Claim: Let M be a symmetric $s \times s$-matrix and $\langle \cdot, \cdot \rangle$ the scalar product of $\mathbb{R}^n$.
Then M is non-negative definite, if and only if

$$\sum_{i=1}^{s} \sum_{j=1}^{s} m_{ij} \langle v_i, v_j \rangle \geq 0 \quad \text{for all } v_i \in \mathbb{R}^n.$$

Proof:

$$V := (v_1 \ v_2 \cdots v_{s-1} \ v_s), \quad V^T V = (\langle v_i, v_j \rangle)_{ij},$$

$$MV^T V = \begin{pmatrix} \sum\limits_{j=1}^{s} m_{1j} \langle v_j, v_1 \rangle & \cdots & \sum\limits_{j=1}^{s} m_{1j} \langle v_j, v_s \rangle \\ \vdots & \ddots & \vdots \\ \sum\limits_{j=1}^{s} m_{sj} \langle v_j, v_1 \rangle & \cdots & \sum\limits_{j=1}^{s} m_{sj} \langle v_j, v_s \rangle \end{pmatrix}.$$

$$M = Q^T D Q \text{ with diagonal matrix } D \text{ and orthogonal matrix } Q, \quad VQ^T =: \tilde{V}$$

$$\Longrightarrow \sum_{i=1}^{s} \sum_{j=1}^{s} m_{ij} \langle v_i, v_j \rangle = \operatorname{tr}(MV^T V) = \operatorname{tr}(Q^T D \tilde{V}^T \tilde{V} Q) = \operatorname{tr}(D \tilde{V}^T \tilde{V}) = \sum_{j=1}^{s} \lambda_j \langle \tilde{v}_j, \tilde{v}_j \rangle$$

with λ_j being the eigenvalues of M and $\tilde{v}_j$ being the columns of $\tilde{V}$.

"$\Rightarrow$": Let M be non-negative definite. That means $\lambda_j \geq 0$ for all $j \in \{1, ...s\}$. Now the result follows from the definiteness of the inner product on the right-hand side.

"$\Leftarrow$": Let the inequality hold. So we can choose $v_i, i \in \{1, ...s\}$ such that $\langle \tilde{v}_j, \tilde{v}_j \rangle = 0$ for all $j \in \{1, ..., k-1, k+1, ..., s\}$, $\langle \tilde{v}_k, \tilde{v}_k \rangle = 1$. $\Longrightarrow \lambda_k \geq 0$.

$$\tilde{V} = (0 \ ... \ e_k \ ... \ 0) = VQ^T \Longrightarrow 0 \leq \operatorname{tr}(D \tilde{V}^T \tilde{V}) = \lambda_k$$

We can construct this for every $k \in \{1, ..., s\}$, so we know all eigenvalues are non-negative. So M is non-negative definite. $\square$