UiO : **Department of Mathematics**
University of Oslo

Join of hexagons and Calabi–Yau threefolds

Public defence

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Outline of the thesis

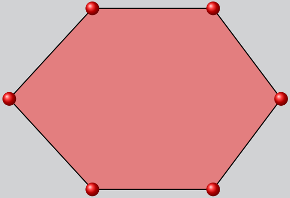
- Unsuccessful attempts? to find new hyper-Kähler varieties.
  - Strategy: Naïvely perturb monomial equations of Stanley–Reisner ideals of known triangulations of $\mathbb{CP}^2$.
- The topology of smoothings of $C(\mathrm{dP}_6)$.
  - Identify the smoothings as hyperplane complements in other spaces, and use exact sequences from algebraic topology.
- New Calabi–Yau varieties and potential mirror partners.
  - Found by smoothing a certain Stanley–Reisner sphere.

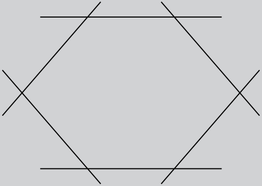
Stanley–Reisner schemes

- Given a simplicial complex $\mathcal{F}$, we get a Stanley–Reisner scheme $\mathbb{P}(\mathcal{F})$.
- Is a union of projective spaces $\mathbb{P}^{\dim f}$, where f is a face of $\mathcal{F}$.

Example

From a simplicial complex to a union of $\mathbb{P}^1$'s. Is the figure supposed to be asymmetric?





Stanley–Reisner schemes

- Join of two subschemes X and Y: The (closure of) the union of all lines between X and Y.
- Join of two Stanley–Reisner schemes $\mathbb{P}(\mathcal{F})$ and $\mathbb{P}(\mathcal{G})$ is $\mathbb{P}(\mathcal{F} * \mathcal{G})$, where the faces of $\mathcal{F} * \mathcal{G}$ are $f \sqcup g$ for $f \in \mathcal{F}$, $g \in \mathcal{G}$.

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Smoothings of Stanley–Reisner schemes

- Given a basis for $T^1(S_{\mathbb{P}(\mathcal{K})}, k)_0$, we can try to find a smoothing of $X_0 = \mathbb{P}(\mathcal{K})$.
- A smoothing X of X_0 will have many of the same properties:
  - The same Hilbert polynomial.
  - By semicontinuity, if X_0 is a sphere, X will be Calabi–Yau.

Calabi–Yau manifolds

Definition

A Calabi–Yau variety is a smooth projective scheme $X/\mathbb{C}$ of dimension 3 satisfying:

- $H^0(X, \mathscr{O}_X) = H^3(X, \mathscr{O}_X) = k$ and $h^1(X, \mathscr{O}_X) = h^2(\mathscr{O}_X) = 0$.
- The canonical sheaf is trivial: $\omega_X \simeq \mathscr{O}_X$.

- Easiest invariants are the Euler characteristic and the Hodge numbers.
- We always have $\chi = 2(h^{11} - h^{12})$.

$$
\begin{array}{ccccccc}
 & & & h^{00} & & & \\
 & & h^{01} & & h^{10} & & \\
 & h^{02} & & h^{11} & & h^{20} & \\
h^{03} & & h^{12} & & h^{21} & & h^{30} \\
 & h^{13} & & h^{22} & & h^{31} & \\
 & & h^{23} & & h^{32} & & \\
 & & & h^{33} & & &
\end{array}
$$

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$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{11} & & 0 & \\
1 & & h^{12} & & h^{12} & & 1 \\
 & 0 & & h^{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Hodge numbers

The quintic $X = V(f) \subset \mathbb{P}^4$ is the canonical example of a Calabi–Yau. It has Hodge numbers $h^{11} = 1$ and $h^{12} = 101$.

Remark (Heuristic)

The number h^{12} is the dimension of the "space of parameters" of X. The following heuristic will give us the correct Hodge number:

- The space of degree 5 polynomials $H^0\left(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(5)\right)$ in $\mathbb{P}^4$ is $\binom{4+5}{5} = \binom{9}{4} = 126$-dimensional. Hence $\mathbb{P}\left(H^0\left(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(5)\right)\right) = \mathbb{P}^{125}$.
- This is not unique, but we can act by $\mathrm{PGL}(5)$ to identify isomorphic quintics. We have $\dim \mathrm{PGL}(5) = 25 - 1 = 24$.
- In total: $125 - 24 = 101$, which is $h^{12}(X)$.

Mirror symmetry

- Calabi–Yau threefolds seem to “always” have “mirror partners”.
- Mirror partner X° to X has “mirrored Hodge diamond”.
- Hence $\chi(X^\circ) = -\chi(X)$.

$$
\begin{array}{ccccccc}
 & & & X & & & \\
\hline
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
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\end{array}
$$

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$$\begin{array}{ccccccc} & & & X & & & \\ \hline & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{ccccccc} & & & X^\circ & & & \\ \hline & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The orbifold heuristic

Sometimes the following method produces a mirror manifold of a Calabi–Yau X:

1. Suppose X has a natural degeneration X_0 with a finite automorphism group G.
2. Find a family $\pi \colon \mathscr{X} \to S$ on which G act, and such that the general fiber X_t has only isolated singularities.
3. There might be a finite subgroup H of the big torus acting. A mirror candidate is then a crepant resolution of X_t/H.

