

# Exercises

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March 24, 2015

I solve and type exercises from different places (read *books*).

## 1 Algebraic Geometry - Hartshorne

### 1.1 Chapter I - Varieties

**Exercise 1** (Exercise 1.1). a) Let  $Y$  be the plane curve  $y = x^2$ . Show that  $A(Y)$  is isomorphic to a polynomial ring in one variable over  $k$ .

b) Let  $Z$  be the plane curve  $xy = 1$ . Show that  $A(Z)$  is not isomorphic to a polynomial ring in one variable over  $k$ .

c) Let  $f$  be any irreducible quadratic polynomial in  $k[x, y]$ , and let  $W$  be the conic defined by  $f$ . Show that  $A(W)$  is isomorphic to  $A(Y)$  or  $A(Z)$ . Which one is it when?



**Solution 1.** a) We have  $A(Y) = k[x, y]/(y - x^2)$ . An isomorphism  $A(Y) \rightarrow k[t]$  is given by  $x \mapsto t$  and  $y \mapsto t^2$ .

b) We have  $A(Z) = k[x, y]/(xy - 1) \simeq k[x, \frac{1}{x}]$ . So we must show that  $k[x, \frac{1}{x}] \not\simeq k[x]$ . It can be computed that the first one has automorphisms given by  $x \mapsto cx^n$  for  $c$  nonzero and  $n \neq 0$ . The second has as automorphisms  $ax + b$  ( $a \neq 0$ ). So the first one have an abelian automorphism group, the second has not.

c) What is special about  $A(Y)$  and  $A(Z)$ ? Staring at pictures, we see that any line in  $\mathbb{A}^2$  intersects  $Y$  in at least one point, but in the case of  $Z$ , there exist two lines which do not intersect  $Z$ . We claim that this is the only two things that can happen.

First we claim that if we are in the second situation, that is, if there exist a pair of lines  $\ell, \ell'$  such that  $W \cap \ell = W \cap \ell' = \emptyset$ , then  $W \simeq Z$ .

A general quadric can be written as

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Suppose now  $\ell \cap W = \emptyset$ . This is equivalent to  $I(f, \ell^\vee) = (1)$ . Without loss of generality, we can assume  $\ell = \{x = 0\}$ . Then

$$I(f, \ell) = (cy^2 + ey + f, x).$$

This generates  $k[x, y]$  if and only if  $c = e = 0$  and  $f \neq 0$ . Thus  $f$  must be of the form

$$ax^2 + bxy + dx + f = 0$$

with  $f \neq 0$ . But this can be written as

$$x(ax + by + d) + f = 0.$$

Put  $y' = ax + by + d$ . Then  $I(W)$  takes the form  $(xy' + f = 0)$ , which is clearly isomorphic to  $Z$  after a linear change of coordinates. Note that the other line not meeting  $W$  is the line given by  $y' = ax + by + d = 0$ .

Assume now that we are in the other situation, namely that *every* line in  $\mathbb{A}^2$  meets  $W$ . Now pick a tangent line  $\ell$  of  $W$ . Without loss of generality, we can assume that  $\ell$  is  $\{y = 0\}$ . This is a tangent line if and only if it meets  $W$  doubly, meaning that  $I(W) + (\ell^\vee)$  takes the form  $(l^2, y)$  for some linear form  $l$ . We can also assume that  $\ell \cap W = (0, 0)$ , so that  $I(W) + (\ell^\vee) = (x^2, y)$ . But this means that

$$\begin{aligned} I(W) + I(\ell) &= (ax^2 + bxy + cy^2 + dx + ey + f, y) \\ &= (ax^2 + dx + f, y) \end{aligned}$$

We want  $ax^2 + dx + f = x^2$ . This can happen only if  $d = f = 0$  and  $a \neq 0$ . Thus the quadric takes the form

$$ax^2 + bxy + cy^2 + ey = 0.$$

Now we claim that there exist one line at each point of  $W$  that intersect  $W$  transversally in exactly one point. This is the case for  $Y$ . Consider the pencil of lines through  $(0, 0)$  defined by  $x = \lambda y$ . We want to find  $\lambda$  such that the intersection is transversal and only one point. We have

$$(ax^2 + bxy + cy^2 + ey, x - \lambda y) = ((a\lambda^2 + b\lambda + c)y^2 + ey, x - \lambda y).$$

This have exactly one solution if and only if  $a\lambda^2 + b\lambda + c = 0$ . This is solvable since  $a \neq 0$  and since all lines intersect  $W$ . Thus choose  $\lambda$  as above. We can rotate this line such that it becomes  $x = 0$ . Then the equation takes the form

$$ax^2 + bxy + ey = 0.$$

We have still not arrived at  $y = x^2$ . Let now  $y = \lambda x$  be a general line through the origin. We demand that this intersect  $W$  twice for every  $\lambda$  such that the line is not tangent. We get that the intersection is given by

$$ax^2 + b\lambda x + ex = x((a + \lambda b)x + e) = 0.$$

For this to have two solutions for every  $\lambda$  we must have  $a + \lambda b \neq 0$  for all  $\lambda$ . But this requires  $b = 0$ . Thus the equation is

$$ax^2 + ey = 0$$

which is the conic we were looking for.

♡

**Exercise 2** (Exercise 1.2, the twisted cubic curve). Let  $Y \subseteq \mathbb{A}^3$  be the set  $\{(t, t^2, t^3) \mid t \in k\}$ . Show that  $Y$  is an affine variety of dimension 1. Find generators for the ideal  $I(Y)$ . Show that  $A(Y)$  is isomorphic to a polynomial ring in one variable over  $k$ . We say that  $Y$  is given by the *parametric equation*  $x = t, y = t^2, z = t^3$ . ♠

**Solution 2.** An affine variety is by definition a closed irreducible subset of  $\mathbb{A}^3$ . So we must find an irreducible ideal  $I$  such that  $Z(I) = Y$  (forgive the abuse of notation).

I claim that  $I(Y) = \langle x^2 - y, x^3 - z \rangle$ . Clearly, every  $P \in Y$  satisfies these equations. This shows the inclusion  $Y \subset Z(I)$ . Now suppose  $P \in Z(I)$ , that is,  $f(P) = 0$  for all  $f \in I$ . In particular  $(x^2 - y)(P) = 0$  and  $(x^3 - z)(P) = 0$ . Thus  $y = x^2$  and  $z = x^3$ . So if  $P = (a, b, c) \in k^3$ , then  $P = (a, a^2, a^3)$ , so  $P \in Y$ . This shows that  $Z(I) = Y$ . If we can show that  $I$  is prime, then it follows that  $I(Y) = I$  and that  $Y$  is a variety.

In fact, we claim that  $k[x, y, z]/I \simeq k[t]$ , implying that  $I$  is prime. The map  $\varphi$  is given by  $x \mapsto t, y \mapsto t^2, z \mapsto t^3$ . Then clearly  $I \subseteq \ker \varphi$ . We must show equality. So suppose  $\varphi(f) = 0$ .

First we claim that any  $f \in k[x, y, z]$  can be written as  $f = R(x) + S(x)y + T(x)z + i(x, y, z)$  where  $i$  is a polynomial in  $I$ . We prove this by induction on  $\deg f$ . If  $\deg f = 1$ , this is trivially true. The rest of the proof proceeds by tedious induction. ♡

## 1.2 Chapter II - Schemes

**Exercise 3** (Exercise 1.2). a) For any morphism of sheaves  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ , show that for each point  $P$ ,  $(\ker \varphi)_P = \ker(\varphi_P)$  and  $(\operatorname{im} \varphi)_P = \operatorname{im}(\varphi_P)$ .

b) Show that  $\varphi$  is injective (resp. surjective) if and only if the induced map on the stalks  $\varphi_P$  is injective (resp. surjective) for all  $P$ .

c) Show that a sequence  $\dots \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \rightarrow \dots$  of sheaves and morphisms is exact if and only if for each  $P \in X$ , the corresponding sequence of stalks is exact as a sequence of abelian groups.



