



UiO : **Department of Mathematics**  
University of Oslo

# Join of hexagons and Calabi–Yau threefolds

Public defence

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**November 8, 2017**

# Outline of the thesis

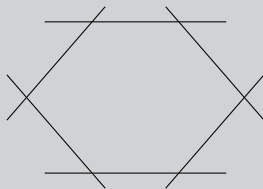
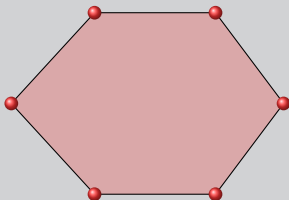
- Unsuccessful attempt to find new hyper-Kähler varieties.
  - Strategy: naïvely perturb monomial equations of Stanley–Reisner ideals of known triangulations of  $\mathbb{CP}^2$ .
- The topology of smoothings of  $C(dP_6)$ .
  - Identify the smoothings as hyperplane complements in other spaces, and use exact sequences from algebraic topology.
- New Calabi–Yau varieties and potential mirror partners.
  - Found by smoothing a certain Stanley–Reisner sphere.

# Stanley–Reisner schemes

- Given a simplicial complex  $\mathcal{F}$ , we get a Stanley–Reisner scheme  $\mathbb{P}(\mathcal{F})$ .
- Is a union of projective spaces  $\mathbb{P}^{\dim f}$ , where  $f$  is a face of  $\mathcal{F}$ .

## Example

From a simplicial complex to a union of  $\mathbb{P}^1$ 's.



The ideal is generated by  $x_i x_{i+2} = x_i x_{i+3} = 0$  ( $i = 0, \dots, 5$ ).

# Stanley–Reisner schemes

- *Join* of two subschemes  $X$  and  $Y$ : the (closure of) the union of all lines between  $X$  and  $Y$ .
- *Join* of two Stanley–Reisner schemes  $\mathbb{P}(\mathcal{F})$  and  $\mathbb{P}(\mathcal{G})$  is  $\mathbb{P}(\mathcal{F} * \mathcal{G})$ , where the faces of  $\mathcal{F} * \mathcal{G}$  are  $f \sqcup g$  for  $f \in \mathcal{F}$ ,  $g \in \mathcal{G}$ .

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## Smoothings of Stanley–Reisner schemes

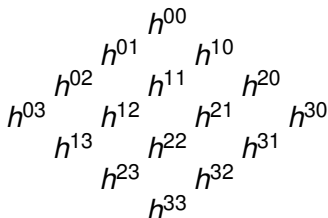
- Given a basis for  $T^1(S_{\mathbb{P}(\mathcal{K})}, k)_0$ , we can try to find a smoothing of  $X_0 = \mathbb{P}(\mathcal{K})$ .
- A smoothing  $X$  of  $X_0$  will have many of the same properties:
  - The same Hilbert polynomial.
  - By semicontinuity, if  $X_0$  is a sphere,  $X$  will be Calabi–Yau.

# Calabi–Yau manifolds

## Definition (Calabi–Yau variety)

A Calabi–Yau variety is a smooth projective scheme  $X/\mathbb{C}$  of dimension 3 satisfying:

- $H^0(X, \mathcal{O}_X) = H^3(X, \mathcal{O}_X) = k$  and  $h^1(X, \mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$ .
  - The canonical sheaf is trivial:  $\omega_X \simeq \mathcal{O}_X$ .
- 
- Easiest invariant are the Euler characteristic and the Hodge numbers.
  - We always have  $\chi = 2(h^{11} - h^{12})$ .



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$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & 0 & & 0 \\
 & & 0 & & h^{11} & & 0 \\
 1 & & h^{12} & & & & h^{12} & 1 \\
 & 0 & & h^{11} & & 0 & & \\
 & & 0 & & 0 & & & \\
 & & & & 1 & & & 
 \end{array}$$

# Hodge numbers

- The quintic  $X = V(f) \subset \mathbb{P}^4$  is the canonical example of a Calabi–Yau. It has Hodge numbers  $h^{11} = 1$  and  $h^{12} = 101$ .

