Exam questions theoretical

$$\frac{D\mathbf{u}}{Dt} + f\mathbf{k} \times \mathbf{u} = -g\nabla\eta \tag{1}$$

$$\frac{Dh}{Dt} + h\nabla \cdot \mathbf{u} = 0 \tag{2}$$

Where (x, y, z) are the zonal, meridional and vertical coordinates with unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, respectively, the gradient operator is here $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ and t is the time. The **total height** (**thickness**, **depth**, **vertical extent**) of the fluid is $h = H + \eta - \eta_B$ where $\eta(x, y, t)$ is a surface displacement from the **constant** mean fluid column height H and $\eta_B(x, y)$ is the bottom topography. Horizontal velocity vector $\mathbf{u}$ has components in zonal and meridional directions (u, v), respectively, g is the constant gravitational acceleration and f is the Coriolis parameter. The density of the fluid is constant.

Question 1

a) State the assumptions needed to derive the SWE.

b) Derive the hydrostatic relation from the vertical momentum equation (state the assumptions necessary to do that) and use it to find the RHS of eq. 1.

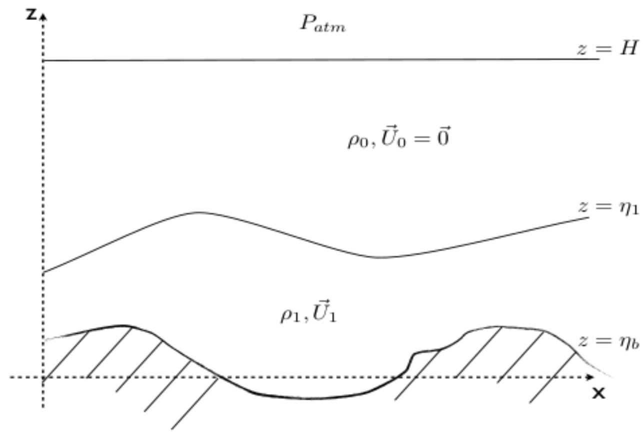
c) Write down a version of the model defined by Eqs. 1-2 for a 2-layer reduced gravity model with a rigid lid. What assumption leads to the form of the equations for the two models being isomorphic (i.e., have same form)?

d) What is the relation of the surface displacement and the interface displacement in a 2-layer SWE with a reduced gravity?

e) Define the Rossby radius of deformation for the 1-layer SWE (constant density) and the 2-layer SWE reduced gravity, rigid-lid model. What is the meaning of the Rossby radius of deformation? What are the typical values of the Rossby radius of deformation in the ocean and atmosphere and how we can related them to the assumption used in (c)?

e) Explain the notion of "fast mode(s)" and "slow mode(s)" as well as "filtering of fast models" implicit in QG approximation applied to the SWE.

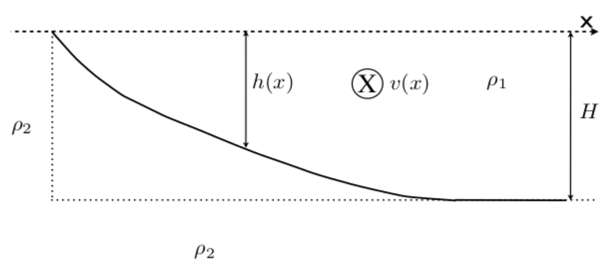
Question 2



Dense bottom currents in the ocean can result from cold water overflowing sills. They are often modelled as a single moving layer of heavy fluid on the bottom, underlying a layer of light fluid at rest. Derive the appropriate shallow water equations for this model, assuming the density to be ρ_1 in the dense layer and ρ_0 in the upper layer, where $\rho_0 < \rho_1$. Show that as $\rho_0/\rho_1 \to 0$ the usual one-layer shallow water equations are obtained.

