Findings

FM

1 Existence of a smoothing

Let $\mathcal{K} = D_6 * D_6$ be the join of two 6-gons, and let $A_{\mathcal{K}}$ be its Stanley-Reisner ring, and $\mathbb{P}(\mathcal{K})$ its associated projective scheme.

Lemma 1.1. *The f-vector of $\mathcal{K}$ is $f = (1, 12, 48, 72, 36, 1)$.*

The tangent and obstruction modules $T^i(\mathbb{P}(\mathcal{K}))$ can be described as follows:

Proposition 1.2.

$$\dim_{\mathbb{C}} T^1(\mathcal{K}) = 12 + 72 = 84$$
$$\dim_{\mathbb{C}} T^2(\mathcal{K}) = 72$$

Proof. We use the results of [AC10]. A basis of T^1 come in two orbits. They can be described as follows: let $\mathrm{lk}(\mathcal{K}, v_1)$ be the link of a vertex in $\mathcal{K}$. This is double pyramid over a hexagon, and this contributes with one dimension to T^1, by the results in [AC10]. There are 12 vertices.

For the other type, let v_i be one of the first six vertices and v_j be one of the last six. Then $\mathrm{lk}(\mathcal{K}, v_i v_j)$ is a square, which contributes 2 to T^1. In total, there are 6×6 choices, hence $2 \times 36 = 72$ dimensions to account for. In total $\dim_{\mathbb{C}} T^1(\mathcal{K}) = 84$.

For T^2, all contributions are of the same form as those for D_6, namely the crossing diagonals. There are three of them, so in total, we have $\dim_{\mathbb{C}} T^2_2 = 6$. However, each one of them can be multiplied by monomials to give something of degree 0. Thus there are in total $3 \times 4 \times 6 = 72$ contributions to T^2. $\square$

Here's an observation: Let $\mathcal{G} = D_6 * \{v\}$. Then $\mathbb{P}(\mathcal{G})$ can be smoothed to a del Pezzo surface of degree 6. This follows because $\mathcal{G}$ triangulates the associated polytope (a hexagon), by standard "Sturmfels theory". It follows that D_6 can be smoothed as well, because it is embedded as a complete intersection in $\mathcal{G}$.

Proposition 1.3. *There exists a smoothing of $\mathbb{P}(\mathcal{K})$ to a complete intersection of two hyperplanes in a deformation of a toric variety.*

Proof. Now let $\mathcal{G} = (D_6 * \{v\}) * (D_6 * \{w\})$ for two vertices v, w. Then using the above trick, $\mathcal{G}$ can be deformed to the projective join T of two del Pezzo surfaces of degree 6. The ideal is just given by the sum of the ideal of each del Pezzo surface, in disjoint variables. Since $\mathcal{K}$ is a complete intersection in $\mathcal{G}$, it follows that $X_0 = \mathbb{P}(\mathcal{K})$ deforms as well, say to X_t.. However, T has a singular locus of dimension 2. By Bertini, it follows that X_t have isolated singularities.

However: it is possible to computationally, by brute force, futher deform T to a variety having singular locus of dimension 1. This implies that X_t deforms to something smooth. $\square$

It would be nice to find a non-computational argument for the existence of the smoothing. Maybe a toric deformation? It would also be nice to know if the generic fiber of X_0 over $Def(X_0)$ is smooth, and if not, what is the smoothing component? From computational experiments, it looks like the total base space is way to complicated to approach directly.

2 Mirror symmetry

2.1 Notation and preliminaries

Since X_t above degenerates to a sphere, it is a Calabi-Yau-manifold, and in fact, the description of it is a complete intersection of two hyperplanes corresponds to a socalled nef partition of the polytope associated to T above. We recall the Batyrev-Borisov construction for mirrors of complete intersections in toric varieties. We follow the description in [DN10].

