Structure-preserving Finite Element schemes with locally refined patches

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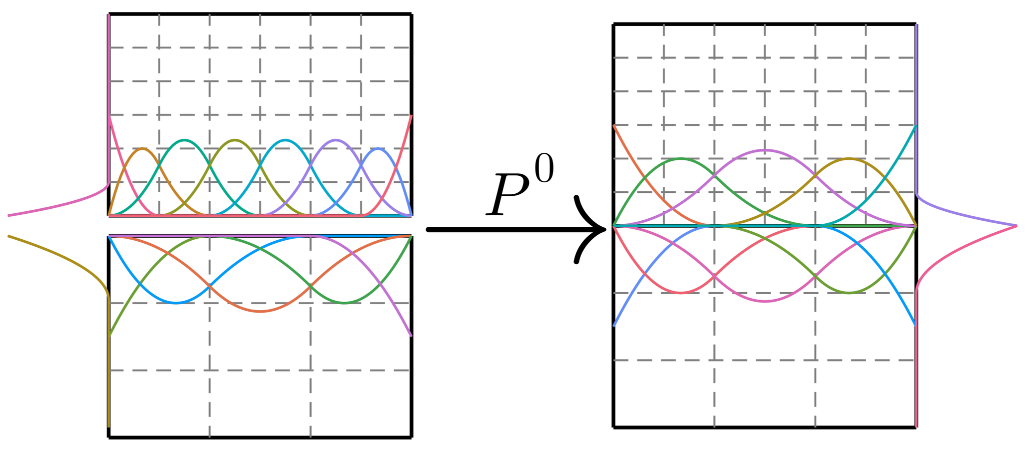
ABSTRACT

Having a **isogeometric multi-patch discretization**[1] with coarse and refined patches allows for a well suited treatment of domains that need high resolution in certain areas, while remaining flexible in the patch-wise discretization and **applications of mappings**. Although this approach is widely used in numerical schemes, its application in **Finite Element Exterior Calculus (FEEC)** methods[2] is virtually nonexistent, mostly because of the inherent conformity requirement of the finite element fields in FEEC theory. This work is an extension of **Conforming/Non-Conforming Galerkin (CONGA)** schemes[3], that will be complemented by properly coupling patch-wise **adaptive** FEEC discretizations[4], that allow for such discontinuities and the applications of mappings. Implementations are done based on the python library **PSYDAC**.

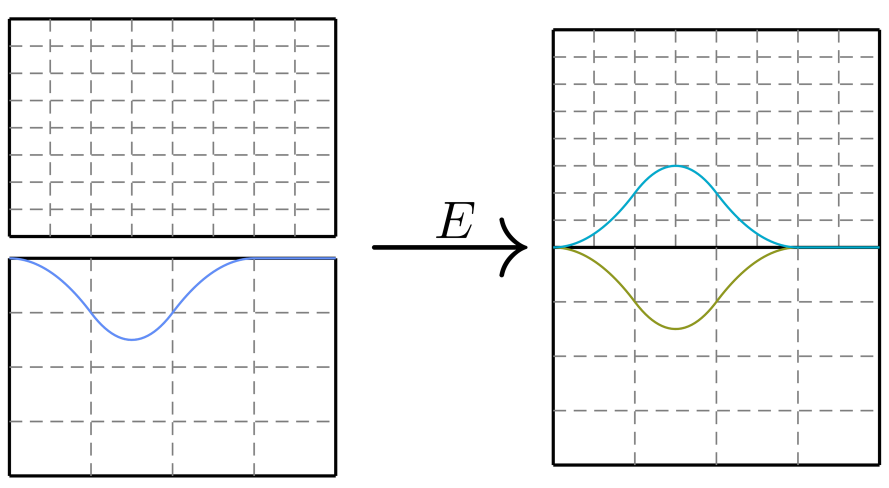
EXAMPLE SETUP

For the construction of the scheme, consider two patches with **nested patch-wise tensor-product** discretizations of basis functions $(\Lambda^{c/f}_{i,j})$. We define the conforming projection P^0



