Non-matching Multi-patch Discretizations in Finite Element Exterior Calculus

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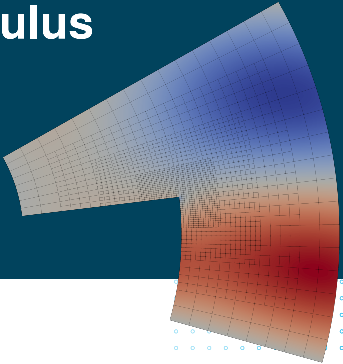
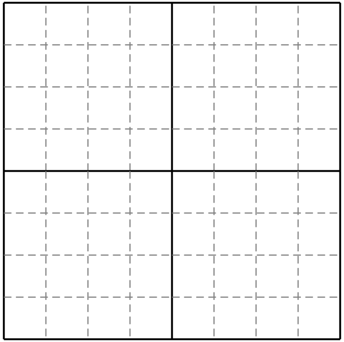


Outline

1. Motivation
2. Conforming Projections
3. L^2-stability analysis
4. Numerical Experiments

Motivation

- Finite Element Exterior Calculus (FEEC)
  - not flexible (strong requirements)
  - can be costly
- Special CONGA scheme
  - local refinements (e.g. singular geometries)
  - handle hanging nodes
- Keep the patch-wise approach
  - local tensor-product structure
  - mappings from reference cells



Joint work with Martin Campos Pinto, Yaman Güçlü and Said Hadjout.

References I

- L^2-stable Projectors for FEEC
  - D. N. Arnold, R. S. Falk, and R. Winther, “Finite element exterior calculus, homological techniques, and applications,” Acta Numerica, vol. 15, May 2006.
  - D. N. Arnold, R. Falk, and R. Winther, “Finite element exterior calculus: from Hodge theory to numerical stability,” Bulletin of the American Mathematical Society, vol. 47, Art. no. 2, Jan. 2010.
  - R. S. Falk and R. Winther, “Double complexes and local cochain projections,” Numerical Methods for Partial Differential Equations, Oct. 2014.
- Broken FEEC
  - M. Campos Pinto and Y. Güçlü, “Broken-FEEC discretizations and Hodge Laplace problems,” preprint, Oct. 2022.
  - M. Campos Pinto,Y. Güçlü and S. Hadjout, A broken Feec framework for electromagnetic problems on mapped multipatch domains,” preprint, Oct. 2022.

References II

- Construction of B-Spline FEEC on tensor-product meshes

  - A. Buffa, G. Sangalli, and R. Vázquez, “Isogeometric analysis in electromagnetics: B-splines approximation,” Computer Methods in Applied Mechanics and Engineering, 2010.
  - A. Buffa, J. Rivas, G. Sangalli, and R. Vázquez, “Isogeometric Discrete Differential Forms in Three Dimensions,” SIAM Journal on Numerical Analysis, 2011.
  - A. Buffa, J. Dölz, S. Kurz, S. Schöps, R. Vázquez, and F. Wolf, “Multipatch approximation of the de Rham sequence and its traces in isogeometric analysis,” Numerische Mathematik, 2019.

- Hierarchical B-Splines

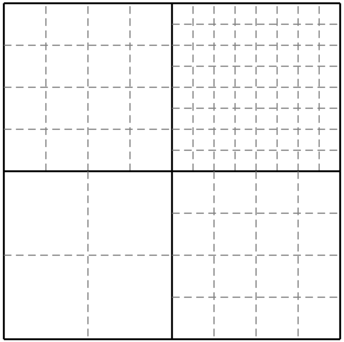
  - J. A. Evans, M. A. Scott, K. M. Shepherd, D. C. Thomas, and R. V. Hernández, “Hierarchical B-spline complexes of discrete differential forms,” IMA Journal of Numerical Analysis, 2018.

