Wiederholung $\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$

$$g_j = y_0 + h \sum_{j=1}^{s} a_{ij} f(x_0 + c_j h, g_j)$$

$$y_1 = y_0 + h \sum_{i=1}^{s} b_i f(x_0 + c_i h, g_i)$$

§ 1 Beispiel Prothero & Robinson

$$\boxed{y' = \lambda(y - \varphi(x)) + \varphi'(x), \quad y(x_0) = \varphi(x_0), \quad \operatorname{Re}\lambda < 0} \quad (PR)$$

Lös: $y(x) = \varphi(x)$

$$g_i = y_0 + h \sum_{j=1}^{s} a_{ij} \left(\lambda \underbrace{(g_j - \varphi(x_0 + c_j h)}_{=0}) + \varphi'(x_0 + c_j h) \right)$$

$$y_1 = y_0 + h \sum_{i=1}^{s} b_i \left(\lambda \underbrace{(g_i - \varphi(x_0 + c_i h))}_{=0} + \varphi'(x_0 + c_i h) \right)$$

$g_j \rightsquigarrow \varphi(x_0 + c_j h)$

$y_0 \rightsquigarrow \varphi(x_0)$

$y_1 \rightsquigarrow \varphi(x_0 + h)$

$$\varphi(x_0 + c_i h) = \varphi(x_0) + h \sum_{j=1}^{s} a_{ij} \varphi'(x_0 + c_j h) + \Delta_{i,h}(x_0) \quad \overset{\text{Taylor}}{\rightsquigarrow} \quad \mathcal{O}(h^{p+1})$$

$$\varphi(x_0 + h) = \varphi(x_0) + h \sum_{i=1}^{s} b_i \varphi'(x_0 + c_i h) + \Delta_{0,h}(x_0) \qquad \mathcal{O}(h^{q+1})$$

$\min(p,q)$ Stage order

Notation: $\varphi_i := \varphi(x_0 + c_i h)$ $\quad \varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_s \end{pmatrix}$ $\quad g = \begin{pmatrix} g_1 \\ \vdots \\ g_s \end{pmatrix}$ $\quad \Delta_h(x) = \begin{pmatrix} \Delta_{1,h}(x) \\ \vdots \\ \Delta_{s,h}(x) \end{pmatrix}$, $\quad z = \lambda \cdot h$

$\varphi_0 := \varphi(x_0)$, $y_0 = g(x_0)$

$$g - \varphi = (g_0 - \varphi_0)\mathbb{1} + z \cdot A(g - \varphi) - \Delta_h(x_0) \rightsquigarrow g - \varphi = (I - zA)^{-1}\mathbb{1}(g_0 - \varphi_0) - (I - zA)^{-1}\Delta_h(x_0)$$

$$\Rightarrow y_1 - \varphi(x_0 + h) = (g_0 - \varphi_0) + z b^T (g - \varphi) - \Delta_{0,h}(x_0) = \overbrace{R(z)}^{1 + z b^T (I - zA)^{-1}\mathbb{1}} \underbrace{(g_0 - \varphi_0)}_{=0} \underbrace{\boxed{- b^T (I - zA)^{-1}\Delta_h(x_0) - \Delta_{0,h}(x_0)}}_{:= \delta_h(x_0)}$$ Lokaler Fehler

$x_0 \rightsquigarrow x_n$: $\quad y_{n+1} - \varphi(x_{n+1}) = R(z)(y_n - \varphi(x_n)) + \delta_h(x_n)$

Globaler Fehler:

$$y_{n+1} - \varphi(x_{n+1}) = R(z)^{n+1}(y_0 - \varphi_0) + \sum_{j=0}^{n} R(z)^{n-j}\delta_h(x_j)$$

Radau IIA $\quad a_{si} = b_i \quad \forall i \in \{1, \dots, s\}$

$$e_s^T (g - \varphi) = e_s^T (I - zA)^{-1} \underbrace{\Delta_h(x)}_{\mathcal{O}(h^{s+1})} \rightsquigarrow \mathcal{O}(z^{-1} h^{s+1})$$

