LETF Options Pricing

Frederick Vu

This document outlines our methodology for estimating prices of European style options for a single stock, 'ABC', and for a 3x leveraged ETF, 'LABC', tracking stock ABC, over a one-week period when given an incomplete dataset of options prices for a single point in time.

1 ABC Options

We first used the Black-Scholes formula to convert the given price data to implied volatilities (IV). We then used the so-called raw Stochastic Volatility Inspired (SVI) model of Gatheral as described in [GJ] to parameterize the volatility curves on days 8, 10, and 12, varying over several initial seed parameters and using root-mean-square error to determine the best fit for each curve. We opted to create separate models for call and put options due to our observation that the IV of ITM call options appeared to deviate siginificantly from that of the put options at the same strike for day 8 (see 1). Let us briefly introduce some notation and describe the SVI model here.

Let $S_{\text{abc}} = 100$ denote the spot price of ABC, and $F_T := e^{rT}$ the forward price of S_{abc} at time T in years with constant annualized risk-free rate $r = 0.05$. For any strike K and time T, we write $k = k(K,T) = \ln(K/F_T)$ for the (forward) log-moneyness of K at T. The SVI model, which actually is meant to model the square of the implied volatility, is described by five real parameters $\chi = (a, b, \rho, m, \sigma)$

$$w_\chi(k) = a + b\left(\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right) \tag{1}$$

where $k \in \mathbb{R}$, with the constraints that $b \geq 0$, $|\rho| \leq 1$, $\sigma > 0$, and $a + b\sigma\sqrt{1-\rho^2} \geq 0$, the last of which guarantees w_χ is nonnegative.

To construct the volatility surface from these disjoint curves with fitted parameters χ_8, χ_{10}, and χ_{12}, we considered the collection of log-moneyness values

$$k \in \{k(K,T) | K \in \{80, \ldots, 110\}, T \in \{T_8 = 1/365, T_{10} = 3/365, T_{12} = 5/365\}\},$$

and linearly interpolated the *total* implied variance values

$$\{w_{\chi_8}(k) \cdot T_8, w_{\chi_{10}}(k) \cdot T_{10}, w_{\chi_{12}}(k) \cdot T_{12}\}$$

to get piecewise linear functions of time t, $L_k(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. From this, we obtained an estimate of the IV $\sigma_{BS}(k,t) \approx \sqrt{L_k(t)/t}$ for $t \in \{T_9 = 2/365, T_{11} = 4/365\}$ and thus also the option price via the Black-Scholes formula.

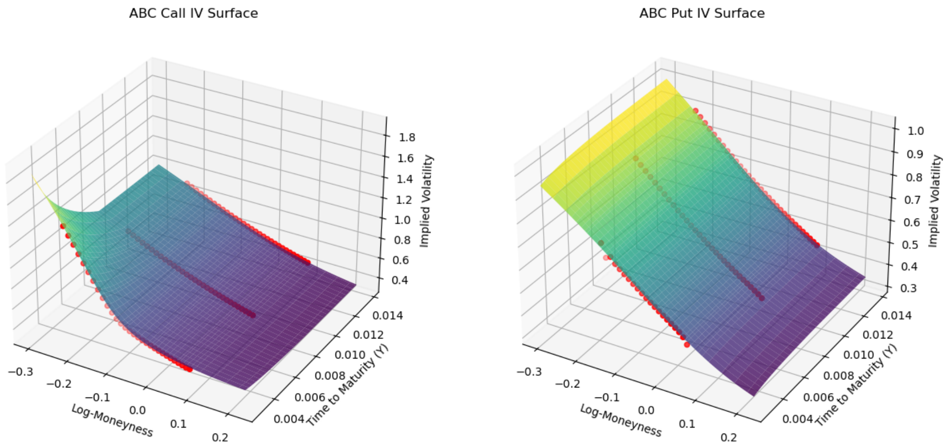

Figure 1: Interpolated IV surface from SVI models plotted with observed IVs.

2 LABC Options

To obtain LABC options prices, we used a method of Leung and Sircar in [LS] which relates the prices of *continuously rebalanced* leveraged ETF options to those of options on unleveraged ETFs (tracking the same underlying). In brief, the idea is that the log-moneyness of a leveraged option can be scaled formulaically to a log-moneyness value for an unleveraged option, and rational pricing theory implies that we should price the leveraged option as we would the unleveraged option. We make the simplifying assumption of continuous rebalancing as it has been shown that difference between daily and continuous balancing on options prices is often negligible [Sa, §4.2.2].

