Universal Lindblad equation for open quantum systems

Frederik Nathan and Mark S. Rudner

Center for Quantum Devices, Niels Bohr Institute, University of Copenhagen, 2100 Copenhagen, Denmark

We develop a Markovian master equation in the Lindblad form that enables the efficient study of a wide range of open quantum many-body systems that would be inaccessible with existing methods. The validity of the master equation is based entirely on properties of the bath and the system-bath coupling, without any requirements on the level structure within the system itself. The master equation is derived using a Markov approximation that is distinct from that used in earlier approaches. We provide a rigorous bound for the error induced by this Markov approximation; the error is controlled by a dimensionless combination of intrinsic correlation and relaxation timescales of the bath. Our master equation is accurate on the same level of approximation as the Bloch-Redfield equation. In contrast to the Bloch-Redfield approach, our approach ensures preservation of the positivity of the density matrix. As a result, our method is robust, and can be solved efficiently using stochastic evolution of pure states (rather than density matrices). We discuss how our method can be applied to static or driven quantum many-body systems, and illustrate its power through numerical simulation of a spin chain that would be challenging to treat by existing methods.

The theoretical description of a quantum system interacting with an environment is an important problem of both fundamental and practical interest. The problem arises in a diverse array of settings, from chemistry to atomic, molecular, and optical physics, as well as condensed matter physics, high-energy physics, and quantum information processing [1–8]. Due to the importance and long history of the problem, there exists a wide range of well-established approaches for describing the dynamics of open quantum systems, see, e.g., Refs. [9–17].

The Nakajima-Zwanzig (NZ) approach [11,12] provides a systematic framework for describing the evolution of open quantum systems. Although formally exact in its most general form, in practice there are many challenges associated with application of the NZ equation, even in approximate form. For example, the Bloch-Redfield (BR) equation, which emerges as a lowest-order approximation to the time-convolutionless NZ equation, is not guaranteed to preserve positivity of the density matrix of the system and may therefore yield unphysical solutions for long time evolution, with negative or diverging probabilities. Moreover, solving these (NZ or BR) equations requires working with the density matrix of the system, whose dimension is the square of that of the system's Hilbert space. This requirement may make their numerical solution prohibitively expensive, even for moderately sized quantum systems [9].

For Markovian systems where the correlation (or "memory") time of the bath is sufficiently short, Lindblad-form master equations provide an alternative to the NZ approach [9]. The Lindblad form is the most general form of a time-local evolution equation that is guaranteed to preserve the trace and positivity of the density matrix [18,19]. Importantly, the Lindblad form also admits efficient numerical solution via stochastic evolution of *pure states* [9,20–22], thus avoiding the computational cost of working with density matrices. However, derivations of Lindbladian master equations, such as the quantum optical master equation [15], typically require stringent conditions on the level spacing of the system itself, thus limiting their applicability to specific classes of systems. In particular, the quantum optical master equation relies on the rotating wave approximation (RWA), and hence is only valid when the level broadening arising from bath-induced transitions is small compared with the smallest level spacing in the system. While this condition is well satisfied in many important cases, for example in atomic physics and quantum optics, many types of systems (including many-body systems with dense spectra) and physical phenomena (such as Fano resonances) cannot be described through this approach.

Our motivation in the present work is based on the following notion: when the correlation time of the bath is much shorter than a characteristic timescale of system-bath interactions, we heuristically expect that the evolution of the system should be generated by a Markovian master equation. Hence, Markovianity should be a property of environment alone, independent of details of the system itself. Noting that a Markovian master equation for the density matrix must be in the Lindblad form, we thus seek to systematically derive a Lindbladian master equation without reference to any details of the system other than the operator(s) through which it couples to its environment.

The main result of this paper is the derivation of a "universal Lindblad equation" (ULE) that can be applied to *any* open quantum system whose bath satisfies a particular Markovianity condition that is defined in terms of the bath spectral function and the system-bath coupling strength. In particular, the derivation of the ULE does not rely on the

rotating wave approximation or any other assumption about the energy level spacings of the system. We provide explicit expressions for the jump operators, and discuss their evaluation for static, Floquet, and arbitrarily driven many-body systems. Importantly, the number of jump operators is equal to the number of independent terms (referred to as quantum noise channels) that couple the system and bath, independent of the details of the system. As a result, for many cases, the ULE features only one or a few jump operators. The jump operators are straightforward to compute, either through exact diagonalization, or controlled expansions.

The principle underlying our derivation of the ULE is that there is no unique way of implementing a Markov approximation in the evolution of the density matrix. Instead, there exists a continuous family of distinct approximations that result in Markovian dynamics of the system, all with error bounds of the same order in a dimensionless Markovianity parameter (see below). One particular choice out of this family of comparable Markov approximations leads to the Bloch-Redfield equation. In this paper we employ a different Markov approximation from within this family which directly leads to a Lindblad-form master equation without any further assumptions about the nature of the system.

We provide rigorous bounds on the relative error induced by making the Markovian approximation that results in the ULE. The error is controlled by a dimensionless "Markovianity" parameter, defined from a combination of correlation and relaxation timescales that we identify from the bath and its coupling to the system. We show that this error is of the same order as that incurred in deriving the Bloch-Redfield equation. Unlike the BR equation, however, the ULE preserves the physicality (i.e., positivity and normalization) of the density matrix. Hence it is intrinsically robust and amenable to solution using efficient stochastic methods [9,20–22].

The universal Lindblad equation that we present here can be used for a wide range of physical situations. In particular, it can be used to efficiently simulate the dynamics of open and noisy quantum many-body systems (i.e., systems with large Hilbert space dimension and small level spacing). In addition, it provides a straightforward, general approach for describing the dynamics of driven systems coupled to Markovian baths.

A master equation of the same form as we derive here was previously employed with phenomenological justification in Ref. [23]. More recently, a similar master equation was also heuristically obtained in Ref. [24]. Here we provide a systematic, rigorous derivation of the universal Lindblad equation, and in particular show that it captures the dynamics of the system at the same level of error as the Bloch-Redfield equation. (Some of our arguments appeared in a preliminary, heuristic derivation in the Ph.D. thesis of one of the present authors [25].) The ansatz in Ref. [23] applies to systems coupled to independent bath observables, with static or weakly time-dependent Hamiltonians, such that the jump operators can be computed within a quasistatic approximation for the system Hamiltonian. Our approach covers systems with arbitrary time dependence and system-bath couplings, and in particular applies beyond the regime where the quasistatic approximation is valid.

Recently, another group of authors has also obtained a Lindblad-form master equation for open quantum systems

whose validity is independent of the details of the system [17]. The master equation of Ref. [17] is distinct from the ULE that we obtain, and was derived using a time–coarse-graining approach that is of a fundamentally different nature from the Markov approximation that we employ here. Interestingly, the error bounds obtained by the authors of Ref. [17] were defined in terms of a closely related Markovianity parameter to the one we identify here (see Appendix B). The ULE we derive is thus valid on an equivalent level of approximation as the master equation of Ref. [17]. The simultaneous validity of these two distinct master equations reflects the nonuniqueness of the Markov approximation discussed above.

The rest of this paper is organized as follows. In the main text we discuss the essential ideas of our work, while we provide technical details and derivations in several appendixes. In Sec. I we provide a summary of our main results. In Sec. II we formally introduce the general model of open quantum systems that we study, review existing approaches to analyzing the dynamics of this class of systems, and present important auxiliary results that are used to derive the ULE. In Sec. III we derive the ULE, allowing for multiple baths and arbitrarily time-dependent system Hamiltonians. In Sec. IV we discuss how to calculate and implement the jump operators of the ULE for a range of relevant special cases, including systems with time-independent Hamiltonians, periodically driven systems, and systems where exact diagonalization of the Hamiltonian is not feasible. In Sec. V we demonstrate our approach via numerical simulations of a spin chain coupled to two baths at different temperatures. We conclude with a discussion in Sec. VI.

I. SUMMARY OF RESULTS

In this section we summarize the main ideas and results of this paper. We investigate the dynamics of a quantum system $\mathcal{S}$ connected to an external environment (bath) $\mathcal{B}$. For simplicity, in this section we illustrate our results for the case where the system's Hamiltonian $H_\mathcal{S}$ is time independent, and the system and bath are connected through a single term $H_{\text{int}} = \sqrt{\gamma} X B$ in the combined system-environment Hamiltonian. Here X and B are observables of the system and bath, respectively, and the energy γ denotes the system-bath coupling strength, normalized such that X has unit spectral norm [26,27]. The results we present below for this system are derived and discussed in detail in Secs. II and III, where we also extend our results to general system-bath couplings and time-dependent system Hamiltonians $H_\mathcal{S}(t)$.

In Sec. III we seek conditions on the bath under which the time evolution of the reduced density matrix of the system ρ takes the Lindblad form [18,19,28]:

$$\partial_t \rho = -i[H_\mathcal{S} + \Lambda, \rho] - \tfrac{1}{2}\{L^\dagger L, \rho\} + L\rho L^\dagger. \quad (1)$$

In the above, L is known as the jump operator, and determines the dissipative component of the system's evolution, while the Hermitian Lamb shift Λ accounts for the renormalization of the Hamiltonian due to the system-bath coupling. Note that we set $\hbar = 1$ throughout.

Importantly, the conditions for the Lindblad-form master equation that we identify in Sec. III are formulated purely in terms of properties of the bath and the system-bath coupling

strength. This situation stands in contrast to the quantum optical master equation, which is also a Lindblad-form master equation but is only valid under additional stringent requirements on the level spacing of the *system* itself [9].

In a wide range of situations, all information of the bath $\mathcal{B}$ required to determine the evolution of ρ is contained in the bath spectral function $J(\omega)$ [9] (see Sec. II A for definition). The conditions we obtain in Sec. III for the Lindblad-form master equation are also expressed in terms of this function: from $J(\omega)$ we identify an energy scale Γ and timescale τ whose dimensionless product $\Gamma\tau$ serves as a measure of Markovianity. As the main result of this paper, we show that, when $\Gamma\tau \ll 1$, the time evolution of ρ is accurately described by a (Markovian) master equation in the Lindblad form [Eq. (1)], with the single jump operator

$$L = \sum_{mn} \sqrt{2\pi \gamma J(E_n - E_m)} X_{mn} |m\rangle\langle n|. \quad (2)$$

Here $\{|n\rangle\}$ and $\{E_n\}$ denote the eigenstates and energies of the system Hamiltonian H_S (not including the Lamb shift), respectively, while $X_{mn} \equiv \langle m|X|n\rangle$. The Lamb shift Λ is proportional to γ and is defined from X and the bath spectral function in Eq. (34) below. Stated more precisely, as we show in Sec. III, the time derivative of ρ is given by Eqs. (1) and (2), up to a correction of order $\Gamma^2\tau$. In comparison, the magnitude of the right-hand side of Eq. (1) is typically well estimated by Γ (hence the correction is smaller by a factor $\Gamma\tau$). Due to its system-independent applicability, we refer to the master equation in Eqs. (1) and (2), along with its generalizations in Sec. IV, as the *universal Lindblad equation*.

The two quantities Γ and τ that determine the accuracy of the universal Lindblad equation [Eqs. (1) and (2)] are associated with the bath spectral function $J(\omega)$ and system-bath coupling γ. Specifically, Γ and τ are derived from a related function $g(t)$ that we call the "jump correlator." The jump correlator is defined via its Fourier transform $g(\omega)$ as the square root of the spectral function: $J(\omega) = 2\pi[g(\omega)]^2$. In time domain this gives

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \sqrt{J(\omega)} e^{-i\omega t}. \quad (3)$$

From this jump correlator, Γ and τ are given by

$$\Gamma = 4\gamma \left[\int_{-\infty}^{\infty} dt \, |g(t)| \right]^2, \quad \tau = \frac{\int_{-\infty}^{\infty} dt \, |g(t)t|}{\int_{-\infty}^{\infty} dt \, |g(t)|}, \quad (4)$$

where γ denotes the system-bath coupling strength. As we explain in Sec. II B, the timescale τ can be seen as a measure of the characteristic correlation time of the bath observable B, while Γ sets an upper bound for the rate of bath-induced evolution of the system, independent of any approximation. In the limit $\Gamma\tau \ll 1$, where the ULE is valid, the correlations of the bath decay rapidly on the characteristic timescale of system-bath interactions Γ^{-1}. In this case, the standard heuristic arguments behind the Markov-Born approximation suggest that the dynamics of the system should be effectively Markovian [9]. The results we obtain here hence put this intuition on rigorous footing, independent of properties of the system itself.

We note that the bath-induced terms in Eq. (1) scale linearly with the system-bath coupling strength γ. In contrast, the correction to Eqs. (1) and (2) we identified above is of order $\Gamma^2\tau$, and thus scales as γ^2. Hence, when the coupling-independent quantities Γ/γ and τ are finite, a small enough value of γ can in principle always be found such that the system's dynamics are Markovian and well described by the universal Lindblad equation in Eq. (1). In this way, the condition $\Gamma\tau \ll 1$ gives a well-defined notion of the weak-coupling limit.

We demonstrate in Sec. II A 2 that the Bloch-Redfield equation is also valid up to a correction of order $\Gamma^2\tau$. In this sense, the ULE is valid on an equivalent level of approximation as the Bloch-Redfield equation.

The universal Lindblad equation in Eqs. (1) and (2) is consistent with existing results for open quantum systems [9]. In particular, the ULE naturally reduces to the quantum optical master equation in the limit where the latter is valid, namely when the rate of system-bath interactions Γ is much smaller than any energy level spacing of the Hamiltonian H_S (i.e., when the rotating wave approximation is valid). However, in contrast to the quantum optical master equation, the derivation of the ULE does not rely on the rotating wave approximation (or any other assumptions about the energy levels of the system); hence it can also be applied beyond the regime where the quantum optical master equation is valid. In addition to being consistent with the quantum optical master equation as explained above, Eqs. (1) and (2) reproduce Fermi's golden rule: Fermi's golden rule states that the transition rate between two energy levels of the Hamiltonian, m and n, is given by $\Gamma_{n\to m} = 2\pi |X_{mn}|^2 J(E_n - E_m)$. This result follows from Eqs. (1) and (2) by identifying $\Gamma_{n\to m} = \langle m|\partial_t \rho(t)|m\rangle|_{t=0}$ when taking $\rho(0) = |n\rangle\langle n|$.

We note that the expression for the jump operator L in Eq. (2) was previously hypothesized in Ref. [23]. Reference [23] showed that the master equation in Eqs. (1) and (2) was consistent with Fermi's golden rule, and reproduced the quantum optical master equation in the regime $\gamma \to 0$ where the latter is valid. Based on these results and numerical demonstrations, Ref. [23] conjectured that the master equation in Eqs. (1) and (2) could accurately describe the evolution of open quantum systems. In this work, by rigorous derivation we recover the hypothesis of Ref. [23], and identify the precise conditions under which the universal Lindblad equation [Eqs. (1) and (2)] holds. Crucially, the conditions we identify rely solely on the properties of the bath and system-bath coupling, and hold well beyond the regime where the quantum optical master equation is valid. In addition, our results generalize the hypothesized master equation from Ref. [23] to arbitrary system-bath couplings and time-dependent Hamiltonians.

II. OPEN SYSTEM DYNAMICS: FORMULATION AND CHARACTERISTIC TIMESCALES

We now set out to derive the universal Lindblad equation by rigorous means, for general open quantum systems. As a first step, in this section we define the model we study and review standard theory for open quantum systems. We moreover present two auxiliary results that play an important

role for the derivation of the ULE: we establish a rigorous upper bound for the correction to the Bloch-Redfield equation (Sec. II A 2, see also Ref. [17]), and obtain an upper bound for the rate of bath-induced quantum evolution (a so-called "quantum speed limit") [Eq. (13)]. These results, which may also be of interest on their own, are derived in Appendix A. The concepts and basic assumptions described in this section will form the foundation for the derivation of the ULE in the next section.

The system we consider in this paper consists of a quantum (sub)system $\mathcal{S}$ which is connected to an external system, referred to as the bath $\mathcal{B}$. The subsystem $\mathcal{S}$ may be anything from a two-level spin to a many-body system, while the bath $\mathcal{B}$ is typically a large system with a dense energy spectrum, such as a phononic or electromagnetic environment, or the fermionic modes in an electronic lead. The bath $\mathcal{B}$ can also consist of several "subbaths" with distinct physical origins and properties. Without loss of generality, the Hamiltonian H of the full system $\mathcal{SB}$ (including the bath) takes the form

$$H = H_{\mathcal{S}} + H_{\mathcal{B}} + H_{\text{int}}, \tag{5}$$

where $H_{\mathcal{S}}$ and $H_{\mathcal{B}}$ are the Hamiltonians of the subsystem and bath, respectively, while H_{int} contains all terms in the Hamiltonian that couple the two. In the following we allow $H_{\mathcal{S}}$ to depend on time, while we assume $H_{\mathcal{B}}$ and H_{int} to be time independent.

It is useful to decompose H_{int} as follows:

$$H_{\text{int}} = \sqrt{\gamma} \sum_{\alpha} X_{\alpha} B_{\alpha}, \tag{6}$$

where, for each α, X_{α} is a dimensionless Hermitian operator on the subsystem $\mathcal{S}$, B_{α} is a Hermitian operator acting on the bath $\mathcal{B}$, with units of $[\text{Energy}]^{1/2}$, and the energy γ parametrizes the system-bath coupling strength (see Ref. [27]). We normalize γ and B_{α} such that X_{α} has unit spectral norm for each α [26]. While γ can still be absorbed into the operators $\{B_{\alpha}\}$, and thus in principle remains arbitrary, we include it in Eq. (6) to highlight the scaling of various quantities with respect to the system-bath coupling in the discussion below. We note that the decomposition above is always possible with a sufficiently high, but finite, number of terms in the sum N. We refer to each such term as a (quantum) noise channel in the following.

For simplicity, in the remainder of this section, and in the derivation of the ULE in Sec. III A, we consider the case where the sum in Eq. (6) consists of a single term, and refer to the system and bath operators as X and B, respectively. In Sec. III C we generalize our results to the case where the sum in Eq. (6) contains multiple terms.

To describe the dynamics of observables in the system $\mathcal{S}$, it is sufficient to know the evolution of the reduced density matrix of $\mathcal{S}$,

$$\rho(t) \equiv \text{Tr}_{\mathcal{B}}[\rho_{\mathcal{SB}}(t)]. \tag{7}$$

Here $\text{Tr}_{\mathcal{B}}$ traces out all the degrees of freedom in $\mathcal{B}$, and $\rho_{\mathcal{SB}}(t)$ denotes the density matrix of the combined system $\mathcal{SB}$. Crucially, it is possible to obtain an equation of motion for $\rho(t)$ which depends only on $H_{\mathcal{S}}$, X, and the statistical properties of the bath. Such an equation of motion is known as a master equation. There exists several approximation schemes for obtaining master equations for ρ (see, for exam-

ple, Refs. [9–15]). While useful in their respective regimes of applicability, each of these methods has its limitations on which cases they may be applied (either due to physical limitations on the regime of applicability, or practical issues associated with numerical implementation). The goal of our paper is to derive a Markovian master equation that can be applied to a wider range of cases, which unifies and extends some of these previous approaches.

A. Born-Markov approximation

Before deriving the universal Lindblad equation, we review one of the existing approaches to obtaining a master equation for ρ, namely the Born-Markov approximation. This standard approach leads to a master equation for ρ known as the Bloch-Redfield (BR) equation. The concepts introduced here will be used in the derivation of the Universal Lindblad equation in Sec. III.

1. Derivation of Bloch-Redfield equation

To derive the BR equation, we assume that the bath was in a steady state at some arbitrary time t_0 in the remote past. Specifically, we assume that $\rho_{\mathcal{SB}}(t_0) = \rho_{\mathcal{S}}(t_0) \otimes \rho_{\mathcal{B}}$, where $\rho_{\mathcal{B}}$ describes a steady state of the bath: $[H_{\mathcal{B}}, \rho_{\mathcal{B}}] = 0$. The bath state $\rho_{\mathcal{B}}$ can for example describe a thermal equilibrium state with a specific temperature and chemical potential. If $\mathcal{B}$ consists of several subbaths, $\rho_{\mathcal{B}}$ can also be a direct product of thermal states out of equilibrium with each other. Due to its macroscopic size, the state of the bath remains practically unaffected by the system $\mathcal{S}$ at later times, except for short-lived fluctuations arising from the system's evolution in the recent past. Without loss of generality, we may assume that each bath operator B_{α} has vanishing expectation value in the bath state $\rho_{\mathcal{B}}$: $\text{Tr}_{\mathcal{B}}(B_{\alpha}\rho_{\mathcal{B}}) = 0$, since nonzero expectation values can be eliminated by appropriate redefinition of $H_{\mathcal{S}}$ and B_{α} in Eqs. (5) and (6).

We note that, due to the finite memory and relaxation times of the bath and of the system, respectively, the evolution of $\rho(t)$ should be independent of the details of the initialization in the remote past. Supporting this, in Appendix A 6 we show that the evolution of the system is independent of the details of the state $\rho_{\mathcal{S}}(t_0)$ and the exact value of t_0, when t_0 is sufficiently far in the past.

