# IMPOSSIBILITY THEOREM FOR TWO-TIER ELECTORAL SYSTEMS

SEBASTIAN TIM HOLDUM AND FREDERIK RAVN KLAUSEN

Abstract. Two-tier electoral systems aim to balance regional representation and proportionality. Recently, the mathematical mechanisms of several Northern European two-tier systems have been challenged by increasing political party fragmentation, as evidenced by the expanding German parliament and the majority-deciding disproportionality of Denmark's 2022 election. This study develops a mathematical framework for two-tier systems, regionality, and proportionality, which identifies the common cause behind these complications through the formulation of an impossibility theorem: no two-tier system with fixed parliament size can guarantee both proportional and geographic representation. Finally, a new method *geographically ranked guaranteed proportionality* is proposed to circumvent the impossibility theorem.

## 1. Introduction

Electoral systems are fundamental to democracies, transforming votes into political power. For an electoral system to be perceived as legitimate, it must be seen as fair. Fairness can take many forms, but it often includes some level of representation. Representation can be measured by various criteria, including regionality and proportionality.

Two-tier systems attempt to align regional representation with proportionality. This paper defines these concepts mathematically and uses the definitions to prove an impossibility theorem, highlighting the trade-offs any two-tier electoral system faces.

Typically, two-tier electoral systems initially allocate seats within regional subdivisions, followed by a secondary process aimed at improving proportionality. This two-layered approach seeks to combine the best aspects of regional representation with overall proportionality [39]. However, the complexity of these systems raises important questions about their efficacy and fairness, particularly concerning the tension between regional interests and proportional outcomes.

This paper develops a rigorous mathematical framework and an impossibility theorem to reveal the conflict between regionality and proportionality in two-tier electoral systems. The results are relevant for mathematically inclined political scientists interested in electoral system design and its implications for democratic representation.

The trade-off has practical implications in real-world examples. For instance, the United Kingdom's first-past-the-post (FPTP) system serves as a stark example of disproportionality. In the 2024 general election, the Reform UK party received 14.3% of the national vote but secured only five seats of the total 650, while the Labour party won 411 seats with 33.7% of the vote [48], showcasing the extreme regionality of the system. According to The Economist [44] the 2024 election was the least proportional UK general election ever, measured by the Gallagher index [19].

Conversely, Germany's mixed-member proportional system was designed to ensure proportional representation in addition to the regionality provided by the constituencies. However, this system faced challenges such as the increasing size of the Bundestag due to overhang and compensatory seats. The Bundestag grew from the standard 598 seats to 736 in the 2021 federal election [13], showcasing the complexity and potential unintended consequences of strict proportionality within a two-tier framework. The debate sparked by these issues led to the 2023 electoral reform [12].

Another noteworthy example is Sweden, which, after experiencing slight disproportionality due to the lack of leveling seats in 2010, debated and reformed its election law to address issues related to proportionality and regional representation [34]. The Swedish system allocates 310 of its 349 parliamentary seats based on regional constituencies, with the remaining compensatory 39 seats used to achieve (a higher degree of) proportional representation [46].

Similarly, discussions have occurred in other Northern European two-tier systems. In Norway, parties have won regional seats with very few votes [49, sec. 5.2.6.5] such as in 2005, when the party Venstre received 1 out of 5 seats in Finnmark, despite only getting 2.2% of the votes there [41]. In the Danish 2022 general election a regional seat was not leveled out by the compensatory seats for the first time since 1947 [17]. This seat decided the majority in parliament, leading to a situation where, in traditional left-right terminology, the left bloc won despite receiving fewer votes than the right bloc [16] - a dramatic event in a two-tier system traditionally labeled as proportional.

These discussions arise because it is impossible to fully balance geographic and proportional representation with a fixed parliament size and no bound on the number of parties. This is the content of the impossibility theorem proven in Section 3. While the mathematics behind this theorem is straightforward, certain electoral laws currently in operation omit addressing these specific cases, suggesting that the trade-off between proportionality and regionality is not always intuitive in practice. An example is the Danish electoral law, which lacks clarification in special cases where the conditions clash and even contains conflicting provisions [18].

In social choice theory there is a rich tradition of proving impossibility theorems starting with Arrow's [1], over Gibbard's [21] to the Balinski-Young theorem [4]. This paper extends that tradition by introducing an impossibility result in the context of complex electoral systems, emphasizing the inherent trade-offs between regionality and proportionality in two-tier electoral structures through a rigorous mathematical framework. While this mathematical perspective may obscure other valuable insights from political science (see e.g., [43, 10, 40, 24]) we hope that it nonetheless contributes to the ongoing discourse on electoral design and evaluation.

1.1. **Omissions and clarifications.** Electoral systems in use are very diverse. To build a simple mathematical framework encapsulating the important aspects of real-world systems simplifying assumptions are necessary. We omit individual candidates and the large set of specialized local rules related to minority parties, race, religious groupings, disability, age or gender quotas etc. The mathematical framework used does not encompass preferential or transferable votes or ties.

We caution the reader that we consciously deviate from some of the traditional classifications in the literature [36]. This includes, but is not limited to the distinction between single- and multi-member constituencies, which for our purposes are equivalent. For the aim of the paper it is also unnecessary to distinguish between *parallel voting, coexistence, party-list proportional representation, mixed-member proportional representation*, and similar classifications, as they will be treated equally in mathematical terms. While this approach may overlook certain granular distinctions, it allows precise mathematical definitions of key concepts.

## 2. Defining electoral systems

2.1. **Election outcomes, amalgamations and seat distributions.** Define an election outcome as a matrix of votes cast. For convenience, label all electable entities as 'parties'. Throughout, let $p$ denote the number of parties and $c$ the number of constituencies. These constituencies can be both multi- and single-member and no distinction is made.

**Definition 2.1** (Election outcome)**.** For positive integers $p$ and $c$ a $(p, c)$-*election outcome* is an $(p + 1) \times c$ matrix $V$ where the entries $V[i, j]$ are non-negative integers, that should be interpreted as the votes that the $i$th party gets in the $j$th constituency. The extra row $V[p + 1, j]$ is the blank[1] and invalid votes in the $j$th constituency. Let $\mathcal{V}_{p,c}$ be the set of all $(p, c)$-election outcomes, $\mathcal{V}_c$ be the set of all election outcomes with fixed $c$, and $\mathcal{V}$ be the set of all election outcomes.

Electoral districts vary in significance within real-world electoral systems. Some districts serve purely administrative purposes, such as organizing voting and counting votes. In such cases, the final seat distribution does not depend on whether a vote is cast in electoral district 1 or 2. However, for other subdivisions of districts, the electoral rules can be such that it does matter in which district the vote is cast.

In some countries, such as Russia, Germany, and Japan, voters receive two ballots — one for local candidates and another for (national or regional) party lists. In those cases, the election outcome can be represented with two columns for each constituency, one for each ballot type. Electoral systems in countries like Zimbabwe and Italy[2] also aggregate votes from each individual constituency at the regional or national level, effectively treating the entire jurisdiction as an additional constituency with reserved seats. Thus, for simplicity and flexibility in certain special cases, it is helpful to consider an *amalgamated election outcome* where appropriate constituencies can be made up of multiple electoral districts. That is, an amalgamated election outcome consists of columns that are the sum of columns in an election outcome.

Suppose that the most fine-grained constituencies (i.e. precincts) are labeled $1, \ldots, c$ and the new resummed by $1, \ldots, \tilde{c}$. Then the amalgamation is a function $A : \mathcal{V}_c \to \mathcal{V}_{\tilde{c}}$ that sums up the corresponding columns of votes[3]. In the simple case, the amalgamation merges constituencies, but generally the new constituencies are allowed to overlap.

One can imagine the constituencies of the amalgamated election outcome $V$ as the coarsest possible decomposition, where the location of the vote cast matters and with possible additional columns for regional or national tallies. As amalgamated constituencies are not necessarily disjoint, the total vote count may not be preserved. The overlaps most often arise when the entire national total is considered a constituency as in the real world example below.

