# Algebraic Geometry Buzzlist

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## 1 Algebraic Geometry

## Contents

## 1.1 General terms

### 1.1.1 Cartier divisor

Let $\mathcal{K}_X$ be the *sheaf of total quotients* on $X$, and let $\mathscr{O}_X^*$ be the sheaf of non-zero divisors on $X$. We have an exact sequence

$$1 \to \mathscr{O}_X^* \to \mathcal{K}_X \to \mathcal{K}_X / \mathscr{O}_X^* \to 1.$$

Then a **Cartier divisor** is a global section of the quotient sheaf at the right.

### 1.1.2 Categorical quotient

Let $X$ be a scheme and $G$ a group. A **categorical quotient** is a morphism $\pi : X \to Y$ that satisfies the following two properties:

1. It is invariant, in the sense that $\pi \circ \sigma = \pi \circ p_2$ where $\sigma : G \times X \to X$ is the group action, and $p_2 : G \times X \to X$ is the projection. That is, the following diagram should commute:

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "p_2"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

2. The map $\pi$ should be *universal*, in the following sense: If $\pi' : X \to Z$ is any morphism satisfying the previous condition, it should uniquely factor through $\pi$. That is:

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "\pi'"'] & Y \\ Z \arrow[ur, "\exists! h"'] & \end{tikzcd}$$

Note: A categorical quotient need not be surjective.

### 1.1.3 Chow group

Let $X$ be an algebraic variety. Let $Z_r(X)$ be the group of $r$-dimension cycles on $X$, a *cycle* being a $\mathbb{Z}$-linear combination of $r$-dimensional subvarieties of $X$. If $V \subset X$ is a subvariety of dimension $r+1$ and $f : X \dashrightarrow \mathbb{A}^1$ is a rational function on $X$, then there is an integer $\mathrm{ord}_W(f)$ for each codimension one subvariety of $V$, the order of vanishing of $f$. For a given $f$, there will only be finitely many subvarieties $W$ for which this number is non-zero. Thus we can define an element $[\mathrm{div}(f)]$ in $Z_r(X)$ by $\sum \mathrm{ord}_W(f)[W]$.

We say that two $r$-cycles $U_1, U_2$ are *rationally equivalent* if there exist $r+1$-dimensional subvarieties $V_1, V_2$ together with rational functions $f_1 \colon V_1 \dashrightarrow \mathbb{A}^1$, $f_2 \colon V_2 \dashrightarrow \mathbb{A}^1$ such that $U_1 - U_2 = \sum_i [\mathrm{div}(f_i)]$. The quotient group is called the **Chow group** of $r$-dimensional cycles on $X$, and denoted by $A_r(X)$.

### 1.1.4 Complete variety

Let $X$ be an integral, separated scheme over a field $k$. Then $X$ is **complete** if is proper.

Then $\mathbb{P}^n$ is proper over any field, and $\mathbb{A}^n$ is never proper.

### 1.1.5 Crepant resolution

A **crepant resolution** is a resolution of singularities $f : X \to Y$ that does not change the canonical bundle, i.e. such that $\omega_X \simeq f^*\omega_Y$.

### 1.1.6 Dominant map

A rational map $f : X \dashrightarrow Y$ is **dominant** if its image (or precisely: the image of one of its representatives) is dense in $Y$.

### 1.1.7 Étale map

A morphism of schemes of finite type $f : X \to Y$ is **étale** if it is smooth of dimension zero. This is equivalent to $f$ being flat and $\Omega_{X/Y} = 0$. This again is equivalent to $f$ being flat and unramified.

### 1.1.8 Genus

The **geometric genus** of a smooth, algebraic variety, is defined as the number of sections of the canonical sheaf, that is, as $H^0(V, \omega_X)$. This is often denoted $p_X$.

### 1.1.9 Geometric quotient

Let $X$ be an algebraic variety and $G$ an algebraic group. Then a **geometric quotient** is a morphism of varieties $\pi : X \to Y$ such that

1. For each $y \in Y$, the fiber $\pi^{-1}(y)$ is an orbit of $G$.
2. The topology of $Y$ is the quotient topology: a subset $U$ of $Y$ is open if and only if $\pi^{-1}(U)$ is open.
3. For any open subset $U \subset Y$, $\pi^* : k[U] \to k[\pi^{-1}(U)]^G$ is an isomorphism of $k$-algebras.

The last condition may be rephrased as an isomorphism of strcture sheaves: $\mathscr{O}_Y \simeq (\pi_* \mathscr{O}_X)^G$.

### 1.1.10 Hodge numbers

If $X$ is a complex manifold, then the **Hodge numbers** $h^{pg}$ of $X$ are defined as the dimension of the cohomology groups $H^p(X, \Omega_X^q)$.

### 1.1.11 Linear series

A **linear series** on a smooth curve $C$ is the data $(\mathcal{L}, V)$ of a line bundle on $C$ and a vector subspace $V \subseteq H^0(C, \mathcal{L})$. We say that the linear series $(\mathcal{L}, V)$ have *degree* $\deg \mathcal{L}$ and *rank* $\dim V - 1$.

### 1.1.12 Log structure

A **prelog structure** on a scheme $X$ is given by a pair $(X, M)$, where $X$ is a scheme and $M$ is a sheaf of monoids on $X$ (on the Ètale site) together with a morphisms $\alpha : M \to \mathscr{O}_X$. It is a **log structure** if the map $\alpha : \alpha^{-1} \mathscr{O}_X^* \to \mathscr{O}_X^*$ is an isomorphism.

See [5].

### 1.1.13 Néron-Severi group

Let $X$ be a nonsingular projective variety of dimension $\geq 2$. Then we can define the subgroup $\mathrm{Cl}^\circ X$ of $\mathrm{Cl}\, X$, the subgroup consisting of divisor classes algebraically equivalent to zero. Then $\mathrm{Cl}\, X / \mathrm{Cl}^\circ X$ is a finitely-generated group. It is denoted by $\mathrm{NS}(X)$.

