UiO : University of Oslo

FREDRIK MEYER

Join of hexagons and Calabi–Yau threefolds

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Contents

Introduction

The work leading up to this thesis started with a naïve idea concerning smoothings of certain Stanley–Reisner schemes. Stanley–Reisner schemes are highly singular projective schemes, whose components are all projective spaces. They are constructed from a simplicial complex, in such a way that the components correspond to the maximal faces of the simplicial complex.

If the simplicial complex is homeomorphic to S^1, a circle, then a smoothing of the Stanley–Reisner scheme yields an elliptic curve. Similarly, if the simplicial complex is a sphere, a smoothing of the Stanley–Reisner scheme will give a K3 surface. Many properties of the simplicial complex correspond to properties of the Stanley–Reisner scheme and its smoothings.

The mentioned naïve idea was this: what if the simplicial complex is a triangulated $\mathbb{CP}^2$? A smoothing of the associated Stanley–Reisner scheme would then give us an (algebraic) hyper-Kähler variety, as we explain in Chapter 2. This would be interesting, since there are very few known families of hyper-Kähler varieties.

Unfortunately, given a triangulation of $\mathbb{CP}^2$ with few vertices, a smoothing of the Stanley–Reisner scheme turned out to be too difficult to find. Even the existence of smoothings are in most cases unclear. However, one particular triangulation of $\mathbb{CP}^2$ led us to study the problems in Chapter 3 and 4. This triangulation, found by Gaifullin [Gai09], is the union of three 4-balls, all of which are suspensions over joins of hexagons. Leaving the idea of studying triangulations of $\mathbb{CP}^2$, we began studying a triangulation of the 3-sphere.

The join of two hexagons is a triangulated 3-sphere. A smoothing of the associated Stanley–Reisner scheme X_0 is a Calabi–Yau variety. Finding new Calabi–Yau varieties has become a small industry, which we did not hesitate to join. This decision turned out to be profitable. The scheme X_0 deforms to several interesting varieties, and three of them are smooth. One of its deformations, which we have denoted by X_Y, is a singular Calabi–Yau variety, whose singularities are all locally-analytically cones over del Pezzo-surfaces. This discovery motivates the third chapter, in which we study this singularity and its two smoothings. We prove that they are topologically different, and calculate their Betti numbers.

We construct three smoothings of X_0. To define them, recall the definition of *join* of two algebraic varieties. It is the closure of the union of all lines between them. Let M be the join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2$ (embedded in disjoint projective spaces). Let N be the join of two copies of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and let W be the join of $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Define X_1 to be M intersected by a codimension 6 hyperplane. Let X_2 be N intersected by a codimension 4 hyperplane, and let X_3 be W intersected by a codimension 5 hyperplane.

We show that X_i $(i = 1, 2, 3)$ are all smooth Calabi–Yau manifolds, and that they are deformations of X_0. They have Euler characteristics -72, -48, and -60, respectively.

To our knowledge, these three Calabi–Yau's have not been previously described. There are many connections to the physics literature, and to works by other mathematicians. Let us explain some of them.

In [Kap15], Kapustka compiles a list of smooth Calabi–Yau varieties with $\operatorname{Pic} X = \mathbb{Z}$. One of the elements of the list is a Calabi–Yau in $\mathbb{P}^{11}$ with the same Hilbert polynomial as our X_1, and with the same Euler characteristic. This Calabi–Yau was however only conjectured to exist, based on the conjecture that to every differential equation of "Calabi–Yau type", there should exist a one parameter family of smooth Calabi–Yau varieties having that equation as its Picard–Fuchs differential equation. A list of such equations has been computed by van Enckevort and van Straten in [ES06].

All of these equations have been made searchable in the online database [Str]. Entering the invariants $H^3 = 36$, $H \cdot c_2 = 72$ and $\dim |H| = 12$, yield exactly three matches, corresponding to Calabi–Yau varieties with Euler characteristics -72, -60 and -48, respectively. These numbers are exactly the Euler characteristics of our X_i $(i = 1, 2, 3)$.

Furthermore, their differential operators are Hadamard products, $c * c$, $a * a$ and $a * c$, which according to van Straten (personal communication) is "mirror dual" to join.

This seems like a perfect match, confirming the existence predicted by the conjecture. The only problem is that our varieties seem to have $h^{11} > 1$, as discussed in Chapter 4.

Several questions arise: can our X_i still correspond to these differential equations, without having $h^{11} = 1$? If not, what is their connection to the conjecture?

There also seem to be connections with discoveries made by physicists. For example, Braun–Candelas–Davis describe in [BCD10] a Calabi–Yau with small Hodge numbers, whose mirror dual lies in the same deformation family as our X_i's.

We did not have the time to ponder these questions, but would very much like to see them answered in the future.

Finally, there is the phenomenon of *mirror symmetry*, which is a sort of duality between different Calabi–Yau manifolds. Producing mirror candidates of Calabi–Yau manifolds is a hard problem, and there are many ways to do this. One heuristic which often works is this: suppose you have a family $\pi : \mathscr{X} \to S$ of Calabi–Yau manifolds, and that some central fiber has a large automorphism group. One can consider the (often singular) sub-family invariant under this group. It is then often the case that a resolution of singularities of an invariant fiber is a mirror to the general fiber of π. This technique is called *orbifolding*. We give a brief introduction to mirror symmetry and orbifolding in the first chapter.

By using the technique of orbifolding, we produce mirror candidates for X_1 and X_2.

The organization of the thesis is as follows:

- In the first chapter, we gather background material which is relevant for the next chapters. We have erred on the side of *too much* background information rather than too little, serving as a motivation for both myself and potential young readers. We end with a give a brief sketch of some of the ideas from mirror symmetry.

- In the second chapter we motivate the original naïve idea about smoothing triangulations of $\mathbb{CP}^2$ to find new hyper-Kähler varieties.

  We comment on four already known triangulations of $\mathbb{CP}^2$, with the number of vertices ranging from 9 to 15, and describe the obstacles encountered in trying to smooth them. We also compute their associated Stanley–Reisner schemes, and the dimensions of their cotangent modules. Their obstruction spaces are in all cases large.

- The third chapter is devoted to a special toric singularity, namely the affine cone $C(\mathrm{dP}_6)$ over the del Pezzo surface dP_6. This singularity has two topologically different smoothings, and we compute their singular homology groups using techniques from toric geometry.

  We start the chapter by discussing dP_6 in some generality. We discuss its Picard group and two natural embeddings in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2 \times \mathbb{P}^2$, respectively.

  It is well known that $C(\mathrm{dP}_6)$ has two smoothing components. We identify them as hyperplane complements of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and of $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$, and use this fact to compute their singular homology groups. In the final computation, we use theorems from algebraic topology, such as Poincaré duality and Lefschetz duality.

- The final chapter is devoted to the construction of new Calabi–Yau varieties and their mirror candidates.

  We start the chapter by discussing the Stanley–Reisner scheme X_0, which comes from the simplicial complex that is the join of two hexagons. We compute its Hilbert polynomial, and explain how it deforms to a special singular Calabi–Yau variety X_Y.

  Then we explain the construction of three topologically different smoothings X_i ($i = 1, 2, 3$) of X_Y (and hence of X_0). They are topologically different, which we prove using a `Macaulay2` computation: it shows that their topological Euler characteristics are different. The construction is very similar to that of Rødland [Rød00].

  Then we explain the existence of special singular subfamilies of X_1 and X_2 which are invariant under a finite subgroup of the big torus. Using orbifolding and a formula by Roan [Roa89], we propose conjectural mirror candidates for X_1 and X_2.

  We end with many open questions, which we hope to see answered in the future.

In the last appendix we include some computations on triangulations of spheres with 8 vertices. Grünbaum and Sreedharan have computed all such triangulations [GS67], and we used their list to compute deformation theoretic invariants for each of the associated Stanley–Reisner schemes of the spheres with 8 vertices. Unfortunately, there seem to be a few typographical errors in their article, as some of the complexes in their list turn out not to be spheres.

The source code of the thesis and all computer computations are available on GitHub at

`https://github.com/FredrikMeyer/JoinsOfHexagonsAndCalabiThreefolds`.

Notation

If V is a vector space, we denote by $\mathbb{P}(V)$ its projectivisation. We write k for a field, which is almost always assumed to be $\mathbb{C}$. If X is a projective variety, we write $S(X)$ for its homogeneous coordinate ring (if the embedding is implicit). If X is a scheme over k, we write X/k. We will write $h^i(X, \mathscr{F})$ for $\dim_k H^i(X, \mathscr{F})$. All schemes are noetherian. We will often write $\triangleq$ for definitions (instead of ":=", common in computer science literature). Unless otherwise stated, we use the definitions from [Har77].

CHAPTER 1

Preliminaries

In this chapter we introduce the notation and results which will be used later. Some of the material in this chapter plays the rôle of motivation rather than preliminary results.

1.1 The join of projective varieties

There are many ways to define the join of two projective varieties X and Y. We will define it in a particularly general and beautiful way, as described by Altman and Kleiman in [AK75]. Then we will specialize to our situation.

Fix a base scheme S. Let $\mathscr{C}$ be the category of graded, quasi-coherent $\mathscr{O}_S$-algebras, generated in degree 1. The tensor product of two $\mathscr{O}_S$-algebras $\mathscr{R}$ and $\mathscr{S}$ is naturally graded: the degree d part is given by

$$(\mathscr{R} \otimes_{\mathscr{O}_S} \mathscr{S})_d = \bigoplus_{p+q=d} \mathscr{R}_p \otimes \mathscr{S}_q.$$

Let $X = \operatorname{Proj} \mathscr{R}$ and $Y = \operatorname{Proj} \mathscr{S}$. Then we define the *join* of the graded $\mathscr{O}_S$-algebras to be

$$X * Y \triangleq \operatorname{Proj}(\mathscr{R} \otimes_{\mathscr{O}_S} \mathscr{S}).$$

If X and Y are projective varieties over S, they come with graded $\mathscr{O}_S$-algebras $\mathscr{R} = \operatorname{Sym}_S \mathscr{O}_X(1)$ and $\mathscr{S} = \operatorname{Sym}_S \mathscr{O}_Y(1)$. Then we define the join of X and Y to be join of these algebras.

The join construction is a contravariant functor in two variables from the category of graded $\mathscr{O}_S$-algebras and surjective maps to the category of projective varieties.

Example 1.1.1. Let $X = \mathbb{P}(E)$ and $Y = \mathbb{P}(F)$, where E, F are quasi-coherent $\mathscr{O}_S$-modules. Then we have the equality $\mathbb{P}(E) * \mathbb{P}(F) = \mathbb{P}(E \oplus F)$, because of the linear algebra fact that $\operatorname{Sym}(E) \otimes \operatorname{Sym}(F) = \operatorname{Sym}(E \oplus F)$. ♡

The algebra $\mathscr{R} \otimes_{\mathscr{O}_S} \mathscr{S}$ contains the ideal $\mathscr{R} \otimes \mathscr{S}_+$. The associated subscheme is denoted by V_X, and it is isomorphic to $X = \operatorname{Proj} \mathscr{R}$. We define V_Y similarly. We call V_X and V_Y the *fundamental subschemes* of $X * Y$.

There is a geometric definition of the join, as described in section (C11) in [AK75]. Let E, F be quasi-coherent $\mathscr{O}_S$-modules[1]. Suppose X and Y are closed subschemes of $\mathbb{P}(E)$ and $\mathbb{P}(F)$, respectively. Then $X * Y$ is a closed subscheme of $\mathbb{P}(E \oplus F)$. Identify X and Y with their fundamental subschemes in $X * Y$. Then it is not difficult to see that $X * Y$ is the (closure of the) locus of points lying on the lines of $\mathbb{P}(E \oplus F)$ determined by pairs of points from X and Y.

Proposition 1.1.2. *Suppose $X/k \subset \mathbb{P}^n$ and $Y/k \subset \mathbb{P}^m$ are smooth projective schemes. Then their join, $X * Y$ has dimension* $\dim X + \dim Y + 1$. *The singular locus is of dimension* $\max\{\dim X, \dim Y\}$ *and consists of the disjoint union of V_X and V_Y.*

Proof. Let $S_X = \bigoplus_{d \geq 0} H^0(X, \mathscr{O}_X(d))$ and $S_Y = \bigoplus_{d \geq 0} H^0(Y, \mathscr{O}_Y(d))$ be the homogeneous coordinate rings of X and Y, respectively. We have that $X * Y \subset \mathbb{P}^{n+m+1}$.

Denote by $C(X * Y)$ the scheme $\operatorname{Spec}(S_X \otimes_k S_Y)$, which is the affine cone over $X * Y$. It is a general fact that if A, B are two algebraic varieties, then the singular locus of the product is equal to the union $\operatorname{Sing}(A) \times B \cup A \times \operatorname{Sing}(B)$. It follows that the singular locus of $C(X * Y) = C(X) \times C(Y)$ is equal to

$$\operatorname{Sing} C(X) \times C(Y) \bigcup C(X) \times \operatorname{Sing} C(Y).$$

Since X and Y are smooth, the only singular point on the affine cones are the origins. Hence

$$\operatorname{Sing}\left(C(X * Y)\right) = \{0\} \times \operatorname{Sing}(C(Y)) \bigcup \operatorname{Sing}(C(X)) \times \{0\}.$$

Projectivizing, we find that $\operatorname{Sing}(X * Y) = V_X \sqcup V_Y$, since $(0, \ldots, 0)$ is the only common point of the affine cones. ∎

Recall that a scheme X is *Cohen–Macaulay* if all its local rings $\mathscr{O}_{X,x}$ are Cohen–Macaulay. This means that depth and codimension agree everywhere on X. One implication of being Cohen–Macaulay is that X will have a dualizing sheaf ω_X. If the dualizing sheaf is a line bundle, then we say that X is *Gorenstein.*

If the homogeneous coordinate ring of a projective variety X is a Gorenstein ring, we say that X is *arithmetically Gorenstein.* In that case, the canonical sheaf can be computed as the sheaf associated to the graded module

$$\operatorname{Ext}_R^{\operatorname{codim} X}\left(S_X, S_X(-\dim N - 1)\right) = S_X(-d),$$

[1] In our case, $S = \operatorname{Spec} k$ always. So E, F are just vector spaces.

where R is the homogeneous coordinate ring of projective space. The last equality is true by definition of Gorenstein graded rings (see [Eis95, page 550]). The number d is the degree of the anticanonical embedding.

If X and Y are two arithmetically Gorenstein schemes, then their join is also arithmetically Gorenstein. Furthermore, we can compute the canonical sheaf in terms of the canonical sheaves of X and Y.

Proposition 1.1.3. *Let $X = \operatorname{Proj} R$ and $Y = \operatorname{Proj} S$ be arithmetically Gorenstein projective schemes with dualizing sheaves ω_X, ω_Y, respectively (here R, S are graded k-algebras). Let Ω_X, Ω_Y be R- and S-modules corresponding to ω_X and ω_Y, respectively.*

Then $X * Y$ is arithmetically Gorenstein with dualizing sheaf ω_{X*Y} given by the sheaf associated to the $R \otimes_k S$-module $j_1^* \Omega_X \otimes_{R \otimes_k S} j_2^* \Omega_Y$ (the homomorphism $j_1 : R \to R \otimes_k S$ is given by $r \mapsto r \otimes 1$, and similarly j_2).

Proof. The statement follows from Theorem 4.2 in [HHS16], where the authors prove that the canonical module of a tensor product is the tensor product of the canonical modules. ■

Remark 1.1.4. If X and Y are arithmetically Gorenstein projective schemes, their canonical modules are $\mathscr{O}_X(n)$ and $\mathscr{O}_Y(m)$ for some m, n, respectively. It follows from the above proposition that $\omega_{X*Y} = \mathscr{O}_{X*Y}(m+n)$. ◇

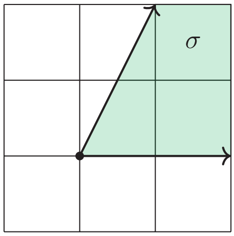
1.2 Toric geometry

Toric geometry lies somewhere in the intersection between algebraic geometry, combinatorics and convex geometry. Toric varieties and their geometry can be described completely in terms of explicit finite combinatorial data. This makes toric geometry well suited for examples and explicit computations. In this section we give a quick and dirty introduction to toric geometry.

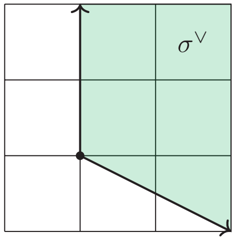
Definition 1.2.1. A *toric variety* is an irreducible normal variety containing the torus $T = (\mathbb{C}^*)^n$ as a dense subset, such that the action of the torus on itself extends to an action on the variety.

We fix some notation that will be used throughout. Details and proofs can be found in [CLS11; Ful93]. Each toric variety comes with two dual lattices. The *lattice of 1-parameter subgroups* N and the *character lattice* M. A one-parameter subgroup is a morphism $\lambda : \mathbb{C}^* \to T$ that is a group homomorphism. The set of one-parameter subgroups is a lattice isomorphic to $\mathbb{Z}^n$. A *character* is a morphism $\chi : T \to \mathbb{C}^*$ that is a group homomorphism. The set of characters is a lattice M isomorphic to $\mathbb{Z}^n$ which is naturally dual to N.

Let V be an $\mathbb{R}$-vector space. Let $V^\vee$ be the dual vector space. A *convex polyhedral cone* is a subset σ of V of the form



(a) The cone $\sigma = \langle (1,0),(1,2)\rangle$.



(b) The dual cone $\sigma^\vee$.

Figure 1.1: A cone and its dual cone, defining an affine toric variety.

$$\sigma = \{r_1 v_1 + \cdots + r_s v_s \mid r_i \geq 0 \text{ for all } i\},$$

where the v_i's are a finite set of vectors in V and the r_i's are real numbers. A *rational polyhedral cone* is a cone such that the vectors v_i can be taken to have rational coordinates.

The *dual cone* $\sigma^\vee$ lives in $V^\vee$, and is defined as the set of functionals that are positive on σ:

$$\sigma^\vee \triangleq \{u \in V^\vee \mid \langle u, v\rangle \geq 0, v \in \sigma\}.$$

Cones have two descriptions: either as the positive hull of a finite set of vectors (as above), or implicitly, as the intersection of finitely many half-spaces. If the u_i's generate $\sigma^\vee$, then it is true that

$$\sigma = \sigma^{\vee\vee} = \{v \in V \mid \langle u_i, v\rangle \geq 0 \text{ for all } i\}.$$

The vectors u_i are the inner normal vectors of the facets of σ.

A *(commutative) semigroup* is a set S with an associative, commutative binary operation $S \times S \to S$, together with an identity element $0 \in S$. Given a cone $\sigma \subset N$, we can form a semigroup $S \triangleq \sigma^\vee \cap M \subseteq M$. From this semigroup S, we can form the *semigroup algebra* $\mathbb{C}[S]$: it is the algebra generated by the elements of S, with multiplicative structure inherited from S. We then define U_σ as $\operatorname{Spec}\mathbb{C}[\sigma^\vee \cap M]$, and call it the *affine toric variety associated to* σ.

We thus have a contravariant functor from the category of cones to the category of affine toric varieties, sending σ to U_σ. This is an equivalence of categories.

Example 1.2.2. Let $\sigma = \langle (1,0),(1,2)\rangle \subset \mathbb{R}^2$. Then

$$\sigma^\vee = \langle (2,-1),(0,1)\rangle \subset \mathbb{R}^2.$$

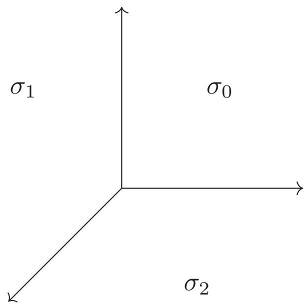


Figure 1.2: The fan corresponding to the toric variety $\mathbb{P}^2$.

See Figure 1.1. Then the semigroup ring S_σ is $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[x, y, x^2/y]$, where we have identified x and y with the standard basis of $\mathbb{R}^2$. This ring is isomorphic to $\mathbb{C}[a, b, c]/(a^2 - bc)$, which is a quadric cone. ♡

General toric varieties are described using collections of cones called *fans*. A set Σ of cones is called a *fan* if it closed under intersections and faces of cones: if $\sigma, \sigma' \in \Sigma$, then we also have $\sigma \cap \sigma' \in \Sigma$, and if $\sigma' \subset \sigma$ is a face with $\sigma \in \Sigma$, then $\sigma' \in \Sigma$ also. Thus, given a fan Σ, we get a collection of affine toric varieties U_σ for each cone $\sigma \in \Sigma$. We have inclusions $U_{\sigma\cap\sigma'} \subset U_\sigma$, and using these inclusions we may glue the affine open sets U_σ to get a separated toric variety.

If the fan is *complete* (meaning that the union of its cones is equal to N), the corresponding toric variety is complete. A toric variety is smooth if and only if all of its cones are smooth, and we say that a cone is smooth if it is generated by part of a $\mathbb{Z}$-basis for N.

Remark 1.2.3. Note that since the matrix formed by $(1, 0)$ and $(1, 2)$ have determinant 2 ($\neq 1$), we can observe directly (without computing the dual cone) that the variety in Example 1.2.2 is singular. ◇

Remark 1.2.4. The category of fans and morphisms between them is equivalent to the category of toric varieties and torus-invariant morphisms. ◇

Example 1.2.5. Consider Figure 1.2. This is the fan corresponding to the toric variety $\mathbb{P}^2$. The dual cones $\sigma_i^\vee$ give rise to the algebras $\mathbb{C}[x, y]$, $\mathbb{C}[\frac{1}{x}, \frac{y}{x}]$ and $\mathbb{C}[\frac{x}{y}, \frac{1}{y}]$. Their spectra glue to form $\mathbb{P}^2$. More complicated fans give rise to exponents in the monomial generators. ♡

Projective toric varieties can be constructed from lattice polytopes. We describe the procedure here. Let Δ be a lattice polytope in $M \simeq \mathbb{Z}^n$. Let $M' = M \oplus \mathbb{Z}$, and embed Δ in M' by sending v to $(v, 1)$. Let $C(\Delta)$ be the cone over Δ in M'. Then $\mathbb{C}[C(\Delta) \cap M']$ is a $\mathbb{Z}$-graded algebra. We let $X_\Delta \stackrel{\Delta}{=} \operatorname{Proj} \mathbb{C}[C(\Delta) \cap M']$ be the associated projective variety.

If Δ is a *normal polytope*, the projective variety X_Δ is a toric variety. The defining fan is the normal fan of Δ. This is described in Chapter 2 of [CLS11].

Note that X_Δ comes with an ample line bundle $\mathscr{O}_\Delta(1)$. The global sections correspond to the lattice points of Δ.

