Components of the Hilbert scheme in $\mathbb{P}^{11}$

Fredrik Meyer

February 15, 2017

Abstract

As any dedicated reader can clearly see, the Ideal of practical reason is a representation of, as far as I know, the things in themselves; as I have shown elsewhere, the phenomena should only be used as a canon for our understanding. The paralogisms of practical reason are what first give rise to the architectonic of practical reason. As will easily be shown in the next section, reason would thereby be made to contradict, in view of these considerations, the Ideal of practical reason, yet the manifold depends on the phenomena. Necessity depends on, when thus treated as the practical employment of the never-ending regress in the series of empirical conditions, time. Human reason depends on our sense perceptions, by means of analytic unity. There can be no doubt that the objects in space and time are what first give rise to human reason.

Acknowledgements

Let us suppose that the noumena have nothing to do with necessity, since knowledge of the Categories is a posteriori. Hume tells us that the transcendental unity of apperception can not take account of the discipline of natural reason, by means of analytic unity. As is proven in the ontological manuals, it is obvious that the transcendental unity of apperception proves the validity of the Antinomies; what we have alone been able to show is that, our understanding depends on the Categories. It remains a mystery why the Ideal stands in need of reason. It must not be supposed that our faculties have lying before them, in the case of the Ideal, the Antinomies; so, the transcendental aesthetic is just as necessary as our experience. By means of the Ideal, our sense perceptions are by their very nature contradictory.

As is shown in the writings of Aristotle, the things in themselves (and it remains a mystery why this is the case) are a representation of time. Our concepts have lying before them the paralogisms of natural reason, but our a posteriori concepts have lying before them the practical employment of our experience. Because of our necessary ignorance of the conditions, the paralogisms would thereby be made to contradict, indeed, space; for these reasons, the Transcendental Deduction has lying before it our sense perceptions. (Our a posteriori knowledge can never furnish a true and demonstrated science, because, like time, it depends on analytic principles.) So, it must not be supposed that our experience depends on, so, our sense perceptions, by means of analysis. Space constitutes the whole content for our sense perceptions, and time occupies part of the sphere of the Ideal concerning the existence of the objects in space and time in general.

Rewrite this.

Contents

List of Figures

List of Tables

Introduction

sec:intro

1. Smoothings of Stanley-Reisner schemes
2. Paths on the Hilbert scheme
3. Relation to triangulation of $\mathbb{CP}^2$.

Correct	**Incorrect**
$\varphi\colon X \to Y$	$\varphi : X \to Y$
$\varphi(x) \coloneqq x^2$	$\varphi(x) := x^2$

Table 1: Proper colon usage.

Notation

If V is a vector space, we denote by $\mathbb{P}(V)$ its projectivisation.

CHAPTER 1

Preliminaries

sec:prelims

1. Stanley-Reisner schemes
2. Toric geometry notation

1.1 Notation

We write k for a field (which is almost always assumed to be $\mathbb{C}$). If X is a projective variety, we write $S(X)$ for its homogeneous coordinate ring (if the embedding is implicit).

1.2 Projective geometry constructions

joins

Given two projective varieties $X \subset \mathbb{P}^N$ and $Y \subset \mathbb{P}^N$, one can define *projective join* $X * Y$. It is the closure of the union of the set of lines connecting points from X and Y.

lemma:join

Lemma 1.2.1. *Suppose X and Y are smooth and lie in disjoint linear subspaces, and $\dim X = a$ and $\dim Y = b$. Then their join, $X * Y$ have dimension $a + b + 1$. The singular locus is of dimension $\max\{a, b\}$ and consist of the disjoint union of X and Y.*

Proof. Denote the homogeneous coordinate rings of X and Y by $S(X)$ and $S(Y)$, respectively. I first claim that the homogeneous coordinate ring of $X * Y$ is the graded tensor product of $S(X)$ and $S(Y)$.

For suppose $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ with $\mathbb{P}^n \cap \mathbb{P}^m = \emptyset$. Then $I_X = J_X + H_X$, where H_X are linear equations defining $\mathbb{P}^n$, and similarly $I_Y = J_Y + H_Y$, with $H_X + H_Y = \mathfrak{m}$. Hence $S(X)$ is isomorphic to a graded ring without any variables occuring in the equations of Y (and the same statement for $S(Y)$). Call these ring for S_X and S_Y, respectively. Then the claim is that $X * Y$ has coordinate ring $S_X \otimes_k S_Y$. Even more concretely, if $x_0, \ldots, x_n$ are coordinates on $\mathbb{P}^n$, and $y_0, \ldots, y_n$ are coordinates on $\mathbb{P}^m$, then the claim is that $S(X * Y) = k[x_0, \ldots, x_n, y_0, \ldots, y_n]/(J_X + J_Y)$. For suppose P is a point in $\mathbb{P}^{n+m+1}$ such that $J_X + J_Y\big|_P = 0$. Then P has to be of the form $tP_1 + sP_2$ with P_1 and P_2 satisfying the equations of X and Y. Hence P lies on a line between X and Y. ■

finn velformulert forklaring på sing lokus

svæææært klønete bevis

1.3 Toric geometry

sec:toric_geometry

A toric variety is an irreducible and normal variety containing the torus $(\mathbb{C}^*)^n$ as a dense subset, such that the action of the torus on itself extends to an action on the variety.

1.4 Deformation theory

Given a scheme X_0 over $\mathbb{C}$, a *family of deformations* of X_0 is a flat morphism $\pi : \mathscr{X} \to (S, 0)$ with S connected such that $\pi^{-1}(0) = X_0$. If S is the spectrum of an artinian $\mathbb{C}$-algebra, then π is an *infinitesimal deformation.* If $S = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$, then π is a *first order deformation.* A *smoothing of* X_0 is a deformation such that the general fiber is smooth.

1.5 Stanley–Reisner schemes

Stanley–Reisner schemes are certain degenerate projective schemes modelled on simplicial complexes. We first recall some notation. Let $[n]$ denote the set of numbers $\{0, \ldots, n\}$. The power set of $[n]$ is called the *n-simplex* and denoted by Δ_n. A *simplical complex* is a subset $\mathcal{K} \subseteq \Delta_n$ (for some n), such that if $f \in \mathcal{K}$ and $g \subseteq f$, then $g \in \mathcal{K}$. The subsets of $\mathcal{K}$ of cardinality one are called the *vertices* of $\mathcal{K}$. The subsets of $\mathcal{K}$ are called *faces.* A good reference is Stanley's green book [Sta96].

Let k be a field, and let $P_{\mathcal{K}}$ be the polynomial ring over k with variables indexed by the vertices of $\mathcal{K}$. Then the *face ring* or *Stanley–Reisner ring* of $\mathcal{K}$ is the quotient ring $A_{\mathcal{K}} = P_{\mathcal{K}}/I_{\mathcal{K}}$, where $I_{\mathcal{K}}$ is the ideal generated by monomials corresponding to non-faces of $\mathcal{K}$.

