Broken-FEEC on multipatch domains with local refinements

RICAM Special Semester 2025
Advances in Isogeometric Analysis

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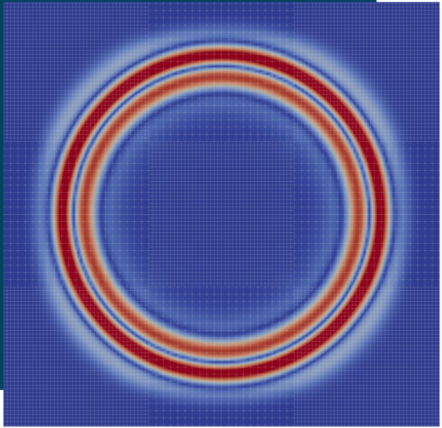

Outline

Motivation

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Solve Hodge-Laplace problems (Poisson, Maxwell, ...) on various domains.

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Key Questions:

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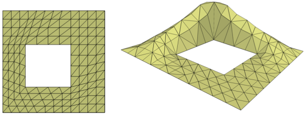
Q1: How to handle complex domains? On multiple patches (of different resolution)?

Q2: How to obtain a priori convergence / stability / spectral correctness results?

Pitfalls of continuous Finite Elements[1]: holes and corners

- Hodge-Laplace eigenvalue problem: $(\mathrm{curl}_0\, \mathrm{curl} - \mathrm{grad}\, \mathrm{div}_0)\boldsymbol{u} = \lambda \boldsymbol{u}$



# Elements	Degree 1 λ_1	Degree 1 λ_2	Degree 3 λ_1	Degree 3 λ_2
256	2.270	2.360	1.896	1.970
1,024	2.050	2.132	1.854	1.925
4,096	1.940	2.016	1.828	1.897
16,384	1.879	1.952	1.812	1.880
65,536	1.843	1.914	1.802	1.870
262,144	1.821	1.890	1.796	1.863

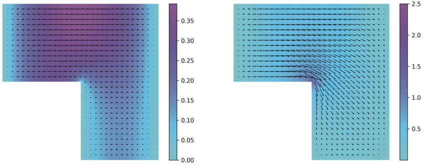
Continuous FEM

# Elements	Degree 1 λ_1	Degree 1 λ_2	Degree 3 λ_1	Degree 3 λ_2
256	0.000	0.638	0.000	0.619
1,024	0.000	0.625	0.000	0.618
4,096	0.000	0.620	0.000	0.617
16,384	0.000	0.618	0.000	0.617
65,536	0.000	0.618	0.000	0.617
262,144	0.000	0.617	0.000	0.617

FEEC

- ▷ with holes: non trivial harmonic fields ($\lambda = 0$), dimension = topological invariant and missed by continuous FEM

- Hodge-Laplace source problem
- ▷ with corners: singular solutions $\boldsymbol{u} \notin H^1$, no convergence with continuous FEM



Continuous FEM FEEC

[1] examples taken from Arnold (18)

Structure-preserving Finite Elements / FEEC

- sequence of spaces with commuting projection operators[1]

$$
\begin{array}{ccccccc}
H^1_0 & \xrightarrow{\text{grad}} & H_0(\text{curl}) & \xrightarrow{\text{curl}} & H_0(\text{div}) & \xrightarrow{\text{div}} & L^2 \\
\downarrow \Pi^0 & & \downarrow \Pi^1 & & \downarrow \Pi^2 & & \downarrow \Pi^3 \\
V_h^0 & \xrightarrow{\text{grad}} & V_h^1 & \xrightarrow{\text{curl}} & V_h^2 & \xrightarrow{\text{div}} & V_h^3
\end{array}
$$

- FEEC: unified analysis $\rightarrow$ key element are the stable commuting projections[2] Π^ℓ

- eigenvalue problem:
  - convergence of eigenmodes
  - correct dim of discrete harmonic fields
  - discrete Hodge-Helmholtz decomp.

- source problem:
  - convergence (even for singular sol.)
  - structure preservation (Gauss laws)
  - time domain: long time stability

[1] Whitney (57) Nédélec (80) Bossavit (88-98) Hiptmair (99) Arnold-Falk-Winther (06) Buffa et al (11) ...

[2] Schöberl (05) AFW (06) Christiansen-Winther (07) Falk-Winther (14) Arnold-Guzman (21)

Locally refined meshes I

- Local treatment of computationally demanding regions (e.g. reentrant corners)
- Reduction of computation time

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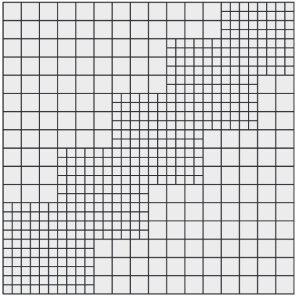
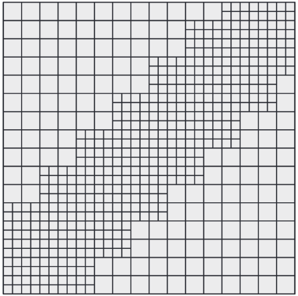


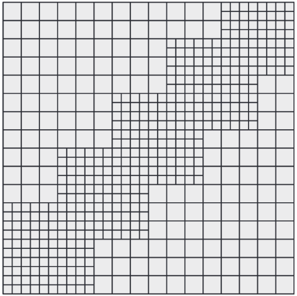


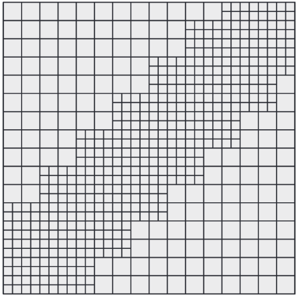
Evans, J et al. (18): Constructs a framework of hierarchical B-spline complexes of discrete differential forms, which offer general refinements but the conditions for spectral correctness (and proper cohomology) are not fully understood yet.

Locally refined meshes I

- Local treatment of computationally demanding regions (e.g. reentrant corners)
- Reduction of computation time

Assumption 1: ✓
Assumption 2: X
Exact sequence: X
Spec. correct: ✓





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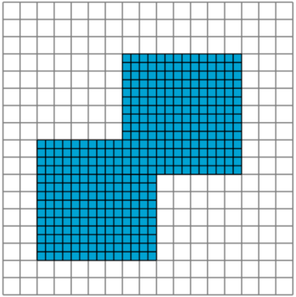

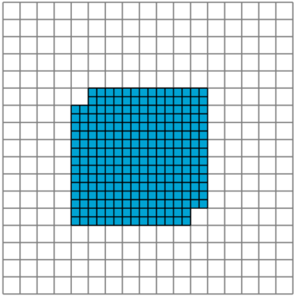

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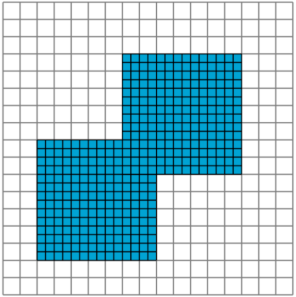

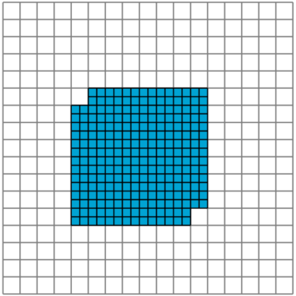

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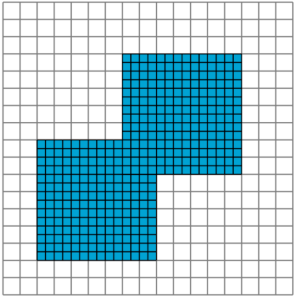

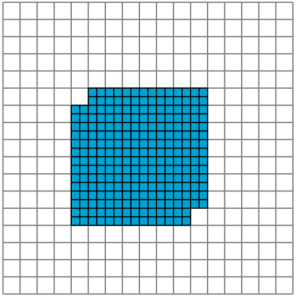

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$\Longrightarrow$ Both focus on the identification of discrete spaces that form an exact sequence.
We try to show the existence of commuting projections to ensure stability.

The commuting projections of Buffa-Rivas-Sangalli-Vázquez (11)

- construction on tensor-product spaces (here in 2D)

$$V_h^0 = \mathbb{S}_p \otimes \mathbb{S}_p \xrightarrow{\text{grad}} V_h^1 = \begin{pmatrix} \mathbb{S}_{p-1} \otimes \mathbb{S}_p \\ \mathbb{S}_p \otimes \mathbb{S}_{p-1} \end{pmatrix} \xrightarrow{\text{curl}} V_h^3 = \mathbb{S}_{p-1} \otimes \mathbb{S}_{p-1}$$

- On V_h^0:

$$\Pi^0 \phi := \sum_{\boldsymbol{i} \in \mathcal{I}^0} c_{\boldsymbol{i}}^0(\phi) \Lambda_{\boldsymbol{i}}^0 \qquad \text{with} \qquad c_{\boldsymbol{i}}^0(\phi) := \langle \Theta_{\boldsymbol{i}}^0, \phi \rangle$$

- On V_h^1 :

$$\Pi^1 \boldsymbol{u} := \sum_{d \in \{1,2\}} \nabla_d \Pi^0 \Phi_d(\boldsymbol{u}) \qquad \text{with} \qquad \begin{cases} \Phi_1(\boldsymbol{u})(\boldsymbol{x}) := \int_0^{x_1} u_1(z, x_2)\,\mathrm{d}z \\ \Phi_2(\boldsymbol{u})(\boldsymbol{x}) := \int_0^{x_2} u_2(x_1, z)\,\mathrm{d}z \end{cases}$$

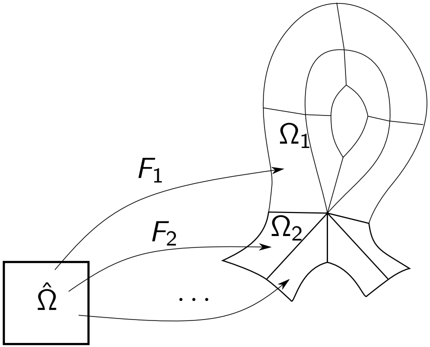
- On V_h^2 :

$$\Pi^2 \rho := \partial_1 \partial_2 \Pi^0 \Psi(\rho) \qquad \text{with} \qquad \Psi(\rho)(\boldsymbol{x}) := \int_0^{x_1} \int_0^{x_2} \rho(z_1, z_2)\,\mathrm{d}z_2\,\mathrm{d}z_1$$

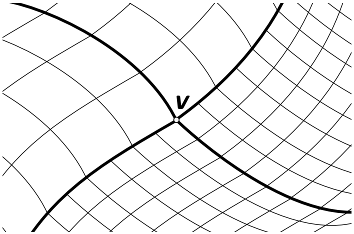
Broken-FEEC

Multi-patch domains with non-matching interfaces

Adjacent patches may have different mappings . . .



