Lecture summery - Waves and Instabilities

Shallow water equations

Knudsen number: $K_n = \frac{\text{mean free path}}{\text{smallest length scale}}$

Scaling: Used for hydrostatic approximation
1) Use icompressabiliy to obtain $w \ll u, v$
2) Use verical momentum equation, with horizontal scaling

Shallow water equations: Assumption $H \ll L$ and Boussinesq (→ homogeneous $\rho = \text{const}$)
1) Use scaling and NS to obtain hydrostatic equilibrium and incompressibility
2) Obtain $\nabla_H p = \rho g \nabla_H \eta$
3) Use continuity and BC to obtain $\frac{Dh}{Dt} + h \nabla_H \cdot \vec{u} = 0$

Two layer model: Assumption same as before, note $\nabla_H p_0 = 0$.
1) Write NS equations with new continuity.
2) First layer same as before
3) The second layer uses continuous pressure. $p_2(z) = p_0 + \rho_1 g(\eta_0 - \eta_1) + \rho_2 g(\eta_1 - z)$
4) Obtain reduced gravity $g' = g\frac{\rho_2 - \rho_1}{\rho_1}$

Reduced gravity model: (aka 1.5) Assume $H_2 \gg H_1$.
1) Notice that shallow part is much faster, $\vec{u}_2 \approx 0$
2) Obtain relation $\nabla_H \eta_0 \approx -\frac{g'}{g} \nabla_H \eta_1$
3) Set $h_1 = \eta_0 - \eta_1$ which gives g' and h_1 for upper layer → $\text{RHS} \approx -g' \nabla_H h_1$.

Potential vorticity: Material invariance, assume shallow. Use $\zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$.
1) Start by taking the curl: $\frac{\partial(\text{eq2})}{\partial x} - \frac{\partial(\text{eq1})}{\partial y}$
2) Solve for ζ and f
3) Write out $\frac{D}{Dt}\left(\frac{\zeta + f}{H}\right)$ to become easier
4) Use continuity
   Note: All functions of Q are material invariants.

Wave theory

Linear wave solution: Assume 1.5 SWE, $f = f_0$ and $\vec{u} = \varepsilon \tilde{\vec{u}}$, $\vec{v} = \varepsilon \tilde{\vec{v}}$, $\eta = H + \varepsilon \tilde{\eta}$.
1) Neglect all ε^2 terms.
2) Apply Fourier ansatz
3) Write (u, v, h) and write SWE on form $A \cdot \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
4) Unique solution for $\|A\| = 0$ → solve for $\omega(f_0, k, l)$
5) Two solutions:
   i. $\omega = 0$ → Geostrophic
   ii. $\omega \neq 0$ → Poincaré waves: Generally dispersive, but not in SW limit with Poincaré disspersion relation $\omega^2 = f_0^2 + c^2 k^2$
6) Use $c = \sqrt{gH}$ and Rossby radius of deformation $L_D = \frac{c}{f_0} = \frac{\sqrt{gH}}{f_0}$.
   i. If SW: $\omega \gg f_0 \rightarrow \omega^2 = c^2(k^2 + l^2)$
   ii. if LW: $\omega \approx f_0$ "Internal oscillations"

Geostrophic adjustment: Assume linearized waves
1) Use equations to derive expression for $\zeta(Q)$
2) Begin by doing $\frac{\partial(\text{eq } 1)}{\partial x} + \frac{\partial(\text{eq } 2)}{\partial y} = -g \nabla H^2 \eta$
   i. Insert ζ when possible
   ii. Use continuity as $\frac{\partial^2 \eta}{\partial t^2} = -H \frac{\partial \eta}{\partial t}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right]$
3) Split solution into homogeneous and particular as $\eta = \eta_w(t) + \eta_p$
4) Use the starting scenario to determine Q, use relation from before.
5) Use particular solution to derive η_p and determine geostrophic wind.

Quasi-geostrophic: Linearize PV-equation and get $\frac{\partial}{\partial t}\left(\nabla^2 \psi - \frac{\psi}{L_D^2}\right) + J(\psi, \nabla^2 \psi) + \beta \frac{\partial \psi}{\partial x} = 0$
1) Assumptions:
   - $\text{Ro} \ll 1$
   - $\frac{\beta y}{f} \ll 1$
   - $\frac{\eta}{H} \ll 1$
2) Go from PV-equation and extract f_0 / H
3) Taylor expand and neglect order ϵ^2
4) Introduce stream-function

Rossby waves with no flow: Start from quasi-geostrophic.
1) Reference state $\psi = \epsilon \psi'$ and $-\partial_x \psi = u$ and neglect ϵ^2
2) Fourier ansatz
3) Obtain dispersion relation $\omega = -\frac{\beta k}{k^2 + l^2 + 1/L_D{}^2}$
   - SW: stationary
   - LW: Non-dispersive, westward

Rossby waves with background flow: Assume $\vec{u} = (u_0, 0)$
1) Reference state $\psi = \epsilon \psi'$, $u = u_0 + \epsilon u'$ and $v = \epsilon v'$ and neglect ϵ^2
2) Fourier ansatz
3) Obtain Rossby dispersion relation $\omega = u_{0k} - k\frac{\beta + u_0 / L_D{}^2}{k^2 + l^2 + 1/L_D{}^2}$
   - SW: background flow
   - LW: Non-dispersive, westward

Kelvin waves: Wave propagation along a fix surface (like coast)
1) Assume $\text{Ro} \ll 1$ and $\vec{u} = (u, 0)$
2) Linerize equation according to $u = \epsilon u$ and $h = H + \epsilon \eta$
3) Fourier ansatz: $\begin{bmatrix} \eta \\ u \end{bmatrix} = \begin{bmatrix} \hat{\eta}(x) \\ \hat{u}(x) \end{bmatrix} e^{i(kx - \omega t)}$ and solve for dispersion
4) Take time-derivative of eq 2 to obtain amplitude.