### 1.1.14 Normal crossings divisor

Let $X$ be a smooth variety and $D \subset X$ a divisor. We say that $D$ is a **simple normal crossing divisor** if every irreducible component of $D$ is smooth and all intersections are transverse. That is, for every $p \in X$ we can choose local coordinates $x_1, \cdots, x_n$ and natural numbers $m_1, \cdots, m_n$ such that $D = (\prod_i x_i^{m_i} = 0)$ in a neighbourhood of $p$.

Then we say that a divisor is **normal crossing** (without the "simple") if the neighbourhood above can is allowed to be chosen locally analytically or as a formal neighbourhood of $p$.

Example: the nodal curve $y^2 = x^3 + x^2$ is a a normal crossing divisor in $\mathbb{C}^2$, but not a simple normal crossing divisor.

This definition is taken from [6].

### 1.1.15 Normal variety

A variety $X$ is **normal** if all its local rings are normal rings.

### 1.1.16 Picard number

The **Picard number** of a nonsingular projective variety is the rank of Néron-Severi group.

### 1.1.17 Proper morphism

A morphism $f : X \to Y$ is **proper** if it separated, of finite type, and universally closed.

### 1.1.18 Resolution of singularities

A morphism $f : X \to Y$ is a **resolution of singularities of** $Y$ if $X$ is non-singular and $f$ is birational and proper.

### 1.1.19 Separated

Let $f : X \to Y$ be a morphism of schemes. Let $\Delta : X \to X \times_Y X$ be the diagonal morphism. We say that $f$ is **separated** if $\Delta$ is a closed immersion. We say that $X$ is **separated** if the unique morphism $f : X \to \operatorname{Spec} \mathbb{Z}$ is separated.

This is equivalent to the following: for all open affines $U, V \subset X$, the intersection $U \cap V$ is affine and $\mathscr{O}_X(U)$ and $\mathscr{O}_X(V)$ generate $\mathscr{O}_X(U \cap V)$. For example: let $X = \mathbb{P}^1$ and let $U_1 = \{[x : 1]\}$ and $U_2 = \{[1 : y]\}$. Then

$\mathscr{O}_X(U_1) = \operatorname{Spec} k[x]$ and $\mathscr{O}_X(U_2) = \operatorname{Spec} k[y]$. The glueing map is given on the ring level as $x \mapsto \frac{1}{y}$. Then $\mathscr{O}_X(U_1 \cap U_2) = k[y, \frac{1}{y}]$.

### 1.1.20 Unirational variety

A variety $X$ is **unirational** if there exists a generically finite dominant map $\mathbb{P}^n \dashrightarrow X$.

## 1.2 Moduli theory and stacks

### 1.2.1 Étale site

Let $S$ be a scheme. Then the **small étale site over** $S$ is the site, denoted by $\acute{\mathsf{E}}\mathsf{t}(S)$ that consists of all étale morphisms $U \to S$ (morphisms being commutative triangles). Let $\operatorname{Cov}(U \to S)$ consist of all collections $\{U_i \to U\}_{i \in I}$ such that

$$\coprod_{i \in I} U_i \to U$$

is surjective.

### 1.2.2 Grothendieck topology

Let $\mathcal{C}$ be a category. A **Grothendieck topology** on $\mathcal{C}$ consists of a set $\operatorname{Cov}(X)$ of sets of morphisms $\{X_i \to X\}_{i \in I}$ for each $X$ in $\operatorname{Ob}(\mathcal{C})$, satisfying the following axioms:

1. If $V \xrightarrow{\approx} X$ is an isomorphism, then $\{V \to X\} \in \operatorname{Cov}(X)$.

2. If $\{X_i \to X\}_{i \in I} \in \operatorname{Cov}(X)$ and $Y \to X$ is a morphism in $\mathcal{C}$, then the fiber products $X_i \times_X Y$ exists and $\{X_i \times_X Y \to Y\}_{i \in I} \in \operatorname{Cov}(Y)$.

3. If $\{X_i \in X\}_{i \in I} \in \operatorname{Cov}(X)$, and for each $i \in I$, $\{V_{ij} \to X_i\}_{j \in J} \in \operatorname{Cov}(X_i)$, then

$$\{V_{ij} \to X_i \to X\}_{i \in I, j \in J} \in \operatorname{Cov}(X).$$

The easiest example is this: Let $\mathcal{C}$ be the category of open sets on a topological space $X$, the morphisms being only the inclusions. Then for each $U \in \operatorname{Ob}(\mathcal{C})$, define $\operatorname{Cov}(U)$ to be the set of all coverings $\{U_i \to U\}_{i \in I}$ such that $U = \bigcup_{i \in I} U_i$. Then it is easily checked that this defines a Grothendieck topology.

### 1.2.3 Site

A **site** is a category equipped with a Grothendieck topology.

## 1.3 Results and theorems

### 1.3.1 Adjunction formula

Let $X$ be a smooth algebraic variety $Y$ a smooth subvariety. Let $i : Y \hookrightarrow X$ be the inclusion map, and let $\mathcal{I}$ be the corresponding ideal sheaf. Then $\omega_Y = i^*\omega_X \otimes_{\mathscr{O}_X} \det(\mathcal{I}/\mathcal{I}^2)^\vee$, where $\omega_Y$ is the canonical sheaf of $Y$.

In terms of canonical classes, the formula says that $K_D = (K_X + D)\big|_D$.

