



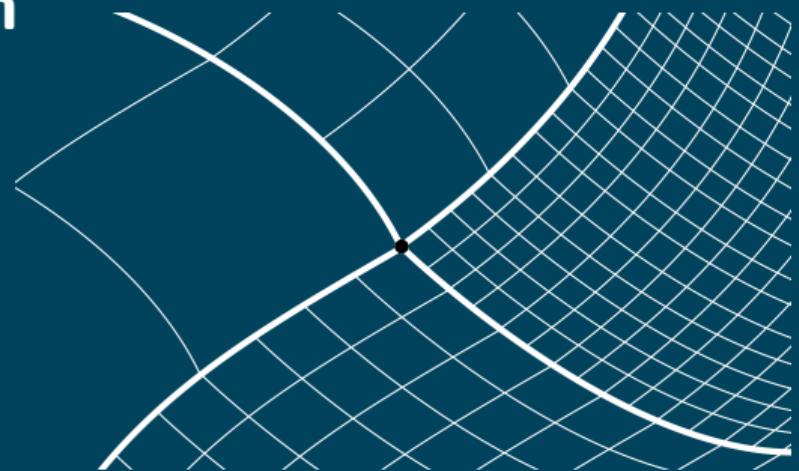
Development and analysis of structure-preserving finite element methods on non-conforming meshes

Doctoral defense

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Outline

1. Motivation
2. Broken-FEEC
3. Construction of numerical schemes
4. Numerical Experiments
5. Conclusion

Motivation



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Solve Hodge-Laplace problems (Poisson, Maxwell, ...) on various two-dimensional domains.



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[Overview](#): From geometry to discretization

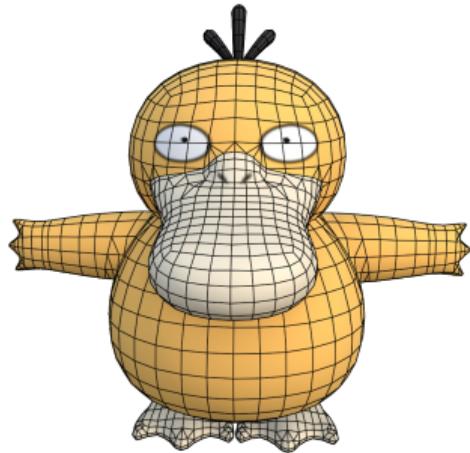




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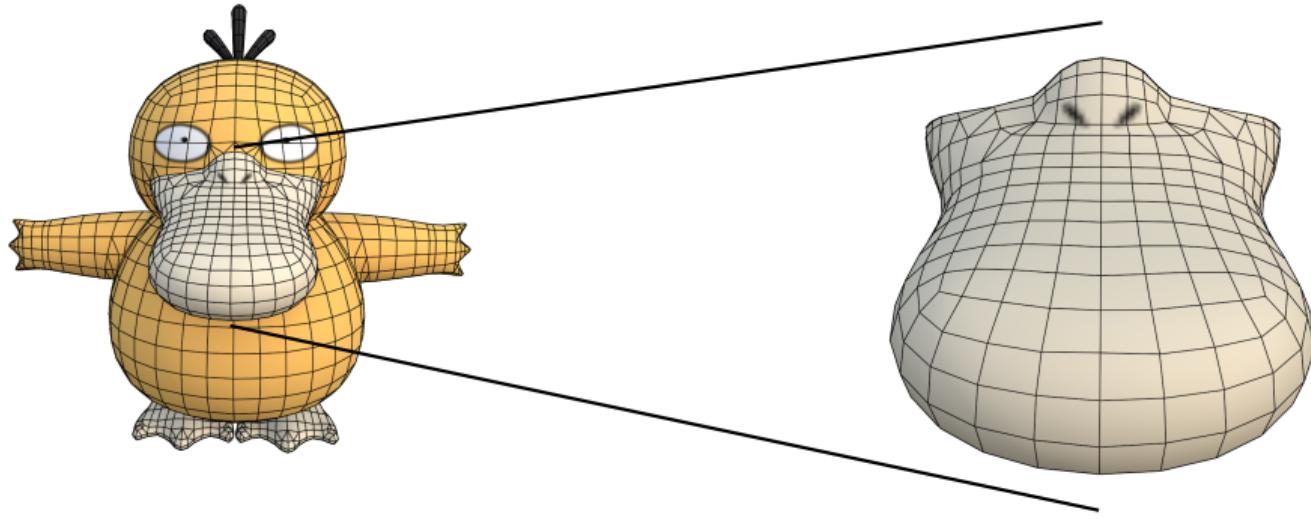




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Solve Hodge-Laplace problems (Poisson, Maxwell, ...) on various two-dimensional domains.

Overview: From geometry to discretization

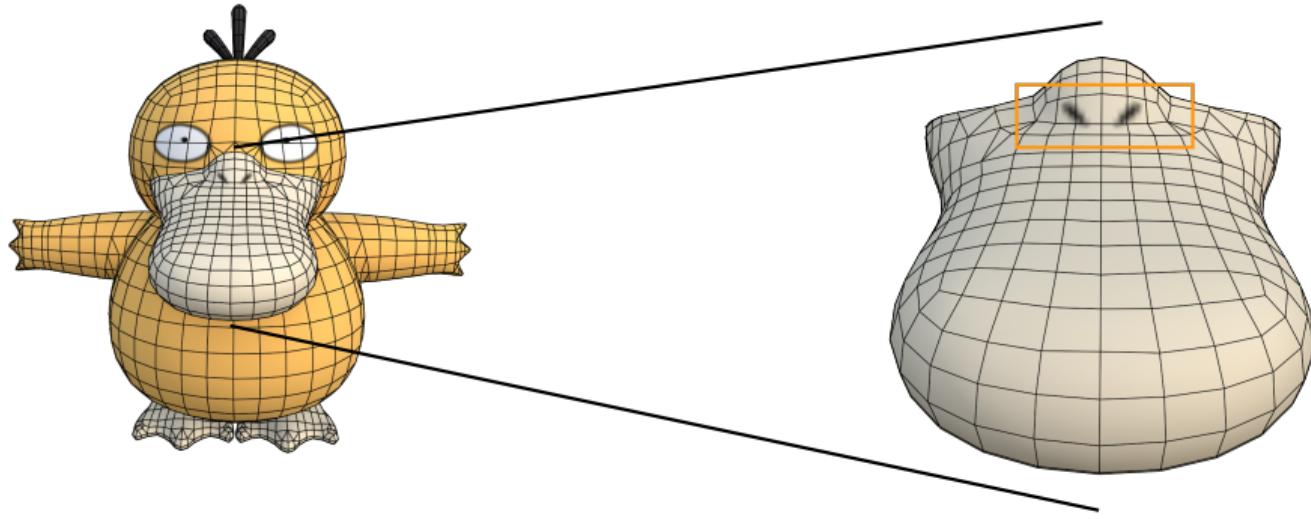




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Overview: From geometry to discretization



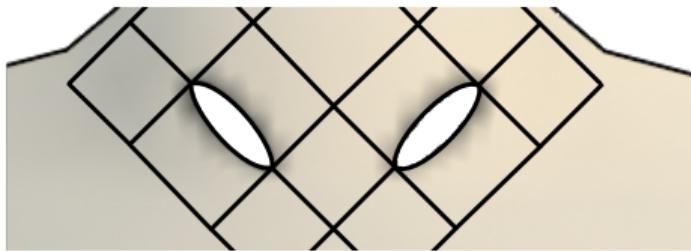
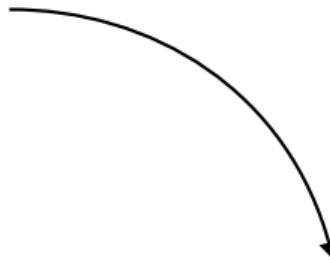
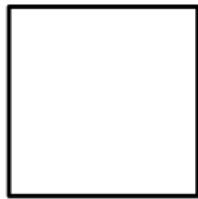


How to build a mesh and do calculations on it?



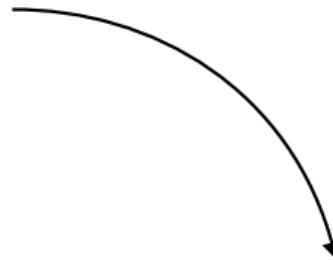
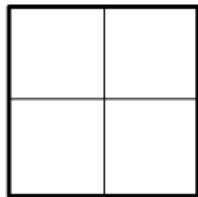


Discretize the domain by several patches



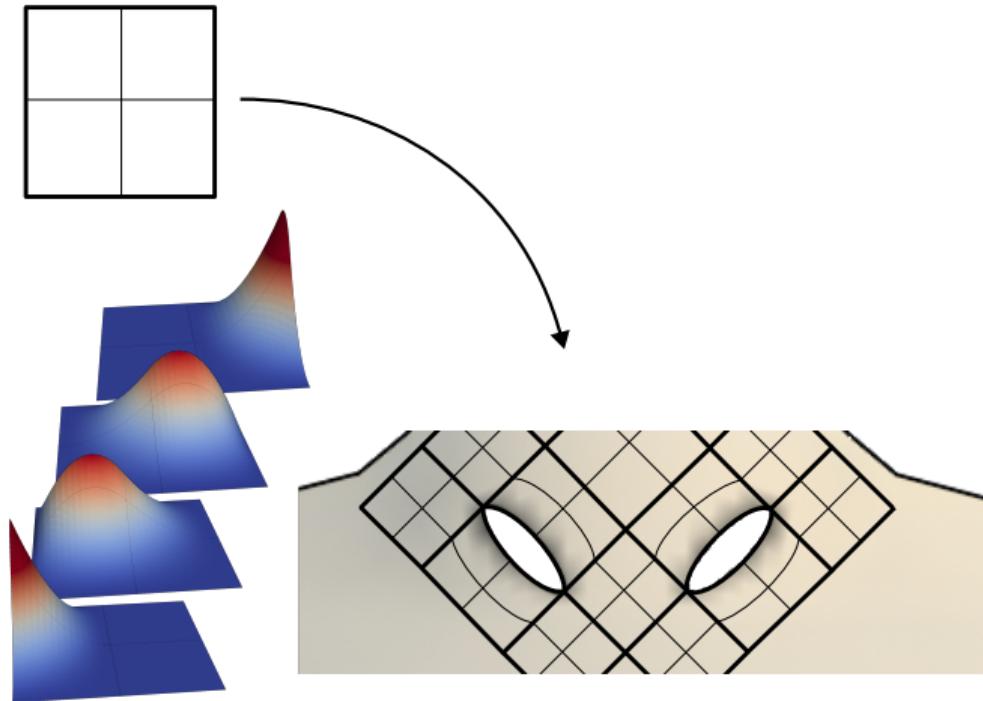


Prescribe discrete function spaces on patches



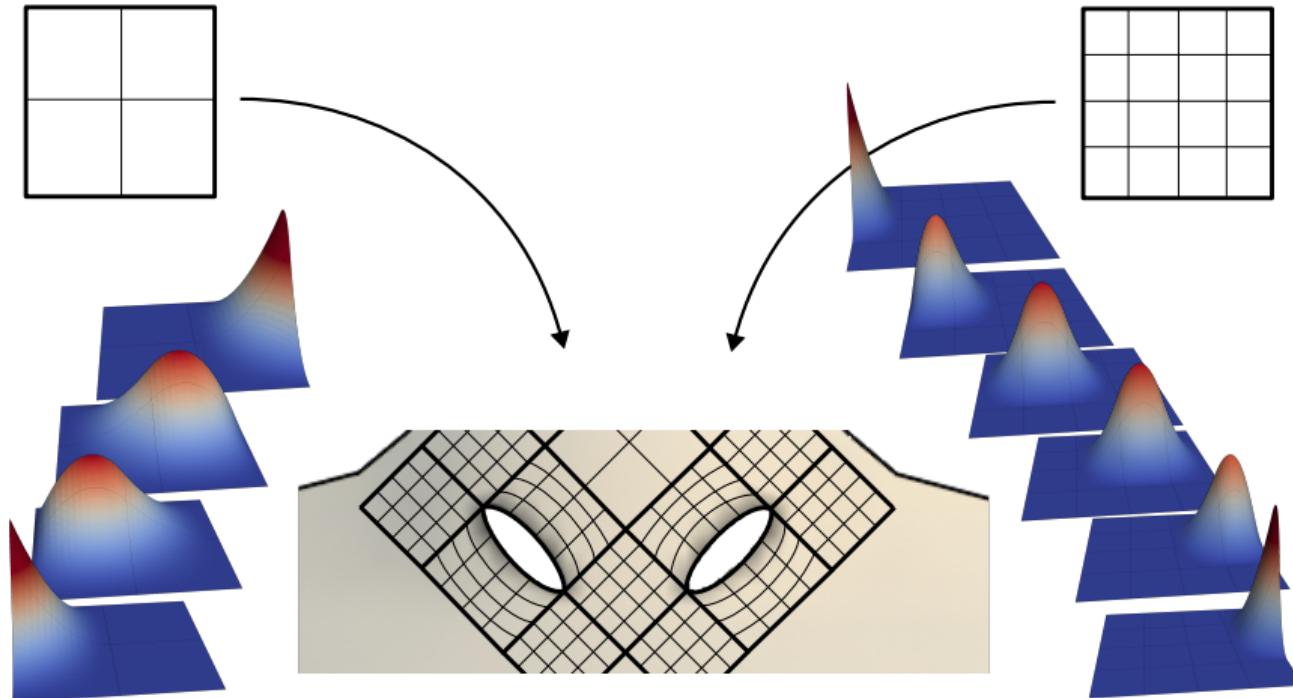


Suitable basis functions enable computation (Finite Elements)



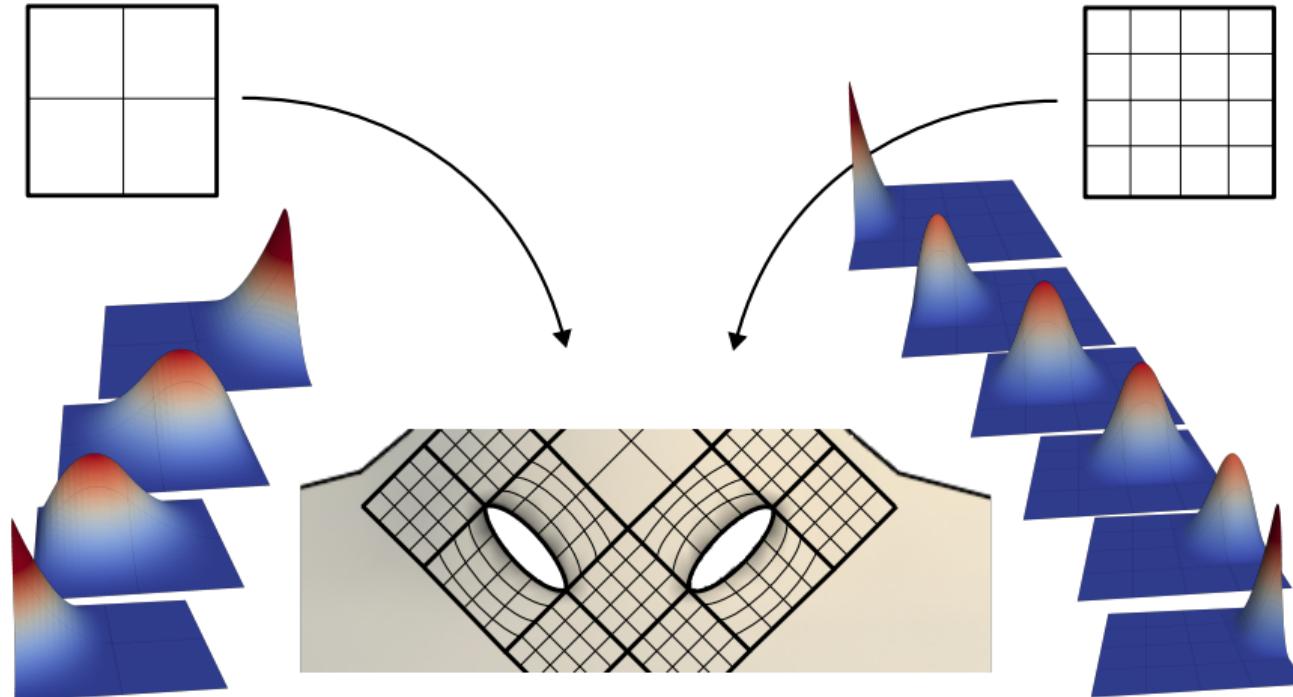


Different resolutions on different patches? (non-conforming meshes)





Different resolutions on different patches? (non-conforming meshes)

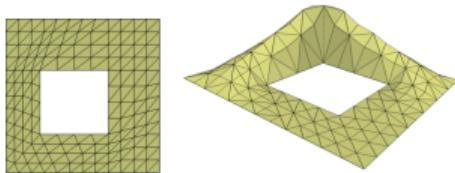


→ Q1: How to handle complex domains? With patches of different resolution?



