

# Lecture summary - Waves and Instabilities

## Shallow water equations

Knudsen number:  $K_n = \frac{\text{mean free path}}{\text{smallest length scale}}$

Shallow water equations: Assumption  $H \ll L$  and homogeneous ( $\rho = \text{const}$ )

- 1) Use scaling to obtain hydrostatic equilibrium and incompressibility
- 2) Obtain  $\nabla_H p = \rho g \nabla_H \eta$
- 3) Use continuity and BC to obtain  $\frac{Dh}{Dt} + h \nabla_H \cdot \vec{u} = 0$

Two layer model: Assumption same as before, note  $\nabla_H p_0 = 0$ .

- 1) Write NS equations with new continuity.
- 2) First layer same as before
- 3) The second layer uses continuous pressure.  $p_2(z) = p_0 + \rho_1 g(\eta_0 - \eta_1) + \rho_2 g(\eta_1 - z)$
- 4) Obtain reduced gravity  $g' = \frac{\rho_2 - \rho_1}{\rho_1}$

Reduced gravity model: (aka 1.5) Assume  $H_2 \gg H_1$ .

- 1) Notice that shallow part is much faster,  $\vec{u}_2 \approx 0$
- 2) Obtain relation  $\nabla_H \eta_0 \approx -\frac{g'}{g} \nabla_H \eta_1$
- 3) Set  $h_1 = \eta_1 - \eta_0$  which gives  $g'$  and  $h_1$  for upper layer.

Potential vorticity: Material invariance, assume shallow. Use  $\zeta = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$ .

- 1) Start by  $\frac{\partial (eq2)}{\partial x} - \frac{\partial (eq1)}{\partial y}$
- 2) Solve for  $\zeta$  and  $\frac{D}{Dt} \zeta$
- 3) Use continuity

Note: All functions of  $Q$  are material invariants.

## Wave theory

Linear wave solution: Assume SWE,  $f = f_0$  and  $\vec{u} = \epsilon \tilde{\vec{u}}$ ,  $\vec{v} = \epsilon \tilde{\vec{v}}$ ,  $\eta = H + \epsilon \tilde{\eta}$ .

- 1) Neglect all  $\epsilon^2$  terms.
- 2) Apply Fourier ansatz
- 3) Solve for  $\omega(f_0, k, l)$
- 4) Two solutions:
  - i.  $\omega = 0 \rightarrow$  Geostrophic
  - ii.  $\omega \neq 0 \rightarrow$  Pointcarré waves: Generally dispersive, but not in
- 5) Use  $c = \sqrt{gH}$  and Rossby radius of deformation  $L_D = \frac{c}{f_0} = \frac{\sqrt{gH}}{f_0}$ .
  - i. If SW:  $\omega \gg f_0 \rightarrow \omega^2 = c^2(k^2 + l^2)$
  - ii. if LW:  $\omega \approx f_0$  "Internal oscillations"

Geostrophic adjustment: Assume linearized waves

- 1) Use equations to derive expression for  $\zeta(Q)$
- 2) Begin by doing  $\frac{\partial (eq1)}{\partial x} + \frac{\partial (eq2)}{\partial y} = -g \nabla H^2 \eta$ 
  - i. Insert  $\zeta$  when possible
  - ii. Use continuity as  $\frac{\partial^2 \eta}{\partial t^2} = -H \frac{\partial \eta}{\partial t} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right]$
- 3) Split solution into homogeneous and particular as  $\eta = \eta_w(t) + \eta_p$
- 4) Use the starting scenario to determine  $Q$ , use relation from before.
- 5) Use particular solution to derive  $\eta_p$  and determine geostrophic wind.

Quasi-geostrophic: Linearize PV-equation and get  $\frac{\partial}{\partial t} \left( \nabla^2 \psi - \frac{\psi}{L_D^2} \right) + J(\psi, \nabla^2 \psi) + \beta \frac{\partial \psi}{\partial x} = 0$

- 1) Assumptions:
  - $\text{Ro} \ll 1$
  - $\beta y / f \ll 1$
  - $\eta / H \ll 1$
- 2) Go from PV-equation and extract  $f_0 / H$
- 3) Taylor expand and neglect order  $\epsilon^2$
- 4) Introduce stream-function

Rossby waves with no flow: Start from quasi-geostrophic.