Instabilitets

Barotropic instability: aka Rossby waves with changing background flow $\vec{u} = [u(y), 0]$
1) Start from reference state and perturb $u_0 = -\partial_y \psi_0 \rightarrow \psi(x, y, t) = \psi_0(y) + \epsilon \psi'(x, y, t)$
2) Linearize QG-equation → LQG-equation
3) Use Fourier ansatz to get Rayleigh equation
4) Integrate and use tricks (like $\vec{v}(y_1) = v(y_2) = 0$) to get criterion for stability

Criterion for barotropic instability:
- Rayleigh-Kuo: For instability $\frac{\partial q}{\partial y}$ must change sign in the domain.
- Fjörtoft: For instability $\frac{u_s - u_0}{\frac{dq_0}{dy}} > 0$.

2 layer model: Assume rigid lid but surface pressure
1) Determine the pressure from continuity and 2 layer SWE
2) Introduce stream-function form geostrophic flow
3) Expand PV-conservation by approximations
4) For solutions define
   - Barotropic mode: $\bar{\psi} = \frac{F_1 \psi_2 + F_2 \psi_1}{F_1 F_2}$
   - Baroclinic mode: $\tau = \frac{1}{2}[\psi_2 - \psi_1]$

Phillips model: Assume the three standard assumptions and $H_1 = H_2$ and $u_1 = -u_2 = u$
1) Start from PV-equation for two layers $\frac{D}{Dt}(q_i) = 0$
2) Use reference state $\bar{u}_i = u_1 + \epsilon u' = \frac{d\psi_i}{dy} + \epsilon \frac{d\psi_i'}{dy}$ and $\bar{v}_i = \epsilon v' = \epsilon \frac{d\psi_i}{dx}$ and $\bar{\psi}_i = \psi(y) + \epsilon \psi'(x, y, t)$
3) Use Fourier ansatz $\psi'_{1,2} = \hat{\psi}_{1,2} e^{ikx + ily - ik\hat{c}t} = \hat{\psi}_{1,2} e^{ikx + ily + \sigma t}$
4) Use math to solve for $\hat{c}$
5) Obtain three cases:
   i. $U = 0$ and $\beta \neq 0$ → Rossby waves (Two roots give baroclinic vs barotropic)
   ii. $U \neq 0$ and $\beta = 0$→ High value *cut-of* Unstable if $\frac{\lambda}{2\pi} < \frac{L_D}{\sqrt{8}}$. No low value *cut-of*
   iii. $U \neq 0$ and $\beta \neq 0$ → *Cut-of* $\frac{\beta}{\sqrt{2} U_0} < k < k_d$

Baroclinic instability:
- Thermal wind $\frac{\partial \vec{u}}{\partial z} = \frac{1}{f_0} \hat{z} \times \nabla b$
- Potential energy decreases when lapse rate is not constant
- Slanted convection: From $\Delta PE = g \Delta \rho \Delta z$ and Taylor→ displacement if $\phi > \alpha > 0$
  - Slope of displacement: $\alpha = \Delta z / \Delta x, y$
  - Slope of isopycnals: $\alpha = -\frac{\partial \rho}{\partial y} / \frac{\partial \rho}{\partial z}$

Equatorial dynamics

Equatorial normal conditions: Assume steady, linear, frictionless and zonal wind.
1) Start from staionary LSWE $-fv = -g' \frac{\partial h}{\partial x} + \frac{\tau^x}{\rho_1 D}$
2) Take curl to see $u = v = 0$
3) Obtain thermocline $\partial_x h < 0$

Equatorial Kelvin waves: Kelvin wave propagation when $f \approx 0 + \beta y = \beta y$
1) Assume $\text{Ro} \ll 1$ and $\vec{u} = (u, 0)$
2) Take x-derivative of eq 1 and t-derivative of eq 3 → same form
3) Fourier ansatz: $\eta = \hat{\eta}(y) e^{i(kx - \omega t)}$ and solve for dispersion
4) Take time-derivative of eq 2 to obtain amplitude.

Equatorial Rossby waves: Rossby wave propagation when $f \approx \beta y$
1) Apply LQG-equation and express L_D with βy
2) Introduce Fourier ansatz with $\hat{\psi}(y)$
3) Rewrite on Hermit form by using variable substitution
4) Obtain dispersion relation and see LW limit is the same as before.

Important numbers

Radius of Earth: 6371 km

Rossby radii: $L_D^{\text{atmosphere}} = 1000\,\text{km}$ and $L_D^{\text{ocean}} = 50\,\text{km}$

Reduced gravity: $g'_{\text{atmosphere}} = 0.3$ and $g'_{\text{ocean}} = 0.01$

Flows: $\langle u_{\text{ocean}} \rangle = 0.1\, ms^{-1}$