**Solution 3.** a) An element of  $(\ker \varphi)_P$  is represented by a pair  $(U, f)$  with  $f \in \mathcal{F}(U)$  satisfying  $\varphi(U)(f) = 0$ . We have  $(U, f) \simeq (V, g)$  if there is a neighbourhood  $W$  of  $p$  contained in  $U \cap V$  such that  $f|_W = g|_W$  (then automatically  $\varphi(W)(f) = 0$ , since  $\varphi(W) = \varphi(U)|_W$ ).

On the other hand, an element of  $\ker \varphi_P$  is represented by a pair  $(U, f)$  satisfying the same conditions.

A similar argument works for  $\operatorname{im} \varphi$ . Alternatively, one can show that finite limits commute with direct limits.

b) Suppose  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  is injective. Then by definition all  $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  are injective, hence  $(\ker \varphi)_P = \ker \varphi_P = 0$ , hence  $\varphi_P$  is injective. Suppose  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  is surjective. By definition, this means that  $\operatorname{im} \varphi = \mathcal{G}$ , hence  $\mathcal{G}_P = (\operatorname{im} \varphi)_P = \operatorname{im} \varphi_P$ , so the stalks are surjective.

c) Exactness means that  $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$ . Taking stalks, gives one implication. Assume that the stalks are exact. Then the same argument works.



**Exercise 4.** a) Let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of presheaves such that  $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is injective for each  $U$ . Show that the induced map  $\varphi^+ : \mathcal{F}^+ \rightarrow \mathcal{G}^+$  of associated sheaves is injective.

b) Use part a) to show that if  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  is a morphism of sheaves, then  $\operatorname{im} \varphi$  can be naturally identified with a subsheaf of  $\mathcal{G}$ , as mentioned in the text.



**Solution 4.** a) From the universal property of the sheafification functor, we have a commutative square:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\ \theta \downarrow & & \downarrow \theta \\ \mathcal{F}^+ & \xrightarrow{\exists!} & \mathcal{G}^+ \end{array}$$

The lower arrow follows from the universal property of sheafification applied to  $\theta \circ \varphi$ . Taking stalks induced the identity map on the vertical arrows, and since a map is injective if it is injective on stalks, the statement follows.

b)  $\text{im } \varphi$  is the sheafification of  $(\text{im } \varphi)_{\text{pre}}(U) = \{U \mapsto \varphi(U)\}$ . We have  $\text{im } \varphi(U) \subset \mathcal{G}(U)$  for all  $U$ , hence  $\text{im } \varphi_P \subset \mathcal{G}_P$  for all  $P$ , hence  $\text{im } \varphi \rightarrow \mathcal{G}$  is injective.

♡

**Exercise 5** (Exercise 1.14, Support). Let  $\mathcal{F}$  be a sheaf on  $X$ , and let  $s \in \mathcal{F}(U)$  be a section over an open set  $U$ . The *support of  $s$*  denoted  $\text{Supp}(s)$ , is defined to be the set  $\{P \in U \mid s_P \neq 0\}$  where  $s_P$  denotes the germ of  $s$  in the stalk  $s_P$ . Show that  $\text{Supp}(s)$  is a closed subset of  $U$ . We define the *support of  $\mathcal{F}$*  by  $\text{Supp } \mathcal{F}$  to be  $\{P \in X \mid \mathcal{F}_P \neq 0\}$ . It need not be a closed subset. ♠

**Solution 5.** Showing that  $\text{Supp}(s)$  is closed is equivalent to showing that the complement is open. So let  $P \in X \setminus \text{Supp}(s)$ . Then  $s_P = 0$ . But every germ is represented by a pair  $(s, U)$  (with  $(s', U') \simeq (s, U)$  if  $s|_W = s'|_W$  for some open  $W \subset U \cap U'$ ). But since  $s_P = 0$ , there must be some neighbourhood  $U$  such that  $s_P$  is represented by  $s = 0$ , hence  $X \setminus \text{Supp}(s)$  can be covered by those open  $U$ 's.

To see that  $\text{Supp } \mathcal{F}$  need not be closed, let  $X = \mathbb{A}_k^1$  with  $k$  an infinite field. Let  $\mathbb{Z}$  be the constant sheaf and let  $\mathcal{L}$  be the direct sum of infinitely many skyscraper sheaves, but not everyone. Let  $\mathcal{F} = \mathcal{L}$  be the quotient. This has support on the infinitely many points chosen, which is not closed. ♡

**Exercise 6** (Exercise 1.16, Flabby/flasque sheaves). A sheaf  $\mathcal{F}$  on a topological space  $X$  is *flasque* (flabby) if for every inclusion  $U \subseteq V$ , the restriction map  $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$  is surjective.

a) Show that a constant sheaf on an irreducible topological space is flasque.

- b) If  $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$  is an exact sequence of sheaves, and if  $\mathcal{F}'$  is flabby, then for any open set  $U$ , the sequence

$$0 \rightarrow \mathcal{F}'(U) \rightarrow \mathcal{F}(U) \rightarrow \mathcal{F}''(U) \rightarrow 0$$

is exact.

- c) Same as above, but suppose  $\mathcal{F}'$  and  $\mathcal{F}$  are flabby. Show that  $\mathcal{F}''$  is flabby.
- d) If  $f : X \rightarrow Y$  is a continuous map, and  $\mathcal{F}$  is a flabby sheaf on  $X$ , then  $f_* \mathcal{F}$  is flabby on  $Y$ .
- e) Let  $\mathcal{F}$  be any sheaf on  $X$ . We define a new sheaf  $\mathcal{G}$ , called the *sheaf of discontinuous sections of  $\mathcal{F}$* , as follows: For each open set  $U \subset X$ ,  $\mathcal{G}(U)$  is the set of maps  $s : U \rightarrow \cup_{P \in U} \mathcal{F}_P$ , such that for all  $P \in U$ ,  $s(P) \in \mathcal{F}_P$ . Show that  $\mathcal{G}$  is a flabby sheaf, and that there is a natural injective morphism from  $\mathcal{F}$  to  $\mathcal{G}$ .



**Solution 6.** a) Every open set in  $X$  is irreducible and dense, and dense sets are connected. Hence a constant sheaf is actually constant, and all the restriction maps are identities (except if one of them is the empty set).

- b) The sheaf axiom for a sheaf  $\mathcal{F}$  is equivalent to the following: for every covering  $\{U_i\}$  of  $U$ , the following sequence is exact:

$$0 \rightarrow \mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightarrow \prod_{ij} \mathcal{F}(U_{ij}),$$

where  $U_{ij} = U_i \cap U_j$ . The first map sends a section  $s$  to the product of all its restrictions, and the second map sends  $(s_i) \mapsto (s_i - s_j)_{ij \in I \times I}$ .