- 1) Reference state  $\psi = \epsilon \psi'$  and  $-\partial_x \psi = u$  and neglect  $\epsilon^2$
- 2) Fourier ansatz
- 3) Obtain dispersion relation  $\omega = \frac{-\beta k}{k^2 + l^2 + 1/L_D^2}$ 
  - SW: stationary
  - LW: Non-dispersive, westward

Rossby waves with background flow: Assume  $\vec{u} = (u_0, 0)$

- 1) Reference state  $\psi = \epsilon \psi'$ ,  $u = u_0 + \epsilon u'$  and  $v = \epsilon v'$  and neglect  $\epsilon^2$
- 2) Fourier ansatz
- 3) Obtain dispersion relation  $\omega = u_{0k} - k \frac{\beta + u_0 / L_D}{k^2 + l^2 + 1 / L_D^2}$ 
  - SW: background flow
  - LW: Non-dispersive, westward

Kelvin waves: Wave propagation along a fix surface (like coast)

- 1) Assume  $\text{Ro} \ll 1$  and  $\vec{u} = (u, 0)$
- 2) Take x-derivative of eq 1 and t-derivative of eq 3  $\rightarrow$  same form
- 3) Fourier ansatz:  $\eta = \hat{\eta}(y) e^{ikx - \omega t}$  and solve for dispersion
- 4) Take time-derivative of eq 2 to obtain amplitude.

## Instabilitets

Rossby waves with changing background flow: Assume  $\vec{u} = [u(y), 0]$

- 1)

2 layer model: Assume rigid lid but surface pressure

- 1) Determine the pressure from continuity and 2 layer SWE
- 2) Introduce stream-function from geostrophic flow
- 3) Expand PV-conservation by approximations
- 4) For solutions define
  - Barotropic mode:  $\bar{\psi} = \frac{F_1 \psi_2 + F_2 \psi_1}{F_1 F_2}$
  - Baroclinic mode:  $\tau = \frac{1}{2} [\psi_2 - \psi_1]$

Criterion for barotropic instability:

- Rayleigh-Kuo: For instability  $\frac{\partial q}{\partial y}$  must change sign in the domain.
- Fjörtoft: For instability  $\frac{u_s - u_0}{d q_0} > 0$

Phillips model: Assume the three standard assumptions and  $H_1 = H_2$  and  $u_1 = -u_2 = u$

- 1) Start from PV-equation for two layers  $\frac{D}{Dt} (q_i) = 0$
- 2) Use reference state  $\bar{u}_i = u_i + \epsilon u' = \frac{d \psi_i}{dy} + \epsilon \frac{d \psi'_i}{dy}$  and  $\bar{v}_i = \epsilon v' = \epsilon \frac{d \psi_i}{dx}$  and  $\bar{\psi}_i = \psi_i(y) + \epsilon \psi'(x, y, t)$
- 3) Use Fourier ansatz  $\psi'_{1,2} = \hat{\psi}_{1,2} e^{ikx + iy - ik\hat{c}t} = \hat{\psi}_{1,2} e^{ikx + iy + \sigma t}$
- 4) Use math to solve for  $\hat{c}$
- 5) Obtain three cases:
  - i.  $U=0$  and  $\beta \neq 0 \rightarrow$  Rossby waves (Two roots give baroclinic vs barotropic)
  - ii.  $U \neq 0$  and  $\beta=0 \rightarrow$  High value cut-off Unstable if  $\frac{\lambda}{2\pi} < \frac{L_D}{\sqrt{8}}$ . No low value cut-off
  - iii.  $U \neq 0$  and  $\beta \neq 0 \rightarrow$  Cut-off  $\frac{\beta}{\sqrt{2} U_0} < k < k_d$

Baroclinic instability:

- Thermal wind  $\frac{\partial \vec{u}}{\partial z} = \frac{1}{f_0} \hat{z} \times \nabla b$
- Potential energy decreases when lapse rate is not constant
- Slanted convection: From  $\Delta PE = g \Delta \rho \Delta z$  and Taylor  $\rightarrow$  displacement if  $\phi > \alpha > 0$ 
  - Slope of displacement:  $\alpha = \Delta z / \Delta x, y$
  - Slope of isopycnals:  $\alpha = -\frac{\partial \rho}{\partial y} / \frac{\partial \rho}{\partial z}$

## Equatorial dynamics

Equatorial Kelvin waves: Kelvin wave propagation when  $f \approx 0 + \beta y = \beta y$

- 5) Assume  $\text{Ro} \ll 1$  and  $\vec{u} = (u, 0)$
- 6) Take x-derivative of eq 1 and t-derivative of eq 3  $\rightarrow$  same form
- 7) Fourier ansatz:  $\eta = \hat{\eta}(y) e^{ikx - \omega t}$  and solve for dispersion
- 8) Take time-derivative of eq 2 to obtain amplitude.