



UiO : **Department of Mathematics**
University of Oslo

Join of hexagons and Calabi–Yau threefolds

Public defence

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Outline of the thesis

- Attempts to find new hyper-Kähler varieties.
 - Strategy: Naïvely perturb monomial equations of Stanley–Reisner ideals of known triangulations of \mathbb{CP}^2 . **Did not work.**

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 - Identify the smoothings as hyperplane complements in other spaces, and use exact sequences from algebraic topology.

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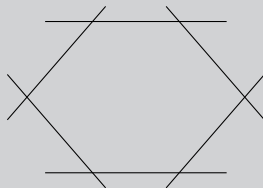
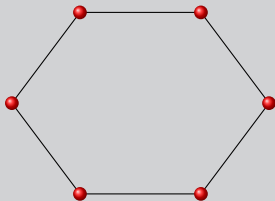
- Attempts to find new hyper-Kähler varieties.
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- The topology of smoothings of $C(dP_6)$.
 - Identify the smoothings as hyperplane complements in other spaces, and use exact sequences from algebraic topology.
- New Calabi–Yau varieties and potential mirror partners.
 - Found by smoothing a certain Stanley–Reisner sphere.

Stanley–Reisner schemes

- Given a simplicial complex \mathcal{F} , we get a Stanley–Reisner scheme $\mathbb{P}(\mathcal{F})$.
- Is a union of projective spaces $\mathbb{P}^{\dim f}$, where f is a face of \mathcal{F} .

Example

From a simplicial complex to a union of \mathbb{P}^1 's.



The ideal is generated by $x_i x_{i+2} = x_i x_{i+3} = 0$ ($i = 0, \dots, 5$).

Stanley–Reisner schemes

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Smoothings of Stanley–Reisner schemes:

- Given a basis for $T^1(S_{\mathbb{P}(\mathcal{K})}/k, S_{\mathbb{P}(\mathcal{K})}_0)$, we can try to find a smoothing of $X_0 = \mathbb{P}(\mathcal{K})$.
- A smoothing X of X_0 will have many of the same properties:
 - The same Hilbert polynomial.
 - By semicontinuity, if X_0 is a sphere, X will be Calabi–Yau.

Calabi–Yau varieties

Definition

A **Calabi–Yau variety** is an irreducible, smooth, projective scheme X/\mathbb{C} of dimension 3 satisfying:

- $H^0(X, \mathcal{O}_X) = H^3(X, \mathcal{O}_X) = \mathbb{C}$ and $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.
- The canonical sheaf is trivial: $\omega_X \simeq \mathcal{O}_X$.

- Easiest invariants are the Euler characteristic and the Hodge numbers, $h^{ij} = h^j(X, \Omega_{X/\mathbb{C}}^i)$.
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$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{11} & & 0 & \\
 1 & & h^{12} & & h^{12} & & 1 \\
 & 0 & & h^{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
 \end{array}$$

Hodge numbers

The quintic $X = V(f) \subset \mathbb{P}^4$ is the canonical example of a Calabi–Yau. It has Hodge numbers $h^{1,1} = 1$ and $h^{1,2} = 101$.

Remark (Heuristic)

The number $h^{1,2}$ is the dimension of the “space of parameters” of X . The following heuristic will give us the correct Hodge number:

- The space of degree 5 polynomials $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5))$ in \mathbb{P}^4 is $\binom{4+5}{5} = \binom{9}{4} = 126$ -dimensional. Hence $\mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5))) = \mathbb{P}^{125}$.

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Mirror symmetry

- Calabi–Yau threefolds seem to “always” have “mirror partners”.
- Mirror partner X° to X has “mirrored Hodge diamond”.
- Hence $\chi(X^\circ) = -\chi(X)$.

X					
<hr/>					
		1			
	0		0		
0		1		0	
1	101		101		1
	0	1		0	
	0		0		
		1			

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X°					
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	0	101		0	
1	1		1		1
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	0		0		
		1			

The orbifold heuristic

Sometimes the following method produces a mirror manifold of a Calabi–Yau X :

- 1 Suppose X has a natural degeneration X_0 with a finite automorphism group G .
- 2 Find a family $\pi: \mathcal{X} \rightarrow S$ on which G act, and such that the general fiber X_t has only isolated singularities.
- 3 There might be a finite subgroup H of the big torus acting. A mirror candidate is then a crepant resolution of X_t/H .

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We can use **Roan's formula** to compute the Euler characteristic:

Theorem (Roan's formula)

$$\chi(\widetilde{X_t/H}) = \frac{1}{|H|} \sum_{g,h \in H} \chi(X_t^g \cap X_t^h)$$

The cone over dP_6

- Let $dP_6 \subset \mathbb{P}^6$ be an anticanonically embedded del Pezzo surface of degree 6. Let $C(dP_6)$ be its affine cone in \mathbb{A}^7 .
- The equations are

$$\begin{vmatrix} y & x_1 & x_2 \\ x_4 & y & x_3 \\ x_5 & x_6 & y \end{vmatrix} \leq 1.$$

The origin is an isolated singularity.