1.2.1 Divisors and Picard groups of toric varieties

Recall that a *Weil divisor* is a formal linear combination of codimension 1 subvarieties of a scheme X (satisfying the "star" condition in Hartshorne [Har77]). The group of Weil divisors modulo linear equivalence is the *class group* of X, and is denoted by $\mathrm{Cl}(X)$. The group of line bundles modulo isomorpism is the *Picard group* of X, and is denoted by $\mathrm{Pic}(X)$. The two groups coincide for smooth varieties. They are in general very hard to compute, but for toric varieties the computation is exceptionally easy, relying only on the structure of the rays in the fan Σ defining the toric variety.

We describe the divisors on toric varieties. The description will be used in Chapter 3, where we work out the geometry of the two smoothings of the affine cone over the del Pezzo surface of degree 6.

Let X be a smooth toric variety, and let $\Sigma(1)$ denote the set of one-dimensional cones (called *rays*) in the fan Σ defining X. For each ray ρ, let $u_\rho \in N$ denote the primitive ray generator of ρ. Then one can show that the torus-invariant divisors on X are in one-to-one correspondence with the rays $\rho \in \Sigma(1)$. Furthermore, every divisor on X is linearly equivalent to a torus-invariant divisor. Using these two facts, one can prove the following:

There is an exact sequence:

$$0 \longrightarrow M \xrightarrow{C} \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Pic}(X) \longrightarrow 0,$$

where the rows of the matrix C are the vectors u_ρ. See [CLS11], Chapter 4, for a proof.

There is also a description of the Cartier divisors on X in terms of *support functions* on N: a support function is a function $\varphi : |\Sigma| \to \mathbb{R}$ such that the restriction $\varphi\big|_\sigma$ of φ to each cone in Σ is linear. A support function is *integral with respect to* N if $\varphi(|\Sigma| \cap N) \subset \mathbb{Z}$. This means that for each cone σ, there is an $m_\sigma \in M$, such that $\varphi(v) = \langle v, m_\sigma \rangle$ if $v \in \sigma$.

The set of support functions is an abelian group under addition, and by Theorem 4.2.12 in [CLS11], there is an isomorphism between the group of integral support functions on Σ and the torus invariant Cartier divisors on X.

Here is how one associates a support function to a divisor on a toric variety Y (we assume that the fan of the toric variety is full-dimensional and complete). Let

$D = \sum a_\rho D_\rho$ be a Cartier divisor on Y. For each maximal cone $\sigma \in \Sigma(\dim Y)$, one can show that there is an $m_\sigma \in M$ such that

$$\langle m_\sigma, u_\rho \rangle = -a_\rho$$

for all $\rho \in \sigma(1)$. The collection $\{m_\sigma\}_{\sigma \in \Sigma(n)}$ is called the *Cartier data* of D.

Given Cartier data of a divisor, we can define a convex function by the rule $u \mapsto \varphi_D(u) = \langle m_\sigma, u \rangle$ if $u \in \sigma$.

1.3 Deformation theory and the Hilbert scheme

Deformation theory is the infinitesimal study of algebro-geometric objects varying in families. Examples of such objects can be families of schemes, families of projective schemes (respecting the embedding), families of vector bundles, and so on.

In this section we will review some notation and motivation from deformation theory. Although results from deformation theory are not central in this thesis, many of the methods and objects have roots from or connections with deformation theory. A reference for deformation theory is the book by Hartshorne [Har10]. For a leisurely popular account connecting deformation theory to other parts of mathematics, the article [Maz04] by Mazur is a nice read.

Definition 1.3.1. Given a scheme X_0 over $\mathbb{C}$, a *family of deformations* of X_0 is a flat morphism $\pi : \mathscr{X} \to (S, 0)$ with S connected such that $\pi^{-1}(0) = X_0$. If S is the spectrum of an artinian $\mathbb{C}$-algebra, then π is an *infinitesimal deformation.* If $S = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$, then π is a *first order deformation.* An *embedded deformation* of an embedded scheme $X_0 \subset \mathbb{P}^n$ is a deformation $\pi : \mathscr{X} \to (S, 0)$ with $\mathscr{X} \subset \mathbb{P}^n \times S$ such that π is the restriction of the projection $\pi : \mathbb{P}^n \times S \to S$. A deformation is *trivial* if it is isomorphic to the projection $X_0 \times S \to S$.

A *smoothing* of X_0 is a deformation of X_0 over a curve, such that the general fiber is smooth.

The sets of first-order embedded deformations have interpretations in terms of "familiar" objects. See the first chapter of [Har10] for proofs.

Proposition 1.3.2. *The set of all first order embedded deformations of a projective scheme X is in one-to-one correspondence with the group $H^0(X, \mathcal{N}_{X/\mathbb{P}^n})$, where $\mathcal{N}_{X/\mathbb{P}^n}$ is the normal sheaf of X in $\mathbb{P}^n$.*[2]

Proposition 1.3.3. *The set of all first order deformations of a smooth scheme X is in one-to-one correspondence with the group $H^1(X, \mathcal{T}_X)$.*

[2] Recall that this is by definition $\mathcal{H}om_{\mathscr{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathscr{O}_X)$, where $\mathcal{I}$ is the ideal sheaf of X.

Remark 1.3.4. The intuition behind this result is the following. From the normal sequence

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}}\big|_X \to \mathcal{N}_{X/\mathbb{P}^n} \to 0,$$

we get a surjection (for $n \geq 2$):

$$H^0(X, \mathcal{T}_{\mathbb{P}}\big|_X) \to H^0(X, \mathcal{N}_{X/\mathbb{P}^n}) \to H^1(X, \mathcal{T}_X) \to 0.$$

The interpretation is that abstract deformations correspond to embedded deformations modulo infinitesimal automorphisms of $\mathbb{P}^n$. ◇

If we denote by $\mathrm{Def}(X)$ (resp. $\mathrm{EmbDef}(X)$) the “space” of all (resp. embedded) deformations of a scheme X, then the above proposition tells us that $H^1(X, \mathcal{T}_X)$ (resp. $H^0(X, \mathcal{N}_{X/\mathbb{P}^n})$) is the tangent space of the point $[X]$ in $\mathrm{Def}(X)$ (resp. $\mathrm{EmbDef}(X)$).

There is a complex, called the *cotangent complex*, associated to A-algebras B and B-modules M, that measures various deformation theoretic aspects of $\operatorname{Spec} B$. These are modules $T^i(B/A, M)$ for $i \geq 0$. Only the first three will be relevant to us, and we will present some ad hoc definitions.

Let B be an A-algebra, where A is a commutative ring. Let R be a polynomial ring surjecting onto B and let I be the kernel. Let F be a free R-module surjecting onto I, and let Q be its kernel. Then we have an exact sequence

$$0 \to Q \to F \xrightarrow{j} I \to 0.$$

There is a “Koszul” submodule F_0 of F generated by the elements $aj(b) - bj(a)$, for $a, b \in F$. Note that $j(F_0) = 0$, which implies that $F_0 \subset Q$. Let $L_2 \triangleq Q/F_0$. Let $L_1 = F \otimes_R B$, and let $L_0 = \Omega^1_{R/A} \otimes_R B$. These are the first few terms of the cotangent complex:

$$L_\bullet : L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \xrightarrow{0} 0.$$

The map d_2 is induced by the inclusion $Q \to F$. The map d_1 is the composition of $j : F \to I$ with the derivation $R \to \Omega^1_{R/A}$.

For any R-module M, we now define

$$T^i(B/A, M) \triangleq H^i(\mathrm{Hom}_B(L_\bullet, M)).$$

There are many things to be checked, but the details are all in [Har10].

We list a few of the important properties of the T^i-functors here:

- We have an equality $T^0(B/A, M) = \mathrm{Der}_A(B, M)$. If $M = B$, this is the *tangent module* of B over A.

- If $A = k[x_1, \ldots, x_n]$ and $B = A/I$, then we have an exact sequence

$$\mathrm{Hom}(\Omega^1_{A/k}, M) \to \mathrm{Hom}(I/I^2, M) \to T^1(B/k, M) \to 0. \tag{1.1}$$

  This gives us a way to compute $T^1(B/k, M)$ which is amenable to computer algebra software. Algorithms for computing $T^i(B/k, B)$ for $i = 0, 1, 2$ are implemented in the `Macaulay2` package `VersalDeformations` written by Nathan Ilten [Ilt12].

- The module $T^1(B/k, B)$ classifies first order deformations of the affine scheme $\operatorname{Spec} B$. It is a finite-dimensional k-vector space if $\operatorname{Spec} B$ has only isolated singularities. Both $T^1(B/k, B)$ and $T^2(B/k, B)$ are zero if B is smooth.

- The module $T^2(B/k, B)$ contains "obstructions" for lifting infinitesimal deformations to larger artinian rings.

- If B and M are graded, then $T^i(B/A, M)$ are graded as well.

If X is a projective variety and S_X its homogeneous coordinate ring, let U_X denote $\operatorname{Spec} S_X$. Then the deformation theory of the affine cone and X itself is closely related. This is studied for example in Schlessinger's article [Sch73], from which the following useful result can be deduced:

Proposition 1.3.5. *Let X/k be a smooth projective Calabi–Yau variety, and let S_X be its homogeneous coordinate ring. Then we have an isomorphism*

$$T^1(S_X/k, S_X)_0 \simeq H^1(X, \mathcal{T}_X),$$

where $\mathcal{T}_X$ is the tangent sheaf of X, and the subscript denotes the degree zero part of the module.

Proof. This is a combination of Theorem 2.5 and Corollary 2.6 in [DFF15], using the fact that X is Calabi–Yau and smooth. ■

This result makes computing Hodge numbers of projective smooth Calabi–Yau's amenable to computer calculations.

We include a somewhat lengthy example of how to compute the T^i modules for a relatively simple ring.

Example 1.3.6. Let $B = k[x, y]/(x^2, xy, y^2)$ be the coordinate ring of the double point in $\mathbb{A}^2$. We want to compute $T^i(B/k, B)$ for $i = 0, 1, 2$.

We have that $T^0(B/k, B) = \mathrm{Der}_k(B, B)$, and this can be identified with the kernel of the map

$$\mathrm{Hom}_B(\Omega_{k[x,y]/k}, B) \xrightarrow{\varphi} \mathrm{Hom}_B(I/I^2, B).$$

See Proposition 3.10 in [Har10]. The map φ can be identified with the transpose of the Jacobian matrix of I. The module to the left is free, generated by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Up to scalars, φ is given by

$$\varphi = \begin{pmatrix} x & 0 \\ y & x \\ 0 & y \end{pmatrix}.$$

Thus $T^0(B/k, B)$ is equal to the set of $(f, g) \in R^2$ annihilated by the ideal $\mathfrak{m} = (x, y)$ in B. But since R is $k[x, y]/\mathfrak{m}^2$, this is equal to $\mathfrak{m} \oplus \mathfrak{m}$. Thus $\dim_k \operatorname{Der}_k(B, B) = 4$, corresponding to the fact that a fat point can move by moving its support and also by moving its "tangent arrow".

We can use the exact sequence (1.1) to compute $T^1(B/k, B)$. We see that $T^1(B/k, B)$ is the cokernel of φ. We must first identify $\operatorname{Hom}_B(I/I^2, B)$.

To compute this module, we start with a free resolution of I over $P = k[x, y]$:

$$0 \to P^2 \xrightarrow{d_1 = \begin{pmatrix} y & 0 \\ -x & y \\ 0 & -x \end{pmatrix}} P^3 \xrightarrow{d_0 = \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}} I \to 0.$$

It is then true that $\operatorname{Hom}_B(I/I^2, B)$ can be identified with $\ker(d_1^\vee \otimes B)$. An easy argument shows that this is $\mathfrak{m} \oplus \mathfrak{m} \oplus \mathfrak{m}$.

But the image of φ is a two-dimensional subset of $\operatorname{Hom}_B(I/I^2, B)$. Hence $\dim_k T^1(B/k, B) = 6 - 2 = 4$.

The computation of $T^2(B/k, B)$ is usually the hardest. We can identify $T^2(B/k, B)$ with $\operatorname{Hom}_B(Q/F_0, B)/\operatorname{im}(d_1 \otimes B)^\vee$, where F_0 is the module of Koszul relations and $Q = \operatorname{im} d_1$. Let us first compute $\operatorname{Hom}_B(Q/F_0, B)$.

We start with finding a presentation for Q/F_0. The module F_0 is the submodule of $F = P^3$ generated by the columns of the matrix

$$\psi = \begin{pmatrix} y^2 & xy & 0 \\ 0 & -x^2 & y^2 \\ -x^2 & 0 & xy \end{pmatrix}.$$

The image of d_1 is isomorphic to R^2. Using this isomorphism, Q/F_0 fits into an exact sequence

$$R^3 \xrightarrow{\begin{pmatrix} x & y & 0 \\ 0 & x & y \end{pmatrix}} R^2 \to Q/F_0 \to 0.$$

Applying $\operatorname{Hom}_B(-, B)$ is left-exact, so we get an exact sequence:

$$0 \to \operatorname{Hom}_B(Q/F_0, B) \to B^2 \xrightarrow{\begin{pmatrix} x & 0 \\ y & x \\ 0 & y \end{pmatrix}} B^3$$

It follows that $\mathrm{Hom}_B(Q/F_0, B) = \mathfrak{m} \oplus \mathfrak{m}$. The image of $d_1^\vee \otimes B$ kills off three of the four generators, so that $T^2(B/k, B)$ is a $4 - 3 = 1$-dimensional vector space over k. This reflects the fact that the fat point correspond to a singular point in its Hilbert scheme.

As we can see, already for this small example, there is a lot of computation involved. Especially the computation of a free resolution is resource demanding when the ideal have more than just two generators. Therefore computer algebra software is invaluable when doing experiments in deformation theory. ♡

1.3.1 A few words about Hilbert schemes

The *Hilbert scheme* $\mathscr{H}_{P(t)}$ parametrizes projective schemes with a given Hilbert polynomial $P(t)$. The proof of its existence is non-trivial, and was first given by Grothendieck in [Gro95]. The proof was later simplified by Mumford [Mum66]. It is often just as easy to work with the functorial description of the Hilbert scheme – namely with the functor it represents rather than the scheme itself.

The functor that the Hilbert scheme represents is the following: $h_{P(t)}(S)$ is the set of all flat families $\mathscr{X} \subset S \times \mathbb{P}^n \to S$ where the fibers have Hilbert polynomial $P(t)$. With this definition, it is not difficult to show for example that the tangent space of $\mathscr{H}_{P(t)}$ at a point corresponding to a scheme X is given by $H^0(X, \mathscr{N}_{X/\mathbb{P}^n})$, where $\mathcal{N}_{X/\mathbb{P}^n}$ is the normal sheaf of X. Thus for a "generic" scheme, the dimension of the component on the Hilbert scheme on which it lies, is given by $h^0(X, \mathcal{N}_{X/\mathbb{P}^n})$.

Note that two different points on $\mathscr{H}_{P(t)}$ might represent isomorphic schemes. Two schemes are different if they occupy different points in $\mathbb{P}^n$. We often write $\mathrm{Hilb}(X)$ for the component of the Hilbert scheme containing a scheme X. With this notation, allowing deformations outside $\mathbb{P}^n$ corresponds to applying the forgetful functor $\mathrm{Hilb}(X) \to \mathrm{Def}(X)$, where $\mathrm{Def}(X)$ is the "space" of all deformations of X.

1.4 Simplicial complexes and Stanley–Reisner schemes

Stanley–Reisner schemes are certain degenerate projective schemes modelled on simplicial complexes.

Let $[n]$ denote the set of numbers $\{0, \ldots, n\}$. The power set of $[n]$ is called the *n-simplex* and is denoted by Δ_n.

Definition 1.4.1. A *simplicial complex* is a subset $\mathcal{K} \subseteq \Delta_n$ (for some n), such that if $f \in \mathcal{K}$ and $g \subseteq f$, then $g \in \mathcal{K}$. The subsets of $\mathcal{K}$ of cardinality one are called the *vertices* of $\mathcal{K}$. The subsets of codimension one are called facets of $\mathcal{K}$. The subsets of $\mathcal{K}$ are called *faces*. The *dimension* of a face f is equal to $|f| - 1$.

It is often convenient to organize the number of faces of various dimensions in the *f-vector*. It is a tuple $(f_0, f_1, \ldots, f_d)$, where f_i is the number of i-dimensional faces of $\mathcal{K}$.

To every simplicial complex we can associate a *Stanley–Reisner scheme* as follows.

Let k be a field, and let $P_\mathcal{K}$ be the polynomial ring over k with variables indexed by the vertices of $\mathcal{K}$. Then the *face ring* or *Stanley–Reisner ring* of the simplicial complex $\mathcal{K}$ is the quotient ring $A_\mathcal{K} = P_\mathcal{K}/I_\mathcal{K}$, where $I_\mathcal{K}$ is the ideal generated by monomials corresponding to non-faces of $\mathcal{K}$. Note that $A_\mathcal{K}$ is generated as an algebra by monomials corresponding to faces of $\mathcal{K}$.

The ideal $I_\mathcal{K}$ is graded since it is defined by monomials. This leads us to define the *Stanley–Reisner scheme* $\mathbb{P}(\mathcal{K})$ as $\operatorname{Proj} A_\mathcal{K}$.

Remark 1.4.2. The ideal $I_\mathcal{K}$ is generated by the non-faces of $\mathcal{K}$, but it is minimally generated by the *minimal non-faces* of $\mathcal{K}$, just as a simplicial complex is determined by its *maximal* facets. ◇

Example 1.4.3. Let $\mathcal{K}$ be the triangle with vertices $\{v_1, v_2, v_3\}$. Its maximal faces are v_1v_2, v_2v_3 and v_1v_3. The Stanley–Reisner ring is $k[v_1, v_2, v_3]/(v_1v_2v_3)$.

Note that $\operatorname{Proj}(A_\mathcal{K})$ deforms to a smooth cubic curve. ♡

Example 1.4.4. Let $\mathcal{K}$ be a hexagon with vertices $\{v_1, \ldots, v_6\}$, indexed cyclically. The minimal non-faces are the edges v_iv_{i+2} and v_iv_{i+3} (indices taken modulo 6). Thus the Stanley–Reisner ring is $k[v_1, \ldots, v_6]/\left(v_iv_{i+2}, v_iv_{i+3}\right)_{i=1,\ldots,6}$. Its Proj is a degenerate elliptic curve. ♡

The *join* of two simplicial complexes $\mathcal{K}$ and $\mathcal{K}'$ is defined as

$$\mathcal{K} * \mathcal{K}' \triangleq \{f \sqcup g \mid f \in \mathcal{K}, g \in \mathcal{K}'\},$$

where $\sqcup$ denotes the disjoint union. We have that $\mathbb{P}(\mathcal{K} * \mathcal{K}') = \mathbb{P}(\mathcal{K}) * \mathbb{P}(\mathcal{K}')$, where the second star means the join of two projective varieties.

If $f \subset \mathcal{K}$ is a face, the *link of f in $\mathcal{K}$* is the simplicial complex defined by

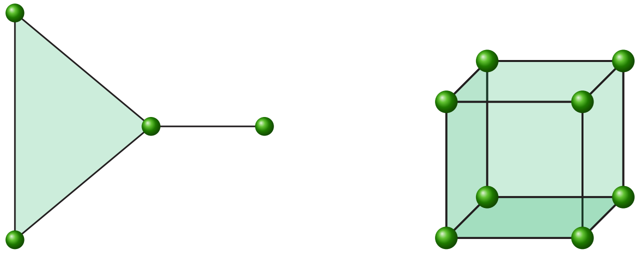
$$\operatorname{lk}(f, \mathcal{K}) \triangleq \{g \in \mathcal{K} \mid f \cap g = \emptyset, f \cup g \in \mathcal{K}\}.$$

If $D_+(x_f) \subset \mathbb{P}(\mathcal{K})$ denotes the distinguished open set corresponding to the monomial x^f, we have that $D_+(x_f) = \mathbb{A}(\operatorname{lk}(f, \mathcal{K})) \times (k^*)^{\dim f}$.

Every simplicial complex has a *geometric realization*, which as a set is defined as follows:

$$|\mathcal{K}| \triangleq \left\{\alpha : [n] \to [0, 1] \mid \operatorname{supp}(\alpha) \in \mathcal{K}, \sum_{i=1}^{n} \alpha(i) = 1\right\}.$$

This is an example of a piecewise linear manifold. For more on piecewise linear manifolds and combinatorial topology, we refer the reader to one of [Gla70; Spa66; Hud69].



(a) A non-manifold simplicial complex. The link at the two right vertices are not spheres.

(b) A simplicial manifold.

Figure 1.3: Two examples of simplicial complexes.

Motivated by this, we single out a class of simplicial complexes:

Definition 1.4.5. A simplicial complex $\mathcal{K}$ is called a *manifold* if the geometric realization of every link $\operatorname{lk}(\mathcal{K}, v)$ (v is a vertex) is homeomorphic to a sphere.

See Figure 1.3 for a non-example and an example of simplicial manifolds.

A good reference for more on simplicial complexes is Stanley's green book [Sta96].

1.4.1 Smoothings of Stanley–Reisner schemes

Because many properties of smooth varieties are easier read off from their degenerations, it is an interesting problem to study smoothings of Stanley–Reisner-schemes (or conversely: degenerations of smooth schemes to Stanley–Reisner schemes). They are highly singular, but their ideal structure is much simpler than that of smooth schemes.

We state a few lemmas to give a feel for how the theory of simplicial complexes relate to their deformations.

Lemma 1.4.6. *If $\mathcal{K}$ is a simplicial complex, then $H^i(\mathcal{K}; k) \simeq H^i(\mathbb{P}(\mathcal{K}), \mathcal{O}_{\mathbb{P}(\mathcal{K})})$.*

The lemma is essentially due to Hochster, and is proved (in a different form) in Stanley's book [Sta96]. This is true essentially because the Čech complex computing the simplicial cohomology and the Čech complex computing sheaf cohomology look exactly the same.

Lemma 1.4.7. *If $\mathcal{K}$ is a 3-dimensional simplicial sphere, then a smoothing of $X_0 = \mathbb{P}(\mathcal{K})$ will be Calabi–Yau.*

Proof. Let $\pi : \mathscr{X} \to S$ be a smoothing. Since $\mathcal{K}$ is a sphere, it follows from Lemma 1.4.6 that $H^i(X_0, \mathscr{O}_{X_0}) = k$ for $i = 0, 3$, and zero for $i \neq 0, 3$. The triviality of the canonical bundle is proved in Theorem 6.1 in [BE91]. Since $H^1(\mathcal{K}; k) = H^2(\mathcal{K}; k) = 0$, it follows from the semicontinuity theorem (Theorem 12.8 in Chapter III in [Har77]) that $H^i(X_t, \mathscr{O}_{X_t}) = 0$ for all $t \in S$. Similarly, if $\omega_0 \simeq \mathscr{O}_{X_0}$, all nearby fibers must have trivial canonical bundle as well. ∎

It is an important fact that since Stanley–Reisner rings are defined by monomial ideals, their coordinate rings and all important modules associated to them are *multigraded*, meaning that they are graded not only by $\mathbb{Z}$, but by $\mathbb{Z}^n$. If M is a multigraded module, we write $M_{\mathbf{a}}$ for the component of M in degree $\mathbf{a} \in \mathbb{N}$. Given a weight vector $\mathbf{c} \in \mathbb{Z}^n$, we can write $\mathbf{c}$ uniquely as $\mathbf{a} - \mathbf{b}$ with $\mathbf{a}, \mathbf{b} \in \mathbb{N}_0$, such that $\mathbf{a}$ and $\mathbf{b}$ have disjoint supports[3]. The *support* $[\mathcal{K}]$ *of a simplicial complex* $\mathcal{K}$ is defined by:

$$[\mathcal{K}] = \{i \in [n] \mid \{i\} \in \mathcal{K}\}.$$

The following is a result by Altmann and Christophersen ([AC10, Theorem 4.6]. It expresses the deformation theory of Stanley–Reisner schemes purely in terms of their combinatorial data. We refer the reader to the original article for the details.

