

# Lecture summery - Waves and Instabilities

## Shallow water equations

Knudsen number:  $K_n = \frac{\text{mean free path}}{\text{smallest length scale}}$

Shallow water equations: Assumption  $H \ll L$  and Boussinesq ( $\rightarrow$  homogeneous  $\rho = \text{const}$ )

- 1) Use scaling and NS to obtain hydrostatic equilibrium and incompressibility
- 2) Obtain  $\nabla_H p = \rho g \nabla_H \eta$
- 3) Use continuity and BC to obtain  $\frac{Dh}{Dt} + h \nabla_H \cdot \vec{u} = 0$

Two layer model: Assumption same as before, note  $\nabla_H p_0 = 0$ .

- 1) Write NS equations with new continuity.
- 2) First layer same as before
- 3) The second layer uses continuous pressure.  $p_2(z) = p_0 + \rho_1 g (\eta_0 - \eta_1) + \rho_2 g (\eta_1 - z)$
- 4) Obtain reduced gravity  $g' = g \frac{\rho_2 - \rho_1}{\rho_1}$

Reduced gravity model: (aka 1.5) Assume  $H_2 \gg H_1$ .

- 1) Notice that shallow part is much faster,  $\vec{u}_2 \approx 0$
- 2) Obtain relation  $\nabla_H \eta_0 \approx -\frac{g'}{g} \nabla_H \eta_1$
- 3) Set  $h_1 = \eta_0 - \eta_1$  which gives  $g'$  and  $h_1$  for upper layer  $\rightarrow \text{RHS} \approx -g' \nabla_H h_1$ .

Potential vorticity: Material invariance, assume shallow. Use  $\zeta = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$ .

- 1) Start by taking the curl:  $\frac{\partial(\text{eq2})}{\partial x} - \frac{\partial(\text{eq1})}{\partial y}$
- 2) Solve for  $\zeta$  and  $f$
- 3) Write out  $\frac{D}{Dt} \left( \frac{\zeta + f}{H} \right)$  to become easier
- 4) Use continuity

Note: All functions of  $Q$  are material invariants.

## Wave theory

Linear wave solution: Assume 1.5 SWE,  $f = f_0$  and  $\vec{u} = \varepsilon \vec{u}$ ,  $\vec{v} = \varepsilon \vec{v}$ ,  $\eta = H + \varepsilon \tilde{\eta}$ .

- 1) Neglect all  $\varepsilon^2$  terms.
- 2) Apply Fourier ansatz
- 3) Write  $(u, v, h)$  and write SWE on form  $A \cdot \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- 4) Unique solution for  $\|A\| = 0 \rightarrow$  solve for  $\omega(f_0, k, l)$
- 5) Two solutions:
  - i.  $\omega = 0 \rightarrow$  Geostrophic
  - ii.  $\omega \neq 0 \rightarrow$  **Poincaré waves: Generally dispersive, but not in SW limit**
- 6) Use  $c = \sqrt{gH}$  and Rossby radius of deformation  $L_D = \frac{c}{f_0} = \frac{\sqrt{gH}}{f_0}$ .
  - i. If SW:  $\omega \gg f_0 \rightarrow \omega^2 = c^2(k^2 + l^2)$
  - ii. if LW:  $\omega \approx f_0$  "Internal oscillations"

Geostrophic adjustment: Assume linearized waves

- 1) Use equations to derive expression for  $\zeta(Q)$
- 2) Begin by doing  $\frac{\partial(\text{eq 1})}{\partial x} + \frac{\partial(\text{eq 2})}{\partial y} = -g \nabla H^2 \eta$ 
  - i. Insert  $\zeta$  when possible
  - ii. Use continuity as  $\frac{\partial^2 \eta}{\partial t^2} = -H \frac{\partial \eta}{\partial t} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right]$
- 3) Split solution into homogeneous and particular as  $\eta = \eta_w(t) + \eta_p$
- 4) Use the starting scenario to determine  $Q$ , use relation from before.
- 5) Use particular solution to derive  $\eta_p$  and determine geostrophic wind.

Quasi-geostrophic: Linearize PV-equation and get  $\frac{\partial}{\partial t} \left( \nabla^2 \psi - \frac{\psi}{L_D^2} \right) + J(\psi, \nabla^2 \psi) + \beta \frac{\partial \psi}{\partial x} = 0$

- 1) Assumptions:
  - **$Ro \ll 1$**
  - **$\frac{\beta y}{f} \ll 1$**
  - **$\frac{\eta}{H} \ll 1$**
- 2) Go from PV-equation and extract  $f_0/H$
- 3) Taylor expand and neglect order  $\varepsilon^2$
- 4) Introduce stream-function

Rossby waves with no flow: Start from quasi-geostrophic.

- 1) Reference state  $\psi = \varepsilon \psi'$  and  $-\partial_x \psi = u$  and neglect  $\varepsilon^2$
- 2) Fourier ansatz
- 3) Obtain dispersion relation  $\omega = -\frac{\beta k}{k^2 + l^2 + 1/L_D^2}$ 
  - SW: stationary
  - LW: Non-dispersive, westward

Rossby waves with background flow: Assume  $\vec{u} = (u_0, 0)$

- 1) Reference state  $\psi = \varepsilon \psi'$ ,  $u = u_0 + \varepsilon u'$  and  $v = \varepsilon v'$  and neglect  $\varepsilon^2$
- 2) Fourier ansatz
- 3) Obtain dispersion relation  $\omega = u_0 k - k \frac{\beta + u_0/L_D^2}{k^2 + l^2 + 1/L_D^2}$ 
  - SW: background flow
  - LW: Non-dispersive, westward

Kelvin waves: Wave propagation along a fix surface (like coast)

- 1) Assume  $Ro \ll 1$  and  $\vec{u} = (u, 0)$
- 2) Linerize equation according to  $u = \varepsilon u$  and  $h = H + \varepsilon \eta$
- 3) Fourier ansatz:  $\begin{bmatrix} \eta \\ u \end{bmatrix} = \begin{bmatrix} \hat{\eta}(x) \\ \hat{u}(x) \end{bmatrix} e^{i(kx - \omega t)}$  and solve for dispersion
- 4) Take time-derivative of eq 2 to obtain amplitude.