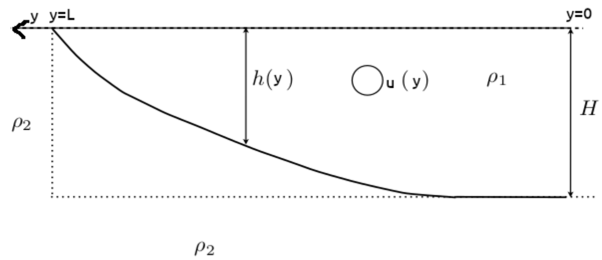
Question 3 (Exam 2023 ++ versions)

a) A shallow layer of warm water with density ρ_1 lies on top of deep cold water with density ρ_2 (see the figure below). The shallow layer exists only in $x > 0$, and the interface outcrops in $x = 0$. Its potential vorticity is uniform in $x > 0$, and its thickness is H as $x \to \infty$. Assume that $(\rho_2 - \rho_1)/\rho_1 \ll 1$ and that f is constant. The flow is stationary.



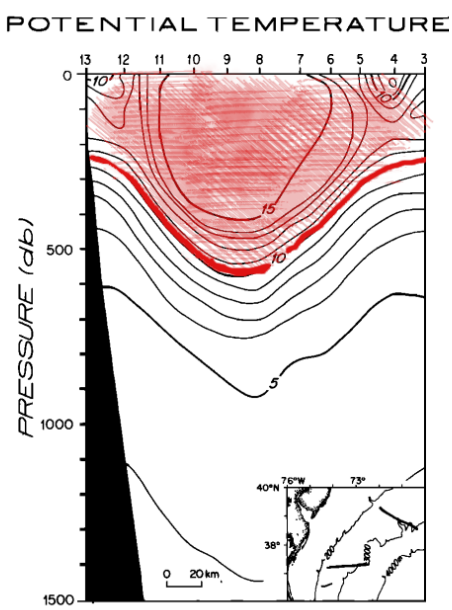
Task: Calculate the thickness $h(x)$ of the shallow layer, and the velocity $v(x)$ along the front. Also calculate the mass flux in the frontal jet.
Hint: The potential vorticity is uniform means that it is equal at x, where the thickness is $h(x)$, and at $x \to \infty$, where the thickness is H. Use this to find an equation for $\frac{\partial v}{\partial x}$ and combine with the reduced gravity momentum equation to get a differential equation of $h(x)$ to solve with suitable boundary conditions.

b) A shallow layer of warm water with density ρ_1 lies on top of deep cold water with density ρ_2 (see the figure below). The shallow layer exists only in $y \in (0, L)$, and the interface outcrops in $y = L$. Its potential vorticity is uniform in $y \in (0, L)$, and its thickness is H as $y \to 0$. Assume that $(\rho_2 - \rho_1)/\rho_1 \ll 1$ and that f is constant. The flow is stationary.



Task: Calculate the thickness $h(y)$ of the shallow layer, and the velocity $u(y)$ along the front. Also calculate the mass flux in the frontal jet.

c) The figure below shows temperature sections through an oceanic eddy (a Gulf Stream ring). Assume that the lower layer is infinitely deep and the densities of the upper and lower layers are 1025 and 1028 $kg\,m^{-3}$, respectively. Sketch the shape of the sea level anomaly and the surface currents associated with this eddy. Estimate the magnitude of the sea level anomaly at the very centre of this eddy. ($1\,\text{db} \approx 1\,\text{m}$ depth).



Question 4 The Shallow Water Equations (SWE) are given by Eqs. 1-2.

a) Linearize Eqs. 1–2 around a rest state with constant depth H **and flat bottom** ($\eta_B = 0$), constant Coriolis parameter $f = f_o = const$ and assuming that the perturbation to the system is uniform in y direction.

