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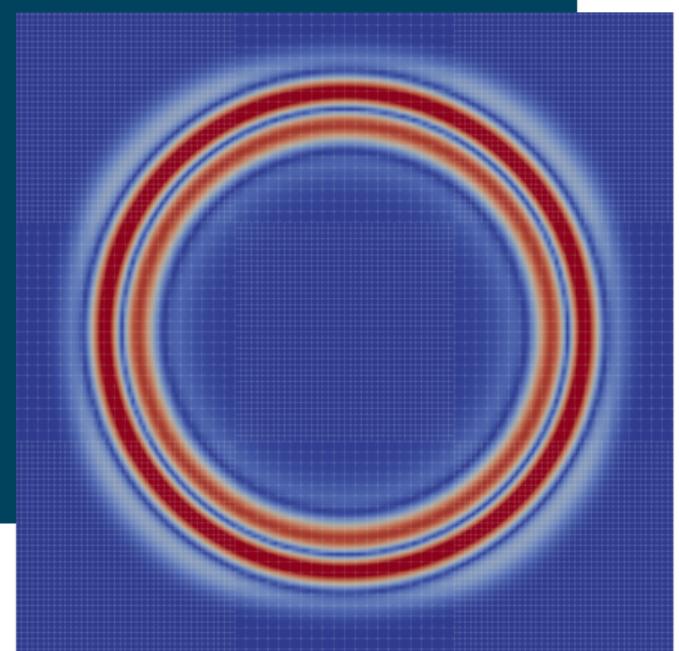
Broken-FEEC with locally refined meshes

NumKin 2024

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Outline

1. Motivation
2. Broken-FEEC
3. Construction of numerical schemes
4. Numerical Experiments
5. Conclusion

Motivation

Key Questions

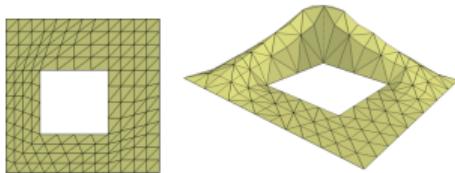


- Q1:** How to handle "complex" domains? On multiple patches (of different resolution)?
- Q2:** How to obtain a priori convergence / stability / spectral correctness results?



Pitfalls of continuous Finite Elements¹: holes and corners

- Hodge-Laplace eigenvalue problem: $(\operatorname{curl}_0 \operatorname{curl} - \operatorname{grad} \operatorname{div}_0) \mathbf{u} = \lambda \mathbf{u}$



# Elements	Degree 1		Degree 3	
	λ_1	λ_2	λ_1	λ_2
256	2.270	2.360	1.896	1.970
1,024	2.050	2.132	1.854	1.925
4,096	1.940	2.016	1.828	1.897
16,384	1.879	1.952	1.812	1.880
65,536	1.843	1.914	1.802	1.870
262,144	1.821	1.890	1.796	1.863

Continuous FEM

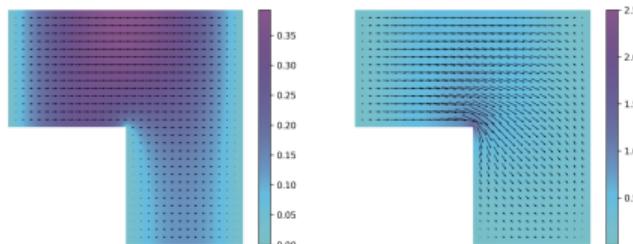
# Elements	Degree 1		Degree 3	
	λ_1	λ_2	λ_1	λ_2
256	0.000	0.638	0.000	0.619
1,024	0.000	0.625	0.000	0.618
4,096	0.000	0.620	0.000	0.617
16,384	0.000	0.618	0.000	0.617
65,536	0.000	0.618	0.000	0.617
262,144	0.000	0.617	0.000	0.617

FEEC

- with holes: non trivial harmonic fields ($\lambda = 0$), dimension = topological invariant and missed by continuous FEM

- Hodge-Laplace source problem

- with corners: singular solutions $\mathbf{u} \notin H^1$, no convergence with continuous FEM



Continuous FEM

FEEC

¹examples taken from Arnold (18)



Structure-preserving Finite Elements / FEEC

- sequence of spaces with commuting projection operators¹

$$\begin{array}{ccccccc} H_0^1 & \xrightarrow{\text{grad}} & H_0(\text{curl}) & \xrightarrow{\text{curl}} & H_0(\text{div}) & \xrightarrow{\text{div}} & L^2 \\ \downarrow \Pi^0 & & \downarrow \Pi^1 & & \downarrow \Pi^2 & & \downarrow \Pi^3 \\ V_h^0 & \xrightarrow{\text{grad}} & V_h^1 & \xrightarrow{\text{curl}} & V_h^2 & \xrightarrow{\text{div}} & V_h^3 \end{array}$$

- FEEC: unified analysis → key element are the **stable commuting projections**² Π^ℓ
- eigenvalue problem:
 - convergence of eigenmodes
 - correct dim of discrete **harmonic fields**
 - discrete Hodge-Helmholtz decomp.
- source problem:
 - convergence (even for singular sol.)
 - structure preservation (Gauss laws)
 - time domain: long time stability

¹Whitney (57) Nédélec (80) Bossavit (88-98) Hiptmair (99) Arnold-Falk-Winther (06) Buffa et al (11) ...

²Schöberl (05) AFW (06) Christiansen-Winther (07) Falk-Winther (14) Arnold-Guzman (21)



The commuting projections of Buffa-Rivas-Sangalli-Vázquez (11)

- construction on tensor-product spaces (here in 2D)

$$V_h^0 = \mathbb{S}_p \otimes \mathbb{S}_p \quad \xrightarrow{\text{grad}} \quad V_h^1 = \begin{pmatrix} \mathbb{S}_{p-1} \otimes \mathbb{S}_p \\ \mathbb{S}_p \otimes \mathbb{S}_{p-1} \end{pmatrix} \quad \xrightarrow{\text{curl}} \quad V_h^3 = \mathbb{S}_{p-1} \otimes \mathbb{S}_{p-1}$$

- On V_h^0 :

$$\Pi^0 \phi := \sum_{\mathbf{i} \in \mathcal{I}^0} c_{\mathbf{i}}^0(\phi) \Lambda_{\mathbf{i}}^0 \quad \text{with} \quad c_{\mathbf{i}}^0(\phi) := \langle \Theta_{\mathbf{i}}^0, \phi \rangle$$

- On V_h^1 :

$$\Pi^1 \mathbf{u} := \sum_{d \in \{1,2\}} \nabla_d \Pi^0 \Phi_d(\mathbf{u}) \quad \text{with} \quad \begin{cases} \Phi_1(\mathbf{u})(\mathbf{x}) := \int_0^{x_1} u_1(z, x_2) dz \\ \Phi_2(\mathbf{u})(\mathbf{x}) := \int_0^{x_2} u_2(x_1, z) dz \end{cases}$$

- On V_h^2 :

$$\Pi^2 \rho := \partial_1 \partial_2 \Pi^0 \Psi(\rho) \quad \text{with} \quad \Psi(\rho)(\mathbf{x}) := \int_0^{x_1} \int_0^{x_2} \rho(z_1, z_2) dz_2 dz_1$$



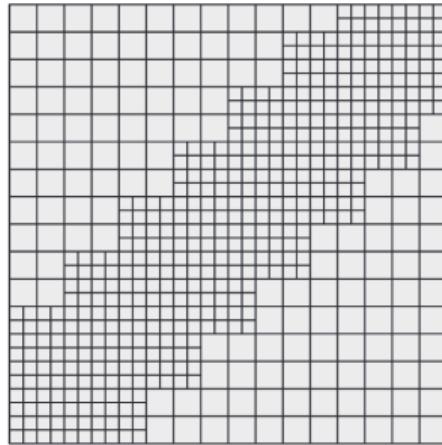
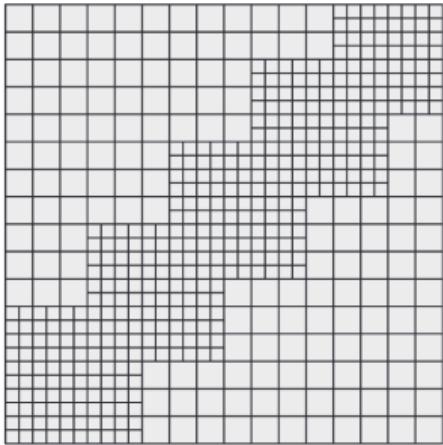
Locally refined meshes I

- Local treatment of computationally demanding regions (e.g. reentrant corners)
- Reduction of computation time



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Evans, J et al. (18): Constructs a framework of hierarchical B-spline complexes of discrete differential forms, which offer general refinements but the conditions for spectral correctness (and proper cohomology) are not well understood yet.



Locally refined meshes I

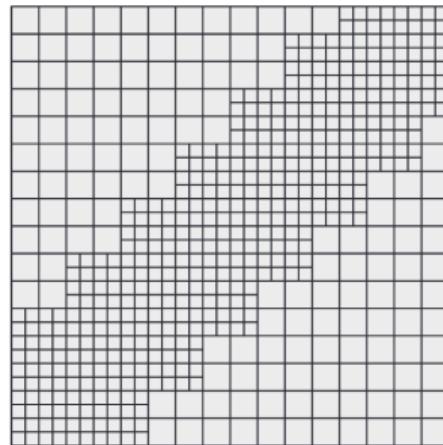
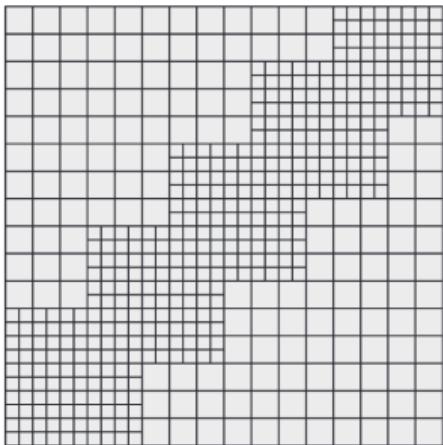
- Local treatment of computationally demanding regions (e.g. reentrant corners)
- Reduction of computation time

Assumption 1: ✓

Assumption 2: X

Exact sequence: X

Spec. correct: ✓



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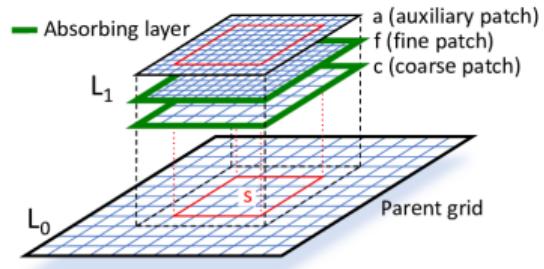
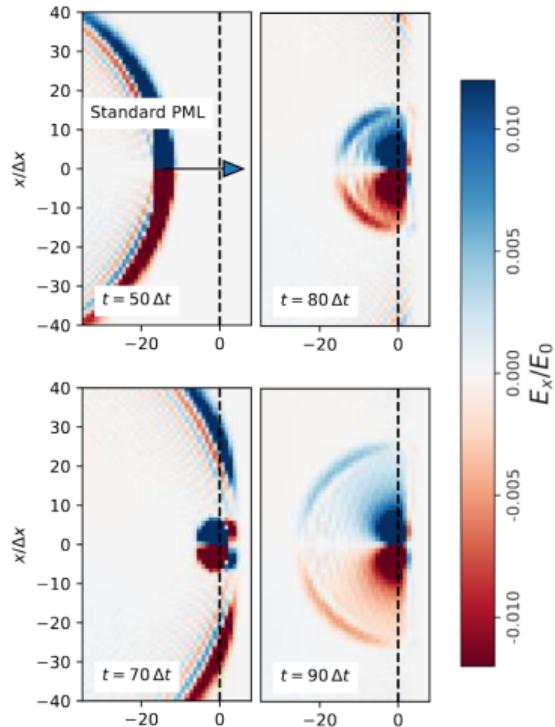
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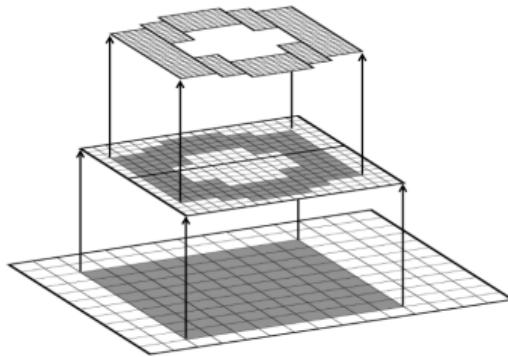
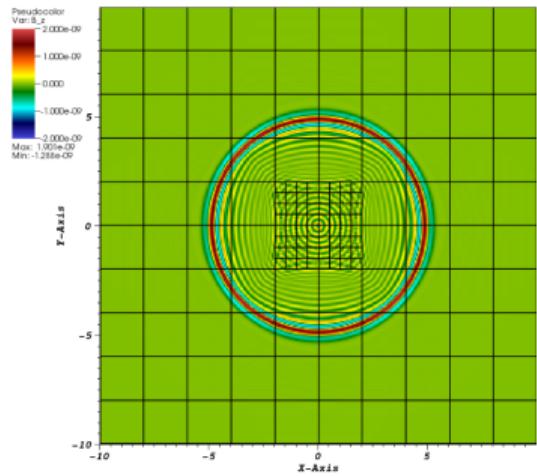
Locally refined meshes II



Vay, J. L. et al. (12, 22): Uses finite difference schemes (FDTD) with absorbing layers (PML) between non-matching grids.



