CY 3-folds and sheaf counting

Fredrik Meyer

April 23, 2016

Abstract

These are notes from the "summer school" at IMPA, Warzaw, held by Balázs Szendrői.

1 Lecture 1 - Calabi-Yau 3-folds

We first cover the basics. That is, the definition:

Let X be a smooth projective variety over $\mathbb{C}$. We call X a *strict* Calabi-Yau 3-fold (CY3) if $\omega_X \simeq \mathscr{O}_X$, and $H^1(X, \mathscr{O}_X) = 0$.

Note that these conditions imply that also $H^2(X, \mathscr{O}_X) = 0$ by Serre duality. They also imply that $H^0(\Omega^1_X) = 0$ by Hodge theory ($H^0(\Omega^1_X$ is the complex dual to $H^1(X, \mathscr{O}, X)$).

Also note that by Hodge decomposition, this implies that $H^1(X, \mathbb{Q}) = 0$, since $H^1(X, \mathbb{C}) = H^1(X, \mathbb{Q}) \otimes \mathbb{C} = H^0(\Omega^1_X) \oplus H^1(\mathscr{O}_X)$. Note also that $H^2(X, \mathbb{C}) = H^{1,1}(X) = H^1(\Omega^1_X) = \mathrm{Pic}(X) \otimes \mathbb{C}$.

Thus we have two interesting Hodge numbers, namely h^{11} and $h^{12} = h^{21}$ (these two are equal by complex conjugation).

We also have a intersection form $S^3 H^2(X, \mathbb{Z}) \to \mathbb{Z}$ given by triple intersection of divisors. We also have a Chern class map $c_2 : H^2(X, \mathbb{Z}) \to \mathbb{Z}$ given by intersecting with the first Chern class (what is this??).

Now we list some examples of Calabi-Yaus:

Example 1.1. "Obvious" ones such as the quintic in $\mathbb{P}^4$. Also $X_{3,3} \subset \mathbb{P}^5$, $X_{(3,3)} \subset \mathbb{P}^2 \times \mathbb{P}^2$, $X_{(2,4)} \subset \mathbb{P}^1 \times \mathbb{P}^3$.

Also the double covering $X \xrightarrow{2:1} \mathbb{P}^3$ branched along a smooth octic surface. ★

Remark. *Some words about weighted projective spaces. Given non-negative natural numbers $a_0, \dots, a_n \in \mathbb{N}_{>0}$, we define*

$$\mathbb{P}^n[a_0, \dots, a_n] := \mathbb{C}^{n+1} \backslash \{0\} / \mathbb{C}^*(a_0, \dots, a_n),$$

where the torus act by the prescribed weights. We say that a weighted projective space is well-formed *if no n of the $n+1$ numbers $a_0, \dots, a_n$ have a common factor.*

Example 1.2. Also hypersurfaces and complete intersections in weighted projective spaces. For example, let X_8 be a degree 8 hypersurface in $\mathbb{P}[1, 1, 1, 1, 4]$. Let the coordinates be $x_1, \dots, x_4, y$. Then we can complete the square, so that X_8 is given by a polynomial of the form $y^2 + f_8(x_i) = 0$.

We have a $2:1$ map to $\mathbb{P}^3$ given by projecting to the first four coordinates. It is ramified exactly over the octic surface $f_8 = 0$. ★

Example 1.3. Another class of examples comes from considering hypersurfaces or complete intersections in special toric varieties. Let Δ be a reflexive polytope. Then the associated toric variety $\mathbb{P}_\Delta$ is Fano. Then elements of $|\omega_{\mathbb{P}_\Delta}|$ are Calabi-Yau. But these are often singular. So one has to find a crepant resolution of singularities $X \to \overline{X} \subset \mathbb{P}_\Delta$. ★

1.1 Quasiprojective case

We say that X is a (weak) CY3 if it is quasiprojective with $\omega_X \simeq \mathscr{O}_X$. Some examples:

1. $X = \mathbb{A}^3$.

2. Let $G \triangleleft \mathrm{SL}_3(\mathbb{C})$. This group act on $\mathbb{A}^3$. Then $\overline{X} = \mathbb{A}^3/G$ has Gorenstein singularities, and we have a non-unique crepant resolution $X \to \overline{X}$.

3. Let $G = \mathbb{Z}/3(1, 1, 1)$ be the subgroup of $\mathrm{SL}_3(\mathbb{C})$ acting by multiplication by a third root of unity. Then $\mathbb{A}^3/G$ has a singularity at the origin. Then one can see that a resolution of singularities is given by the total space of $\mathscr{O}_{\mathbb{P}^2}(-3)$, the zero section being the exceptional divisor, mapping down to the origin.