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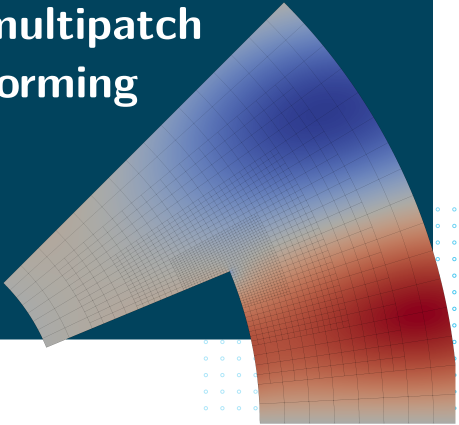
FÜR PLASMAPHYSIK

Broken-FEEC schemes on multipatch spline spaces with non-conforming refinement

ECCOMAS CONGRESS 2024



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Outline

1. Motivation: FEEC

2. Broken FEEC

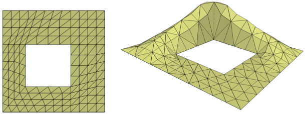
3. Non-matching Interfaces

4. Numerical Experiments

5. Outlook

Pitfalls of continuous Finite Elements[1]: holes and corners

- Hodge-Laplace eigenvalue problem: $(\mathrm{curl}_0\,\mathrm{curl} - \mathrm{grad}\,\mathrm{div}_0)\boldsymbol{u} = \lambda \boldsymbol{u}$



# Elements	Degree 1 λ_1	Degree 1 λ_2	Degree 3 λ_1	Degree 3 λ_2
256	2.270	2.360	1.896	1.970
1,024	2.050	2.132	1.854	1.925
4,096	1.940	2.016	1.828	1.897
16,384	1.879	1.952	1.812	1.880
65,536	1.843	1.914	1.802	1.870
262,144	1.821	1.890	1.796	1.863

Continuous FEM

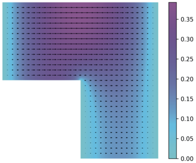
# Elements	Degree 1 λ_1	Degree 1 λ_2	Degree 3 λ_1	Degree 3 λ_2
256	0.000	0.638	0.000	0.619
1,024	0.000	0.625	0.000	0.618
4,096	0.000	0.620	0.000	0.617
16,384	0.000	0.618	0.000	0.617
65,536	0.000	0.618	0.000	0.617
262,144	0.000	0.617	0.000	0.617

FEEC

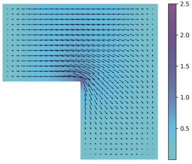
▷ with holes: non trivial harmonic fields ($\lambda = 0$), dimension = topological invariant and missed by continuous FEM

- Hodge-Laplace source problem

▷ with corners: singular solutions $\boldsymbol{u} \notin H^1$, no convergence with continuous FEM



Continuous FEM



FEEC

[1]examples taken from Arnold ('18)

Structure-preserving Finite Elements / FEEC

- sequence of spaces with commuting projection operators[1]

$$
\begin{array}{ccccccc}
H^1_0 & \xrightarrow{\text{grad}} & H_0(\text{curl}) & \xrightarrow{\text{curl}} & H_0(\text{div}) & \xrightarrow{\text{div}} & L^2 \\
\downarrow \Pi^0 & & \downarrow \Pi^1 & & \downarrow \Pi^2 & & \downarrow \Pi^3 \\
V^0_h & \xrightarrow{\text{grad}} & V^1_h & \xrightarrow{\text{curl}} & V^2_h & \xrightarrow{\text{div}} & V^3_h
\end{array}
$$

- FEEC: unified analysis $\to$ key element are the stable commuting projections[2] Π^ℓ
- eigenvalue problem:
  - convergence of eigenmodes
  - correct dim of discrete harmonic fields
  - discrete Hodge-Helmholtz decomp.
- source problem:
  - convergence (even for singular sol.)
  - structure preservation (Gauss laws)
  - time domain: long time stability

