PYTHON FOR THE FINANCIAL ECONOMIST, ORDINARY EXAM 2022
PORTFOLIO DIVERSIFICATION AND OPTIMAL PORTFOLIOS

Copenhagen Business School
19. December 2022

2 weeks, home assignment

The home assignment is to be answered in groups of two students (maximum of 25 A4-pages) or individually (maximum of 15 A4-pages). The students must individualize the assignment.

The take-home assignment should take the form of an academic report written in either Word or Latex converted to pdf format. It is expected that students present relevant formulas, present results using visualizations and tables, include relevant references, etc. The overall impression of how the results are presented will count in the assessment.

The analysis should be performed using Python. Please attach code / Jupyter notebooks.

If you think that you do not have all the necessary information to answer a problem, make the necessary assumptions in order to proceed and state these assumptions in the solution.

Good luck!

Portfolio diversification

Diversification is key in asset allocation, but how do we measure it? The following will ask you to explore different ways of measuring diversification and how we may use diversification in our asset allocation process. We will be using the 49 Industry Portfolios from the Kenneth R. French data library from the beginning of 1970.

A simplistic or naive view of portfolio diversification is to look at the number of constituents in the portfolios or the portfolio weights. One possibility is to use the norm of the portfolio weight vector, $\mathbf{w}$, to determine the Effective Number of Constituents (ENC) (valid for any $\alpha > 0$ if weights are all positive)

$$\mathrm{ENC}_\alpha(\mathbf{w}) = \|\mathbf{w}\|_\alpha^{\frac{\alpha}{1-\alpha}} = \left(\sum_{i=1}^N w_i^\alpha\right)^{\frac{1}{1-\alpha}}, \quad \alpha > 0, \; \alpha \neq 1$$

where N denote the number of assets. If $\alpha = 2$, the diversification measure is equal to the inverse of the Herfindahl index

$$\mathrm{ENC}_2(\mathbf{w}) = \frac{1}{\sum_{i=1}^N w_i^2}$$

In the limiting case when $\alpha \to 1$, we obtain the exponential of the entropy of the distribution of the portfolio weight vector

$$\mathrm{ENC}_1 = \exp\left(-\sum_{i=1}^N w_i \ln w_i\right)$$

Based on monthly data, calculate the market cap weights (sector weights) for the industry portfolios for each month and visualize the development of market cap weights over time. Based on the sector weights, calculate ENC_1 and ENC_2 for each month and visualize the development of the diversification measures over time. Comment on your findings, e.g. which sectors dominate and how far are the values of the diversification measures from their largest possible value.

The problem with the ENC measure is that it does not take potentially large differences in volatilities (or other risk measures) and the correlation structure into account. Let $\boldsymbol{\Sigma}$, $\mathbf{v}$ and $\mathbf{C}$ denote respectively the covariance matrix, vector of volatilities, and the correlation matrix of asset returns. To understand the effect of differences in risk across industry portfolios, we want to plot the relative risk constributions over time for an investment strategy that allocates equally to the industry portfolios. Remember that we by using the Euler principle can decompose the portfolio standard deviation as (see e.g. Bruder and Roncalli (2012))

$$\sigma_P(\mathbf{w}) = \mathbf{w}^\top \frac{\boldsymbol{\Sigma}\mathbf{w}}{\sqrt{\mathbf{w}^\top\boldsymbol{\Sigma}\mathbf{w}}} = \sum_{i=1}^N w_i \frac{(\boldsymbol{\Sigma}\mathbf{w})_i}{\sqrt{\mathbf{w}^\top\boldsymbol{\Sigma}\mathbf{w}}}$$

Discuss the choice of data frequency, initial sample length, covariance estimator and how to update the estimates when adding more data. Apply your recipe and visualize the evolution of the relative risk contribution over time. Comment on your findings.

