Solution of quadratic systems of bosons

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Here we derive the solution to a finite quadratic systems of bosons.

In these notes we seek to find the solution to a quadratic system coupled to a Gaussian bath. We consider the Hamiltonian

$$H_S = \sum_{mn} H_{mn} A_m A_n \tag{1}$$

where $\{A_m\}$ are Hermitian operators with scalar commutation relations: $[A_m, A_n] = C_{mn}$, where C is an imaginary scalar. Without loss of generality, we may take C to be i times the symplectic matrix, and H to be real and symmetric. I.e. the As can be divided into canonically-conjugate pairs of positions and momenta. We work in this basis below.

We seek to solve the system by obtaining the Heisenberg picture evolution of A, $A(t) \equiv U^\dagger(t) A U(t)$, where $U(t)$ denotes the time-evolution operator of the system. Here and in the following, $\mathcal{O}(t)$ denotes the Hesienberg time evolution of the (Schrodinger picture) operator $\mathcal{O}$: $\mathcal{O}(t) \equiv U^\dagger(t)\mathcal{O}U(t)$. We use this to obtain the correlation functions and evolutions of the system.

First, we consider the equation of motion for A_n.

$$\partial_t A_n(t) = iU^\dagger(t)[H_S, A_n]U(t). \tag{2}$$

Note that $[H_S, A_n] = \sum_{kl} H_{kl}(C_{kn}A_l + C_{ln}A_k)$. Using that H is symmetric and C is complex and antisymmetric, $[H_S, A_n] = -2\sum_{km} C_{nl}H_{lm}A_m$. Then

$$[H_S, A_n(t)] = -2i\sum_m (CH)_{nm} A_m(t), \tag{3}$$

Thus, with $A = (A_1, \dots A_N)$,

$$\partial_t A(t) = -2iCH(t)A(t) \tag{4}$$

This equation has the solution

$$A(t) = G(t)A(0), \quad G(t) \equiv \mathcal{T} e^{-2i\int_0^t \mathrm{d}t' CH(t')} \tag{5}$$

Note that the propagator (or evolution operator) $G(t)$ is not unitary, since CH is not Hermitian. However, G is real-valued, since H is real and C is imaginary. For positive definite Hamiltonians (in a canonical basis), it seems (though we cannot show it) that G has complex phases as eigenvalues, but its eigenvectors are not orthogonal. Equivalently, for positive definite systems, it seems that CH has real eigenvalues only.

Now let us look at the correlation functions, expectation values and spectral functions. Since the system is quadratic, everything is determined by the 1- and 2-point correlation functions.

A. Expectation values (1-point function).

We find

$$\langle A(t)\rangle = G(t)\langle A(0)\rangle. \tag{6}$$

Note that in a thermal state, the right-hand side is independent of temperature.

B. 2-point function

Next, consider the correlation function:

$$\langle A_m(t_1)A_n(t_2)\rangle = \sum_{kl} G_{mk}(t_1)G_{nl}(t_2)\langle A_k(0)A_l(0)\rangle \tag{7}$$

Letting $K(t_1,t_2) \equiv \langle A(t_1) \circ A(t_2)\rangle$ denote the matrix above (here $\circ$ denotes the outer vector product), such that $K_{mn}(t_1,t_2) \equiv \langle A_m(t_1) A_n(t_2)\rangle$, we thus find

$$K(t_1,t_2) = G(t_1) K_0 G^T(t_2) \tag{8}$$

with $K_0 \equiv K(0,0)$ encoding the initial conditions.

C. Correlation function

Next, we consider the correlation function, which is the 2-point function after transients has been removed:

$$J(t) \equiv \lim_{t_0\to\infty} \frac{1}{t_0} \int_0^{t_0} \mathrm{d}t'\, K(t'+t,t'), \tag{9}$$

We consider this for time-independent systems only, where $G(t) = e^{-2iCHt}$. Letting $\{|v_n\rangle\}$ denote the right-eigenvectors of CH, and $\{|\bar{v}_n\rangle\}$ the dual vectors, while ε_n denotes the eigenvalue,

$$G(t) = \sum_n |v_n\rangle\langle\bar{v}_n| e^{-i\varepsilon_n t} \tag{10}$$

where for positive-definite Hamiltonians, ε_n is real-valued. Since G is real-valued, for each eigenvector $|v_n\rangle$, its complex conjugate $|v_n^*\rangle$ is also an eigenvector with energy $-\varepsilon_n$. Thus we find

$$K(t_0+t,t_0) = \sum_{nm} |v_n\rangle\langle\bar{v}_n| M_0 (|v_m\rangle\langle\bar{v}_m|)^T e^{-i(\varepsilon_n+\varepsilon_m)t_0 - i\varepsilon_n t} \tag{11}$$

Using the symmetry of the eigenvectors and spectrum described above,

$$K(t_0+t,t_0) = \sum_{nm} |v_n\rangle\langle\bar{v}_n| M_0 |\bar{v}_m\rangle\langle v_m| e^{-i(\varepsilon_n-\varepsilon_m)t_0 - i\varepsilon_n t} \tag{12}$$

where we used $(|v_m\rangle\langle\bar{v}_m|)^\dagger = |\bar{v}_m\rangle\langle v_m|$.

For large t_0 the terms with $m \neq n$ acquire random phases and cancel out. Equivalently, they have zero time-average. Thus we see that only terms with $n = m$ survive when for large t_0/averaged t_0:

$$J(t) = \sum_n |v_n\rangle\langle\bar{v}_n| M_0 |\bar{v}_n\rangle\langle v_n| e^{-i\varepsilon_n t} \tag{13}$$

D. Spectral function

Finally there's the spectral function. The spectral function is the Fourier transform of the correlation function:

$$J(\omega) = \sum_n |v_n\rangle\langle\bar{v}_n| M_0 |\bar{v}_n\rangle\langle v_n| \delta(\omega - \varepsilon_n) \tag{14}$$

This concludes these notes.