. . . and different grids



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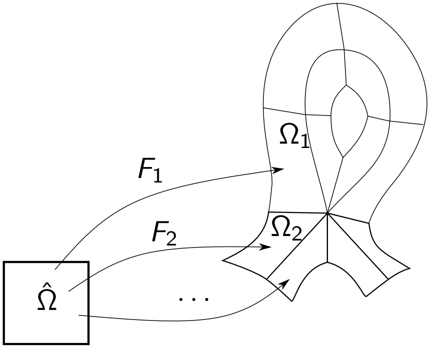

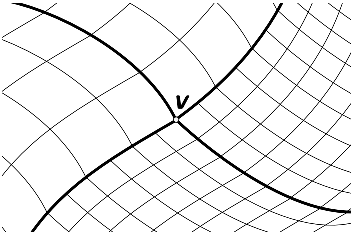

Challenge: Break of the tensor-product structure near the interfaces

$\rightarrow$ How to couple neighboring patches while preserving the FEEC structure?

Broken-FEEC approach

- broken (patch-wise) spaces: $V^\ell_{\mathrm{pw}} := \{v \in L^2(\Omega) : v|_{\Omega_k} \in V^\ell_k \text{ for } k \in \mathcal{K}\}$
- patch-wise differential operators

$$V^0_{\mathrm{pw}} \xrightarrow{\mathrm{grad}_{\mathrm{pw}}} V^1_{\mathrm{pw}} \xrightarrow{\mathrm{curl}_{\mathrm{pw}}} V^2_{\mathrm{pw}}$$

- conforming subspaces

$$V^0_h = V^0_{\mathrm{pw}} \cap H^1(\Omega) \xrightarrow{\mathrm{grad}} V^1_h = V^1_{\mathrm{pw}} \cap H(\mathrm{curl};\Omega) \xrightarrow{\mathrm{curl}} V^2_h = V^2_{\mathrm{pw}} \cap L^2(\Omega) = V^2_{\mathrm{pw}}.$$

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- for example on V^0_h, an L^2-stable projection is given by

$$\Pi^0 := P^0 \Pi^0_{\mathrm{pw}} \quad \text{with} \quad \begin{cases} \Pi^0_{\mathrm{pw}} = \sum_k \Pi^0_k : L^p \to V^0_{\mathrm{pw}} & \text{(stable, patch-wise)} \\ \qquad\qquad P^0 : V^0_{\mathrm{pw}} \to V^0_h & \text{(averages interface dofs)} \end{cases}$$

Broken commuting diagram with Splines in 2D

$$\begin{tikzcd}
V^0 = H^1(\Omega) \arrow[r, "\mathrm{grad}"] \arrow[d, "\Pi^0_{\mathrm{pw}}"] \arrow[dd, bend right, "\Pi^0"'] & V^1 = H(\mathrm{curl}, \Omega) \arrow[r, "\mathrm{curl}"] \arrow[d, "\Pi^1_{\mathrm{pw}}"] \arrow[dd, bend left, "\Pi^1"] & V^2 = L^2(\Omega) \arrow[d, "\Pi^2_{\mathrm{pw}}"] \arrow[dd, bend left, "\Pi^2"] \\
V^0_{\mathrm{pw}} = \mathbb{S}^p \otimes \mathbb{S}^p \arrow[r, "\mathbb{G}_{\mathrm{pw}}"] \arrow[d, "P^0"] & V^1_{\mathrm{pw}} = \begin{pmatrix} \mathbb{S}^{p-1} \otimes \mathbb{S}^p \\ \mathbb{S}^p \otimes \mathbb{S}^{p-1} \end{pmatrix} \arrow[r, "\mathbb{C}_{\mathrm{pw}}"] \arrow[d, dashed, "P^1"] & V^2_{\mathrm{pw}} = \mathbb{S}^{p-1} \otimes \mathbb{S}^{p-1} \arrow[d, dashed, "\mathrm{id}"] \\
V^0_{\mathrm{h}} = V^0_{\mathrm{pw}} \cap H^1(\Omega) \arrow[r, "\mathbb{G}"] & V^1_{\mathrm{h}} = V^1_{\mathrm{pw}} \cap H(\mathrm{curl}, \Omega) \arrow[r, "\mathbb{C}"] & V^2_{\mathrm{h}} = V^2_{\mathrm{pw}}
\end{tikzcd}$$

Broken commuting diagram with Splines in 2D

$$
\begin{tikzcd}
V^0 = H^1(\Omega) \arrow[r, "\mathrm{grad}"] \arrow[d, "\Pi^0_{\mathrm{pw}}"] \arrow[dd, bend right=60, "\Pi^0"'] & V^1 = H(\mathrm{curl}, \Omega) \arrow[r, "\mathrm{curl}"] \arrow[d, "\Pi^1_{\mathrm{pw}}"] \arrow[dd, bend left=60, "\Pi^1"] & V^2 = L^2(\Omega) \arrow[d, "\Pi^2_{\mathrm{pw}}"] \arrow[dd, bend left=60, "\Pi^2"] \\
V^0_{\mathrm{pw}} = \mathbb{S}^p \otimes \mathbb{S}^p \arrow[r, "\mathbb{G}_{\mathrm{pw}}"] \arrow[d, "P^0"] & V^1_{\mathrm{pw}} = \begin{pmatrix} \mathbb{S}^{p-1} \otimes \mathbb{S}^p \\ \mathbb{S}^p \otimes \mathbb{S}^{p-1} \end{pmatrix} \arrow[r, "\mathbb{C}_{\mathrm{pw}}"] \arrow[d, dashed, "P^1"] & V^2_{\mathrm{pw}} = \mathbb{S}^{p-1} \otimes \mathbb{S}^{p-1} \arrow[d, dashed, "\mathrm{id}"] \\
V^0_{\mathrm{h}} = V^0_{\mathrm{pw}} \cap H^1(\Omega) \arrow[r, "\mathbb{G}"] & V^1_{\mathrm{h}} = V^1_{\mathrm{pw}} \cap H(\mathrm{curl}, \Omega) \arrow[r, "\mathbb{C}"] & V^2_{\mathrm{h}} = V^2_{\mathrm{pw}}
\end{tikzcd}
$$

Careful: $\Pi^1 \neq P^1 \Pi^1_{\mathrm{pw}}$.

Existence of (non-matching) stable commuting projections

Theorem[1] If interior vertices are shared by four patches with nested discretization, there exist commuting projections $\Pi^\ell : L^p \to V_h^\ell$, local and stable for $1 \le p \le \infty$.

Broken-FEEC construction: patchwise projections + interface corrections

- on V_h^0 we define

$$\Pi^0 := P\Pi^0_{\mathrm{pw}} \quad \text{with} \quad \begin{cases} \Pi^0_{\mathrm{pw}} = \sum_{k\in\mathcal{K}} \Pi^0_k : L^p \to V^0_{\mathrm{pw}} & \text{(stable, patch-wise)} \\ P : V^0_{\mathrm{pw}} \mapsto V^0_h & \text{(averages interface dofs)} \end{cases}$$

- on V_h^1 we set

$$\Pi^1 := \Pi^1_{\mathrm{pw}} + \sum_{e\in\mathcal{E}} \tilde{\Pi}^1_e + \sum_{\boldsymbol{v}\in\mathcal{V}} \tilde{\Pi}^1_{\boldsymbol{v}} + \sum_{(e,\boldsymbol{v})\in\mathcal{E}\times\mathcal{V}} \tilde{\Pi}^1_{e,\boldsymbol{v}} \quad \text{with}$$

$$\tilde{\Pi}^1_g \boldsymbol{u} := \sum_{d\in\{\|,\perp\}} \nabla^g_d Q_g \Pi^0_{\mathrm{pw}} \Phi^g_d(\boldsymbol{u}) \quad \text{with} \quad \begin{cases} \Phi^g_d : L^p \to L^p & \text{(antiderivative)} \\ Q_g : V^0_{\mathrm{pw}} \mapsto V^0_{\mathrm{pw}} & \text{(interface jump)} \end{cases}$$

- similar construction on V_h^2

[1] Campos Pinto, FS (25)

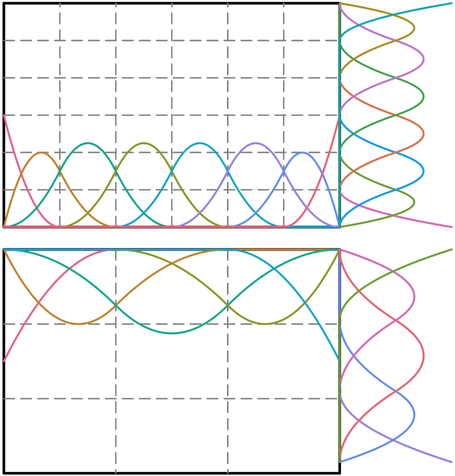
Construction of numerical schemes

Numerical schemes - preliminaries

Given: Multiple patches, with different grids but single-patch FEEC spaces V^ℓ_{pw}.

Goal: Efficient way to map onto the conforming subspaces V^ℓ_h which have the structure-preserving properties established by the stable commuting projections.

