Results so far

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1 Introduction

In the article [Gaĭ09], it is proven that there exists a triangulation of $\mathbb{CP}^2$ with 15 vertices and other nice properties [[expand]]. The triangulation is constructed by "glueing" cones of 3-spheres along a triangle. Call this triangulation for $\mathcal{S}$.

A smoothing of the associated Stanley-Reisner scheme $\mathbb{P}(\mathcal{S})$ would have interesting properties. In particular, it would be a hyper-Kähler variety [[EXPLAIN]]

The Hilbert polynomial of $\mathbb{P}(\mathcal{S})$ is $(9/2)t^4 + (15/2)t^2 + 3$. The simplicial complex $\mathcal{S}$ have f-vector $(15, 90, 240, 270, 108)$. In particular, the degree of $\mathbb{P}(\mathcal{S})$ is 108.

Here are a few computations:

Lemma 1.1. *The module of first-order deformations of* $\mathbb{P}(\mathcal{S})$ *is* 90*-dimensional, i.e.* $\dim_k T^1(\mathbb{P}(\mathcal{S})) = 90$.

Lemma 1.2. *The obstruction module have* $\dim_k T^2(\mathbb{P}(\mathcal{K})) = 306$.

The link of $\mathcal{S}$ at one of its vertices is a particularly simple 3-sphere, namely the join of the boundaries of two hexagons. Call this $\mathcal{K}$. Then $\mathcal{K}$ have f-vector $(1, 12, 48, 72, 36)$. In particular the Stanley-Reisner scheme $\mathbb{P}(\mathcal{K})$ have degree 36.

Consider now $\mathcal{K} * \Delta^1$. This a cone over a ball, so it is topologically a 5-dimensional ball. In fact, it is the join of two hexagons. So it is a 5-dimensional ball. Consider now the polytope $P * P$ that is the convex hull of the columns of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

This polytope is the join of two *actual* hexagons in $\mathbb{R}^5$. Thus, by standard Sturmfels theory, there is a Gröebner degeneration of the associated toric variety (whose fan is the polar polytope of P).

2 Deformations of the 5-dimensional toric variety

The 5-dimensional toric variety T had 2-dimensional singularities (actually two disjoint copies of dP_6).

Theorem 2.1. *There exists a flat deformation of $\mathbb{P}(\mathcal{K} * \Delta^1)$, $\mathfrak{X} \to S$, such that $\mathfrak{X}_{t_1} = T$ for some $t_1 \in S$ and such that the general fiber $\widetilde{T}$ have one-dimensional singularities.*

Proof. As per now, the proof is purely by computer. The technique is this: First, consider the monomial degeneration of T to the Stanley-Reisner ring $A(\mathcal{K} * \Delta^1)$ (recall that $\mathcal{K} = D_6 * D_6$). Choose deformation parameters t_i perturbing the equations "in the direction of T", meaning that we only choose parameters introducing terms already occuring in the equations of T.

Now, using the package `VersalDeformations`, it is possible to produce a flat family $\mathfrak{X} \to S'$ with $\mathfrak{X}_0 = \mathbb{P}(\mathcal{K} * \Delta^1)$.

The base space is a union of toric varieties of dimensions $14, 13, 13, 12$, respectively. Call the largest one for S. The equations are

$$\begin{vmatrix} t_1 & t_2 & t_2 & \cdots & t_6 \\ t_7 & t_8 & t_9 & \cdots & t_{12} \end{vmatrix} \leq 1 \qquad \begin{vmatrix} t_{13} & t_{14} & t_{15} & \cdots & t_{18} \\ t_{19} & t_{20} & t_{21} & \cdots & t_{24} \end{vmatrix} \leq 1$$

By restriction, we get the claimed family $\mathfrak{X} \to T$. It is cheched by setting $t_i = i$ for $i = 1, \ldots, 12$ and $t_i = 2i$ for $i = 13, \ldots, 24$ that the resulting fiber have one-dimensional singularities. The reason we don't set all $t_i = 1$, is that this point lies in the intersection of the components of S. $\square$

Corollary 2.2. *The Stanley-Reisner scheme $\mathbb{P}(\mathcal{K})$ smooths to a smooth Calabi-Yau variety X.*

Proof. The scheme $\mathbb{P}(\mathcal{K})$ sits as a complete intersection in $\mathbb{P}(\mathcal{K} * \Delta^1)$. Complete intersections deform together with the ambient variety, so $\mathbb{P}(\mathcal{K} * \Delta^1)$ deforms to a general complete intersection in $\widetilde{T}$. Since $\widetilde{T}$ have curve singularities, it follows by two applications of Bertini's theorem [Har77, Theorem II.8.18], that X is smooth. $\square$

Now what we would really like to do is to compute the Hodge numbers $h^{ij} = \dim_k H^j(X, \Omega_X^i)$ of X.