Pitfalls of continuous Finite Elements¹: holes and corners

- Hodge-Laplace eigenvalue problem: $(\operatorname{curl}_0 \operatorname{curl} - \operatorname{grad} \operatorname{div}_0) \mathbf{u} = \lambda \mathbf{u}$



# Elements	Degree 1		Degree 3	
	λ_1	λ_2	λ_1	λ_2
256	2.270	2.360	1.896	1.970
1,024	2.050	2.132	1.854	1.925
4,096	1.940	2.016	1.828	1.897
16,384	1.879	1.952	1.812	1.880
65,536	1.843	1.914	1.802	1.870
262,144	1.821	1.890	1.796	1.863

Continuous FEM

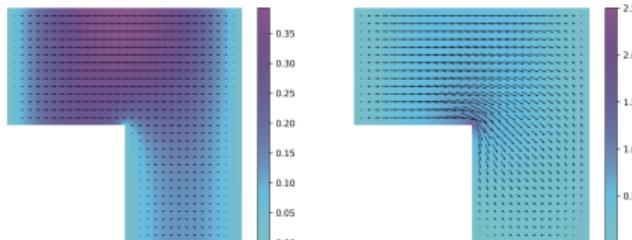
# Elements	Degree 1		Degree 3	
	λ_1	λ_2	λ_1	λ_2
256	0.000	0.638	0.000	0.619
1,024	0.000	0.625	0.000	0.618
4,096	0.000	0.620	0.000	0.617
16,384	0.000	0.618	0.000	0.617
65,536	0.000	0.618	0.000	0.617
262,144	0.000	0.617	0.000	0.617

FEEC

- ▷ with holes: non trivial harmonic fields ($\lambda = 0$), dimension = topological invariant and missed by continuous FEM

- Hodge-Laplace source problem

- ▷ with corners: singular solutions $\mathbf{u} \notin H^1$, no convergence with continuous FEM



Continuous FEM

FEEC

¹examples taken from Arnold (18)



Structure-preserving Finite Elements / FEEC

- sequence of spaces with commuting projection operators¹

$$\begin{array}{ccccccc} H_0^1 & \xrightarrow{\text{grad}} & H_0(\text{curl}) & \xrightarrow{\text{curl}} & H_0(\text{div}) & \xrightarrow{\text{div}} & L^2 \\ \downarrow \Pi^0 & & \downarrow \Pi^1 & & \downarrow \Pi^2 & & \downarrow \Pi^3 \\ V_h^0 & \xrightarrow{\text{grad}} & V_h^1 & \xrightarrow{\text{curl}} & V_h^2 & \xrightarrow{\text{div}} & V_h^3 \end{array}$$

- FEEC: unified analysis → key element are the stable commuting projections² Π^ℓ
- eigenvalue problem:
 - convergence of eigenmodes
 - correct dim of discrete harmonic fields
 - discrete Hodge-Helmholtz decomp.
- source problem:
 - convergence (even for singular sol.)
 - structure preservation (Gauss laws)
 - time domain: long time stability

¹Whitney (57) Nédélec (80) Bossavit (88-98) Hiptmair (99) Arnold-Falk-Winther (06) Buffa et al (11) ...

²Schöberl (05) AFW (06) Christiansen-Winther (07) Falk-Winther (14) Arnold-Guzman (21)



Structure-preserving Finite Elements / FEEC

- sequence of spaces with commuting projection operators¹

$$\begin{array}{ccccccc} H_0^1 & \xrightarrow{\text{grad}} & H_0(\text{curl}) & \xrightarrow{\text{curl}} & H_0(\text{div}) & \xrightarrow{\text{div}} & L^2 \\ \downarrow \Pi^0 & & \downarrow \Pi^1 & & \downarrow \Pi^2 & & \downarrow \Pi^3 \\ V_h^0 & \xrightarrow{\text{grad}} & V_h^1 & \xrightarrow{\text{curl}} & V_h^2 & \xrightarrow{\text{div}} & V_h^3 \end{array}$$

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→ Q2: How to obtain a priori convergence / stability / spectral correctness results?

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The commuting projections of Buffa-Rivas-Sangalli-Vázquez (11)

- construction on (**single patch**) tensor-product spaces (here in 2D)

$$V_h^0 = \mathbb{S}_p \otimes \mathbb{S}_p \quad \xrightarrow{\text{grad}} \quad V_h^1 = \begin{pmatrix} \mathbb{S}_{p-1} \otimes \mathbb{S}_p \\ \mathbb{S}_p \otimes \mathbb{S}_{p-1} \end{pmatrix} \quad \xrightarrow{\text{curl}} \quad V_h^3 = \mathbb{S}_{p-1} \otimes \mathbb{S}_{p-1}$$

- On V_h^0 :

$$\Pi^0 \phi := \sum_{i \in \mathcal{I}^0} c_i^0(\phi) \Lambda_i^0 \quad \text{with} \quad c_i^0(\phi) := \langle \Theta_i^0, \phi \rangle$$

- On V_h^1 :

$$\Pi^1 \mathbf{u} := \sum_{d \in \{1,2\}} \nabla_d \Pi^0 \Phi_d(\mathbf{u}) \quad \text{with} \quad \begin{cases} \Phi_1(\mathbf{u})(\mathbf{x}) := \int_0^{x_1} u_1(z, x_2) dz \\ \Phi_2(\mathbf{u})(\mathbf{x}) := \int_0^{x_2} u_2(x_1, z) dz \end{cases}$$

- On V_h^2 :

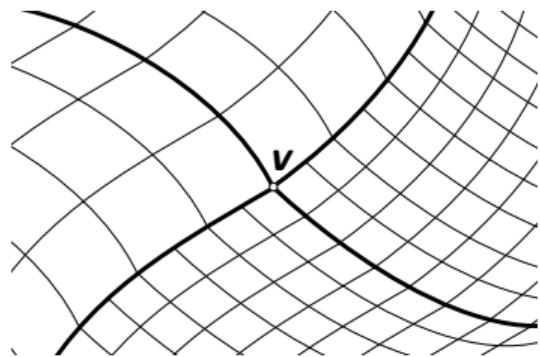
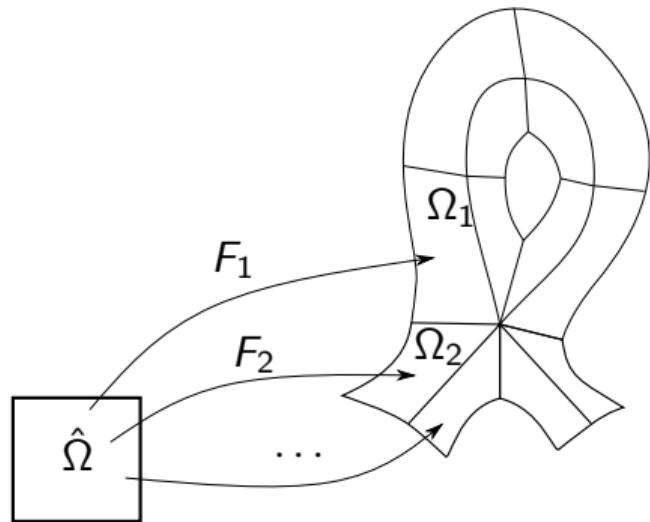
$$\Pi^2 \rho := \partial_1 \partial_2 \Pi^0 \Psi(\rho) \quad \text{with} \quad \Psi(\rho)(\mathbf{x}) := \int_0^{x_1} \int_0^{x_2} \rho(z_1, z_2) dz_2 dz_1$$



Multi-patch domains with non-matching interfaces

Adjacent patches may have different mappings ...

... and different grids

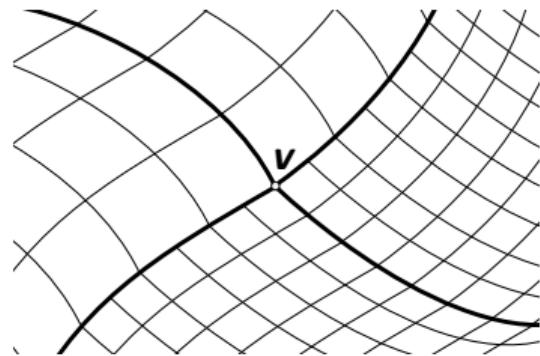
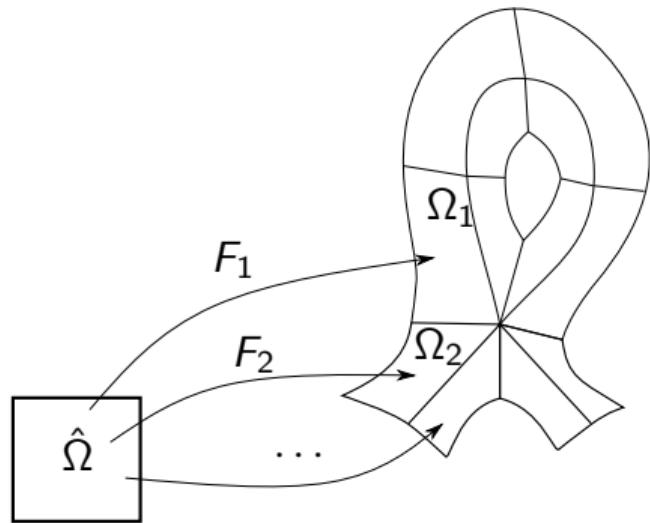




Multi-patch domains with non-matching interfaces

Adjacent patches may have different mappings ...

... and different grids



Challenge: Break of the tensor-product structure near the interfaces
→ How to couple neighboring patches while preserving the FEEC structure?

Broken-FEEC



Broken-FEBC approach

- broken (patch-wise) spaces: $V_{\text{pw}}^\ell := \{v \in L^2(\Omega) : v|_{\Omega_k} \in V_k^\ell \text{ for } k \in \mathcal{K}\}$
- patch-wise differential operators

$$V_{\text{pw}}^0 \xrightarrow{\text{grad}_{\text{pw}}} V_{\text{pw}}^1 \xrightarrow{\text{curl}_{\text{pw}}} V_{\text{pw}}^2$$

- conforming subspaces

$$V_h^0 = V_{\text{pw}}^0 \cap H^1(\Omega) \xrightarrow{\text{grad}} V_h^1 = V_{\text{pw}}^1 \cap H(\text{curl}; \Omega) \xrightarrow{\text{curl}} V_h^2 = V_{\text{pw}}^2 \cap L^2(\Omega) = V_{\text{pw}}^2.$$



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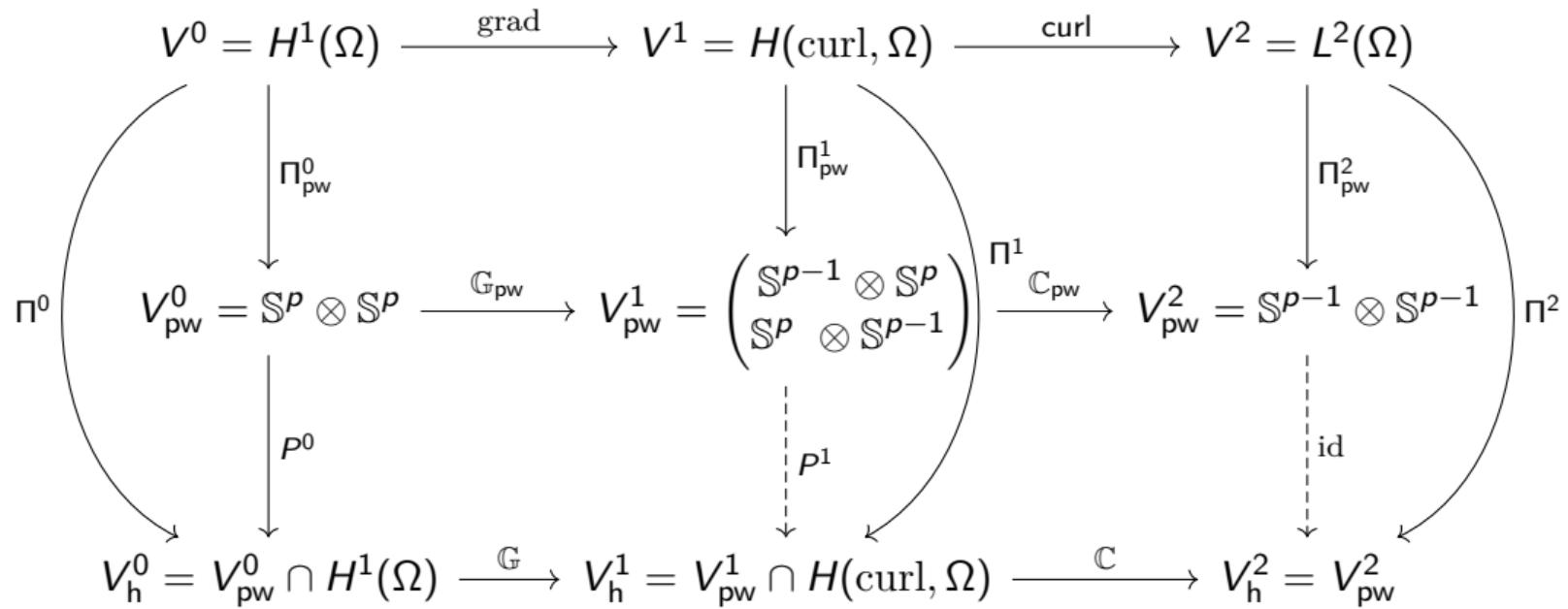
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- for example on V_h^0 , an L^2 -stable projection is given by

$$\Pi^0 := P^0 \Pi_{\text{pw}}^0 \quad \text{with} \quad \begin{cases} \Pi_{\text{pw}}^0 = \sum_k \Pi_k^0 : L^p \rightarrow V_{\text{pw}}^0 & (\text{stable, patch-wise}) \\ P^0 : V_{\text{pw}}^0 \rightarrow V_h^0 & (\text{averages interface dofs}) \end{cases}$$

