## Exercises deformation theory

FM

## 1 Chapter 1.1

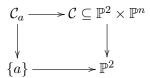
**Exercise 1.** Existence of the Hilbert scheme for curves in  $\mathbb{P}^2$ . Here a *curve* is a closed subscheme of  $\mathbb{P}^2_k$  defined by any homogeneous polynomial of degree d in S = k[x, y, z] (so curves are in 1-1 correspondence with points in  $\mathbb{P}(S^dV)$  where V is a 3-dim k-vector space).

Write f has  $a_0 x_0^d + \ldots + a_n z^d$  with  $a_i \in k$  and  $n = {d+2 \choose 2} - 1$ . Consider  $(a_0, \dots, a_n)$  as a point in  $\mathbb{P}_k^n$ .

- 1. Curves in  $\mathbb{P}^2$  of degree d are in 1-1 correspondence with points in  $\mathbb{P}^n$  by this correspondence.
- 2. Define  $\mathcal{C} \subseteq \mathbb{P}^2 \times \mathbb{P}^n$  by  $a_0 x^d + \cdots + a_n + x^d = 0$ . Show that the correspondence in a) is given by  $a \in \mathbb{P}^n$  goes to the fiber  $\mathcal{C}_a \subseteq \mathbb{P}^2$  over the point a. We call  $\mathcal{C}$  the tautological family.
- 3. For any finitely generated k-algebra A, we define a family of curves of degree d in  $\mathbb{P}^2$  over A to be a closed subscheme  $X \subseteq \mathbb{P}^2_A$ , flat over A, such that the fibers over closed points of Spec A are curves of degree d in  $\mathbb{P}^2$ . Show that the ideal  $I_X \subseteq A[x,y,z]$  is generated by a single homogeneous polynomial f of degree d in A[x,y,z].

Solution 1. 1. Obvious.

2. Let  $a \in \mathbb{P}^n$ . Then  $\mathcal{C}_a$  is precisely the subscheme  $\subseteq \mathbb{P}^2$  cut out by f = 0.



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3. By lifting, we can assume that  $A=k[b_1,\cdots,b_l]$  for some l. Then the question is equivalent to: Suppose  $I\subseteq k[b_1,\cdots,b_l]\otimes k[x,y,z]$  is such that  $I\otimes_k A/\mathfrak{m}=\langle f\rangle$  for some  $f\in k[x,y,z]$  for all  $\mathfrak{m}\in\operatorname{Spec} A$ . Suppose in addition that A[x,y,z]/I is a flat A-module. Then  $I=\langle \tilde{f}\rangle$  for some  $\tilde{f}\in A[x,y,z]$  such that  $\tilde{f}\otimes A/\mathfrak{m}=f$ .

This should follow from the equational criterium for flatness. In particular: in each fibre,  $I \otimes A/\mathfrak{m}$  is generated by a single polynomial, and this lifts to a generator of I, together with the trivial relation. If I had more than one generator, there would be a relation that is trivial in all fibers. But then it must be trivial everywhere.

 $\Diamond$