Problem 2. (Dirac input)

Consider the Dirac delta $\delta(t-a)$, with $a \geq 0$. In this exercise, you are asked to *use the Laplace transform* to solve various initial value problems, some of them containing Dirac inputs.

a) Solve the ODE $y'' + y = \delta(t)$, with zero as initial conditions, and show that it has the same solution as the ODE $y'' + y = 0$, with $y(0) = 0$ and $y'(0) = 1$.
Remark: this result illustrates the important fact that, in a second-order system, the application of a Dirac impulse is equivalent to enforcing an "initial velocity".

$$L\{y'\} = sL\{y\} - y_0$$

$$L\{y''\} = s^2 L\{y\} - s y_0 - y_0'$$

$$L\{\delta\} = 1$$

$$\Rightarrow \quad s^2 L\{y\} - s y_0 - y_0' + sL\{y\} - y_0 = 1$$

$$y_0 = 0 \qquad y_0' = 0$$

$$\Rightarrow \quad s^2 L\{y\} + sL\{y\} = 1$$

$$L\{y\}(s^2 + s) = 2$$

$$L\{y\} = \frac{2}{s(s+1)}$$

$$L\{y\} = 2\left(\frac{1}{s} \cdot \frac{1}{s+1}\right)$$

$$y = 2 \cdot (1 * e^{-t})$$

$$= 2 \int_0^t 1 \, e^{\tau} \, d\tau$$

$$= 2 \, e^{\tau} \Big|_0^t$$

$$= \underline{\underline{2(1 - e^{t})}}$$

$$s^2 L\{y\} - s y_0 - y_0' + s L\{y\} - y_0 = 1$$

$$y_0 = 0 \quad y_0' = 1$$

$$\Rightarrow \quad L\{y\} = \frac{2}{s(s+1)}$$

$$\Rightarrow \quad y = \underline{\underline{2(1 - e^{t})}}$$

Q.E.D

b) Solve the ODE $y''' + 8y = 0$, with $y(0) = y'(0) = 0$ and $y''(0) = 1$.
Hint: remember the identity $s^3 + a^3 = (s+a)(s^2 - as + a^2)$.

$$s^3 L\{y\} - s^2 y_0 - s y_0' - y_0'' + 8 L\{y\} = 0$$

$$L\{y\} = \frac{1}{s^3 + 2^3}$$

$$= \frac{1}{(s+2)(s^2 - 2s + 2^2)}$$

$$= \frac{1}{s+2} \cdot \frac{1}{(s-1)^2 + 3}$$

$$y = e^{-2t} * \frac{1}{\sqrt{3}} e^t \sin(\sqrt{3}\, t)$$

$$= \int_0^t e^{-2(t-\tau)} \frac{1}{\sqrt{3}} e^{\tau} \sin(\sqrt{3}\,\tau)\, d\tau$$

$$= \frac{e^{-2t}}{\sqrt{3}} \int e^{3\tau} \sin(\sqrt{3}\,\tau)\, d\tau$$

$$= \frac{e^{-2t}}{\sqrt{3}} \left(\sin(\sqrt{3}\tau) \frac{e^{3\tau}}{3} - \int \frac{e^{3\tau}}{3} \cos(\sqrt{3}\,\tau) \sqrt{3}\, d\tau \right) \Bigg|_0^t$$

$$= \frac{e^{-2t}}{3\sqrt{3}} \left(\sin(\sqrt{3}\,\tau)\, e^{3\tau} - \sqrt{3} \int e^{3\tau} \cos(\sqrt{3}\,\tau)\, d\tau \right) \Bigg|_0^1$$

$$= \frac{e^{-2t}}{3\sqrt{3}} \Big(\sin(\sqrt{3}\,\tau)\, e^{3\tau} - \sqrt{3} \Big(e^{3\tau} \cos(\sqrt{3}\tau)$$

$$+ \sqrt{3} \int e^{3\tau} \sin(\sqrt{3}\,\tau)\, d\tau \Big) \Big) \Bigg|_0^t$$

$$\cancel{\frac{e^{-2t}}{\sqrt{3}}} \int_0^t e^{3\tau} \sin(\sqrt{3}\,\tau)\, d\tau = \cancel{\frac{e^{-2t}}{3\sqrt{3}}} \Big(\sin(\sqrt{3}\,\tau)\, e^{3\tau} - \frac{\sqrt{3}}{3} \Big(e^{3\tau} \cos(\sqrt{3}\tau)$$