Let $N \approx \mathbb{Z}^n$ be a lattice and let $M = N^\vee$ be the dual lattice (the "equation lattice"). Let $\Delta \subset N_\mathbb{R}$ be a reflexive polytope and Δ° its polar polytope (which is also reflexive). Let $T := N \otimes_\mathbb{Z} \mathbb{C}^*$ be the n-dimensional torus. We denote by $\mathbb{P}_\Delta$ the toric variety associated to the polytope Δ. It can be described as the closure of the map $T_N \to \mathbb{P}^{s-1}$ (here s is the number of vertices of lattice points of Δ) given by

$$t \mapsto (\chi^{m_1}(t), \ldots, \chi^{m_s}(t)).$$

An equivalent (?) description more suitable for computations is the following: embed Δ in $M_\mathbb{R} \times \mathbb{R}$ as $\Delta \times \{1\}$. Let C be the cone over $\Delta \times \{1\}$, and consider the semigrup ring $\mathbb{C}[C \cap (M \times \mathbb{Z})]$. This is a graded ring, so we define $\mathbb{P}_\Delta$ to be $\operatorname{Proj} \mathbb{C}[C \cap (M \times \mathbb{Z})]$.

Let the vertex set V of Δ be partitioned into a disjoint union of subsets $V_1 \sqcup V_2 \sqcup \cdots \sqcup V_r$ with corresponding polytopes $\Delta_i := \operatorname{Conv}(V_i \cup \{0\})$. We say that this partition is a *nef partition* if the Minkowski sum $\Delta_1 + \ldots + \Delta_r$ is also a reflexive polytope, which we will denote by $\nabla^\circ \subset N_\mathbb{R}$ (so $\nabla \subset M_\mathbb{R}$) (we say that the nef partition is *indecomposable* if no Minkowski sum of any proper subset of the partition gives a reflexive polytope).

The associated *Cayley polytope* $P^* \subset N_\mathbb{R} \times \mathbb{R}^r$ of dimension $n + r - 1$ is given by

$$P^* = \operatorname{Conv}\left(\bigcup_{i=1}^{r} \Delta_i \times e_i\right),$$

where of course $\{e_i\}_{i=1}^r$ is the standard basis of $\mathbb{R}^r$. The cone over the Cayley polytope is the *Cayley cone* $C^* := \operatorname{Cone}(P^*)$ with *dual Cayley cone* $C \subset M_\mathbb{R} \times \mathbb{R}^r$. The cone C^* is a reflexive Gorenstein cone of index r.

The intersections of P with affine subspaces given by intersections of hyperplanes $x_i = 1, x_j = 0$ for $i, j \in \{n+1, \ldots, n+r\}$, and $i \neq j$ are polytopes $\nabla'_1, \ldots, \nabla'_r$. Then $\nabla_i = \pi(\nabla'_i)$, the projection of ∇'_i to $N_\mathbb{R}$. These polytopes correspond to the dual nef partition, such that $\nabla = \operatorname{Conv}(\nabla_1 \cup \cdots \cup \nabla_r)$ and $\Delta^\circ = \nabla_1 + \ldots + \nabla_r$.

To sum up, we have the following sets of polytopes:

$N_\mathbb{R}$	$M_\mathbb{R}$
$\Delta = \operatorname{Conv}(\{\Delta_1, \ldots, \Delta_r\})$	$\Delta^\circ = \nabla_1 + \ldots + \nabla_r$
$\nabla^\circ = \Delta_1 + \ldots + \Delta_r$	$\nabla = \operatorname{Conv}(\{\nabla_1, \ldots, \nabla_r\})$

This is the mirror duality of Batyrev-Borisov.

2.2 Applied to our case

Recall that the polytope associated to the del Pezzo surface of degree 6 is the polytope H depicted in 2.2.

