Project in Mathematical Modeling

Submitted to Serge Lukasiewicz by Gilad Zusman and Ariel Fuxman

2022

2 Qualitative Information

2.1) Distributing the right-hand side of equation (1),

$$\frac{dB}{dt} = r_B B \left(1 - \frac{B}{K_B}\right) = r_B B - \frac{r_B}{K_B} B^2,$$

and since r_B and K_B are constants (during the analysis on p.316), it is readily seen that equation (1) is the Logistic equation with vital coefficients $a = r_B$ and $b = \frac{r_B}{K_B}$. By what we learned in class about the Logistic equation,

$$\lim_{t\to\infty} B(t) = \frac{a}{b} = \frac{r_B}{\frac{r_B}{K_B}} = K_B,$$

which explains why the parameter K_B is called the carrying capacity, as it is the maximal amount of budworm density the ecological system can sustain.

2.2) 0 is always an equilibrium of equation (3), since for all r_B, K_B, α, β, the number $B = 0$ is a root of the equation

$$r_B B \left(1 - \frac{B}{K_B}\right) - \beta \frac{B^2}{\alpha^2 + B^2} = 0.$$

0 is a repeller, since $\frac{dB}{dt} > 0$ for B slightly greater than 0; indeed, we observe that

$$\lim_{B\to 0^+} \left(r_B \left(1 - \frac{B}{K_B}\right) - \beta \frac{B}{\alpha^2 + B^2}\right) = r_B > 0,$$

which implies that

$$\frac{dB}{dt} = r_B B \left(1 - \frac{B}{K_B}\right) - \beta \frac{B^2}{\alpha^2 + B^2} = B \left(r_B \left(1 - \frac{B}{K_B}\right) - \beta \frac{B}{\alpha^2 + B^2}\right)$$

is positive at some right neighborhood of $B = 0$.

2.3) We use the free online graphing tool called "Desmos". For values of Q smaller than a certain value Q_C, there is a single intersection for all values of R. We fix a small $Q \leq Q_C$, and study the intersections as illustrated in Figures 1 and 2.

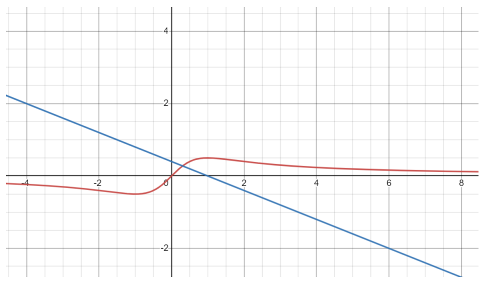

Figure 1: A plot of the curves with $Q \leq Q_C$, R small ($Q = 1, R = 0.4$). There is only a single intersection.

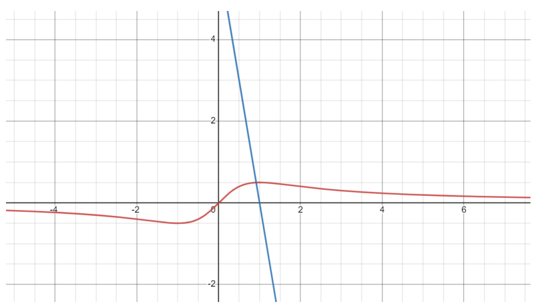

Figure 2: A plot of the curves with $Q \leq Q_C$ and R big ($Q = 1, R = 6$). There is still a single intersection, even as R increases.

Then, when the values of Q exceed value Q_C, there is a possibility of having one to three intersections. We fix a certain $Q > Q_C$ and study the intersections as R varies, illustrated in Figures 3, 4, 5.

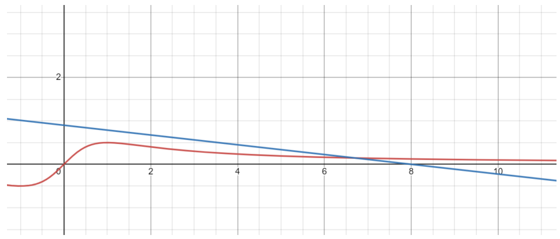

Figure 3: A plot of the curves with $Q > Q_C$ and R small ($Q = 8, R = 0.3$). There is still a single intersection.