The BR equation is most easily derived in the interaction picture. We transform the problem to the interaction picture by applying a rotating frame transformation generated by the Hamiltonian $H_{\mathcal{S}}(t) + H_{\mathcal{B}}$. After this transformation, the Hamiltonian of the combined system $\mathcal{SB}$ in the interaction picture is given by

$$\tilde{H}(t) = \sqrt{\gamma}\tilde{X}(t)\tilde{B}(t). \tag{8}$$

Here $\tilde{X}(t) \equiv U^{\dagger}(t)XU(t)$, and $\tilde{B}(t) \equiv e^{iH_{\mathcal{B}}t}Be^{-iH_{\mathcal{B}}t}$, where $U(t) \equiv \mathcal{T}e^{-i\int_0^t dt' H_{\mathcal{S}}(t')}$ is the time-evolution operator of the subsystem $\mathcal{S}$ relative to an arbitrary origin of time, and $\mathcal{T}$ is the time-ordering operation. We let $\tilde{\rho}(t)$ denote the reduced density matrix of $\mathcal{S}$ in the interaction picture. Specifically, $\tilde{\rho}(t) \equiv \text{Tr}_{\mathcal{B}}[\tilde{\rho}_{\mathcal{SB}}(t)]$, where $\tilde{\rho}_{\mathcal{SB}}(t)$ denotes the state of the combined system $\mathcal{SB}$ when time evolved with $\tilde{H}(t)$ from the state $\tilde{\rho}_{\mathcal{SB}}(0) = \rho_{\mathcal{SB}}(0)$. From $\tilde{\rho}(t)$, one can straightforwardly

obtain the time evolution of the system in the Schrödinger picture through the relation $\rho(t) = U(t)\tilde{\rho}(t)U^\dagger(t)$.

After transforming to the interaction picture, the system's dynamics occur on a timescale which is set by the system-bath coupling γ. When this coupling is sufficiently weak, $\tilde{\rho}(t)$ can be assumed static on the intrinsic correlation timescale of the bath (see Sec. I). This so-called weak-coupling limit forms the basis for the derivation of the BR equation, using the Born-Markov approximation [9]. To employ the Born-Markov approximation, we integrate the von Neumann equation $\partial_t \tilde{\rho}_{\mathcal{SB}}(t) = -i[\tilde{H}(t), \tilde{\rho}_{\mathcal{SB}}(t)]$ once, obtaining $\partial_t \tilde{\rho}_{\mathcal{SB}}(t) = -\int_{t_0}^t dt'[\tilde{H}(t), [\tilde{H}(t'), \tilde{\rho}_{\mathcal{SB}}(t')]]$. (Here we exploited the fact that $\mathrm{Tr}_{\mathcal{B}}[\tilde{B}(t_0)\rho_{\mathcal{SB}}] = \mathrm{Tr}_{\mathcal{B}}(B\rho_{\mathcal{SB}})$ vanishes by assumption as described above, to eliminate the term arising from the boundary term of the integration). The Born approximation amounts to setting $\tilde{\rho}_{\mathcal{SB}}(t') \approx \tilde{\rho}(t') \otimes \rho_{\mathcal{B}}$ inside the integral. The next step is to take the partial trace over the bath, to obtain an equation of motion for the reduced density matrix of the system $\tilde{\rho}(t)$. Using the fact that $\tilde{X}(t)$ acts only on the system, while $\tilde{B}(t)$ acts only on the bath, we obtain $\partial_t \tilde{\rho}(t) \approx -\gamma \int_{t_0}^t dt' J(t-t')[\tilde{X}(t), \tilde{X}(t')\tilde{\rho}(t')] + \mathrm{H.c.}$, where we introduced the (two-point) bath correlation function

$$J(t - t') \equiv \mathrm{Tr}_{\mathcal{B}}(\tilde{B}(t)\tilde{B}(t')\rho_{\mathcal{B}}). \quad (9)$$

Finally, the Markov approximation is implemented by assuming that $\tilde{\rho}(t')$ is stationary over the characteristic decay time of the bath correlation function $J(t)$ (see below for discussion). By making the replacement $\tilde{\rho}(t') \approx \tilde{\rho}(t)$ inside the integral over the history of the system (t'), and taking the limit $t_0 \to -\infty$, we obtain [9]

$$\partial_t \tilde{\rho}(t) = \mathcal{D}_{\mathrm{R}}(t)[\tilde{\rho}(t)] + \xi(t), \quad (10)$$

where

$$\mathcal{D}_{\mathrm{R}}(t)[\rho] \equiv -\gamma \int_{-\infty}^t dt' J(t-t')[\tilde{X}(t), \tilde{X}(t')\rho] + \mathrm{H.c.}, \quad (11)$$

and $\xi(t)$ denotes the correction arising from the Born and Markov approximations above. The Bloch-Redfield equation is obtained by assuming the error induced by the Born-Markov approximation $\xi(t)$ to be negligible in Eq. (10). In Appendix A we derive an upper bound for this correction, thus obtaining rigorous conditions for the validity of the BR equation. See Sec. II A 2 for a further discussion.

Note that the last approximation in the above derivation resulted in an equation of motion for $\tilde{\rho}(t)$ which is *Markovian*: in Eq. (10) the time derivative $\partial_t \tilde{\rho}(t)$ depends only on the value of $\tilde{\rho}(t)$ at the the same time t. As we demonstrate in Sec. III (see Sec. III B for discussion), the Born-Markov approximation above is not the only way of approximating $\partial_t \tilde{\rho}$ by a Markovian master equation in the weak-coupling limit. In Sec. III we will develop a different Markovian approximation for $\partial_t \tilde{\rho}$ which is valid under the same conditions as the standard Markov-Born approximation above, but, unlike the former, leads to a master equation in the Lindblad form.

The bath correlation function in Eq. (9), or equivalently its Fourier transform $J(\omega) \equiv \frac{1}{2\pi} \int_{-\infty}^\infty dt\, J(t)e^{i\omega t}$, known as the bath spectral function, plays a crucial role for describing the dynamics of the system $\mathcal{S}$: in the BR equation [Eq. (10)], $J(t)$ contains all information of the bath required to determine the

evolution of $\tilde{\rho}$. Importantly, even without the Born-Markov approximation, a wide class of baths (so-called Gaussian baths) are fully characterized by the two-point correlation function $J(t)$. This situation for instance arises if the bath consists of a large collection of decoupled subsystems, such as continua of independent fermionic or bosonic modes. While we note that our approach below can also be applied to cases where higher-order bath correlations are relevant, in this paper, we assume for simplicity that the bath is Gaussian.

The spectral function $J(\omega)$, which is real and non-negative, can in many cases be computed or phenomenologically assumed [9] [see for example Sec. V, where we calculate $J(\omega)$ for a bath of bosonic modes]. While in the above $\tilde{B}(t)$ emerged as a time-evolved observable in a quantum mechanical bath, the results in this paper also apply to the case where $\tilde{B}(t)$ is a classical noise signal, and the bath trace is replaced by the statistical average over noise realizations. In this case, $J(\omega)$ is symmetric in ω and gives the spectral density of the classical noise signal.

2. Correction to Bloch-Redfield equation

As an important secondary result, in this paper we derive a rigorous upper bound for the correction to the Bloch-Redfield equation $\xi(t)$, which is independent of the details of the system Hamiltonian $H_{\mathcal{S}}$. This error bound is used in the next section, where we derive the ULE.

To derive the error bound, we assume that the bath is Gaussian, and that the bath and system were decoupled at some point in the remote past (see beginning of this subsection). Using these assumptions, in Appendix A we systematically expand the time derivative of $\tilde{\rho}$ in powers of the dimensionless "Markovianity parameter" $\Gamma\tau$, where the bath timescales Γ^{-1} and τ were defined from the bath spectral function and the system-bath coupling strength γ in Eqs. (3) and (4) (in Sec. II B, we further discuss the physical meaning of these timescales). As we show in Appendix A, truncation of the expansion of $\partial_t \tilde{\rho}$ to leading order in $\Gamma\tau$ yields $\mathcal{D}_{\mathrm{R}}[\tilde{\rho}(t)]$. This truncation is thus is equivalent to making the Born and Markov approximations, while the correction $\xi(t)$ corresponds to the sum of all subleading terms in the expansion. In Appendix A we obtain a bound for this subleading correction:

$$\|\xi(t)\| \leqslant \Gamma^2\tau, \quad (12)$$

where (here and in the following) $\|\cdot\|$ refers to the spectral norm (see Ref. [26]).

Note that consistent correction bounds for the BR equation were recently obtained elsewhere [17,29]. Our derivation of Eq. (12) holds in the general case where the system and bath are connected through multiple noise channels, with the generalized definitions Γ and τ given in Eq. (26) below (see Ref. [9] or Appendix A for the multichannel generalization of the BR equation).

In Eq. (13) below, we also show that the spectral norm of $\partial_t \tilde{\rho}$ on the left-hand side of Eq. (10) is bounded by $\Gamma/2$. Comparing this bound with Eq. (12) above, we conclude that $\Gamma\tau \ll 1$ is a necessary condition for the Born-Markov approximation to be justified by our arguments.

B. Characteristic timescales of the bath

Above we found that the validity of the Born-Markov approximation is determined from the characteristic timescales Γ^{-1} and τ, which are intrinsic to the bath (and its coupling to the system). In Sec. III we show that Γ and τ also determine the accuracy of the universal Lindblad equation. Here we briefly discuss the nature of these quantities. As a demonstration, we moreover explicitly calculate the jump correlator $g(t)$ and the correlation time τ for the case of an Ohmic bath.

To highlight the physical meaning of the timescale Γ^{-1}, in Appendix A 3 we show that Γ provides a strict bound on the rate of change of $\tilde{\rho}$,

$$\|\partial_t \tilde{\rho}\| \leqslant \Gamma/2. \tag{13}$$

Note that Eq. (13) is exact, and is derived without any approximations, other than the assumption of a Gaussian bath. In this way, the rate $\Gamma/2$ can be seen as a "quantum speed limit" for dissipative quantum evolution [30,31]. Heuristically, Γ^{-1} thus characterizes the (shortest) typical interval between real or virtual system-bath interaction events, such as, e.g., photon emission or absorption. Note that the inequality in Eq. (13) extends to the case of multiple noise channels, with Γ as defined in Eq. (26) below (see Appendix A 3).

The timescale τ captures the characteristic decay time of correlations in the bath. To see this, note from Eq. (4) that τ gives the mean value of $|t|$ associated with the normalized distribution $|g(t)|/C$, where $C \equiv \int_{-\infty}^{\infty} dt\,|g(t)|$. The existence of a finite value of τ requires that $g(t)$ effectively decays faster than $C\tau/t^2$ for $|t| \gg \tau$ [32]. Noting from Eq. (3) that the bath correlation function $J(t)$ is given by the convolution of the jump correlator with itself, $J(t - t') = \int_{-\infty}^{\infty} ds\, g(t - s)g(s - t')$, the bath correlation function hence must also decay on a timescale of magnitude τ. The conditions described above hold under the assumption that τ takes a finite value; a divergent value of τ indicates that long-term memory is present in the bath; in this case, the system cannot be well described by a Markovian master equation.

To illustrate the above relationship between $J(t)$, $g(t)$, and the correlation time τ, we explicitly compute $J(t)$, $g(t)$, and τ, for an Ohmic bath. The Ohmic bath consists of a continuum of bosonic modes with Hamiltonian $H_B = \int_0^{\infty} d\omega\, \omega b^{\dagger}(\omega)b(\omega)$, where $b(\omega)$ denotes the annihilation operator of modes with frequency ω, satisfying $[b(\omega), b^{\dagger}(\omega')] = \delta(\omega - \omega')$ and $\delta(\omega)$ denotes the Dirac delta function. In the framework of Eqs. (5) and (6), the bath operator B is given by the bosonic field operator $\int_0^{\infty} d\omega\, \sqrt{S(\omega)}[b(\omega) + b^{\dagger}(\omega)]$, where $S(\omega)$ denotes the effective spectral density of the bath, including the frequency dependence of the system-bath coupling. The class of models above is commonly used in the literature [9,10], and can for example describe a phononic or electromagnetic environment of an electronic system.

We consider the Ohmic spectral density $S(\omega) = \omega e^{-\omega^2/2\Lambda^2}/\omega_0$, with an ultraviolet energy cutoff set by the scale Λ. The energy scale ω_0 is introduced to keep $S(\omega)$ dimensionless. Assuming the bath is in equilibrium at temperature T, a straightforward calculation [9,10] yields the bath spectral function

$$J(\omega) = \frac{1}{\omega_0} \frac{\omega e^{-\frac{\omega^2}{2\Lambda^2}}}{1 - e^{-\omega/T}}, \tag{14}$$
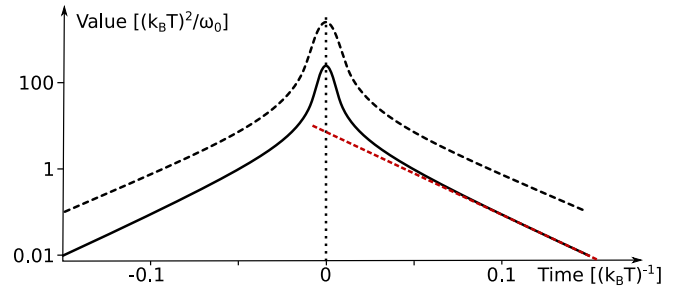

FIG. 1. Absolute value of the jump correlator $g(t)$ (solid) and bath correlation function $J(t)$ (black dashed line) for the Ohmic bath studied in Sec. II B. Red line indicates the fit used to obtain the exponential decay constant for g (see Sec. II B).

where we work in units where $k_B = 1$.

Using Eq. (3), we numerically compute the jump correlator $g(t)$ from the spectral function in Eq. (14), for the case where $\Lambda = 50\,T$. By explicit computation [see Eq. (4)], we find for this case $\tau \approx 0.007\,T^{-1}$. In Fig. 1 we plot $|g(t)|$ on a logarithmic scale (solid line), along with the bath correlation function $|J(t)|$ (dashed line). As Fig. 1 shows, both $J(t)$ and $g(t)$ decay exponentially, at approximately the same rate, after a sharp initial drop at short times. We confirm numerically (data not shown here) that the short-time peak arises from high-energy modes in the bath, and is controlled by the cutoff Λ. By linear regression (red dashed line in Fig. 1) we find the slope of $\log g(t)$ outside this initial decrease to be given by approximately $0.023\,T^{-1}$. The difference between the exponential decay constant from τ is caused by the short-time peak of $g(t)$, and is thus controlled by Λ.

III. UNIVERSAL LINDBLAD EQUATION

While useful, the standard Born-Markov approximation discussed in Sec. II A has some shortcomings. In particular, the Bloch-Redfield equation in Eq. (10) is not in the Lindblad form. As a result, as was explained in the introduction, integration of the BR equation may yield negative or diverging probabilities, and can be impractical to implement numerically even for moderately sized quantum systems. In this section we derive a master equation for $\tilde{\rho}$ which is valid under the same conditions as the BR equation, but will be in the Lindblad form and thus free of the limitations above. Specifically, our new master equation is accurate up to a correction of the same magnitude as the correction bound $\Gamma^2\tau$ we identified for the BR equation in Sec. II A 2. The new master equation, which we term the universal Lindblad equation (ULE), constitutes the main result of our paper. Crucially, the ULE does not require any special conditions on the system to be valid; rather, its validity relies solely on the properties of the *bath* itself (along with its coupling to the system).

To make the physical basis for the ULE most transparent, in Sec. III A we derive the ULE on an intuitive level of argumentation, focusing on the case where the system and bath are coupled through a single noise channel. In Appendix C we provide a rigorous derivation of these results that also holds for general system-bath couplings. In Sec. III B we

comment on the principle underlying our derivation, i.e., the existence of distinct but equivalently valid Markov approximations. The results for general system-bath couplings are given in Sec. III C, and expressed in the Schrödinger picture in Sec. III D.

A. Single noise channel

In this subsection we heuristically derive the universal Lindblad equation for the case of a single quantum noise channel. As a first step in our derivation, we identify an alternative form of Markov approximation, which is valid at the same level of approximation as the standard Born-Markov approximation (i.e., up to a correction of order $\Gamma^2\tau$). Subsequently, we demonstrate that this Markov approximation results in a master equation in the Lindblad form (in contrast, the standard Born-Markov approximation does not lead to a Lindblad-form master equation).

The starting point for our derivation is the BR equation [Eq. (10)], whose error bounds we obtained in Sec. II A 2 above. Below we apply additional manipulations to the BR equation, which induce errors of the same magnitude as those inherent in the Born-Markov approximation used in deriving Eq. (10). These additional steps hence lead to a master equation that is different from the BR equation, but is valid on the same level of approximation. Unlike the BR equation, our master equation is crucially in the Lindblad form. Our modification procedure is equivalent to employing a distinct Markovian approximation from the standard Markov approximation (reviewed in Sec. II A) that is used to obtain the BR equation [note that our derivation still makes use of the "conventional" Born approximation, as described in the paragraph above Eq. (9)]. In Sec. III B below, we discuss the diversity of possible Markov approximations in more detail.

As the first step of our derivation, we decompose the bath correlation function $J(t-t')$ [Eq. (9)] as a convolution using "jump correlator" $g(t)$ defined in Eq. (3): $J(t-t') = \int_{-\infty}^{\infty} ds\, g(t-s)g(s-t')$. Using this decomposition, the BR equation [Eq. (10)] can be (exactly) rewritten as

$$\partial_t \tilde{\rho}(t) \approx \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} ds\, \mathcal{F}(t,s,t')[\tilde{\rho}(t)], \qquad (15)$$

where

$$\mathcal{F}(t,s,t')[\tilde{\rho}]$$
$$= \gamma\theta(t-t')g(t-s)g(s-t')[\tilde{X}(t), \tilde{\rho}\tilde{X}(t')] + \text{H.c.} \quad (16)$$

The approximate equality in Eqs. (10) and (15) captures the correction to the BR equation $\xi(t)$, whose bound (with respect to the spectral norm) we identified in Sec. II A 2. For brevity we do not include this correction in the derivation below. Note that $\mathcal{F}(t,s,t')$ is a linear operator acting on system operators.

Next, we integrate Eq. (15) with respect to t to compute the change of $\tilde{\rho}$ over a finite time interval from t_1 to t_2, that we will choose much longer than τ:

$$\tilde{\rho}(t_2) - \tilde{\rho}(t_1) \approx \int_{t_1}^{t_2} dt \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} ds\, \mathcal{F}(t,s,t')[\tilde{\rho}(t)]. \qquad (17)$$

We now argue that the weak-coupling limit $\Gamma\tau \ll 1$ allows us to apply two approximations to the right-hand side above,

which yield a new expression that is valid on an equivalent level of approximation as the standard Bloch-Redfield equation in Eq. (10).

To make the first approximation, we note that, in the limit $\Gamma\tau \ll 1$, the condition $\|\partial_t\tilde{\rho}\| \leqslant \Gamma/2$ in Eq. (13) ensures that $\tilde{\rho}(t) = \tilde{\rho}(s) + \mathcal{O}(\Gamma\tau)$ for $|t-s| \lesssim \tau$. Additionally, since $g(t)$ decays on the timescale τ (see Sec. II B), $\mathcal{F}(t,s,t')$ is suppressed when the difference between any two of its time arguments is much larger than τ [see Eq. (16)] [in particular, note that $g(t-s)g(s-t')$, and thus $\mathcal{F}(t,s,t')$, must be small when $t-t' \gg \tau$]. These two results suggest that we may replace $\tilde{\rho}(t)$ by $\tilde{\rho}(s)$ on the right-hand side of Eq. (17) when $\Gamma\tau \ll 1$. In Appendix C we implement this substitution in a systematic way, and prove that replacing $\tilde{\rho}(t)$ by $\tilde{\rho}(s)$ in Eqs. (15) and (17) results in a correction to $\partial_t\tilde{\rho}$ of order $\Gamma^2\tau$.

To make our second approximation, we again use the fact that $\mathcal{F}(t,s,t')$ decays when the difference between any of its time arguments exceeds τ. Thus, since $\tau \ll t_2 - t_1$ by assumption, most of the contribution to the integral in Eq. (17) comes from the region where all three integration variables t, s, t' are located in the interval $[t_1, t_2]$. As a result, the right-hand side of Eq. (17) is approximately unaffected if we change the integration domain from $-\infty < (s,t') < \infty$, $t_1 \leqslant t \leqslant t_2$ to the domain $-\infty < (t,t') < \infty$, $t_1 \leqslant s \leqslant t_2$. Indeed, in Appendix C we show that this change of integration domain results in a correction to $\tilde{\rho}$ which is bounded by $\Gamma\tau$.