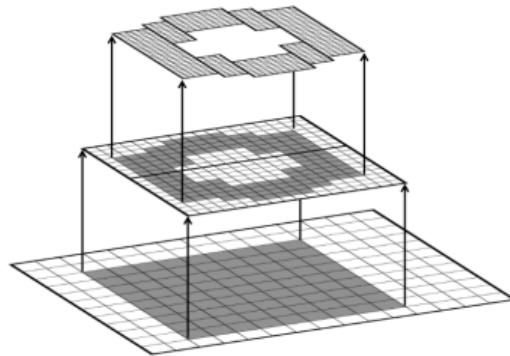
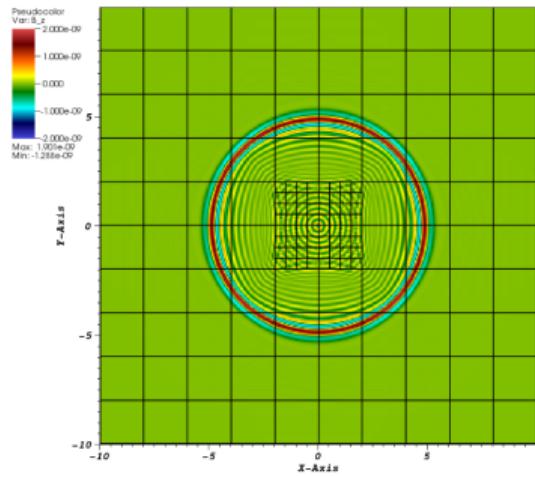
Locally refined meshes III



Chilton, S (13): Uses finite volume schemes and RK4, with damping and dissipation in transition layers between non-matching grids.



Locally refined meshes III



Chilton, S (13): Uses finite volume schemes and RK4, with damping and dissipation in transition layers between non-matching grids.

→ Q3: How to solve wave problems and **avoid spurious reflections?**

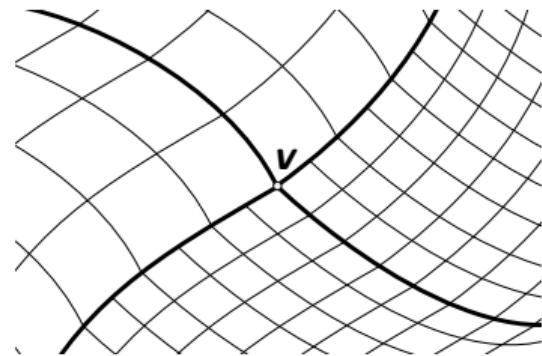
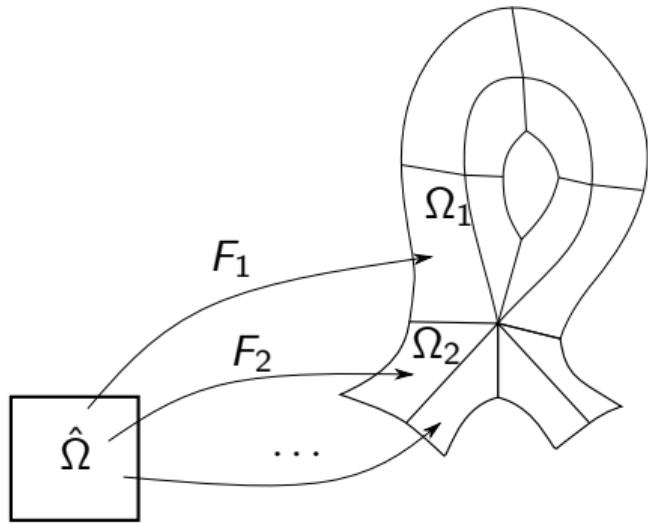
Broken-FEEC



Multi-patch domains with non-matching interfaces

Adjacent patches may have different mappings ...

... and different grids



Challenge: Break of the tensor-product structure near the interfaces
→ How to couple neighboring patches while preserving the FEEC structure?



Broken-FEBC approach

- broken (patch-wise) spaces: $V_{\text{pw}}^\ell := \{v \in L^2(\Omega) : v|_{\Omega_k} \in V_k^\ell \text{ for } k \in \mathcal{K}\}$
- patch-wise differential operators

$$V_{\text{pw}}^0 \xrightarrow{\text{grad}_{\text{pw}}} V_{\text{pw}}^1 \xrightarrow{\text{curl}_{\text{pw}}} V_{\text{pw}}^2$$

- conforming subspaces

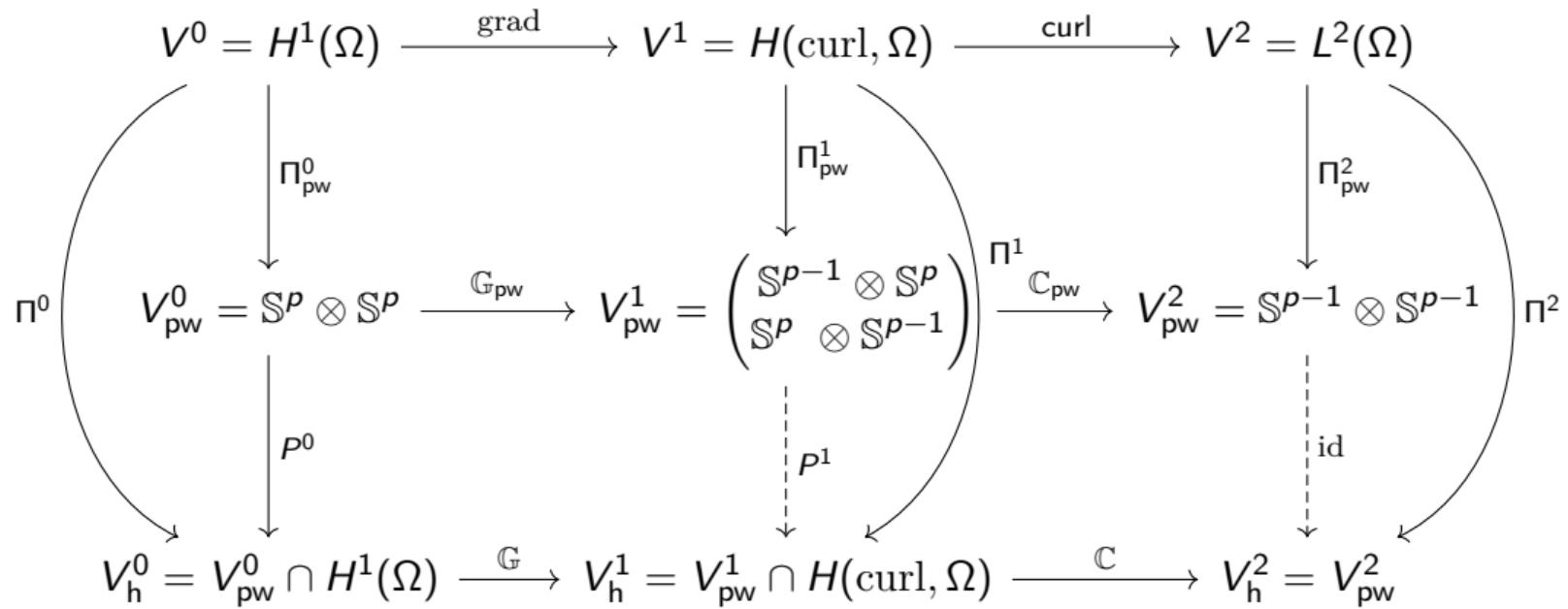
$$V_h^0 = V_{\text{pw}}^0 \cap H^1(\Omega) \xrightarrow{\text{grad}} V_h^1 = V_{\text{pw}}^1 \cap H(\text{curl}; \Omega) \xrightarrow{\text{curl}} V_h^2 = V_{\text{pw}}^2 \cap L^2(\Omega) = V_{\text{pw}}^2.$$

- for example on V_h^0 , an L^2 -stable projection is given by

$$\Pi^0 := P^0 \Pi_{\text{pw}}^0 \quad \text{with} \quad \begin{cases} \Pi_{\text{pw}}^0 = \sum_k \Pi_k^0 : L^p \rightarrow V_{\text{pw}}^0 & (\text{stable, patch-wise}) \\ P^0 : V_{\text{pw}}^0 \rightarrow V_h^0 & (\text{averages interface dofs}) \end{cases}$$



Broken commuting diagram with Splines in 2D



Careful: $\Pi^1 \neq P^1 \Pi_{\text{pw}}^1$.



Existence of (non-matching) stable commuting projections

Theorem¹ If interior vertices are shared by four patches with nested discretization, there exist commuting projections $\Pi^\ell : L^p \rightarrow V_h^\ell$, local and stable for $1 \leq p \leq \infty$.

Broken-FEEC construction: patchwise projections + interface corrections

- on V_h^0 we define

$$\Pi^0 := P\Pi_{\text{pw}}^0 \quad \text{with} \quad \begin{cases} \Pi_{\text{pw}}^0 = \sum_{k \in \mathcal{K}} \Pi_k^0 : L^p \rightarrow V_{\text{pw}}^0 & \text{(stable, patch-wise)} \\ P : V_{\text{pw}}^0 \mapsto V_h^0 & \text{(averages interface dofs)} \end{cases}$$

- on V_h^1 we set

$$\Pi^1 := \Pi_{\text{pw}}^1 + \sum_{e \in \mathcal{E}} \tilde{\Pi}_e^1 + \sum_{v \in \mathcal{V}} \tilde{\Pi}_v^1 + \sum_{(e,v) \in \mathcal{E} \times \mathcal{V}} \tilde{\Pi}_{e,v}^1 \quad \text{with}$$

$$\tilde{\Pi}_g^1 \mathbf{u} := \sum_{d \in \{\parallel, \perp\}} \nabla_d^g Q_g \Pi_{\text{pw}}^0 \Phi_d^g(\mathbf{u}) \quad \text{with} \quad \begin{cases} \Phi_d^g : L^p \rightarrow L^p & \text{(antiderivative)} \\ Q_g : V_{\text{pw}}^0 \mapsto V_{\text{pw}}^0 & \text{(interface jump)} \end{cases}$$

- similar construction on V_h^2

¹Campos Pinto, FS (23) <https://arxiv.org/abs/2303.14449>

Construction of numerical schemes

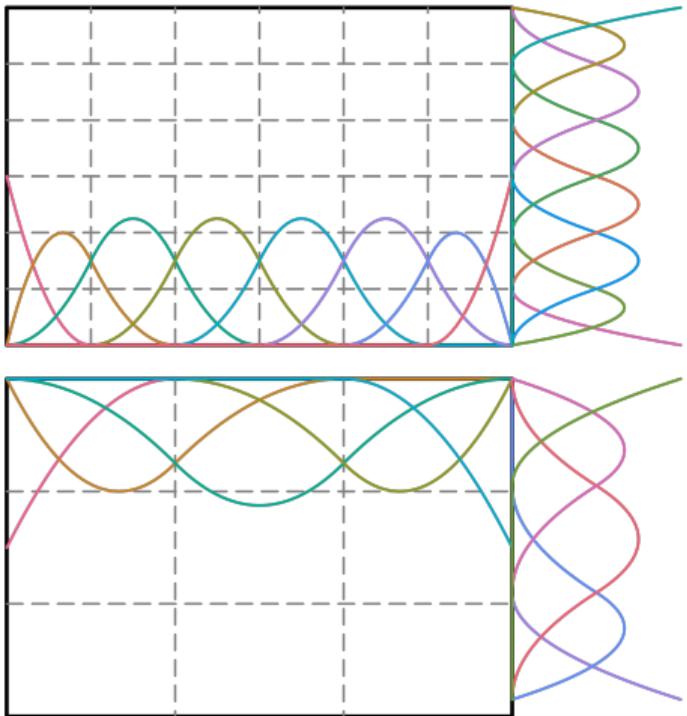


Numerical schemes - preliminaries

Given: Multiple patches, with different grids but single-patch FEEC spaces V_{pw}^{ℓ} .