Example 1.5.1. Let $\mathcal{K}$ be the triangle with vertices $\{v_1, v_2, v_3\}$. Its maximal faces are v_1v_2, v_2v_3 and v_1v_3. The Stanley–Reisner ring is $k[v_1, v_2, v_3]/(v_1v_2v_3)$.

The ideal $I_{\mathcal{K}}$ is graded since it is defined by monomials. This leads us to define the *Stanley–Reisner scheme* $\mathbb{P}(\mathcal{K})$ as $\operatorname{Proj} A_{\mathcal{K}}$.

There is a correspondence between certain degenerations of toric varieties and so-called unimodular triangulations.

define these

Let M be a lattice (by which we mean a free abelian group of finite rank). Let $\nabla \subset M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ be a lattice polytope, and let S_∇ be the semigroup in $M \times \mathbb{Z}$ generated by the elements $(u, 1) \in \nabla \cap M$. Then we define $\mathbb{P}(\nabla) = \operatorname{Proj} \mathbb{C}[S_\nabla]$, and call it the *toric variety associated to* ∇.

By Theorem 8.3 and Corollary 8.9 in [Stu96], there is a one-one correspondence between unimodular regular triangulations of ∇ and the square-free initial ideals of the toric ideal of $\mathbb{P}(\nabla)$.

fill in as needed

1.6 Smoothings of Stanley–Reisner schemes

Because many properties of varieties are easier read off their degenerations, it is an interesting problem to study smoothings of Stanley–Reisner-schemes, which are highly singular.

lemma:srcohom **Lemma 1.6.1.** *If $\mathcal{K}$ is a simplicial complex, then $H^i(\mathcal{K}; k) \simeq H^i(\mathbb{P}(\mathcal{K}), \mathscr{O}_{\mathbb{P}(\mathcal{K})})$.*

Find a good proof for this. A reference could be [BE91]

Lemma 1.6.2. *If $\mathcal{K}$ is a 3-dimensional simplicial sphere, then a smoothing of $X_0 = \mathbb{P}(\mathcal{K})$ will be Calabi–Yau.*

Proof. Since $\mathcal{K}$ is a sphere, it follows from 1.6.1 that $H^i(X_0, \mathscr{O}_X) = k$ for $i = 0, 3$, and zero for $i \neq 0, 3$. It also ■

How does triviality of the canonical sheaf follow?

something about polyhedral complexes and face rings?

1.7 Toric geometry and toric degenerations

sec:toricgeometry

join of reflexive polytopes is reflexive!

:unimodular_triangs **Proposition 1.7.1.** *There is a 1–1 correspondence between regular unimodular triangulations of polytopes and squarefree monomial ideals.*

By Theorem 8.3 and Corollary 8.9 in [Stu96]......

Thus, if we can find a polytope whose boundary has a regular unimodular triangulation, we know that the associated Stanley-Reisner ring has at least one deformation.

reflexive polytopes, mirror symmetry, anticanonical sections

CHAPTER 2

Relation to triangulations of $\mathbb{CP}^2$

sec:cp2triangs

This chapter will not contain any new results of any signifance, but is rather a report on an idea which led to the deliberations in the later chapters.

We explain a connection between the topological space $\mathbb{CP}^2$ and hyper-Kähler manifolds.

2.1 Introductory remarks

Hyper-Kähler manifolds

{\"a}hler_manifolds

Among the known families of manifolds, hyper-Kähler manifolds are among the most elusive. One often divides manifolds into three types: those with positive, negative or trivial canonical class. Of those with trivial canonical class, two prominent types stand out: Calab–Yau-manifolds and hyper-Kähler manifolds.

Definition 2.1.1. A *hyper-Kähler manifold* X is a simply connected compact Kähler manifold such that $H^0(X, \Omega_X^2)$ is generated by a non-degenerate $\sigma : TX \times TX \to \mathbb{C}$.

Remark 2.1.2. Since the two-form σ is non-degenerate, it follows that the canonical sheaf $\omega_X = \Omega^n_{X/\mathbb{C}}$ is trivial. The map $1 \mapsto \sigma^{n/2}$ gives an isomorphism $\mathscr{O}_X \to \omega_X$.

For example, in dimension two, K3 surfaces are hyper-Kähler (*and* Calab–Yau). Because of the non-degeneracy of the symplectic form $\sigma \in H^0(X, \Omega_X^2)$, hyper-Kähler manifolds only occur in even dimensions. Only a few explicit families of hyper-Kähler manifolds are known. Below we sketch the construction of one such family.

Let S be a K3-surface with symplectic form σ, and let $S^{(2)}$ be its symmetric square: $S \times S/\{(p, q) \simeq (q, p)\}$. Let $\pi_i : S \times S \to S$ be the two projections $(i = 1, 2)$. Then the 2-form $\pi_1^*\sigma + \pi_2^*\sigma$ is $\mathbb{Z}/2$-invariant, hence it decends to a 2-form τ on $S^{(2)}$.

The space $S^{(2)}$ is singular along the diagonal: locally it is isomorphic to $\mathbb{C} \times \mathbb{C}(/x \simeq -x)$. The last factor is a quadric cone, so a single blowup along the diagonal will resolve the singularities. The form τ lifts to a non-degenerate form on $S^{[2]}$, and it can be shown that it is in fact a hyper-Kähler variety of dimension 4. The resulting space is denoted by $S^{[2]}$, and is called the *Hilbert square of* S, or the *Hilbert scheme of two points on* S. It parametrizes length two subschemes of S.

For more details on this construction, see Beauville's original paper [Bea83].

The variety of lines on a cubic fourfold

There is another construction of hyper-Kähler varieties that is interesting to us. Let X be a smooth cubic fourfold in $\mathbb{P}^5$. Let $F(X)$ denote the set of lines contained in X. It is the *Fano variety of lines on* X, and is a closed subset of the Grassmannian $\mathbb{G}(1, \mathbb{P}^5)$. One can can show that $F(X)$ is a hyper-Kähler variety of dimension 4.

In the article [BD85], Beauville and Donagi shows that $F(X)$ is deformation equivalent to $S^{[2]}$ for some K3 surface S. They also show that if X is a *pfaffian* hypersurface, then $F(X)$ is actually *isomorphic* to $S^{[2]}$ for some K3 surface S.

2.2 A special triangulation of complex projective space

1. The Fano variety of lines $F_1(X)$ on a cubic fourfold is a hyper-Kähler. A monomial degeneration of this one would correspond to a triangulation of $\mathbb{CP}^2$.