After making the two approximations described above [i.e., setting $\rho(t) \approx \rho(s)$ in Eq. (17), and subsequently changing the domain of integration], we obtain

$$\tilde{\rho}(t_2) - \tilde{\rho}(t_1) \approx \int_{t_1}^{t_2} ds \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, \mathcal{F}(t,s,t')[\tilde{\rho}(s)]. \qquad (18)$$

By taking the derivative with respect to t_2, and renaming the variables of integration, we obtain the (time-local) master equation

$$\partial_t \tilde{\rho}(t) \approx \mathcal{L}(t)[\tilde{\rho}(t)], \quad \mathcal{L}(t) = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds'\, \mathcal{F}(s,t,s'). \qquad (19)$$

In Appendix C we put the above line of arguments on rigorous footing: we identify a slightly modified density matrix $\rho'(t)$ whose norm distance to $\tilde{\rho}(t)$ remains bounded by $\Gamma\tau$ at all times. Assuming that the bath is Gaussian, and that the bath and system were decoupled at some point in the remote past, we show that ρ' evolves according to the master equation

$$\partial_t \rho'(t) = \mathcal{L}(t)[\rho'(t)] + \xi'(t), \qquad (20)$$

where $\mathcal{L}(t)$ is defined by Eqs. (16) and (19), and $\|\xi'(t)\| \leqslant 2\Gamma^2\tau$ for all times t. In the Markovian limit $\Gamma\tau \ll 1$, $\rho'(t)$ is nearly identical to $\tilde{\rho}(t)$, and the evolution of the system is thus well described by $\rho'(t)$. The same condition $\Gamma\tau \ll 1$ is already required for the BR equation to be valid by our arguments (see Sec. II A 2) and hence does not impose additional constraints on the system. Consistent with Eq. (13), we show in Appendix C that $\|\mathcal{L}(t)[\rho]\| \leqslant \Gamma/2$. Hence, by the same arguments as in Sec. II A 2, $\Gamma\tau \ll 1$ is also a necessary condition for error ξ' above to be negligible.

As a final step, we verify that the master equation in Eq. (19) is in the Lindblad form. By decomposing the step

function $\theta(t - t')$ in Eq. (16) into its symmetric and antisymmetric components, $\theta(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t)$, we find through a straightforward computation (see Appendix C 3 for details) that

$$\mathcal{L}(t)[\tilde{\rho}] = -i[\tilde{\Lambda}(t), \tilde{\rho}] - \frac{1}{2}\{\tilde{L}^\dagger(t)\tilde{L}(t), \tilde{\rho}\} + \tilde{L}(t)\tilde{\rho}\tilde{L}^\dagger(t), \tag{21}$$

where the jump operator $\tilde{L}(t)$ is given by

$$\tilde{L}(t) = \sqrt{\gamma} \int_{-\infty}^{\infty} ds\, g(t - s)\tilde{X}(s) \tag{22}$$

and

$$\tilde{\Lambda}(t) = \frac{\gamma}{2i} \int_{-\infty}^{\infty} ds\, ds'\, \tilde{X}(s)g(s - t)g(t - s')\tilde{X}(s')\,\text{sgn}(s - s'). \tag{23}$$

The Lamb shift $\tilde{\Lambda}(t)$ is by construction Hermitian, due to the symmetry of the jump correlator $g(t) = g^*(-t)$, which results from its definition in Eq. (3).

Comparing with the BR equation [Eq. (10)], we see that the universal Lindblad equation [Eqs. (20)–(23) with $\xi'(t)$ neglected] yields an expression for $\partial_t \rho'(t)$ which is accurate up to a correction bounded by the same value as the correction for the BR equation (up to a factor of 2). In this sense, the ULE and the BR equation are valid on an equivalent level of approximation [33].

To summarize this section, we showed that, in the weak-coupling limit $\Gamma\tau \ll 1$, the interaction picture density matrix of the subsystem $\mathcal{S}$, $\tilde{\rho}(t)$ evolves according to the Lindblad-form master equation in Eqs. (19)–(21). At each time t, the error in $\partial_t\tilde{\rho}$ is of the same magnitude as that of the Bloch-Redfield equation. Thus the ULE is valid over the same regimes as the previously developed Markovian master equations, while offering important gains in usability and applicability.

B. Equivalence of Markov approximations

Above we derived a time-local master equation for $\tilde{\rho}(t)$ that is distinct from the Bloch-Redfield equation, but is valid on an equivalent level of approximation. As we explain here, the existence of distinct but equivalently valid time-local master equations reflects the existence of a class of distinct but equivalently valid Markov approximations. We refer to two approximations as being "equivalently valid" if they are both valid up to an error of the same order in the Markovianity parameter $\Gamma\tau$.

We demonstrate the existence of equivalently valid Markov approximations by means of a simple example. Consider the master equation for $\tilde{\rho}(t)$ that results from the Born approximation [see text above Eq. (9)]: $\partial_t\tilde{\rho}(t) = -\int_{t_0}^{t} dt'\, J(t - t')[X(t), [X(t'), \tilde{\rho}(t')]] + \text{H.c.}$ As explained in Sec. II A, the standard Markov approximation amounts to approximating $\tilde{\rho}(t') \approx \rho(t)$ in this expression. This approximation is justified when the bath correlation time τ is much shorter than the characteristic timescale of system-bath interactions Γ^{-1}. By the same arguments, however, instead of setting $\tilde{\rho}(t') \approx \tilde{\rho}(t)$, we just as well could have approximated $\tilde{\rho}(t')$ by any weighted average of $\tilde{\rho}(s)$ within a window of times s near $s = t'$, as long as the width of the time window is much smaller than Γ^{-1}. These different choices of weight functions result in distinct, but equivalently valid, time-local master equations.

The infinite family of suitable weight functions can thus be seen as generating a class of distinct Markov approximations.

As we show in Appendix C, there are also other classes of equivalent Markov approximations of more subtle origin than the simple example above. These other classes of equivalent approximations can be identified using similar approaches as above. The approximations in Eqs. (15)–(19) constitute such an alternative Markov approximation. A rigorous definition and discussion of this approximation is given in Appendix C.

C. General system-bath couplings

In Sec. III A we derived the universal Lindblad equation for the case where the system-bath coupling H_{int} in Eq. (6) holds a single noise channel. In this subsection we extend our results to the most general case of system-bath couplings, namely the case where H_{int} contains an arbitrarily high (but finite) number of noise channels N: $H_{\text{int}} = \sqrt{\gamma} \sum_{\alpha=1}^{N} X_\alpha B_\alpha$. Here, for each α, X_α, and B_α, are observables of the system $\mathcal{S}$ and bath $\mathcal{B}$, respectively. These are normalized such that $\|X_\alpha\| = 1$, while the bath operators $\{B_\alpha\}$ may have different scales of magnitude.

For general system-bath coupling, the ULE can be derived through straightforward generalization of the single-channel case in Sec. III A. Because of this, the derivation of the ULE in Appendix C that we quoted in Sec. III A considers the case of multiple noise channels. Here we present the results from Appendix C.

As for the single-channel case, the validity of the ULE is determined solely by properties of the bath correlation (or, equivalently, spectral) functions. In the case where the system and bath are connected through N quantum noise channels, the bath correlation function introduced in Eq. (9) takes values as an $N \times N$ matrix $\boldsymbol{J}(t)$ with matrix elements

$$J_{\alpha\beta}(t - s) \equiv \text{Tr}_{\mathcal{B}}[\tilde{B}_\alpha(t)\tilde{B}_\beta(s)\rho_{\mathcal{B}}]. \tag{24}$$

Here $\tilde{B}_\alpha(t) \equiv e^{iH_{\mathcal{B}}t} B_\alpha e^{-iH_{\mathcal{B}}t}$ denotes the interaction picture version of the bath operator B_α, and the indices α and β label the noise channels, taking values $1, \ldots, N$. Using the definition above, one can verify that the bath spectral function $\boldsymbol{J}(\omega) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, \boldsymbol{J}(t)e^{i\omega t}$ forms a positive-semidefinite matrix for all ω. The fact that $\boldsymbol{J}$ is positive semidefinite generalizes the single-channel result that the scalar-valued bath spectral function $J(\omega)$ is non-negative (see Sec. II A).

To establish the conditions under which the ULE holds, we generalize the jump correlator $g(t)$ from Eq. (3) to the multiple-channel case. Using the fact that $\boldsymbol{J}(\omega)$ is positive semidefinite, we define the matrix-valued jump correlator $\boldsymbol{g}(t)$ as follows:

$$\boldsymbol{g}(t) = \int_{-\infty}^{\infty} d\omega\, \boldsymbol{g}(\omega)e^{-i\omega t}, \quad \boldsymbol{g}(\omega) = \sqrt{\boldsymbol{J}(\omega)/2\pi}. \tag{25}$$

Here the square root in the second equation denotes the matrix square root; i.e., $J_{\alpha\beta}(\omega) = \frac{1}{2\pi} \sum_\lambda g_{\alpha\lambda}(\omega)g_{\lambda\beta}(\omega)$, where $\{g_{\alpha\beta}(\omega)\}$ denote the matrix elements of $\boldsymbol{g}(\omega)$. Since $\boldsymbol{J}(\omega)$ is positive semidefinite, the Fourier transform of the jump correlator $\boldsymbol{g}(\omega)$ is itself a well-defined positive-semidefinite matrix for all values of ω.

From the multichannel jump-correlator $\boldsymbol{g}(t)$, we define the quantities Γ and τ from the mult-channel jump correlator $\boldsymbol{g}(t)$

as follows:

$$\Gamma = 4\gamma \left[\int_{-\infty}^{\infty} dt \, \|\boldsymbol{g}(t)\|_{2,1} \right]^2, \quad \tau = \frac{\int_{-\infty}^{\infty} dt \, \|\boldsymbol{g}(t)t\|_{2,1}}{\int_{-\infty}^{\infty} dt \, \|\boldsymbol{g}(t)\|_{2,1}}. \quad (26)$$

Here for any matrix $\boldsymbol{M}$ with elements $M_{\alpha\lambda}$, $\|\boldsymbol{M}\|_{2,1} \equiv \sum_{\lambda} (\sum_{\alpha} |M_{\alpha\lambda}|^2)^{1/2}$. The matrix norm $\|\cdot\|_{2,1}$ is also known

as the $L_{2,1}$ matrix norm [34], and is identical to the trace norm for diagonal matrices. As we require, for the special case of a single noise channel ($N = 1$), the definitions of Γ and τ above are identical to the definitions in Eq. (4).

In Appendix C we show that when $\Gamma\tau \ll 1$ [with Γ and τ as defined in Eq. (26)], the system's dynamics are effectively Markovian, and $\tilde{\rho}$ evolves according to the following Lindblad-form master equation:

$$\partial_t \tilde{\rho}(t) = -i[\tilde{\Lambda}(t), \tilde{\rho}(t)] + \sum_{\lambda=1}^{N} \left(\tilde{L}_{\lambda}(t)\tilde{\rho}(t)\tilde{L}_{\lambda}^{\dagger}(t) - \frac{1}{2}\{\tilde{L}_{\lambda}^{\dagger}(t)\tilde{L}_{\lambda}(t), \tilde{\rho}(t)\} \right) + \xi'(t), \quad (27)$$

where $\|\xi'(t)\| \leqslant 2\Gamma^2\tau$, and $\tilde{L}_{\lambda}(t)$ is the jump operator associated with the emergent noise channel λ (which may involve operators from multiple baths). Explicitly, $\tilde{L}_{\lambda}(t)$ is given by

$$\tilde{L}_{\lambda}(t) = \sqrt{\gamma} \sum_{\alpha} \int_{-\infty}^{\infty} ds \, g_{\lambda\alpha}(t - s)\tilde{X}_{\alpha}(s), \quad (28)$$

while the multiple-channel Lamb shift $\tilde{\Lambda}(t)$ is given by

$$\tilde{\Lambda}(t) = \frac{\gamma}{2i} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds' \sum_{\alpha\beta} \tilde{X}_{\alpha}(s)\tilde{X}_{\beta}(s')\phi_{\alpha\beta}(s - t, s' - t). \quad (29)$$

Here $\{\phi_{\alpha\beta}(t, s)\}$ denote the matrix elements of the $N \times N$ matrix $\boldsymbol{\phi}(t, s) \equiv \boldsymbol{g}(t)\boldsymbol{g}(-s)\mathrm{sgn}(t - s)$. The above expressions for $\tilde{L}_{\lambda}(t)$ and $\tilde{\Lambda}(t)$ simplify further in the case of independent noise channels [i.e., when the bath spectral function $\boldsymbol{J}(\omega)$ is diagonal], since $\boldsymbol{g}(t)$ in this case is diagonal.

Analogous to the single-channel case, Eqs. (27)–(29) hold up to a correction of order $2\Gamma^2\tau$ for a density matrix ρ' whose norm distance to $\tilde{\rho}$ remains bounded by $\Gamma\tau$ at all times. Hence, we expect Eq. (27) to accurately describe the evolution of $\tilde{\rho}$ in the weak-coupling limit $\Gamma\tau \ll 1$ (see discussion in Sec. III A).

D. Schrödinger picture

We conclude this section by expressing the universal Lindblad equation [Eqs. (27)–(29)] in the Schrödinger picture. Using the transformation between the interaction and Schrödinger pictures below Eq. (8), we obtain the master equation for the reduced density matrix of the system in the Schrödinger picture ρ:

$$\partial_t \rho(t) = -i[H_{\mathcal{S}}(t) + \Lambda(t), \rho(t)] + \sum_{\lambda=1}^{N} \mathcal{D}_{\lambda}[\rho(t), t], \quad (30)$$

where we suppressed the correction of order $\Gamma^2\tau$ from Eq. (27). In the above, $\mathcal{D}_{\lambda}$ denotes the dissipator associated with emergent noise channel λ, and is given by

$$\mathcal{D}_{\lambda}[\rho, t] \equiv L_{\lambda}(t)\rho L_{\lambda}^{\dagger}(t) - \frac{1}{2}\{L_{\lambda}^{\dagger}(t)L_{\lambda}(t), \rho\}. \quad (31)$$

Here the Schrödinger picture jump operators and Lamb shift are given by $L_{\lambda}(t) = U(t)\tilde{L}_{\lambda}(t)U^{\dagger}(t)$ and $\Lambda(t) = U(t)\tilde{\Lambda}(t)U^{\dagger}(t)$, where $U(t)$ denotes the unitary evolution

operator of the system $\mathcal{S}$, while $\tilde{\Lambda}(t)$ and $\tilde{L}_{\lambda}(t)$ were given in Sec. III C above. By direct computation, we find, in particular,

$$L_{\lambda}(t) = \sqrt{\gamma} \sum_{\alpha} \int_{-\infty}^{\infty} ds \, g_{\lambda\alpha}(t - s)U(t, s)X_{\alpha}U^{\dagger}(t, s), \quad (32)$$

where $U(t, s) \equiv \mathcal{T}e^{-i\int_s^t dt' H_{\mathcal{S}}(t')}$ denotes the time-evolution operator of the system from time s to time t, defined such that $U(s, t) = U^{\dagger}(t, s)$. An analogous expression can be obtained for $\Lambda(t)$. As we show in Sec. IV A, the jump operators $\{L_{\lambda}(t)\}$ and Lamb shift $\Lambda(t)$ above are time independent when the system Hamiltonian $H_{\mathcal{S}}(t)$ is time independent.

IV. PRACTICAL IMPLEMENTATION

In this section we discuss how to implement the universal Lindblad equation [Eq. (32)] in practice. We separate our discussion into three often-arising cases: in Sec. IV A we consider systems with time-independent Hamiltonians, in Sec. IV B we consider systems with time-dependent Hamiltonians, and in Sec. IV C we demonstrate how the ULE can be implemented in cases where exact diagonalization of the system Hamiltonian is not feasible (such as, e.g., quantum many-body systems).

A. Systems with time-independent Hamiltonians

We first consider the case of time-independent Hamiltonians. We moreover assume that the system's Hamiltonian can be efficiently diagonalized, either analytically or numerically. In this case the jump operators and Lamb shift can be easily computed from the Hamiltonian's eigenstates and energies. In Sec. IV C we discuss an efficient approximate implementation for cases where exact diagonalization is not practically possible.

When the system $\mathcal{S}$ has a time-independent Hamiltonian $H_{\mathcal{S}}$, the time-evolution operator of the system is given by $U(t, s) = \sum_n |n\rangle\langle n|e^{-iE_n(t-s)}$ where $\{|n\rangle\}$ and $\{E_n\}$ denote the eigenstates and energy spectrum of $H_{\mathcal{S}}$. Inserting this result into Eq. (32), we obtain the following simple expression for the system's jump operators:

$$L_{\lambda} = 2\pi\sqrt{\gamma} \sum_{m,n,\alpha} g_{\lambda\alpha}(E_n - E_m)X_{mn}^{(\alpha)}|m\rangle\langle n|, \quad (33)$$

where $X_{mn}^{(\alpha)} \equiv \langle m|X_{\alpha}|n\rangle$. The result above holds for an arbitrary number of quantum noise channels, and was quoted in

Sec. I for the single-channel case. Note that the jump operators are time independent, as required by the time-translation symmetry in this case of the problem.

The Lamb shift Λ can be expressed in similar terms as above, and also inherits the time independence of H_S: in Appendix D we find

$$\Lambda = \sum_{l,m,n} f_{\alpha\beta}(E_m - E_l, E_n - E_l) X_{ml}^{(\alpha)} X_{ln}^{(\beta)} |m\rangle\langle n|. \quad (34)$$

Here the functions $\{f_{\alpha\beta}(E_1, E_2)\}$ denote the elements of the matrix $\boldsymbol{f}(E_1, E_2) \equiv 2\pi\gamma \, \mathcal{P} \int_{-\infty}^{\infty} d\omega \, \omega^{-1} \boldsymbol{g}(\omega - E_1)\boldsymbol{g}(\omega + E_2)$, with $\mathcal{P}\int$ denoting the Cauchy principal value integral.

We confirm that the master equation above reproduces previous results for open quantum systems in the limit of large level spacing and weak γ (i.e., in the regime where the quantum optical master equation is valid) [9]: in this limit, standard arguments [9] show that the rotating wave approximation can be applied to Eq. (31). Using the jump operator in Eq. (33), this approximation reduces Eq. (30) to the quantum optical master equation. Similarly, to first order in γ, the Lamb shift Λ renormalizes each energy level of the system E_n by the amount $\delta E_n = \langle n|\Lambda|n\rangle$. Using $\boldsymbol{g}(\omega)^2 = 2\pi\boldsymbol{J}(\omega)$ in Eq. (34), one can verify that δE_n is identical to previous expressions for the Lamb-shift renormalization of energy levels in the small-γ limit [9].

B. Systems with time-dependent Hamiltonians

We now consider the situation where the system's Hamiltonian $H_S(t)$ varies with time. While in this case one can always obtain the jump operators from Eq. (32), here we obtain more convenient expressions in two important situations of wide applicability.

The first case we consider arises when the time dependence of H_S is slow on the bath correlation timescale τ. In this case, $H_S(t)$ may be assumed constant in Eq. (32), and the jump operators $\{L_\lambda(t)\}$ and Lamb shift $\Lambda(t)$ can be calculated from the energies and eigenstates of the instantaneous Hamiltonian $H_S(t)$. Specifically, as we show in Appendix E, calculating the jump operator from the instantaneous Hamiltonian as above yields a correction of order up to $\sqrt{\Gamma} \, \tau^2 \|\partial_t H_S\|$ (recall that $L_\lambda(t)$ has units of $[\text{Energy}]^{1/2}$). See Appendix E for further details. The above result confirms that $L_\lambda(t)$ may be calculated from the eigenstates and energies of the instantaneous Hamiltonian when its time-derivative $\partial_t H_S$ is sufficiently small compared to $1/\tau^2$. The approach in Appendix E can also be used to identify similar corrections for the Lamb shift.

The second situation where the universal Lindblad equation simplifies is the special case of periodically driven systems, where $H_S(t) = H_S(t + T)$ for some driving period T. In this case, $L_\lambda(t)$ and $\Lambda(t)$ can be exactly computed from the time-periodic Floquet states $|\phi_n(t)\rangle = |\phi_n(t + T)\rangle$ and quasienergies of the system [35] ε_n: for periodically driven systems, the evolution operator of the system is given by $U(t, s) = \sum_n |\phi_n(t)\rangle\langle\phi_n(s)|e^{-i\varepsilon_n(t-s)}$. Using this in Eq. (32), one can verify by straightforward computation that

$$L_\lambda(t) = \sum_{m,n} \sum_{z=-\infty}^{\infty} L_{mn;z}^{(\lambda)} |\phi_m(t)\rangle\langle\phi_n(t)|e^{-i\Omega z t},$$

where

$$L_{mn;z}^{(\lambda)} \equiv \sum_\alpha \int_0^T \frac{dt}{T} \langle\phi_m(t)|X_\alpha|\phi_n(t)\rangle e^{i\Omega z t} g_{\alpha\lambda}(\varepsilon_{nm}^z). \quad (35)$$

Here $\Omega \equiv 2\pi/T$, while $\varepsilon_{nm}^z \equiv \varepsilon_n - \varepsilon_m + z\Omega$. Note that the jump operators inherit the time periodicity of the Hamiltonian, as required by discrete time-translation symmetry: $L_\lambda(t) = L_\lambda(t + T)$. The Lamb shift $\Lambda(t)$ has a similar expression in terms of the Floquet states and also satisfies $\Lambda(t + T) = \Lambda(t)$. Interestingly, when the time dependence of $H_S(t)$ is slow compared to the bath correlation time τ (which may be very short), the results above show that the jump operators in Eq. (35) are equivalent to the jump operators generated from the instantaneous eigenstate basis of the Hamiltonian $H_S(t)$ through Eq. (33). The above form of the jump operators was used by the one of the authors to numerically simulate the dynamics of a driven-dissipative quantum cavity in Ref. [36].

