UiO : Department of Mathematics
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Join of hexagons and Calabi–Yau threefolds

Public defence

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Outline of the thesis

- Unsuccessful attempt to find new hyper-Kähler varieties.
  - Strategy: naïvely perturb monomial equations of Stanley–Reisner ideals of known triangulations of $\mathbb{CP}^2$.

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  - Identify the smoothings as hyperplane complements in other spaces, and use exact sequences from algebraic topology.

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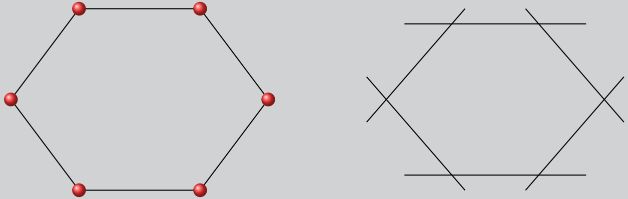
- The topology of smoothings of $C(\mathrm{dP}_6)$.
  - Identify the smoothings as hyperplane complements in other spaces, and use exact sequences from algebraic topology.
- New Calabi–Yau varieties and potential mirror partners.
  - Found by smoothing a certain Stanley–Reisner sphere.

Stanley–Reisner schemes

- Given a simplicial complex $\mathcal{F}$, we get a Stanley–Reisner scheme $\mathbb{P}(\mathcal{F})$.
- Is a union of projective spaces $\mathbb{P}^{\dim f}$, where f is a face of $\mathcal{F}$.

Example

From a simplicial complex to a union of $\mathbb{P}^1$'s.



The ideal is generated by $x_i x_{i+2} = x_i x_{i+3} = 0$ $(i = 0, \ldots, 5)$.

Stanley–Reisner schemes

- *Join* of two subschemes X and Y: the (closure of) the union of all lines between X and Y.
- *Join* of two Stanley–Reisner schemes $\mathbb{P}(\mathcal{F})$ and $\mathbb{P}(\mathcal{G})$ is $\mathbb{P}(\mathcal{F} * \mathcal{G})$, where the faces of $\mathcal{F} * \mathcal{G}$ are $f \sqcup g$ for $f \in \mathcal{F}$, $g \in \mathcal{G}$.

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Smoothings of Stanley–Reisner schemes:

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Smoothings of Stanley–Reisner schemes:

- Given a basis for $T^1(S_{\mathbb{P}(\mathcal{K})}/k, S_{\mathbb{P}(\mathcal{K})})_0$, we can try to find a smoothing of $X_0 = \mathbb{P}(\mathcal{K})$.
- A smoothing X of X_0 will have many of the same properties:
  - The same Hilbert polynomial.
  - By semicontinuity, if X_0 is a sphere, X will be Calabi–Yau.

Calabi–Yau varieties

Definition (Calabi–Yau variety)

A Calabi–Yau variety is an irreducible smooth projective scheme $X/\mathbb{C}$ of dimension 3 satisfying:

- $H^0(X, \mathscr{O}_X) = H^3(X, \mathscr{O}_X) = \mathbb{C}$ and $H^1(X, \mathscr{O}_X) = H^2(X, \mathscr{O}_X) = 0$.
- The canonical sheaf is trivial: $\omega_X \simeq \mathscr{O}_X$.

- Easiest invariants are the Euler characteristic and the Hodge numbers. $h^{ij} = h^j(X, \Omega^i_{X/\mathbb{C}})$.
- We always have $\chi = 2(h^{11} - h^{12})$.

$$
\begin{array}{ccccccc}
 & & & h^{00} & & & \\
 & & h^{01} & & h^{10} & & \\
 & h^{02} & & h^{11} & & h^{20} & \\
h^{03} & & h^{12} & & h^{21} & & h^{30} \\
 & h^{13} & & h^{22} & & h^{31} & \\
 & & h^{23} & & h^{32} & & \\
 & & & h^{33} & & &
\end{array}
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$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{11} & & 0 & \\
1 & & h^{12} & & h^{12} & & 1 \\
 & 0 & & h^{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Hodge numbers

- The quintic $X = V(f) \subset \mathbb{P}^4$ is the canonical example of a Calabi–Yau. It has Hodge numbers $h^{11} = 1$ and $h^{12} = 101$.

