UiO : Department of Mathematics
University of Oslo

Join of hexagons and Calabi–Yau threefolds

Public defence

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  - Found by smoothing a certain Stanley–Reisner sphere.

I will now focus on the last point — the construction of the new Calabi–Yau's.

Stanley–Reisner schemes

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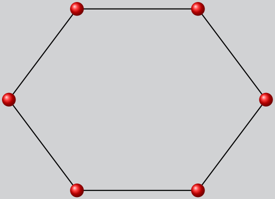
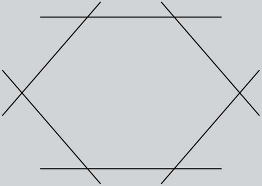
Example

From a simplicial complex to a union of $\mathbb{P}^1$'s.





The ideal is generated by $x_i x_{i+2} = x_i x_{i+3} = 0$ $(i = 0, \ldots, 5)$.

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Smoothings of Stanley–Reisner schemes:

- Given a basis for $T^1(S_{\mathbb{P}(\mathcal{K})}/k, S_{\mathbb{P}(\mathcal{K})})_0$, we can try to find a smoothing of $X_0 = \mathbb{P}(\mathcal{K})$.

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- Given a basis for $T^1(S_{\mathbb{P}(\mathcal{K})}/k, S_{\mathbb{P}(\mathcal{K})})_0$, we can try to find a smoothing of $X_0 = \mathbb{P}(\mathcal{K})$.
- A smoothing X of X_0 will have many of the same properties:
  - The same Hilbert polynomial.
  - By semicontinuity, if X_0 is a sphere, X will be Calabi–Yau.

Calabi–Yau varieties

Definition

A Calabi–Yau variety is an irreducible, smooth, projective scheme $X/\mathbb{C}$ of dimension 3 satisfying:

- $H^0(X, \mathscr{O}_X) = H^3(X, \mathscr{O}_X) = \mathbb{C}$ and $H^1(X, \mathscr{O}_X) = H^2(X, \mathscr{O}_X) = 0$.
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- Easiest invariants are the Euler characteristic and the Hodge numbers, $h^{ij} = h^j\left(X, \Omega^i_{X/\mathbb{C}}\right)$.

$$\begin{array}{ccccccc}
 & & & h^{00} & & & \\
 & & h^{01} & & h^{10} & & \\
 & h^{02} & & h^{11} & & h^{20} & \\
h^{03} & & h^{12} & & h^{21} & & h^{30} \\
 & h^{13} & & h^{22} & & h^{31} & \\
 & & h^{23} & & h^{32} & & \\
 & & & h^{33} & & &
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$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{11} & & 0 & \\
1 & & h^{12} & & h^{12} & & 1 \\
 & 0 & & h^{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
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Hodge numbers heuristic

The quintic $X \stackrel{\Delta}{=} V(f) \subset \mathbb{P}^4$ is the canonical example of a Calabi–Yau. It has Hodge numbers $h^{11} = 1$ and $h^{12} = 101$.

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- The space of degree 5 polynomials $H^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(5))$ in $\mathbb{P}^4$ is $\binom{4+5}{4} = \binom{9}{4} = 126$-dimensional. Hence $\mathbb{P}(H^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(5))) = \mathbb{P}^{125}$.

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- This is not unique, but we can act by $\mathrm{PGL}(5)$ to identify isomorphic quintics. We have $\dim \mathrm{PGL}(5) = 25 - 1 = 24$.
- In total: $125 - 24 = 101$, which is $h^{12}(X)$. ■

Mirror symmetry

- Calabi–Yau threefolds seem to “always” have “mirror partners”.

$$\begin{array}{ccccccc} & & & X & & & \\ \hline & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

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- Mirror partner X° to X has "mirrored Hodge diamond".

			X			
			1			
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	0		1		0	
1		101		101		1
	0		1		0	
		0		0		
			1			

			X°			
			1			
		0		0		
	0		101		0	
1		1		1		1
	0		101		0	
		0		0		
			1			

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- Mirror partner X° to X has “mirrored Hodge diamond”.
- Hence $\chi(X^\circ) = -\chi(X)$.

$$X$$

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
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The mirror construction Ansatz

Sometimes the following method produces a mirror manifold of a Calabi–Yau X:

1. Suppose X has a natural degeneration X_0 with a finite automorphism group G.
2. Find a family $\pi\colon \mathscr{X} \to S$ on which G acts, and such that the general fiber X_t has only isolated singularities.
3. There might be a finite subgroup H of the big torus acting. A mirror candidate is then a crepant resolution of X_t/H.

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We can use Roan's formula to compute the Euler characteristic:

Theorem (Roan's formula)

$$\chi\left(\widetilde{X_t/H}\right) = \frac{1}{|H|}\sum_{g,h\in H}\chi\left(X_t^g \cap X_t^h\right)$$

The cone over dP_6

- Let $\mathrm{dP}_6 \subset \mathbb{P}^6$ be an anticanonically embedded del Pezzo surface of degree 6. Let $C(\mathrm{dP}_6)$ be its affine cone in $\mathbb{A}^7$.
- The equations are

$$\begin{vmatrix} y & x_1 & x_2 \\ x_4 & y & x_3 \\ x_5 & x_6 & y \end{vmatrix} \leq 1 .$$

The origin is an isolated singularity.

