Assignment 1

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April 1, 2020

1

For a fixed point $x_* \neq 0$ of the logistic map, we have by definition

$$\mu x_*(1-x_*) = x_* \implies \mu - \mu x_* = 1 \implies \mu - 1 = \mu x_* \implies \frac{\mu-1}{\mu} = x_* \,.$$

Thus the fixed points are $x_* = 0$ and $x_* = \frac{\mu-1}{\mu}$.

2

Let $f_\mu(x)$ be any map, and $F(x,\mu) = f_\mu(x) - x$. Then if x_* is a fixed point of f_μ , $F(x_*,\mu) = 0$ and the curve $\{(x(\mu),\mu) \in \mathbb{R}^2 : f_\mu(x(\mu)) = x(\mu)\}$ is a level curve of F. Thus

$$\nabla F \cdot (x'(\mu), 1) = \frac{\partial F}{\partial x} x'(\mu) + \frac{\partial F}{\partial \mu} = 0 \implies x'(\mu) = -\frac{F_\mu}{F_x} \,,$$

so we must have $F_x \neq 0$, as stated by the implicit function theorem (in the two-dimensional case). The criterion for existence of a function $x(\mu)$ in a neighbourhood around x_* is therefore $f'_\mu(x_*) - 1 \neq 0$.

3

Let x_* be a fixed point of a smooth map $f(x)$, and suppose $|f'(x_*)| < 1$. Then, for sufficiently small h,

$$f(x_* + h) = f(x_*) + f'(x_*)h + O(h^2) \approx f(x_*) + f'(x_*)h \implies |f(x_* + h) - x_*| \approx |f'(x_*)h| < |h| \,.$$

Thus $f(x_* + h)$ is closer to x_* than $x_* + h$, and $|f^k(x_* + h) - x_*| < |h^k|$. So for all x in a neighbourhood around x_*, the orbit of x will converge to x_*. Similarly, if $|f'(x_*)| > 1$, it will diverge from x_*. For the logistic map we have $f'_\mu(x) = \mu - 2\mu x$, and

$$f'_\mu(0) = \mu$$
$$f'_\mu\big(1 - \mu^{-1}\big) = \mu - 2\mu(1 - \mu^{-1}) = 2 - \mu$$

Thus $x_* = 0$ is stable when $\mu \in [0, 1)$, and $x_* = 1 - \mu^{-1}$ is stable when $\mu \in (1, 3)$.

4

With $0 < \mu < 1$ we expect to see convergence to $x_* = 0$, since this fixed point will be a stable attractor. This is illustrated in figure 1 and 8. When $1 < \mu < 3$ we instead get convergence to $x_* = 1 - \mu^{-1}$, which is illustrated in figure 2, 3 and 8. In figure 4 we see how $2 < \mu < 3$ causes oscillating convergence. With $\mu > 3$ we get orbits of increasing period, and eventually chaotic behaviour. This is shown in figure 5, 6 and 8.

