A Dynamic Approach to Starlike and Spirallike Functions

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Abstract

The purpose of this essay is to give a complete characterization of spirallike and starlike functions on the unit circle, using a dynamic approach. Nevanlinna's condition and other similar results are presented as corollaries. Part one gives the necessary background in semigroups and their generators. Part two introduces the concepts of spirallikeness and starlikeness. Part three shows a dynamic approach for characterization of spirallike and starlike functions.

Part I

Background in Complex Dynamics

Definition 1. A ***domain*** is a non-empty connected open set in a topological space.

Remark. Regarding the definition above, remember that whenever the notion of openness is present in our space, there is a topology. The complex plane $\mathbb{C}$ together with the Euclidean metric induces a topology.

Definition 2. A ***simply connected domain*** in $\mathbb{C}$ is a domain such that every path between two points can be continuously transformed into any other such path while preserving the two endpoints in question.

Remark. Regarding the definition above, we restricted ourselves to $\mathbb{C}$ in order to prevent a discussion on path-connectedness. In fact, open subsets of $\mathbb{R}^n$ are connected if and only if they are path-connected.

Definition 3. Let $D \subset \mathbb{C}$ be a domain. A family $\mathcal{S} = \{F_t\}_{t\geq 0} \subset \mathrm{Hol}\,(D)$ of holomorphic self-mappings of D is called a ***one-parameter continuous semigroup*** if

(1) $F_{t+s} = F_t \circ F_s$ for all $t, s \geq 0$;

(2) $F_0(z) = z$ for all $z \in D$; that is, F_0 is the identity mapping on D;

(3) $\lim_{t\to s} F_t(z) = F_s(z)$ for all $s \geq 0$ and $z \in D$.

We like to think of $F_t(z)$ as the solution of some initial value problem, with z indicating the initial condition and t indicating the progress in time. This comes with the assumption that the solution is defined for all $t \geq 0$ and uniqueness of the solution (so that property (1) is satisfied). Later we will see a justification of this. Sometimes F is called a "flow" in the context of dynamical systems. Continuity here refers to two points

- The parameter t which is continuous, as opposed to discrete.
- Condition (3).

The term semigroup here is used since $(S, \circ)$ (that is, the family together with the composition operation) is an additive semigroup, since property (1) implies associativity and commutativity. What is special about our semigroup is that it has an identity element F_0. It is, however, not a group, since we cannot invert elements. This is since our time variable t does not admit negative values.

The concept of generators is introduced by a theorem.

Theorem 4. *Let $D \subset \mathbb{C}$ be a simply connected domain. Let $\mathcal{S} = \{F_t\}_{t\geq 0}$ be a semigroup, such that for all $z \in D$,*

$$\lim_{t\to 0^+} F_t(z) = z. \tag{1}$$

Then, for all $z \in D$ there exists the limit

$$\lim_{t\to 0^+} \frac{z - F_t(z)}{t} = f(z), \tag{2}$$

which is a holomorphic function on D. Moreover, the semigroup $\mathcal{S}$ can be defined as the unique solution to the initial value problem:

$$\begin{cases} \frac{\partial F_t(z)}{\partial t} + f\left(F_t(z)\right) = 0, & t \geq 0 \\ F_0(z) = z, & \forall z \in D \end{cases} \tag{3}$$

so a semigroup satisfying eq. (1) is differentiable in parameter and thus continuous

Definition 5. The function $f \in \mathrm{Hol}\,(D, \mathbb{C})$ defined by the limit (2) is called the ***generator*** of the continuous semigroup $\mathcal{S} = \{F_t\}_{t\geq 0}$.

Definition 6. A function $f \in \mathrm{Hol}\,(D, \mathbb{C})$ is called a ***semi-complete vector field*** if for all initial points $z \in D$, the initial value problem (3) has a unique solution $\{F_t(z)\} \subset D$ defined for all $t \geq 0$.

Thus, Theorem 4 asserts that $f \in \mathrm{Hol}(D, \mathbb{C})$ is a semi-complete vector field if and only if it is a generator of a continuous semigroup on D.

Notation. Denote the class of all holomorphic generators on D by $\mathcal{G}(D)$.

The following theorem discusses how every generator on a domain Ω corresponds to another generator on a biholomorphically equivalent set D.

Theorem 7. *Let $D, \Omega \subset \mathbb{C}$ be domains such that $h(D) = \Omega$ for some biholomorphic (a bijective holomorphic mapping such that its inverse is also holomorphic) function h. Then, there is a linear invertible operator T on the space $Hol(\Omega, \mathbb{C})$ onto the space $Hol(D, \mathbb{C})$, which takes the set $\mathcal{G}Hol(\Omega) \subset Hol(\Omega, \mathbb{C})$ onto the set $\mathcal{G}Hol(D)$. Moreover, T is given by the formula*

$$T(\varphi)(\cdot) = \left(h'(\cdot)\right)^{-1} \varphi(h(\cdot))$$

where $\varphi \in \mathcal{G}Hol(\Omega)$.