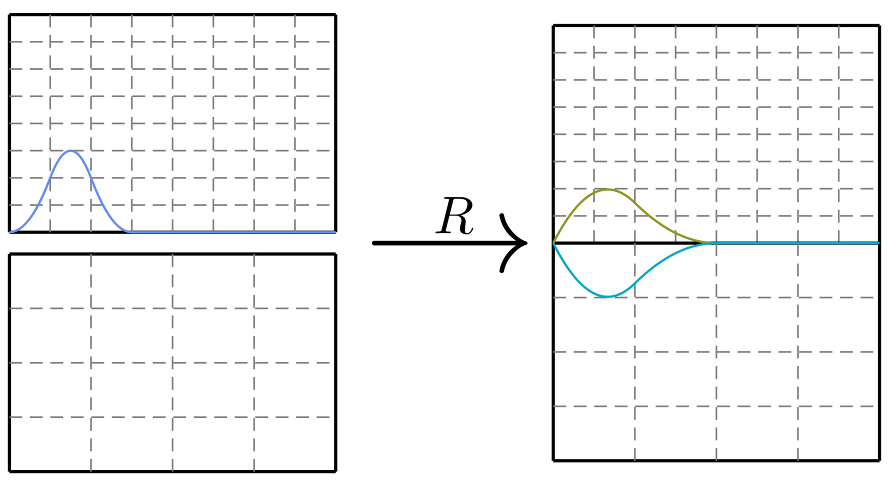
as a combination of the **extension** operator

$$E : V^{c,\Gamma} \to V^{\Gamma} : \lambda^{0,c}_i \mapsto \lambda^{0,c}_i = \sum_{j=0}^{n^f} \mathbb{E}_{j,i} \lambda^{0,f}_j,$$



i.e. a **change of basis**, and the **coarsening** operator

$$R : V^{f,\Gamma} \to V^{\Gamma} : \lambda^{0,f}_i \mapsto \sum_{j=0}^{n^c} \mathbb{R}_{j,i} \lambda^{0,c}_j = \sum_{j=0}^{n^f} (\mathbb{E}\mathbb{R})_{j,i} \lambda^{0,f}_j,$$



that corresponds to a **local projection** to the coarse space.

IMPLEMENTATION

- Extension matrix $\mathbb{E}$ by **knot-insertion** of B-Splines to calculate the interpolation coefficients/ change of basis.
- Coarsening matrix as a **left inverse** $\mathbb{R} = (\mathbb{E}^T\mathbb{E})^{-1}\mathbb{E}^T$ or by a local B-Spline projection.

COMMUTING DIAGRAM

$$\begin{tikzcd}
V^0 = H^1(\Omega) \arrow[r, "\mathrm{grad}"] \arrow[d, "\Pi^0_{\text{pw}}"] \arrow[dd, bend right, "\Pi^0"'] & V^1 = H(\mathrm{curl}, \Omega) \arrow[d, "\Pi^1_{\text{pw}}"] \arrow[dd, bend left, "\Pi^1"] \\
V^0_{\text{pw}} = \mathbb{S}^p \otimes \mathbb{S}^p \arrow[r, "\mathbb{G}_{\text{pw}}"] \arrow[d, "P^0"] & V^1_{\text{pw}} = \begin{pmatrix} \mathbb{S}^{p-1} \otimes \mathbb{S}^p \\ \mathbb{S}^p \otimes \mathbb{S}^{p-1} \end{pmatrix} \arrow[d, dashed, "P^1"] \\
V^0_{\text{conf}} = V^0_{\text{pw}} \cap H^1(\Omega) \arrow[r, "\mathbb{G}"] & V^1_{\text{conf}} = V^1_{\text{pw}} \cap H(\mathrm{curl}, \Omega)
\end{tikzcd}$$

The first line describes the continuous spaces and operator. The second line consists of the discrete **patch-wise** defined spaces and a gradient block-matrix. Here, we want to perform our calculations in practice as all objects are defined in a patch-wise manner, that leads to **block-wise** mass matrices for example. The third line is the discrete conforming space, that gives raise to the **theoretically proven** global FEEC structure, which provides the desired **structure-preserving** properties. The **patch-wise** projections $\Pi^0_{\text{pw}}, \Pi^1_{\text{pw}}$ and global **projections** Π^0, Π^1 are defined to be L^2**-stable** and **commuting**. The conforming projections P^0, P^1 are only used for computation and **do not** need to be commuting or L^2-stable, but for certain applications should be moment-preserving.