Choueifaty and Coignard (2008) suggest the following diversification measure

$$\mathrm{CC}(\mathbf{w}, \boldsymbol{\Sigma}) = \frac{\mathbf{w}^\top \mathbf{v}}{\sqrt{\mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}}} = \frac{\sum_{i=1}^{N} w_i \sigma_i}{\sqrt{\mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}}}$$

Explain the intuition behind the formula and implement a function able to calculate the diversification measure. Plot the evolution of the diversification measure over time for an equally weighted portfolio. Explain the intuition behind your findings.

Choueifaty and Coignard (2008) suggest to select portfolios by maximizing the diversification ratio, i.e. creating the most diversified portfolio. Consider the maximization problem

$$\begin{aligned} \max_{\mathbf{w}} \quad & \mathrm{CC}(\mathbf{w}, \boldsymbol{\Sigma}) \\ \text{s.t.} \quad & \mathbf{w}^\top \mathbf{1} = 1 \\ & w_i \geq 0, \ i = 1, ..., N \end{aligned}$$

where we restrict ourselves to long-only portfolios. We want to compare the most diversified portfolios with the minimum-variance portfolios, where the latter are given by solution to the minimization problem

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^\top \mathbf{1} = 1 \\ & w_i \geq 0, \ i = 1, ..., N \end{aligned}$$

Solve the optimization problems numerically and visualize the evolution of the most diversified and minimum variance portfolio weights over time. Also compare the diversification measures for the three portfolios (most diversified, minimum variance, equal weighted) over time. Comment on your findings.

Finally, we want to perform a simple evaluation of the three portfolio optimization strategies: Most diversified, minimum variance, equal weighted portfolios. Plot the cumulative return indices of the portfolios. Calculate the average return, standard deviation, information ratio (compared to a zero return benchmark)

$$\mathrm{IR} = \frac{\mathrm{E}[R_p]}{\sigma_P}$$

where R_p is the portfolio return and σ_P is the portfolio standard deviation (need to be replaced with estimates), and turn-over

$$\text{Turn-Over} = \frac{1}{T_{OOS} - 1} \sum_{t=1}^{T_{OOS}-1} \sum_{i=1}^{N} |\tilde{w}_{i,t+1} - w_{i,t+1}|$$

where $T_{OOS} - 1$ is the number of re-balancing dates and

$$\tilde{w}_{i,t+1} = \frac{w_{i,t}(1 + R_{i,t+1})}{1 + R_{p,t+1}}$$

is the portfolio weight of asset i immediately before re-balancing. Discuss your findings.

Portfolio Optimization

Consider an investor endowed with an initial wealth $W_0 = 5,000,000$ €. The investor can invest in 10 different assets: Government bonds, Investment-grade bonds, High-yield bonds, Emerging markets gov. bonds, Equities (developed markets), Equities (Emerging markets), Private equity, Infrastructure, Real Estate, and Hedgefunds. The yearly compound / log-returns are assumed to be i.i.d. multivariate normally distributed

$$\mathbf{r}_{t+1,1} \sim MVN(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

with correlation matrix

$$\mathbf{C} = \begin{bmatrix} 1.0 & 0.6 & 0.1 & 0.3 & -0.1 & -0.1 & -0.2 & -0.1 & -0.1 & -0.1 \\ 0.6 & 1.0 & 0.6 & 0.6 & 0.2 & 0.2 & 0.2 & 0.1 & 0.1 & 0.3 \\ 0.1 & 0.6 & 1.0 & 0.7 & 0.7 & 0.6 & 0.6 & 0.4 & 0.3 & 0.7 \\ 0.3 & 0.6 & 0.7 & 1.0 & 0.5 & 0.6 & 0.4 & 0.2 & 0.2 & 0.5 \\ -0.1 & 0.2 & 0.7 & 0.5 & 1.0 & 0.7 & 0.8 & 0.4 & 0.4 & 0.8 \\ -0.1 & 0.2 & 0.6 & 0.6 & 0.7 & 1.0 & 0.7 & 0.4 & 0.4 & 0.7 \\ -0.2 & 0.2 & 0.6 & 0.4 & 0.8 & 0.7 & 1.0 & 0.4 & 0.4 & 0.7 \\ -0.1 & 0.1 & 0.4 & 0.2 & 0.4 & 0.4 & 0.4 & 1.0 & 0.3 & 0.4 \\ -0.1 & 0.1 & 0.3 & 0.2 & 0.4 & 0.4 & 0.4 & 0.3 & 1.0 & 0.4 \\ -0.1 & 0.3 & 0.7 & 0.5 & 0.8 & 0.7 & 0.7 & 0.4 & 0.4 & 1.0 \end{bmatrix},$$