Theorem 7 has a lot of applications due to ***Riemann mapping theorem***, which states that every simply connected domain, excluding $\mathbb{C}$ itself, is biholomorphically equivalent to Δ. One of the applications is determining which dynamical characteristics (for example, fixed points) are preserved under a biholomorphism. Later this theorem will be used is Part 3.

We present an important characterization called "Berkson-Porta parametric representation" of generators on Δ.

Theorem 8. *Let $f \in Hol(\Delta, \mathbb{C})$. Then, $f \in \mathcal{G}(\Delta)$ if and only if there is a point $\tau \in \overline{\Delta}$ and a function $p \in Hol(\Delta, \mathbb{C})$ with $Re\, p(z) \geq 0$, $\forall z \in \Delta$ such that*

$$f(z) = (z - \tau)(1 - z\overline{\tau})\, p(z), \quad \forall z \in \Delta.$$

Moreover, this representation is unique.

Part II

Starlike and Spirallike Functions

The definition of what it means for a real valued function $h : \mathbb{R}^n \to \mathbb{R}$ to be convex, relies on the definition of what it means for a subset $D \subset \mathbb{R}^n$ to be convex. Similarly, before introducing starlike functions, we have to understand what a starlike domain is.

Definition 9. A domain $\Omega \subset \mathbb{C}$ is said to be ***starlike*** (with respect to the origin) if for every point $w \in \Omega$, the linear segment joining w to zero,

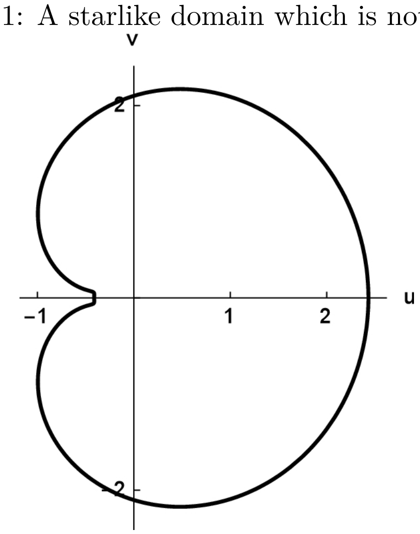
$$(0, w] := \{tw : t \in (0, 1]\},$$

lies entirely in Ω.

We can easily expand this definition to define what it means to be stalike with respect to another point $a \in \mathbb{C}$ instead, by examining the segment $(a, w]$ instead of the segment $(0, w]$.

There is a similarity between *Definition 9* and the definition of a convex domain. The obvious difference is that for a domain to be convex, the linear segment joining **every two points** must lie entirely in the domain. So if $D \subset \mathbb{C}$ is a convex domain such that $0 \in D$, then D is starlike. However, the converse is not true as seen in the following figure:

Figure 1: A starlike domain which is not convex



A subtlety is that in Definition 9, we examine the linear segment $(0, w]$ and not $[0, w]$, so it is not necessary that $0 \in \Omega$ (for example, $\Delta \setminus \{0\}$ is starlike). This leads us to the following proposition:

Proposition 10. *If Ω is a starlike domain, then $0 \in \overline{\Omega}$, where $\overline{\Omega}$ is the closure of Ω.*

Proof. If $0 \notin \overline{\Omega}$, then there exists an open ball $B_r(0)$ such that $B_r(0) \cap \Omega = \emptyset$. Ω is a domain, so it is non empty, so there exists $x \in \Omega$. Since Ω is starlike, $(0, x] \subset \Omega$, which implies that $\frac{r}{2|x|}x \in \Omega$, but $\frac{r}{2|x|}x \in B_r(0)$ which is a contradiction. $\square$

After defining what a starlike domain is, we are ready to define what a starlike function is.

Definition 11. $h \in \operatorname{Univ}(D)$ is called a ***starlike function*** on D if $h(D)$ is a starlike domain.

1. If $0 \in h(D)$, then h is said to be starlike with respect to an ***interior*** point.
2. If $0 \notin h(D)$, then h is said to be starlike with respect to an ***boundary*** point (sometimes called fanlike).

Note that these two above are the only possibilites, as proved in Proposition 10.

Definition 12. A domain $\Omega \subset \mathbb{C}$ is said to be ***$\mu-$spirallike*** (with respect to the origin) if there exists $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > 0$, such that for all $w \in \Omega$, the spiral curve

$$\left\{e^{-t\mu}w : \ t \geq 0\right\}$$

lies entirely in Ω.