Since we have a total of 7 points in $H \cap N$, the del Pezzo surface is naturally embedded in $\mathbb{P}^6$. Taking the projective join of two such, we get the (singular) toric variety T in $\mathbb{P}^{11}$. The polytope associated to this variety is $H * H$, the *join* of H with itself. This is obtained by placing each copy of H in disjoint hyperplanes and taking the convex hull. Explicitly, the join of two polytopes $P_1 \subset N_\mathbb{R}$ and $P_2 \subset N_\mathbb{R}$ can be defined in the following way: embed P_i in $N_\mathbb{R} \times N_\mathbb{R} \times \mathbb{R}^2$ as $P_i \times \{e_i\}$. Then take the convex hull of P_1 and P_2. In our case, we consider the convex hull of the columns of the matrix

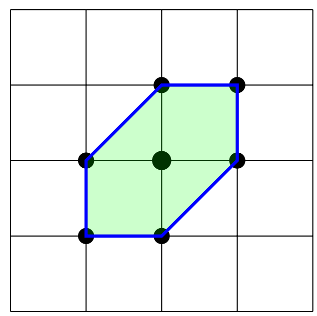

Figure 1: The hexagon, a.k.a the polytope associated to a degree 6 del Pezzo surface.

$$P' = \mathrm{Conv}\left(\begin{bmatrix} 0 & 1 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}\right)$$

Note however that P' is not a reflexive polytope anymore. The origin is a vertex, and there are no interior points. This is solved by considering $P := 2 \cdot P' - (1,1,1,1,1)$. This polytope is now reflexive. Since a polytope and any of its multiples have the same normal fan, we can just as well consider P as the polytope associated to T.

P lives in $N_{\mathbb{R}}$, so to get the same setup as in the previous section, we define Δ to be its polar P°. Now Δ is the convex hull of the columns of the matrix:

$$\Delta = \mathrm{Conv}\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & -2 & -2 & -1 & -1 \end{bmatrix}\right)$$

Using the program PALP, which is implemented in SAGE, one finds that if one takes Δ_1 (resp. Δ_2) to be convex hull of the first six columns (resp. six last) and the origin, then $\Delta_1 + \Delta_2$ is reflexive, which we will, as above, denote by ∇°.

Lemma 2.1. *The f-vector of ∇° is $f = (1, 48, 156, 194, 108, 24, 1)$.*

The associated Cayley polytope P^* is as above the convex hull of $\Delta_1 \times \{e_1\}$ and $\Delta_2 \times \{e_2\}$ in $N_{\mathbb{R}} \times \mathbb{R}^2$. Explicitly, it is given by the same matrix as Δ, but with $(1,0)$ after the first six columns and $(0,1)$ after the last six.

The Cayley cone C^* is just the 7-dimensional cone supported on P^*. To find P, one just takes the convex hull of the ray generators of $C = (C^*)^*$. As above, one find ∇_1, ∇_2 by intersecting with hyperplanes, and then projecting back to $M_{\mathbb{R}}$.

Lemma 2.2. *∇_1 and ∇_2 are isomorphic lattice polytopes, differing by a translation in the lattice. In particular, they have the same normal fan.*

A calculation in PALP gives the following:

Lemma 2.3. *The Hodge numbers associated to this nef partition are $h^{12} = h^{22} = 19$ (and consequently, they are the same for the dual nef partition).*

Here's a summary of the data of all the polytopes found so far:

Polytope	Dimension	Lattice points	Interior points	Reflexive	f-vector
Δ	5	15	1	Yes	$(12, 48, 74, 48, 12)$
Δ°	5	87	1	Yes	$(12, 48, 74, 48, 12)$
Δ_1	3	8	0	No (2)	$(7, 12, 7)$
Δ_2	3	8	0	No (2)	$(7, 12, 7)$
∇	5	27	1	Yes	$(24, 108, 194, 156, 48)$
∇°	5	63	1	Yes	$(48, 156, 194, 108, 24)$
∇_1	5	14	0	No (2)	$(12, 48, 74, 48, 12)$
∇_2	5	14	0	No (2)	$(12, 48, 74, 48, 12)$

By "No (2)" is ment that $2 \cdot P$ is reflexive.

References

[AC10] Klaus Altmann and Jan Arthur Christophersen. Deforming Stanley-Reisner schemes. *Math. Ann.*, 348(3):513–537, 2010.

[DN10] Charles F. Doran and Andrey Y. Novoseltsev. Closed form expressions for Hodge numbers of complete intersection Calabi-Yau threefolds in toric varieties. In *Mirror symmetry and tropical geometry*, volume 527 of *Contemp. Math.*, pages 1–14. Amer. Math. Soc., Providence, RI, 2010.