We can however compute the Hodge numbers of $\widetilde{T}$. The hope is that there is some sort of Lefschetz theorem giving us the Hodge numbers of X.

Theorem 2.3. *We have $h^{11}(\widetilde{T}) = 1$ and $h^{12}(\widetilde{T}) = 12$.*

Proof. Again, this is purely computational. We use long exact sequences together with sheaf cohomology computations in `Macaulay2`.

Since the ideal of $\widetilde{T}$ is rather complicated, doing this naïvely does not work. The trick is to choose the right term order. Since we know that $\widetilde{T}$ has a nice degeneration, we would like to find a term order such that its initial ideal is precisely the Stanley-Reisner ideal.

The `Macaulay2` package `gfanInterface` provides an interface with `gfan`, which is a program that can compute weight vectors given polynomials with prescribed initial terms. The weight vector is

$$\omega = (1, 1, 4, 7, 7, 4, 1, 1, 4, 7, 7, 4, 1, 1).$$

With this term order, giving a very small Gröbner basis (18 elements), the computations are much faster than with the standard term order. We are able to compute resolutions of all the relevant modules within a few minutes in total.

We have an exact sequence of sheaves on $\widetilde{T}$:

$$0 \to \mathscr{T}_1 \hookrightarrow \mathcal{I}/\mathcal{I}^2 \xrightarrow{d} \Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}} \to \Omega^1_{\widetilde{T}} \to 0.$$

This sequence can be broken into two short exact sequences. The relevant one is this:

$$0 \to \operatorname{im} d \to \Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}} \to \Omega^1_{\widetilde{T}} \to 0. \tag{1}$$

We also have the restriced Euler sequence:

$$0 \to \Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}} \to \mathscr{O}_{\widetilde{T}}(-1)^{14} \to \mathscr{O}_{\widetilde{T}} \to 0. \tag{2}$$

We first compute h^{11}. From (1) we get a long exact sequence

$$\ldots \to H^1(\operatorname{im} d) \to H^1(\Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}}) \to H^1(\Omega^1_{\widetilde{T}}) \to H^2(\operatorname{im} d) \to \ldots$$

The cohomology of $H^1(\operatorname{im} d)$ and $H^2(\operatorname{im} d)$ was computed with `Macaulay2` to be both zero. Thus $H^1(\Omega^1_{\widetilde{T}}) \simeq H^1(\Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}})$. From the Euler sequence we get

$$\ldots \to H^0(\mathscr{O}_{\widetilde{T}}(-1)^{14}) \to H^0(\mathscr{O}_{\widetilde{T}}) \to H^1(\Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}}) \to H^1(\mathscr{O}_{\widetilde{T}}(-1)^{14}) \to \ldots$$

But the left and right terms are both zero. Hence $h^{11} = 1$. We now compute h^{12}.

From (1) we again get

$$0 \to H^2(\Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}}) \to H^2(\Omega^1_{\widetilde{T}}) \to H^3(\operatorname{im} d) \to H^3(\Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}}) \to \ldots,$$

where we have used that $H^2(\operatorname{im} d) = 0$. But from the Euler sequence we get that the right term is also zero. Thus $H^2(\Omega^1_{\widetilde{T}}) \simeq H^3(\operatorname{im} d)$. This last group can be computed in `Macaulay2` to be 12-dimensional. $\square$

By `Macaulay2` computations we find that

$$h^i(\widetilde{\operatorname{im} d}) = \begin{cases} 12 & i = 3 \\ 2 & i = 4 \\ 0 & \text{else,} \end{cases}$$

and

$$h^i(\widetilde{\operatorname{im} d}(-1)) = \begin{cases} 0 & i = 0,1,2 \\ 24 & i = 3 \\ 12 & i = 4 \\ 18 & i = 5, \end{cases}$$

and

$$h^i(\widetilde{\operatorname{im} d}(-2)) = \begin{cases} 0 & i = 0,1,2 \\ 36 & i = 3 \\ 24 & i = 4 \\ 218 & i = 5, \end{cases}$$

In the same manner we find that:

Proposition 2.4. *We have $H^3(\Omega^1_Y) = 2$. The other Hodge groups $H^i(\Omega^1_Y) = 0$ (for $i = 0,4,5$).*

By twisting all the exaxt sequences above, we can also calculate:

Proposition 2.5. *We have*

$$h^i(\Omega^1_Y(-1)) = \begin{cases} 0 & i = 0 \\ 0 & i = 1 \\ 24 & i = 2 \\ 12 & i = 3, \end{cases}$$

and also $h^4(\Omega^1_Y(-1)) - h^5(\Omega^1_Y(-1)) = 4$.

