Part 1:

```
  -- abstract syntax of set expressions with variables of type v
data TERM v = Empty
  | Singleton (TERM v)
  | Union        (TERM v) (TERM v)
  | Intersection (TERM v) (TERM v)
  | Var          v
  deriving Show
  -- predicates over pure set expressions
data PRED v = Elem (TERM v) (TERM v)
  | Subset          (TERM v) (TERM v)
  | And             (PRED v) (PRED v)
  | Or              (PRED v) (PRED v)
  | Implies (PRED v) (PRED v)
  | Not             (PRED v)
  deriving Show
```

Part 2:

```
eval :: Eq v => Env v Set -> TERM v -> Set
eval env term = case term of
  Empty          -> S []
  Singleton t    -> S [eval env t]
  Union     t t1 -> S $ f t `union` f t1
  Intersection t t1 -> S $ f t `intersect` f t1
  Var       v    -> fromJust $ lookup v env
  where
  f = (\(S xs) -> xs) . eval env
  g Nothing = error "variable is not in enviroment"
  g (Just s) = s

check :: Eq v => Env v Set -> PRED v -> Bool
check env pred = case pred of
  Elem   t t1 -> eval env t `elem` g t1
  Subset t t1 -> all (`elem` g t1) $ g t
  And    p p1 -> f p && f p1
  Or     p p1 -> f p || f p1
  Implies p p1 -> not (f p) || f p1
  Not    p    -> not $ f p
  where
  f = check env
  g = (\(S xs) -> xs) . eval env
```

Part 3:

```
vonNeumann :: Int -> TERM v
vonNeumann 0 = Empty
```

$$
\begin{array}{l}
\mathit{vonNeumann}\ n = \mathit{Union}\ (\mathit{vonNeumann}\ \$\ n - 1)\ (\mathit{Singleton}\ (\mathit{vonNeumann}\ \$\ n - 1)) \\
\mathit{claim} :: \mathit{Int} \to \mathit{Int} \to \mathit{Bool} \\
\mathit{claim}\ i\ \mathit{i1}\ \mid (\#)\ n \leqslant (\#)\ n = \mathit{check}\ \mathit{env}\ (\mathit{Subset}\ n\ \mathit{n1}) \\
\quad \textbf{where} \\
\quad (n, \mathit{n1}) = (\mathit{vonNeumann}\ i, \mathit{vonNeumann}\ \mathit{i1}) \\
\quad (\#) \qquad = (\lambda(S\ \mathit{xs}) \to \mathit{length}\ \mathit{xs}) \circ \mathit{eval}\ \mathit{env} \\
\mathit{claim1} :: \mathit{Int} \to \mathit{Bool} \\
\mathit{claim1}\ \ i = n\ i \equiv S\ [n\ j \mid j \leftarrow [0\,..\,i - 1]] \\
\quad \textbf{where} \\
\quad n = \mathit{eval}\ \mathit{env} \circ \mathit{vonNeumann}
\end{array}
$$