Goal: Efficient way to map onto the conforming subspaces V_h^{ℓ} which have the structure-preserving properties established by the stable commuting projections.

Idea: Establish the correct conformity depending on V^{ℓ} in a geometric way. (On vertices and edges sequentially for $V^0 = H^1$, tangential to edges for $V^1 = H(\text{curl})$, ...)





Vertex conformity: P_v

Let $\Lambda_i^k = \lambda_{i_1}^k \otimes \lambda_{i_2}^k$ denote the B-Spline basis functions on patch k . Fix a vertex $v \in \mathcal{V}$.

$$P_v : \Lambda_i^k \mapsto \begin{cases} \frac{1}{|\mathcal{K}(v)|} \sum_{k' \in \mathcal{K}(v)} \Lambda_{i^{k'}(v)}^{k'} & \text{if } i = i^k(v), \\ \Lambda_i^k & \text{else,} \end{cases}$$

where

$\mathcal{K}(v)$ is the collection of patches adjacent to the vertex v ,

$i^k(v)$ is the index of the single vertex basis function on patch k with support on v .



Vertex conformity: $P_{\mathcal{V}}$

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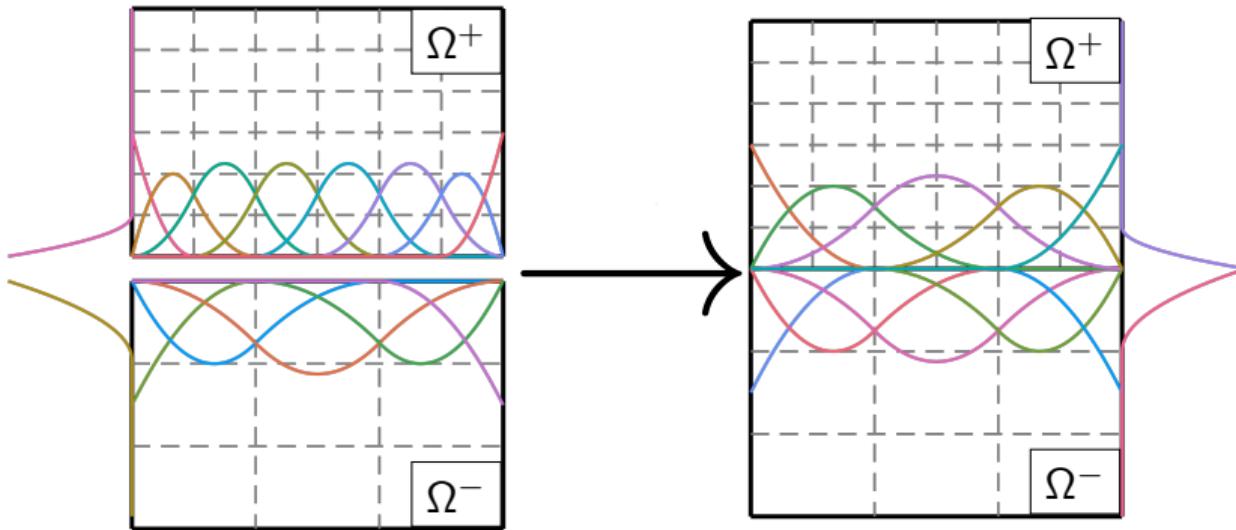
$$P_{\mathcal{V}} = \prod_{v \in \mathcal{V}} P_v$$

→ Only modifies vertex DOF, leaves interior and edge-interior DOF unchanged.



Edge conformity

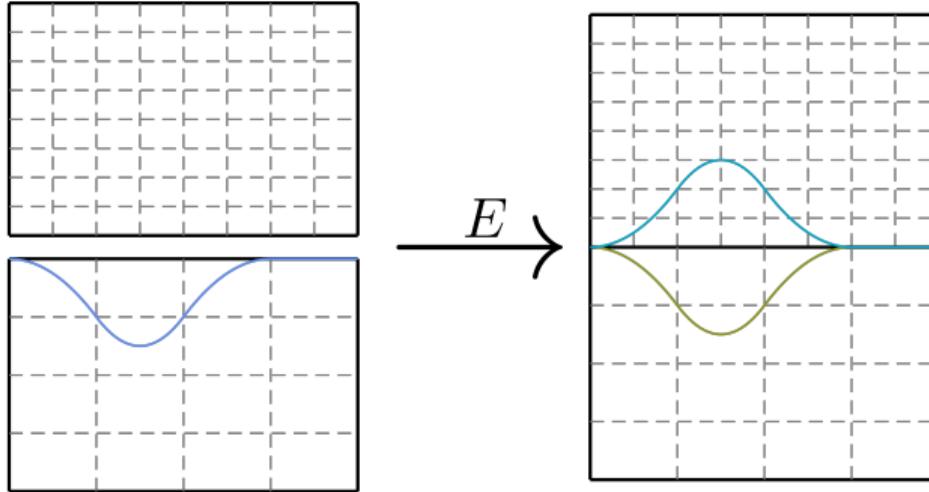
Define the projection P_e at the edge $e \in \mathcal{E}$,



that maps the basis functions on the interface to the conforming space
→ leave interior (and vertex) DOF unchanged.



Extension operator

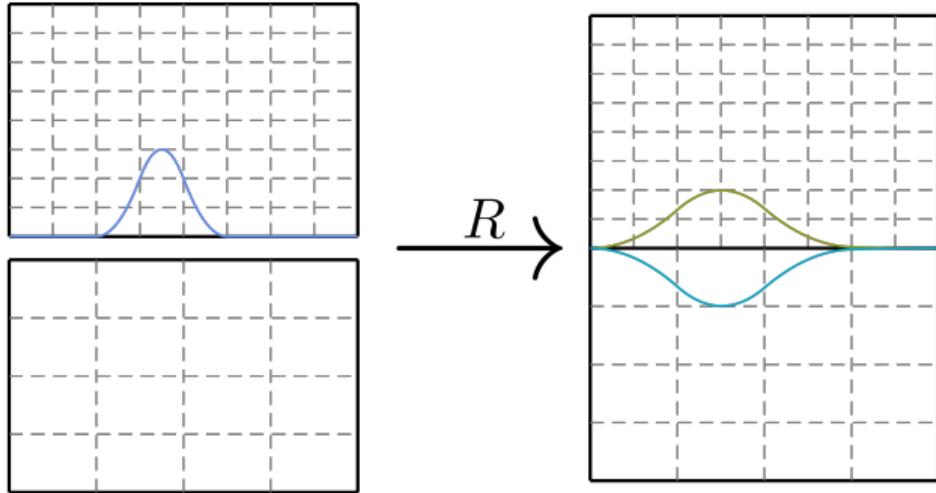


$$E : \lambda_i^- \otimes \lambda_0^- \mapsto \underbrace{\lambda_i^-}_{\sum_{j=0}^{n^+-1} \mathbb{E}_{i,j} \lambda_j^+} \otimes \frac{1}{2} (\lambda_0^- + \lambda_0^+)$$

Change of basis, with interpolation coefficients $\mathbb{E}_{i,j}$ obtained by knot-insertion.



Restriction operator



$$\begin{aligned} R : \lambda_i^+ \otimes \lambda_0^+ &\mapsto \underbrace{\mathcal{R}\lambda_i^+}_{= \sum_{j=0}^{n^- - 1} \mathbb{R}_{i,j} \lambda_j^-} \otimes \frac{1}{2} (\lambda_0^- + \lambda_0^+) \\ &= \sum_{j=0}^{n^+ - 1} (\mathbb{E}\mathbb{R})_{i,j} \lambda_j^+ \end{aligned}$$

E.g. modified L^2 -projection of a fine spline into the coarse spline space.



Edge conformity: $P_{\mathcal{E}}$

Assume $\mathbf{j} = (j_1, j_2) = (j_{\parallel(e)}, j_{\perp(e)})$ for a fixed edge $e \in \mathcal{E}$.

$$P_e : \begin{cases} \Lambda_j^- \mapsto \begin{cases} \lambda_j^- \otimes \frac{1}{2} (\lambda_{i_\perp^-(e)}^- + \lambda_{i_\perp^+(e)}^+) & \text{if } 0 < j_1 = j < n^-, j_2 = i_\perp^-(e), \\ \Lambda_j^- & \text{else.} \end{cases} \\ \Lambda_j^+ \mapsto \begin{cases} \mathcal{R} \lambda_j^+ \otimes \frac{1}{2} (\lambda_{i_\perp^-(e)}^+ + \lambda_{i_\perp^+(e)}^-) & \text{if } 0 < j_1 = j < n^+, j_2 = i_\perp^+(e), \\ \Lambda_j^+ & \text{else.} \end{cases} \end{cases}$$

where $i_\perp^\pm(e)$ is the index of the perpendicular basis function on patch k^\pm with support on e .



Edge conformity: $P_{\mathcal{E}}$

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where $i_\perp^\pm(e)$ is the index of the perpendicular basis function on patch k^\pm with support on e .

$$P_{\mathcal{E}} = \prod_{e \in \mathcal{E}} P_e$$

→ Only modifies edge-interior DOF, leaves interior and vertex DOF unchanged



Non-matching conforming projections

Finally:

$$P^0 = P_{\mathcal{E}} P_{\mathcal{V}} : V_{\text{pw}}^0 \rightarrow V_{\text{pw}}^0$$

is a projection onto the conforming space V_h^0 .



Non-matching conforming projections

Finally:

$$P^0 = P_{\mathcal{E}} P_{\mathcal{V}} : V_{\text{pw}}^0 \rightarrow V_{\text{pw}}^0$$

is a projection onto the conforming space V_h^0 .

The construction for V^1 is similar, but we only enforce conformity on the tangential traces (on the whole edge **including vertex DOF**):

$$\mathbf{P}^1 = \sum_{e \in \mathcal{E}} \mathbf{P}_e^1 : V_{\text{pw}}^1 \rightarrow V_{\text{pw}}^1,$$

is a projection onto the conforming subspace V_h^1 .

Numerical Experiments



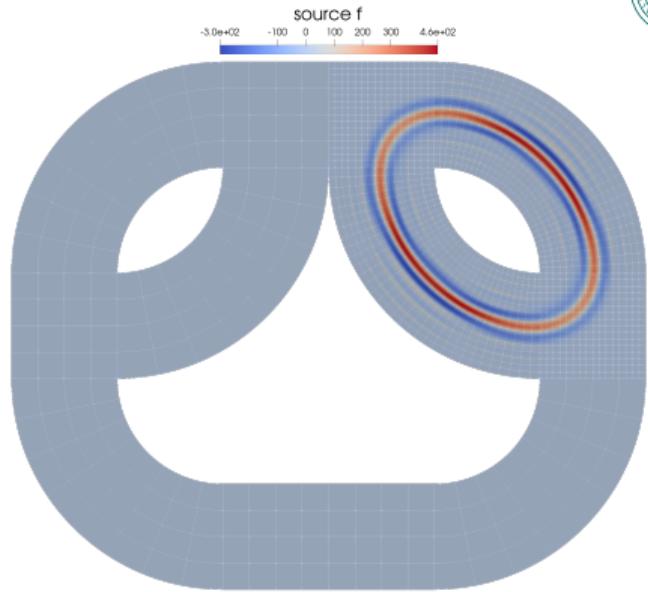
Poisson problem (pretzel domain)

Find $\phi \in V^0 = H_0^1(\Omega)$ such that

$$-\Delta\phi = f,$$

where $f \in L^2(\Omega)$.

The domain consists of 18 patches, we only refine the top-right handle (6 patches).