[1] Whitney (57) Nédélec (80) Bossavit (88-98) Hiptmair (99) Arnold-Falk-Winther (06) Buffa et al (11) ...
[2] Schöberl (05) AFW (06) Christiansen-Winther (07) Falk-Winther (14) Arnold-Guzman (21)

The commuting projections of Buffa-Rivas-Sangalli-Vázquez (2011)

- construction on tensor-product spaces (here in 2D)

$$V_h^0 = \mathbb{S}_p \otimes \mathbb{S}_p \xrightarrow{\text{grad}} V_h^1 = \begin{pmatrix} \mathbb{S}_{p-1} \otimes \mathbb{S}_p \\ \mathbb{S}_p \otimes \mathbb{S}_{p-1} \end{pmatrix} \xrightarrow{\text{curl}} V_h^3 = \mathbb{S}_{p-1} \otimes \mathbb{S}_{p-1}$$

- On V_h^0:

$$\Pi^0 \phi := \sum_{\boldsymbol{i} \in \mathcal{I}^0} c_{\boldsymbol{i}}^0(\phi) \Lambda_{\boldsymbol{i}}^0 \qquad \text{with} \qquad c_{\boldsymbol{i}}^0(\phi) := \langle \Theta_{\boldsymbol{i}}^0, \phi \rangle$$

- On V_h^1 :

$$\Pi^1 \boldsymbol{u} := \sum_{d \in \{1,2\}} \nabla_d \Pi^0 \Phi_d(\boldsymbol{u}) \qquad \text{with} \qquad \begin{cases} \Phi_1(\boldsymbol{u})(\boldsymbol{x}) := \int_0^{x_1} u_1(z, x_2) \, \mathrm{d}z \\ \Phi_2(\boldsymbol{u})(\boldsymbol{x}) := \int_0^{x_2} u_2(x_1, z) \, \mathrm{d}z \end{cases}$$

- On V_h^2 :

$$\Pi^2 \rho := \partial_1 \partial_2 \Pi^0 \Psi(\rho) \qquad \text{with} \qquad \Psi(\rho)(\boldsymbol{x}) := \int_0^{x_1} \int_0^{x_2} \rho(z_1, z_2) \, \mathrm{d}z_2 \, \mathrm{d}z_1$$

Broken FEEC approach[1]

- broken (patch-wise) spaces: $V^\ell_{\rm pw} := \{v \in L^2(\Omega) : v|_{\Omega_k} \in V^\ell_k \text{ for } k \in \mathcal{K}\}$
- patch-wise differential operators

$$V^0_{\rm pw} \xrightarrow{\ \mathrm{grad}_{\rm pw}\ } V^1_{\rm pw} \xrightarrow{\ \mathrm{curl}_{\rm pw}\ } V^2_{\rm pw}$$

- conforming subspaces

$$V^0_h = V^0_{\rm pw} \cap H^1(\Omega) \xrightarrow{\ \mathrm{grad}\ } V^1_h = V^1_{\rm pw} \cap H(\mathrm{curl};\Omega) \xrightarrow{\ \mathrm{curl}\ } V^2_h = V^2_{\rm pw} \cap L^2(\Omega) = V^2_{\rm pw}.$$

- for example on V^0_h, an L^2-stable projection is given by

$$\Pi^0 := P^0 \Pi^0_{\rm pw} \qquad \text{with} \qquad \begin{cases} \Pi^0_{\rm pw} = \sum_k \Pi^0_k : L^p \to V^0_{\rm pw} & \text{(stable, patch-wise)} \\ \qquad\quad P^0 : V^0_{\rm pw} \to V^0_h & \text{(averages interface dofs)} \end{cases}$$

[1] Güçlü, Hadjout, Campos Pinto (22)