$$+ \sqrt{3} \int e^{3\tau} \sin(\sqrt{3}\,\tau)\, d\tau \Big) \Big) \Bigg|_0^t$$

$$\int_0^t e^{3\tau} \sin(\sqrt{3}\,\tau)\, d\tau = \left(\frac{\sin(\sqrt{3}\,\tau)\, e^{3\tau}}{3} - \left(\frac{\sqrt{3}\, e^{3\tau} \cos(\sqrt{3}\tau) + 3 \int e^{3\tau} \sin(\sqrt{3}\,\tau)\, d\tau}{9} \right) \right) \Bigg|_0^t$$

$$9\int_0^t e^{3\tau}\sin(\sqrt{3}\tau)\,d\tau = 3\sin(\sqrt{3}\tau)e^{3\tau} - \sqrt{3}\cos(\sqrt{3}\tau)e^{3\tau} - 3\int e^{3\tau}\sin(\sqrt{3}\tau)\,d\tau\Big|$$

$$\int_0^t e^{3\tau}\sin(\sqrt{3}\tau)\,d\tau = \frac{1}{12}\left(3\sin(\sqrt{3}\tau)e^{3\tau} - \sqrt{3}\cos(\sqrt{3}\tau)e^{3\tau}\right)\Big|_0^t$$

$$= \frac{1}{12}\left(3\sin(\sqrt{3}t)e^{3t} - \sqrt{3}\cos(\sqrt{3}t)e^{3t} + \sqrt{3}\right)$$

$$\Rightarrow \frac{e^{-2t}}{\sqrt{3}}\int_0^t e^{3\tau}\sin(\sqrt{3}\tau)\,d\tau$$

$$= \frac{e^{-2t}}{\sqrt{3}\cdot 12}\left(3\sin(\sqrt{3}t)e^{3t} - \sqrt{3}\cos(\sqrt{3}t)e^{3t} + \sqrt{3}\right)$$

$$= \frac{\sin(\sqrt{3}t)e^{t}}{4\sqrt{3}} - \frac{\cos(\sqrt{3}t)e^{t}}{12} + \frac{e^{-2t}}{12}$$

$$= \frac{1}{12}\left(\sqrt{3}\sin(\sqrt{3}t)e^{t} - \cos(\sqrt{3}t)e^{t} + e^{-2t}\right)$$

b) Solve the ODE $y''' + 8y = 0$, with $y(0) = y'(0) = 0$ and $y''(0) = 1$.
Hint: remember the identity $s^3 + a^3 = (s + a)(s^2 - as + a^2)$.

c) Solve the ODE $y''' + 8y = \delta(t-1)$, with $y(0) = y'(0) = y''(0) = 0$.
Hint: see if you can reuse some of the computations done for item b).

$$s^3 L\{y\} - s^2 y_0 - s y_0' - y_0'' + 8L\{y\} = e^{-s}$$

$$L\{y\} = \frac{e^{-s}}{s^3 + 2^2}$$

$$\Rightarrow \quad y = \frac{1}{12}\left(-\sqrt{3}\,\sin(-\sqrt{3}(t-1))\,e^{t-1} - \cos(-\sqrt{3}(t-1))\,e^{t-1} + e^{-2t+2}\right.$$

Problem 3. (Numerical differentiation - 4D only)

Let

$$u'(x) = \frac{3u(x) - 4u(x-h) + u(x-2h)}{2h} + e(h).$$

a) Find an expression for the error $e(h)$.

$$u(x-h) = u(x) - h\,u'(x) + \frac{h^2 u''(x)}{2} - \frac{h^3 u^{(3)}(x)}{6} + O(h^4)$$

$$u(x-2h) = u(x) - 2h\,u'(x) + 2h^2 u''(x) - \frac{8h^3 u^{(3)}(x)}{6} + O(h^4)$$

$$\Rightarrow \cancel{u'}(x) = \frac{2h\cancel{u'}(x) - \frac{\cancel{2}h^3 u^{(3)}(x)}{3}}{\cancel{2}h} + e(h)$$

$$\Rightarrow e(h) = \boxed{\frac{h^2 u^{(3)}(x)}{3} + O(h^3)}$$

b) Let $u(x) = x\cos(x)$. Use the expression above to find approximations to $u'(x)$ at $x = \pi/2$, using $h = 0.1$. How large is the error $|e(h)|$ in this case?

