Prym varieties — basic constructions and ~~applications~~ central results

1

- First remind the audience about abelian varieties.
- They got their name from Niels Henrik Abel (1802–1829) who studied integrals of the form

$$\int_\gamma \frac{dz}{\sqrt{f(z)}}$$

where γ is a path in $\mathbb{C}$ and f is a polynomial w/ distinct roots.

- ~~In general~~, If $d=3$ or 4, the integral can be solved using elliptic functions. If however $d \geq 5$, no integration is known. The problem is the multi-valuedness of $\omega = \frac{dz}{\sqrt{f(z)}}$. This is remedied by instead integrating on the associated Riemann surface $C = \{w^2 = f(z)\} \subseteq \mathbb{C}^2$.

$$\begin{array}{c} C \\ {\scriptstyle 2:1}\downarrow \\ \mathbb{C} \end{array}$$

ω holomorphic here

- The strategy is this: Let $\omega_i = z^{i-1}\frac{dz}{\sqrt{f(z)}}$, $i = 1, \ldots, g = \left\lfloor \frac{d-1}{2} \right\rfloor$ be a basis for the differentials $H^0(C, \omega_C)$ on C.

Fix a point $P_0 \in C$, and consider the map ②

$$p \longmapsto \left(\int_{P_0}^{P} \omega_1 , \ldots , \int_{P_0}^{P} \omega_g \right)$$

defined on a small $U \ni P_0$. We cannot extend this to a map on all of C, because the value depend on the path chosen.

However, the value is unique modulo closed paths, that is, modulo $H_1(C, \mathbb{Z})$. Thus we get a well-defined map

[Albanese] $$C \xrightarrow{\alpha} H^0(C, \omega_C)^* \big/ H_1(C, \mathbb{Z}) = \mathbb{C}^g \big/ H_1(C, \mathbb{Z}) \overset{\text{top.}}{\cong} (S^1)^{2g}$$

The space $H^0(C, \omega_C)^* / H_1(C, \mathbb{Z})$ is the <u>Jacobian</u> of C.

It is a complex torus, which can be shown to be algebraic as well. Thus it is an example of a <u>compact algebraic group variety</u>, which means that it is an <u>abelian variety</u>. ← def

- Not all abelian varieties are Jacobians, however.
- In general, an abelian variety over $\mathbb{C}$ can be described as $X = \mathbb{C}^n / \Lambda$, where Λ is a lattice of full rank (meaning $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}^n$) such that Λ satisfies the <u>Riemann relations</u>: let Π be a $g \times 2g$-matrix s.t. $\Lambda = \Pi \mathbb{Z}^{2g}$.

Then X is abelian iff $\exists\, A \in M_{2g}(\mathbb{Z})$ s.t.
① A non-deg, alternating
② $\Pi (A^{-1})^{T} \Pi^{T} = 0$
③ $i\, \Pi (A^{-1})^{T} \bar{\Pi}^{T} \geq 0$

- Can show that X then is projective.
- By linear algebra, we can define the dual abelian variety $\hat{X}$ to be ~~$\left(H^0(C, \omega_C)^{**} \big/ H_1(C,\mathbb{Z}) \right)^* =$~~ $\mathbb{C}^{*} / H_1(C,\mathbb{Z})$.

This can be identified with $\mathrm{Pic}^0(X) = \{$line bundles $\mathcal{L}$ w/ $c_1(\mathcal{L}) = 0\}/_{\text{iso}}$

means topologically trivial

algebraic def

$= \{$ inv. sheaves $\mathcal{L}$ s.t. $t_a^* \mathcal{L} \approx \mathcal{L}$ $\forall a \in X \}$ (assume $k = \bar{k}$)

$\left(= \{ L \in \mathrm{Pic}\, X \mid \varphi_L = L \} \right)$

- Given an invertible sheaf on X $\mathcal{L}$ we get a map $X \xrightarrow{\varphi_{\mathcal{L}}} \hat{X}$ given by $X \ni x \longmapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1} \in \mathrm{Pic}^0(X) = \hat{X}$
- A polarization on X is a morphism $p : X \longrightarrow \hat{X}$ s.t. $p = \varphi_{\mathcal{L}}$ for some ample $\mathcal{L}$. It is principal if it is an isomorphism. This implies that $H^0(X, \mathcal{L}) = \vartheta\, \mathbb{C}$. Then divisor of zeroes$(\vartheta)$ is the associated Θ divisor.