Since the sequence of sheaves in the exercise is exact, for small enough  $U_i$ , the sequence  $0 \rightarrow \mathcal{F}'(U_i) \rightarrow \mathcal{F}(U_i) \rightarrow \mathcal{F}''(U_i) \rightarrow 0$  is exact (for sheaves,

exactness is a local property). Hence we can form the following diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}'(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{F}''(U) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_i \mathcal{F}'(U_i) & \longrightarrow & \prod_i \mathcal{F}(U_i) & \longrightarrow & \prod_i \mathcal{F}''(U_i) \longrightarrow 0 \\
& & \downarrow f & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_{ij} \mathcal{F}'(U_{ij}) & \longrightarrow & \prod_{ij} \mathcal{F}(U_{ij}) & \longrightarrow & \prod_{ij} \mathcal{F}''(U_{ij}) \longrightarrow 0 \\
& & \downarrow & & & & \\
& & \text{coker } f & & & & 
\end{array}$$

Hm! If  $f$  was surjective, we could apply the snake lemma!! But  $f$  is not surjective. (...) All proofs I've found use Zorns lemma...

- c) Use the same diagram. The middle column is surjective at the bottom, and by commutativity, the right column must be as well.
- d) This is obvious, since  $f_* \mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$ .
- e) It is clear that  $\mathcal{G}$  is a sheaf. If  $U \subset V$ , let  $s \in \mathcal{G}(U)$  be given. Then define  $s' \in \mathcal{G}(V)$  as follows:  $s'(P) = s(P)$  if  $P \in U$  and zero elsewhere. This element will be sent to  $s$ .

The injective morphism from  $\mathcal{F}$  to  $\mathcal{G}$  is defined as follows: send  $s \in \mathcal{F}(U)$  to the function  $s(P) = s_P$  in  $\mathcal{G}(U)$ .

♡

**Exercise 7** (Exercise 2.19). Let  $A$  be a ring. The following are equivalent:

1.  $\text{Spec } A$  is disconnected.
2. There exists nonzero elements  $e_1, e_2 \in A$  such that  $e_1 e_2 = 0$ ,  $e_1^2 = e_1$ ,  $e_2^2 = e_2$  and  $e_1 + e_2 = 1$  (these are called *orthogonal idempotents*).
3.  $A$  is isomorphic to a direct product  $A_1 \times A_2$  of two nonzero rings.

♠

**Solution 7.**  $1 \Rightarrow 3$ : Let  $U$  be a nonempty connected component of  $X = \text{Spec } A$ . Let  $V = X \setminus U$  be its complement, and let  $i_1 : U \rightarrow X$  and  $i_2 : V \rightarrow X$  be the natural inclusions on topological spaces. This can be extended to a map of schemes as well: we need to give a morphism  $f^\# : \mathcal{O}_X \rightarrow f_* \mathcal{O}_U$ . But  $f_* \mathcal{O}_U(W) = \mathcal{O}_X(W \cap U)$ , so  $f_* \mathcal{O}_U = \mathcal{O}_X|_U$ . Hence we just choose  $f^\# : \mathcal{O}_X \rightarrow \mathcal{O}_U$  to be the natural map provided by the sheaf axioms.

We now have two morphisms  $i_1 : U \rightarrow X$  and  $i_2 : V \rightarrow X$  which are closed immersions, hence the induced ring morphisms  $A \rightarrow A_1$  and  $A \rightarrow A_2$  are surjective. Also, the universal property for products hold because the universal property for coproducts hold in the category of affine schemes. Hence  $A \simeq A_1 \times A_2$ . (a bit clumsy??)

$2 \Rightarrow 3$ : Let  $\pi_i : A \rightarrow A_i$  be given by multiplication by  $e_i$  and let  $A_i$  be its image. Then  $\ker \pi_1 = A_2$ , because if  $e_1 f$  then  $f = e_2 f$ , so  $f \in A_2$ . The splitting maps are the natural inclusions.

$3 \Rightarrow 2$ : If  $A = A_1 \times A_2$ , let  $e_i = \pi_i(1)$ .

$3 \Rightarrow 1$ : Similar to the first argument, just opposite.

♡

**Exercise 8** (Exercise 7.1). Let  $(X, \mathcal{O}_X)$  be a locally ringed space and let  $f : \mathcal{L} \rightarrow \mathcal{M}$  be a surjective map of invertible sheaves on  $X$ . Show that  $f$  is an isomorphism. ♠

**Solution 8.** Since  $\mathcal{L}, \mathcal{M}$  are invertible, we have isomorphisms  $\mathcal{L}_x \simeq \mathcal{O}_{X,x}$  and  $\mathcal{M}_x \simeq \mathcal{O}_{X,x}$  for each  $x \in X$ .

But  $\text{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{X,x}) = \mathcal{O}_{X,x}$ , that is, all homomorphisms are given by multiplication by some  $h \in \mathcal{O}_{X,x}$ . But since  $f$  was surjective, we conclude that  $h$  is outside  $\mathfrak{m}_x$ , the maximal ideal of  $\mathcal{O}_{X,x}$ . But then  $h$  is a unit, so  $f$  is an isomorphism. ♡

### 1.3 Chapter III - Cohomology

**Exercise 9** (Exercise 2.1). a) Let  $X = \mathbb{A}_k^1$  be the affine line over an infinite field  $k$ . Let  $P, Q$  be distinct closed points on  $X$  and let  $U = X - \{P, Q\}$ . Show that  $H^1(X, \mathbb{Z}_U) \neq 0$ . ♠

**Solution 9.** a) We have an exact sequence

$$0 \rightarrow \mathbb{Z}_U \rightarrow \mathbb{Z} \rightarrow i_*(\mathbb{Z}|_Z) \rightarrow 0,$$

where  $Z = P \cup Q$ . The last sheaf is equal to the skyscraper sheaf  $\mathbb{Z}_P \oplus \mathbb{Z}_Q$ . Since  $\mathbb{Z}$  is flabby, we have  $H^1(X, \mathbb{Z}) = 0$ . Hence the long exact sequence



reads

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow H^1(\mathbb{Z}_U) \rightarrow 0.$$

It follows that  $H^1(\mathbb{Z}_U) = \mathbb{Z}^2$ . In fact, this should please us, because if  $k = \mathbb{C}$ , we have that  $U$  is the complex plane minus two points, which is homotopic to the figure eight, which indeed have  $H_{sing}^1(U, \mathbb{C}) = \mathbb{C}^2$ .

♡

**Exercise 10** (Exercise 4.3). Let  $X = \mathbb{A}_k^2 = \text{Spec } k[x, y]$  and let  $U = X \setminus \{(0, 0)\}$ . Use a suitable open cover of  $X$  by open affine subsets to show that  $H^1(U, \mathcal{O}_U)$  is isomorphic to the  $k$ -vector space spanned by  $\{x^i y^j \mid i, j < 0\}$ . In particular, it is infinite-dimensional, and so  $U$  cannot be affine (not projective either). ♠

**Solution 10.** We can cover  $U$  by  $U_1 = \mathbb{A}^2 \setminus \{x = 0\}$  and  $U_2 = \mathbb{A}^2 \setminus \{y = 0\}$ . We have  $U_1 \cap U_2 = \mathbb{A}^2 \setminus \{xy = 0\}$ . Also,  $\mathcal{O}(U_1) = k[x, y, \frac{1}{x}]$  and  $\mathcal{O}(U_2) = k[x, y, \frac{1}{y}]$  and  $\mathcal{O}(U_1 \cap U_2) = k[x, y, \frac{1}{xy}]$ . Then the Čech complex takes the form

$$0 \rightarrow k[x, y, \frac{1}{x}] \times k[x, y, \frac{1}{y}] \xrightarrow{d} k[x, y, \frac{1}{xy}] \rightarrow 0,$$

the differential being difference. Then  $H^1(U, \mathcal{O}_U)$  can be computed as the homology at the second term. But nothing on the left side can hit anything of the form  $x^i y^j$  with  $i, j < 0$ . Anything else is hit. Thus we have

$$H^1(U, \mathcal{O}_U) \simeq \{x^i y^j \mid i, j < 0\}$$

as  $k$ -vector spaces.