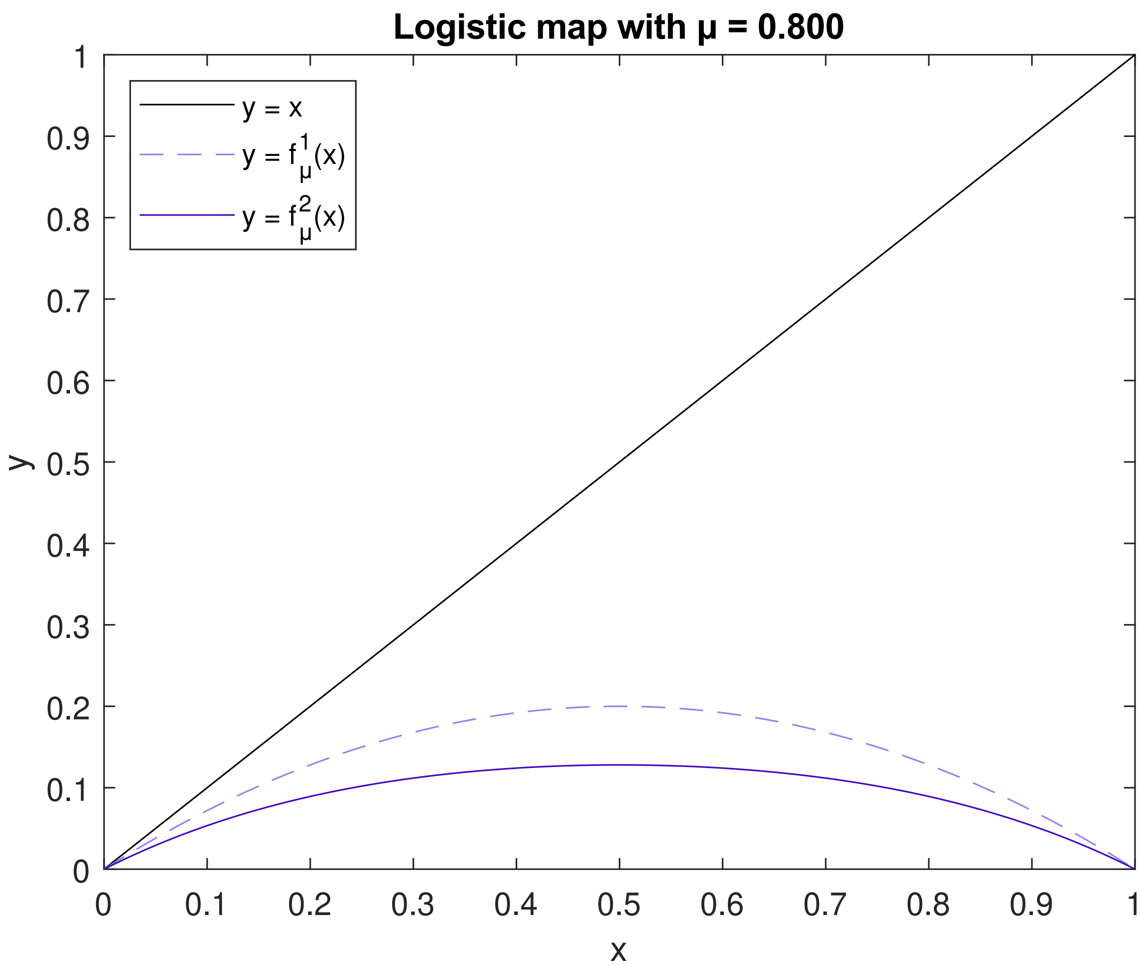

Figure 1: $f_\mu(x)$ and $f_\mu^2(x)$, $\mu = 0.8$

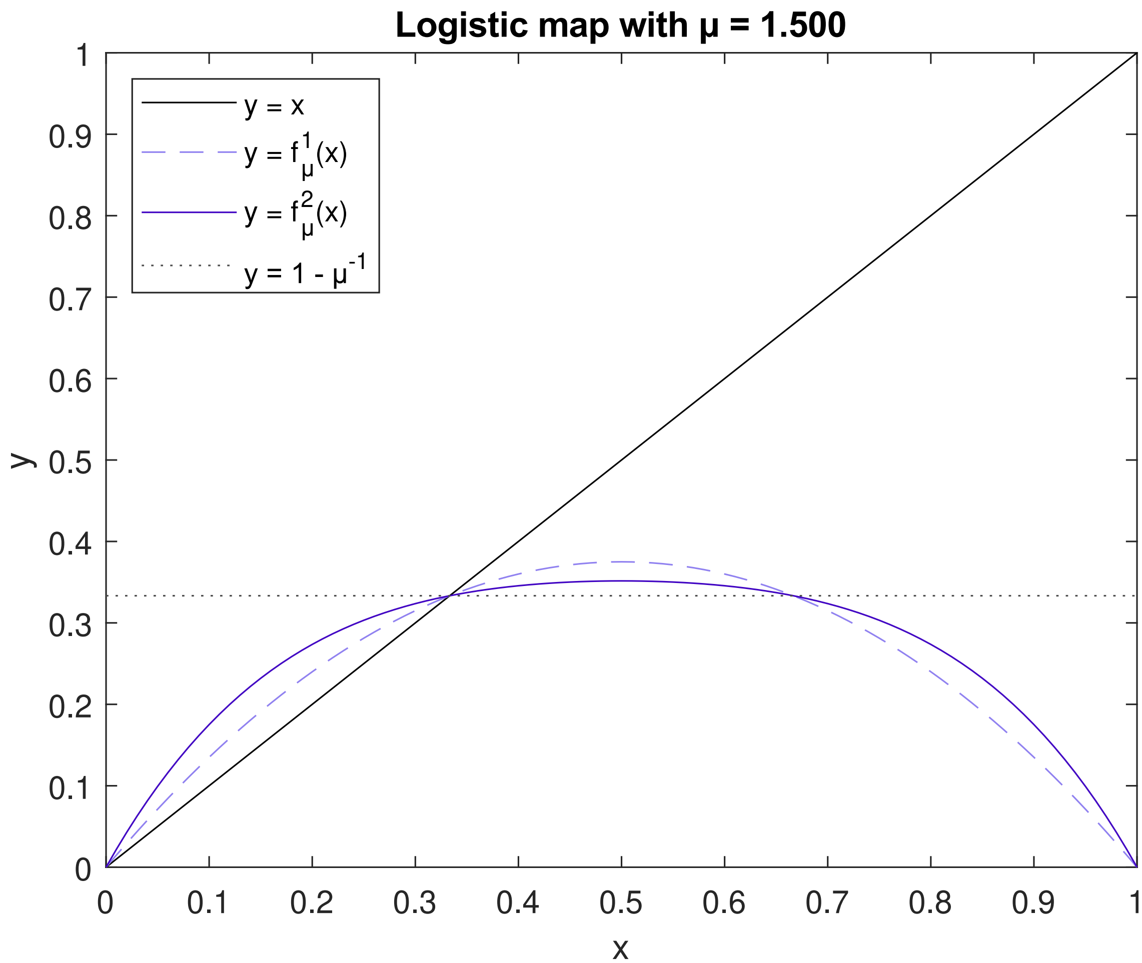