- Principally polarized abelian var's (pp av's) are nice because they have finite automorphism groups $\Rightarrow$ well suited for moduli problems.

More about Jacobians

- Recall that $J(C) = \dfrac{H^0(C, \omega_C)^\vee}{H_1(C, \mathbb{Z})} = \mathrm{Pic}^0(C)$.

The intersection form $H_1(C,\mathbb{Z}) \otimes H_1(C,\mathbb{Z}) \to \mathbb{Z}$ is unimodular and positive definite, and induces a principal polarization P_C on $J(C)$ and hence a θ-divisor $\Theta_C \subseteq J(C)$.

Geometrically: Consider the Abel-Jacobi morphism

Let $c \in C$.

considered as divisors on C.

$$\begin{aligned} \varphi_d : C^{(d)} &\longrightarrow J(C) \\ (x_1, \dots, x_d) &\longmapsto [x_1 + \dots + x_d - dc] \end{aligned}$$

Thm (Riemann-Kempf) ① $d \geq g$: φ_d surjective

② $d = g-1$: $\varphi_{g-1}(C^{(g-1)}) \subseteq J(C)$ is the θ (up til translation)

IF TIME

Universal property of Jacobians:

Assume $C \xrightarrow{\varphi} X$ sends $c \in C$ to 0.
abelian variety

"abelianization"

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\alpha_c"'] & X \\ J(C) \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

Corollary: Every abelian X is a quotient of $J(C)$ for some C.

Pf ① By Bertini: exists smooth $C \subseteq X$.

② $C \hookrightarrow X$ induces $J(C) \xrightarrow{f} X$ by universal property.

③ Show f is surjective.

Prym varieties

- ~~PAKA~~ Start w/ a étale double cover

$$\tilde{C} \xrightarrow{\pi} C$$

Note
On C then genus $g+1$,
So then $\tilde{C}$ genus $2g+1$.

- ~~(whatever)~~

Apply the $J(-)$ Jacobian functor to get a morphism

$$J(\tilde{C}) \xrightarrow{Nm} J(C).$$

$$\mathcal{O}_{\tilde{C}}(\tilde{D}) \mapsto \mathcal{O}_C(\pi_* \tilde{D})$$

(or equiv. $\sum a_i P_i \mapsto \sum a_i \pi(P_i)$)

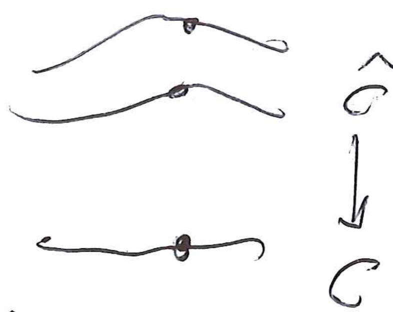
The kernel $\ker Nm$ has two connected components, and

$$Prym = (\ker Nm)_0 .$$

It is a theorem that this is also equal to $Im(1-\tilde{\sigma})$, where σ is the involution $\tilde{C} \to \tilde{C}$ interchanging the sheets of the covering.

"odd part of $Jac(\tilde{C})$"

Prop $(\ker Nm)_0 = Im(1-\sigma)$



Pf One direction is easy.

Suppose $D = \sum a_i P_i - \sum a_i \overline{P_i}$ is in the image of σ. Then $\pi_* D = 0$ since $P_i, \overline{P_i} \to P_i$.

$\boxtimes$

We also have the following

<u>Fact</u> The induced polarization on P is twice a principal polarization.