Theorem 1.4.8. *If $\mathcal{K}$ is a simplicial manifold, and $\mathbf{c} = \mathbf{a} - \mathbf{b}$ (with disjoint supports a and b), then*

$$\dim_k T^1 \left(A_{\mathcal{K}}/k, A_{\mathcal{K}}\right)_{\mathbf{c}} = \begin{cases} 1 & \text{if } a \in \mathcal{K} \text{ and } b \in \mathcal{B}(\mathrm{lk}(a, \mathcal{K})) \\ 0 & \text{otherwise.} \end{cases}$$

Here $\mathcal{B}(\mathcal{K})$ is defined as follows:

Definition 1.4.9. The set $\mathcal{B}(\mathcal{K})$ is the set of $b \subseteq [\mathcal{K}]$ with $|b| \geq 2$ such that

1. $\mathcal{K} = L * \partial b$, where $|L|$ is an $(n - |b| + 1)$-sphere, if $b \notin \mathcal{K}$.

2. $\mathcal{K} = L * \partial b \cup \partial L * \bar{b}$ where $|L|$ is an $(n - |b| + 1)$-ball, if $b \in \mathcal{K}$.

The theorem is useful in that it says that certain faces of a simplicial complex contribute more than others to the space of deformations. There is also a similar result for $T^2(A_{\mathcal{K}}/k, A_{\mathcal{K}})$, saying that certain kinds of faces contribute to the obstruction space.

[3]The *support* of a vector $\mathbf{a}$ is the set of its non-zero coordinates.

1.5 Calabi–Yau manifolds and mirror symmetry

The main contribution of this thesis is concerned with the construction of new Calabi–Yau manifolds. In this chapter we define what they are, and give examples on how to construct them.

Definition 1.5.1. A *Calabi–Yau manifold* is an irreducible complex projective variety X such that $\omega_X \simeq \mathscr{O}_X$ and $H^i(X, \mathscr{O}_X) = 0$ for $i = 1, \dots, \dim X - 1$.

We will always have $\dim X = 3$. Beware that the literature often requires Calabi–Yau manifolds to be smooth, or to have only certain kinds of singularities.

Mathematically, Calabi–Yau varieties are interesting because they are among the varieties having Kodaira dimension zero. This means that they have trivial canonical models, making them harder to study.

Before the 90's there were only sporadic constructions of Calabi–Yau varieties, but after the advent of toric geometry and the construction of Batyrev in [Bat94], thousands of new examples were found, all of which were anticanonical sections in Fano toric varieties.

Let Ω^1_X be the sheaf of holomorphic one-forms on X, and assume that $\dim X = 3$. Let h^{ij} denote the dimension of $H^j(X, \Omega^i_X)$. Here Ω^i_X is by definition the wedge product $\wedge^i \Omega^1_X$. Then we can form the *Hodge diamond* of X:

$$\begin{array}{ccccccc}
 & & & h^{00} & & & \\
 & & h^{01} & & h^{10} & & \\
 & h^{02} & & h^{11} & & h^{20} & \\
h^{03} & & h^{12} & & h^{21} & & h^{30} \\
 & h^{13} & & h^{22} & & h^{31} & \\
 & & h^{23} & & h^{32} & & \\
 & & & h^{33} & & &
\end{array}$$

Because of the Calabi–Yau condition, we have that $h^{j0} = 0$ for $0 < j < 3$, and also that $h^{00} = h^{0d} = 1$. It follows by Serre duality (see [Har77, Corollary 7.7, Chapter III]) that $h^{ij} = h^{3-i,3-j}$. Note that this amounts to a horizontal symmetry of the Hodge diamond. Since X was assumed to be a complex manifold, it follows by complex conjugation that $h^{ij} = h^{ji}$. This amounts to vertical symmetry of the Hodge diamond. It follows that for 3-dimensional Calabi–Yau varieties, the Hodge diamond simplifies to

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{11} & & 0 & \\ 1 & & h^{12} & & h^{12} & & 1 \\ & 0 & & h^{11} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} .$$

The *Hodge decomposition* theorem [Voi02, page 142] states that the singular cohomology groups decomposes as

$$H^k(X, \mathbb{C}) = \bigoplus_{i+j=k}^{\dim X} H^i(X, \Omega_X^j).$$

The *topological Euler characteristic* is defined as

$$\chi(X) = \sum_{k=0}^{2 \dim X} (-1)^k \dim_{\mathbb{C}} H^k(X, \mathbb{C}).$$

For 3-dimensional Calabi–Yau varieties, it follows from the above discussion that $\chi(X)$ can be computed as $2(h^{11} - h^{12})$.

Example 1.5.2. The canonical example of a Calabi–Yau variety is the quintic in $\mathbb{P}^4$. Let $X = V(f)$ be the zero locus of a general element in $H^0\left(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(5)\right)$. Then X is a smooth threefold, and by the adjunction formula we have

$$\omega_X = \omega_{\mathbb{P}^4}\big|_X \otimes \det\left((f)/(f)^2\right)^\vee = \omega_{\mathbb{P}^4}\big|_X \otimes \mathscr{O}_X(5) = \mathscr{O}_X(-5) \otimes \mathscr{O}_X(5) = \mathscr{O}_X,$$

so the canonical bundle is trivial. By the ideal sheaf sequence, we find that $H^i(X, \mathscr{O}_X) \simeq H^i(X, \mathscr{O}_{\mathbb{P}^4}(-5))$, for $i \geq 0$, which by [Har77, Theorem 5.1, Chapter III] implies the required vanishing of the structure sheaf cohomology groups.

The Euler characteristic can be computed as the degree of the top Chern class of X. If Y is a degree d hypersurface in $\mathbb{P}^n$, the following formula holds:

$$c_{n-1}(T_X) = h^{n-1}\left(\binom{n+1}{n-1} - d\binom{n+1}{n-2} + d^2\binom{n+1}{n-3} + \dots\right),$$

where h is the class of a hyperplane. Putting $n = 4$ and $d = 5$, we find that $\chi(X) = -200$.

To compute h^{11}, we consider the conormal sequence:

$$0 \to \mathscr{O}_X(-5) \to \Omega^1_{\mathbb{P}^4}\big|_X \to \Omega^1_X \to 0.$$

Then we see that $H^1(X, \Omega^1_X) \simeq H^1(X, \Omega^1_{\mathbb{P}^4}\big|_X)$. Finally, consider the restricted Euler sequence:

$$0 \to \Omega^1_{\mathbb{P}^4}\big|_X \to \mathscr{O}_X(-1)^5 \to \mathscr{O}_X \to 0.$$

By considering the associated long exact sequence, we see easily that $h^{11} = 1$, and since $\chi(X) = 2(h^{11} - h^{12})$ we find that $h^{12} = 101$. ♡

In general it is very hard to compute the Hodge numbers of Calabi–Yau varieties, with the exception of hypersurfaces in four-dimensional toric varieties. Often the best one can hope for is the topological Euler characteristic $\chi(X)$, which is much easier to compute.

A variety Y is *Fano* if the anticanonical line bundle ω_Y^{-1} is ample. Recall the statement of Kodaira vanishing, which says that if $\mathscr{L}$ is an ample invertible sheaf, then $H^q(Y, \mathscr{L} \otimes \Omega^p_Y) = 0$ for $p + q > d$, where $d = \dim Y$. Putting $\mathscr{L} = \omega_Y^{-1}$, and $p = d$, we find that $H^q(Y, \mathscr{O}_Y) = 0$ for $q > 0$. This fact will be used in the proof below.

Remark 1.5.3. Kodaira vanishing only holds for smooth varieties, but since $\dim_k H^q(Y, \mathscr{O}_Y)$ is upper semi-continuous, it follows that all *smoothable* Fano varieties have $H^q(Y, \mathscr{O}_Y) = 0$ as well. ◇

Given a Fano variety, there is an associated family of complete intersection Calabi–Yau varieties:

Proposition 1.5.4. *Let $Y \subset \mathbb{P}^N$ be an n-dimensional Fano variety with $\omega_Y = \mathscr{O}_Y(-k)$. Suppose $n > 1$. Then a general section X of $\mathscr{O}_Y(1)^{\oplus k}$ is an $n - k$-dimensional Calabi–Yau variety.*

Proof. The triviality of the canonical bundle follows from the adjunction formula, which says that

$$\omega_X = \omega_Y\big|_X \otimes \bigwedge^k (\mathcal{I}_X/\mathcal{I}_X^2)^\vee.$$

A general section of $\mathscr{O}_Y(1)^{\oplus k}$ is a complete intersection, and the normal bundle is then equal to $\mathscr{O}_X(1)^{\oplus k}$. It is then true that $\wedge^k \mathscr{O}_X(1)^{\oplus k} = \mathscr{O}_X(k)$, from which it follows that the canonical bundle is trivial.

From Remark 1.5.3 we have that the cohomology groups $H^i(Y, \mathscr{O}_Y) = 0$ for $i > 0$ when Y is a Fano variety. The vanishing of the cohomology groups $H^i(X, \mathscr{O}_X)$ for $i = 1, \ldots, n - k$ can be seen as follows. The structure sheaf $\mathscr{O}_X$ has a Koszul resolution of the form

$$0 \to \mathscr{O}_Y(-k) \to \mathscr{O}_Y(-k+1)^{\oplus \binom{k}{k-1}} \to \ldots \to \mathscr{O}_Y{}^{\oplus k}(-1) \to \mathcal{I}_X \to 0.$$

Note that all terms $\mathscr{O}_Y(-j)$ with $0 < j < k$ are cohomologically trivial, in the sense that $H^*(Y, \mathscr{O}_Y(-j)) = 0$. An induction argument then shows that

$$H^p(Y, \mathcal{I}_X) \simeq H^{p+k-1}(Y, \mathscr{O}_Y(-k)) \tag{1.2}$$

for all p. Consider the ideal sheaf sequence

$$0 \to \mathcal{I}_X \to \mathscr{O}_Y \to \mathscr{O}_X \to 0.$$

The beginning of the associated long exact sequence is

$$0 \to H^0(Y, \mathcal{I}_X) \to H^0(Y, \mathscr{O}_Y) \to H^0(X, \mathscr{O}_X) \to H^1(Y, \mathcal{I}_X).$$

It follows by (1.2) that the first group is equal to

$$H^{k-1}(Y, \mathscr{O}_Y(-k)) \overset{\text{Serre}}{\simeq} H^{n-k+1}(Y, \mathscr{O}_Y) = 0$$

for $k > 0$. The right term is equal to

$$H^k(Y, \mathscr{O}_Y(-k)) \overset{\text{Serre}}{\simeq} H^{n-k}(Y, \mathscr{O}_Y) = 0.$$

Now assume $i > 0$. Then we find that $H^i(X, \mathscr{O}_X) \simeq H^{i+1}(Y, \mathcal{I}_X)$. From the observation above, this group is non-zero only when $n - k - i = 0$. Thus

$$H^i(X, \mathscr{O}_X) = \begin{cases} k & i = 0 \\ 0 & i < n-k \\ k & i = n-k. \end{cases}$$

Since X has dimension $n - k$, we have now proved that X is Calabi–Yau, since we have checked the triviality of the canonical sheaf and the vanishing of the middle cohomology groups. ■

1.5.1 Mirror symmetry

After the invention of string theory in the late 60's, Calabi–Yau varieties caught the attention of theoretical physicists. They predict that space-time is really 10-dimensional, and locally looks like $\mathbb{R}^4 \times X$, where X is a Calabi–Yau manifold of complex dimension 3.

They predicted that every Calabi–Yau manifold X has a "mirror partner" X° in such a way that there is a natural isomorphism between the moduli space of complex structures on X (whose dimension is $h^{11}(X)$), and the moduli space of Kähler structures on X° (whose dimension is $h^{12}(X^\circ)$), and vice versa. It follows that their Hodge numbers satisfy $h^{11}(X) = h^{12}(X^\circ)$ and $h^{12}(X) = h^{11}(X^\circ)$.

This correspondence was named *mirror symmetry* because by going from X to X°, the Hodge diamond is "mirrored" horizontally.

In the 90's, Candelas et al. constructed the mirror of the general quintic [Can+91]. They calculated certain Hodge theoretic invariants on the mirror, and used them to count *rational curves* of all degrees on the general quintic. This greatly surprised the mathematical community, because earlier this computation had only been done for low degree curves.

The mathematical proof of this curve counting led to the invention of Gromov–Witten-invariants and homological mirror symmetry. Today mirror symmetry is often best understood as an equivalence between two derived categories.

Mirror symmetry is a fascinating and notoriously technical topic. There are several good introductions, depending upon taste and technical proficiency. Two of the most comprehensive introductions are [CK99] and [Hor+03].

Explicitly constructing mirrors of Calabi–Yau manifolds have become a small industry in the mathematics community. In the last chapter of this thesis, we propose mirror candidates for two of our Calabi–Yau constructions.

1.5.2 The mirror construction Ansatz

In many cases of interest, given a construction of a Calabi–Yau manifold, the following Ansatz produces a mirror.

Let $\mathcal{K}$ be a simplicial complex, with associated Stanley–Reisner scheme X_0. Let G be the automorphism group (or a subgroup of the automorphism group) of $\mathcal{K}$. Then G induces an action on $T^1_{X_0} \triangleq T^1(S_{X_0}/k, S_{X_0})$ in the following way: each element of $T^1_{X_0}$ can be represented by a $\phi \in \mathrm{Hom}(I/I^2, A)$, and then $g \cdot \phi$ is given by $(g \cdot \phi)(f) = g \cdot \phi(g^{-1} \cdot f)$.

There is an action of $T_n = (\mathbb{C}^*)^{n+1}/\mathbb{C}^*$ on $\mathbb{P}^n$, and since I_{X_0} is generated by monomials, the action restricts to an action on X_0 as well.

Given a smoothing family with general fiber X and special fiber X_0, we can consider a subfamily with only isolated singularities on which G act. Let $H \subset T_n$ be the subgroup of the torus acting on this family. Then the mirror candidate to X is given by a crepant resolution of $Y_t = X_t/H$.

Though it is often overlooked (or stated differently) in the literature, even the mirror construction of the famous quintic arises this way. Briefly, the quintic Calabi–Yau is given by the zero locus of a general element in $H^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(5))$. The special quintic given by the zeroes of $f = x_0x_1x_2x_3x_4$ is the Stanley–Reisner scheme associated to the 3-simplex. The automorphism group is S_5, and an invariant 1-parameter family is given by $f_t = \sum_{i=0}^4 x_i^5 + t x_0x_1x_2x_3x_4$. The fiber at $t = \infty$ is the Stanley–Reisner scheme.

There is an $H \triangleq (\mathbb{Z}/5)^5/\mathbb{Z}^5$-action on $X_t = Z(f_t)$ given by coordinate-wise multiplication by fifth roots of unity. Thus H is a subgroup of T_5. The general element of the family X_t is smooth, so the only singularities of the quotient $Y_t = X_t/H$ comes from points with non-trivial stabilizer. These can be resolved by methods from toric geometry. For details, see for example the first chapter of Ingrid Fausk's thesis [Fau12].

In the last chapter of this thesis, we use this Ansatz to produce mirror candidates for two of our Calabi–Yau constructions.

CHAPTER 2

Hyper-Kähler manifolds and triangulations of $\mathbb{CP}^2$

This chapter will not contain any new results of any significance, but is rather a report on an idea which branched off to the explorations in the next two chapters.

We explain an interesting connection between the topological space $\mathbb{CP}^2$ and degenerations of hyper-Kähler manifolds.

2.1 Hyper-Kähler manifolds

One often divides varieties into three types: those with positive, negative or trivial canonical class. Of those with trivial canonical class, three prominent types stand out: Calabi–Yau-manifolds, hyper-Kähler manifolds and complex tori.

Calabi–Yau manifolds look cohomologically like spheres (in the sense that $H^i(X, \mathscr{O}_X) \simeq H^{2i}(S^n; k)$). Complex tori (which are $\mathbb{C}^n$ modulo a lattice), have structure sheaf cohomology $H^i(X, \mathscr{O}_X) = \wedge^i \mathbb{C}^n$, and trivial tangent bundle. Hyper-Kähler manifolds have trivial fundamental group, as do Calabi–Yau manifolds, but non-trivial structure sheaf cohomology, as do complex tori.

Definition 2.1.1. A *hyper-Kähler manifold* X is a simply connected compact Kähler[1] complex manifold such that $H^0(X, \Omega_X^2)$ is generated by a non-degenerate 2-form $\sigma : TX \times TX \to \mathbb{C}$.

Remark 2.1.2. Because of the non-degeneracy of the symplectic form $\sigma \in H^0(X, \Omega_X^2)$, hyper-Kähler manifolds only occur in even dimensions: the determinant of the skew-symmetric form σ is $\det \sigma = (-1)^n \det \sigma$, implying $(-1)^n = 1$, so that n has to be even. ◇

[1]Recall that a complex manifold is *Kähler* if it is equipped with a Hermitian metric h whose associated two-form σ is closed. The two-form σ is defined by $\sigma(u, v) = \Re h(iu, v)$.

Remark 2.1.3. Since the two-form σ is non-degenerate, it follows that the canonical sheaf $\omega_X = \Omega^n_{X/\mathbb{C}}$ is trivial. The map $1 \mapsto \sigma^{n/2}$ gives an isomorphism $\mathscr{O}_X \to \omega_X$. ◇

Remark 2.1.4. In dimension 2, there is no difference between Calabi–Yau varieties and hyper-Kähler manifolds. These are the K3 surfaces[2]. ◇

For our purposes it will be useful to define a class of varieties similar to the class of hyper-Kähler manifolds.

Definition 2.1.5. Suppose X is a smooth projective variety over $\mathbb{C}$ satisfying

1. $H^1(X, \mathscr{O}_X) = 0$ and
2. $H^0(X, \Omega^2_X)$ is generated by a non-degenerate 2-form $\sigma : TX \times TX \to \mathbb{C}$.

Then we call X an *algebraic hyper-Kähler manifold*.

The first condition is an algebraic condition mimicking the $\pi_1(X)$-condition for hyper-Kähler manifolds.

Proposition 2.1.6. *If X is a projective hyper-Kähler manifold, then X is an algebraic hyper-Kähler manifold.*

Proof. By Hodge decomposition, we have $H^1(X;\mathbb{C}) = H^0(X, \Omega^1_X) \oplus H^1(X, \mathscr{O}_X)$. The left group is zero because it is equal to $\pi_1(X)/[\pi_1(X), \pi_1(X)] \otimes_{\mathbb{Z}} \mathbb{C}$, which by definition is trivial. It follows that both terms on the right-hand side are zero as well. ∎

Only a few explicit families of hyper-Kähler manifolds are known. Below we sketch the construction of two such families.

2.1.1 The Hilbert square $S^{[2]}$

Let S be a K3 surface with symplectic form σ, and let $S^{(2)}$ be its symmetric square: $S \times S/\{(p, q) \sim (q, p)\}$. Let $\pi_i : S \times S \to S$ be the two projections $(i = 1, 2)$. Then the 2-form $\pi_1^*\sigma + \pi_2^*\sigma$ is $\mathbb{Z}/2$-invariant. It follows that it descends to a 2-form τ on $S^{(2)}$.

The space $S^{(2)}$ is singular along the diagonal: locally it is isomorphic to $\mathbb{C}^2 \times \big(\mathbb{C}^2/(x \sim -x)\big)$. The second factor is a quadric cone, so a single blowup along the diagonal will resolve the singularities. The form τ lifts to a non-degenerate form on the blowup $\mathrm{Bl}_\Delta S^{(2)}$, which we denote by $S^{[2]}$. It can be shown that it is in fact a hyper-Kähler variety of dimension 4. The resulting space is denoted by $S^{[2]}$, and is called the *Hilbert square of S*, or the *Hilbert scheme of two points on S*. It parametrizes length two subschemes of S.

For more details on this construction, see Beauville's original paper [Bea83].

[2] K3 surfaces are named after Kummer, Kähler and Kodaira.

2.1.2 Lines on hypersurfaces

There is another construction of hyper-Kähler varieties that is relevant to us. Let X be a smooth cubic fourfold in $\mathbb{P}^5$. Let $F(X)$ denote the set of lines contained in X. It is the *Fano variety of lines on* X, and is a closed subset of the Grassmannian $\mathbb{G}(1, \mathbb{P}^5)$.

Proposition 2.1.7. *If X is a smooth cubic fourfold in $\mathbb{P}^5$, then $F(X)$ is a 4-dimensional (algebraic) hyper-Kähler variety.*

In the article [BD85], Beauville and Donagi shows that $F(X)$ is deformation equivalent to $S^{[2]}$ for some K3 surface S. They also show that if X is a *pfaffian* hypersurface, then $F(X)$ is actually *isomorphic* to $S^{[2]}$ for some K3 surface S. Furthermore, the family $\{F(X)\}$ obtained this way is 19-dimensional, and is a hypersurface in the deformation space of $S^{[2]}$.

For more details on hyper-Kähler manifolds and their constructions, we recommend the lecture notes by Lehn [Leh04].

2.2 Connection to the complex projective plane

Let X be a topological space. Recall that the symmetric product $X^{(2)}$ is defined as follows:

$$X^{(2)} \triangleq X \times X / \{(x, y) \sim (y, x)\}.$$

If $X = S^2$, we have that $X^{(2)}$ is naturally isomorphic to $\mathbb{CP}^2$, which can be seen as follows: S^2 can be identified with $\mathbb{P}^1_{\mathbb{C}}$. Unordered pairs of points in $\mathbb{P}^1$ correspond to degree 2 polynomials up to scalar multiplication. Hence we have identifications

$$(S^2)^{(2)} = (\mathbb{P}^1)^{(2)} = \{(P, Q) \in \mathbb{P}^1 \times \mathbb{P}^1\}/\mathbb{Z}_2 = \mathbb{P}\left(H^0(\mathscr{O}_{\mathbb{P}^1}(2))\right) = \mathbb{CP}^2.$$

Here is an observation.