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   Crepant resolutions of X_t/H exists, and is a mirror.

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We can use Roan's formula to compute the Euler characteristic:

Theorem (Roan's formula)

$$\chi\left(\widetilde{X_t/H}\right) = \frac{1}{|H|}\sum_{g,h\in H}\chi\left(X_t^g \cap X_t^h\right)$$

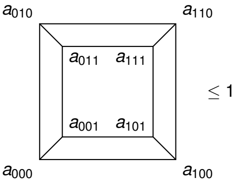
The cone over dP_6

- Let $\mathrm{dP}_6 \subset \mathbb{P}^6$ be an anticanonically embedded del Pezzo surface of degree 6. Let $C(\mathrm{dP}_6)$ be its affine cone in $\mathbb{A}^7$.
- The equations are

$$\begin{vmatrix} y & x_1 & x_2 \\ x_4 & y & x_3 \\ x_5 & x_6 & y \end{vmatrix} \leq 1.$$

The origin is an isolated singularity.

- There are two smoothing components.
- They come from perturbations of different forms of writing the equation.
- Can also write the equations as:



The two smoothing components

TODO: trengs denne?

Construction of a new Calabi–Yau: X_1

- Let E be a vector space with basis e_1, e_2, e_3. Consider

$$\mathbb{P}^{17} = \mathbb{P}(E \otimes E \oplus E \otimes E).$$

  The elements are pairs of 3×3 matrices, equal up to scalar multiplication.
- Consider the set of pairs M of matrices (A, B) with rank $1 + 1$.
- Intersect M with a generic $\mathbb{P}^{11} \subset \mathbb{P}^{17}$. Let $X_1 \triangleq M \cap \mathbb{P}^{11}$.

Theorem

X_1 is a smooth Calabi–Yau with Euler characteristic -72.

Construction of a new Calabi–Yau: X_2

- Let F be a vector space with basis f_1, f_2. Consider

$$\mathbb{P}^{15} = \mathbb{P}(F \otimes F \otimes F \oplus F \otimes F \otimes ????).$$

  The elements are pairs of $2 \times 2 \times 2$-tensors, equal up to scalar multiplication.
- Consider the set of pairs N of tensors (A, B) with rank $1 + 1$.
- Intersect N with a generic $\mathbb{P}^{11} \subset \mathbb{P}^{15}$. Let $X_2 \triangleq N \cap \mathbb{P}^{11}$.

Theorem

X_2 is a smooth Calabi–Yau with Euler characteristic -48.

Construction of a new Calabi–Yau: X_3

- Let E and F be as before. Consider

$$\mathbb{P}^{16} = \mathbb{P}(E \otimes E \oplus F \otimes F \otimes ????).$$

- Consider the set of pairs W of tensors (A, B) with rank $1 + 1$.
- Intersect W with a generic $\mathbb{P}^{11} \subset \mathbb{P}^{16}$. Let $X_3 \triangleq W \cap \mathbb{P}^{11}$.

Theorem

X_3 is a smooth Calabi–Yau with Euler characteristic -60.

Hodge number heuristics

Conjecture

X_1 has Hodge numbers $h^{11} = 3$ and $h^{12} = 39$.

“Reason”.

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5. Hence $h^{12} = 72 - \left(\dim \prod_{i=1}^{4} \mathrm{GL}(E) - 3\right) = 72 - 33 = 39.$ ∎

Mirror candidate for X_1

Using the mirror Ansatz, we propose mirror candidates for X_1 and X_2.

- There is an $H = \mathbb{Z}/3$-action on E defined by $e_i \mapsto \omega^i e_i$.
- Another $\mathbb{Z}/3$-action $e_i \mapsto e_{i+1}$.
- Extends to actions on $\mathbb{P}(E \otimes E \oplus E \otimes E) = \mathbb{P}^{17}$.
- Choose invariant $\mathbb{P}^{11}$: Defined by

$$f_{ij}^{\alpha} = e_{ij}^{\alpha} + t_{-i-j}^{\alpha} e_{-i-j,-i-j}^{\alpha+1}$$

  for $i, j \in \mathbb{Z}/3 \times \mathbb{Z}/3$ $(i \neq j)$ and $\alpha = 0, 1$.
- The resulting $X_{H_t} \stackrel{\Delta}{=} \mathbb{P}^{11} \cap M$ is singular with 48 isolated double point singularities.

Mirror candidate for X_1

- We divide out by the H-action and resolve: $X_1^\circ \overset{\Delta}{=} ? \widetilde{X_{H_t}/H}$.
- Roan's formula gives:

$$\chi(X_1^\circ) = \frac{1}{3}(24 + 8 \cdot 24) = 72.$$

Based on this calculation and the mirror heuristic, we conjecture:

Conjecture

X_1° is a mirror of X_1.

Remark

A very similar construction gives a mirror candidate for X_2.

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