Figure 2: $f_\mu(x)$ and $f_\mu^2(x)$, $\mu = 1.5$

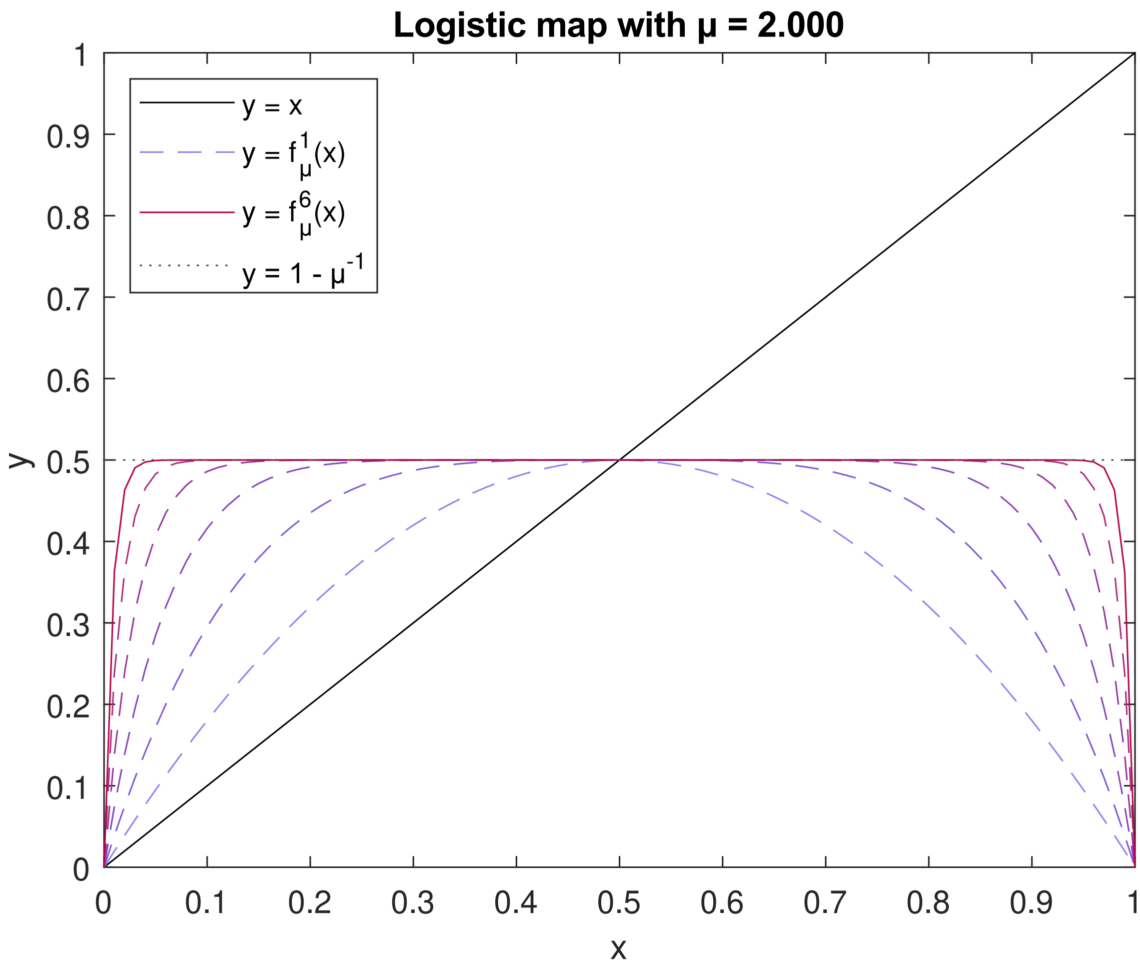

Figure 3: Convergence on stable fixed point, $f_\mu(x), \ldots, f_\mu^6(x)$, $\mu = 2$