The results above reproduce the generalization of the quantum optical master equation to periodically driven systems, for example derived in Refs. [37–39]. In these works, a Lindblad-form master equation is obtained for the system using a rotating wave approximation (RWA) which assumes the relaxation rate (Γ) much smaller than the smallest possible level spacing in the system's quasienergy spectrum $\delta\varepsilon_{\min} = \min_{m \neq n}(\varepsilon_n - \varepsilon_m + z\Omega)$. The approaches we present above do not rely on such a rotating wave approximation, and hence are valid for a wider class of systems. We note that both approaches presented here are equivalent to the above RWA master equations in the limit $\Gamma \ll \delta\varepsilon$ where the latter are valid. In this limit, one can verify that the steady state of the system is diagonal in the Floquet state basis.

C. Obtaining jump operators without diagonalization

We finally show how the universal Lindblad equation can be implemented in cases where diagonalization of the system Hamiltonian H_S is not feasible, such as for large quantum many-body systems. In this case, the jump operators and Lamb shift of the ULE cannot be obtained from the eigenstate decompositions presented above. Instead, as we show here, these operators can be easily obtained through a systematic, convergent expansion of Eq. (32) in powers of τ/τ_X, where τ is the bath correlation time, and τ_X denotes the characteristic timescale for the dynamics of the system observables $\{X_\alpha\}$ (see below for definition).

For simplicity we consider here the time-independent single-channel case. The approach we present below generalizes straightforwardly to time-dependent Hamiltonians and multiple noise channels. We moreover focus on computing the jump operator L (we suppress λ since we consider the single-channel case); the Lamb shift can be obtained through a similar approach.

To compute the jump operator L, we note that, for a time-independent Hamiltonian H_S, $U(t, s) = e^{-i(t-s)H_S}$. Next, we note that $e^{iH_S t} X e^{-iH_S t} = \sum_{n=0}^{\infty} (it)^n (\text{ad}_{H_S})^n [X]/n!$ where ad_{H_S} denotes the commutation operation by H_S, i.e., $\text{ad}_{H_S}[O] = [H_S, O]$. Using these results in Eq. (32), we obtain

$$L = \sqrt{\gamma} \sum_{n=0}^{\infty} c_n \, (\text{ad}_{H_S})^n [X], \quad c_n \equiv \frac{i^n}{n!} \int_{-\infty}^{\infty} dt \, g(t) t^n. \quad (36)$$
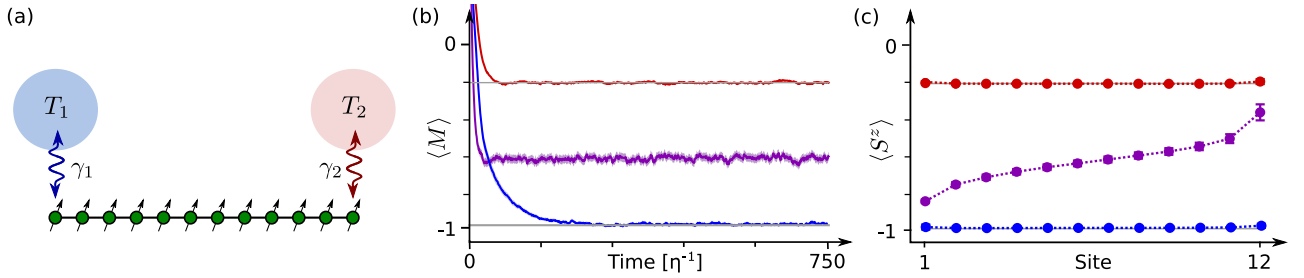
FIG. 2. Simulation of the open Heisenberg spin chain model in Sec. V (see main text for further details): (a) Schematic depiction of the system. (b) Average z magnetization in the chain, as a function of time, for the cases where the chain is connected to bath 2 (red), to bath 1 (blue), and to both baths (purple). Gray lines: Expectation values of the z magnetization in the Gibbs states at the temperatures of baths 1 (lower) and 2 (upper). Shaded areas surrounding curves (only visible for purple curve) indicates the uncertainty due to the finite number of sampling states. (c) Average z magnetization in the chain $\langle S_n^z \rangle$ for the final duration $150\eta^{-1}$ of the simulation, as a function of site index n [using same coloring scheme as in (b)]. Error bars indicate the uncertainty due to the finite number of sampling states in the simulation.

Crucially, the coefficients $\{c_n\}$ are system independent, and can be easily computed from the jump correlator.

The convergence of the series in Eq. (36) can be ensured by introducing a temporal cutoff $\tau_{\max}$, such that $g(t)$ is set to zero for $|t| \geqslant \tau_{\max}$. The error in L resulting from this approximation is bounded by $2 \int_{\tau_{\max}}^{\infty} dt |g(t)|$ [see Eq. (32)], and can thus be made arbitrarily small by choosing $\tau_{\max}$ sufficiently large [40]. In particular, we expect the error to be negligible when $\tau_{\max} \gg \tau$. With the temporal cutoff imposed, $c_n \sim \tau_{\max}^n / n!$ for large n. As a result, the series in Eq. (36) converges at order $\tau_{\max} / \tau_X$, where τ_X denotes the typical timescale associated with the dynamics of X, such that $\|(\mathrm{ad}_{H_S})^n [X]\| \sim 1/\tau_X^n$.

The expansion of the jump operator simplifies further when the Hamiltonian H_S is composed of an easily diagonalizable term H_0 (such as a quadratic term) and a weak nonintegrable perturbation V (such as an interaction term): $H_S = H_0 + V$. By transforming to the interaction picture with respect to H_0, we find $e^{-iH_S t} = e^{-iH_0 t} U'(t)$, where $U'(t) \equiv \mathcal{T} e^{-i\int_0^t dt' \tilde{V}(t')}$, and $\tilde{V}(t) = e^{iH_0 t} V e^{-iH_0 t}$. Using this result in Eq. (32), and expanding $U'(t) X [U'(t)]^\dagger$ in powers of $\tilde{V}(t)$ as above Eq. (36), we obtain a series expansion, where the nth order term is bounded by $(\tau_{\max}/\tilde{\tau}_X)^n / n!$, where $\tilde{\tau}_X$ denotes the timescale for the dynamics of X induced by the perturbation $\tilde{V}$, such that $\|\mathrm{ad}_{\tilde{V}}^n X\| \lesssim \tilde{\tau}_X^{-n}$. As a result, only terms up to order $\tau_{\max}/\tilde{\tau}_X$ contribute in the expansion of the jump operator. Importantly, $\tilde{\tau}_X$ is inversely proportional to the strength of the perturbation V.

Thus, in the limit of weak perturbations, where $\tau_{\max} \ll \tilde{\tau}_X$, the expansion above can be truncated at order zero. In this case, we may thus ignore the nonintegrable perturbation in the calculation of the jump operator, and obtain L from the spectrum and eigenstates of the integrable Hamiltonian H_0. In the case where V is small, but not completely negligible, the jump operator may still be efficiently approximated by including the first few terms of the expansion discussed above.

V. NUMERICAL DEMONSTRATION: HEISENBERG SPIN CHAIN

In this section we demonstrate how the universal Lindblad equation can be used in a numerical simulation. We consider a ferromagnetic spin-1/2 Heisenberg chain coupled to two

Ohmic baths that are out of equilibrium with each other, as schematically depicted in Fig. 2(a). By numerically solving the ULE, we obtain the nontrivial steady states and transport properties of the spin chain, along with its transient relaxation dynamics.

The system we consider cannot be easily simulated by current master equation techniques, and hence our demonstration highlights the utility of the ULE. For instance, the quantum optical master equation can only be employed when the system's relaxation rate is small compared to the level spacing of the system Hamiltonian. For the spin chain we consider, this level spacing is exponentially suppressed in the number of spins N, and hence, even for moderately sized chains, the quantum optical master equation only works for extremely weak system-bath couplings. In contrast, the validity of the ULE is independent of the level spacing in the system. Thus the ULE is valid for system-bath couplings many orders of magnitude larger than allowed by the quantum optical master equation.

Another common master equation approach, the Bloch Redfield equation, is also ill-suited for the spin chain we consider: the BR equation is often not stable, and may yield unphysical results, as discussed in the beginning of Sec. III. In contrast, the ULE is in the Lindblad form, and thus inherently robust. Even without instabilities, integration of the BR equation is numerically expensive, since it requires evolving the $D \times D$ density matrix of the system ρ, where $D = 2^N$ is the Hilbert space dimension of the system. On the other hand, Lindblad-form master equations can be integrated with the stochastic-Schrödinger equation, which only requires evolving a D-component state vector. This significantly reduces the computational cost, with the relative gain scaling exponentially with the size of the system.

The spin chain Hamiltonian is given by

$$H_S = -B_z \sum_{n=1}^{N} S_n^z - \eta \sum_{n=1}^{N-1} \mathbf{S}_n \cdot \mathbf{S}_{n+1}, \qquad (37)$$

where B_z denotes the strength of a uniform Zeeman field, η is the nearest-neighbor coupling strength, and $\mathbf{S}_n = (S_n^x, S_n^y, S_n^z)$, where S_n^μ denotes the spin-μ operator on site n in the chain.

For the simulations below, we take $N = 12$ sites. The system is connected to two baths, $\mathcal{B}_1$ and $\mathcal{B}_2$, via spins $\mathbf{S}_1$ and $\mathbf{S}_N$ at the opposite ends of the chain, as schematically depicted

in Fig. 2(a). For demonstration we couple the baths to the spins through their x components, S_1^x and S_N^x, with coupling strengths γ_1 and γ_2. The baths $\mathcal{B}_1$ and $\mathcal{B}_2$ are modeled as Ohmic baths in thermal equilibrium, with spectral functions given in Eq. (14). For the simulations below we take the baths to have the same values of the cutoff Λ and ω_0, but distinct temperatures, T_1 and T_2. The system-bath coupling of the system is hence given by

$$H_{\text{int}} = \sqrt{\gamma_1} S_1^x B_1' + \sqrt{\gamma_2} S_N^x B_2', \qquad (38)$$

where, for $\alpha = 1, 2$, B_α' is a bosonic field operator in the Ohmic bath $\mathcal{B}_\alpha$ with spectral function $J_\alpha(\omega)$ given by Eq. (14) with $T = T_\alpha$.

To obtain the master equation for the system, we cast the above system-bath coupling into the form given in Eq. (6). Within this framework, the bath consists of two noise channels with $X_1 = S_1^x$ and $X_2 = S_N^x$. The corresponding bath operators are given by $B_\alpha = \sqrt{\tilde{\gamma}_\alpha} B_\alpha'$ for $\alpha = 1, 2$, where $\tilde{\gamma}_\alpha \equiv \gamma_\alpha/\gamma$ denotes the relative system-bath coupling strength, and γ denotes the redundant energy scale we introduced in Eq. (6) to parametrize the overall system-bath coupling [see discussion below Eq. (6)]. Straightforward calculations [9] show that the elements of the 2×2 spectral function matrix $\mathbf{J}(\omega)$ are given by $J_{\alpha\beta}(\omega) = \delta_{\alpha\beta} \tilde{\gamma}_\alpha J_\alpha(\omega)$, where $\delta_{\alpha\beta}$ denotes the Kronecker symbol and $J_\alpha(\omega)$ denotes the spectral function of bath operator B_α' (see above). Note that the coefficients $\tilde{\gamma}_\alpha$ appear in the spectral function due the parametrization of H_{int} in Eq. (6), in which a single common coupling scale γ is factored out of the coupling Hamiltonian.

A. Relaxation to thermal steady state

We first seek to verify that, when connected only to bath 2, the system relaxes to a steady state in thermal equilibrium with the bath, as we expect from basic thermodynamics. In our simulation, we therefore set $\gamma_1 = 0$, $\gamma_2 = 0.02\eta$. The remaining parameters are set to $B_z = 8\eta$, $\Lambda = 100\eta$, $\omega_0 = 2\eta$ (for both baths), while $T_2 = 20\eta$, and $T_1 = 2\eta$. All parameters except for γ_1 and γ_2 are given by the same values throughout this section.

As a first step, we compute the characteristic bath timescales Γ^{-1} and τ, which define the regime of applicability of the universal Lindblad equation. Using Eq. (26), we find $\Gamma = 0.41\eta$ and $\tau = 0.00013/\eta$, resulting in $\Gamma\tau = 0.00053$. Thus, following the discussion in Sec. III, we expect the ULE to be valid. In particular, the ULE correctly describes the rate of change of the system's density matrix $\partial_t \rho$ up to a correction bounded by $2\Gamma^2\tau = 0.0043\eta$. This error bound is several orders of magnitude smaller than the other energy scales of the model, and we thus expect the ULE to faithfully capture the system's evolution and steady states [33].

To solve the ULE, we computed the system's jump operators $\{L_\lambda\}$ by exact diagonalization of H_S, using Eq. (33). Note that Eq. (36) can be used if diagonalization is not feasible. We excluded the Lamb shift from the simulation, since this term only weakly perturbs H_S; thus we do not expect it to affect the system's dynamics significantly [9]. In contrast, the jump operators, no matter how weak, break the unitarity of time evolution, and hence cannot be neglected in the master equation. We initialized the system in the state with all spins aligned against the uniform field B_z, and integrated the ULE

numerically using the stochastic Schrödinger equation, with an ensemble of 100 states [20–22].

In Fig. 2(b) we plot the expectation value of the average z magnetization in the chain $M = \frac{1}{N}\sum_{n=1}^N S_n^z$ as a function of time (red line). The uncertainty of the expectation value $\langle M \rangle \equiv \text{Tr}[\rho(t)M]$ arising from the finite number of ensemble states is smaller than the thickness of the line. As Fig. 2(b) shows, $\langle M \rangle$ reaches a stationary value after a transient relaxation period of approximate duration $50\eta^{-1}$. The steady-state value of $\langle M \rangle$ is identical to the expectation value of M in a Gibbs state at temperature T_2 (upper gray line), up to the accuracy of the simulation. A similar result arises in the case where the chain is connected only to bath $\mathcal{B}_1$: $\gamma_1 = 0.1\eta$ and $\gamma_2 = 0$ [blue curve in Fig. 2(b)]. Thus we confirm that the universal Lindblad equation reproduces the expected equilibrium steady states, further supporting its validity.

B. Nonequilibrium steady state with two baths

We now consider the case where the spin chain is simultaneously connected to both baths, $\mathcal{B}_1$ and $\mathcal{B}_2$, with $\gamma_1 = 0.1\eta$ and $\gamma_2 = 0.02\eta$. In this case, due to the temperature difference between the baths, we expect the system to reach a nonequilibrium steady state characterized by nonzero transport of energy and magnetization between the baths. With the parameters above, the characteristic timescales as defined in Eqs. (4) evaluate to $\Gamma \approx 3.6\eta$ and $\tau \approx 0.0032\eta^{-1}$. Thus, $\Gamma\tau \approx 0.011$ and $2\Gamma^2\tau \approx 0.079\eta$, indicating that the universal Lindblad equation should accurately describe the system's dynamics.

In Fig. 2(b) we plot the the magnetization in the chain $\langle M \rangle$ as a function of time (purple), obtained with the universal Lindblad equation. Similar to the two equilibrium cases, the magnetization settles to a steady-state value after a transient relaxation period of duration $\sim 50\eta^{-1}$. However, the relaxed system is not in a Gibbs state, but rather a more complicated nonequilibrium steady state: to demonstrate this, in Fig. 2(c) we show the site-resolved magnetization $\langle S_n^z(t) \rangle$, averaged over a time window of length $150\eta^{-1}$ at the end of the simulation. As Fig. 2(c) clearly shows, the local magnetization of the system is not uniform, but gradually increases from the left to the right end of the chain, indicative of a nonequilibrium steady state. In contrast, for the two cases where only a single bath is connected to the chain (red and blue), the local magnetization is uniform throughout the chain, consistent with a thermal Gibbs state at temperatures T_2 and T_1 of the connected baths [horizontal gray lines in Fig. 2(c)]. The skewed magnetization profile in the nonequilibrium case above reflects a nonzero transport of heat and magnetization (magnons) between the two baths through the chain. By direct computation (see Appendix F for details), we compute the average rate of heat transfer $\bar{I}_E$ and magnetization transfer $\bar{I}_M$ from bath 2 to bath 1 over a time window of duration $75\eta^{-1}$ at the end of the simulation, finding $\bar{I}_E = 2.1\eta^2 \pm 0.2\eta^2$ and $\bar{I}_M = 0.33\eta \pm 0.02\eta$.

VI. DISCUSSION

In this paper we derived a Lindblad-form master equation for open quantum many-body systems: the universal Lindblad equation (ULE). We identified rigorous upper bounds

for the correction to the ULE, expressed in terms of the intrinsic timescales of the bath and the system-bath coupling. Crucially, the correction bounds we obtained for the ULE are independent of the details of the system, and are of the same magnitude as the error bounds we obtained for the Bloch-Redfield (BR) equation, which is not in the Lindblad form. In this sense, the ULE is valid on an equivalent level of approximation as the BR equation.

The universal Lindblad equation opens up new possibilities for systematically studying a wide class of open quantum systems. These classes of systems include quantum many-body systems, and general driven quantum systems with dense energy spectra, for which the stringent conditions of the quantum optical master equation are not met. In addition, the ULE can be implemented with lower computational cost and greater stability than the BR equation, since by construction it preserves the positivity and trace of the reduced density matrix of the system.

We have demonstrated the utility of the ULE in numerical simulations of an open Heisenberg spin chain, where we used it to extract the transport characteristic of the system's steady state in a nonequilibrium setting. We expect the ULE can be used to easily infer other nonequilibrium characteristics of the chain, such as, e.g., the correlations of magnetization or heat current fluctuations, without adding any additional cost in the simulation. In addition to the spin chain model we considered here for demonstration, the universal Lindblad was recently used by one of the authors to simulate the dynamics of a periodically driven cavity-spin system in Ref. [36], and by our collaborators to study readout of topological qubits in Ref. [41]. The universal Lindblad equation was also implemented in numerical simulations in Ref. [23], in order to support the hypothesized master equation there (see Sec. I).

The principle underlying our derivation of the ULE is that there does not exist a unique Markov approximation in the Markovian regime $\Gamma\tau \ll 1$. Rather there exists an infinite family of Markov approximations yielding distinct time-local master equations for the system that each are valid on an equivalent level of approximation. From this family of equivalent master equations, we identified a master equation in the Lindblad form, the ULE.

An interesting avenue of future studies is the mathematical exploration of this equivalence class of Markov approximations; in particular, it will be interesting to investigate whether the above freedom of choice can be exploited further, to obtain master equations that are even more efficient or accurate, or perhaps explicitly respect desired symmetries or conservation laws. Another relevant question along this direction of research is whether higher-order bath correlations and non-Markovian corrections can also be incorporated in the framework we develop here, and yield efficient and accurate master equations for the system.

As stimulus for another direction of future work, we speculate that the correction bounds we obtained can be improved further. In particular, while we do not show it here, for Ohmic baths, the energy scale Γ scales linearly with the high-energy cutoff of the bath (see Sec. II B). However, this divergence arises from ultrashort (i.e., effectively time-local) correlations and reflects a divergent renormalization of the Hamiltonian through the Lamb shift. Hence, adding a correction to the

bare system Hamiltonian to compensate the divergent terms, we speculate that much better bounds can be obtained for the correction the ULE. Often, such a correcting counterterm is physically well motivated. We believe further analysis of the problem using this principle can lead to significant improvement of the error bounds for the ULE.

As important secondary results, in this work, we obtained rigorous error bounds for the Bloch-Redfield equation, and established a "quantum speed limit" for the rate of bath-induced evolution of open quantum systems. These results may also be relevant for future work. The results were established using a perturbative approach in Appendix A, in which the time derivative of the reduced density matrix of the system is systematically expanded in orders of the dimensionless number $\Gamma\tau$. We speculate that this approach may be used in the future to obtain master equations that are valid at higher orders in $\Gamma\tau$.

In summary, we have rigorously derived a Lindblad-form master equation for open quantum systems that offers several advantages over previously existing methods. We expect that the efficiency, wide applicability, and simplicity of our method opens up new possibilities for future studies of open quantum systems.

APPENDIX A: CORRECTION TO THE BLOCH-REDFIELD EQUATION

Here we derive the rigorous upper bounds for the correction to the Bloch-Redfield (BR) equation that were quoted in Sec. II A 2 in the main text. We derive the bounds for the general case where multiple noise channels connect the system and the bath. As a part of our derivation, in Appendix A 3 below we establish the upper bound for the rate of bath-induced evolution in the system $\|\partial_t\tilde{\rho}\|$ that we quoted in Sec. II B of the main text.

In this Appendix we work exclusively in the interaction picture (see Appendix II A). To avoid cumbersome notation, we therefore use different notation here than in the main text, and neglect the $\tilde{\ }$ accent on all interaction picture operators. Thus, throughout this Appendix, $\rho(t)$, $\rho_{\mathcal{SB}}(t)$, $H(t)$, $X_\alpha(t)$, and $B_\alpha(t)$ denote the interaction picture operators $\tilde{\rho}(t)$, $\tilde{\rho}_{\mathcal{SB}}(t)$, $\tilde{H}(t)$, $\tilde{X}_\alpha(t)$, and $\tilde{B}_\alpha(t)$ from the main text, respectively.

