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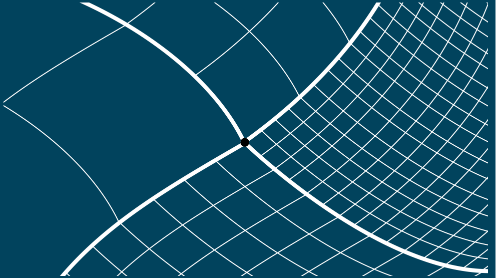
# Bounded commuting projections for non-matching interfaces

## 29th Biennial Numerical Analysis Conference 2023



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# Outline

1. Motivation
2. Known (single-patch) projectors
3. Multi-patch domains: the broken-FEEC approach
4. Construction of the Projectors
5. Summary

# Structure-preserving Finite Elements / FEEC

- sequence of spaces with commuting projection operators[1]

$$
\begin{tikzcd}
H_0^1 \arrow[r, "\mathrm{grad}"] \arrow[d, "\Pi^0"] & H_0(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\Pi^1"] & H_0(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi^2"] & L^2 \arrow[d, "\Pi^3"] \\
V_h^0 \arrow[r, "\mathrm{grad}"] & V_h^1 \arrow[r, "\mathrm{curl}"] & V_h^2 \arrow[r, "\mathrm{div}"] & V_h^3
\end{tikzcd}
$$

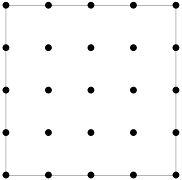
- FEEC: unified analysis, key element are $L^2$-stable commuting projections $\Pi^\ell$: see [2]

- eigenvalue problem:
  - convergence of eigenmodes
  - correct dim of discrete harmonic fields
  - discrete Hodge-Helmholtz decomp.

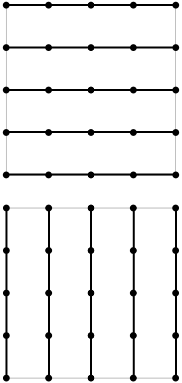
- source problem:
  - convergence (even for singular sol.)
  - structure preservation (Gauss laws)
  - time domain: long time stability

[1] Whitney (57) Nédélec (80) Bossavit (88-98) Hiptmair (99) Arnold-Falk-Winther (06) Buffa et al (11) ...
[2] Schöberl (05) AFW (06) Christiansen-Winther (07) Falk-Winther (14) Arnold-Guzman (21)

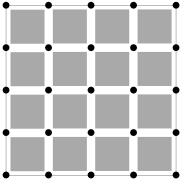
# Projections with “geometric” degrees of freedom (not $L^2$-stable)



$V_h^0$ (nodes)



$V_h^1$ (edges)



$V_h^2$ (faces)

# The geometric commuting projections

- On $V_h^0 = \mathrm{Span}(\{\Lambda_i^0 : \boldsymbol{v}_i \in \mathcal{V}\})$:

$$\Pi^0\phi(\boldsymbol{x}) := \sum_{\boldsymbol{v}_i \in \mathcal{V}} c_i^0(\phi)\Lambda_i^0(\boldsymbol{x}) \qquad \text{with} \qquad c_i^0(\phi) := \phi(\boldsymbol{v}_i)$$

- On $V_h^1 = \mathrm{Span}(\{\Lambda_{ij}^1 : e_{ij} \in \mathcal{E}\})$:

$$\Pi^1\boldsymbol{u}(\boldsymbol{x}) := \sum_{e_{ij} \in \mathcal{E}} c_{ij}^1(\boldsymbol{u})\Lambda_{ij}^1(\boldsymbol{x}) \qquad \text{with} \qquad c_{ij}^1(\boldsymbol{u}) := \int_{\boldsymbol{v}_i}^{\boldsymbol{v}_j} \boldsymbol{\tau} \cdot \boldsymbol{u}\, \mathrm{d}s$$

- On $V_h^2 = \mathrm{Span}(\{\Lambda_{ijk}^2 : f_{ijk} \in \mathcal{F}\})$:

$$\Pi^2\boldsymbol{J}(\boldsymbol{x}) := \sum_{f_{ijk} \in \mathcal{F}} c_{ijk}^2(\boldsymbol{J})\Lambda_{ijk}^2(\boldsymbol{x}) \qquad \text{with} \qquad c_{ijk}^2(\boldsymbol{J}) := \iint_{\sigma_{ijk}(\boldsymbol{y})} \boldsymbol{n} \cdot \boldsymbol{J}\, \mathrm{d}^2 s$$

where $\sigma_{ijk} := [\boldsymbol{v}_i, \boldsymbol{v}_j, \boldsymbol{v}_k]$

# The commuting projections of Buffa-Rivas-Sangalli-Vázquez (2011)

- construction on tensor-product spaces (here in 2D)

$$V_h^0 = \mathbb{S}_p \otimes \mathbb{S}_p \quad \xrightarrow{\text{grad}} \quad V_h^1 = \begin{pmatrix} \mathbb{S}_{p-1} \otimes \mathbb{S}_p \\ \mathbb{S}_p \otimes \mathbb{S}_{p-1} \end{pmatrix} \quad \xrightarrow{\text{curl}} \quad V_h^3 = \mathbb{S}_{p-1} \otimes \mathbb{S}_{p-1}$$