## Remark (Heuristic)

The number  $h^{12}$  is the dimension of the “space of parameters” of  $X$ . The following heuristic will give us the correct Hodge number:

- The space of degree 5 polynomials  $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5))$  in  $\mathbb{P}^4$  is  $\binom{4+5}{5} = \binom{9}{4} = 126$ -dimensional. Hence  $\mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5))) = \mathbb{P}^{125}$ .
- This is not unique, but we can act by  $\mathrm{PGL}(5)$  to identify isomorphic quintics. We have  $\dim \mathrm{PGL}(5) = 25 - 1 = 24$ .
- In total:  $125 - 24 = 101$ , which is  $h^{12}(X)$ .



# Mirror symmetry

- Calabi–Yau threefolds seem to “always” have “mirror partners”.
- Mirror partner  $X^\circ$  to  $X$  have “mirrored Hodge diamond”.
- Hence  $\chi(X^\circ) = -\chi(X)$ .

$X$					
<hr/>					
		1			
	0		0		
0		1		0	
1	0 1	1 0 1		1 0 1	1
	0	1		0	
	0		0		
		1			

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0		1		0
1	0	1	0	1
	1		1	
	0		0	
	0		0	
		1		

$X^\circ$				
		1		
	0		0	
0		1		0
1	0	1	0	1
	1		1	
	0		0	
	0		0	
		1		

# The orbifold heuristic

Sometimes the following method produces a mirror manifold of a Calabi–Yau  $X$ .

- 1 Suppose  $X$  has a natural degeneration  $X_0$  with a finite automorphism group  $G$ .
- 2 Find a family  $\pi : \mathcal{X} \rightarrow S$  on which  $G$  act, and such that the general fiber  $X_t$  have only isolated singularities.
- 3 There might be a finite subgroup  $H$  of the big torus acting. A mirror candidate is then a crepant resolution of  $X_t/H$ .

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We can use *Roan's formula* to compute the Euler characteristic.

Theorem (Roan's formula)

$$\chi(\widetilde{X_t/H}) = \frac{1}{|H|} \sum_{g,h \in H} \chi(X_t^g \cap X_t^h).$$

# The cone over $dP_6$

- Let  $dP_6 \subset \mathbb{P}^6$  be an anticanonically embedded del Pezzo surface of degree 6. Let  $C(dP_6)$  be its affine cone in  $\mathbb{A}^7$ .
- The equations are

$$\begin{vmatrix} y & x_1 & x_2 \\ x_4 & y & x_3 \\ x_5 & x_6 & y \end{vmatrix} \leq 1$$

The origin is an isolated singularity.

- There are two smoothing components.
- They come from perturbations of different form of writing the equation.
- Can also write the equations as:

$$\begin{array}{ccc} & a_{010} & a_{110} \\ & \swarrow & \searrow \\ a_{011} & & a_{111} \\ \swarrow & & \searrow \\ a_{001} & & a_{101} \\ \swarrow & & \searrow \\ a_{000} & & a_{100} \end{array} \leq 1$$



# The two smoothing components

TODO: trengs denne?

# Construction of a new Calabi–Yau: $X_1$

- Let  $E$  be a vector space with basis  $e_1, e_2, e_3$ . Consider

$$\mathbb{P}^{17} = \mathbb{P}(E \otimes E \oplus E \otimes E).$$

The elements are pairs of  $3 \times 3$  matrices, equal up to scalar multiplication.

- Consider the set of pairs  $M$  of matrices  $(A, B)$  with rank  $1 + 1$ .
- Intersect  $M$  with a generic  $\mathbb{P}^{11} \subset \mathbb{P}^{17}$ . Let  $X_1 \triangleq M \cap \mathbb{P}^{11}$ .

## Theorem

$X_1$  is a smooth Calabi–Yau with Euler-characteristic  $-72$ .