Idea: Establish the correct conformity depending on V^ℓ in a geometric way. (On vertices and edges sequentially for $V^0 = H^1$, tangential to edges for $V^1 = H(\mathrm{curl})$, ...)



Vertex conformity: $P_{\mathcal{V}}$

Let $\Lambda_{\boldsymbol{i}}^k = \lambda_{i_1}^k \otimes \lambda_{i_2}^k$ denote the B-Spline basis functions on patch k. Fix a vertex $\mathrm{v} \in \mathcal{V}$.

$$P_{\mathrm{v}} : \Lambda_{\boldsymbol{i}}^k \mapsto \begin{cases} \dfrac{1}{|\mathcal{K}(v)|} \displaystyle\sum_{k' \in \mathcal{K}(v)} \Lambda_{\boldsymbol{i}^{k'}(v)}^{k'} & \text{if } \boldsymbol{i} = \boldsymbol{i}^k(v), \\ \Lambda_{\boldsymbol{i}}^k & \text{else,} \end{cases}$$

where

- $\mathcal{K}(v)$ is the collection of patches adjacent to the vertex v,
- $\boldsymbol{i}^k(v)$ is the index of the single vertex basis function on patch k with support on v.

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where

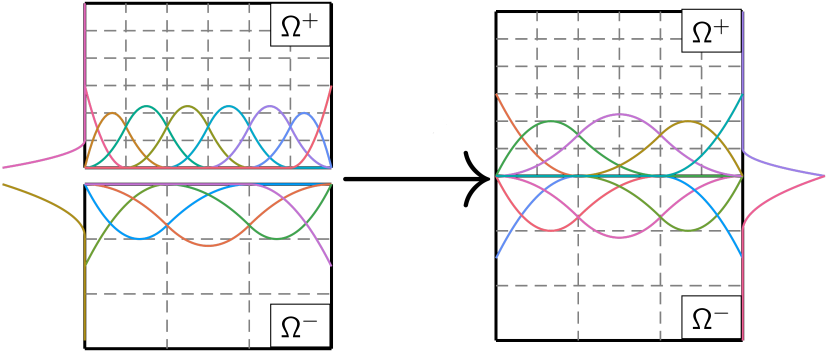
- $\mathcal{K}(v)$ is the collection of patches adjacent to the vertex v,
- $\boldsymbol{i}^k(v)$ is the index of the single vertex basis function on patch k with support on v.

$$P_{\mathcal{V}} = \prod_{\mathrm{v} \in \mathcal{V}} P_{\mathrm{v}}$$

$\rightarrow$ Only modifies vertex DOF, leaves interior and edge-interior DOF unchanged.

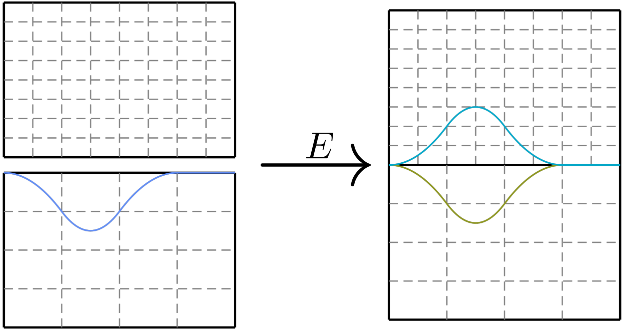
Edge conformity

Define the projection P_e at the edge $e \in \mathcal{E}$,



that maps the basis functions on the interface to the conforming space
$\rightarrow$ leave interior (and vertex) DOF unchanged.

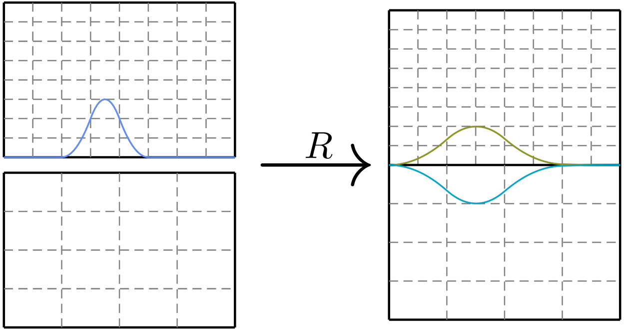
Extension operator



$$E : \lambda_i^- \otimes \lambda_0^- \mapsto \underbrace{\lambda_i^-}_{\sum_{j=0}^{n^+-1} \mathbb{E}_{i,j}\lambda_j^+} \otimes \frac{1}{2}\left(\lambda_0^- + \lambda_0^+\right)$$

Change of basis, with interpolation coefficients $\mathbb{E}_{i,j}$ obtained by knot-insertion.

Restriction operator



$$R : \lambda_i^+ \otimes \lambda_0^+ \mapsto \underbrace{\mathcal{R}\lambda_i^+}_{= \sum_{j=0}^{n^- -1} \mathbb{R}_{i,j}\lambda_j^- = \sum_{j=0}^{n^+ -1} (\mathbb{E}\mathbb{R})_{i,j}\lambda_j^+} \otimes \frac{1}{2}\left(\lambda_0^- + \lambda_0^+\right)$$

E.g. modified L^2-projection of a fine spline into the coarse spline space.

Edge conformity: $P_{\mathcal{E}}$

Assume $\boldsymbol{j} = (j_1, j_2) = (j_{\|(\mathrm{e})}, j_{\perp(\mathrm{e})})$ for a fixed edge $\mathrm{e} \in \mathcal{E}$.

$$P_{\mathrm{e}} : \begin{cases} \Lambda^-_{\boldsymbol{j}} \mapsto \begin{cases} \lambda^-_j \otimes \frac{1}{2}\left(\lambda^-_{i^-_\perp(\mathrm{e})} + \lambda^+_{i^+_\perp(\mathrm{e})}\right) & \text{if } 0 < \boldsymbol{j}_1 = j < n^-,\ \boldsymbol{j}_2 = i^-_\perp(\mathrm{e}), \\ \Lambda^-_{\boldsymbol{j}} & \text{else.} \end{cases} \\ \Lambda^+_{\boldsymbol{j}} \mapsto \begin{cases} \mathcal{R}\lambda^+_j \otimes \frac{1}{2}\left(\lambda^+_{i^-_\perp(\mathrm{e})} + \lambda^-_{i^+_\perp(\mathrm{e})}\right) & \text{if } 0 < \boldsymbol{j}_1 = j < n^+,\ \boldsymbol{j}_2 = i^+_\perp(\mathrm{e}), \\ \Lambda^+_{\boldsymbol{j}} & \text{else.} \end{cases} \end{cases}$$

where $i^\pm_\perp(\mathrm{e})$ is the index of the perpendicular basis function on patch $k^\pm$ with support on e.

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where $i_\perp^{\pm}(\mathrm{e})$ is the index of the perpendicular basis function on patch $k^{\pm}$ with support on e.

$$P_{\mathcal{E}} = \prod_{\mathrm{e} \in \mathcal{E}} P_{\mathrm{e}}$$

$\rightarrow$ Only modifies edge-interior DOF, leaves interior and vertex DOF unchanged

Non-matching conforming projections

Finally:

$$P^0 = P_{\mathcal{E}} P_{\mathcal{V}} : V^0_{\mathrm{pw}} \to V^0_{\mathrm{pw}}$$

is a projection onto the conforming space V^0_h.

Non-matching conforming projections

Finally:

$$P^0 = P_{\mathcal{E}} P_{\mathcal{V}} : V^0_{\mathrm{pw}} \to V^0_{\mathrm{pw}}$$

is a projection onto the conforming space V^0_h.

The construction for V^1 is similar, but we only enforce conformity on the tangential traces (on the whole edge including vertex DOF):

$$\boldsymbol{P}^1 = \sum_{\mathrm{e}\in\mathcal{E}} \boldsymbol{P}^1_{\mathrm{e}} : V^1_{\mathrm{pw}} \to V^1_{\mathrm{pw}},$$

is a projection onto the conforming subspace V^1_h.

Numerical Experiments

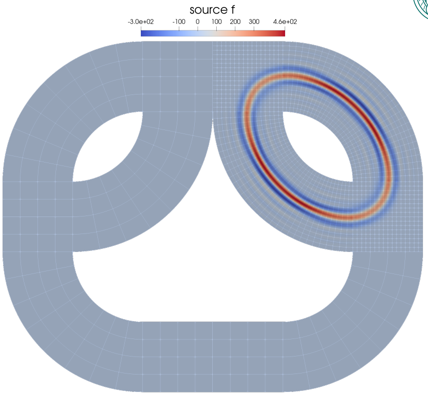
Poisson problem (pretzel domain)

Find $\phi \in V^0 = H_0^1(\Omega)$ such that

$$-\Delta\phi = f,$$

where $f \in L^2(\Omega)$.

The domain consists of 18 patches, we only refine the top-right handle (6 patches).