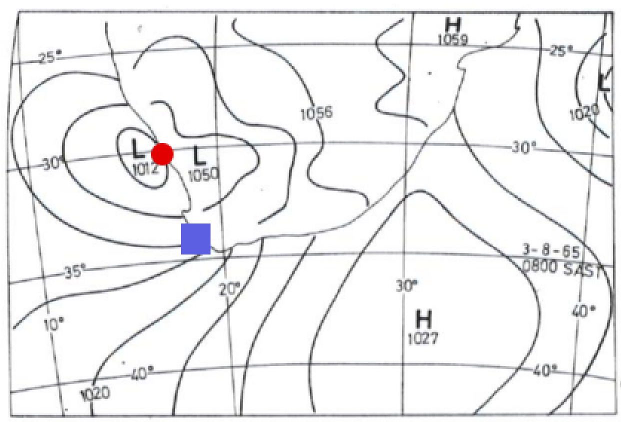
b) Derive the dispersion relation for Poincaré waves for the linearized equations derived in (a). Sketch the dispersion relation on the frequency-wavevector plane and calculate the phase (propagation) velocity.

c) Define the Rossby radius of deformation and explain its meaning for the asymptotic cases of long and short wave. Explain the difference between the barotropic and baroclinic Rossby radi. What are the orders of magnitudes of corresponding barotropic and baroclinic wave speeds in the ocean?

d) Define the group velocity for the Poincaré waves. Explain the difference between the phase (**propagation**) speed, group velocity and (**wave-induced oscillatory flow**, orbital, *i.e., (u, v)*) velocity.

e) Show that the dispersion relation for equations (1-2) is the same if we consider $h = H - \eta$ instead.

f) (Similar questions for different orientations of the coast possible). Consider an atmospheric low pressure system hitting the South African coast at 30^{o}S (red dot, see figure below). Describe in qualitative terms evolution of the perturbation due to this low pressure system: what kind of waves are excited? Linearize Eqs. 1-2 and define appropriate boundary condition for a wave solution in case of a coast aligned in the y (meridional) direction and derive the dispersion relation for this system.



g) For the case (f) derive a solution for the perturbation η in function of space and time. How long will it take for the peak of initial perturbation at 30^{o}S (red dot) to reach Cape Town (blue dot) which is located 400 km away? Assume $H = 10$km.

h) At what distance from the coast will the amplitude of the wave be 30% of the amplitude at the coast?

Question 6 The linear equation for Poincaré waves on the equatorial beta plane is:

$$\frac{\partial^2 \eta}{\partial t^2} + f^2 \eta - c^2 \nabla^2 \eta = 0 \tag{3}$$

The linear equation for Rossby waves is:

$$\frac{\partial}{\partial t}\left(\nabla^2 \psi - \frac{\psi}{L_d^2}\right) + \beta \frac{\partial \psi}{\partial x} = 0 \tag{4}$$

where on the equatorial plane we make an approximation: $f = f(y) = \beta y$ and $c = \sqrt{gH}$. These equations describe two important limits of the complete set of solutions to the unforced equatorial beta–plane SWE.

a) Using an appropriate Ansatz, show that both equations lead to a parabolic guide equation describing equatorially trapped waves:

$$f'' + (a - b^2 y^2) f = 0, \tag{5}$$

which solutions have a form: $f(y) = H_n(\sqrt{b}y)e^{-\frac{by^2}{2}}$ for $\frac{a}{b} = 2n + 1$ for $n = 0, 1, 2, \ldots$.

b) Derive dispersion relation for both types of waves using appropriate values of a and b. Sketch the dispersion relation. What is the main difference with respect to the mid-latitude waves?

c) Calculate the phase speed for the equatorial Rossby waves (c_p^x). Show that the short waves are nearly stationary and show that for the long wave limit the phase speed always negative and of a smaller magnitude than the phase speed of the equatorial Kelvin waves.

d) Derive the group velocity for the equatorial long Rossby Waves.

Question 7

Consider a form of Eqs. 1-2 valid on the equatorial beta plane and for a reduced gravity model representative of the equatorial Pacyfik: a varying depth of the upper layer (depth of the thermocline) is $h(x, y, t)$, mean depth of the upper layer is $H = 120\,\mathrm{m}$ and the density difference between the upper (ρ) and the lower layer ($\rho + \Delta\rho$) scales as $\frac{\Delta\rho}{\rho} = 0.006$.

a) Write the form of Eqs. 1-2 valid for the reduced gravity model stated above. Define the beta parameter and write the expression for f in the vicinity of the equator (zero latitude, $y = 0$). Hint: equatorial beta plane.

b) Linearize the system of equations you got in a).

