Impossibility Theorem for Two-Tier Electoral Systems

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Abstract

Two-tier electoral systems aim to balance regional representation and proportionality. Recently, the mathematical mechanisms of several Northern European two-tier systems have been challenged by increasing political party fragmentation, as evidenced by the expanding German parliament and the majority-deciding disproportionality of Denmark's 2022 election. This study develops a mathematical framework for two-tier systems, regionality, and proportionality, which identifies the common cause behind these complications through the formulation of an impossibility theorem: no two-tier system with fixed parliament size can guarantee both proportional and geographic representation. Finally, a new method—geographically ranked guaranteed proportionality is proposed to circumvent the impossibility theorem.

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1. Introduction

Electoral systems are fundamental to democracies, transforming votes into political power. For an electoral system to be perceived as legitimate, it must be seen as fair. Fairness can take many forms, but it often includes some level of representation. Representation can be measured by various criteria, including but not limited to regional balance and proportionality.

Two-tier systems attempt to align regional representation with proportionality. This paper aims to define these concepts mathematically and use the definitions to prove an impossibility theorem, highlighting the inherent trade-offs between regionality and proportionality that any two-tier electoral system faces.

Typically, two-tier electoral systems initially allocate seats within regional subdivisions, followed by a secondary process aimed at improving proportionality. This two-layered approach seeks to combine the best aspects of regional representation with overall proportionality [33]. However, the complexity of these systems raises important questions about

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their efficacy and fairness, particularly concerning the tension between regional interests and proportional outcomes.

The purpose of this paper is to illuminate the trade-offs between regionality and proportionality in two-tier electoral systems. We provide a rigorous mathematical framework and an impossibility theorem, aiming to enhance understanding of how these systems operate and the inherent compromises they entail. The study is particularly relevant for mathematically inclined political scientists seeking to deepen their understanding of electoral system design and its implications for democratic representation.

To illustrate the practical implications of this trade-off, we examine several specific examples. The United Kingdom's first-past-the-post (FPTP) system serves as a stark example of disproportionality. In the 2024 general election, the Reform UK party received 14.3% of the national vote but secured only five seats of the total 650, while the Labour party won 411 seats with 33.7% of the vote [43]. This discrepancy demonstrates the extreme regionality of the FPTP system, constituency based system, that can lead to significant disproportionality at the national level. According to the Economist [38] the 2024 election was the least proportional UK general election ever, according to the index developed by Michael Gallagher [16].

Conversely, Germany's mixed-member proportional system was designed to ensure proportional representation in addition to the regionality provided by the constituencies. However, this system faced challenges such as the increasing size of the Bundestag due to overhang and compensatory seats. The Bundestag grew from the standard 598 seats to 736 in the 2021 federal election [10], showcasing the complexity and potential unintended consequences of strict proportionality within a two-tier framework. These issues sparked significant debate about the necessity of electoral reform in Germany [9], resulting in the 2023 reform.

Another noteworthy example is Sweden, which, after experiencing overrepresentation in 2010, debated and reformed its election law to address issues related to proportionality and regional representation [29]. The Swedish system allocates 310 of its 349 parliamentary seats based on regional constituencies, with the remaining compensatory 39 seats used to achieve (a higher degree of) proportional representation [40]. Despite these efforts, debates continue about the effectiveness of this approach in balancing regional and proportional interests.

Similarly, discussions have occurred in other Northern European two-tier systems. In Norway, parties have won regional seats with very few votes [44, sec. 5.2.6.5] such as in 2005, when the party Venstre received 1 out of 5 seats in Finnmark, despite only getting 2.2% of the votes there [35]. In the Danish 2022 general election a regional seat was not leveled out by the compensatory seats for the first time since 1947 [14]. This seat decided the majority in parliament, leading to a situation where, in traditional left-right terminology, the left bloc won despite receiving fewer votes than the right bloc [13] - a dramatic event in a two-tier system traditionally labeled as proportional.

These discussions come about because it is impossible to fully balance geographic and proportional representativity with a fixed parliament size. This is the content of the impossibility theorem that we prove in Section 3. While the mathematics behind this theorem is straightforward, certain electoral laws currently in operation omit addressing these specific cases, suggesting that the trade-off between proportionality and regionality is not always intuitive in practice. An example is the Danish electoral law, which lacks clarification on special cases where these conditions clash and even contains conflicting provisions [15].

In social choice theory there is a rich tradition for proving impossibility theorems starting with Arrow's [1], over Gibbard's [17] to the Balinski-Young theorem [3]. This paper extends that tradition by introducing an impossibility result in the context of complex electoral systems, emphasizing the inherent trade-offs between regionality and proportionality in two-tier electoral structures through a rigorous mathematical framework. While this mathematical perspective may obscure other valuable insights from political science (see, [37, 8, 34, 20] and many others) which have provided seminal analyses of electoral systems, we hope that our study nonetheless contributes to the ongoing discourse on electoral design and evaluation.

1.1. Omissions and clarifications

Electoral systems throughout the world are very diverse. To make a simple mathematical framework encapsulating the important aspects of real-world systems quite a few simplifying assumptions are necessary. Most important is arguably our omission of individual candidates as we only focus on parties. Another more mathematically problematic issue is omission of the large set of specialized local rules related to minority parties, race, religious groupings, disability, age or gender quota etc. The mathematical framework we present is also not able to deal with prioritized or transferable votes. Tie-breaking mechanisms are beyond the scope of this study, as they do not impact the core issue. Many of our results are applicable to both two-round systems and apparentments, but they were not our motivating examples.

We caution the reader that we consciously deviate from some of traditional classifications in the literature (see [31] for an overview of these concepts). This includes, but is not limited to the distinction between single- and multi-member constituencies, which for our purposes are equivalent. For the aim of the paper it is also unnecessary to distinguish between *parallel voting, coexistence, party-list proportional representation, mixed-member proportional representation*, and similar classifications, as they will be treated equally in mathematical terms. Instead, we will focus primarily on segmenting *mixed electoral systems* by whether they include *compensatory seat allocation* or not. While this approach may overlook certain granular distinctions, it allows us to provide precise mathematical definitions of key concepts, thereby minimizing ambiguity.

For an apportionment method `M`, such as Hare-LR, D'Hondt etc., we say that an electoral system agrees with `M` if for all possible elections the seats (on the top-tier) are distributed with respect to the method `M`. This is a mathematical property of the electoral system itself, not of any individual election. For the purposes here it is beneficial to introduce the new notion of guaranteed proportionality, if there is any (of the usual) `M` such that the electoral system agrees with `M`, (see cf. Definition 16 for details). Many well-functioning two-tier systems traditionally labeled as *proportional representation*, are not guaranteed proportional even if all past elections have been consistent with the method. We have chosen this deviation from the literature because the definition, and the resulting impossibility theorem offers the most appropriate lens to understand the mathematical mechanisms behind the recent debates mentioned above.

2. Defining electoral systems

In this section, election outcomes and seat distributions are defined mathematically and the theory of apportionment methods is briefly reviewed. Afterwards, we give a general definition of an electoral system, define one and two-tier systems and discuss how they are instances of the general definition.

2.1. Election outcomes and seat distributions

We begin by defining an election outcome as a matrix of votes cast. For convenience, we label all electable entities as 'parties'. Some electoral systems have different rules for parties, independent candidates, multi-party alliances and ethnic minorities. These criteria can be included in the apportionment method. In the formalism which is to be developed, let p denote the number of parties and c the number of constituencies. These constituencies can be both multi- and single member and we will make no distinction between the two cases.

Definition 1 (Election outcome)**.** For positive integers p and c a (p, c)-*election outcome* is an $(p + 1) \times c$ matrix V where the entries $V[i, j]$ are non-negative integers, that should be interpreted as the votes that the ith party gets in the jth constituency. The extra row $V[p + 1, j]$ is the blank and invalid votes in the jth constituency.

Let $\mathcal{V}_{p,c}$ be the set of all (p, c)-election outcomes, $\mathcal{V}_c$ be the set of all election outcomes with fixed c, and $\mathcal{V}$ be the set of all election outcomes.

In the following, note that even though the setup can account for blank and invalid votes, discussions of these are omitted.

Electoral districts vary in significance within real-world electoral systems. Some districts serve purely administrative purposes, such as organizing voting and counting votes. In such cases, the final seat distribution does not depend on whether a vote is cast in electoral district 1 or 2. However, other subdivisions of district can impact the election results.

