

Lecture summary - Waves and Instabilities

Shallow water equations

Knudsen number: $K_n = \frac{\text{mean free path}}{\text{smallest length scale}}$

Scaling: Used for hydrostatic approximation

- 1) Use incompressibility to obtain $w \ll u, v$
- 2) Use vertical momentum equation, with horizontal scaling

Shallow water equations: Assumption $H \ll L$ and Boussinesq (\rightarrow homogeneous $\rho = \text{const}$)

- 1) Use scaling and NS to obtain hydrostatic equilibrium and incompressibility
- 2) Obtain $\nabla_H p = \rho g \nabla_H \eta$
- 3) Use continuity and BC to obtain $\frac{Dh}{Dt} + h \nabla_H \cdot \vec{u} = 0$

Two layer model: Assumption same as before, note $\nabla_H p_0 = 0$.

- 1) Write NS equations with new continuity.
- 2) First layer same as before
- 3) The second layer uses continuous pressure. $p_2(z) = p_0 + \rho_1 g(\eta_0 - \eta_1) + \rho_2 g(\eta_1 - z)$
- 4) Obtain reduced gravity $g' = g \frac{\rho_2 - \rho_1}{\rho_1}$

Reduced gravity model: (aka 1.5) Assume $H_2 \gg H_1$.

- 1) Notice that shallow part is much faster, $\vec{u}_2 \approx 0$
- 2) Obtain relation $\nabla_H \eta_0 \approx -\frac{g'}{g} \nabla_H \eta_1$
- 3) Set $h_1 = \eta_0 - \eta_1$ which gives g' and h_1 for upper layer $\rightarrow \text{RHS} \approx -g' \nabla_H h_1$.

Potential vorticity: Material invariance, assume shallow. Use $\zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$.

- 1) Start by taking the curl: $\frac{\partial(\text{eq2})}{\partial x} - \frac{\partial(\text{eq1})}{\partial y}$
- 2) Solve for ζ and f
- 3) Write out $\frac{D}{Dt} \left(\frac{\zeta + f}{H} \right)$ to become easier
- 4) Use continuity

Note: All functions of Q are material invariants.

Wave theory

Linear wave solution: Assume 1.5 SWE, $f = f_0$ and $\vec{u} = \epsilon \tilde{\vec{u}}, \vec{v} = \epsilon \tilde{\vec{v}}, \eta = H + \epsilon \tilde{\eta}$.

- 1) Neglect all ϵ^2 terms.
- 2) Apply Fourier ansatz
- 3) Write (u, v, h) and write SWE on form $A \cdot \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- 4) Unique solution for $\|A\|=0 \rightarrow$ solve for $\omega(f_0, k, l)$
- 5) Two solutions:
 - i. $\omega = 0 \rightarrow$ Geostrophic
 - ii. $\omega \neq 0 \rightarrow$ Pointcarré waves: Generally dispersive, but not in SW limit
- 6) Use $c = \sqrt{gH}$ and Rossby radius of deformation $L_D = \frac{c}{f_0} = \frac{\sqrt{gH}}{f_0}$.
 - i. If SW: $\omega \gg f_0 \rightarrow \omega^2 = c^2(k^2 + l^2)$
 - ii. if LW: $\omega \approx f_0$ "Internal oscillations"

Geostrophic adjustment: Assume linearized waves

- 1) Use equations to derive expression for $\zeta(Q)$
- 2) Begin by doing $\frac{\partial(\text{eq 1})}{\partial x} + \frac{\partial(\text{eq 2})}{\partial y} = -g \nabla H^2 \eta$
 - i. Insert ζ when possible
 - ii. Use continuity as $\frac{\partial^2 \eta}{\partial t^2} = -H \frac{\partial \eta}{\partial t} \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right]$
- 3) Split solution into homogeneous and particular as $\eta = \eta_w(t) + \eta_p$
- 4) Use the starting scenario to determine Q , use relation from before.
- 5) Use particular solution to derive η_p and determine geostrophic wind.

Quasi-geostrophic: Linearize PV-equation and get $\frac{\partial}{\partial t} \left(\nabla^2 \psi - \frac{\psi}{L_D^2} \right) + J(\psi, \nabla^2 \psi) + \beta \frac{\partial \psi}{\partial x} = 0$

- 1) Assumptions:
 - o $\text{Ro} \ll 1$
 - o $\frac{\beta y}{f} \ll 1$
 - o $\frac{\eta}{H} \ll 1$
- 2) Go from PV-equation and extract f_0/H
- 3) Taylor expand and neglect order ϵ^2
- 4) Introduce stream-function

Rossby waves with no flow: Start from quasi-geostrophic.

- 1) Reference state $\psi = \epsilon \psi'$ and $-\partial_x \psi = u$ and neglect ϵ^2
- 2) Fourier ansatz
- 3) Obtain dispersion relation $\omega = -\frac{\beta k}{k^2 + l^2 + 1/L_D^2}$
 - o SW: stationary
 - o LW: Non-dispersive, westward

Kelvin waves: Wave propagation along a fix surface (like coast)

- 1) Assume $\text{Ro} \ll 1$ and $\vec{u} = (u, 0)$
- 2) Linearize equation according to $u = \epsilon u'$ and $h = H + \epsilon \eta$
- 3) Fourier ansatz: $\begin{bmatrix} \eta \\ u \end{bmatrix} = \begin{bmatrix} \hat{\eta}(x) \\ \hat{u}(x) \end{bmatrix} e^{i(kx - \omega t)}$ and solve for dispersion
- 4) Take time-derivative of eq 2 to obtain amplitude.