Similarly:

$$h^i(\Omega^1_Y(-2)) = \begin{cases} 0 & i = 0 \\ 0 & i = 1 \\ 36 & i = 2 \\ 24 & i = 3, \end{cases}$$

and $h^4(\Omega^1_Y(-2)) - h^5(\Omega^1_Y(-2)) = 23$.

Remark. *The reader may wonder we just didn't ask* `Macaulay2` *to compute the cohomology sheaf* $\Omega^1_{\widetilde{T}}$ *directly, e.g. by the command* `HH^i(cotangentSheaf Proj A)`. *The reason is that* `Macaulay2`*'s algorithims actually compute the* sheaf, *and not just the dimension, and this is too computationally intensive.*

Remark (Question)**.** *A computation reveals that* $H^i(\mathcal{I}/\mathcal{I}^2) \simeq H^i(\operatorname{im} d)$ *for* $i \geq 2$. *This could be because the singularities are of dimension* 1. *Is there a theoretical result to this effect?*

We can compute it `Macaulay2` that:

Lemma 2.6. *The first cotangent module of* $\widetilde{T}$ *has* $\dim_k T^1(\widetilde{T}/k) = 26$.

We have that $h^0(\mathbb{P}(\mathcal{K} * \Delta^1), \mathcal{T}) = 14$ by Theorem 5.2 in [AC10]. Thus these numbers fit in the narrative that we should have $T^1_{X_0} = h^1(\mathcal{T}_{X_t}) + h^0(\mathcal{T}_{X_0})$. (IS THERE ANY HEURISTIC FOR THIS??)

3 Computing the Hodge numbers of X

Since X is a complete intersection of two hyperplanes in Y, we have an exact sequence

$$0 \to \mathscr{O}_Y(-2) \to \mathscr{O}_Y(-1)^2 \to \mathscr{I}_{X/Y} \to 0,$$

where $\mathscr{I}_{X/Y}$ is the ideal sheaf of X in Y. We also have the sequence

$$0 \to \mathscr{I}_{X/Y} \to \mathscr{O}_Y \to i^* \mathscr{O}_X \to 0, \tag{3}$$

where $i : X \to Y$ is the inclusion. This allows us to compute the Hodge numbers of X:

Theorem 3.1. *There exists a non-singular Calabi-Yau with X with $\chi(\Omega^1_X) = 36$.*

Proof. Since X is a complete intersection in Y, we have $\mathscr{I}_{X/Y}/\mathscr{I}^2_{X/Y} \simeq \mathscr{O}_X(-1)^2$ as $\mathscr{O}_X$-modules. Hence we have an exact sequence

$$0 \to \mathscr{O}_X(-1)^2 \to \Omega^1_Y \otimes \mathscr{O}_X \to \Omega^1_X \to 0.$$

The first sheaf have cohomology only in $H^3(\mathscr{O}_X(-1)) = H^0(\mathscr{O}_X(1)) = 12$, which can be computed from its Stanley-Reisner degeneration. Hence the Euler characteristics are related by $\chi(\Omega^1_X) = \chi(\Omega^1_Y\big|_X) + 24$.

Now tensor the exact sequence (3) with Ω^1_Y to get

$$0 \to \mathscr{I}_{X/Y} \otimes \Omega^1_Y \to \Omega^1_Y \to \Omega^1_Y \otimes \mathscr{O}_X \to 0.$$

Tensoring with Ω^1_Y is exact because the singularities of Y lie outside X (recall that the sheaf on the right is extended by zero outside X). Do the same with the $\mathscr{O}_Y$-resolution of $\mathscr{I}_{X/Y}$ to get

$$0 \to \Omega^1_Y(-2) \to \Omega_Y(-1)^2 \to \mathscr{I}_{X/Y} \otimes \Omega_Y \to 0.$$

Taking Euler characteristics, we find that $\chi(\mathscr{I}_{X/Y} \otimes \Omega_Y) = -3$. Since $\chi(\Omega^1_Y)$ was computed to be 9, it follows from the first exact sequence that $\chi(\Omega^1_X) = 36$. $\square$

Remark. *The standard toric construction used on T gives a Calabi-Yau X' with $\chi(\Omega^1_{X'}, X') = -36$ with Hodge numbers $(44, 8)$, so we would* really *want our Calabi-Yau to have Hodge numbers $(8, 44)$. In that case, it would be an example of an* extremal transition*, in the sense of Morrison.*

To actually find the Hodge numbers, we need a few lemmas.