In some countries, such as Russia and Japan, voters receive two ballots — one for local candidates and another for (national or regional) party lists. In those cases, the election outcome can be represented with two columns for each constituency, one for each ballot type.

Electoral systems in countries like Zimbabwe and Italy[1] also aggregate votes from each individual constituency at the regional or national level, effectively treating the entire jurisdiction as an additional constituency with reserved seats. Thus, for simplicity and flexibility in certain special cases, it is helpful to consider an *amalgamated election outcome* where appropriate constituencies can be made up of multiple electoral districts. That is an amalgamated election outcome consists of columns that are the sum of columns in an election outcome.

Suppose that the most fine-grained constituencies (i.e. precincts) are labeled $1, \dots, c$ and the new resummed by $1, \dots, \tilde{c}$. Then the amalgamation is a function $A : \mathcal{V}_c \to \mathcal{V}_{\tilde{c}}$ that

[1] Excluding Aosta Valley and the overseas constituencies.

sums up the corresponding columns of votes[2]. In the simple case, the amalgamation merges constituencies, but generally the new constituencies are allowed to overlap.

One can imagine the constituencies of the amalgamated election outcome V as the coarsest possible decomposition, where the location of the vote cast matters and with possible additional columns for regional or national tallies. As amalgamated constituencies are not necessarily disjoint, the total vote count may not be preserved. The overlaps most often arise when the entire national total is considered a constituency as in the real world example below.

Example 2. Elections for *Lasanble Nasyonal* in the Seychelles consists of 26 constituencies, each electing a local candidate. However, for the 2020 general election, 44 separate polling stations were established to accommodate elderly voters, hospital patients, and residents of outlying islands [32]. Additionally, the Seychelles employ a system where party votes are also aggregated at the national level and additional seats are rewarded. The election featured four competing parties [12]. Consequently, the following 4×27 amalgamation can be beneficial to work with

$$A\begin{bmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,43} & v_{1,44} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,43} & v_{2,44} \\ v_{3,1} & v_{3,2} & \cdots & v_{3,43} & v_{3,44} \\ v_{4,1} & v_{4,2} & \cdots & v_{4,43} & v_{4,44} \end{bmatrix} = \begin{bmatrix} v_{1,1}+v_{1,2} & v_{1,3} & \cdots & v_{1,41} & v_{1,42}+v_{1,43}+v_{1,44} & \sum_j v_{1,j} \\ v_{2,1}+v_{2,2} & v_{2,3} & \cdots & v_{2,41} & v_{2,42}+v_{2,43}+v_{2,44} & \sum_j v_{2,j} \\ v_{3,1}+v_{3,2} & v_{3,3} & \cdots & v_{3,41} & v_{3,42}+v_{3,43}+v_{3,44} & \sum_j v_{3,j} \\ v_{4,1}+v_{4,2} & v_{4,3} & \cdots & v_{4,41} & v_{4,42}+v_{4,43}+v_{4,44} & \sum_j v_{4,j} \end{bmatrix}.$$

Here the 46 columns are summed up to 26 new columns corresponding to the constituencies and a new column with the vote totals is added. The electoral system is discussed in further detail in Example 9iv).

Next, we turn to seat distributions.

Definition 3 (Seat distribution)**.** A (p, c)-*seat distribution* is a $p \times c$ array of non-negative integers S, where the matrix entry $S[i, j]$, is the number of seats allocated for the ith party from the jth constituency for $1 \leq i \leq p$ and $1 \leq j \leq c$. The set of all (p, c)-seat distributions we denote by $\mathcal{S}_{p,c}$, the set of all seat distributions for fixed c by $\mathcal{S}_c$ and we let $\mathcal{S}$ denote the set of all seat distributions.

2.2. Apportionment methods

Apportionment methods determine how votes are converted into seats within individual constituencies. Formally, they can be defined as a mapping from a column in the election outcome to a corresponding column in the seat distribution. These methods serve as fundamental building blocks for more complex electoral systems.

[2]Mathematically, we can define an amalgamation, using a function $a : \mathcal{P}(\{1, \ldots, c\}) \to \{1, \ldots, \tilde{c}, \perp\}$ with the property that each of the elements $\{1\}, \ldots, \{\tilde{c}\}$ has a unique preimage $a^{-1}(\{r\})$ in $\mathcal{P}(\{1, \ldots, c\})$. Here $\perp$ is the outcome if that combination of columns is not used in the amalgamation. Now, $A(V)[i, j] = \sum_{l \in a^{-1}(\{j\})} V[i, l]$, meaning we have summed the columns in V that a combines.

Definition 4 (Apportionment method)**.** An *apportionment method* is a function $\mathtt{M} : \mathcal{V}_{p,1} \to \mathcal{S}_{p,1}$, that for every tuple of votes $v = (v_1, \ldots, v_p, v_\perp)$ outputs a tuple of seats

$$\mathtt{M}(v) = (m_1, \ldots m_p).$$

Below are some usual characteristics of apportionment methods.

Definition 5 (Apportionment method properties)**.** An apportionment method $\mathtt{M}$ satisfies

Party invariance (cf. Anonymity [27, Sec. 4.3]): if the outcome of the apportionment method does not depend on the ordering of the parties.

Monotonicity (cf. Concordance [27, Sec. 4.3]): if party 1 receives more votes than party 2 ($v_1 > v_2$) then the number of seats for party 1 is larger or equal to the number of seats for party 2 ($m_1 \geq m_2$).

Fixed seat count: if there is a fixed number k of seats available in a given constituency independently of votes cast v. We will use the shorthand notation $\mathtt{M}^k$, for the apportionment method $\mathtt{M}$ distributing k seats.

In the following, we will assume that the apportionment methods $\mathtt{M}$ are party invariant and monotone[3]. Notice that monotonicity implies that the largest party always gets at least one seat. We refer to that property as the *winner-take-one rule.*

Common examples of apportionment methods that satisfy all these assumptions are the D'Hondt method ($\mathtt{DH}$) for allocating k seats in a given constituency, "first-past-the-post" ($\mathtt{FPTP}$), Hare-LR ($\mathtt{LR}$) (the method of largest remainders) and the Sainte-Laguë method ($\mathtt{SL}$) as well as suitably modified Sainte-Laguë methods and their generalizations (see for example the β-linear divisor method and γ-quota methods defined in [27, 22]).

Let us give some examples of apportionment methods that illustrate our notation and the diversity of electoral systems that we aim to capture.

Example 6 (Apportionment methods)**.**

(i) (United States *Electoral College* for presidential elections) The first-past-the-post is mostly used for single-member constituencies that is $k_i = 1$ seats, but is also used for $k_i > 1$ in all US states except Maine and Nebraska for the electoral college. The algorithm is simple. Suppose $v = (v_1, \ldots, v_p)$ is *size-ordered* (such that $v_1 > v_2 \geq v_3 \cdots \geq v_p$) then
$$\mathtt{FPTP}^k(v) = (k, 0, \ldots, 0).$$

(ii) (Cameroons lower house of parliament *L'Assemblée Nationale*) All of Cameroon's 58 single- and multi-member constituencies use the following apportionment method: Any list that gains a majority of the vote wins all of the seats. Otherwise, the first-placed list receives one half of the seats rounded up to the nearest whole number. The remaining seats are then distributed among the other parties above the threshold using

[3]We refrain from embarking on the program of abstractly characterizing suitable methods $\mathtt{M}$, another property in such a program could be *homogeneity* (cf. Decency [27, Sec. 4.3]) , that is for every positive integer N then $\mathtt{M}(Nv, k) = \mathtt{M}(v, k)$.

the apportionment method Hare-LR (LR). For $v = (v_1, \dots v_n)$ size-ordered with sum v, then

$$\mathtt{M}^k_{\text{Cameroon}}(v) = \begin{cases} (k, 0, \dots, 0) & \text{if } v_1 \geq 0.5v \\ \left(\lceil \frac{k}{2} \rceil, \mathtt{LR}^{\lfloor \frac{k}{2} \rfloor}(v_2, \dots, v_p)\right) & \text{else.} \end{cases}$$

Here and in the following $\lceil\cdot\rceil, \lfloor\cdot\rfloor$ respectively denote rounding up and rounding down.

A less conventional example of an apportionment method without a fixed seat count is the initial phase[4] of the electoral system used in Armenia [30].

