



UiO : **Department of Mathematics**  
University of Oslo

# Join of hexagons and Calabi–Yau threefolds

Public defence

**Fredrik Meyer**

**November 8, 2017**

# Outline of the thesis

- Unsuccessful attempts<sup>s?</sup> to find new hyper-Kähler varieties.
- The topology of  $C(dP_6)$ .
- New Calabi–Yau varieties and potential mirror partners.

# Calabi–Yau manifolds

## Definition

A **Calabi–Yau variety** is a smooth projective scheme  $X/\mathbb{C}$  of dimension 3 satisfying:

- $H^0(X, \mathcal{O}_X) = H^3(X, \mathcal{O}_X) = k$  and  $h^1(X, \mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$ .
- The canonical sheaf is trivial:  $\omega_X \simeq \mathcal{O}_X$ .

- Easiest invariants are the Euler characteristic and the Hodge numbers.
- We always have  $\chi = 2(h^{11} - h^{12})$ .

The diagram shows the 27 Hodge numbers  $h^{p,q}$  for a 3-dimensional Calabi-Yau manifold. The numbers are arranged in a hexagonal pattern. The top row has  $h^{00}$ ,  $h^{01}$ ,  $h^{10}$ ,  $h^{02}$ ,  $h^{11}$ ,  $h^{20}$ ,  $h^{03}$ ,  $h^{12}$ ,  $h^{21}$ ,  $h^{13}$ ,  $h^{22}$ ,  $h^{30}$ ,  $h^{23}$ ,  $h^{31}$ ,  $h^{32}$ ,  $h^{33}$ .

# Calabi–Yau manifolds

## Definition

A **Calabi–Yau variety** is a smooth projective scheme  $X/\mathbb{C}$  of dimension 3 satisfying:

- $H^0(X, \mathcal{O}_X) = H^3(X, \mathcal{O}_X) = k$  and  $h^1(X, \mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$ .
  - The canonical sheaf is trivial:  $\omega_X \simeq \mathcal{O}_X$ .
- 
- Easiest invariants are the Euler characteristic and the Hodge numbers.
  - We always have  $\chi = 2(h^{11} - h^{12})$ .

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{11} & & 0 & \\
 1 & & h^{12} & & h^{12} & & 1 \\
 & 0 & & h^{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & & 
 \end{array}$$

# Hodge numbers

The quintic  $X = V(f) \subset \mathbb{P}^4$  is the canonical example of a Calabi–Yau. It has Hodge numbers  $h^{11} = 1$  and  $h^{12} = 101$ .

## Remark (Heuristic)

The number  $h^{12}$  is the dimension of the “space of parameters” of  $X$ . The following heuristic will give us the correct Hodge number:

- The space of degree 5 polynomials  $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5))$  in  $\mathbb{P}^4$  is  $\binom{4+5}{5} = \binom{9}{4} = 126$ -dimensional. Hence  $\mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5))) = \mathbb{P}^{125}$ .
- This is not unique, but we can act by  $\mathrm{PGL}(5)$  to identify isomorphic quintics. We have  $\dim \mathrm{PGL}(5) = 25 - 1 = 24$ .
- In total:  $125 - 24 = 101$ , which is  $h^{12}(X)$ .

# Mirror symmetry

- Calabi–Yau threefolds seem to “always” have “mirror partners”.
- Mirror partner  $X^\circ$  to  $X$  has “mirrored Hodge diamond”.
- Hence  $\chi(X^\circ) = -\chi(X)$ .

| $X$   |     |   |     |   |   |
|-------|-----|---|-----|---|---|
| <hr/> |     |   |     |   |   |
|       |     | 1 |     |   |   |
|       | 0   |   | 0   |   |   |
| 0     |     | 1 |     | 0 |   |
| 1     | 101 |   | 101 |   | 1 |
|       | 0   | 1 |     | 0 |   |
|       | 0   |   | 0   |   |   |
|       |     | 1 |     |   |   |

# Mirror symmetry

- Calabi–Yau threefolds seem to “always” have “mirror partners”.
- Mirror partner  $X^\circ$  to  $X$  has “mirrored Hodge diamond”.
- Hence  $\chi(X^\circ) = -\chi(X)$ .

| $X$ |   |     |   |     |   |
|-----|---|-----|---|-----|---|
|     |   | 1   |   |     |   |
|     | 0 |     | 0 |     |   |
|     | 0 | 1   |   | 0   |   |
| 1   |   | 101 |   | 101 | 1 |
|     | 0 | 1   |   | 0   |   |
|     | 0 |     | 0 |     |   |
|     |   | 1   |   |     |   |

| $X^\circ$ |   |     |   |   |   |
|-----------|---|-----|---|---|---|
|           |   | 1   |   |   |   |
|           | 0 |     | 0 |   |   |
|           | 0 | 101 |   | 0 |   |
| 1         |   | 1   |   | 1 | 1 |
|           | 0 | 101 |   | 0 |   |
|           | 0 |     | 0 |   |   |
|           |   | 1   |   |   |   |

