Hyper-Kähler manifolds

Fredrik Meyer

April 20, 2016

Abstract

These are notes from the "summer school" at IMPA, Warzaw, held by Kieran O'Grady.

1 Lecture 1 - Introduction

We will first motivate the definition of hyper-Kähler by looking at K3 surfaces.

By definition, a K3 surface is compact Kähler 2-dimensional complex manifold that is simply connected and has trivial canonical bundle ($K_X \simeq \mathscr{O}_X$).

Example 1.1. Let $X \subset \mathbb{P}^3$ be a smooth quartic surface. Then by Lefschetz:

$$\pi_(X, *) \xrightarrow{\sim} \pi_1(\mathbb{P}^3, *) = \{1\},$$

so X is simply-connected. By adjunction, we have:

$$K_x \simeq (K_{\mathbb{P}^3}\big|_{\times}) \otimes \mathcal{N}_{X/\mathbb{P}^3} = \mathscr{O}_X(-4) \otimes \mathscr{O}_X(4) = \mathscr{O}_X .$$

★

We list some of the know results about K3's:

1. Any two K3's are deformation equivalent (Kodaira).
2. There is a Hodge-theoretic description of the Kähler cone of a K3 (after having chosen one Kähler class).
3. There is a *Global Torelli Theorem* (Shafarevich and Piateski-Shapiro). Namely, a Hodge structure on $H^2(K3; \mathbb{C})$ and a lattice structure on $H^2(K3, \mathbb{Z})$ determines X up to isomorphism.

We now state the definition of a hyper-Kähler manifold:

A hyper-Kähler manifold X is a compact Kähler manifold X, simply connected, such that $H^0(X, \Omega^2, X) = \mathbb{C}\sigma$, where σ is *symplectic*, meanig that $T_xX \times T_xX \xrightarrow{\sigma(x)} \mathbb{C}$ is non-degenerate for all $x \in X$.

Remark. *A HK must have even dimension: the skew-symmetric form σ can be represented by a $n \times n$-matrix A that is skew-symmetric with non-zero determinant. Skew-symmety of A means that $A^T = -A$. Hence $\det A = \det A^T = (-1)^n \det A$. This forces n to be even if we want $\det A \neq 0$.*

In dimension 2, K3's are hyper-Kähler.

To motivate hyper-Käher manifolds, we state the Beaville-Bogomolov decomposition theorem:

Theorem 1.2. *Let Z be a compact Kähler manifold with $c_1(Z) = 0$ (what does this mean?) in $H^2(X, \mathbb{Q})$. Then there exists a finite étale cover $\widetilde{Z} \to Z$ such that*

$$\widetilde{Z} = \mathbb{C}^d/\Lambda \times \prod_i X_i \times \prod_j Y_i$$

where the first factor is a compact torus. The second factor is a product of hyper-Kähler manifolds, and the second is a product of Calabi-Yau manifolds (that is, manifolds with $K_{Y_i} \simeq \mathcal{O}_{Y_i}$ and $h^0(\Omega^p_{Y_i}) = 0$ for all $0 < p < \dim Y_i$).

1.1 First example of a HK

Now we define some higher-dimensional examples of HK's. Terminology: when we say "HK variety", we shall mean a *projective* HK manifold.

Let S be a K3 surface. Then let $S^{(2)}$ be the symmetric square of S (that is, $S \times S/(p,q) \sim (q,p)$). This space comes equipped with two projections, π_1, π_2, and the form $\pi_1^*\sigma + \pi_2^*\sigma \in H^0(\Omega^2_{S\times S})$ is τ-invariant, hence it descends to a holomorphic 2-form on $S^{(2)}$.

But the symmetric square is singular along the (image of the) diagonal, where two points come together. In fact, one can see that it locally looks like $\mathbb{C} \times \mathbb{C}/x \sim -x$. The last factor is a quadric cone, hence a single blowup along the diagonal will resolve the singularities in this case.

Let D be the diagonal. We have a diagram

$$\begin{tikzcd} Bl_D(S^2) \arrow[r, "\widetilde{\rho}"] \arrow[d] & S^{[2]} \arrow[d, "\gamma"] \\ S^2 \arrow[r, "\rho"] & S^{(2)} \end{tikzcd}$$

The top right space $S^{[2]}$ can be thought of in two ways: first, notice that the involution on S^2 act on the blowup as well. Hence we can take the quotient of the blowup. This is one description of $S^{[2]}$. Secondly, one can think of $S^{[2]}$ as the blowup of $S^{(2)}$ along the points $2P$ (the image of the diagonal).