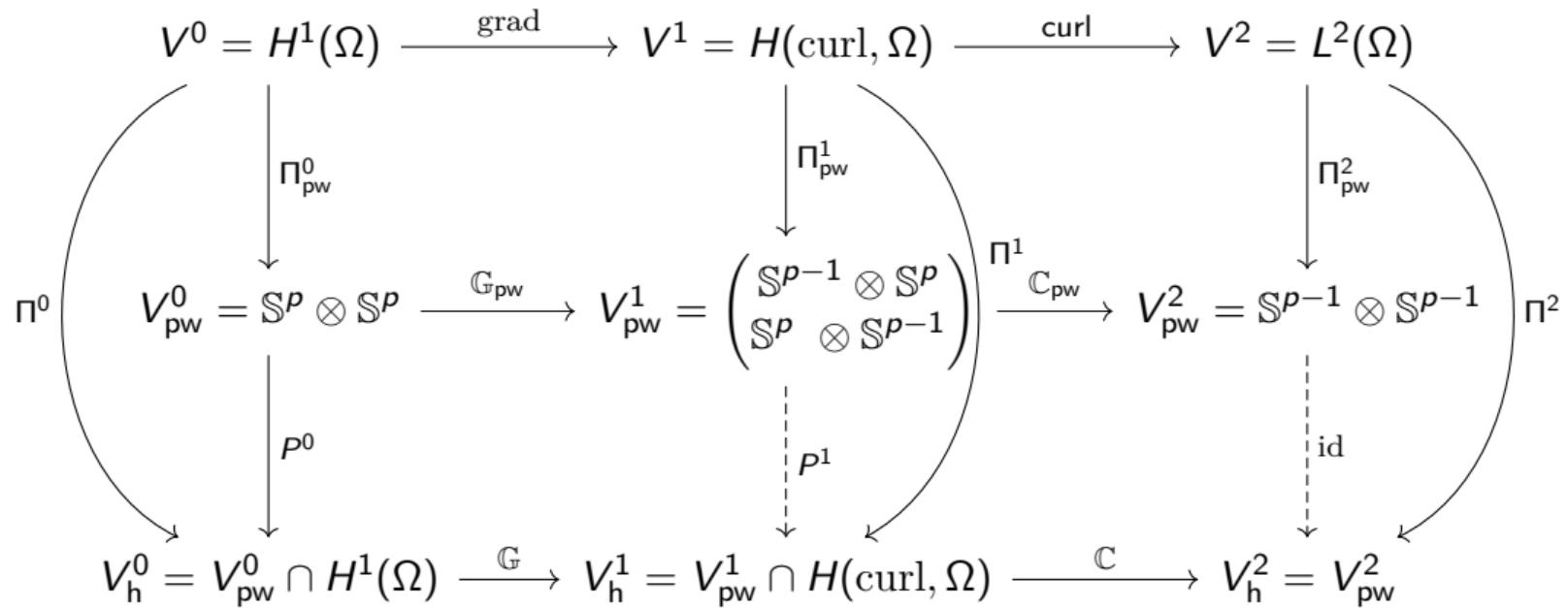


Broken commuting diagram with Splines in 2D





Broken commuting diagram with Splines in 2D



Careful: $\Pi^1 \neq P^1 \Pi_{\text{pw}}^1$.



Existence of (non-matching) stable commuting projections

Theorem¹ If interior vertices are shared by four patches with nested discretization, there exist commuting projections $\Pi^\ell : L^p \rightarrow V_h^\ell$, local and stable for $1 \leq p \leq \infty$.

Broken-FEEC construction: patchwise projections + interface corrections

- on V_h^0 we define

$$\Pi^0 := P\Pi_{\text{pw}}^0 \quad \text{with} \quad \begin{cases} \Pi_{\text{pw}}^0 = \sum_{k \in \mathcal{K}} \Pi_k^0 : L^p \rightarrow V_{\text{pw}}^0 & \text{(stable, patch-wise)} \\ P : V_{\text{pw}}^0 \mapsto V_h^0 & \text{(averages interface dofs)} \end{cases}$$

- on V_h^1 we set

$$\Pi^1 := \Pi_{\text{pw}}^1 + \sum_{e \in \mathcal{E}} \tilde{\Pi}_e^1 + \sum_{v \in \mathcal{V}} \tilde{\Pi}_v^1 + \sum_{(e,v) \in \mathcal{E} \times \mathcal{V}} \tilde{\Pi}_{e,v}^1 \quad \text{with}$$

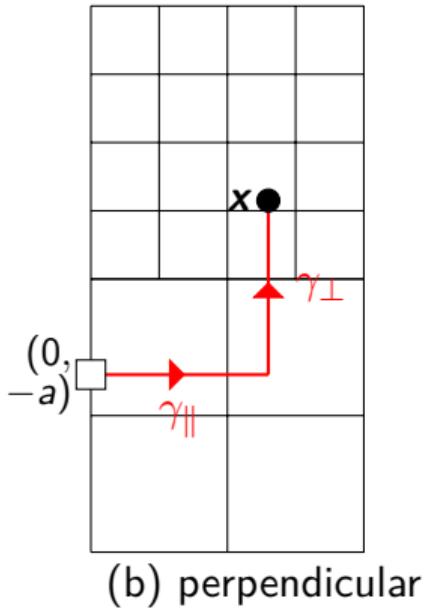
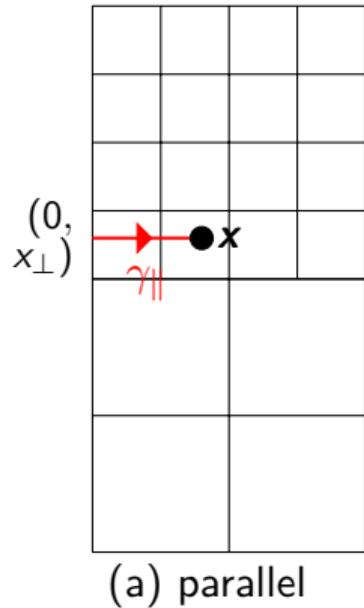
$$\tilde{\Pi}_g^1 \mathbf{u} := \sum_{d \in \{\parallel, \perp\}} \nabla_d^g Q_g \Pi_{\text{pw}}^0 \Phi_d^g(\mathbf{u}) \quad \text{with} \quad \begin{cases} \Phi_d^g : L^p \rightarrow L^p & \text{(antiderivative)} \\ Q_g : V_{\text{pw}}^0 \mapsto V_{\text{pw}}^0 & \text{(interface jump)} \end{cases}$$

- similar construction on V_h^2

¹Campos Pinto, FS (25)



Construction on two patches



$$\tilde{\Pi}_e^1 \mathbf{u} = \sum_{d \in \{\parallel, \perp\}} \nabla_d^e (P^e - I^e) \Pi_{\text{pw}}^0 \Phi_d^e(\mathbf{u}),$$

with

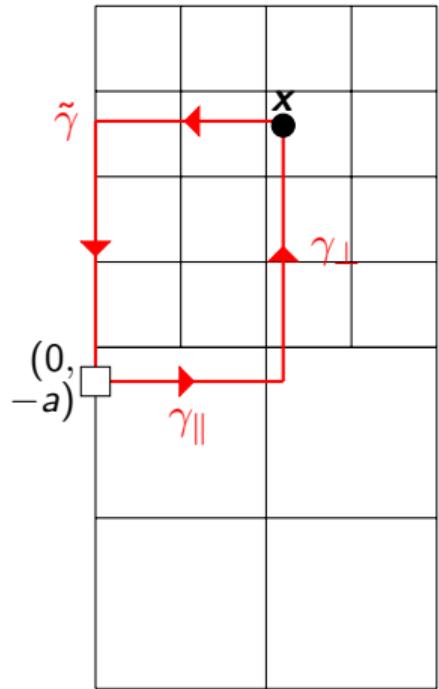
$$\Phi_\parallel^e(\mathbf{u})(\mathbf{x}) = \int_0^{\hat{x}_\parallel} u_\parallel(z, \hat{x}_\perp) dz,$$

$$\begin{aligned} \Phi_\perp^e(\mathbf{u})(\mathbf{x}) = & \int_0^{h_e} \left(\int_0^{\hat{x}_\parallel} u_\parallel(z_\parallel, -a) dz_\parallel \right. \\ & \left. + \int_{-a}^{\hat{x}_\perp} u_\perp(\hat{x}_\parallel, z_\perp) dz_\perp \right) da \end{aligned}$$

Idea: $\Phi(\nabla\phi)(\mathbf{x}) = \phi(\mathbf{x}) + \text{const}_\parallel.$



Construction on two patches



(c) enclosing

$$\tilde{\Pi}_e^2 f = D^{2,e} (P^e - I^e) \Pi_{\text{pw}}^0 \Psi^e(f),$$

with

$$\Psi^e(f)(\mathbf{x}) = \int_0^{h_e} \iint_{\sigma^e(\mathbf{x}, a)} f(\mathbf{z}) d\mathbf{z} da,$$

$$\text{where } \partial\sigma^e(\mathbf{x}, a) = \gamma_{\perp}(\mathbf{x}, a) - \gamma_{||}(\mathbf{x}) + \tilde{\gamma}(\mathbf{x}, a).$$

$$\text{Idea: } \Psi^e(\operatorname{curl} \mathbf{u}) = \Phi_{\perp}^e(\mathbf{u}) - \Phi_{||}^e(\mathbf{u}) + \text{const}_{||}.$$

Construction of numerical schemes

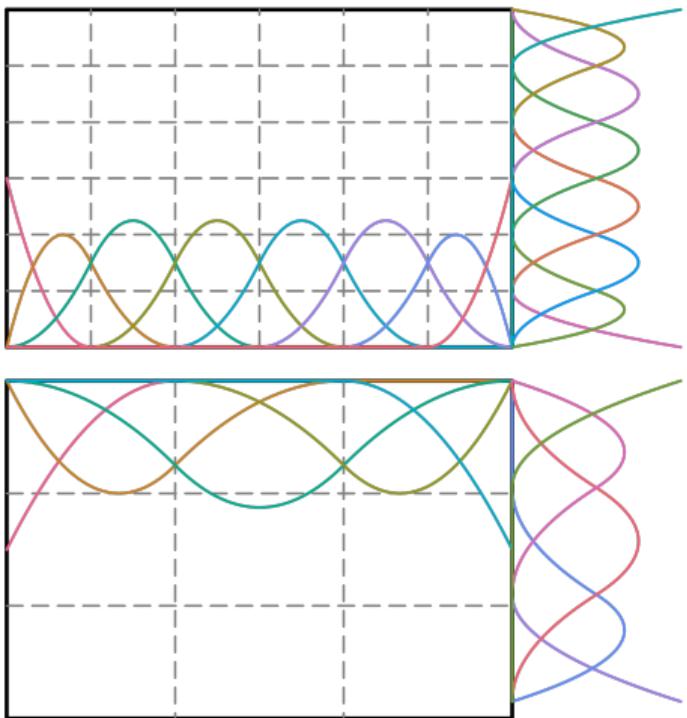


Numerical schemes - preliminaries

Given: Multiple patches, with different grids but single-patch FEEC spaces V_{pw}^{ℓ} .

Goal: Efficient way to map onto the conforming subspaces V_h^{ℓ} which have the structure-preserving properties established by the stable commuting projections.

Idea: Establish the correct conformity depending on V^{ℓ} in a geometric way. (On vertices and edges sequentially for $V^0 = H^1$, tangential to edges for $V^1 = H(\text{curl})$, ...)





Vertex conformity: P_v

Let $\Lambda_i^k = \lambda_{i_1}^k \otimes \lambda_{i_2}^k$ denote the B-Spline basis functions on patch k . Fix a vertex $v \in \mathcal{V}$.

$$P_v : \Lambda_i^k \mapsto \begin{cases} \frac{1}{|\mathcal{K}(v)|} \sum_{k' \in \mathcal{K}(v)} \Lambda_{i^{k'}(v)}^{k'} & \text{if } i = i^k(v), \\ \Lambda_i^k & \text{else,} \end{cases}$$

where

$\mathcal{K}(v)$ is the collection of patches adjacent to the vertex v ,

$i^k(v)$ is the index of the single vertex basis function on patch k with support on v .



Vertex conformity: $P_{\mathcal{V}}$

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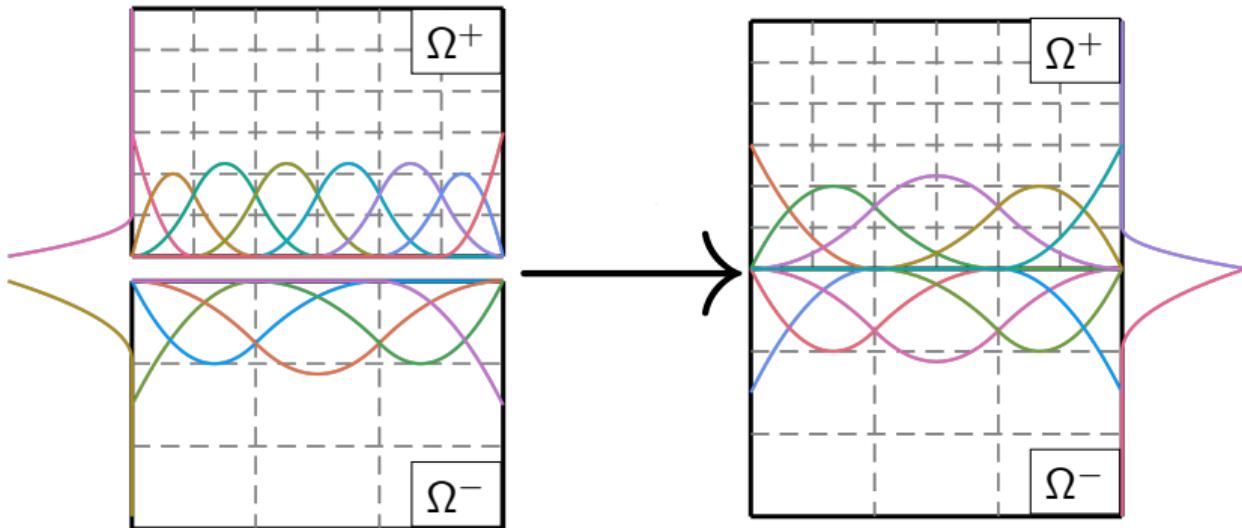
$$P_{\mathcal{V}} = \prod_{v \in \mathcal{V}} P_v$$

→ Only modifies vertex DOF, leaves interior and edge-interior DOF unchanged.