Idea

- Keep the patch wise approach
  with local FEEC structure on each patch

- Allow for different meshes on each patch,
  that allow for a shared interface space

- Additionally to geometrical averaging,
  introduce coarsening and extension operators
  when defining the conforming projections

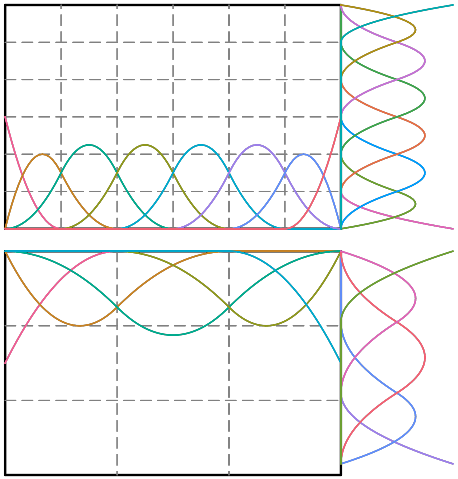


Commuting diagram

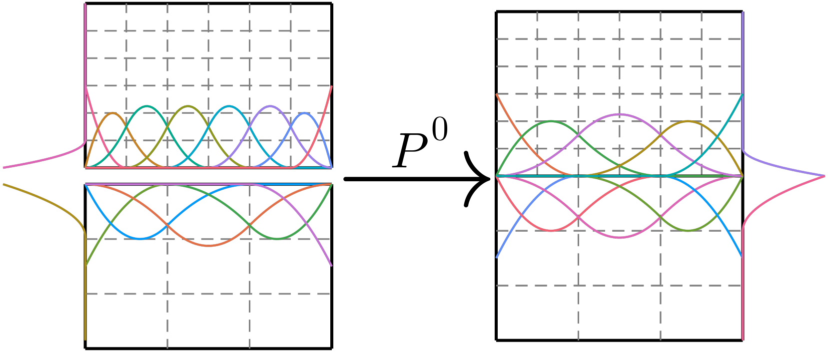
$$
\begin{tikzcd}
V^0 = H^1(\Omega) \arrow[rr, "\mathrm{grad}"] \arrow[d, "\Pi^0_{\mathrm{pw}}"] \arrow[dd, bend right=60, "\Pi^0"'] & & V^1 = H(\mathrm{curl}, \Omega) \arrow[d, "\Pi^1_{\mathrm{pw}}"] \arrow[dd, bend left=60, "\Pi^1"] \\
V^0_{\mathrm{pw}} = \mathbb{S}^p \otimes \mathbb{S}^p \arrow[rr, "\mathbb{G}_{\mathrm{pw}}"] \arrow[d, "P^0"] & & V^1_{\mathrm{pw}} = \begin{pmatrix} \mathbb{S}^{p-1} \otimes \mathbb{S}^p \\ \mathbb{S}^p \otimes \mathbb{S}^{p-1} \end{pmatrix} \arrow[d, dashed, "P^1"] \\
V^0_{\mathrm{conf}} = V^0_{\mathrm{pw}} \cap H^1(\Omega) \arrow[rr, "\mathbb{G}"] & & V^1_{\mathrm{conf}} = V^1_{\mathrm{pw}} \cap H(\mathrm{curl}, \Omega)
\end{tikzcd}
$$

Example setup

- Two patches (2d and no mapping for now)
- Different meshes
  $\Longrightarrow$ How should they relate?
- Local FEEC structure
  $\Longrightarrow$ How to couple it?
- What should be the interface space?



Conforming projections

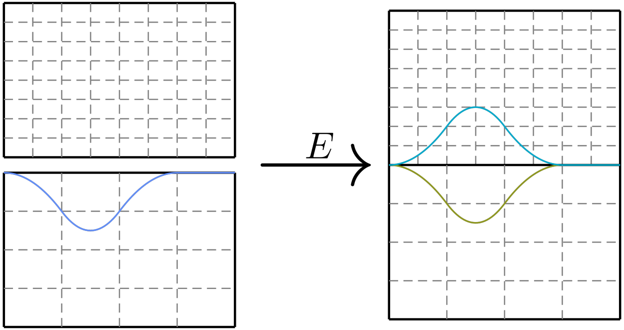


Goal: Define a projection

$$P^0 : V^0_{\mathrm{pw}} \longrightarrow V^0_{\mathrm{conf}},$$

from the patch-wise to the conforming space, that acts as the identity on the *interior* basis functions and maps *interface* basis functions to the *interface function space*.