$h \to 0 \qquad h \gg \frac{1}{|\lambda|}$ & $\lambda h = z \to \infty$

$R(z) \to 0$ for $z \to \infty$

Verfahren	klassische Ordnung	lokaler Fehler	globaler Fehler	stifly accurate	B-Konv.
Gauss	$2s$	h^{s+1}	h^s		s
Radau IA	$2s-1$	h^s	h^s		$s-1$
Radau IIA	$2s-1$	$z^{-1}h^{s+1}$	$z^{-1}h^{s+1}$	✓	s
Lobatto IIIA	$2s-2$	$z^{-1}h^{s+1}$	$z^{-1}h^{s}$	✓	/
Lobatto IIIB	$2s-2$	zh^{s-1}	zh^{s-2}		/
Lobatto IIIC	$2s-2$	$z^{-1}h^{s+1}$	$z^{-1}h^{s}$	✓	$s-1$ $(\nu<0)$

§ 2. Lokaler Fehler

2.1 Definition Für $y' = f(x,y)$: $\langle f(x,y) - f(x,z), y-z \rangle \leq \nu \|y-z\|^2$ (ELS)

2.2 Theorem (Dekker)

$$g_i \longrightarrow \hat{g}_i = g_i + \delta_i \quad \frown \quad y_1 \longrightarrow \hat{y}_1$$

$$\Rightarrow \|g - \hat{g}\| \leq C \|\delta\| \quad \frown \quad \|y_1 - \hat{y}_1\| \leq C' \|\delta\|$$

für A inv., (ELS) und $h\nu < \alpha$ ($\rightsquigarrow$ Verfahren, berechenbar)

2.3 Prop. (ELS), A inv., stage order ρ

Falls $\alpha \geq 0$:

$$\|\delta_h(x)\| \leq C\, h^{\rho+1} \max_{\xi \in [x, x+h]} \|y^{(\rho+1)}(\xi)\| \quad \text{für } h\nu < \alpha$$

nur vom Verf. abh.

Beweis (Idee):

Betrachte $y(x+c_i h) =: \hat{g}_i = g_i + \delta_i \curvearrowright y_1 \longrightarrow \hat{y}_1$

$$\|\delta_h(x)\| = \|y_1 - y(x+h)\| \leq \underbrace{\|y_1 - \hat{y}_1\|}_{2.2} + \underbrace{\|\hat{y}_1 - y(x+h)\|}_{\text{Taylor}} \qquad \boxtimes$$

§ 3. Fehlerfortpflanzung

3.1 Definition (Butcher 1975): B-Stabil

$$\langle f(x,y) - f(x,z), y-z \rangle \leq 0$$

$$\Rightarrow \|y_1 - \hat{y}_1\| \leq \|y_0 - \hat{y}_0\| \quad \text{f.a. } h \geq 0$$

$y_1, \hat{y}_1$ num. Lös. zu $y' = f(x,y)$ mit AW $y_0, \hat{y}_0$

3.2. Definition Algebraisch Stabil:

i) $b_i \geq 0 \quad \forall i \in \{1, \dots, s\}$,

ii) $M = (m_{ij}) = (b_i a_{ij} + b_j a_{ji} - b_i b_j)_{i,j=1}^{s}$ positiv semi definit.

3.3 Theorem: Algebraisch Stabil $\Longleftrightarrow$ B-Stabil

Beweis: Notation: $\Delta y_0 = y_0 - \hat{y}_0$, $\Delta y_1 = y_1 - \hat{y}_1$, $\Delta g_i = g_i - \hat{g}_i$, $\Delta f_i = h\left(f(x_0 + c_i h, g_i) - f(x_0 + c_i h, \hat{g}_i)\right)$

$$\Delta y_1 = \Delta y_0 + \sum_{j=1}^{s} b_i \Delta f_i \Rightarrow \|\Delta y_1\|^2 = \|\Delta y_0\|^2 + 2\sum_{i=1}^{s} b_i \langle \Delta f_i, \underline{\Delta y_0} \rangle + \sum_{i,j=1}^{s} b_i b_j \langle \Delta f_i, \Delta f_j \rangle$$