Here's an example: Let $X$ be a smooth quartic surface in $\mathbb{P}^3$. Then $H^1(X, \mathscr{O}_X) = 0$. The divisor class group of $\mathbb{P}^3$ is generated by the class of a hyperplane, and $\mathcal{K}_{\mathbb{P}^3} = -4H$. The class of $X$ is then $4H$ since $X$ is of degree 4. $X$ corresponds to a smooth divisor $D$, so by the adjunction formula, we have that

$$K_D = (K_{\mathbb{P}^3} + D)\big|_D = -4H + 4H\big|_D = 0.$$

Thus $X$ is an example of a K3 surface.

### 1.3.2 Bertini's Theorem

Let $X$ be a nonsingular closed subvariety of $\mathbb{P}^n_k$, where $k = \bar{k}$. Then the set of of hyperplanes $H \subseteq \mathbb{P}^n_k$ such that $H \cap X$ is regular at every point) and such that $H \not\subseteq X$ is a dense open subset of the complete linear system $|H|$. See [4, Thm II.8.18].

### 1.3.3 Chow's lemma

Chow's lemma says that if $X$ is a scheme that is proper over $k$, then it is "fairly close" to being projective. Specifically, we have that there exists a projective $k$-scheme $X'$ and morphism $f : X' \to X$ that is birational.

So every scheme proper over $k$ is birational to a projective scheme. For a proof, see for example the Wikipedia page.

### 1.3.4 Euler sequence

If $A$ is a ring and $\mathbb{P}^n_A$ is projective $n$-space over $A$, then there is an exact sequence of sheaves on $X$:

$$0 \to \Omega_{\mathbb{P}^n_A/A} \to \mathscr{O}_{\mathbb{P}^n_A}(-1)^{n+1} \to \mathscr{O}_{\mathbb{P}^n_A} \to 0.$$

See [4, Thm II.8.13].

### 1.3.5 Genus-degree formula

If $C$ is a smooth plane curve, then its genus can be computed as

$$g_C = \frac{(d-1)(d-2)}{2}.$$

This follows from the adjunction formula. In particular, there are no curves of genus 2 in the plane.

### 1.3.6 Hirzebruch-Riemann-Roch formula

Let $X$ be a nonsingular variety and let $\mathscr{T}_X$ be its tangent bundle. Let $\mathscr{E}$ be a locally free sheaf on $X$. Then

$$\chi(\mathscr{E}) = \deg\left(\mathrm{ch}(\mathscr{E}) \cdot \mathrm{td}(\mathscr{T})\right)_n,$$

where $\chi$ is the Euler characteristic, ch denotes the Chern class, and td denotes the Todd class. See [4, Appendix A].

### 1.3.7 Hurwitz' formula

Let $X, Y$ be smooth curves in the sense of Hartshorne. That is, they are integral 1-dimensional schemes, proper over a field $k$ (with $\bar{k} = k$), all of whose local rings are regular.

Then Hurwitz' formula says that if $f : X \to Y$ is a separable morphism and $n = \deg f$, then

$$2(g_X - 1) = 2n(g_Y - 1) + \deg R,$$

where $R$ is the ramification divisor of $f$, and $g_X, g_Y$ are the genera of $X$ and $Y$, respectively. See Example 6.1.1.

### 1.3.8 Kodaira vanishing

If $k$ is a field of characteristic zero, $X$ is a smooth and projective $k$-scheme of dimension $d$, and $\mathcal{L}$ is an ample invertible sheaf on $X$, then $H^q(X, \mathcal{L} \otimes_{\mathscr{O}_X} \Omega^p_{X/k}) = 0$ for $p + q > d$. In addition, $H^q(X, \mathcal{L}^{-1} \otimes_{\mathscr{O}_X} \Omega^p_{X/k}) = 0$ for $p + q < d$.

### 1.3.9 Lefschetz hyperplane theorem

Let $X$ be an $n$-dimensional complex projective algebraic variety in $\mathbb{P}^n_{\mathbb{C}}$ and let $Y$ be a hyperplane section of $X$ such that $U = X \backslash Y$ is smooth. Then the natural map $H^k(X, \mathbb{Z}) \to H^k(Y, \mathbb{Z})$ in singular cohomology is an isomorphism for $k < n - 1$ and injective for $k = n - 1$.

### 1.3.10 Riemann-Roch for curves

The **Riemann-Roch theorem** relates the number of sections of a line bundle with the genus of a smooth proper curve $C$. Let $\mathcal{L}$ be a line bundle $\omega_C$ the canonical sheaf on $C$. Then

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^{-1} \otimes_{\mathscr{O}_C} \omega_C) = \deg(\mathcal{L}) + 1 - g.$$

This is [4, Theorem IV.1.3].

### 1.3.11 Semi-continuity theorem

Let $f : X \to Y$ be a projective morphism of noetherian schemes, and let $\mathscr{F}$ be a coherent sheaf on $X$, flat over $Y$. Then for each $i \geq 0$, the function $h^i(y, \mathscr{F}) = \dim_{k(y)} H^i(X_y, \mathscr{F}_y)$ is an upper semicontinuous function on $Y$. See [4, Chapter III, Theorem 12.8].

### 1.3.12 Serre duality

Let $X$ be a projective Cohen-Macaulay scheme of equidimension $n$. Then for any locally free sheaf $\mathcal{F}$ on $X$ there are natural isomorphisms

$$H^i(X, \mathcal{F}) \simeq H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ).$$

Here $\omega_X^\circ$ is a *dualizing sheaf* for $X$. In the case that $X$ is nonsingular, we have that $\omega_X^\circ \simeq \omega_X$, the canonical sheaf on $X$ (see [4, Chapter III, Corollary 7.12]).

### 1.3.13 Serre vanishing

One form of Serre vanishing states that if $X$ is a proper scheme over a noetherian ring $A$, and $\mathcal{L}$ is an ample sheaf, then for any coherent sheaf $\mathscr{F}$ on $X$, there exists an integer $n_0$ such that for each $i > 0$ and $n \geq n_0$ the group $H^i(X, \mathscr{F} \otimes_{\mathscr{O}_X} \mathcal{L}^n) = 0$ vanishes. See [4, Proposition III.5.3].