(iii) (Armenias unicameral parliament ***Ազգային ժողով*** (*Azgayin Zhoghov*)) The electoral system of Armenia[5] has a single constituency. Initially, 101 seats are distributed, that is the ith party is awarded $\mathtt{LR}^{101}_{\text{Armenia}}(v)[i]$, where the apportionment method $\mathtt{LR}_{\text{Armenia}}$ is a modified variant of the largest remainders method, where the remainder's of the largest parties are rounded up. Afterwards, the seat distribution is amended: If a party secures a majority of the seats but not 54%, it is allocated additional seats until it reaches 54%. If any party wins over 67% of the seats, the remaining parties are given extra seats so that the largest party holds at most 67%.

2.3. Electoral systems

Let us define an electoral system generally and abstractly, as this will enable us to capture most real-world systems in our impossibility theorem. The motivation for this rather abstract definition is that many real world electoral systems have intermediate calculations (e.g. assigning regional seats) that are done independently and used as input for the next calculation. Recall that $\mathcal{S}_c$ is the set of seat distributions with c constituencies and $\mathcal{V}_c$ is corresponding set of vote distributions.

Definition 7 (Electoral system)**.** An electoral system ES is a function

$$\mathtt{ES} : \mathcal{S}_c \times \mathcal{V}_c \to \mathcal{S}$$

such that there exists S, V_1 and V_2 where $\mathtt{ES}(S, V_1) \neq \mathtt{ES}(S, V_2)$.

In other words, an electoral system is an algorithm that takes a seat distribution and an election outcome as input. At first glance it may seem odd that the electoral system takes a seat distribution as input, but we insist on this definition as it will allow us to define two-tier systems.

[4]If no party (or a coalition of parties formed within six days after the election) can obtain a majority, a second round of elections will be held in which the two best-performing political forces will participate. All seats received in the first round will be retained. The party (or newly formed coalition) that wins the second round will be allocated additional seats to reach 54% of all seats.

[5]In this example, the electoral threshold and the reserved seats for Assyrians, Kurds, Russians, and Yazidis are disregarded.

In the simplest two-tier case, the input seat matrix S is the regional seats and input vote matrix V is the national vote totals and the function $\texttt{ES}$ is the (sometimes rather complicated) function that calculates the compensatory seats.

However, the one-tier systems are even simpler, since the electoral system does not depend on any input seat distribution. So we begin with them and return to the unpacking Definition 7 in the case of two-tier systems.

2.4. One-tier electoral systems

A both simple and important example of an electoral system is a one-tier system. For electoral systems, the output does not depend on the input seat distribution which can therefore be omitted. For matrix M we denote the ith row as $M[i,\cdot]$ and $M[\cdot,j]$ the jth column.

Definition 8 (One-tier electoral system)**.** An electoral system $\texttt{ES}_{\text{1-Tier}}$ is *one-tier* if there exists an amalgamation $A : \mathcal{V}_c \to \mathcal{V}_{\tilde{c}}$ and a tuple of apportionment methods $\texttt{M} = (\texttt{M}_1, \dots, \texttt{M}_{\tilde{c}})$ such that

$$\texttt{ES}_{\text{1-Tier}}(A(V))[\cdot,j] = \texttt{M}_j(A(V)[\cdot,j]). \tag{1}$$

In other words, one-tier systems are those in which the resulting seat distribution can be determined by applying the apportionment method independently to the votes for each constituency $\tilde{c}$, without any interaction between them. One-tier systems are common throughout the world, cf. Figure 1, especially among former British territories where $\texttt{FPTP}$ is often used.

Example 9 (One-tier systems)**.** The following parliament chambers are all examples of one-tier electoral systems.

(i) (The United Kingdoms *House of Commons*) First-past-the-post voting is used in every 650 single-member constituencies (as of 2024) corresponds for every $1 \le j \le 650$ to the apportionment method $\texttt{M}_j = \texttt{FPTP}^1$.

(ii) (The Netherlands *Tweede Kamer der Staten-Generaal*) The Dutch electoral system uses the D'Hondt method to distribute 150 seats in a single constituency. Thus, it can be viewed as a one-tier system with a single apportionment method ($\texttt{DH}^{150}$).

(iii) (Russia's Государственная дума (*State Duma*)) Each voter is given two votes. The first vote is used to elect a local candidate in one of the 225 single-member constituencies using the first-past-the-post method. The second vote is cast for a party list and is tallied nationally, with another 225 seats distributed using the Hare Quota rule (with a 5% electoral threshold) $\texttt{LR}_{5\%}$, thus the overall method is $(\texttt{M}_1, \dots, \texttt{M}_{226}) = (\texttt{FPTP}^1, \dots \texttt{FPTP}^1, \texttt{LR}^{225}_{5\%})$.

(iv) (The Seychelles *Lasanble Nasyonal*) The Seychellois electoral system distributes 26 seats using first-past-the-post in single-member constituencies. In addition, up to 10 additional seats are allocated nationally - one per 10% of the total national vote received by a party. Therefore, under an appropriate amalgamation as the one given in Example 2, the electoral system can be classified as a one-tier system in the sense of Definition 8, with

$$(\mathtt{M}_1, \ldots, \mathtt{M}_{27}) = (\mathtt{FPTP}^1, \ldots \mathtt{FPTP}^1, \mathtt{M}_{\text{Seychellois Bonus}}),$$

where $\mathtt{M}_{\text{Seychellois Bonus}}(v_i) = \lfloor \frac{10 v_i}{\Sigma v_i} \rfloor$.

Remark 10 (Parallel vs. mixed compensatory system)**.** The definition of a one-tier electoral system can be used to provide a mathematical definition of what is described as parallel voting in [31]. A system can be defined as *parallel* if it is one-tier and employs different apportionment methods $\mathtt{M}$ across different constituencies.

An example of such a system is the electoral college for US presidential elections due to Maine and Nebraska not using $\mathtt{FPTP}$ (cf. Example 6). Another example of this is the European parliament elections, which consist of separate national elections held by each of the 27 member states, without interaction between them and with different methods $\mathtt{M}$ (which in the case of Italy and Poland are in themselves not one-tier systems and therefore not quite apportionment methods, see e.g. [28]).

The notion of parallel system is to be contrasted with *mixed compensatory* systems, where different methods interact in a way that is not captured by the one-tier framework.

2.5. Multi- and two-tier systems

The impossibility theorem holds for our general definition of electoral system, but our work is especially motivated by the issues present in real world two- and multi-tier systems which are common (see Figure 1 for an illustration of the current status.

Multi-tier systems are built on top of a one-tier system with additional mixing. That is the input seat matrix S is the result of a one-tier calculation. Thus,

$$\mathtt{ES}_{\text{Multi-Tier}}(S, V) = \mathtt{ES}(\mathtt{ES}_{\text{1-Tier}}(A(V)), V).$$

We now consider two-tier systems the special subset of multi-tier systems which can be computed column-wise. To give a definition also consider amalgamated seat distributions, where the columns of seats are summed up.

Definition 11 (Two-tier system)**.** Let $A : \mathcal{V}_c \to \mathcal{V}_{\tilde{c}}$ be an amalgamation. An electoral system is *two-tier* if it is not one-tier and is of the form

$$\mathtt{ES}_{\text{2-Tier}}(S, V)[\cdot, j] = \mathtt{ES}_j(A(\mathtt{ES}_{\text{1-Tier}}(V))[\cdot, j], A(V)[\cdot, j]), \tag{2}$$

for each $1 \leq j \leq \tilde{c}$ for a one-tier system $\mathtt{ES}_{\text{1-Tier}}$ and $\tilde{c}$ electoral systems $\mathtt{ES}_j$.