CONFORMING PROJECTIONS

Following the construction in the left column:

$$P^0\Lambda^{0,f}_{(i,j)} = \begin{cases} \Lambda^{0,f}_{(i,j)}, & j > 0, \\ \frac{1}{2}\left(\sum_k \mathbb{R}_{k,i}\Lambda^{0,c}_{k,n^c} + \sum_k (\mathbb{E}\mathbb{R})_{k,i}\Lambda^{0,f}_{k,0}\right), & j = 0, \end{cases}$$

$$P^0\Lambda^{0,c}_{(i,j)} = \begin{cases} \frac{1}{2}\left(\Lambda^{0,c}_{i,n^c} + \sum_k \mathbb{E}_{k,i}\Lambda^{0,f}_{k,0}\right), & j = n^c, \\ \Lambda^{0,c}_{(i,j)}, & j < n^c, \end{cases}$$

or more abstractly written as the **block-matrix** between the coarse and fine interface degrees of freedom:

$$P^0|_{\Gamma} = \frac{1}{2}\begin{pmatrix} I & R \\ E & ER \end{pmatrix} \otimes I, \quad \text{and} \quad P^0|_{\text{int}(\Omega)} = I.$$

We define P^1 analogously, which will be **not** commuting .

COMMUTING PROJECTIONS

In order to **theoretically prove** the **structure-preserving and stability** properties of our adaptive discretization, we need to **couple the patch-wise FEEC** structure. This is achieved by defining projectors, which are commuting and L^2-stable.[4]
In our particular example, this takes the form of

$$\Pi^0 = P^0\Pi^0_{\text{pw}}, \quad \Pi^1 = \Pi^1_{\text{pw}} + \tilde{\Pi}^1_{\Gamma}, \quad \text{where} \quad \Pi_{\text{pw}} = \Pi^c + \Pi^f,$$

with the V^1 patch-wise projection and edge correction defined as

$$\Pi^1_k \boldsymbol{u} = \sum_{d\in\{1,2\}} \nabla^k_d \Pi^0_k \Phi^k_d(\boldsymbol{u}), \quad \tilde{\Pi}^1_e \boldsymbol{u} := \sum_{d\in\{\parallel,\perp\}} \nabla^e_d (P_e - I_e)\Pi^0_{\text{pw}}\Phi^e_d(\boldsymbol{u})$$

and suitable antiderivative operators Φ.

NUMERICAL EXPERIMENTS

Curl curl eigenvalue problem (curved L-shape domain)

