Geophysical Fluid Dynamics
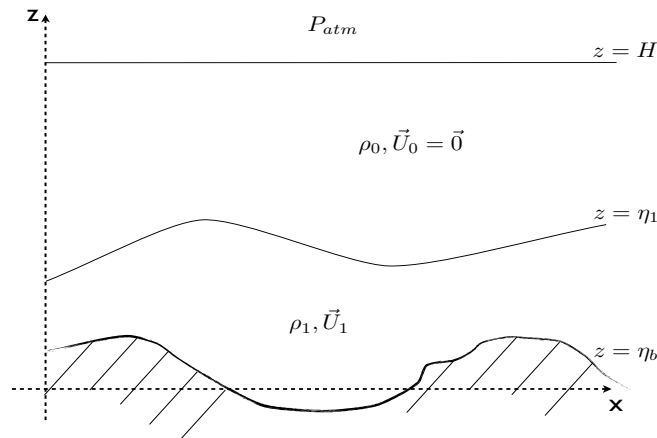
Department of Meteorology - Stockholm University

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Exercise 1 :

Dense bottom currents in the ocean can result from cold water overflowing sills. They are often modelled as a single moving layer of heavy fluid on the bottom, underlying a layer of light fluid at rest. Derive the appropriate shallow water equations for this model, assuming the density to be ρ_1 in the dense layer and ρ_0 in the upper layer, where $\rho_0 < \rho_1$. Show that as $\rho_0/\rho_1 \to 0$ the usual one-layer shallow water equations are obtained.
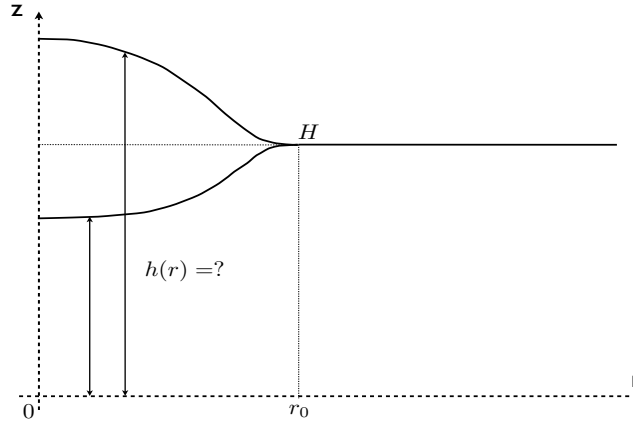
Exercise 2 : Consider a 3D incompressible and inviscid fluid flow in a rotating system that can be described by the one-layer shallow water equations. The bottom topography is variable. Show that the vertical velocity within the layer is given by

$$w = \frac{z - \eta_b}{h} \frac{Dh}{Dt} + \frac{D\eta_b}{Dt}, \qquad \text{(eq. 2)}$$

where w is the vertical velocity, η_b the height of the bottom topography relative to a reference level and h the thickness of the fluid layer. Interpret this result, showing that it gives sensible answers at the top and the bottom of the fluid layer.

Hint: The vertical velocity can be found by vertically integrating the 3D mass continuity equation from η_b to z. Note that $w(z) \equiv \frac{Dz}{Dt}$.

A vortex flow is described by the expression

$$\mathbf{u} = U_\phi(r)\mathbf{e}_\phi, \qquad \text{(eq. 3-1)}$$

where $\mathbf{u}$ is the velocity vector in cylindrical coordinates, and

$$U_\phi(r) = \begin{cases} Ar & , & r < r_0 \\ 0 & , & r > r_0 \end{cases} \qquad \text{(eq. 3-2)}$$
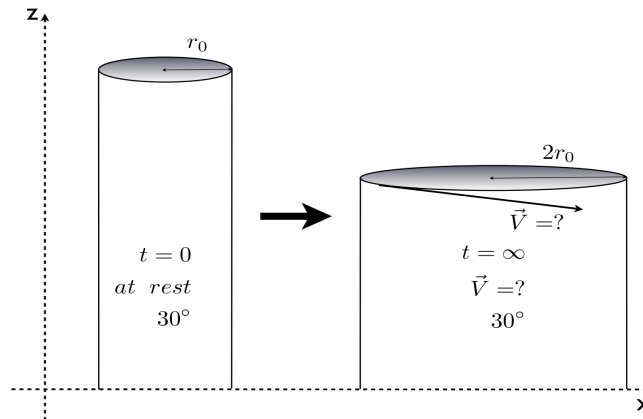
Here $A > 0$ gives a cyclone and $A < 0$ an anticyclone. Determine the corresponding $h(r)$ from the shallow water equations. Also determine the maximum flow for a low-pressure vortex and a high-pressure vortex. *Assumptions*: steady state, $f = $ const and $h = H = $ const outside the vortex.