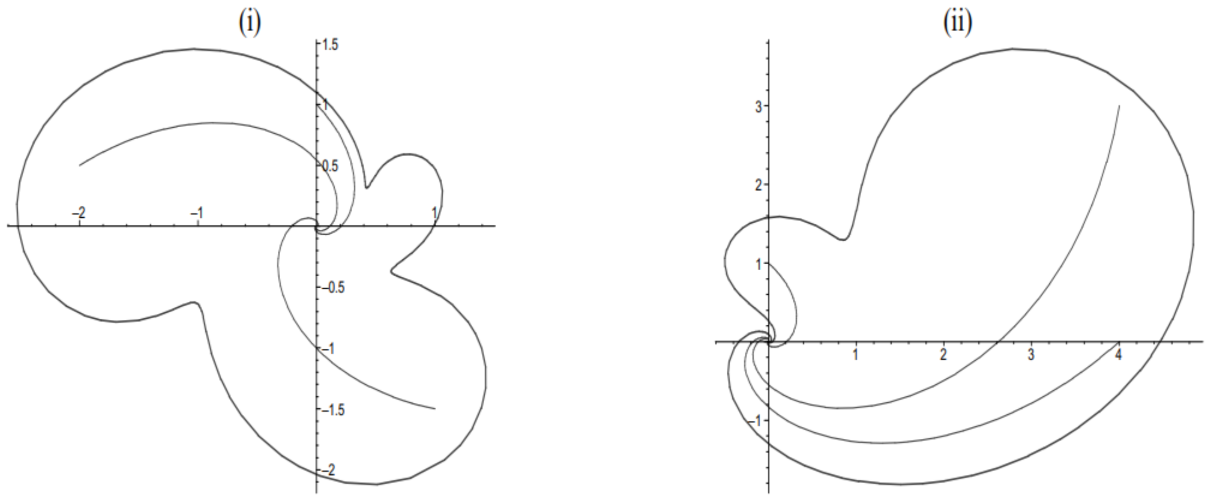
To get a better understanding of the curve in Definition 12, suppose $\mu = x + iy$. Then,

$$e^{-t\mu} = e^{-tx}e^{-ity}.$$

- Multplication by e^{-xt} corresponds to exponential decay of the norm of w, with x determining the decay rate.
- Multiplication by e^{-ity} corresponds to clockwise rotation of the angle of w, with y determining the frequencey of rotation.
- We see that if there is no rotation, there is just a straight convergence towards zero. In other words, a starlike function is a spirallike function with parameter μ such that $\operatorname{Im}\mu = 0$.

We indeed obtain images of spirals.

Figure 2: Examples of spirallike functions



There seems to be clockwise rotation. Also the decay seems exponential because it is fast; there is fast convergence towards zero, since the spiral doesn't even have time to complete a full cycle.

Due to the convergence towards zero, it is clear that $0 \in \overline{\Omega}$ for any spirallike domain. Analogusly to Definition 11, we arrive at the follwing definition.

Definition 13. $h \in \operatorname{Univ}(D)$ is called a ***spirallike function*** on D if $h(D)$ is a spirallike domain.

1. If $0 \in h(D)$, then h is said to be spirallike with respect to an ***interior*** point.

2. If $0 \notin h(D)$, then h is said to be spirallike with respect to an ***boundary*** point (sometimes called snaillike).

Part III

A Dynamic Approach

Proposition 14. *Let D be a spirallike domain with parameter μ. Then,*

$$F_t(z) = e^{-t\mu} z$$

is a continuous semigroup.

Proof. Due to spirallikeness, F_t is indeed a self-mapping of D for all $t \geq 0$. Now, for all $t, s \geq 0$ and $z \in D$,

$$F_{t+s}(z) = e^{-\mu(t+s)} z = e^{-\mu t} e^{-\mu s} z = e^{-\mu t} F_s(z) = F_t\left(F_s(z)\right) = \left(F_t \circ F_s\right)(z).$$

In addition, for all $z \in D$,

$$F_0(z) = e^{-\mu \cdot 0} z = z.$$

Lastly, for all $s \geq 0$ and $z \in D$,

$$\lim_{t \to s} F_t(z) = \lim_{t \to s} e^{-\mu t} z = e^{-st} z = F_s(z).$$

□

Corollary 15. *The generator of the semigroup in Proposition 14 is*

$$f(z) = \lim_{t \to 0^+} \frac{z - F_z(t)}{t} = -\frac{\partial}{\partial t} e^{-t\mu} z = \mu z.$$

Now we start moving towards a characterization of spirallike functions.

