1/2} + o(1).$$

where we have used that $(\Lambda_U^*)''(\mu_U) = 1/\sigma_U^2$ in the last step. In the degenerate case, one can follow the same idea to obtain for $n\to\infty$, with $C=\psi_\infty(\kappa)(2\pi)^{-\frac{1}{2}}$

$$x_{\alpha,n} = \mu_U + \left(\frac{2\sigma_U^2[-\log(1-\alpha) - \frac{1}{2}\log n + \log C]}{n}\right)^{\frac{1}{2}} + o(1).$$

For large n, the approximation of $x_{\alpha,n}$ is meaningful only when α is close to one, since our large deviation results concern the extremal behavior, where the tails are dominated by the deviations from U. For other cases, one can expand upon this idea and then follow the results in Collamore, de Silva and Vidyashankar (2022) to obtain a corresponding estimate.

E.2. Expected Shortfall

Expected Shortfall at level $\alpha \in (0,1)$ is

$$\mathrm{ES}_\alpha\left(\frac{L_n}{n}\right) = \frac{1}{1-\alpha}\,\mathbf{E}\left[\frac{L_n}{n}\,\mathbf{1}\left\{\frac{L_n}{n} \geq x_{\alpha,n}\right\}\right].$$

Notice that from Section E.1, for large n, $x_{\alpha,n} > \mathbf{E}[U]$ is close to $\mathbf{E}[U]$, and $\Lambda_U^*(x)$ is strictly increasing in $x \in [x_{\alpha,n},\infty)$ (see Section 3). Now applying the Laplace method (Olver (1997), Fukuda and Kagaya (2025)), we obtain for $n \to \infty$,

$$\int_{x_{\alpha,n}}^{\infty} e^{-n\Lambda_U^*(x)}\mathrm{d}x = \frac{e^{-n\Lambda_U^*(x_{\alpha,n})}}{n(\Lambda_U^*)'(x_{\alpha,n})} + o(1).$$

Thus as $n \to \infty$, with $C = \psi_\infty$,

$$\mathrm{ES}_\alpha\left(\frac{L_n}{n}\right) = x_{\alpha,n} + \frac{Ce^{-n\phi_n(-\tilde{M}_n)}n^{-\frac{1}{2}}[H_n(-\tilde{M}_n)]^{-1}}{1-\alpha}\cdot\frac{e^{-n\Lambda_U^*(x_{\alpha,n})}}{n(\Lambda_U^*)'(x_{\alpha,n})} + o(1) = x_{\alpha,n} + \frac{1}{n\theta_{x_{\alpha,n}}} + o(1).$$

Turning to the degenerate case, using Theorem 2.2 and similar calculations, with $C = \psi_\infty(\kappa)(2\pi)^{-\frac{1}{2}}$, we have for $n \to \infty$,

$$\mathrm{ES}_\alpha\left(\frac{L_n}{n}\right) = x_{\alpha,n} + \frac{Cn^{-\frac{1}{2}}}{1-\alpha}\cdot\frac{e^{-n\Lambda_U^*(x_{\alpha,n})}}{n(\Lambda_U^*)'(x_{\alpha,n})} + o(1) = x_{\alpha,n} + \frac{1}{n\theta_{x_{\alpha,n}}} + o(1).$$

E.3. Numerical experiments

We now provide numerical illustrations of VaR_α and ES_α under different distributional assumptions for the idiosyncratic factor ϵ and the common factor $\mathcal{Z}$. In all examples, we fix $U \sim U(0,1)$, $b = 0.5$, and $v = 0$.

Case 1: Gaussian ϵ and Gaussian $\mathcal{Z}$. Table E.1 reports VaR and ES estimates for the setting $\epsilon \sim N(0,1)$ and $\mathcal{Z} \sim N(0,1)$.

Case 2: Gaussian ϵ and Pareto $\mathcal{Z}$. Table E.2 presents the results when $\epsilon \sim N(0,1)$ and $\mathcal{Z} \sim \mathrm{Pareto}(x_m = 1, \alpha = 2)$. Compared with the Gaussian case, the heavy-tailed distribution of $\mathcal{Z}$ yields systematically larger values of both VaR_α and ES_α, reflecting the heavier tail risk.

For comparison, the central limit approximation (cf. Proposition C.2) would suggest $\mathrm{VaR}_\alpha \approx 0.5$ uniformly in α, missing the systematic inflation of tail risk. Sharp large deviation estimates capture this effect, with the correction especially pronounced in the Pareto case.