1. Superoperator formalism

Our derivation of error bounds for the BR equation exploits the fact that linear operators on a Hilbert space (such as density matrices) can themselves be seen as vectors, or "kets." To

make this vector nature of operators explicit, in the following we use double brackets $|\cdot\rangle\rangle$ to indicate operators acting on the Hilbert spaces $\mathbb{H}_{\mathcal{S}}$, $\mathbb{H}_{\mathcal{B}}$, and $\mathbb{H}_{\mathcal{SB}}$ of the system $\mathcal{S}$, bath $\mathcal{B}$, or the combined system $\mathcal{SB}$. In this way, any operator which is denoted by $\mathcal{O}$ in standard notation is denoted by the ket $|\mathcal{O}\rangle\rangle$ in the derivation below. The vector space of operator kets that act on Hilbert space $\mathbb{H}_i$ (with $i = \{\mathcal{S}, \mathcal{B}, \mathcal{SB}\}$) defines an operator Hilbert space $\mathbb{H}_i^2$, defined with the inner product $\langle\langle R|S\rangle\rangle \equiv \text{Tr}(R^\dagger S)$. This notation is commonly used in the literature, see, e.g., Ref. [43] for a recent example. Note that the operator space of the combined system $\mathcal{SB}$, $\mathbb{H}_{\mathcal{SB}}^2$, inherits the tensor product structure of the standard Hilbert space of $\mathcal{SB}$, $\mathbb{H}_{\mathcal{SB}}$: $\mathbb{H}_{\mathcal{SB}}^2 \cong \mathbb{H}_{\mathcal{S}}^2 \otimes \mathbb{H}_{\mathcal{B}}^2$, where $\mathbb{H}_{\mathcal{S}}^2$ and $\mathbb{H}_{\mathcal{B}}^2$ denote the operator spaces of the system $\mathcal{S}$ and bath $\mathcal{B}$, respectively.

In the superoperator notation above, the von-Neumann equation for the density matrix of the combined system (in the interaction picture) $\partial_t \rho_{\mathcal{SB}}(t) = -i[H(t), \rho_{\mathcal{SB}}(t)]$ translates to a linear Schrödinger-type equation:

$$\partial_t |\rho_{\mathcal{SB}}(t)\rangle\rangle = -i\hat{\mathcal{H}}(t)|\rho_{\mathcal{SB}}(t)\rangle\rangle, \tag{A1}$$

where $\hat{\mathcal{H}}(t)$ denotes the commutator with $H(t)$: $\hat{\mathcal{H}}(t)|\mathcal{O}\rangle\rangle = |[H(t), \mathcal{O}]\rangle\rangle$. Note that $\hat{\mathcal{H}}(t)$ acts linearly on $|\rho_{\mathcal{SB}}\rangle\rangle$, and hence it can be represented as a matrix acting on the operator space $\mathbb{H}_{\mathcal{SB}}^2$. Below we furthermore show that $\hat{\mathcal{H}}(t)$ is Hermitian, and hence can be seen as a "Hamiltonian" acting on $\mathbb{H}_{\mathcal{SB}}^2$. We refer to $\hat{\mathcal{H}}(t)$, and other linear transformations on operator kets, as superoperators. To make notation unambiguous, in the following we use the "hat" accent ($\hat{\ }$) to indicate superoperators.

A useful class of superoperators which we employ extensively in the following is left and right multiplication by some given operator: for any operator $|A\rangle\rangle$ in $\mathbb{H}_i^2$ (for $i = \{\mathcal{S}, \mathcal{B}, \mathcal{SB}\}$), we define the left- and right-multiplication superoperators $\hat{A}^l$ and $\hat{A}^r$ as

$$\hat{A}^l|\mathcal{O}\rangle\rangle = |A\mathcal{O}\rangle\rangle, \quad \hat{A}^r|\mathcal{O}\rangle\rangle = |\mathcal{O}A\rangle\rangle. \tag{A2}$$

From the above definition one can verify that the superoperator $\hat{\mathcal{H}}(t)$ in Eq. (A1) is given by $\hat{H}^l(t) - \hat{H}^r(t)$, where, for any time-dependent operator $A(t)$, and for $m = \{l, r\}$, we use $\hat{A}^m(t)$ as shorthand for $\widehat{A(t)}^m$ to avoid cumbersome notation. The right- and left-multiplication superoperators have a few useful properties that we use below: first, we note that, by associativity, $\hat{A}^l\hat{B}^l = \widehat{(AB)}^l$, while $\hat{A}^r\hat{B}^r = \widehat{(BA)}^r$. Moreover, the Hermitian conjugate of the superoperator $\hat{A}^m$ [i.e., $(\hat{A}^m)^\dagger$] is given by $(\hat{A}^\dagger)^m$. This follows from Eq. (A2), along with the definition of the inner product $\langle\langle\cdot|\cdot\rangle\rangle$: $\langle\langle\mathcal{O}_1|\hat{A}^l|\mathcal{O}_2\rangle\rangle = \langle\langle\mathcal{O}_1|A\mathcal{O}_2\rangle\rangle = \langle\langle A^\dagger\mathcal{O}_1|\mathcal{O}_2\rangle\rangle$. For this reason, in the following, we let $\hat{A}^{m\dagger}$ simply refer to $(\hat{A}^m)^\dagger = (\hat{A}^\dagger)^m$. From these results, it follows that $\hat{\mathcal{H}}(t)$ is Hermitian: $\hat{\mathcal{H}}(t) = \hat{\mathcal{H}}^\dagger(t)$, as we claimed above.

In deriving the error bounds for the Bloch-Redfield equation, we will make use of the norms of superoperators. We define the norm of the superoperator $\hat{A}$ acting on $\mathbb{H}_i^2$ as

$$\|\hat{A}\| \equiv \sup_{|\mathcal{O}\rangle\rangle \in \mathbb{H}_i^2} \frac{\|\hat{A}|\mathcal{O}\rangle\rangle\|}{\||\mathcal{O}\rangle\rangle\|}, \tag{A3}$$

where, here and in the following, $\||\mathcal{O}\rangle\rangle\|$ denotes the spectral norm of the operator $|\mathcal{O}\rangle\rangle$. Note that $\||\mathcal{O}\rangle\rangle\|$ is *not* identical to $\sqrt{\langle\langle\mathcal{O}|\mathcal{O}\rangle\rangle}$; rather, $\sqrt{\langle\langle\mathcal{O}|\mathcal{O}\rangle\rangle}$ gives the Frobenius norm of $|\mathcal{O}\rangle\rangle$.

From the definition above, it follows that the superoperator norm is submultiplicative: $\|\hat{A}\hat{B}\| \leqslant \|\hat{A}\|\|\hat{B}\|$. Moreover, using the submultiplicativity of the spectral norm along with the definitions in Eq. (A2), we conclude that, for any operator $|\mathcal{O}\rangle\rangle$ and for $m = \{l, r\}$, $\|\hat{\mathcal{O}}^m\| = \||\mathcal{O}\rangle\rangle\|$.

Using the superoperator notation above, we now consider the evolution of the reduced density matrix of the system in the interaction picture $|\rho(t)\rangle\rangle$. Recalling that the superoperator $\hat{\mathcal{H}}(t)$ in Eq. (A1) is Hermitian, the "Schrödinger equation" for the density matrix of the combined system, Eq. (A1), has the well-known solution

$$|\rho_{\mathcal{SB}}(t)\rangle\rangle = \hat{\mathcal{U}}(t, s)|\rho_{\mathcal{SB}}(s)\rangle\rangle, \tag{A4}$$

where $\hat{\mathcal{U}}(t, s)$ denotes the unitary evolution superoperator of the combined system, given by $\hat{\mathcal{U}}(t, s) = \mathcal{T}e^{-i\int_s^t dt' \hat{\mathcal{H}}(t')}$. By taking the time derivative, one can verify that $\hat{\mathcal{U}}(t, s) = \hat{U}_{\mathcal{SB}}^l(t, s)\hat{U}_{\mathcal{SB}}^{r\dagger}(t, s)$, where $U_{\mathcal{SB}}(t, s) = \mathcal{T}e^{-i\int_s^t dt' H(t')}$ denotes the (ordinary) time-evolution operator of the combined system. Thus $|\rho_{\mathcal{SB}}(t)\rangle\rangle = |U_{\mathcal{SB}}(t, s)\rho_{\mathcal{SB}}(s)U_{\mathcal{SB}}^\dagger(t, s)\rangle\rangle$. Using the properties of the superoperator norm below Eq. (A3), we conclude that that $\|\hat{\mathcal{U}}(t, s)\| = 1$.

In the superoperator notation, the partial trace Tr_B over the bath degrees of freedom can be expressed as the dual vector (bra) of the bath identity operator $|I_{\mathcal{B}}\rangle\rangle$. Here, as for ordinary bra-ket notation, $\langle\langle Y_{\mathcal{B}}|$ is understood as the linear mapping $\mathbb{H}_{\mathcal{SB}}^2 \to \mathbb{H}_{\mathcal{S}}^2$, such that, for $|M_{\mathcal{SB}}\rangle\rangle = \sum_{a,b} M_{ab}|a_{\mathcal{S}}\rangle\rangle|b_{\mathcal{B}}\rangle\rangle$, $\langle\langle Y_{\mathcal{B}}|M_{\mathcal{SB}}\rangle\rangle = \sum_{a,b} M_{ab}|a_{\mathcal{S}}\rangle\rangle\langle\langle Y_{\mathcal{B}}|b_{\mathcal{B}}\rangle\rangle$, where a and b label orthonormal bases for the operator spaces on $\mathcal{S}$ and $\mathcal{B}$, respectively. (Recall that the operator space $\mathcal{H}_{\mathcal{SB}}^2$ inherits the tensor product structure of $\mathcal{H}_{\mathcal{SB}}$.) As a result, we may write the reduced density matrix of the system $\mathcal{S}$ as $|\rho(t)\rangle\rangle = \langle\langle I_{\mathcal{B}}|\rho_{\mathcal{SB}}(t)\rangle\rangle$. Inserting the above result into Eq. (A4), and using our assumption that $|\rho_{\mathcal{SB}}(t_0)\rangle\rangle = |\rho_0\rangle\rangle|\rho_{\mathcal{B}}\rangle\rangle$ for some time t_0 in the remote past (see Sec. II in the main text), we find

$$|\rho(t)\rangle\rangle = \langle\langle I_{\mathcal{B}}|\hat{\mathcal{U}}(t, t_0)|\rho_{\mathcal{B}}\rangle\rangle|\rho_0\rangle\rangle. \tag{A5}$$

To obtain a master equation for $|\rho(t)\rangle\rangle$, we explicitly take the time derivative in Eq. (A5), obtaining

$$\partial_t|\rho(t)\rangle\rangle = -i\langle\langle I_{\mathcal{B}}|\hat{\mathcal{H}}(t)\hat{\mathcal{U}}(t, t_0)|\rho_{\mathcal{B}}\rangle\rangle|\rho_0\rangle\rangle. \tag{A6}$$

Using the decomposition $H(t) = \sqrt{\gamma} \sum_\alpha X_\alpha(t)B_\alpha(t)$ [Eq. (6) in the main text, translated to the interaction picture], we find $\hat{\mathcal{H}}(t) = \sqrt{\gamma} \sum_{m,\alpha} v_m\hat{X}_\alpha^m(t)\hat{B}_\alpha^m(t)$, where $m = \{l, r\}$, with $v_l = 1$ and $v_r = -1$. Recalling that (for each α) $X_\alpha(t)$ acts trivially on the bath degrees of freedom, we obtain

$$\partial_t|\rho(t)\rangle\rangle = -i\sqrt{\gamma} \sum_{m,\alpha} v_m\hat{X}_\alpha^m(t)\langle\langle I_{\mathcal{B}}|\hat{B}_\alpha^m(t)\hat{\mathcal{U}}(t, t_0)|\rho_{\mathcal{B}}\rangle\rangle|\rho_0\rangle\rangle. \tag{A7}$$

2. Statistical properties of the bath

To obtain a convenient expression for the bath expectation value $\langle\langle I_{\mathcal{B}}|\hat{B}_\alpha^m(t)\hat{\mathcal{U}}(t, t_0)|\rho_{\mathcal{B}}\rangle\rangle$ in Eq. (A7), we make use of our assumption that the bath is Gaussian. For simplicity, in this section we assume that all bath operators are bosonic. Similar considerations can be applied for fermionic bath operators.

For a Gaussian bath, the expectation value of any bath operator can be computed from the two-point correlation

function using Wick's theorem. In the superoperator notation we use, with $\hat{B}_j \equiv \hat{B}_{\alpha_j}^{m_j}(t_j)$ (where $m_j = \{l, r\}$, while α_j refers to the noise channel index), Wick's theorem applied to a product of k bath operators takes the form

$$\langle\langle \hat{B}_1 \cdots \hat{B}_k \rangle\rangle = \sum_{j=2}^{k} \langle\langle \hat{B}_1 \hat{B}_j \rangle\rangle \langle\langle \hat{A}_{2,j-1} \hat{A}_{j+1,k} \rangle\rangle, \quad (A8)$$

where $\hat{A}_{i,j} = \prod_{n=i}^{j} \hat{B}_n$ for $j \geqslant i$, $\hat{A}_{i,j} = 1$ for $j < i$, and we introduced the shorthand $\langle\langle \hat{\mathcal{O}} \rangle\rangle \equiv \langle\langle I_B | \hat{\mathcal{O}} | \rho_B \rangle\rangle$ to simplify notation. Wick's theorem for superoperators, as stated in Eq. (A8), can be proven by direct computation using the definitions of the superoperators $\{\hat{B}_j\}$, along with Wick's theorem for the (nonsuper) operators $\{B_{\alpha_j}(t_j)\}$ [9,10].

By iteration of Wick's theorem [Eq. (A8)], it is straightforward to show that the expectation value $\langle\langle (\cdot) \rangle\rangle$ of any polynomial functional of the bath superoperators $\{\hat{B}_\alpha(t)\}$ can be expressed fully in terms of the (two-point) bath superoperator correlation functions

$$J_{\alpha\beta}^{mn}(t - t') \equiv \langle\langle \hat{B}_\alpha^m(t) \hat{B}_\beta^n(t') \rangle\rangle. \quad (A9)$$

The bath superoperator correlation functions hold the same information as the ordinary bath correlation function $\boldsymbol{J}(t - s)$ [see Eq. (24) in the main text]: letting $\boldsymbol{J}^{mn}(t)$ denote the matrix with elements $\{J_{\alpha\beta}^{mn}(t)\}$, and using the cyclic property of the

trace, one can verify that, for $m = \{l, r\}$, $\boldsymbol{J}^{ml}(t) = \boldsymbol{J}(t)$, while $\boldsymbol{J}^{mr}(t) = \boldsymbol{J}^\dagger(t)$.

Importantly, the unitary evolution superoperator of the combined system $\mathcal{SB}$, $\hat{\mathcal{U}}(t, s)$, is analytic, and hence can be expanded as a polynomial of the bath superoperators $\{\hat{B}_\alpha^m(t)\}$. By using this expansion along with Wick's theorem [Eq. (A8)], one can then verify that

$$\langle\langle \hat{B}_\alpha^m(t) \hat{\mathcal{U}}(t, s) \rangle\rangle = \int_{-\infty}^{\infty} dt' \sum_{\beta, n} J_{\alpha\beta}^{mn}(t - t') \left\langle\!\!\left\langle \frac{\delta \hat{\mathcal{U}}(t, s)}{\delta \hat{B}_\beta^n(t')} \right\rangle\!\!\right\rangle, \quad (A10)$$

where $\delta / \delta \hat{B}_\beta^n(t')$ denotes the functional derivative with respect to $\hat{B}_\beta^n(t')$. Specifically, $\delta \hat{B}_\alpha^m(t) / \delta \hat{B}_\beta^n(t') = \delta_{\alpha\beta} \delta_{mn} \delta(t - t')$, where δ_{ij} denotes the Kronecker symbol, and $\delta(t)$ is the Dirac delta function.

Using the Trotter decomposition of $\hat{\mathcal{U}}(t, s)$ along with $\hat{\mathcal{H}}(t) = -i\sqrt{\gamma} \sum_{m,\alpha} v_m \hat{X}_\alpha^m(t) \hat{B}_\alpha^m(t)$, one can verify that, for t' in the interval between s and t,

$$\frac{\delta \hat{\mathcal{U}}(t, s)}{\delta \hat{B}_\beta^n(t')} = -i\sqrt{\gamma} v_n \mathcal{U}(t, t') \hat{X}_\beta^n(t') \hat{\mathcal{U}}(t', s), \quad (A11)$$

while $\delta \hat{\mathcal{U}}(t, s) / \delta \hat{B}_\beta^n(t') = 0$ when t' is outside the interval between s and t. Inserting Eqs. (A10) and (A11) into Eq. (A7) gives

$$\partial_t |\rho(t)\rangle\rangle = -\gamma \sum_{m,n;\alpha,\beta} v_m v_n \hat{X}_\alpha^m(t) \int_{t_0}^{t} ds\, J_{\alpha\beta}^{mn}(t - s) \langle\langle I_\mathcal{B} | \hat{\mathcal{U}}(t, s) \hat{X}_\beta^n(s) \hat{\mathcal{U}}(s, t_0) | \rho_\mathcal{B} \rangle\rangle | \rho_0 \rangle\rangle. \quad (A12)$$

Equation (A12) is a crucial result, and forms the basis for the derivation below. Importantly, the result is exact for Gaussian baths, and does not rely on any other approximations or assumptions. Equation (A12) can be generalized to non-Gaussian baths by expanding the left-hand side of Eq. (A10) in terms of the (nonvanishing) higher-order correlation functions of the bath. While such an extension to non-Gaussian baths is in principle straightforward, in this Appendix we restrict ourselves for simplicity to the case of Gaussian baths.

3. Upper bound for rate of bath-induced evolution

While Eq. (A12) looks somewhat complicated, we may already use it in its present form to infer important facts about the evolution of the system. Specifically, in this subsection, using Eq. (A12), we identify an upper limit for the rate of bath-induced evolution in the system $\|\partial_t |\rho\rangle\rangle \|$. This result was quoted in Sec. II B of the main text [recall that $|\rho(t)\rangle\rangle$ in this Appendix is identical to $\tilde{\rho}(t)$ in the main text]. The arguments and concepts we use here will also be used in the following subsections, when we derive error bounds for the Bloch-Redfield equation.

To derive an upper bound for $\|\partial_t |\rho\rangle\rangle \|$, we take the (spectral) norm on both sides in Eq. (A12). Using the triangle inequality along with $\|\hat{X}_\alpha^m(t) | \mathcal{O} \rangle\rangle \| \leqslant \| |\mathcal{O}\rangle\rangle \|$ (this follows from the properties of the superoperator norm listed in Appendix A 1 and the fact that the operators X_α are assumed to have unit

spectral norm), we thereby obtain

$$\| \partial_t |\rho(t)\rangle\rangle \| \leqslant \gamma \sum_{m,n;\alpha,\beta} \int_{t_0}^{t} ds\, \big| J_{\alpha\beta}^{mn}(t - s) \big| k_\beta^n(t, s), \quad (A13)$$

where $k_\beta^n(t, s) \equiv \| \langle\langle I_\mathcal{B} | \hat{\mathcal{U}}(t, s) \hat{X}_\beta^n(s) \hat{\mathcal{U}}(s, t_0) | \rho_\mathcal{B} \rangle\rangle | \rho_0 \rangle\rangle \|$. We now prove that $k_\beta^n(t, s) \leqslant 1$. To establish this bound, it is simplest to consider the cases $n = l$ and $n = r$ separately. Specifically, below we prove that $k_\beta^l(t, s) \leqslant 1$. The proof for $k_\beta^r(t, s) \leqslant 1$ proceeds along the same lines.

To establish that $k_\beta^l(t, s) \leqslant 1$, we write $k_\beta^l(t, s) = \| |Q\rangle\rangle \|$, where

$$|Q\rangle\rangle \equiv \langle\langle I_\mathcal{B} | \hat{\mathcal{U}}(t, s) \hat{X}_\beta^l(s) \hat{\mathcal{U}}(s, t_0) | \rho_\mathcal{B} \rangle\rangle | \rho_0 \rangle\rangle \quad (A14)$$

represents an operator on the system $\mathcal{S}$. We now note that $\hat{\mathcal{U}}(s, t_0) | \rho_\mathcal{B} \rangle\rangle | \rho_0 \rangle\rangle = | \rho_{\mathcal{SB}}(s) \rangle\rangle$, since $| \rho_{\mathcal{SB}}(t_0) \rangle\rangle = | \rho_\mathcal{B} \rangle\rangle | \rho_0 \rangle\rangle$. Thus $|Q\rangle\rangle = \langle\langle I_\mathcal{B} | \hat{\mathcal{U}}(t, s) \hat{X}_\beta^l(s) | \rho_{\mathcal{SB}}(s) \rangle\rangle$.