Hint: Write the momentum equation with $\mathbf{u}$ according to eq. eq. 3-1. Note that $\frac{\partial}{\partial \phi}\mathbf{e}_\phi = -\mathbf{e}_r$. The maximum flow is found when considering $h(r = 0) = 0$ and $h(r = 0)$ as large as possible.
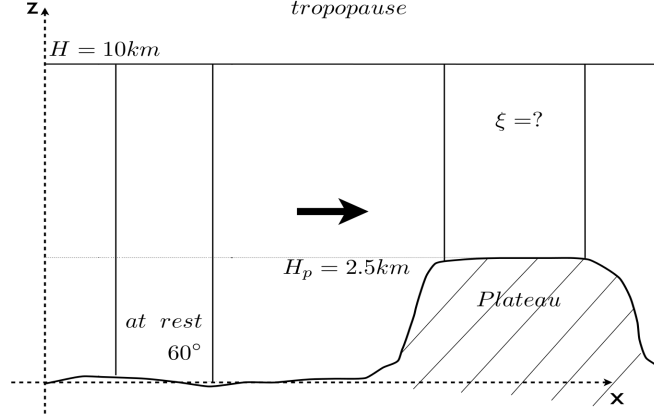
In an adiabatic shallow water fluid in a rotating reference frame the conservation law for potential vorticity is

$$\frac{D}{Dt}\left(\frac{\zeta + f}{\eta - \eta_b}\right) = 0, \qquad \text{(eq. 4)}$$

where η is the height of the free surface and η_b is the height of the bottom topography, both relative to the same reference level.
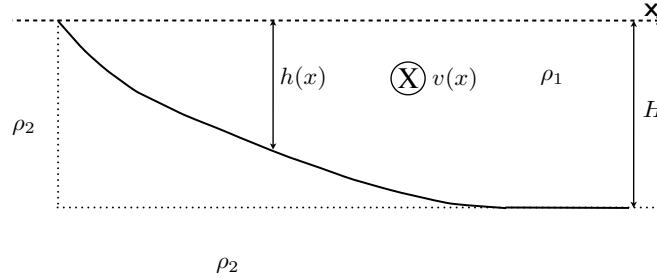

a) A cylindrical column of air at $30°$ latitude with radius 100 km expands horizontally to twice its original radius. If the air is initially at rest, what is the mean tangential velocity at the perimeter after

the expansion?

b) An air column at $60°$ N with zero relative vorticity ($\zeta = 0$) stretches from the surface to the tropopause, which we assume is a rigid lid, at 10 km. The air column moves zonally onto a plateau that is 2.5 km high. What is its relative vorticity? Suppose it moves southwards to $30°$ N and that it is not on the plateau anymore. What is its relative vorticity? *Assumption*: the density is constant.
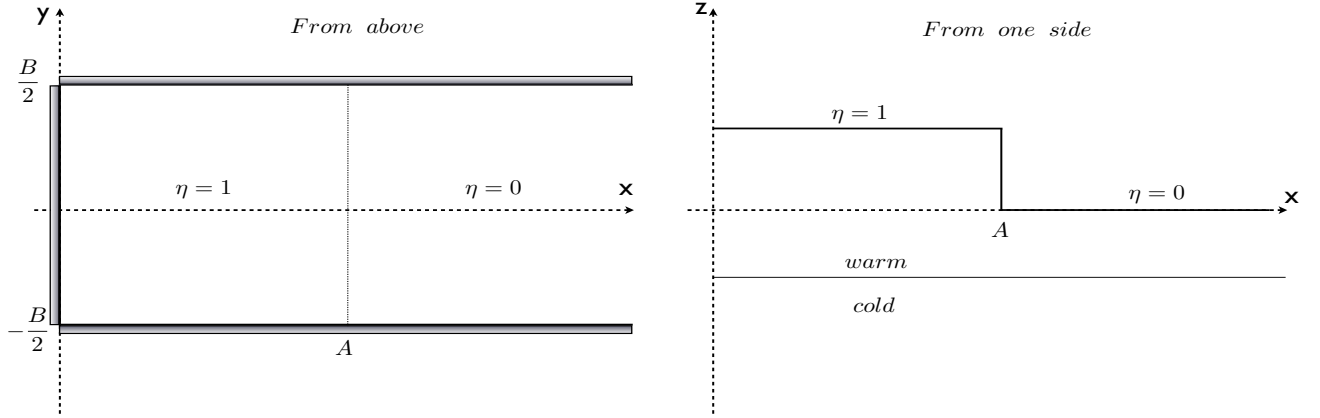
Exercise 5 :