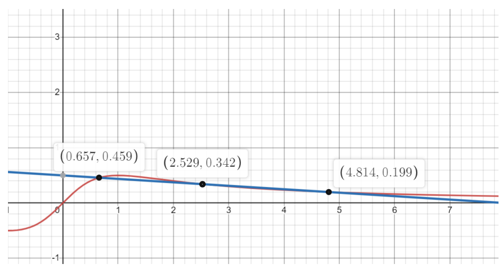

Figure 4: A plot of the curves with $Q > Q_C$ and R medium ($Q = 8, R = 0.5$). There are three intersections.

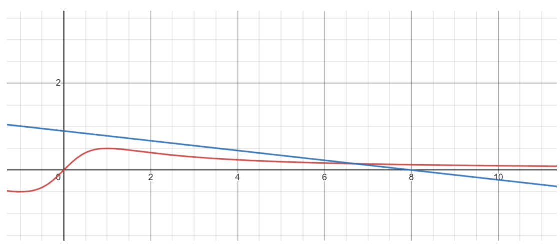

Figure 5: A plot of the curves with $Q > Q_C$ and R Big ($Q = 8, R = 0.9$). There is again a single intersection.

In addition, there exist $R_1 \in (0.3, 0, 5), R_2 \in (0.5, 0.9)$ such that there are two intersections. What we have shown proves that there are always 1 - 3 intersection points, and our results are compatible with Figure 2 in the paper.

2.4) Equations (3) and (9) have the same stability at corresponding equilibrium points, because when $B > 0$, then at matching points of the equations they both have the same sign, since equation (9) is obtained from equation (3) by multiplying by $\frac{\alpha}{\beta B}$ (A positive factor) and the new variable μ is just B but scaled (by a positive factor, so the direction doesn't change).

2.5) Rearranging equation (9), we see that the function which defines the autonomous differential equation is

$$h(\mu) := R\left(1-\frac{\mu}{Q}\right)-\frac{\mu}{1+\mu^2}=\frac{1}{1+\mu^2}\left(\left(1+\mu^2\right)\left(R-\frac{\mu}{Q}\right)-\mu\right).$$

Denote $P(\mu)=\left(1+\mu^2\right)\left(R-\frac{\mu}{Q}\right)-\mu$. Since $1+\mu^2>0$, the sign and the roots of $h(\mu)$ are determined by the cubic polynomial $P(\mu)$. Let $\mu_1,\mu_2,\mu_3\in\mathbb{C}$ be its roots. If there are exactly two distinct real roots, then since the coefficients of $P(\mu)$ are real, one of the roots must be a double root (For if $\mu_1,\mu_2\in\mathbb{R},\mu_1\neq\mu_2$ and $\mu_3\notin\mathbb{R}$, then also $\overline{\mu_3}\notin\mathbb{R}$ is a root, and since $\mu_3\neq\overline{\mu_3}$, there are four distinct roots, which contradicts the Fundamental Theorem of Algebra). Thus, without loss of generality, we can decompose $P(\mu)=a\left(\mu-\mu_1\right)^2\left(\mu-\mu_2\right)$, where $a<0$ is the leading coefficient, showing that $P(\mu_1)=P'(\mu_1)=0$. Using the quotient rule,

$$\frac{dh}{d\mu}=\frac{d}{d\mu}\left(\frac{P(\mu)}{1+\mu^2}\right)=\frac{P'(\mu)\left(1+\mu^2\right)-2\mu P(\mu)}{\left(1+\mu^2\right)^2},$$

which shows that μ_1 is a non-hyperbolic point.
The stabilities for all the cases are:

- When there is one intersection, then from the property given to us in the question, the point is an attractor (Since $B=0$ is a repeller of equation (1) and it is hyperbolic).
- When there are two intersections, we use the decomposition

$$h(\mu)=\frac{a\left(\mu-\mu_1\right)^2\left(\mu-\mu_2\right)}{1+\mu^2}.$$

  It is seen that μ_1 is a saddle point, since, locally, left to it and right to it the sign it the same. In addition, since $a<0$, μ_2 is an attractor, since, locally, left to it the sign is positive and right to it the sign is negative.
- When there are three intersections, then similar to our deduction in the case of one intersection, since there are only hyperbolic points, μ_- is an attractor, μ_C is a repeller and μ_+ is an attractor.