Figure 1 gives a graphical overview of the electoral systems captured by the definition. To unpack the formula, it is instructive to first consider the most standard two-tier system where the amalgamation $A_{total} : \mathcal{V}_c \to \mathcal{V}_1$ is *total*, meaning that it sums up all constituencies and there is only one j. Those two-tier systems are of the form

$$\mathtt{ES}_{\text{2-Tier}}(S, V) = \mathtt{ES}(A_{\text{total}}(\mathtt{ES}_{\text{1-Tier}}(V)), A_{\text{total}}(V)) \tag{3}$$

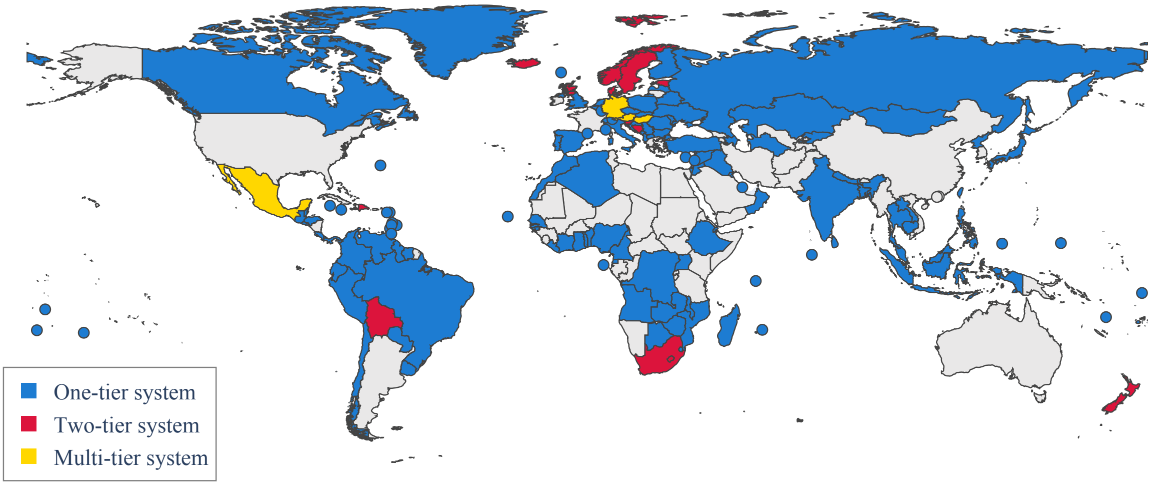

Figure 1: An overview of countries that as of 2024 employ one-tier, two-tier or multi-tier systems for elections to the lower (or unicameral) house without taking specialized rules for minority quotas or electoral thresholds into account[6]. Other systems include two-round systems, preferential votes, transferable votes, non-democracies without elections etc.

so that the remaining seats only depend on the total number of votes and the total number of seats in the one-tier system. At this point, an example of one of the simplest non-trivial two-tier system in use is instructive.

Example 12 (Icelands Unicameral parliament *Alþingi*)**.** The Icelandic parliament has 63 members. 54 of these members are elected directly in multi-member constituencies using the D'Hondt method and the remaining nine seats are compensatory. In our notation, this electoral system is a two-tier system: The first tier has six MMCs distributing 7, 9, 9, 9, 9 and 11 regional seats [39, Art. 8] using the D'Hondt method (see Figure 2). Thus, the one-tier basis of the system can be written as $\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}} = (\mathtt{DH}^7, \mathtt{DH}^9, \mathtt{DH}^9, \mathtt{DH}^9, \mathtt{DH}^9, \mathtt{DH}^{11})$.

Now, the compensatory seats are distributed [39, Art.108] in the following way[7]: The D'Hondt method is applied to the national vote totals to distribute all 63 seats. However, the 54 quotients corresponding to the regional seats are assigned before determining the compensatory seats. Crucially, this algorithm depends on the distribution of regional seats. If a party performs well in multiple MMCs, it may gain an additional seat beyond its national allocation, potentially preventing the 63rd largest quotient (or more) from securing a seat.

Let us use this example to unpack the mathematical definitions explicitly: For each election outcome V, the seat array $\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}}(V)$ corresponds to the distribution of regional

[6]The map is constructed with data from a variety of sources, including (but not limited to) national election laws such as [30, 6, 39, 40], already compiled tables of electoral systems such as [31, 21, 41]. The biggest challenges in constructing such a map is the ever-evolving electoral systems used around the world and the need to compare and verify those mentioned in the literature to ensure they were not outdated. This map will also become outdated soon.

[7]We only consider the national distribution and not how they are distributed to the MMCs. We also made a slight simplification for pedagogical purposes.

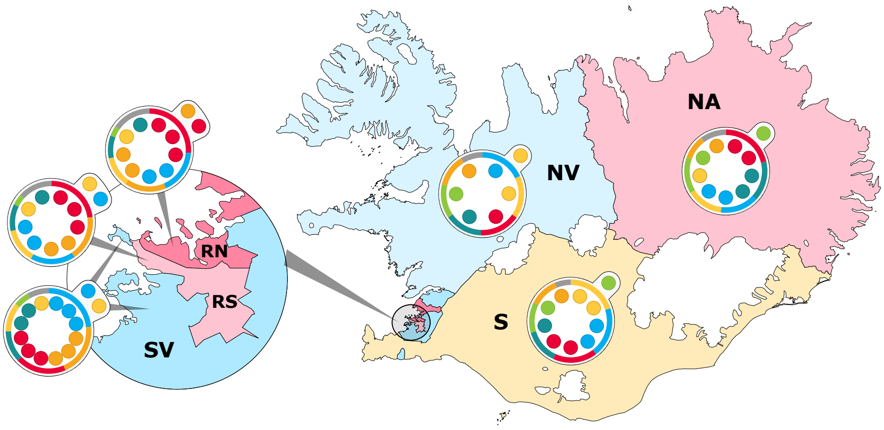

Figure 2: Results of the 2024 Icelandic Parliamentary Election. The six multi-member constituencies are shown, where the colored inner circles represent each party's regional seats, and the outer circles indicate the compensatory seats [5].

seats. In the next step, the election is summed up completely using a total amalgamation A, that is $A(V)$ is the list of the total national vote for all parties and $A(\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}}(V))$ is the total number of regional seats. The final distribution $\mathtt{ES}^{\text{Iceland}}$ depends both on the distribution of regional seats $A(\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}}(V))$ and on the national votes. Thus, the final distribution has the form of Equation (3):

$$\mathtt{ES}^{\text{Iceland}}(V) = \widetilde{\mathtt{DH}}(A(\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}}(V)), A(V)),$$

where $\widetilde{\mathtt{DH}}$ is the modified D'Hondt process that distributes the seats already distributed in the seat matrix first.

Notice how the Icelandic two-tier system is distinct from a parallel system due to the interaction between the votes distributed at the regional and national levels. This way, two-tier systems are usually better at evening out disproportionalities from the regional level than parallel systems and as a result they perform better at the Gallagher index [16].

Many of the Northern European systems fit approximately into the framework of Equation (3). For example, elections for the German parliament (*Bundestag*) are built on top of a one-tier system with first-past-the-post, the Danish unicameral parliament (*Folketinget*) is built on top of a one-tier system with the D'Hondt method. The resulting seat distributions are then used as input in a further calculation as elaborated on in Section 3.3.

We will refer to seats distributed in the first round of calculation as *regional seats*. In many two-tier systems the regional seats are final (meaning that the parties obtaining the seats in the first calculation keep them no matter what) in which case we say that the system satisfies *regionality* (cf. Definition 15). Such regional seats are sometimes perceived differently than the compensatory seats [26].

The electoral system for elections to the Swedish parliament (*Riksdagen*) is an example that does not satisfy regionality, where a regionally distributed mandate can be 'taken back' to ensure national proportionality (cf. the discussion in Section 3.3).

Any two-tier system striving for proportionality and satisfying regionality must have some mathematical mechanism for the case when one or more parties obtains more regional seats than can be leveled out using additional tiers. As we saw in the Icelandic case, the national seats are ordered by the size of their D'Hondt divisor. Systems allocating national seats based on Hare Quota do not have this luxury. Therefore, the electoral system of many countries including Denmark, Norway, South Africa, and Bosnia-Herzegovina build on iteration where new Hare Quota are calculated omitting overrepresented parties.

A disadvantage of iterative solutions is their increased complexity, with specialized rules, which can become so overwhelming that it is difficult to maintain a clear overview. Since electoral systems are typically defined using natural language in legal texts, it is possible that there exist election outcomes for which the algorithm is ambiguous or not well-defined. We have demonstrated this issue in the Danish electoral system [15], and we suspect that similar ambiguities could exist in other electoral systems as well.

2.6. Parallel two-tier and multi-tier systems

Let us now turn our attention back to the general case in Definition 11 when $\tilde{c} > 1$. One should think of the system as a parallel two-tier system, similarly to the one-tier parallel electoral system of the Russian Duma in Example 9iii). In other words, if we have two two-tier systems that do not interact at all then we still consider the system two-tier. A relevant example of this is the electoral system of Bosnia-Herzegovina .

Example 13 (Bosnia-Herzegovina 's parliaments lower house *Dom Bosne*)**.** The electoral system of the country of Bosnia-Herzegovina consists of two standard two-tier systems of the form Equation (3) corresponding to the two autonomous entities, *the Federation of Bosnia and Herzegovina* and *Republika Srpska*, not interacting at all. That is $\tilde{c} = 2$.