- On $V_h^0$:

$$\Pi^0 \phi := \sum_{\boldsymbol{i} \in \mathcal{I}^0} c_{\boldsymbol{i}}^0(\phi) \Lambda_{\boldsymbol{i}}^0 \qquad \text{with} \qquad c_{\boldsymbol{i}}^0(\phi) := \langle \Theta_{\boldsymbol{i}}^0, \phi \rangle$$

- On $V_h^1$ :

$$\Pi^1 \boldsymbol{u} := \sum_{d \in \{1,2\}} \nabla_d \Pi^0 \Phi_d(\boldsymbol{u}) \qquad \text{with} \qquad \begin{cases} \Phi_1(\boldsymbol{u})(\boldsymbol{x}) := \int_0^{x_1} u_1(z, x_2)\, \mathrm{d}z \\ \Phi_2(\boldsymbol{u})(\boldsymbol{x}) := \int_0^{x_2} u_2(x_1, z)\, \mathrm{d}z \end{cases}$$

- On $V_h^2$ :

$$\Pi^2 \rho := \partial_1 \partial_2 \Pi^0 \Psi(\rho) \qquad \text{with} \qquad \Psi(\rho)(\boldsymbol{x}) := \int_0^{x_1} \int_0^{x_2} \rho(z_1, z_2)\, \mathrm{d}z_2\, \mathrm{d}z_1$$

# The commuting projections of Buffa-Rivas-Sangalli-Vázquez (2011) – cont'd

- projection property:
  - for $\Pi^0$, as usual $\langle \Theta^0_{\boldsymbol{i}}, \Lambda^0_{\boldsymbol{j}} \rangle = \delta_{\boldsymbol{i},\boldsymbol{j}}$
  - for $\Pi^1$, follows from $\int_0^{x_1} u_1(z, x_2)\,\mathrm{d}z \in V^0_h$ (same argument for $\Pi^2$)
- commuting property: for the grad,

$$(\Pi^1 \nabla \phi(\boldsymbol{x}))_1 = \partial_1 \big(\Pi^0 \Phi^1_1(\nabla\phi)(\boldsymbol{x})\big) = \partial_1 \Pi^0 \big(\phi(\boldsymbol{x}) - \phi(0, x_2)\big) = \partial_1 \Pi^0 \phi(\boldsymbol{x}) \qquad \text{(same for the curl)}$$

- $L^2_h$ stability:
  - on $V^0_h$, follows from locally stable dofs
  - on $V^1_h$, follows from a localization argument: on a local domain $\omega_{\boldsymbol{i}}$, we have

$$\nabla_1 \Pi^0 \Phi^1_1(\boldsymbol{u}) = \nabla_1 \Pi^0 \Phi^1_{1,\boldsymbol{i}}(\boldsymbol{u}) \quad \text{with} \quad \Phi^1_{1,\boldsymbol{i}}(\boldsymbol{u})(\boldsymbol{x}) := \int_{x_1(\omega_{\boldsymbol{i}})}^{x_1} u_1(z, x_2)\,\mathrm{d}z$$

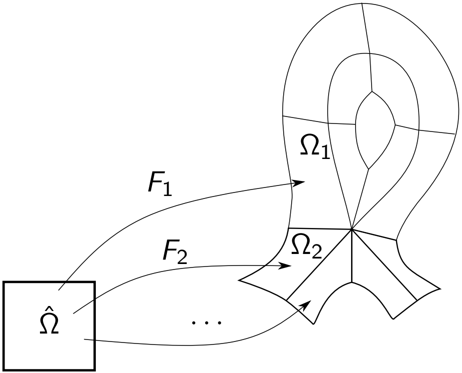
  and inverse / local stability estimates

$$\|\nabla_1 \Pi^0 \Phi^1_{1,\boldsymbol{i}}(\boldsymbol{u})\|_{L^2(\omega_{\boldsymbol{i}})} \lesssim h^{-1} \|\Pi^0 \Phi^1_{1,\boldsymbol{i}}(\boldsymbol{u})\|_{L^2(E_h(\omega_{\boldsymbol{i}}))} \lesssim h^{-1} \|\Phi^1_{1,\boldsymbol{i}}(\boldsymbol{u})\|_{L^2(E^2_h(\omega_{\boldsymbol{i}}))} \lesssim \|\boldsymbol{u}\|_{L^2(E^3_h(\omega_{\boldsymbol{i}}))}$$

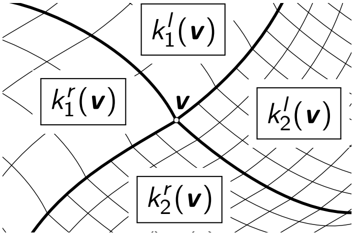
  - on $V^2_h$, the same principle applies
- Note: as long as $\mathrm{supp}(\Theta^0_{\boldsymbol{i}}) \approx \mathrm{supp}(\Lambda^0_{\boldsymbol{i}})$ these operators are local

# Multi-patch domains with non-matching interfaces

Adjacent patches may have different mappings . . .