## 1.4 Sheaves and bundles

### 1.4.1 Ample line bundle

A line bundle $\mathcal{L}$ is **ample** if for any coherent sheaf $\mathscr{F}$ on $X$, there is an integer $n$ (depending on $\mathscr{F}$) such that $\mathscr{F} \otimes_{\mathscr{O}_X} \mathcal{L}^{\otimes n}$ is generated by global sections. Equivalently, a line bundle $\mathcal{L}$ is ample if some tensor power of it is very ample.

### 1.4.2 Invertible sheaf

A locally free sheaf of rank 1 is called **invertible**. If $X$ is normal, then, invertible sheaves are in $1-1$ correspondence with line bundles.

### 1.4.3 Anticanonical sheaf

The **anticanonical sheaf** $\omega_X^{-1}$ is the inverse of the canonical sheaf $\omega_X$, that is $\omega_X^{-1} = \mathscr{H}om_{\mathscr{O}_X}(\omega_X, \mathscr{O}_X)$.

### 1.4.4 Canonical class

The **canonical class** $K_X$ is the class of the canonical sheaf $\omega_X$ in the divisor class group.

### 1.4.5 Canonical sheaf

If $X$ is a smooth algebraic variety of dimension $n$, then the canonical sheaf is $\omega := \wedge^n \Omega^1_{X/k}$ the $n$'th exterior power of the cotangent bundle of $X$.

### 1.4.6 Nef divisor

Let $X$ be a normal variety. Then a Cartier divisor $D$ on $X$ is **nef** (*numerically effective*) if $D \cdot C \geq 0$ for every irreducible complete curve $C \subseteq X$. Here $D \cdot C$ is the intersection product on $X$ defined by $\deg(\phi^* \mathscr{O}_X(D))$. Here $\phi : C' \to C$ is the normalization of $C$.

### 1.4.7 Sheaf of holomorphic p-forms

If $X$ is a complex manifold, then the **sheaf of of holomorphic $p$-forms** $\Omega^p_X$ is the $p$-th wedge power of the cotangent sheaf $\wedge^p \Omega^1_X$.

### 1.4.8 Normal sheaf

Let $Y \hookrightarrow X$ be a closed immersion of schemes, and let $\mathcal{I} \subseteq \mathscr{O}_X$ be the ideal sheaf of $Y$ in $X$. Then $\mathcal{I}/\mathcal{I}^2$ is a sheaf on $Y$, and we define the sheaf $\mathcal{N}_{Y/X}$ by $\mathscr{H}om_{\mathscr{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathscr{O}_Y)$.

### 1.4.9 Rank of a coherent sheaf

Given a coherent sheaf $\mathscr{F}$ on an irreducible variety $X$, form the sheaf $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}_X$. Its global sections is a finite dimensional vector space, and we say that $\mathscr{F}$ has rank $r$ if $\dim_k \Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}_X) = r$.

### 1.4.10 Reflexive sheaf

A sheaf $\mathscr{F}$ is **reflexive** if the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorpism. Here $\mathscr{F}^{\vee}$ denotes the sheaf $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$.

### 1.4.11 Very ample line bundle

A line bundle $\mathcal{L}$ is **very ample** if there is an embedding $i : X \hookrightarrow \mathbb{P}^n_S$ such that the pullback of $\mathscr{O}_{\mathbb{P}^n_S}(1)$ is isomorphic to $\mathcal{L}$. In other words, there should be an isomorphism $i^* \mathscr{O}_{\mathbb{P}^n_S}(1) \simeq \mathcal{L}$.

## 1.5 Toric geometry

### 1.5.1 Chow group of a toric variety

The Chow group $A_{n-1}(X)$ of a toric variety can be computed directly from its fan. Let $\Sigma(1)$ be the set of rays in $\Sigma$, the fan of $X$. Then we have an exact sequence

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(X) \to 0.$$

The first map is given by sending $m \in M$ to $(\langle m, v_p\rangle)_{\rho\in\Sigma(1)}$, where $v_p$ is the unique generator of the semigroup $\rho \cap N$. The second map is given by sending $(a_\rho)_{\rho\in\Sigma(1)}$ to the divisor class of $\sum_\rho a_\rho D_\rho$.

### 1.5.2 Generalized Euler sequence

The **generalized Euler sequence** is a generalization of the Euler sequence for toric varieties. If $X$ is a smooth toric variety, then its cotangent bundle $\Omega^1_X$ fits into an exact sequence

$$0 \to \Omega^1_X \to \oplus_\rho \mathscr{O}_X(-D_\rho) \to \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathscr{O}_X \to 0.$$

Here $D_\rho$ is the divisor corresponding to the ray $\rho \in \Sigma(1)$. See [2, Chapter 8].

### 1.5.3 Polarized toric variety

A toric variety equipped with an ample $T$-invariant divisor.

### 1.5.4 Toric variety associated to a polytope

There are several ways to do this. Here is one: Let $\Delta \subset M_{\mathbb{R}}$ be a convex polytope. Embed $\Delta$ in $M_R \times \mathbb{R}$ by $\Delta \times \{1\}$ and let $C_\Delta$ be the cone over

$\Delta \times \{1\}$, and let $\mathbb{C}[C_\Delta \cap (M \times \mathbb{Z})]$ be the corresponding semigroup ring. This is a semigroup ring graded by the $\mathbb{Z}$-factor. Then we define $\mathbb{P}_\Delta = \operatorname{Proj} \mathbb{C}[C_\Delta \cap (M \times \mathbb{Z})]$ to be the toric variety associated to a polytope.