**Example 2.2.** Elections for *Lasanble Nasyonal* in the Seychelles consists of 26 constituencies, each electing a local candidate. However, for the 2020 general election, 44 separate polling stations were established [38]. Additionally, the Seychelles employs a system where party votes are also aggregated at the national level and additional seats are awarded. The election featured four competing parties [15]. Consequently, the following $4 \times 27$ amalgamation is natural

$$A \begin{bmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,43} & v_{1,44} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,43} & v_{2,44} \\ v_{3,1} & v_{3,2} & \cdots & v_{3,43} & v_{3,44} \\ v_{4,1} & v_{4,2} & \cdots & v_{4,43} & v_{4,44} \end{bmatrix} = \begin{bmatrix} v_{1,1}+v_{1,2} & v_{1,3} & \cdots & v_{1,41} & v_{1,42}+v_{1,43}+v_{1,44} & \sum_j v_{1,j} \\ v_{2,1}+v_{2,2} & v_{2,3} & \cdots & v_{2,41} & v_{2,42}+v_{2,43}+v_{2,44} & \sum_j v_{2,j} \\ v_{3,1}+v_{3,2} & v_{3,3} & \cdots & v_{3,41} & v_{3,42}+v_{3,43}+v_{3,44} & \sum_j v_{3,j} \\ v_{4,1}+v_{4,2} & v_{4,3} & \cdots & v_{4,41} & v_{4,42}+v_{4,43}+v_{4,44} & \sum_j v_{4,j} \end{bmatrix}.$$

Here the 44 columns are summed up to 26 new columns corresponding to the constituencies and a new column with the vote totals is added. The electoral system is discussed in Example 2.8iv). To get there, turn to seat distributions.

[1] In the following, note that even though the setup can account for blank and invalid votes, discussions of these are omitted.

[2] Excluding Aosta Valley and the overseas constituencies.

[3] Mathematically, we can define an amalgamation, using a function $a : \mathcal{P}(\{1, \ldots, c\}) \to \{1, \ldots, \tilde{c}, \perp\}$ with the property that each of the elements $\{1\}, \ldots, \{\tilde{c}\}$ has a unique preimage $a^{-1}(\{r\})$ in $\mathcal{P}(\{1, \ldots, c\})$. Here $\perp$ is the outcome if that combination of columns is not used in the amalgamation. Now, $A(V)[i, j] = \sum_{l \in a^{-1}(\{j\})} V[i, l]$, meaning we have summed the columns in $V$ that $a$ combines.

**Definition 2.3** (Seat distribution)**.** A $(p, c)$-*seat distribution* is a $p \times c$ array of non-negative integers $S$, where the matrix entry $S[i, j]$, is the number of seats allocated for the $i$th party from the $j$th constituency for $1 \leq i \leq p$ and $1 \leq j \leq c$. The set of all $(p, c)$-seat distributions we denote by $\mathcal{S}_{p,c}$, the set of all seat distributions for fixed $c$ by $\mathcal{S}_c$ and we let $\mathcal{S}$ denote the set of all seat distributions.

2.2. **Apportionment methods.** Apportionment methods determine how votes are converted into seats within individual constituencies and serve as building blocks for more complex electoral systems. Below $v_\perp$ are blank/invalid votes.

**Definition 2.4** (Apportionment method)**.** An *apportionment method* is a function $\mathtt{M} : \mathcal{V}_{p,1} \to \mathcal{S}_{p,1}$ that for every tuple of votes $v = (v_1, \dots, v_p, v_\perp)$ outputs a tuple of seats $\mathtt{M}(v) = (m_1, \dots m_p)$.

An apportionment method $\mathtt{M}$ satisfies *party invariance* (also called *neutrality*) if the outcome of the apportionment method does not depend on the ordering of the parties. It satisfies *monotonicity (cf. Concordance* [32, Sec. 4.3]*)* if party 1 receives more votes than party 2 ($v_1 > v_2$) then the number of seats for party 1 is greater or equal to the number of seats for party 2 ($m_1 \geq m_2$). If there is a fixed number $k$ of seats available in a given constituency independently of votes cast $v$, say that the method has *fixed seat count* and use the shorthand notation $\mathtt{M}^k$, for the apportionment method $\mathtt{M}$ distributing $k$ seats. In the following, it is always assumed that the apportionment methods are party invariant and monotone.

Common examples of apportionment methods that satisfy all these assumptions are the D'Hondt method (DH), "first-past-the-post" (FPTP), Hare-LR (LR) (the method of largest remainders[4]) and the Sainte-Laguë method (SL). Standard apportionment methods typically fall into two classes, divisor methods and quota methods and one way to phrase the Balinski-Young theorem is that these classes are irreconcilable [4]. For a general definition of the methods in these classes discussion see e.g., [32, 26]. Let us give some less standard examples of apportionment methods that illustrate notation and the diversity of electoral systems. The examples are not essential for understanding the rest of the paper.

**Example 2.5** (Apportionment methods)**.**

(i) (United States *Electoral College* for presidential elections) The first-past-the-post is mostly used for single-member constituencies that is $k_i = 1$ seats, but is also used for $k_i > 1$ in all US states except Maine and Nebraska for the electoral college. The algorithm is simple. Suppose $v = (v_1, \dots, v_p)$ is *size-ordered* (such that $v_1 > v_2 \geq v_3 \cdots \geq v_p$) then
$$\mathtt{FPTP}^k(v) = (k, 0, \dots, 0).$$

(ii) (Cameroon's lower house of parliament *L'Assemblée Nationale*) All of Cameroon's 58 single- and multi-member constituencies use the following apportionment method: Any list that gains a majority of the vote wins all of the seats. Otherwise, the first-placed list receives one half of the seats rounded up to the nearest whole number. The remaining seats are then distributed among the other parties above the threshold using Hare-LR (LR). For $v = (v_1, \dots v_n)$ size-ordered with sum $v$, then (with $\lceil\cdot\rceil, \lfloor\cdot\rfloor$ denoting rounding up and down)
$$\mathtt{M}^k_{\text{Cameroon}}(v) = \begin{cases} (k, 0, \dots, 0) & \text{if } v_1 \geq 0.5v \\ \left(\lceil \frac{k}{2} \rceil, \mathtt{LR}^{\lfloor \frac{k}{2} \rfloor}(v_2, \dots, v_p)\right) & \text{else.} \end{cases}$$

(iii) (Armenia's unicameral parliament ***Ազգային ժողով*** (*Azgayin Zhoghov*) initial phase[5] [35]) The electoral system of Armenia[6] has a single constituency. Initially, 101 seats are distributed using a modified variant of the largest remainders method, where the remainders of the largest parties are rounded up. Afterwards, the seat distribution is amended: If a party secures a majority of the seats but not 54%, it is allocated additional seats until it reaches 54%. If any party wins over 67% of the seats, the remaining parties are given extra seats so that the largest party holds at most 67%. Note that this method does not have fixed seat count.

2.3. **One-tier electoral systems.** To state and prove the impossibility theorem in as simple a framework as possible we define a simple electoral system. In Section 4 general electoral systems are defined which enable us to define two-tier systems and give a better handle on electoral systems with intermediate calculations.

**Definition 2.6** (Simple electoral system)**.** A simple electoral system ES is a function $\mathtt{ES} : \mathcal{V}_c \to \mathcal{S}$.

For matrix $M$, denote the $i$th row as $M[i, \cdot]$ and the $j$th column as $M[\cdot, j]$. One-tier electoral systems are simple.

**Definition 2.7** (One-tier electoral system)**.** A simple electoral system $\mathtt{ES}_{\text{1-Tier}}$ is *one-tier* if there exists an amalgamation $A : \mathcal{V}_c \to \mathcal{V}_{\tilde{c}}$ and a tuple of apportionment methods $\mathtt{M} = (\mathtt{M}_1, \dots, \mathtt{M}_{\tilde{c}})$ such that

(1) $$\mathtt{ES}_{\text{1-Tier}}(A(V))[\cdot, j] = \mathtt{M}_j(A(V)[\cdot, j]).$$

[4]Hare quota with largest remainders $\mathtt{LR}^k$: Define $q_i = k\, v_i / \sum_j v_j$, allocate $\lfloor q_i \rfloor$, then give remaining seats to largest $r_i = q_i - \lfloor q_i \rfloor$.