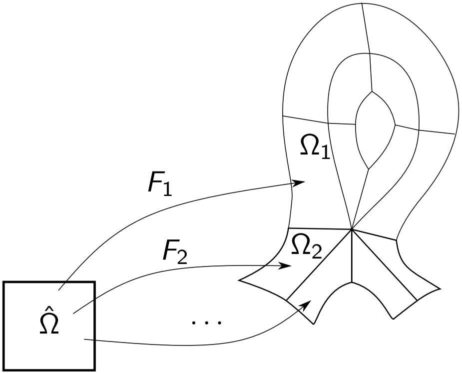
Broken commuting diagram with Splines in 2D

$$
\begin{tikzcd}
V^0 = H^1(\Omega) \arrow[r, "\mathrm{grad}"] \arrow[d, "\Pi^0_{\mathrm{pw}}"] \arrow[dd, bend right, "\Pi^0"'] & V^1 = H(\mathrm{curl}, \Omega) \arrow[r, "\mathrm{curl}"] \arrow[d, "\Pi^1_{\mathrm{pw}}"] \arrow[dd, bend left, "\Pi^1"] & V^2 = L^2(\Omega) \arrow[d, "\Pi^2_{\mathrm{pw}}"] \arrow[dd, bend left, "\Pi^2"] \\
V^0_{\mathrm{pw}} = \mathbb{S}^p \otimes \mathbb{S}^p \arrow[r, "\mathbb{G}_{\mathrm{pw}}"] \arrow[d, "P^0"] & V^1_{\mathrm{pw}} = \begin{pmatrix} \mathbb{S}^{p-1} \otimes \mathbb{S}^p \\ \mathbb{S}^p \otimes \mathbb{S}^{p-1} \end{pmatrix} \arrow[r, "\mathbb{C}_{\mathrm{pw}}"] \arrow[d, dashed, "P^1"] & V^2_{\mathrm{pw}} = \mathbb{S}^{p-1} \otimes \mathbb{S}^{p-1} \arrow[d, dashed, "\mathrm{id}"] \\
V^0_h = V^0_{\mathrm{pw}} \cap H^1(\Omega) \arrow[r, "\mathbb{G}"] & V^1_h = V^1_{\mathrm{pw}} \cap H(\mathrm{curl}, \Omega) \arrow[r, "\mathbb{C}"] & V^2_h = V^2_{\mathrm{pw}}
\end{tikzcd}
$$

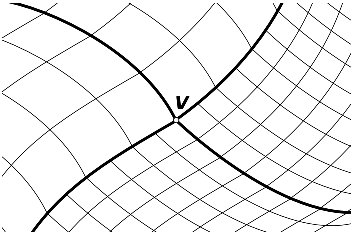
Careful: $\Pi^1 \neq P^1 \Pi^1_{\mathrm{pw}}$.

Multi-patch domains with non-matching interfaces

Adjacent patches may have different mappings . . .



. . . and different grids



Challenge: Break of the tensor-product structure near the interfaces

Existence of non-matching commuting projections

Theorem[2] If interior vertices are shared by four patches with nested discretization, there exist commuting projections $\Pi^\ell : L^p \to V_h^\ell$, local and stable for $1 \le p \le \infty$.

Broken-FEEC construction: patchwise projections + interface corrections

- on V_h^0 we define

$$\Pi^0 := P\Pi^0_{\mathrm{pw}} \quad \text{with} \quad \begin{cases} \Pi^0_{\mathrm{pw}} = \sum_{k\in\mathcal{K}} \Pi^0_k : L^p \to V^0_{\mathrm{pw}} & \text{(stable, patch-wise)} \\ P : V^0_{\mathrm{pw}} \mapsto V^0_h & \text{(averages interface dofs)} \end{cases}$$

- on V_h^1 we set

$$\Pi^1 := \Pi^1_{\mathrm{pw}} + \sum_{e\in\mathcal{E}} \tilde{\Pi}^1_e + \sum_{\boldsymbol{v}\in\mathcal{V}} \tilde{\Pi}^1_{\boldsymbol{v}} + \sum_{(e,\boldsymbol{v})\in\mathcal{E}\times\mathcal{V}} \tilde{\Pi}^1_{e,\boldsymbol{v}} \quad \text{with}$$

$$\tilde{\Pi}^1_g \boldsymbol{u} := \sum_{d\in\{\parallel,\perp\}} \nabla^g_d Q_g \Pi^0_{\mathrm{pw}} \Phi^g_d(\boldsymbol{u}) \quad \text{with} \quad \begin{cases} \Phi^g_d : L^p \to L^p & \text{(antiderivative)} \\ Q_g : V^0_{\mathrm{pw}} \mapsto V^0_{\mathrm{pw}} & \text{(interface jump)} \end{cases}$$