Within the broken-FEEC framework¹, this equation is discretized as

$$\left(\left(\mathbb{G}_{\text{pw}} \mathbb{P}^0 \right)^T \mathbb{M}^1 \mathbb{G}_{\text{pw}} \mathbb{P}^0 + \underbrace{\left(\mathbb{I} - \mathbb{P}^0 \right)^T \mathbb{M}^0 \left(\mathbb{I} - \mathbb{P}^0 \right)}_{\text{jump stabilization}} \right) \phi_h = (\mathbb{P}^0)^T f_h.$$

¹All numerical examples are performed using <https://github.com/pyccel/psydac>
| MARTIN CAMPOS PINTO, FREDERIK SCHNACK | NOVEMBER 9, 2024

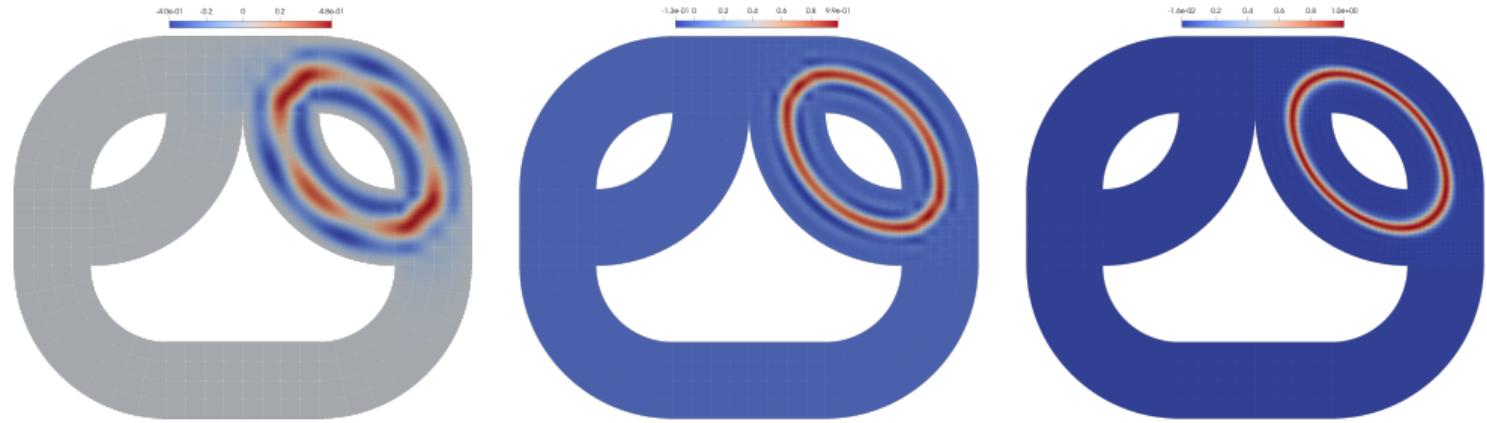


Figure: Numerical solution ϕ_h for degree 3 and refinement of the top-right handle.

Cells p.p.	Degree	uniform refinement				refined handle			
		2	3	4	5	2	3	4	5
4	6.71637	0.86527	0.34468	0.18317	6.71637	0.86527	0.34468	0.18317	
8	0.31598	0.04601	0.08671	0.05566	0.31596	0.04606	0.08675	0.05573	
16	0.01516	0.00489	0.00575	0.00361	0.01516	0.00489	0.00575	0.00361	

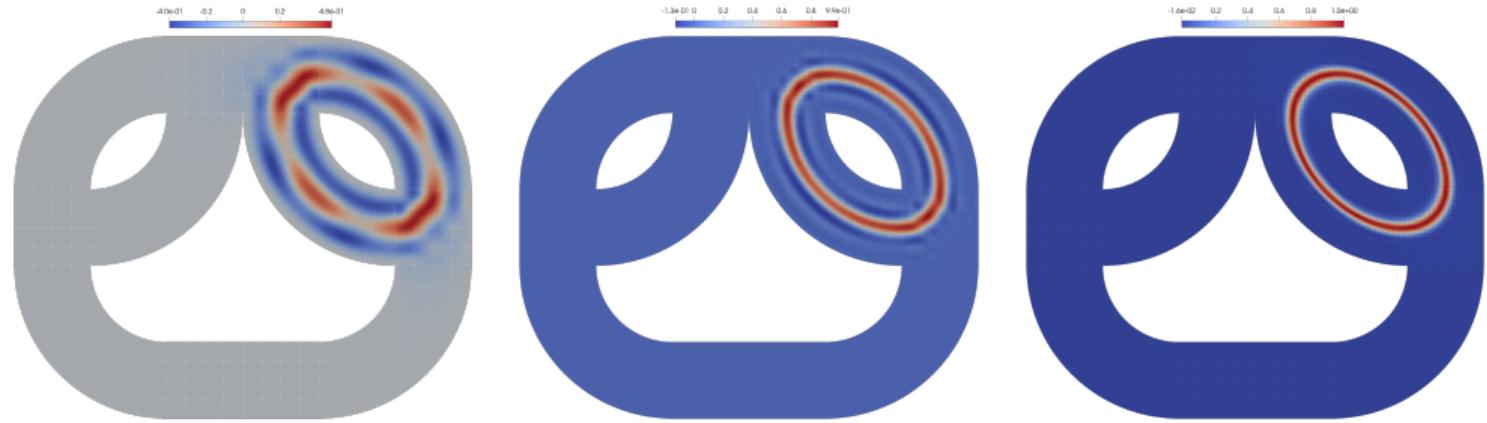


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8	8	0.31598	0.04601	0.08671	0.05566	0.31596	0.04606	0.08675	0.05573
16	16	0.01516	0.00489	0.00575	0.00361	0.01516	0.00489	0.00575	0.00361

→ On the finest level only half the DOF needed for the same accuracy.



Time-harmonic Maxwell equation (pretzel domain)

For $\omega \in \mathbb{R}$ and $\mathbf{J} \in L^2(\Omega)$, solve

$$-\omega^2 \mathbf{E} + \mathbf{curl} \mathbf{curl} \mathbf{E} = \mathbf{J},$$

for $\mathbf{E} \in H(\mathbf{curl}, \Omega)$.

In our framework, that reads

$$\left(-\omega^2 (\mathbb{P}^1)^T \mathbb{M}^1 \mathbb{P}^1 + (\mathbb{C}_{\text{pw}} \mathbb{P}^1)^T \mathbb{M}^2 \mathbb{C}_{\text{pw}} \mathbb{P}^1 + \underbrace{(\mathbf{I} - \mathbb{P}^1)^T \mathbb{M}^1 (\mathbf{I} - \mathbb{P}^1)}_{\text{jump stabilization}} \right) \mathbf{E}_h = (\mathbb{P}^1)^T \mathbf{J}_h.$$

Manufactured solution: $\boldsymbol{J} = \begin{pmatrix} -\pi^2 \sin(\pi y) \cos(\pi x) \\ 0 \end{pmatrix}, \quad \boldsymbol{E} = \begin{pmatrix} \sin(\pi y) \\ \sin(\pi x) \cos(\pi y) \end{pmatrix}.$

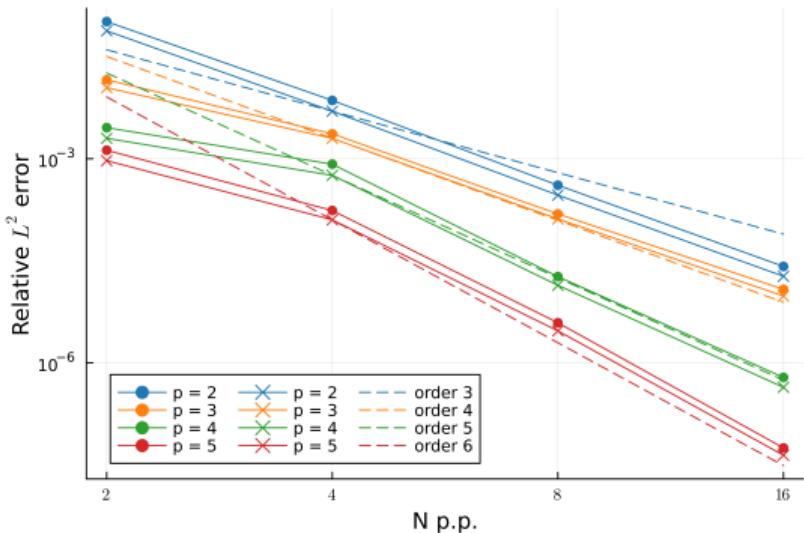


Figure: Convergence curves for matching (circle) and refined (cross) grids, where every other grid was of different resolution.

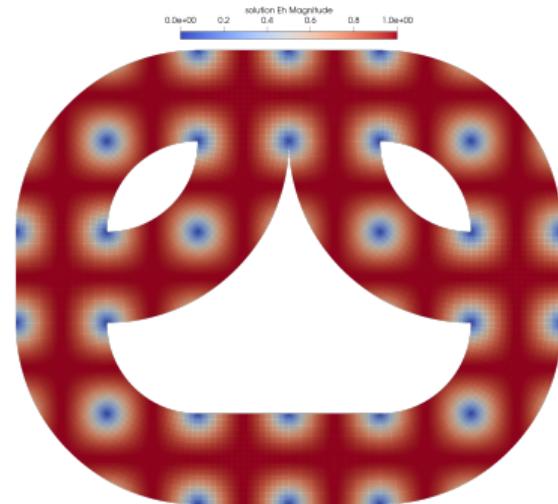


Figure: Numerical solution $\|\boldsymbol{E}_h\|$ for a refined grid between $N = 8, 16$ and $p = 3$.



Curl curl eigenvalue problem (curved L-shape domain)

Find $\lambda > 0$ such that

$$\operatorname{curl} \operatorname{curl} \mathbf{E} = \lambda \mathbf{E},$$

where $\mathbf{E} \in H_0(\operatorname{curl}, \Omega)$.

Within the broken-FEEC framework, this equation is discretized as

$$\left(\mathbb{C}_{\text{pw}} \mathbb{P}^1 \right)^T \mathbb{M}^2 \mathbb{C}_{\text{pw}} \mathbb{P}^1 \mathbf{E}_h = \lambda \left(\underbrace{(\mathbb{I} - \mathbb{P}^1)^T \mathbb{M}^1 (\mathbb{I} - \mathbb{P}^1) + (\mathbb{P}^1)^T \mathbb{M}^1 \mathbb{P}^1}_{\text{jump stabilization}} \right) \mathbf{E}_h,$$

where we refine layers around the reentrant corner and compare it to a uniform solution with similar amount of DOF.

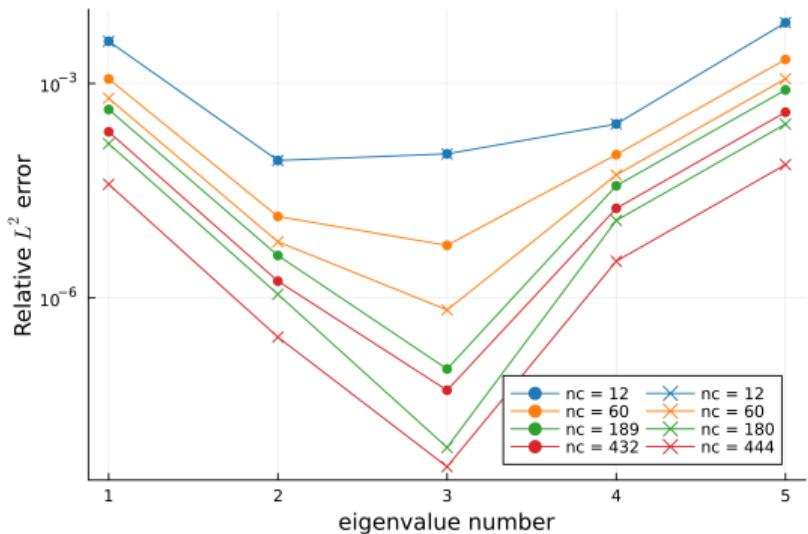


Figure: Error of each eigenvalue for matching (circle) and refined (cross) patches.