Edge conformity

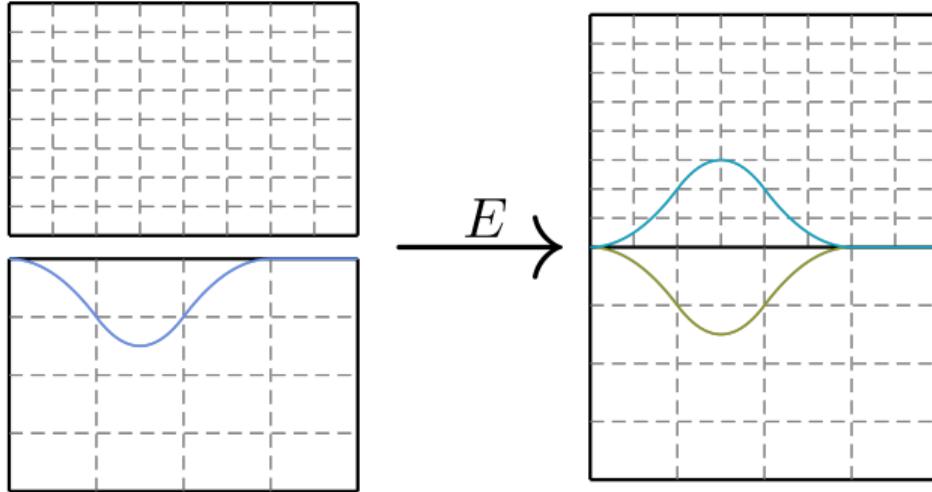
Define the projection P_e at the edge $e \in \mathcal{E}$,



that maps the basis functions on the interface to the conforming space
→ leave interior (and vertex) DOF unchanged.



Extension operator

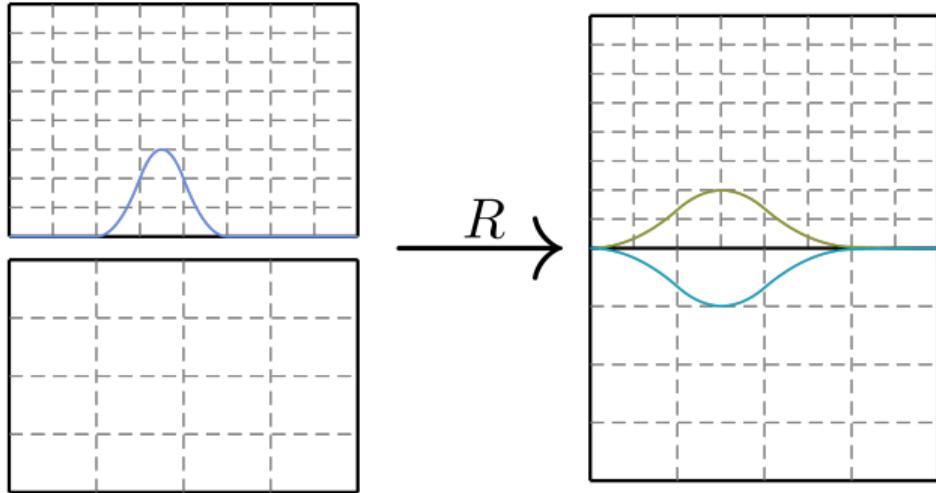


$$E : \lambda_i^- \otimes \lambda_0^- \mapsto \underbrace{\lambda_i^-}_{\sum_{j=0}^{n^+-1} \mathbb{E}_{i,j} \lambda_j^+} \otimes \frac{1}{2} (\lambda_0^- + \lambda_0^+)$$

Change of basis, with interpolation coefficients $\mathbb{E}_{i,j}$ obtained by knot-insertion.



Restriction operator



$$\begin{aligned} R : \lambda_i^+ \otimes \lambda_0^+ &\mapsto \underbrace{\mathcal{R}\lambda_i^+}_{= \sum_{j=0}^{n^- - 1} \mathbb{R}_{i,j} \lambda_j^-} \otimes \frac{1}{2} (\lambda_0^- + \lambda_0^+) \\ &= \sum_{j=0}^{n^+ - 1} (\mathbb{E}\mathbb{R})_{i,j} \lambda_j^+ \end{aligned}$$

E.g. modified L^2 -projection of a fine spline into the coarse spline space.



Edge conformity: $P_{\mathcal{E}}$

Assume $\mathbf{j} = (j_1, j_2) = (j_{\parallel(e)}, j_{\perp(e)})$ for a fixed edge $e \in \mathcal{E}$.

$$P_e : \begin{cases} \Lambda_j^- \mapsto \begin{cases} \lambda_j^- \otimes \frac{1}{2} (\lambda_{i_\perp^-(e)}^- + \lambda_{i_\perp^+(e)}^+) & \text{if } 0 < j_1 = j < n^-, j_2 = i_\perp^-(e), \\ \Lambda_j^- & \text{else.} \end{cases} \\ \Lambda_j^+ \mapsto \begin{cases} \mathcal{R} \lambda_j^+ \otimes \frac{1}{2} (\lambda_{i_\perp^-(e)}^+ + \lambda_{i_\perp^+(e)}^-) & \text{if } 0 < j_1 = j < n^+, j_2 = i_\perp^+(e), \\ \Lambda_j^+ & \text{else.} \end{cases} \end{cases}$$

where $i_\perp^\pm(e)$ is the index of the perpendicular basis function on patch k^\pm with support on e .



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where $i_\perp^\pm(e)$ is the index of the perpendicular basis function on patch k^\pm with support on e .

$$P_{\mathcal{E}} = \prod_{e \in \mathcal{E}} P_e$$

→ Only modifies edge-interior DOF, leaves interior and vertex DOF unchanged



Non-matching conforming projections

Finally:

$$P^0 = P_{\mathcal{E}} P_{\mathcal{V}} : V_{\text{pw}}^0 \rightarrow V_{\text{pw}}^0$$

is a projection onto the conforming space V_h^0 .



Non-matching conforming projections

Finally:

$$P^0 = P_{\mathcal{E}} P_{\mathcal{V}} : V_{\text{pw}}^0 \rightarrow V_{\text{pw}}^0$$

is a projection onto the conforming space V_h^0 .

The construction for V^1 is similar, but we only enforce conformity on the tangential traces (on the whole edge **including vertex DOF**):

$$\mathbf{P}^1 = \sum_{e \in \mathcal{E}} \mathbf{P}_e^1 : V_{\text{pw}}^1 \rightarrow V_{\text{pw}}^1,$$

is a projection onto the conforming subspace V_h^1 .

Numerical Experiments



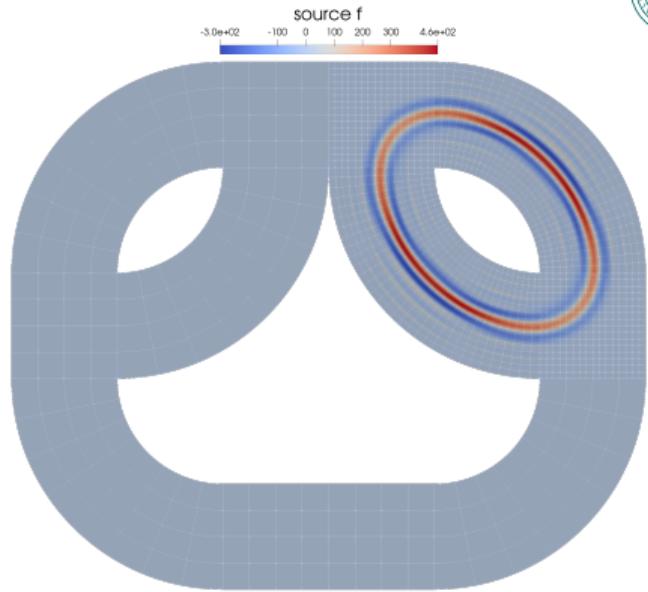
Poisson problem (pretzel domain)

Find $\phi \in V^0 = H_0^1(\Omega)$ such that

$$-\Delta\phi = f,$$

where $f \in L^2(\Omega)$.

The domain consists of 18 patches, we only refine the top-right handle (6 patches).



Within the broken-FEBC framework¹, this equation is discretized as

$$\left(\left(\mathbb{G}_{pw} \mathbb{P}^0 \right)^T \mathbb{M}^1 \mathbb{G}_{pw} \mathbb{P}^0 + \underbrace{\left(\mathbb{I} - \mathbb{P}^0 \right)^T \mathbb{M}^0 \left(\mathbb{I} - \mathbb{P}^0 \right)}_{\text{jump stabilization}} \right) \phi_h = (\mathbb{P}^0)^T f_h.$$

¹All numerical examples are performed using <https://github.com/pyccel/psydac>
| FREDERIK SCHNACK | OCTOBER 28, 2025



Poisson problem in psydac

```
# setup domain
domain = build_multipatch_domain(domain_name='pretzel')
domain_h = discretize(domain, ncells=ncells)

# setup derham sequence
derham = Derham(domain, ["H1", "Hcurl", "L2"])
derham_h = discretize(derham, domain_h, degree=degree)
V0h, V1h, V2h = derham_h.spaces

# patch-wise derivative operators
bD0, bD1 = derham_h.derivatives(kind='linop')

# patch-wise mass matrices
H0 = derham_h.hodge_operator(space='V0', kind='linop')
H1 = derham_h.hodge_operator(space='V1', kind='linop')
dH0 = derham_h.hodge_operator(space='V0', kind='linop', dual=True)

# conforming projectors
cP0, cP1, cP2 = derham_h.conforming_projectors(kind='linop', hom_bc = True)
```





Poisson problem in psydac

```
# div grad
DG = - cP0.T @ bD0.T @ H1 @ bD0 @ cP0
# jump penalization
I0 = IdentityOperator(V0h)
JP0 = (I0 - cP0).T @ H0 @ (I0 - cP0)

A = DG + JP0

# obtain source and exact solution
f, u_ex = get_source_and_solution_h1(domain_h=domain_h)
df = cP0.T @ get_dual_dofs(Vh=V0h, f=f, domain_h=domain_h)

# solve the system
solver = inverse(A, solver='cg', tol=1e-8)
u = solver.solve(df)
```



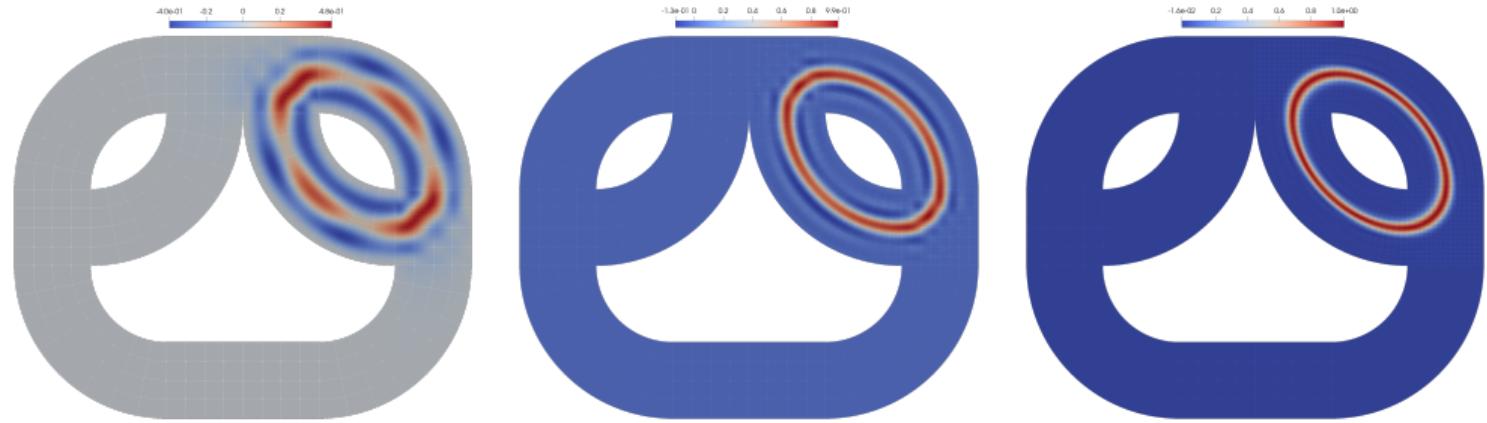


Figure: Numerical solution ϕ_h for degree 3 and refinement of the top-right handle.

Cells p.p.	Degree	uniform refinement				refined handle			
		2	3	4	5	2	3	4	5
4	4	6.71637	0.86527	0.34468	0.18317	6.71637	0.86527	0.34468	0.18317
8	8	0.31598	0.04601	0.08671	0.05566	0.31596	0.04606	0.08675	0.05573
16	16	0.01516	0.00489	0.00575	0.00361	0.01516	0.00489	0.00575	0.00361

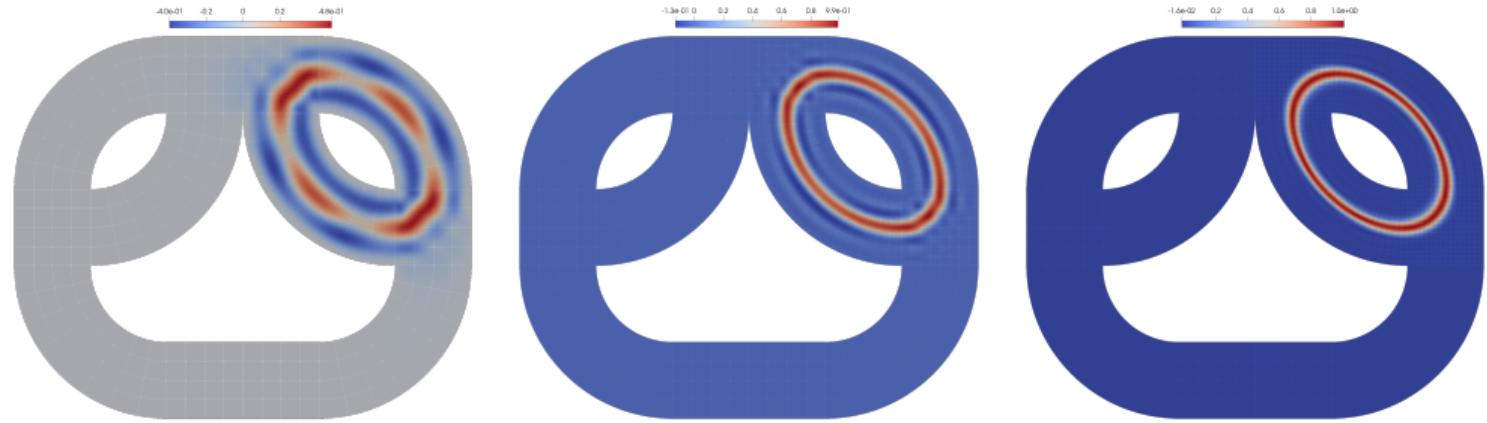


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16	16	0.01516	0.00489	0.00575	0.00361	0.01516	0.00489	0.00575	0.00361

→ On the finest level only half the DOF needed for the same accuracy.



Time-harmonic Maxwell equation (pretzel domain)

For $\omega \in \mathbb{R}$ and $\mathbf{J} \in L^2(\Omega)$, solve

$$-\omega^2 \mathbf{E} + \mathbf{curl} \mathbf{curl} \mathbf{E} = \mathbf{J},$$

for $\mathbf{E} \in H(\mathbf{curl}, \Omega)$.