2.6) We proceed by proving the contrapositive. If $f(y_0, a_0, b_0) \neq 0$, then since $f \in C^1$, it is continuous, hence the inequality holds also in a neighborhood of (y_0, a_0, b_0), meaning that around (y_0, a_0, b_0) there are zero equilibrium points, and thus (y_0, a_0, b_0) is not a bifurcation point. If $\frac{\partial f}{\partial y}(y_0, a_0, b_0) \neq 0$ and $f(y_0, a_0, b_0) = 0$, then since $f \in C^1$ we apply the implicit function theorem to the equation $f(y, a, b) = 0$ and conclude that around (y_0, a_0, b_0), $y := y(a, b)$ is defined implicitly as a function of a, b, meaning that locally there is only one equilibrium point.

2.7)

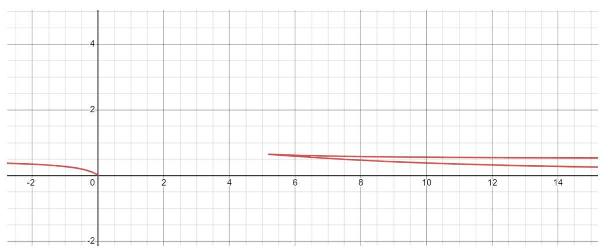

Figure 6: Our plot of the bifurcation diagram in the (Q, R) plane. The left part of the curve can be discarded since it corresponds to negative Q values.

A double root occurs if

$$\begin{cases} \frac{df}{d\mu} = \frac{dg}{d\mu} \\ f(\mu) = g(\mu) \end{cases}$$

We have $f(\mu) = \frac{R}{Q}(Q - \mu)$, $g(\mu) = \frac{\mu}{1+\mu^2}$, so $\frac{df}{d\mu} = -\frac{R}{Q}$, $\frac{dg}{d\mu} = \frac{1-\mu^2}{(1+\mu^2)^2}$. Then, we get:

$$\frac{\mu^2 - 1}{\left(1 + \mu^2\right)^2} (Q - \mu\ ) = \frac{\mu}{1 + \mu^2}$$

$$\implies (\mu^2 - 1)(Q - \mu) = \mu(1 + \mu^2)$$

$$\implies Q = \frac{\mu + \mu^3 + \mu^3 - \mu}{\mu^2 - 1} = \frac{2\mu^3}{\mu^2 - 1}$$

$$\implies R = \frac{Q\left(\mu^2 - 1\right)}{\left(1 + \mu^2\right)^2} = \frac{2\mu^2}{\left(1 + \mu^2\right)^2}$$

To calculate the cusp point, we find the root of $\frac{d^2 g}{d\mu^2}$:

$$\frac{d^2 g}{d\mu^2} = \frac{2\mu\left(\mu^2 - 3\right)}{\left(\mu^2 + 1\right)^3} = 0$$

The positive root, which we are interested in, is $\mu = \sqrt{3}$. Substituting this μ into the expressions for Q, R, we obtain that the cusp point is $(Q, R) = (3^{3/2}, \frac{3^{3/2}}{8})$.

2.8) We explained in question 2.3 why inside the region there are three equilibrium points and why outside there is one equilibrium point, by splitting into the two cases $Q \leq Q_C$ and $Q > Q_C$ and considering R small, medium and big. Crossing the bifurcation curve corresponds to a bifurcation since the number of equilibrium points changes (from one to three or from three to one). The region of the outbreak is where $Q > Q_C$ and (Q, R) is above the upper bifurcation curve. The region between the bifurcation curves is called bistable because there are 2 stable equilibrium points there (μ_- and μ_+). The region of refuge is where $Q > Q_C$ and (Q, R) is below the lower bifurcation curve or where Q is small.