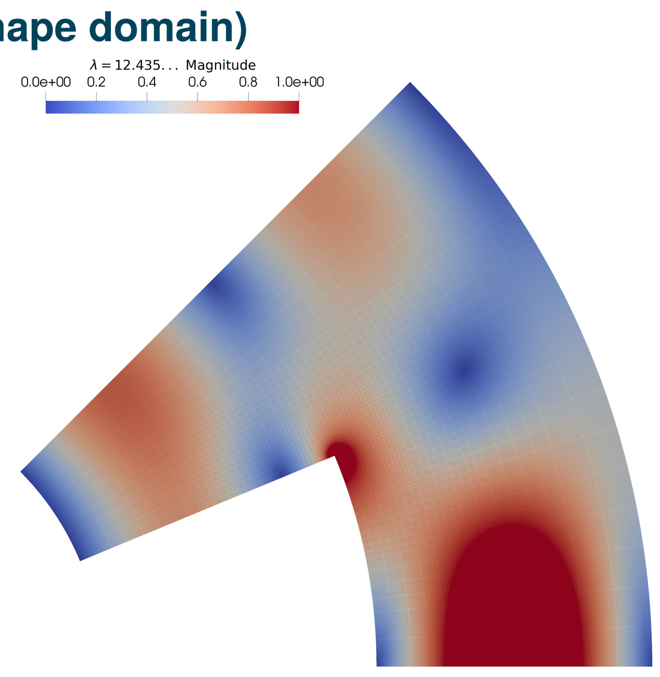
Find $\lambda > 0$

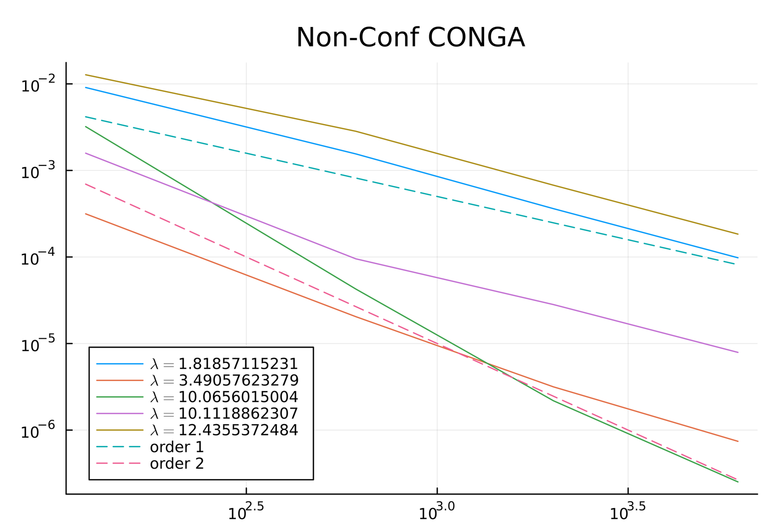
$$\mathbf{curl}\ \mathrm{curl}\ \boldsymbol{E} = \lambda \boldsymbol{E},$$

where $\boldsymbol{E} \in H_0(\mathrm{curl}, \Omega)$. Discretized in this framework, it reads

$$(\mathbb{P}^1)^T \mathbb{C}^T_{\text{pw}} \mathbb{M}^2 \mathbb{C}_{\text{pw}} \mathbb{P}^1 \ E_h = \lambda_h (\mathbb{P}^1)^T \mathbb{M}^1 \mathbb{P}^1 \ E_h,$$

where $\mathbb{P}^1$ is the matrix of the conforming projection P^1 and the patch-wise curl matrix $\mathbb{C}_{\text{pw}}$ and mass-block-matrices $\mathbb{M}^1, \mathbb{M}^2$.





Time-harmonic Maxwell equation (Pretzel domain)

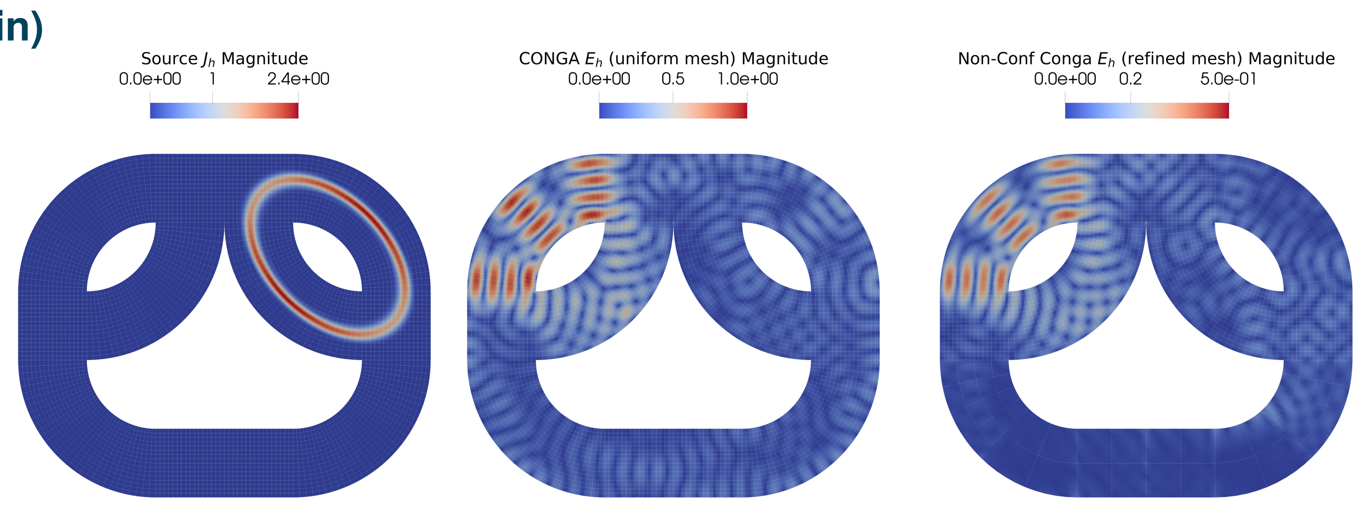
For $\omega \in \mathbb{R}$ and $\boldsymbol{J} \in L^2(\Omega)$, solve