We now convert the above expression for $|Q\rangle\rangle$ into standard (nonsuperoperator) notation for the corresponding operator Q that acts on system $\mathcal{S}$:

$$Q = \text{Tr}_\mathcal{B}[U_{\mathcal{SB}}(t, s) X_\beta(s) \rho_{\mathcal{SB}}(s) U_{\mathcal{SB}}^\dagger(t, s)], \quad (A15)$$

where $U_{\mathcal{SB}}(t, s) = \mathcal{T} e^{-i \int_s^t dt' H(t')}$ denotes the standard (i.e, nonsuper) unitary evolution operator of the combined system in the interaction picture [see discussion below Eq. (A4)]. We recall that the spectral norm of Q, also denoted $\| |Q\rangle\rangle \|$, is given by the maximal value of $|\langle \phi | Q | \psi \rangle|$ for any two normalized states $|\psi\rangle$, $|\phi\rangle$ in the system Hilbert space $\mathbb{H}_\mathcal{S}$. To bound this

number, we exploit the cyclic property of the trace along with Eq. (A15) to write

$$\langle \phi | Q | \psi \rangle = \mathrm{Tr}_{\mathcal{SB}}[C \rho_{\mathcal{SB}}(s)], \qquad (A16)$$

where $C \equiv U_{\mathcal{SB}}^{\dagger}(t, s)(|\psi\rangle\langle\phi| \otimes I_{\mathcal{B}})U_{\mathcal{SB}}(t, s)X_{\beta}(s)$, with $I_{\mathcal{B}}$ denoting the identify operator on the bath Hilbert space $\mathbb{H}_{\mathcal{B}}$.

Next, we use the spectral decomposition of $\rho_{\mathcal{SB}}(s)$, $\rho_{\mathcal{SB}}(s) = \sum_i |n_i\rangle\langle n_i| p_i$, where $\{|n_i\rangle\}$ form an orthonormal basis for the Hilbert space of the combined system $\mathbb{H}_{\mathcal{SB}}$, and the eigenvalues $\{p_i\}$ are non-negative and have unit sum. Inserting this into Eq. (A16), we find $\langle \phi | Q | \psi \rangle = \sum_i \langle n_i | C | n_i \rangle p_i$. Using the triangle inequality, along with $|\langle n_i | C | n_i \rangle| \leqslant \|C\|$, where $\|\cdot\|$ denotes the spectral norm, we find

$$|\langle \phi | Q | \psi \rangle| \leqslant \|C\|, \qquad (A17)$$

where we also exploited the non-negativity and unit sum of the eigenvalues $\{p_i\}$. Using the submultiplicativity of the spectral norm and the fact that $\||\psi\rangle\langle\phi| \otimes I_{\mathcal{B}}\| \leqslant 1$ when $|\psi\rangle$ and $|\phi\rangle$ are normalized, one can verify that $\|C\| \leqslant 1$. Thus, for any normalized states $|\psi\rangle$ and $|\phi\rangle$, $|\langle\psi|Q|\phi\rangle| \leqslant 1$. We thus conclude that $\||Q\rangle\rangle\| = k_{\beta}^l(t, s)$ must be smaller than or equal to 1. The same line of arguments shows that $k_{\beta}^r(t, s) \leqslant 1$. Recalling that $k_{\beta}^n(t, s)$ by construction cannot be negative, we thus conclude

$$0 \leqslant k_{\beta}^n(t, s) \leqslant 1. \qquad (A18)$$

We now use Eq. (A18) in Eq. (A13) to obtain

$$\|\partial_t |\rho(t)\rangle\rangle\| \leqslant \gamma \sum_{m,n;\alpha,\beta} \int_{t_0}^t ds \, |J_{\alpha\beta}^{mn}(t - s)|. \qquad (A19)$$

Evaluating the sum, using the results below Eq. (A9), we obtain

$$\|\partial_t |\rho(t)\rangle\rangle\| \leqslant 4\gamma \int_{t_0}^t ds \, \|\mathbf{J}(t - s)\|_1, \qquad (A20)$$

where $\|\cdot\|_1$ denotes the entrywise matrix 1-norm, such that for any matrix $\mathbf{M}$ with elements $\{M_{\alpha\beta}\}$, $\|\mathbf{M}\|_1 \equiv \sum_{\alpha\beta} |M_{\alpha\beta}|$.

To obtain Eq. (A20), we used the relation $\|\mathbf{M}\|_1 = \|\mathbf{M}^{\dagger}\|_1$, which follows from the definition above.

Extending the lower limit of integration in Eq. (A20) to $-\infty$ and changing integration variables, we finally obtain

$$\|\partial_t |\rho(t)\rangle\rangle\| \leqslant \Gamma_0, \quad \Gamma_0 \equiv 4\gamma \int_0^{\infty} dt \, \|\mathbf{J}(t)\|_1. \qquad (A21)$$

Thus, the energy scale Γ_0 sets an upper bound for the rate of bath-induced evolution in the system, $\||\partial_t \rho(t)\rangle\rangle\|$. This timescale was also identified in Ref. [17] (see main text and Appendix B for further discussion). In Appendix B we further show that $\Gamma_0 \leqslant \Gamma/2$, where Γ was given in Eq. (26) in the main text [see Eq. (4) for the special case of a single noise channel]. Thus, recalling that $|\rho(t)\rangle\rangle$ corresponds to $\tilde{\rho}(t)$ in the main text, we have shown that

$$\|\partial_t \tilde{\rho}(t)\| \leqslant \Gamma/2. \qquad (A22)$$

This was the result quoted in Sec. II B of the main text.

4. Error induced by the Born approximation

Until now, our derivation has been exact, with our only assumptions being that the bath is Gaussian, and that the system and bath were decoupled at some point t_0 in the remote past (see Sec. II A in the main text). At this point, exact manipulations cannot take us further, and we thus need to make our first approximation: the Born approximation.

To make the Born approximation, we integrate the equation of motion for the evolution superoperator of the combined system $\partial_t \hat{\mathcal{U}}(t, s) = -i\hat{\mathcal{H}}(t)\hat{\mathcal{U}}(t, s)$:

$$\hat{\mathcal{U}}(t, s) = 1 - i \int_s^t dt' \, \hat{\mathcal{H}}(t')\hat{\mathcal{U}}(t', s). \qquad (A23)$$

By directly substituting this expression in for the factor of $\hat{\mathcal{U}}(t, s)$ in Eq. (A12), we obtain

$$\partial_t |\rho(t)\rangle\rangle = -\gamma \sum_{m,n;\alpha,\beta} \nu_m \nu_n \int_{t_0}^t ds \, \hat{X}_{\alpha}^m(t) J_{\alpha\beta}^{mn}(t - s) \left(\hat{X}_{\beta}^n(s)|\rho(s)\rangle\rangle - i \int_s^t dt' \, \langle\langle I_{\mathcal{B}}|\hat{\mathcal{H}}(t')\hat{\mathcal{U}}(t', s)\hat{X}_{\beta}^n(s)\hat{\mathcal{U}}(s, t_0)|\rho_{\mathcal{B}}\rangle\rangle|\rho_0\rangle\rangle \right). \qquad (A24)$$

For the first term in the parentheses above we used that $\hat{X}_{\beta}^n(t)$ acts trivially on the bath, along with $\hat{\mathcal{U}}(s, t_0)|\rho_0\rangle\rangle|\rho_{\mathcal{B}}\rangle\rangle = |\rho_{\mathcal{SB}}(s)\rangle\rangle$ and $\langle\langle I_{\mathcal{B}}|\rho_{\mathcal{SB}}(s)\rangle\rangle = |\rho(s)\rangle\rangle$, such that $\langle\langle I_{\mathcal{B}}|\hat{X}_{\beta}^n(s)\hat{\mathcal{U}}(s, t_0)|\rho_{\mathcal{B}}\rangle\rangle|\rho_0\rangle\rangle = \hat{X}_{\beta}^n(s)|\rho(s)\rangle\rangle$.

Next, we separate the two terms in the parentheses in Eq. (A24). Referring to the second term in the resulting expression as $|\xi_{\mathrm{B}}(t)\rangle\rangle$ (we discuss this term in further detail below), we find

$$\partial_t |\rho(t)\rangle\rangle = -\gamma \int_{t_0}^t ds \, \hat{\Delta}_{\mathrm{B}}(t, s)|\rho(s)\rangle\rangle + |\xi_{\mathrm{B}}(t)\rangle\rangle,$$

$$\text{where} \quad \hat{\Delta}_{\mathrm{B}}(t, s) \equiv -\gamma \sum_{m,n;\alpha,\beta} \nu_m \nu_n \hat{X}_{\alpha}^m(t)\hat{X}_{\beta}^n(s)J_{\alpha\beta}^{mn}(t - s).$$

$$(A25)$$

By applying the definitions of the quantities $\hat{X}_{\alpha}^m(t)$, ν_m, and $J_{\alpha\beta}^{mn}(t)$, one can verify that the first term on the right-hand side of Eq. (A25) is identical to the master equation for $|\rho(t)\rangle\rangle$ in the Born approximation [9] [see text above Eq. (9) in the main text for the single-channel case]. Hence, the Born approximation is equivalent to neglecting the term $|\xi_{\mathrm{B}}(t)\rangle\rangle$ in the above, and we identify $|\xi_{\mathrm{B}}(t)\rangle\rangle$ as the error induced by the Born approximation. Note that the Born-approximated master equation for $|\rho(t)\rangle\rangle$ [Eq. (A25) with the correction $|\xi_{\mathrm{B}}(t)\rangle\rangle$ neglected] can also be obtained through other approaches than the one we use here. For example, this result may also be obtained using the Nakajima-Zwanzig equation (see, e.g., Refs. [9,44]).

We now seek a bound for the norm of $|\xi_{\mathrm{B}}(t)\rangle\rangle$, i.e., the norm of the error in $\partial_t |\rho(t)\rangle\rangle$ induced by the Born approximation.

Matching Eqs. (A24) and (A25), we see that

$$
|\xi_{\mathrm{B}}(t)\rangle\!\rangle = i\gamma \sum_{m,n;\alpha,\beta} \nu_m \nu_n \int_{t_0}^t ds\, J_{\alpha\beta}^{mn}(t-s)\hat{X}_\alpha^m(t)
$$

$$
\times \int_s^t dt' \langle\!\langle I_\mathcal{B}|\hat{\mathcal{H}}(t')\hat{\mathcal{U}}(t',s)\hat{X}_\beta^n(s)\hat{\mathcal{U}}(s,t_0)|\rho_\mathcal{B}\rangle\!\rangle|\rho_0\rangle\!\rangle.
$$

$$
\text{(A26)}
$$

To obtain a bound for $\||\xi_{\mathrm{B}}(t)\rangle\!\rangle\|$, we take the norm on both sides of Eq. (A26) above. Using the triangle inequality and submultiplicativity of the superoperator norm, along with $\|\hat{X}_\alpha^m(t)\| = 1$, we find

$$
\||\xi_{\mathrm{B}}(t)\rangle\!\rangle\| \leqslant \gamma \sum_{m,n;\alpha,\beta} \int_{t_0}^t ds\, |J_{\alpha\beta}^{mn}(t-s)| \int_s^t dt'\, q_\beta^n(t,t',s),
$$

$$
\text{(A27)}
$$

where

$$
q_\beta^n(t,t',s) \equiv \|\langle\!\langle I_\mathcal{B}|\hat{\mathcal{H}}(t')\hat{\mathcal{U}}(t',s)\hat{X}_\beta^n(s)\hat{\mathcal{U}}(s,t_0)|\rho_\mathcal{B}\rangle\!\rangle|\rho_0\rangle\!\rangle\|.
$$

Following the same line of arguments that showed that the number $k_\beta^n(t,s)$ in Appendix A 3 was bounded by 1, one can verify that

$$
0 \leqslant q_\beta^n(t,t',s) \leqslant \Gamma_0, \qquad \text{(A28)}
$$

where Γ_0 was defined in Eq. (A21). Substituting this result into Eq. (A27) and evaluating the integral over t', we find

$$
\||\xi_{\mathrm{B}}(t)\rangle\!\rangle\| \leqslant 4\gamma\,\Gamma_0 \int_{t_0}^t ds\, \|\boldsymbol{J}(t-s)\|_1 \cdot |t-s|, \qquad \text{(A29)}
$$

where the matrix norm $\|\cdot\|_1$ was defined in Appendix A 3. Extending the lower limit of integration to $-\infty$ and using the definition of Γ_0 in Eq. (A21), we obtain

$$
\||\xi_{\mathrm{B}}(t)\rangle\!\rangle\| \leqslant \Gamma_0^2 \tau_0, \qquad \tau_0 \equiv \frac{\int_0^\infty dt\, t\, \|\boldsymbol{J}(t)\|_1}{\int_0^\infty dt\, \|\boldsymbol{J}(t)\|_1}. \qquad \text{(A30)}
$$

The timescale τ_0, which was also identified in Ref. [17], can be seen as a measure for the characteristic decay timescale of correlations in the bath, and we expect it to typically be comparable to the timescale τ from the main text (see Appendix B and Sec. II B in the main text for further discussion). Importantly, in Appendix B we show that $\Gamma_0\tau_0 \leqslant \Gamma\tau$.

Using the above results, along with $\Gamma_0 \leqslant \Gamma/2$ (see Appendix A 3), we conclude that

$$
\||\xi_{\mathrm{B}}(t)\rangle\!\rangle\| \leqslant \Gamma^2 \tau/2. \qquad \text{(A31)}
$$

Recalling that $|\xi_{\mathrm{B}}(t)\rangle\!\rangle$ gives the correction to the Born-approximated master equation for the system [Eq. (A25)], we conclude that the Born approximation induces an error in the expression for $\partial_t \tilde{\rho}(t)$ whose spectral norm is no greater than $\Gamma^2\tau/2$.

5. Error induced by the Markov approximation

We now implement the Markov approximation, which is the second approximation necessary to derive the Bloch-Redfield equation. Below, we show that Markov approximation induces an error in the expression for $|\partial_t\rho(t)\rangle\!\rangle$ [i.e., $\partial_t\tilde{\rho}(t)$ in the main text] whose spectral norm is bounded by $\Gamma^2\tau/2$. This bound is identical to the error bound we obtained

for the Born approximation in Appendix A 4. In this sense, the Markov approximation is valid on an equivalent level of approximation as the Born approximation: the validity of one approximation by our arguments implies the validity of the other.

To implement the Markov approximation, we insert $|\rho(s)\rangle\!\rangle = |\rho(t)\rangle\!\rangle + [|\rho(s)\rangle\!\rangle - |\rho(t)\rangle\!\rangle]$ into Eq. (A25), thereby obtaining

$$
\partial_t|\rho(t)\rangle\!\rangle = \int_{t_0}^t ds\, \hat{\Delta}_{\mathrm{B}}(t,s)|\rho(t)\rangle\!\rangle + |\xi_{\mathrm{M}}(t)\rangle\!\rangle + |\xi_{\mathrm{B}}(t)\rangle\!\rangle,
$$

$$
\text{(A32)}
$$

where

$$
|\xi_{\mathrm{M}}(t)\rangle\!\rangle = \int_{t_0}^t ds\, \hat{\Delta}_{\mathrm{B}}(t,s)[|\rho(s)\rangle\!\rangle - |\rho(t)\rangle\!\rangle]. \qquad \text{(A33)}
$$

We note that neglecting the terms $|\xi_{\mathrm{B}}(t)\rangle\!\rangle$ and $|\xi_{\mathrm{M}}(t)\rangle\!\rangle$ in Eq. (A32) results in a Markovian master equation for the system. Recalling that $|\xi_{\mathrm{B}}(t)\rangle\!\rangle$ arises from the Born approximation, we hence identify $|\xi_{\mathrm{M}}(t)\rangle\!\rangle$ as the error induced by the Markov approximation. The Bloch-Redfield equation [9] [see Eq. (10) in the main text for the single-channel case] is obtained by neglecting these two terms, and subsequently taking the limit $t_0 \to -\infty$, i.e., using our assumption that t_0 was in the remote past. In Appendix A 6 we discuss the physical justification for this assumption, and provide a bound for the correction that arises when this limit is not taken.

To obtain an upper bound for the error induced by the Markov approximation $|\xi_{\mathrm{M}}(t)\rangle\!\rangle$, we take the norm on both sides in Eq. (A33) and use the triangle inequality. Recalling from Appendix A 3 that $\|\partial_t|\rho(t)\rangle\!\rangle\| \leqslant \Gamma_0$, we have $\||\rho(s)\rangle\!\rangle - |\rho(t)\rangle\!\rangle\| \leqslant \Gamma_0|t-s|$, where Γ_0 was defined in Eq. (A21). Moreover, we note $\|\hat{\Delta}_{\mathrm{B}}(t,s)\| \leqslant 4\gamma\|\boldsymbol{J}(t-s)\|_1$; this can be shown using the triangle inequality in Eq. (A25). Combining these inequalities, we find

$$
\||\xi_{\mathrm{M}}(t)\rangle\!\rangle\| \leqslant 4\Gamma_0\gamma \int_{t_0}^t ds\, \|\boldsymbol{J}(t-s)\|_1|t-s|. \qquad \text{(A34)}
$$

Extending the lower limit of integration to $-\infty$, and using the definitions of Γ_0 and τ_0 in Eqs. (A21) and (A30), we conclude that $\||\xi_{\mathrm{M}}(t)\rangle\!\rangle\| \leqslant \Gamma_0^2\tau_0$. Recalling that $\Gamma_0^2\tau_0 \leqslant \Gamma^2\tau/2$ (see Appendixes A 4 and B), we thus find

$$
\||\xi_{\mathrm{M}}(t)\rangle\!\rangle\| \leqslant \Gamma^2\tau/2. \qquad \text{(A35)}
$$

The result in Eq. (A35) shows that the error in the expression for $\partial_t|\rho(t)\rangle\!\rangle$ [corresponding to $\partial_t\tilde{\rho}(t)$ in the main text] induced by the Markov approximation $|\xi_{\mathrm{M}}(t)\rangle\!\rangle$, has spectral norm no greater than $\Gamma^2\tau/2$, as we claimed.

Based on the derivation above, we conclude that the density matrix of the system evolves according to the Markovian master equation

$$
\partial_t|\rho(t)\rangle\!\rangle = \int_{t_0}^t ds\, \hat{\Delta}_{\mathrm{B}}(t,s)|\rho(t)\rangle\!\rangle + |\xi(t)\rangle\!\rangle, \qquad \text{(A36)}
$$

where $\hat{\Delta}_{\mathrm{B}}(t,s)$ is given in Eq. (A25), and $|\xi(t)\rangle\!\rangle \equiv |\xi_{\mathrm{B}}(t)\rangle\!\rangle + |\xi_{\mathrm{M}}(t)\rangle\!\rangle$ denotes the error induced by the Born-Markov approximation. From our results above that $|\xi_{\mathrm{B}}(t)\rangle\!\rangle, |\xi_{\mathrm{M}}(t)\rangle\!\rangle \leqslant \Gamma^2\tau/2$, we hence conclude that the total error induced by the Markov and Born approximations is bounded by $\Gamma^2\tau$, as we claimed in the main text.

6. Transient correction from initialization at t_0

As a final step in our derivation, here we show that when t_0 is in the remote past, the error induced by extending t_0 to $-\infty$ in Eq. (A36) is negligible compared to the error induced by the Born-Markov approximation, $|\xi(t)\rangle\rangle \sim \mathcal{O}(\Gamma^2\tau)$. Specifically, we show that the spectral norm of this error is bounded by $\Gamma\tau/(t-t_0)$, and hence is negligible when $t - t_0 \gg \Gamma^{-1}$. By setting $t_0 \to -\infty$ in Eq. (A36), we obtain

$$\partial_t |\rho(t)\rangle\rangle = \hat{\mathcal{D}}_R(t)|\rho(t)\rangle\rangle + |\xi(t)\rangle\rangle, \tag{A37}$$

where $\hat{\mathcal{D}}_R(t) \equiv \int_{-\infty}^{t} ds\, \hat{\Delta}_B(t,s)$. This is the Bloch-Redfield equation [9] [including the error induced by the Born-Markov approximation, see Eq. (10) in the main text for the single-channel case].

To establish a bound for the error induced by setting $t_0 \to -\infty$ in Eq. (A36), we rewrite Eq. (A36) as follows:

$$\partial_t |\rho(t)\rangle\rangle = [\hat{\mathcal{D}}_R(t) + \hat{\mathcal{D}}_T(t)]|\rho(t)\rangle\rangle + |\xi(t)\rangle\rangle, \tag{A38}$$

where $\hat{\mathcal{D}}_T(t) \equiv -\int_{-\infty}^{t_0} ds\, \hat{\Delta}_B(t,s)$. This term can be seen as the transient correction to the BR equation induced by the absence of system-bath correlations in our assumed initial state at time t_0, $|\rho_{\mathcal{SB}}(t_0)\rangle\rangle = |\rho_0\rangle\rangle|\rho_B\rangle\rangle$. This "correction" is thus an artifact of our choice of initial state (see Sec. II A). Below, we show that $\|\hat{\mathcal{D}}_T(t)|\rho(t)\rangle\rangle\| \leqslant \Gamma\tau/(t-t_0)$. Thus, when $t - t_0 \gg \Gamma^{-1}$, i.e., after a time long enough for weak correlations to be established between the system and the bath, the transient correction $\hat{\mathcal{D}}_T(t)$ is negligible compared to the bound we obtained for the error induced by the Born-Markov approximation $\Gamma^2\tau$. As a result, the BR equation [Eq. (A37)] accurately describes the system's evolution in this limit.