<u>Note</u> The <u>induced polarization</u> comes from the line bundle $i^*\Theta$, where $i : P \hookrightarrow J(\tilde{C})$ is the inclusion.

<u>"Analytic/topological" proof"</u> (from "Complex Abelian Varieties" by Lange–Birkenhake)

Recall the description of $J(\tilde{C})$ as $H^0(\tilde{C}, \omega_{\tilde{C}})^* / H_1(\tilde{C}, \mathbb{Z})$.

In these terms, we have

$$\text{Prym } P = \frac{\left(H^0(\tilde{C}, \omega_{\tilde{C}})^-\right)^*}{H_1(\tilde{C}, \mathbb{Z})^-}$$

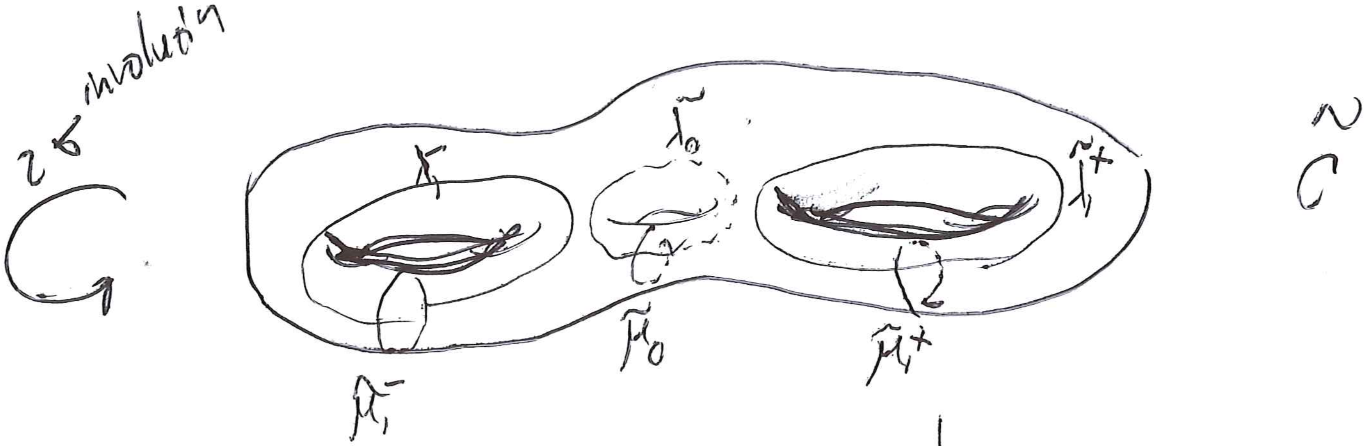
where $(-)^-$ denotes the (-1)-eigenspace of the induced action.

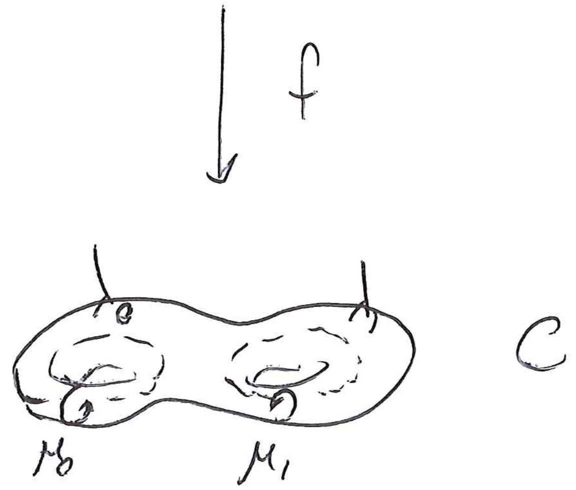
We want to compute the induced polarization from this topological description.

(1) Choose a symplectic basis $\lambda_0, \lambda_1, \dots, \lambda_g$, $\mu_1, \dots, \mu_g$ of $H_1(C, \mathbb{Z})$.