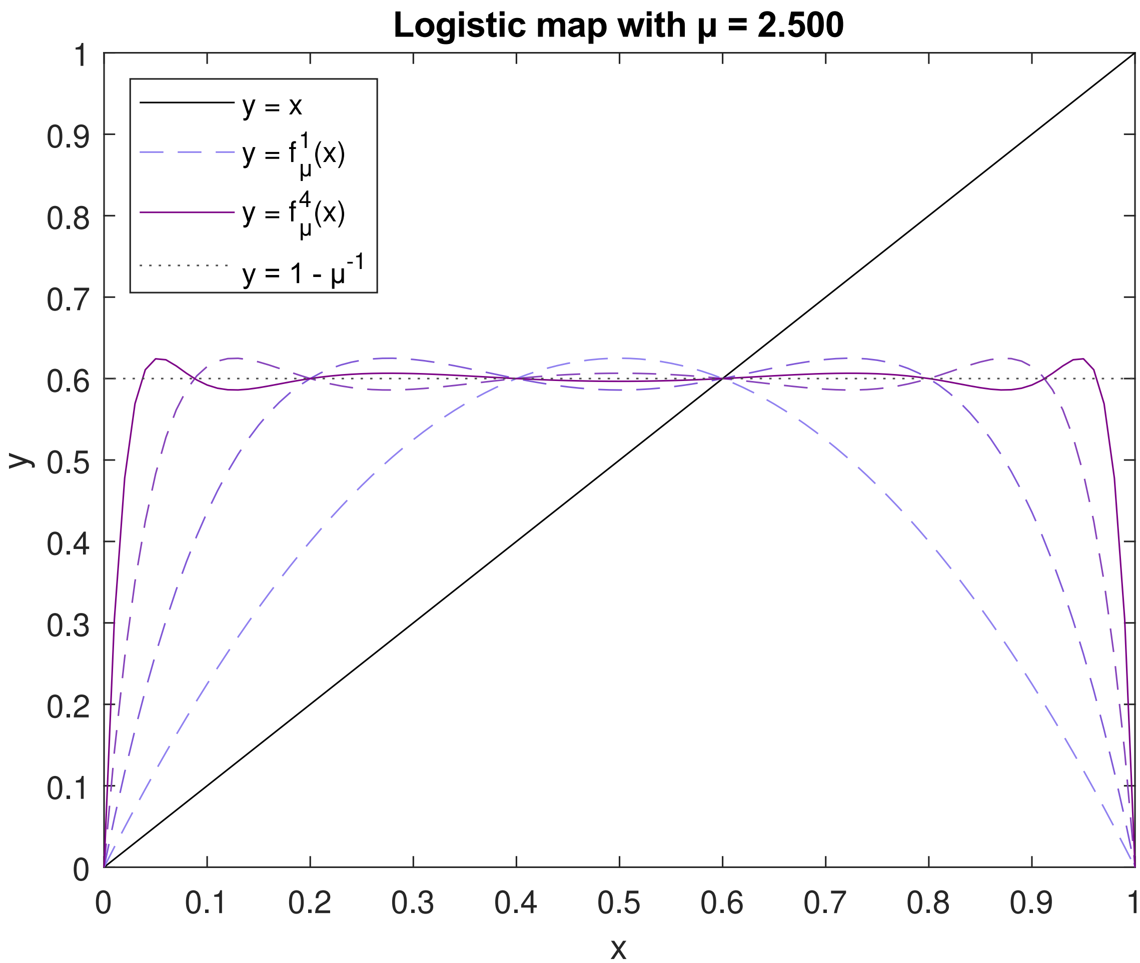

Figure 4: Oscillating convergence, $f_\mu(x), \ldots, f_\mu^4(x)$, $\mu = 2.5$

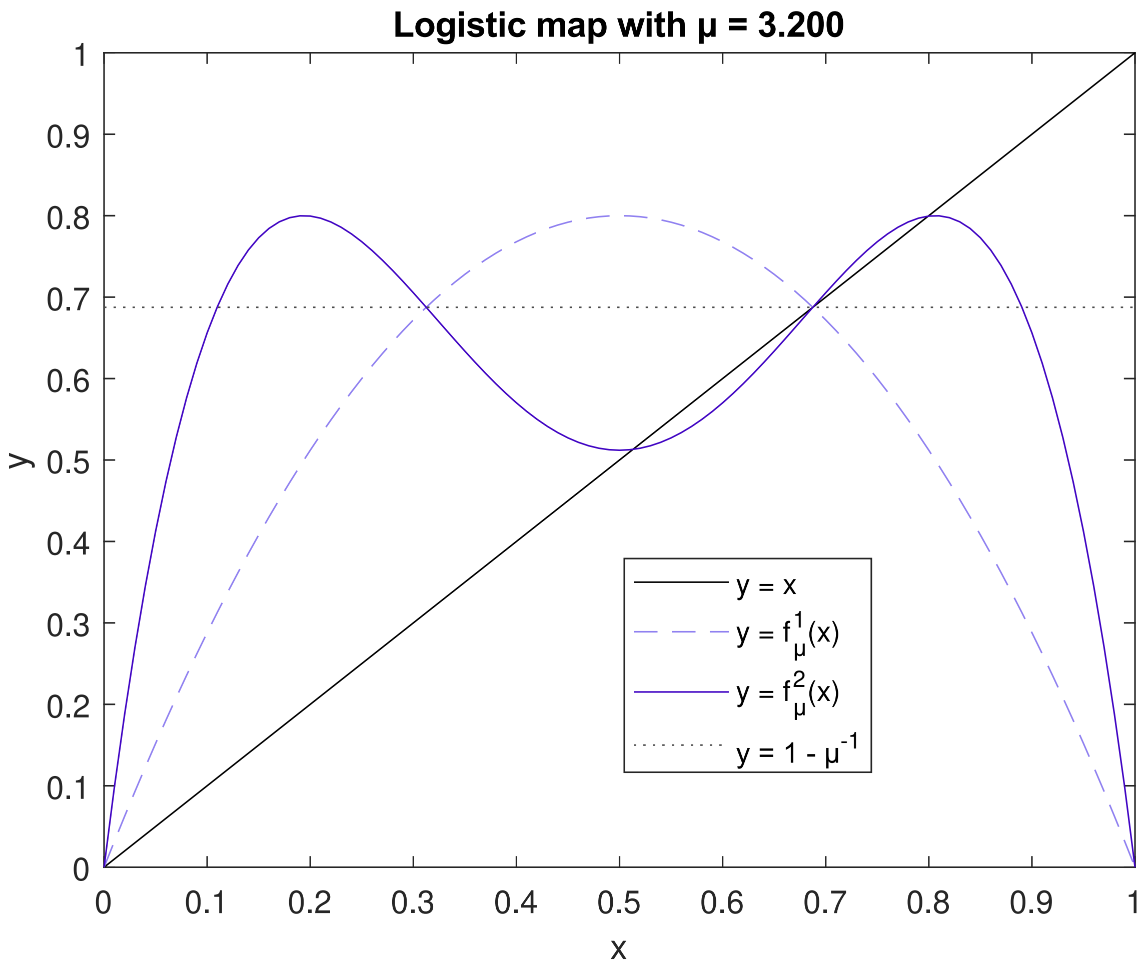

Figure 5: $f_\mu(x)$ and $f_\mu^2(x)$, $\mu = 3.2$