Finally, we conclude with two examples of multi-tier systems that do not qualify as two-tier under Definition 11. The first example is the electoral system for the Austrian parliaments lower house *Der Nationalrat.* Austria has nine federal states which each form multi-member constituencies, which are further divided into sub-constituencies, all assigned seats from the outset. First, for each federal state the election number (i.e., the number of votes per seat, ge. *Wahlzahl*) is calculated. Then, in each sub-constituency, seats are distributed using the largest remainders method based on the election number, rounding down. Secondly, on the state level another round of the largest remainders method, also rounded down. Finally, on the national level additional seats are distributed using the D'Hondt method. Because interactions occur across all three levels, this system does not meet the criteria of Definition 11.

The second example is the electoral system for the Hungarian unicameral parliament *Országgyűlés*[8], which is similar to the now-abolished Italian *Scorporo system.* It is based on first-past-the-post voting in the 106 individual constituencies, where votes for unelected candidates and excess votes for elected candidates are pooled together for redistribution.

[8]Ignoring preferential minority seats.

The remaining seats are then distributed with the D'Hondt method on the redistribution pool. However, under Definition 11, only amalgamations (i.e., sum the columns) are allowed, so this system does not qualify as two-tier under that definition. Nevertheless, it is still considered an electoral system under Definition 7.

2.7. Two-round systems and additional considerations

Our definition of electoral systems in Definition 7 is flexible enough to accommodate two-round systems (TRS) as the input seat distribution S could be the output of a previous election V_1 (that again could depend on an earlier seat distribution S_1) as

$$\mathtt{ES}_{\text{2-Round}}(S, V_2) = \mathtt{ES}_{\text{2-Round}}(\mathtt{ES}(S_1, V_1), V_2).$$

Thus, the two-round-system takes the seat distribution $S = \mathtt{ES}(S_1, V_1)$ as input. The framework can fit the French parliamentary elections where S_1 is empty and V_1 is the first election outcome. The first electoral system $\mathtt{ES}(S_1, V_1)$ determines the seats of the directly elected candidates. The remaining seats are then distributed with the electoral system $\mathtt{ES}_{\text{2-Round}}$ depending on the election outcome V_2. Note, the outcome V_1 also determine which candidates can run in the second round, and thus which outcomes V_2 are possible.

Beyond two-round systems, this framework can also model staggered elections, such as those used for the US *Senate*. In this case, the already elected members serve as the input $\mathtt{ES}(S_1, V_1)$, which itself depends on the previously elected members S_1.

Additionally, the definition extends to countries such as Italy, Burundi, and the Democratic Republic of the Congo, which currently have five, two, and one lifetime-appointed senators, respectively.

3. Impossibility theorem

In the following, properties of electoral systems are defined. While the definitions are primarily motivated by two- and multi-tier systems currently in use, the discussion is more broadly applicable. The two-tier electoral systems defined above all try to balance regionality and proportionality. These properties are desirable as they ensure representativity and therefore increases the legitimacy of the democratic system. A third desirable property is a fixed parliament size, which contributes to simplicity, prevents the inflation of legislative power, helps control the costs associated with variable-sized parliaments. Additionally, maintaining a fixed and odd number of seats can help preventing legislative deadlocks by ensuring a decisive majority in votes, further strengthening the stability and functionality of the system. Defining these properties enables us to state the impossibility theorem in Section 3.2 and then some examples are discussed in Section 3.3.

3.1. Properties of electoral systems

The simplest property is fixed parliament size. To write it neatly, for a seat array S let $\mathbf{N}(S) = \sum_{i=1}^{p} \sum_{j=1}^{c} S[i, j]$ be the total number of seats of S.

Definition 14 (Fixed parliament size)**.** An electoral system $\texttt{ES}$ has *fixed parliament size* if the total number of distributed seats $\mathbf{N}(\texttt{ES}(S, V))$ is constant for all $S \in \mathcal{S}_c$ and $V \in \mathcal{V}_c$.

In practice, most electoral systems have fixed parliament size. A notable exception is the parliament of New Zealand *Pāremata Aotearoa* and the one-tier system of the Seychelles Example 9iv). In some cases, seats may be left vacant when the list of candidates is insufficient to accommodate the allocated seats.[9] A curious example occurred in the 2021 Kyrgyz parliamentary election, where two single-member constituency seats remained unfilled because the *against all* option received the most votes. While such cases exist, they fall outside the scope of our discussion.

For two p-tuples $m = (m_1, \ldots, m_p)$ and $m' = (m'_1, \ldots, m'_p)$ say that $m \leq m'$ if $m_i \leq m'_i$ for all $1 \leq i \leq p$. This definition enables us to succinctly define regionality.

Definition 15 (Regionality)**.** An election system $\texttt{ES}$ with $c \geq 2$ *respects regionality* according to an electoral system $\texttt{ES}_{\text{1-Tier}}$ of the form $(\texttt{M}_i^{k_i})_{1 \leq i \leq c}$ if for all $V \in \mathcal{V}_c$ and $S \in \mathcal{S}_c$ it holds that

$$\texttt{ES}_{\text{1-Tier}}(V) \leq \texttt{ES}(S, V).$$

Note that the definition of regionality only reflects whether parties keep the seats they initially won in the constituencies with $\texttt{ES}_{\text{1-Tier}}$, but not whether the total number of seats won in different regions is fixed. Most two and multi-tier electoral systems respect regionality, cf. Table 1. In fact, $\texttt{ES}$ is usually built from an $\texttt{ES}_{\text{1-Tier}}$ with some additional way of distributing compensatory seats.
Additionally remark that the electoral system $\texttt{ES}_{\text{1-Tier}}$ is not completely general as it assumes that all of the apportionment methods $\texttt{M}^{k_i}$ have a fixed seat count. Two-tier systems that do not satisfy this requirement include the current Polish subelectoral system in the European parliamentary elections [28] and the system employed at Rumanian parliamentary elections from 2008-2012 [18].

Following Balinski-Young [3, Section 2] we say that an apportionment method $\texttt{M}^k$ is *weakly proportional* if it divides proportionally, whenever possible, that is when there are no fractional parts in the largest remainders method. Formally, if $(v_1, \ldots, v_p)$ is a vector of votes and v is their sum. Then $\texttt{M}^k$ is weakly proportional if $\texttt{M}^k(v_1, \ldots, v_p) = (\frac{kv_1}{v}, \ldots \frac{kv_p}{v})$ whenever $\frac{kv_i}{v}$ is an integer for all $1 \leq i \leq p$.
Balinski and Young show that all the traditional apportionment methods Hare-LR ($\texttt{LR}$) (Hamilton), D'Hondt ($\texttt{DH}$) (Jefferson), Lowndes, Sainte-Laguë ($\texttt{SL}$) (Webster), Adams, Dean, and Hill are weakly proportional [2, Proposition 2.1.].

Definition 16 (Guaranteed Proportionality)**.** An electoral system $\texttt{ES}$ is *guaranteed proportional* if there exists a weakly proportional apportionment method $\texttt{M}$ satisfying Definition 5 such that $\texttt{ES}$ agrees with $\texttt{M}$ for all elections V.

[9] An example is electoral system for South Korea's National Assembly where such a case in considered in ([23], article 189 (5)).

Mathematically, for all elections V where v_i is the total number of votes for party i, then there exists a seat number m such that for each $i \in \{1, \ldots, p\}$,

$$\sum_{j=1}^{c} \mathtt{ES}(S, V)[i, j] = \mathtt{M}^m(v_1, \ldots, v_p)[i]. \tag{4}$$

or, equivalently, in terms of a total amalgamation A_{total}

$$A_{\text{total}}(\mathtt{ES}(S, V)) = \mathtt{M}^m(A_{\text{total}}(V)). \tag{5}$$

Notice in the definition it is important that the total number of seats in parliament, m, can depend on the election V. Many well-functioning two-tier systems traditionally labeled as *proportional systems*, are not guaranteed proportional even if all previous elections have been consistent with the proportional method used.

3.2. Impossibility theorem

With the necessary definitions established we turn to the title character of the story.