3 Bifurcation Diagram

3.1)

Listing 1: Our MATLAB code used to plot the implicit surface.

```
f = @(x,y,z) x * (1 - z / y) - z / (1 + z^2)
interval = [0 1 0 10 0 10]
fimplicit3(f, interval)
xlabel('R-axis')
ylabel('Q-axis')
zlabel('\mu-axis')
```

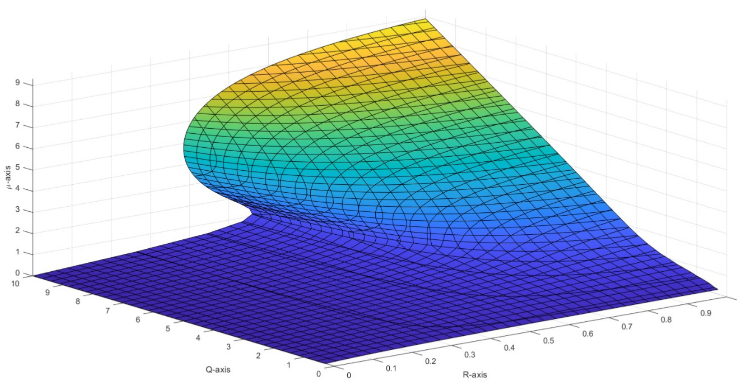

Figure 7: The graph of the implicit surface, outputted by our code from above

3.2) The part of the surface which is projected orthogonally on the bifurcation curve in the plane (Q, R) is the set of all points (Q, R, μ) where there are exactly two solutions to equation (9), union with the point on the surface corresponding to the cusp.

3.3) All the points on the upper and lower bifurcation curves are tipping points. To demonstrate it, suppose (Q, R) is in the bistable region and μ is around the attractor μ_-. Afterwards, if R is increased and the point crosses the upper bifurcation curve, then μ_- and μ_C abruptly disappear and μ starts converging to μ_+. The sharp increase that is caused is irreversible; if μ had enough time to increase, then even if R returns to its initial value, μ will stay at relatively high values at between μ_C and μ_+. In fact, to really reverse the change, R will have to decrease substantially and cross the lower bifurcation curve. This shows that the system exhibits hysteresis, since the future dynamics depend on the previous state of the variables. The paper describes the possibility of a hysteresis cycle; if B starts at a low value around B_-, then since B is small the forest variable E, S might increase so much that the point crosses the upper bifurcation curve, causing B to suddenly increase rapidly. Now, since B is large, the forest variables E, S might decrease such that the lower bifurcation curve is crossed, causing B to plummet back to the initial value. This is the outbreak/decline hysteresis cycle of the budworm/balsam interaction.