## 1.6 Types of varieties

### 1.6.1 Abelian variety

A variety $X$ is an **abelian variety** if it is a connected and complete algebraic group over a field $k$. Examples include elliptic curves and for special lattices $\Lambda \subset \mathbb{C}^{2g}$, the quotient $\mathbb{C}^{2g}/\Lambda$ is an abelian variety.

### 1.6.2 Calabi-Yau variety

In algebraic geometry, a **Calabi-Yau** variety is a smooth, proper variety $X$ over a field $k$ such that the canonical sheaf is trivial, that is, $\omega_X \simeq \mathscr{O}_X$, and such that $H^j(X, \mathscr{O}_X) = 0$ for $1 \leq j \leq n-1$.

### 1.6.3 del Pezzo surface

A **del Pezzo** surface is a 2-dimensional Fano variety. In other words, they are complete non-singular surfaces with ample anticanonical bundle. The *degree* of the del Pezzo surface $X$ is by definition the self intersection number $K.K$ of its canonical class $K$.

### 1.6.4 Elliptic curve

An **elliptic curve** is a smooth, projective curve of genus 1. They can all be obtained from an equation of the form $y^2 = x^3 + ax + b$ such that $\Delta = -2^4(4a^3 + 27b^2) \neq 0$.

### 1.6.5 Fano variety

A variety $X$ is **Fano** if the anticanonical sheaf $\omega_X^{-1}$ is ample.

### 1.6.6 Jacobian variety

Let $X$ be a curve of genus $g$ over $k$. The **Jacobian variety** of $X$ is a scheme $J$ of finite type over $k$, together with an element $\mathcal{L} \in \operatorname{Pic}^\circ(X/J)$, with the following universal property: for any scheme $T$ of finite type over $k$ and for any $\mathcal{M} \in \operatorname{Pic}^\circ(X/T)$, there is a unique morphism $f : T \to J$ such that $f^*\mathcal{L} \simeq \mathcal{M}$ in $\mathrm{Pic}^\circ(X/T)$. This just says that $J$ represents the functor $T \mapsto \mathrm{Pic}^\circ(X/T)$.

If $J$ exists, its closed points are in $1-1$ correspondence with elements of $\mathrm{Pic}^\circ(X)$.

It can be checked that $J$ is actually a group scheme. For details, see [4, Ch. IV.4].

#### 1.6.7 K3 surface

A **K3 surface** is a complex algebraic surface $X$ such that the canonical sheaf is trivial, $\omega_X \simeq \mathscr{O}_X$, and such that $H^1(X, \mathscr{O}_X) = 0$. These conditions completely determine the Hodge numbers of $X$.

#### 1.6.8 Stanley-Reisner scheme

A **Stanley-Reisner scheme** is a projective variety associated to a simplicial complex as follows. Let $\mathcal{K}$ be a simplicial complex. Then we define an ideal $I_\mathcal{K} \subseteq k[x_v \mid v \in V(\mathcal{K})] = k[\mathbf{x}]$ (here $V(\mathcal{K})$ denotes the vertex set of $\mathcal{K}$) by

$$I_\mathcal{K} = \langle x_{v_{i_1}} x_{v_{i_2}} \cdots x_{v_{i_k}} \mid v_{i_1} v_{i_2} \cdots v_{i_k} \notin \mathcal{K} \rangle.$$

We get a projective scheme $\mathbb{P}(\mathcal{K})$ defined by $\mathrm{Proj}\,(k[\mathbf{x}]/I_\mathcal{K})$, together with an embedding into $\mathbb{P}^{\#V(\mathcal{K})-1}$. It can be shown that $H^p(\mathbb{P}(\mathcal{K}), \mathscr{O}_{\mathbb{P}(\mathcal{K})}) \simeq H^p(\mathcal{K}; k)$, where the right-hand-side denotes the cohomology group of the simplicial complex.

#### 1.6.9 Toric variety

A **toric variety** $X$ is an integral scheme containing the torus $(k^*)^n$ as a dense open subset, such that the action of the torus on itself extends to an action $(k^*)^n \times X \to X$.

# 2 Commutative algebra

## 2.1 Modules

#### 2.1.1 Depth

Let $R$ be a noetherian ring, and $M$ a finitely-generated $R$-module and $I$ an ideal of $R$ such that $IM \neq M$. Then the $I$-depth of $M$ is (see Ext):

$$\inf\{i \mid \mathrm{Ext}^i_R(R/I, M) \neq 0\}.$$

This is also the length of a maximal $M$-sequence in $I$.

### 2.1.2 Kähler differentials

Let $A \to B$ be a ring homomorphism. The **module of Kähler differentials** $\Omega_{B/A}$ is the module together with a map $d : B \to \Omega_{B/A}$ satisfying the following universal property: if $D : B \to M$ is any $A$-linear derivation (an element of $\mathrm{Der}_A(B, M)$), then there is a unique module homomorphism $\widetilde{D} : \Omega_{B/A} \to M$ such that

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "D"'] & \Omega_{B/A} \arrow[d, "\widetilde{D}"] \\ & M \end{tikzcd}$$

is commutative. Thus we have a natural isomorphism $\mathrm{Der}_A(B, M) = \mathrm{Hom}_B(\Omega_{B/A}, M)$. In the language of category theory, this means that $\mathrm{Der}_A(B, -)$ is *corepresented* by $\Omega_{B/A}$.

A concrete construction of $\Omega_{B/A}$ is given as follows. Let $M$ be the free $B$-module generated by all symbols $df$, where $f \in B$. Let $N$ be the submodule generated by $da$ if $a \in A$, $d(f + g) - df - dg$ and the Leibniz rule $d(fg) - fdg - gdf$. Then $M/N \simeq \Omega_{B/A}$ as $B$-modules.