Lemma 2.2.1. *If $\mathcal{K}$ is a simplicial complex that is a* manifold*, isomorphic to S^2, then a smoothing of $\mathcal{K}$ is K3 surface.*

Proof. See the article [BE91] by Eisenbud–Bayer. ∎

Stanley–Reisner degenerations of K3 surfaces correspond to triangulated 2-spheres. Since the symmetric square of a sphere is $\mathbb{CP}^2$, a Stanley–Reisner degeneration of the symmetric square of a K3 surface should correspond to a triangulated $\mathbb{CP}^2$.

Thus a naïve idea is this: since $F(X)$ is deformation equivalent to $S^{[2]}$, we would like to find the ideal of $F(X)$, and then find a square-free monomial degeneration of $F(X)$. This would correspond to a Stanley–Reisner triangulation of $\mathbb{CP}^2$:

Proposition 2.2.2. *Suppose $\mathcal{K}$ is a triangulation of $\mathbb{CP}^2$ and $X_0 = \mathbb{P}(\mathcal{K})$ is its associated Stanley–Reisner-scheme. Then a smoothing X of X_0 will be an algebraic hyper-Kähler manifold.*

Proof. The dimensions of the groups $H^i(X, \mathscr{O}_X)$ are in this case constant in flat families. Because of the triviality of the canonical bundle, we have that $h^0 = h^4 = 1$. Also, $h^1 = h^3 = 0$, and by semi-continuity h^0 and h^4 can not drop. Since $H^0(X, \Omega_X^2) = H^2(X, \mathscr{O}_X) = H^2(\mathcal{K}; \mathbb{C}) = \mathbb{C}$ (the first equality is complex conjugation), we have that $H^0(X, \Omega_X^2)$ is generated by a single 2-form. It is non-degenerate since $\omega_X \simeq \mathscr{O}_X$.

It follows that X is an algebraic hyper-Kähler manifold. ∎

2.3 Smoothing Stanley–Reisner schemes associated to triangulations of $\mathbb{CP}^2$

If $\mathcal{K}$ is a triangulation of $\mathbb{CP}^2$ and $\mathbb{P}(\mathcal{K})$ is the associated Stanley–Reisner-scheme, a smoothing of $\mathbb{P}(\mathcal{K})$ will give an algebraic hyper-Kähler manifold. Using this idea, and the `Macaulay2` package `VersalDeformations` (by Nathan Ilten, see [Ilt12]), we tried to find potentially new hyper-Kähler varieties. Unfortunately, it looks like all the triangulations we experimented with were not smoothable.

In the next four subsections we describe four different triangulations of $\mathbb{CP}^2$, their ideal structure, and compute some of their deformation theoretic invariants. In all cases we conclude that the corresponding Stanley–Reisner scheme is probably not smoothable.

Before we go on to describe the triangulations, we recall some basic facts about combinatorial manifolds.

We can decompose $\mathbb{CP}^2$ into three four-dimensional closed balls B_j, whose pairwise intersections are solid tori $\Pi_{ij} \triangleq B_i \cap B_j$, and whose triple intersection is a two-dimensional torus T. The closed ball B_0 is defined as

$$B_0 = \left\{[x_0 : x_1 : x_2] \in \mathbb{CP}^2 \mid x_0\overline{x_0} \geq x_1\overline{x_1},\, x_0\overline{x_0} \geq x_2\overline{x_2}\right\},$$

and similarly for B_1 and B_2. This is called the *equilibrium decomposition* of the complex projective plane.

A triangulation of $\mathbb{CP}^2$ is *equilibrium* if the closed balls, the solid tori, and the torus T are subcomplexes of the triangulation. Several of the triangulations below are equilibrium.

2.3.1 The 15-vertex triangulation

A very interesting triangulation $\mathcal{T}$ of $\mathbb{CP}^2$ is discovered in [Gai09] by Alexander Gaifullin. Gaifullin describes a triangulation of $\mathbb{CP}^2$ using 15 vertices. One reason it is interesting is that the corresponding Stanley–Reisner scheme $\mathbb{P}(\mathcal{T})$

has the same Hilbert-polynomial as $F_1(X)$, the Fano variety of lines on a cubic hypersurface. This means that they live in the *same* Hilbert scheme, and one could naively hope that they live in the same component as well, meaning that there exists a degeneration of $F_1(X)$ to $\mathbb{P}(\mathcal{T})$.

We will spend some time describing this triangulation, since parts of it inspired our construction of the Calabi–Yau's in the last chapter. We cite the definition *ad verbatim* from [Gai09].

Definition 2.3.1. Let $V_4 \subset S_4$ be the Klein four group. The vertex set of $\mathcal{T}$ is defined as

$$V = (V_4 \backslash \{e\}) \sqcup (\{1,2,3,4\} \times \{1,2,3\}) .$$

Thus the vertices of $\mathcal{T}$ are the permutations (12)(34),(13)(24) and (14)(23) and the pairs of integers (a,b) with $1 \leq a \leq 4$ and $1 \leq b \leq 3$. The maximal faces are spanned by the sets

$$\nu, (1, b_1), (2, b_2), (3, b_3), (4, b_4)$$

with $\nu \in V_4 \backslash \{e\}$ and $1 \leq b_a \leq 3$ $(a = 1,2,3,4)$ such that $b_{\nu(a)} \neq b_a$ for $a = 1,2,3,4$.

See Appendix A.4 for a SAGE [Wil17] script for computing the maximal facets of $\mathcal{T}$. The face-vector is $(15, 90, 240, 270, 108)$.

The triangulation $\mathcal{T}$ is the union over the cones over three 3-spheres S_j, so that $\mathcal{T}$ is an equilibrium triangulation. Each S_j is a very simple 3-sphere. It is the join of two hexagons (recall that $S^1 * S^1 \approx S^3$).

Remark 2.3.2. It is the Stanley–Reisner-scheme of S_j and some if its deformations that is studied in Chapter 4, leading to constructions of some new Calabi–Yau manifolds. ◇

We compute some deformation-theoretic invariants of $\mathbb{P}(\mathcal{T})$, the Stanley–Reisner scheme associated to $\mathcal{T}$.

Proposition 2.3.3. *We have:*

$$\dim_{\mathbb{C}} T^1(S_{\mathbb{P}(\mathcal{T})}/k, S_{\mathbb{P}(\mathcal{T})})_0 = 90$$
$$\dim_{\mathbb{C}} T^2(S_{\mathbb{P}(\mathcal{T})}/S_{\mathbb{P}(\mathcal{T})})_0 = 306.$$

The normal sheaf $\mathcal{N}_{\mathbb{P}(\mathcal{T})/\mathbb{P}^{14}}$ *has* 300 *global sections.*

The proof is a computation in `Macaulay2`. We remark that since $\mathbb{P}(\mathcal{T})$ is not Cohen–Macaulay, some standard comparison theorems does not hold. In our case we only have an inclusion $T^1(S_{\mathbb{P}(\mathcal{T})}/k, S_X)_0 \hookrightarrow T^1$, where the right module parametrizes all first-order deformations. See the article of Kleppe [Kle79] and his Theorem 3.9. This means that there might be deformations of $\mathbb{P}(\mathcal{T})$ that are not induced from the ambient projective space.

Because of the high number of parameters, we have not been able to say anything meaningful regarding the deformations of $\mathbb{P}(\mathcal{T})$. However, it is possible to deform $\mathbb{P}(\mathcal{T})$ into the union of three toric varieties, each being deformations of the Stanley–Reisner scheme $\mathbb{P}(B_j)$. This is not surprising, since B_j is a triangulation of the normal polyhedron of the corresponding toric variety. This deformation reduces the number of components of $\mathbb{P}(\mathcal{T})$ from 108 to 3.

It is not clear however if this union of toric varieties can be further deformed.

2.3.2 Kühnel's 9-vertex triangulation

The minimal triangulation $\mathcal{T}_9$ of $\mathbb{CP}^2$ is a 9-vertex triangulation with f-vector $(9, 36, 84, 90, 36)$. This implies that the associated Stanley–Reisner scheme $\mathbb{P}(\mathcal{T}_9)$ lives in $\mathbb{P}^8$ and is of degree 36. The automorphism group of $\mathcal{T}_9$ is a group of order 54, and it can be realized as a semidirect product $(\mathbb{Z}_3 \times \mathbb{Z}_3) \ltimes \mathbb{Z}_3 \ltimes \mathbb{Z}_2$. For a very readable account of the construction and motivation of this triangulation, consult the article [KB83] by Kühnel–Banchoff.

The ideal has a resolution of the form (in `Macaulay2` format):

```
        0  1  2  3  4 5 6
total: 1 36 90 84 37 9 1
    0: 1  .  .  .  . . .
    1: .  .  .  .  . . .
    2: .  .  .  .  . . .
    3: . 36 90 84 36 9 1
    4: .  .  .  .  . . .
    5: .  .  .  .  1 . .
```

This means that the ideal of $\mathbb{P}(\mathcal{T}_9)$ is generated by 36 cubic monomials, and there are 90 relations between them, lying in $\mathcal{O}_{\mathbb{P}(\mathcal{T}_9)}(-5)$, et cetera. Since the resolution is not symmetric, we see immediately that $\mathbb{P}(\mathcal{T}_9)$ is not arithmetically Gorenstein.

Proposition 2.3.4. *We have*

$$\begin{aligned} \dim_{\mathbb{C}} T^1(S_{\mathbb{P}(\mathcal{T})}/k, S_{\mathbb{P}(\mathcal{T})})_0 &= 21 \\ \dim_{\mathbb{C}} T^2(S_{\mathbb{P}(\mathcal{T})}/k, S_{\mathbb{P}(\mathcal{T})})_0 &= 126. \end{aligned}$$

The normal sheaf $\mathcal{N}_{\mathbb{P}(\mathcal{T}_9)/\mathbb{P}^8}$ has 93 global sections.

We can compute the action of the automorphism group on T^1. Using SAGE, we find that the 21 deformation parameters split in two orbits, one of size 3 and one of size 18.

We have not been able to lift any first-order deformation of $\mathbb{P}(\mathcal{T}_9)$ to a family over $\operatorname{Spec} \mathbb{C}[t]$.

2.3.3 The minimal equilibrium triangulation

In [BK92], Banchoff and Kühnel construct a 10 vertex equilibrium triangulation $\mathcal{T}_{10}$ of $\mathbb{CP}^2$. They start with the minimal 7-vertex triangulation of the torus, and then they construct $\mathcal{T}_{10}$ by taking cones over unions of three tori.

The automorphism group is of order 42, and comes from the symmetries of the torus.

The Betti table of the resolution of the ideal of $\mathbb{P}(\mathcal{T}_{10})$ is the following:

```
            0  1   2   3   4  5  6 7
     total: 1 38 128 177 123 46 10 1
         0: 1  .   .   .   .  .  . .
         1: .  3   2   .   .  .  . .
         2: .  .   .   .   .  .  . .
         3: . 35 126 175 120 45 10 1
         4: .  .   .   2   3  .  . .
         5: .  .   .   .   .  1  . .
```

Again we see that the ideal is not Gorenstein.

Proposition 2.3.5. *We have*

$$\dim_{\mathbb{C}} T^1(S_{\mathbb{P}(\mathcal{T})}/k, S_{\mathbb{P}(\mathcal{T})})_0 = 42$$
$$\dim_{\mathbb{C}} T^2(S_{\mathbb{P}(\mathcal{T})}/k, S_{\mathbb{P}(\mathcal{T})})_0 = 105.$$

The normal sheaf $\mathcal{N}_{\mathbb{P}(\mathcal{T}_{10})/\mathbb{P}^8}$ *has* 132 *global sections.*

In fact, it is possible to lift the versal family of deformation parameters to an honest family over $\operatorname{Spec}\mathbb{C}[t_1, \cdots, t_{42}]$, using the `VersalDeformations` package. Surprisingly, even though the T^2-module is big, there are no obstructions in the family (in the sense that the base space is $\mathbb{A}^{42}$). However, the generic member of this family is reducible (verified in `Macaulay2` for "random" values of the deformation parameters), implying that $\mathbb{P}(\mathcal{T}_{10})$ is not smoothable.

The automorphism group act transitively on the natural basis of T^1, so that $\dim_{\mathbb{C}} T^1(S_{\mathbb{P}(\mathcal{T})}/k, S_{\mathbb{P}(\mathcal{T})})_0^G = 1$.

2.3.4 The Bagchi–Datta triangulation

There is another 10-vertex triangulation $\mathcal{T}_{BD}$ of $\mathbb{CP}^2$, which is obtained as a $\mathbb{Z}/2$-quotient of a triangulation of $S^2 \times S^2$. It is described in the article [BD11] by Bagchi–Datta. The automorphism group is the alternating group A_4. The face-vector is $(10, 45, 110, 120, 48)$.

The triangulation is bistellarly equivalent to both the 9-vertex triangulation and the 10-vertex triangulation above.

Proposition 2.3.6. *We have*

$$\dim_{\mathbb{C}} T^1(S_{\mathbb{P}(\mathcal{T})}/k, S_{\mathbb{P}(\mathcal{T})})_0 = 41$$
$$\dim_{\mathbb{C}} = T^2(S_{\mathbb{P}(\mathcal{T})}/k, S_{\mathbb{P}(\mathcal{T})})_0 = 180.$$

The normal sheaf $\mathcal{N}_{\mathbb{P}(\mathcal{T}_{BD})/\mathbb{P}^8}$ *has* 131 *global sections.*

We have not been able to find any meaningful lifting of the first-order deformations here either.

2.4 Naïve attempt to degenerate

Degenerating the ideal of $F_1(X) \subset \mathbb{P}^N$ to a square-free monomial ideal should give a triangulation of $\mathbb{CP}^2$. Since $F_1(X)$ sits inside $\mathbb{G}(1,5)$, and there are many known degenerations of $\mathbb{G}(1,5)$, we hoped that maybe $F_1(X)$ would degenerate inside $\mathbb{G}(1,5)$. Unfortunately, we did not succeed, mainly because we could not see any structure in the ideal of $F_1(X)$.

It was possible to explicitly compute $F_1(X)$ for some hypersurfaces, both pfaffian and non-pfaffian. However, the ideals were too complicated and the Gröbner bases too big to find any initial ideals with only square-free generators (and even their existence is unclear).

2.5 Conclusion

It would be interesting to study other triangulations of $\mathbb{CP}^2$. One way to proceed would be to start with existing triangulations, and analyze which faces of a given triangulation corresponds to basis elements of the T^2 module, perhaps using the results by Altmann–Christophersen from [AC10]. Then one can do bistellar flips away from these combinations, ideally obtaining triangulations corresponding to unobstructed Stanley–Reisner schemes.

This is an interesting and very hard question. Even with an unobstructed triangulation, it is not clear how to proceed to smooth it in a computationally feasible way. Already with Gröbner bases with 50 elements (for deformations of the 15-vertex triangulation, they had around 70 elements), computations take far too long (and consume too much memory) to be feasible to work with.

Without the presence of any good parallel processing Gröbner basis algorithms (which would allow the use of clustered super–computers), there is need for either more patience or smarter solutions to computational algebra problems.

CHAPTER 3

The two smoothings of $C(\mathrm{dP}_6)$

In this chapter we study the toric singularity that is the cone over the del Pezzo surface of degree 6. It has two topologically different smoothings, which we haven't seen studied in some detail before.

We describe the smoothings and show that they are topologically different. We also compute their singular cohomology groups in the classical topology, using techniques from toric geometry.

3.1 The del Pezzo surface dP_6

We start this chapter by talking about the del Pezzo surface of degree 6 in some generality. We first recall the definition of a del Pezzo surface:

Definition 3.1.1. A *del Pezzo surface* is a smooth surface with ample anti-canonical bundle. In other words, it is a 2-dimensional Fano variety.

Denote by dP_6 the blow-up of $\mathbb{P}^2$ in three non-collinear points. These points can be chosen to be the coordinate points $(1:0:0),(0:1:0)$ and $(0:0:1)$. Since the coordinate points are invariant under the natural torus action on $\mathbb{P}^2$, it follows that the dP_6 is a toric variety.

As a toric variety, it can be described as the toric variety defined by the planar hexagon depicted in Figure 3.1(a). The normal fan is in Figure 3.1(b).

The class of the anti-canonical sheaf is $-K = 3H - E_1 - E_2 - E_3$. It is proved in Hartshorne that this divisor is ample. Thus dP_6 is in fact a del Pezzo surface. Computing $(-K)^2$, we find that it has degree 6.

3.1.1 The Picard group

We will need a description of the Picard group of dP_6. By the description of dP_6 as a blowup in three points P_i of a projective space, it follows that it is generated by the hyperplane section H and the three exceptional divisors E_i $(i = 1, 2, 3)$, so that $\operatorname{Pic}\mathrm{dP}_6 \simeq \mathbb{Z}^4$.

If we order the basis of $\operatorname{Pic}\mathrm{dP}_6 = \mathbb{Z}^4$ as $\{H, E_1, E_2, E_3\}$, then the matrix of the intersection form $(D, D') \mapsto D \cdot D'$ is given by

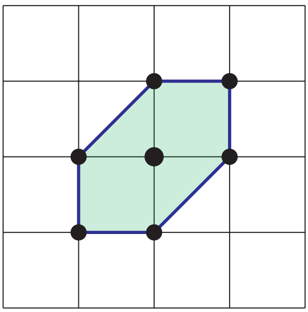
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

There are three other (-1)-curves on dP_6. Let L_{ij} be the line connecting P_i and P_j. By abuse of notation, denote by L_{ij} also the pullback of L_{ij} in the blowup. See Figure 3.2.

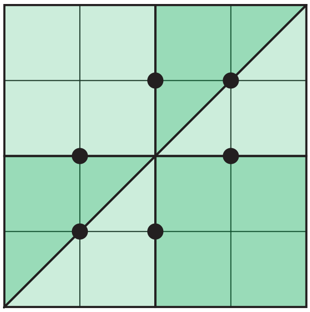
Since L_{ij} intersects P_i and P_j exactly once, it intersects E_i and E_j exactly once in the blowup. Thus $L_{ij} = H - E_i - E_j$ in the Picard group, and we can compute that the self-intersection L_{ij}^2 of L_{ij} is -1, where we have read off the coefficients from the intersection matrix.

Here is an interesting calculation (which we won't use later, but we found it interesting). There is an automorphism of dP_6 which is induced from the Cremona transformation $(x_0 : x_1 : x_2) \mapsto \left(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right)$ on $\mathbb{P}^2$. It induces a permutation of the lines in Figure 3.2: the exceptional divisors $\{E_1, E_2, E_3\}$ are switched with the lines L_{ij}.

This induces a linear automorphism of the Picard group, which in matrix



(a) The hexagon corresponding to dP_6.



(b) The fan over the polar polytope.

Figure 3.1: Toric description of dP_6.

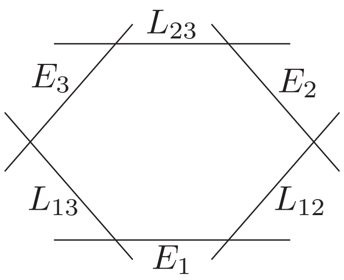


Figure 3.2: The six (-1)-lines in dP_6.

form is

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}.$$

The effect on the hexagon is a horizontal reflection.

3.1.2 Embedding in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Blowing up is "transitive", in the sense that blowing up two points is the same as blowing up one point, and then blowing up the inverse image of the second point. It follows that one way to find equations describing dP_6, is to blow up each point separately. Let x_0, x_1, x_2 be coordinates of $\mathbb{P}^2$. Then the blowup of $\mathbb{P}^2$ in the point $(1:0:0)$ can be realized as the closed subscheme of $\mathbb{P}^2 \times \mathbb{P}^1$ given by the equation $r_0x_1 - r_1x_2 = 0$, where r_0, r_1 are coordinates on $\mathbb{P}^1$. We can repeat this procedure on the two other points $(0:1:0)$ and $(0:0:1)$ to obtain similar equations. Collecting these, we see that dP_6 is given by the matrix equation

$$M\vec{x} = \begin{pmatrix} 0 & r_0 & -r_1 \\ s_1 & 0 & -s_0 \\ -t_0 & t_1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0$$

in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Since $\vec{x}$ is non-zero, it follows that we must have $\det M = 0$. It is not difficult to see that M cannot have rank 1 or lower, because that would force some of the $\mathbb{P}^1$-coordinates to be all zero. Consider the projection forgetting the $\mathbb{P}^2$-factor:

$$\pi : \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

The image of dP_6 is the hypersurface $\det M = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Any solution to this equation gives a unique solution to the equation $M\vec{x} = 0$: if

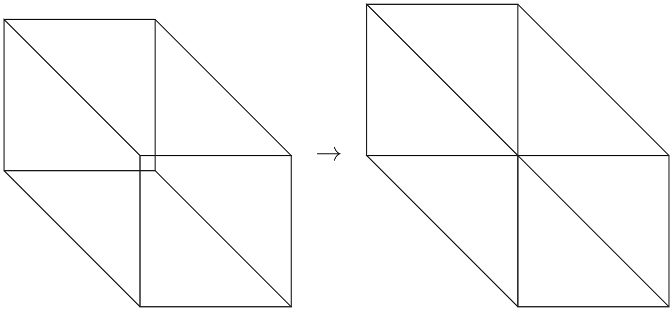


Figure 3.3: The projection of a cube onto a hexagon.

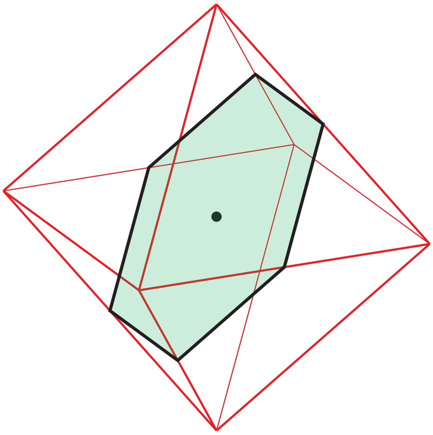


Figure 3.4: The inclusion of a hexagon in an octahedron.

$\det M = 0$, we must have that M is of rank 2. Thus there is a line of solutions, spanned by (x_0, y_0, z_1). Projectivizing, this correspond to a unique point in $\mathbb{P}^2$. Thus the restriction of π to dP_6 is an isomorphism onto the hypersurface $\det M = r_0 s_0 t_0 - r_1 s_1 t_1 = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Hence we have proven that dP_6 naturally embeds in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

It is also interesting to see how this embedding arises from a toric perspective using polytopes. Since $\mathbb{P}^1$ is the toric variety associated with the interval $[-1, 1] \subset \mathbb{R}$, it follows that $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the toric variety associated with the cube $\Delta = [-1, 1]^3 \subset M_{\mathbb{R}} = \mathbb{R}^3$. The inclusion of dP_6 in M induces a surjection of coordinate rings $\mathbb{C}[M] \to \mathbb{C}[\mathrm{dP}_6]$. This corresponds to the fact that there is a lattice projection of the cube onto the hexagon. See Figure 3.3.

Conversely, if N_1 is the fan of dP_6, and N_2 is the fan of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, we have an inclusion of lattices $N_1 \hookrightarrow N_2$, which is induced by an inclusion of convex polytopes, as in Figure 3.4.