A shallow layer of warm water with density ρ_1 lies on top of deep cold water with density ρ_2. The shallow layer exists only in $x > 0$, and the interface outcrops in $x = 0$. Its potential vorticity is uniform in $x > 0$, and its thickness is H as $x \to \infty$. Assume that $(\rho_2 - \rho_1)/\rho_1 \ll 1$ and that f is constant. Calculate the thickness $h(x)$ of the shallow layer, and the velocity $v(x)$ along the front. Also calculate the mass flux in the frontal jet. The flow is stationary.

Hint: That the potential vorticity is uniform means that it is equal at x, where the thickness is $h(x)$, and at $x \to \infty$, where the thickness is H. Use this to find an equation for $\frac{\partial v}{\partial x}$ and combine with the reduced gravity momentum equation to get a differential equation of $h(x)$ to solve with suitable boundary conditions.

Exercise 6 : A shallow layer of fluid is bounded by vertical walls at $x = 0$ and $x = L$. The bottom is flat and the layer thickness is h_0 at $x = 0$ and h_L at $x = L$. Calculate the northward mass transport between the walls. Suppose that we would like to apply this result in order to calculate the mass transport across an ocean basin. What are the main reasons why this may give an incorrect result? *Assumption*: steady state and $\mathbf{u} = v(x)\hat{y}$.

Exercise 7 : Consider an ocean basin with coast at $x = 0$ and $y = \pm B/2$, i.e. water in $x > 0$ and $-B/2 < y < B/2$ (cf. Fig. hereafter). There is a layer of warm water on top of a much thicker layer of cold water, and the baroclinic Rossby radius is small compared to the basin, $L_d \ll A$ and $L_d \ll B$. At $t = 0$ the warm layer is thicker in $0 < x < A$ than in $x > A$. We model the flow by the linearized reduced gravity shallow water equations on an f-plane, and thus set $\eta = 1$ in $0 < x < A$ and $\eta = 0$ in $x > A$ at $t = 0$.

a) Describe qualitatively the asymptotic geostrophically balanced state at $t = \infty$.

b) Describe qualitatively the evolution toward the asymptotic state. What waves are involved?

Exercise 8 : Explain qualitatively why Kelvin waves propagate with the coast on the right-hand side in the Northern Hemisphere.

 Hint: The dynamics in the direction along the coast are similar to a non-rotating gravity wave. Therefore start by finding the phase relation between u and η in a gravity wave.

Exercise 9 : Derive the dispersion relation for Rossby waves when there is a uniform background flow $U_0\mathbf{x}$. Start from the quasigeostrophic vorticity equation. Examine in particular how background flow affects long and short waves.

Exercise 10 : Derive the quasigeostrophic vorticity equation for a case with topography $\eta_b(x,y)$. Start from the conservation law for potential vorticity, assume that $\eta_b/H \sim$ Ro, and use the β-plane approximation.

 Hint: Rewrite the PV conservation law to a suitable expression to use the Maclaurin series expansion $\frac{1}{1+\epsilon} = 1 - \epsilon + \mathcal{O}(\epsilon^2)$, where $\epsilon = \mathcal{O}(RO)$, on.

Exercise 11 : Consider a quasigeostrophic flow over a localized mountain. The horizontal length scale is much smaller than the Rossby radius, so that we can assume a rigid lid. The quasigeostrophic vorticity equation is then

$$(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla)(\nabla^2\psi + \beta y + b) = 0, \tag{eq. 11-1}$$

where $\mathbf{u} = \mathbf{z} \times \nabla\psi$, and the topography is described by

$$b(x,y) = \frac{f_0}{H}\eta_b(x,y). \tag{eq. 11-2}$$

Suppose that the uniform background flow far from the mountain is $\mathbf{u}_\infty = U_0\mathbf{x}$. Derive a linear equation for ψ in the steady case that must be satisfied on all open streamlines extending to infinity. What kind of solutions does this equation have if U_0 is positive and negative, respectively?