Theorem 17 (Impossibility theorem)**.** Suppose that $\mathtt{ES}_{\text{1-Tier}}$ is a one-tier electoral system where the corresponding apportionment methods $(\mathtt{M}_i^{k_i})_{1 \le i \le c}$ have $c \geq 2$ and $k_i \geq 1$ for at least two distinct i and satisfy the apportionment method properties of Definition 5. For any electoral system ES there exists an election outcome where at least one of the following desiderata fails:

(i) Regionality: ES respects regionality with respect to $\mathtt{ES}_{\text{1-Tier}}$.
(ii) Fixed parliament size: The system ES has fixed parliament size.
(iii) Proportionality: ES is guaranteed proportional.

Proof. Assume for contradiction that all three properties are satisfied and that the fixed parliament size is m. Then there are $\sum_{i=1}^{c} k_i$ regional seats and $T = m - \sum_{i=1}^{c} k_i$ additional seats. Consider an election outcome with only two parties P_1 and P_2 running. The first party P_1 is regionally strong and gets $a(T + k_1 + 1)$ votes in the first constituency for some positive integer a and no votes elsewhere. The second party P_2 gets a total of $a\,(m - T - k_1 - 1)$ votes across all constituencies. The party P_1 obtains at most k_1 regional seats in the first constituency. By weak proportionality of the apportionment method that ES is weakly proportional with respect to it must be the case that P_1 obtains

$$\frac{a(T + k_1 + 1)}{a(T + k_1 + 1) + a\,(m - T - k_1 - 1)} m = T + k_1 + 1 \tag{6}$$

seats in total. By the same arguments P_2 expects $m - T - k_1 - 1$ seats in total. However, this leads to a contradiction as P_1 can get at most $T + k_1$ seats even if awarded all T leveling seats. $\square$

The theorem was straightforward to prove without any restrictions on the distributions of the regional seats available in each constituency, as in the proof the votes for P_1 could just be increased arbitrarily. This does indeed seem to be a real world issue. Examples of countries where the number of seats per voter varies significantly across constituencies include Norway and Denmark, with Finnmark and Bornholm respectively being overrepresented relative to what their sparse populations would suggest. For current operational two-tier systems, k_j is often determined in advance based on various methods. Examples currently used include the number of votes cast in the constituency in the last election, the population in the constituency at the last census, the surface area of the constituency, and politically negotiated minimum number of representatives. But the impossibilty of the theorem does not rest on this discrepancy in the ratio of votes per seat across constituencies. To attempt to salvage a potential two-tier system, one could try an impose a stricter correspondence between the number of seats and votes in each constituency. The strongest possible restriction is to insist that the number of regional seats are apportioned by weakly proportional method according to the vote-share of each constituency. Even in this case the imposibility arises:

Theorem 18. The impossibility theorem holds even if $(k_1, \dots, k_c)$ is distributed weakly proportionally according the sum of the votes across the constituencies.

Proof. An election outcome V is constructed, which leads to an impossibility whenever the number of parties p satisfies the inequality

$$p > \frac{T}{\tilde{c} - 1} + \max(k_1, \dots, k_{\tilde{c}}) + 1, \tag{7}$$

where T is the number of non-regional seats and $\tilde{c}$ is the number of constituencies with $k_i \geq 1$. (If there are any constituencies without any regional seats weak proportionality can be satisfied without distributing any votes there.)

For the construction, suppose there are $r = \max(k_1, \dots, k_{\tilde{c}}) + 1$ large parties $P_1, \dots, P_r$, where the first k_j parties $P_1, \dots, P_{k_j}$ receive exactly $k_j(a+1)$ votes in the jth constituency for $1 \leq j \leq \tilde{c} - 1$. In the last constituency, the $k_{\tilde{c}}$ parties $P_2, \dots, P_{k_{\tilde{c}}+1}$ each receive $k_j(a+1)$ votes. Furthermore, there are n small parties, each receiving $k_j a$ votes within every constituency. Schematically, the vote distribution can be visualized as follows:

$$V = \left[\begin{array}{cccccc|ccccc}
\multicolumn{6}{c|}{\overbrace{\hspace{12em}}^{r \text{ parties}}} & \multicolumn{5}{c}{\overbrace{\hspace{12em}}^{n \text{ parties}}} \\
k_1(a+1) & * & * & \cdots & * & 0 & k_1 a & k_1 a & \cdots & k_1 a & k_1 a \\
k_2(a+1) & * & * & \cdots & * & 0 & k_2 a & k_2 a & \cdots & k_2 a & k_2 a \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
k_{\tilde{c}-1}(a+1) & * & * & \cdots & * & 0 & k_{\tilde{c}-1} a & k_{\tilde{c}-1} a & \cdots & k_{\tilde{c}-1} a & k_{\tilde{c}-1} a \\
0 & * & * & \cdots & * & * & k_{\tilde{c}} a & k_{\tilde{c}} a & \cdots & k_{\tilde{c}} a & k_{\tilde{c}} a
\end{array}\right]$$

where $*$ is either 0 or $k_j(a+1)$ depending on the number of seats in that particular constituency.

In the jth constituency the total votes are $v_j = k_j(n(a+1) + ar)$ thus satisfying the assumption of weakly proportionally distributed regional seats. With the vote distribution described, assume for contradiction that all three desiderata in the theorem are satisfied.

Since the first large party P_1 is the largest party in each of the $\tilde{c}-1$ first constituencies by winner-takes-one for each of the methods $\mathtt{M}_i$ it will win at least $\tilde{c}-1$ regional seats with a total of $(a+1)\sum_{1\leq j\leq \tilde{c}-1} k_j$ votes. Since $\mathtt{ES}$ satisfies regionality with respect to $\mathtt{ES}_{\text{1-Tier}}$ defined by $(\mathtt{M}_i)_{1\leq i\leq c}$, the party P_1 obtains at least $\tilde{c}-1$ seats in $\mathtt{ES}(V)$.

The small parties do not win any regional seats, but each of them obtain in total $a\sum_{1\leq j\leq \tilde{c}} k_j$ votes. Letting the free parameter a satisfy $a \geq \sum_{1\leq j\leq \tilde{c}-1} \frac{k_j}{k_{\tilde{c}}}$ this is more votes than P_1 have so by monotonicity of the apportionment method that $\mathtt{ES}$ is guaranteed proportional with respect to, they can also expect to win at least $\tilde{c}-1$ seats in total.

Now, since we assume fixed parliament size m, we can define the non-regional seats as $T = m - \sum_{i=1}^{\tilde{c}} k_i$. That each of the n small parties get at least $\tilde{c}-1$ of these seats implies that $T \geq (\tilde{c}-1)n$. For a sufficiently large n this contradicts the fixed parliament size constraint since $T = m - \sum_{i=1}^{\tilde{c}} k_i$ is a fixed value that cannot grow with n. Therefore, it is impossible to simultaneously satisfy all three desiderata. The bound (7) follows by noticing that $p = r + n$. $\square$

At this point, a couple of remarks are in order.

Remark 19 (Many parties complicates proportionality and regionality)**.** The second proof hinges on the possibility of many (small) parties. This highlights that the impossibility can arise in several different ways: skewed districts, many parties, the apportionment methods employed, and combinations thereof. The mechanism of many parties has a suggestive parallel with the recent increased number of parties in European politics, with 4 parties achieving more than 10% in the UK, with the CSU and Die Linke making it to the Bundestag in Germany in 2021 only on direct seats [10]. Similarly, the Danish parliamentary election in 2022 saw a record-breaking number of parties entering parliament [13]. This suggests that this proof, in contrast to the orginal one, elucidates the mathematical mechanism between these developments. While these examples are recent, the effective number of parties has been increasing for many years, see Figure 3 - thereby increasing the relevance of the trade-off exhibited in Theorem 17.

Remark 20 (Mircomega rule)**.** The bound (7) can be viewed as an incarnation of the micromega rule, which describes the effect whereby large parties benefit from smaller constituencies (they obtain overrepresentation without adjustments). For a fixed total number of regional seats, if the constituencies become smaller, the number of constituencies increases and the number of parties needed to construct the counterexample decreases.