4 Numerical Simulation

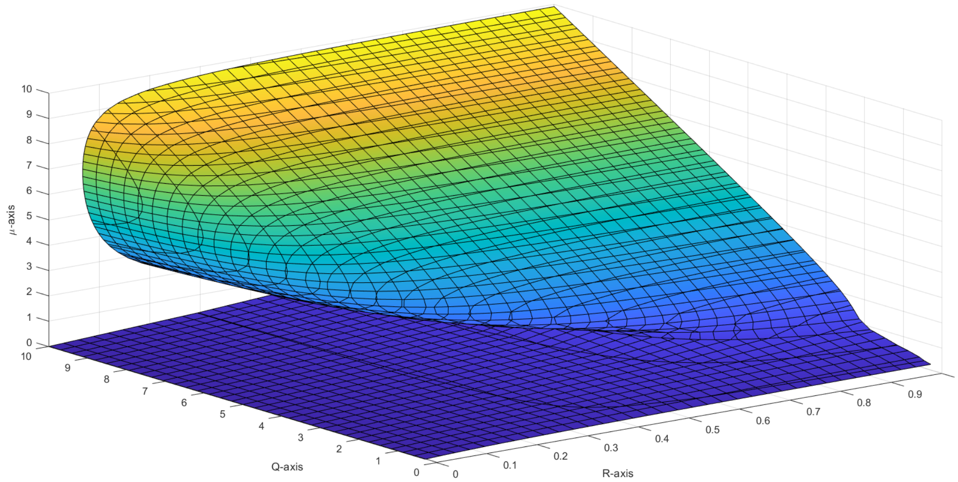

Figure 8: The new bifurcation surface

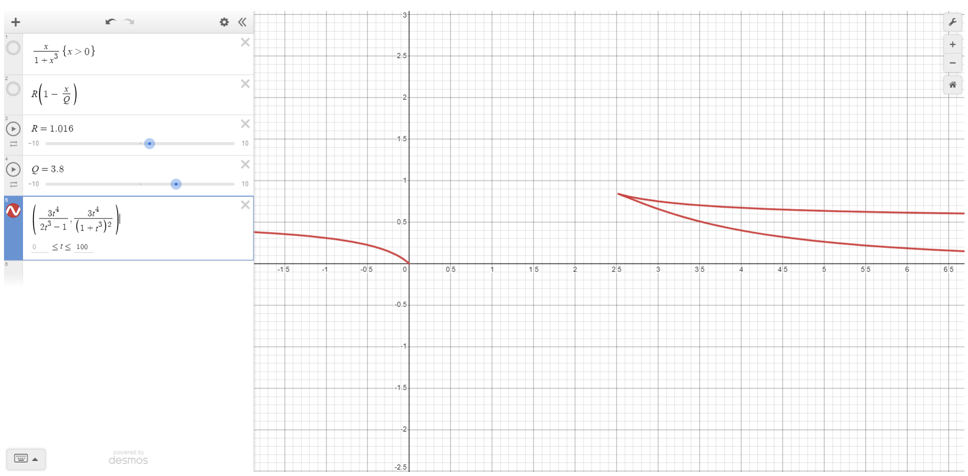

Figure 9: The new bifurcation plane curve

Now we iterate over the relevant parts of section 2 and answer them with respect to the scaled predation term.

4.3) The analysis is identical to section 2. Again, there exists Q_C (different value), such that if $Q < Q_C$ there is a single intersection.