 Hint: Starting from the steady state of eq. 11-1, motivate that the quasigeostrophic vorticity q is constant along streamlines, i.e. that q can be written as a function of ψ: $q = \nabla^2\psi + \beta y + b = F(\psi)$. Find the function F by looking at the streamlines extending to infinity.

Exercise 12 : The quasigeostrophic two-layer equations are

$$\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = 0, \tag{eq. 12-1}$$

$$\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = 0, \tag{eq. 12-2}$$

where, using the rigid-lid approximation, the potential vorticity is given by

$$q_1 = \beta y + \nabla^2\psi_1 + \frac{f_0^2}{g'H_1}(\psi_2 - \psi_1) \tag{eq. 12-3}$$

$$q_2 = \beta y + \nabla^2 \psi_2 + \frac{f_0^2}{g' H_2}(\psi_1 - \psi_2) \qquad \text{(eq. 12-4)}$$

Derive the dispersion relation for the linear waves from these equations. How do you think the result would change if the rigid-lid approximation was not made? And what is the expression obtained for the baroclinic Rossby radius? Compare this with the expression that would be obtained from the one-layer reduced gravity model.

Exercise 13 : Consider a flow governed by the quasigeostrophic vorticity equation:

$$\frac{\partial}{\partial t}(\nabla^2 \psi - \frac{\psi}{L_d^2}) + J(\psi, \nabla^2 \psi) + \beta \frac{\partial \psi}{\partial x} = 0 \qquad \text{(eq. 13-1)}$$

Suppose that there is a localized perturbation on an infinite plane (a vortex for example). We may define the "center of mass" $(\bar{x}, \bar{y})$ of the perturbation by

$$\bar{x} = \frac{\int x \psi dS}{\int \psi dS} \qquad \text{(eq. 13-2)}$$

$$\bar{y} = \frac{\int y \psi dS}{\int \psi dS} \qquad \text{(eq. 13-3)}$$

Show that:

$$\frac{d\bar{x}}{dt} = -\beta L_d^2 \qquad \text{(eq. 13-4)}$$

$$\frac{d\bar{y}}{dt} = 0 \qquad \text{(eq. 13-5)}$$

Hint: Multiply eq. 13-1 by x and integrate over the entire plane S, assuming that ψ goes quickly to zero at the boundaries. Note that eq. 13-1 integrated over S gives $\frac{d}{dt}\int \psi dS = 0$. Repeat when eq. 13-1 is instead multiplied by y and note the difference.

Exercise 14 : The equatorial Rossby radius is defined by $L_e = \sqrt{c/2\beta}$, where $c = \sqrt{gH}$. Show that the magnitude of L_e (except the factor $1/\sqrt{2}$) can be obtained by adapting the usual Rossby radius L_d to the equatorial β-plane.

Exercise 15 : The linear equation for Poincaré waves is

$$\frac{\partial^2 \eta}{\partial t^2} + f^2 \eta - c^2 \nabla^2 \eta = 0 \qquad \text{(eq. 15-1)}$$

where $c^2 = gH$. Assume that $f(y)$ varies as on the equatorial β-plane and derive an ordinary differential equation that describes equatorially trapped Poincaré waves. Solve this equation using suitable boundary conditions, and determine the turning point y_t for mode n.

Hint: the equation

$$\frac{d^2 \Phi}{dy^2} + (a - b^2 y^2)\Phi = 0 \qquad \text{(eq. 15-2)}$$

has the solutions

$$\Phi(y) = H_n(\sqrt{b}y)e^{-by^2/2} \quad \text{for} \quad \frac{a}{b} = 2n+1, \quad n = 0, 1, ... \qquad \text{(eq. 15-3)}$$

where H_n is the nth Hermite polynomial.

Exercise 16 : The dispersion relation for equatorial Rossby waves is

$$\omega = \frac{-\beta k}{k^2 + (2n+1)\frac{\beta}{c}}, \quad n = 0, 1, 2, ... \qquad \text{(eq. 16)}$$

Determine the maximum westward phase velocity for each mode number n, and compare with the velocity of equatorial Kelvin waves. Also determine the maximum frequency and compare with the typical value of the inertial frequency f in the region within an equatorial Rossby radius from the equator.