♡

**Exercise 11** (Exercise 4.7). Let  $X$  be the subscheme of  $\mathbb{P}_k^2$  defined by a single homogeneous polynomial  $f(x_0, x_1, x_2) = 0$  of degree  $d$ . Assume that  $(1, 0, 0)$  is not on  $X$ . Then show that  $X$  can be covered by the two open affine subsets  $U = X \cap \{x_1 \neq 0\}$  and  $V = X \cap \{x_2 \neq 0\}$ . Now calculate the Čech complex

$$\Gamma(U, \mathcal{O}_X) \oplus \Gamma(V, \mathcal{O}_X) \rightarrow \Gamma(U \cap V, \mathcal{O}_X)$$

explicitly, and thus show that

$$\begin{aligned} \dim_k H^0(X, \mathcal{O}_X) &= 1 \\ \dim_k H^1(X, \mathcal{O}_X) &= \frac{1}{2}(d-1)(d-2). \end{aligned}$$

♠

**Solution 11.**  $X$  can be covered by just two open affines since  $\mathbb{P}^2 \setminus (U \cup V) = \{(1 : 0 : 0)\}$ , which was assumed not to lie on the curve.

The open affine subset  $\Gamma(U, \mathcal{O}_X)$  can be identified with the polynomial ring  $k[u, v]/\langle f(u, 1, v) \rangle$ , and  $\Gamma(V, \mathcal{O}_X) = k[x, y]/f(x, y, 1)$ . The differential is then given by

$$(g(u, v), h(x, y)) \mapsto g(xy^{-1}, y^{-1}) - h(x, y) \in k[x, y, \frac{1}{y}].$$

We can assume that  $f = x_0^d$ , since what really matters is the degree, and we are just doing linear algebra.

We first calculate  $H^0(X, \mathcal{O}_X)$ . So suppose  $g(xy^{-1}, y^{-1}) - h(x, y) = 0$  in  $k[x, y, y^{-1}]/\langle f(x, y, 1) \rangle$ . By definition this means that

$$g(xy^{-1}, y^{-1}) - h(x, y) = f(x, y, 1) \cdot \tilde{f}(x, y, \frac{1}{y})$$

for some polynomial  $\tilde{f}$ . Write  $\tilde{f}$  as  $\tilde{f}_0 + \tilde{f}_1$ , where  $\tilde{f}_0 = \sum_{j < 0} a_{ij} x^i y^j$  and  $\tilde{f}_1 \in k[x, y]$ . Then we have the equality

$$g(xy^{-1}, y^{-1}) - h(x, y) = \sum_{j < 0} a_{ij} x^{i+d} y^j + \sum_{j \geq 0} x^{i+d} y^j.$$

First of all, we see that the constant terms of  $g$  and  $h$  must be equal, because there are no constant terms on the right hand side. Secondly,  $g(xy^{-1}, y^{-1})$  consists solely of terms with  $j < 0$ . Thus the non constant terms of  $g(xy^{-1}, y^{-1})$  must be equal to the left term of the right hand side above. But both terms of the right hand side are zero modulo  $f$ , so the constant terms of  $g(xy^{-1}, y^{-1})$  are also zero mod  $f$ . The same holds for  $h(x, y)$ . Thus  $H^0(X, \mathcal{O}_X) = \{(c, c) \mid c \in k\} \simeq k$ .

Now we compute  $H^1(X, \mathcal{O}_X)$ . Consider a monomial  $x^i y^j$  in the target. If both  $i, j \geq 0$ , then it is hit by  $(0, -x^i y^j)$ . Likewise, if  $j \geq i$ , then  $(x^i y^{j-i}, 0) \mapsto x^i x^{-j}$ . Thus all monomials  $x^i y^{-j}$  with  $j \geq i$  is zero in the cokernel. Further, if  $i \geq d$ , then  $x^i y^j$  is already zero! Thus, we can draw the non-zero monomials in the cokernel as points in the lattice  $\mathbb{Z}^2$ . This is a triangle of length  $d - 2$ . Thus the dimension of  $H^1(X, \mathcal{O}_X)$  is

$$1 + 2 + \dots + d - 3 + d - 2 = \frac{1}{2}(d - 2)(d - 2 + 1) = \frac{1}{2}(d - 2)(d - 1).$$

♡

## 1.4 Chapter IV - Curves

**Exercise 12** (Exercise 1.1). Let  $X$  be a curve and  $P \in X$  a point. Show that there exists a nonconstant rational function  $f \in K(X)$  which is regular everywhere except at  $P$ . ♠

**Solution 12.** Let  $D$  be the divisor  $D = nP$ . The linear system

$$\{E = D + f \geq 0\}$$

consists of all divisors linearly equivalent to  $D$ . But these are classified by those  $f$  with  $(f) \geq -nP$ , i.e. those  $f$  with at most poles of order  $n$  at  $P$ .

By Riemann-Roch we have

$$l(D) - l(K - D) = \deg D + 1 - g = n + 1 - g.$$

If  $n$  is large enough,  $K - D$  will have negative degree, so  $l(K - D) = 0$ . Thus for large  $n$ , we can get  $l(D)$  as big as we want. ♡

## 2 Categories for the working mathematician - Saunders MacLane

### 2.1 Chapter 1.1

**Exercise 13** (1.1). Show that each of the following constructions can be regarded as a functor: The field of quotients of an integral domain; the Lie algebra of a Lie group. ♠

**Solution 13.** The first is a functor  $\text{Rings} \rightarrow \text{Fields}$ , since "obviously" the assignment  $R \mapsto K(R)$  is functorial, since a morphism is determined by what it does to elements of  $R$ .

The second is a functor from the category of Lie groups to the category of Lie algebras ♡

## 3 Calculus on Manifolds - Spivak

### 3.1 Functions on Euclidean Space

**Exercise 14** (Exercise 1.1). Prove that  $|x| \leq \sum_{i=1}^n |x^i|$ . ♠

**Solution 14.** By induction, one can prove that  $\sqrt{\sum_i a_i} \leq \sum_i \sqrt{a_i}$ . The claim then follows trivially. ♡

**Exercise 15** (Exercise 1.7). A linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is *norm preserving* if  $|T(x)| = |x|$  for all  $x \in \mathbb{R}^n$ . It is *inner product preserving* if  $\langle Tx, Ty \rangle = \langle x, y \rangle$ .

- a) Prove that  $T$  is norm preserving if and only if it is inner product preserving.
- b) Prove that such a linear transformation is 1 – 1 and  $T^{-1}$  is of the same sort.

