Two constructions

We give two smoothings of a singular Calabi-manifold X_0, both with actions of D_6, the symmetries of the hexagon.

Lemma 0.1. *There is an isomorphism $S_3 \times \mathbb{Z}_2 \simeq D_6$.*

Proof. Let $D_6 = \langle \rho, \sigma \mid \rho^6 = \sigma^2 = s\rho s\rho \rangle$. Then $S_3 \simeq \langle \rho^2, \sigma \rangle$ and $\mathbb{Z}^2 \simeq \langle \rho^3 \rangle$. $\square$

0.1 Construction 1

Let E be a 3-dimensional vector space. Let $\{e_1, e_2, e_3\}$ be a basis for E. Then S_3 act on E by $e_i \mapsto e_{\sigma(i)}$. It also act on $(E \otimes E)^{\oplus 2)} \approx k^{18}$. There is a $\mathbb{Z}_2$-action switching the factors. Let $\mathbb{P} = \mathbb{P}(k^{18})$. Then $S_3 \times \mathbb{Z}_2$ act on $\mathbb{P}$.

The elements of $\mathbb{P}$ are pairs of 3×3-matrices, not both zero. Let M be the closure of the set of pairs (A, B) where $\operatorname{rank} A = \operatorname{rank} B = 1$.

If $\mathbb{P}$ have coordinates $x_1, \ldots, x_{18}$, let M_1, M_2 be generic matrices:

$$M_1 = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} \text{ and } M_1 = \begin{pmatrix} x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} \\ x_{16} & x_{16} & x_{17} \end{pmatrix}.$$

Then M is defined by the zeroes of the 2×2-minors of M_1 and M_2. Note that M is the projective join of $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$.

The variety M is 9-dimensional: the affine cone over M, $C(M)$, is equal to $C(\mathbb{P}^2 \times \mathbb{P}^2) \times C(\mathbb{P}^2 \times \mathbb{P}^2)$. This variety has dimension $5 + 5 = 10$, hence its projectivization M is 9-dimensional.

The singular locus of M consists of the pairs $(0, B)$, and $(A, 0)$, where $\operatorname{rank} A = \operatorname{rank} B = 1$, hence $\dim \operatorname{Sing} M = 4$.

Intersecting M with a codimension 6 hyperplane gives a smooth Calabi-Yau variety X_1. It has topological Euler characteristic -72.

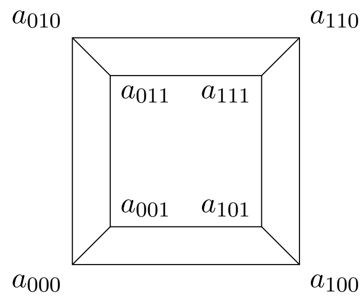

Figure 1: A $2 \times 2 \times 2$-tensor.

0.2 Construction 2

Let E be a 2-dimensional vector space with basis $\{e_1, e_2\}$. Let $\mathbb{P} = \mathbb{P}\left((E \otimes E \otimes E)^{\oplus 2}\right)$. Then $\mathbb{P} = \mathbb{P}^{15}$. There is an action of S_3 on $E \otimes E \otimes E$ given by permuting the tensor factors. Combining this with a $\mathbb{Z}_2$ switching A and B, we get a $S_3 \times \mathbb{Z} \simeq D_6$-action on $\mathbb{P}$.

The elements of $\mathbb{P}$ are pairs (A, B) of $2 \times 2 \times 2$-tensors, not both zero.

Let N be the closure of set of pairs (A, B) where both A and B have tensor rank 1[1]. A pure $2 \times 2 \times 2$-tensor can be visualized as a box in $\mathbb{Z}^3$ of unit volume. Let the variables on $\mathbb{P}$ be a_{ijk} and b_{ijk} for $i, j, k = 0, 1$. See the diagram in Figure 1.

The equations of the set of rank 1 tensors are obtained as the "minors" along the 6 sides together with the minors along the 4 long diagonals, giving a total of 9 binomial equations.

Note that N is the projective join of two copies of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

The singular locus of N consists of the pairs $(A, 0)$ and $(0, B)$ where both A, B have rank 1. Hence the singular locus is of dimension 3.

Intersecting N with a codimension 4-hyperplane gives a smooth variety X_2. It is Calabi-Yau and has topological Euler characteristic -48.

0.3 A Calabi-Yau with isolated singularities

Let $\mathrm{dP}_6 \subset \mathbb{P}^6$ be the del Pezzo surface of degree 6 anticanonically embedded in $\mathbb{P}^6$.