. . . and different grids




Challenge: Break of the tensor-product structure near the interfaces

▷ no (stable) commuting projections known so far

# Our result

**Theorem** If interior vertices are shared by four patches with nested discretization, there exist commuting projections $\Pi^\ell : L^p \to V_h^\ell$, local and stable for $1 \le p \le \infty$.

broken-FEEC construction: patchwise projections + interface correction

- on $V_h^0$ we define

$$\Pi^0 := P\Pi^0_{\mathrm{pw}} \quad \text{with} \quad \begin{cases} \Pi^0_{\mathrm{pw}} = \sum_{k\in\mathcal{K}} \Pi^0_k : L^p \to V^0_{\mathrm{pw}} & \text{(stable, patch-wise)} \\ P : V^0_{\mathrm{pw}} \mapsto V^0_h & \text{(averages interface dofs)} \end{cases}$$

- on $V_h^1$ we set $\Pi^1 := \Pi^1_{\mathrm{pw}} + \sum_{e\in\mathcal{E}} \tilde{\Pi}^1_e + \sum_{\boldsymbol{v}\in\mathcal{V}} \tilde{\Pi}^1_{\boldsymbol{v}} + \sum_{(e,\boldsymbol{v})\in\mathcal{E}\times\mathcal{V}} \tilde{\Pi}^1_{e,\boldsymbol{v}}$ with

$$\tilde{\Pi}^1_g \boldsymbol{u} := \sum_{d\in\{\parallel,\perp\}} \nabla^g_d Q_g \Pi^0_{\mathrm{pw}} \Phi^g_d(\boldsymbol{u}) \quad \text{with} \quad \begin{cases} \Phi^g_d : L^p \to L^p & \text{(antiderivative)} \\ Q_g : V^0_{\mathrm{pw}} \mapsto V^0_{\mathrm{pw}} & \text{(interface jump)} \end{cases}$$

- similar construction on $V_h^2$

# Broken FEEC approach: the projection $\Pi^0$

- broken (patch-wise) spaces: $V^\ell_{\mathrm{pw}} := \{v \in L^2(\Omega) : v|_{\Omega_k} \in V^\ell_k \text{ for } k \in \mathcal{K}\}$
- patch-wise differential operators

$$V^0_{\mathrm{pw}} \xrightarrow{\mathrm{grad}_{\mathrm{pw}}} V^1_{\mathrm{pw}} \xrightarrow{\mathrm{curl}_{\mathrm{pw}}} V^2_{\mathrm{pw}}$$

- conforming subspaces

$$V^0_h = V^0_{\mathrm{pw}} \cap H^1(\Omega) \xrightarrow{\mathrm{grad}} V^1_h = V^1_{\mathrm{pw}} \cap H(\mathrm{curl};\Omega) \xrightarrow{\mathrm{curl}} V^2_h = V^2_{\mathrm{pw}} \cap L^2(\Omega) = V^2_{\mathrm{pw}}.$$

- on $V^0_h$, an $L^2$ stable projection is given by

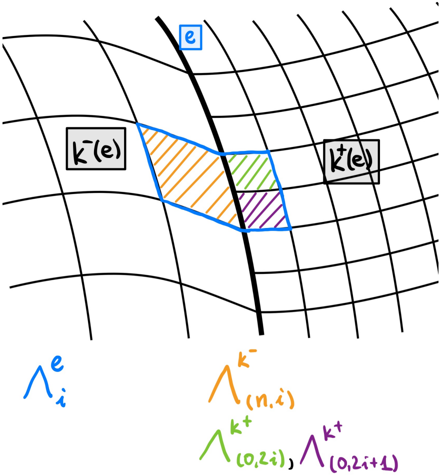
$$\Pi^0 := P\Pi^0_{\mathrm{pw}} \qquad \text{with} \qquad \Pi^0_{\mathrm{pw}}\phi = \sum_k \Pi^0_k \phi = \sum_k \sum_{\boldsymbol{i} \in \mathcal{I}^k} \langle \Theta^{0,k}_{\boldsymbol{i}}, \phi \rangle \Lambda^{0,k}_{\boldsymbol{i}}$$

where $P : V^0_{\mathrm{pw}} \mapsto V^0_h$ performs local averages of interface degrees of freedom

# Broken commuting diagram

$$\begin{tikzcd}
V^0 = H^1(\Omega) \arrow[r, "\mathrm{grad}"] \arrow[d, "\Pi^0_{\mathsf{pw}}"] \arrow[dd, bend right, "\Pi^0"'] & V^1 = H(\mathrm{curl}, \Omega) \arrow[r, "\mathsf{curl}"] \arrow[d, "\Pi^1_{\mathsf{pw}}"] \arrow[dd, bend left, "\Pi^1"] & V^2 = L^2(\Omega) \arrow[d, "\Pi^2_{\mathsf{pw}}"] \arrow[dd, bend left, "\Pi^2"] \\
V^0_{\mathsf{pw}} \arrow[r, "\mathbb{G}_{\mathsf{pw}}"] \arrow[d, "P^0"] & V^1_{\mathsf{pw}} \arrow[r, "\mathbb{C}_{\mathsf{pw}}"] \arrow[d, dashed, "P^1"] & V^2_{\mathsf{pw}} \arrow[d, dashed, "\mathrm{id}"] \\
V^0_{\mathsf{h}} = V^0_{\mathsf{pw}} \cap H^1(\Omega) \arrow[r, "\mathbb{G}"] & V^1_{\mathsf{h}} = V^1_{\mathsf{pw}} \cap H(\mathrm{curl}, \Omega) \arrow[r, "\mathbb{C}"] & V^2_{\mathsf{h}} = V^2_{\mathsf{pw}}
\end{tikzcd}$$