♠

**Solution 15.** a). The direction  $\Leftarrow$  is trivial. For the other direction, choose a basis  $\{x_1, \dots, x_n\}$  of  $\mathbb{R}^n$  such that  $x = x_1$  and  $y = \sum a_i x_i$ . Then  $\langle Tx, a_i x_i \rangle = a_i \langle Tx, x_i \rangle = 0$  if  $i \neq 1$  and  $a_1$  else. Then since  $T(0) = 0$  it follows that

$$\langle Tx, Ty \rangle = \langle Tx, T(a_1 x_1) \rangle = a_1 \langle Tx_1, Tx_1 \rangle = a_1 |Tx_1|^2 = a_1 |x_1|^2 = a_1 \langle x_1, x_1 \rangle.$$

b). Suppose  $T(x) = 0$ . Then  $0 = \langle Tx, Tx \rangle = \langle x, x \rangle$ , but this happens if and only if  $x = 0$ . Also  $\langle T^{-1}y, T^{-1}y \rangle = \langle T^{-1}T(x), T^{-1}T(x) \rangle = \langle TT^{-1}(x), TT^{-1}(x) \rangle = \langle T^{-1}x, T^{-1}x \rangle = \langle y, y \rangle$ . ♡

## 4 Commutative Algebra - Eisenbud

### 4.1 Chapter 16 - Modules of Differentials

**Exercise 16** (Exercise 16.1). Show that if  $b \in S$  is an idempotent ( $b^2 = b$ ), and  $d : S \rightarrow M$  is any derivation, then  $db = 0$ . ♠

**Solution 16.** This is trivial.  $db = d(b^2) = 2db$ . If  $2 = 0$ , then the statement is automatically true. If not, then  $db = 0$  by subtraction. ♡

## 5 Deformation Theory - Hartshorne

### 5.1 Chapter I.3 - The $T^i$ functors

**Exercise 17** (Exercise 3.1). Let  $B = k[x, y](xy)$ . Show that  $T^1(B/k, M) = M \otimes k$  and  $T^2(B/k, M) = 0$  for any  $B$ -module  $M$ . ♠

**Solution 17.** Since  $B$  is defined by a principal ideal in  $P = k[x, y]$ , it follows that  $L_2 = 0$  in the cotangent complex. Thus  $T^2(B/k, M)$  is automatically zero.

We have that  $L_1 = B$  and  $L_0 = Bdx \oplus Bdy$  with  $d_1$  being  $f \mapsto (fy, fx)$ . Applying  $\text{Hom}(-, M)$ , we get  $\text{Hom}(L_0, M) = M \oplus M$  and  $\text{Hom}(L_1, M) = M$ .

We have  $\text{Hom}(B \oplus B, M) \simeq M \oplus M$  by  $\phi \mapsto (\phi(1, 0), \phi(0, 1))$ . We have a diagram

$$\begin{array}{ccc} \text{Hom}(B \oplus B, M) & \xrightarrow{\psi^*} & \text{Hom}(B, M) \\ \simeq \downarrow & & \downarrow \simeq \\ M \oplus M & \longrightarrow & M \end{array}$$

Under these isomorphisms, it is easy to see that the bottom map is given by

$$(\phi(1, 0), \phi(0, 1)) \mapsto y\phi(1, 0) + x\phi(0, 1).$$

Thus since  $T^1$  is the cokernel of this map, we must have  $T^1(B/k, M) = M \otimes k$ .  $\heartsuit$

**Exercise 18** (Exercise 3.3). Let  $B = k[x, y]/(x^2, xy, y^2)$ . Show that  $T^0(B/k, B) = k^4$ ,  $T^1(B/k, B) = k^4$  and  $T^2(B/k, B) = k$ .  $\spadesuit$

**Solution 18.** Let's compute  $L_2$  first. For that we need part of a resolution of  $I$ . We have in fact

$$0 \rightarrow \text{im} \begin{pmatrix} -y & 0 \\ x & -y \\ 0 & x \end{pmatrix} \rightarrow P(-2)^3 \rightarrow I \rightarrow 0.$$

The Koszul relations are given by

$$\text{im} \begin{pmatrix} -y^2 & -xy & 0 \\ 0 & x^2 & -y^2 \\ x^2 & 0 & xy \end{pmatrix}.$$

Let's compute  $Q/F_0$  (relations modulo Koszul relations). Since  $Q$  is generated in degree 3, and  $F_0$  is of degree 4, we have  $\dim_k(Q/F_0)_3 = 2$ . Let's consider degree 4. As a  $k$ -vector space  $Q_4$  is spanned by the four elements

$$\begin{pmatrix} -y^2 \\ xy \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -y^2 \\ xy \end{pmatrix}, \begin{pmatrix} -yx \\ x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -yx \\ x^2 \end{pmatrix}.$$

The two in the middle are already Koszul relations, so that  $(Q/F_0)_4$  have dimension  $\leq 2$ . But we also have

$$\begin{pmatrix} -y^2 \\ xy \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ yx \\ -x^2 \end{pmatrix} + \begin{pmatrix} -y^2 \\ 0 \\ x^2 \end{pmatrix}.$$

Thus  $\dim_k(Q/F_0)_4 = 1$ , since the second term above is a Koszul relation. Similarly we find that  $\dim_k(Q/F_0)_5 = 0$ . Hence,  $L_2$  is the 3-dimensional  $k$ -vector space spanned by  $Q_3$  and one more relation.  $L_1$  is  $F \otimes B = B^3$ , and  $L_0$  is  $B \oplus B$ , spanned by  $dx, dy$ .

Taking duals, we get that  $L_2 = \text{Hom}(Q/F_0, B)$ . This set can be identified with

$$\begin{aligned} \text{Hom}(Q/F_0, B) &= \{\varphi : Q \rightarrow B \mid \varphi|_{F_0} = 0\} \\ &= \{\varphi : Q \rightarrow P \mid \text{im } \varphi|_{F_0} \subseteq I\} \end{aligned}$$

Thus, since  $I = \mathfrak{m}^2$ , we must have that  $\varphi$  sends the two generators of  $Q$  to something of degree 1 (degree 0 is not ok, since then  $F_0$  would be sent outside  $I$ ). Thus  $\text{Hom}(Q/F_0, B)$  is  $2 \times 2 = 4$ -dimensional, spanned by

$$\text{im} \begin{pmatrix} y & x & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}.$$

But  $d_2$  is the dual of the inclusion  $Q \rightarrow F$  from the exact sequence above. The dual is given by transposing, and we are left with one column - in conclusion,  $T^2(B/k, B)$  is one-dimensional.

The Jacobian of  $I$  is given by

$$\begin{pmatrix} 2x & y & 0 \\ 0 & x & 2y \end{pmatrix},$$

and it is easily seen that the kernel of  $\text{Jac} \otimes B$  is given by  $\mathfrak{m} \oplus \mathfrak{m} \oplus \mathfrak{m} \subset B^3$ . The two relations kill off two dimensions, so  $\dim_k T^1(B/k, B) = \dim_k \mathfrak{m}^{\oplus 3} - 2 = 6 - 2 = 4$ .

Also  $T^0(B/k, B)$  is  $B^2$  modulo the image of the Jacobian. The constants are left untouched, so  $\dim_k T^0(B/k, B) = 2 + 2 + 2 - 3 = 3$ . A basis is given by  $(1, 0)$ ,  $(0, 1)$  and  $(x, y)$ . (thus Hartshorne is wrong?)  $\heartsuit$

## 6 Introduction to Differential Geometry - Spivak

### 6.1 Chapter 1 - Manifolds

**Exercise 19** (Exercise 3). a) Every manifold is locally compact.

- b) Every manifold is locally pathwise connected, and a connected manifold is pathwise connected.
- c) A connected manifold is arcwise connected. (*Arcwise connected* means that any two points can be connected by a  $1 - 1$  path.)



**Solution 19.** a) Indeed, let  $x : \mathbb{R}^n \rightarrow U \subset M$  be a homeomorphism of an open subset of  $M$  with  $\mathbb{R}^n$ . Then the image of  $[0, 1]^n$  is compact in  $M$ .

- b) The first part follows in the same way, since  $M$  is locally homeomorphic to  $\mathbb{R}^n$ . Now assume that  $M$  is connected. Fix  $q \in M$ . Let  $V$  be the set of all points in  $M$  such that there is a path from  $q$  to  $p$ . Clearly  $V$  is non-empty, by the first part of the exercise.

$V$  is also open: For let  $p \in V$ . Choose a neighbourhood  $U$  around  $p$  homeomorphic to  $\mathbb{R}^n$ . By composing paths, any point in  $U$  can be reached as well. Hence  $V$  is open.