c) Define appropriate boundary condition for a wave solution to a perturbation at the equator that leads to a generation of Equatorial Kelvin wave.

d) Derive the dispersion relation for this type of wave. Is the wave dispersive?

e) Derive the full solution for this type of wave ($h(x, y, t), u(x, y, t), v(x, y, t)$).

f) How long will it take for the wave crest to cross the Pacyfik (14,000 km)?

g) At what (meridional) distance from the equator the amplitude of the wave will decrease to 10% of the amplitude of the equator?

h) The crest of the wave have been observed by the satellite altimetry as the sea level anomaly of $\eta_o = +\,5\,\mathrm{cm}$ (a positive anomaly). What is the corresponding change in h and what is the corresponding thermocline displacement from the mean depth H? (Draw a schematic to avoid sign confusion!).

Question 9 Derive the **Potential Vorticity (PV) conservation equation** for the Shallow Water system described by Eqs 1-2 for the following cases (*You can do it using cross-differentiation like in we did in L3 or you can also derive it in a vector form*):

a) Full nonlinear version of the equations: free surface and topography (bottom is not flat, **but is small compared to mean depth**) and the beta plane approximation (*Note the material derivative in Eqs 1-2 - don't forget advection term and $f = f(y)$ in the derivation!*)

b) Full nonlinear version of the equations, rigid lid and flat bottom (in that case the RHS of Eq.1 is $-g'\nabla h$ and the equations describe the motion of the upper layer), beta plane approximation.

c) Linear version of the equations, rigid lid and flat bottom (in that case the RHS of Eq.1 is $-g'\nabla h$ and the equations describe the motion of the upper layer), beta plane approximation.

d) Equations (1-2) with the friction terms $-r\mathbf{u}$ on the RHS of the momentum equations.

e) Derive an expression for the Lagrangian (material) invariants for this system. How many of them are there? (*See the end of L3.pdf*).

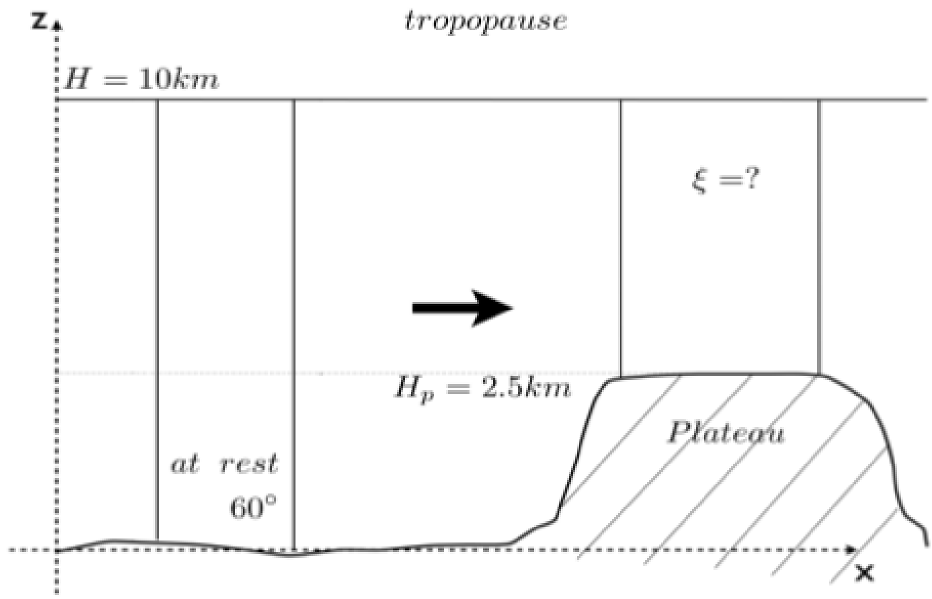
f) Show that the PV conservation can be obtained from the Kelvin circulation theorem and explain the terms: vortex line and vortex tube (*Hint: the second part of the exercise you don't need to derive the Kelvin circulation theorem but to show how it applies for a vortex tube of a crossection A and height h*).