[5]If no party (or a coalition of parties formed within six days after the election) can obtain a majority, a second round of elections will be held in which the two best-performing political forces will participate. All seats received in the first round will be retained. The party (or newly formed coalition) that wins the second round will be allocated additional seats to reach 54% of all seats.

[6]In this example, the electoral threshold and the reserved seats for Assyrians, Kurds, Russians, and Yazidis are disregarded.

In other words, one-tier systems are those in which the resulting seat distribution can be determined by applying the apportionment method independently to the votes for each constituency $\tilde{c}$, without any interaction between them.

A one-tier system where the methods $\mathtt{M}_j$ are all equal is called *homogeneous* and if they are different the one-tier system is called *parallel* [36]. One-tier systems are common throughout the world, cf. Figure 2, especially among former British territories where FPTP is often used.

**Example 2.8** (One-tier systems)**.**

(i) (The United Kingdom's *House of Commons*) First-past-the-post voting is used in all 650 single-member constituencies (as of 2024) corresponds for every $1 \le j \le 650$ to the apportionment method $\mathtt{M}_j = \mathtt{FPTP}^1$.

(ii) (The Netherlands *Tweede Kamer der Staten-Generaal*) The Dutch electoral system uses the D'Hondt method to distribute 150 seats in a single constituency. Thus, it is a one-tier system with apportionment method $(\mathtt{DH}^{150})$.

(iii) (Russia's Государственная дума (*State Duma*)) Each voter is given two votes. The first vote is used to elect a local candidate in one of the 225 single-member constituencies using the first-past-the-post method. The second vote is cast for a party list and is tallied nationally, with another 225 seats distributed using the Hare quota rule (with a 5% electoral threshold) $\mathtt{LR}_{5\%}$, thus the overall method is $(\mathtt{M}_1, \dots, \mathtt{M}_{226}) = (\mathtt{FPTP}^1, \dots \mathtt{FPTP}^1, \mathtt{LR}^{225}_{5\%})$.

(iv) (The Seychelles *Lasanble Nasyonal*) The Seychellois electoral system distributes 26 seats using first-past-the-post in single-member constituencies. In addition, up to 10 additional seats are allocated nationally - one per 10% of the total national vote received by a party, that is $\mathtt{M}_{\text{Seychellois Bonus}}(v_i) = \lfloor \frac{10 v_i}{\Sigma v_i} \rfloor$. So with the amalgamation in Example 2.2, the electoral system is one-tier, with $(\mathtt{M}_1, \dots, \mathtt{M}_{27}) = (\mathtt{FPTP}^1, \dots \mathtt{FPTP}^1, \mathtt{M}_{\text{Seychellois Bonus}})$.

(v) (The United States *Electoral College*) for US presidential elections might at first sight look parallel due to Maine and Nebraska not using FPTP (cf. Example 2.5). In the two states, the largest party in each congressional district is awarded a seat and the statewide largest party is awarded two additional seats. So, upon amalgamation, the method is $(\mathtt{FPTP}^1, \dots, \mathtt{FPTP}^1, \mathtt{FPTP}^2)$ and the entire electoral college is homogeneous.

(vi) (The *European Parliament*) electoral system consists of separate national elections held by each of the 27 member states, without interaction between them and with different methods M (which in the case of Italy and Poland are in themselves not one-tier systems and therefore not quite apportionment methods, see e.g., [33]).

(vii) (The United States *House of Representatives*) uses FPTP across the constituencies in 46 states, but run-off systems are employed in Alaska, Maine, Georgia and Louisiana, so it is not one-tier.

## 3. Impossibility theorem

To state the impossibility theorem, regionality, proportionality and fixed parliament size are defined. The definitions are primarily motivated by two- and multi-tier systems (to be defined in Section 4) that often aim to balance regionality and proportionality. Fixed parliament size contributes to simplicity and helps control costs[7].

3.1. **Properties of electoral systems.** The simplest property is fixed parliament size. To write it neatly, for a seat array $S$ let $\mathbf{N}(S) = \sum_{i=1}^{p} \sum_{j=1}^{c} S[i,j]$ be the total number of seats of $S$.

**Definition 3.1** (Fixed parliament size)**.** A simple electoral system ES has *fixed parliament size* if the total number of distributed seats $\mathbf{N}(\mathtt{ES}(V))$ is constant for all $V \in \mathcal{V}_c$.

In practice, most electoral systems have fixed parliament size. Notable exceptions[8] are the parliament of New Zealand *Pāremata Aotearoa* and the one-tier system of the Seychelles (Example 2.8iv).

Regionality reflects whether parties always keep the *regional seats* they initially won in the constituencies with $\mathtt{ES}_{\text{1-Tier}}$. To define it say for two $p$-tuples $m = (m_1, \dots, m_p)$ and $m' = (m'_1, \dots, m'_p)$ that $m \le m'$ if $m_i \le m'_i$ for all $1 \le i \le p$.

**Definition 3.2** (Regionality)**.** A simple election system ES with $c \ge 2$ *respects regionality* according to an electoral system $\mathtt{ES}_{\text{1-Tier}}$ of the form $(\mathtt{M}_i^{k_i})_{1 \le i \le c}$ if for all $V \in \mathcal{V}_c$, $\mathtt{ES}_{\text{1-Tier}}(V) \le \mathtt{ES}(V)$.

For electoral systems satisfying regionality say the regional seats are *final*. Such regional seats are sometimes perceived differently than the compensatory seats [30]. Most two and multi-tier electoral systems respect regionality, cf. Table 1 in Section 5. In fact, ES is usually[9] built from an $\mathtt{ES}_{\text{1-Tier}}$ with an additional way of distributing compensatory seats.

[7] Additionally, maintaining a fixed and odd number of seats can help preventing legislative deadlocks by ensuring a decisive majority in votes, further strengthening the stability and functionality of the system.

[8] There are other exceptions. In some cases, seats may be left vacant when the list of candidates is insufficient to accommodate the allocated seats, which is the case for South Korea's National Assembly considered in [28, article 189 (5)]. A curious example occurred in the 2021 Kyrgyz parliamentary election, where two single-member constituency seats remained unfilled because the *against all* option received the most votes. While such cases exist, they fall outside the scope of our discussion.

[9] Additionally, note that the electoral system $\mathtt{ES}_{\text{1-Tier}}$ is not completely general as it assumes that all of the apportionment methods $\mathtt{M}^{k_i}$ have a fixed seat count. Two-tier systems that do not satisfy this requirement include the current Polish subelectoral system in the European parliamentary elections [33] and the system employed at Romanian parliamentary elections from 2008-2012 [22].

Following Balinski-Young [4, Section 2] say that an apportionment method $\mathtt{M}^k$ is *weakly proportional* if it divides proportionally, whenever possible, that is when there are no fractional parts in the largest remainders method. Formally, if $(v_1, \dots, v_p)$ is a vector of votes and $v$ is their sum, $\mathtt{M}^k$ is weakly proportional if $\mathtt{M}^k(v_1, \dots, v_p) = (\frac{kv_1}{v}, \dots \frac{kv_p}{v})$ whenever $\frac{kv_i}{v}$ is an integer for each $i$. All the (traditional) apportionment methods Hare-LR ($\mathtt{LR}$) (Hamilton), D'Hondt ($\mathtt{DH}$) (Jefferson), Lowndes, Sainte-Laguë ($\mathtt{SL}$) (Webster), Adams, Dean, and Hill are weakly proportional [3, Proposition 2.1].

**Definition 3.3** (Guaranteed Proportionality)**.** A simple electoral system $\mathtt{ES}$ is *guaranteed proportional* if there exists a weakly proportional apportionment method $\mathtt{M}$ satisfying Definition 2.4 such that $\mathtt{ES}$ agrees with $\mathtt{M}$ for all elections $V$.

