Mandatory assignment

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Exercise 1. For $c \in \mathbb{R}\backslash\{0\}$, consider the set $C \subset \mathbb{R}^2$ defined by

$$C = \{(x, y) \mid x^3 + xy + y^3 = c\}$$

1. Show that for $c \neq \frac{1}{27}$, the set C is a closed one-dimensional submanifold of $\mathbb{R}^2$.

2. Prove or disprove that for $c = \frac{1}{27}$, the set C is an embedded submanifold of $\mathbb{R}^2$.

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Solution 1. 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^3 + xy + y^3$. Then $C = f^{-1}(c)$. By the inverse function theorem, we are interested in for which values of c the Jacobian of f have maximal rank for all points of C. In this case, the Jacobian is

$$\nabla(f) = \left(3x^2 + y, 3y^2 + x\right).$$

This has maximal rank if and only if not both of the components are zero. So suppose they are. Then $y = -3x^2$, and hence $0 = 3y^2 + x = 27x^4 + x$ by the second component. Thus $x = 0$ or $x^3 = -\frac{1}{27}$. If $x = 0$, then also $y = 0$, but since $(x, y) \in C$, this forces $c = 0$, which was not an option. So $x \neq 0$, so $x = -\frac{1}{3}$. Then $y = -\frac{1}{3}$ also because of the symmetric form the equation.

Hence
$$c = -\frac{1}{27} + \frac{1}{9} - \frac{1}{27} = \frac{-1 + 3 - 1}{27} = \frac{1}{27}.$$

Thus for $c \neq \frac{1}{27}$, the inverse function theorem holds, and C is a one-dimensional submanifold of $\mathbb{R}^2$. It is closed because f is continous.

2. Now suppose $c = \frac{1}{27}$. Then one can check that

$$x^3 + y^3 + xy - \frac{1}{27} = (x + y - \frac{1}{3})(x^2 - xy + y^2 + \frac{1}{3}x + \frac{1}{3}y + \frac{1}{9}).$$

   Thus C is the union of the two zero sets defined by the two components. Thus we must check two things: the two components must not intersect, and they must both be smooth. Any line is smooth, so the first component is okay.

   Solving the quadratic in terms of x (treating the y as a constant), we see that the discriminant is $-3(y+\frac{1}{3})^2$. Thus if we want real solutions, the only hope is $y = -\frac{1}{3}$. By symmetry, we must have $x = -\frac{1}{3}$ as well. Thus C is the disjoint union of a line and a point, and thus C is a non-equidimensional manifold.

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Exercise 2. Let X be a smooth vector field on a manifold M. Suppose there exists an $\epsilon > 0$ such that for every $p \in M$ there is an integral curve $\gamma : (-\epsilon, \epsilon) \to M$ of X such that $\gamma(0) = p$. Then the maximal integral curves of X are defined on the whole line $\mathbb{R}$. $\spadesuit$

Solution 2. The condition says that there is a single ϵ that works for every point $p \in M$.

Let $p \in M$ and let $\gamma : (-\epsilon, \epsilon) \to M$ be an integral curve starting at p. Then let $q = \gamma(\frac{\epsilon}{2})$. Then there is an integral curve $\tilde{\gamma} : (-\epsilon, \epsilon) \to M$.

Then by uniqueness of integral curves we must have $\tilde{\gamma}(s) = \gamma(t + \epsilon/2)$ (they both satisfy the same differential equation). In this way we can extend the domain of γ by $\frac{\epsilon}{2}$, and we continue this process indefinitely. Since $\mathbb{R}$ is the increasing union of the intervals $(-\epsilon - n\frac{\epsilon}{2}, \epsilon + \frac{\epsilon}{2})$, this will define γ on all of $\mathbb{R}$. $\heartsuit$

Exercise 3. Let G be a Lie group. Denote by $l_g(h) = gh$ left-multplication by g. We say that a vector field on G is *left-invariant* if $(d_h l_g)(X_h) = X_{gh}$ for all $g, h \in G$. Denote by X^v the vector field with the value v at $e \in G$. Then $X^v_g = (d_e l_g)(v)$.

i) For $v \in \mathfrak{g}$, let γ_v be the maximal integral curve of X^v such that $\gamma_v(0) = e$. Show that for every $g \in G$, the curve $\gamma(t) = g\gamma_v(t)$ is an integral curve of X^v such that $\gamma(0) = g$. Conclude that $\gamma_v(t)$ is defined for all $t \in \mathbb{R}$, and the flow $(\phi^v_t)_t$ defined by X^v is given by $\phi^v_t(g) = g\gamma_v(t)$.

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Solution 3. i) Note that $\gamma(t)$ is the composition of $\gamma_v(t)$ and l_g. Since taking differentials is functorial, we

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