

Smoothing a Calabi-Yau manifold

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Chapter 1

Preliminary definitions

We work over \mathbb{C} .

1.1 Stanley-Reisner basics

Given a simplicial complex \mathcal{K} , one can associate to it a projective scheme $\mathbb{P}(\mathcal{K})$ defined as follows. Let P be the polynomial ring with one variable for each vertex of \mathcal{K} . Then the *Stanley-Reisner ideal* $I_{\mathcal{K}}$ corresponding to \mathcal{K} is generated by the monomials corresponding to *non-faces* of \mathcal{K} . Then we define the *Stanley-Reisner scheme* to be $\text{Proj } P/I_{\mathcal{K}}$.

Example 1.1.1. Let \mathcal{K} be the square, with vertices v_0, v_1, v_2, v_3 . Then the Stanley-Reisner ideal is generated by $v_0 v_2$ and $v_1 v_3$. ★

Some of the topology of the simplicial complex is encoded in the scheme structure of $\mathbb{P}(\mathcal{K})$. In particular, the simplicial (co)homology groups of \mathcal{K} can be computed as the sheaf cohomology of $\mathbb{P}(\mathcal{K})$

Lemma 1.1.2. *Let $(K; \mathbb{C})$ denote the singular cohomology groups of \mathcal{K} . Then there are isomorphisms $H^i(K; \mathbb{C}) = H^i(\mathbb{P}(\mathcal{K}), \mathcal{O}_{\mathbb{P}(\mathcal{K})})$ for all i .*

Proof. ———to come——— □

Corollary 1.1.3. *We have isomorphisms $H^i(K, \mathbb{C}) \simeq H^{2i}(\mathbb{P}(\mathcal{K}); \mathbb{C})$ of singular cohomology groups.*

Proof. Something about i -cells in even dimensions □

1.2 Calabi-Yau basics

Definition 1.2.1. A *Calabi-Yau variety* is a smooth projective variety satisfying the following two conditions:

1. $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$.
2. The canonical sheaf is trivial: $\omega_X \simeq \mathcal{O}_X$.

■

The classical example of a Calabi-Yau manifold is the quintic threefold in \mathbb{P}^5 . Another example is the following:

Example 1.2.2. Let X be the double cover of \mathbb{P}^3 ramified along a smooth octic. The projection map is affine, so the conditions on $H^i(X, \mathcal{O}_X)$ are fulfilled. To see that the canonical sheaf is trivial, we use the adjunction formula, which says that $K_X = 2K_{\mathbb{P}^3}|_X + \deg R$, where R is the ramification divisor. Then, since $\omega_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-4)$, it follows that $K_X = 0$. ★

If \mathcal{K} is a simplicial sphere, then a smoothing of $\mathbb{P}(\mathcal{K})$ will give a Calabi-Yau manifold.

— ref: bayer-eisenbud graph curves.

1.3 Deformation theory

Chapter 2

Two topologically distinct smoothings

Denote by dP_6 the del Pezzo surface of degree 6 embedded in \mathbb{P}^6 . This can be realized as the blow-up of \mathbb{P}^2 in three points not lying on a line. Let X denote the affine cone over dP_6 . Then it has long been known that X has two smoothing components, and we show here that they are topologically distinct.

Recall that a *del Pezzo* surface is a surface such that the anti-canonical bundle is ample. The degree is the degree given by the anticanonical embedding. It is a classical result that every del Pezzo surface is obtained either by blowing up \mathbb{P}^2 in $r = 0, \dots, 6$ points in suitable positions, or as the 2-uple embedding of a quadric surface in \mathbb{P}^3 .

2.1 Different embeddings of dP_6

We first obtain the equations of dP_6 directly from the description of it as blow-up. Let x_0, x_1, x_2 be coordinates of \mathbb{P}^2 . Recall that the blowup of \mathbb{P}^2 in the point $(1 : 0 : 0)$ can be realized as the closed subscheme of $\mathbb{P}^2 \times \mathbb{P}^1$ given by the equation $r_0 x_1 - r_1 x_2 = 0$, where r_0, r_1 are coordinates on \mathbb{P}^1 . We can repeat this process on the points $(0 : 1 : 0)$ and $(0 : 0 : 1)$ to obtain similar equations. Collecting

these, we see that dP_6 is given by the matrix equation

$$M\vec{x} = \begin{pmatrix} 0 & r_0 & -r_1 \\ s_1 & 0 & -s_0 \\ -t_0 & t_1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0.$$

in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Here r_i, s_i and t_i ($i = 0, 1$) are of course coordinates on \mathbb{P}^1 .