Mathematically, for all elections $V$ where $v_i$ is the total number of votes for party $i$, then there exists a seat number $m$ such that for each $i \in \{1, \dots, p\}$,

$$\sum_{j=1}^{c} \mathtt{ES}(V)[i,j] = \mathtt{M}^m(v_1, \dots, v_p)[i]. \tag{2}$$

or, equivalently, $A_{\text{total}}(\mathtt{ES}(V)) = \mathtt{M}^m(A_{\text{total}}(V))$, where $A_{\text{total}}$ is a total amalgamation.

For two-vote systems a bit of flexibility should be allowed and guaranteed proportionality should be defined only with respect to one vote type (column in an amalgamation).

The total number of seats in parliament, $m$, can depend on the election $V$. Many well-functioning two-tier systems traditionally labeled as *proportional systems*, are not guaranteed proportional even if all previous elections have been consistent with the proportional method used.

3.2. **Impossibility theorem.** A mismatch between the number of seats and votes in each constituency relatively easily causes problems with having regionality, proportionality and fixed parliament size[10].

This does indeed seem to be a real world issue. Examples of countries where the number of seats per voter varies significantly across constituencies include Norway and Denmark, with Finnmark and Bornholm respectively being over-represented relative to what their sparse populations would suggest. For current operational two-tier systems, $k_j$ is often determined in advance based on various methods[11]. To avoid this mismatch, the strongest possible restriction is to insist that the number of regional seats $(k_1, \dots, k_c)$ is distributed weakly proportionally according to the sum of the votes across the constituencies. Call such election outcomes *vote-seat consistent.* Even in this case the impossibility arises:

**Theorem 3.4** (Impossibility theorem)**.** Suppose that $\mathtt{ES}_{\text{1-Tier}}$ is a one-tier electoral system where the corresponding apportionment methods $(\mathtt{M}_i^{k_i})_{1 \le i \le c}$ have $c \ge 2$ constituencies with $k_i \ge 1$ mandates for at least two distinct $i$. For any electoral system $\mathtt{ES}$ there exists a vote-seat consistent election outcome where at least one of the following desiderata fails:

(i) Regionality: $\mathtt{ES}$ respects regionality with respect to $\mathtt{ES}_{\text{1-Tier}}$.
(ii) Fixed parliament size: The system $\mathtt{ES}$ has fixed parliament size.
(iii) Proportionality: $\mathtt{ES}$ is guaranteed proportional.

*Proof.* A vote-seat consistent election outcome $V$ is constructed, which leads to an impossibility whenever

$$p > \frac{T}{\tilde{c} - 1} + \max(k_1, \dots, k_{\tilde{c}}) + 1, \tag{3}$$

where $p$ is the total number of parties, $T = m - \sum_{i=1}^{\tilde{c}} k_i$ is the number of non-regional seats, and $\tilde{c}$ is the number of constituencies with $k_i \ge 1$.

For the construction, suppose there are $p_r = \max(k_1, \dots, k_{\tilde{c}}) + 1$ regional parties $P_1, \dots, P_{p_r}$ and $p_n$ national parties $N_1, \dots, N_{p_n}$. In constituency $j$, each national party gets $k_j a$ votes where $a$ is to be specified later. The first regional party $P_1$ receives $k_j(a+2)$ votes in each constituency $1 \le j \le \tilde{c} - 1$ and 0 votes when $j = \tilde{c}$. The regional parties $P_2, \dots, P_{p_r - 1}$ get $k_j(a+1)$ votes for each $j$. The last regional party $P_{p_r}$ gets 0 in every constituency except the last one, where it gets $k_{\tilde{c}}(a+2)$. Schematically, the (transposed) vote distribution can be visualized as follows:

$$V^T = \left[\begin{array}{ccccc|cccc}
\multicolumn{5}{c|}{\overbrace{\hspace{18em}}^{p_r \text{ regional parties}}} & \multicolumn{4}{c}{\overbrace{\hspace{14em}}^{p_n \text{ national parties}}} \\
k_1(a+2) & k_1(a+1) & \cdots & k_1(a+1) & 0 & k_1 a & k_1 a & \cdots & k_1 a & k_1 a \\
k_2(a+2) & k_2(a+1) & \cdots & k_2(a+1) & 0 & k_2 a & k_2 a & \cdots & k_2 a & k_2 a \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
k_{\tilde{c}-1}(a+2) & k_{\tilde{c}}(a+1) & \cdots & k_{\tilde{c}}(a+1) & 0 & k_{\tilde{c}-1} a & k_{\tilde{c}-1} a & \cdots & k_{\tilde{c}-1} a & k_{\tilde{c}-1} a \\
0 & k_{\tilde{c}}(a+1) & \cdots & k_{\tilde{c}}(a+1) & k_{\tilde{c}}(a+2) & k_{\tilde{c}} a & k_{\tilde{c}} a & \cdots & k_{\tilde{c}} a & k_{\tilde{c}} a
\end{array}\right]$$

[10]Indeed, even with only two parties and three constituencies. If one party is the largest party in two constituencies they get at least 2 seats with a total of $v_1$ votes, but by having many votes in the last constituency the second party may in total get $(m-1)v_1$ votes, where $m$ is the fixed parliament size. By weak proportionality the mandates should be distributed $(1, m-1)$.

[11]Examples currently used include the number of votes cast in the constituency in the last election, the population in the constituency at the last census, the surface area of the constituency, and politically negotiated minimum number of representatives.

In the $j$th constituency the total number of votes is $v_j = k_j\Big((p_r-1)(a+1)+1+ap_n\Big)$, so $V$ is vote-seat consistent.

With the vote distribution described, assume for contradiction that all three desiderata in the theorem are satisfied.

Since the first regional party $P_1$ is the largest party in each of the first $\tilde{c}-1$ constituencies, by winner-take-one for each of the methods $\mathtt{M}_i$ it will win at least $\tilde{c}-1$ regional seats, with a total of $(a+2)\sum_{1\le j\le \tilde{c}-1} k_j$ votes. Since $\mathtt{ES}$ satisfies regionality with respect to $\mathtt{ES}_{\text{1-Tier}}$ defined by $(\mathtt{M}_i)_{1\le i\le \tilde{c}}$, the party $P_1$ obtains at least $\tilde{c}-1$ seats in $\mathtt{ES}(V)$.

Now look at the national parties. In each constituency $j$, none of the national parties $N_1,\dots,N_{p_n}$ are among the $k_j$ largest parties, so by monotonicity of the method $\mathtt{M}^j$, each of them receives 0 regional seats. However, each national party gets in total $a\sum_{1\le j\le \tilde{c}} k_j$ votes. Choosing the free parameter $a$ such that $a \ge \sum_{1\le j\le \tilde{c}-1} \frac{2k_j}{k_{\tilde{c}}}$ then each national party has strictly more total votes than $P_1$. By monotonicity of the apportionment method that $\mathtt{ES}$ is guaranteed proportional with respect to, each national party must then receive at least $\tilde{c}-1$ seats in total.

Now, since we assume fixed parliament size $m$, define the non-regional seats as $T = m - \sum_{i=1}^{\tilde{c}} k_i$. We have just argued that each of the $p_n$ national parties gets at least $\tilde{c}-1$ of these non-regional seats. Hence $T \ge (\tilde{c}-1)p_n$. For sufficiently large $p_n$, this contradicts the fixed parliament size constraint, since $T = m - \sum_{i=1}^{\tilde{c}} k_i$ is fixed and cannot grow with $p_n$.

Therefore, it is impossible to satisfy all three desiderata simultaneously. The bound (3) follows since $p = p_r + p_n$. $\square$

At this point, a couple of remarks are in order.

**Remark 3.5** (Many parties complicates proportionality and regionality)**.** The proof hinges on the possibility of many (small) parties. This highlights that the impossibility can arise in several different ways: distorted districts, many parties, the apportionment methods employed, and combinations thereof. The mechanism of many parties has a suggestive parallel with the recent increased number of parties in European politics, with 4 parties achieving more than 10% in the UK, with the CSU and Die Linke making it to the Bundestag in Germany in 2021 only on direct seats [13]. Similarly, the Danish parliamentary election in 2022 saw a record-breaking number of parties entering parliament [16]. This suggests that the proof elucidates the mathematical mechanism between these developments. While these examples are recent, the effective number of parties has been increasing for many years, increasing the relevance of the trade-off exhibited in Theorem 3.4.