# Edge Conforming basis functions




$$\Lambda^e_i(\boldsymbol{x}) := \begin{cases} \hat\lambda^-_i(\hat x^-_\parallel)\hat\lambda^-_{i^-_\perp(e)}(\hat x^-_\perp) & \text{on } \Omega_{k^-}, \\ \hat\lambda^-_i(\eta_e(\hat x^+_\parallel))\hat\lambda^+_{i^+_\perp(e)}(\hat x^+_\perp) & \text{on } \Omega_{k^+}, \\ 0 & \text{else.} \end{cases}$$

- Projection on the local edge-broken space:

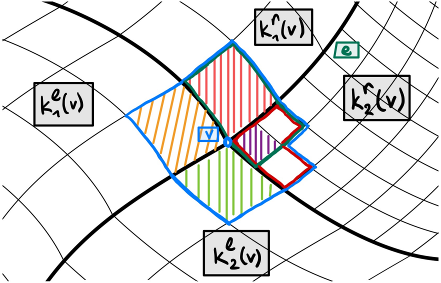
$$I_e : V^0_{\mathrm{pw}} \to V^0_{\mathrm{pw}}, \quad \Lambda^k_{\boldsymbol{i}} \mapsto \begin{cases} \Lambda^k_{\boldsymbol{i}} & \text{if } i_2 = i^k_2(e) \\ 0 & \text{otherwise} \end{cases}$$

- Projection on the local edge-conforming space:

$$P_e : V^0_{\mathrm{pw}} \to V^0_h, \quad \Lambda^k_{\boldsymbol{i}} \mapsto \begin{cases} \Lambda^e_{i_1} & \text{if } k = -, \; i_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

# Vertex conforming basis functions

$$\Lambda^{\boldsymbol{v}} := \sum_{e \in \mathcal{E}(\boldsymbol{v})} \Lambda^e_{\boldsymbol{v}} - \sum_{k \in \mathcal{K}(\boldsymbol{v})} \Lambda^k_{\boldsymbol{v}}.$$




- Projection on the local vertex-broken space:

$$I_{\boldsymbol{v}} : V^0_{\mathrm{pw}} \to V^0_{\mathrm{pw}}, \quad \Lambda^k_{\boldsymbol{i}} \mapsto \begin{cases} \Lambda^k_{\boldsymbol{i}} & \text{if } k \in \mathcal{K}(\boldsymbol{v}),\ \boldsymbol{i} = \boldsymbol{i}^k(\boldsymbol{v}) \\ 0 & \text{otherwise} \end{cases}$$

- Projection on the local vertex-conforming space:

$$P_{\boldsymbol{v}} : V^0_{\mathrm{pw}} \to V^0_h, \quad \Lambda^k_{\boldsymbol{i}} \mapsto \begin{cases} \Lambda^{\boldsymbol{v}} & \text{if } k = k^*(\boldsymbol{v}),\ \boldsymbol{i} = \boldsymbol{i}^k(\boldsymbol{v}) \\ 0 & \text{otherwise} \end{cases}$$

$\Lambda_v$ $\quad \Lambda_v^{k_1^e}$

$\Lambda_v|_{\Omega_{k_2^r}}$ $\quad \Lambda_v^{k_2^e}$

$\Lambda^e_v$ $\quad \Lambda_v^{k_1^r}$

$\Lambda_v^{k_2^r}$

## Conforming projection $P$

For any broken $\phi \in V^0_{\mathrm{pw}}$, we have the decomposition

$$\phi = \Big(\sum_{k} I_0^k + \sum_{e\in\mathcal{E}} I_{e,0} + \sum_{v\in\mathcal{V}} I_{\boldsymbol{v}}\Big)\phi \quad \text{and} \quad P\phi := \Big(\sum_{k\in\mathcal{K}} I_0^k + \sum_{e\in\mathcal{E}} P_{e,0} + \sum_{\boldsymbol{v}\in\mathcal{V}} P_{\boldsymbol{v}}\Big)\phi.$$

## Building Blocks of $\Pi^1$

**Strategy:** Start from the patch-wise commuting projections and add correction terms.

Patch-wise projections $\Pi^1_k$:

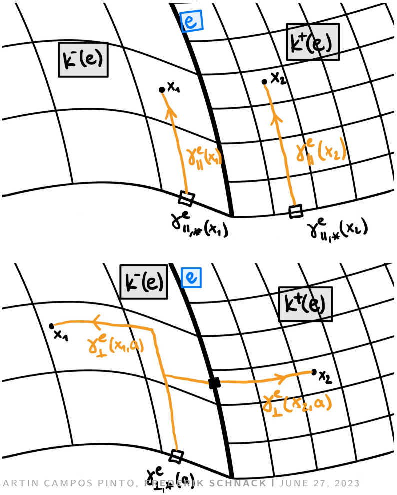
$$\Pi^1_k \boldsymbol{u} = \sum_{d\in\{1,2\}} \nabla^k_d \Pi^0_k \Phi^k_d(\boldsymbol{u}),$$

where the antiderivatives

$$\Phi^k_1(\boldsymbol{u})(\boldsymbol{x}) = \int_0^{x_1} \boldsymbol{u}_1(z, x_2)\,\mathrm{d}z, \qquad \Phi^k_2(\boldsymbol{u})(\boldsymbol{x}) = \int_0^{x_2} \boldsymbol{u}_2(x_1, z)\,\mathrm{d}z$$

are defined for $\boldsymbol{x} \in \Omega_k$ such that $\Phi^k_d(\boldsymbol{u}) \in V^0_{\mathrm{pw}}$ for $\boldsymbol{u} \in V^1_h$, $d = 1, 2$.