Theorem 16. *Let h be a $\mu-$spirallike function on D. Then, there exists a generator $f \in \mathcal{G}Hol(D)$ of a continuous semigroup such that*

$$\mu h(z) = h'(z) f(z), \quad \forall z \in D.$$

Proof. Denote the image of h by $\Omega = h(D)$. Since Ω is spirallike, by Corollary 15, $g(z) = \mu z$ is a generator on Ω. Using Theorem 7 (a surjective univalent function is biholomorphic),

$$f(z) = (Tg)(z) = \mu \left(h'(z)\right)^{-1} h(z)$$

belongs to $f \in \mathcal{G}\mathrm{Hol}\,(D)$. Multiplying both sides by $h'(z)$, we get

$$\mu h(z) = h'(z) f(z), \quad \forall z \in D.$$

□

Now we move to the converse theorem.

Theorem 17. *Suppose that $h \in Hol\,(D)$ satisfies*

$$\mu h(z) = h'(z) f(z), \quad \forall z \in D,$$

with $\mu \in \mathbb{C}$ $Re\,\mu > 0$ and $f \in \mathcal{G}Hol\,(D)$. Then the set $\Omega = h(D)$ is spirallike. Moreover, if D is bounded and h has a null point $\tau \in D$ with $h'(\tau) \neq 0$, then h is univalent, hence $\mu-$spirallike with respect to an interior point.

Using Theorem 8, we get a complete characterization of spirallike functions on Δ.

Corollary 18. *Let $h \in Univ\,(\Delta, \mathbb{C})$. Then, h is $\mu-$spirallike if and only if*

$$\mu h(z) = h'(z)\,(z-\tau)\,(1 - z\overline{\tau})\,p(z), \quad \forall z \in \Delta, \tag{4}$$

where $\tau \in \overline{\Delta}$, $\mu \in \mathbb{C}$ with $Re\,\mu > 0$ and $p \in Hol\,(\Delta, \mathbb{C})$ with $Re\,p(z) \geq 0$ for all $z \in \Delta$.

To arrive at Nevanlinna's condition, we need to consider the special case $\tau = 0$, $\mu \in \mathbb{R}$.

Corollary 19. *Let $h \in Univ\,(\Delta)$ such that $h(0) = 0$. Then, h is starlike if and only if*

$$Re \frac{zh'(z)}{h(z)} > 0, \quad \forall z \in \Delta.$$

Proof. Assume h is starlike. Consider eq. (4) and its attached conditions. Since h is univalent, its derivative doesn't vanish (in fact, if a function is locally univalent at some point if and only if its derivative does not vanish there). Also, since $\tau \in \overline{\Delta}$ and $z \in \Delta$, the term $(1 - z\overline{\tau})$ does not vanish. The real part of

$p(z)$ is non-negative, so $p(z)$ also never vanishes. Together with $h(0) = 0$, we conclude that $\tau = 0$. So the eq. (4) transforms into

$$\mu h(z) = zh'(z)p(z).$$

Dividing both sides by $zh'(z)$, we get

$$p(z) = \frac{\mu h(z)}{zh'(z)}.$$

Note that $p(z)$ is holomorphic and thus continuous, so $p(0)$ is actually given by

$$\lim_{z\to 0} \frac{\mu h(z)}{zh'(z)} = \lim_{z\to 0} \frac{\mu}{h'(z)} \underbrace{\lim_{z\to 0} \frac{h(z)}{z}}_{h'(0)} = \mu.$$

Now, looking at the real part of $p(z)$, for all $z \in \Delta$ we have

$$\operatorname{Re} p(z) > 0 \Longrightarrow \operatorname{Re} \frac{\mu}{p(z)} > 0 \Longrightarrow \operatorname{Re} \frac{zh'(z)}{h(z)} > 0.$$

Now, suppose that

$$\operatorname{Re} \frac{zh'(z)}{h(z)} > 0, \quad \forall z \in \Delta.$$

Then, eq. (4) and its attached conditions hold with $p(z) = \frac{\mu h(z)}{zh'(z)}$, $\tau = 0$, $\mu = 1$. □

We can also get a simiar charactization for spirallike functions.

Corollary 20. *Let $h \in Univ(\Delta)$ such that $h(0) = 0$. Then, h is spirallike with angle of rotation θ if and only if*

$$Re\, e^{-i\theta} \frac{zh'(z)}{h(z)} > 0, \quad \forall z \in \Delta.$$

Proof. The corollary can be proved similarly to Corollary 19, considering that such suitable μ has the representation

$$\mu = re^{i\theta},$$

with $r > 0$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. □

In our assay we have showed the intrinsic connection between complex dynamics and the geometry of spirallike and starlike functions. We went through defining the necessary background in complex dynamics such as the definition of semigroups and generators, introduced the concept starlike and spirallike functions, and showed the connection between these two theories.