Within the broken-FEEC framework[1], this equation is discretized as

$$\left(\left(\mathbb{G}_{\mathrm{pw}}\mathbb{P}^0\right)^T \mathbb{M}^1 \mathbb{G}_{\mathrm{pw}}\mathbb{P}^0 + \underbrace{(\mathbb{I}-\mathbb{P}^0)^T\mathbb{M}^0(\mathbb{I}-\mathbb{P}^0)}_{\text{jump stabilization}} \right) \phi_h = (\mathbb{P}^0)^T f_h.$$

[1] All numerical examples are performed using `https://github.com/pyccel/psydac`

Poisson problem in psydac

```
# setup domain
domain = build_multipatch_domain(domain_name='pretzel')
domain_h = discretize(domain, ncells=ncells)

# setup derham sequence
derham = Derham(domain, ["H1", "Hcurl", "L2"])
derham_h = discretize(derham, domain_h, degree=degree)
V0h, V1h, V2h = derham_h.spaces

# patch-wise derivative operators
bD0, bD1 = derham_h.derivatives(kind='linop')

# patch-wise mass matrices
H0 = derham_h.hodge_operator(space='V0', kind='linop')
H1 = derham_h.hodge_operator(space='V1', kind='linop')
dH0 = derham_h.hodge_operator(space='V0', kind='linop', dual=True)

# conforming projectors
cP0, cP1, cP2 = derham_h.conforming_projectors(kind='linop', hom_bc = True)
```

Poisson problem in psydac

```python
# div grad
DG = - cP0.T @ bD0.T @ H1 @ bD0 @ cP0
# jump penalization
I0 = IdentityOperator(V0h)
JP0 = (I0 - cP0).T @ H0 @ (I0 - cP0)

A = DG + JP0

# obtain source and exact solution
f, u_ex = get_source_and_solution_h1(domain_h=domain_h)
df = cP0.T @ get_dual_dofs(Vh=V0h, f=f, domain_h=domain_h)

# solve the system
solver = inverse(A, solver='cg', tol=1e-8)
u = solver.solve(df)
```

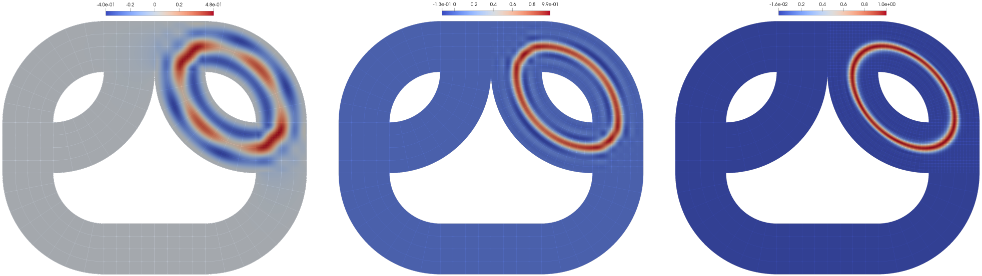
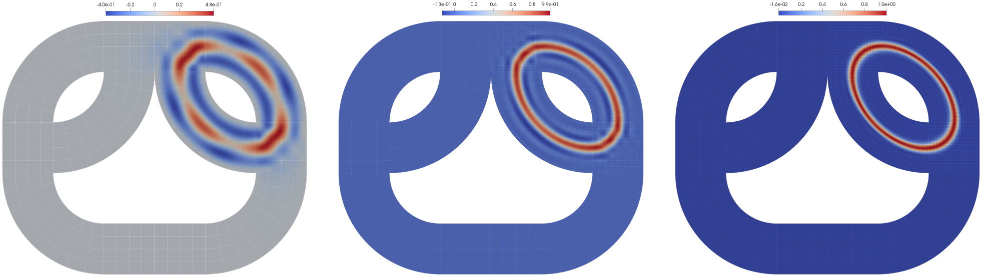


Figure: Numerical solution ϕ_h for degree 3 and refinement of the top-right handle.

Relative L^2-errors for:

Cells p.p. \ Degree	uniform refinement 2	3	4	5	refined handle 2	3	4	5
4	6.71637	0.86527	0.34468	0.18317	6.71637	0.86527	0.34468	0.18317
8	0.31598	0.04601	0.08671	0.05566	0.31596	0.04606	0.08675	0.05573
16	0.01516	0.00489	0.00575	0.00361	0.01516	0.00489	0.00575	0.00361



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16	0.01516	0.00489	0.00575	0.00361	0.01516	0.00489	0.00575	0.00361

$\rightarrow$ On the finest level only half the DOF needed for the same accuracy.

Time-harmonic Maxwell equation (pretzel domain)

For $\omega \in \mathbb{R}$ and $\boldsymbol{J} \in L^2(\Omega)$, solve

$$-\omega^2 \boldsymbol{E} + \mathbf{curl}\ \mathrm{curl}\ \boldsymbol{E} = \boldsymbol{J},$$

for $\boldsymbol{E} \in H(\mathrm{curl}, \Omega)$.

In our framework, that reads

$$\left(-\omega^2 (\mathbb{P}^1)^T \mathbb{M}^1 \mathbb{P}^1 + (\mathbb{C}_{\mathsf{pw}} \mathbb{P}^1)^T \mathbb{M}^2 \mathbb{C}_{\mathsf{pw}} \mathbb{P}^1 + \underbrace{(\mathbb{I} - \mathbb{P}^1)^T \mathbb{M}^1 (\mathbb{I} - \mathbb{P}^1)}_{\text{jump stabilization}} \right) \boldsymbol{E}_h = (\mathbb{P}^1)^T \boldsymbol{J}_h.$$

Manufactured solution: $\quad \boldsymbol{J} = \begin{pmatrix} -\pi^2 \sin(\pi y)\cos(\pi x) \\ 0 \end{pmatrix}, \qquad \boldsymbol{E} = \begin{pmatrix} \sin(\pi y) \\ \sin(\pi x)\cos(\pi y) \end{pmatrix}.$

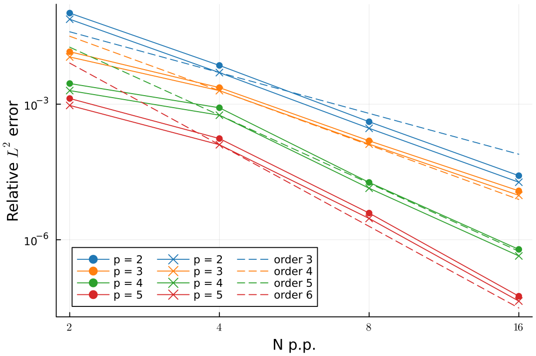


Figure: Convergence curves for matching (circle) and refined (cross) grids, where every other grid was of different resolution.

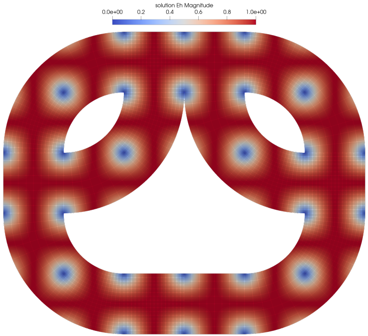


Figure: Numerical solution $\|\boldsymbol{E_h}\|$ for a refined grid between $N = 8, 16$ and $p = 3$.

Curl curl eigenvalue problem (curved L-shape domain)

Find $\lambda > 0$ such that

$$\mathbf{curl}\ \mathrm{curl}\ \boldsymbol{E} = \lambda \boldsymbol{E},$$

where $\boldsymbol{E} \in H_0(\mathrm{curl}, \Omega)$.

Within the broken-FEEC framework, this equation is discretized as

$$\left(\mathbb{C}_{\mathsf{pw}}\mathbb{P}^1\right)^T \mathbb{M}^2\mathbb{C}_{\mathsf{pw}}\mathbb{P}^1\ \boldsymbol{E}_h = \lambda \left(\underbrace{(\mathbb{I} - \mathbb{P}^1)^T\mathbb{M}^1(\mathbb{I} - \mathbb{P}^1)}_{\text{jump stabilization}} + (\mathbb{P}^1)^T\mathbb{M}^1\mathbb{P}^1 \right)\ \boldsymbol{E}_h,$$

where we refine layers around the reentrant corner and compare it to a uniform solution with similar amount of DOF.

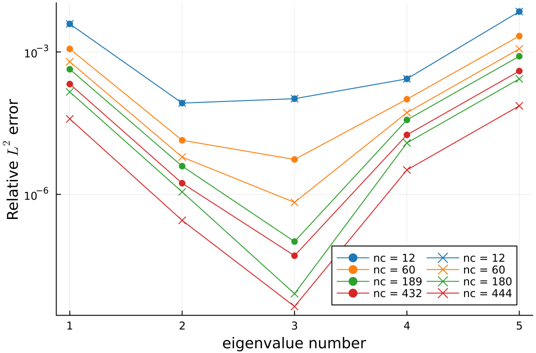


Figure: Error of each eigenvalue for matching (circle) and refined (cross) patches.

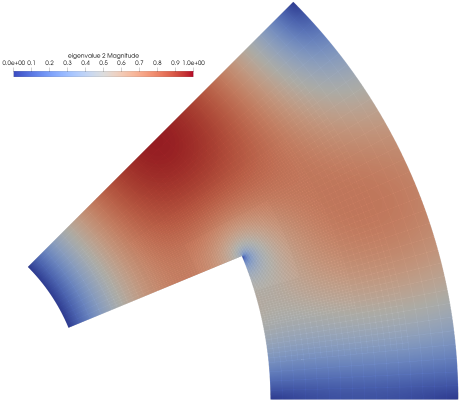
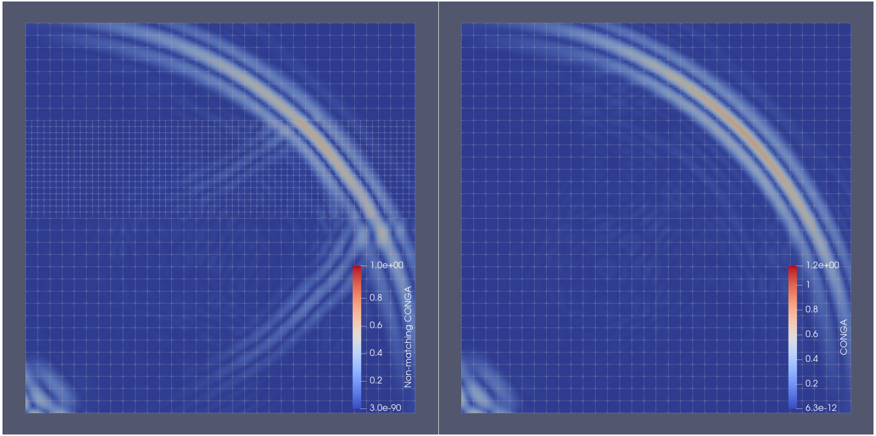


Figure: Absolute value of the eigenfunction corresponding to the eigenvalue λ_3 on the most refined grid.