We show that  $V$  is closed: let  $\{p_i\}$  be a convergent sequence of points  $p_i \in M$  with all  $p_i \in V$ . We want to show that the limit point is contained in  $V$ . Choose a compact neighbourhood around  $\lim p_i = p$ , which we can assume to be  $\approx [0, 1]^n$ . Then  $p \in [0, 1]^n$ , and this is path connected. Hence  $V$  is closed.

- c) For  $n > 2$ , one can always homotope away from the points of non-injectivity. For  $n = 2$ , one can do “Reidemeister” moves.



**Exercise 20** (Exercise 4). A space  $X$  is called locally connected if for each  $x \in X$ , it is the case that every neighbourhood of  $x$  contains a connected neighbourhood.

- a) Connectedness does not imply local connectedness.
- b) An open subset of a locally connected space is locally connected.
- c)  $X$  is locally connected if and only if components of open sets are open, so every neighbourhood of a point in a locally connected space contains an *open* connected neighbourhood.

- d) A locally connected space is homeomorphic to the disjoint union of its components.
- e) Every manifold is locally connected, and consequently homeomorphic to the disjoint union of its components, which are open submanifolds.



**Solution 20.** a) Consider the topologist's sine curve. Every neighbourhood of 0 is disconnected.

b) This is “trivial”. Let  $U$  be the said open subset. The open subsets of  $U$  are intersections  $U \cap V$  where  $V$  is open in  $X$ . Hence local connectedness is trivially inherited.

c) Suppose  $X$  is locally connected. Let  $U \subset X$  be an open set, and let  $U = \cup_i U_i$  be its decomposition into its components. We want to show that each  $U_i$  is open. So let  $x \in U_i$ . Then  $U_i$  contains a connected neighbourhood containing  $x$ , by definition, hence  $U_i$  is open.

Conversely, assume components of open sets are open. Let  $x \in X$ , and let  $U$  be an open neighbourhood of  $x$ . Then  $U_i$  as above is connected and can be chosen to contain  $x$ , hence  $x$  is locally connected.

- d) This is trivial, since the components are open.
- e) Pathwise local connectedness implies local connectedness.



**Exercise 21** (Exercise 15). a) Show that  $\mathbb{P}^1$  is homeomorphic to  $S^1$ . (in fact, diffeomorphic)

b) Show that  $\mathbb{P}^n \setminus \mathbb{P}^{n-1}$  is homeomorphic to the interior  $D^n = \{x \in \mathbb{R}^n \mid d(x, 0) < 1\}$ .



**Solution 21.** a) Both  $\mathbb{P}^1$  and  $S^1$  can be covered by two open subsets homeomorphic to  $\mathbb{R}$ , and it can be checked that in both cases, the transition mapping is given by  $x \mapsto \frac{1}{x}$ , hence they are glued in the same way, hence they must be diffeomorphic.

b) By using homogeneous coordinates, we see that  $\mathbb{P}^n \setminus \mathbb{P}^{n-1}$  is homeomorphic to  $\mathbb{R}^n$  which is again homeomorphic to the interior of a disc.





## 7 Introduction to Commutative Algebra - Atiyah-MacDonald

### 7.1 Chapter 1 - Rings and ideals

**Exercise 22.** Let  $x$  be a nilpotent element of a ring  $A$ . Show that  $1 + x$  is a unit of  $A$ . Deduce that the sum of a nilpotent element and a unit is a unit. ♠

**Solution 22.** Suppose  $x^{n+1} = 0$  and that  $x^n \neq 0$ . Consider

$$s = 1 - x + x^2 - x^3 + \dots + x^n$$

Then

$$sx = x - x^2 + x^3 - x^4 + \dots - x^n$$

since  $x^{n+1} = 0$ . But then  $s + sx = 1$ , so that  $s(1 + x) = 1$ . Hence  $1 + x$  is a unit. To prove that the sum of any unit and any nilpotent is a unit, note that if  $u$  is any unit, then  $u^{-1}x$  is still nilpotent. So since  $u + x = u(1 + u^{-1}x)$  and product of units are units, the claim follows. ♥

**Exercise 23** (Exercise 11). A ring  $A$  is *Boolean* if  $x^2 = x$  every  $x \in A$ . In a Boolean ring  $A$ , show that

- i)  $2x = 0$  for all  $x \in A$ .
- ii) Every prime ideal  $\mathfrak{p}$  is maximal, and  $A/\mathfrak{p}$  is a field with two elements.
- iii) Every finitely generated ideal in  $A$  is principal.

♠

**Solution 23.** i) We have  $4x = 4x^2 = (2x)^2 = 2x$ , hence  $2x = 0$ .

- ii) Consider  $A/\mathfrak{p}$ . This is an integral domain in which  $x^2 = x$  for all  $x \in A/\mathfrak{p}$ . But then  $x^2 - x = x(x - 1) = 0$ . Hence either  $x = 0$  or  $x = 1$ , hence  $A/\mathfrak{p}$  can have only two elements. Thus it is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  which is a field, hence  $\mathfrak{p}$  is maximal.
- iii) Let  $I = (a_1, \dots, a_r)$ . Every ideal is contained in a maximal ideal  $\mathfrak{m}$ . Consider the image of  $I$  in  $A/\mathfrak{m}$ .
- iv) By induction we can assume that  $I$  is generated by two elements, say  $I = (a_1, a_2)$ . Then I claim that  $I = (a_1 + a_2)$ . Clearly  $(a_1 + a_2) \subseteq (a_1, a_2)$ . The other direction will follow if we can see that  $a_1 a_2 = 0$  (or

they can be assumed to satisfy this), because  $a_1a_2 + a_1 \in (a_1 + a_2)$ .  
 [????????????????]

♡

**Exercise 24** (Exercise 12). A local ring contains no nontrivial idempotents.

♠

**Solution 24.** Suppose  $x \neq 0, 1$  and that  $x^2 = x$ . Then  $x^2 - x = x(x - 1) = 0$ . Both  $x$  and  $x - 1$  cannot be contained in  $\mathfrak{m}$  since they generate  $A$ . Hence one of the is unit. Hence either  $x = 0$  or  $x = 1$ , contradiction. ♡

**Exercise 25** (Exercise 15, The prime spectrum of a ring). Let  $A$  be a ring and let  $X$  be the set of prime ideals of  $A$ . For each subset  $E$  of  $A$ , let  $V(E)$  denote the set of prime ideals of  $A$  which contain  $E$ . Prove that

1. If  $\mathfrak{a}$  is the ideal generated by  $E$ , then  $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))^1$ .
2.  $V(0) = X$  and  $V(1) = \emptyset$ .
3. If  $(E_i)_{i \in I}$  is a family of subsets of  $A$ , then

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i).$$

4.  $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$  for all ideals  $\mathfrak{a}, \mathfrak{b}$  of  $A$ .

These results show that the sets  $V(E)$  satisfy the axioms for closed sets in a topological space. The resulting topology is called the *Zariski topology*. The topological space  $X$  is called the *prime spectrum of  $A$*  and denoted  $\text{Spec } A$ . ♠

**Solution 25.** We do these one by one.

1. Clearly  $\mathfrak{p} \supset \langle E \rangle \supset E$ , where the brackets denote the ideal generated by  $E$ . Hence  $V(\mathfrak{a}) \subset V(E)$ . But if  $\mathfrak{p} \supset E$ , we must have  $\mathfrak{p} \supset \mathfrak{a}$  since  $\langle \mathfrak{p} \rangle = \mathfrak{p}$ . Thus the first equality is established.