In our framework, that reads

$$\left(-\omega^2 (\mathbb{P}^1)^T \mathbb{M}^1 \mathbb{P}^1 + (\mathbb{C}_{\text{pw}} \mathbb{P}^1)^T \mathbb{M}^2 \mathbb{C}_{\text{pw}} \mathbb{P}^1 + \underbrace{(\mathbb{I} - \mathbb{P}^1)^T \mathbb{M}^1 (\mathbb{I} - \mathbb{P}^1)}_{\text{jump stabilization}} \right) \mathbf{E}_h = (\mathbb{P}^1)^T \mathbf{J}_h.$$

Manufactured solution: $\mathbf{J} = \begin{pmatrix} -\pi^2 \sin(\pi y) \cos(\pi x) \\ 0 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} \sin(\pi y) \\ \sin(\pi x) \cos(\pi y) \end{pmatrix}.$

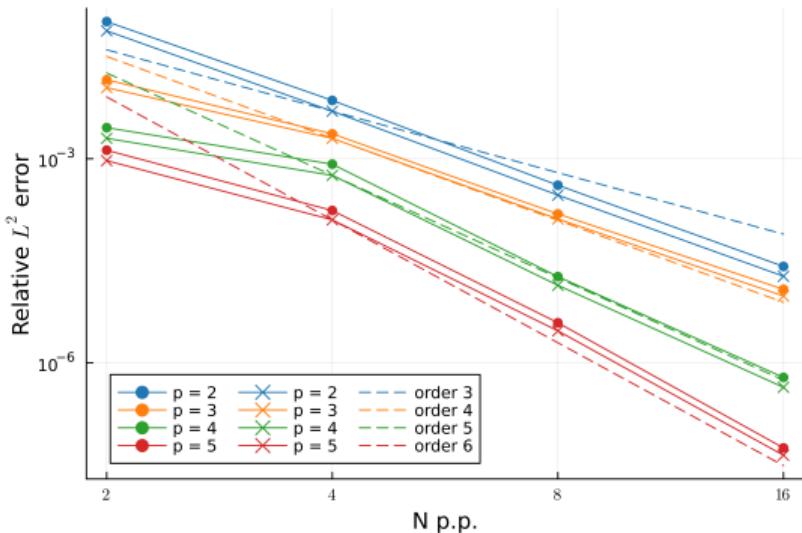


Figure: Convergence curves for matching (circle) and refined (cross) grids, where every other grid was of different resolution.

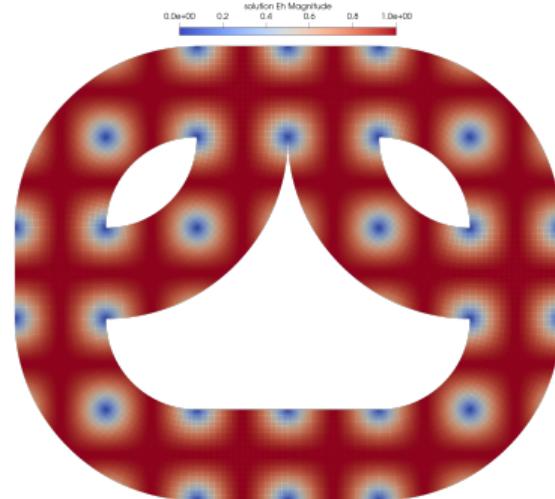


Figure: Numerical solution $\|\mathbf{E}_h\|$ for a refined grid between $N = 8, 16$ and $p = 3$.



Curl curl eigenvalue problem (curved L-shape domain)

Find $\lambda > 0$ such that

$$\operatorname{curl} \operatorname{curl} \boldsymbol{E} = \lambda \boldsymbol{E},$$

where $\boldsymbol{E} \in H_0(\operatorname{curl}, \Omega)$.

Within the broken-FEEC framework, this equation is discretized as

$$\left(\mathbb{C}_{\text{pw}} \mathbb{P}^1 \right)^T \mathbb{M}^2 \mathbb{C}_{\text{pw}} \mathbb{P}^1 \boldsymbol{E}_h = \lambda \left(\underbrace{(\mathbb{I} - \mathbb{P}^1)^T \mathbb{M}^1 (\mathbb{I} - \mathbb{P}^1) + (\mathbb{P}^1)^T \mathbb{M}^1 \mathbb{P}^1}_{\text{jump stabilization}} \right) \boldsymbol{E}_h,$$

where we refine layers around the reentrant corner and compare it to a uniform solution with similar amount of DOF.

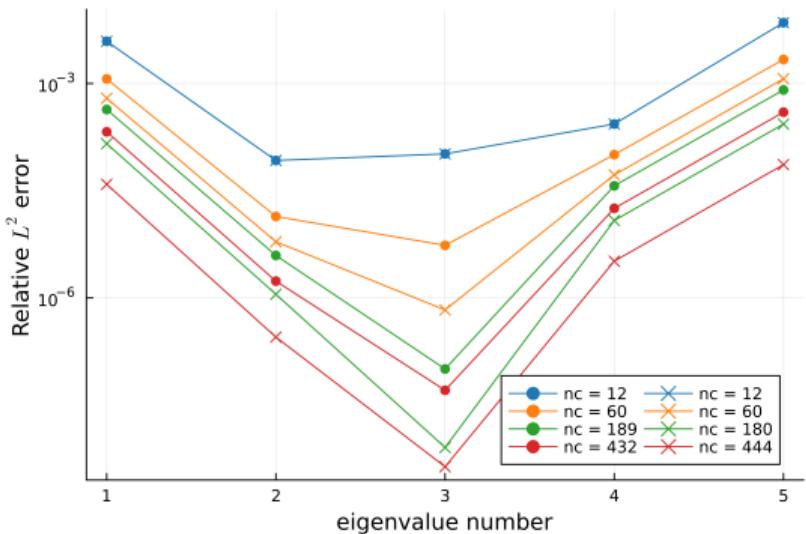


Figure: Error of each eigenvalue for matching (circle) and refined (cross) patches.

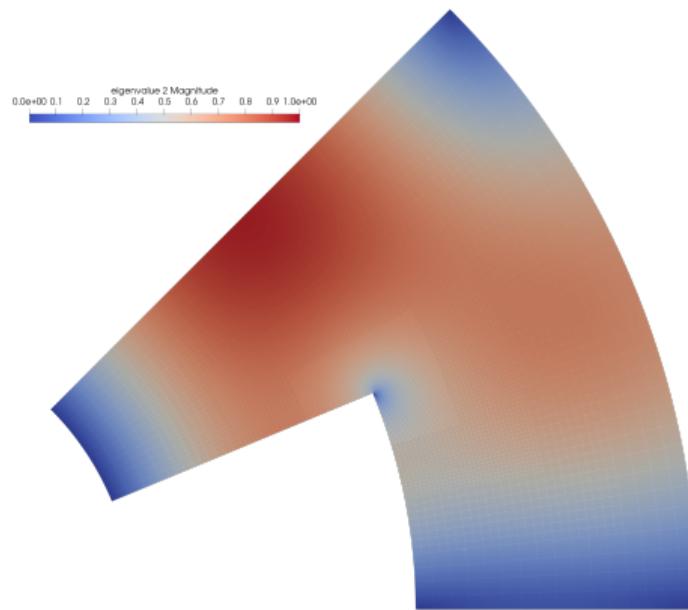
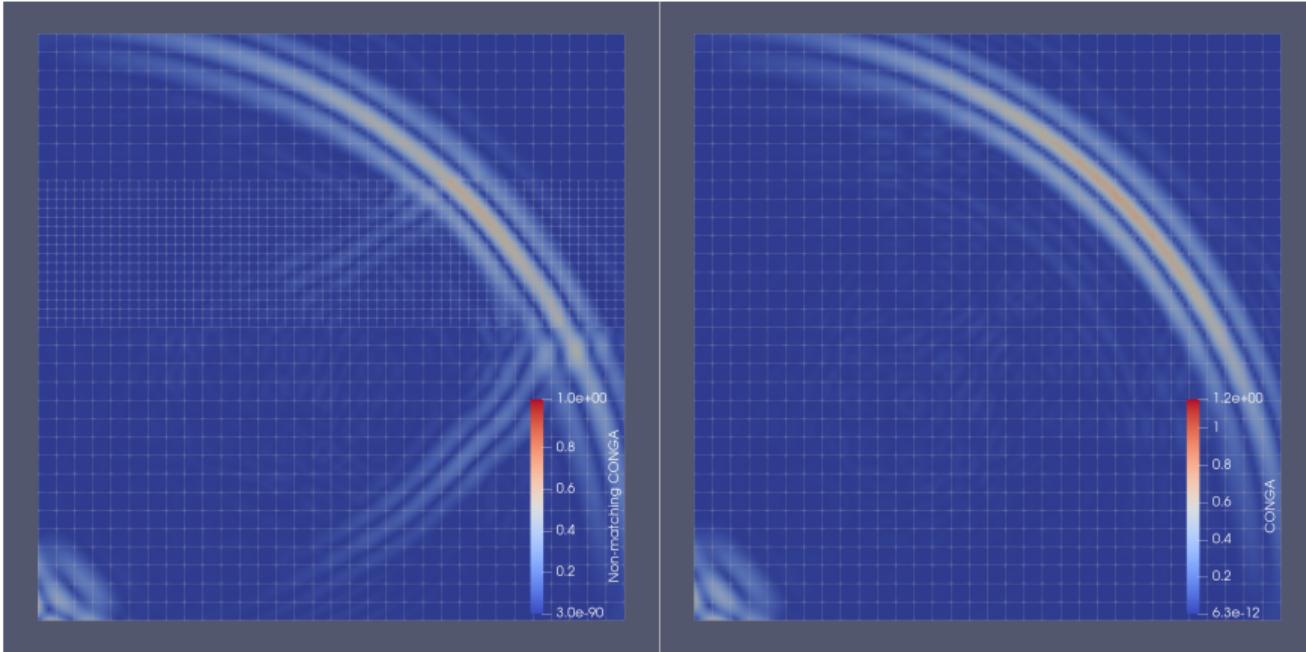


Figure: Absolute value of the eigenfunction corresponding to the eigenvalue λ_3 on the most refined grid.



Time domain Maxwell

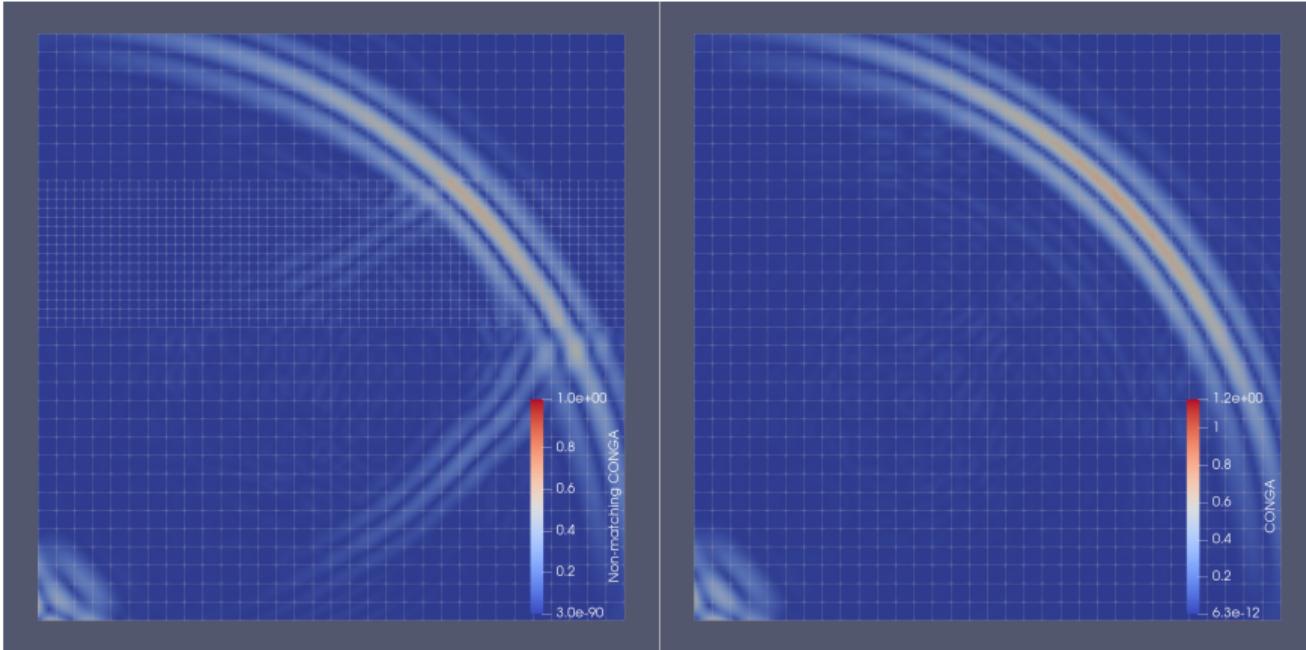
non-matching grid vs. matching grid





Time domain Maxwell

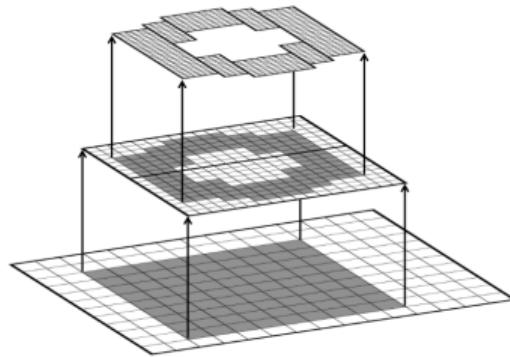
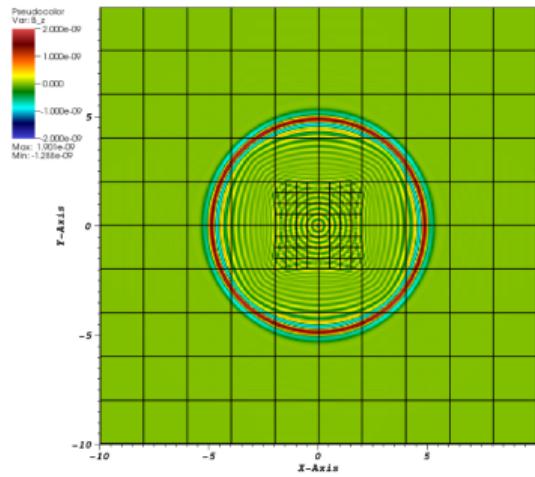
non-matching grid vs. matching grid



Spurious reflections at non-matching patch interfaces.



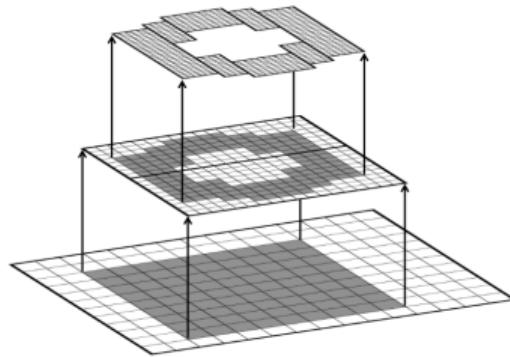
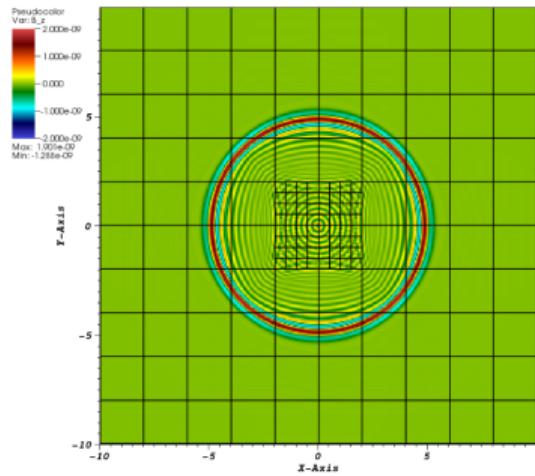
Locally refined meshes



Chilton (13): Uses finite volume schemes and RK4, with damping and dissipation in transition layers between non-matching grids.