Remark 21 (Bounding the number of required leveling seats in terms of the number of parties)**.** A natural next mathematical question is whether the bound (7) on the number of parties p can be tightened by constructing more sophisticated election outcomes. Additionally, one can attempt to construct bounds that link other quantities, such as T and $\sum k_j$. This direction for further work could also involve assumptions about the relation between

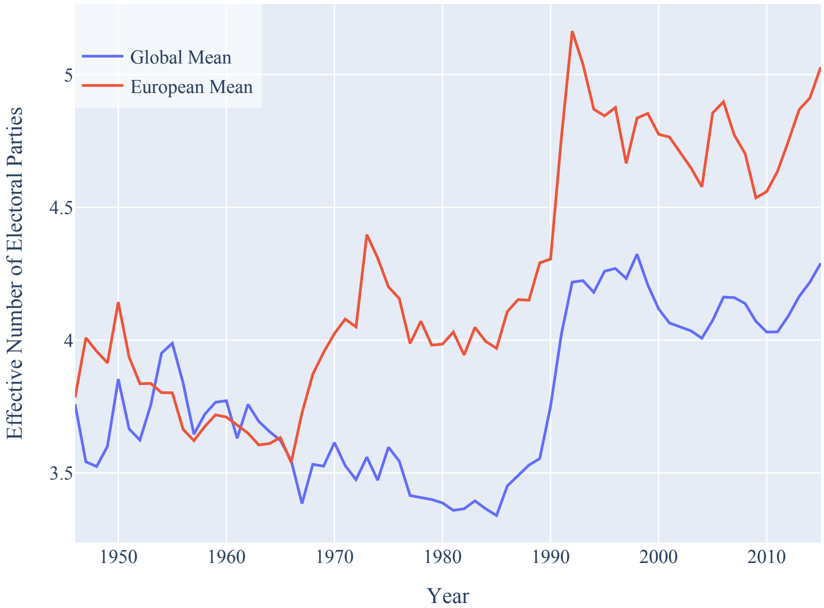

Figure 3: The global and European political effective number of parties over time. The effective number of parties is given $\frac{1}{\sum_{i=1}^{p} v_i^2}$ where v_i is the vote share of party i [25]. The plot is based on data from the World Banks Prosperity Data 360 [42] which has a data cut-off in 2016. In many of the two-tier systems of interest in the context of the impossibility theorem the effective number of parties has increased further since 2016 (see e.g. [7, Figure 5.1]) .

the number of votes and seats in each constituency. In addition, the mathematically attentive reader might notice that only the monotonicity of `ES` was used and the full power of weak proportionality was not needed. Better bounds can be constructed by assuming that `ES` is guaranteed proportional with respect to for example `SL` or `LR`.

3.3. Real world examples

Our main motivation for the impossibility theorem is the view it gives on real world examples and present discussions (cf. Section 1). In Table 1 a classification of multi-tier systems is shown exhibiting how all sides of the trade-off have been explored.

The theorem also applies to one-tier systems, which usually satisfy regionality and fixed parliament size, but not guaranteed proportionality. In a certain sense, the trivial Dutch electoral system described in Example 9ii), satisfies all three properties, but since there is only a single constituency it does not satisfy the requirement that $c \geq 2$ and $k_i \geq 1$ for at least two i.

The 2023 reforms to the German electoral law are summarized in Table 1. Previously, the parliament size that was not fixed. The increasing number of parties winning representation in the Bundestag, led to an increase in the number of additional regional seats that some parties obtained without appropriate nationwide votes to back them up (ge. *Überhang-*

Country	**Regionality**	**Guaranteed proportionality**	**Fixed parliament size**
Germany_{2021}	✓	✓	×
Sweden, Germany_{2025}	×	✓	✓
Austria, Bolivia, Bosnia-Herzegovina, Denmark, Dominican Republic, Estonia, Iceland, Lesotho, Norway, Scotland Slovenia, South Africa, $\text{South Korea}_{2020}$	✓	×	✓
New Zealand	✓	×	×
$\text{Romania}_{2008-2012}$	×	✓	×

Table 1: Broad overview of the properties of some multi-tier electoral systems for national parliaments, excluding the electoral threshold (discussed in Section 4) and other specialized rules, with the table not capturing all system details[10]. Only multi-tier systems with a compensatory element are listed (other non-compensatory multi-tier systems such as Hungary and Mexico are left out). Here the two German systems are guaranteed proportional with respect to Sainte-Laguë/Schepers method, the Romanian to D'Hondt method and the Swedish to a modified Sainte-Laguë method.

mandate, en. *Overhang* seats) and the seats that other parties were additionally awarded to compensate for that (ge. *Ausgleichsmandate*). Consequentially, the German parliament expanded in size from the standard 598 to 736.

In the German system introduced in 2023 (see [6]), the overhang seats (that is seats without *Zweitstimmendeckung* in new German terminology) are not redistributed to the next candidate in the constituency and the algorithm for choosing which seats to give up is complicated and depends on coarser structure of the Bundesländer (see for example the fictious use of the law on the 2021 election [11]). This complication makes the system multi-tier in contrast to the former German system which was two-tier. It is a central feature of the algorithm for retracting regional seats that it prioritizes the seats after the percentage of the votes the candidate obtained. In Section 5.1 we devise another algorithm for choosing which seats to redistribute.

Any electoral system that build on a one-tier system, which is guaranteed proportional and have a fixed parliament size needs a mechanism for redistribution of regional mandates. In the Swedish system the overhang seats are redistributed locally in each MMC one by one starting with the seat corresponding to the least quotient [36, Kap. 14].

With these examples at hand, it is also easy to understand the electoral system of New Zealand. Here the overhang seats are not redistributed and thus regionality is satisfied. Parties still obtain the number of seats corresponding to their vote share with respect to the initial size of the parliament (using the Sainte-Laguë method) - potentially leading to an increasing size of the parliament. In the most recent 2023, this meant that the parliament increased by two seats. However, parties without overhang seats are not compensated and hence the system does not satisfy guaranteed proportionality.

[10] Germany_{2025} does not have fixed parliament size if one party gets more than 50% of the vote in which case extra seats are added to ensure that they obtain at least 50% of the seats. In Slovenia two additional deputies are elected by the Italian and Hungarian minorities.

Finally, the impossibility theorem means that any two-tier system with fixed parliament size and regionality that nevertheless strives for proportionality using some form of compensatory seats must have a way of deciding which compensatory seats cannot be distributed in case of overhang seats. As discussed in Section 2.4 this can be done using rankings for divisor methods (employed by Iceland, Austria, etc). However, since the quota methods do not rank the leveling seats, systems distributing leveling seats using for example Hare Quota often rely on iteration (employed by Denmark, Norway, South Africa etc.).

Looking beyond national elections there are still some two-tier systems in use. Most notably, elections for the regional parliaments in the German federal states exhibit a variety combinations for mitigating the trade-off in the impossibility theorem as shown in Figure 4. In contrast to the new federal German system the only criteria that they all satisfy is regionality.

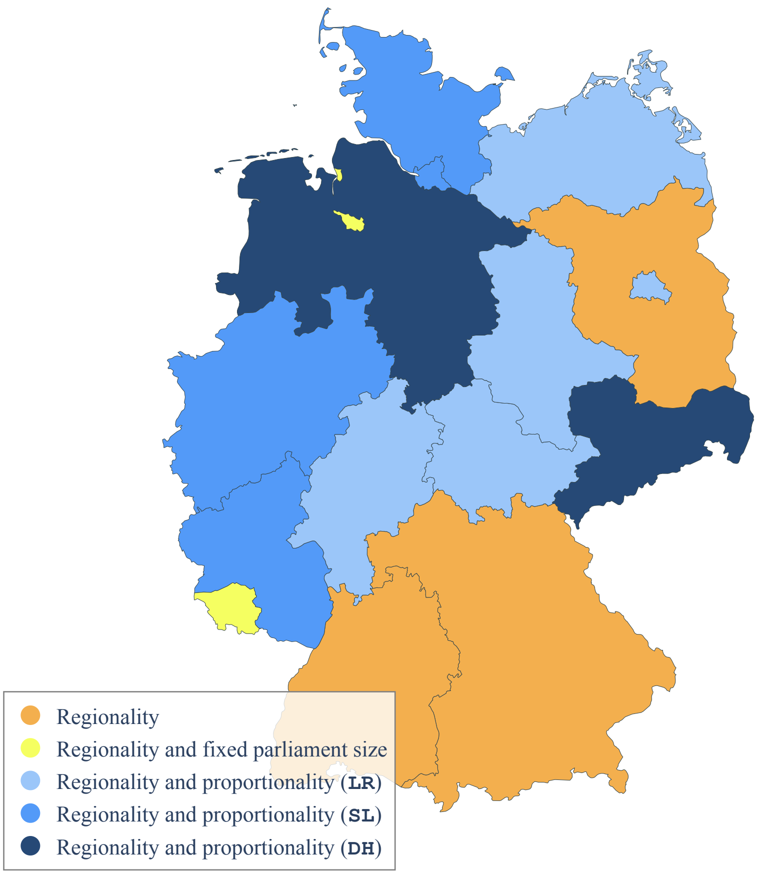

Figure 4: Overview of the electoral systems applied for regional German parliaments (ge. *Landtage*) and the apportionment methods that some are guaranteed proportional with respect to according to [24].