We can do more than this however.

Lemma 2.1.1. *We can also realize dP_6 embedded in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with equation $r_0 s_0 t_0 = r_1 s_1 t_1$.*

Proof. Note that the matrix cannot have rank 1 or lower.

Now consider the projection onto the last three factors:

$$\pi : \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Each point P in the product on the right-hand side gives a matrix M_P of rank 2. Thus there is a line of solutions, which correspond exactly to a point in \mathbb{P}^2 .

Hence the restriction of π to dP_6 is an isomorphism onto the hypersurface given by $\det M = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. \square

Another way to realize blow-ups is this: let \mathfrak{d} be the linear system of quadrics with assigned basepoints $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$ in \mathbb{P}^2 . We can choose a basis given by $x_0 x_1$, $x_0 x_2$ and $x_1 x_2$. This gives a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. The closure of the graph of this map is a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$ defined by two bilinear equations. Each of the projections correspond to the blowup.

Explicitly, if we let y_0, y_1, y_2 be coordinates on the other \mathbb{P}^2 , then the equations are $x_1 y_0 - x_0 y_1 = x_1 y_1 - x_2 y_2 = 0$.

We also have a natural embedding in \mathbb{P}^6 as follows. Denote by E_1, E_2, E_3 the exceptional divisors on the blowup. Let L be a line in \mathbb{P}^2 . Then the divisor $\pi^* 3L - E_1 - E_2 - E_3$ is ample, and gives an embedding in \mathbb{P}^6 (see [2, Chapter V, Theorem 4.6]). A basis for the corresponding linear system is given by all monomials in $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ except x^3, y^3 and z^3 .

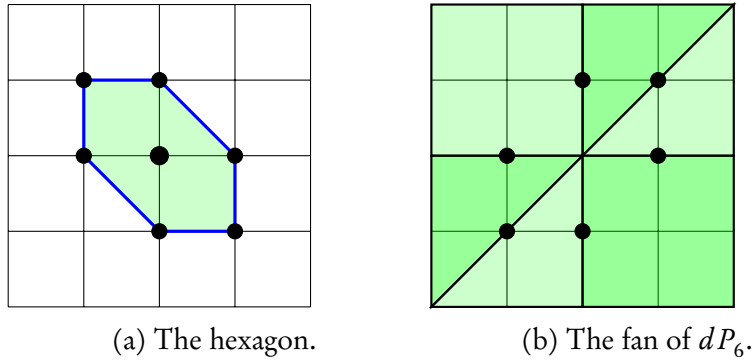


Figure 2.1: Toric description of dP_6 .

The equations can be arranged in a particularly symmetric form: let y, x_1, \dots, x_6 be coordinates on \mathbb{P}^6 . Then the equations of dP_6 are the 2×2 minors of the matrix

$$\begin{pmatrix} x_1 & y & x_6 \\ x_2 & x_3 & y \\ y & x_4 & x_5 \end{pmatrix}.$$

This gives 9 equations, which can be compactly written as $x_i x_{i+2} - y x_i = 0$ and $x_i x_{i+3} - y^2 = 0$, for $i = 1, \dots, 6$ (where i is taken modulo 6). Note that the equations have a visible D_6 -symmetry, where D_6 denotes the dihedral group.

2.1.1 As a toric variety

There is a nice combinatorial description of dP_6 as a toric variety associated to a polytope. Namely, let P denote the hexagon in Figure 2.1a. Then the normal fan of this polytope defines a fan in $N_{\mathbb{R}}$, defining a toric variety.

The polytope is reflexive, implying that the normal fan of P is the face fan over the same polytope. See Figure 2.1b. From standard toric geometry, it is clear that dP_6 is the blowup of \mathbb{P}^2 in the three torus-fixed points.

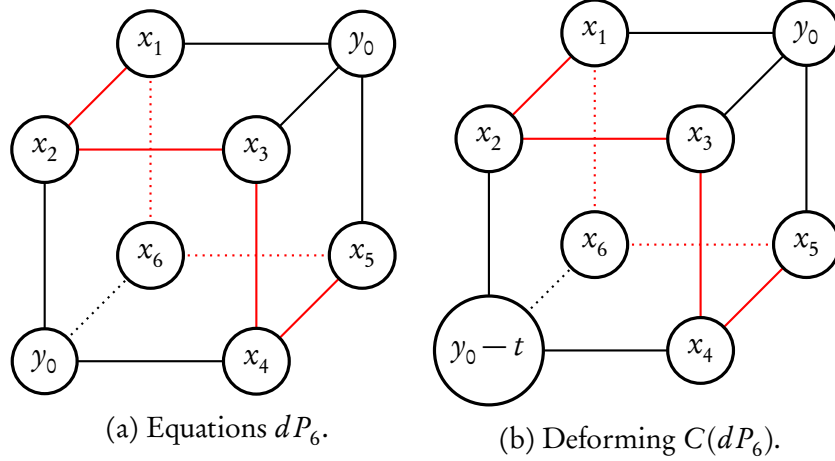


Figure 2.2: Forms of equations.