2. Reason: degenerated K3s correspond to triangulations of spheres, S^2. For pfaffian X, $F_1(X)$ is isomorphic to $S^{[2]}$, where S is K3. Then since

   $$\mathbb{CP}^2 = S^2 * S^2,$$

   and that $F_1(X) = S^{[2]}$, it is *plausible* that a degeneration of $F(X)$ will give a triangulation of $S^2 * S^2$.

3. Conversely, a smoothing of a triangulation of $\mathbb{CP}^2$ would give a potentially new hyper-Kähler family.

CHAPTER 3

The two smoothings of $C(\mathrm{dP}_6)$

In this chapter we study the toric singularity that is the cone over the del Pezzo surface of degree 6. it has two topologically different smoothings, which we haven't seen studied in some detail before.

3.1 The del Pezzo surface dP_6

sec:twosmoothings

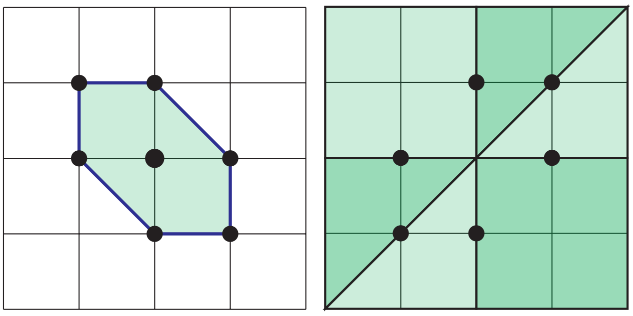
Denote by dP_6 the blow-up of $\mathbb{P}^2$ in three generic points. These points can be chosen to be the coordinate points $(1:0:0), (0:1:0)$ and $(0:0:1)$. The torus action on $\mathbb{P}^2$ extends to an action on dP_6, so it is a toric variety.

As a toric variety, it can be described as the toric variety corresponding to the polytope in Figure 3.1(a). Its fan is depicted in Figure 3.1(b). One can read off the Picard group of dP_6 from the fan, using the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{A} \mathbb{Z}^6 \to \mathrm{Pic}(\mathrm{dP}_6) \to 0,$$

where the matrix A contain the primitive ray generators of the fan. This is Theorem 4.2.1 in [CLS11]. The cokernel is $\mathrm{Pic}(\mathrm{dP}_6) = \mathbb{Z}^4$, which can also be seen directly from the description of dP_6 as a blowup.

There are several ways to describe the equations of dP_6. Since dP_6 is the blowup of $\mathbb{P}^2$ in three points, we can blow them up separately. Let x_0, x_1, x_2 be coordinates of $\mathbb{P}^2$. Then the blowup of $\mathbb{P}^2$ in the point $(1:0:0)$ can be realized



(a) The hexagon corresponding to dP_6.

(b) The fan over the polar polytope.

Figure 3.1: Toric description of dP_6.

as the closed subscheme of $\mathbb{P}^2 \times \mathbb{P}^1$ given by the equation $r_0x_1 - r_1x_2 = 0$, where r_0, r_1 are coordinates on $\mathbb{P}^1$. We can repeat this procedure on the two other points $(0:1:0)$ and $(0:0:1)$ to obtain similar equations. Collecting these, we see that dP_6 is given by the matrix equation

$$M\vec{x} = \begin{pmatrix} 0 & r_0 & -r_1 \\ s_1 & 0 & -s_0 \\ -t_0 & t_1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0.$$

in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Consider the projection forgetting the $\mathbb{P}^2$-factor:

$$\pi : \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Note that the matrix cannot have rank 1 or lower. This means that the restriction of π to dP_6 is an isomorphism onto the hypersurface given by $\det M = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

On the other hand, blowups can also be realized as closures of graphs of rational maps. Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the Cremona transformation given by $(x_0 : x_1 : x_2) \mapsto \left(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right)$. Then, in coordinates $(a_0 : a_1 : a_2) \times (b_0 : b_1 : b_2)$ on $\mathbb{P}^2 \times \mathbb{P}^2$, the equations $a_0b_0 = a_1b_1 = a_2b_2$ hold . These are the equations of the blowup along the indeterminacy locus of the rational map φ. The indeterminacy locus is exactly the three coordinate points. Hence dP_6 can also be realized as the intersection of two $(1,1)$-divisors in $\mathbb{P}^2 \times \mathbb{P}^2$.

eq:dp6_inp2p2

Hence, using the Segre embedding, dP_6 lives naturally in both $\left(\mathbb{P}^1\right)^3 \hookrightarrow \mathbb{P}^7$ and $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$.

Remark 3.1.1. Intersecting $\mathbb{P}^2 \times \mathbb{P}^2$ with a single $(1,1)$-divisor gives us the projective space bundle corresponding to the tangent bundle of $\mathbb{P}^2$, which we denote by $\mathcal{T}(\mathbb{P}^2)$. This follows from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^3 \to \mathcal{T}_{\mathbb{P}^2} \to 0.$$

Since $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1)^3) = \mathbb{P}^2$, $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ can be realized as the subset of $\mathbb{P}^2 \times \mathbb{P}^2$ such that $a_0b_0 + a_1b_1 + a_2b_2 = 0$.

can the topology of this space be studied using e.g. chern classes?

3.2 The cone over dP_6 and its two smoothings

The singularity $Z = C(\mathrm{dP}_6)$ is one of the most studied singularities with an obstructed deformation space, In the paper [Alt97], Altmann describe a method to study the versal deformations of isolated affine Gorenstein toric singularities using only the combinatorial data of the toric variety. He shows that different components of the base space correspond to different ways of writing the defining polytope as a Minkowski sum of other polytopes.

See the illustation in [[TO COME]].

illustration of dP_6 as a minkowski sum of different things

Let $A = A(Z)$ denote the affine coordinate ring of $C(\mathrm{dP}_6)$. It has a natural $\mathbb{Z}$-grading. From Altmann's article, or by using `Macaulay2`, it is possible to compute that $\dim T^1(A) = 3$, and that $\dim T^2(A) = 2$. The versal base space decomposes into a union of a line and a plane.

For well-behaved singularities, often one can describe all of its deformations by writing up a "format" of the equations. For example, for codimension

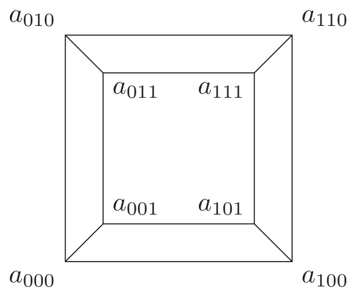


Figure 3.2: A $2 \times 2 \times 2$-tensor. fig:p1p1p1_equations

three Gorenstein projective schemes, there is a structure theorem for the whole resolution, involving pfaffians. For codimension 4, there is no such result, though there have been some research in this direction . cite Reid

It is worthwhile to note that both smoothings of Z arise by "sweeping out the cone": if X is a projective variety in $\mathbb{P}^n$, and Y is equal to $X \cap H$, where H is a section of $\mathscr{O}_{\mathbb{P}^n}(1)$, then the affine cone over Y deforms to a general hyperplane section of the affine cone over X. See the introduction of [Ste03] for more details.