(i.e. s.t. $\lambda_i \cdot \lambda_j = 0 = \mu_i \cdot \mu_j$ and $\lambda_i \cdot \mu_j = \delta_{ij}$.)

(2) A degree 2 cover is determined by a cycle in $H_1(C, \mathbb{Z}/2)$.





A symplectic basis for $H_1(\tilde C, \mathbb{Z})$ is $\tilde\mu_i^-$, μ_0, μ_0, $\tilde\mu_i^+$

(so $\tilde C$ have genus $2g+1$)

A basis for the (-1)-eigenspace $H_1(C, \mathbb{Z})^-$ is then:

$$\alpha_i = \lambda_i^+ - \lambda_i^-, \quad \beta_i = \mu_i^+ - \mu_i^- \quad i = 1, \dots, g.$$

The restriction of $E: H_1(\tilde C, \mathbb{Z}) \times H_1(\tilde C, \mathbb{Z}) \to \mathbb{Z}$ to the basis $\{\alpha_i, \beta_i\}$ is then

$$E(\alpha_i, \beta_j) = 2\delta_{ij} \quad E(\alpha_i, \alpha_j) = E(\beta_i, \beta_j) = 0.$$

... so the induced polarization is twice a principal polarization.

[Rk More is true: there is a stronger relation between $\tilde{\Theta}$ and Ξ. Known as the Schottky-Jung-relations.

- From this we also see that $\dim P = g$.

We have now (finally) defined Prym varieties. Let us say a few words about ~~their~~ some central results in their theory.

- Mumford studied the singularities of the Θ-divisor Ξ on P. He finds that

$$\dim \mathrm{Sing}(\Xi) \geq g-6.$$

Maps between moduli spaces

- Let $\mathcal{A}_g$ denote the moduli space of principally polarized abelian varieties of dim g. It is a complex-analytic ~~quasi-projective~~ space of dimension $\frac{g(g+1)}{2}$.
- Let M_g be the moduli space of curves of ~~degree~~ genus g.
- Let R_g be the space of double covers $\left\{ \begin{array}{c} \tilde{C} \\ \downarrow 2:1 \\ C \end{array} \right\}$ where C has genus g. Equivalently it is $\{ \text{pairs } [C, \eta] \mid \eta \in \mathrm{Pic}(C) \setminus \{0\}, \eta^{\otimes 2} = \mathcal{O}_C \}$

(relative Spec-construction) ...... finite covering of degree $2^{2g}-1$.

- Then we get maps

$$\begin{tikzcd} & R_g \arrow[dl, "\pi"'] \arrow[dr, "Pr_g"] & \\ M_g & & \mathcal{A}_{g-1} \end{tikzcd}$$

The structure of the map

$$Pr_g : R_g \longrightarrow A_{g-1}$$

is very interesting. (analogue of the Torelli map $J : M_g \to A_g$)

<u>We have</u>

"Generic Torelli" ~~For~~ $g \geq 7$, the Prym map is generically injective (injective on a dense open).

It is however <u>never</u> injective, because of the following <u>tetragonal construction</u>. (recall: a curve is tetragonal if it admits a g^1_4, i.e. a 4:1 morphism $C \longrightarrow \mathbb{P}^1$.)

The tetragonal construction associates to any double covering $\begin{matrix}\tilde{C}\\ \downarrow\\ C\end{matrix}$ (tetragonal), two other coverings with the same Prym. We sketch quickly the construction: start w/ a double covering $\tilde{C} \to C$, where C is tetragonal. Consider the symmetric product (4 times) of $\tilde{C}$ with itself:

$$\text{unordered 4-tuples} = \tilde{C}^{(4)} \subseteq Pic^4(\tilde{C})$$

Then look at the commutative diagram

$$\begin{tikzcd}
\{(D,p) \mid f^{(4)}(Nm(D)) = \tilde{f}(p)\} \arrow[r] \arrow[dd, "\tilde{f}"'] \arrow[ddr, phantom, "\ulcorner"] & \tilde{C}^{(4)} \arrow[d, "Nm"] \\
& C^{(4)} \arrow[d, "f^{(4)}"] \\
\mathbb{P}^1 \arrow[r, "\Delta"] & (\mathbb{P}^1)^{(4)} = \mathbb{P}^4
\end{tikzcd}$$

~~Then I claim the following (Donagi):~~

Note 1) The involution of $\tilde{C}$ acts on $f_*\tilde{C}$ as well.