Time domain Maxwell

non-matching grid vs. matching grid



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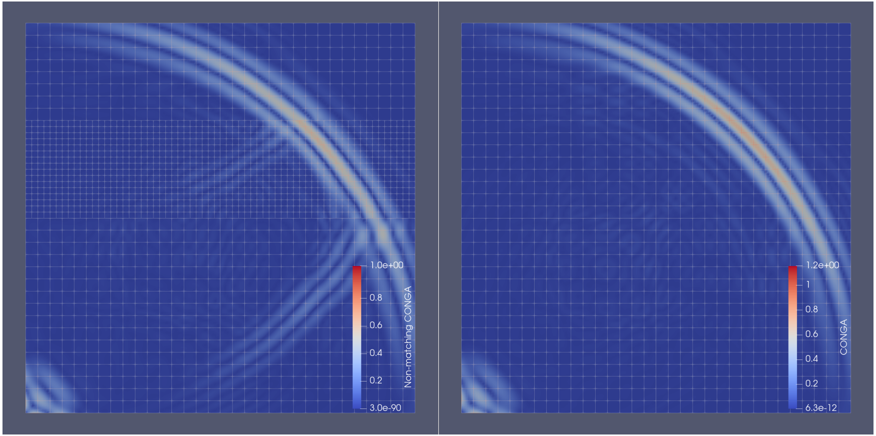
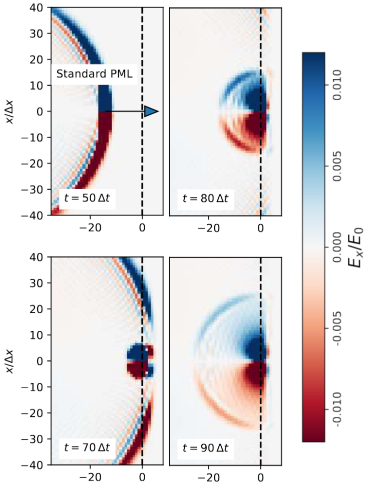
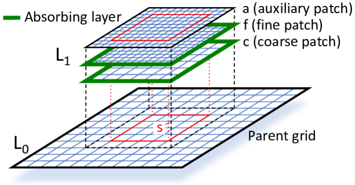


Spurious reflections at non-matching patch interfaces.

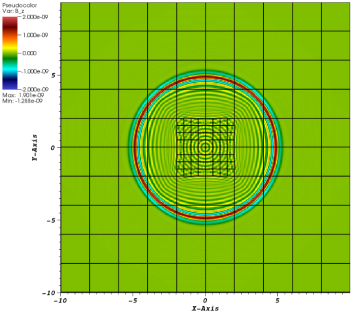
Locally refined meshes II

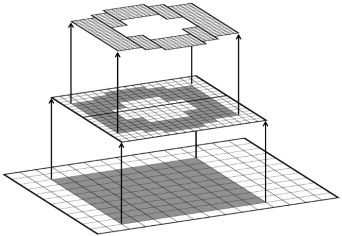




Vay, J. L. et al. (12, 22): Uses finite difference schemes (FDTD) with absorbing layers (PML) between non-matching grids.

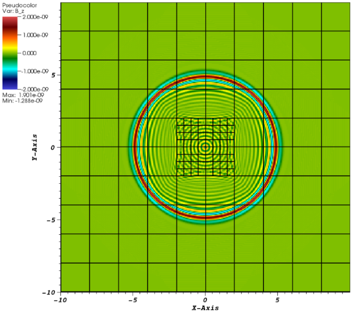
Locally refined meshes III

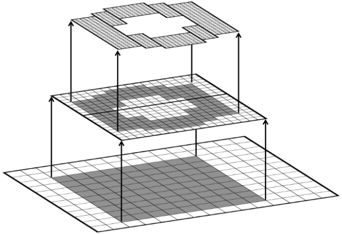




Chilton, S (13): Uses finite volume schemes and RK4, with damping and dissipation in transition layers between non-matching grids.

Locally refined meshes III





Chilton, S (13): Uses finite volume schemes and RK4, with damping and dissipation in transition layers between non-matching grids.

→ Q3: How to solve wave problems and avoid spurious reflections?

Motivation: Weak Divergence

$$\begin{tikzcd}
H^1 \arrow[r, "\nabla"] \arrow[d, "\Pi^0_{\mathrm{pw}}"] & H(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\Pi^1_{\mathrm{pw}}"] & L^2 \arrow[d, "\Pi^2_{\mathrm{pw}}"] \\
V^0_{\mathrm{pw}} \arrow[r, "\mathrm{grad}\, P^0"] & V^1_{\mathrm{pw}} \arrow[r, "\mathrm{curl}\, P^1"] & V^2_{\mathrm{pw}}
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{V}^2_{\mathrm{pw}} & \tilde{V}^1_{\mathrm{pw}} \arrow[l, "\mathrm{div}_h"'] & \tilde{V}^0_{\mathrm{pw}} \arrow[l, "\mathbf{curl}_h"'] \\
L^2 \arrow[u, "\tilde{\Pi}^2_{\mathrm{pw}}"'] & H(\mathrm{div}) \arrow[l, "\mathrm{div}"'] \arrow[u, "\tilde{\Pi}^1_{\mathrm{pw}}"'] & H(\mathbf{curl}) \arrow[l, "\mathbf{curl}"'] \arrow[u, "\tilde{\Pi}^0_{\mathrm{pw}}"']
\end{tikzcd}$$

Idea: if

$$\mathrm{div}_h = -(\mathrm{grad}\, P^0)^*,$$

then

$$\mathrm{div}\, u \approx \mathrm{div}_h\, Q^1_{\mathrm{pw}} u \quad \forall u \in H_0(\mathrm{div})$$

is of high order as long as P^0 is preserving polynomial moments.

Preservation of polynomial moments

Preserving polynomial moments is defined as:

$$\int P^\ell \phi q_j = \int \phi q_j, \qquad \forall \phi \in V_{\mathrm{pw}}^\ell, q_j \in \mathbb{P}_j, \quad j = 0, \dots, r$$

$\longrightarrow$ Introduces a set of linear constraints on the coefficients of P^0 and $\boldsymbol{P}^1$.

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$\longrightarrow$ Introduces a set of linear constraints on the coefficients of P^0 and $\boldsymbol{P}^1$.

We introduce the one-dimensional correction coefficients γ solving the linear system:

$$\sum_{m=1}^{r} \gamma_m \int \lambda_m q_j = \int \lambda_0 q_j, \quad \forall j$$

which are independent of the mesh resolution.

Leading to the modified vertex-based operator:

$$P_{\mathrm{v}} : \Lambda^k_{\boldsymbol{j}} \mapsto \begin{cases} \bar{\Lambda}^{\mathrm{v}} + \displaystyle\sum_{\boldsymbol{m}_1=1}^{r} \sum_{\boldsymbol{m}_2=1}^{r} \boldsymbol{a}_{\boldsymbol{m}} \gamma_{\boldsymbol{m}} \Lambda^k_{\boldsymbol{m}} & \\ \qquad - \dfrac{1}{\#\mathrm{v}} \displaystyle\sum_{k' \in \mathcal{K}(\mathrm{v})} \sum_{\boldsymbol{m}_1=1}^{r} \sum_{\boldsymbol{m}_2=1}^{r} \boldsymbol{a}_{\boldsymbol{m}} \gamma_{\boldsymbol{m}} \Lambda^{k'}_{\boldsymbol{m}} & \text{if } \boldsymbol{j} \text{ at } \mathrm{v}, \\ \Lambda^k_{\boldsymbol{j}} & \text{else,} \end{cases}$$

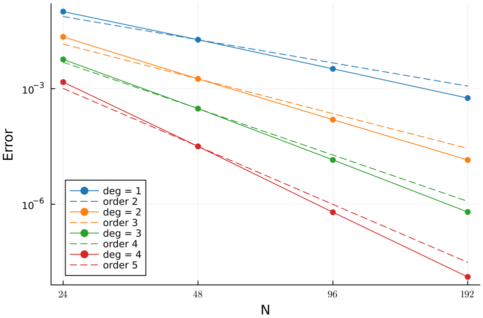
...and the modified edge-based operator:

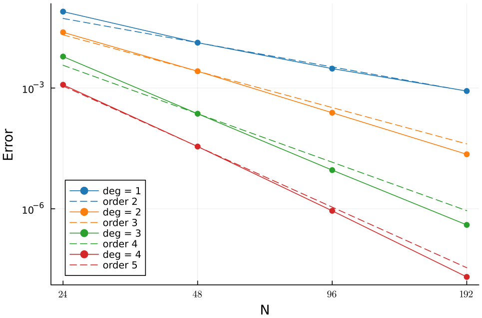
$$P_{\mathrm{e}}: \begin{cases} \Lambda_{\boldsymbol{j}}^{-} \mapsto \begin{cases} \lambda_{j_1}^{-} \otimes \left(\bar{\lambda}_{\perp}^{\mathrm{e}} + \frac{1}{2} \left(\sum_{m=1}^{r} \gamma_m \lambda_m^{-} - \sum_{m=1}^{r} \gamma_m \lambda_m^{+} \right) \right) & \text{if } \boldsymbol{j} \text{ on } \mathrm{e} \text{ and not at } \mathrm{v}, \\ \lambda_{j_1}^{-} \otimes \bar{\lambda}_{\perp}^{\mathrm{e}} + \mu_{\mathrm{v}}^{-} \otimes \frac{1}{2} \left(\sum_{m=1}^{r} \gamma_m \lambda_m^{-} - \sum_{m=1}^{r} \gamma_m \lambda_m^{+} \right) & \text{if } \boldsymbol{j} \text{ on } \mathrm{e} \text{ and at } \mathrm{v}, \\ \Lambda_{\boldsymbol{j}}^{-} & \text{else}, \end{cases} \\ \Lambda_{\boldsymbol{j}}^{+} \mapsto \begin{cases} \mathcal{R}\lambda_{j_1}^{+} \otimes \left(\bar{\lambda}_{\perp}^{\mathrm{e}} - \frac{1}{2} \left(\sum_{m=1}^{r} \gamma_m \lambda_m^{-} - \sum_{m=1}^{r} \gamma_m \lambda_m^{+} \right) \right) & \text{if } \boldsymbol{j} \text{ on } \mathrm{e} \text{ and not at } \mathrm{v}, \\ \mathcal{R}\lambda_{j_1}^{+} \otimes \bar{\lambda}_{\perp}^{\mathrm{e}} - \mathcal{R}\mu_{\mathrm{v}}^{+} \otimes \frac{1}{2} \left(\sum_{m=1}^{r} \gamma_m \lambda_m^{-} - \sum_{m=1}^{r} \gamma_m \lambda_m^{+} \right) & \text{if } \boldsymbol{j} \text{ on } \mathrm{e} \text{ and at } \mathrm{v} \\ \Lambda_{\boldsymbol{j}}^{+} & \text{else}, \end{cases} \end{cases}$$

where $\mu_{\mathrm{v}}^{+} = \sum_{i=1}^{r} \gamma_i^{+} \lambda_i^{+}$ and $\mu_{\mathrm{v}}^{-} = \lambda_0^{-} + \mathcal{R}\left(\mu_{\mathrm{v}}^{+} - \lambda_0^{+}\right)$.