Question 10

Now derive the **PV evolution equation** for the Shallow Water system given by Eqs 1-2 (full version) but with a Rayleigh friction term $-r\mathbf{u}$ or forcing term $+R\mathbf{u}$ added to the RHS of Eq.1. *Repeat the steps as in the tasks above and keep the friction term on RHS side and all the other terms on the LHS side of the PV equation.* Interpret the result: will the PV be constant, rise or decrease in time, and how? Explain why we cannot call it "PV conservation equation". Derive an expression for the Lagrangian (material) invariants for the system without friction. How many of them are there? How many are there for the case with friction? *The last question can be tricky in interpretation.*

Question 11

Consider an air column at 60^{o}N with zero relative vorticity (ζ_o= 0) and of a vertical extent from the surface to the tropopause, which we assume is a rigid lid at 10 km. The air column moves zonally onto a plateau that is 2.5 km high. What is its relative vorticity? Suppose it moves southwards to 30^{o}N and that it is not on the plateau anymore. What is its relative vorticity? Assumption: the density is constant.



Question 12

Consider a flipped version of Task 8 applicable for the oceanic case of the Denmark Strait Overflow: a water column at a latitude $\theta_o = 67^o$N with initial zero relative vorticity (ζ_o= 0) sinks from a depth of H_o= 1000m to a new depth $H_1 = H_o - \eta_B$ = -2000m, and at the same time moves south to a new latitude $\theta_o = 66.5^o$N. What is its relative vorticity ζ_1? Assume a rigid lid and a constant density. Discuss the case when the water sinks periodically every 2-5 days and there is also a mean southward flow: what will be the flow look like then? *OBS. "flipped" - the orientation of the coordinate system wrt of the topography variation - the thickness of the layer after sinking is 3000 m.*

Question 13

a) What assumptions must be made in order to derive the quasigeostrophic vorticity equation from Eqs. 1–2?

b) Starting from the conservation of potential vorticity (PV) valid for the SWE (*Hint: this is* $\frac{D}{Dt}\left(\frac{f+\zeta}{h}\right) = 0$), derive the quasigeostrophic vorticity equation (equation for one variable: ψ) for a flat bottom and for a beta plane approximation. Make sure to justify clearly all steps in your derivation.

c) Starting from the conservation of potential vorticity (PV) valid for the SWE (*Hint: this is* $\frac{D}{Dt}\left(\frac{f+\zeta}{h}\right) = 0$), derive the quasigeostrophic vorticity equation (equation for one variable: ψ) for a case with topography $\eta_B(x, y)$ where the topographic variations are much smaller than the mean depth H, and for a beta plane approximation. Make sure to justify clearly all steps in your derivation. *Hint: small variations in topography you correspond to* $\frac{\eta_B(x,y)}{H} \sim Ro$ *which makes the derivation similar to the case (b).*

d) What is the difference between the shallow-water equations and the quasigeostrophic vorticity equation in terms of the phenomena these two models can describe? What assumption leads to this difference?

Question 14

The quasigeostrophic vorticity equation can be written as:

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - \frac{\psi}{L_d^2}\right) + J\left(\psi, \nabla^2\psi\right) + \beta\frac{\partial\psi}{\partial x} = 0, \tag{6}$$

where ψ is the geostrophic streamfunction and $L_d^2 = \frac{gH}{f_o^2}$ (1-layer QG) or $L_d^2 = \frac{g'H_1}{f_o^2}$ (reduced 2-layer model with a rigid lid and an active layer of mean depth H_1, where a reduced gravity is $g' = [(\rho_2 - \rho_1)/\rho_1]g$.

a) Derive the **dispersion relation**[1] for Rossby waves by linearizing Eq. **??** around a state of no mean flow and sketch the solution on a (k, ω) plane. Calculate the **phase (propagation) velocity** c_p (both, c_p^x and c_p^y) and interpret the propagation direction. Explain the **dispersive properties**[2] of Rossby waves for the short wave- and the long wave limits. **OBS! Be clear if both directions considered or we take $l \to 0$.**

b) Calculate the group velocity $\mathbf{c_G} = (c_G^x, c_G^y)$ for the Rossby wave in case of no mean flow. What is the direction of **wave energy propagation**[3] for the short wave and long wave limits?