The inclusion $N_1 \hookrightarrow N_2$ can be seen to be given by the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \tag{3.1}$$

Note that there are essentially four inclusions of the hexagon into the octahedron, because each inclusion is given by choosing a line through opposite faces of the cube (the line spanned by the normal vector of the hexagon), and there are 8 faces, hence 4 lines through opposite faces.

3.1.3 Embedding in $\mathbb{P}^2 \times \mathbb{P}^2$

On the other hand, blowups can also be realized as closures of graphs of rational maps. Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the Cremona transformation given by

$$(x_0 : x_1 : x_2) \mapsto \left(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2} \right).$$

Let $\Gamma \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the closure of the graph of φ. Then, in coordinates $(a_0 : a_1 : a_2) \times (b_0 : b_1 : b_2)$ on $\mathbb{P}^2 \times \mathbb{P}^2$, the equations $a_0 b_0 = a_1 b_1 = a_2 b_2$ hold on Γ. These are the equations of the blowup along the indeterminacy locus of the rational map φ. The indeterminacy locus is exactly the three coordinate points. Hence dP_6 can also be realized as the intersection of two $(1,1)$-divisors in $\mathbb{P}^2 \times \mathbb{P}^2$.

There is also in this case a description in terms of polytopes. The polytope associated with $\mathbb{P}^2 \times \mathbb{P}^2$ is $\Delta^2 \times \Delta^2$, the product of two 2-simplices. Also in this case, there is a projection onto a hexagon in $\mathbb{R}^2$. This is harder to visualize, but can be described as follows: if we order the vertices of Δ^2 by v_1, v_2, v_3, then the vertices of $\Delta^2 \times \Delta^2$ are of the form (v_i, v_j). The projection is then given by identifying the vertices (v_i, v_i).

Hence, using the Segre embedding, dP_6 lives naturally in both $\left(\mathbb{P}^1\right)^3 \hookrightarrow \mathbb{P}^7$ and $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$.

Remark 3.1.2. Intersecting $\mathbb{P}^2 \times \mathbb{P}^2$ with a single $(1,1)$-divisor gives us the projective space bundle corresponding to the tangent bundle of $\mathbb{P}^2$, which we denote by $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$. This follows from the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2} \xrightarrow{\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}} \mathscr{O}_{\mathbb{P}^2}(1)^3 \to \mathcal{T}_{\mathbb{P}^2} \to 0.$$

Since $\mathbb{P}(\mathscr{O}_{\mathbb{P}^2}(1)^3) = \mathbb{P}^2 \times \mathbb{P}^2$, the projective bundle $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ can be realized as the subset of $\mathbb{P}^2 \times \mathbb{P}^2$ such that $a_0 b_0 + a_1 b_1 + a_2 b_2 = 0$. The space $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ is a non-toric Fano 3-fold. ◇

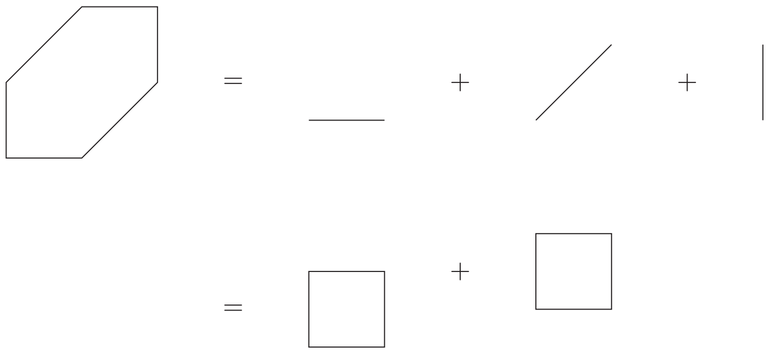


Figure 3.5: Minkowski-decompositions of the hexagon.

3.2 The cone over dP_6 and its two smoothings

The singularity $Z \triangleq C(\mathrm{dP}_6)$ is one of the most studied singularities with an obstructed deformation space. For example, in the paper [Alt97], Klaus Altmann describe a method to study the versal deformations of isolated affine Gorenstein toric singularities using only the combinatorial data of the toric singularity. He shows that different components of the base space correspond to different ways of writing the defining polytope as a Minkowski sum of smaller polytopes. See the illustration in Figure 3.5 for a decomposition of the hexagon.

Let S_Z denote the affine coordinate ring of $C(\mathrm{dP}_6)$. It has a natural $\mathbb{Z}$-grading, coming from the embedding in $\mathbb{P}^6$. From Altmann's article, or by using `Macaulay2`, ones computes that $\dim_k T^1(S_Z/k, S_Z) = 3$, and that $\dim T^2_k(S_Z/k, S_Z) = 2$. The versal base space decomposes into a union of a line and a plane. Both components are smoothing components.

It is worthwhile to note that both smoothings of Z arise by "sweeping out the cone": if X is a projective variety in $\mathbb{P}^n$, and Y is equal to $X \cap H$, where H is a section of $\mathscr{O}_{\mathbb{P}^n}(1)$, then the affine cone over Y deforms to a general hyperplane section of the affine cone over X. See the introduction of [Ste03] for more details.

3.2.1 Equations of smoothings

Using the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ and substituting from the linear equations in the description from Section 3.1.3, we can write the equations of dP_6 inside $\mathbb{P}^6$ as

$$\begin{vmatrix} y & x_1 & x_2 \\ x_4 & y & x_3 \\ x_5 & x_6 & y \end{vmatrix} \leq 1, \tag{3.2}$$

where ≤ 1, means taking all 2×2-minors.

On the other hand, dP_6 can be realized as a subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as well, as we described in Section 3.1.2. The equations can be described as follows: draw a cube, and let each vertex correspond to a variable. Then the equations of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding are given by taking all "minors" along all sides of the cube together with the three long diagonals. See Figure 3.6, in which we look at the cube from the front face. To get dP_6, one identifies two opposite corners, corresponding to the equation $a_{000} = a_{111}$ inside $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Thus in total there are $8 - 1 = 7$ variables, just as above.

The first smoothing is obtained by perturbing the equations of dP_6 as a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$. It can be described by perturbing two of the entries of the matrix shown below:

$$\begin{vmatrix} y & x_1 & x_2 \\ x_4 & y + t_1 & x_3 \\ x_5 & x_6 & y + t_2 \end{vmatrix} \leq 1. \tag{3.3}$$

For $t_1 = t_2 = 0$, we get the cone over dP_6, while for generic t_i, we get a smooth variety. In fact, we can compute that the discriminant locus (the set of points in $\mathbb{A}^2_{t_1,t_2}$ with singular fiber) are the t_1-axis, the t_2-axis and the line $t_1 = t_2$. Notice that the total space is equal to the cone over $\mathbb{P}^2 \times \mathbb{P}^2$.

Call (any) smooth fiber Z_2.

Lemma 3.2.1. *Let $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ be the projective space bundle associated to the tangent sheaf on $\mathbb{P}^2$. Then the smoothing Z_2 is isomorphic to $M \setminus \mathrm{dP}_6$.*

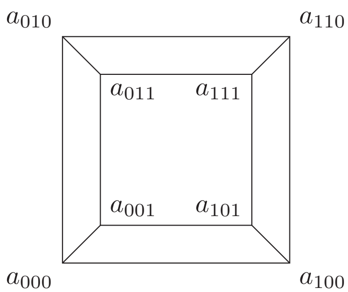


Figure 3.6: A $2 \times 2 \times 2$-tensor.

Proof. First homogenize the equations (3.3) with respect to a new variable y_1. Call the homogenized variety N. Put $y_0' = y$, $y_1' = y + ty_1$ and $y_2' = y + t_2 y_1$. Then we have the relation

$$h \triangleq (t_1 - t_2)y_0 + t_2 y_1' - t_1 y_2' = 0.$$

Hence we see that $N = \mathbb{P}^2 \times \mathbb{P}^2 \cap V(h)$. Let $\mathbb{P}^2 \times \mathbb{P}^2$ have coordinates z_0, z_1, z_2 and z_0', z_1', z_2'. Then h can be written as

$$(t_1 - t_2)z_0 z_0' + t_2 z_1 z_1' - t_1 z_2 z_2' = 0 = (z_0, z_1, z_2) \cdot ((t_1 - t_2)\, z_0', t_2 z_2', t_1 z_2').$$

in $\mathbb{P}^2 \times \mathbb{P}^2$. As long as $t_1 \neq t_2$ and $t_1, t_2 \neq 0$, we can do a change of coordinates in $\mathbb{P}^2_{z_0' z_1' z_2'}$, so that h transforms to

$$(z_0, z_1, z_2) \cdot (z_0', z_1', z_2') = 0.$$

Hence we see that M is isomorphic to the total space of the Grassmannian of lines in $\mathbb{P}^2$ (each point in one of the $\mathbb{P}^2$'s gives a line in the other $\mathbb{P}^2$). This is in turn isomorphic to $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$, since each tangent vector through a point determines a line through it.

What have we gained by homogenizing? The divisor at infinity is $y_1 = 0$, which is a dP_6 again. In our new coordinates this is equivalent to $y_1' = y_2' = y_0'$. ■

The other smoothing is obtained by replacing one of the corners of the cube in Figure 3.6 with $a_{000}' = a_{000} + t$. The total space is now the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Call this smoothing Z_1.

Lemma 3.2.2. *The smoothing Z_1 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathrm{dP}_6$.*

Proof. The proof is almost identical to the previous proof. ■

The following fact is well-known, and follows from the above two lemmas.

Proposition 3.2.3. *The two smoothings are topologically different.*

Proof. The Euler characteristic of $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ is 6, which follows from the next lemma.

This lets us calculate the Euler characteristics of the smoothings. Note that $\chi(\mathbb{P}^1) = 2$. By the Künneth formula, the Euler characteristic is multiplicative on products, so that $\chi(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 8$. By additivity of Euler characteristics we have $\chi(Z_1) = 2$ and $\chi(Z_2) = 0$, since $\chi(\mathrm{dP}_6) = 6$.

It follows that the two smoothing components correspond to topologically different smoothings, since the Euler characteristic is a topological invariant. ■

3.2.2 Topology of the smoothings

In this final section, we compute the simplicial homology groups of the two smoothings.

Lemma 3.2.4. *The cohomology ring of* $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ *is* $\mathbb{Z}[x,y]/(x^3, y^2+3y+3)$, *where* x *and* y *have degree* 2. *In particular, the cohomology of* $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ *is given by* $(1,0,2,0,2,0,1)$.

Proof. The first claim follows from the Leray-Hirch theorem. See [BT82, page 270]. The next claim follows since x and y both have degree 2. ∎

We first need a preliminary lemma from toric geometry. We state it in a general form, since we could not find a proper reference.

Lemma 3.2.5. *Let* $Y \overset{i}{\hookrightarrow} X$ *be a closed immersion of smooth toric varieties, corresponding to a map of fans* $\Sigma_1 \xrightarrow{A} \Sigma_2$. *Let* M_1 *and* M_2 *be the corresponding character lattices. Then we have a commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & M_2 \arrow[r, "R_1"] \arrow[d, "A^T"] & \mathbb{Z}^{\Sigma_2(1)} \arrow[r] \arrow[d, "C^T"] & \mathrm{Pic}(X) \arrow[r] \arrow[d, "i^*"] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "R_2"] & \mathbb{Z}^{\Sigma_1(1)} \arrow[r] & \mathrm{Pic}(Y) \arrow[r] & 0
\end{tikzcd}$$

Here $i^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ *is the map of Picard groups induced by the closed embedding.*

Proof. The horizontal rows are well-known. See for example Theorem 4.1.3 in [CLS11].

The matrix C^T is defined as follows: each primitive ray generator of cones in $\Sigma_1(1)$ can be thought of as lying in N_2 via the embedding A. The image lies in a unique minimal cone in $\Sigma_2(1)$, and as such, can be written as a unique linear combination of primitive ray generators from this cone. Let the columns of C be the coefficients of this linear combination. Then, by definition, the first square commutes.

It follows that there is an induced map of Picard groups. We must show that the induced map is exactly the one induced by the closed embedding. To see this, note that what the map C does, is to write divisors on Y as a linear combination of divisors on X, which correspond to the restriction to X (which is what the map i^* is).

Alternatively, consider the commutative diagram dual to the diagram in the lemma:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathrm{Pic}\, Y, \mathbb{Z}) \arrow[r] \arrow[d] & \mathbb{Z}^{\Sigma_1(1)} \arrow[r, "R_1^T"] \arrow[d, "C"] & N_1 \arrow[r] \arrow[d, hook, "A"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{Z}) \arrow[r] & \mathbb{Z}^{\Sigma_2(1)} \arrow[r, "R_2^T"] & N_1 \arrow[r] & 0
\end{tikzcd}$$

In Proposition 6.2.7 in [CLS11], there is a description of the induced Cartier divisor in terms of support functions. The proposition says that given a support function $\varphi : N_1 \to \mathbb{R}$ corresponding to a divisor D, the support function corresponding to $i^*(D)$ is given by composition with A. Our C is exactly a lift of the map A, and globally linear functions are trivial in $\mathrm{Hom}(\mathrm{Pic}\, Y, \mathbb{Z})$. Thus the statement that i^* is the induced function on Picard groups is just a reformulation of the Proposition in [CLS11] in terms of divisors (instead of support functions). ■

Example 3.2.6. Let us see how we can use Lemma 3.2.5 to find an explicit form of the induced map on Picard groups coming from the inclusion $\mathrm{dP}_6 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We use the matrix A from Equation (3.1). The rows of R_1 are the coordinates of the primitive ray generators of the rays of the fan of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. They are also the vertices of the octahedron in Figure 3.4.

The rows in R_2 are the coordinates of the hexagon in Figure 3.1(a).

In order to compute explicit cokernels, we need to find splittings of $\mathbb{Z}^6$ as $\mathbb{Z}^6 = \mathbb{Z}^3 \oplus \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ and $\mathbb{Z}^6 = \mathbb{Z}^2 \oplus \mathrm{Pic}(\mathrm{dP}_6)$, respectively.

This can be done explicitly by Gaussian elimination. We illustrate this with the first map. We start with the matrix (R_1, I_6), and after Gaussian elimination (row operations), we get the matrix (R_1', B).

$$\left(\begin{array}{ccc|cccccc}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right) \sim \left(\begin{array}{ccc|cccccc}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right)$$

The last three rows of B give a map $\pi_1 : \mathbb{Z}^6 \to \mathbb{Z}^3$ with kernel equal to the image of R_1. We do the same with the pair (R_2, I_6).

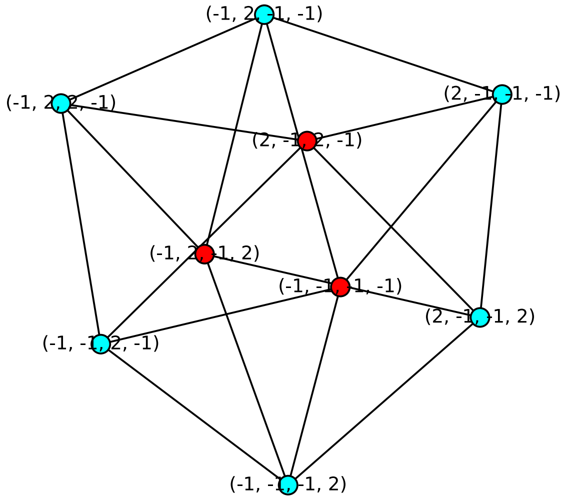


Figure 3.7: The edge graph of $\Delta \times \Delta$. The red vertices are the diagonal vertices v_{ii}.

We find that the induced map $i^* : \mathbb{Z}^3 \to \mathbb{Z}^4$ is given by the matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The matrix Q represents an injective map with non-torsion cokernel. We will use this information below in the proof of the next theorem. $\heartsuit$

Example 3.2.7. We repeat the previous example, with the embedding $\mathrm{dP}_6 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ instead. On the level of coordinate rings, it is induced by a projection of polytopes $\Delta^2 \times \Delta^2 \to H$ (where H denotes the hexagon in Figure 4.1).

The anticanonical polytope of $\mathbb{P}^2$ is the convex hull of the points $v_1 = (-1, 2)$, $v_2 = (-1, -1)$ and $v_3 = (2, -1)$. It follows that the anticanonical polytope of $\mathbb{P}^2 \times \mathbb{P}^2$ is the convex hull of the 9 vertices $v_{ij} \stackrel{\Delta}{=} v_i \times v_j \in \mathbb{R}^4$.

We want a projection sending the vertices v_{ii} $(i = 1, 2, 3)$ to the origin in $\mathbb{R}^2$. In Figure 3.7, we have visualized the edge graph of $\Delta^2 \times \Delta^2$. The three vertices that are sent to zero are marked in red.

By demanding that $v_{12} \mapsto (1, 0)$ and $v_{23} \mapsto (0, 1)$, together with $v_{ii} \mapsto (0, 0)$, we get a system of 8 linear equations, corresponding to a unique map $\mathbb{R}^4 \to \mathbb{R}^2$ with the required properties. We get:

$$A'^T = \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

The image generates a sublattice $\frac{1}{3}\mathbb{Z}^2 \subset \mathbb{Z}^2$. Replace A' by $A \triangleq 3A'$, and consider only the sublattice.

The images of the rays of the fan of dP_6 under A^T are exactly the 6 rays of the fan of $\mathbb{P}^2 \times \mathbb{P}^2$. This means that the map C^T in the Lemma is the identity matrix I_6, and we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^4 \arrow[r, "R_1"] \arrow[d, "A^T"] & \mathbb{Z}^6 \arrow[r] \arrow[d, "I_6"] & \mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2) = \mathbb{Z}^2 \arrow[r] \arrow[d, "i^*"] & 0 \\ 0 \arrow[r] & \mathbb{Z}^2 \arrow[r, "R_2"] & \mathbb{Z}^6 \arrow[r] & \mathrm{Pic}(\mathrm{dP}_6) = \mathbb{Z}^4 \arrow[r] & 0 \end{tikzcd} \tag{3.4}$$

It follows from the snake lemma that i^* is injective with zero cokernel. ♡

We are now ready to compute the (ranks of) singular cohomology groups of the two smoothings.

Theorem 3.2.8. *The two affine smoothings are topologically different. The homology groups are:*

Group	0	1	2	3	4	5	6	*Euler-characteristic*
$H_i(Z_1, \mathbb{Z})$	1	0	2	1	0	0	0	*2*
$H_i(Z_2, \mathbb{Z})$	1	0	1	2	0	0	0	*0*

Proof. The singular cohomology of $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is given by $(1, 0, 3, 0, 3, 0, 1)$, which can be computed by the Künneth formula (see [Hat02], page 275). The cohomology of dP_6 is given by $(1, 0, 4, 0, 1)$.

We will use the Lefschetz duality theorem [Spa66], which in this case says that $H_q(M \backslash dP_6; \mathbb{Z}) \simeq H^{6-q}(M, dP_6; \mathbb{Z})$. The long exact sequence of the pair (M, dP_6) ([Hat02], page 200) takes the form:

$$\begin{tikzcd}
0 \arrow[r] & H^0(M,D;\mathbb{Z}) \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[dll, "\delta"] \\
 & H^1(M,D;\mathbb{Z}) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, "\delta"] \\
 & H^2(M,D;\mathbb{Z}) \arrow[r] & \mathbb{Z}^3 \arrow[r] & \mathbb{Z}^4 \arrow[dll, "\delta"] \\
 & H^3(M,D;\mathbb{Z}) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, "\delta"] \\
 & H^4(M,D;\mathbb{Z}) \arrow[r] & \mathbb{Z}^3 \arrow[r] & \mathbb{Z} \arrow[dll, "\delta"] \\
 & H^5(M,D;\mathbb{Z}) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, "\delta"] \\
 & H^6(M,D;\mathbb{Z}) \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

From the exactness of the sequence, we immediately find $H^0(Z_1;\mathbb{Z}) = \mathbb{Z}$. Also, since $H^0(M;\mathbb{Z}) \to H^0(\mathrm{dP}_6;\mathbb{Z})$ is an isomorpism (both are connected), it follows that $H^6(Z_1;\mathbb{Z}) = H^5(Z_1;\mathbb{Z}) = 0$.

The other groups depend upon the explicit form of the maps

$$H^2(M;\mathbb{Z}) \to H^2(\mathrm{dP}_6;\mathbb{Z}) \text{ and } \ H^4(M;\mathbb{Z}) \to H^4(\mathrm{dP}_6,\mathbb{Z}).$$

The map $H^2(M;\mathbb{Z}) \to H^2(\mathrm{dP}_6;\mathbb{Z})$ can be identified with the map

$$i^*\colon\ \mathrm{Pic}(M) \to \mathrm{Pic}(\mathrm{dP}_6)$$

induced by the inclusion. This map was computed in Example 3.2.6. It is an injective map with torsion-free cokernel, and it follows from the long-exact sequence and the Lefschetz theorem that $H_3(Z_1;\mathbb{Z}) \simeq H^3(M, dP_6;\mathbb{Z}) \simeq \mathbb{Z}$, and also that $H_4(Z_1;\mathbb{Z}) = 0$.

To compute the map $H^4(M;\mathbb{Z}) \to H^4(\mathrm{dP}_6;\mathbb{Z})$, note that $H^4(M;\mathbb{Z})$ is Poincaré dual to $H_2(M;\mathbb{Z})$, and this group is generated by $\mathbb{P}^1 \times \{pt\} \times \{pt\}$ (and permutations). Also, $H^4(\mathrm{dP}_6;\mathbb{Z}) \simeq H_0(\mathrm{dP}_6,\mathbb{Z}) = \mathbb{Z}$. In this description, pullback corresponds to intersection, and one sees that the map is given by $(a,b,c) \mapsto a+b+c$, since the three $\mathbb{P}^1$'s intersect dP_6 in a single point each[1]. This map has two-dimensional kernel, and we conclude that $H_2(Z_1;\mathbb{Z}) \simeq H^4(M,\mathrm{dP}_6;\mathbb{Z}) = \mathbb{Z}^2$, and that $H^1(Z_1;\mathbb{Z}) = 0$.

The computations for Z_2 are similar. We first note that the Picard group of $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ is generated by the pullbacks F, G of the generators of

$$\mathrm{Pic}(\mathbb{P}^2_{x_0x_1x_2} \times \mathbb{P}^2_{y_0y_1y_2}).$$

Say F is represented by $V(x_0)$ and G is represented by $V(y_0)$.

[1]We thank the math.stackexchange user *nefertiti* for this argument.