## 2.2 Results and theorems

### 2.2.1 The conormal sequence

The **conormal sequence** is a sequence relating Kähler differentials in different rings. Specifically, if $A \to B \to 0$ is a surjection of rings with kernel $I$, then we have an exact sequence of $B$-modules:

$$I/I^2 \xrightarrow{d} B \otimes_A \Omega_{B/A} \xrightarrow{D\pi} \Omega_{T/R} \to 0$$

The map $d$ sends $f \mapsto 1 \otimes df$, and $D\pi$ sends $c \otimes db \mapsto cdb$. For proof, see [3, Chapter 16].

### 2.2.2 The Unmixedness Theorem

Let $R$ be a ring. If $I = \langle x_1, \cdots, x_n \rangle$ is an ideal generated by $n$ elements such that $\operatorname{codim} I = n$, then all minimal primes of $I$ have codimension $n$. If in addition $R$ is Cohen-Macaulay, then every associated prime of $I$ is minimal over $I$. See the discussion after [3, Corollary 18.14] for more details.

### 2.3 Rings

#### 2.3.1 Cohen-Macaulay ring

A local Cohen-Macaulay ring (CM-ring for short) is a commutative noetherian local ring with Krull dimension equal to its depth. A ring is Cohen-Macaulay if its localization at all prime ideals are Cohen-Macaulay.

#### 2.3.2 Depth of a ring

The depth of a ring $R$ is is its depth as a module over itself.

#### 2.3.3 Gorenstein ring

A commutative ring $R$ is Gorenstein if each localization at a prime ideal is a Gorenstein local ring. A Gorenstein local ring is a local ring with finite injective dimension as an $R$-module. This is equivalent to the following: $\mathrm{Ext}^i_R(k, R) = 0$ for $i \neq n$ and $\mathrm{Ext}^n_R(k, R) \simeq k$ (here $k = R/\mathfrak{m}$ and $n$ is the Krull dimension of $R$).

#### 2.3.4 Normal ring

An integral domain $R$ is **normal** if all its localizations at prime ideals $\mathfrak{p} \in \operatorname{Spec} R$ are integrally closed domains.

## 3 Convex geometry

### 3.1 Cones

#### 3.1.1 Gorenstein cone

A strongly convex cone $C \subset M_{\mathbb{R}}$ is **Gorenstein** if there exists a point $n \in N$ in the dual lattice such that $\langle v, n \rangle = 1$ for all generators of the semigroup $C \cap M$.

#### 3.1.2 Reflexive Gorenstein cone

A cone $C$ is **reflexive** if both $C$ and its dual $C^{\vee}$ are Gorenstein cones. See for example [1].

### 3.1.3 Simplicial cone

A cone $C$ generated by $\{v_1, \cdots, v_k\} \subseteq N_{\mathbb{R}}$ is **simplicial** if the $v_i$ are linearly independent.

## 3.2 Polytopes

### 3.2.1 Dual (polar) polytope

If $\Delta$ is a polyhedron, its dual $\Delta^\circ$ is defined by

$$\Delta^\circ = \{x \in N_{\mathbb{R}} \mid \langle x, y\rangle \geq -1 \,\forall\, y \in \Delta\}\,.$$

### 3.2.2 Gorenstein polytope of index r

A lattice polytope $P \subset \mathbb{R}^{d+r-1}$ is called a **Gorenstein polytope of index** $r$ if $rP$ contains a single interior lattice point $p$ and $rP - p$ is a reflexive polytope.

### 3.2.3 Nef partition

Let $\Delta \subset M_{\mathbb{R}}$ be a $d$-dimensional reflexive polytope, and let $m = \mathrm{int}(\Delta) \cap M$. A Minkowski sum decomposition $\Delta = \Delta_1 + \ldots + \Delta_r$ where $\Delta_1, \ldots, \Delta_r$ are lattice polytopes is called a **nef partition of $\Delta$ of length** $r$ if there are lattice points $p_i \in \Delta_i$ for all $i$ such that $p_1 + \cdots + p_r = m$. The nef partition is called *centered* if $p_i = 0$ for all $i$.

This is equivalent to the toric divisor $D_j = \mathscr{O}(\Delta_i) = \sum_{\rho \in \Delta_i} D_\rho$ being a Cartier divisor generated by its global sections. See [1, Chapter 4.3].

### 3.2.4 Reflexive polytope

A polytope $\Delta$ is **reflexive** if the following two conditions hold:

1. All facets $\Gamma$ of $\Delta$ are supported by affine hyperplanes of the form $\{m \in M_{\mathbb{R}} \mid \langle m, v_\Gamma\rangle = -1\}$ for some $v_\Gamma \in N$.

2. The only interior point of $\Delta$ is 0, that is: $\mathrm{Int}(\Delta) \cap M = \{0\}$.

It can be proved that a polytope $\Delta$ is reflexive if and only if the associated toric variety $\mathbb{P}_\Delta$ is Fano.

# 4 Homological algebra

## 4.1 Classes of modules

### 4.1.1 Projective modules

Projective modules are those satisfying a universal lifting property. A module $P$ is **projective** if for every epimorphism $\alpha : M \to N$ and every map, $\beta : P \to N$, there exists a map $\gamma : P \to M$ such that $\beta = \alpha \circ \gamma$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\gamma"'] \arrow[d, "\beta"] & \\ M \arrow[r, two heads, "\alpha"] & N \arrow[r] & 0 \end{tikzcd}$$

These are the modules $P$ such that $\mathrm{Hom}(P, -)$ is exact.

## 4.2 Derived functors

### 4.2.1 Ext

Let $R$ be a ring and $M, N$ be $R$-modules. Then $\mathrm{Ext}^i_R(M, N)$ is the right-derived functors of the $\mathrm{Hom}(M, -)$-functor. In particular, $\mathrm{Ext}^i_R(M, N)$ can be computed as follows: choose a projective resolution $C_.$ of $N$ over $R$. Then apply the left-exact functor $\mathrm{Hom}_R(M, -)$ to the resolution and take homology. Then $\mathrm{Ext}^i_R(M, N) = h^i(C_.)$.