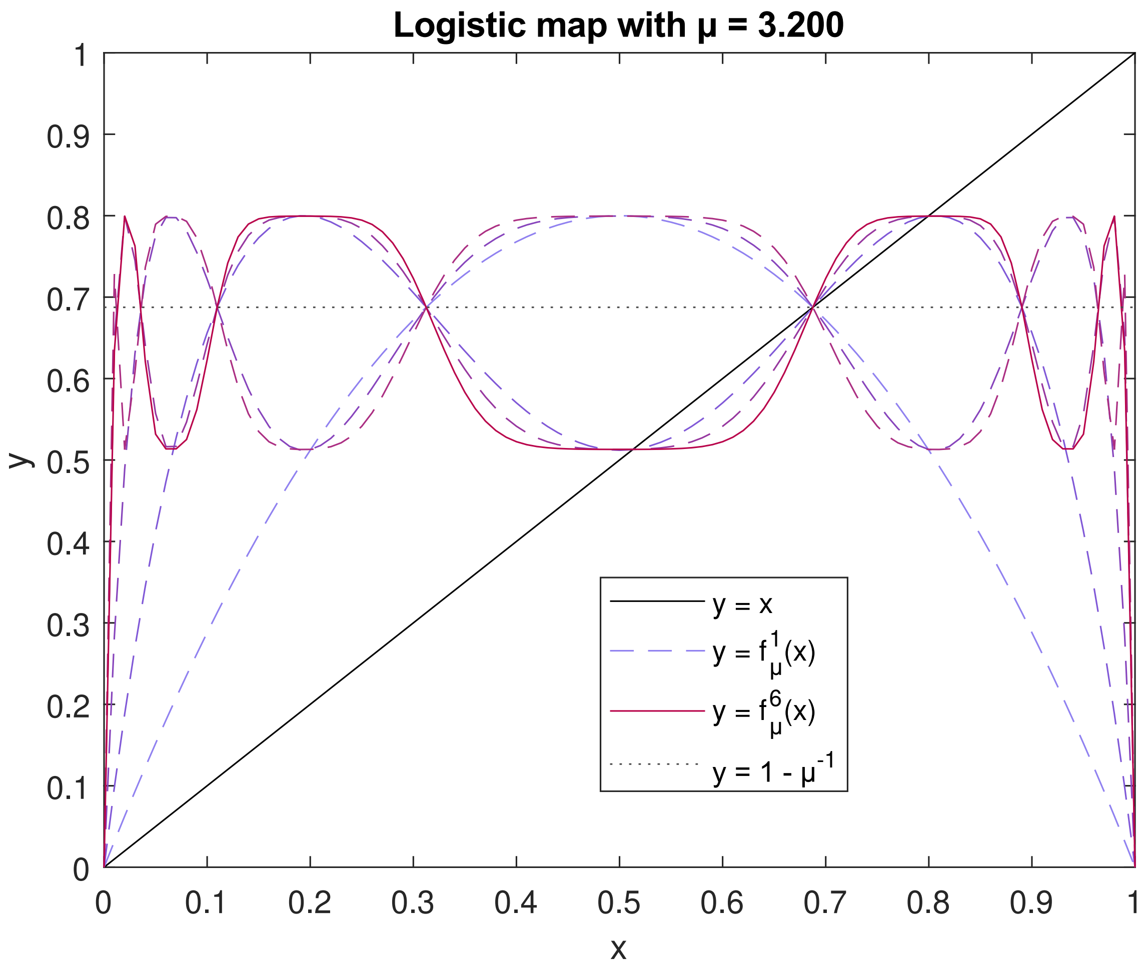

Figure 6: Periodic orbits, $f_\mu(x), \ldots, f_\mu^6(x)$, $\mu = 3.2$

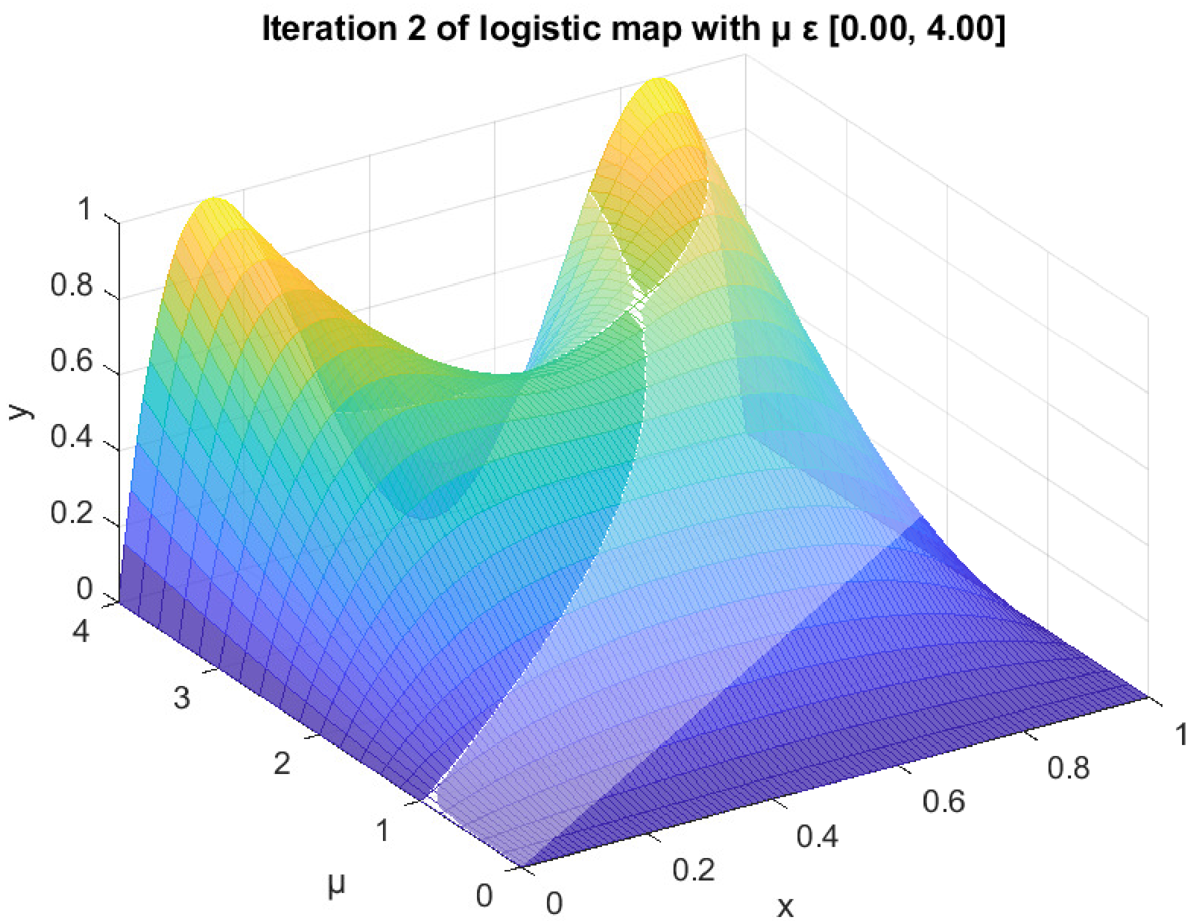

Figure 7: Mesh plot of $f_{\mu}^{2}(x, \mu)$