**Remark 3.6** (Micromega rule)**.** The bound (3) can be viewed as an incarnation of the micromega rule, which describes the effect whereby large parties benefit from smaller constituencies (they obtain over-representation without adjustments). For a fixed total number of regional seats, if the constituencies become smaller, the number of constituencies increases and the number of parties needed to construct the counterexample decreases.

**Remark 3.7** (Relation to the impossibility result of Demange [11])**.** In [11], Demange considered the related problem of allocating seats across both parties and constituencies (bi-allocation). Demange introduced the concept of district decentralization, meaning that the seat distribution in each constituency depends solely on the votes cast in that constituency. District decentralization is a much stronger requirement than regionality, which only requires some of the seats to be determined locally. The impossibility result [11, Proposition 2] in the setup of real numbers is that if the district allocation is distorted (not vote-seat consistent over the reals) then any bi-allocation method cannot be both district decentralized and party proportional (perfectly proportional over the reals). Demange further illustrates through the German and Italian examples that introducing integer requirements cannot solve these problems. From one perspective, Theorem 3.4 is a formalized stronger version of this statement: over the integers, no party-proportional method that distributes even some of the seats district decentralized exists, regardless of whether district seat totals match district vote totals. See Section 6 for further discussion.

**Remark 3.8** (Bounding the number of required leveling seats in terms of the number of parties)**.** The impossibility theorem refines the discussion in the literature about the number of compensatory seats necessary to ensure proportional representation [40]. The theorem gives an answer, namely that if there is no bound to the number of parties, then there is no definite number. A natural next mathematical question is whether the bound (3) on the number of parties $p$ can be tightened by constructing more sophisticated election outcomes. Additionally, one can attempt to construct bounds that link other quantities, such as $T$ and $\sum k_j$. This direction for further work could also involve assumptions about the relation between the number of votes and seats in each constituency. In addition, the mathematically attentive reader might notice that only the monotonicity of $\mathtt{ES}$ was used and the full power of weak proportionality was not needed. Better bounds can be constructed by assuming that $\mathtt{ES}$ is guaranteed proportional with respect to for example $\mathtt{SL}$ or $\mathtt{LR}$.

## 4. Multi- and two-tier systems

Now, define a general electoral system:

**Definition 4.1** (Electoral system)**.** An electoral system $\mathtt{ES}$ is a function $\mathtt{ES} : \mathcal{S}_c \times \mathcal{V}_c \to \mathcal{S}$.

The motivation for this generality is that many real world electoral systems have intermediate calculations (e.g., assigning regional seats) that are done independently and used as input for the next calculation, see Appendix A.2 where this is worked out for two-round systems. Apart from being instrumental in the definition of two-tier systems this additional definition does not play a mathematical role as an empty seat distribution can be given if it's not relevant, but should be thought of as a guide for the mind when thinking about two- and multi-tier systems. If one were to write pseudo-code for the electoral algorithms, these intermediate steps would be the obvious natural building blocks. On the other hand, one could define a single monolithic function that takes the full election outcome as input and returns the final seat distribution; however, that black-box formulation offers little insight into the system's mechanics and makes meaningful comparisons across systems harder.

4.1. **Multi- and two-tier systems.** The definitions in Section 3 straightforwardly extend to general electoral systems and the impossibility theorem holds as well. Indeed, it is especially motivated by the issues present in real world two- and multi-tier systems which are common (see Figure 2 for an illustration of the current status). A central contribution of this section is the precise mathematical definition of two-tier systems, which is used somewhat loosely in the literature [39].

Multi-tier systems are built on top of a one-tier system with additional mixing. That is, the input seat matrix $S$ is the result of a one-tier calculation. Thus, an electoral system is multi-tier if it is not one-tier and of the form:

$$\mathtt{ES}_{\text{Multi-Tier}}(S, V) = \mathtt{ES}(\mathtt{ES}_{\text{1-Tier}}(A(V)), V). \tag{4}$$

Two-tier systems are a special subset of multi-tier systems which can be computed column-wise up to amalgamation. First, consider the simplest case.

**Definition 4.2** (Irreducible two-tier system)**.** An electoral system $\mathtt{ES}$ is an irreducible two-tier system if,

$$\mathtt{ES}_{\text{2-Tier}}(S, V) = \mathtt{ES}(A_{\text{total}}(\mathtt{ES}_{\text{1-Tier}}(V)), A_{\text{total}}(V)) \tag{5}$$

where the amalgamation $A_{\text{total}} : \mathcal{V}_c \to \mathcal{V}_1$ is *total*, meaning that it sums up all constituencies and there is only one $j$ so that the remaining seats only depend on the total number of votes and the total number of seats in the one-tier system. At this point, an example of one of the simplest non-trivial two-tier systems in use is instructive.

**Example 4.3** (Iceland's Unicameral parliament *Alþingi*)**.** The Icelandic parliament has 63 members. 54 of these members are elected directly in multi-member constituencies using the D'Hondt method and the remaining nine seats are compensatory. In our notation, this electoral system is a two-tier system: The first tier has six constituencies distributing 7, 9, 9, 9, 9 and 11 regional seats [45, Art. 8] using the D'Hondt method (see Figure 1). Thus, the one-tier basis of the system can be written as $\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}} = (\mathtt{DH}^7, \mathtt{DH}^9, \mathtt{DH}^9, \mathtt{DH}^9, \mathtt{DH}^9, \mathtt{DH}^{11})$.

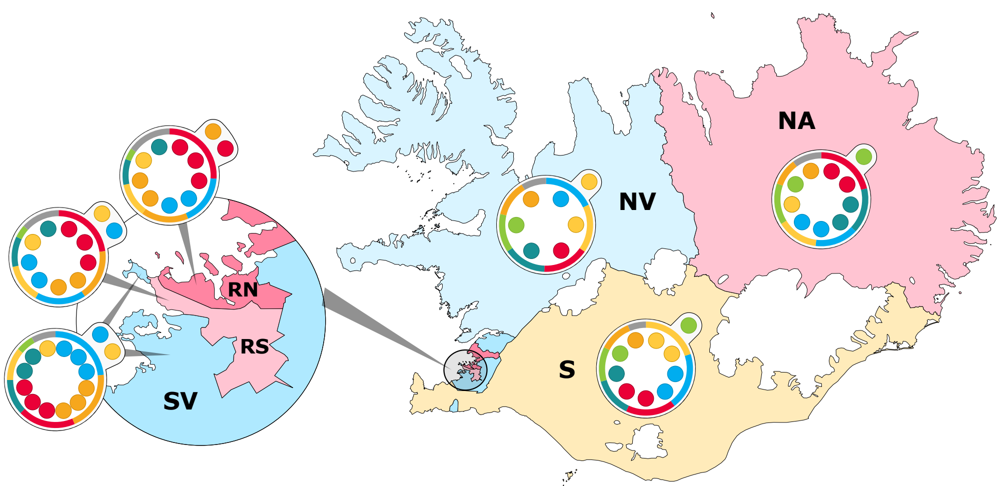



FIGURE 1. Results of the 2024 Icelandic Parliamentary Election. The six multi-member constituencies are shown, where the colored inner circles represent each party's regional seats, and the outer circles indicate the compensatory seats [6].

Now, the compensatory seats are distributed [45, Art.108] in the following way[12]: The national list of D'Hondt quotients is generated[13] and ordered by size. The 54 regionally distributed seats correspond to 54 entries on this list. Mark these entries as final. The remaining 9 seats are distributed to the 9 largest quotients that are not already marked.

If all the 54 initially marked quotients are among the 63 largest the national seat totals will be distributed according to the D'Hondt method, but otherwise they are not, so the system is not guaranteed proportional, but it does satisfy regionality and fixed parliament size.

[12]We only consider the national distribution and not how they are distributed to the constituencies. We also made a slight simplification for pedagogical purposes.