Weak Divergence II

Convergence curves for the weak divergence (left) and the weak curl (right) operator on a curved L-shaped domain consisting of three patches with non-matching grids.





Time domain Maxwell equations (Square domain)

For $\boldsymbol{J} \in L^2(\Omega)$, solve

$$
\begin{aligned}
\partial_t \boldsymbol{E} - \mathbf{curl}\ B &= -\boldsymbol{J}, \\
\partial_t B + \operatorname{curl}\ \boldsymbol{E} &= 0,
\end{aligned}
$$

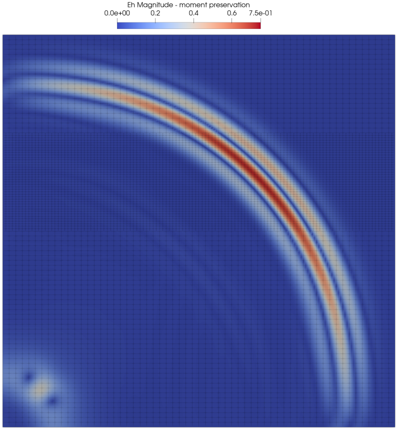
where $\boldsymbol{E} \in H_0(\operatorname{curl}, \Omega)$ and $B \in L^2(\Omega)$.

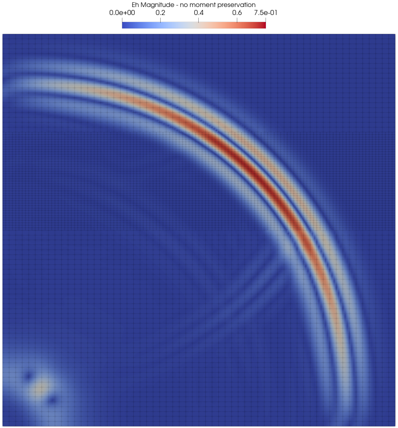
Cast into a simple leap-frog time-stepping scheme, this reads

$$
\begin{aligned}
B_h^{n+\frac{1}{2}} &= B_h^n - \frac{\Delta t}{2} \mathbb{C}_{\mathrm{pw}} \mathbb{P}^1 E_h^n, \\
\mathbb{M}^1 E_h^{n+1} &= \mathbb{M}^1 E_h^n + \Delta t \left((\mathbb{C}_{\mathrm{pw}} \mathbb{P}^1)^T \mathbb{M}^2 B_h^{n+\frac{1}{2}} - \mathbb{P}^1 J_h^{n+\frac{1}{2}} \right), \\
B_h^{n+1} &= B_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \mathbb{C}_{\mathrm{pw}} \mathbb{P}^1 E_h^{n+1}.
\end{aligned}
$$

Redemption of results

Moment preservation vs. no moment preservation



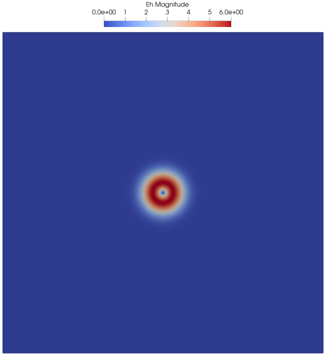


Electromagnetic wave (without source)

In our example, we choose the initial conditions

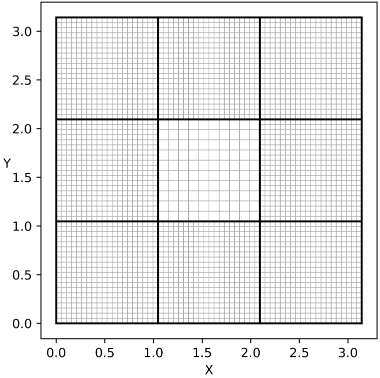
$$\boldsymbol{E}(t=0;\boldsymbol{x}) = \boldsymbol{p}\ \cos(\boldsymbol{k}\cdot\boldsymbol{x})\ e^{-\frac{x^2}{2\sigma^2}},$$
$$\boldsymbol{B}(t=0;\boldsymbol{x}) = \operatorname{curl}\boldsymbol{E},$$
$$\boldsymbol{J}(t,\boldsymbol{x}) \equiv 0.$$

See the initial magnitude of E_h^0 on the right.

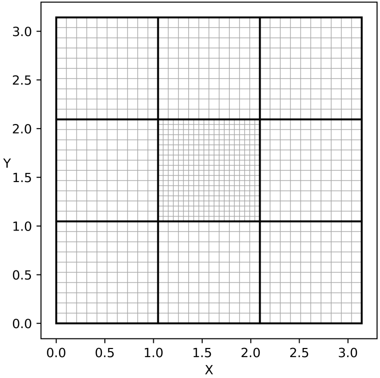


Electromagnetic wave: domains

Coarse to fine

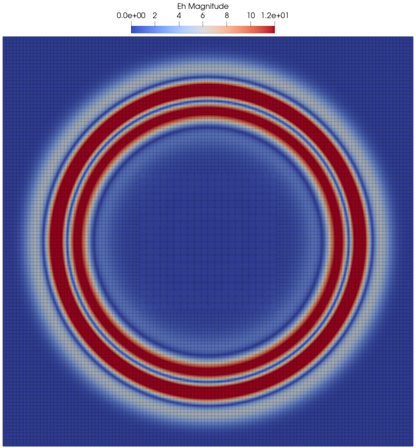


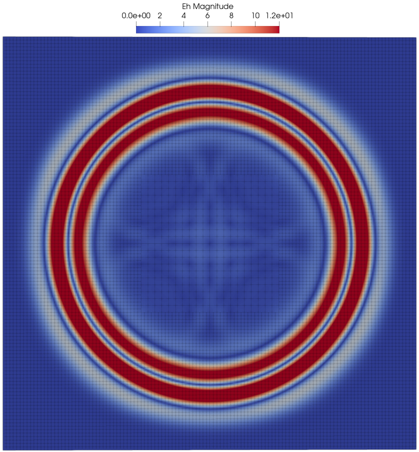
Fine to coarse



Electromagnetic wave: Coarse to fine

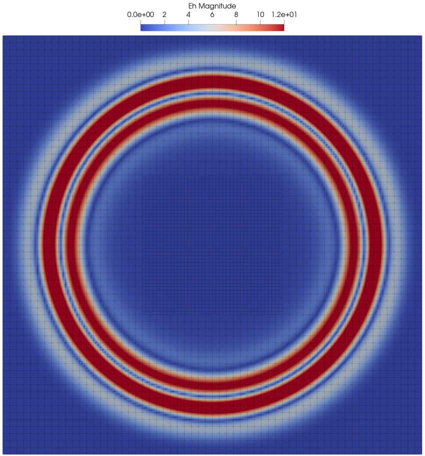
Moment preservation vs. no moment preservation

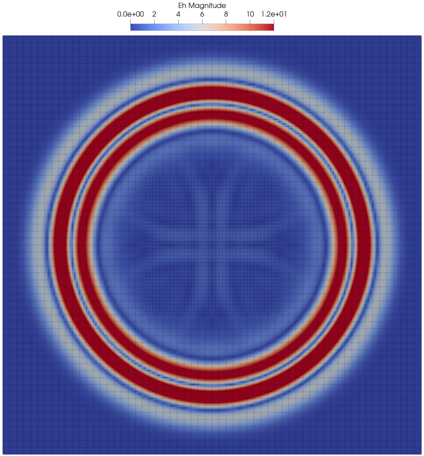




Electromagnetic wave: Fine to coarse

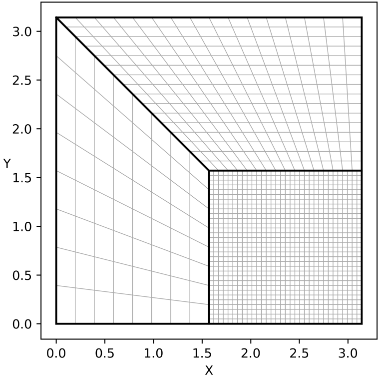
Moment preservation vs. no moment preservation



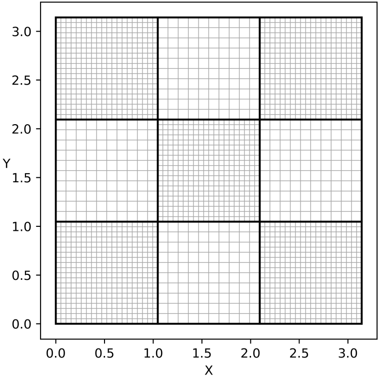


Restrictions by theory

Curl-curl eigenvalue problem



Time-domain Maxwell



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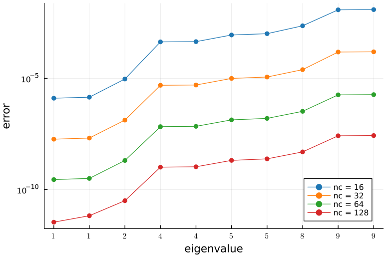
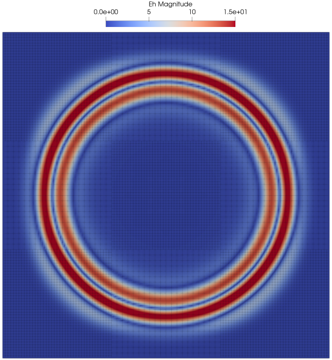


Time-domain Maxwell



…which work well in practice.