Locally refined meshes

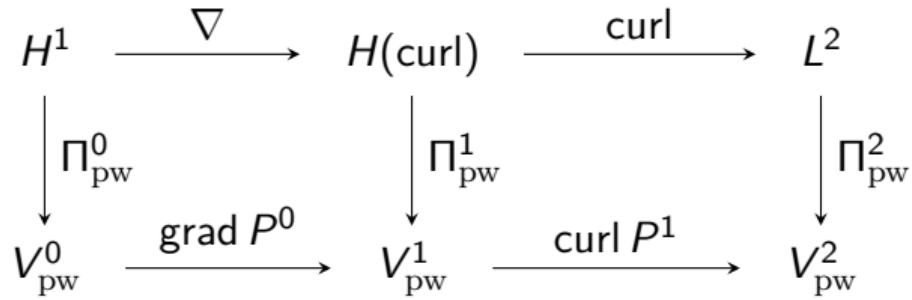


Chilton (13): Uses finite volume schemes and RK4, with damping and dissipation in transition layers between non-matching grids.

→ Q3: How to solve wave problems and **avoid spurious reflections?**



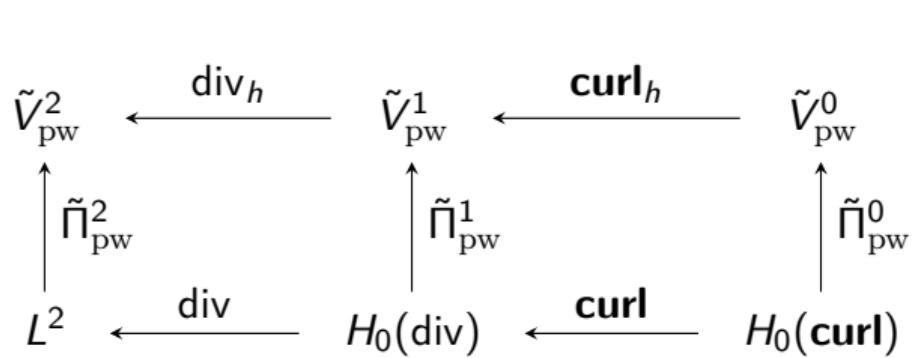
Motivation: Weak Divergence



Idea: if

$$\text{div}_h = -(\text{grad } P^0)^*,$$

then



$$\text{div } u \approx \text{div}_h Q_{\text{pw}}^1 u \quad \forall u \in H_0(\text{div})$$

is of high order as long as P^0 is
preserving polynomial moments.



Preservation of polynomial moments

Preserving polynomial moments is defined as:

$$\int P^\ell \phi q_j = \int \phi q_j, \quad \forall \phi \in V_{\text{pw}}^\ell, q_j \in \mathbb{P}, \quad j = 0, \dots, r$$

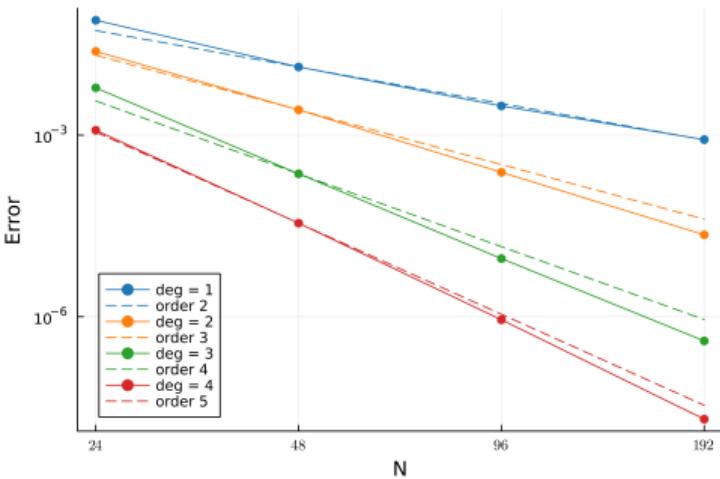
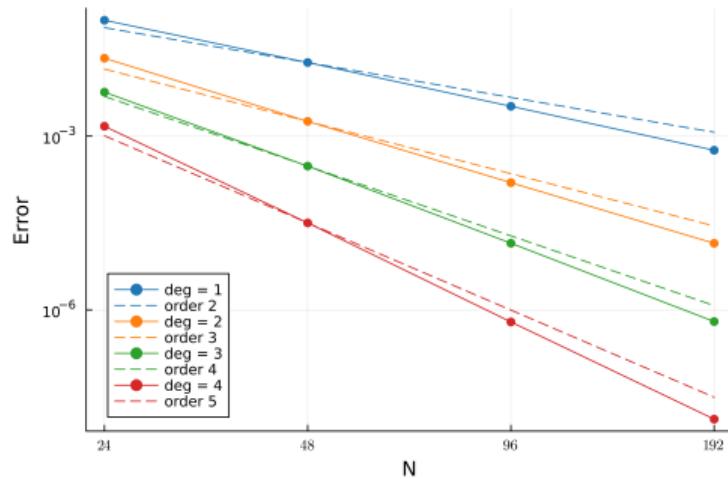
→ Introduces a set of linear constraints on the coefficients of P^0 and P^1 .

This property ensures high-order approximation of the adjoint operator $(P^\ell)^*$.



Weak Divergence II

Convergence curves for the **weak divergence** (left) and the **weak curl** (right) operator on a curved L-shaped domain consisting of three patches with non-matching grids.





Time domain Maxwell equations (Square domain)

For $\mathbf{J} \in L^2(\Omega)$, solve

$$\begin{aligned}\partial_t \mathbf{E} - \mathbf{curl} \mathbf{B} &= -\mathbf{J}, \\ \partial_t \mathbf{B} + \mathbf{curl} \mathbf{E} &= 0,\end{aligned}$$

where $\mathbf{E} \in H_0(\mathbf{curl}, \Omega)$ and $\mathbf{B} \in L^2(\Omega)$.

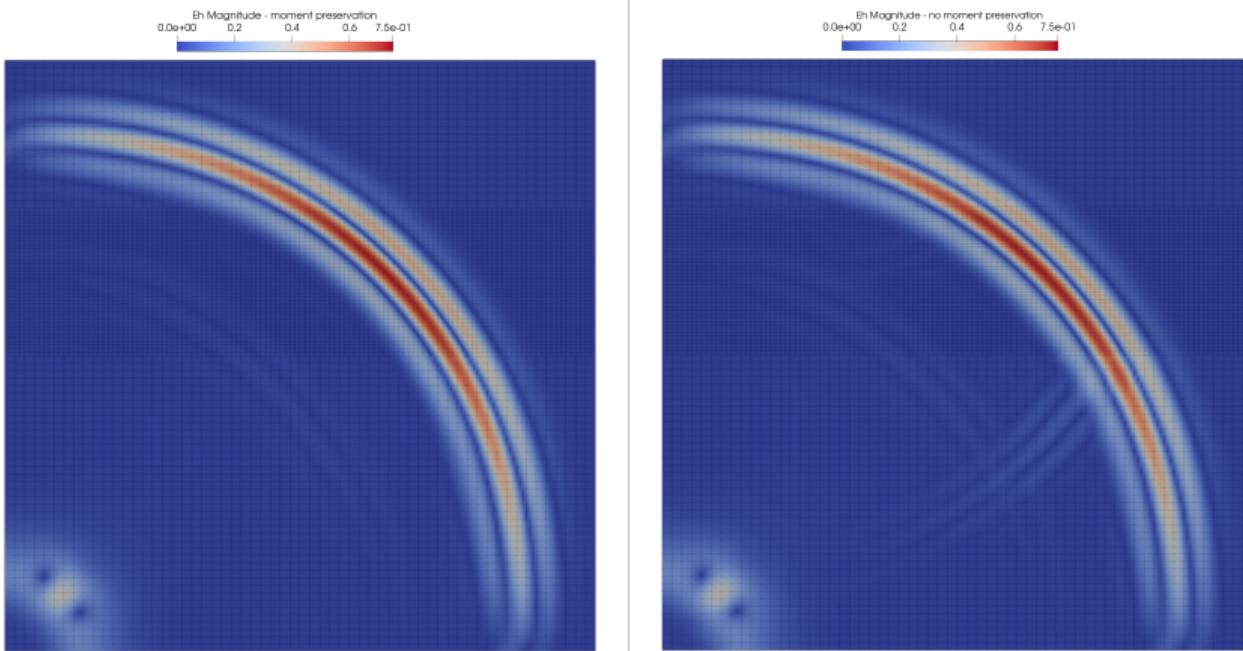
Cast into a simple leap-frog time-stepping scheme, this reads

$$\begin{aligned}B_h^{n+\frac{1}{2}} &= B_h^n - \frac{\Delta t}{2} \mathbb{C}_{\text{pw}} \mathbb{P}^1 E_h^n, \\ \mathbb{M}^1 E_h^{n+1} &= \mathbb{M}^1 E_h^n + \Delta t \left((\mathbb{C}_{\text{pw}} \mathbb{P}^1)^T \mathbb{M}^2 B_h^{n+\frac{1}{2}} - \mathbb{P}^1 J_h^{n+\frac{1}{2}} \right), \\ B_h^{n+1} &= B_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \mathbb{C}_{\text{pw}} \mathbb{P}^1 E_h^{n+1}.\end{aligned}$$



Redemption of results

Moment preservation vs. no moment preservation





Electromagnetic wave (without source)

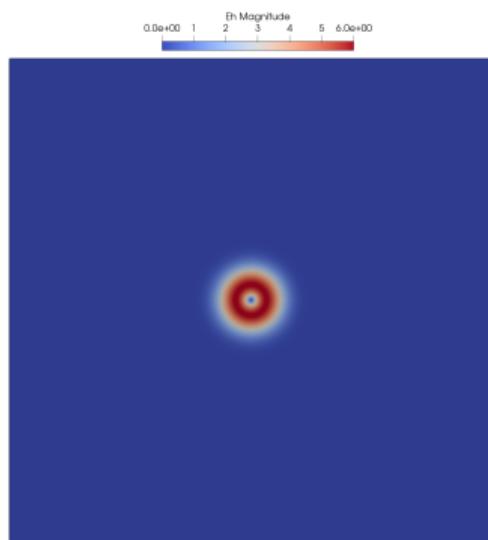
In our example, we choose the initial conditions

$$\mathbf{E}(t = 0; \mathbf{x}) = \mathbf{p} \cos(\mathbf{k} \cdot \mathbf{x}) e^{-\frac{x^2}{2\sigma^2}},$$

$$\mathbf{B}(t = 0; \mathbf{x}) = \text{curl } \mathbf{E},$$

$$\mathbf{J}(t, \mathbf{x}) \equiv 0.$$

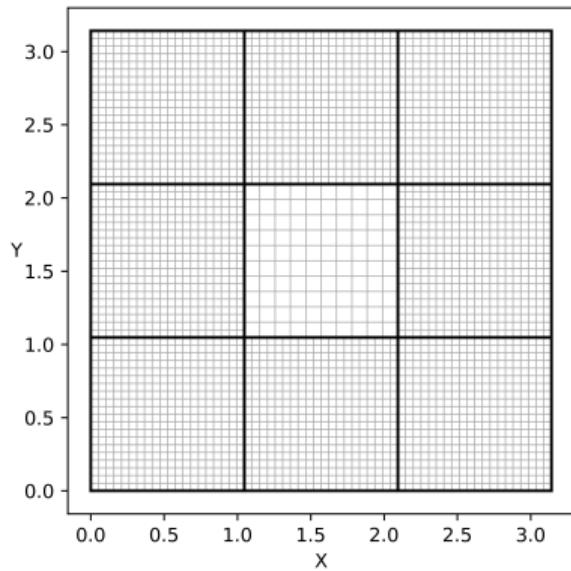
See the initial magnitude of E_h^0 on the right.