Remark (Heuristic)

The number h^{12} is the dimension of the "space of parameters" of X. The following heuristic will give us the correct Hodge number:

- The space of degree 5 polynomials $H^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(5))$ in $\mathbb{P}^4$ is $\binom{4+5}{5} = \binom{9}{4} = 126$-dimensional. Hence $\mathbb{P}\left(H^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(5))\right) = \mathbb{P}^{125}$.
- This is not unique, but we can act by $\mathrm{PGL}(5)$ to identify isomorphic quintics. We have $\dim_{\mathbb{C}} \mathrm{PGL}(5) = 25 - 1 = 24$.
- In total: $125 - 24 = 101$, which is $h^{12}(X)$.

Mirror symmetry

- Calabi–Yau threefolds seem to “always” have “mirror partners”.
- Mirror partner X° to X have “mirrored Hodge diamond”.
- Hence $\chi(X^\circ) = -\chi(X)$.

$$
\begin{array}{ccccccc}
\multicolumn{7}{c}{X} \\
\hline
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
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1 & & 1 & & 1 & & 1 \\
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 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
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The orbifold heuristic

Sometimes the following method produces a mirror manifold of a Calabi–Yau X.

1. Suppose X has a natural degeneration X_0 with a finite automorphism group G.
2. Find a family $\pi : \mathscr{X} \to S$ on which G act, and such that the general fiber X_t have only isolated singularities.
3. There might be a finite subgroup H of the big torus acting. A mirror candidate is then a crepant resolution of X_t/H.

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We can use *Roan's formula* to compute the Euler characteristic.

Theorem (Roan's formula)

$$\chi(\widetilde{X_t/H}) = \frac{1}{|H|} \sum_{g,h \in H} \chi\left(X_t^g \cap X_t^h\right).$$

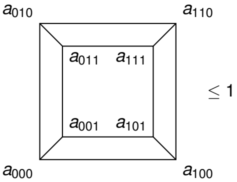
The cone over dP_6

- Let $\mathrm{dP}_6 \subset \mathbb{P}^6$ be an anticanonically embedded del Pezzo surface of degree 6. Let $C(\mathrm{dP}_6)$ be its affine cone in $\mathbb{A}^7$.
- The equations are

$$\begin{vmatrix} y & x_1 & x_2 \\ x_4 & y & x_3 \\ x_5 & x_6 & y \end{vmatrix} \leq 1$$

The origin is an isolated singularity.

- There are two smoothing components.
- They come from perturbations of differents form of writing the equation.
- Can also write the equations as:

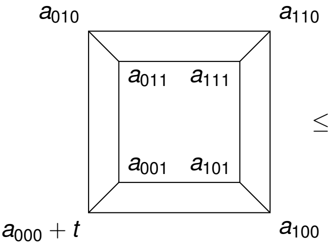


The two smoothing compoments of dP_6

- We can identify this component with $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})\backslash H$, where $\mathcal{T}_{\mathbb{P}^2}$ is the tangent bundle of $\mathbb{P}^2$.

$$\begin{vmatrix} y & x_1 & x_2 \\ x_4 & y+t_1 & x_3 \\ x_5 & x_6 & y+t_2 \end{vmatrix} \leq 1$$

- The second component can be identified with $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)\backslash H$, where H is a $(1,1,1)$-divisor.



≤ 1

Construction of a new Calabi–Yau: X_1

- Let E be a vector space with basis e_1, e_2, e_3. Consider

$$\mathbb{P}^{17} = \mathbb{P}(E \otimes E \oplus E \otimes E).$$

The elements are pairs of 3×3 matrices, equal up to scalar multiplication.