# Edge-correction $\tilde{\Pi}^1_e$



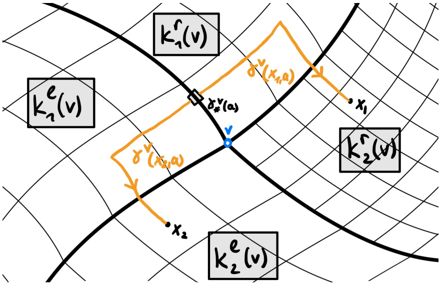

$$\tilde{\Pi}^1_e \boldsymbol{u} := \sum_{d\in\{\parallel,\perp\}} \nabla^e_d (P_e - I_e)\Pi^0_{\mathrm{pw}} \Phi^e_d(\boldsymbol{u}),$$

where the antiderivatives are defined as

$$\Phi^e_d \boldsymbol{u}(\boldsymbol{x}) = \int_{\gamma^e_d(\boldsymbol{x})} \boldsymbol{u}\cdot \mathrm{d}l, \qquad d \in \{\parallel,\perp\}$$

such that for $\boldsymbol{u} \in V^1_h$ and for all $e \in \mathcal{E}$, $\Phi^e_\parallel(\boldsymbol{u})$ and $\Phi^e_\perp(\boldsymbol{u})$ belong to $V^0_{\mathrm{pw}}$ and are continuous across $e$ which implies that $(P_e - I_e)$ is equal to zero in the correction term.

# Vertex-correction $\tilde{\Pi}^1_v$



$$\tilde{\Pi}^1_{\boldsymbol{v}} \boldsymbol{u} := \nabla_{\mathrm{pw}} (P_{\boldsymbol{v}} - \bar{I}_{\boldsymbol{v}}) \Pi^0_{\mathrm{pw}} \Phi^{\boldsymbol{v}}(\boldsymbol{u}),$$

where the antiderivative is defined as

$$\Phi^{\boldsymbol{v}} \boldsymbol{u}(\boldsymbol{x}) = \int_{\gamma^{\boldsymbol{v}}(\boldsymbol{x})} \boldsymbol{u} \cdot \mathrm{d}l, \qquad d \in \{\|, \perp\}$$

such that for $\boldsymbol{u} \in V^1_h$ and for all $\boldsymbol{v} \in \mathcal{V}$, $\Phi^{\boldsymbol{v}}(\boldsymbol{u})$ belongs to $V^0_{\mathrm{pw}}$ and is continuous across every $e \in \mathcal{E}(\boldsymbol{v})$ which implies that $(P_{\boldsymbol{v}} - \bar{I}_{\boldsymbol{v}})$ is equal to zero in the correction term.

# Full projection $\Pi^1$

$$\Pi^1 := \sum_{k\in\mathcal{K}} \Pi^1_k + \sum_{e\in\mathcal{E}} \tilde{\Pi}^1_e + \sum_{\boldsymbol{v}\in\mathcal{V}} \tilde{\Pi}^1_{\boldsymbol{v}} + \sum_{\boldsymbol{v}\in\mathcal{V}, e\in\mathcal{E}(\boldsymbol{v})} \tilde{\Pi}^1_{e,\boldsymbol{v}}$$

with edge, vertex and edge-vertex correction operators defined as

$$\begin{cases} \tilde{\Pi}^1_e \boldsymbol{u} := \displaystyle\sum_{d\in\{\parallel,\perp\}} \nabla^e_d (P_e - I_e)\Pi^0_{\mathrm{pw}} \Phi^e_d(\boldsymbol{u}) \\ \tilde{\Pi}^1_{\boldsymbol{v}} \boldsymbol{u} := \nabla_{\mathrm{pw}}(P_{\boldsymbol{v}} - \bar{I}_{\boldsymbol{v}})\Pi^0_{\mathrm{pw}} \Phi^{\boldsymbol{v}}(\boldsymbol{u}) \\ \tilde{\Pi}^1_{e,\boldsymbol{v}} \boldsymbol{u} := \displaystyle\sum_{d\in\{\parallel,\perp\}} \nabla^e_d (\bar{I}_{e,\boldsymbol{v}} - P_{e,\boldsymbol{v}})\Pi^0_{\mathrm{pw}} \Phi^{\boldsymbol{v},e}_d(\boldsymbol{u}) \end{cases}$$

**Main result:** $\Pi^1$ is a projection on $V^1_h$, $L^p$ stable for $1 \le p \le \infty$ and commuting:

$$\Pi^1 \nabla\phi = \nabla \Pi^0 \phi \qquad \phi \in H^1(\Omega)$$

## Commuting property

- let $\boldsymbol{u} = \nabla\phi$ and write $\phi_h = \Pi^0_{\rm pw}\phi \in V^0_{\rm pw}$
  - by single-patch construction: $\sum_k \Pi^1_k \boldsymbol{u} = \nabla_{\rm pw}\phi_h$
  - by preservation of $\|$ invariance along $e$: $\nabla^e_{\|}(P_e - I_e)\Pi^0_{\rm pw}\Phi^e_{\|}(\boldsymbol{u}) = \nabla^e_{\|}(P_e - I_e)\phi_h$
  - by cancellation of constants in $(P_e - I_e)$: $(P_e - I_e)\Pi^0_{\rm pw}\Phi^e_{\perp}(\boldsymbol{u}) = (P_e - I_e)\phi_h$
  - and similarly for the $\boldsymbol{v}$ and $e, \boldsymbol{v}$ terms.
  - thus we have $\phi_h = \Big(\sum_k I^k_0 + \sum_e I_{e,0} + \sum_{\boldsymbol{v}} I_{\boldsymbol{v}}\Big)\phi_h$ and $\Pi^1\boldsymbol{u} = \nabla_{\rm pw}\psi_h$ with