To show that $\|\hat{\mathcal{D}}_T(t)|\rho(t)\rangle\rangle\| \leqslant \Gamma\tau/(t-t_0)$, we consider the superoperator norm of $\hat{\mathcal{D}}_T(t)$ [see Eq. (A3)]. Noting that $\|\hat{\Delta}_B(t,s)\| \leqslant 4\gamma\|J(t-s)\|_1$ [this can be shown using the triangle inequality in Eq. (A25)], we find

$$\|\hat{\mathcal{D}}_T(t)\| \leqslant 4\gamma \int_{-\infty}^{t_0} ds\, \|J(t-s)\|_1. \tag{A39}$$

Using the fact that $t > t_0$, we have that $|t-s| \geqslant |t-t_0|$ for all $s \leqslant t_0$. Thus,

$$\|\hat{\mathcal{D}}_T(t)\| \leqslant 4\gamma \int_{-\infty}^{t} ds\, \|J(t-s)\|_1 \frac{|t-s|}{|t-t_0|}. \tag{A40}$$

Changing variables of integration and using the definitions of Γ_0 and τ_0 in Eqs. (A21) and (A30), we conclude

$$\|\hat{\mathcal{D}}_T(t)\| \leqslant \frac{\tau_0 \Gamma_0}{t-t_0}. \tag{A41}$$

Using the fact that that $\Gamma_0\tau_0 \leqslant \Gamma\tau$ (see Appendix B) along with the definition of the superoperator norm, we conclude that $\|\hat{\mathcal{D}}_T(t)|\rho(t)\rangle\rangle\| \leqslant \frac{\Gamma\tau}{t-t_0}$, as we claimed.

APPENDIX B: RELATIONSHIP BETWEEN BATH TIMESCALES

In this Appendix we discuss the relationship between the bath timescales Γ and τ introduced in Eq. (26) of the main text, and the timescales Γ_0^{-1}, τ_0 identified in Eqs. (A21) and

(A30) of Appendix A:

$$\Gamma_0 = 4\gamma \int_0^\infty dt\, \|J(t)\|_1, \quad \tau_0 = \frac{\int_0^\infty dt\, t\, \|J(t)\|_1}{\int_0^\infty dt\, \|J(t)\|_1}, \tag{B1}$$

where $J(t)$ denotes the matrix-valued bath correlation function (see Sec. III C), and $\|M\|_1 \equiv \sum_{\alpha\beta} |M_{\alpha\beta}|$ refers to the entrywise matrix 1-norm of a matrix M with elements $\{M_{\alpha\beta}\}$ (see Appendix A). The above timescales Γ_0^{-1} and τ_0 were also identified in Ref. [17].

Like the timescales Γ^{-1} and τ, Γ_0^{-1} and τ_0 serve as measures for the characteristic timescales for bath-induced evolution, and the decay bath correlations, respectively. In contrast to Γ^{-1} and τ, which are defined in terms of the "jump correlator" $g(t)$ [see Eqs. (3) and (25) of the main text], the timescales Γ_0^{-1} and τ_0^{-1} above are defined directly from the bath correlation function $J(t)$. However, as discussed in Sec. II B and demonstrated in Fig. 1, we expect these two distinct ways of characterizing the timescales of the bath to give comparable results in most cases. Further supporting this point, in this Appendix, we rigorously prove the following inequalities between the timescales $\{\Gamma_0^{-1}, \tau_0\}$ and $\{\Gamma, \tau\}$:

$$\Gamma_0 \leqslant \Gamma/2 \quad \text{and} \quad \Gamma_0\tau_0 \leqslant \Gamma\tau. \tag{B2}$$

These inequalities were used in Appendix A.

We first show that $\Gamma_0 \leqslant \Gamma/2$. We note from the definition of $J(t)$ in Eq. (24) that $J(t) = J^\dagger(-t)$. Using $\|M\|_1 = \|M^\dagger\|_1$, we thus have $\|J(t)\|_1 = \|J(-t)\|_1$. Using this result in Eq. (B1), we find

$$\Gamma_0 = 2\gamma \int_{-\infty}^{\infty} dt\, \|J(t)\|_1. \tag{B3}$$

To obtain a bound for $\|J(t)\|_1$ we note that $J(t)$ is related to the jump correlator $g(t)$ through the convolution $J(t) = \int_{-\infty}^{\infty} ds\, g(t-s)g(s)$. This result follows from the definition of $g(t)$ in Sec. III C, and is the multichannel generalization of the result quoted above Eq. (15) in the main text. Using the triangle inequality, we obtain

$$\|J(t)\|_1 \leqslant \int_{-\infty}^{\infty} ds\, \|g(t-s)g(s)\|_1. \tag{B4}$$

To rewrite the integrand above, we now prove that, for any two matrices A and B,

$$\|AB\|_1 \leqslant \|A^\dagger\|_{2,1}\|B\|_{2,1}, \tag{B5}$$

where the matrix norm $\|\cdot\|_{2,1}$ was defined below Eq. (26) in the main text: $\|M\|_{2,1} \equiv \sum_\beta (\sum_\alpha |M_{\alpha\beta}|^2)^{1/2}$ for a matrix M with elements $\{M_{\alpha\beta}\}$. To prove Eq. (B5), we recall that $\|AB\|_1 = \sum_{\alpha\beta\gamma} |A_{\alpha\beta}B_{\beta\gamma}|$. We consider the sum over the index β first. Using the Cauchy-Schwartz inequality, we find

$$\sum_\beta |A_{\alpha\beta}B_{\beta\gamma}| \leqslant \left(\sum_\beta |A_{\alpha\beta}|^2\right)^{1/2} \left(\sum_{\beta'} |B_{\beta'\gamma}|^2\right)^{1/2}. \tag{B6}$$

Using this inequality in the expression for $\|AB\|_1$ in Eq. (B5), we conclude $\|AB\|_1 \leqslant c_A c_B$, where $c_A = \sum_\alpha (\sum_\beta |A_{\alpha\beta}|^2)^{1/2}$ and $c_B = \sum_\gamma (\sum_{\beta'} |B_{\beta'\gamma}|^2)^{1/2}$. Recalling the definition of the norm $\|\cdot\|_{2,1}$, we identify $c_A = \|A^\dagger\|_{2,1}$ and $c_B = \|B\|_{2,1}$. Thus Eq. (B5) holds.

Combining Eqs. (B3)–(B5) we obtain

$$\Gamma_0 \leqslant 2\gamma \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \, \|\boldsymbol{g}^{\dagger}(t-s)\|_{2,1} \|\boldsymbol{g}(s)\|_{2,1}. \quad (B7)$$

The Hermiticity of $\boldsymbol{g}(\omega)$ implies that $\boldsymbol{g}(t) = \boldsymbol{g}^{\dagger}(-t)$ (see Sec. III C). Using this result in the above and shifting the variables of integration, we obtain

$$\Gamma_0 \leqslant 2\gamma \left[\int_{-\infty}^{\infty} dt \, \|\boldsymbol{g}(t)\|_{2,1} \right]^2. \quad (B8)$$

Comparing this result with the definition of Γ in Eq. (26) in the main text, we conclude that $\Gamma_0 \leqslant \Gamma/2$.

We now prove the second inequality in Eq. (B2), $\Gamma_0 \tau_0 \leqslant \Gamma\tau$. Using the fact that $\|\boldsymbol{J}(t)\|_1 = \|\boldsymbol{J}(-t)\|_1$ [see text above Eq. (B3)], along with the definitions of Γ_0 and τ_0 in Eq. (B1), we have

$$\Gamma_0 \tau_0 = 2\gamma \int_{-\infty}^{\infty} dt \, |t| \cdot \|\boldsymbol{J}(t)\|_1. \quad (B9)$$

Using Eqs. (B4) and (B5) along with $\boldsymbol{g}(t) = \boldsymbol{g}^{\dagger}(-t)$ [see text above Eq. (B8)], we obtain

$$\Gamma_0 \tau_0 \leqslant 2\gamma \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds \, |t| k(s-t) k(s), \quad (B10)$$

where we introduced the shorthand $k(t) \equiv \|\boldsymbol{g}(t)\|_{2,1}$. Using that $|t| \leqslant |s-t| + |s|$ and shifting variables of integration, one can then verify that $\Gamma_0 \tau_0 \leqslant 2\gamma \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} ds \, (|t'| + |s|) k(t') k(s)$. Exploiting the symmetry of this expression under exchange of t' and s, we find

$$\Gamma_0 \tau_0 \leqslant 4\gamma \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} ds \, |t'| k(t') k(s). \quad (B11)$$

Recalling that $k(t) \equiv \|\boldsymbol{g}(t)\|_{2,1}$, and comparing with the definitions of Γ and τ in Eq. (26), we identify the right-hand side above as $\Gamma\tau$. Thus, $\Gamma_0 \tau_0 \leqslant \Gamma\tau$, and Eq. (B2) holds. This was what we wanted to prove and concludes this Appendix.

APPENDIX C: DERIVATION OF THE ULE

In this Appendix we rigorously derive the universal Lindblad equation (ULE) in the interaction picture [Eq. (27) of the main text].

As in Appendixes A and B, we consider here the case of arbitrary system-bath coupling H_{int}, such that the system and bath are connected through multiple noise channels (see Sec. II in the main text). In the main text we heuristically derived the ULE for the case of a single noise channel [Eq. (21)]. This result is a special case of the more general result that we rigorously prove here, and hence this Appendix also serves as a proof of Eq. (21).

As discussed in the main text, the ULE [Eq. (27)] holds for a modified density matrix $\rho'(t)$ whose spectral norm distance to the exact density matrix $\tilde{\rho}(t)$ (in the interaction picture) remains bounded by $\Gamma\tau$ at all times. Here the bath timescales Γ^{-1} and τ were defined in Eq. (26) in the main text. In the Markovian limit, $\Gamma\tau \ll 1$, which is required for the ULE to be valid (see Secs. II A 2 and III A in the main text), the modified density matrix ρ' is thus nearly identical to the true density matrix $\tilde{\rho}$, and accurately describes the state of the system.

Our derivation below proceeds in three steps. In Appendix C 1 we define the modified density matrix $\rho'(t)$ [see Eq. (C8)] and prove that its spectral norm distance to $\tilde{\rho}(t)$ remains bounded by $\Gamma\tau$ at all times. Subsequently, in Appendix C 2 we show that $\rho'(t)$ evolves according to the master equation

$$\partial_t \rho'(t) = \mathcal{L}(t)[\rho'(t)] + \xi'(t),$$
$$\mathcal{L}(t) \equiv \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds' \, \mathcal{F}(s,t,s'), \quad (C1)$$

where the spectral norm of $\xi'(t)$ is bounded by $2\Gamma^2\tau$, and, for any operator A, we have defined

$$\mathcal{F}(s,t,s')[A] = \gamma \sum_{\alpha,\beta,\lambda} \theta(s-s')(g_{\alpha\lambda}(s-t)g_{\lambda\beta}(t-s')$$
$$\times [\tilde{X}_{\alpha}(s), A\tilde{X}_{\beta}(s')] + g_{\alpha\lambda}^*(s-t)g_{\lambda\beta}^*(t-s')$$
$$\times [\tilde{X}_{\beta}(s')A, \tilde{X}_{\alpha}(s)]). \quad (C2)$$

Note that the definitions above generalize the superoperators $\mathcal{L}(t)$ and $\mathcal{F}(s,t,s')$ in Sec. III A to cases with multiple noise channels [45]. As the third and final step of our derivation, in Appendix C 3 we show that the superoperator $\mathcal{L}(t)$ takes the Lindblad form in Eq. (27). Thereby we reach the goal of this Appendix, proving that $\rho'(t)$ evolves according to the Lindblad-form master equation in Eq. (27) of the main text.

1. Modified density matrix

Here we define the modified density matrix $\rho'(t)$, and prove that $\|\rho'(t) - \tilde{\rho}(t)\| \leqslant \Gamma\tau$ at all times. Our approach is to identify a transformation $\rho'(t) \equiv [1 + \mathcal{M}(t)]\tilde{\rho}(t)$ such that, if $\tilde{\rho}(t)$ satisfies the Bloch-Redfield equation [Eq. (A36)], then, up to an error of order $\Gamma^2\tau$, $\rho'(t)$ evolves according to Eq. (C1) (which can then be expressed in Lindblad form). We will bound the norm distance between $\rho'(t)$ and $\tilde{\rho}(t)$ using the explicit form of this transformation.

To motivate our definition of $\rho'(t)$, we note that the multichannel Bloch-Redfield (BR) equation [Eq. (A36) in Appendix A] can be written as

$$\partial_t \tilde{\rho}(t) = \int_{-\infty}^{\infty} ds' \int_{-\infty}^{\infty} ds \, \mathcal{F}(t,s,s')[\tilde{\rho}(t)] + \xi(t), \quad (C3)$$

where $\xi(t)$ denotes the error induced by the Born-Markov approximation, with norm bounded by $\Gamma^2\tau$. The expression above generalizes the single-channel result in Eq. (15) in the main text to the case of multiple noise channels. It is convenient to rewrite the right-hand side above in terms of the superoperator

$$\mathcal{G}(t,s) \equiv \int_{-\infty}^{\infty} ds' \, \mathcal{F}(t,s,s'). \quad (C4)$$

Specifically, we express the BR equation [Eq. (C3)] as

$$\partial_t \tilde{\rho}(t) = \int_{-\infty}^{\infty} ds \, \mathcal{G}(t,s)[\tilde{\rho}(t)] + \xi(t). \quad (C5)$$

Similarly, we may rewrite Eq. (C1) (our target equation of motion for the modified density matrix ρ') as

$$\partial_t \rho'(t) = \int_{-\infty}^{\infty} ds \, \mathcal{G}(s,t)[\rho'(t)] + \xi'(t). \quad (C6)$$

We will identify the precise form of the correction $\xi'(t)$ in the derivation below.

Note that, when neglecting the corrections $\xi(t)$ and $\xi'(t)$, the only difference between Eqs. (C5) and (C6) is the order of the arguments in the superoperator $\mathcal{G}$. The modified density matrix $\rho'(t)$ is obtained from a (time-local) linear operation on $\tilde{\rho}(t)$ that transforms Eq. (C5) into Eq. (C6). As we show in Appendix C 2 below, such a linear transformation is generated by the superoperator $[1 + \mathcal{M}(t)]$, where

$$\mathcal{M}(t) \equiv \int_t^\infty ds \int_{-\infty}^t ds' [\mathcal{G}(s, s') - \mathcal{G}(s', s)]. \quad \text{(C7)}$$

Specifically, we define $\rho'(t)$ as follows:

$$\rho'(t) = [1 + \mathcal{M}(t)][\tilde{\rho}(t)]. \quad \text{(C8)}$$

In Appendix C 2 we show that $\rho'(t)$, as defined above, evolves according to Eq. (C6). Before proving this, we show here that ρ' deviates from $\tilde{\rho}$ by a correction whose spectral norm is bounded by $\Gamma\tau$ at all times: $\|\rho'(t) - \tilde{\rho}(t)\| \leqslant \Gamma\tau$.

To show that $\|\rho'(t) - \tilde{\rho}(t)\| \leqslant \Gamma\tau$, we prove below that, for any operator A,

$$\|\mathcal{M}(t)[A]\| \leqslant \Gamma\tau \|A\|. \quad \text{(C9)}$$

By the definition of $\rho'(t)$ in Eq. (C8), this result in particular implies that $\|\rho'(t) - \tilde{\rho}(t)\| \leqslant \Gamma\tau$, since $\|\tilde{\rho}(t)\| \leqslant 1$. We will also use Eq. (C9) for other purposes in Appendix C 2.

To prove Eq. (C9), we use the triangle inequality in Eq. (C7) to obtain

$$\|\mathcal{M}(t)[A]\| \leqslant \int_t^\infty ds \int_{-\infty}^t ds' \{\|\mathcal{G}(s, s')[A]\| + \|\mathcal{G}(s', s)[A]\|\}. \quad \text{(C10)}$$

Using the triangle inequality in Eq. (C4), we have

$$\|\mathcal{G}(t, s)[A]\| \leqslant \int_{-\infty}^\infty ds' \|\mathcal{F}(t, s, s')[A]\|. \quad \text{(C11)}$$

From the definition of $\mathcal{F}$ in Eq. (C2), using the triangle inequality and the submultiplicativity of the spectral norm, one can verify that $\|\mathcal{F}(t, s, s')[A]\| \leqslant 4\gamma \|\boldsymbol{g}(t - s)\boldsymbol{g}(s - s')\|_1 \|A\| \theta(t - s)$, where the 1-matrix norm $\|\cdot\|_1$ is defined in Appendix A 3, and we used that $\|\tilde{X}_\alpha(t)\| \leqslant 1$. In Appendix B we established that $\|\boldsymbol{g}(t)\boldsymbol{g}(s)\|_1 \leqslant \|\boldsymbol{g}(-t)\|_{2,1} \|\boldsymbol{g}(s)\|_{2,1}$, where the matrix norm $\|\cdot\|_{2,1}$ is defined in Sec. III C. Thus,

$$\|\mathcal{F}(t, s, s')[A]\| \leqslant 4\gamma \|\boldsymbol{g}(s - t)\|_{2,1} \|\boldsymbol{g}(s - s')\|_{2,1} \theta(t - s') \|A\|.$$

Using this result in Eq. (C11), we find

$$\|\mathcal{G}(t, s)[A]\| \leqslant G(t - s)\|A\|, \quad \text{(C12)}$$

where

$$G(t) \equiv 4\gamma \|\boldsymbol{g}(-t)\|_{2,1} \int_{-\infty}^t ds \|\boldsymbol{g}(-s)\|_{2,1}. \quad \text{(C13)}$$

Using Eq. (C12) in Eq. (C10) we obtain

$$\|\mathcal{M}(t)[A]\| \leqslant \int_t^\infty ds \int_{-\infty}^t ds' [G(s - s') + G(s' - s)]\|A\|. \quad \text{(C14)}$$

To rewrite Eq. (C14), we note that, for any function $f(s)$, $\int_t^\infty ds \int_{-\infty}^t ds' f(s - s') = \int_0^\infty ds \, sf(s)$ (this can be verified

by change of integration variables). Using this result in Eq. (C14), we obtain

$$\|\mathcal{M}(t)[A]\| \leqslant \int_{-\infty}^\infty dt \, |t| G(t) \|A\|. \quad \text{(C15)}$$

We now seek a convenient bound for $G(t)$. Extending the upper limit of integration in Eq. (C13) to ∞, and using the definition of Γ in Eq. (26), we obtain

$$G(t) \leqslant \sqrt{4\gamma\Gamma} \|\boldsymbol{g}(-t)\|_{2,1}. \quad \text{(C16)}$$

Using this result in Eq. (C15) along with the definitions of Γ and τ in Eq. (26), we conclude that the right-hand side of Eq. (C15) is bounded by $\Gamma\tau\|A\|$. Thus, Eq. (C9) holds. By the arguments below Eq. (C9), we hence conclude $\|\rho'(t) - \tilde{\rho}(t)\| \leqslant \Gamma\tau$. This was what we wanted to show.

2. Master equation for modified density matrix

We now show that $\rho'(t)$, as defined in Eq. (C8), evolves according to the master equation in Eq. (C1). To establish this result, we explicitly take the time derivative of $\rho'(t)$ in Eq. (C8), obtaining

$$\partial_t \rho'(t) = \partial_t \tilde{\rho}(t) + \partial_t \mathcal{M}(t)[\tilde{\rho}(t)] + \mathcal{M}(t)[\partial_t \tilde{\rho}(t)], \quad \text{(C17)}$$

where we exploited the linear dependence of $\mathcal{M}(t)[\tilde{\rho}]$ on $\tilde{\rho}$. We consider the second term first in the above. Using the definition of $\mathcal{M}(t)$ in Eq. (C7), one can verify by explicit computation that

$$\partial_t \mathcal{M}(t) = \int_{-\infty}^\infty ds \, \mathcal{G}(s, t) - \int_{-\infty}^\infty ds \, \mathcal{G}(t, s), \quad \text{(C18)}$$

Inserting this result into Eq. (C17), and using Eq. (C5) along with $\mathcal{L}(t) = \int_{-\infty}^\infty ds \, \mathcal{G}(s, t)$ [see Eqs. (C1) and (C4)], we obtain

$$\partial_t \rho'(t) = \mathcal{L}(t)[\tilde{\rho}(t)] + \xi'_1(t) + \xi(t), \quad \text{(C19)}$$

where $\xi'_1(t) \equiv \mathcal{M}(t)[\partial_t \tilde{\rho}(t)]$. Noting that $\|\partial_t \tilde{\rho}(t)\| \leqslant \Gamma/2$ (see Sec. II B), and that $\|\mathcal{M}(t)[A]\| \leqslant \Gamma\tau\|A\|$ [Eq. (C9)], we conclude that $\|\xi'_1(t)\| \leqslant \Gamma^2\tau/2$.