Again we compute the intersections of F and G with dP_6. Intersecting with F is computed by decomposing the ideal $(x_0, x_1y_0 - x_2y_1, x_1y_0 - x_0y_2)$ in $k[x_0, x_1, x_2, y_0, y_1, y_2]$ and saturating by (x_0, x_1, x_2) and (y_0, y_1, y_2). This can either be done by hand or by using `Macaulay2`. Either way, we find that $F\big|_{dP_6} = E_3 + L_{23} + E_2 = H$, using the notation from earlier this chapter. Similarly $G\big|_{dP_6} = L_{23} + L_{12} + E_2 = 2H - E_1 - E_2 - E_3$. Hence the map on cohomology is given by the matrix

$$H^2(M;\mathbb{Z}) \simeq H_4(M;\mathbb{Z}) \simeq \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}} \mathbb{Z}^4 \simeq H_2(\mathrm{dP}_6;\mathbb{Z}) \simeq H^2(\mathrm{dP}_6;\mathbb{Z}).$$

This is an injective map, and as above, we conclude that $H_3(Z_2;\mathbb{Z}) \simeq H^3(M, \mathrm{dP}_6;\mathbb{Z}) \simeq \mathbb{Z}^2$, and also that $H_4(Z_1;\mathbb{Z}) = 0$.

Another way to see this, is to consider the composition

$$\operatorname{Pic}\left(\mathbb{P}^2 \times \mathbb{P}^2\right) \to \operatorname{Pic}\mathbb{P}(\mathcal{T}_{\mathbb{P}^2}) \to \operatorname{Pic}\mathrm{dP}_6\,.$$

The first map is just the identity map. The composition is the map from Example 3.2.7. It follows from the example that the map

$$H_4(M;\mathbb{Z}) \simeq \operatorname{Pic} M \to \operatorname{Pic}\mathrm{dP}_6 \simeq H_2(\mathrm{dP}_6;\mathbb{Z})$$

is injective. ■

Remark 3.2.9. In fact, the Andreotti–Frankel theorem [AF59] states the following: if V is any smooth affine variety of complex dimension n, then it has the homotopy type of a CW complex of real dimension n. Thus it comes for free that $H^j(Z_i, \mathbb{Z}) = 0$ for $j > 3$. ◇

CHAPTER 4

New Calabi–Yau varieties and mirror symmetry

In this chapter we describe the construction of three topologically different smoothings of a singular Calabi-Yau manifold. They correspond to different components of the Hilbert scheme of threefolds in $\mathbb{P}^{11}$ with Hilbert polynomial $p(t) = 6t^3 + 6$.

We first describe a degenerate Calabi–Yau X_0 in the form of a Stanley–Reisner scheme $\mathbb{P}(\mathcal{K})$, which has a quite large symmetry group. We show that X_0 has several topologically distinct smoothings X_i ($i = 1, 2, 3$), which lie on different components of the Hilbert scheme in $\mathbb{P}^{11}$.

In the last section, we propose mirror candidates for two of the constructions, based on orbifolding. We end with some open questions.

4.1 A Stanley–Reisner sphere

Let E_6 be the hexagon as a simplicial complex. The associated Stanley–Reisner scheme $\mathbb{P}(E_6)$ is a degenerate elliptic curve in $\mathbb{P}^5$. If $\mathbb{P}^5$ have coordinates $x_0, \ldots, x_5$, the equations of $\mathbb{P}(E_6)$ are $x_i x_{i+2} = x_i x_{i+3} = 0$, where i is taken modulo 6. This gives a total of 9 quadratic equations.

Lemma 4.1.1. *The Hilbert polynomial of* $\mathbb{P}(E_6)$ *is* $h(t) = 6t$.

Proof. We want to count the dimension of $S_{\mathbb{P}(E_6)}$ in degree t. Any monomial in $S_{\mathbb{P}(E_6)}$ has support on the simplicial complex E_6, so its support is either a vertex or an edge. In the first case, the monomial has the form x_i^t, so there are six of these.

In the other case, it has the form $x_i^a x_{i+1}^b$, with $a + b = t$ and $a, b \neq 0$. Counting, there are $6(t-1)$ of these monomials. In total, the dimension is $6 + 6(t-1) = 6t$. ■

Remark 4.1.2. Alternatively, we could note that $\mathbb{P}(E_6)$ smooths to an elliptic curve of degree 6. Since Hilbert polynomials are constant in flat families, it follows from the Riemann–Roch theorem that

$$h(t) = \deg \mathscr{O}_{\mathbb{P}(E_6)}(6t) - 1 + 1 = 6t.$$

$\diamond$

Note that the Hilbert polynomial only differs from the Hilbert function for $t = 0$, since $h(0) = 0$, while $\dim_{\mathbb{C}} \left(S_{\mathbb{P}(E_6)}\right)_0 = 1$.

We now introduce the central fiber in the discussions onward. Let $\mathcal{K}$ be the simplicial complex $E_6 * E_6$. It is a triangulation of the 3-sphere.

Denote the vertices of the left E_6 by $x_1, \ldots, x_6$, and the vertices of the right E_6 by $z_1, \ldots, z_6$. Then the maximal faces of $\mathcal{K}$ are of the form $x_i x_{i+i} z_j z_{j+1}$, where $i, j \in \mathbb{Z}_6$. The number of i-faces are easy to compute:

Lemma 4.1.3. *The f-vector of $\mathcal{K}$ is* $(12, 48, 36)$.

Proof. There are 12 vertices, and $6 \times 6 = 36$ maximal facets. Since $\mathcal{K}$ is a 3-sphere, it follows that $12 - f_1 + 36 = \chi(S^3) = 0$ so that $f_1 = 48$.[1] ■

Lemma 4.1.4. *The Hilbert polynomial of $\mathbb{P}(\mathcal{K})$ is $h(t) = 6t^3 + 6t$.*

Proof. The homogeneous coordinate ring $S_{\mathbb{P}(\mathcal{K})} = \bigoplus_{t \geq 0} S_t$ of $\mathbb{P}(\mathcal{K})$ is the graded tensor product of $S_{\mathbb{P}(E_6)}$ with itself. It follows from Lemma 4.1.1 that

$$\dim S_t = \sum_{i+j=t, ij \neq 0} 36ij + 12t,$$

where the last term is a correction term because $h(t) \neq 1$. It is now a routine computation using formulas for sums of squares to verify the claim. ■

Corollary 4.1.5. *Any smoothing of $\mathbb{P}(\mathcal{K})$ satisfy $\dim |H| = 12$, $c_2 \cdot H = 72$, and $H^3 = 36$.*

Proof. All these invariants can be read off from the Hilbert polynomial. ■

Either by using `Macaulay2` or by using the more combinatorial description of the T^i-modules from [AC10], we can compute to give the following result:

Proposition 4.1.6. *We have that*

$$\dim_k T^1(S_{\mathbb{P}(\mathcal{K})}/k, S_{\mathbb{P}(\mathcal{K})})_0 = 84$$
$$\dim_k T^2(S_{\mathbb{P}(\mathcal{K})}/k, S_{\mathbb{P}(\mathcal{K})})_0 = 72.$$

[1] Here we used that in a cell complex, the Euler characteristic is also the alternating sum of the number of cells in each dimension. This is Theorem 2.44 in [Hat02].

Proof. We will prove this using the techniques and notation from [AC10]. Our goal is to compute the degree zero part of $T^1_{A_\mathcal{K}}$. We will do this using Theorem 1.4.8.

First notice that all links of vertices of $\mathcal{K} = E_6 * E_6$ are double suspensions over hexagons (they are denoted by ΣE_6 in [AC10]).

According to Table 1 in Christophersen's and Altmann's article, double suspensions over hexagons contribute with one dimension to $T^1_{A_\mathcal{K}}$, namely in degree $x_i^2/x_{i-1}x_{i+1}$ (if $\mathbf{a} = x_i^2$). In total there are $6 + 6 = 12$ contributions of this form.

Taking the link at the vertex $x_i z_j$ produces a square with vertices x_{i+1},z_{j+1},x_{i-1}, and z_{j-1} (in that order). According to Table 1 in the article, these links contribute with dimension 2 to $T^1_{A_\mathcal{K}}$. The contributions have degrees $x_i z_j/x_{i+1}x_{i-1}$ and $x_i z_j/z_{j+1}z_{j-1}$. There are $2 \cdot 6 \cdot 6 = 72$ contributions of this form.

Thus, in total, $T^1_{A_\mathcal{K}}$ have $\mathbb{C}$-dimension 84.

We now compute $T^2_{A_\mathcal{K}}$. The contributions come from choosing $\mathbf{a} = x_i^2$ and $\mathbf{a} = x_i x_{i+1}$, respectively. If $|a| = 1$ (as in the first case), the results from the article imply that $L_b := \cap_{b' \subset b} \mathrm{lk}(b', \mathrm{lk}(x_i, \mathcal{K}))$ must have more than one connected component (the contribution comes from $\widetilde{H}^0(L_b, \mathbb{C})$). This is the case if b consist of two opposite vertices in the suspended circle. In total there are $2 \cdot 6 \cdot 3 = 36$ contributions of this form.

If $|a| = 2$, the contributing links are hexagons, and in this case the contributions come from b such that $L_b = \emptyset$. Again choosing b to consist of opposite vertices of the hexagon, we find three pairs b with $L_b = \emptyset$ for each hexagon. Thus in total there are $2 \cdot 6 \cdot 3 = 36$ contributions of this form.

In sum, $T^2_{A_\mathcal{K}}$ is $36 + 36$-dimensional. ∎

The automorphism group of $\mathcal{K}$ is $D_6 \times D_6 \times \mathbb{Z}_2$, and have order $12 \cdot 12 \cdot 2 = 288$. It is not difficult to see that the induced action on the basis of $T^1(S_{X_0}/k, S_{X_0})$ have two orbits under $\mathrm{Aut}(\mathcal{K})$, corresponding to first order deformations of the form $x_i x_{i-2} + t x_{i+1} z_j$ and $x_{i-1}x_{i+1} + t x_i^2$, respectively.

4.2 A partial smoothing of $\mathbb{P}(\mathcal{K})$

Consider Figure 4.1. It is the 2-dimensional polytope associated to the del Pezzo surface of degree 6. The fan over this polytope correspond to a unimodular regular triangulation of the polytope, and it follows by Theorem 8.3 in [Stu96], that dP_6 degenerates to the Stanley–Reisner scheme $\mathbb{P}(E_6 * \{pt\})$, where $\{pt\}$ correspond to the origin. Concretely, the equations of dP_6 are given by $x_i x_{i+2} - y x_{i+1} = x_i x_{i+3} - y^2 = 0$ inside $\mathbb{P}^6$. The degeneration to $\mathbb{P}(E_6 * \{pt\})$ is given by setting the second terms equal to zero.

Now form the join of two copies of dP_6, to get a new variety $Y \subset \mathbb{P}^{13}$. By Proposition 1.1.2, this is a $2 + 2 + 1 = 5$-dimensional toric variety with singular locus consisting of two copies of dP_6. Since the coordinate ring is just the tensor product of two copies of $S(\mathrm{dP}_6)$, it follows that Y degenerates to $\mathbb{P}(E_6 * \{pt\} * E_6 * \{pt\}) = \mathbb{P}(\mathcal{K} * \Delta^1)$.

The following holds:

Proposition 4.2.1. *There is a deformation of the Stanley–Reisner scheme X_0 to an irreducible Calabi–Yau variety $X_Y \subset Y$ with 12 isolated singularities. The singularities are locally isomorphic to cones over del Pezzo surfaces. More precisely: let (U, p_i) be the germ of X_Y at p_i in the analytic topology. Then $(U, p_i) \simeq (C(\mathrm{dP}_6), 0)$.*

Proof. Since X_0 is a complete intersection inside $\mathbb{P}(\mathcal{K} * \Delta^1)$, it follows that X_0 deforms to a complete intersection inside any deformation of $\mathbb{P}(\mathcal{K} * \Delta^1)$. We explained above that $\mathbb{P}(\mathcal{K} * \Delta^1)$ deforms to the join Y of two del Pezzo surfaces, and it follows that X_0 deforms to Y intersected with two generic hyperplanes.

Since Y has singular locus of dimension 2 and is of degree $6 + 6 = 12$, it follows by Bertini's theorem [Har77, Chapter II, Theorem 8.18] that X_Y has twelve isolated singularities p_i.

To see how the singularities look locally, we argue as follows. Locally, Y looks like $\mathbb{A}^2_{a_1,a_2} \times C(\mathrm{dP}_6)_{x_i}$, where the subscripts refer to the coordinates.

The claim now follows from two applications of Theorem 3.1.5 in [Bat94], which says that the singularities on Σ-regular toric hypersurfaces are inherited from the ambient toric variety. ■

Since the cone over dP_6 deforms in two topologically different ways, we might expect that X_Y does so too. This is indeed true.

4.3 Three different smoothings of X_Y

By embedding dP_6 in different spaces, we obtain different smoothings of X_Y as subvarieties of the join of these spaces.

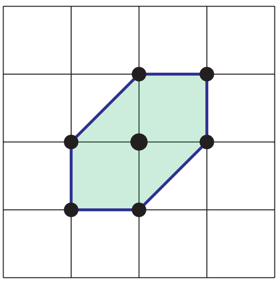


Figure 4.1: A hexagon.

4.3.1 The block matrix construction

We are inspired by the construction in Rødland's thesis [Rød00][2].

Let E be a 3-dimensional vector space. Let $\{e_1, e_2, e_3\}$ be a basis for E. Then we can form the vector space $V = (E \otimes E) \oplus (E \otimes E)$, which has dimension 18. Let $\mathbb{P}^{17} = \mathbb{P}(V)$. Choose coordinates $x_1, \ldots, x_{18}$ on $\mathbb{P}^{17}$.

Thinking of $E \otimes E$ as 3×3-matrices, we can think of the elements of $\mathbb{P}^{17}$ as pairs of 3×3-matrices up to a scalar, not both zero. Concretely, two pairs of matrices (A', B') and (A, B) are equivalent if $(A', B') = (\lambda A, \lambda B)$ for some $\lambda \in \mathbb{C}^*$.

We can also interpret $\mathbb{P}^{17}$ as the geometric join of $\mathbb{P}(E \otimes E)$ with itself. This is the set of all lines connecting pairs of 3×3-matrices.

There is a natural rational map $\pi : \mathbb{P}^{17} \dashrightarrow \mathbb{P}^8 \times \mathbb{P}^8$, which is the identity on coordinates, given by dividing out by the antidiagonal $\mathbb{C}^*$-action: $\lambda' \cdot (A, B) = (\lambda', \lambda'^{-1} B)$.

Remark 4.3.1. Denote by V_1 and V_2 the subspaces $x_1 = \ldots = x_9 = 0$ and $x_{10} = \ldots = x_{18} = 0$, respectively. Blow up $\mathbb{P}^{17}$ in $V_1 \cup V_2$, to get $\widetilde{\mathbb{P}^{17}}$. The spaces V_i are exactly the indeterminacy locus of π, so π extends to a map $\pi : \widetilde{\mathbb{P}^{17}} \to \mathbb{P}^8 \times \mathbb{P}^8$. Denote by π_1 and π_2 the two natural projections to $\mathbb{P}^8$. Then it is true that $\widetilde{\mathbb{P}^{17}} = \mathbb{P}_{\mathbb{P}^8 \times \mathbb{P}^8}(\pi_1^* \mathscr{O}_{\mathbb{P}^8}(1) \oplus \pi_2^* \mathscr{O}_{\mathbb{P}^8}(1)) = \mathbb{P}(\mathscr{O}_{\mathbb{P}^8 \times \mathbb{P}^8} \oplus \mathscr{O}_{\mathbb{P}^8 \times \mathbb{P}^8}(1, -1))$. This is explained further in Section C7 in [AK75]. ◇

Let M be the closure of the set of pairs (A, B) where $\operatorname{rank} A = \operatorname{rank} B = 1$.

Proposition 4.3.2. *The variety M is the join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, and has singular locus $\mathbb{P}^2 \times \mathbb{P}^2 \subset V_i$ of dimension 4.*

The canonical sheaf is $\omega_M = \mathscr{O}_M(-6)$, so that M is a Fano toric variety.

Proof. If $\mathbb{P}^{17}$ have coordinates $x_1, \ldots, x_{18}$, let M_1 and M_2 be the matrices

$$M_1 = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} \\ x_{16} & x_{17} & x_{18} \end{pmatrix}.$$

Then M is defined by the zeros of the 2×2-minors of M_1 and M_2. Then it is clear that M is the projective join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8 \subset \mathbb{P}^{17}$, since the sets of variables are disjoint.

The variety M is 9-dimensional: the affine cone over M, $C(M)$, is equal to $C(\mathbb{P}^2 \times \mathbb{P}^2) \times C(\mathbb{P}^2 \times \mathbb{P}^2)$. This variety has dimension $5 + 5 = 10$, hence its projectivization M is 9-dimensional.

The singular locus of M consists of the pairs $(0, B)$, and $(A, 0)$, where $\operatorname{rank} A = \operatorname{rank} B = 1$, hence $\dim \operatorname{Sing} M = \dim \left(\mathbb{P}^2 \times \mathbb{P}^2\right) = 4$. See also Proposition 1.1.2.

[2]Rødland's construction is a linear subvariety of $\mathbb{P}(E \wedge E)$, where E is 7-dimensional.

By Remark 1.1.4, it follows that $\omega_M = \mathscr{O}_M(-6)$, since

$$\omega_{\mathbb{P}^2\times\mathbb{P}^2} = \mathscr{O}_{\mathbb{P}^8}(-3)\big|_{\mathbb{P}^2\times\mathbb{P}^2}.$$

■

Here comes our first construction. Let X_1 be the intersection of M with a generic $\mathbb{P}^{11}$. Then the following is true.

Proposition 4.3.3. *X_1 is a smooth Calabi–Yau variety with $\chi(X_1) = -72$.*

Proof. The singularities of M are of dimension 4. By Bertini's theorem, intersecting M with a codimension 6 hyperplane gives a smooth variety X_1.

The fact that X_1 is Calabi–Yau follows from Proposition 1.5.4.

To find the topological Euler characteristic, we compute in `Macaulay2`. Computing the whole cotangent sheaf of X_1 is infeasible with current computer technology[3]. Instead we make use of standard exact sequences. Let $\mathscr{I}$ be the ideal sheaf of M in $\mathbb{P}^{17}$. First, we have the exact sequence

$$0 \to \mathscr{I}/\mathscr{I}^2\big|_{X_1} \to \Omega^1_{\mathbb{P}^{17}}\big|_{X_1} \to \Omega^1_M\big|_{X_1} \to 0.$$

The restriction to X_1 is exact since $\mathscr{I}/\mathscr{I}^2$ is locally free on the smooth locus.

The `Macaulay2` command `eulers` computes the Euler characteristics of generic linear sections of a sheaf $\mathscr{F}$ (behind the scene, this is equivalent to computing the Koszul resolution of the relative ideal sheaf $\mathscr{I}_{X_1/M}$). Using this command, we find that $\chi(\mathscr{I}/\mathscr{I}^2\big|_{X_1}) = -180$. Using the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^{17}}\big|_{X_1} \to \mathscr{O}_{X_1}(-1)^{18} \to \mathscr{O}_{X_1} \to 0,$$

we find that the Euler characteristic of $\Omega^1_{\mathbb{P}^{17}}\big|_{X_1}$ is $-216 = -12 \cdot 18$. It follows from the first exact sequence that $\Omega^1_M\big|_{X_1}$ has Euler characteristic -36.

Since X_1 is a complete intersection in M, the conormal sequence is

$$0 \to \mathscr{O}_{X_1}(-1)^6 \to \Omega_M\big|_{X_1} \to \Omega^1_{X_1} \to 0.$$

Hence $\chi(\Omega^1_X) = -36 + 72 = 36$.

It follows that the topological Euler characteristic is $\chi(X_1) \stackrel{\Delta}{=} \chi(\mathcal{T}_{X_1}) = -2\chi(\Omega^1_{X_1}) = -72$. ■

Remark 4.3.4. We can give explicit equations for a flat family with special fiber X_Y and general fiber X_1. Let $y_0 = h_1(x_1, \ldots, x_{12})$ and $y_1 = h_2(x_1, \ldots, x_{12})$ be the generic linear forms in $\mathbb{P}^{13}$ defining X_Y as a subscheme of Y. Let g_i (for

[3] An external computer has been trying to compute this sheaf for several months now without terminating.

$i = 1, \dots, 6$) be generic linear forms in $\mathbb{P}^{11}$. Then such a flat family is defined by the 2×2-minors of the two matrices below:

$$A_1 = \begin{pmatrix} h_1 + tg_1 & x_2 & x_3 \\ x_4 & h_1 + tg_2 & x_6 \\ x_7 & x_8 & h_1 + tg_3 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} h_2 + tg_4 & x_{11} & x_{12} \\ x_{13} & h_2 + tg_5 & x_{15} \\ x_{16} & x_{17} & h_2 + tg_6 \end{pmatrix}.$$

For $t = 0$, we get X_Y. Note that the subscheme defined by the minors of

$$A_1 = \begin{pmatrix} y_1 & x_2 & x_3 \\ x_4 & y_1 & x_6 \\ x_7 & x_8 & y_1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} y_2 & x_{11} & x_{12} \\ x_{13} & y_2 & x_{15} \\ x_{16} & x_{17} & y_2 \end{pmatrix}$$

is the join of dP_6 with itself. Since $X_Y \subset \mathrm{dP}_6 * \mathrm{dP}_6$, we see that X_1 lies in a deformation of $\mathrm{dP}_6 * \mathrm{dP}_6$. ◇

Remark 4.3.5. We have not been able to rigorously compute the Hodge numbers of X_1. However, over several finite fields we have computed the dimension of the degree zero part of the T^1-module in `Macaulay2`. By Proposition 1.3.5, we have that $(T^1(S_{X_1}/k, S_{X_1}))_0 = H^1(X_1, \Omega^2_X)$.

After about a week of computation on a modern desktop computer, the answer turns out to be $\dim_{\mathbb{F}_p}(T^1(S_{X_1}/\mathbb{F}_p, S_{X_1}))_0 = 39$ for several large primes p.

This is plausible because of the following heuristic moduli count: X_1 is parametrized by the Grassmannian $\mathbb{G}(12, (E \otimes E)^{\oplus 2})$, which has dimension $(18-12) \cdot 12 = 72$. Each E-factor is acted upon by $\mathrm{GL}(E)$. There are four of these factors, so we have an action of $\prod^4 \mathrm{GL}(E)$ on the Grassmannian. There is a torus subgroup $(\mathbb{C}^*)^4$ acting by $(v \otimes w, r \otimes s) \mapsto (t_1 t_2 v \otimes w, t_3 t_4 r \otimes s)$ on $\mathbb{P}^{17}$. Elements of $(\mathbb{C}^*)^4$ satisfying $t_1 t_2 = t_3 t_4$ act trivially, forming an isotropy subgroup K. Hence we have an action of the quotient group $G \triangleq \left(\prod_{i=1}^4 \mathrm{GL}(4)\right)/K$ on the Grassmannian. This quotient group has dimension $9 \cdot 4 - 3 = 33$.

We form the quotient space $\mathbb{G}(12, (E \otimes E)^{\oplus 2})/G$, which has dimension $72 - 33 = 39$.