$$\begin{aligned}\psi_h &= \phi_h + \Big(\sum_e (P_e - I_e) + \sum_{\boldsymbol{v}} (P_v - \bar{I}_v) + \sum_{e,\boldsymbol{v}} (\bar{I}_{e,\boldsymbol{v}} - P_{e,\boldsymbol{v}})\Big)\phi_h \\ &= \Big(\sum_k I^k_0 + \sum_e (I_{e,0} + P_e - I_e) + \sum_{\boldsymbol{v}} (I_{\boldsymbol{v}} - P_v - \bar{I}_v) + \sum_{e,\boldsymbol{v}} (\bar{I}_{e,\boldsymbol{v}} - P_{e,\boldsymbol{v}})\Big)\phi_h = \cdots = P\phi_h\end{aligned}$$

## $L^2$ stability

- for each term, stabilty follows from
  - local inverse estimates for gradients
  - the stability of the local projections
  - the stability of the antiderivatives and localization arguments

# Projection property

- range property: $(\Pi^1 \boldsymbol{u} \in V_h^1)$
  - clearly, $\Pi^1 \boldsymbol{u} \in V_{\mathrm{pw}}^1$
  - the antiderivatives and the local conforming / broken projections are such that

$$(\boldsymbol{\tau} \cdot \Pi^1 \boldsymbol{u})|_{\Omega_{k^-(e)}} = (\boldsymbol{\tau} \cdot \Pi^1 \boldsymbol{u})|_{\Omega_{k^+(e)}}, \qquad e \in \mathcal{E}$$

- projection property: $(\Pi^1 \boldsymbol{u} = \boldsymbol{u}$ for $\boldsymbol{u} \in V_h^1)$
  - patch-wise operator is a projection:

$$\sum_{k \in \mathcal{K}} \Pi_k^1 \boldsymbol{u} = \sum_{k \in \mathcal{K}} \boldsymbol{u}|_{\Omega_k} = \boldsymbol{u}, \qquad \boldsymbol{u} \in V_h^1 \subset V_{\mathrm{pw}}^1$$

  - for $\boldsymbol{u} \in V_h^1$, the continuity properties of the antiderivatives yield

$$\begin{cases} (P_e - I_e)\Pi_{\mathrm{pw}}^0 \Phi_d^e(\boldsymbol{u}) = 0, & d \in \{\|, \perp\} \\ (P_{\boldsymbol{v}} - \bar{I}_{\boldsymbol{v}})\Pi_{\mathrm{pw}}^0 \Phi^{\boldsymbol{v}}(\boldsymbol{u}) = 0 & \\ (P_{e,\boldsymbol{v}} - \bar{I}_{e,\boldsymbol{v}})\Pi_{\mathrm{pw}}^0 \Phi_d^{e,\boldsymbol{v}}(\boldsymbol{u}) = 0, & d \in \{\|, \perp\} \end{cases}$$

# Conclusion and Outlook

## Summary

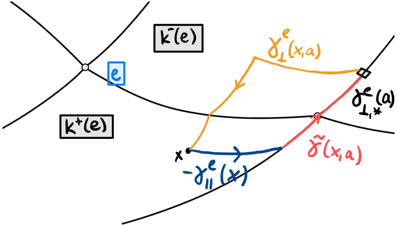
- Existence of bounded commuting projectors
  $\Longrightarrow$ structure-preserving schemes
- broken-FEEC
  $\Longrightarrow$ local operators

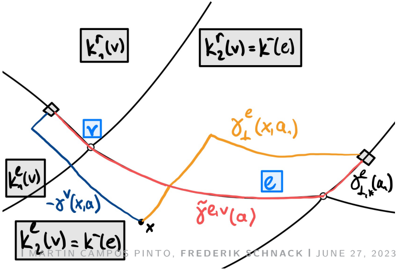
## Perspectives

- numerical schemes on locally refined patches
- four-patch assumption needed?
- extension to 3D

# Backup Slides

# Projection $\Pi^2$




$$\Pi^2 := \sum_{k\in\mathcal{K}} \Pi_k^2 + \sum_{e\in\mathcal{E}} \tilde\Pi_e^2 + \sum_{\boldsymbol{v}\in\mathcal{V}, e\in\mathcal{E}(\boldsymbol{v})} \tilde\Pi_{e,\boldsymbol{v}}^2$$

with edge correction terms

$$\tilde\Pi_e^2 : \begin{cases} L^p(\Omega) \to V_{\mathrm{pw}}^2, \\ f \mapsto D^{2,e}(P^e - I^e)\Pi_{\mathrm{pw}}^0 \Psi^e(f) \end{cases}$$

and edge-vertex corrections

$$\tilde\Pi_{e,\boldsymbol{v}}^2 : \begin{cases} L^p(\Omega) \to V_{\mathrm{pw}}^2, \\ f \mapsto D^{2,e}(\bar{I}_{\boldsymbol{v}}^e - P_{\boldsymbol{v}}^e)\Pi_{\mathrm{pw}}^0 \Psi^{\boldsymbol{v},e}(f). \end{cases}$$

$$\Psi(\operatorname{curl}\boldsymbol{u}) = \Phi_\perp(\boldsymbol{u}) - \Phi_\parallel(\boldsymbol{u}) + \tilde\Phi(\boldsymbol{u})$$