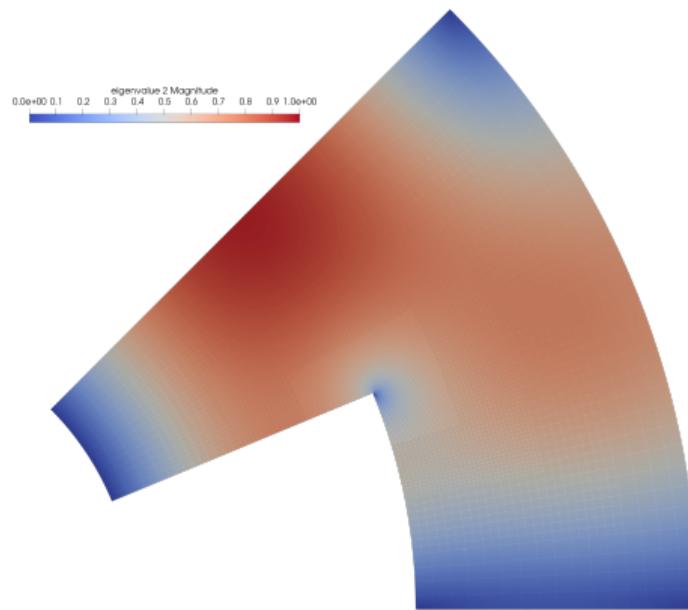


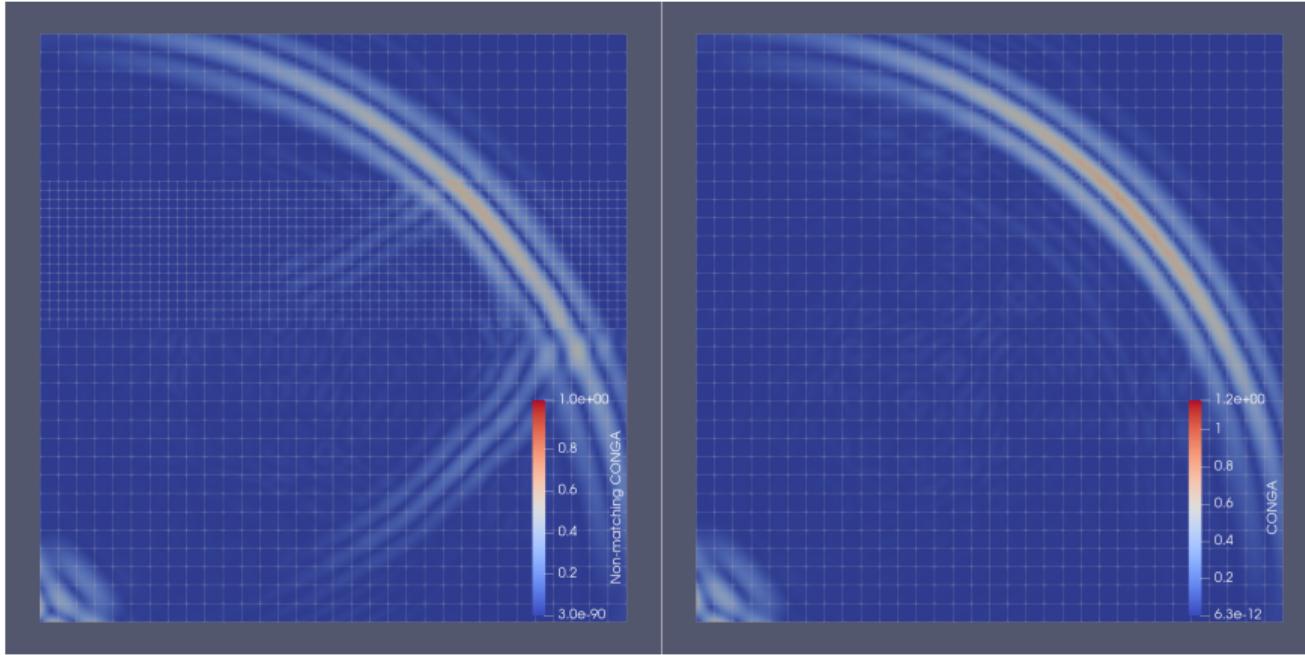
Figure: Absolute value of the eigenfunction corresponding to the eigenvalue λ_2 on the most refined grid.

Time domain Maxwell: Open problems from the previous NumKin talk



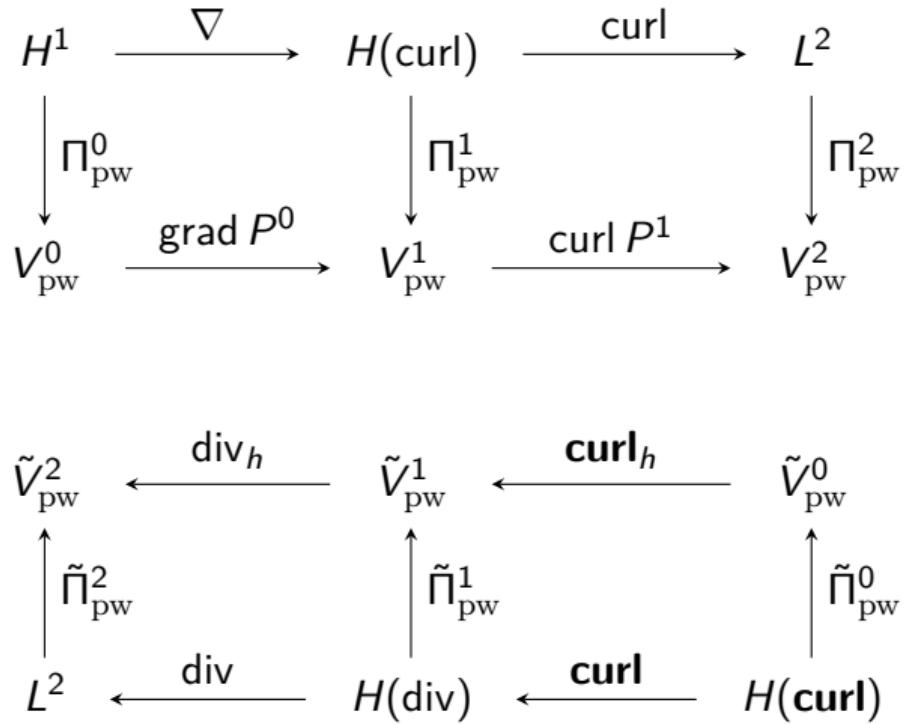
Spurious reflections at non-matching patch interfaces.

non-matching grid vs. matching grid





Motivation: Weak Divergence



Idea: if

$$\text{div}_h = -(\text{grad } P^0)^*,$$

then

$$\text{div } u \approx \text{div}_h Q_{\text{pw}}^1 u \quad \forall u \in H_0(\text{div})$$

is of high order as long as P^0 is preserving polynomial moments.



Preservation of polynomial moments

erienw Some properties of the projectors are improved by preserving polynomial moments:

$$\int P^\ell \phi q_j = \int \phi q_j, \quad \forall \phi \in V_{\text{pw}}^\ell, q_j \in \mathbb{P}_j, \quad j = 0, \dots, r$$

→ Introduces a set of linear constraints on the coefficients of P^0 and \mathbf{P}^1 .



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→ Introduces a set of linear constraints on the coefficients of P^0 and P^1 .

This property also ensures high-order approximation of the adjoint operator:

Let $u \in V_{\text{pw}}^\ell$, then

$$\langle (P^\ell)^* u, v \rangle = \langle u, P^\ell v \rangle = \langle u, v \rangle \quad \forall v \in V_{\text{pw}}^\ell.$$

We introduce the correction coefficients γ solving the linear system:

$$\sum_{m=1}^r \gamma_m \int \lambda_m q_j = \int \lambda_0 q_j, \quad \forall j$$

which establish preservation of polynomial moments and are independent of the mesh resolution.

Leading to the modified vertex-based operator:

$$P_v : \Lambda_j^k \mapsto \begin{cases} \bar{\Lambda}^v + \sum_{m_1=1}^r \sum_{m_2=1}^r \mathbf{a}_m \gamma_m \Lambda_m^k & \text{if } j \text{ at } v, \\ -\frac{1}{\#v} \sum_{k' \in \mathcal{K}(v)} \sum_{m_1=1}^r \sum_{m_2=1}^r \mathbf{a}_m \gamma_m \Lambda_m^{k'} & \text{else,} \\ \Lambda_j^k & \end{cases}$$

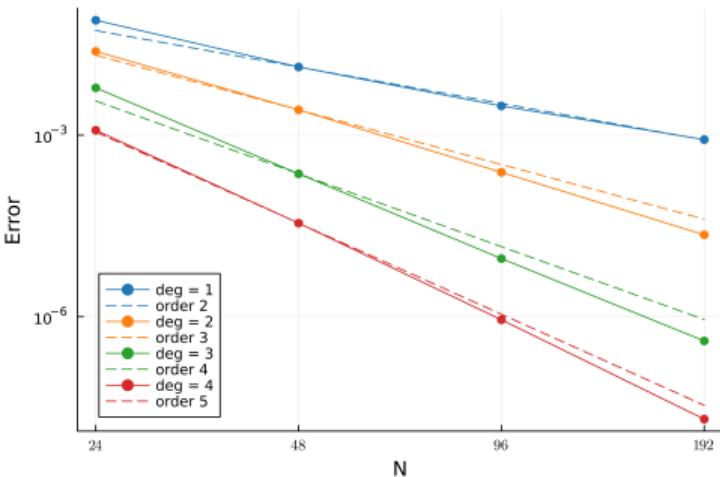
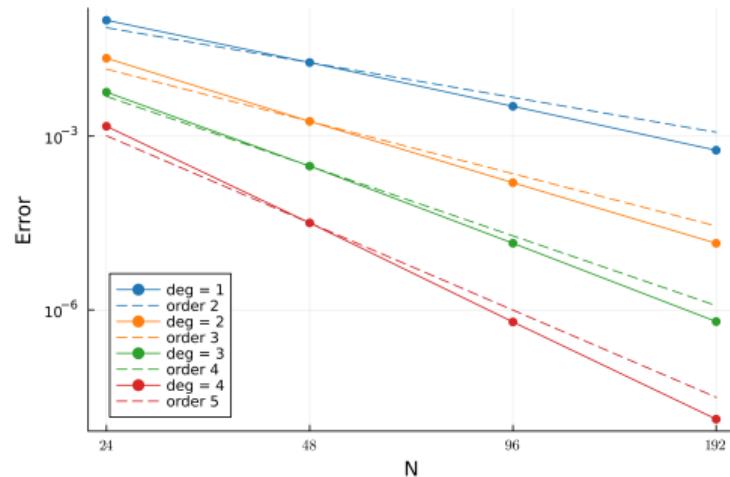
$$P_e : \begin{cases} \Lambda_j^- \mapsto \begin{cases} \lambda_{j_1}^- \otimes \left(\bar{\lambda}_\perp^e + \frac{1}{2} \left(\sum_{m=1}^r \gamma_m \lambda_m^- - \sum_{m=1}^r \gamma_m \lambda_m^+ \right) \right) & \text{if } j \text{ on e and not at v,} \\ \lambda_{j_1}^- \otimes \bar{\lambda}_\perp^e + \mu_v^- \otimes \frac{1}{2} \left(\sum_{m=1}^r \gamma_m \lambda_m^- - \sum_{m=1}^r \gamma_m \lambda_m^+ \right) & \text{if } j \text{ on e and at v,} \\ \Lambda_j^- & \text{else,} \end{cases} \\ \Lambda_j^+ \mapsto \begin{cases} \mathcal{R} \lambda_{j_1}^+ \otimes \left(\bar{\lambda}_\perp^e - \frac{1}{2} \left(\sum_{m=1}^r \gamma_m \lambda_m^- - \sum_{m=1}^r \gamma_m \lambda_m^+ \right) \right) & \text{if } j \text{ on e and not at v,} \\ \mathcal{R} \lambda_{j_1}^+ \otimes \bar{\lambda}_\perp^e - \mathcal{R} \mu_v^+ \otimes \frac{1}{2} \left(\sum_{m=1}^r \gamma_m \lambda_m^- - \sum_{m=1}^r \gamma_m \lambda_m^+ \right) & \text{if } j \text{ on e and at v} \\ \Lambda_j^+ & \text{else,} \end{cases} \end{cases}$$

where $\mu_v^+ = \sum_{i=1}^r \gamma_i^+ \lambda_i^+$ and $\mu_v^- = \lambda_0^- + \mathcal{R} (\mu_v^+ - \lambda_0^+)$.



Weak Divergence II

Convergence curves for the **weak divergence** (left) and the **weak curl** (right) operator on a curved L-shaped domain consisting of three patches with non-matching grids.