- similar construction on V_h^2

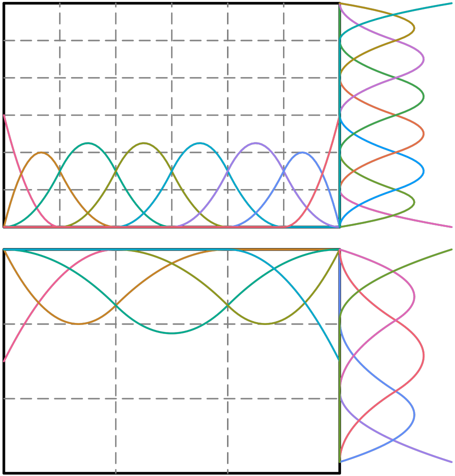
[2] Campos Pinto, S. (23) `https://arxiv.org/abs/2303.14449`

Numerical schemes - preliminaries

Given: Multiple patches, with different grids but single-patch FEEC spaces V^ℓ_{pw}.

Goal: Efficient way to map onto the conforming subspaces V^ℓ_h which have the structure-preserving properties established by the stable commuting projections above.

Idea: Establish the correct conformity depending on V^ℓ in a geometric way. (On vertices and edges sequentially for $V^0 = H^1$, tangential to edges for $V^1 = H(\mathrm{curl})$, ...)



Vertex conformity: P_{vertex}

Let $\Lambda_{\boldsymbol{i}}^k = \lambda_{i_1} \otimes \lambda_{i_2}$ denote the B-Spline basis functions on patch k.

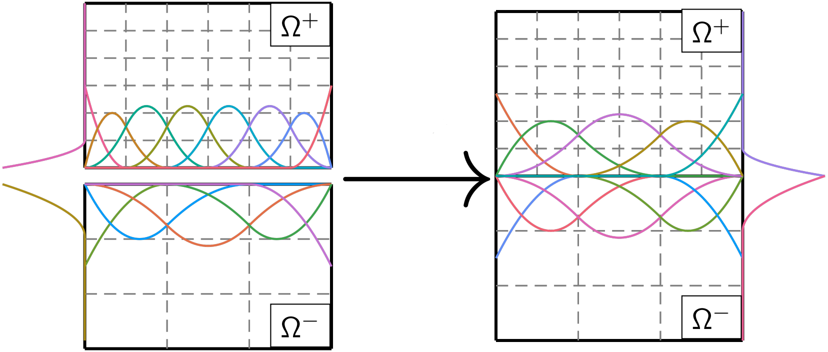
$$P_{\text{vertex}} : \Lambda_{\boldsymbol{i}}^k \mapsto \begin{cases} \dfrac{1}{|\mathcal{K}(v)|} \displaystyle\sum_{k' \in \mathcal{K}(v)} \Lambda_{\boldsymbol{i}^{k'}(v)}^{k'} & \text{if } \boldsymbol{i} = \boldsymbol{i}^k(v) \text{ for any vertex } v, \\ \Lambda_{\boldsymbol{i}}^k & \text{else,} \end{cases}$$

where $\mathcal{K}(v)$ is the collection of patches adjacent to a vertex v and $\boldsymbol{i}^k(v)$ describes the index of the basis function on patch k with support on v.

$\rightarrow$ Only modifies vertex dof, leaves interior and edge-interior unchanged.

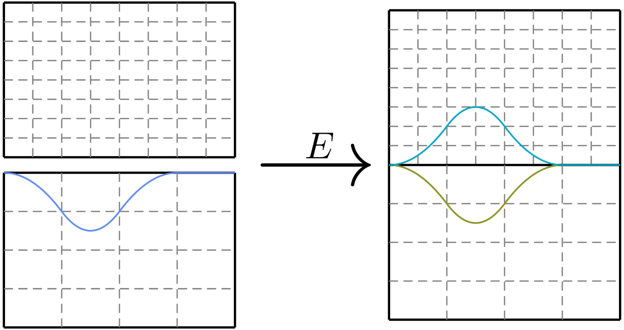
Edge conformity

Define a projection P_{edge},



that maps the basis functions on the interface to the conforming space, while leaving interior (and vertex) dof unchanged.