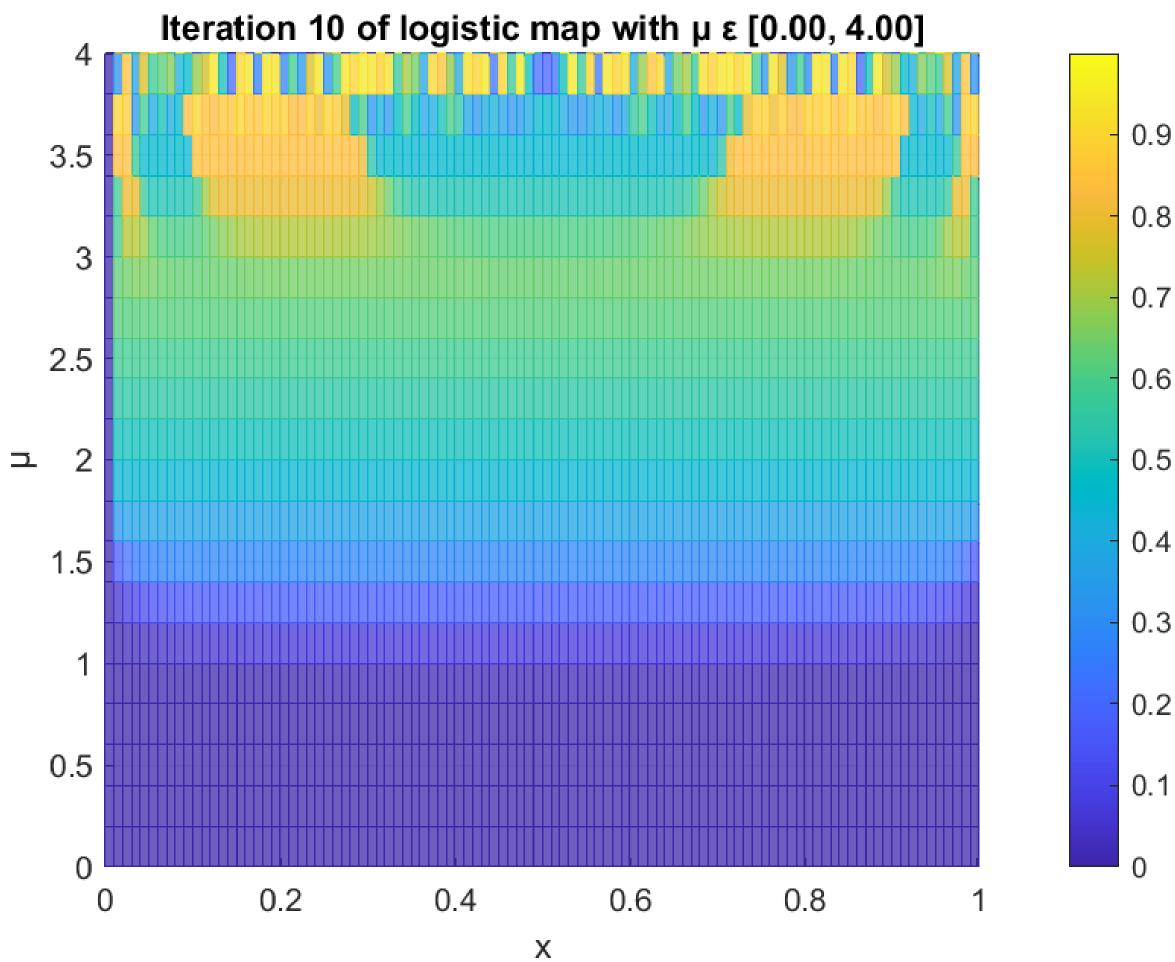

Figure 8: Color plot of $f_\mu^{10}(x, \mu)$ showing transition to instability and eventually chaos

5

Let $F(x) = f^n(x) - x = (f \circ f \circ \ldots \circ f)(x) - x$. Then $F'(x) = f'(f^{n-1}(x)) f^{n-1\,\prime}(x) - 1$ by chain rule. So we have a recurrence relationship. Thus

$$F'(x) = -1 + \prod_{i=0}^{n-1} f'(f^i(x)) \; ,$$

where $f^0(x) = x$.