Let us describe two embeddings of dP_6 into $\mathbb{P}^2 \times \mathbb{P}^2$ and $(\mathbb{P}^1)^{\times 3}$, respectively. The surface dP_6 is the blowup of $\mathbb{P}^2$ in three points. Assume these are $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$ and $P_3 = (0 : 0 : 1)$.

[1] An elements of $E^{\otimes}3$ have rank 1 if it is a pure tensor. It has rank k if it can be written as a sum of k pure tensors.

Then we can realize dP_6 as the solutions of the matrix equation

$$\begin{pmatrix} 0 & a_0 & -a_1 \\ b_1 & 0 & -b_0 \\ c_0 & -c_1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0$$

in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Since the r_i, s_i are variables in $\mathbb{P}^1$, we see that the matrix cannot have rank 1. Hence the matrix have rank exactly 2, and that the matrix equation is equivalent to the equation $\det M = a_0b_0c_0 - a_1b_1c_1 = 0$ in $(\mathbb{P}^1)^3$. Hence dP_6 is the set of zeroes of a section of $\mathscr{O}_{(\mathbb{P}^1)^3}(1,1,1)$.

We can also describe dP_6 as the closure of the graph of a rational map. The ideal of the three points is (x_1x_2, x_0x_2, x_1x_2), and this gives a rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ by $(x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_0x_2 : x_0x_1)$. The blowup is defined as the closure of the graph of this map, and lives inside $\mathbb{P}^2_{x_0x_1x_2} \times \mathbb{P}^2_{y_0y_1y_2}$ as the solutions of the equation $x_0y_0 = x_1y_1 = x_2y_2$. Hence dP_6 is the zero locus of a section of $\mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1)^{\oplus 2}$.

In these descriptions, we can think of dP_6 as either the set of pure $2\times2\times2$ tensors with two opposite corners (recall Figure 1) equal, or as the set of rank 1 matrices with all the diagonal entries equal.

Now consider M_1 from above. It is the join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2$. Now let $h_1 = x_1 - x_5, h_2 = x_1 - x_9, h_3 = x_{10} - x_{14}$ and $h_4 = x_{10} - x_{17}$. By the description above, we see that variety $M_1 \cap V(h_1, h_2, h_3, h_4)$ is the join of two copies of dP_6. The singular locus is the disjoint union of these two. Intersecting with two more generic hyperplanes we get a Calabi-Yau X_0 with 12 isolated singularities. A local calculation shows that these singularities are locally isomorphic to $C(\mathrm{dP}_6)$, the affine cone over a del Pezzo.

Consider now M_2 from above. It is the join of two copies of $(\mathbb{P}^1)^3$. Let $g_1 = a_{000} - a_{111}$ and $g_2 = b_{000} - b_{111}$. Then $M_2 \cap V(g_1, g_2)$ is again the join of two copies of dP_6, isomorphic to the one above by a coordinate renaming.

Perturbing the h_i or the g_i by generic linear equations we get X_1 and X_2.

Hence we see that X_1, X_2 both degenerate to the same singular Calabi-Yau variety.

0.4 Euler characteristic heuristics

Let $\hat{X}$ be the blowup of X_0 in the 12 singular points. The exceptional divisors are del Pezzo surfaces.

One can show that $C(\mathrm{dP}_6)$ has two topological different smoothings V_1, V_2, where $V_1 = \mathbb{P}^2 \times \mathbb{P}^2 \setminus \mathrm{dP}_6$ and $V_2 = (\mathbb{P}^1)^3 \setminus \mathrm{dP}_6$. The Euler characteristics are 0 and 2, respectively.

Let us for the moment assume without proof that X_1 is what is obtained when we replace all the singularities in X_0 by V_1, and similarly for X_2.

Let $U = X_0 \setminus \mathrm{Sing}(X_0)$. Then in the first case, we have $\chi(X_1) = \chi(U) + 12\chi(V_1) = \chi(U) + 12 \cdot 0 = \chi(U) = -72$.

One the other hand, we have $\chi(X_2) = \chi(U) + 12\chi(V_2) = \chi(U) + 12 \cdot 2 = -48$, implying $\chi(U) = -72$, so the calculations are at least consistent.

In both cases, we find that $\chi(\hat{X}) = 0$.

It would be interesting to see if we can find a crepant resolution. Also, can we use this resolution to find the Hodge numbers of X_1 and X_2?