Now $S^{[2]}$ is smooth projective and "$\pi_1^*\sigma + \pi_2^*\sigma$" is symplectic.

Then $H^2(S^{[2]})$ is spanned by this symplectic 2-form. We can show that $S^{[2]}$ is simply-connected as well:

$$\pi_1(S^{[2]}\backslash D, *) \twoheadrightarrow \pi_1(S^{[2]}, *).$$

The first group is $\mathbb{Z}/2$, generated by a loop around D (we abuse notation: D denotes the image of the diagonal in $S^{[2]}$), and it is $\mathbb{Z}/2$ because $S^2\backslash D$ is a double cover of $S^{[2]}\backslash D$.

We call $S^{[2]}$ the *Hilbert square* of S. It is a HK variety of dimension 4. It can also be realized as the Hilbert scheme parametrizing length 2 subschemes of S.

In general,

$$Y^{[n]} := \{Z \subset Y \mid l(Z) = n\},$$

is smooth, irreducible of dimension $2n$ (if Y is a surface).

Beauville shows that if S is a K3, then $S^{[n]}$ is always a HK variety of dimension $2n$. Hence we have examples of hyper-Kählers in each even dimension!

If $n \geq 2$, then $b_2(S^{[2]}) = 23$, which is $22+1$, the last divisor coming from the blowup.

1.2 Second example of a HK

The analogue of $S^{[2]}$ with S replaced by A, an abelian surface.

The space $A^{[2]}$ has a holomorphic symplectic form, but is far from being simply connected. But consider the maps

$$A^{[2]} \xrightarrow{\gamma_2} A^{(2)} \xrightarrow{s_2} A.$$

The first map sends a points z to the sum $\sum_{p\in A} l(\mathscr{O}_{Z,P}\, P$. The second map sends $P+Q$ to $(p)+(q) \in A$, where the parenthesis means that we actually consider the sum in the *group* A.

Then the composition $s_2 \circ \gamma_2$ is a locally trivial fibration in the étale/analytic topology. It follows that the cohomology group $H^1(A) = \mathbb{C}^4 \hookrightarrow H^1(A^{[2]})$.

Now look at the fiber $s_2 \circ \gamma_2^{-1}(0)$. This is a smooth Kummer surface $\approx A/\langle -1\rangle$. These are K3 surfaces!

In general we look at the sequence

$$A^{[n+1]} \xrightarrow{\gamma_{n+1}} A^{(n+1)} \xrightarrow{s_{n+1}} A$$

defined analogously, and we define the *generalized Kummer* to be $Kum^{[n]}(A) := (s_{n+1} \circ \gamma_{n+1})^{-1}(0)$.

Beauville proved that $K^{[n]}$ is a HK variety of dimension $2n$. For $n \geq 2$, we have $b_2(K^{[n]}(A)) = 7 = b_2(A) + 1$, so we do actually have two topologically distinct families.

1.3 Third example, lines on a cubic 4-fould

Let $Y \subset \mathbb{P}^n$ be some algebraic variety. Let $X = F(Y)$ be the set of lines contained in Y. It is a closed subset of the Grassmannian $\mathbb{G}(1, \mathbb{P}^n)$, which we can think of as embedded via Plücker in $\mathbb{P}^{\binom{n-1}{2}-1}$.

Theorem 1.3. *Let $Y \subset \mathbb{P}^n$ be a smooth cubic hypersurface. Then $X = F(Y)$ is a smooth connected variety of dimension $2n - 6$ and $K_X \simeq \mathscr{O}_X(5 - n)$.*

Since we are interested in HK varieties, we put $n = 5$.

Remark. *All HK's have trivial canonical bundle: the $n/2$th power of the symplectic form gives a trivialization of $K_X = \Omega_X^n$.*

We want to look at a incidence correspondence $\mathcal{I} \subset Y \times X$:

$$\begin{tikzcd} & \mathcal{I} = \{(y, L) \in Y \times X \mid y \in L\} \arrow[dl, "\rho"'] \arrow[dr, "\pi"] & \\ Y & & X \end{tikzcd}$$

The fibers of ρ are $\mathbb{P}^1$s (how to see this?). In general we get a map

$$H^{n-1}(Y) \xrightarrow{c} H^{n-3}(X)$$

given by $\alpha \mapsto \pi_*(\rho^*\alpha)$. So for $n = 5$, we get a map $H^4(Y) \to H^2(X)$.