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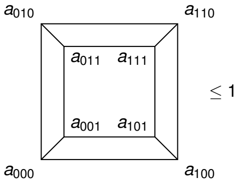

The two smoothing compoments of dP_6

We can identify one component with $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2}) \setminus H$, where $\mathcal{T}_{\mathbb{P}^2}$ is the tangent bundle of $\mathbb{P}^2$.

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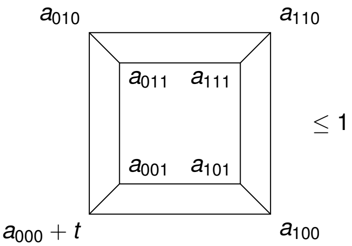
The second component can be identified with $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \setminus H$, where H is a $(1, 1, 1)$-divisor.



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$$\mathbb{P}^{17} = \mathbb{P}\big((E \otimes E) \oplus (E \otimes E)\big).$$

The elements are pairs of 3×3 matrices, equal up to scalar multiplication.

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Theorem

X_1 is a smooth Calabi–Yau with Euler characteristic -72.

Construction of a new Calabi–Yau: X_2

- Let F be a vector space with basis f_1, f_2. Consider

$$\mathbb{P}^{15} = \mathbb{P}\big((F \otimes F \otimes F) \oplus (F \otimes F \otimes F)\big).$$

  The elements are pairs of $2 \times 2 \times 2$-tensors, equal up to scalar multiplication.
- Consider the set N of pairs of tensors (A, B) with rank $1 + 1$.
- Intersect N with a generic $\mathbb{P}^{11} \subset \mathbb{P}^{15}$. Let $X_2 \triangleq N \cap \mathbb{P}^{11}$.

Theorem

X_2 is a smooth Calabi–Yau with Euler characteristic -48.

Construction of a new Calabi–Yau: X_3

- Let E and F be as before. Consider

$$\mathbb{P}^{16} = \mathbb{P}\big((E \otimes E) \oplus (F \otimes F \otimes F)\big).$$

- Consider the set W of pairs of tensors (A, B) with rank $1 + 1$.
- Intersect W with a generic $\mathbb{P}^{11} \subset \mathbb{P}^{16}$. Let $X_3 \triangleq W \cap \mathbb{P}^{11}$.

Theorem

X_3 is a smooth Calabi–Yau with Euler characteristic -60.

Hodge number heuristics

Conjecture

X_1 has Hodge numbers $h^{11} = 3$ and $h^{12} = 39$.

"Reason".

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4. The subgroup $\{t_1 t_2 = t_3 t_4\} \subset (\mathbb{C}^*)^4 \subset \prod_{i=1}^4 \mathrm{GL}(E)$ acts trivially.
5. Hence $h^{12} = 72 - \left(\dim \prod_{i=1}^4 \mathrm{GL}(E) - 3\right) = 72 - 33 = 39.$ ∎

Mirror candidate for X_1

Using the mirror Ansatz, we propose mirror candidates for X_1 and X_2.

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- Extend to actions on $\mathbb{P}\big((E \otimes E) \oplus (E \otimes E)\big) = \mathbb{P}^{17}$.
- Choose invariant $\mathbb{P}^{11}$: Defined by

$$f_{ij}^{\alpha} = e_{ij}^{\alpha} + t_{-i-j}^{\alpha} e_{-i-j,-i-j}^{\alpha+1}$$

for $i, j \in \mathbb{Z}/3 \times \mathbb{Z}/3$ ($i \neq j$) and $\alpha = 0, 1$.

Mirror candidate for X_1

Using the mirror Ansatz, we propose mirror candidates for X_1 and X_2.

- There is an $H \stackrel{\Delta}{=} \mathbb{Z}/3$-action on E defined by $e_i \mapsto \omega^i e_i$.
- Another H-action $e_i \mapsto e_{i+1}$.
- Extend to actions on $\mathbb{P}\big((E \otimes E) \oplus (E \otimes E)\big) = \mathbb{P}^{17}$.
- Choose invariant $\mathbb{P}^{11}$: Defined by

  $$f_{ij}^{\alpha} = e_{ij}^{\alpha} + t_{-i-j}^{\alpha} e_{-i-j,-i-j}^{\alpha+1}$$

  for $i, j \in \mathbb{Z}/3 \times \mathbb{Z}/3$ $(i \neq j)$ and $\alpha = 0, 1$.
- The resulting $X_{H_t} \stackrel{\Delta}{=} \mathbb{P}^{11} \cap M$ is singular with 48 isolated double point singularities.

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$$\chi(X_1^\circ) = \frac{1}{3}(24 + 8 \cdot 24) = 72.$$

Based on this calculation and the mirror heuristic, we conjecture:

Conjecture

X_1° is a mirror of X_1.

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Remark

A very similar construction gives a mirror candidate for X_2.

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Join of hexagons and Calabi–Yau threefolds

Public defence