vector of volatilities

$$\mathbf{v} = \left[0.037, 0.055, 0.119, 0.107, 0.153, 0.217, 0.204, 0.14, 0.108, 0.094\right]^\top,$$

and expected log return

$$\boldsymbol{\mu} = \left[0.019, 0.022, 0.049, 0.043, 0.061, 0.083, 0.102, 0.056, 0.041, 0.038\right]^\top - \frac{1}{2}\text{Diag}(\boldsymbol{\Sigma}),$$

The investor currently hold an equally weighted portfolio of the assets such that

$$W_0 = \boldsymbol{\alpha}_0^\top \mathbf{P}_0$$

where $\boldsymbol{\alpha}_0$ is a vector of asset holdings and $\mathbf{P}_0 = \mathbf{1}$ is the vector of initial asset prices (assumed to all be equal to one €). Thus, $\alpha_{0,i} = 500,000,\ i = 1, ..., N$.

The investor seeks to choose an optimal allocation $\boldsymbol{\alpha} \geq \mathbf{0}$ which is held as a buy-and-hold portfolio for the next 5 years. Based on the above information what is the distribution of the vector of asset prices in 5 years, $\mathbf{P}_5$? What is the expected value and covariance of the asset prices in 5 years? What is the distribution of the wealth in 5 years $W_5(\boldsymbol{\alpha}) = \boldsymbol{\alpha}^\top \mathbf{P}_5$? Visualize the evolution of wealth for the next five years using the initial allocation.

The investor is faced with linear transaction costs defined by the function

$$T(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}) = \mathbf{k}^\top |\boldsymbol{\alpha}_0 - \boldsymbol{\alpha}|$$

where $\mathbf{k}$ is a vector of positive entries. Assume that $k_i = 0.02$ for all $i = 1, ..., N$. Plot the transaction costs for a single asset.

The investor has a power utility function with $\gamma = -9$

$$U(W_5(\boldsymbol{\alpha})) = \frac{W_5(\boldsymbol{\alpha})^\gamma}{\gamma}$$

such that we can define the satisfaction measure (the certainty equivalent)

$$S(\boldsymbol{\alpha}) = \left(\gamma \mathrm{E}\left[\frac{W_5(\boldsymbol{\alpha})^\gamma}{\gamma}\right]\right)^{1/\gamma}$$

To solve the problem, the investor wants to use mean-variance optimization to solve for the optimal allocation, e.g. maximize the expected wealth for a given variance / standard deviation target or minimize the variance / standard deviation for a given wealth target. Write up the optimization problem when taking the relevant constraints listed above into account. Solve the optimization problem for a range of relevant standard deviation / variance targets or wealth targets. Explain how to calculate the satisfaction measure for each optimized portfolio and perform the calculations. What is the optimal allocation given the investor's preferences? Present relevant intuition and visualizations.

Visualize the evolution of wealth for the next five years for the optimal portfolio. How does it compare with the visualization for the initial portfolio?

How can we calculate the risk contribution of each asset using standard deviation and CVaR of final wealth? Perform the calculations and comment on your findings.

References

Bruder, B. and T. Roncalli (2012): "Managing Risk Exposures Using the risk Budgeting Approach," *SSRN working paper.*

Choueifaty, Y. and Y. Coignard (2008): "Toward Maximum Diversification," *The Journal of Portfolio Management.*