Time domain Maxwell equations (Square domain)

For $\mathbf{J} \in L^2(\Omega)$, solve

$$\begin{aligned}\partial_t \mathbf{E} - \mathbf{curl} \mathbf{B} &= -\mathbf{J}, \\ \partial_t \mathbf{B} + \mathbf{curl} \mathbf{E} &= 0,\end{aligned}$$

where $\mathbf{E} \in H_0(\mathbf{curl}, \Omega)$ and $\mathbf{B} \in L^2(\Omega)$.

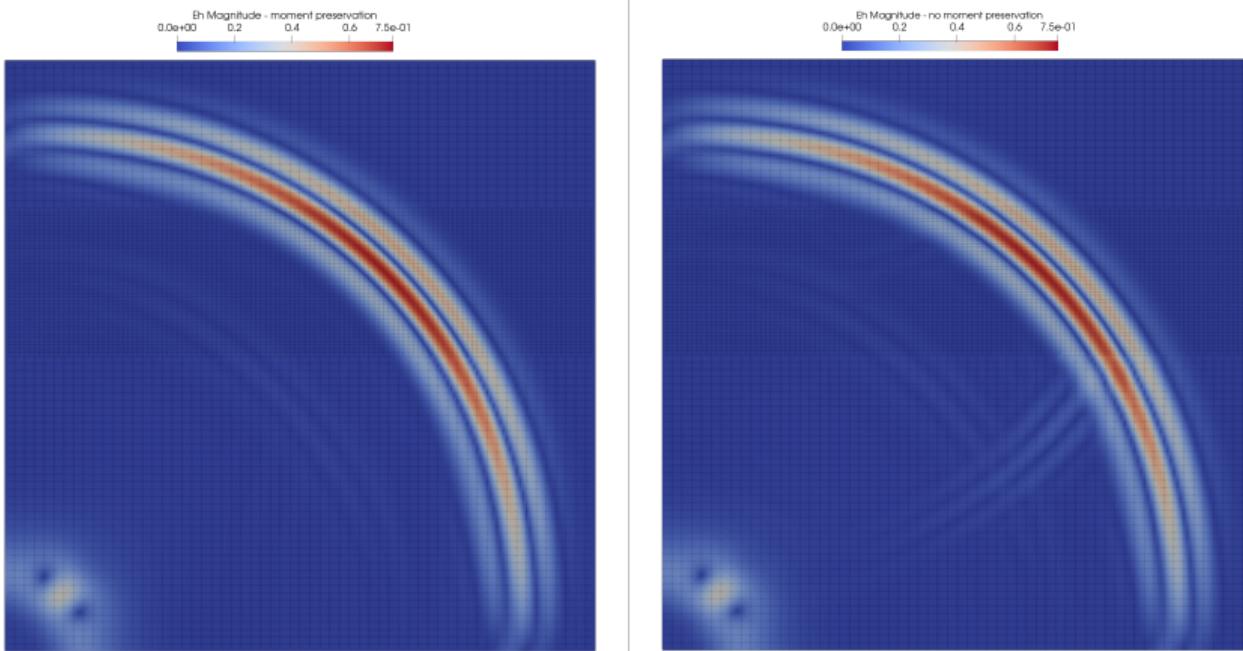
Cast into a simple leap-frog time-stepping scheme, this reads

$$\begin{aligned}B_h^{n+\frac{1}{2}} &= B_h^n - \frac{\Delta t}{2} \mathbb{C}_{\text{pw}} E_h^n, \\ \mathbb{M}^1 E_h^{n+1} &= \mathbb{M}^1 E_h^n + \Delta t \left((\mathbb{C}_{\text{pw}} \mathbb{P}^1)^T \mathbb{M}^2 B_h^{n+\frac{1}{2}} - \mathbb{P}^1 J_h^{n+\frac{1}{2}} \right), \\ B_h^{n+1} &= B_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \mathbb{C}_{\text{pw}} E_h^{n+1}.\end{aligned}$$



Redemption of results

Moment preservation vs. no moment preservation





Electromagnetic wave (without source)

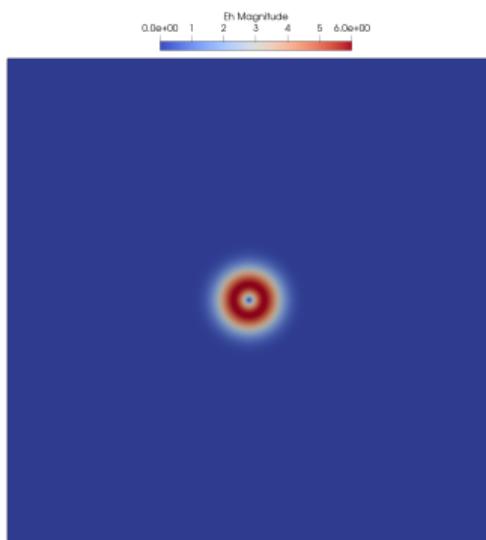
In our example, we choose the initial conditions

$$\mathbf{E}(t = 0; \mathbf{x}) = \mathbf{p} \cos(\mathbf{k} \cdot \mathbf{x}) e^{-\frac{x^2}{2\sigma^2}},$$

$$\mathbf{B}(t = 0; \mathbf{x}) = \text{curl } \mathbf{E},$$

$$\mathbf{J}(t, \mathbf{x}) \equiv 0.$$

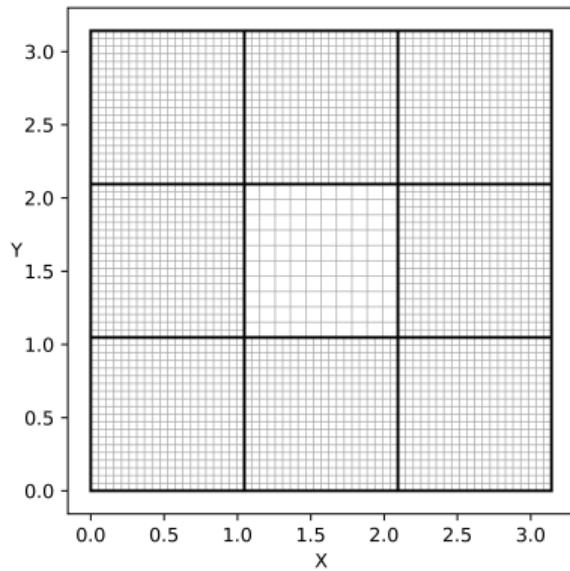
See the initial magnitude of E_h^0 on the right.



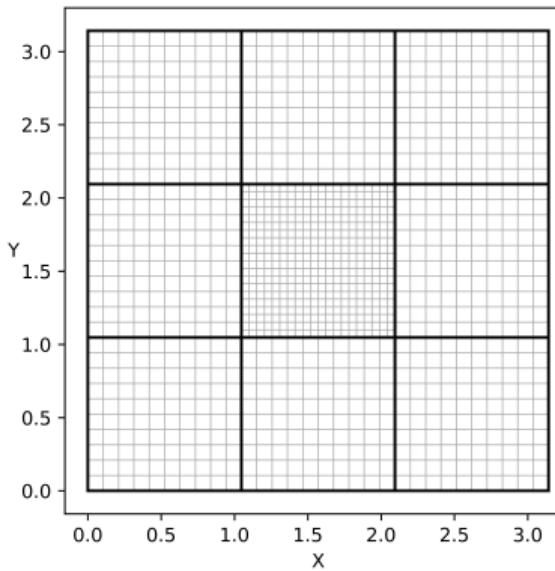


Electro-magnetic wave: domains

Coarse to fine



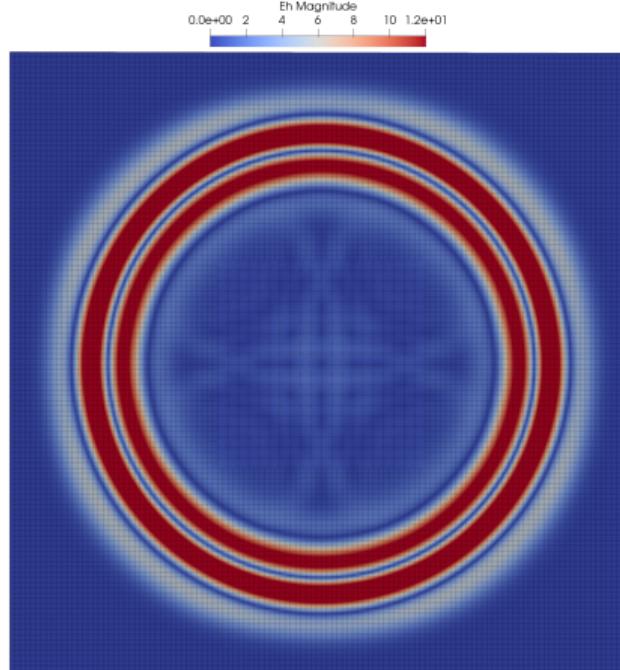
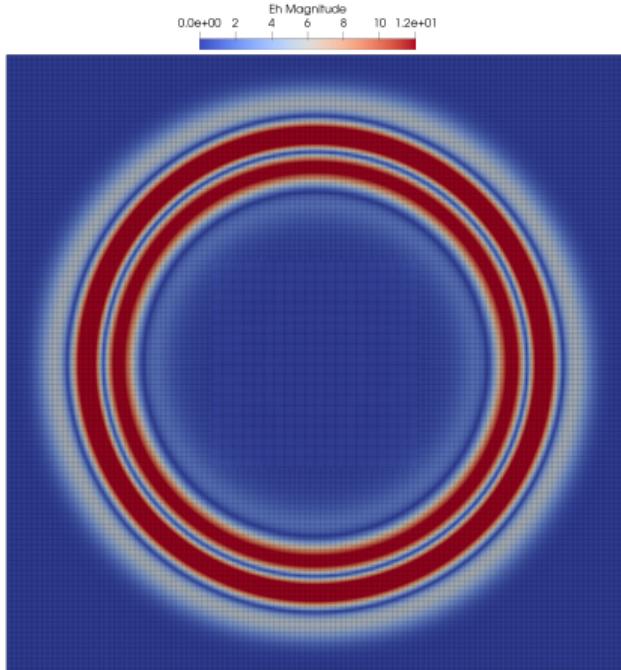
Fine to coarse





Electro-magnetic wave: Coarse to fine

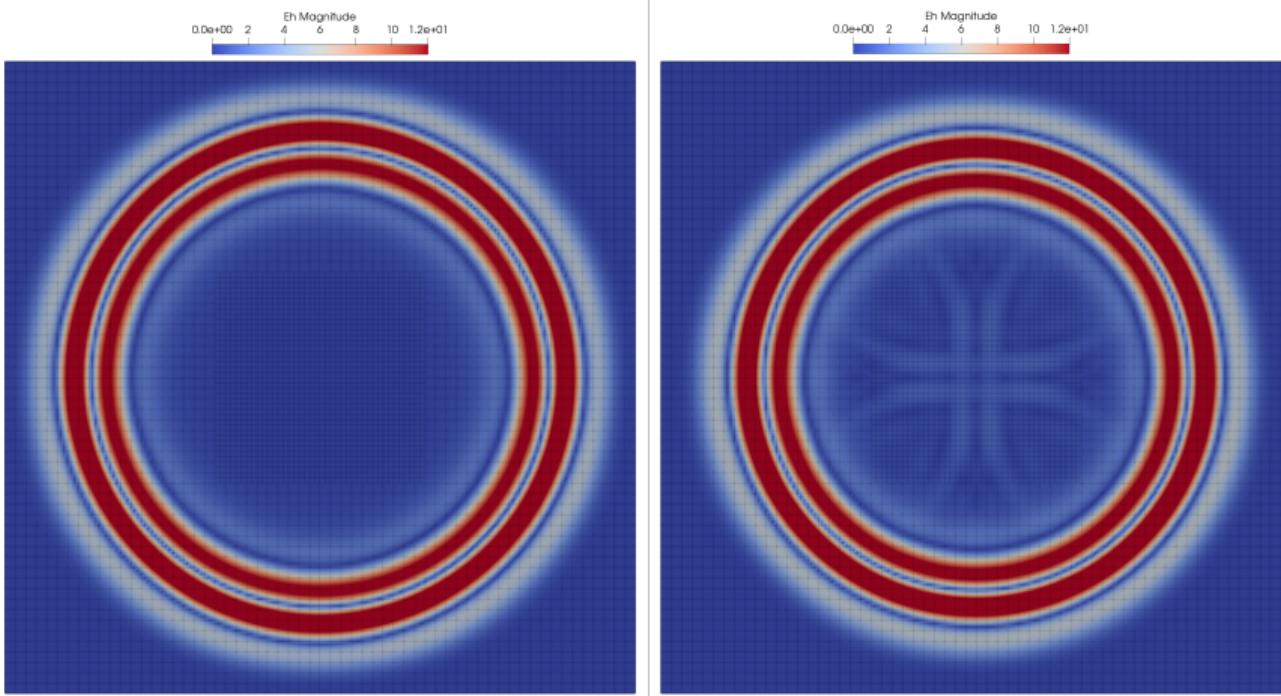
Moment preservation vs. no moment preservation