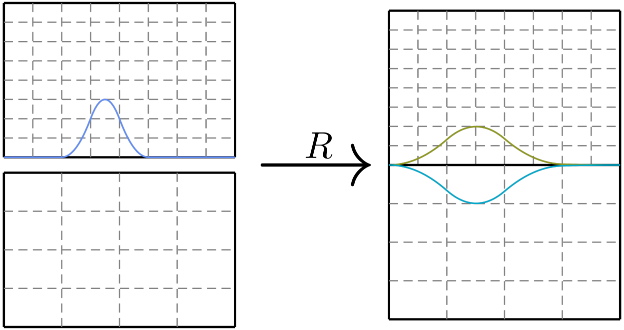
Extension



$$E : V^{c,\Gamma} \to V^{\Gamma} : \lambda_i^{0,c} \mapsto \lambda_i^{0,c} = \sum_{j=0}^{n^f-1} E_{i,j} \lambda_j^{0,f},$$

or in other words a change of basis, where $E_{i,j}$ are the interpolation coefficients, given the spaces are **nested**.

Coarsening



$$R : V^{f,\Gamma} \to V^{\Gamma} : \lambda_i^{0,f} \mapsto \sum_{j=0}^{n^c-1} R_{i,j}\lambda_j^{0,c} = \sum_{j=0}^{n^f-1} (ER)_{i,j}\lambda_j^{0,f},$$

i.e. a local projection of the fine basis functions to the coarse ones, for example with coefficients given by $R_{i,j} = \langle \theta_i^{0,c}, \lambda_j^{0,f} \rangle_{L^2}$. It acts as a **left inverse** for the extension E.

Definition of P^0

$$P^0 \underbrace{\Lambda^{0,s}_{(i,j)}}_{=\lambda^{0,s}_i \otimes \lambda^{0,s}_j} = \begin{cases} \Lambda^{0,c}_{(i,j)}, & s = c, \forall i,j < n^c - 1, \\ \frac{1}{2}\left(\lambda^{0,c}_i \otimes \lambda^{0,c}_{n^c-1} + \sum_k E_{i,k} \lambda^{0,f}_k \otimes \lambda^{0,f}_0\right), & s = c, \forall i, j = n^c - 1, \\ \frac{1}{2}\left(\sum_k R_{i,k} \lambda^{0,c}_k \otimes \lambda^{0,c}_{n^c-1} + \sum_k (ER)_{i,k} \lambda^{0,f}_k \otimes \lambda^{0,f}_0\right), & s = f, \forall i, j = 0, \\ \Lambda^{0,c}_{(i,j)}, & s = f, \forall i, j > 0, \end{cases}$$

or more abstractly written as the block matrix between the coarse and fine interface degrees of freedom:

$$P^0|_\Gamma = \frac{1}{2}\begin{pmatrix} I & R \\ E & ER \end{pmatrix} \otimes I,$$

and $P^0|_{\mathrm{int}(\Omega)} = I$.

Definition of P^1

P^1 is defined the same way, but with extension and coarsening based on the derived basis functions $\lambda^{1,s}$. Clearly, this will **not** result in a commuting diagram.

Since we only aim for continuity across the tangential traces, we only apply the previous construction to interfaces along the tangential vector.

Global projections

$$\Pi^0 v = P^0 \Pi^0_{\mathrm{pw}} v, \quad \text{where } \Pi^0_{\mathrm{pw}} = \Pi^{0,c} + \Pi^{0,f},$$

$$\begin{aligned}\Pi^1 u = &\begin{pmatrix} \partial_x \Pi^0_{\mathrm{pw}} \left(\int_x u_x(\tilde{x}, y) \, \mathrm{d}\tilde{x} \right) \\ \partial_y \Pi^0_{\mathrm{pw}} \left(\int_y u_y(x, \tilde{y}) \, \mathrm{d}\tilde{y} \right) \end{pmatrix} \\ + &\begin{pmatrix} \partial_x (P^0 - I) \Pi^0_{\mathrm{pw}} \left(\int_x u_x(\tilde{x}, y) \, \mathrm{d}\tilde{x} \right) \\ \partial_y (P^0 - I) \Pi^0_{\mathrm{pw}} \left(\fint_{-\Delta x}^{\Delta x} \left(\int_0^x u_x(\tilde{x}, a) \, \mathrm{d}\tilde{x} + \int_a^y u_y(x, \tilde{y}) \, \mathrm{d}\tilde{y} \right) \mathrm{d}a \right) \end{pmatrix}.\end{aligned}$$