- Consider the set of pairs M of matrices (A, B) with rank $1 + 1$.
- Intersect M with a generic $\mathbb{P}^{11} \subset \mathbb{P}^{17}$. Let $X_1 \triangleq M \cap \mathbb{P}^{11}$.

Theorem

X_1 is a smooth Calabi–Yau with Euler-characteristic -72.

Construction of a new Calabi–Yau: X_2

- Let F be a vector space with basis f_1, f_2. Consider

$$\mathbb{P}^{15} = \mathbb{P}(F \otimes F \otimes F \oplus F \otimes F \otimes),$$

  The elements are pairs of $2 \times 2 \times 2$-tensors, equal up to scalar multiplication.
- Consider the set of pairs N of tensors (A, B) with rank $1 + 1$.
- Intersect N with a generic $\mathbb{P}^{11} \subset \mathbb{P}^{15}$. Let $X_2 \triangleq N \cap \mathbb{P}^{11}$.

Theorem

X_2 is a smooth Calabi–Yau with Euler-characteristic -48.

Construction of a new Calabi–Yau: X_3

- Let E and F be as before. Consider

$$\mathbb{P}^{16} = \mathbb{P}(E \otimes E \oplus F \otimes F \otimes).$$

- Consider the set of pairs W of tensors (A, B) with rank $1 + 1$.
- Intersect W with a generic $\mathbb{P}^{11} \subset \mathbb{P}^{16}$. Let $X_3 \triangleq W \cap \mathbb{P}^{11}$.

Theorem

X_3 is a smooth Calabi–Yau with Euler-characteristic -60.

Hodge number heuristics

Conjecture

X_1 have Hodge numbers $h^{11} = 3$ and $h^{12} = 39$.

"Reason".

1. We know the Euler characteristic, so it is enough to find h^{12}.
2. The Grassmannian of $\mathbb{P}^{11}$'s in $\mathbb{P}^{17}$ is 72-dimensional.
3. We can act with automorphisms from $\prod_{i=1}^{4} \mathrm{GL}(E)$.
4. The subgroup $\{t_1 t_2 = t_3 t_4\} \subset (\mathbb{C}^*)^4 \subset \prod_{i=1}^{4} \mathrm{GL}(E)$ acts trivially.
5. Hence $h^{12} = 72 - (\dim \prod_{i=1}^{4} \mathrm{GL}(E) - 3) = 72 - 33 = 39$.

∎

Mirror candidate for X_1

Using the mirror Ansatz, we propose mirror candidates for X_1 and X_2.

- There is a $H = \mathbb{Z}/3$-action on E defined by $e_i \mapsto \omega^i e_i$.
- Another $\mathbb{Z}/3$-action $e_i \mapsto e_{i+1}$.
- Extends to actions on $\mathbb{P}(E \otimes E \oplus E \otimes E) = \mathbb{P}^{17}$.
- Choose invariant $\mathbb{P}^{11}$: defined by

$$f_{ij}^{\alpha} = e_{ij}^{\alpha} + t_{-i-j}^{\alpha} e_{-i-j,-i-j}^{\alpha+1}$$

  for $i, j \in \mathbb{Z}/3 \times \mathbb{Z}/3$ ($i \neq j$) and $\alpha = 0, 1$.
- The resulting $X_{H_t} \triangleq \mathbb{P}^{11} \cap M$ is singular with 48 isolated double point singularities.

Mirror candidate for X_1

- We divide out by the H-action and resolve: $X_1^\circ = \widetilde{X_{H_t}/H}$.
- Roan's formula gives:

$$\chi(X_1^\circ) = \frac{1}{3}(24 + 8 \cdot 24) = 72.$$

Based on this calculation and the mirror heuristic, we conjecture:

Conjecture

X_1° is a mirror of X_1.

Remark

A very similar construction gives a mirror candidate for X_2.

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