$$u(x) = x\cos(x)$$

$$x = \frac{\pi}{2}, \quad h = 0{,}1$$

$$u'(x) = \cos(x) - x\sin(x)$$

$$u''(x) = -\sin(x) - \sin(x) - x\cos(x)$$

$$u^{(3)}(x) = -2\cos(x) - \cos(x) + x\sin(x)$$

$$= -3\cos(x) + x\sin(x)$$

$$u^{(3)}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\Rightarrow e(0{,}1) = \frac{(0{,}1)^2}{3} \cdot \frac{\pi}{2} = \underline{\underline{\frac{1}{600}\pi}} \approx 5{,}235 \cdot 10^{-3}$$

Problem 4. (Convolution)

For two functions $f(t)$ and $g(t)$, consider the convolution defined as

$$f * g := \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau = \int_0^t g(\tau)f(t-\tau)\,\mathrm{d}\tau\,.$$

a) Compute $\mathrm{e}^{-at} * \cos\omega t$, with a and ω being two (given) real constants.

$$e^{-at} * \cos(w t)$$

$$= \int_0^t e^{-a(t-\tau)} \cos(w\tau)\, d\tau$$

$$= e^{-at} \int_0^t e^{a\tau} \cos(w\tau)\, d\tau$$

$$\int e^{a\tau} \cos(w\tau)\, d\tau$$

$$= \left(\frac{e^{a\tau}}{a} \cos(w\tau) - \frac{w}{a}\left(\frac{e^{a\tau}}{a} \sin(w\tau) - \frac{w}{a} \int e^{a\tau} \cos w\tau \right)\right)\Bigg|_0^t$$

$$= \left(\frac{e^{a\tau} \cos w\tau}{a} - \frac{w e^{a\tau} \sin(w\tau)}{a^2} + \frac{w^2}{a^2} \int e^{a\tau} \cos w\tau\, d\tau \right)\Bigg|_0^t$$

$$\Rightarrow \int_0^t e^{a\tau}\cos(w\tau)\,d\tau = \frac{ae^{at}\cos wt - we^{at}\sin(wt) - ae^{at}}{a^2\left(1-\frac{w^2}{a}\right)}$$

$$\Rightarrow e^{-at} * \cos(wt) = \cancel{e^{-at}}\left(\frac{\cancel{e^{at}}a\cos wt - w\cancel{e^{at}}\sin(wt) - a}{a^2 - w^2}\right)$$

$$= \frac{a\cos(wt) - w\sin(wt) - ae^{-at}}{a^2 - w^2}$$

b) Compute $t * t^n$, with n being a given natural number.

$$t * t^n = \int_0^t (t-\tau)\tau^n \, d\tau$$

$$= t\int_0^t \tau^n \, d\tau - \int_0^t \tau^{n+1} \, d\tau$$

$$= t\left(\frac{\tau^{n+1}}{n+1}\right)\Bigg|_0^t - \frac{\tau^{n+2}}{n+2}\Bigg|_0^t$$

$$= \frac{t^{n+2}}{n+1} - \frac{t^{n+2}}{n+2}$$

$$= \frac{t^{n+2}}{n^2+3n+2}$$

c) Using convolution, find the inverse Laplace transform of $F(s) = \frac{s}{(s^2+1)^2}$

$$= \frac{s}{s^2+1} \cdot \frac{1}{s^2+1}$$

$$\Rightarrow y = \cos(t) * \sin(t)$$

$$= \int_0^t \cos(t-\tau)\sin(\tau)\, d\tau$$

$$= \cos(t) \int_0^t \cos(\tau)\sin(\tau)\, d\tau + \sin(t) \int_0^t \sin^2(\tau)\, d\tau$$

$$= \cos(t) \left(\frac{\sin^2(\tau)}{2}\right)\Bigg|_0^t + \frac{\sin t}{2} \int_0^t 1 - \cos(2\tau)\, d\tau$$

$$= \frac{\cos(t)\sin^2(t)}{2} + \frac{\sin(t)}{2}\left(t - \frac{\sin(2t)}{2}\right)$$

$$= \frac{\cos(t)\sin^2 t + \sin(t)\, t}{2} - \frac{\sin(t)\sin(2t)}{4}$$

$$= \frac{\cancel{\cos(t)\sin^2(t)} + \sin(t)\,t}{2} - \frac{\cancel{2\sin^2(t)\cos t}}{\cancel{4}}$$

$$= \underline{\underline{\frac{\sin(t)\,t}{2}}}$$