Conclusion

Take home messages

- FEEC: Good for domains with corners or holes
- Broken-FEEC + Conforming projections: Good for (non-matching) multi-patch (local, modular schemes) on complex domains

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  $\longrightarrow$ Q2

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- Bounded Commuting projections: Yield a priori convergence and stability results
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- Moment-preserving schemes: Good for higher-order adjoint differential operators, useful wave equations
  $\longrightarrow$ Q3

Thank you for your attention!

Backup Slides

THEORY

The commuting projections of Buffa-Rivas-Sangalli-Vázquez (2011) – cont'd

- projection property:
  - for Π^0, as usual $\langle \Theta^0_{\boldsymbol{i}}, \Lambda^0_{\boldsymbol{j}} \rangle = \delta_{\boldsymbol{i},\boldsymbol{j}}$
  - for Π^1, follows from $\int_0^{x_1} u_1(z, x_2)\,\mathrm{d}z \in V^0_h$ (same argument for Π^2)
- commuting property: for the grad,

$$(\Pi^1 \nabla \phi(\boldsymbol{x}))_1 = \partial_1 \big(\Pi^0 \Phi^1_1(\nabla \phi)(\boldsymbol{x})\big) = \partial_1 \Pi^0 \big(\phi(\boldsymbol{x}) - \phi(0, x_2)\big) = \partial_1 \Pi^0 \phi(\boldsymbol{x}) \qquad \text{(same for the curl)}$$

- L^2_h stability:
  - on V^0_h, follows from locally stable dofs
  - on V^1_h, follows from a localization argument: on a local domain $\omega_{\boldsymbol{i}}$, we have

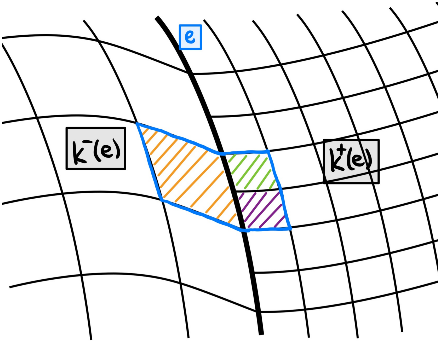
$$\nabla_1 \Pi^0 \Phi^1_1(\boldsymbol{u}) = \nabla_1 \Pi^0 \Phi^1_{1,\boldsymbol{i}}(\boldsymbol{u}) \quad \text{with} \quad \Phi^1_{1,\boldsymbol{i}}(\boldsymbol{u})(\boldsymbol{x}) := \int_{x_1(\omega_{\boldsymbol{i}})}^{x_1} u_1(z, x_2)\,\mathrm{d}z$$

  and inverse / local stability estimates

$$\|\nabla_1 \Pi^0 \Phi^1_{1,\boldsymbol{i}}(\boldsymbol{u})\|_{L^2(\omega_{\boldsymbol{i}})} \lesssim h^{-1} \|\Pi^0 \Phi^1_{1,\boldsymbol{i}}(\boldsymbol{u})\|_{L^2(E_h(\omega_{\boldsymbol{i}}))} \lesssim h^{-1} \|\Phi^1_{1,\boldsymbol{i}}(\boldsymbol{u})\|_{L^2(E^2_h(\omega_{\boldsymbol{i}}))} \lesssim \|\boldsymbol{u}\|_{L^2(E^3_h(\omega_{\boldsymbol{i}}))}$$

  - on V^2_h, the same principle applies
- Note: as long as $\mathrm{supp}(\Theta^0_{\boldsymbol{i}}) \approx \mathrm{supp}(\Lambda^0_{\boldsymbol{i}})$ these operators are local

Edge Conforming basis functions



Λ^e_i

$\Lambda^{k^-}_{(n,i)}$

$\Lambda^{k^+}_{(0,2i)}$, $\Lambda^{k^+}_{(0,2i+1)}$

$$\Lambda^e_i(\boldsymbol{x}) := \begin{cases} \hat\lambda^-_i(\hat x^-_\parallel)\hat\lambda^-_{i^-_\perp(e)}(\hat x^-_\perp) & \text{on } \Omega_{k^-}, \\ \hat\lambda^-_i(\eta_e(\hat x^+_\parallel))\hat\lambda^+_{i^+_\perp(e)}(\hat x^+_\perp) & \text{on } \Omega_{k^+}, \\ 0 & \text{else.} \end{cases}$$

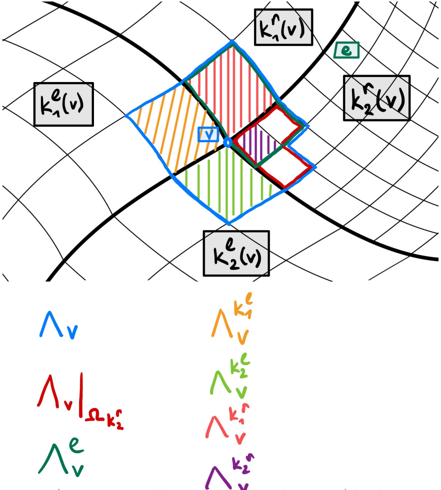
- Projection on the local edge-broken space:

$$I_e : V^0_{\mathrm{pw}} \to V^0_{\mathrm{pw}}, \quad \Lambda^k_{\boldsymbol{i}} \mapsto \begin{cases} \Lambda^k_{\boldsymbol{i}} & \text{if } i_2 = i^k_2(e) \\ 0 & \text{otherwise} \end{cases}$$

- Projection on the local edge-conforming space:

$$P_e : V^0_{\mathrm{pw}} \to V^0_h, \quad \Lambda^k_{\boldsymbol{i}} \mapsto \begin{cases} \Lambda^e_{i_1} & \text{if } k = -, \ i_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

Vertex conforming basis functions



$$\Lambda^{\boldsymbol{v}} := \sum_{e\in\mathcal{E}(\boldsymbol{v})} \Lambda^e_{\boldsymbol{v}} - \sum_{k\in\mathcal{K}(\boldsymbol{v})} \Lambda^k_{\boldsymbol{v}}.$$

- Projection on the local vertex-broken space:

$$I_{\boldsymbol{v}} : V^0_{\mathrm{pw}} \to V^0_{\mathrm{pw}}, \quad \Lambda^k_{\boldsymbol{i}} \mapsto \begin{cases} \Lambda^k_{\boldsymbol{i}} & \text{if } k \in \mathcal{K}(\boldsymbol{v}),\ \boldsymbol{i} = \boldsymbol{i}^k(\boldsymbol{v}) \\ 0 & \text{otherwise} \end{cases}$$

- Projection on the local vertex-conforming space:

$$P_{\boldsymbol{v}} : V^0_{\mathrm{pw}} \to V^0_h, \quad \Lambda^k_{\boldsymbol{i}} \mapsto \begin{cases} \Lambda^{\boldsymbol{v}} & \text{if } k = k^*(\boldsymbol{v}),\ \boldsymbol{i} = \boldsymbol{i}^k(\boldsymbol{v}) \\ 0 & \text{otherwise} \end{cases}$$

Conforming projection P

For any broken $\phi \in V^0_{\mathrm{pw}}$, we have the decomposition

$$\phi = \Big(\sum_k I^k_0 + \sum_{e\in\mathcal{E}} I_{e,0} + \sum_{v\in\mathcal{V}} I_{\boldsymbol{v}}\Big)\phi \quad \text{and} \quad P\phi := \Big(\sum_{k\in\mathcal{K}} I^k_0 + \sum_{e\in\mathcal{E}} P_{e,0} + \sum_{\boldsymbol{v}\in\mathcal{V}} P_{\boldsymbol{v}}\Big)\phi.$$

Building Blocks of Π^1

Strategy: Start from the patch-wise commuting projections and add correction terms.

Patch-wise projections Π^1_k:

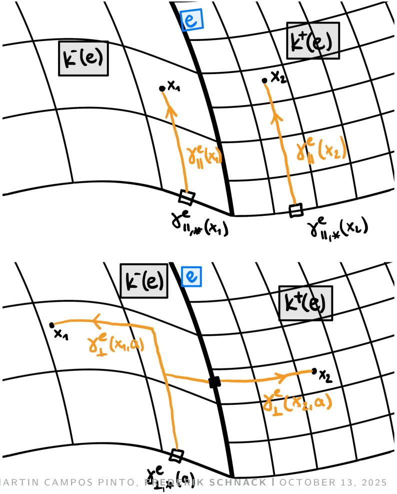
$$\Pi^1_k \boldsymbol{u} = \sum_{d\in\{1,2\}} \nabla^k_d \Pi^0_k \Phi^k_d(\boldsymbol{u}),$$

where the antiderivatives

$$\Phi^k_1(\boldsymbol{u})(\boldsymbol{x}) = \int_0^{x_1} \boldsymbol{u}_1(z, x_2)\,\mathrm{d}z, \qquad \Phi^k_2(\boldsymbol{u})(\boldsymbol{x}) = \int_0^{x_2} \boldsymbol{u}_2(x_1, z)\,\mathrm{d}z$$

are defined for $\boldsymbol{x} \in \Omega_k$ such that $\Phi^k_d(\boldsymbol{u}) \in V^0_{\mathrm{pw}}$ for $\boldsymbol{u} \in V^1_h, d = 1, 2$.