# The orbifold heuristic

Sometimes the following method produces a mirror manifold of a Calabi–Yau  $X$ :

- 1 Suppose  $X$  has a natural degeneration  $X_0$  with a finite automorphism group  $G$ .
- 2 Find a family  $\pi: \mathcal{X} \rightarrow S$  on which  $G$  act, and such that the general fiber  $X_t$  has only isolated singularities.
- 3 There might be a finite subgroup  $H$  of the big torus acting. A mirror candidate is then a crepant resolution of  $X_t/H$ .



# The orbifold heuristic

Sometimes the following method produces a mirror manifold of a Calabi–Yau  $X$ :

- 1 The general quintic degenerates to the singular scheme  $V(x_0x_1x_2x_3x_4)$ .
- 2 Find a family  $\pi: \mathcal{X} \rightarrow S$  on which  $G$  act, and such that the general fiber  $X_t$  has only isolated singularities.
- 3 There might be a finite subgroup  $H$  of the big torus acting. A mirror candidate is then a crepant resolution of  $X_t/H$ .

# The orbifold heuristic

Sometimes the following method produces a mirror manifold of a Calabi–Yau  $X$ :

- 1 The general quintic degenerates to the singular scheme  $V(x_0x_1x_2x_3x_4)$ .
- 2 The family defined by  $f_t = x_0x_1x_2x_3x_4 + t \sum_{i=1}^5 x_i^5$  is  $S_5$ -invariant.
- 3 There might be a finite subgroup  $H$  of the big torus acting. A mirror candidate is then a crepant resolution of  $X_t/H$ .

# The orbifold heuristic

Sometimes the following method produces a mirror manifold of a Calabi–Yau  $X$ :

- 1 The general quintic degenerates to the singular scheme  $V(x_0x_1x_2x_3x_4)$ .
- 2 The family defined by  $f_t = x_0x_1x_2x_3x_4 + t \sum_{i=1}^5 x_i^5$  is  $S_5$ -invariant.
- 3 There is an action of  $H \triangleq (\mathbb{Z}/5)^5/(\mathbb{Z}/5)$  on  $X_t$ .  
Crepant resolutions of  $X_t/H$  exists, and is a mirror.

# The orbifold heuristic

Sometimes the following method produces a mirror manifold of a Calabi–Yau  $X$ :

- 1 The general quintic degenerates to the singular scheme  $V(x_0x_1x_2x_3x_4)$ .
- 2 The family defined by  $f_t = x_0x_1x_2x_3x_4 + t \sum_{i=1}^5 x_i^5$  is  $S_5$ -invariant.
- 3 There is an action of  $H \triangleq (\mathbb{Z}/5)^5/(\mathbb{Z}/5)$  on  $X_t$ .  
Crepan resolutions of  $X_t/H$  exists, and is a mirror.

We can use **Roan's formula** to compute the Euler characteristic:

Theorem (Roan's formula)

$$\chi(\widetilde{X_t/H}) = \frac{1}{|H|} \sum_{g,h \in H} \chi(X_t^g \cap X_t^h)$$

# The cone over $dP_6$

- Let  $dP_6 \subset \mathbb{P}^6$  be an anticanonically embedded del Pezzo surface of degree 6. Let  $C(dP_6)$  be its affine cone in  $\mathbb{A}^7$ .
- The equations are

$$\begin{vmatrix} y & x_1 & x_2 \\ x_4 & y & x_3 \\ x_5 & x_6 & y \end{vmatrix} \leq 1.$$

The origin is an isolated singularity.

- There are two smoothing components.
- They come from perturbations of different forms of writing the equation.
- Can also write the equations as:

$$\begin{matrix} a_{010} & & a_{110} \\ \square & & \\ a_{000} & & a_{100} \end{matrix} \leq 1$$

# The two smoothing components

TODO: trengs denne?

# Construction of a new Calabi–Yau: $X_1$

- Let  $E$  be a vector space with basis  $e_1, e_2, e_3$ . Consider

$$\mathbb{P}^{17} = \mathbb{P}(E \otimes E \oplus E \otimes E).$$

The elements are pairs of  $3 \times 3$  matrices, equal up to scalar multiplication.

- Consider the set of pairs  $M$  of matrices  $(A, B)$  with rank  $1 + 1$ .
- Intersect  $M$  with a generic  $\mathbb{P}^{11} \subset \mathbb{P}^{17}$ . Let  $X_1 \triangleq M \cap \mathbb{P}^{11}$ .