As the final step in our derivation, we show that we may replace the argument $\tilde{\rho}(t)$ of $\mathcal{L}(t)$ in Eq. (C19) by $\rho'(t)$, at the cost of a correction $\xi'_2(t)$ whose spectral norm is bounded by $\Gamma^2\tau/2$. To show this, we exploit the linearity of $\mathcal{L}(t)$ to write

$$\mathcal{L}(t)[\tilde{\rho}(t)] = \mathcal{L}(t)[\rho'(t)] + \mathcal{L}(t)[\Delta\rho(t)], \quad \text{(C20)}$$

where $\Delta\rho(t) \equiv \tilde{\rho}(t) - \rho'(t)$. We now show that $\|\mathcal{L}(t)[A]\| \leqslant \Gamma\|A\|/2$, such that the second term in Eq. (C20) is bounded by $\Gamma^2\tau/2$ [recall that $\|\Delta\rho(t)\| \leqslant \Gamma\tau$, see Appendix C 1]. To prove this result, we use $\mathcal{L}(t) = \int_{-\infty}^\infty ds \, \mathcal{G}(s, t)$ along with Eq. (C12) to obtain

$$\|\mathcal{L}(t)[A]\| \leqslant \int_{-\infty}^\infty dt \, G(t) \|A\|, \quad \text{(C21)}$$

where $G(t)$ was defined in Eq. (C13). By explicit computation, using the definition of Γ in Eq. (26), one can verify that $\int_{-\infty}^\infty dt \, G(t) \leqslant \Gamma/2$. Thus $\|\mathcal{L}(t)[A]\| \leqslant \Gamma\|A\|/2$. We conclude that

$$\mathcal{L}(t)[\tilde{\rho}(t)] = \mathcal{L}(t)[\rho'(t)] + \xi'_2(t), \quad \text{(C22)}$$

where $\|\xi'_2(t)\| \leqslant \Gamma^2\tau/2$.

Using the relation in Eq. (C22) in Eq. (C19), we conclude that $\rho'(t)$, as defined in Eq. (C8), evolves according to Eq. (C1), with $\xi'(t) = \xi_1'(t) + \xi_2'(t) + \xi(t)$. Since the spectral norm of $\xi(t)$ is bounded by $\Gamma^2\tau$, while the spectral norms of $\xi_1'(t)$ and $\xi_2'(t)$ are both bounded by $\Gamma^2\tau/2$, we conclude that $\|\xi'(t)\| \leqslant 2\Gamma^2\tau$. Note that the bound for the error $\xi_1'(t) + \xi_2'(t)$ induced by the modified Markov approximation described above is identical to the bound for the error induced by the Born-Markov approximation $\xi(t)$.

3. Lindblad form of master equation

As the final step in our derivation, we now show that the right-hand side of the master equation for ρ' in Eq. (C1) is identical to the right-hand side of the universal Lindblad equation [Eq. (27) in the main text]. To prove this result, we first modify the expression for the superoperator $\mathcal{L}(t)$ that was defined in Eqs. (C1) and (C2). By decomposing the step function in Eq. (C2) into its symmetric and antisymmetric components: $\theta(s - s') = \frac{1}{2}[1 + \text{sgn}(s - s')]$, we find

$$\mathcal{L}(t) = \mathcal{L}_S(t) + \mathcal{L}_A(t), \qquad \mathcal{L}_i(t) \equiv \int_{-\infty}^{\infty} ds' \int_{-\infty}^{\infty} ds\, \mathcal{F}_i(s, t, s'), \quad i = \{S, A\}. \tag{C23}$$

For any density matrix ρ, we have defined

$$\mathcal{F}_S(s, t, s')[\rho] = -\frac{\gamma}{2} \sum_{\alpha,\beta,\lambda} [g_{\alpha\lambda}(s - t)\tilde{X}_\alpha(s), g_{\lambda\beta}(t - s')\tilde{X}_\beta(s')\rho] + \text{H.c.}, \tag{C24}$$

$$\mathcal{F}_A(s, t, s')[\rho] = -\frac{\gamma}{2} \sum_{\alpha,\beta} \phi_{\alpha\beta}(s - t, s' - t)[\tilde{X}_\alpha(s), \tilde{X}_\beta(s')\rho] + \text{H.c.}, \tag{C25}$$

where $\{\phi_{\alpha\beta}(s, t)\}$ denote the matrix elements of the $N \times N$ matrix $\boldsymbol{\phi}(t, s) \equiv \boldsymbol{g}(t)\boldsymbol{g}(-s)\text{sgn}(t - s)$ that was defined below Eq. (29). Below, we show that the superoperator $\mathcal{L}_S$ in Eq. (C23) generates the dissipative component of the ULE, while $\mathcal{L}_A$ generates the Lamb shift.

We consider the term $\mathcal{L}_S$ first. By direct computation, one can verify that

$$\mathcal{L}_S(t)[\rho] = -\frac{1}{2} \sum_{\lambda} [\tilde{L}_\lambda^\dagger(t), \tilde{L}_\lambda(t)\rho] + \text{H.c.}, \tag{C26}$$

where $\tilde{L}_\lambda(t)$ denotes the interaction picture jump operator defined in Eq. (28) in the main text. Here we used that $\tilde{L}_\lambda^\dagger(t) = \sqrt{\gamma} \int_{-\infty}^{\infty} ds'\, g_{\alpha\lambda}(s - t)\tilde{X}_\alpha(s)$, which follows from the relation $\boldsymbol{g}(t) = \boldsymbol{g}^\dagger(-t)$, along with the definition of $\tilde{L}_\lambda(t)$. Writing out all terms in Eq. (C26), we obtain

$$\mathcal{L}_S(t)[\rho] = \sum_{\lambda} \left[\tilde{L}_\lambda(t)\rho\tilde{L}_\lambda^\dagger(t) - \frac{1}{2}\{\tilde{L}_\lambda^\dagger(t)\tilde{L}_\lambda(t), \rho\} \right]. \tag{C27}$$

Hence $\mathcal{L}_S(t)$ is in the Lindblad form and generates the dissipative part of the ULE.

Next, we consider the term $\mathcal{L}_A$ in Eq. (C23). By expanding the commutator in Eq. (C25), we obtain $\mathcal{F}_A(s, t, s')[\rho] = T_1(s, t, s') - T_2(s, t, s') + \text{H.c.}$, where

$$T_1(s, t, s') \equiv \frac{\gamma}{2} \sum_{\alpha,\beta} \phi_{\alpha\beta}(s - t, s' - t)\tilde{X}_\beta(s')\rho\tilde{X}_\alpha(s),$$

$$T_2(s, t, s') \equiv \frac{\gamma}{2} \sum_{\alpha,\beta} \phi_{\alpha\beta}(s - t, s' - t)\tilde{X}_\alpha(s)\tilde{X}_\beta(s')\rho.$$

We now show that $T_1(s, t, s') = -T_1^\dagger(s', t, s)$. This implies that the net contribution to $\mathcal{L}_A(t)$ from T_1 and its Hermitian conjugate vanishes: $\int_{-\infty}^{\infty} ds' \int_{-\infty}^{\infty} ds\, [T_1(s, t, s') + T_1^\dagger(s, t, s')] = 0$, and hence [see Eq. (C23)]

$$\mathcal{L}_A(t)[\rho] = -\int_{-\infty}^{\infty} ds'ds\, [T_2(s, t, s') + T_2^\dagger(s, t, s')]. \tag{C28}$$

To prove that $T_1(s, t, s') = -T_1^\dagger(s', t, s)$, we note that $\boldsymbol{\phi}(t, s) = -\boldsymbol{\phi}^\dagger(s, t)$ [this follows from the definition of $\boldsymbol{\phi}$ below Eq. (C25) along with $\boldsymbol{g}(t) = \boldsymbol{g}^\dagger(-t)$]. Using this identity in the definition of T_1 above, we find, after a relabeling of summation variables,

$$T_1(s, t, s') = -\frac{\gamma}{2} \sum_{\alpha,\beta} \phi_{\alpha\beta}^*(s' - t, s - t)\tilde{X}_\alpha(s')\rho\tilde{X}_\beta(s).$$

We identify the right-hand side as $-T_1^\dagger(s', t, s)$ (see definition of T_1 above). Thus, $T_1(s, t, s') = -T_1^\dagger(s', t, s)$, and hence Eq. (C28) holds.

We finally note that $\int_{-\infty}^{\infty} ds' \int_{-\infty}^{\infty} ds\, T_2(s, t, s') = i\tilde{\Lambda}(t)\rho$, where

$$\tilde{\Lambda}(t) = \frac{\gamma}{2i} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds' \sum_{\alpha\beta} \tilde{X}_\alpha(s)\tilde{X}_\beta(s')\phi_{\alpha\beta}(s - t, s' - t)$$
$$\tag{C29}$$

denotes the Lamb shift from Eq. (23) in the main text. Hence the antisymmetric component $\mathcal{L}_A$ generates the Lamb shift in the ULE, as we claimed:

$$\mathcal{L}_A(t)[\rho] = -i[\tilde{\Lambda}(t), \rho]. \tag{C30}$$

Combining Eqs. (C23), (C27), and (C30), we obtain

$$\mathcal{L}(t)[\rho] = -i[\tilde{\Lambda}(t), \tilde{\rho}(t)]$$
$$+ \sum_{\lambda} \left[\tilde{L}_\lambda(t)\tilde{\rho}(t)\tilde{L}_\lambda^\dagger(t) - \frac{1}{2}\{\tilde{L}_\lambda^\dagger(t)\tilde{L}_\lambda(t), \tilde{\rho}(t)\} \right]. \tag{C31}$$

Thus, the superoperator $\mathcal{L}(t)$ is in the Lindblad form. Using this result in Eq. (C1), we conclude that the modified density matrix $\rho'(t)$, as defined in Eq. (C8), evolves according to the ULE in Eq. (27), with the correction term $\xi'(t)$ being bounded by $2\Gamma^2\tau$. Proving this was the goal of this Appendix.

APPENDIX D: LAMB SHIFT FOR STATIC HAMILTONIANS

In this Appendix we derive the expression for the Lamb shift in Eq. (34) of the main text, which holds for cases where the system Hamiltonian H_S is time independent.

Equation (34) is most conveniently derived in the interaction picture. We recall from Eq. (23) that, in the interaction picture, the Lamb shift is given by

$$\tilde{\Lambda}(t) = \frac{\gamma}{2i} \int_{-\infty}^{\infty} ds' \int_{-\infty}^{\infty} ds \sum_{\alpha\beta} \tilde{X}_\alpha(s) \tilde{X}_\beta(s') \phi_{\alpha\beta}(s-t, s'-t), \tag{D1}$$

where $\{\phi_{\alpha\beta}(s, s')\}$ denote the elements of the matrix $\boldsymbol{\phi}(t, s) \equiv \boldsymbol{g}(t) \boldsymbol{g}(-s) \operatorname{sgn}(t - s)$, and $\boldsymbol{g}(t)$ denotes the matrix-valued jump correlator defined in Eq. (25) in the main text.

As a first step in our derivation, we decompose the time-evolved system operator $\tilde{X}_\alpha(t)$ in terms of the eigenstates $\{|n\rangle\}$ and energies $\{E_n\}$ of the system Hamiltonian H_S:

$$\tilde{X}_\alpha(t) = \sum_{m,n} X_{mn}^{(\alpha)} e^{-iE_{nm}t} |m\rangle\langle n|, \tag{D2}$$

where, as in the main text, $X_{mn}^{(\alpha)} \equiv \langle m|X_\alpha|n\rangle$, while $E_{nm} \equiv E_n - E_m$. Inserting Eq. (D2) into Eq. (D1), shifting variables of integration, and using $E_{lm} + E_{nl} = E_{nm}$ along with the definition of $\boldsymbol{\phi}_{\alpha\beta}(t, s)$, we obtain

$$\tilde{\Lambda}(t) = \sum_{mnl;\alpha\beta} X_{ml}^{(\alpha)} X_{ln}^{(\beta)} f_{\alpha\beta}(E_{lm}, E_{nl}) e^{-iE_{nm}t} |m\rangle\langle n|, \tag{D3}$$

where $\{f_{\alpha\beta}(p, q)\}$ denote the elements of the matrix

$$\boldsymbol{f}(p, q) = \frac{\gamma}{2i} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds \operatorname{sgn}(s - t) e^{-i(pt+qs)} \boldsymbol{g}(t) \boldsymbol{g}(-s). \tag{D4}$$

Note that $\boldsymbol{f}(p, q)$ is the Fourier transform of $\boldsymbol{\phi}(t, s)$, up to a constant prefactor.

We now express the jump correlator in terms of its Fourier transform: $\boldsymbol{g}(t) = \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} \boldsymbol{g}(\omega)$. After factoring out the integrals over t and s, we obtain

$$\boldsymbol{f}(p, q) = \frac{\gamma}{2i} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \, \boldsymbol{g}(\omega) \boldsymbol{g}(\omega') k(p + \omega, q - \omega'), \tag{D5}$$

where $k(p, q) \equiv \int_{-\infty}^{\infty} ds' \int_{-\infty}^{\infty} ds \operatorname{sgn}(s - s') e^{-i(ps+qs')}$. By explicit computation, one can verify that

$$k(p, q) = -4\pi i \delta(p + q) \operatorname{Re}\left(\frac{1}{p - i0^+}\right), \tag{D6}$$

where $\delta(x)$ denotes the Dirac delta function. Using this result in Eq. (D5), integrating out ω', and subsequently shifting variables of integration, we find

$$\boldsymbol{f}(p, q) = -2\pi\gamma \int_{-\infty}^{\infty} d\omega \, \boldsymbol{g}(\omega - p) \boldsymbol{g}(\omega + q) \operatorname{Re}\left(\frac{1}{\omega - i0^+}\right).$$

We can rewrite this to the following:

$$\boldsymbol{f}(p, q) = -2\pi\gamma \, \mathcal{P} \int_{-\infty}^{\infty} d\omega \, \frac{\boldsymbol{g}(\omega - p) \boldsymbol{g}(\omega + q)}{\omega}, \tag{D7}$$

where $\mathcal{P}$ denotes the Cauchy principal value.

As a final step in our derivation, we use the expression for $\tilde{\Lambda}$ in Eqs. (D7) and (D3) to compute the Lamb shift in the Schrödinger picture Λ. We recall that $\Lambda = U(t) \tilde{\Lambda}(t) U^\dagger(t)$

[see below Eq. (31) in the main text], where $U(t) = e^{-iH_S t}$ denotes the unitary evolution operator generated by the system Hamiltonian H_S. Noting that for time-independent system Hamiltonians, $U(t)|m\rangle\langle n|U^\dagger(t) = e^{iE_{nm}t}|m\rangle\langle n|$, this implies that

$$\Lambda = \sum_{m,n,l} \sum_{\alpha\beta} X_{mn}^{(\alpha)} X_{nl}^{(\beta)} f_{\alpha\beta}(E_{mn}, E_{nl}) |m\rangle\langle l|, \tag{D8}$$

where the matrix $\boldsymbol{f}(p, q)$ is defined in Eq. (D7). This was the result quoted in the main text.

We note that the above line of arguments can be generalized to periodically driven systems with a few modifications. However, for the sake of brevity, we do not provide such a derivation here.

APPENDIX E: CONDITIONS FOR SLOW TIME DEPENDENCE

In this Appendix we identify the conditions on the time dependence of the system Hamiltonian $H_S(t)$, under which the Schrödinger picture jump operators $\{L_\lambda(t)\}$ and Lamb shift $\Lambda(t)$ can be computed from the eigenstates and energies of the instantaneous Hamiltonian $H_S(t)$, using Eq. (33) of the main text.

To show this explicitly for the jump operator $L_\lambda(t)$, we note that $U(t, s) = e^{-i(t-s)H_S(t)} + \mathcal{O}[v(t-s)^2]$, where $v = \sup_{s \leq t' \leq t} \|\partial_{t'} H_S(t')\|$ denotes the maximal rate of change of H_S [46]. Using this form of $U(t, s)$ in Eq. (32) of the main text, along with the results from Sec. IV A, we conclude that $L_\lambda(t)$ can be computed from the spectrum and eigenstates of $H_S(t)$ through Eq. (34), up to a correction of order $\sqrt{\Gamma} v(\tau_2)^2$ [note from Eq. (1) that the jump operators have units of $(\text{Energy})^{1/2}$]. Here $(\tau_2)^2 \equiv \int_{-\infty}^{\infty} dt \, \|\boldsymbol{g}(t) t^2\|_{2,1} / \mathcal{N}$, where $\mathcal{N} \equiv \sqrt{\Gamma/4\gamma}$ (see Sec. III C in the main text for the definition of the matrix norm $\|\cdot\|_{2,1}$). The timescale τ_2 gives the square root of the second moment of the normalized distribution $\|\boldsymbol{g}(t)\|_{2,1} / \mathcal{N}$ [see definition of Γ in Eq. (26)], and we expect it to typically be comparable to the first moment τ. Thus, when $H_S(t)$ changes slowly on the correlation timescale of the bath τ, i.e., $\partial_t H_S(t)(\tau_2)^2 \ll 1$, the jump operators of the system $\{L_\lambda(t)\}$ can be computed from the instantaneous Hamiltonian $H_S(t)$ using Eq. (33). A similar result holds for the Lamb shift $\Lambda(t)$.

APPENDIX F: CALCULATION OF TRANSPORT PROPERTIES

Here we define the heat and magnetization currents computed for the nonequilibrium spin chain in Sec. V. The average heat current $\bar{I}_E$ can be identified from the equation of motion for the energy in the spin chain: $\partial_t \langle E(t) \rangle = \operatorname{Tr}[H \partial_t \rho]$. Using the universal Lindblad equation [Eqs. (30) and (31)], along with $[H, H] = 0$, we find $\partial_t \langle E(t) \rangle = \sum_\lambda \langle I_E^{(\lambda)} \rangle$, where $I_E^{(\lambda)} = L_\lambda^\dagger H L_\lambda - \frac{1}{2}\{L_\lambda^\dagger L_\lambda, H\}$. For $\lambda = 1, 2$, we identify $I_E^{(\lambda)}$ as the heat current flowing into the system from bath λ. Since the energy of the chain is bounded, the time-averaged heat current from bath 1 must exactly compensate the average heat current from bath 2. Hence, we identify $\bar{I}_E$ as the time-averaged expectation value of $-I_E^{(1)}$. The magnetization current $\bar{I}_M$ can be obtained similarly from the equation of motion for the magnetization M, using $[H, M] = 0$.

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[26] Here the spectral norm is defined as the maximum singular value norm: $\|X\| = \sup_{\psi,\phi} |\langle\psi|X|\phi\rangle|$, where the supremum is taken over all normalized states. We note that, to apply our framework to a system where X may be unbounded, some additional physically justified truncation of the Hilbert space is needed.

[27] The square root is introduced for convenience, since the "bare" system-bath coupling $\sqrt{\gamma}$ only appears in even powers in the master equations we obtain. As a result of this parametrization, B has dimensions of $[\text{Energy}]^{1/2}$. These units of B are a natural choice when the bath has a continuous energy spectrum [9], such as is the case for the Ohmic bath in Sec. II B.

[28] To be precise, a general Lindblad form allows the time derivative of ρ to be given by a sum of multiple terms on the form in Eq. (1), where the Lamb shift and each jump operator may be time dependent.

[29] In Appendix A we obtain a stricter bound, namely $\|\xi(t)\| \leqslant 2\Gamma_0\tau_0$, where Γ_0^{-1} and τ_0 are distinct, but typically comparable, measures for the characteristic timescales of bath-induced evolution and bath correlations. These quantities are defined in Appendix B, where we also show that $2\Gamma_0^2\tau_0 \leqslant \Gamma^2\tau$. The timescales Γ_0^{-1} and τ_0 were also identified in Ref. [17], where analogous bounds for the trace norm of the correction $\xi(t)$ were derived in terms of these timescales. While we could have used the timescales Γ_0^{-1} and τ_0 to express the bound for $\|\xi(t)\|$ in the BR equation, the steps leading to the ULE induce errors whose bounds we can only express in terms of Γ and τ. To simplify the discussion, in the main text we therefore use the (looser) bound $\Gamma^2\tau$ in Eq. (12).

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[32] Specifically, the relative weight of the jump correlator $|g(t)|$ beyond a particular time t, $\int_t^\infty dt'|g(t')|/C$, is bounded by τ/t [this is straightforward to verify from Eq. (4)]. Note also that for many physically relevant cases, such as for the Ohmic bath discussed below, the jump correlator decays much faster than by this power law (often exponentially). In particular, if the bath spectral function is smooth in a way such that, for some n, $I_n \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega |(\partial_\omega)^n \sqrt{J(\omega)}|$ is a finite number, one can show that $|g(t)|$ is always bounded by $I_n/|t|^n$. This can be straightforwardly shown by using the definition of $g(t)$ from Eq. (3), along with the triangle inequality.

[33] In principle, the error induced by neglecting $\xi'(t)$ in the ULE may accumulate over time and result in inaccurate values of $\rho'(t)$ for $t \gtrsim (\Gamma^2\tau)^{-1}$. While the bound we obtained for $\|\xi'(t)\|$ can be used to infer rigorous results for the error to $\rho'(t)$ (using, e.g., the spectral gap of the Liouvillian), such a discussion is beyond the scope of this paper. We expect it is often a good strategy to simply compare the correction bound $2\Gamma^2\tau$ to the other relevant energy scales of the physical model and from this comparison determine whether the correction $\xi'(t)$ can safely be neglected using physical arguments. We expect this approach will include a much wider range of models than those allowed by rigorous mathematical results.

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