# Commuting diagram

$$
\begin{tikzcd}
V^0 = H^1(\Omega) \arrow[r, "\mathrm{grad}"] \arrow[d, "\Pi^0_{\mathrm{pw}}"] \arrow[dd, bend right=60, "\Pi^0"'] & V^1 = H(\mathrm{curl}, \Omega) \arrow[r, "\mathrm{curl}"] \arrow[d, "\Pi^1_{\mathrm{pw}}"] \arrow[dd, bend left=60, "\Pi^1"] & V^2 = L^2(\Omega) \arrow[d, "\Pi^2_{\mathrm{pw}}"] \arrow[dd, bend left=60, "\Pi^2"] \\
V^0_{\mathrm{pw}} = \mathbb{S}^p \otimes \mathbb{S}^p \arrow[r, "\mathbb{G}_{\mathrm{pw}}"] \arrow[d, "P^0"] & V^1_{\mathrm{pw}} = \begin{pmatrix} \mathbb{S}^{p-1} \otimes \mathbb{S}^p \\ \mathbb{S}^p \otimes \mathbb{S}^{p-1} \end{pmatrix} \arrow[r, "\mathbb{C}_{\mathrm{pw}}"] \arrow[d, dashed, "P^1"] & V^2_{\mathrm{pw}} = \mathbb{S}^{p-1} \otimes \mathbb{S}^{p-1} \arrow[d, dashed, "\mathrm{id}"] \\
V^0_h = V^0_{\mathrm{pw}} \cap H^1(\Omega) \arrow[r, "\mathbb{G}"] & V^1_h = V^1_{\mathrm{pw}} \cap H(\mathrm{curl}, \Omega) \arrow[r, "\mathbb{C}"] & V^2_h = V^2_{\mathrm{pw}}
\end{tikzcd}
$$

# The geometric commuting projections – cont'd

- projection property comes from the definition of the basis functions:

$$c_i^0(\Lambda_j^0) = \delta_{i,j} \qquad \Longrightarrow \qquad \pi^0 \Lambda_j^0 = \Lambda_j^0$$

- commuting property comes from the geometric nature of the degrees of freedom:

$$c_{ij}^1(\nabla \phi) = \int_{\boldsymbol{v}_i}^{\boldsymbol{v}_j} \boldsymbol{\tau} \cdot \nabla \phi \,\mathrm{d}s = \phi(\boldsymbol{v}_j) - \phi(\boldsymbol{v}_i) = c_j^0(\phi) - c_i^0(\phi)$$

so that

$$\pi^1 \nabla \phi = \sum_{e_{ij} \in \mathcal{E}} \big(c_j^0(\phi) - c_i^0(\phi)\big) \Lambda_{ij}^1 = \sum_{\boldsymbol{v}_i \in \mathcal{V}} c_i^0(\phi) \nabla \Lambda_i^0 = \nabla \pi^0 \phi$$

- however $\pi^0$ cannot be $L^2$ stable:
  indeed there exists $(\phi_n)_{n \geq 0}$ with $\|\phi_n\|_{L^2} = 1$ and $\phi_n(0) \to \infty$
- another construction: Nédélec degrees of freedom, commuting but also not $L^2$ stable

# The commuting projection of Schöberl (2005)

- On $V_h^0 = \mathrm{Span}(\{\Lambda_i^0 : \boldsymbol{v}_i \in \mathcal{V}\})$:

$$\pi^0\phi(\boldsymbol{x}) := \sum_{\boldsymbol{v}_i\in\mathcal{V}} c_i^0(\phi)\Lambda_i^0(\boldsymbol{x}) \qquad \text{with} \qquad c_i^0(\phi) := \int_{\omega_i} w_i(\boldsymbol{y})\phi(m_i(\boldsymbol{y}))\,\mathrm{d}\boldsymbol{y}$$

- On $V_h^1 = \mathrm{Span}(\{\Lambda_{ij}^1 : e_{ij} \in \mathcal{E}\})$:

$$\pi^1\boldsymbol{u}(\boldsymbol{x}) := \sum_{e_{ij}\in\mathcal{E}} c_{ij}^1(\boldsymbol{u})\Lambda_{ij}^1(\boldsymbol{x}) \qquad \text{with} \qquad c_{ij}^1(\boldsymbol{u}) := \iint_{\omega_i\times\omega_j} w_i(\boldsymbol{y}_1)w_j(\boldsymbol{y}_2)\int_{m_i(\boldsymbol{y}_1)}^{m_j(\boldsymbol{y}_2)} \boldsymbol{\tau}\cdot\boldsymbol{u}\,\mathrm{d}s\,\mathrm{d}^2\boldsymbol{y}$$