$$\begin{tikzcd}
V^0 \arrow[r, "\mathrm{grad}"] \arrow[d, "\Pi^0_{\mathrm{pw}}"] \arrow[dd, bend right, "\Pi^0"'] & V^1 \arrow[d, "\Pi^1_{\mathrm{pw}}"] \arrow[dd, bend left, "\Pi^1"] \\
V^0_{\mathrm{pw}} \arrow[r, "\mathbb{G}_{\mathrm{pw}}"] \arrow[d, "P^0"] & V^1_{\mathrm{pw}} \arrow[d, dashed, "P^1"] \\
V^0_{\mathrm{conf}} \arrow[r, "\mathbb{G}"] & V^1_{\mathrm{conf}}
\end{tikzcd}$$

Properties:

L^2-stability, Commuting property, Projection property.

Implementation

Implement P^0 and P^1:

- Extension matrix $\mathbb{E}$ by knot-insertion of B-Splines to calculate the interpolation coefficients.

- Coarsening matrix as a left inverse:

$$\mathbb{R} = (\mathbb{E}^T \mathbb{E})^{-1} \mathbb{E}^T.$$

Assemble the patch-wise spaces as usual and project with the conforming projections when needed.

Note that the current code can already handle mappings and multiple patches.
The theoretical side for mapped non-matching interfaces is currently in preparation and will soon be published.

Curl curl eigenvalue problem (curved L-shape domain)

Find $\lambda > 0$

$$\textbf{curl}\,\mathrm{curl}\,\boldsymbol{E} = \lambda \boldsymbol{E},$$

where $\boldsymbol{E} \in H_0(\mathrm{curl}, \Omega)$.

Within the CONGA framework, this equation is discretized as

$$(\mathbb{P}^1)^T \mathbb{C}_{\mathrm{pw}}^T \mathbb{M}^2 \mathbb{C}_{\mathrm{pw}} \mathbb{P}^1 \; E_h = \lambda_h \left((I - \mathbb{P}^1)^T \mathbb{M}^1 (I - \mathbb{P}^1) + (\mathbb{P}^1)^T \mathbb{M}^1 \mathbb{P}^1 \right) \; E_h,$$

where $\mathbb{P}^1$ is the matrix form of our conforming projection P^1 and the usual patch-wise curl matrix $\mathbb{C}_{\mathrm{pw}}$ and mass-matrices $\mathbb{M}^{1/2}$. Note that we also added a jump stabilization term.

We compare the results to an Discontinuous Galerkin method implemented in psydac.[1]

[1] A. Buffa and I. Perugia, “Discontinuous Galerkin Approximation of the Maxwell Eigenproblem”, 2006.