Electro-magnetic wave: Fine to coarse

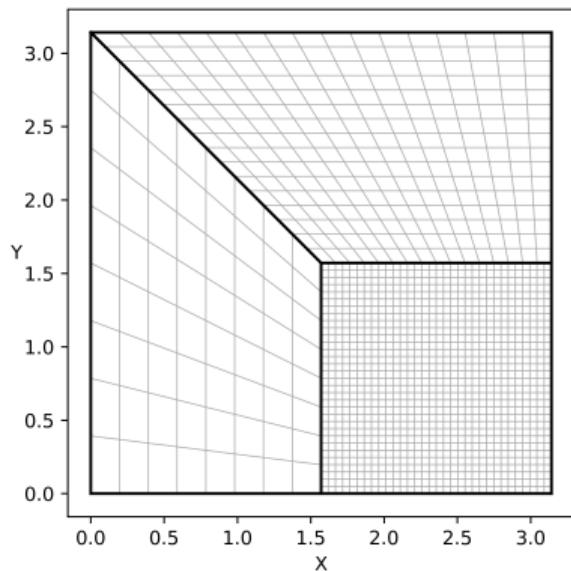
Moment preservation vs. no moment preservation



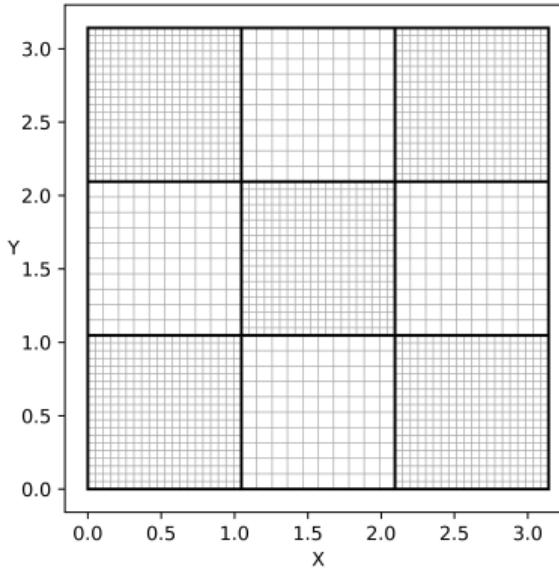


Restrictions by theory

Curl-curl eigenvalue problem



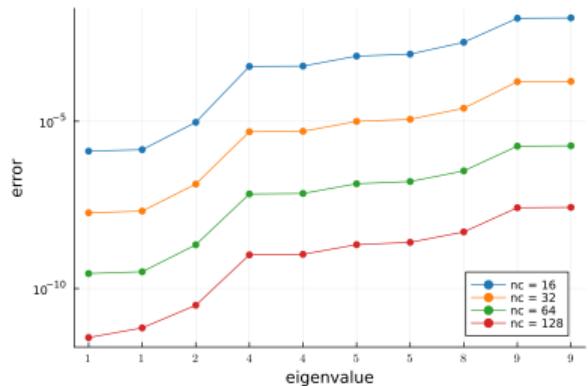
Time-domain Maxwell



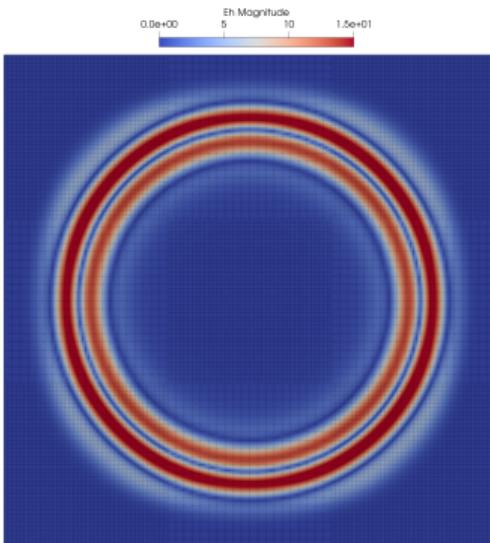


Restrictions by theory

Curl-curl eigenvalue problem



Time-domain Maxwell



...which work well in practice.

Conclusion



Take home messages

- FEEC: Good for domains with corners or holes
- Broken-FEEC + Conforming projections: Good for (non-matching) multi-patch (local, modular schemes) on complex domains



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Take home messages

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Take home messages

- FEEC: Good for domains with corners or holes
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→ Q3

Thank you for your attention!

Backup Slides



THEORY





The commuting projections of Buffa-Rivas-Sangalli-Vázquez (2011) – cont'd

- projection property:
 - for Π^0 , as usual $\langle \Theta_i^0, \Lambda_j^0 \rangle = \delta_{i,j}$
 - for Π^1 , follows from $\int_0^{x_1} u_1(z, x_2) dz \in V_h^0$ (same argument for Π^2)
- commuting property: for the grad,

$$(\Pi^1 \nabla \phi(\mathbf{x}))_1 = \partial_1 (\Pi^0 \Phi_1^1(\nabla \phi)(\mathbf{x})) = \partial_1 \Pi^0 (\phi(\mathbf{x}) - \phi(0, x_2)) = \partial_1 \Pi^0 \phi(\mathbf{x}) \quad (\text{same for the curl})$$

- L_h^2 stability:
 - on V_h^0 , follows from locally stable dofs
 - on V_h^1 , follows from a localization argument: on a local domain ω_i , we have

$$\nabla_1 \Pi^0 \Phi_1^1(\mathbf{u}) = \nabla_1 \Pi^0 \Phi_{1,i}^1(\mathbf{u}) \quad \text{with} \quad \Phi_{1,i}^1(\mathbf{u})(\mathbf{x}) := \int_{x_1(\omega_i)}^{x_1} u_1(z, x_2) dz$$

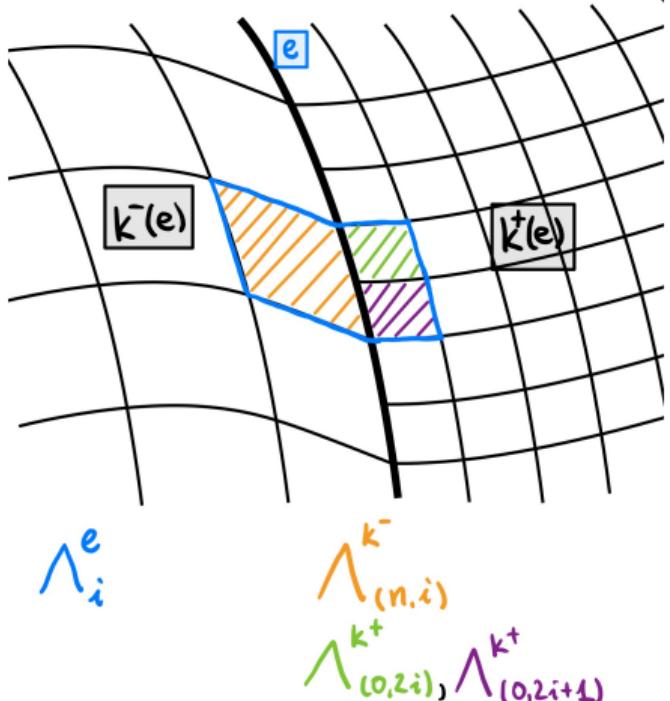
and inverse / local stability estimates

$$\|\nabla_1 \Pi^0 \Phi_{1,i}^1(\mathbf{u})\|_{L^2(\omega_i)} \lesssim h^{-1} \|\Pi^0 \Phi_{1,i}^1(\mathbf{u})\|_{L^2(E_h(\omega_i))} \lesssim h^{-1} \|\Phi_{1,i}^1(\mathbf{u})\|_{L^2(E_h^2(\omega_i))} \lesssim \|\mathbf{u}\|_{L^2(E_h^3(\omega_i))}$$

- on V_h^2 , the same principle applies
- Note: as long as $\text{supp}(\Theta_i^0) \approx \text{supp}(\Lambda_i^0)$ these operators are local



Edge Conforming basis functions



$$\Lambda_i^e(\mathbf{x}) := \begin{cases} \hat{\lambda}_i^-(\hat{x}_{||}^-)\hat{\lambda}_{i_{\perp}^-(e)}^-(\hat{x}_{\perp}^-) & \text{on } \Omega_{k-}, \\ \hat{\lambda}_i^-(\eta_e(\hat{x}_{||}^+))\hat{\lambda}_{i_{\perp}^+(e)}^+(\hat{x}_{\perp}^+) & \text{on } \Omega_{k+}, \\ 0 & \text{else.} \end{cases}$$

- Projection on the local edge-broken space:

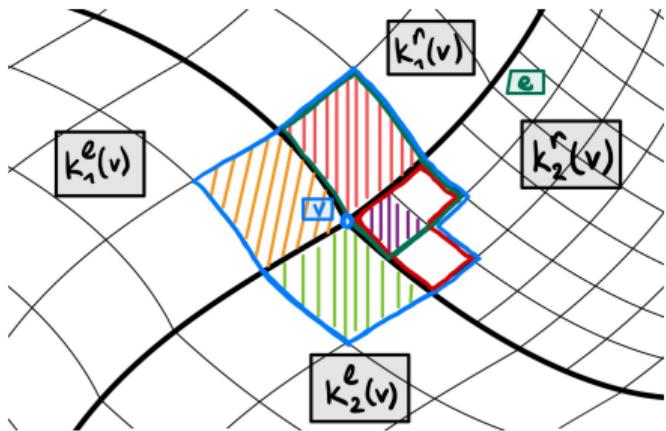
$$I_e : V_{\text{pw}}^0 \rightarrow V_{\text{pw}}^0, \quad \Lambda_i^k \mapsto \begin{cases} \Lambda_i^k & \text{if } i_2 = i_2^k(e) \\ 0 & \text{otherwise} \end{cases}$$

- Projection on the local edge-conforming space:

$$P_e : V_{\text{pw}}^0 \rightarrow V_h^0, \quad \Lambda_i^k \mapsto \begin{cases} \Lambda_{i_1}^e & \text{if } k = -, \ i_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$



Vertex conforming basis functions



$$\Lambda^v := \sum_{e \in \mathcal{E}(v)} \Lambda_v^e - \sum_{k \in \mathcal{K}(v)} \Lambda_v^k.$$

- Projection on the local vertex-broken space:

$$I_v : V_{pw}^0 \rightarrow V_{pw}^0, \quad \Lambda_i^k \mapsto \begin{cases} \Lambda_i^k & \text{if } k \in \mathcal{K}(v), \ i = i^k(v) \\ 0 & \text{otherwise} \end{cases}$$

- Projection on the local vertex-conforming space:

$$P_v : V_{pw}^0 \rightarrow V_h^0, \quad \Lambda_i^k \mapsto \begin{cases} \Lambda^v & \text{if } k = k^*(v), \ i = i^k(v) \\ 0 & \text{otherwise} \end{cases}$$

Conforming projection P

For any broken $\phi \in V_{\text{pw}}^0$, we have the decomposition

$$\phi = \left(\sum_k I_0^k + \sum_{e \in \mathcal{E}} I_{e,0} + \sum_{v \in \mathcal{V}} I_v \right) \phi \quad \text{and} \quad P\phi := \left(\sum_{k \in \mathcal{K}} I_0^k + \sum_{e \in \mathcal{E}} P_{e,0} + \sum_{v \in \mathcal{V}} P_v \right) \phi.$$

Building Blocks of Π^1

Strategy: Start from the patch-wise commuting projections and add correction terms.