- There are two smoothing components.
- They come from perturbations of different forms of writing the equation.
- Can also write the equations as:

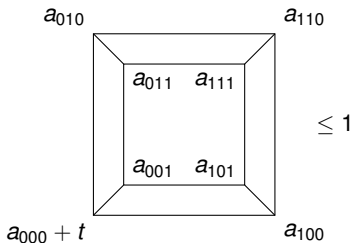
$$\begin{matrix} a_{010} & & a_{110} \\ \square & & \\ a_{011} & a_{111} & \\ \square & & \\ a_{001} & a_{101} & \\ \square & & \\ a_{000} & & a_{100} \end{matrix} \leq 1$$

The two smoothing components of dP_6

We can identify this component with $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2}) \setminus H$, where $\mathcal{T}_{\mathbb{P}^2}$ is the tangent bundle of \mathbb{P}^2 .

$$\begin{vmatrix} y & x_1 & x_2 \\ x_4 & y + t_1 & x_3 \\ x_5 & x_6 & y + t_2 \end{vmatrix} \leq 1$$

The second component can be identified with $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \setminus H$, where H is a $(1, 1, 1)$ -divisor.



Construction of a new Calabi–Yau: X_1

- Let E be a vector space with basis e_1, e_2, e_3 . Consider

$$\mathbb{P}^{17} = \mathbb{P}(E \otimes E \oplus E \otimes E).$$

The elements are pairs of 3×3 matrices, equal up to scalar multiplication.

- Consider the set of pairs M of matrices (A, B) with rank $1 + 1$.
- Intersect M with a generic $\mathbb{P}^{11} \subset \mathbb{P}^{17}$. Let $X_1 \triangleq M \cap \mathbb{P}^{11}$.

Theorem

X_1 is a smooth Calabi–Yau with Euler characteristic -72 .

Construction of a new Calabi–Yau: X_2

- Let F be a vector space with basis f_1, f_2 . Consider

$$\mathbb{P}^{15} = \mathbb{P}(F \otimes F \otimes F \oplus F \otimes F \otimes F).$$

The elements are pairs of $2 \times 2 \times 2$ -tensors, equal up to scalar multiplication.

- Consider the set of pairs N of tensors (A, B) with rank $1 + 1$.
- Intersect N with a generic $\mathbb{P}^{11} \subset \mathbb{P}^{15}$. Let $X_2 \triangleq N \cap \mathbb{P}^{11}$.

Theorem

X_2 is a smooth Calabi–Yau with Euler characteristic -48 .

Construction of a new Calabi–Yau: X_3

- Let E and F be as before. Consider

$$\mathbb{P}^{16} = \mathbb{P}(E \otimes E \oplus F \otimes F \otimes F).$$

- Consider the set of pairs W of tensors (A, B) with rank $1 + 1$.
- Intersect W with a generic $\mathbb{P}^{11} \subset \mathbb{P}^{16}$. Let $X_3 \triangleq W \cap \mathbb{P}^{11}$.

Theorem

X_3 is a smooth Calabi–Yau with Euler characteristic -60 .

Hodge number heuristics

Conjecture

X_1 has Hodge numbers $h^{11} = 3$ and $h^{12} = 39$.

“Reason”.

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- 4 The subgroup $\{t_1 t_2 = t_3 t_4\} \subset (\mathbb{C}^*)^4 \subset \prod_{i=1}^4 \mathrm{GL}(E)$ acts trivially.

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- 4 The subgroup $\{t_1 t_2 = t_3 t_4\} \subset (\mathbb{C}^*)^4 \subset \prod_{i=1}^4 \mathrm{GL}(E)$ acts trivially.
- 5 Hence $h^{12} = 72 - \left(\dim \prod_{i=1}^4 \mathrm{GL}(E) - 3 \right) = 72 - 33 = 39$. ■

Mirror candidate for X_1

Using the mirror Ansatz, we propose mirror candidates for X_1 and X_2 .

- There is an $H = \mathbb{Z}/3$ -action on E defined by $e_i \mapsto \omega^i e_i$.
- Another $\mathbb{Z}/3$ -action $e_i \mapsto e_{i+1}$.
- Extends to actions on $\mathbb{P}(E \otimes E \oplus E \otimes E) = \mathbb{P}^{17}$.
- Choose invariant \mathbb{P}^{11} : Defined by

$$f_{ij}^\alpha = e_{ij}^\alpha + t_{-i-j}^\alpha e_{-i-j, -i-j}^{\alpha+1}$$

for $i, j \in \mathbb{Z}/3 \times \mathbb{Z}/3$ ($i \neq j$) and $\alpha = 0, 1$.

- The resulting $X_{H_t} \stackrel{\Delta}{=} \mathbb{P}^{11} \cap M$ is singular with 48 isolated double point singularities.

Mirror candidate for X_1

- We divide out by the H -action and resolve: $X_1^\circ \triangleq \widetilde{X_{H_t}}/H$.
- Roan's formula gives:

$$\chi(X_1^\circ) = \frac{1}{3} (24 + 8 \cdot 24) = 72.$$

Based on this calculation and the mirror heuristic, we conjecture:

Conjecture

X_1° is a mirror of X_1 .

Remark

A very similar construction gives a mirror candidate for X_2 .

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