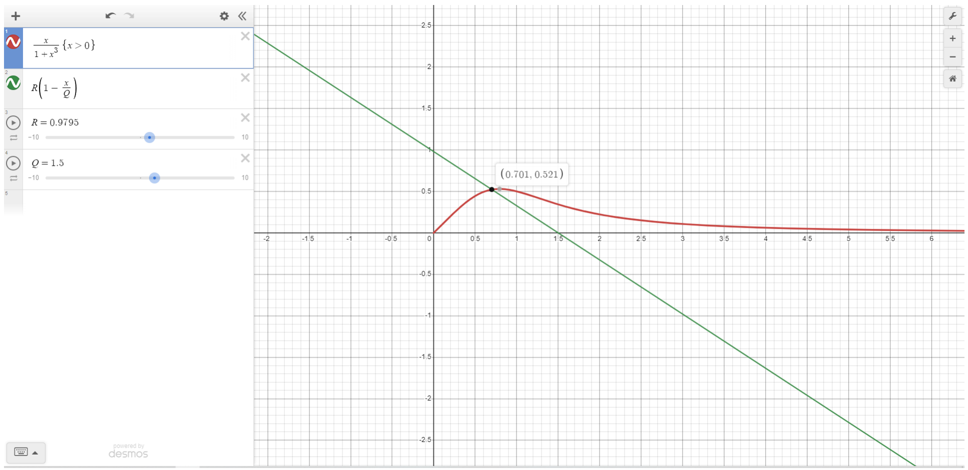

Figure 10: Q small and R small

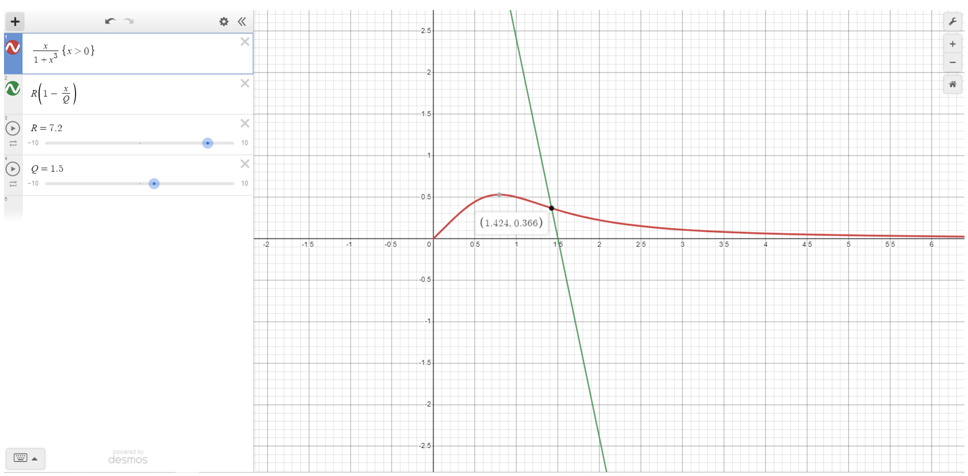

Figure 11: Q small and R big

And when $Q \geq Q_C$ then it is similar to section 2.