Beauville and Donagi showed that if $Y \subset \mathbb{P}^5$ is a smooth cubic hypersurface, then $X = F(Y)$ is a HK variety of type $K3^{[2]}$. Moreover, the restriction of c to the primitive cohomology

$$H^4(Y)_0 := \{\alpha \in H^4(Y) \mid \alpha - c_1(\mathscr{O}_Y(1)) = 0\}$$

is an isomorphism to $H^2(X)_0$ of Hodge structures.

We get an isomorphism $H^{p,q}(Y)_0 \simeq H^{p-1,q-1}(X)_0)$, and there exists a bilinear symmetric form $\langle , \rangle$ on $H^2(X)_0$ such that

$$\langle c(\alpha), c(\beta) \rangle = -\int_Y \alpha \wedge \beta$$

for all $\alpha, \beta \in H^4(Y)_0$.

One consequence: if $Y \subset \mathbb{P}^5$ is very general, then $H^2(X)_0$ has no nonzero integral $(1,1)$-classes. [explanation follows]

Main point: if Y is very general, then $F(Y) \not\simeq S^{[2]}$ (not isomorphic) (S is K3).

Beauville proved however that if Y is a pfaffian cubic

$$\{(t_0 : \cdots : t_5) \in \mathbb{P}^5 \mid Pf(t_0 A_0 + \ldots t_5 A_5)\},$$

where the A_i are skewsymmetric matrices, then $F(Y) \simeq S^{[2]}$.

2 Lecture 2 - some of the main general results

The general theory is developed by Bogomolov, Fujiki, Beauville, Verlotsky, Huybrechts and others. Today's lecture have three main ingredients:

1. The deformation theory if HK's are unobstructed.
2. BBF (Bogomolov-Beauville-Fujiki) quadratic form on $H^2(HK)$.
3. Twistor families of HK's. This leads to the deepest results.

2.1 Deformations of a HK X

Let σ be the given holomorphic symplectic form on X. Contraction with σ defines an isomorphism $L_\sigma : \Theta_X \to \Omega_X^1$. Hence we get isomorphism

$$H^i(L_\sigma) : H^i(X, \Theta_X) \xrightarrow{\sim} H^i(X, \Omega_X^1).$$

In particular, $H^0(X, \Theta_X) = H^0(X, \Omega_X^1) = 0$, since X is simply-connected (to see this: by the Hodge decomposition we have $0 = H^1(X; \mathbb{C}) = H^0(X, \Omega_X^1) \oplus H^1(\mathscr{O}_X))$.

This implies (by general results?) that there exists a universal deformation space of X:

$$\begin{tikzcd} X \simeq X_0 \arrow[r, hook] \arrow[d] & \mathscr{X} \arrow[d, "\pi"] \\ 0 \arrow[r] & B \end{tikzcd}$$

Then we have $T_0B = H^1(X, \Theta_X) = H^1(X, \Omega^1_X)$.

Theorem 2.1 (Bogomolov)**.** *If X is HK, then X is unobstructed. That is, there exists a universal deformation space B of X with B smooth.*

Corollary 2.2.

$$\dim Def(X) = dim B = b_2(X) - 2.$$

Proof. Since

$$Def(X)$$

is smooth, we have

$$\dim Def(X) = h^1(X, \Theta_X) = h^1(X, \Omega^1_X) = b_2(X) - h^{2,0} - h^{0,2} = b_2(X) - 2.$$

□

Remark. *This is because the Kodaira-Spencer map $T_0B \xrightarrow{\kappa} H^1(X, \Theta_X)$ is an isomorphism if $B = Def(X)$.*

Example 2.3. Let $X = S^{[n]}$ and $n \geq 2$. Then $b_2(X) = 23$, hence $\dim Def(X) = 21$. If X is K3, then $dim Def(X) = 20$. ★

2.2 The BBF quadratic form

We will study the *local period map*. Again, let $\pi : \mathscr{X} \to B$ be a family of HK's, with π a proper submersion. If $b \in B$, we write $X_b := \pi^{-1}(b)$.

By choosing B small enough, we can assume that the local system $R^2\pi_*\mathbb{Z}$ is trivial: so we can identify $H^2(X_0, \mathbb{Z}) \simeq H^2(X_{b_2}, \mathbb{Z}) = \Lambda$ for all $b_2 \in B$ and for a fixed finitely generated torsion free abelian group Λ of rank r (i.e $R^2\pi_*\mathbb{Z} \simeq B \times \Lambda$).