[13]That is for every party the total number of votes is divided by 1,2,3,.. etc.

Let us use this example to unpack the mathematical definitions explicitly: For each election outcome $V$, the seat array $\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}}(V)$ corresponds to the distribution of regional seats. In the next step, the election is summed up completely using a total amalgamation $A$, that is $A(V)$ is the list of the total national vote for all parties and $A(\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}}(V))$ is the total number of regional seats. The final distribution $\mathtt{ES}^{\text{Iceland}}$ depends both on the distribution of regional seats $A(\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}}(V))$ and on the national votes. Thus, the final distribution has the form of Equation (5):

$$\mathtt{ES}^{\text{Iceland}}(V) = \tilde{\mathtt{DH}}(A(\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}}(V)), A(V)),$$

where $\tilde{\mathtt{DH}}$ is the modified D'Hondt process that distributes the seats already distributed in the seat vector first.

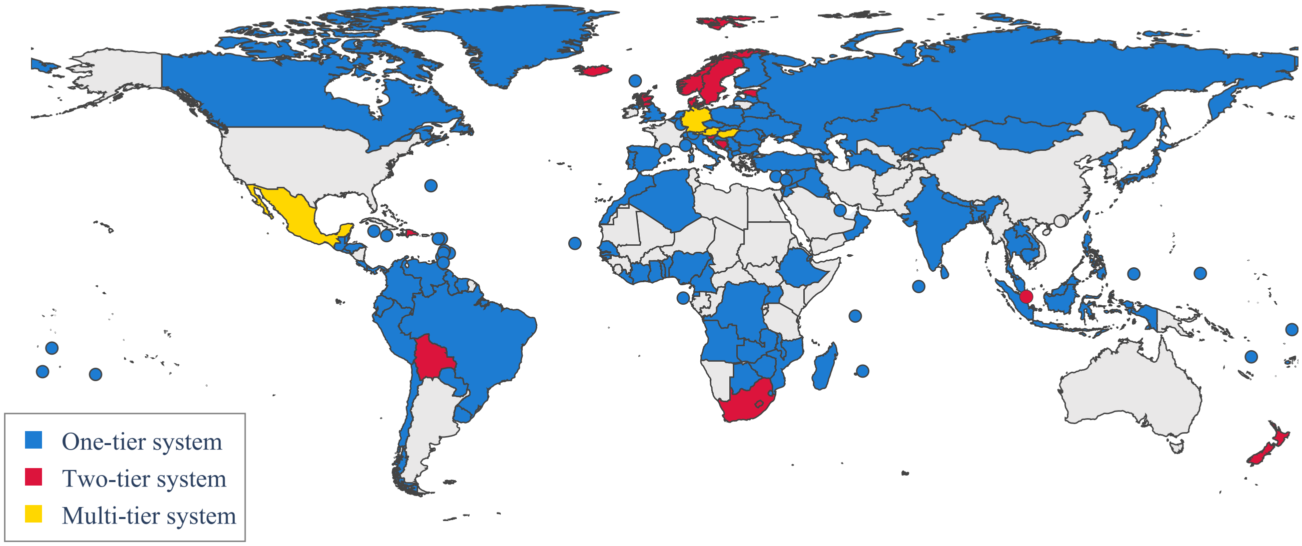



FIGURE 2. An overview of countries that as of 2024 employ one-tier, two-tier or multi-tier systems for elections to the lower (or unicameral) house without taking specialized rules for minority quotas or electoral thresholds into account[14]. Other systems include two-round systems, preferential votes, transferable votes, non-democracies without elections etc.

4.2. **Parallel two-tier and multi-tier systems.** To define general two-tier systems, one should think of the system as a parallel irreducible two-tier system, similarly to the one-tier parallel electoral system of the Russian Duma in Example 2.8iii). In other words, two two-tier systems that do not interact at all should still be considered two-tier.

**Definition 4.4** (Two-tier system)**.** Let $A : \mathcal{V}_c \to \mathcal{V}_{\tilde{c}}$ be an amalgamation. An electoral system is *two-tier* if it is not one-tier and is of the form

$$\mathtt{ES}_{\text{2-Tier}}(S, V)[\cdot, j] = \mathtt{ES}_j(A(\mathtt{ES}_{\text{1-Tier}}(V))[\cdot, j], A(V)[\cdot, j]), \tag{6}$$

for each $1 \leq j \leq \tilde{c}$ for a one-tier system $\mathtt{ES}_{\text{1-Tier}}$ and $\tilde{c}$ electoral systems $\mathtt{ES}_j$.

A relevant example of this is the electoral system of Bosnia-Herzegovina.

**Example 4.5** (Bosnia-Herzegovina's parliament's lower house *Dom Bosne*)**.** The electoral system of the country of Bosnia-Herzegovina consists of two standard two-tier systems of the form Equation (5) corresponding to the two autonomous entities, *the Federation of Bosnia and Herzegovina* and *Republika Srpska*, not interacting at all. That is $\tilde{c} = 2$.

**Example 4.6** (Multi-tier systems)**.** Throughout the world some multi-tier systems are in use, see Figure 2.

(i) The Austrian lower house *Der Nationalrat*. Austria has nine federal states which each form multi-member constituencies that are further divided into sub-constituencies, all assigned seats from the outset. First, for each federal state the election number (i.e., the number of votes per seat, Ger. *Wahlzahl*) is calculated. Then, in each sub-constituency, seats are distributed using the largest remainders method based on the election number, rounding down. Secondly, at the state level another round of the largest remainders method, also rounded down.

[14]The map is constructed with data from a variety of sources, including (but not limited to) national election laws such as [35, 8, 45, 46], already compiled tables of electoral systems such as [36, 25, 47]. The biggest challenges in constructing such a map are the ever-evolving electoral systems used around the world and the need to compare and verify those mentioned in the literature to ensure they were not outdated. This map will also become outdated soon.

Finally, on the national level additional seats are distributed using the D'Hondt method. Because interactions occur across all three levels, this system does not meet the criteria of Definition 4.4.

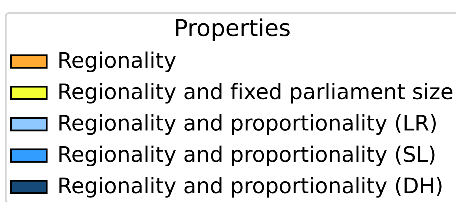
(ii) The Hungarian unicameral parliament *Országgyűlés*[15], similar to the abolished Italian *Scorporo system*, is based on first-past-the-post voting in the 106 individual constituencies, where votes for unelected candidates and excess votes for elected candidates are pooled together for redistribution. The remaining seats are then distributed with the D'Hondt method on the redistribution pool. In Definition 4.4, only amalgamations (i.e., column summation) are allowed, so this system is not two-tier. However, it is an electoral system cf. Definition 4.1.

## 5. Real world examples

Our main motivation for the impossibility theorem is the view it gives on real world examples and present discussions (cf. Section 1). In Table 1 a classification of multi-tier systems shows how all sides of the trade-off have been explored. See also Figure 3 that shows how regional German parliaments exhibit a variety of combinations.

| **Country** | **Regionality** | **Guaranteed proportionality** | **Fixed parliament size** |
|---|---|---|---|
| Germany$_{2021}$ | ✓ | ✓ | × |
| Sweden, Germany$_{2025}$, Denmark$_{1948-1953}$ | × | ✓ | ✓ |
| Austria, Bolivia, Bosnia-Herzegovina, Denmark, Dominican Republic, Estonia, Iceland, Lesotho, Norway, Scotland Slovenia, South Africa, South Korea$_{2020}$ | ✓ | × | ✓ |
| New Zealand, Singapore | ✓ | × | × |
| Romania$_{2008-2012}$ | × | ✓ | × |

Table 1. Broad overview of the properties of some multi-tier electoral systems for national parliaments, excluding the electoral threshold (discussed in Section A.1) and other specialized rules, with the table not capturing all system details[16]. Only multi-tier systems with a compensatory element are listed (other non-compensatory multi-tier systems such as Hungary and Mexico are left out[17]). Here the two German systems are guaranteed proportional with respect to Sainte-Laguë/Schepers method, the Romanian to D'Hondt method and the Swedish to a modified Sainte-Laguë method.