## Instabilitets

Barotropic instability: aka Rossby waves with changing background flow  $\vec{u} = [u(y), 0]$

- 1) Start from reference state and perturb  $u_0 = -\partial_y \psi_0 \rightarrow \psi(x, y, t) = \psi_0(y) + \varepsilon \psi'(x, y, t)$
- 2) Linearize QG-equation  $\rightarrow$  LQG-equation
- 3) Use Fourier ansatz to get Rayleigh equation
- 4) Integrate and use tricks (like  $\vec{v}(y_1) = \vec{v}(y_2) = 0$ ) to get criterion for stability

Criterion for barotropic instability:

- **Rayleigh-Kuo: For instability  $\frac{\partial q}{\partial y}$  must change sign in the domain.**
- **Fjörtoft: For instability  $\frac{u_s - u_0}{\frac{dq_0}{dy}} > 0$ .**

2 layer model: Assume rigid lid but surface pressure

- 1) Determine the pressure from continuity and 2 layer SWE
- 2) Introduce stream-function form geostrophic flow
- 3) Expand PV-conservation by approximations
- 4) For solutions define
  - Barotropic mode:  $\bar{\psi} = \frac{F_1 \psi_2 + F_2 \psi_1}{F_1 F_2}$
  - Baroclinic mode:  $\tau = \frac{1}{2} [\psi_2 - \psi_1]$

Phillips model: Assume the three standard assumptions and  $H_1 = H_2$  and  $u_1 = -u_2 = u$

- 1) Start from PV-equation for two layers  $\frac{D}{Dt}(q_i) = 0$
- 2) Use reference state  $\vec{u}_i = u_1 + \varepsilon u' = \frac{d\psi_i}{dy} + \varepsilon \frac{d\psi_i'}{dy}$  and  $\vec{v}_i = \varepsilon v' = \varepsilon \frac{d\psi_i}{dx}$  and  $\bar{\psi}_i = \psi(y) + \varepsilon \psi'(x, y, t)$
- 3) Use Fourier ansatz  $\psi'_{1,2} = \hat{\psi}_{1,2} e^{ikx + ily - ik\hat{c}t} = \hat{\psi}_{1,2} e^{ikx + ily + \sigma t}$
- 4) Use math to solve for  $\hat{c}$
- 5) Obtain three cases:
  - i.  $U = 0$  and  $\beta \neq 0 \rightarrow$  Rossby waves (Two roots give baroclinic vs barotropic)
  - ii.  $U \neq 0$  and  $\beta = 0 \rightarrow$  High value *cut-off* Unstable if  $\frac{\lambda}{2\pi} < \frac{L_D}{\sqrt{8}}$ . No low value *cut-off*
  - iii.  $U \neq 0$  and  $\beta \neq 0 \rightarrow$  *Cut-off*  $-\frac{\beta}{\sqrt{2}U_0} < k < k_d$

Baroclinic instability:

- Thermal wind  $\frac{\partial \vec{u}}{\partial z} = \frac{1}{f_0} \hat{z} \times \nabla b$
- Potential energy decreases when lapse rate is not constant
- Slanted convection: From  $\Delta PE = g \Delta \rho \Delta z$  and Taylor  $\rightarrow$  displacement if  $\phi > \alpha > 0$ 
  - Slope of displacement:  $\alpha = \Delta z / \Delta x, y$
  - Slope of isopycnals:  $\alpha = -\frac{\partial \rho}{\partial y} / \frac{\partial \rho}{\partial z}$

## Equatorial dynamics

Equatorial normal conditions: Assume steady, linear, frictionless and zonal wind.

- 1) Start from SWE  $-fv = -g' \frac{\partial h}{\partial x} + \frac{\tau^x}{\rho_1 D}$
- 2) Take curl and combine
- 3) Obtain thermocline  $\partial_x h < 0$

Equatorial Kelvin waves: Kelvin wave propagation when  $f \approx 0 + \beta y = \beta y$

- 1) Assume  $Ro \ll 1$  and  $\vec{u} = (u, 0)$
- 2) Take x-derivative of eq 1 and t-derivative of eq 3  $\rightarrow$  same form
- 3) Fourier ansatz:  $\eta = \hat{\eta}(y) e^{i(kx - \omega t)}$  and solve for dispersion
- 4) Take time-derivative of eq 2 to obtain amplitude.

Equatorial Rossby waves: Rossby wave propagation when  $f \approx \beta y$

- 1) Apply LQG-equation and express  $L_D$  with  $\beta y$
- 2) Introduce Fourier ansatz with  $\hat{\psi}(y)$
- 3) Rewrite on Hermit form by using variable substitution
- 4) Obtain dispersion relation and see LW limit is the same as before.

## Important numbers

Radius of Earth: 6371 km

Rossby radii:  $L_D^{\text{atmosphere}} = 1000 \text{ km}$  and  $L_D^{\text{ocean}} = 50 \text{ km}$

Reduced gravity:  $g'_{\text{atmosphere}} = 0.3$  and  $g'_{\text{ocean}} = 0.01$