Lemma 3.2. *Let $\mathcal{N}_{X/\mathbb{P}^{13}}$ be the normal sheaf of X in $\mathbb{P}^{13}$. Then $h^3(\mathcal{I}_X/\mathcal{I}^2_X) = h^0(\mathcal{N}_{X/\mathbb{P}^{13}})$.*

Proof. By Serre duality $h^{3-i}(\mathcal{I}_X/\mathcal{I}^2_X) = h^i((\mathcal{I}_X/\mathcal{I}^2_X)^\vee \otimes \omega)$, where ω is the dualizing sheaf. But X is Calabi-Yau, so $\omega \simeq \mathscr{O}_X$. The dual of $\mathcal{I}_X/\mathcal{I}^2_X$ is by definition the normal bundle. $\square$

Consider the Euler sequence

$$0 \to \Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X \to \mathscr{O}_X(-1)^{14} \to \mathscr{O}_X \to 0.$$

Since X is a deformation of a Stanley-Reisner sphere, we know the cohomology of $\mathscr{O}_X$. So we can extract the cohomology of $\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X$.

Lemma 3.3. *We have*

$$H^i(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) = \begin{cases} 0 & \textit{if } i = 0 \\ 1 & \textit{if } i = 1 \\ 0 & \textit{if } i = 2 \\ 167 & \textit{if } i = 3. \end{cases}$$

Proof. The full long exact sequence is:

$$\begin{aligned} 0 \to H^0(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) &\to H^0(\mathscr{O}_X(-1)^{14}) \to H^0(\mathscr{O}_X) \to \\ H^1(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) \to H^1(\mathscr{O}_X(-1)^{14}) &\to H^1(\mathscr{O}_X) \to \\ H^2(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) \to H^2(\mathscr{O}_X(-1)^{14}) &\to H^2(\mathscr{O}_X) \to \\ H^3(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) \to H^3(\mathscr{O}_X(-1)^{14}) &\to H^3(\mathscr{O}_X) \to 0 \end{aligned}$$

Inserting the dimensions we know, we get:

$$\begin{aligned} 0 \to H^0(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) &\qquad 0 \to 1 \to \\ H^1(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) \to 0 &\qquad \to 0 \to \\ H^2(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) \to 0 &\qquad \to 0 \to \\ H^3(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) \to 168 &\qquad \to 1 \to 0 \end{aligned}$$

Hence we conclude. $\square$

Since X is smooth, the conormal sequence is exact, so we have

$$0 \to \mathcal{I}_X/\mathcal{I}_X^2 \to \Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_X \to \Omega^1_X \to 0.$$

Lemma 3.4. *We have*

$$h^{21}(X) = h^0(\mathcal{N}_{X/\mathbb{P}^{13}}) - 167.$$

where χ denotes the Euler characteristic.

Proof. Write up the long exact sequence coming from the conormal sequence of X and use Lemma 3.2. $\square$

Lemma 3.5. *There is an exact sequence*

$$0 \to \mathscr{O}_X(1)^2 \to \mathcal{N}_{X/\mathbb{P}^{13}} \to \mathcal{N}_{Y/\mathbb{P}^{13}}\big|_X \to 0$$

Proof. First note that there is an exact sequence of conormal sheaves:

$$0 \to \mathcal{I}_Y/\mathcal{I}_Y^2\big|_X \to \mathcal{I}_X/\mathcal{I}_X^2 \to \mathscr{O}_X(-1)^2 \to 0.$$

The last term is $\mathcal{N}_{X/Y}$, since X is a complete intersection in Y. Dualizing is exact even though Y is not smooth, because by the long exact sequence of $\mathscr{E}xt$ sheaves, we must have $\mathscr{E}xt^1(\mathscr{O}_X(-1), \mathscr{O}_X) = 0$. But this is true, since both of these are locally free. □

Proposition 3.6. *We have*

$$h^{21}(X) = 39.$$

Hence $h^{11}(X) = 3$.

Proof. By lemma 3.4, we need to compute $h^0(\mathcal{N}_{X/\mathbb{P}^{13}})$. By the previous lemma, we have $h^0(\mathcal{N}_{X/\mathbb{P}^{13}}) = \mathcal{N}_{Y/\mathbb{P}^{13}}\big|_X + 24$.

We have an exact sequence

$$0 \to \mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}} \to \mathcal{N}_{Y/\mathbb{P}^{13}} \to \mathcal{N}_{Y/\mathbb{P}^{13}}\big|_X \to 0.$$

And also an exact sequence:

$$0 \to \mathcal{N}_{Y/\mathbb{P}^{13}}(-2) \to \mathcal{N}_{Y/\mathbb{P}^{13}}(-1)^{\oplus 2} \to \mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}} \to 0.$$

The cohomology of $\mathcal{N}_{Y/\mathbb{P}^{13}}(-i)$ is possible to compute in `Macaulay2`, and it follows from the long exact sequence that $h^0(\mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}}) = 36$, and $h^1(\mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}}) = 0$. Hence it follows from the same computation that $h^0(\mathcal{N}_{Y/\mathbb{P}^{13}}\big|_X) = 182$ and that $h^0(\mathcal{N}_{X/\mathbb{P}^{13}}) = 206$. We conclude that $h^{21} = 206 - 167 = 39$. □

References

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