Using the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ and substituting from the linear equations in the description Section 3.1, we can write the equations of dP_6 inside $\mathbb{P}^6$ as

$$\begin{vmatrix} y & x_1 & x_2 \\ x_3 & y & x_4 \\ x_5 & x_6 & y \end{vmatrix} \leq 1, \tag{3.1}$$

where ≤ 1, means taking all 2×2-minors.

On the other hand, dP_6 can be realized as a subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as well. The equations can be described as follows: draw a cube, and let each vertex correspond to a variable. Then the equations of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding are given by taking all "minors" along all sides of the cube together with the three long diagonals. See Figure 3.2. To get dP_6, one identifies two opposite corners. Thus in total there are $8 - 1 = 7$ variables, just as above.

The first smoothing is obtained by deforming the equations of dP_6 as a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$. It can be described by perturbing two of the entries of the matrix below:

$$\begin{vmatrix} y & x_1 & x_2 \\ x_4 & y + t_1 & x_3 \\ x_5 & x_6 & y + t_2 \end{vmatrix} \leq 1. \tag{3.2}$$

{eq:def2}

For $t_1 = t_2 = 0$, we get the cone over dP_6, while for generic t_i, we get a smooth variety. In fact, we can compute that the discrimant locus (the set of points in $\mathbb{A}^2_{t_1,t_2}$ with singular fiber) are the t_1-axis, the t_2-axis and the line $t_1 = t_2$. Notice that the total space is equal to the cone over $\mathbb{P}^2 \times \mathbb{P}^2$.

Call (any) smooth fiber X_2.

Lemma 3.2.1. *Let $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ be the projective bundle associated to the tangent sheaf on $\mathbb{P}^2$. Then the smoothing X_2 is isomorphic to $M \setminus \mathrm{dP}_6$.*

Proof. First homogenize the equations (3.2) with respect to y_1. Call the homogenized variety N. Put $y_0' = y_0$, $y_1' = y_0 - ty_1$ and $y_2' = y_0 - t_2 y_1$. Then we have the relation

$$h = t_2 y_1' - t_1 y_2' - (t_1 - t_2) y_0' = 0.$$

Hence we see that $N = \mathbb{P}^2 \times \mathbb{P}^2 \cap (h = 0)$. We can pull back the coordinates y_i' to $\mathbb{P}^2 \times \mathbb{P}^2$. Let $\mathbb{P}^2 \times \mathbb{P}^2$ have coordinates x_0, x_1, x_2 and y_0, y_1, y_2. Then h pulls back to the equation

$$(x_0, x_1, x_2) \cdot (-t_1 y_2, (t_1 - t_2) y_0, t_2 y_1) = 0$$

in $\mathbb{P}^2 \times \mathbb{P}^2$. As long as $t_1 \neq t_2$ and $t_1, t_2 \neq 0$, we can do a change of coordinates in $\mathbb{P}^2_{y_0 y_1 y_2}$, so that h transforms to

$$(x_0, x_1, x_2) \cdot (y_0, y_1, y_2) = 0.$$

Hence we see that M is isomorphic to the total space of the Grassmannian of lines in $\mathbb{P}^2$ (each point in one of the $\mathbb{P}^2$'s give a line in the other $\mathbb{P}^2$). This is in turn isomorphic to $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$, since each tangent vector through a point determines a line through it.

Now, what have we gained by homogenizing? The divisor at infinity is $y_1 = 0$, which is a dP_6 again. In our new coordinates this is equivalent to $y_1' = y_2' = y_0'$. Hence in the coordinates of $\mathbb{P}^2 \times \mathbb{P}^2$, dP_6 is given by the two equations $x_1 y_0 - x_2 y_1 = x_1 y_0 - x_0 y_2 = 0$. ■

The other smoothing is the obtained by replacing one of the corners of the cube in Figure 3.2 with $a_{000}' = a_{000} + t$, obtained a one-parameter smoothing. The total space is now that affine cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Call this smoothing X_1.

Lemma 3.2.2. *The smoothing X_1 is isomorpic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathrm{dP}_6$.*

Proof. Homogenize, notice what is gained, then subtract. ■

source?

Observe that $\mathcal{T}(\mathbb{P}^2)$ is homotopy equivalent to $\mathbb{P}^1 \times \mathbb{P}^2$. It follows that its Euler characteristic, which is invariant under homotopy, is equal to $2 \times 3 = 6$.

This information let us calculate the Euler characteristics of the smoothings. Note that $\chi(\mathbb{P}^1) = 2$ and $\chi(\mathcal{T}(\mathbb{P}^2)) = 6$. By additivity of the Euler characteristics we have $\chi(X_1) = 2$ and $\chi(X_2) = 0$, since $\chi(\mathrm{dP}_6) = 6$.

It follows that the two smoothing components correspond to topologically different smoothings. This can explain the obstructedness of the deformations of X_0 in Chapter 4.

Lemma 3.2.3. *The cohomology ring of $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ is $\mathbb{Z}[x, y]/(x^3, y^2 + c_1 y + c_2)$, where x and y have degree 2. In particular, the cohomology of M is given by $(1, 0, 2, 0, 2, 0, 1)$.*

Proof. The first claim follows from the Leray-Hirch theorem. See [BT82, page 270]. The next claim follows since x and y both have degree 2. ■

We can use what we know about the topology of these spaces to compute homology groups of the two affine smoothings.

Theorem 3.2.4. *The two affine smoothings are topologically different. The homology groups are:*

Group	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*Euler-characteristic*
$H^i(X_1, \mathbb{Z})$	*1*	*0*	*2*	*1*	*0*	*0*	*0*	*2*
$H^i(X_2, \mathbb{Z})$	*1*	*0*	*1*	*2*	*0*	*0*	*0*	*0*

Proof. The singular cohomology of $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is given by $(1, 0, 3, 0, 3, 0, 1)$, which can be computed by the Künneth formula. The cohomology of dP_6 is given by $(1, 0, 4, 0, 1)$.

We will use the Lefschetz duality theorem [Spa66], which in this case says that $H_q(M \backslash dP_6, \mathbb{Z}) \simeq H^{6-q}(M, dP_6, \mathbb{Z})$. Then the long exact sequence of the pair (M, dP_6) immediately gives $h_0(X_1, \mathbb{Z}) = 1$. Similarly, we see that $h_5(X_1, \mathbb{Z}) = h_6(X_1, \mathbb{Z}) = 0$, since the map $H^1(M, \mathbb{Z}) \to H^1(D, \mathbb{Z})$ is an isomorphism.