$$\Delta g_i = \underline{\Delta y_0} + \sum_{j=1}^{s} a_{ij} \Delta f_j$$

$$\Rightarrow \|\Delta y_1\|^2 = \|\Delta y_0\|^2 + 2\sum_{i=1}^{s} \underbrace{b_i}_{\geq 0} \underbrace{\langle \Delta f_i, \Delta g_i \rangle}_{\leq 0} - \underbrace{\sum_{i,j=1}^{s} m_{ij} \langle \Delta f_i, \Delta f_j \rangle}_{\geq 0 \rightarrow \text{Aufgabe}} \Rightarrow \|\Delta y_1\| \leq \|\Delta y_0\| \quad (1)$$

(*)

3.4 Theorem

(ELS), Alg. stabil, A inv. $\Rightarrow \exists\, C$:

$$\|y_1 - \hat{y}_1\| \leq (1 + C\nu h)\, \|y_0 - \hat{y}_0\| \quad \text{für} \quad 0 \leq h\nu < \alpha$$

Beweis:

(*)
$$2\sum_{i=1}^{s} b_i \langle \Delta f_i, \Delta g_i \rangle \leq 2h\nu \sum_{i=1}^{s} b_i \underbrace{\|\Delta g_i\|}_{\overset{\text{e.e.}}{\leq} C\|y_0 - \hat{y}_0\|} \leq C h\nu \|y_0 - \hat{y}_0\| \quad \text{in } (1)$$

$\boxtimes$

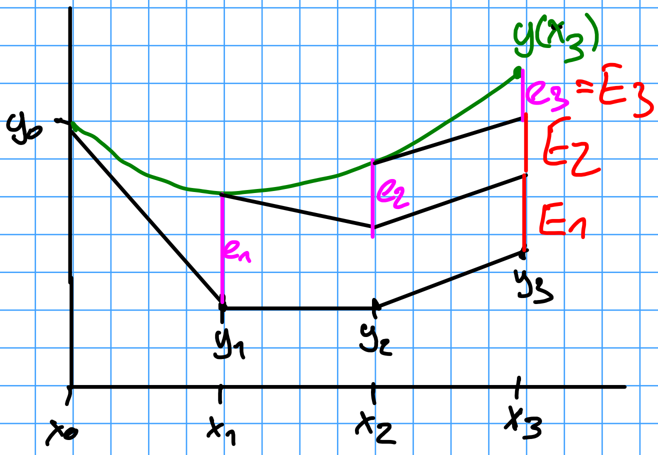
§ 4. B-Konvergenz

4.1 Theorem Alg. Stab., A inv., stage order p, (ELS) $\quad (h = \max_i h_i)$

$$\|y_n - y(x_n)\| \le h^p \gamma(x_n - x_0, \nu) \max_{x \in [x_0, x_n]} \|y^{(p+1)}(x)\| \quad \text{für } h\nu < \alpha$$

$$\|y_n - y(x_n)\| \le h^q \underbrace{e^{\Lambda(x_n - x_0)}}_{\to \infty \text{ für große Steifheit}} \max_{x \in [x_0, x_n]} \|y^{(q+1)}(x)\|$$

Beweis (Idee):



$$\|y_n - y(x_n)\| \le \sum_{i=1}^{n} \|E_i\|$$

$$\|E_i\| \underset{3.4}{\le} (1 + C\nu(x_n - x_j)) \|\delta_{j-1}(x_{j-1})\|$$

$$\le e^{C\nu(x_n - x_j)} \underbrace{\|\delta_{j-1}(x_{j-1})\|}_{\le C h^{p+1} \max \ldots}$$

B-Konvergenz der Ord. r: $\|y_n - y(x_n)\| \le h^r \gamma(x_n - x_0, \nu) \max_\ell \max_x \|y^{(\ell)}(x)\|$

((ELS)) erfüllt

für $h\nu < \alpha$