c) Describe the mechanism for Rossby waves (*Draw a figure similar to Fig. 6.3. in Vallis (2019) and explain how the westward propagation emerges*). Are Rossby waves also possible solutions to the nonlinear Shallow Water system (Eqs. 1–2)?

d) Now linearize Eq. **??** around a state of a mean constant zonal flow U_o and derive the dispersion relation for Rossby waves. How does the background flow affect long and short Rossby waves, respectively? Find the cut-off wavenumber, i.e., wavenumber k_s for which the Rossby Wave is stationary for a given mean flow U_o. Find the speed of the mean flow U_s that makes a wave of a given wavenumber k stationary. Interpret this result giving examples from atmospheric phenomena e.g. how the tropospheric jet stream will affect the propagation of short and long waves? **OBS! Be clear if both directions considered or we take $l \to 0$.**

e) Calculate the group velocity for the Rossby wave in case of a constant mean flow U_o. How the mean flow affects the direction of **wave energy propagation** for the short wave and long wave limits?

f) The QG equation (**??**) is valid for a 1-layer model ($L_d^2 = \frac{gH}{f_o^2}$) as well as for a reduced gravity model with a rigid lid where $L_d^2 = \frac{g'H_1}{f_o^2}$. Calculate typical values of Rossby wave periods $T = 2\pi/\omega$ and phase speeds c_p^x for a mid-latitude ($\theta_o = 50^o$N): atmosphere (H_a=10 km, $g' = 0.3g$) and ocean (H_a=4000 m, $g' = 0.01g$) and argue that the reduced gravity model (describing baroclinic Rossby waves propagating in the troposphere and in the ocean upper layer, respectively) is more relevant in discussing the weather and ocean dynamics than the barotropic case.

[1] Dispersion relation $\omega(\mathbf{k})$ describes the relation between the frequency and the wavenumber.

[2] For dispersive waves the phase (propagation) speed is a function of the wavenumber: $c_p = c_p(\mathbf{k})$.

[3] The wave energy propagates with a group velocity $\mathbf{c_G}$.

Question 14 (exam 2022 version)

The quasigeostrophic vorticity equation for a 2-layer, reduced gravity, rigid lid model is given by eq. (**??**), where ψ is the geostrophic streamfunction and $L_d^2 = \frac{g' H_1}{f_o^2}$ where H_1 is the mean depth of the upper layer. The reduced gravity g' is related to fluid densities in the two layers as $g' = [(\rho_2 - \rho_1)/\rho_1]g$ where ρ_1 is the upper layer density and ρ_2 is the lower layer density.

a) Linearize eq. (**??**) around a state of a mean constant zonal background flow $U_o\mathbf{i}$ and derive the dispersion relation for Rossby waves.

b) Consider the asymptotic long and short wave limits and show how the background flow affects the dispersion properties of the long and short Rossby waves, respectively. For simplicity consider only the zonal direction taking $l = 0$.
Hint: Calculate the phase velocity to solve this.

c) Find the cut-off wavenumber k_s and the wavelength λ_s for which the Rossby Wave becomes stationary for a given mean flow speed U_o.

Question 15 Consider a quasigeostrophic flow over a localized mountain. The horizontal length scale is much smaller than the Rossby radius, so that we can assume a rigid lid. The quasigeostrophic vorticity equation is then:

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\left(\nabla^2\psi + \beta y + b\right) = 0, \tag{7}$$

where $\mathbf{u} = \mathbf{k}\times\nabla\psi$, and the topography is described by:

$$b(x, y) = \frac{f_o}{H}\eta_b(x, y). \tag{8}$$

a) Suppose that the uniform background flow far from the mountain is $\mathbf{u}_\infty = U_o\mathbf{x}$. Derive a linear equation for in the steady case that must be satisfied on all open streamlines extending to infinity. What kind of solutions does this equation have if U_o is positive and negative, respectively?
Hint: Starting from the steady state of Eq. **??** motivate that the quasigeostrophic vorticity q is constant along streamlines, i.e. that q can be written as $q = \nabla^2\psi + \beta y + b = F(\psi)$. Find the function F by looking at the streamlines extending to infinity.

b) What would be the qualitative solution to this problem in case of a constant Coriolis parameter?

c) Do we consider barotropic or baroclinic Rossby radius in formulating condition on the length scale of this flow?