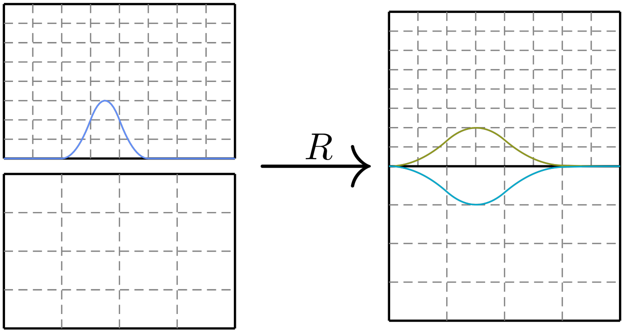
Extension operator



$$E : \lambda_i^- \otimes \lambda_0^- \mapsto \underbrace{\lambda_i^-}_{\sum_{j=0}^{n^+-1} \mathbb{E}_{i,j}\lambda_j^+} \otimes \frac{1}{2}\left(\lambda_0^- + \lambda_0^+\right)$$

Change of basis, with interpolation coefficients $\mathbb{E}_{i,j}$ obtained by knot-insertion.

Restriction operator



$$R : \lambda_i^+ \otimes \lambda_0^+ \mapsto \underbrace{\mathcal{R}\lambda_i^+}_{= \sum_{j=0}^{n^- -1} \mathbb{R}_{i,j}\lambda_j^- = \sum_{j=0}^{n^+ -1} (\mathbb{E}\mathbb{R})_{i,j}\lambda_j^+} \otimes \frac{1}{2}\left(\lambda_0^- + \lambda_0^+\right)$$

E.g. modified L^2-projection of a fine spline into the coarse spline space.

$P_{\mathbf{edge}}$

For ease of notation, assume a horizontal interface Γ:

$$P_{\text{edge}} : \begin{cases} \Lambda_{\boldsymbol{j}}^- \mapsto \begin{cases} \lambda_j^- \otimes \frac{1}{2}\left(\lambda_0^- + \lambda_0^+\right) & \text{if } 0 < \boldsymbol{j}_1 = j < N, \boldsymbol{j}_2 = 0 \text{ for any interface } \Gamma, \\ \Lambda_{\boldsymbol{j}}^- & \text{else.} \end{cases} \\ \Lambda_{\boldsymbol{j}}^+ \mapsto \begin{cases} \mathcal{R}\lambda_j^+ \otimes \frac{1}{2}\left(\lambda_0^+ + \lambda_0^-\right) & \text{if } 0 < \boldsymbol{j}_1 = j < N, \boldsymbol{j}_2 = 0 \text{ for any interface } \Gamma, \\ \Lambda_{\boldsymbol{j}}^+ & \text{else.} \end{cases} \end{cases}$$

$\rightarrow$ Analogous for vertical interfaces.

Non-matching conforming projections

Finally:

$$P^0 = P_{\mathrm{edge}} \circ P_{\mathrm{vertex}}.$$

Non-matching conforming projections

Finally:

$$P^0 = P_{\text{edge}} \circ P_{\text{vertex}}.$$

The construction for V^1 is similar, but we only enforce conformity on the tangential traces (on the whole edge including vertex dof):

$$P^1 = \bar{P}_{\text{edge}}.$$

Preservation of polynomial moments

Some properties of the projectors are improved by preserving polynomial moments:

$$\int P\phi q_j = \int \phi q_j, \qquad \forall \phi \in V_h^0, q_j \in \mathbb{P}_j, \quad j = 0, \dots, r$$

$\longrightarrow$ Introduces a set of linear constraints on the coefficients of P^0 and P^1.