6 Ex. 6, 7 and 9

Figure 9 shows a bifurcation diagram for $\mu \in [2, 3.8]$ with orbits of period up to 128. We see stability of $x_* = 1 - \mu^{-1}$ up to $\mu = 3$, and then orbits of period 2, 4, 8... and then transition to chaotic behaviour at $\mu \approx 3.57$.

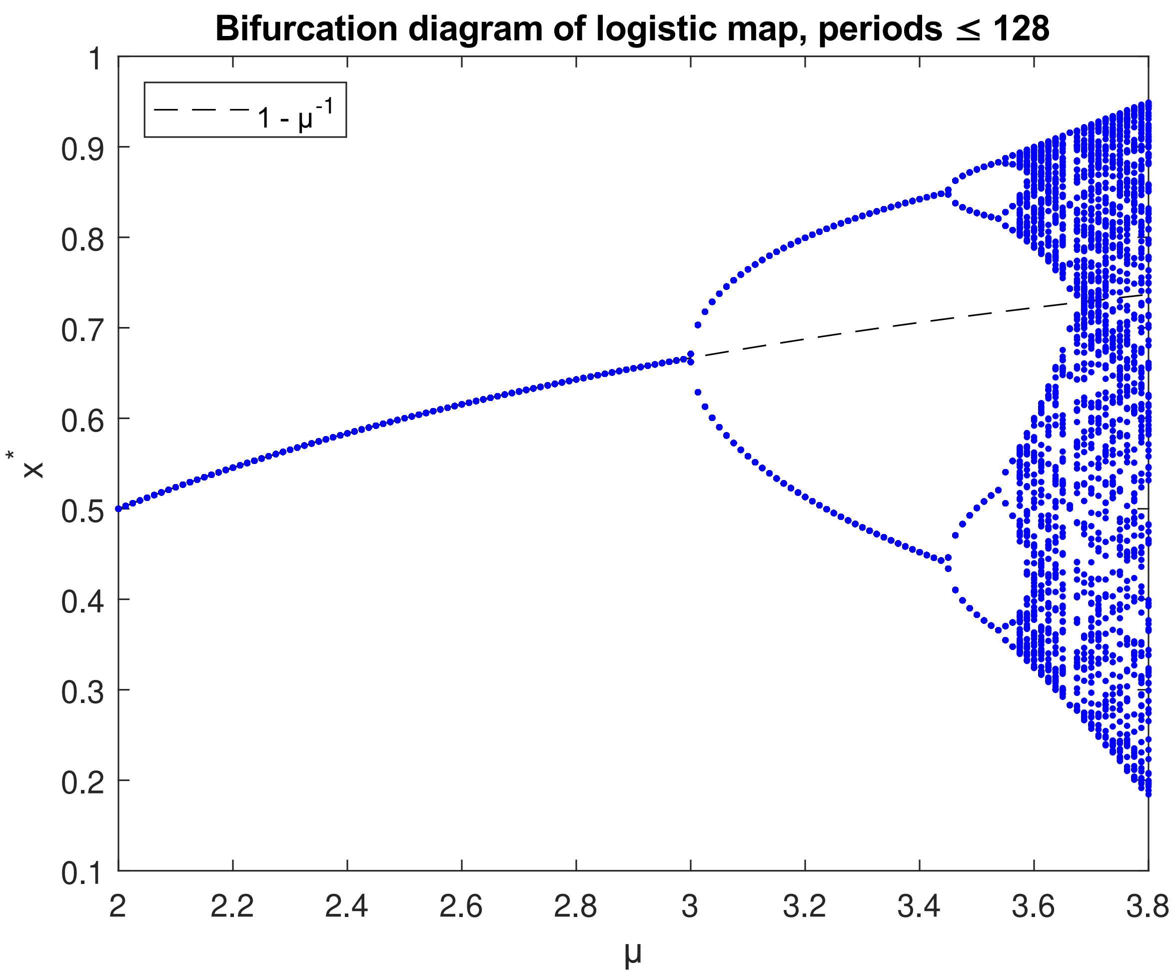

Figure 9: Bifurcation diagram