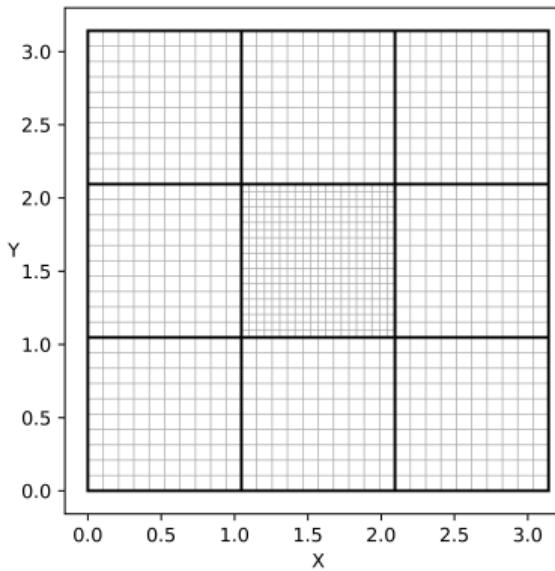


Electromagnetic wave: domains

Coarse to fine



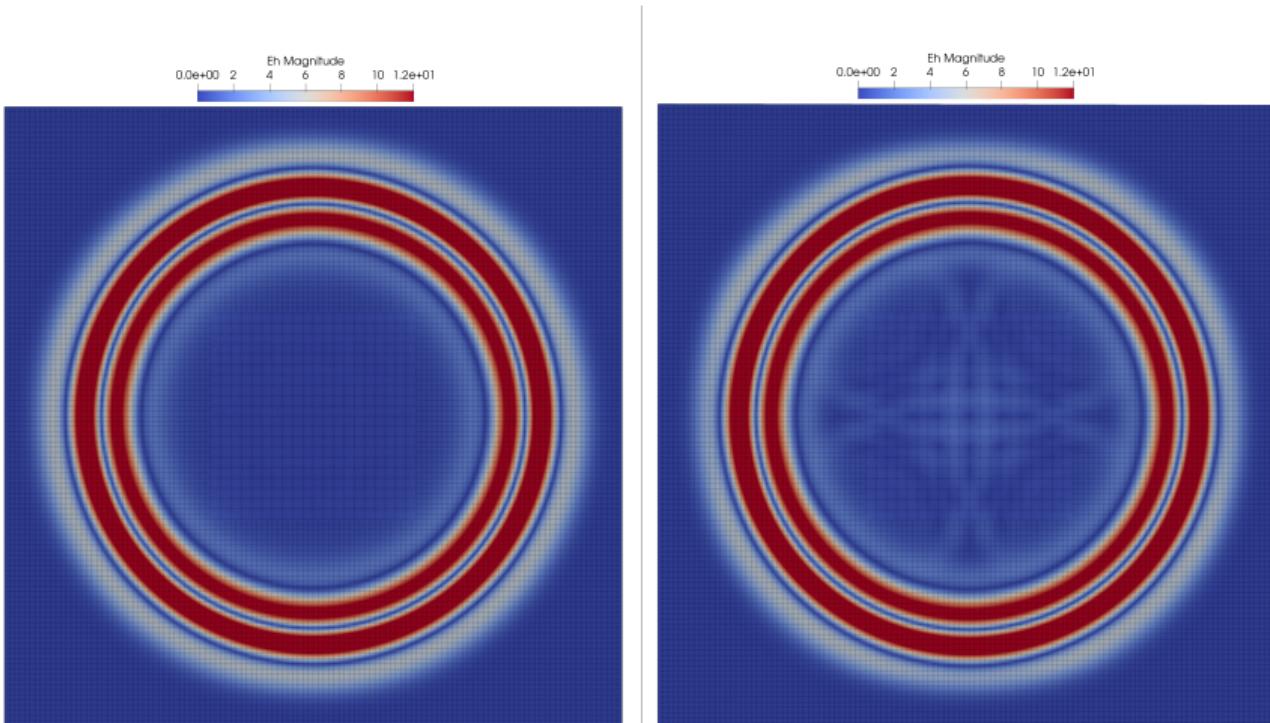
Fine to coarse





Electromagnetic wave: Coarse to fine

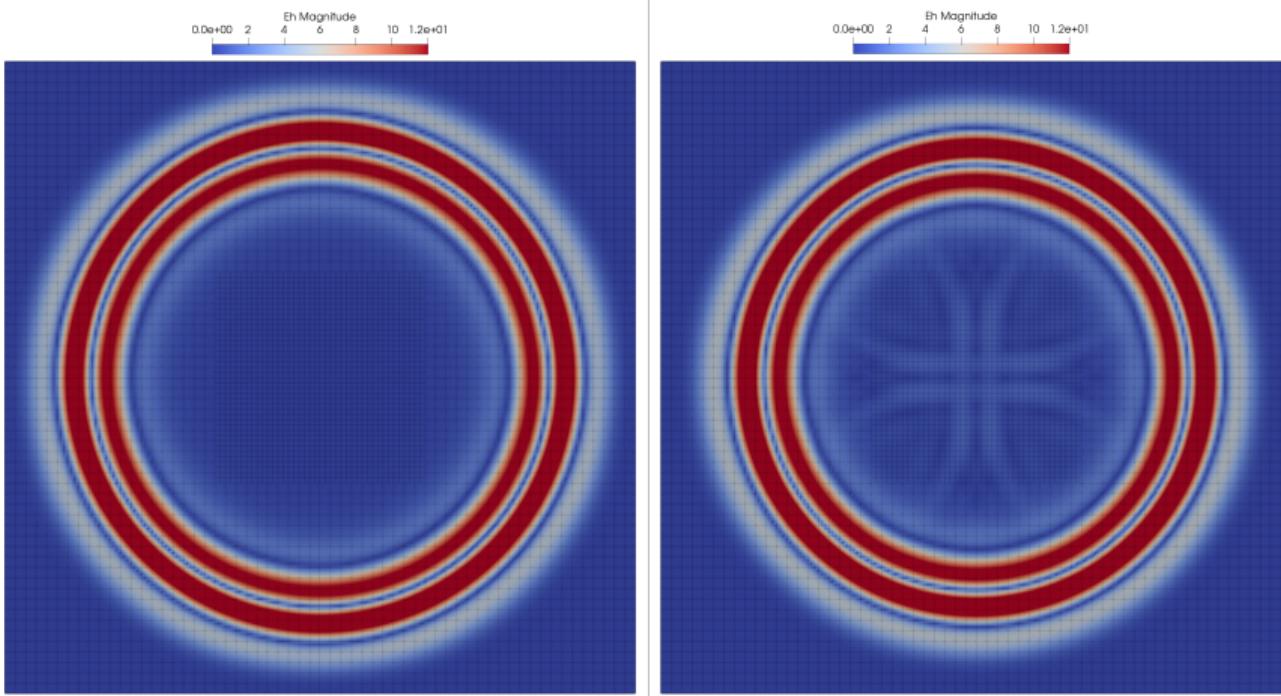
Moment preservation vs. no moment preservation





Electromagnetic wave: Fine to coarse

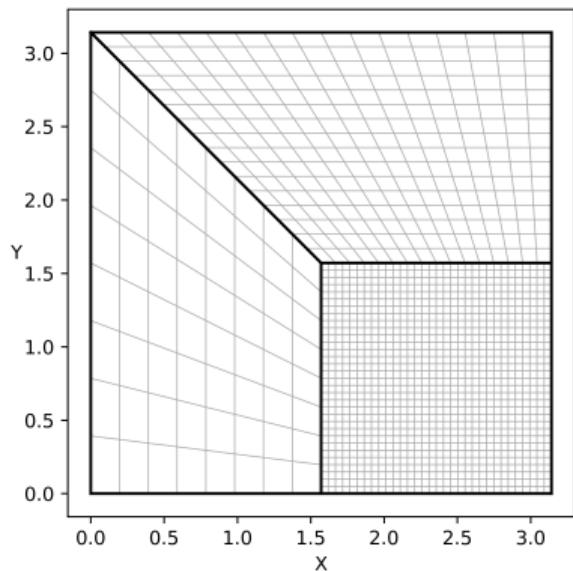
Moment preservation vs. no moment preservation



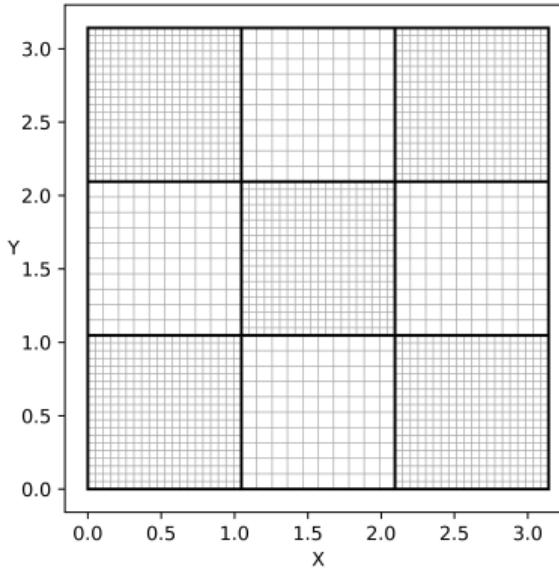


Restrictions by theory

Curl-curl eigenvalue problem



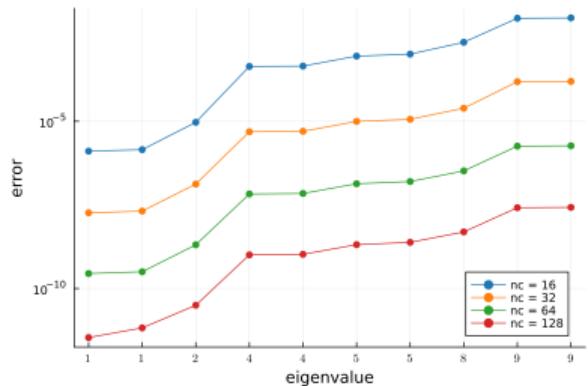
Time-domain Maxwell



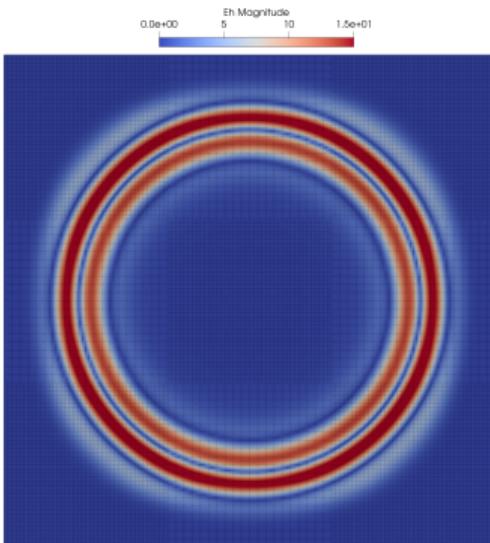


Restrictions by theory

Curl-curl eigenvalue problem



Time-domain Maxwell



...which work well in practice.

Conclusion



Take home messages

- FEEC: Good for domains with corners or holes
- Broken-FEEC + Conforming projections: Good for (non-matching) multi-patch (local, modular schemes) on complex domains



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→ Q1



Take home messages

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- Broken-FEEC + Conforming projections: Good for (non-matching) multi-patch (local, modular schemes) on complex domains
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- Bounded Commuting projections: Yield a priori convergence and stability results



Take home messages

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→ Q1
- Bounded Commuting projections: Yield a priori convergence and stability results
→ Q2



Take home messages

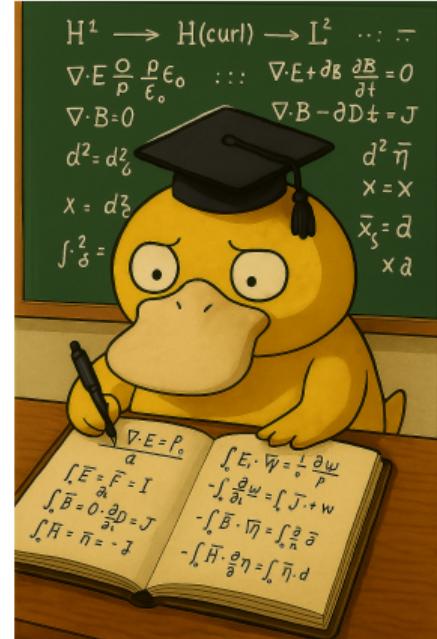
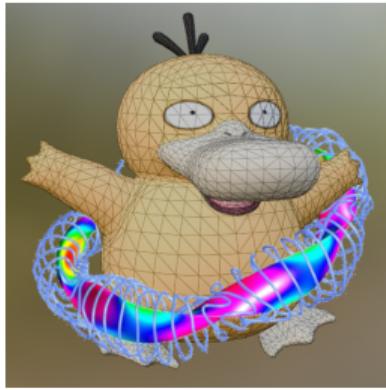
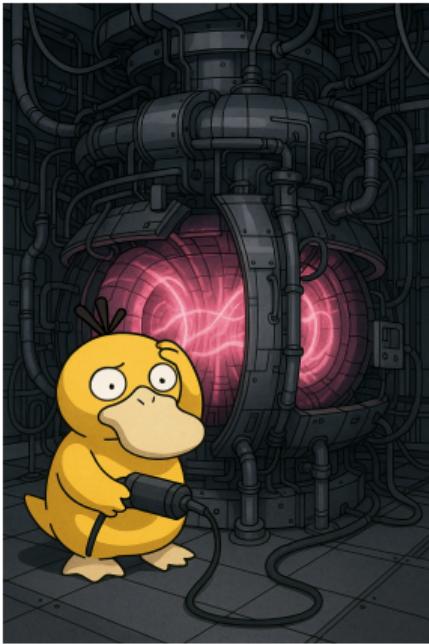
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Take home messages

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Thank you for your attention!



Backup Slides



THEORY





The commuting projections of Buffa-Rivas-Sangalli-Vázquez (2011) – cont'd

- projection property:
 - for Π^0 , as usual $\langle \Theta_i^0, \Lambda_j^0 \rangle = \delta_{i,j}$
 - for Π^1 , follows from $\int_0^{x_1} u_1(z, x_2) dz \in V_h^0$ (same argument for Π^2)

- commuting property: for the grad,

$$(\Pi^1 \nabla \phi(\mathbf{x}))_1 = \partial_1 (\Pi^0 \Phi_1^1(\nabla \phi)(\mathbf{x})) = \partial_1 \Pi^0 (\phi(\mathbf{x}) - \phi(0, x_2)) = \partial_1 \Pi^0 \phi(\mathbf{x}) \quad (\text{same for the curl})$$

- L_h^2 stability:

- on V_h^0 , follows from locally stable dofs
- on V_h^1 , follows from a localization argument: on a local domain ω_i , we have

$$\nabla_1 \Pi^0 \Phi_1^1(\mathbf{u}) = \nabla_1 \Pi^0 \Phi_{1,i}^1(\mathbf{u}) \quad \text{with} \quad \Phi_{1,i}^1(\mathbf{u})(\mathbf{x}) := \int_{x_1(\omega_i)}^{x_1} u_1(z, x_2) dz$$

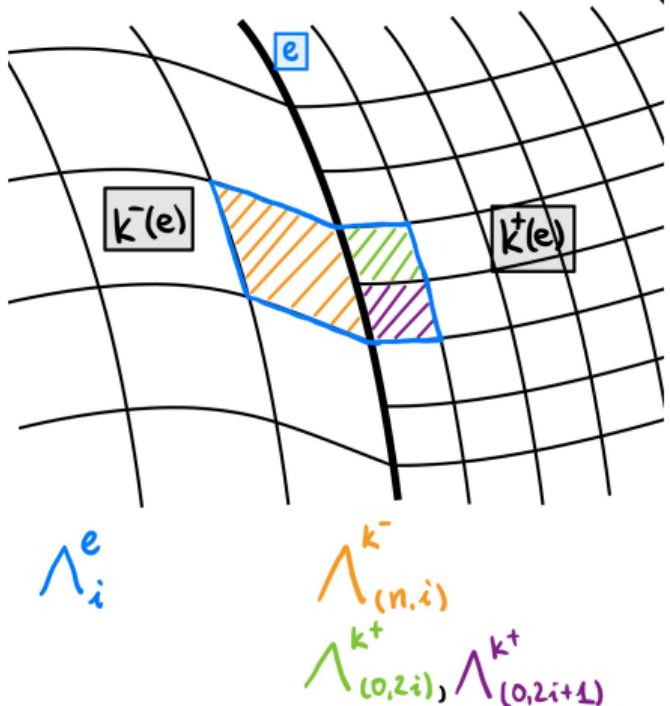
and inverse / local stability estimates

$$\|\nabla_1 \Pi^0 \Phi_{1,i}^1(\mathbf{u})\|_{L^2(\omega_i)} \lesssim h^{-1} \|\Pi^0 \Phi_{1,i}^1(\mathbf{u})\|_{L^2(E_h(\omega_i))} \lesssim h^{-1} \|\Phi_{1,i}^1(\mathbf{u})\|_{L^2(E_h^2(\omega_i))} \lesssim \|\mathbf{u}\|_{L^2(E_h^3(\omega_i))}$$

- on V_h^2 , the same principle applies
- Note: as long as $\text{supp}(\Theta_i^0) \approx \text{supp}(\Lambda_i^0)$ these operators are local



Edge Conforming basis functions



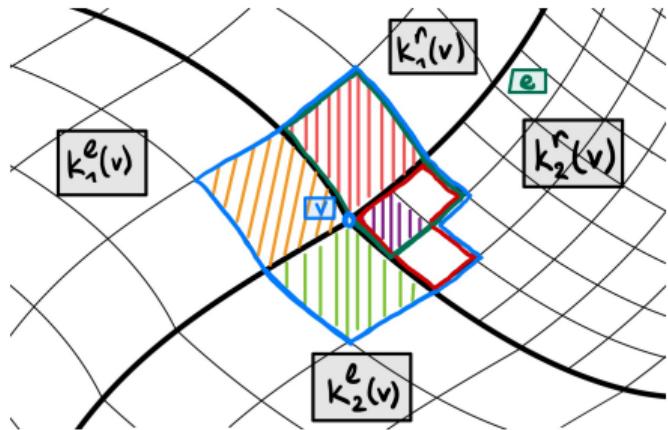
$$\Lambda_i^e(\mathbf{x}) := \begin{cases} \hat{\lambda}_i^-(\hat{x}_{||}^-)\hat{\lambda}_{i_\perp^-(e)}^-(\hat{x}_\perp^-) & \text{on } \Omega_{k^-}, \\ \hat{\lambda}_i^-(\eta_e(\hat{x}_{||}^+))\hat{\lambda}_{i_\perp^+(e)}^+(\hat{x}_\perp^+) & \text{on } \Omega_{k^+}, \\ 0 & \text{else.} \end{cases}$$