		$n = 10$	$n = 50$	$n = 100$	$n = 500$	$n = 1000$
VaR	$\alpha = 0.95$	0.565	0.522	0.514	0.505	0.503
	$\alpha = 0.99$	0.630	0.550	0.534	0.513	0.509
	$\alpha = 0.999$	0.711	0.589	0.561	0.526	0.518
ES	$\alpha = 0.95$	0.692	0.597	0.572	0.537	0.528
	$\alpha = 0.99$	0.692	0.583	0.558	0.526	0.518
	$\alpha = 0.999$	0.745	0.607	0.574	0.532	0.522

Table E.1. VaR_α and ES_α under Gaussian ϵ and Gaussian $\mathcal{Z}$.

		$n = 10$	$n = 50$	$n = 100$	$n = 500$	$n = 1000$
VaR	$\alpha = 0.95$	0.664	0.560	0.539	0.514	0.510
	$\alpha = 0.99$	0.718	0.587	0.559	0.524	0.516
	$\alpha = 0.999$	0.777	0.619	0.582	0.535	0.524
ES	$\alpha = 0.95$	0.712	0.587	0.561	0.526	0.518
	$\alpha = 0.99$	0.752	0.606	0.573	0.531	0.521
	$\alpha = 0.999$	0.801	0.632	0.592	0.540	0.528

Table E.2. VaR and ES under Gaussian ϵ and Pareto $\mathcal{Z}$.

Appendix F: Appendix

Remark F.1 (Allowing U to depend on X). All results extend when the distribution of U depends on $X \in \{0, 1\}$. Conditional on $\mathcal{Z}_n$, let $(U_i^{(n)}, X_i^{(n)})$ be i.i.d., with

$$X_i^{(n)} = \mathbf{1}\{\mathcal{Z}_n + b^{(n)}\epsilon_i^{(n)} \le v\}, \qquad U_i^{(n)} \mid \{X_i^{(n)} = x\} \sim P_{U^{(n)}|X=x}, \quad x \in \{0, 1\}.$$

Since $e^{\theta UX} \equiv 1$ when $X = 0$, the conditional moment generating function in all formulas is replaced by

$$\Lambda_n(\theta; z) = \log\Big(p_n(z)\lambda_1^{(n)}(\theta) + 1 - p_n(z)\Big), \qquad \lambda_1^{(n)}(\theta) := \mathbf{E}[e^{\theta U_1^{(n)}} \mid X_1^{(n)} = 1],$$

and every occurrence of $\lambda_{U_1^{(n)}}$ (resp. λ_U) is replaced by $\lambda_1^{(n)}$ (resp. λ_1), with

$$\lambda_1(\theta) := \mathbf{E}[e^{\theta U} \mid X = 1], \quad \Lambda_1(\theta) := \log \lambda_1(\theta), \quad \theta_x \text{ solving } \Lambda_1'(\theta_x) = x.$$

Assumption (L2) (third tilted moments and nonlattice) is imposed for $U^{(n)} \mid X = 1$ (uniform in n), which suffices for the CLT, Berry-Esseen, and Bahadur-Rao steps. Turning to *Gibbs limits*, in the unbounded case ($\kappa = -\infty$), the product limit in the theorem holds with P_{θ_x} replaced by the tilt of $U \mid X = 1$:

$$\mathbf{P}^{\theta_x}_{U|X=1}(\cdot) = e^{\theta_x u - \Lambda_1(\theta_x)}\,\mathbf{P}_{U|X=1}(\cdot), \qquad X \equiv 1.$$

In the boundary case ($\kappa > -\infty$), use the same one-step formula as in the theorem, but with U replaced by $U \mid X$ and

$$\Lambda(\theta; \kappa) = \log\left((1 - p_\kappa) + p_\kappa \lambda_1(\theta)\right), \qquad p_\kappa = F_\epsilon\left(\frac{v - \kappa}{b}\right), \quad \bar{\theta}_{x,\kappa} : \partial_\theta \Lambda(\bar{\theta}_{x,\kappa}, \kappa) = x.$$

All total-variation conclusions (for fixed k and for $k_n = o(n)$) are unchanged.

References

BILLINGSLEY, P. (2013). *Convergence of probability measures*. John Wiley & Sons.

CHOW, Y. S. and TEICHER, H. (2012). *Probability Theory: Independence, Interchangeability, Martingales*. Springer Science & Business Media.

COLLAMORE, J. F., DE SILVA, H. and VIDYASHANKAR, A. N. (2022). Sharp Probability Tail Estimates for Portfolio Credit Risk. *Risks* **10** 239.

FUKUDA, I. and KAGAYA, Y. (2025). Higher-order asymptotic expansions for Laplace's integral and their error estimates. *arXiv preprint arXiv:2504.00801*.

OLVER, F. (1997). *Asymptotics and Special Functions*. AK Peters/CRC Press.

SWEETING, T. (1989). On conditional weak convergence. *Journal of Theoretical Probability* **2** 461–474.