Convergence eigenvalue problem

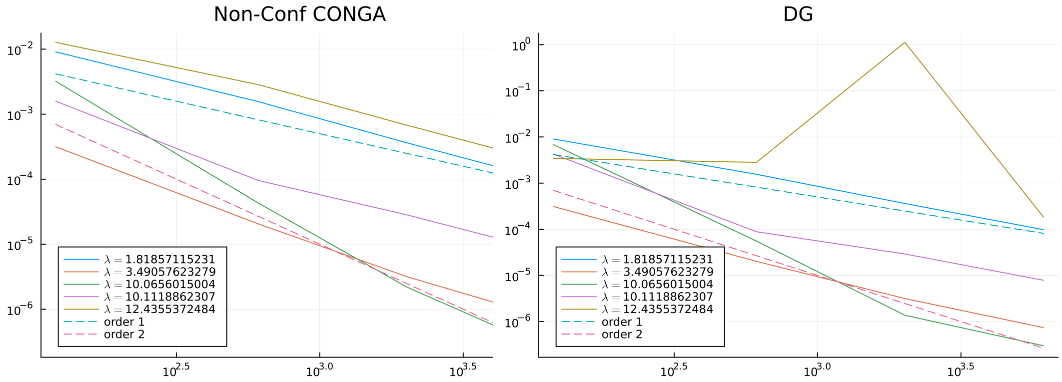
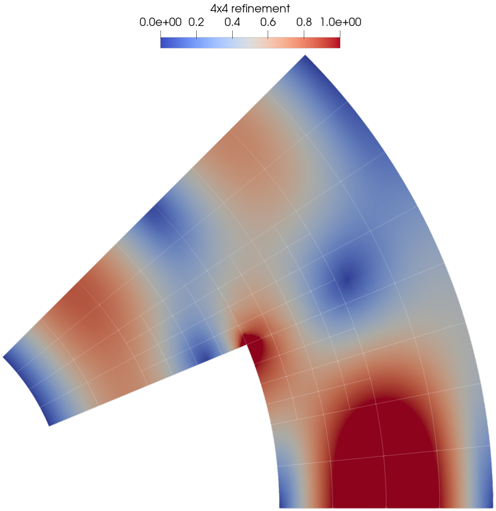
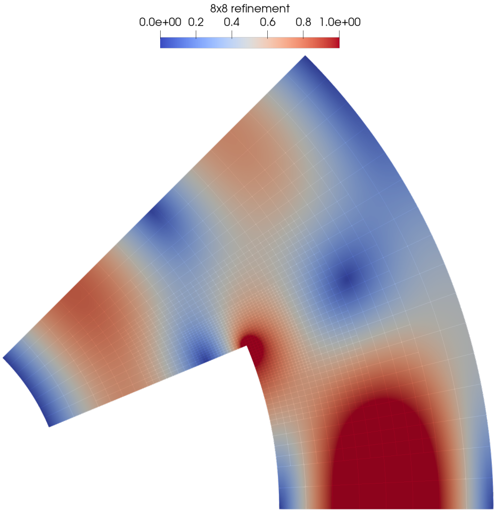


Figure: Error of the discrete eigenvalues of the Conga (left) and DG (right) schemes with $p = 3$ over the DOF, for $\dim(V^1_{\text{pw}}) = 120, 612, 2016, 6132$.

Comparison non-matching Conga eigenfunction for $\lambda = 12.43\ldots$



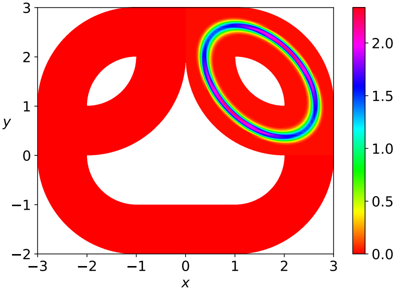


Time-harmonic Maxwell equation (Pretzel domain)

For $\omega \in \mathbb{R}$ and $\boldsymbol{J} \in L^2(\Omega)$, solve

$$-\omega^2 \boldsymbol{E} + \mathbf{curl}\, \mathrm{curl}\, \boldsymbol{E} = \boldsymbol{J},$$

where $\boldsymbol{E} \in H_0(\mathrm{curl}, \Omega)$.