### 4.2.2 Local cohomology

Let $R$ be a ring and $I \subset R$ an ideal. Let $\Gamma_I(-)$ be the following functor on $R$-modules:

$$\Gamma_I(M) = \{f \in M \mid \exists n \in \mathbb{N}, s.t. I^n f = 0\}.$$

Then $H^i_I(-)$ is by definition the $i$th right derived functor of $\Gamma_I$. In the case that $R$ is noetherian, we have $H^i_I(M) = \varinjlim \mathrm{Ext}^i_R(R/I_n, M)$.

See [3] and [7] for more details.

### 4.2.3 Tor

Let $R$ be a ring and $M, N$ be $R$-modules. Then $\mathrm{Tor}^i_R(M, N)$ is the right-derived functors of the $- \otimes_R N$-functor. In particular $\mathrm{Tor}^i_R(M, N)$ can be computed by taking a projective resolution of $M$, tensoring with $N$, and then taking homology.

# 5 Differential and complex geometry

## 5.1 Definitions and concepts

### 5.1.1 Almost complex structure

An **almost complex structure** on a manifold $M$ is a map $J : T(M) \to T(M)$ whose square is $-1$.

### 5.1.2 Connection

Let $E \to M$ be a vector bundle over $M$. A **connection** is a $\mathbb{R}$-linear map $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$ such that the Leibniz rule holds:

$$\nabla(f\sigma) = f\nabla(\sigma) + \sigma \otimes \mathrm{d}f$$

for all functions $f : M \to \mathbb{R}$ and sections $\sigma \in \Gamma(E)$.

### 5.1.3 Hermitian manifold

A *Hermitian metric* on a complex vector bundle $E$ over a manifold $M$ is a positive-definite Hermitian form on each fiber. Such a metric can be written as a smooth section $\Gamma(E \otimes \bar{E})^*$, such that $h_p(\eta, \bar{\zeta}) = h_p\left(\bar{\zeta}, \bar{\eta}\right)$ for all $p \in M$, and such that $h_p(\eta, \bar{\eta}) > 0$ for all $p \in M$. A **Hermitian manifold** is a complex manifold with a Hermitian metric on its holomorphic tangent space $T^{(1,0)}(M)$.

### 5.1.4 Kähler manifold

A **Kahler manifold** is ????

### 5.1.5 Symplectic manifold

A $2n$-dimensional manifold $M$ is **symplectic** if it is compact and oriented and has a closed real two-form $\omega \in \bigwedge^2 T^*(M)$ which is nondegenerate, in the sense that $\wedge^n \omega\big|_p \neq 0$ for all $p \in M$.

### 5.2 Results and theorems

# 6 Worked examples

## 6.1 Algebraic geometry

### 6.1.1 Hurwitz formula and Kähler differentials

Let $X$ be the conic in $\mathbb{P}^2$ given with ideal sheaf $\langle xz - y^2 \rangle$. Let $Y$ be $\mathbb{P}^1$, and consider the map $f : X \to Y$ given by projection onto the $xz$-line. $X$ is covered by two affine pieces, namely $X = U_x \cup U_z$, the spectra of the homogeneous localizations at $x, z$, respectively. Let $U_x = \operatorname{Spec} A$ for $A = k[z]$ and $U_z = \operatorname{Spec} B$ for $B = k[x]$. Then the map is locally given by $A \to k[y,z]/(z - y^2)$ where $z \mapsto \bar{z}$, and similarly for $B$. We have an isomorphism $k[y,z]/(z-y^2) \simeq k[t]$, given by $y \mapsto t$ and $z \mapsto t^2$, so that locally the map is given by $k[z] \to k[t], z \mapsto t^2$.

This is a map of smooth projective curves, so we can apply Hurwitz' formula. Both $X, Y$ are $\mathbb{P}^1$, so both have genus zero. Hence Hurwitz formula says that

$$-2 = -n \cdot 2 + \deg R,$$

where $R$ is the ramification divisor and $n$ is the degree of the map. The degree of the map can be defined locally, and it is the degree of the field extension $k(Y) \hookrightarrow k(X)$. But (the image of) $k(Y) = k(t^2)$ and $k(X) = k(t)$, so that $[k(Y) : k(X)] = 2$. Hence by Hurwitz' formula, we should have $\deg R = 2$. Since $R = \sum_{P \in X} \operatorname{length} \Omega_{X/Y\,P} \cdot P$, we should look at the sheaf of relative differentials $\Omega_{Y/X}$.

First we look in the chart $U_z$. We compute that $\Omega_{k[t]/k[t^2]} = k[t]/(t)$. This follows from the relation $d(t^2) = 2dt$, implying that $dt = 0$ in $\Omega_{k[t]/k[t^2]}$. This module is zero localized at all primes but $(t)$, where it is $k$. Thus for $P = (0:0:1)$, we have $\operatorname{length} \Omega_{X/Y\,P} = 1$.

The situation is symmetric with $z \leftrightarrow x$, so that we have $R = (0:0:1) + (1:0:0)$, confirming that $\deg R = 2$.

In fact, the curve $C$ is isomorphic to $\mathbb{P}^1$ via the map $\mathbb{P}^1 \to C$ given by $(s:t) \mapsto (s^2 : st : t^2)$. Identifying $C$ with $\mathbb{P}^1$, we thus see that $C \to \mathbb{P}^1$ correspond to the map $\mathbb{P}^1 \to \mathbb{P}^1$ given by $(s:t) \mapsto (s^2 : t^2)$.