- Projection on the local **edge-broken space**:
- $$I_e : V_{pw}^0 \rightarrow V_{pw}^0, \quad \Lambda_i^k \mapsto \begin{cases} \Lambda_i^k & \text{if } i_2 = i_2^k(e) \\ 0 & \text{otherwise} \end{cases}$$
- Projection on the local **edge-conforming space**:

$$P_e : V_{pw}^0 \rightarrow V_h^0, \quad \Lambda_i^k \mapsto \begin{cases} \Lambda_{i_1}^e & \text{if } k = -, \ i_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$



Vertex conforming basis functions



$$\Lambda_v$$

$$\Lambda_v|_{\Omega_k}$$

$$\Lambda_v^e$$

$$\Lambda_v^{k_1^e}$$

$$\Lambda_v^{k_2^e}$$

$$\Lambda_v^{k_1^r}$$

$$\Lambda_v^{k_2^r}$$

$$\Lambda^\nu := \sum_{e \in \mathcal{E}(\nu)} \Lambda_\nu^e - \sum_{k \in \mathcal{K}(\nu)} \Lambda_\nu^k.$$

- Projection on the local vertex-broken space:

$$I_\nu : V_{\text{pw}}^0 \rightarrow V_{\text{pw}}^0, \quad \Lambda_i^k \mapsto \begin{cases} \Lambda_i^k & \text{if } k \in \mathcal{K}(\nu), \ i = i^k(\nu) \\ 0 & \text{otherwise} \end{cases}$$

- Projection on the local vertex-conforming space:

$$P_\nu : V_{\text{pw}}^0 \rightarrow V_h^0, \quad \Lambda_i^k \mapsto \begin{cases} \Lambda^\nu & \text{if } k = k^*(\nu), \ i = i^k(\nu) \\ 0 & \text{otherwise} \end{cases}$$

Conforming projection P

For any broken $\phi \in V_{\text{pw}}^0$, we have the decomposition

$$\phi = \left(\sum_k I_0^k + \sum_{e \in \mathcal{E}} I_{e,0} + \sum_{v \in \mathcal{V}} I_v \right) \phi \quad \text{and} \quad P\phi := \left(\sum_{k \in \mathcal{K}} I_0^k + \sum_{e \in \mathcal{E}} P_{e,0} + \sum_{v \in \mathcal{V}} P_v \right) \phi.$$

Building Blocks of Π^1

Strategy: Start from the patch-wise commuting projections and add correction terms.

Patch-wise projections Π_k^1 :

$$\Pi_k^1 \mathbf{u} = \sum_{d \in \{1, 2\}} \nabla_d^k \Pi_k^0 \Phi_d^k(\mathbf{u}),$$

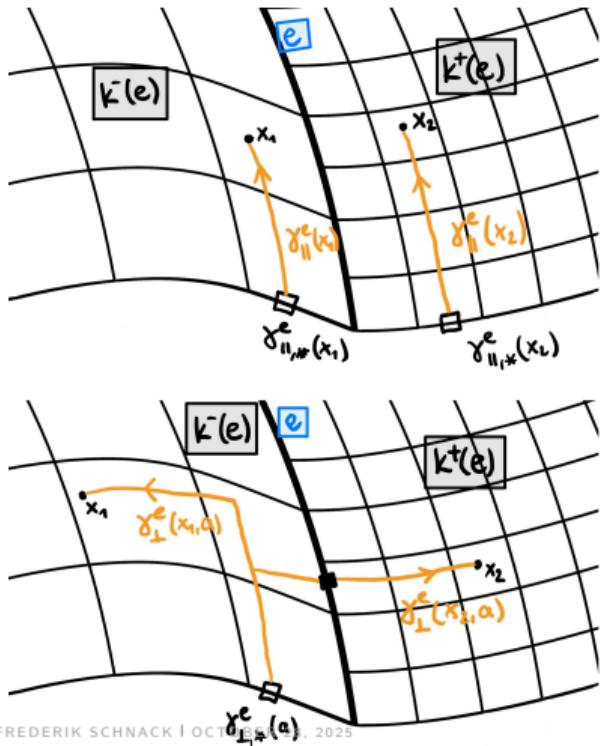
where the antiderivatives

$$\Phi_1^k(\mathbf{u})(\mathbf{x}) = \int_0^{x_1} \mathbf{u}_1(z, x_2) dz, \quad \Phi_2^k(\mathbf{u})(\mathbf{x}) = \int_0^{x_2} \mathbf{u}_2(x_1, z) dz$$

are defined for $\mathbf{x} \in \Omega_k$ such that $\Phi_d^k(\mathbf{u}) \in V_{\text{pw}}^0$ for $\mathbf{u} \in V_h^1$, $d = 1, 2$.



Edge-correction $\tilde{\Pi}_e^1$



$$\tilde{\Pi}_e^1 \mathbf{u} := \sum_{d \in \{\parallel, \perp\}} \nabla_d^e (P_e - I_e) \Pi_{\text{pw}}^0 \Phi_d^e(\mathbf{u}),$$

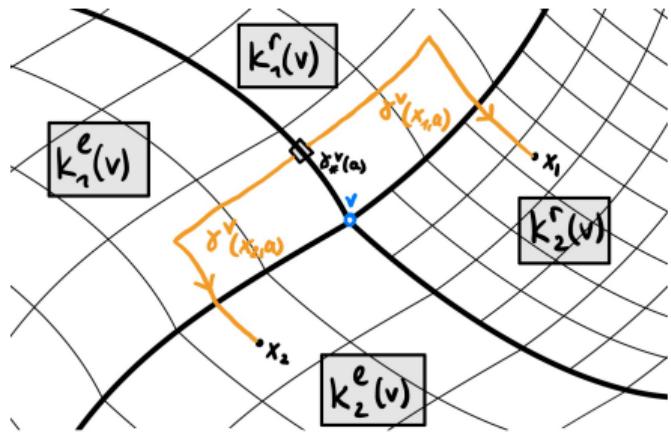
where the antiderivatives are defined as

$$\Phi_d^e \mathbf{u}(\mathbf{x}) = \int_{\gamma_d^e(\mathbf{x})} \mathbf{u} \cdot dI, \quad d \in \{\parallel, \perp\}$$

such that for $\mathbf{u} \in V_h^1$ and for all $e \in \mathcal{E}$, $\Phi_\parallel^e(\mathbf{u})$ and $\Phi_\perp^e(\mathbf{u})$ belong to V_{pw}^0 and are continuous across e which implies that $(P_e - I_e)$ is equal to zero in the correction term.



Vertex-correction $\tilde{\Pi}_v^1$



$$\tilde{\Pi}_v^1 \mathbf{u} := \nabla_{\text{pw}} (P_v - \bar{I}_v) \Pi_{\text{pw}}^0 \Phi^v(\mathbf{u}),$$

where the antiderivative is defined as

$$\Phi^v \mathbf{u}(x) = \int_{\gamma^v(x)} \mathbf{u} \cdot dI, \quad d \in \{\parallel, \perp\}$$

such that for $\mathbf{u} \in V_h^1$ and for all $v \in \mathcal{V}$, $\Phi^v(\mathbf{u})$ belongs to V_{pw}^0 and is continuous across every $e \in \mathcal{E}(v)$ which implies that $(P_v - \bar{I}_v)$ is equal to zero in the correction term.



Full projection Π^1

$$\Pi^1 := \sum_{k \in \mathcal{K}} \Pi_k^1 + \sum_{e \in \mathcal{E}} \tilde{\Pi}_e^1 + \sum_{v \in \mathcal{V}} \tilde{\Pi}_v^1 + \sum_{v \in \mathcal{V}, e \in \mathcal{E}(v)} \tilde{\Pi}_{e,v}^1$$

with edge, vertex and edge-vertex correction operators defined as

$$\begin{cases} \tilde{\Pi}_e^1 \mathbf{u} := \sum_{d \in \{\parallel, \perp\}} \nabla_d^e (P_e - I_e) \Pi_{\text{pw}}^0 \Phi_d^e(\mathbf{u}) \\ \tilde{\Pi}_v^1 \mathbf{u} := \nabla_{\text{pw}} (P_v - \bar{I}_v) \Pi_{\text{pw}}^0 \Phi_v(\mathbf{u}) \\ \tilde{\Pi}_{e,v}^1 \mathbf{u} := \sum_{d \in \{\parallel, \perp\}} \nabla_d^e (\bar{I}_{e,v} - P_{e,v}) \Pi_{\text{pw}}^0 \Phi_d^{v,e}(\mathbf{u}) \end{cases}$$

Main result: Π^1 is a projection on V_h^1 , L^p stable for $1 \leq p \leq \infty$ and commuting:

$$\Pi^1 \nabla \phi = \nabla \Pi^0 \phi \quad \phi \in H^1(\Omega)$$

Commuting property

- let $\mathbf{u} = \nabla\phi$ and write $\phi_h = \Pi_{\text{pw}}^0 \phi \in V_{\text{pw}}^0$
 - by single-patch construction: $\sum_k \Pi_k^1 \mathbf{u} = \nabla_{\text{pw}} \phi_h$
 - by preservation of \parallel invariance along e : $\nabla_{\parallel}^e (P_e - I_e) \Pi_{\text{pw}}^0 \Phi_{\parallel}^e(\mathbf{u}) = \nabla_{\parallel}^e (P_e - I_e) \phi_h$
 - by cancellation of constants in $(P_e - I_e)$: $(P_e - I_e) \Pi_{\text{pw}}^0 \Phi_{\perp}^e(\mathbf{u}) = (P_e - I_e) \phi_h$
 - and similarly for the \mathbf{v} and e, \mathbf{v} terms.
 - thus we have $\phi_h = \left(\sum_k I_0^k + \sum_e I_{e,0} + \sum_{\mathbf{v}} I_{\mathbf{v}} \right) \phi_h$ and $\Pi^1 \mathbf{u} = \nabla_{\text{pw}} \psi_h$ with

$$\begin{aligned}\psi_h &= \phi_h + \left(\sum_e (P_e - I_e) + \sum_{\mathbf{v}} (P_{\mathbf{v}} - \bar{I}_{\mathbf{v}}) + \sum_{e,\mathbf{v}} (\bar{I}_{e,\mathbf{v}} - P_{e,\mathbf{v}}) \right) \phi_h \\ &= \left(\sum_k I_0^k + \sum_e (I_{e,0} + P_e - I_e) + \sum_{\mathbf{v}} (I_{\mathbf{v}} - P_{\mathbf{v}} - \bar{I}_{\mathbf{v}}) + \sum_{e,\mathbf{v}} (\bar{I}_{e,\mathbf{v}} - P_{e,\mathbf{v}}) \right) \phi_h = \dots = P\phi_h\end{aligned}$$

L^2 stability

- for each term, stability follows from
 - local inverse estimates for gradients
 - the stability of the local projections
 - the stability of the antiderivatives and localization arguments



Projection property

- range property: $(\Pi^1 \mathbf{u} \in V_h^1)$
 - clearly, $\Pi^1 \mathbf{u} \in V_{\text{pw}}^1$
 - the antiderivatives and the local conforming / broken projections are such that

$$(\boldsymbol{\tau} \cdot \Pi^1 \mathbf{u})|_{\Omega_{k^-}(e)} = (\boldsymbol{\tau} \cdot \Pi^1 \mathbf{u})|_{\Omega_{k^+}(e)}, \quad e \in \mathcal{E}$$

- projection property: $(\Pi^1 \mathbf{u} = \mathbf{u} \text{ for } \mathbf{u} \in V_h^1)$
 - patch-wise operator is a projection:

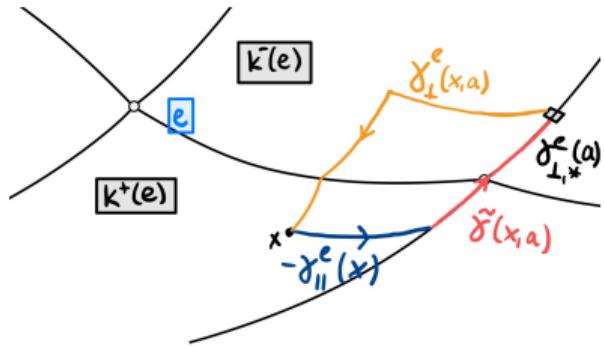
$$\sum_{k \in \mathcal{K}} \Pi_k^1 \mathbf{u} = \sum_{k \in \mathcal{K}} \mathbf{u}|_{\Omega_k} = \mathbf{u}, \quad \mathbf{u} \in V_h^1 \subset V_{\text{pw}}^1$$

- for $\mathbf{u} \in V_h^1$, the continuity properties of the antiderivatives yield

$$\begin{cases} (P_e - I_e) \Pi_{\text{pw}}^0 \Phi_d^e(\mathbf{u}) = 0, & d \in \{\parallel, \perp\} \\ (P_v - \bar{I}_v) \Pi_{\text{pw}}^0 \Phi^v(\mathbf{u}) = 0 \\ (P_{e,v} - \bar{I}_{e,v}) \Pi_{\text{pw}}^0 \Phi_d^{e,v}(\mathbf{u}) = 0, & d \in \{\parallel, \perp\} \end{cases}$$



Projection Π^2



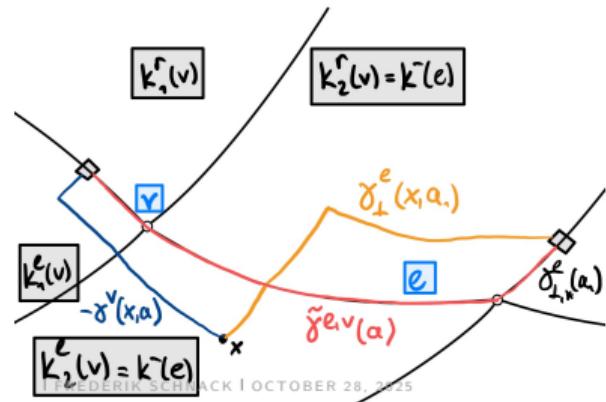
$$\Pi^2 := \sum_{k \in \mathcal{K}} \Pi_k^2 + \sum_{e \in \mathcal{E}} \tilde{\Pi}_e^2 + \sum_{v \in \mathcal{V}, e \in \mathcal{E}(v)} \tilde{\Pi}_{e,v}^2$$

with edge correction terms

$$\tilde{\Pi}_e^2 : \begin{cases} L^p(\Omega) \rightarrow V_{\text{pw}}^2, \\ f \mapsto D^{2,e}(P^e - I^e)\Pi_{\text{pw}}^0 \Psi^e(f) \end{cases}$$

and edge-vertex corrections

$$\tilde{\Pi}_{e,v}^2 : \begin{cases} L^p(\Omega) \rightarrow V_{\text{pw}}^2, \\ f \mapsto D^{2,e}(\bar{I}_v^e - P_v^e)\Pi_{\text{pw}}^0 \Psi^{v,e}(f). \end{cases}$$



$$\Psi(\operatorname{curl} \mathbf{u}) = \Phi_{\perp}(\mathbf{u}) - \Phi_{\parallel}(\mathbf{u}) + \tilde{\Phi}(\mathbf{u})$$