Patch-wise projections Π_k^1 :

$$\Pi_k^1 \mathbf{u} = \sum_{d \in \{1, 2\}} \nabla_d^k \Pi_k^0 \Phi_d^k(\mathbf{u}),$$

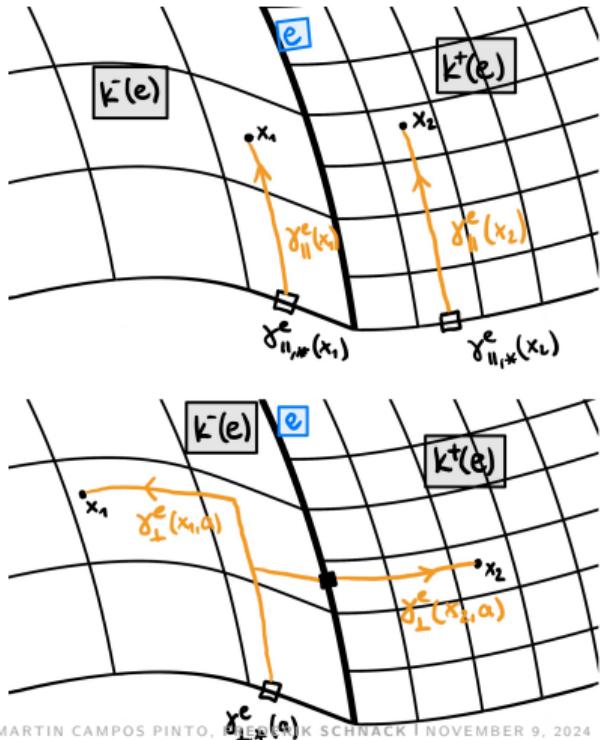
where the antiderivatives

$$\Phi_1^k(\mathbf{u})(\mathbf{x}) = \int_0^{x_1} \mathbf{u}_1(z, x_2) dz, \quad \Phi_2^k(\mathbf{u})(\mathbf{x}) = \int_0^{x_2} \mathbf{u}_2(x_1, z) dz$$

are defined for $\mathbf{x} \in \Omega_k$ such that $\Phi_d^k(\mathbf{u}) \in V_{\text{pw}}^0$ for $\mathbf{u} \in V_h^1$, $d = 1, 2$.



Edge-correction $\tilde{\Pi}_e^1$



$$\tilde{\Pi}_e^1 \mathbf{u} := \sum_{d \in \{\parallel, \perp\}} \nabla_d^e (P_e - I_e) \Pi_{\text{pw}}^0 \Phi_d^e(\mathbf{u}),$$

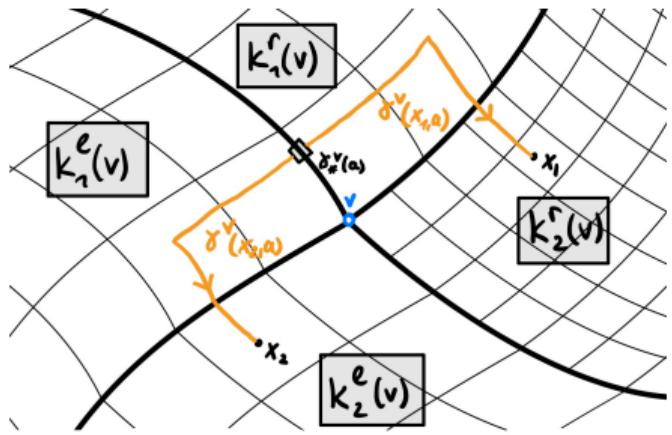
where the antiderivatives are defined as

$$\Phi_d^e \mathbf{u}(\mathbf{x}) = \int_{\gamma_d^e(\mathbf{x})} \mathbf{u} \cdot dI, \quad d \in \{\parallel, \perp\}$$

such that for $\mathbf{u} \in V_h^1$ and for all $e \in \mathcal{E}$, $\Phi_\parallel^e(\mathbf{u})$ and $\Phi_\perp^e(\mathbf{u})$ belong to V_{pw}^0 and are continuous across e which implies that $(P_e - I_e)$ is equal to zero in the correction term.



Vertex-correction $\tilde{\Pi}_v^1$



$$\tilde{\Pi}_v^1 \mathbf{u} := \nabla_{\text{pw}} (P_v - \bar{I}_v) \Pi_{\text{pw}}^0 \Phi^v(\mathbf{u}),$$

where the antiderivative is defined as

$$\Phi^v \mathbf{u}(x) = \int_{\gamma^v(x)} \mathbf{u} \cdot dI, \quad d \in \{\parallel, \perp\}$$

such that for $\mathbf{u} \in V_h^1$ and for all $\mathbf{v} \in \mathcal{V}$, $\Phi^v(\mathbf{u})$ belongs to V_{pw}^0 and is continuous across every $e \in \mathcal{E}(\mathbf{v})$ which implies that $(P_v - \bar{I}_v)$ is equal to zero in the correction term.



Full projection Π^1

$$\Pi^1 := \sum_{k \in \mathcal{K}} \Pi_k^1 + \sum_{e \in \mathcal{E}} \tilde{\Pi}_e^1 + \sum_{v \in \mathcal{V}} \tilde{\Pi}_v^1 + \sum_{v \in \mathcal{V}, e \in \mathcal{E}(v)} \tilde{\Pi}_{e,v}^1$$

with edge, vertex and edge-vertex correction operators defined as

$$\begin{cases} \tilde{\Pi}_e^1 \mathbf{u} := \sum_{d \in \{\parallel, \perp\}} \nabla_d^e (P_e - I_e) \Pi_{\text{pw}}^0 \Phi_d^e(\mathbf{u}) \\ \tilde{\Pi}_v^1 \mathbf{u} := \nabla_{\text{pw}} (P_v - \bar{I}_v) \Pi_{\text{pw}}^0 \Phi^v(\mathbf{u}) \\ \tilde{\Pi}_{e,v}^1 \mathbf{u} := \sum_{d \in \{\parallel, \perp\}} \nabla_d^e (\bar{I}_{e,v} - P_{e,v}) \Pi_{\text{pw}}^0 \Phi_d^{v,e}(\mathbf{u}) \end{cases}$$

Main result: Π^1 is a projection on V_h^1 , L^p stable for $1 \leq p \leq \infty$ and commuting:

$$\Pi^1 \nabla \phi = \nabla \Pi^0 \phi \quad \phi \in H^1(\Omega)$$

Commuting property

- let $\mathbf{u} = \nabla\phi$ and write $\phi_h = \Pi_{\text{pw}}^0 \phi \in V_{\text{pw}}^0$
 - by single-patch construction: $\sum_k \Pi_k^1 \mathbf{u} = \nabla_{\text{pw}} \phi_h$
 - by preservation of \parallel invariance along e : $\nabla_{\parallel}^e (P_e - I_e) \Pi_{\text{pw}}^0 \Phi_{\parallel}^e(\mathbf{u}) = \nabla_{\parallel}^e (P_e - I_e) \phi_h$
 - by cancellation of constants in $(P_e - I_e)$: $(P_e - I_e) \Pi_{\text{pw}}^0 \Phi_{\perp}^e(\mathbf{u}) = (P_e - I_e) \phi_h$
 - and similarly for the \mathbf{v} and e, \mathbf{v} terms.
 - thus we have $\phi_h = \left(\sum_k I_0^k + \sum_e I_{e,0} + \sum_{\mathbf{v}} I_{\mathbf{v}} \right) \phi_h$ and $\Pi^1 \mathbf{u} = \nabla_{\text{pw}} \psi_h$ with

$$\begin{aligned}\psi_h &= \phi_h + \left(\sum_e (P_e - I_e) + \sum_{\mathbf{v}} (P_{\mathbf{v}} - \bar{I}_{\mathbf{v}}) + \sum_{e,\mathbf{v}} (\bar{I}_{e,\mathbf{v}} - P_{e,\mathbf{v}}) \right) \phi_h \\ &= \left(\sum_k I_0^k + \sum_e (I_{e,0} + P_e - I_e) + \sum_{\mathbf{v}} (I_{\mathbf{v}} - P_{\mathbf{v}} - \bar{I}_{\mathbf{v}}) + \sum_{e,\mathbf{v}} (\bar{I}_{e,\mathbf{v}} - P_{e,\mathbf{v}}) \right) \phi_h = \dots = P\phi_h\end{aligned}$$

L^2 stability

- for each term, stability follows from
 - local inverse estimates for gradients
 - the stability of the local projections
 - the stability of the antiderivatives and localization arguments



Projection property

- range property: $(\Pi^1 \mathbf{u} \in V_h^1)$
 - clearly, $\Pi^1 \mathbf{u} \in V_{\text{pw}}^1$
 - the antiderivatives and the local conforming / broken projections are such that

$$(\boldsymbol{\tau} \cdot \Pi^1 \mathbf{u})|_{\Omega_{k^-}(e)} = (\boldsymbol{\tau} \cdot \Pi^1 \mathbf{u})|_{\Omega_{k^+}(e)}, \quad e \in \mathcal{E}$$

- projection property: $(\Pi^1 \mathbf{u} = \mathbf{u} \text{ for } \mathbf{u} \in V_h^1)$
 - patch-wise operator is a projection:

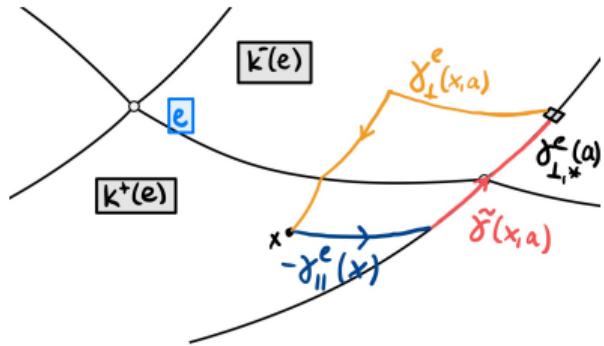
$$\sum_{k \in \mathcal{K}} \Pi_k^1 \mathbf{u} = \sum_{k \in \mathcal{K}} \mathbf{u}|_{\Omega_k} = \mathbf{u}, \quad \mathbf{u} \in V_h^1 \subset V_{\text{pw}}^1$$

- for $\mathbf{u} \in V_h^1$, the continuity properties of the antiderivatives yield

$$\begin{cases} (P_e - I_e) \Pi_{\text{pw}}^0 \Phi_d^e(\mathbf{u}) = 0, & d \in \{\parallel, \perp\} \\ (P_v - \bar{I}_v) \Pi_{\text{pw}}^0 \Phi^v(\mathbf{u}) = 0 \\ (P_{e,v} - \bar{I}_{e,v}) \Pi_{\text{pw}}^0 \Phi_d^{e,v}(\mathbf{u}) = 0, & d \in \{\parallel, \perp\} \end{cases}$$



Projection Π^2



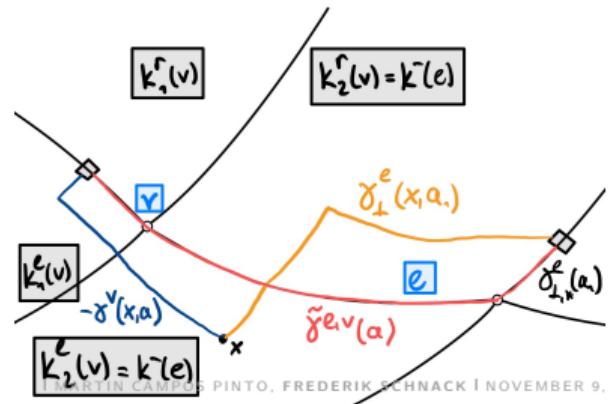
$$\Pi^2 := \sum_{k \in \mathcal{K}} \Pi_k^2 + \sum_{e \in \mathcal{E}} \tilde{\Pi}_e^2 + \sum_{v \in \mathcal{V}, e \in \mathcal{E}(v)} \tilde{\Pi}_{e,v}^2$$

with edge correction terms

$$\tilde{\Pi}_e^2 : \begin{cases} L^p(\Omega) \rightarrow V_{\text{pw}}^2, \\ f \mapsto D^{2,e}(P^e - I^e)\Pi_{\text{pw}}^0 \Psi^e(f) \end{cases}$$

and edge-vertex corrections

$$\tilde{\Pi}_{e,v}^2 : \begin{cases} L^p(\Omega) \rightarrow V_{\text{pw}}^2, \\ f \mapsto D^{2,e}(\bar{I}_v^e - P_v^e)\Pi_{\text{pw}}^0 \Psi^{v,e}(f). \end{cases}$$



$$\Psi(\operatorname{curl} \mathbf{u}) = \Phi_{\perp}(\mathbf{u}) - \Phi_{\parallel}(\mathbf{u}) + \tilde{\Phi}(\mathbf{u})$$