Since  $r(\mathfrak{a}) \subset \mathfrak{a}$ , we have  $V(\mathfrak{a}) \subset V(r(\mathfrak{a}))$ . Suppose  $\mathfrak{p} \supset r(\mathfrak{a})$  and suppose  $a \in \mathfrak{a}$ . We want to show  $a \in \mathfrak{p}$ . We know that  $a^n \in r(\mathfrak{a})$  for some  $n$ , hence  $a^n \in \mathfrak{p}$ . But  $\mathfrak{p}$  is a prime ideal, so  $a \in \mathfrak{p}$  also. Hence equality is established.

2. Every ideal contains the zero ideal and (1) is not a prime ideal.

---

<sup>1</sup>Here  $r(\mathfrak{a})$  denotes the radical of  $\mathfrak{a}$

3. Suppose  $\mathfrak{p} \supset \cup E_i$ . Then  $\mathfrak{p} \supset E_i$  for all  $i$ , so  $\mathfrak{p} \in \cap V(E_i)$ . Thus this is just a formal consequence of the contravariant nature of  $V(-)$ .
4. Since  $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$ , we automatically have  $V(\mathfrak{a} \cap \mathfrak{b}) \subset V(\mathfrak{a}\mathfrak{b})$ . So suppose  $\mathfrak{p} \supset \mathfrak{a}\mathfrak{b}$  and let  $a \in \mathfrak{a} \cap \mathfrak{b}$ . Then  $a^2 \in \mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ , but then  $a \in \mathfrak{p}$  since  $\mathfrak{p}$  is prime.

Now suppose  $\mathfrak{p} \supset \mathfrak{a}$  or  $\mathfrak{p} \supset \mathfrak{b}$ . Then if  $a \in \mathfrak{a} \cap \mathfrak{b}$ , we have  $a \in \mathfrak{p}$ , so  $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subset V(\mathfrak{a} \cap \mathfrak{b})$ . Now suppose  $\mathfrak{p} \supset \mathfrak{a} \cap \mathfrak{b}$ . Then by Proposition 1.11, we have  $\mathfrak{p} \supset \mathfrak{a}$  or  $\mathfrak{p} \supset \mathfrak{b}$ .

♡

**Exercise 26** (Exercise 17). For each  $f \in A$ , let  $X_f$  denote the complement of  $V(f)$  in  $X = \text{Spec } A$ . The sets  $X_f$  are open. Show that they form a basis for the Zariski topology, and that

1.  $X_f \cap X_g = X_{fg}$ .
2.  $X_f = \emptyset \Leftrightarrow f$  is nilpotent.
3.  $X_f = X \Leftrightarrow f$  is a unit.
4.  $X_f = X_g \Leftrightarrow r((f)) = r((g))$ .
5.  $X$  is quasi-compact.
6. More generally, each  $X_f$  is quasi-compact.
7. An open subset of  $X$  is quasi-compact if and only if it is a finite union of the sets  $X_f$ .

The sets  $X_f$  are called *basic open sets* of  $X = \text{Spec } A$ .

♠

**Solution 26.** We need to show that the sets  $X_f$  forms a basis for the Zariski topology on  $X$ . This means that each open in  $X$  can be written as a union of the  $X_f$ . An open in  $X$  have the form

$$U(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec } A \mid \mathfrak{p} \not\supset \mathfrak{a}\}.$$

The sets  $X_f$  have the form

$$X_f = \{\mathfrak{p} \in \text{Spec } A \mid f \notin \mathfrak{p}\}.$$

Let  $\{f_i\}_{i \in I}$  generate  $\mathfrak{a}$ . I claim that  $\bigcup X_{f_i} = U(\mathfrak{a})$ . Let  $\mathfrak{p}$  be an element of the left hand side. This means by definition that  $f_i \notin \mathfrak{p}$  for some  $i$ . But  $f_i$  is an element of  $\mathfrak{a}$ , so  $\mathfrak{a} \not\subset \mathfrak{p}$ , hence  $\mathfrak{p} \in U(\mathfrak{a})$ .

Conversely, suppose  $\mathfrak{p} \not\supset \mathfrak{a}$ . Then some generator  $f_i$  of  $\mathfrak{a}$  is not contained in  $\mathfrak{p}$ . Hence  $\mathfrak{p} \in X_{f_i}$ .

1. We have

$$X_f \cap X_g = \{\mathfrak{p} \mid f, g \notin \mathfrak{p}\} = \{\mathfrak{p} \mid fg \notin \mathfrak{p}\},$$

since  $\mathfrak{p}$  is a prime ideal: for suppose  $f, g \notin \mathfrak{p}$ , then  $fg \notin \mathfrak{p}$  also, because if  $fg \in \mathfrak{p}$ , primality implies either  $f \in \mathfrak{p}$  or  $g \in \mathfrak{p}$ . Conversely, suppose  $fg \notin \mathfrak{p}$ . Then neither  $f, g$  can be in  $\mathfrak{p}$  by definition of ideals.

2. Suppose  $X_f$  is empty. Then there are no prime ideals with  $f \notin \mathfrak{p}$ . But that means that  $f$  is contained in every prime ideal, hence  $f$  is nilpotent.
3. Suppose  $X_f = X$ . Then for all prime ideals,  $f \notin \mathfrak{p}$ , hence  $f$  generates the unit ideal, hence  $f$  is a unit. For if  $f$  did not generate the unit ideal,  $f$  would be contained in some maximal ideal  $\mathfrak{m}$ , and maximal ideals are prime.
4. Suppose  $X_f = X_g$ . By definition, this means that for every prime  $\mathfrak{p}$  with  $f \notin \mathfrak{p}$ , we have  $g \notin \mathfrak{p}$  (and conversely). The contrapositive of this is  $g \in \mathfrak{p} \Leftrightarrow f \in \mathfrak{p}$ . Hence we have

$$r((f)) = \bigcap_{\mathfrak{p} \supset (f)} \mathfrak{p} = \bigcap_{\mathfrak{p} \ni f} \mathfrak{p} = \bigcap_{\mathfrak{p} \ni g} \mathfrak{p} = r((g)).$$

5. Let  $\{X_f\}_{f \in I}$  be a covering of  $X$  by basic opens, that is,  $X = \bigcup_{f \in I} X_f$ . This means that for every  $\mathfrak{p} \in X$ , there is some  $f \in I$  with  $f \notin \mathfrak{p}$ . I claim that the  $f_i$  generate the unit ideal: for if not,  $\langle f_i \rangle$  would be contained in some prime ideal, but by the above, this is not the case. Hence there is an equation of the form  $1 = \sum g_i f_i$  with  $g_i \in A$ , which is a *finite* sum. Hence these finitely many  $f_i$  suffice.
6. ...

♡

## 7.2 Chapter 2 - Modules

**Exercise 27** (Exercise 1). Show that  $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n = 0$  if  $m, n$  are coprime.

♠

**Solution 27.** Write  $1 = am + bn$ . Then

$$\begin{aligned} 1 \otimes 1 &= (am + bn) \otimes 1 = am \otimes 1 + bn \otimes 1 \\ &= 0 + bn \otimes 1 = 1 \otimes bn = 1 \otimes 0 = 0. \end{aligned}$$

And we are done.