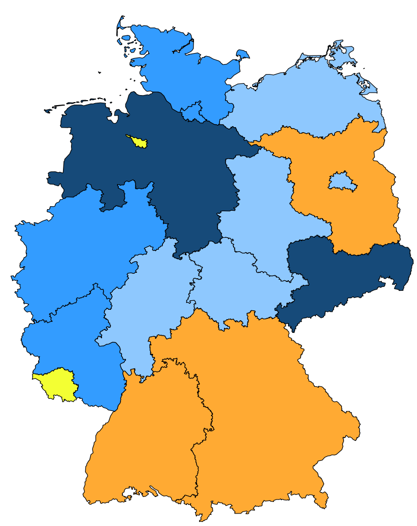



Figure 3. Overview of the electoral systems applied for regional German parliaments (Ger. *Landtage*) and the apportionment methods that some are guaranteed proportional with respect to. In contrast to the new federal German system they all satisfy regionality. Data source: [29]

The theorem also applies to one-tier systems, which usually satisfy regionality and fixed parliament size, but not guaranteed proportionality. In a sense, the trivial Dutch electoral system described in Example 2.8ii), satisfies all three properties, but there is only a single constituency so it is not true that $c \geq 2$ and $k_i \geq 1$ for at least two $i$.

[15]Ignoring preferential minority seats.

[16]Germany$_{2025}$ does not have fixed parliament size if one party gets more than 50% of the vote in which case extra seats are added to ensure that they obtain at least 50% of the seats. In Slovenia two additional deputies are elected by the Italian and Hungarian minorities.

[17]Singapore is classified as two-tier due to the *non-constituency member of parliament system*, where up to 12 non-winning candidates will be elected and enjoy all of the privileges of ordinary members of Parliament, apart from a substantially lower salary [9].

Any two-tier system striving for proportionality and satisfying regionality must have some mathematical mechanism for the case when one or more parties obtain more regional seats than can be leveled out using additional tiers. The redistribution comes in two flavors, if the regional seats are final the national ones must be redistributed and if the system is guaranteed proportional some regional seats must be "taken back".

5.1. **The need for regional seat redistribution under guaranteed proportionality and fixed parliament size.** The 2023 reforms to the German electoral law are summarized in Table 1. Previously, the parliament size was not fixed. The increasing number of parties winning representation in the Bundestag, led to an increase in the number of additional regional seats that some parties obtained without appropriate nationwide votes to back them up (Ger. *Überhangmandate*, Eng. *Overhang* seats) and the seats that other parties were additionally awarded to compensate for that (Ger. *Ausgleichsmandate*). Consequently, the German parliament expanded in size from the standard 598 to 736.

In the German system introduced in 2023 (see [8]), the overhang seats (that is seats without *Zweitstimmendeckung* in new German terminology) are not redistributed to the next candidate in the constituency and the algorithm for choosing which seats to give up is complicated and depends on coarser structure of the Bundesländer (see for example the fictitious use of the law on the 2021 election [14]). This complication makes the system multi-tier in contrast to the former German system which was two-tier. It is a central feature of the algorithm for retracting regional seats that it prioritizes the seats by the percentage of the votes the candidate obtained.

Similarly, for the electoral system of the Swedish parliament (*Riksdagen*) regionally distributed mandates can be 'taken back' to ensure national proportionality so that system does not satisfy regionality. Here, the overhang seats are redistributed locally in each constituency one by one starting with the seat corresponding to the least quotient [42, Kap. 14]. In Section 6.1 yet another redistribution algorithm is devised.

5.2. **National seat redistribution under regionality and fixed parliament size.** In case the regional seats are final, but the system nevertheless aims at but does not achieve guaranteed proportionality the system needs a mechanism for redistributing national seats. The natural choice depends on whether the system aims at guaranteed proportionality with respect to a divisor or a quota method.

*Divisor methods:* In the Icelandic case (Example 4.3), the national seats are ordered by the size of their D'Hondt divisor. Naturally, the national seats that are left out are the ones with the lowest quotients.

*Quota methods:* Systems allocating national seats based on Hare quota do not have a natural ordering of the quotients[18]. Therefore, the electoral systems of many countries including Denmark, Norway, South Africa, and Bosnia-Herzegovina are built on an iterative procedure where new Hare quotas are calculated omitting overrepresented parties. A disadvantage of iterative solutions is their increased complexity, which can become so overwhelming that it is difficult to maintain a clear overview. Since electoral systems are typically defined using natural language in legal texts, it is possible that there exist election outcomes for which the algorithm is ambiguous or not well-defined [31, 37]. This issue was demonstrated for the Danish electoral system in [18], and we suspect that similar ambiguities could exist in other electoral systems as well.

In New Zealand the overhang seats are not redistributed and thus regionality is satisfied. Parties still obtain the number of seats corresponding to their vote share with respect to the initial size of the parliament (using the Sainte-Laguë method) - potentially leading to an increasing size of the parliament (two seats in the 2023 election). Since parties without overhang seats are not compensated, the system is not guaranteed proportional.

## 6. Circumventing the impossibility theorem

The physicist John Bell said in a different context: "*...what is proved by impossibility proofs is lack of imagination*" [5]. In that spirit, an electoral system that escapes the constraints of the impossibility theorem is constructed. It does not have a one-tier base, so it is mixed, but not multi-tier in the sense of (4). Afterwards, the related method of bi-proportionality is discussed. In contrast to the previous chapters, these electoral systems also distribute seats on constituencies.

6.1. **Geographically ranked guaranteed proportionality.** A guaranteed proportional (with respect to a weakly proportional apportionment method that satisfies Definition 2.4) electoral system with fixed parliament size is constructed. The system will break the regionality constraint, which is in a sense obsolete in this case, since there is no preliminary allocation of seats. However, it still attains a strong form of geographic representation.

The algorithm can be described as follows: Distribute the national seats using the weakly proportional apportionment method. This determines, right from the beginning, how many seats each party gets and the method aims at distributing the seats to the constituencies in the best possible way. Here is the overall idea of what we suggested, for details see [17].

- Construct rankings for each party in each constituency (e.g. the Sainte-Laguë quotients).
- Construct an overall ranking of the parliamentary seats coming from each constituency. That is if there are seats $k_1, \dots k_c$ in each constituency. Then formally consider the set of seats as the set of pairs $(i, j)$ for $1 \leq i \leq c$ and

[18] If one tries to use the size of the remainders one can run out of remainders.

$1 \leq j \leq k_i$. Now, prioritize the pairs ordered in some way, i.e. put on a list. The simplest way to do this is to list the quotients by size.
- Distribute the number of seats that each party is entitled to, to the quotients on the list starting from the top.

If there are no overhang seats this system is equivalent to a regional system, where the seats are first distributed regionally followed by a leveling procedure. Table 2 shows geographically ranked guaranteed proportionality deployed on the Belgian senate elections dataset that was used to evaluate bi-proportional methods in [11, 20].

6.2. **Double proportionality and dual-member proportionality.** Bi-proportional systems are a class of electoral systems that circumvent the impossibility theorem, studied axiomatically by Balinski and Demange in [2]. For these systems the regional seat distribution depend on the votes in all constituencies (as is the case with GRGP above). One such system was described by Pukelsheim in [32, Chap. 14-15] and was implemented in several cantonal electoral systems in Switzerland. Both national and constituency seat counts are fixed.