- On $V_h^2 = \mathrm{Span}(\{\Lambda_{ijk}^2 : f_{ijk} \in \mathcal{F}\})$:

$$\pi^2\boldsymbol{J}(\boldsymbol{x}) := \sum_{f_{ijk}\in\mathcal{F}} c_{ijk}^2(\boldsymbol{J})\Lambda_{ijk}^2(\boldsymbol{x}) \qquad \text{with} \qquad c_{ijk}^2(\boldsymbol{J}) := \iiint_{\omega_i\times\omega_j\times\omega_k} w_i(\boldsymbol{y}_1)w_j(\boldsymbol{y}_2)w_k(\boldsymbol{y}_3)\iint_{\sigma_{ijk}(\boldsymbol{y})} \boldsymbol{n}\cdot\boldsymbol{J}\,\mathrm{d}^2s\,\mathrm{d}^3\boldsymbol{y}$$

where $\sigma_{ijk}(\boldsymbol{y}) := [m_i(\boldsymbol{y}_1), m_j(\boldsymbol{y}_2), m_k(\boldsymbol{y}_3)]$

# The commuting projection of Schöberl (2005) – cont'd

- $L^2$ stability comes from the local averages: we have $\|\Lambda_i^0\|_L^2 \lesssim |m_i(\omega_i)|^{\frac{1}{2}} \sim h_i^{\frac{3}{2}}$ and

$$|c_i^0(\phi)| \le \int_{\omega_i} |w_i(\boldsymbol{y})\phi(m_i(\boldsymbol{y}))| \,\mathrm{d}\boldsymbol{y} \le \|w_i\|_{L^2(\omega_i)}\|\phi\|_{L^2(m_i(\omega_i))} h_i^{-\frac{3}{2}} \lesssim \|\phi\|_{L^2(m_i\omega_i))} h_i^{-\frac{3}{2}}$$

so that

$$\|\pi^0\phi\|_{L^2} \le \sum_{\boldsymbol{v}_i\in\mathcal{V}} |c_i^0(\phi)|\|\Lambda_i^0\|_L^2 \lesssim \sum_{\boldsymbol{v}_i\in\mathcal{V}} \|\phi\|_{L^2(m_i\omega_i))} \lesssim \|\phi\|_{L^2}$$

- commuting property comes from the geometric degrees of freedom:

$$c_{ij}^1(\nabla\phi) = \iint_{\omega_i\times\omega_j} w_i(\boldsymbol{y}_1)w_j(\boldsymbol{y}_2)\phi(m_j(\boldsymbol{y}_2)) - \phi(m_i(\boldsymbol{y}_1))\,\mathrm{d}^2\boldsymbol{y} = c_j^0(\phi) - c_i^0(\phi)$$

so that

$$\pi^1\nabla\phi = \sum_{e_{ij}\in\mathcal{E}} \left(c_j^0(\phi) - c_i^0(\phi)\right)\Lambda_{ij}^1 = \sum_{\boldsymbol{v}_i\in\mathcal{V}} c_i^0(\phi)\nabla\Lambda_i^0 = \nabla\pi^0\phi$$

- however: $\pi^\ell$ is a quasi-interpolation on $V_h^\ell$, but not a projection

# The commuting projection of Schöberl (2005) – cont'd (again)

- additional step: define

$$\Pi^0\phi := \left(\pi^0|_{V_h^0}\right)^{-1}\pi^0\phi$$

- in other words,

$$\Pi^0\phi := \phi_h \in V_h^0 \qquad \text{such that} \qquad \pi^0\phi_h = \pi^0\phi$$

  this is well posed because $\pi^0 \approx I$ on $V_h^0$.
- others $\Pi^\ell$ are defined similarly
- projection property is immediate
- $L^2$ stability is preserved (requires a priori estimates on $\|I - \pi^0\|$)
- commuting property is preserved, indeed we observe that

$$\boldsymbol{u}_h := \nabla\phi_h \in V_h^1 \qquad \text{and} \qquad \pi^1\boldsymbol{u}_h = \pi^1\nabla\phi_h = \nabla\pi^0\phi_h = \nabla\pi^0\phi = \pi^1\nabla\phi$$

  so that

$$\Pi^1\nabla\phi = \boldsymbol{u}_h = \nabla\phi_h = \nabla\Pi^0\phi$$

## Arnold-Falk-Winther (2006)/Christiansen-Winther (2007)/Ern-Guermond (2016):

- extend to simplicial polynomial spaces generalizing Raviart-Thomas-Nedelec
- General form: $\Pi^\ell = J_h^{\ell,\varepsilon} I_h^\ell R_h^{\ell,\varepsilon}$ with:
  - $R_h^{\ell,\varepsilon}$: local smoothing operator on carefully crafted domains around subsimplices
  - $I_h^\ell$: finite element interpolation for the subsimplex dofs
  - $J_h^{\ell,\varepsilon} := \big((I_h^\ell R_h^{\ell,\varepsilon})|_{V_h^\ell}\big)^{-1}$, following Schöberl
- commuting projections with uniform $L^2$ stability, but non-local

## Falk-Winther (2014):

- recursive definition: $\Pi^0\phi = \Pi_n^0\phi = \Pi_{n-1}^0\phi + \sum_{f\in\Delta(\mathcal{T}_h)} E_f^0 \mathrm{tr}_f P_f^0(\phi - \Pi_{n-1}^0\phi)$ with:
  - $P_f^0$: $H^1$ orthogonal projection on local space $V_h^0(\Omega_f)$
  - $\mathrm{tr}_f$: trace operator on subsimplex $f$
  - $E_f^0$: harmonic extension operator
- local commuting projections, and bounded in $H^1$, $H(\mathrm{curl})$, ... but not in $L^2$

## Arnold-Guzman (2021):

- extend the Falk-Winther approach
- local commuting projections with uniform $L^2$ stability