If this is true, then X_1 has Hodge numbers $h^{11} = 3$ and $h^{12} = 39$, since we have computed the Euler characteristic. It is not clear which other divisors there are besides the hyperplane divisor. ◇

Remark 4.3.6. Since X_1 avoids the fundamental subscheme $V_1 \cup V_2$, the inverse image $\pi^{-1}(X_1) \subset \widetilde{\mathbb{P}^{17}}$ is isomorphic to X_1. Thus we can realize X_1 as a subvariety of a *smooth* variety. Unfortunately, X_1 is cut out by non-ample divisors in $\widetilde{\mathbb{P}^{17}}$. ◇

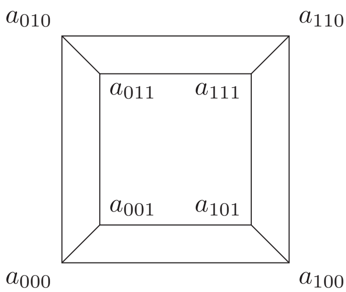


Figure 4.2: A $2 \times 2 \times 2$-tensor, seen from "above".

4.3.2 The three-tensor construction

The construction in the previous section used the embedding of dP_6 in $\mathbb{P}^2 \times \mathbb{P}^2$ to deform X_Y. There is also the embedding of dP_6 in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to exploit. The construction is similar.

Let F be a 2-dimensional vector space with basis $\{f_0, f_1\}$. Then we can form the vector space $V = (F \otimes F \otimes F)^{\oplus 2}$. Let $\mathbb{P}^{15} = \mathbb{P}(V)$. Choose coordinates $a_{ijk} = (f_i \otimes f_j \otimes f_k, 0)$ and $b_{ijk} = (0, f_i \otimes f_j \otimes f_k)$ $(i, j, k = 0, 1)$ for $\mathbb{P}^{15}$.

The elements of $\mathbb{P}^{15}$ are pairs (A, B) of $2 \times 2 \times 2$ tensors, not both zero. There is also in this case a natural map $\pi : \mathbb{P}^{15} \to \mathbb{P}^7 \times \mathbb{P}^7$, given by dividing out by the antidiagonal $\mathbb{C}^*$-action.

Remark 4.3.7. Just as above, let V_1 and V_2 be the subspaces $A = 0$ and $B = 0$, respectively. Let $\widetilde{\mathbb{P}^{15}}$ be the blowup of $\mathbb{P}^{15}$ in $V_1 \cup V_2$. The V_i's are exactly the indeterminacy locus of π, so π extends to a morphism $\pi : \widetilde{\mathbb{P}^{15}} \to \mathbb{P}^8 \times \mathbb{P}^8$, which is a $\mathbb{P}^1$-bundle. In this case it is also true that $\widetilde{\mathbb{P}^{15}} = \mathbb{P}\left(\mathscr{O}_{\mathbb{P}^7 \times \mathbb{P}^7} \oplus \mathscr{O}_{\mathbb{P}^7 \times \mathbb{P}^7}(1, -1)\right)$. $\diamond$

Let N be the closure of the set of pairs (A, B) where both A and B have tensor rank 1[4].

Proposition 4.3.8. *The variety N is the join of two copies of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$, and has singular locus $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset V_i$ of dimension 3.*

The canonical sheaf is $\omega_N = \mathscr{O}_N(-4)$, so that N is a Fano toric variety.

Proof. A pure $2 \times 2 \times 2$-tensor can be visualized as a cube with vertices a_{ijk}. See the diagram in Figure 4.2.

The equations of the set of rank 1 tensors in $\mathbb{P}(F \otimes F \otimes F)$ are obtained as the "minors" along the 6 sides of the cube, together with the minors along

[4] An element of $F^{\otimes 3}$ has rank 1 if it is a pure tensor. It has rank $\leq k$ if it can be written as a sum of k pure tensors.

with the 3 long diagonals, giving a total of 9 binomial equations. We write this symbolically as $[a_{ijk}] \leq 1$.

Hence the equations for N are given by $[a_{ijk}] \leq 1$, together with $[b_{ijk}] \leq 1$. Since these are equations in a disjoint set of variables, it is clear that $N = \left(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\right)^{*2}$.

The claim about the singular locus and the canonical sheaf follows as in the proof of Proposition 4.3.2. ■

Let X_2 be the intersection of N with a general $\mathbb{P}^{11}$.

Proposition 4.3.9. *X_2 is a smooth Calabi–Yau variety with $\chi(X_2) = -48$.*

Proof. The proof is identical to the proof of Proposition 4.3.3. ■

Remark 4.3.10. A heuristic moduli count works also in this case.

X_2 lies in a $\mathbb{P}^{11}$ in $\mathbb{P}((F \otimes F \otimes F)^{\oplus 2})$. Such planes are parametrized by $\mathbb{G}(12, 16)$, the Grassmannian of 12-planes in k^{16}. This space is $12 \cdot (16 - 12) = 48$-dimensional. There is an action of the group $\prod_{i=1}^{6} \mathrm{GL}(F)$ on $(F \otimes F \otimes F)^{\oplus 2}$. There is also in this case a torus subgroup acting trivially, namely the elements satisfying $t_1 t_2 t_3 = t_4 t_5 t_6$. Call this subgroup K. Thus we really have an action of the group $\left(\prod_{i=1}^{6} \mathrm{GL}(F)\right) / K$, which has dimension $6 \cdot 4 - 5 = 19$. Thus in total we have $48 - 19 = 29$ moduli parameters.

Since we know the Euler characteristic, we predict the Hodge numbers to be $(h^{11}, h^{12}) = (5, 29)$. ◇

4.3.3 The mixed smoothing

In the above cases, we formed the join of equal varieties. We mix things up: let $V = (E \otimes E) \oplus (F \otimes F \otimes F)$. Then let $\mathbb{P}^{16} = \mathbb{P}(V)$.

Now let W be the set of "mixed" rank 1 tensors. In a way similar to above, we find that W is a singular Fano toric variety of dimension 8. The singular locus is of dimension 4, so a 5-fold complete intersection is again a smooth Calabi-Yau variety X_3.

Proposition 4.3.11. *X_3 is a smooth Calabi–Yau variety with $\chi(X_3) = -60$.*

Proof. The proof is identical to the proofs above. ■

Remark 4.3.12. Again we give a heuristic moduli count. The Grassmannian in this case is 60-dimensional. The group acting on it is $\prod_{i=1}^{2} \mathrm{GL}(E) \times \prod_{i=1}^{3} \mathrm{GL}(F)$. Here the trivially acting torus subgroup consists of the elements satisfying $t_1 t_2 = t_3 t_4 t_5$. It follows that the parameter space is $60 - (18 + 12 - 4) = 34$-dimensional.

Hence we predict the Hodge numbers to be $(h^{11}, h^{12}) = (4, 34)$. ◇

Remark 4.3.13. Even though we did not find a satisfying proof that the Euler characteristics of the X_i's were -72, -48 and -60, respectively, there is an internal consistency here. The first smoothing, X_1, of X_Y, correspond to smoothing all the cones over del Pezzo-surfaces. They are cut out and replaced by the smoothing corresponding to the largest smoothing component. This smoothing, Z_2, had Euler characteristic zero. If we instead smoothed by glueing in the other smoothing of $C(\mathrm{dP}_6)$, the Euler characteristic would increase by $12 \cdot 2 = 24$. Thus we would have a smoothing with Euler characteristic $-72 + 24 = -48$, which is exactly what happens. $\diamond$

4.4 Degeneration of X_Y

Consider the construction of X_1 from above, and the explicit equations from Remark 4.3.4. Putting $t = 0$ and $h_1 = h_2$, gives a degeneration of X_Y to another, more singular, variety, which we denote by $X_{Y'}$. Explicitly, it is given by the 2×2-minors of the following two matrices, where h is a generic linear form in the variables.

$$A_1 = \begin{pmatrix} h & x_2 & x_3 \\ x_4 & h & x_6 \\ x_7 & x_8 & h \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} h & x_{11} & x_{12} \\ x_{13} & h & x_{15} \\ x_{16} & x_{17} & h \end{pmatrix}.$$

We can realize $X_{Y'}$ as a hypersurface in the toric variety Y' as follows. Introduce a new variable y, and consider the variety defined by the 2×2-minors of

$$A_1 = \begin{pmatrix} y & x_2 & x_3 \\ x_4 & y & x_6 \\ x_7 & x_8 & y \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} y & x_{11} & x_{12} \\ x_{13} & y & x_{15} \\ x_{16} & x_{17} & y \end{pmatrix}.$$

This is a 4-dimensional toric variety. It is the toric variety associated to the polytope Δ with vertices the columns of the matrix

$$\begin{pmatrix} -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \end{pmatrix}.$$

A computation shows that Y' has 1-dimensional singularities, and the singular locus is a graph of $\mathbb{P}^1$'s: take two hexagons, and join each vertex of one of them with all vertices of the other one. This makes in total 48 $\mathbb{P}^1$'s.

The variety Y' is a Fano toric variety, and as such, it has an anticanonical section $X_{Y'}$ which is a singular Calabi–Yau variety. A local computation shows that $X_{Y'}$ has 12 singularities that are locally isomorphic to $C(\mathrm{dP}_6)$, and 36 double points. This can also be seen torically: the cones in the fan of Y'

corresponding to the singular locus comes in two types. The first type is a cone over a hexagon, and the other type is a cone over a square. These give (algebro-geometrically) cones over dP_6 and double points, respectively.

Since Y' is a four-fold, it follows that $X_{Y'}$ has a maximal projective crepant resolution of singularities (a *MPCP-desingularization*), which we denote by $\widetilde{X_{Y'}}$. This is proved in [CK99].

A computation using PALP [KS04] shows that $\widetilde{X_{Y'}}$ has Hodge numbers $(44, 8)$ and Euler-characteristic 72.

Remark 4.4.1. There is a heuristic surgical reason for the Euler characteristic being $+72$. Our $X_{Y'}$ deforms to X_1, which has Euler characteristic -72. This is obtained by starting with $X_{Y'}$, smoothing 36 double points and 12 cones over del Pezzo surfaces. By the inclusion-exclusion principle, it follows that a small resolution of the singularities of $X_{Y'}$ has Euler characteristic $\chi(X_1) + 2 \cdot 36 + 6 \cdot 12 = 72$. ◇

Remark 4.4.2. The variety $X_{Y'}$ has also been described elsewhere. The polar polytope Δ° is equal to the product of two hexagons, and it follows that $\mathbb{P}_{\Delta^\circ}$ is equal to the product of two del Pezzo surfaces. An anticanonical hypersurface in $\mathrm{dP}_6 \times \mathrm{dP}_6$ has Euler characteristic -72 (see for example Theorem 3.1 in [Hüb92]).

In the article [BCD10], Braun et al. study this hypersurface and a group action on it. They also describe, in detail, a crepant resolution of singularities of $X_{Y'}$. ◇

Remark 4.4.3. In [CD10], the authors study Calabi–Yau complete intersections admitting free actions by finite groups, and certain transitions between them (these are similar to Morrison's extremal transitions). They find that there is a Calabi–Yau with Hodge numbers $(3, 39)$ and a Calabi–Yau with Hodge numbers $(8, 44)$ belonging to the same family of transitions, both admitting $\mathbb{Z}/3$-actions. It is not clear to us if their $(3, 39)$-manifold is the same as our X_1. ◇

4.5 Invariant Calabi–Yau's and a mirror construction

In this section, we will explain natural group actions on the X_i's constructed above. Using the mirror construction Ansatz from above, we propose mirror candidates for X_1 and X_2.

4.5.1 Invariant subfamily of X_1

Let us first consider $M = (\mathbb{P}^2 \times \mathbb{P}^2)^{*2}$. Recall that M can be thought of as pairs of rank one 3×3-matrices up to a scalar. We will describe several natural finite group actions on M.

There is a natural $\mathbb{Z}/3$-action on M, defined as follows. If E is a 3-dimensional vector space with basis $\{e_0, e_1, e_2\}$, then we can define $e_i \mapsto \omega^i e_i$, where ω is a fixed third root of unity. This action extends to an action on $E \otimes E$ by the rule $e_{ij} \mapsto \omega^{i+j} e_{ij}$[5]. Furthermore, it extends to an action on $(E \otimes E) \oplus (E \otimes E)$ by $(v, w) \mapsto (gv, gw)$. Call a generator for this group for g.

There is also a non-toric permutation action defined as follows. Let $\langle \sigma \rangle \subset S_3$ be the cyclic permutation action on $\{e_0, e_1, e_2\}$ defined by $e_i \mapsto e_{i+1}$, where σ is a generator for this subgroup. Again, we get an action on $E \otimes E$ by $e_{ij} \mapsto e_{i+1,j+1}$, and by extension an action on $(E \otimes E) \oplus (E \otimes E)$.

Furthermore, there is a $\mathbb{Z}/2$-action switching the $E \otimes E$-factors. Call the generator for this group for τ.

All these groups commute up to a scalar, so we get a $\mathbb{Z}/3 \times \mathbb{Z}/3 \times \mathbb{Z}/2$-action on $\mathbb{P}(E \otimes E \oplus E \otimes E)$. Let G be the abelian group generated by g and σ. Let G' be the group generated by g, σ and τ.

For the G-action to restrict to $X_1 = M \cap H$, we must choose H to be invariant under the group action. We describe a family of G-invariant $\mathbb{P}^{11}$'s: denote a unit matrix in the first factor of $(E \otimes E) \oplus (E \otimes E)$ by e^0_{ij}, and denote a unit matrix in the second factor by e^1_{ij}, where $0, 1$ are taken modulo 2.

Now consider the $H_t = \mathbb{P}^{11}$ spanned by the following matrices:

$$f^\alpha_{ij} = e^\alpha_{ij} + t^\alpha_{i-j} e^{\alpha+1}_{-i-j,-i-j} \in (E \otimes E) \oplus (E \otimes E), \tag{4.1}$$

where $i \neq j \in \mathbb{Z}_3$ and $\alpha \in \mathbb{Z}_2$, and t^α_{i+j} is a parameter. Note that $g \cdot f^\alpha_{ij} = \omega^{i+j} f^\alpha_{ij}$, so that H is spanned by eigenvectors of the $\mathbb{Z}/3$-action. This gives us a 4-parameter family of G-invariant planes. However, multiplying all the t^α_{i-j} by the same number yield isomorphic families, so we really have a 3-parameter family.

Denote the intersection between M and H by X_{H_t}. Denote by P_i the coordinate points $(0 : \ldots : 1 : \ldots : 0)$. Then $\langle \sigma \rangle \simeq \mathbb{Z}/3$ acts without fixed points outside these points (this can be computed in `Macaulay2`). A `Macaulay2` computation also shows that for $t_i \neq 1, 0$, the family has 48 isolated singularities: the P_i, and 36 other points, which come in two orbits under the G-action. These are all double points, which can be verified by local computations.

Lemma 4.5.1. *There exists a minimal resolution of X_{H_t} $(t \neq 0, 1)$, respecting the group action by G, leaving the dualizing sheaf trivial.*

Proof. Analytically, a small resolution is a local operation. The singularities come in 3 orbits under the action, so it is enough to do the resolution on one singularity in each orbit.

Since the singularity is small, the change happens in codimension 2. The holomorphic 3-form on X_{H_t} extends holomorphically to all of the resolution by Hartog's theorem from complex analysis. ∎

[5]We write e_{ij} for $e_i \otimes e_j$.

Lemma 4.5.2. *After resolving the double points as above, the action of g has 24 fixed points on $\widetilde{X_{H_t}}$, two on each of the $\mathbb{P}^1$'s of the initial fixed points.*

Furhermore, the resolution has Euler characteristic 24.

Proof. To see that the $\mathbb{Z}/3$-action has two fixed points on the $\mathbb{P}^1$'s coming from the initial fix points, we find local equations of X_{H_t}. This is done in `Macaulay2`. By writing the equations of X_{H_t} as $x_i u + g = 0$, where u is a unit locally around each fixed point, we can eliminate the variable x_i locally. Doing this repeatedly, we end up with a single local equation for X_{H_t}: (we're now looking in the chart where $x_1 \neq 0$)

$$x_{10}x_{11} - x_8 x_{12} + (\text{higher order terms}).$$

The coordinates of the corresponding $\mathbb{P}^1$ are given by (up to flops):

$$[z_0 : z_1] = [x_{10} : x_8] = [x_{11} : x_{12}].$$

The action of g on the x_i are given by $g \cdot x_8 = \omega^2 x_8$, $g \cdot x_{10} = x_{10}$, $g \cdot x_{11} = \omega^2 x_{11}$ and $g \cdot x_{12} = x_{12}$. This makes $g \cdot [z_0 : z_1] = [z_0 : \omega^2 z_1]$, which shows that the $\mathbb{Z}/3$-action has two fixed points (the points $[1 : 0]$ and $[0 : 1]$).

Similar local equations are given in the eleven other charts.

The Euler characteristic of a small resolution is given by $\chi(\widetilde{X_{H_t}}) = \chi(X_t) + 2s$, where s is the number of double points, and X_t is a smooth member of a smooth smoothing family of X_{H_t} (which we know exists by construction, and is X_1 from above). There are 48 double points, so the Euler characteristic is $-72 + 2 \cdot 48 = 24$. ∎

These resolutions are still Calabi–Yau manifolds. One reference for this fact is [Cle83].

Let $\mathbb{Z}/3$ denote the torus subgroup acting on $\widetilde{X_{H_t}}$.

Theorem 4.5.3. *Let X_1° be a minimal resolution of $\widetilde{X_{H_t}}/(\mathbb{Z}/3)$. Such a resolution exists, and it has Euler-characteristic $+72$, making it a potential mirror for X_1.*

Proof. The existence of a resolution of this kind of quotient singularity is proved in Roan's article [Roa96]. Furthermore, in his article [Roa89], Roan proves a formula for the Euler characteristic of such resolutions (let $V = \widetilde{X_{H_t}}$):

$$\chi\left(\widetilde{V/(\mathbb{Z}/3)}\right) = \frac{1}{3} \sum_{g,h \in \mathbb{Z}/3} \chi(V^g \cap V^h),$$

where V^g refers to the fixed points of g.

For $(g, h) \neq (e, e)$, $\chi(V^g \cap V^h)$ is just the finite set of fixed points. There are 24 of these. For $(g, h) = (e, e)$, $\chi(V^e \cap V^e) = \chi(V)$ is the Euler characteristic of the resolution of X_{H_t}, which is 24.

In sum, we find

$$\chi(X_1^\circ) = \frac{1}{3}(24 + 8 \cdot 24) = 72.$$

■

Remark 4.5.4. We still have the cyclic permutation action σ. Since σ commutes (up to scalar) with g, it acts on the mirror as well. It can be checked that it has no fixed points on X_{H_t}. Thus the induced $\mathbb{Z}/3$-action is free, and we can form the quotients $X_{H_t}/\langle\sigma\rangle$ and $X^\circ_{H_t}/\langle\sigma\rangle$. These will have Euler characteristics 24 and -24, respectively. However, the fundamental group will be non-trivial. ◇

4.5.2 Invariant subfamily of X_2

In this case we are also able to produce a mirror candidate. We start by describing natural group actions on N, and then describe a natural invariant subfamily.

Recall that F is a 2-dimensional vector space with basis f_0, f_1. There is, like above, a natural $\mathbb{Z}/2$-action given by $f_i \mapsto (-1)^i f_i$. Concretely, $\mathbb{Z}/2$ acts by sending f_0 to itself and multiplying f_1 by -1. This action extends in a natural way to an action on $\mathbb{P}(F^{\otimes 3} \oplus F^{\otimes 3})$.

Furthermore, there is another $\mathbb{Z}/2$-action given by $f_i \mapsto f_{i+1}$ (indices taken modulo 2).

Using the same notation as in the previous section, define K_t to be $\mathbb{P}^{11}$ spanned by the following matrices:

$$g^\alpha_{ijk} = e^\alpha_{ijk} + t_{i,j,k} e^{\alpha+1}_{i+j+k,i+j+k,i+j+k} \tag{4.2}$$

for $(i,j,k) \neq (0,0,0),(1,1,1)$ and $\alpha = 0,1$. These matrices span a $\mathbb{P}^{11}$. As above, the g^α_{ijk} are eigenvectors for the $\mathbb{Z}/2$-action.

For $t_{i,j,k} = 1$ for all i,j,k the variety $X_{K_t} \stackrel{\Delta}{=} N \cap K_t$ has 36 double points. Using the same arguments as in the previous section, it follows that a small resolution of X_{K_1} has Euler-characteristic 24 as well. Again, using Roan's formula, we find that a small resolution of the quotient X_{K_1} has Euler characteristic $+48$. Thus we have a mirror candidate for X_2 as well.

Proposition 4.5.5. *There exists a mirror candidate for X_2 as well. More precisely, there exists a Calabi–Yau desingularization X_2° of the quotient $\widetilde{X_{K_t}}/H$ in such a way that the Hodge numbers satisfy $\chi(X_2) = -\chi(X_2^\circ) = -48$.*

4.5.3 Comment about X_3

The same mirror construction does not work for X_3, at least not directly. In the cases of X_1 and X_2, there were natural finite group actions on E and F, respectively. The actions extended to $\mathbb{P}\left((E \otimes E)^{\oplus 2}\right)$ and $\mathbb{P}\left((F \otimes F \otimes F)^{\otimes 2}\right)$, and we intersected with invariant $\mathbb{P}^{11}$'s to get special Calabi–Yau's.

In the case of X_3, there is no natural finite group action on the ambient projective space coming from the join factors, so the same construction does not apply.

4.6 Conclusion and further questions

In this final chapter we constructed several smooth Calabi–Yau manifolds. Three of them, X_1, X_2 and X_3 lie in the same flat family. They are all smoothings of X_Y, a complete intersection in a 5-dimensional toric variety Y. This X_Y has a maximally crepant resolution of singularities which is a smooth Calabi–Yau. We constructed mirror candidates and finite group quotients of X_1 and X_2.

We end with a few open questions that we would like to see answered in the future.

The Calabi–Yau with Hodge numbers $(44, 8)$ in Section 4.4 seem to have some connection with our X_1. Its mirror dual $X_{8,44}$ is a complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, while X_1 is a complete intersection in $(\mathbb{P}^2 \times \mathbb{P}^2) * (\mathbb{P}^2 \times \mathbb{P}^2)$ with the same Euler characteristic. There seem to be some kind of duality going on, which is unfortunately not described (to our knowledge) in the literature.

We have a morphism $\pi : X_1 \to \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by $(v \otimes w, r \otimes s) \mapsto v \otimes w \otimes r \otimes s$. The morphism is generically $1-1$. We have not been able to see what the image is (or if the morphism is an isomorphism).

The same situation occurs with X_2. Here there is a morphism $\pi : X_2 \to (\mathbb{P}^1)^{\times 6}$. We do not know what the image is. Also here there should be a connection with $X_{8,44}$, since $X_{8,44}$ can also be realized as a complete intersection in $(\mathbb{P}^1)^{\times 6}$. See the introduction of [BCD10].

It would also be interesting to find proofs of the Euler characteristics being $-72, -60$ and -48 not involving computer calculations. In all cases the Grassmannian parameterizing the X_i have dimension 72, 60, and 48, though we haven't seen the connection yet.