Table 2. The 1981 senate election in the former Belgian province Brabant: vote distribution and seat allocation. For the overall distribution `DH` is used and the local seats are distributed according to `SL` quotient giving the same result as the bi-proportional methods in [11, 20]. The signs indicate the mandate change if `DH` was used both overall and locally.

| | **CVP** | **FDF-RW** | **PVV** | **PRL** | **PS** | **SP** | **VU** | **UDRT** | **PSC** | **Ecolo** |
|---|---|---|---|---|---|---|---|---|---|---|
| *Vote Distribution* | | | | | | | | | | |
| Br. | 109,377 | 148,928 | 88,645 | 106,920 | 80,644 | 53,409 | 63,807 | 59,730 | 54,549 | 34,966 |
| Le. | 78,280 | 2,048 | 70,273 | 1,765 | 0 | 60,024 | 32,178 | 4,203 | 0 | 0 |
| Ni. | 0 | 17,879 | 707 | 47,579 | 48,965 | 0 | 923 | 16,984 | 24,590 | 13,060 |
| *Seat Allocation under geographically ranked guaranteed proportionality* | | | | | | | | | | |
| Br. | $3^-$ | $3^+$ | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 |
| Le. | $1^+$ | 0 | 1 | 0 | 0 | 1 | $1^-$ | 0 | 0 | 0 |
| Ni. | 0 | $1^-$ | 0 | 1 | 1 | 0 | $0^+$ | 0 | 0 | 0 |
| Total | 4 | 4 | 3 | 3 | 3 | 2 | 2 | 1 | 1 | 1 |

Following [27], say that an apportionment method is *locally accountable* if there is no constituency where a locally smaller party gets more seats than a locally larger party. Since the proof of the impossibility theorem only used that monotonicity of the local apportionment methods building up the underlying one-tier system had a winner-take-one rule and that if there is $k$-seats locally then no party which is not among the $k$ largest can get a seat which both follow from local accountability the same proof shows the following theorem even for electoral systems without a one-tier basis.

**Theorem 6.1.** Any locally accountable electoral system based on $c \geq 2$ constituencies that always assigns at least one local seat cannot be both guaranteed proportional and have fixed parliament size even upon assuming vote-seat consistency.

One could ask whether one could weaken the local accountability assumption to winner-take-one and that is indeed possible but one needs many more parties than in the proof of the impossibility theorem[19]. This means that it is impossible to have a winner-take-one rule for both geographically ranked guaranteed proportionality and bi-proportional methods (if they are guaranteed proportional). In many practical contexts [7], e.g., the Danish island of Bornholm [17, 18], the winner-take-one modification is essential for the legitimacy of the system[20].

Another proposed system is dual-member proportionality suggested by Sean Graham for Canada [23]. The system also has a winner-take-one rule, is guaranteed proportional with respect to the Hare-LR method, and fixed parliament size. Thus, by the impossibility theorem, there must exist distributions of votes where the algorithm is not defined (and indeed this happens if one party wins more "first seats" than their Hare quota).

## 7. Conclusion

Many technical issues in two-tier systems represent trade-offs between regionality and proportionality once these concepts have been given mathematical definitions. It is impossible to guarantee both perfectly for systems with fixed parliament size. The workarounds bi-proportionality and geographically ranked guaranteed proportionality balance geographical and proportional representation and evade the impossibility theorem by not having a regional one-tier base.

[19] If a small party just is the largest in one constituency and $m$ (parliament size) additional parties are larger, but with spread out votes.

[20] For that reason, the 1948-1953 Danish electoral law, which was guaranteed proportional and fixed parliament size removed a party's first seat in each constituency last, see [17, App. C.2].

The impossibility theorem highlights the inherent limitations of two-tier electoral systems, and provides a valuable framework for policymakers and electoral reform advocates to critically evaluate existing systems. The trade-offs between guaranteed proportionality, regional representation, and fixed parliamentary size reveal the need for innovative approaches in electoral design. One potential policy direction is the exploration of mixed or flexible seat allocation models that adapt dynamically to electoral outcomes, with a variable number of regional seats and compensatory seats.

## Acknowledgements

We warmly thank Jørgen Elklit, Svante Janson and Peter Rasmussen for discussions and comments to an early draft of the manuscript and Friedrich Pukelsheim and the referees for feedback. FRK is supported by the Carlsberg Foundation, grant CF24-0466. The authors have no conflicts of interest.

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A.1. **Impossibility theorem in the presence of thresholds.** Thresholds often act as concrete barriers to proportionality. Most common are percentage threshold, where parties below a certain percentage are excluded from (proportional) representation. Some two-tier systems have specialized rules where regional seats act as a *back door* to proportional representation. A party with a certain number of regional seats is allowed proportional representation even if the party does not meet a certain percentage threshold. Examples include Denmark, Germany[21] and New Zealand [36, p.83]. In other two-tier systems, parties failing to meet the threshold may retain their regional seats under specific conditions, but they are not granted access to compensatory seats ensuring national proportionality. An example is Sweden, which has a 4% percentage threshold, and parties below threshold can keep regional seats won with at least 12% of the vote regionally.

There are also examples of *hard thresholds* where parties lose regional seats if they do not make the national threshold. A notable example is Turkey, which previously had a hard electoral threshold of 10%[22].

The three different types of thresholds also highlight three different approaches to balancing proportionality and regionality in the context of thresholds. Each of these threshold types can be incorporated readily in the formalism by considering the threshold as a function on election outcomes that sets columns on parties that do not reach the threshold to zero. For simplicity we restrict the following discussion to the case of hard thresholds.

Considering the case of a hard threshold, one could relax either the regionality (Definition 3.2) or guaranteed proportionality criteria (Definition 3.3) to only include parties above threshold and ask for a modified impossibility theorem.

If guaranteed proportionality is not relaxed, it is easy to see that the electoral system is not proportional. Similarly, if regionality is not relaxed then a party not meeting the threshold may win a regional seat (by having its votes concentrated) and thus the electoral system does not satisfy regionality.

So restrict attention to the case where both regionality and guaranteed proportionality are relaxed to only concern parties reaching the hard threshold. Since the threshold is hard, it puts an effective limit on the number of parties in parliament. Combining this limit with a system that has very few regional seats and a lot of compensatory seats (e.g. a huge parliament), one can construct electoral systems satisfying the two relaxed criteria as well as a fixed parliament size. For this approach to work in general (with no assumptions on the distribution of votes or regional seats) if the hard threshold is at $\alpha\%$ then in general at most $\alpha\%$ seats can be regional (since in the worst case a party right above the threshold could win all the regional seats with only these votes). Real world systems using compensatory seats have much fewer compensatory seats than what is needed for this mathematical guarantee.

A.2. **Two-round systems and additional considerations.** Our definition of electoral systems in Definition 4.1 is flexible enough to accommodate two-round systems (TRS) as the input seat distribution $S$ could be the output of a

[21] The back door was abolished in the law of 2023, but reinstated by the constitutional court in 2024 [8].

[22] De facto thresholds are also often mentioned in the literature. They arise because obtaining representation requires at least one seat (for example $\frac{1}{150} = 0.67\%$ in the Dutch example from Example 2.8ii there are 150 seats in total). Here only superimposed thresholds are considered.

previous election $V_1$ (possibly depending on a yet earlier $S_1$) as $\mathtt{ES}_{\text{2-Round}}(S, V_2) = \mathtt{ES}_{\text{2-Round}}(\mathtt{ES}(S_1, V_1), V_2)$. Thus, the two-round system takes the seat distribution $S = \mathtt{ES}(S_1, V_1)$ as input. The framework can fit the French parliamentary elections where $S_1$ is empty and $V_1$ is the first election outcome. The first electoral system $\mathtt{ES}(S_1, V_1)$ determines the seats of the directly elected candidates. The remaining seats are then distributed with the electoral system $\mathtt{ES}_{\text{2-Round}}$ depending on the election outcome $V_2$ (which is restricted by $V_1$).

Beyond two-round systems, this framework can also model staggered elections, such as those used for the US *Senate*. In this case, the already elected members serve as the input $\mathtt{ES}(S_1, V_1)$, which itself depends on the previously elected members $S_1$. Additionally, the definition extends to countries such as Italy, Burundi, and the Democratic Republic of the Congo, which currently have five, two, and one lifetime-appointed senators, respectively.

Sebastian Tim Holdum, University of Copenhagen, Denmark.
*Email address*: `sebastian.holdum@nbi.dk`

Frederik Ravn Klausen, University of Cambridge, DPMMS, Cambridge, United Kingdom
*Email address*: `frk23@cam.ac.uk`