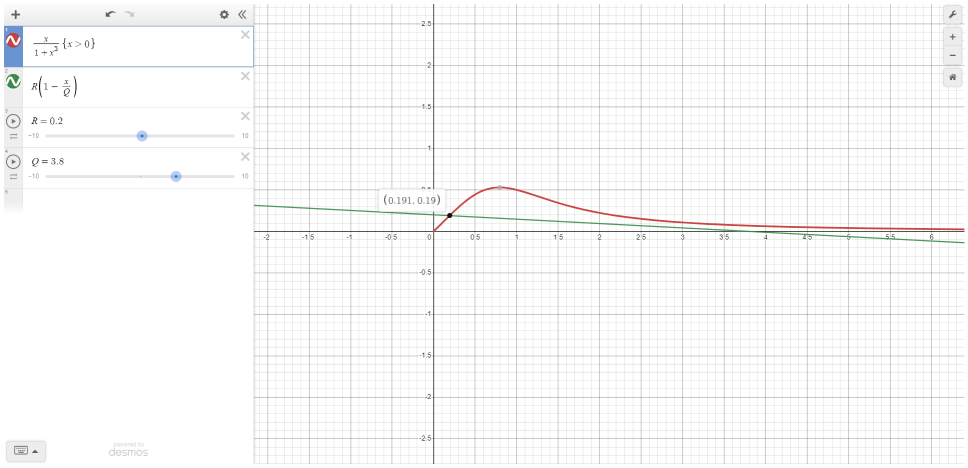

Figure 12: Q big and R small

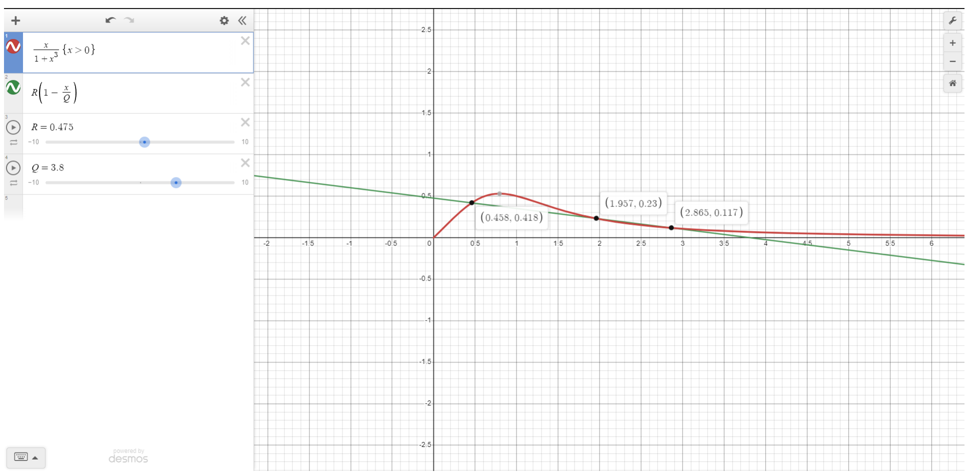

Figure 13: Q big and R medium

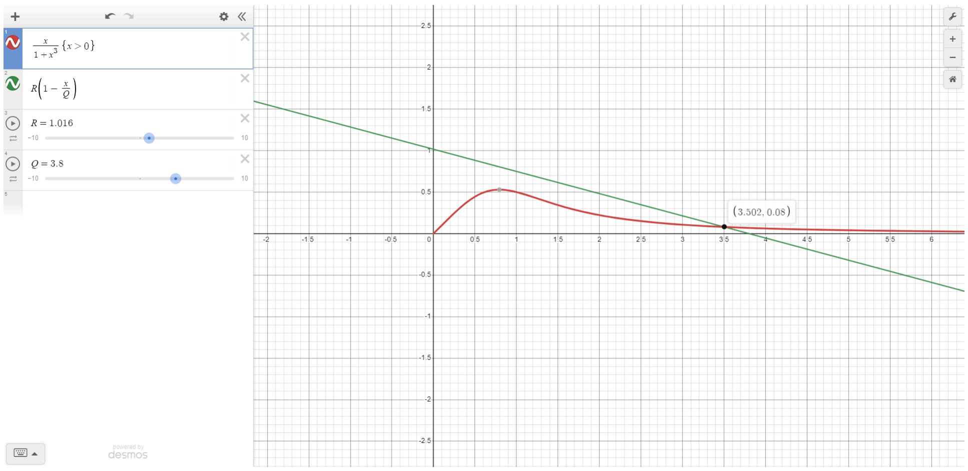

Figure 14: Q big and R big

4.4) Essentially the same argument also applies here, since the scaled predation factor $\frac{\mu}{1+\mu^3}$ is positive when $\mu > 0$.

4.5) This time, we define:

$$h(\mu) := \frac{1}{1+\mu^3}\left(\left(1+\mu^3\right)\left(R-\frac{\mu}{Q}\right)-\mu\right)$$
$$P(\mu) := \left(1+\mu^3\right)\left(R-\frac{\mu}{Q}\right)-\mu$$

We show again that when there are exactly two positive real roots, there exists $\mu > 0$ such that $P(\mu) = P'(\mu) = 0$. This time, since $P(\mu)$ is of degree four with a negative leading coefficient $a < 0$, we are guaranteed the existence of a negative real root $\mu_0 < 0$. Then, one of the roots must be a double root (considering the Fundamental Theorem of Algebra again). So $P(\mu) = a(\mu-\mu_0)(\mu-\mu_1)(\mu-\mu_2)^2$ which tells us that $P(\mu_2) = P'(\mu_2) = 0$ as desired. The stabilities for all the cases are identical to section 2.

4.7)

$$f(\mu) := \frac{R}{Q}(R-\mu)$$
$$g(\mu) := \frac{\mu}{1+\mu^3}$$

$$\begin{cases} \frac{df}{d\mu} = \frac{dg}{d\mu} \\ f(\mu) = g(\mu) \end{cases}$$

$$\begin{aligned}
\frac{df}{d\mu} = \frac{dg}{d\mu} &\implies \frac{-R}{Q} = \frac{1+\mu^3-3\mu^3}{\left(1+\mu^3\right)^2} \implies \frac{R}{Q} = \frac{2\mu^3-1}{\left(1+\mu^3\right)^2} \\
&\implies \frac{2\mu^3-1}{\left(1+\mu^3\right)^2}\left(Q-\mu\right) = \frac{\mu}{1+\mu^3} \\
&\implies Q = \frac{\mu\left(1+\mu^3\right)}{2\mu^3-1} + \mu = \frac{3\mu^4}{2\mu^3-1} \\
&\implies R = \frac{3\mu^4}{2\mu^2-1} \cdot \frac{2\mu^2-1}{\left(1+\mu^3\right)^2} = \frac{3\mu^4}{\left(1+\mu^3\right)^2}
\end{aligned}$$

And now,

$$\begin{aligned}
&\frac{d^2 g}{d\mu^2} = 0 \\
&\frac{6\mu^5-12\mu^2}{\left(1+\mu^3\right)^2} = 0 \\
&\implies \mu = 0 \ \text{ or } \ 6\mu^3 = 12 \\
&\implies \mu = \sqrt[3]{2} \ \text{ and } \ (Q,R) = \left(2^{\frac{4}{3}}, \frac{2^{\frac{4}{3}}}{3}\right)
\end{aligned}$$

See Figure 9 for the new bifurcation curve.

4.8) There is no difference with section 2. The scaled predation factor does not have any significant impact on the behavior of the system. This is since there was no difference in part 4.3).