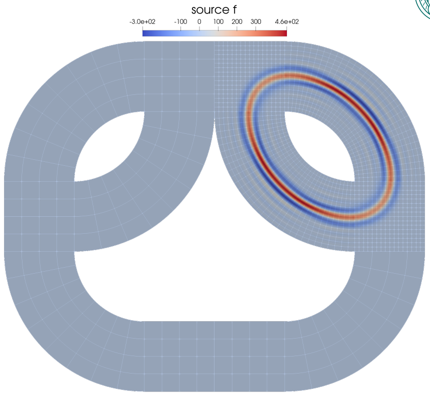
Poisson problem (pretzel domain)

Find $\phi \in V^0 =\in H^1_0(\Omega)$ such that

$$-\Delta\phi = f,$$

where $f \in L^2(\Omega)$.

The domain consists of 18 patches, we only refine the top-right handle (6 patches).



Within the broken-FEEC framework[1], this equation is discretized as

$$\left(\mathbb{G}_{\text{pw}}\mathbb{P}^0\right)^T \mathbb{M}^1\mathbb{G}_{\text{pw}}\mathbb{P}^0 + \underbrace{(\mathbb{I}-\mathbb{P}^0)^T\mathbb{M}^0(\mathbb{I}-\mathbb{P}^0)}_{\text{jump stabilization}}\ \bar{\phi}_h = (\mathbb{P}^0)^T\bar{f}_h.$$

[1] All numerical examples are performed using `https://github.com/pyccel/psydac`

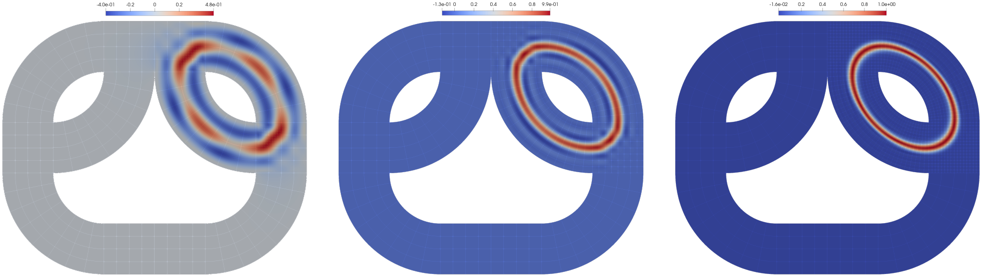


Figure: Numerial solution ϕ_h for degree 3 and refinement of the top-right handle.

Relative L^2-errors for:

Cells p.p. \ Degree	uniform refinement 2	3	4	5	refined handle 2	3	4	5
4	6.71637	0.86527	0.34468	0.18317	6.71637	0.86527	0.34468	0.18317
8	0.31598	0.04601	0.08671	0.05566	0.31596	0.04606	0.08675	0.05573
16	0.01516	0.00489	0.00575	0.00361	0.01516	0.00489	0.00575	0.00361

$\rightarrow$ Only half the dof needed for the same accuracy on the finest level.

Time-harmonic Maxwell equation (pretzel domain)

For $\omega \in \mathbb{R}$ and $\boldsymbol{J} \in L^2(\Omega)$, solve

$$-\omega^2 \boldsymbol{E} + \mathbf{curl}\,\mathrm{curl}\ \boldsymbol{E} = \boldsymbol{J},$$

for $\boldsymbol{E} \in H(\mathrm{curl}, \Omega)$.

In our framework, that reads

$$\left(-\omega^2 (\mathbb{P}^1)^T \mathbb{M}^1 \mathbb{P}^1 + (\mathbb{C}_{\mathrm{pw}} \mathbb{P}^1)^T \mathbb{M}^2 \mathbb{C}_{\mathrm{pw}} \mathbb{P}^1 + \underbrace{(I - \mathbb{P}^1)^T \mathbb{M}^1 (I - \mathbb{P}^1)}_{\text{jump stabilization}} \right) \bar{\boldsymbol{E}}_h = (\mathbb{P}^1)^T \bar{\boldsymbol{J}}_h.$$