Assuming that our, or rather our computer's, calculation of the Hodge numbers of the X_i are correct, what are representatives of the generators of $\operatorname{Pic} X_i$? (being $\mathbb{Z}^3$, $\mathbb{Z}^5$ and $\mathbb{Z}^4$, respectively)

Can our construction via joins be generalized to produce other (potentially new) Calabi–Yau varieties?

Appendices

APPENDIX A

Computer code

Extensive use of computer software such as `Macaulay2` [GS] and SAGE [Wil17] has been invaluable during my work. Especially the `Macaulay2` package `VersalDeformations` [Ilt12] has been useful for experiments (lifting deformations to higher order, looking at base spaces, etc.).

In this Appendix we collect computer code for reproducing some of my calculations. Not everything is reproduced here. For all my code, consult my GitHub account at `https://github.com/FredrikMeyer/m2files`.

A.1 Computing the singular locus

In some cases, equations simplify significantly in affine charts. Therefore, using the naive command `singularLocus` in `Macaulay2` often takes unnecessarily long time (and sometimes the computations never finish), as it computes the minors of a very large Jacobian matrix. Restricting to each affine chart, we can use the command `minimalPresentation` to eliminate variables to produce a new ring isomorphic to the first one, but with fewer equations.

The following code produces a list of the components of the singular locus of the projective scheme with homogeneous ideal I.

```
fastSingularities = I -> (
    R := ring I;
    n := numgens R;
    gensR := gens R;
    singlist := {};
    for i from 0 to (n-1) do {
        affineChart := I + ideal(gensR_i - 1);
        singloc     := singularLocus minimalPresentation affineChart;
        sing        := radical ideal mingens ideal singloc;
        inv         := affineChart.cache.minimalPresentationMap;
        singlist = singlist | {(homogenize(preimage(inv,sing),gensR_i))};
        };
    saturate intersect(singlist)
    )
```

The method works by computing the singular locus in each affine chart, taking the radical, and then pulling back to the homogeneous coordinate ring. Finally, we get a list of singular loci in each affine chart. We return the (saturation of) the intersection of the ideals of the singular loci of each affine chart.

The script is especially fast when computing the singular locus of toric varieties with a low-dimensional singular locus.

The following code finds the singular locus of the projective cone $C(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^9$.

```
R = QQ[x_0..x_8,x_9]
M = genericMatrix(R,3,3)
I = minors(2,M)
time fastSingularities I
time radical ideal singularLocus I
```

Our function performs significantly faster than the native function `singularLocus`. On a modern MacBook Pro, the times are 1.14 seconds versus 4.31 seconds, respectively.

Here is a more involved example. Let Y' be the four-dimensional singular toric variety from Chapter 4. It is defined by the 2×2-minors of two matrices with variables. In `Macaulay2` we can define it as follows:

```
S = QQ[x_1..x_6,z_1..z_6,y]
M1 = matrix{{y,x_1,x_2},{x_4,y,x_3},{x_5,x_6,y}}
M2 = matrix{{y,z_1,z_2},{z_4,y,z_3},{z_5,z_6,y}}
J = minors(2,M1) + minors(2,M2)
```

Here the difference in performance is even more striking. Our function computes the singular locus in 7.29 seconds, but the built-in function `singularLocus` used more than 22 minutes (at which point we interrupted the computation).

A.2 Torus action

The following lines checks whether a projective scheme with ideal sheaf `IX` admits an action of a subtorus of $G = (\mathbb{C}^*)^n \subset \mathbb{P}^n$. To check this, we check if the equations are still valid after a torus action. Since G is abelian, it acts on functions by $\lambda \cdot f(x_0, \dots, x_n) = f(\lambda_0 x_1, \dots, \lambda_n x_n)$.

Lemma A.2.1. *Suppose $\{f_1, \dots, f_r\}$ is a homogeneous generating set for $I_X =$ `IX`. Then the subgroup of G acting on $X \subset \mathbb{P}^n$ is generated by those $\lambda \in G$ such that $\lambda \cdot f_i = cf_i$ for some $c \in \mathbb{C}^*$.*

Proof. Let H be the subgroup of G fixing the ideal I_X. Let H' be the subgroup of $g \in G$ acting on the f_i's by scalar multiplication: $g \cdot f_i = cf_i$. Clearly $H' \subseteq H$. Now suppose $g \in H$. Then

$$g \cdot f_1 = \sum_j a_j f_j$$

for some constants a_j. We have that $g \cdot f_1 = f_1(\lambda_1 x_1, \ldots, \lambda_n x_n)$. Suppose the leading term of f_1 is $x_1^{b_1} \cdots x_n^{b_n}$. Then comparing leading terms in the left hand side and the right hand side, we see that $a_1 = \lambda_1^{b_1} \cdots \lambda_n^{b_n} := \lambda^m$. Hence the right hand side is $\lambda^m f_1 +$ other terms. But there are the same number of terms on each side of the equation, meaning that the "other terms"-part must be zero.

Hence $H = H'$. ■

It follows that to find the subgroup of G acting on X, we have to find the $\lambda \in G$ such that the f_i are simultaneous eigenvectors for them.

Example A.2.2. Let X be defined by $f = x_0 x_1 x_2 x_3 x_4 + \sum_{i=0}^{5} x_i^5$ in $\mathbb{P}^4$. Then for $\mathbb{C}^4$ to act on it, we must have $\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4 = \lambda_0^5 = \ldots = \lambda_4^5$. By setting $\lambda_0 = 1$, we see that all the λ_i's are fifth roots of unity. Hence the subgroup acting on H is the subgroup of $(\mathbb{Z}/5)^5/\mathbb{Z}_5$ given by $\{(a_0, \ldots, a_5) \mid \sum a_i = 0\}$. ♡

The following code find the subtori of G acting on X in this way, by equating terms in the polynomials defining X.

```
loadPackage "Binomials"
torus = ideal apply(flatten apply(
                  apply(
                      apply(flatten entries gens IX, monomials),
                          v ->  flatten entries v),
                              j -> subsets(j,2)),
                                  s -> s_0-s_1)
toruskomps = BPD torus
toruskomps = select(toruskomps, I -> dim I == 1)
```

Explanation. In order to have $g \cdot f = \lambda f$, all terms of the polynomial must be eigenvectors of g. Then as in Example A.2.2, this translates into equating all monomials in the generators. The code first makes a list of all pairs of monomials in generators of `IX`. Then we make the ideal of differences between each pair. Putting all the differences equal to zero, we find the subset of the torus acting on X.

The ideal `torus` is the ideal generated by the differences of terms in the polynomials defining X.

The `Macaulay2` package `Binomials` [Kah12] can decompose binomials over cyclic extensions of $\mathbb{Q}$ with the command `BPD`. In the last line we select the components corresponding to finite subgroups of the torus.

Then we check manually if these actually correspond to non-trivial actions. There will be one component for each generator of the cyclic group acting on X. ■

A.3 Computing fixed points

Computing fixed points of a torus action is often just as easy to do by hand, but to save time and potential for error, we mostly did this in `Macaulay2`.

To check if a point $P \in \mathbb{P}^n$ is a fixed point of a group action, we lift P to $\overline{P} \in \mathbb{C}^{n+1}$. Then P is a fix point if and only if $g \cdot \overline{P} = \lambda \overline{P}$ for some $\lambda \in \mathbb{C}^*$.

To compute all fix points, we consider the ideal generated by $x_i - \lambda(g \cdot x_i)$ for each generator x_i. The fixed locus correspond to a primary decomposition of this ideal.

Below is the code to compute the fixed points of the $\mathbb{Z}/2$-action on the invariant subfamily of X_2. We create the ideal, then saturate by the maximal ideal $(x_1, \ldots, x_n)$ (since not all coordinates are allowed to be zero). Then we use the `decompose` command in `Macaulay2` to get a primary decomposition.

```
S  = R[lambda]
M1 =  matrix{{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_10,x_11,x_12}}
nnnnnM2 =  matrix{{x_1*lambda,x_2*lambda,-x_3*lambda,x_4*lambda,-x_5*lambda,-
     x_6*lambda,-lambda*x_7,-x_8*lambda,lambda*x_9,-lambda*x_10,lambda*x_11,
     lambda*x_12}}

Ifiks = saturate(ideal (M1-M2), sub(ideal gens R,S))
decompose(Ifiks + IX)
```

The result is a list of 12 ideals, corresponding to the 12 fixed points.

A.4 Computing the Gaifullin triangulation

Below is a short SAGE script computing the 15 vertex triangulation of $\mathbb{CP}^2$ as described in [Gai09]. The last line returns a `SimplicialComplex` object in SAGE.

```
#Defines the Klein 4 group.
V4 = PermutationGroup([Permutation("(1,2)(3,4)"),Permutation("(1,3)(2,4)")])

def isValidFace(F):
    '''
    Assumes the first vertex is a permutation.
    Then checks if F satisfies the condition in the
    definition of T.
    '''
    g = F[0]
    for v in (1,2,3,4):
        if (F[g(v)][1] == F[v][1]):
            return False
    return True


# Makes a list of all possible maximal faces of the correct form
candidates = [(g,(1,a1),(2,a2),(3,a3),(4,a4)) for g in V4.list()[1:] for a1 in
      (1,2,3) for a2 in (1,2,3) for a3 in (1,2,3) for a4 in (1,2,3)]

# Filters out the faces not fulfilling the condition
maximalFacets = filter(lambda F: isValidFace(F), candidates)

# Renames the vertices
S = SimplicialComplex(maximalFacets)
vertexSet = S.vertices()
D = dict([(F,i) for i,F in enumerate(vertexSet)])
renamedMaximalFacets = [[D[v] for v in F] for F in maximalFacets]
SS = SimplicialComplex(renamedMaximalFacets)
```

To get the Stanley–Reisner ideal, one can write:

```
list(SS.stanley_reisner_ring().defining_ideal().gens())
```

The returned value is a list of the monomials generating the Stanley–Reisner ideal of $\mathcal{T}$. This can then be copied into `Macaulay2` for further analysis.

A.5 Construction of the X_i

In this section we describe an efficient way to present the Calabi–Yau varieties X_i from Chapter 4 in `Macaulay2`.

A.5.1 Construction of X_1

Recall the construction of X_1: it is the intersection of a toric variety $M \subset \mathbb{P}^{17}$ with a generic $\mathbb{P}^{11}$. The variety X_1 parametrizes pairs of rank $1+1$ tensors in this $\mathbb{P}^{11}$.

We can think of elements of $E \otimes E \oplus E \otimes E$ as pairs of 3×3 matrices, which we denote by (A, B). To span the $\mathbb{P}^{11}$, we choose block matrices $(A, B)_i$ $(i = 1, \dots, 12)$. Then we form the sum

$$A \triangleq \sum_{i=1}^{12} (A, B)_i x_i,$$

with variables x_i. This matrix has rank $1 + 1$ if all the 2×2-minors of A and B vanish, and neither A nor B is zero (which for generic (A, B) won't happen).

Below is a short `Macaulay2` script implementing this construction.

```
kk = ZZ/3001
R = kk[x_1..x_12]

generateX2 = () -> (
    K = random(R^18,R^12);
    a = transpose gens gb K; -- same image
    b = entries a;
    b = apply(0..11, i-> apply(b#i, z -> z*x_(i+1)));
    bb = sum toList b;
    bb1 = bb_{0..8};
    bb2 = bb_{9..17};
    M1 = matrix toList apply(0..2,
            i-> toList apply(0..2, j-> bb1#(3*i+j)));
    M2 = matrix toList apply(0..2,
            i-> toList apply(0..2, j-> bb2#(3*i+j)));
    I1 = minors(2, M1);
    I2 = minors(2, M2);
    I1+I2
    )
```

Listing A.1: Code for X_1

We explain each step. First we create a random 18×12-matrix with coefficients from the field `kk`. Then we replace the random matrix with its Gaussian reduced form, which have the same image in k^{18}, but is much simpler.

Next, we use the matrix to create 18 random linear forms in the variables x_i. These are then inserted into two 3×3 matrices M_1 and M_2. Finally, we return the ideal which is the sum of the ideal of the minors of the two matrices M_1 and M_2. This is the ideal of X_1.

Remark A.5.1. Replacing the matrix K with its Gaussian reduced form is the same as letting $\mathrm{GL}(k^{12})$ act on the left. This *significantly* reduces the size of the resulting Gröbner basis. Without this simplification, the resulting Gröbner basis has 49 elements, but with it, it has 19 elements.

As an example, computing the degree zero part of $T^1(S_{X_1}/k, S_{X_1})$ takes about a week on a modern computer before simplification. With the smaller Gröbner basis, the same computation takes just a couple of hours. $\diamond$

A.5.2 Construction of X_2

The construction of X_2 is very similar. Again, we create 12 random elements of $(F \otimes F \otimes F)^{\oplus 2}$ spanning a $\mathbb{P}^{11}$. This correspond to the 12 columns of the random matrix K.

As with X_1, we replace K with its Gaussian reduced form. This matrix spans the same $\mathbb{P}^{11}$, but has a lot more zeroes.

Then we form the sum

$$\sum_{i=1}^{12} (T_1, T_2)_i x_i,$$

where T_1 and T_2 are $2 \times 2 \times 2$-tensors. We return the ideal generated by the "minors" of this sum.

```
minors222tensor = (L) -> ( -- L is a list of lists of lists
    eqs = {L#0#0#0*L#1#0#1 - L#0#0#1*L#1#0#0,
      L#1#0#0*L#1#1#1 - L#1#1#0*L#1#0#1,
      L#1#1#0*L#0#1#1 - L#1#1#1*L#0#1#0,
      L#0#1#0*L#0#0#1 - L#0#1#1*L#0#0#0,
      L#1#0#1*L#0#1#1 - L#1#1#1*L#0#0#1,
      L#1#0#0*L#0#1#0 - L#1#1#0*L#0#0#0};
    eqs = eqs | {L#0#0#0 * L#1#1#1 - L#0#0#1*L#1#1#0,
      L#1#0#0*L#0#1#1 - L#1#0#1*L#0#1#0,
      L#0#0#1*L#1#1#0 - L#1#0#1*L#0#1#0};
    ideal eqs
    )

generateX2 = () -> (
    K = random(R^16,R^12);
    a = transpose gens gb K;
    b = entries transpose K;
    b = entries a;
    b = apply(0..11, i-> apply(b#i, z -> z*x_(i+1)));
    bb = sum toList b;
    bb1 = bb_{0..7};
    bb2 = bb_{8..15};
    I1 = minors222tensor {{{bb1#0,bb1#1},{bb1#2,bb1#3}},
                          {{bb1#4,bb1#5},{bb1#6,bb1#7}}};
    I2 = minors222tensor {{{bb2#0,bb2#1},{bb2#2,bb2#3}}
                          {{bb2#4,bb2#5},{bb2#6,bb2#7}}};
    I1+I2
    )
```

Listing A.2: Code for X_2

Constructing X_3 in `Macaulay2` is entirely similar to the above two constructions, so we omit the code.

Remark A.5.2. Using a finite field when computing T^1 is essential. Without a limit on the size of the coefficients, the amount of necessary computer RAM is way beyond current technology. ◇

A.6 Constructing the invariant subfamilies

Below is Macaulay2 code for constructing the invariant Calabi–Yau families described in Chapter 4.

A.6.1 Code for X_{H_t}

```
Z = QQ[x_1..x_12]

pars = {2,3,5}
fija = (i,j,a) -> (
    Eij  := (id_(Z^3))_{i} * transpose (id_(Z^3))_{j};
    Eij' :=  (id_(Z^3))_{(-i-j) % 3} * transpose (id_(Z^3))_{(-i-j) % 3};
    if (a == 0) then (
      Eij | pars#((i-j)%3) * Eij'
      )
    else (
      pars#((i-j)%3) * Eij' | Eij
      )
    )

MG = x_1*fija(0,1,0) + x_2*fija(0,2,0) + x_3*fija(1,0,0) +
     x_4*fija(1,2,0) + x_5*fija(2,0,0) + x_6*fija(2,1,0) +
     x_7*fija(0,1,1) + x_8*fija(0,2,1) + x_9*fija(1,0,1) +
     x_10*fija(1,2,1)+ x_11*fija(2,0,1)+ x_12*fija(2,1,1)

IX = minors(2,MG_{0..2}) + minors(2,MG_{3..5})
```

Listing A.3: Code for X_{H_t}

The function fija takes as inputs the indices in the definition of f^{α}_{ij} in Equation (4.1). The elements of the list pars are parameters. Only if the parameters are all equal to 1 do the variety obtain more singularities.

A.6.2 Code for X_{K_t}

For the invariant subfamily of the X_2-family, the code is shorter (but uglier). We manually entered the equations of the invariant $2 \times 2 \times 2$-tensors g^{α}_{ijk} from Equation (4.2), and then computed the $2 \times 2 \times 2$-minors.

```
pars = {2,3,5}
L1 = {{{pars#0 * x_7 + pars#1 * x_8 + pars#2 * x_10,x_1},
      {x_2,x_3}},{{x_4,x_5},
      {x_6,pars#2 *x_9 + pars#1 * x_11 + pars#0 * x_12}}}
L2 = {{{pars#0 * x_1 + pars#1 * x_2 + pars#2 * x_4,x_7},
      {x_8,x_9}},{{x_10,x_11},
      {x_12,pars#2 *x_3 + pars#1 * x_5 + pars#0 * x_6}}}

IX = (minors222tensor L1) + (minors222tensor L2)
```

Listing A.4: Code for X_{K_t}

APPENDIX B

Triangulations of spheres with 8 vertices

In the article [GS67], Grünbaum–Sreedharan enumerates all simplicial 4-polytopes with 7 and 8 vertices. There are 5 combinatorial types of triangulations of the 4-sphere with 7 vertices, and there are 37 combinatorial types of triangulations with 8 vertices.

In her thesis [Fau12], Ingrid Fausk considered the polytopes with 7 vertices, and their associated Stanley–Reisner schemes. She showed that four out of the five possible Stanley–Reisner schemes of triangulations of 4-spheres with seven vertices admit a smoothing. These smoothings correspond to Calabi–Yau varieties with Hodge numbers $(1, 73)$, $(1, 73)$, $(1, 61)$ and $(1, 50)$, respectively. The last one is Rødland's construction.

In this Appendix, we perform deformation theoretic calculations on the 37 triangulations with 8 vertices. Unfortunately, most of them appear to be non-smoothable, at least with naïve techniques.

Unfortunately, there seems to be a mistake in Grünbaum–Sreedharan's list. Two of the spheres listed have $H^3(K; k) = 0$, which should not occur if they were spheres.

In [Kap15], Kapustka compiles a list of smooth Calabi–Yau varieties with $\operatorname{Pic} X = \mathbb{Z}$. Several of the smoothings we find below occur in that list. There is also the paper [Cou+16], where Coughlan–Gołębiowski–Kapustka–Kapustka make a list of arithmetically Gorenstein Calabi–Yau threefolds in $\mathbb{P}^7$, which they conjecture is the complete list of such threefolds. One can ask if all of these are smoothings of one of the Stanley–Reisner schemes from the below list.

B.0.1 Technique

We manually entered the maximal facets from each triangulation P_i^8 (in Grünbaum's notation) into `Macaulay2`. Then we used Nathan Ilten's package [Ilt12] to compute their first order deformations and the obstruction spaces, T^1 and T^2, respectively.

Those with $T^2 = 0$ are perhaps the most interesting, as they correspond to smooth points on the Hilbert scheme. Having $T^2 = 0$ means that all first-order deformations lift to a second-order deformation. In many cases this implies that it lifts automatically to an honest family over $\operatorname{Spec}\mathbb{C}[t_1, \dots, t_N]$ (where $N = \dim_k T^1$).

However, even in non-obstructed cases, we might have power series solutions, meaning that lifting the equations one step at a time will never terminate.

Then we compute the T^i modules for the other triangulations. We also compute their automorphism groups, using SAGE.

B.1 Table of information

Here is the whole table of T^i-dimensions together with some other information. Compare with the list in [Kap15].

Number	degree	$c_2 \cdot H$	T^1	T^2	Aut(T)	Comment
P^8_1	14	–	–	–	–	Not a sphere.
P^8_2	14	68	98	9	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
P^8_3	14	68	108	24	D_6	
P^8_4	15	66	95	17	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
P^8_5	15	64	88	32	$\mathbb{Z}/2 \times D_4$	
P^8_6	15	66	88	9	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
P^8_7	15	–	–	–	–	Not a sphere.
P^8_8	16	64	78	9	1	
P^8_9	16	64	82	17	$\mathbb{Z}/2$	
P^8_{10}	16	64	92	32	$\mathbb{Z}/4$	
P^8_{11}	17	62	74	18	$\mathbb{Z}/2$	
P^8_{12}	17	62	77	25	$\mathbb{Z}/2$	
$\mathbf{P^8_{13}}$	15	66	83	0	$S_3 \times D_5$	Smooths to $X_{113} \subset \mathbb{G}(2,5)$.
P^8_{14}	16	64	80	18	D_4	
P^8_{15}	16	64	88	32	$\mathbb{Z}/2 \times \mathbb{Z}/4$	
$\mathbf{P^8_{16}}$	16	64	72	0	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
$\mathbf{P^8_{17}}$	16	64	72	0	$\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$	
P^8_{18}	17	62	72	17	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
P^8_{19}	17	62	72	17	$\mathbb{Z}/2$	
P^8_{20}	17	62	67	9	$\mathbb{Z}/2$	
P^8_{21}	17	62	80	32	D_4	
$\mathbf{P^8_{22}}$	17	62	62	0	$\mathbb{Z}/2$	
P^8_{23}	18	60	63	17	$\mathbb{Z}/2$	
P^8_{24}	18	60	18	18	$\mathbb{Z}/2$	
P^8_{25}	18	60	67	25	1	
$\mathbf{P^8_{26}}$	17	62	62	0	D_6	

Number	degree	$c_2 \cdot H$	T^1	T^2	Aut(T)	Comment
P^8_{27}	18	60	58	9	$\mathbb{Z}/2$	
P^8_{28}	18	60	58	9	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
P^8_{29}	18	60	58	9	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
P^8_{30}	19	58	63	33	$\mathbb{Z}/2$	
P^8_{31}	19	58	59	26	1	
P^8_{32}	19	58	55	18	$\mathbb{Z}/2$	
P^8_{33}	19	58	60	27	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
$\mathbf{P^8_{34}}$	16	64	72	0	$S_4 \times (\mathbb{Z}/2)^4$	Smooths to $X_{2222} \subset \mathbb{P}^7$.
P^8_{35}	20	56	72	64	D_8	
P^8_{36}	20	56	64	50	$\mathbb{Z}/4$	
P^8_{37}	20	56	61	43	$\mathbb{Z}/2$	
$\mathcal{M}$	20	56	53	27	S_3	

The notations X_{112} and X_{2222} mean a complete intersection of degrees $1, 1, 2$ (resp. $2, 2, 2, 2$) in X.

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