5 A Differential System

5.1)

$$\begin{cases} y^2 = 0 \\ x = 0 \end{cases}$$

so $(0,0)$ is a unique equilibrium point.

Listing 2: Our MATLAB code used to plot phase portrait

```
syms x y
f(x,y) = y^2;
g(x,y) = x;
[x,y] = meshgrid(-5:0.05:5  -5:0.05:5);
f1 = f(x,y);
g1 = g(x,y);
streamslice(x,y,f1,g1, 2);
xlabel(x-axis)
ylabel(y-axis)
```

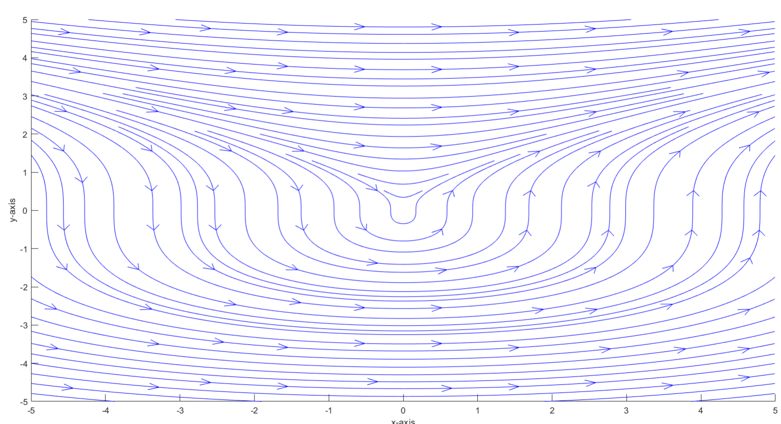

Figure 15: The graph of the phase portrait, outputted by our code from above

5.2)

$$\begin{cases} \frac{dx}{dt} = y^2 \\ \frac{dy}{dt} = x \end{cases} \implies \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{x}{y^2}$$

$$\implies y^2 dy = x dx$$

$$\implies \frac{1}{3}y^3 = \frac{1}{2}x^2 + C$$

$$\implies y = \sqrt[3]{\frac{3}{2}x^2 + C}$$

Imposing that the solution passes through the origin, we set $C = 0$ and draw the curve.

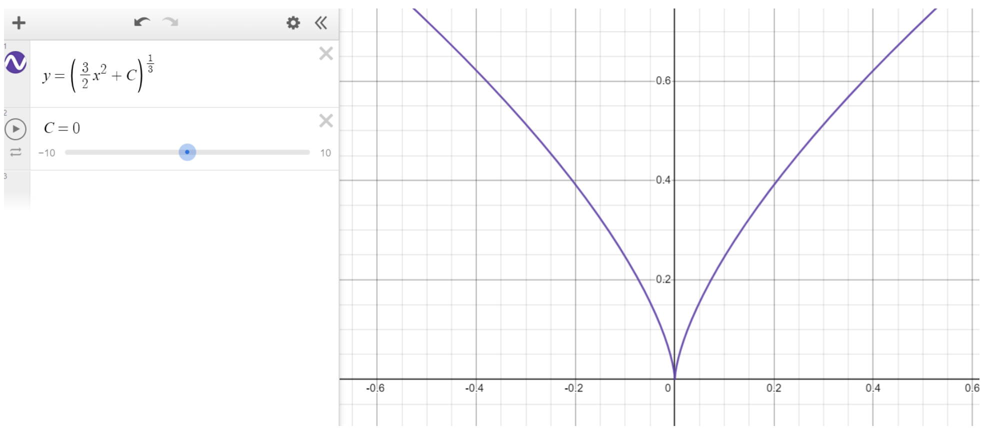

Figure 16: The graph of the alleged solution $y = \sqrt[3]{\frac{3}{2}x^2 + C}$ with $C = 0$

The graph indeed appears to have a non-differentiable point at the origin, and it is easy to confirm it by looking at the definition of the derivative. This doesn't contradict the fact that the solution must be C^1, since what we sketched is actually a union of three different solutions, namely the left branch of the solution, united with the constant solution $(0, 0)$ and united with the right branch of the solution. We achieved this solution since we have divided by $\frac{dx}{dt}$ which vanishes at $(0, 0)$.