Manifactured solution: $\quad \boldsymbol{J} = \begin{pmatrix} -\pi^2 \sin(\pi y)\cos(\pi x) \\ 0 \end{pmatrix}, \qquad \boldsymbol{E} = \begin{pmatrix} \sin(\pi y) \\ \sin(\pi x)\cos(\pi y) \end{pmatrix}.$

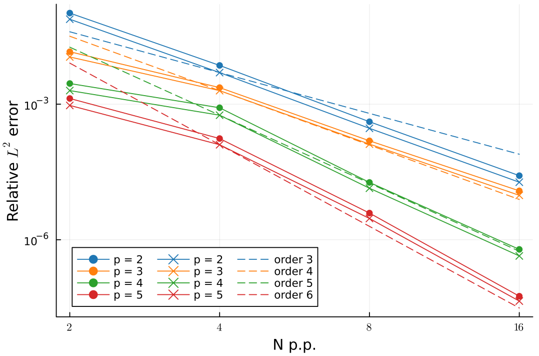


Figure: Convergence curves for matching (circle) and refined (cross) grids, where every other grid was of different resolution.

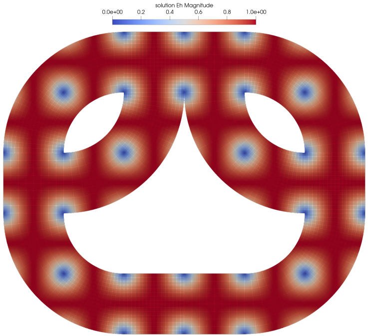


Figure: Numerical solution $\|\boldsymbol{E_h}\|$ for a refined grid between $N = 8, 16$ and $p = 3$.

Curl curl eigenvalue problem (curved L-shape domain)

Find $\lambda > 0$ such that

$$\mathbf{curl}\ \mathrm{curl}\ \mathbf{E} = \lambda \mathbf{E},$$

where $\mathbf{E} \in H_0(\mathrm{curl}, \Omega)$.

Within the broken-FEEC framework, this equation is discretized as

$$\left(\mathbb{C}_{\mathrm{pw}}\mathbb{P}^1\right)^T \mathbb{M}^2 \mathbb{C}_{\mathrm{pw}}\mathbb{P}^1 \ \bar{\mathbf{E}}_h = \lambda \left(\underbrace{(\mathbb{I} - \mathbb{P}^1)^T \mathbb{M}^1 (\mathbb{I} - \mathbb{P}^1)}_{\text{jump stabilization}} + (\mathbb{P}^1)^T \mathbb{M}^1 \mathbb{P}^1 \right) \ \bar{\mathbf{E}}_h,$$

where we refine layers around the reentrant corner and compare it to a uniform solution with similar amount of dof.

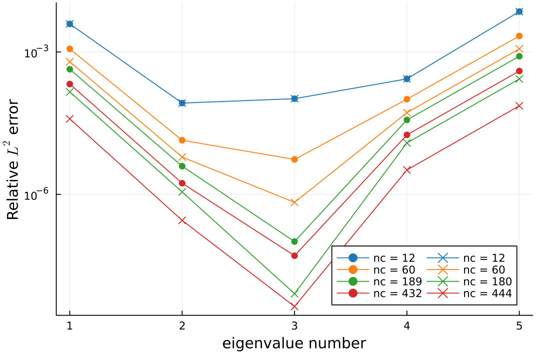


Figure: Error of each eigenvalue for matching (circle) and refined (cross) patches.

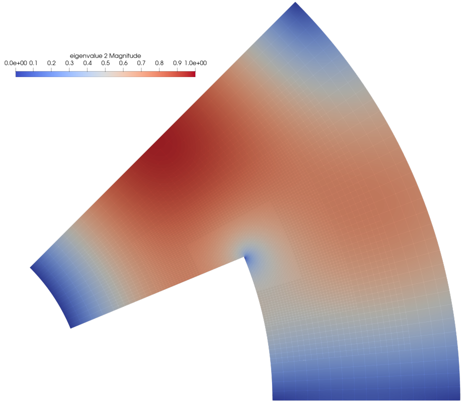


Figure: Absolute value of the eigenfunction corresponding to the eigenvalue λ_2 on the most refined grid.