Edge-correction $\tilde{\Pi}^1_e$



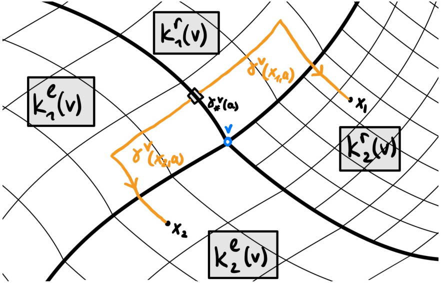
$$\tilde{\Pi}^1_e \boldsymbol{u} := \sum_{d \in \{\parallel, \perp\}} \nabla^e_d (P_e - I_e) \Pi^0_{\mathrm{pw}} \Phi^e_d(\boldsymbol{u}),$$

where the antiderivatives are defined as

$$\Phi^e_d \boldsymbol{u}(\boldsymbol{x}) = \int_{\gamma^e_d(\boldsymbol{x})} \boldsymbol{u} \cdot \mathrm{d}l, \qquad d \in \{\parallel, \perp\}$$

such that for $\boldsymbol{u} \in V^1_h$ and for all $e \in \mathcal{E}$, $\Phi^e_\parallel(\boldsymbol{u})$ and $\Phi^e_\perp(\boldsymbol{u})$ belong to V^0_{pw} and are continuous across e which implies that $(P_e - I_e)$ is equal to zero in the correction term.

Vertex-correction $\tilde{\Pi}^1_v$



$$\tilde{\Pi}^1_{\boldsymbol{v}}\boldsymbol{u} := \nabla_{\mathrm{pw}}(P_{\boldsymbol{v}} - \bar{I}_{\boldsymbol{v}})\Pi^0_{\mathrm{pw}}\Phi^{\boldsymbol{v}}(\boldsymbol{u}),$$

where the antiderivative is defined as

$$\Phi^{\boldsymbol{v}}\boldsymbol{u}(\boldsymbol{x}) = \int_{\gamma^{\boldsymbol{v}}(\boldsymbol{x})} \boldsymbol{u} \cdot \mathrm{d}l, \qquad d \in \{\|, \perp\}$$

such that for $\boldsymbol{u} \in V_h^1$ and for all $\boldsymbol{v} \in \mathcal{V}$, $\Phi^{\boldsymbol{v}}(\boldsymbol{u})$ belongs to V^0_{pw} and is continuous across every $e \in \mathcal{E}(\boldsymbol{v})$ which implies that $(P_{\boldsymbol{v}} - \bar{I}_{\boldsymbol{v}})$ is equal to zero in the correction term.

Full projection Π^1

$$\Pi^1 := \sum_{k \in \mathcal{K}} \Pi^1_k + \sum_{e \in \mathcal{E}} \tilde{\Pi}^1_e + \sum_{\boldsymbol{v} \in \mathcal{V}} \tilde{\Pi}^1_{\boldsymbol{v}} + \sum_{\boldsymbol{v} \in \mathcal{V}, e \in \mathcal{E}(\boldsymbol{v})} \tilde{\Pi}^1_{e,\boldsymbol{v}}$$

with edge, vertex and edge-vertex correction operators defined as

$$\begin{cases} \tilde{\Pi}^1_e \boldsymbol{u} := \sum_{d \in \{\parallel, \perp\}} \nabla^e_d (P_e - I_e) \Pi^0_{\mathrm{pw}} \Phi^e_d(\boldsymbol{u}) \\ \tilde{\Pi}^1_{\boldsymbol{v}} \boldsymbol{u} := \nabla_{\mathrm{pw}} (P_{\boldsymbol{v}} - \bar{I}_{\boldsymbol{v}}) \Pi^0_{\mathrm{pw}} \Phi^{\boldsymbol{v}}(\boldsymbol{u}) \\ \tilde{\Pi}^1_{e,\boldsymbol{v}} \boldsymbol{u} := \sum_{d \in \{\parallel, \perp\}} \nabla^e_d (\bar{I}_{e,\boldsymbol{v}} - P_{e,\boldsymbol{v}}) \Pi^0_{\mathrm{pw}} \Phi^{\boldsymbol{v},e}_d(\boldsymbol{u}) \end{cases}$$

Main result: Π^1 is a projection on V^1_h, L^p stable for $1 \le p \le \infty$ and commuting:

$$\Pi^1 \nabla \phi = \nabla \Pi^0 \phi \qquad \phi \in H^1(\Omega)$$

Commuting property

- let $\boldsymbol{u} = \nabla\phi$ and write $\phi_h = \Pi^0_{\rm pw}\phi \in V^0_{\rm pw}$
  - by single-patch construction: $\sum_k \Pi^1_k \boldsymbol{u} = \nabla_{\rm pw}\phi_h$
  - by preservation of $\|$ invariance along e: $\nabla^e_{\|}(P_e - I_e)\Pi^0_{\rm pw}\Phi^e_{\|}(\boldsymbol{u}) = \nabla^e_{\|}(P_e - I_e)\phi_h$
  - by cancellation of constants in $(P_e - I_e)$: $(P_e - I_e)\Pi^0_{\rm pw}\Phi^e_{\perp}(\boldsymbol{u}) = (P_e - I_e)\phi_h$
  - and similarly for the $\boldsymbol{v}$ and $e, \boldsymbol{v}$ terms.
  - thus we have $\phi_h = \Big(\sum_k I^k_0 + \sum_e I_{e,0} + \sum_{\boldsymbol{v}} I_{\boldsymbol{v}}\Big)\phi_h$ and $\Pi^1\boldsymbol{u} = \nabla_{\rm pw}\psi_h$ with

$$
\begin{aligned}
\psi_h &= \phi_h + \Big(\sum_e (P_e - I_e) + \sum_{\boldsymbol{v}} (P_v - \bar{I}_v) + \sum_{e,\boldsymbol{v}} (\bar{I}_{e,\boldsymbol{v}} - P_{e,\boldsymbol{v}})\Big)\phi_h \\
&= \Big(\sum_k I^k_0 + \sum_e (I_{e,0} + P_e - I_e) + \sum_{\boldsymbol{v}} (I_{\boldsymbol{v}} - P_v - \bar{I}_v) + \sum_{e,\boldsymbol{v}} (\bar{I}_{e,\boldsymbol{v}} - P_{e,\boldsymbol{v}})\Big)\phi_h = \cdots = P\phi_h
\end{aligned}
$$

L^2 stability

- for each term, stability follows from
  - local inverse estimates for gradients
  - the stability of the local projections
  - the stability of the antiderivatives and localization arguments

Projection property

- range property: $(\Pi^1 \boldsymbol{u} \in V_h^1)$
  - clearly, $\Pi^1 \boldsymbol{u} \in V^1_{\mathrm{pw}}$
  - the antiderivatives and the local conforming / broken projections are such that

$$(\boldsymbol{\tau} \cdot \Pi^1 \boldsymbol{u})|_{\Omega_{k^-(e)}} = (\boldsymbol{\tau} \cdot \Pi^1 \boldsymbol{u})|_{\Omega_{k^+(e)}}, \qquad e \in \mathcal{E}$$

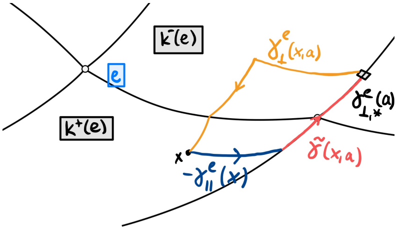
- projection property: $(\Pi^1 \boldsymbol{u} = \boldsymbol{u}$ for $\boldsymbol{u} \in V_h^1)$
  - patch-wise operator is a projection:

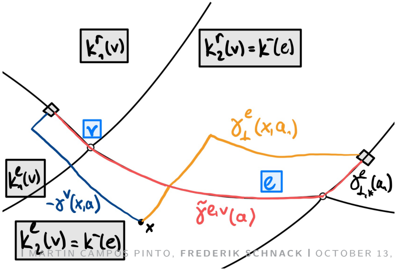
$$\sum_{k \in \mathcal{K}} \Pi^1_k \boldsymbol{u} = \sum_{k \in \mathcal{K}} \boldsymbol{u}|_{\Omega_k} = \boldsymbol{u}, \qquad \boldsymbol{u} \in V_h^1 \subset V^1_{\mathrm{pw}}$$

  - for $\boldsymbol{u} \in V_h^1$, the continuity properties of the antiderivatives yield

$$\begin{cases} (P_e - I_e)\Pi^0_{\mathrm{pw}} \Phi^e_d(\boldsymbol{u}) = 0, & d \in \{\|, \perp\} \\ (P_{\boldsymbol{v}} - \bar{I}_{\boldsymbol{v}})\Pi^0_{\mathrm{pw}} \Phi^{\boldsymbol{v}}(\boldsymbol{u}) = 0 & \\ (P_{e,\boldsymbol{v}} - \bar{I}_{e,\boldsymbol{v}})\Pi^0_{\mathrm{pw}} \Phi^{e,\boldsymbol{v}}_d(\boldsymbol{u}) = 0, & d \in \{\|, \perp\} \end{cases}$$

Projection Π^2





$$\Pi^2 := \sum_{k\in\mathcal{K}} \Pi^2_k + \sum_{e\in\mathcal{E}} \tilde{\Pi}^2_e + \sum_{\boldsymbol{v}\in\mathcal{V}, e\in\mathcal{E}(\boldsymbol{v})} \tilde{\Pi}^2_{e,\boldsymbol{v}}$$

with edge correction terms

$$\tilde{\Pi}^2_e : \begin{cases} L^p(\Omega) \to V^2_{\mathrm{pw}}, \\ f \mapsto D^{2,e}(P^e - I^e)\Pi^0_{\mathrm{pw}}\Psi^e(f) \end{cases}$$

and edge-vertex corrections

$$\tilde{\Pi}^2_{e,\boldsymbol{v}} : \begin{cases} L^p(\Omega) \to V^2_{\mathrm{pw}}, \\ f \mapsto D^{2,e}(\bar{I}^e_{\boldsymbol{v}} - P^e_{\boldsymbol{v}})\Pi^0_{\mathrm{pw}}\Psi^{\boldsymbol{v},e}(f). \end{cases}$$

$$\Psi(\mathrm{curl}\,\boldsymbol{u}) = \Phi_\perp(\boldsymbol{u}) - \Phi_\parallel(\boldsymbol{u}) + \tilde{\Phi}(\boldsymbol{u})$$