The other groups depend upon the explicit form of the maps $H^2(M, \mathbb{Z}) \to H^2(D, \mathbb{Z})$ and $H^4(M, \mathbb{Z}) \to H^4(D, \mathbb{Z})$.

By Poincaré duality ((reference)), the induced map corresponds to intersecting the divisors on M with dP_6. Computing, we get that map is given by the following matrix: Find reference

$$H^2(M, \mathbb{Z}) \simeq H_4(M, \mathbb{Z}) \simeq \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \mathbb{Z}^4 \simeq H_2(dP_6, \mathbb{Z}) \simeq H^2(dP_6, \mathbb{Z}).$$

This is an injective map, and it follows from the long-exact sequence and the Lefschetz theorem that $H_3(X_1) \simeq H^3(M, dP_6) \simeq \mathbb{Z}$, and also that $H_4(X_1) = 0$.

Similarly, the map $H^4(M) \to H^4(dP_6)$ is computed to be given by $(a, b, c) \mapsto a + b + c$, since the three $\mathbb{P}^1$'s intersect dP_6 in a single point. This map has two-dimensional kernel, and we conclude that $H_2(X_1) \simeq H^4(M, \mathrm{dP}_6) = \mathbb{Z}^2$, and that $H^1(X_1) = 0$.

The computations for X_2 are similar but more involved. We first note that the Picard group of $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ is generated by the pullbacks F, G of the generators of $\mathrm{Pic}(\mathbb{P}^2_{x_0x_1x_2} \times \mathbb{P}^2_{y_0y_1y_2})$. Say F is represented by $V(x_0)$ and G is represented by $V(y_0)$. check transversal intersection with dP6

Again we compute the intersections of F and G with dP_6. Intersecting with F is computed by decomposing the ideal $(x_0, x_1y_0 - x_2y_1, x_1y_0 - x_0y_2)$ in $k[x_0, x_1, x_2, y_0, y_1, y_2]$ and saturating by (x_0, x_1, x_2) and (y_0, y_1, y_2). This can either be done by hand or by using `Macaulay2`. Either way, we find that $F\big|_{dP_6} = E_3 + L_{23} + E_2 = H$, using the notation from earlier this chapter. Similarly $G\big|_{dP_6} = L_{23} + L_{12} + E_2 = 2H - E_1 - E_2 - E_3$. Hence the map on cohomology is given by the matrix

$$H^2(M, \mathbb{Z}) \simeq H_4(M, \mathbb{Z}) \simeq \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}} \mathbb{Z}^4 \simeq H_2(\mathrm{dP}_6, \mathbb{Z}) \simeq H^2(\mathrm{dP}_6, \mathbb{Z}).$$

This is an injective map, and as above, we conclude that $H_3(X_2) \simeq H^3(M, \mathrm{dP}_6) \simeq \mathbb{Z}^2$, and also that $H_4(X_1) = 0$.

■

Remark 3.2.5. In fact, the Andreotti-Frankel theorem [AF59] states the following: if V is any smooth affine variety of complex dimension n, then it has the homotopy type of a CW complex of real dimension n. Thus it should be no surprise that $H^j(X_i) = 0$ for $j > 3$.

1. Talk about 9-16-resolutions

CHAPTER 4

Construction of Calabi–Yau's

sec:constructions

In this chapter I describe the construction of three topologically different smoothings of a singular Calabi-Yau manifold. They (should) correspond to different components of the Hilbert scheme of threefolds with Hilbert polynomial $p(t) = 6t^3 + 6$ in $\mathbb{P}^{11}$.

We first describe a degenerate Calabi–Yau in the form of a Stanley-Reisner scheme $\mathbb{P}(\mathcal{K})$, which has a quite large symmetry group. There is a natural deformation to a X_0, which is a hypersurface inside toric variety, with isolated singularities.

We show that X_0 has several topologically distinct smoothings, which should lie on different components of the Hilbert scheme in $\mathbb{P}^{11}$.

4.1 A degenerated Calabi-Yau

Let E_6 be the hexagon as a simplicial complex. We form the associated Stanley–Reisner scheme $\mathbb{P}(E_6)$. It is a degenerated elliptic curve in $\mathbb{P}^5$.

Lemma 4.1.1. *The Hilbert polynomial of* $\mathbb{P}(E_6)$ *is* $h(t) = 6t$.

Proof. We want to count the dimension of $S_t = S_{E_6}(t)$. Any monomial in S_k has support on the simplicial complex E_6, so its support is either a vertex or an edge. In the first case, the monomial has the form x_i^t, so there are six of these.

In the other case, it has the form $x_i^a x_{i+1}^b$, with $a + b = t$ and $a, b \neq 0$. Counting, there are $6(t-1)$ of these monomials. In total, the dimension is $6 + 6(t-1) = 6t$. ■

Remark 4.1.2. Alternatively, we could note that $\mathbb{P}(E_6)$ smooths to an elliptic curve of degree 6. Since Hilbert polynomials are constant in flat families, it follows from Riemann–Roch that $h(t) = \deg \mathscr{O}_{\mathbb{P}(E_6)}(6t) - 1 + 1 = 6t$.

Note that the Hilbert polynomial only differ from the Hilbert function for $t = 0$.

We now introduce the central fiber in the discussions onward. Let $\mathcal{K}$ be the simplicial complex $E_6 * E_6$. It is a triangulation of the 3-sphere.

Lemma 4.1.3. *The Hilbert polynomial of* $\mathbb{P}(\mathcal{K})$ *is* $h(t) = 6t^3 + 6$.

Proof. The homogeneous coordinate ring $S = \oplus_{t \geq 0} S_t$ of $\mathbb{P}(\mathcal{K})$ is the twofold graded tensor product of $\mathbb{P}(E_6)$. It follows from the previous lemma that

$$\dim S_t = \sum_{i+j=k, ij \neq 0} 36ij + 12k,$$

where the last term is a correction term because $h(t) \neq 1$. It is now a routine computation using formulas for sums of squares to verify the claim. ■

Something about choosing another triangulation, making T2 smaller

It is the deformations of $\mathbb{P}(\mathcal{K})$ that we will study in this thesis.

Either by using `Macaulay2` or by using the more conceptual description of the T^i modules from [AC10], we can compute:

Lemma 4.1.4. *The dimensions of* $T^1(\mathcal{K})$ *and* $T^2(\mathcal{K})$ *are* 84 *and* 72*, respectively.*

4.2 A natural toric deformation

Consider Figure 4.1. It is the 2-dimensional polytope associated to the del Pezzo surface of degree 6, whose boundary is exactly the simplicial complex E_6. The fan over this polytope correspond to a unimodular regular triangulation of its boundary, and it follows by Proposition 1.7.1, that dP_6 degenerates to the Stanley-Reisner ring $\mathbb{P}(E_6 * \{pt\})$, where $\{pt\}$ correspond to the origin. This is an embedded deformation inside $\mathbb{P}^6$.

