Questions of the 2022 Project

1. Introduction

First read the text and try to understand it the best as you can. When you answer questions, explain what you are doing and for a proof, give all the steps.

2. Qualitative information

2.1

Prove that the equation (1), p.316 is a logistic equation. Explain what is K_B the carrying capacity. What's the relationship between K_B the carrying capacity and the usual vital coefficients?

2.2

Prove that 0 is always an equilibrium point of equation (3). Which kind of equilibrium point it is? (attractor, repeller or saddle)

2.3

Draw the function $f(x) = \frac{x}{1+x^2}$ with the help of any computer device. Prove graphically that there are one to three equilibrium points in equation (9) (Hint: study intersections of curves for a fixed Q like in the bottom of p.318).

2.4

Explain why the stability of the equilibrium points of equation (3) is the same as the non zero corresponding equilibrium points in equation (9). From now on, we'll study just equation (9)

2.5

Given an autonomous differential equation $y' = f(y)$, an equilibrium point x is said to be hyperbolic if $f'(x) \neq 0$. If the differential equation is autonomous C^1 and all the equilibrium points are hyperbolic, then two successive equilibrium points may only be an attractor and a repeller or a repeller and an attractor. Prove that there exists a non hyperbolic point in the case of two intersections.
Deduce from this the stability of the equilibrium points in the three cases above (one, two or three equilibrium points).

2.6

Let $y' = f(y, a, b)$ be an autonomous differential system with a,b parameters $a, b \in \mathbb{R}$ where f is a C^1 function defined in $\mathbb{R}$.
Prove that a necessary condition for a bifurcation at (y_0, a_0, b_0) is

$$\begin{cases} f(y_0, a_0, b_0) = 0 \\ \dfrac{\partial f}{\partial y}(y_0, a_0, b_0) = 0 \end{cases}$$

2.7

Prove that the bifurcation curve of equation 9 is given by the parametric representation on the plane (Q, R) which is given by the formula in the appendix:

$$\begin{cases} R = \dfrac{2\mu^3}{(1+\mu^2)^2} \\ Q = \dfrac{2\mu^3}{\mu^2 - 1} \end{cases}$$

Draw this curve with the help of any computer device. There is a singular point. This singularity is called a cusp singularity. Compute the coordinates of the singular point as explained in the appendix (p.332).

2.8

Explain why inside the region delimited by the bifurcation curve in the plane (Q, R), there are three equilibrium points and why there is only one outside this region. Explain also why crossing the bifurcation curve corresponds to a bifurcation.
Where is the region of the outbreak? The region between the curves is called bistable, Why? Where is the region (the refuge) without the insect outbreak problem for the trees?

3. Bifurcation diagram

3.1

With the help of Maple or Python, plot the bifurcation diagram (without phase lines) of the scaled differential equation (9). We recall that it is an implicitly defined surface (submanifold precisely) in the space (Q, R, μ)

3.2

What is the part of the surface which is projected orthogonally on the bifurcation curve in the plane (Q, R)?

3.3

Explain why there are turning points in the diagram and what may happen at their location. Is there a possibility of an hysteresis cycle? What happens then to the trees and to the insects? .

4. Numerical simulation

4.1

We now take instead of $f(\mu)$ in (9) the scaled reduced predation function

$$g(\mu) = \frac{\mu}{1+\mu^3}$$

. Draw the function and then do the same steps as in part 2. With the help of Maple or Python, draw the new bifurcation curve in the plane (Q, R) and the new bifurcation surface.

4.2

Compare with the modeling of the paper. Do you see some changes?

5. A Differential system (10 points)

Find all the equilibrium points of the differential systems: $\begin{cases} \dfrac{dx}{dt} = y^2 \\ \dfrac{dy}{dt} = x \end{cases}$

5.1

Plot in maple or Python the phase portrait of this differential system. Explain what you see.

5.2

Compute and draw the trajectory in the phase plane which is going through the origin $(0, 0)$. Notice that there is a cusp in $(0, 0)$ as above in the bifurcation curve of the insect outbreak.
The curve you've drawn has a discontinuous derivative in $(0, 0)$. Why this is not a contradiction with the fact that every solution of a C^1 differential equation must be C^1?