## Construction of a new Calabi–Yau: $X_2$

- Let  $F$  be a vector space with basis  $f_1, f_2$ . Consider

$$\mathbb{P}^{15} = \mathbb{P}(F \otimes F \otimes F \oplus F \otimes F \otimes F),$$

The elements are pairs of  $2 \times 2 \times 2$ -tensors, equal up to scalar multiplication.

- Consider the set of pairs  $N$  of tensors  $(A, B)$  with rank  $1 + 1$ .
- Intersect  $N$  with a generic  $\mathbb{P}^{11} \subset \mathbb{P}^{15}$ . Let  $X_2 \triangleq N \cap \mathbb{P}^{11}$ .

### Theorem

$X_2$  is a smooth Calabi–Yau with Euler-characteristic  $-48$ .

# Construction of a new Calabi–Yau: $X_3$

- Let  $E$  and  $F$  be as before. Consider

$$\mathbb{P}^{16} = \mathbb{P}(E \otimes E \oplus F \otimes F \otimes).$$

- Consider the set of pairs  $W$  of tensors  $(A, B)$  with rank  $1 + 1$ .
- Intersect  $W$  with a generic  $\mathbb{P}^{11} \subset \mathbb{P}^{16}$ . Let  $X_3 \triangleq W \cap \mathbb{P}^{11}$ .

## Theorem

*$X_3$  is a smooth Calabi–Yau with Euler-characteristic  $-60$ .*

# Hodge number heuristics

## Conjecture

$X_1$  have Hodge numbers  $h^{11} = 3$  and  $h^{12} = 39$ .

## “Reason”.

- 1 We know the Euler characteristic, so it is enough to find  $h^{12}$ .
- 2 The Grassmannian of  $\mathbb{P}^{11}$ 's in  $\mathbb{P}^{17}$  is 72-dimensional.
- 3 We can act with automorphisms from  $\prod_{i=1}^4 \mathrm{GL}(E)$ .
- 4 The subgroup  $\{t_1 t_2 = t_3 t_4\} \subset (\mathbb{C}^*)^4 \subset \prod_{i=1}^4 \mathrm{GL}(E)$  acts trivially.
- 5 Hence  $h^{12} = 72 - (\dim \prod_{i=1}^4 \mathrm{GL}(E) - 3) = 72 - 33 = 39$ .

# Mirror candidate for $X_1$

Using the mirror Ansatz, we propose mirror candidates for  $X_1$  and  $X_2$ .

- There is a  $H = \mathbb{Z}/3$ -action on  $E$  defined by  $e_i \mapsto \omega^i e_i$ .
- Another  $\mathbb{Z}/3$ -action  $e_i \mapsto e_{i+1}$ .
- Extends to actions on  $\mathbb{P}(E \otimes E \oplus E \otimes E) = \mathbb{P}^{17}$ .
- Choose invariant  $\mathbb{P}^{11}$ : defined by

$$f_{ij}^\alpha = e_{ij}^\alpha + t_{-i-j}^\alpha e_{-i-j, -i-j}^{\alpha+1}$$

for  $i, j \in \mathbb{Z}/3 \times \mathbb{Z}/3$  ( $i \neq j$ ) and  $\alpha = 0, 1$ .

- The resulting  $X_{H_t} \stackrel{\Delta}{=} \mathbb{P}^{11} \cap M$  is singular with 48 isolated double point singularities.

## Mirror candidate for $X_1$

- We divide out by the  $H$ -action and resolve:  $X_1^\circ = \widetilde{X_{H_t}}/H$ .
- Roan's formula gives:

$$\chi(X_1^\circ) = \frac{1}{3} (24 + 8 \cdot 24) = 72.$$

Based on this calculation and the mirror heuristic, we conjecture:

### Conjecture

$X_1^\circ$  is a mirror of  $X_1$ .

### Remark

A very similar construction gives a mirror candidate for  $X_2$ .

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