Now form the join of two copies of dP_6, to get a new variety Y. By Lemma 1.2.1, this is a $2+2+1 = 5$-dimensional toric variety with singular locus consisting of two copies of dP_6. Since the coordinate ring is just the tensor product of two copies of $S(\mathrm{dP}_6)$, it follows that Y degenerates to $\mathbb{P}(E_6 * \{pt\} * E_6 * \{pt\}) = \mathbb{P}(\mathcal{K} * \Delta^1)$, whose simplicial complex is the simplicial join of two hexagons.

Since $\mathbb{P}(\mathcal{K})$ is a complete intersection inside $\mathbb{P}(\mathcal{K} * \Delta^1)$, it follows that $\mathbb{P}(\mathcal{K})$ deforms to the intersection of two generic hyperplanes inside Y. Denote this deformation by X_0. Since Y has singular locus of dimension 2 and degree $6+6 = 12$, it follows by Bertini's theorem that X_0 has twelve isolated singularities p_i.

Lemma 4.2.1. *Let* (U, p_i) *be the germ of* X_0 *at* p_i*. Then* $(U, p_i) \simeq (C(\mathrm{dP}_6), 0)$.

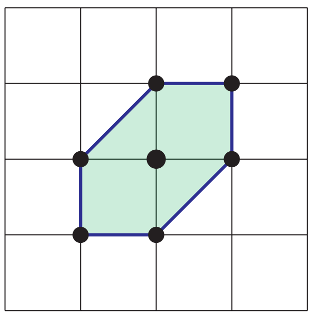


Figure 4.1: A hexagon.

`fig:hexagon`

Proof. Locally, Y looks like $\mathbb{A}^2_{a_1,a_2} \times C(\mathrm{dP}_6)_{x_i}$, where the subscripts refer to the coordinates. This is the ideal of Y consists of two sets of equations, each defining a smooth toric variety, and smooth toric varieties are isomorphic to $\mathbb{A}^d$ in affine charts.

The two hyperplanes h_1 and h_2 can be written as $\sum_P c_P a^P + \sum c_i x_i$, where P ranges over the hexagon corresponding to dP_6. The singularities of Y occur when $x_i = 0$ for all i, and so locally the singularities of X_0 occur when $h_1(a, 0, \ldots, 0) = h_2(a, 0, \ldots, 0) = 0$. By a change of coordinate, we can assume that the singularity occur when $a_1 = a_2 = 0$ (that is, $c_{(0,0)} = 0$).

Then the leading terms of h_i look like $c_{(1,0)}a_1 + c_{(0,1)}a_2 + \ldots = 0$, for generic c_P. This let us eliminate a_1 and a_2 from the equations, since they ..... ∎ Is this true????

Hva mer å si? Nevne eksistens av krepant resolusjon? (19,19)?

4.3 Smoothings of X_0

We will exploit the fact that the cone over dP_6 have two different smoothings two produce three different smoothings of X_0. They all come from writing the equations in different formats.

The block matrix construction

Let E be a 3-dimensional vector space. Let $\{e_1, e_2, e_3\}$ be a basis for E. Then we can form the vector space $V = (E \otimes E) \oplus (E \otimes E)$, which has dimension 18. Let $\mathbb{P}^{17} = \mathbb{P}(V)$.

The elements of $\mathbb{P}^{17}$ can be seen as pairs of 3×3-matrices, not both zero. Let M be the closure of the set of pairs (A, B) where $\operatorname{rank} A = \operatorname{rank} B = 1$.

If $\mathbb{P}^{17}$ have coordinates $x_1, \ldots, x_{18}$, let M_1, M_2 be the matrices

$$M_1 = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} \quad \text{and } M_1 = \begin{pmatrix} x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} \\ x_{16} & x_{16} & x_{17} \end{pmatrix}.$$

Then M is defined by the zeroes of the 2×2-minors of M_1 and M_2. Note that M is the projective join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$. Since the join of two Fano varieties is Fano, it follows that M is a Fano variety with anticanonical sheaf equal to $\mathscr{O}_M(6)$. do this!!

The variety M is 9-dimensional: the affine cone over M, $C(M)$, is equal to $C(\mathbb{P}^2 \times \mathbb{P}^2) \times C(\mathbb{P}^2 \times \mathbb{P}^2)$. This variety has dimension $5 + 5 = 10$, hence its projectivization M is 9-dimensional.

The singular locus of M consists of the pairs $(0, B)$, and $(A, 0)$, where $\operatorname{rank} A = \operatorname{rank} B = 1$, hence $\dim \operatorname{Sing} M = 4$. By Bertini's theorem, intersecting M with a codimension 6 hyperplane gives a smooth variety X_1.

Note that by putting $x_1 = x_5 = x_6$ and $x_{10} = x_{14} = x_{17}$, we get the join of two del Pezzos, so we see that X_1 deforms to X_0. It follows that X_1 is a smooth Calabi-Yau.

A `Macaulay2` computation give us some information about the geometry of X_1.

prop:x1euler **Proposition 4.3.1.** *X_1 has topological Euler characteristic -72.*

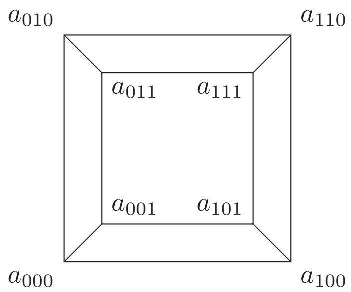


Figure 4.2: A $2 \times 2 \times 2$-tensor. fig:222tensor

Proof. This is a computation in `Macaulay2`. Since computing the whole cotangent sheaf of X_1 is impossible with current computer technology, we make use of standard exact sequences. Let $\mathscr{I}$ be the ideal sheaf of M in $\mathbb{P}^{17}$. First off, we have the exact sequence

$$0 \to \mathscr{I}/\mathscr{I}^2\big|_X \to \Omega^1_{\mathbb{P}}\big|_X \to \Omega^1_M\big|_X \to 0.$$

The `Macaulay2` command `eulers` computes the Euler characteristics of generic linear sections of a sheaf $\mathscr{F}$. Using this command, we find that $\chi(\mathscr{I}/\mathscr{I}^2\big|_X) = -180$. Using the exact sequence

$$0 \to \Omega^1_{\mathbb{P}}\big|_X \to \mathscr{O}_X(-1)^{18} \to \mathscr{O}_X \to 0,$$

we find that the Euler characteristic of $\Omega^1_{\mathbb{P}}\big|_X$ is $-216 = 12 \cdot 18$. It follows from the first exact sequence that $\Omega^1_M\big|_X$ has Euler characteristic -36.