## Theorem

$X_1$  is a smooth Calabi–Yau with Euler characteristic  $-72$ .

## Construction of a new Calabi–Yau: $X_2$

- Let  $F$  be a vector space with basis  $f_1, f_2$ . Consider

$$\mathbb{P}^{15} = \mathbb{P}(F \otimes F \otimes F \oplus F \otimes F \otimes ???).$$

The elements are pairs of  $2 \times 2 \times 2$ -tensors, equal up to scalar multiplication.

- Consider the set of pairs  $N$  of tensors  $(A, B)$  with rank  $1 + 1$ .
- Intersect  $N$  with a generic  $\mathbb{P}^{11} \subset \mathbb{P}^{15}$ . Let  $X_2 \triangleq N \cap \mathbb{P}^{11}$ .

### Theorem

$X_2$  is a smooth Calabi–Yau with Euler characteristic  $-48$ .



## Construction of a new Calabi–Yau: $X_3$

- Let  $E$  and  $F$  be as before. Consider

$$\mathbb{P}^{16} = \mathbb{P}(E \otimes E \oplus F \otimes F \otimes \text{????}).$$

- Consider the set of pairs  $W$  of tensors  $(A, B)$  with rank  $1 + 1$ .
- Intersect  $W$  with a generic  $\mathbb{P}^{11} \subset \mathbb{P}^{16}$ . Let  $X_3 \triangleq W \cap \mathbb{P}^{11}$ .

### Theorem

$X_3$  is a smooth Calabi–Yau with Euler characteristic  $-60$ .

# Hodge number heuristics

## Conjecture

$X_1$  has Hodge numbers  $h^{11} = 3$  and  $h^{12} = 39$ .

## “Reason”.

- 1 We know the Euler characteristic, so it is enough to find  $h^{12}$  (= number of parameters).

# Hodge number heuristics

## Conjecture

$X_1$  has Hodge numbers  $h^{11} = 3$  and  $h^{12} = 39$ .

## “Reason”.

- 1 We know the Euler characteristic, so it is enough to find  $h^{12}$  (= number of parameters).
- 2 The Grassmannian of  $\mathbb{P}^{11}$ 's in  $\mathbb{P}^{17}$  is 72-dimensional.

# Hodge number heuristics

## Conjecture

$X_1$  has Hodge numbers  $h^{11} = 3$  and  $h^{12} = 39$ .

## “Reason”.

- 1 We know the Euler characteristic, so it is enough to find  $h^{12}$  (= number of parameters).
- 2 The Grassmannian of  $\mathbb{P}^{11}$ 's in  $\mathbb{P}^{17}$  is 72-dimensional.
- 3 We can act by automorphism from  $\prod_{i=1}^4 \mathrm{GL}(E)$ .

# Hodge number heuristics

## Conjecture

$X_1$  has Hodge numbers  $h^{11} = 3$  and  $h^{12} = 39$ .

## “Reason”.

- 1 We know the Euler characteristic, so it is enough to find  $h^{12}$  (= number of parameters).
- 2 The Grassmannian of  $\mathbb{P}^{11}$ 's in  $\mathbb{P}^{17}$  is 72-dimensional.
- 3 We can act by automorphism from  $\prod_{i=1}^4 \mathrm{GL}(E)$ .
- 4 The subgroup  $\{t_1 t_2 = t_3 t_4\} \subset (\mathbb{C}^*)^4 \subset \prod_{i=1}^4 \mathrm{GL}(E)$  acts trivially.

# Hodge number heuristics

## Conjecture

$X_1$  has Hodge numbers  $h^{11} = 3$  and  $h^{12} = 39$ .

## “Reason”.

- 1 We know the Euler characteristic, so it is enough to find  $h^{12}$  (= number of parameters).
- 2 The Grassmannian of  $\mathbb{P}^{11}$ 's in  $\mathbb{P}^{17}$  is 72-dimensional.
- 3 We can act by automorphism from  $\prod_{i=1}^4 \mathrm{GL}(E)$ .
- 4 The subgroup  $\{t_1 t_2 = t_3 t_4\} \subset (\mathbb{C}^*)^4 \subset \prod_{i=1}^4 \mathrm{GL}(E)$  acts trivially.
- 5 Hence  $h^{12} = 72 - \left( \dim \prod_{i=1}^4 \mathrm{GL}(E) - 3 \right) = 72 - 33 = 39$ . ■

# Mirror candidates

Using the mirror Ansatz, we propose mirror candidates for  $X_1$  and  $X_2$ .

■ BB

UiO : **Department of Mathematics**  
University of Oslo



**Fredrik Meyer**



**Join of hexagons and  
Calabi–Yau threefolds**  
Public defence