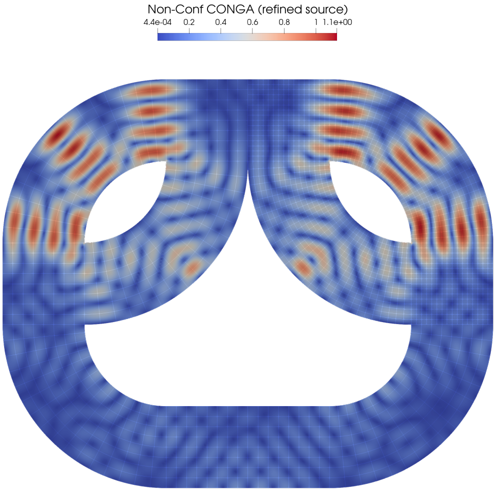


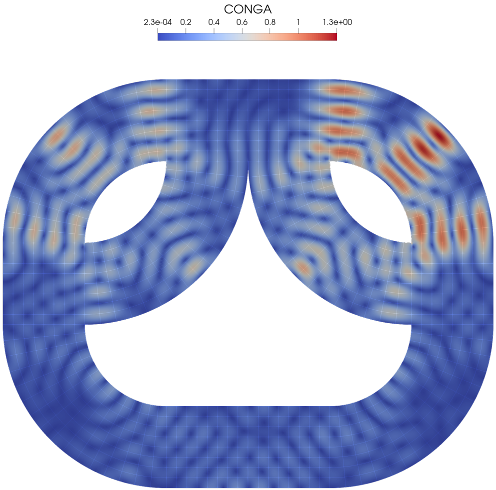
In our framework, that reads

$$\left(-\omega^2 (\mathbb{P}^1)^T \mathbb{M}^1 \mathbb{P}^1 + (\mathbb{P}^1)^T \mathbb{C}_{\mathrm{pw}}^T \mathbb{M}^2 \mathbb{C}_{\mathrm{pw}} \mathbb{P}^1 + \gamma (I - \mathbb{P}^1)^T \mathbb{M}^1 (I - \mathbb{P}^1)\right) E_h = \mathbb{P}^1 J_h,$$

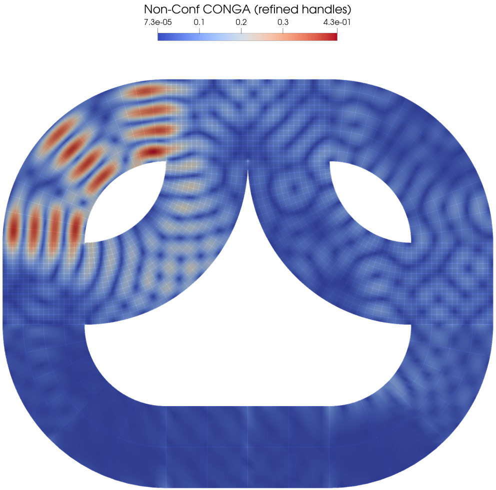
where γ is a constant that weights the added jump penalization and J_h is a projection of $\boldsymbol{J}$ to $V_{\mathrm{pw}}^1(\Omega)$, see the plot above.

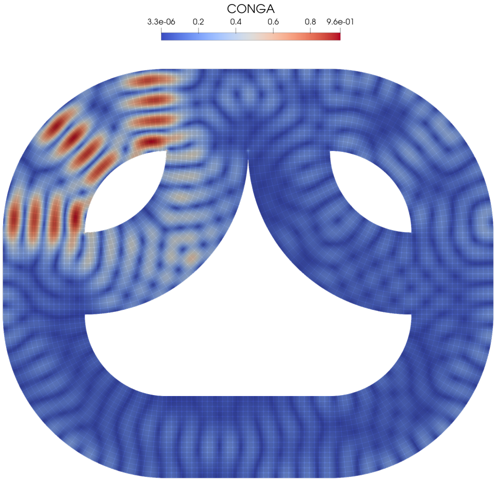
Pretzel refinement I





Pretzel refinement II





Time domain Maxwell equations (Square domain)

For $\boldsymbol{J} \in L^2(\Omega)$, solve

$$\partial_t \boldsymbol{E} - \textbf{curl}\, B = -\boldsymbol{J},$$
$$\partial_t B + \operatorname{curl} \boldsymbol{E} = 0,$$

where $\boldsymbol{E} \in H_0(\mathrm{curl}, \Omega)$ and $B \in L^2(\Omega)$.
Casted into a simple leap-frog time-stepping scheme, this reads

$$B_h^{n+\frac{1}{2}} = B_h^n - \frac{\Delta t}{2} \mathbb{C}_{\mathrm{pw}} E_h^n,$$
$$\mathbb{M}^1 E_h^{n+1} = \mathbb{M}^1 E_h^n + \Delta t \left((\mathbb{P}^1)^T \mathbb{C}_{\mathrm{pw}}^T \mathbb{M}^2 B_h^{n+\frac{1}{2}} - \mathbb{P}^1 J_h^{n+\frac{1}{2}} \right),$$
$$B_h^{n+1} = B_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \mathbb{C}_{\mathrm{pw}} E_h^{n+1}.$$