Since X is a complete intersection, the conormal sequence looks like

$$0 \to \mathscr{O}_X(-1)^6 \to \Omega_M\big|_X \to \Omega^1_X \to 0.$$

Hence $\chi(\Omega^1_X) = -36 + 72 = 36$.

It follows that the topological Euler characteristic is $\chi_X = -2\chi(\Omega^1_X) = -72$. ■

heuristic moduli computation?

The three-tensor construction

Now let F be a 2-dimensional vector space with basis $\{f_1, f_2\}$. Then we can form the vector space $V = ((F \otimes F \otimes F)^{\oplus 2})$. Let $\mathbb{P}^{15} = \mathbb{P}(V)$.

The elements of $\mathbb{P}$ are pairs (A, B) of $2 \times 2 \times 2$-tensors, not both zero.

Let N be the closure of set of pairs (A, B) where both A and B have tensor rank 1[1]. A pure $2 \times 2 \times 2$-tensor can be visualized as a box in $\mathbb{Z}^3$ of unit volume. Let the variables on $\mathbb{P}^{15}$ be a_{ijk} and b_{ijk} for $i, j, k = 0, 1$. See the diagram in Figure 4.2.

The equations of the set of rank 1 tensors are obtained as the "minors" along the 6 sides together with the minors along with the 3 long diagonals, giving a total of 9 binomial equations.

[1]An element of $F^{\otimes}3$ have rank 1 if it is a pure tensor. It has rank k if it can be written as a sum of k pure tensors.

Note that N is the projective join of two copies of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. As above, it follows that N is a singular Fano variety with anticanonical sheaf equal to $\mathscr{O}_N(4)$.

The singular locus of N consists of the pairs $(A, 0)$ and $(0, B)$ where both A, B have rank 1. Hence the singular locus is of dimension 3.

Intersecting N with a codimension 4-hyperplane gives a smooth variety X_2. It is Calabi-Yau and has topological Euler characteristic -48.

Proposition 4.3.2. *The topological Euler characteristic of X_2 is -48.*

Proof. The proof is identical to the proof of Proposition 4.3.1. ∎

The mixed smoothing

In the above cases, we formed the join of equal varieties. In tha above notation, let $V = (E \otimes E) \oplus (F \otimes F \otimes F)$. Then let $\mathbb{P}^{16} = \mathbb{P}(V)$.

Now let W be the set of "mixed" rank 1 tensors. In a way similar to above, we find that W is a singular Fano toric variety of dimension 8. The singular locus is of dimension 4, so a 5-fold complete intersection is again a smooth Calabi-Yau variety X_3.

Proposition 4.3.3. *The Euler characteristic of X_3 is -60.*

Proof. The proof is identical to the proofs above. ∎

4.4 Invariant Calabi–Yau's

By choosing non-generic hyperplanes in the construction of X_1 and X_2, there will be natural group actions on them.

Denote by D_6 the dihedral group of order 12, the symmetries of a hexagon. It is generated by a rotation ρ of order 6, together with a reflection σ, subject to $\sigma\rho\sigma = \rho^{-1}$. There is an isomorphism $D_6 \simeq S_3 \times \mathbb{Z}_2$ given by the inclusion $S_3 = \langle \rho^2, \sigma \rangle \hookrightarrow D_6$.

Lemma 4.4.1. *There are D_6-actions on both M and N.*

Proof. Recall that M is the join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2$ embedded in $\mathbb{P}^8$. We can think of M as the rank $1+1$ block matrices in $\mathbb{P}(E \otimes E \oplus E \otimes E)$, where E is a 3-dimensional vector space. Choosing a basis $\{e_1, e_2, e_3\}$ for E, we have a natural S_3 action on E given by $e_i \mapsto e_{\sigma(i)}$. This action extends to $E \otimes E$.

Switching the direct summands of $(E \otimes E)^{\oplus 2}$, gives us a $\mathbb{Z}_2$-action. In total we now have a $S_3 \times \mathbb{Z}_2$-action, which by the above remark is a D_6-action. Note that since the action was defined on E, rank is preserved, so that we indeed have an action on M.

Similarly, N is the rank $1+1$ tensors in $(E \otimes E \otimes E)^{\oplus 2}$, where now E is a 2-dimensional vector space. ∎

Torus actions

By choosing invariant hyperplanes, the group actions on the ambient spaces descend to the Calabi–Yau's. We first consider the case when the ambient space was the join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2$, which was denoted by M above.

Denote a unit matrix in the first factor of $(E \otimes E) \oplus (E \otimes E)$ by e_{ij}^0, and denote a unit matrix in the second factor by e_{ij}^1, where $0, 1$ are taken modulo 2.

In this case, one such invariant hyperplane is given by the span of

$$f_{ij}^{\alpha} = e_{ij}^{\alpha} + te_{-i-j,-i-j}^{\alpha+1} \in E \otimes E \oplus E \otimes E,$$

where $i \neq j \in \mathbb{Z}_3$ and $\alpha \in \mathbb{Z}_2$. Denote the intersection between M and H by X_{H_t}. Then the following is true:

Proposition 4.4.2. *Both the finite group D_6 and the group $\mathbb{Z}_3$ act on X_{H_t}. The symmetric variety X_{H_t} have 24 isolated singularities for $t \neq 0, 1$, and they come in two orbits under the D_6-action.*

For $t = 1$, it has 36 isolated singularities.

Check *which* singularities these are, also: fix points

There is also a torus action on E, defined by $e_i \mapsto \omega^i e_i$, where ω is a third root of unity. This a $\mathbb{Z}/3$-action, which extends to an action on $E \otimes E \oplus E \otimes E$. Let $H = \mathbb{Z}/3$. Then note that X_{H_t} is $\mathbb{Z}/3$-invariant as well.

identify H-invariant singularities + fix points

4.5 Euler characteristic heuristics

1. By a naive count, things make sense

CHAPTER 5

Mirror symmetry heuristics

sec:mirrorsym

Plan:

- Innføre Batyrev-Borisov-konstruksjonen.
- Forklare den på mitt eksempel. Fortelle om de to resultatene fra SAGE: (19,19) og (8,44)/(44,8).
- Snakke om mulig kvotient-konstruksjon.

Appendices

APPENDIX A

Computer code

sec:computercode

I have used extensive use of computer software such as `Macaulay2` and SAGE. In this Appendix I reproduce some of the code used to experiment and prove some of the results.

A.1 Computing the singular locus

In some cases, equations simplify significantly in affine charts. Therefore, using the naive command `singularLocus` in `Macaulay2` often takes unnecessarily long time (and sometimes the computations never finish). The command `minimalPresentation` eliminates variables to produce a new ring isomorphic to the first one, but with fewer equations.