Exercise 17 : Derive the energy conservation for the linearized shallow water equations,

$$\frac{\partial \mathbf{u}}{\partial t} + f\mathbf{z} \times \mathbf{u} = -g\nabla \eta \qquad \text{(eq. 17-1)}$$

$$\frac{\partial \eta}{\partial t} + H \nabla \cdot \mathbf{u} = 0 \qquad \text{(eq. 17-2)}$$

where f and H are constant. Also derive the ratio between the kinetic energy density and the potential energy density in Poincaré waves. Consider particularly the limits $k^2 + l^2 \gg L_d^{-2}$ and $k^2 + l^2 \ll L_d^{-2}$.

Hint: The ratio between the energies can be written as $\frac{H}{g} \frac{|u|^2 + |v|^2}{\eta^2}$. Make a Fourier ansatz in the linearized shallow water equations to express this ratio for the Poincaré waves.

Exercise 18 : The shallow water equations with a flat bottom, wind stress and bottom friction are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + f \mathbf{z} \times \mathbf{u} = -g \nabla h - r \mathbf{u} + \boldsymbol{\tau} \qquad \text{(eq. 18-1)}$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (\mathbf{u} h) = 0 \qquad \text{(eq. 18-2)}$$

where r is a friction coefficient and τ the wind stress. Derive expressions for the energy flux and for the evolution of the total energy from (18-1) and (18-2).

Exercise 19 : Formulate a two-layer shallow water model with a rigid lid and flat bottom. Show energy conservation for this model.

Hint: Introduce the surface pressure $p_s(x, y, t)$, since the lid takes up a pressure.

Exercice 20 : A stationary shallow water flow satisfies

$$\mathbf{u} \cdot \nabla \mathbf{u} + f \mathbf{z} \times \mathbf{u} = -g \nabla \eta \qquad \text{(eq. 20-1)}$$

$$\nabla \cdot (\mathbf{u} h) = 0 \qquad \text{(eq. 20-2)}$$

$$h = \eta - \eta_b$$

Because of eq. 20-2, the mass flux $\mathbf{u}h$ can be expressed in terms of a stream function ψ. Express the relative vorticity with the help of ψ. Because of the potential vorticity conservation, the potential vorticity q is constant along stream lines. Show that the quantity $B = |\mathbf{u}|^2/2 + g\eta$ is also constant along stream lines.

Hint: Show that if B is constant along stream lines, then $\mathbf{u}h$ and ∇B are perpendicular. That $\mathbf{u}h$ and ∇B are perpendicular can in turn be shown by multiplying eq. 20-1 with $\mathbf{u}h$.

Exercise 21 : Rossby waves are excited by time-dependent perturbations around $x = y = 0$. Assume that the flow is described by the linearized quasigeostrophic vorticity equation:

$$\frac{\partial}{\partial t}\left(\nabla^2 \psi - \frac{\psi}{L_d^2}\right) + \beta \frac{\partial \psi}{\partial x} = 0, \qquad \text{(eq. 21)}$$

a) Do you expect the waves to appear to the east of the source, to the west of the source, or both?
b) Explain how the energy carried away by the waves can be calculated.

Exercise 22 : A flow is governed by the barotropic vorticity equation:

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi) + \beta \frac{\partial \psi}{\partial x} = 0 \qquad \text{(eq. 22-1)}$$

A westward jet is centered at the latitude $60°$ corresponding to $y = 0$, and has the form

$$U = -U_0 e^{-y^2/a^2} \qquad \text{(eq. 22-2)}$$

where $a = 500$ km

a) At what value of U_0 does the jet become unstable ? Where does the instability first occur ?
b) Suppose that the jet is eastward instead of westward, but with the same form. Does it become unstable at a larger or smaller value of U_0 than the westward jet ? Does it occur at the same latitude?

Hint: The conserved potential vorticity is $q = \nabla^2 \psi + \beta y$. Write the expression for and sketch $q(y)$ for the relevant jet and find where and for what parameter values $\frac{dq}{dy}$ will just change sign.

Exercise 23 : A barotropic flow is governed by the equation

$$\frac{Dq}{Dt} = 0 \qquad \text{(eq. 23-1)}$$

where $\mathbf{u} = \mathbf{z} \times \nabla \psi$ and $q = \nabla^2 \psi + b$. Here $b = \frac{f_0 \eta_b}{H}$ and the topography η_b has the form of a zonal ridge

$$\eta_b = A e^{-y^2/a^2}, \qquad A > 0 \qquad \text{(eq. 23-2)}$$

The background flow is a linear shear flow:

$$U_0 = -Sy \qquad \text{(eq. 23-3)}$$

Do you expect this flow to be unstable for $S > 0$ and for $S < 0$?