Algebraiske grupper og moduliteori

Fredrik Meyer

26. september 2014

1 Representations of algebraic groups

Let $G = \operatorname{Spec} A$ be an affine algebraic group over a field k.

Definition 1.1. An *algebraic representation of* G is a pair (V, μ_V) consisting of a k-vector space V and a k-linear map $\mu_V : V \to V \otimes_k A$ satisfying the following two conditions:

1. The diagram
$$\begin{tikzcd} V \arrow[r, "\mu_V"] \arrow[dr, "\mathrm{id}"'] & V \otimes_k A \arrow[d, "\mathrm{id}\otimes\epsilon"] \\ & V \simeq V \otimes_k k \end{tikzcd}$$
is commutative.

2. The diagram
$$\begin{tikzcd} V \arrow[r, "\mu_V"] \arrow[d, "\mu_V"'] & V \otimes_k A \arrow[d, "\mu_V\otimes\mathrm{id}_A"] \\ V \otimes_k A \arrow[r, "\mathrm{id}_V\otimes\mu_A"'] & V \otimes_k A \otimes_k A \end{tikzcd}$$
is commutative. Here μ_A is the coproduct in the coordinate ring of G.

■

We often drop the subcript from μ_V unless confusion may arise. The same comment applies to tensor products. They will always be over the ground field unless otherwise stated. We will sometimes refer to a representation (V, μ_V) sometimes as "a representation $\mu : V \to V \otimes A$" and sometimes as just "a representation V".

Definition 1.2. Let $\mu : V \to V \otimes A$ be a representation of $G = \operatorname{Spec} A$. Then:

1. A vector $x \in V$ is said to be *G-invariant* if $\mu(x) = x \otimes 1$.
2. A subspace $U \subset V$ is called a *subrepresentation* if $\mu(U) \subseteq U \otimes A$.

■

Proposition 1.3. *Every representation V of G is locally finite-dimensional. Precisely: every $x \in V$ is contained in a finite-dimensional subrepresentation of G.*

Bevis. Write $\mu(x)$ as a finite sum $\sum_i x_i \otimes f_i$ for $x_i \in V$ and linearly independent $f_i \in A$. This we can always do, by definition of tensor product and bilinearity. Let U be the subspace of V spanned by the vectors x_i. □