Instabilities

Barotropic instability: aka Rossby waves with changing background flow $\vec{u} = [u(y), 0]$

- 1) Start from reference state and perturb $u_0 = -\partial_y \psi_0 \rightarrow \psi(x, y, t) = \psi_0(y) + \epsilon \psi'(x, y, t)$
- 2) Linearize QG-equation \rightarrow LQG-equation
- 3) Use Fourier ansatz to get Rayleigh equation
- 4) Integrate and use tricks (like $\vec{v}(y_1) = \vec{v}(y_2) = 0$) to get criterion for stability

Criterion for barotropic instability:

- Rayleigh-Kuo: For instability $\frac{\partial q}{\partial y}$ must change sign in the domain.
- Fjörtoft: For instability $\frac{u_s - u_0}{dq_0/dy} > 0$.

2 layer model: Assume rigid lid but surface pressure

- 1) Determine the pressure from continuity and 2 layer SWE
- 2) Introduce stream-function form geostrophic flow
- 3) Expand PV-conservation by approximations
- 4) For solutions define
 - o Barotropic mode: $\bar{\psi} = \frac{F_1 \psi_2 + F_2 \psi_1}{F_1 F_2}$
 - o Baroclinic mode: $\tau = \frac{1}{2} [\psi_2 - \psi_1]$

Phillips model: Assume the three standard assumptions and $H_1 = H_2$ and $u_1 = -u_2 = u$

- 1) Start from PV-equation for two layers $\frac{D}{Dt} (q_i) = 0$
- 2) Use reference state $\bar{u}_i = u_i + \epsilon u' = \frac{d \psi_i}{dy} + \epsilon \frac{d \psi'_i}{dy}$ and $\bar{v}_i = \epsilon v' = \epsilon \frac{d \psi_i}{dx}$ and $\bar{\psi}_i = \psi_i(y) + \epsilon \psi'(x, y, t)$
- 3) Use Fourier ansatz $\psi'_{1,2} = \hat{\psi}_{1,2} e^{ikx + iky - ik\hat{c}t} = \hat{\psi}_{1,2} e^{ikx + iky + \sigma t}$
- 4) Use math to solve for $\hat{\psi}$
- 5) Obtain three cases:
 - i. $U=0$ and $\beta \neq 0 \rightarrow$ Rossby waves (Two roots give baroclinic vs barotropic)
 - ii. $U \neq 0$ and $\beta=0 \rightarrow$ High value cut-off Unstable if $\frac{\lambda}{2\pi} < \frac{L_D}{\sqrt{8}}$. No low value cut-off
 - iii. $U \neq 0$ and $\beta \neq 0 \rightarrow$ Cut-off $\frac{\beta}{\sqrt{2} U_0} < k < k_d$

Baroclinic instability:

- Thermal wind $\frac{\partial \vec{u}}{\partial z} = \frac{1}{f_0} \hat{z} \times \nabla b$
- Potential energy decreases when lapse rate is not constant
- Slanted convection: From $\Delta PE = g \Delta \rho \Delta z$ and Taylor \rightarrow displacement if $\phi > \alpha > 0$
 - Slope of displacement: $\alpha = \Delta z / \Delta x, y$
 - Slope of isopycnals: $\alpha = -\frac{\partial \rho}{\partial y} / \frac{\partial \rho}{\partial z}$

Equatorial dynamics

Equatorial normal conditions: Assume steady, linear, frictionless and zonal wind.

- 1) Start from stationary LSW $-fv = -g' \frac{\partial h}{\partial x} + \frac{\tau^x}{L_D}$
- 2) Take curl to see $u = v = 0$
- 3) Obtain thermocline $\partial_x h < 0$

Equatorial Kelvin waves: Kelvin wave propagation when $f \approx 0 + \beta y = \beta y$

- 1) Assume $\text{Ro} \ll 1$ and $\vec{u} = (u, 0)$
- 2) Take x-derivative of eq 1 and t-derivative of eq 3 \rightarrow same form
- 3) Fourier ansatz: $\eta = \hat{\eta}(y) e^{i(kx - \omega t)}$ and solve for dispersion
- 4) Take time-derivative of eq 2 to obtain amplitude.

Equatorial Rossby waves: Rossby wave propagation when $f \approx \beta y$

- 1) Apply LQG-equation and express L_D with βy
- 2) Introduce Fourier ansatz with $\hat{\psi}(y)$
- 3) Rewrite on Hermit form by using variable substitution
- 4) Obtain dispersion relation and see LW limit is the same as before.

Important numbers

Radius of Earth: 6371 km

Rossby radii: $L_D^{\text{atmosphere}} = 1000 \text{ km}$ and $L_D^{\text{ocean}} = 50 \text{ km}$

Reduced gravity: $g'_{\text{atmosphere}} = 0.3$ and $g'_{\text{ocean}} = 0.01$

Flows: $\langle u_{\text{ocean}} \rangle = 0.1 \text{ m s}^{-1}$