Question 16

The dispersion relation for equatorial Rossby waves is:

$$\omega = \frac{-\beta k}{k^2 + (2n+1)\frac{\beta}{c}}, \quad n = 0, 1, 2, ...$$

Determine the maximum westward phase velocity for each mode number n, and compare with the **phase speed (propagation) velocity** of equatorial Kelvin waves. Also determine the maximum frequency and compare with the typical value of the inertial frequency f at latitude $\theta = 2^o$N, i.e., within an equatorial Rossby radius from the equator.

Question 23

A flow is governed by the barotropic vorticity equation eq. (**??**), under a assumption $\frac{1}{L_d^2} \to 0$. A westward jet is centered at the latitude 60^{o}N corresponding to $y = 0$, and has the form:

$$U = -U_0 e^{-y^2/a^2}$$

where a = 500 km.

a) At what value of U_0 does the jet become unstable ? Where does the instability first occur ?

b) Suppose that the jet is eastward instead of westward, but with the same form. Does it become unstable at a larger or smaller value of U_0 than the westward jet? Does it occur at the same latitude?

c) Consider again the situation (a). What will be the effect of decreasing the width of the jet (a) to the half of the initial value?

Hint: The potential vorticity is $q = \nabla^2\psi + \beta y$. Write the expression for $q(y)$ and sketch it for the relevant jet and find where and for what parameter values dq/dy will change sign.

Question 24

A barotropic flow is governed by the equation

$$\frac{Dq}{Dt} = 0$$

where $\mathbf{u} = \mathbf{z} \times \nabla\psi$ and $q = \nabla^2\psi + b$. Here $b = \frac{f_0\eta_b}{H}$ and the topography η_b has the form of a zonal ridge

$$\eta_b = Ae^{-y^2/a^2}, \qquad A > 0$$

The background flow is a linear shear flow:

$$U_0 = -Sy$$

Do you expect this flow to be unstable for $S > 0$ and for $S < 0$?

Question 25

Take the momentum equations of an inviscid and unforced Navier-Stokes flow on the rotating Earth, and the mass conservation equation:

$$\frac{D\mathbf{v}}{Dt} + f\mathbf{k} \times \mathbf{v} = -\frac{1}{\rho}\nabla p - \mathbf{k}g \tag{9}$$

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{v} \ \ = 0, \tag{10}$$

where (x, y, z) are the zonal, meridional and vertical coordinates with unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, respectively, the gradient operator is here $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ and t is the time. The velocity vector $\mathbf{v}$ has components in zonal, meridional and vertical directions (u, v, w), respectively, g is the constant gravitional acceleration and f is the Coriolis parameter.

a) Show that for shallow water flows at midlatitudes the vertical momentum equation can be reduced to the hydrostatic balance

b) Justify that for atmospheric and oceanic shallow water flows, the mass conservation equation can be reduced to $\nabla \cdot \mathbf{v} = 0$

Question 26

Consider a 3D incompressible and inviscid fluid flow in a rotating system that can be described by the one-layer shallow water equations. No information about the free surface η is given. The bottom topography η_B relative to a reference level is variable.

a) Show that the vertical velocity w within the layer is given by:

$$w = \frac{z - \eta_B}{h}\frac{Dh}{Dt} + \frac{D\eta_B}{Dt}, \tag{11}$$

where h is the thickness of the fluid layer. Interpret this result, showing that it gives sensible answers at the top and the bottom of the fluid layer.

b) Derive the expression for w for the case that no information on η_B is given and the free surface η is variable.

Hint: The vertical velocity can be diagnosed by vertically integrating the 3D mass continuity equation from η_B to z. Note that $w(z) = \frac{Dz}{Dt}$.