For $b \in B$, we get isomorphisms $p_b : H^2(X_0, \mathbb{Z}) \to \Lambda$. Extension of scalars gives a map $p_b : H^2(X_b, \mathbb{C}) \to \Lambda_{\mathbb{C}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$. Note that $H^{2,0}(X_b)$ is contained in the source.

We are now ready to define the period map:

$$B \xrightarrow{\mathcal{P}_\pi} \mathbb{P}(\Lambda_{\mathbb{C}})$$

$$b \longmapsto p_b(H^{2,0}(X))$$

This makes sense, because $H^{2,0}(X)$ is one-dimensional, hence spans a line in Λ_C. We want to compute $d\mathcal{P}_\pi(0) = d\mathcal{P}_\pi$. [1]

Let $v \in T_0B$. A priori $d\mathcal{P}_\pi(v) \in \operatorname{Hom}(H^{2,0}(X), F^1H^2(X)/H^{2,0}(X))$. This follows from Griffiths transversality. Recall $F^1H^2(X) = H^{2,0}(X) \oplus H^{1,1}(X)$. This last Hom group is equal to $\operatorname{Hom}(H^{2,0}(X), H^{1,1}(X))$. Griffiths in this case tells us that $\langle d\mathcal{P}_\pi(v), \sigma\rangle = L_sigma(\kappa(v))$.

We see that (?) the differential of the period map is injective with image $\operatorname{Hom}(H^{2,0}(X), H^{2,1}(X))$. Hence we conclude that the image of the period map is a local analytic hypersurface in $\mathbb{P}(\Lambda_{\mathbb{C}})$.

We now state the theorem of the existence of the *Bogomolov-Beauville-Fujiki quadratic form*:

Theorem 2.4. *There exists a quadratic form (with an associated bilinear form $(,)_X$)*

$$q_X : H^2(X) \to \mathbb{C}$$

which is integral, indivisible[2]*, and a $c_X \in \mathbb{Q}_+$ such that*

$$\int_X \alpha^{2n} = c_X \frac{(2n)!}{n!2^n} q_X \alpha^n \tag{1}$$

for all $\alpha \in H^2(X)$.

Why the factor $\frac{(2n)!}{n!2^n}$? It has to with (1) is equivalent to the assertion

$$\int_X \alpha_1 \wedge \cdots \wedge \alpha_{2n} = c_X \sum_{\tau \in S_n} (\alpha_{\tau(1)}, \alpha_{\tau(2)})_X \cdot \ldots \cdot (\alpha_{\tau(2n-1)}, \alpha_{\tau(2n)})_X,$$

where we sum over permutations "without stupid repetitions" (interpret this).

Now let X be of type $K3^{[n]}$. If S is K3, then $H^2(S^{[n]}, \mathbb{Z}) = im\mu_n \oplus \mathbb{Z}\zeta_n$, where μ_n is the composition

$$H^2(S, \mathbb{Z}) \to H^2(S^{(n)}, \mathbb{Z}) \to H^2(S^{[n]}, \mathbb{Z}).$$

[1] The rest of this section will be quite sketchy, mostly because I didn't understand so much.

[2] What does this mean?

The first map sends a curve C to the $\{Z \in S^{[n]} \mid Z \cap C \neq \emptyset\}$. The ζ_n is the reduced image of the diagonal: one can see that the class of the diagonal in $H^2(S^{[n]}, \mathbb{Z})$ is divisible by two. This is analogous to the fact that the map $Bl_D S^2 \to S^{[2]}$ is $2:1$-ramified along D. However, for general n, the construction of $S^{[n]}$ by a sequence of blowups and blowdowns is quite complicated. But for dimension reasons, we only need to look at points of $S^{(n)}$ where two points come together.

In this case the BBF quadratic form takes the form (no pun...): On $im\mu_n$ it is defined by $(\mu_n(\alpha), \mu_n(\beta)) = \int_S \alpha \wedge \beta$, and on the other summand we have $q(\zeta_n) = -2(n-1)$. And $c_X = 1$.

This is useful, for it lets us compute for example the self-intersection of the diagonal:

$$\Delta_n \cdot \ldots \cdot \Delta_n = \frac{(2n)!}{n!2^n}(-8(n-1))^n.$$

We sum up some of the things we know in a table:

X	$\dim X$	$b_2(X)$	c_X	$H^2(X;\mathbb{Z})$
$K3^{[n]}$	2n	23	1	$U^3 \oplus E_8^2 \oplus (-2(n-1))$
$Kum^{[n]}$	2n	7	n+1	$U^3 \oplus (-2(n-1))$

(he goes on to prove the theorem, but I drew a pig instead)

3 Lecture 3 - more results