3.4. Further perspectives

Compared to the proofs of the impossibility theorems of Arrow [1] and Balinski-Young [3], the mathematics of the proof above is not particularly ground-breaking. But rather than a mathematical discussion, the theorem should be viewed as a lens that can illuminate the struggles of real-world electoral systems and the recent discussions about such systems.

Furthermore, the theorem qualifies the discussion in the literature about the number of compensatory seats necessary to ensure proportional representation [34]. The theorem gives an answer, namely that if there is no bound to the number of parties, then there is no definite number. Following this line of reasoning some mathematical questions for future studies with practical implications arise.

First, given a two-tier electoral system with a fixed parliament size and a fixed number of compensatory seats (and a mathematical guarantee about how the number of votes in each constituency roughly corresponds to the number of seats that has to be specified) for how many parties does the system guarantee proportional representation?

Or conversely, given a two-tier electoral system with a fixed number of parties (and similar assumptions about vote distributions), how many compensatory seats are necessary to ensure proportional representation?

In practice, there is probably a large difference between the number of compensatory seats that needed to get a mathematical guarantee and what would be enough in most cases (since the case where a single party wins last regional seat in all constituencies can be very unlikely). However, if one constructs sensible models of elections it should be possible to answer a question of the type: For a given model of vote distributions, how many compensatory seats are needed to ensure that more than 99.9% of all elections satisfy some criterion of proportionality?

4. Thresholds

So far, we have ignored electoral threshold, although electoral thresholds are common throughout the two-tier systems currently in function. Thresholds often act as concrete barriers to proportionality. A common example of an electoral threshold is a percentage threshold, where parties below a certain percentage are excluded from (proportional) representation.

In some two-tier systems, there are specialized rules where regional seats act as a *back door* to proportional representation, meaning that a party with a certain number of regional seats is allowed proportional representation even if the party does not meet a certain percentage threshold. Examples of two-tier systems with back doors are Denmark, Germany[11] and New Zealand [31, p.83]. In other two-tier systems, parties failing to meet the threshold may retain their regional seats under specific conditions, but they are not granted access to compensatory seats ensuring national proportionality. An example is Sweden, which has a 4% percentage threshold, and where parties below the threshold can keep regional seats if they win at least 12% of the vote regionally.

[11] The back door was abolished in the law of 2023, but reinstated by the constitutional court in 2024 [6].

There are also examples of *hard thresholds* where parties lose regional seats if they do not make the national threshold. A notable example is Turkey, which previously had a hard electoral threshold of 10%[12].

The three different types of thresholds also highlight three different implementations of the trade-off between proportionality and regionality in the context of thresholds. Each of these threshold types can be incorporated readily in the formalism by considering the threshold as a function on election outcomes that sets columns on parties that do not reach the threshold to zero. For simplicity we restrict the following discussion to the case of hard thresholds.

Considering the case of a hard threshold, one could relax either the regionality (Definition 15) or guaranteed proportionality criteria (Definition 16) to only include parties passing the threshold and ask whether a modified impossibility theorem exists.

If guaranteed proportionality is not relaxed, it is easy to see that the electoral system is not proportional. Similarly, if regionality is not relaxed then a party not meeting the threshold may win a regional seat (by having its votes concentrated) and thus the electoral system does not satisfy regionality.

Based on the discussion above we restrict our attention to the case where both regionality and guaranteed proportionality are relaxed to only concern parties reaching the hard threshold. Since the threshold is hard, it puts an effective limit on the number of parties in parliament. Combining this limit with a system that has very few regional seats and a lot of compensatory seats (e.g. a huge parliament), one can construct electoral systems satisfying the two relaxed criteria as well as a fixed parliament size. For this approach to work in general (with no assumptions on the distribution of votes or regional seats) one can see that if the hard threshold is at $\alpha\%$ then in general at most $\alpha\%$ seats can be regional (since in the worst case a party right above the threshold could win all the regional seats with only these votes). Real world systems using compensatory seats have much fewer compensatory seats than what is needed for this mathematical guarantee.

5. Circumventing the impossibility theorem

The physicist John Bell said in a different context: "...what is proved by impossibility proofs is lack of imagination" [4]. In that spirit, we now construct an electoral system that escapes the constraints of the impossibility theorem. The art of the escape is by cutting the cake slightly differently than the Swedish or new German system.

5.1. Geographically ranked guaranteed proportionality

We consider an electoral system with the number of seats in parliament fixed and we will construct a system that is guaranteed proportional with respect to a weakly proportional

[12] De-facto thresholds are also often mentioned in the literature. These arise by the fact that one must be able to obtain at least one of the total number of available seats to gain representation (for example $\frac{1}{150} = 0.67\%$ in the Dutch example from Example 9ii there are 150 seats in total). Here only formal thresholds that are superimposed above the de-facto threshold are considered.

apportionment method that satisfies Definition 5. The system will break the regionality constraint, which is in a sense obsolete in this case, since there is no preliminary allocation of seats. However, it still attains a strong form of geographic representation. The algorithm can be described as follows:

Distribute the national seats using the weakly proportional apportionment method. This determines, right from the beginning, how many seats each party gets and the method aims at distributing the seats to the constituencies in the best possible way. A specific example of such a system was discussed in [14]. Here is the overall idea of what we suggested, for details see [14].

- Construct rankings for each party in each constituency (e.g. the Sainte-Laguë quotients).
- Construct an overall ranking of the parliamentary seats of coming from each constituency. That is if there are seats $k_1, \ldots k_c$ in each constituency. Then formally consider the set of seats as the set of pairs (i, j) for $1 \leq i \leq c$ and $1 \leq j \leq k_i$. Now, prioritize the pairs ordered in some way, i.e. put on a list. The simplest way to do this is to list the quotients by size.
- Distribute the number of seats that each party is entitled to, to the quotients on the list starting from the top.

In case there are no overhang seats this system is equivalent to a regional system, where the seats are first distributed regionally followed by a leveling procedure.

5.2. Double proportionality and dual-member proportionality

The impossibility theorem also provides a strong mathematical lens to gauge proposals for electoral systems.

An interesting proposal is Pukelsheim's double proportionality system [27, Chap. 14-15] which suggests a general way around the impossibility theorem that was implemented in several cantonal electoral systems in Switzerland. For Pukelsheim, the national seat count as well as the number of seats per constituency is given and he devises an algorithm to distribute the seats on the constituencies. The algorithm does not satisfy that the largest party in each constituency gets a seat. However, with a winner-takes-one modification the winner in each constituency always gets a seat in that constituency. As we can see from the proof of the impossibility theorem if the seats in the beginning were distributed using a monotone apportionment method then it is impossible to ensure the winner-takes-one modification. However, in many practical contexts, such as the case of the Danish island of Bornholm [14, 15], the winner-takes-one modification is essential for the legitimacy of the system.

Another proposed system is the dual-member proportional representation suggested by Sean Graham for Canada [19]. The system also has a winner-takes-one rule, is guaranteed proportional with respect to the Hare-LR method, and fixed parliament size. Thus, by the impossibility theorem, there must exists distributions of votes where the algorithm is not defined (and indeed this happens if one party wins more "first seats" than their Hare Quota).

This discussion illustrates that although the mathematical content of the impossibility theorem may be basic; the framework that was built to state it can provide valuable insight.

6. Conclusion

We have discussed how many of the technical issues that current two-tier electoral systems face can be put on a common footing as trade-offs between regionality and proportionality. We have given mathematical definitions that attempt to capture important aspects of electoral systems and used them to demonstrate an impossibility theorem. Subsequently, we have shown that the impossibility still holds for systems that ensure guaranteed proportionality of electoral thresholds. Finally, we gave an example of a system, that balances geographical and proportional representation breaking some of the key assumptions of the theorem.

While the impossibility theorem highlights the inherent limitations of two-tier electoral systems, it also provides a valuable framework for policymakers and electoral reform advocates to critically evaluate existing systems. The trade-offs between guaranteed proportionality, regional representation, and fixed parliamentary size reveals the need for innovative approaches in electoral design. One potential policy direction is the exploration of mixed or flexible seat allocation models that adapt dynamically to electoral outcomes, such as a variable number of regional seats and compensatory seats.

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