Outlook

- Time-dependent problems
  $\rightarrow$ Issues with reflecting waves on refined interfaces

- Higher dimensions
  $\rightarrow$ In theory and practice

- Proper benchmarking against DG methods

Thank you for your attention!

Backup Slides

Time domain Maxwell equations (Square domain)

For $\boldsymbol{J} \in L^2(\Omega)$, solve

$$
\begin{aligned}
\partial_t \boldsymbol{E} - \textbf{curl}\ B &= -\boldsymbol{J}, \\
\partial_t B + \mathrm{curl}\ \boldsymbol{E} &= 0,
\end{aligned}
$$

where $\boldsymbol{E} \in H_0(\mathrm{curl}, \Omega)$ and $B \in L^2(\Omega)$.

Casted into a simple leap-frog time-stepping scheme, this reads

$$
\begin{aligned}
B_h^{n+\frac{1}{2}} &= B_h^n - \frac{\Delta t}{2} \mathbb{C}_{\mathrm{pw}} E_h^n, \\
\mathbb{M}^1 E_h^{n+1} &= \mathbb{M}^1 E_h^n + \Delta t \left((\mathbb{P}^1)^T \mathbb{C}_{\mathrm{pw}}^T \mathbb{M}^2 B_h^{n+\frac{1}{2}} - \mathbb{P}^1 J_h^{n+\frac{1}{2}} \right), \\
B_h^{n+1} &= B_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \mathbb{C}_{\mathrm{pw}} E_h^{n+1}.
\end{aligned}
$$

Electromagnetic wave (without source)

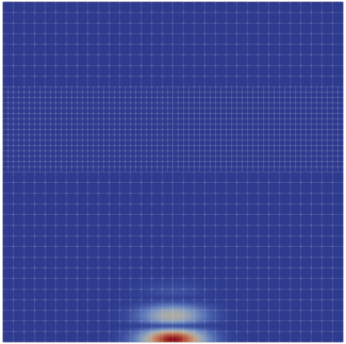
In our example, we choose the initial conditions

$$\boldsymbol{E}(t=0;\boldsymbol{x}) = \boldsymbol{p}\ \cos(\boldsymbol{k}\cdot\boldsymbol{x})\ e^{-\frac{x^2}{2\sigma^2}},$$

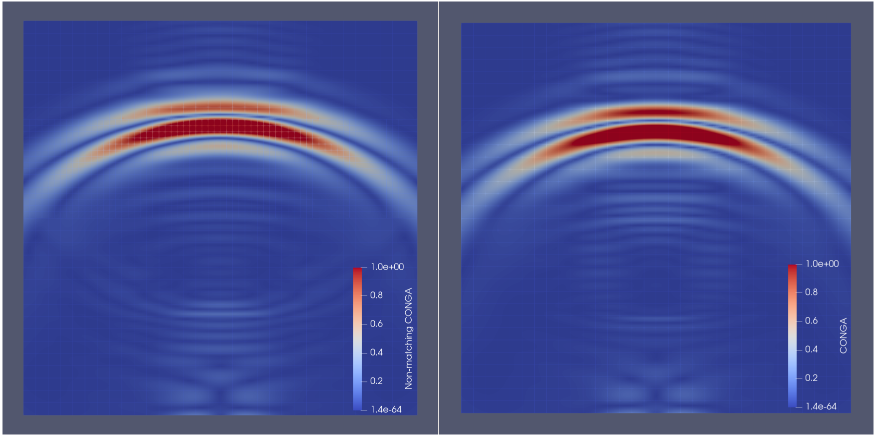
$$\boldsymbol{B}(t=0;\boldsymbol{x}) = \boldsymbol{k}\times\boldsymbol{p}\ \cos(\boldsymbol{k}\cdot\boldsymbol{x})\ e^{-\frac{x^2}{2\sigma^2}},$$

$$\boldsymbol{J}(t,\boldsymbol{x}) \equiv 0.$$

See the initial magnitude of E_h^0 on the right.



Electro-magnetic wave traveling I



Electro-magnetic wave traveling II