Electromagnetic wave (without source)

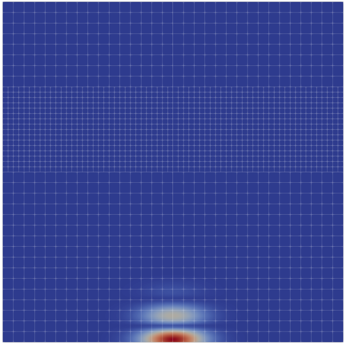
In our example, we choose the initial conditions

$$\boldsymbol{E}(t=0;\boldsymbol{x}) = \boldsymbol{p}\ \cos(\boldsymbol{k}\cdot\boldsymbol{x})\ e^{-\frac{x^2}{2\sigma^2}},$$

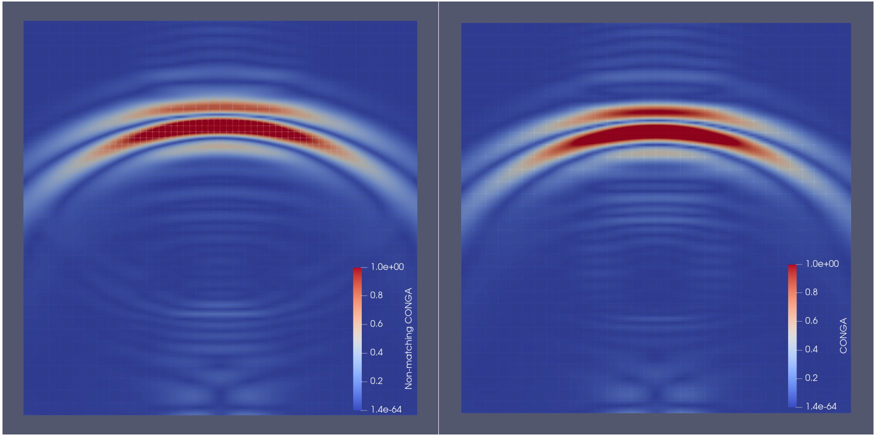
$$\boldsymbol{B}(t=0;\boldsymbol{x}) = \boldsymbol{k}\times\boldsymbol{p}\ \cos(\boldsymbol{k}\cdot\boldsymbol{x})\ e^{-\frac{x^2}{2\sigma^2}},$$

$$\boldsymbol{J}(t,\boldsymbol{x}) \equiv 0.$$

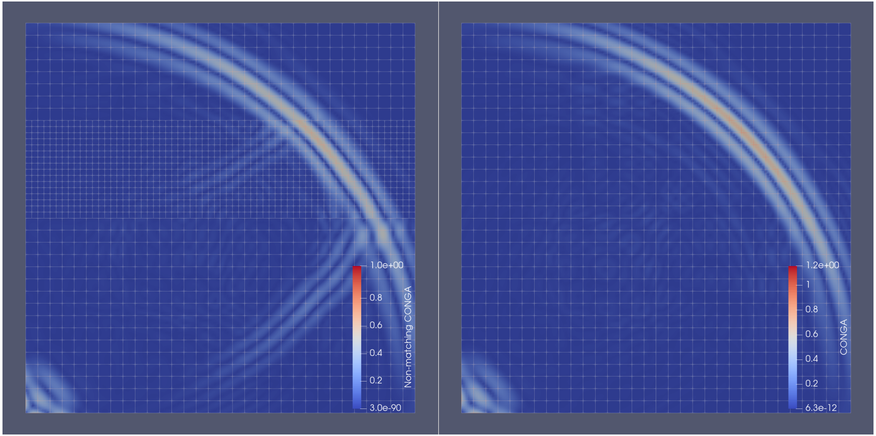
See the initial magnitude of E_h^0 on the right.



Electro-magnetic wave traveling I



Electro-magnetic wave traveling II



Outlook

Current road map

- Finalize the theory on mapped multi-patch domains
- Look into the reflecting waves at patch interfaces
- Properly benchmark against CONGA and DG schemes

Possible expansions

- More general polynomial discretizations instead of only B-Splines
- Projectors independent of dimension

Thank you for your attention!