2.2 Divisors and topology

2.3 The affine cone and its two smoothings

Let X denote the affine cone over dP_6 . It is an affine variety with an isolated singularity at the origin. One can compute that it has two smoothing components: the union of a plane and a line. They both come from different ways of perturbing the equations of dP_6 .

Look at Figure 2.2a. One can read off the equations of dP_6 by taking minors along “faces” and long diagonals of this square. This correspond to a hyperplane cut of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in the Segre embedding. Then the one-dimensional component of the versal deformation of X is obtained by perturbing one of the γ_0 -corners as in Figure 2.2b.

It is clear the corresponding deformation is smooth, since it is a hyperplane cut of cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ outside the origin. Call this smoothing X_1 .

Lemma 2.3.1. *The smoothing X_1 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus dP_6$.*

Proof. Specialize to some $t \neq 0$. Then we can homogenize the equations with respect to y_1 to obtain a projective variety in 8 variables. However, in this form, $y_0 - ty_1$ and y_0 are linearly independent, hence by a change of variables, we see that this variety is in fact isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding.

What we gained by homogenizing is exactly the projective variety given by setting $y_1 = 0$. But then we get back the equations of dP_6 in \mathbb{P}^6 . \square

The second smoothing is obtained by deforming the equations of dP_6 as a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$. Namely, consider the following matrix:

$$\begin{vmatrix} x_1 & y_0 & x_6 \\ x_2 & x_3 & y_0 - t_1 \\ y_0 - t_2 & x_4 & x_5 \end{vmatrix} \leq 1. \quad (2.1)$$

For $t_1 = t_2 = 0$, we get the cone over dP_6 , while for generic t_i , we get a smooth variety. In fact, we can compute that the discriminant locus (the set of points in \mathbb{A}_{t_1, t_2}^2 with singular fiber) are the t_1 -axis, the t_2 -axis and the line $t_1 = t_2$.

Call (any) smooth fiber X_2 .

Lemma 2.3.2. *The variety X_2 is isomorphic to $(\mathbb{P}^2 \times \mathbb{P}^2) \cap H \setminus dP_6$, where H is a hyperplane in \mathbb{P}^8 .*

Proof. The technique is the same as in the previous proof. First homogenize the equations (2.1) with respect to y_1 . If we let t_{ij} ($i, j = 0, 1, 2$) be coordinates of \mathbb{P}^8 , then the homogenized variety is $\mathbb{P}^2 \times \mathbb{P}^2 \cap \{h = 0\}$, where h is the hyperplane $(t(t_{32} - t_{11}) = s(t_{23} - t_{11})$.

Again, what we gained by homogenizing is given by intersecting with $y_1 = 0$. But this is exactly dP_6 again, in the form $\mathbb{P}^2 \times \mathbb{P}^2 \cap \{t_{23} = t_{32} = 0\}$. \square

We can use what we know about the topology of these spaces to compute homology groups of the two affine smoothings.

Theorem 2.3.3. *The two affine smoothings are topologically different. The homology groups are:*

Group	0	1	2	3	4	5	6	Euler-characteristic
$H^i(X_1, \mathbb{Z})$	1	0	2	1	0	0	0	2
$H^i(X_2, \mathbb{Z})$	1	0	1	2	0	0	0	0

Proof. Long exact sequence of a pair + Lefschetz duality \square

Remark. *In fact, the Andreotti-Frankel theorem [1] states the following: if V is any smooth affine variety of complex dimension n , then it has the homotopy type of a CW complex of dimension n .*

Chapter 3

A smooth Calabi-Yau

Consider the hexagon E_6 . The join $E_6 * E_6$ is a 3-dimensional sphere, and so a smoothing of the corresponding Stanley-Reisner scheme would correspond to a smooth Calabi-Yau manifold. In this chapter I prove that there does indeed exist a smoothing, and I describe some of its properties.

Description, singularities, etc.

Bibliography

- [1] Aldo Andreotti and Theodore Frankel. The Lefschetz theorem on hyperplane sections. *Ann. of Math. (2)*, 69:713–717, 1959.
- [2] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.