$$-\omega^2 \boldsymbol{E} + \mathbf{curl}\ \mathrm{curl}\ \boldsymbol{E} = \boldsymbol{J},$$

where $\boldsymbol{E} \in H_0(\mathrm{curl}, \Omega)$. In our framework, that reads

$$\left(-\omega^2 (\mathbb{P}^1)^T \mathbb{M}^1 \mathbb{P}^1 + (\mathbb{P}^1)^T \mathbb{C}^T_{\text{pw}} \mathbb{M}^2 \mathbb{C}_{\text{pw}} \mathbb{P}^1\right) E_h = \mathbb{P}^1 J_h,$$

where J_h is a projection of $\boldsymbol{J}$ to $V^1_{\text{pw}}(\Omega)$.



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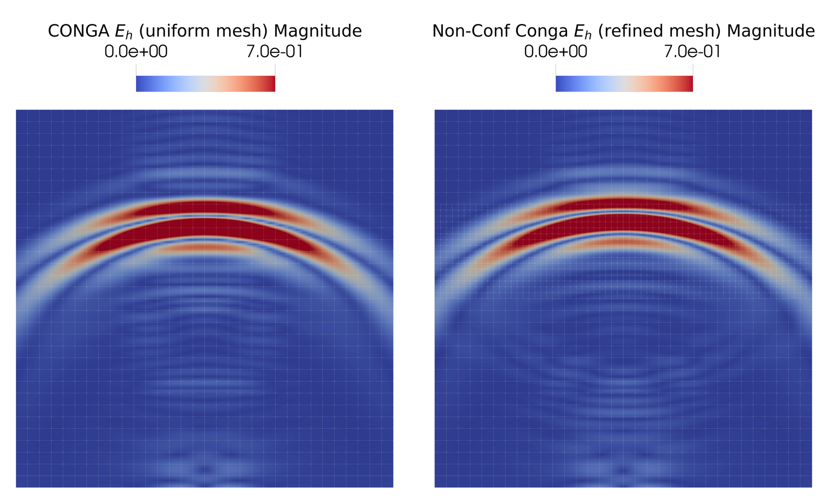
Time domain source-free Maxwell equations (Square domain)

For $\boldsymbol{E} \in H_0(\mathrm{curl}, \Omega)$ and $B \in L^2(\Omega)$ solve

$$\partial_t \boldsymbol{E} - \mathbf{curl}\ B = 0,$$
$$\partial_t B + \mathrm{curl}\ \boldsymbol{E} = 0.$$

With a simple leap-frog time-stepping scheme, the discrete scheme yields

$$B^{n+\frac{1}{2}}_h = B^n_h - \frac{\Delta t}{2}\mathbb{C}_{\text{pw}} E^n_h,$$
$$\mathbb{M}^1 E^{n+1}_h = \mathbb{M}^1 E^n_h + \Delta t (\mathbb{P}^1)^T \mathbb{C}^T_{\text{pw}} \mathbb{M}^2 B^{n+\frac{1}{2}}_h,$$
$$B^{n+1}_h = B^{n+\frac{1}{2}}_h - \frac{\Delta t}{2}\mathbb{C}_{\text{pw}} E^{n+1}_h.$$



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SUMMARY

We showed a brief derivation of our numerical scheme in a simplified setup on two patches, that is motivated by the recent theoretical work in [4]. A first implementation of this method was performed in PSYDAC, which can already handle patch refinements for multiple patches with different configurations on more involved geometries. The numerical results look very promising and let us believe, that the current theoretical work can be extended further.