2) There is an equivalence relation on $f_*\tilde{C}$:

$(D,p) \sim (D',p')$ if $\begin{cases} \#(D \cap D') \equiv 0 \mod 2 \\ p = p'. \end{cases}$

Then I claim (Donagi)

1) $\tilde{f} : f_*\tilde{C} = C_1 \sqcup C_2 \longrightarrow \mathbb{P}^1$ w/ C_i tetragonal

2) $C_i \xrightarrow{\pi_i} C_i/2$ is an unramified double covering.

Furthermore Donagi proves that the Pryms of these coverings are isomorphic.

THM

(so Torelli fails)

The special case $Pr_6 : R_6 \longrightarrow A_5$ is very interesting.

In this case one has $\dim R_6 = \dim A_5 = 15$, so the map

$3\cdot 6 - 3 \qquad \frac{5\cdot 6}{2}$

is generically finite. Its degree has been computed by Donagi-Smith: 27.

QUOTE Wake an algebraic geometer in the dead of night, whisper 27. Chances are, he will respond lines on a cubic surface.

11

The result is this: Two pairs $(\tilde{C}, C)$ and $(\tilde{C}', C')$ are incident if they are related by a tetragonal construction.

[The fibre of $\mathcal{P}_5$ has the structure of the 27 lines on a cubic surface.

- (quite amazing really)
- Gives restrictions on possible degenerations of the fibre.

Final application: irrationality of smooth cubic threefolds.

"Aside" • Schottky problem classify "Jacobian locus" in $\mathcal{A}_g$.

• Can ask same for Pryms.

[Shokurov: If ($g \neq 4$) If $\mathrm{Prym}(\tilde{C}, C) \in$ Jac-locus then C is hyperelliptic, trigonal or C is a plane quintic w/ $h^0(\mathcal{O}_C(1) \otimes \eta) = 0$.

Def Intermediate Jacobian (ad hoc) $X \rightsquigarrow H^{1,2}/H^3(X, \mathbb{Z}) \in \mathcal{A}_g$ (assume $H^{0,3} = 0$) $\boxed{\text{abelian varieties}}$

Thm Mumford: If $X \xrightarrow{\pi} \mathbb{P}^2$ is a conic bundle, then $J(X)$ is Prym. $J(\tilde{C}, C)$, where C is the discriminant curve of π.

These facts are enough to show the irrationality of the cubic threefold:

① Choose $l \subseteq X$ and blow up:
$$\begin{tikzcd} X_l \arrow[d] \\ X \end{tikzcd}$$

② The projection from l is a generic $\mathbb{P}^2$ define a morphism
$$\begin{tikzcd} X_l \arrow[d] \arrow[dr] & \\ X \arrow[r, dashed] & \mathbb{P}^2 \supseteq C \end{tikzcd}$$

It is a conic bundle.

③ Can show $JX = JX_l$

④ The covering $\tilde{C} \to C$, $[l] \mapsto [l]$ satisfies $h^0(\mathcal{O}_C(1) \otimes \eta) = 1$. (???)

⑤ By Mumford (Clemens) $h^0(\mathcal{O}_C(1) \otimes \eta) = 0$, we find that JX is not a Jacobian.

⑥ ~~By the~~ Clemens-Griffiths X rational $\Rightarrow$ $J(X)$ (product of) Jacobian.

X not rational.

(but it is unirational).

$\boxed{\text{FIN}}$