## 6.2 The quintic threefold

Let $Y$ be a the zeroes of a general hypersurface of degree 5 in $\mathbb{P}^4$, or in other words, a section of $\omega_{\mathbb{P}^4}$. We want to compute the cohomology of $Y$ and its Hodge numbers. Let $\mathbb{P} = \mathbb{P}^4$.

We have the ideal sheaf sequence

$$0 \to \mathscr{I} \to \mathscr{O}_{\mathbb{P}} \to i^* \mathscr{O}_Y \to 0,$$

where $i : Y \to \mathbb{P}^4$ is the inclusion. Note that $\mathscr{I} = \mathscr{O}_{\mathbb{P}}(-5)$. Thus we have from the long exact sequence of cohomology that

$$\cdots \to H^i(\mathbb{P}, \mathscr{I}) \to H^i(\mathbb{P}, \mathscr{O}_{\mathbb{P}}) \to H^i(Y, \mathscr{O}_Y) \to H^{i+1}(\mathbb{P}, \mathscr{I}) \to \cdots$$

Note that $H^{i+1}(\mathbb{P}, \mathscr{I}) = 0$ for $i \neq 3$ and 1 for $i = 3$. Also $H^i(\mathbb{P}, \mathscr{O}_{\mathbb{P}}) = 0$ unless $i = 0$ in which case it is 1. Thus we get that $H^i(Y, \mathscr{O}, Y)$ is $k$ for $i = 0$, for $i = 1, 2$ it is 0, and for $i = 3$ it is $k$. For higher $i$ it is zero by Grothendieck vanishing.

The adjunction formula relates the canonical bundles as follows: if $\omega_{\mathbb{P}}$ is the canonical bundle on $\mathbb{P}$, then $\omega_Y = i^*\omega_{\mathbb{P}} \otimes_{\mathscr{O}_{\mathbb{P}}} \det(\mathscr{I}/\mathscr{I}^2)^\vee$. The ideal sheaf is already a line bundle, so taking the determinant does not change anything. Now

$$\begin{aligned}(\mathscr{I}/\mathscr{I}^2)^\vee &= \mathrm{Hom}_Y(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_Y) \\ &= \mathrm{Hom}_{\mathbb{P}}(\mathscr{I}, \mathscr{O}_Y) = \mathrm{Hom}_{\mathbb{P}}(\mathscr{O}_{\mathbb{P}}(-5), \mathscr{O}_Y) = \mathscr{O}_Y(5).\end{aligned}$$

It follows that $\omega_Y = \mathscr{O}_Y(-5) \otimes \mathscr{O}_Y(5) = \mathscr{O}_Y$. Thus the canonical bundle is trivial and we conclude that $Y$ is Calabi-Yau.

It remain to compute the Hodge numbers. We start with $h^{11} = \dim_k H^1(Y, \Omega_Y)$. We have the conormal sequence of sheaves on $Y$:

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_{\mathbb{P}} \otimes \mathscr{O}_Y \to \Omega_Y \to 0,$$

which gives us the long exact sequence:

$$\cdots \to H^i(\mathscr{I}/\mathscr{I}^2) \to H^i(\Omega_{\mathbb{P}} \otimes \mathscr{O}_Y) \to H^i(\Omega_Y) \to H^{i+1}(\mathscr{I}/\mathscr{I}^2) \to \cdots$$

We first compute the cohomology of $\mathscr{I}/\mathscr{I}^2$. We use the short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}}(-10) \to \mathscr{O}_{\mathbb{P}}(-5) \to \mathscr{I}/\mathscr{I}^2 \to 0. \tag{1}$$

we have $H^i(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(-10)) = 0$ for $i = 0, 1, 2, 3$, and for $i = 4$ we have $H^4(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(-10)) = H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(5)) = k^{126}$. Similarly $H^i(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(-5)) = 0$ for $i = 0, 1, 2, 3$ and $H^4(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(-5)) = H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}) = k$. We conclude that $h^i(Y, \mathscr{I}/\mathscr{I}^2) = 0$ for $i = 0, 1, 2$ and 125 for $i = 3$.

In particular $H^1(\Omega_Y) \simeq H^1(\Omega_{\mathbb{P}} \otimes \mathscr{O}_Y)$. We have the Euler sequence:

$$0 \to \Omega_{\mathbb{P}} \to \mathscr{O}_{\mathbb{P}}(-1)^{\oplus 5} \to \mathscr{O}_{\mathbb{P}} \to 0$$

Now $\mathscr{O}_Y = \mathscr{O}_{\mathbb{P}}/\mathscr{I}$ is a flat $\mathscr{O}_{\mathbb{P}}$-module since $\mathscr{I}$ is principal and generated by a non-zero divisor. Thus we can tensor the Euler sequence with $\mathscr{O}_Y$ and get

$$0 \to \Omega_{\mathbb{P}} \otimes \mathscr{O}_Y \to \mathscr{O}_Y(-1)^5 \to \mathscr{O}_Y \to 0,$$

from which it easily follows that $H^1(Y, \Omega_{\mathbb{P}} \otimes \mathscr{O}_Y) \simeq H^0(\mathscr{O}_Y) = k$. We conclude that $h^{11} = 1$.

Now we compute $h^{12} = \dim_k H^1(Y, \Omega_Y^2)$. This is equal to $H^2(Y, \Omega_Y)$ by Serre duality. Again we use the conormal sequence. From the Euler sequence we get that $H^2(Y, \Omega_{\mathbb{P}} \otimes \mathscr{O}_Y) = 0$. We also get that $h^3(Y, \Omega_{\mathbb{P}} \otimes \mathscr{O}_Y) = 24$. NOW $H^3(\Omega_Y) = 0$ (WHY??), and it follows from the above computations that $h^{12} = 125 - 24 = 101$.

This example is extremely important in mirror symmetry.

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