Though the reader should refer to the cited paper for a more thorough detailing of the method, we summarize the pertinent formulas here. Let $(X_t)_{t\geq 0}$ denote the price process of the unleveraged asset and and $(L_t)_{t\geq 0}$ the price process of the leveraged asset with leverage rate $\beta > 0$, and assume that $(X_t)_{t\geq 0}$ is a diffusion process under the risk-neutral measure with constant drift r and stochastic volatility σ_t

$$\frac{dX_t}{X_t} = r\,dt + \sigma_t\,dW_t^*.$$

One may deduce that

$$\frac{dL_t}{L_t} = r\,dt + \beta\sigma_t\,dW_t^*,$$

and moreover that with maturity date T,

$$\ln\left(\frac{L_T}{L_0}\right) = \ln\left(\frac{X_T}{X_0}\right) - r(\beta-1)T - \frac{\beta(\beta-1)}{2}\int_0^T \sigma_t^2\,dt.$$

For any number $LM^{(1)} \in \mathbb{R}$, we define

$$LM^{(\beta)} := \beta LM^{(1)} - r(\beta-1)T - \frac{\beta(\beta-1)}{2}\mathbb{E}^*\left[\int_0^T \sigma_t^2, dt \middle| \ln\left(\frac{X_T}{X_0}\right) = LM^{(1)}\right]. \qquad (2)$$

Leung and Sircar call this *moneyness scaling*, and they interpret $LM^{(\beta)}$ as the best estimate for the terminal log-return of $(L_t)_{t\geq 0}$, given the terminal value $\ln(X_T/X_0) = LM^{(1)}$.

The conditional expectation in (2) is difficult to compute for general stochastic volatility models. To simplify, following Leung and Sircar, we take the rather crude approximation of replacing the random variable σ_t with the average implied volatility of ABC for the given maturity, which we denote by $\overline{\sigma}_T$. That is, we take

$$LM^{(\beta)} \approx \beta LM^{(1)} - r(\beta-1)T - \frac{\beta(\beta-1)}{2}\overline{\sigma}_T^2 T.$$

Since the right-hand side of this equation is an increasing function of $LM^{(1)}$, we can define the function $\phi_T : \mathbb{R} \to \mathbb{R}$ implicitly by

$$\phi_T(LM^{(\beta)}) := LM^{(1)}.$$

For a given LABC option with forward log-moneyness $k = \ln(K/S_{\text{labc}}) - rT$, where $S_{\text{labc}} = 30$ is the spot price of LABC, we price it using the Black-Scholes formula, but with the implied volatility given by our ABC volatility surface of the previous sectin at log-moneyness $\phi_T(k)$ and maturity T, scaled by a factor of β, following [LS, §2.3].

3 Issues and Other Possible Directions

The extremely short time scale renders many common stochastic volatility models less than useful. For instance, we implemented the Heston stochastic volatility model using an vectorized analytic integration following [CDG]. However, we discovered that the Heston model failed to capture the behavior of the volatility surface at extremely short maturities: the model's implied volatility curve actually becomes flat as maturity goes to zero. We considered implementing a Heston model with jumps, which has shown promise in capturing short-term skew. Results of [Sa] and [AHJ] indicate that these jump diffusion models can also be used to price LETF options. However, due to time constraints, we were unable to fully explore this approach.

In light of the observed shortcomings of the Heston model, the SVI model thus seemed like an appropriate choice for modeling the volatility skew, as it was designed for this purpose. However, there are issues with the model: the number of parameters is quite high (five for each curve for a total of fifteen for the entire surface), and the parameters are generally unstable, meaning subtle shifts in the data can lead to large differences in the parameter values, leading to general unreliability of IVs obtained from exptrapolation from given data. The Surface SVI (SSVI) model of [GJ] attempts to parameterize the full volatility surface, leading to more reliable extrapolation results. However, after reading several articles regarding the SSVI model, it was our understanding that the SSVI model is often ill-suited for capturing the implied volatility skew at very short maturities. Extensions of the SSVI model do exist which try to improve the results of the SSVI model at shorter terms, though implementation and calibration complexities again prevented us from exploring this approach fully.

References

[AHJ] A. Ahn, M. Haugh, A. Jain, "Consistent pricing of options on leveraged ETFs", *SIAM Journal on Financial Mathmetics*, Vol. 6, 2015, pp. 559–593.

[CDG] Y. Cui, S. Del Baño Rollin, G. Germano, "Full and fast calibration of the Heston stochastic volatilty model", *European Journal of Operational Research*, Vol. 263, 2017, pp. 625–638.

[GJ] J. Gatheral, A. Jacquier, "Arbitrage-free SVI volatility surfaces", *Quantitative Finance*, Vol. 14, 2014, pp. 59–71.

[LS] T. Leung, R. Sircar, "Implied volatility of leveraged ETF options", *Applied Mathematical Finance*, Vol. 22, 2015, pp. 162-168.

[Sa] M. Santoli, "Methods for pricing pre-earnings equity options and leveraged ETF options", PhD thesis, 2015, Columbia University.