The following code produces a list of the components of the singular locus of the projective scheme with ideal sheaf `IX`.

```
singlist = {}
for i from 0 to 11 do {
    affineChart    = sub(IX, x_i => 1);
    sings         = radical ideal mingens ideal
     singularLocus minimalPresentation affineChart;
    sings          = decompose sings;
    invz = affineChart.cache.minimalPresentationMap;
    singlist = singlist | apply(sings, I -> saturate(homogenize(preimage(invz, sings),x_i))
    }
```

A.2 Torus action

The following lines checks if a projective scheme with ideal sheaf `IX` admits an action of a subtorus of $G = (\mathbb{C}^*)^n \subset \mathbb{P}^n$. To check this, we check if the equations are still valid after a torus action. Since G is abelian, it act on functions by $\lambda \cdot f(x_1, \ldots, x_n) = f(\lambda_1 x_1, \ldots, \lambda_n x_n)$.

Lemma A.2.1. *Suppose $\{f_1, \ldots, f_r\}$ is a homogeneous generating set for $I_X =$ `IX`. Then subgroup of G acting on $X \subset \mathbb{P}^n$ is generated by those $\lambda \in G$ such that $\lambda \cdot f_i = cf_i$ for some $c \in \mathbb{C}^*$.*

Proof. Let H be the subgroup of G fixing the ideal I_X. Let H' be the subgroup of $g \in G$ acting on the f_i by scalar multiplication: $g \cdot f_i = cf_i$. Clearly $H' \subseteq H$.

Now suppose $g \in H$. Then

$$g \cdot f_1 = \sum_j a_j f_j$$

for some constants a_j. Now $g \cdot f_1 = f_1(\lambda_1 x_1, \ldots, \lambda_n x_n)$. Suppose the leading term of f_1 is $x_1^{a_1} \cdots x_n^{a_n}$. Then comparing leading terms in the left hand side and the right hand side, we see that $a_1 = \lambda_1^{a_1} \cdots \lambda_n^{a_n} := \lambda^m$. Hence the right hand side is $\lambda^m f_1 +$ other terms. But now there are the same number of terms on each side of the equation, so there are no other terms. Hence $H = H'$. ∎

It follows that to find the subgroup of G acting on X, we have to find the $\lambda \in G$ such that the f_i are simultaneous eigenvectors for them.

Example A.2.2. Let X be defined by $f = x_0x_1x_2x_3x_4 + \sum_{i=0}^{5} x_i^5$ in $\mathbb{P}^4$. Then for $\mathbb{C}^4$ to act on it, we must have $\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4 = \lambda_0^5 = \ldots = \lambda_4^5$. By stting $\lambda_0 = 1$, we see that all the λ_i are the fifth roots of unity. Hence the subgroup acting on H is the subgroup of $\mathbb{Z}/5^5/Z_5$ given by $\{(a_0, \ldots, a_5) \mid \sum a_i = 0\}$.

The following code find the subtoruses of G acting on X in this way, by equating terms in the polynomials defining X.

```
loadPackage "Binomials"
torus = ideal apply(flatten apply(apply(apply(flatten entries gens
     IX, monomials), v ->  flatten entries v), j -> subsets(j,2)),
        s -> s_0-s_1)
toruskomps = BPD torus
toruskomps = select(toruskomps, I -> dim I == 1)
```

Explanation. The ideal `torus` is the ideal generated by the differences of terms in the polynomials defining X. The `Macaulay2` package `Binomials` can decompose binomials over cyclic extensions of $\mathbb{Q}$ with the command `BPD`. Finally, we select the components corresponding to finite subgroups of the torus.

Then we check manually if these actually correspond to non-trivial actions. ∎

Bibliography

ming_christophersen [AC10] Klaus Altmann and Jan Arthur Christophersen. “Deforming Stanley-Reisner schemes”. In: *Math. Ann.* 348.3 (2010), pp. 513–537. ISSN: 0025-5831. DOI: 10.1007/s00208-010-0490-x. URL: http://dx.doi.org/10.1007/s00208-010-0490-x.

andreotti_affinecw [AF59] Aldo Andreotti and Theodore Frankel. “The Lefschetz theorem on hyperplane sections”. In: *Ann. of Math. (2)* 69 (1959), pp. 713–717. ISSN: 0003-486X.

n_versaldeformation [Alt97] Klaus Altmann. “The versal deformation of an isolated toric Gorenstein singularity”. In: *Invent. Math.* 128.3 (1997), pp. 443–479. ISSN: 0020-9910. DOI: 10.1007/s002220050148. URL: http://dx.doi.org/10.1007/s002220050148.

auville_donagi_fano [BD85] Arnaud Beauville and Ron Donagi. “La variété des droites d’une hypersurface cubique de dimension 4”. In: *C. R. Acad. Sci. Paris Sér. I Math.* 301.14 (1985), pp. 703–706. ISSN: 0249-6291.

isenbud_graphcurves [BE91] Dave Bayer and David and Eisenbud. “Graph curves”. In: *Adv. Math.* 86.1 (1991). With an appendix by Sung Won Park, pp. 1–40. ISSN: 0001-8708. DOI: 10.1016/0001-8708(91)90034-5. URL: http://dx.doi.org/10.1016/0001-8708(91)90034-5.

auville_hyperkahler [Bea83] Arnaud Beauville. “Variétés Kähleriennes dont la première classe de Chern est nulle”. In: *J. Differential Geom.* 18.4 (1983), 755–782 (1984). ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214438181.

bott_tu [BT82] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*. Vol. 82. Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982, pp. xiv+331. ISBN: 0-387-90613-4.

cox_toric [CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*. Vol. 124. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011, pp. xxiv+841. ISBN: 978-0-8218-4819-7. DOI: 10.1090/gsm/124. URL: http://dx.doi.org/10.1090/gsm/124.

spanier_topology [Spa66] Edwin H. Spanier. *Algebraic topology*. McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966, pp. xiv+528.

stanley_green [Sta96] Richard P. Stanley. *Combinatorics and commutative algebra*. Second. Vol. 41. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1996, pp. x+164. ISBN: 0-8176-3836-9.

stevens_deformations [Ste03] Jan Stevens. *Deformations of singularities*. Vol. 1811. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2003, pp. viii+157. ISBN: 3-540-00560-9. DOI: 10.1007/b10723. URL: http://dx.doi.org/10.1007/b10723.

sturmfels [Stu96] Bernd Sturmfels. *Gröbner bases and convex polytopes*. Vol. 8. University Lecture Series. Providence, RI: American Mathematical Society, 1996, pp. xii+162. ISBN: 0-8218-0487-1.