♡

**Exercise 28** (Exercise 2). Let  $A$  be a ring,  $\mathfrak{a}$  an ideal, and  $M$  an  $A$ -module. Then  $(A/\mathfrak{a}) \otimes_A M$  is isomorphic to  $M/\mathfrak{a}M$ . ♠

**Solution 28.** Start with

$$0 \rightarrow \mathfrak{a} \rightarrow A \rightarrow A/\mathfrak{a} \rightarrow 0.$$

Tensoring with  $M$  gives

$$\mathfrak{a} \otimes M \rightarrow M \rightarrow A/\mathfrak{a} \otimes_A M \rightarrow 0.$$

But  $\mathfrak{a} \otimes_A M \simeq \mathfrak{a}M$ , so that the sequence reads  $A/\mathfrak{a} \otimes M \simeq M/\mathfrak{a}M$ . ♡

**Exercise 29** (Exercise 3). Let  $A$  be a local ring,  $M, N$  finitely generated  $A$ -modules. Prove that if  $M \otimes N = 0$ , then  $M = 0$  or  $N = 0$ . ♠

**Solution 29.** First a counterexample if  $A$  is not a local ring. Let  $A = k[x]$  and  $M = k[x]/(x-1)$  and  $N = k[x]/(x)$ . We can write  $1 = -(x-1) + x$ . Then  $M \otimes_A N = 0$  by the same method as in Exercise 1 ( $1 \otimes 1 = (-x+1+x) \otimes 1 = x \otimes 1 = 1 \otimes x = 0$ ).

Let  $M_k := M \otimes k = M/\mathfrak{m}M$ . By Nakayama's lemma,  $M_k = 0 \Rightarrow M = 0$ .

So suppose  $M \otimes_A N = 0$ . Then  $(M \otimes_A N)_k = 0$ . But this is isomorphic to  $M_k \otimes_A N_k$  since  $k \otimes_A k = k$ . But  $M_k \otimes_A N_k \simeq M_k \otimes_k N_k$ , as  $k$ -modules, since everything in  $\mathfrak{m}$  acts trivially on  $M_k$ . But these are vector spaces over a field, now we must have  $M_k = 0$  or  $N_k = 0$ , and by Nakayama we are done. ♡

**Exercise 30** (Exercise 4). Let  $M_i$  ( $i \in I$ ) be any family of  $A$ -modules, and let  $M$  be their direct sum. Then  $M$  is flat if and only if each  $M_i$  is flat. ♠

**Solution 30.** Let

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

be any exact sequence. Then tensoring with  $M$  gives

$$0 \rightarrow N' \otimes_A M \rightarrow N \otimes_A M \rightarrow N'' \otimes_A M \rightarrow 0.$$

We only need to check that the left map is injective. But we have  $N' \otimes_A M \simeq \bigoplus_i N' \otimes_A M_i$  and  $N \otimes_A M \simeq \bigoplus_i N \otimes_A M_i$ , and thus the left map is just the direct sum of all the maps

$$0 \rightarrow N' \otimes_A M_i \rightarrow N \otimes_A M_i,$$

which is injective if and only if each  $M_i$  is flat. ♡

**Exercise 31** (Exercise 5). Let  $A[x]$  be the ring of polynomials in one indeterminate over a ring  $A$ . Prove that  $A[x]$  is flat  $A$ -algebra. ♠

**Solution 31.** We have  $A[x] = \bigoplus_{i=0}^{\infty} x^i A$  as an  $A$ -module. Now use Exercise 4. ♡

**Exercise 32** (Exercise 24). If  $M$  is an  $A$ -module, the following are equivalent:

- i)  $M$  is flat.
- ii)  $\text{Tor}_n^A(M, N) = 0$  for all  $n > 0$  and  $A$ -modules  $N$ .
- iii)  $\text{Tor}_1^A(M, N) = 0$  for all  $A$ -modules  $N$ .

**Solution 32.** To compute  $\text{Tor}_A^n(M, N)$ , one takes an  $A$ -resolution of  $N$  and tensor it with  $M$  and take homology. But  $M$  is flat, so the sequence stays exact, so the homology is zero. This shows  $i) \Rightarrow ii)$ .

The implication  $ii) \Rightarrow iii)$  is trivial.

Now let

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

be any exact sequence of  $A$ -modules. Then by properties of the Tor functor, we have an exact sequence

$$\text{Tor}_1(M, N'') \rightarrow N' \otimes M \rightarrow N \otimes M \rightarrow N'' \otimes M \rightarrow 0.$$

But  $\text{Tor}_1(M, N'') = 0$ , so the sequence is short exact. Hence  $M$  is flat. ♡

**Exercise 33** (Exercise 25). Let

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

be an exact sequence with  $N''$  flat. Then  $N'$  is flat if and only if  $N$  is flat. ♠

**Solution 33.** We have from the Tor exact sequence

$$0 \rightarrow \text{Tor}_1(N', M) \rightarrow \text{Tor}_1(N, M) \rightarrow 0$$

since  $\text{Tor}_2(N'', M) = \text{Tor}_1(N'', M) = 0$ . The statement follows. ♡

♠

### 7.3 Chapter III - Rings and modules of fractions

**Exercise 34** (Exercise 1). Let  $S$  be a multiplicatively closed subset of a ring  $A$ , and let  $M$  be a finitely-generated  $A$ -module. Prove that  $S^{-1}M = 0$  if and only if there exists  $s \in S$  such that  $sM = 0$ . ♠

**Solution 34.** Suppose there exists such  $s$ . Let  $m/s' \in S^{-1}M$ . This is zero if and only if there exists  $s \in M$  such that  $s(s'm) = 0$ . But  $ss'm = s'sm = s'0 = 0$ . So  $m = 0$  in  $S^{-1}M$ . (note that we did not use finite generation)

Now let  $m_1, \dots, m_r$  be a set of generators for  $M$  and suppose that  $S^{-1}M = 0$ . Then for each  $i$  ( $i = 1, \dots, r$ ), there exists  $s_i$  such that  $s_i m_i = 0$ . Since every element of  $M$  is an  $A$ -linear combination of the  $m_i$ , it follows that the product  $s_1 s_2 \cdots s_r$  makes  $sM = 0$ . ♥

### 7.4 Chapter 5 - Integral dependence and valuations

**Exercise 35** (Exercise 1). Let  $f : A \rightarrow B$  be an integral morphism of rings. Show that  $f^* : \text{Spec } B \rightarrow \text{Spec } A$  is a closed mapping. ♠

**Solution 35.** The map  $f^*$  is by definition given by  $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p}) = \mathfrak{p} \cap A$ . A closed subset of  $\text{Spec } B$  is by definition

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec } B \mid \mathfrak{p} \supset \mathfrak{a}\}$$

for some ideal  $\mathfrak{a} \subset B$ .

Then the image of  $V(\mathfrak{a})$  is the set

$$f^*(V(\mathfrak{a})) = \{\mathfrak{p} \cap A \mid \mathfrak{p} \in \text{Spec } B, \mathfrak{p} \supset \mathfrak{a}\}$$

I claim that this is equal to

$$V(\mathfrak{a} \cap A) = \{\mathfrak{q} \in \text{Spec } A \mid \mathfrak{q} \supset \mathfrak{a} \cap A\},$$

which clearly is a closed subset of  $\text{Spec } A$ .

One direction is obvious: let  $\mathfrak{p} \cap A$  be an element of  $f^*(V(\mathfrak{a}))$ . This is a point of  $\text{Spec } A$ , and clearly  $\mathfrak{p} \cap A \supset \mathfrak{a} \cap A$  since  $\mathfrak{p} \supset \mathfrak{a}$ .

The other direction needs the going up Theorem 5.10. Suppose  $\mathfrak{q} \in V(\mathfrak{a} \cap A)$ . Then by Going Up, there exists  $\mathfrak{p} \in \text{Spec } B$  with  $\mathfrak{p} \cap A = \mathfrak{q}$ . But we need to check that  $\mathfrak{p} \supset \mathfrak{a}$ . That is, we need to prove the assertion that if  $\mathfrak{q} = \mathfrak{p} \cap A$  and  $\mathfrak{q} \supset \mathfrak{a} \cap A$ , then  $\mathfrak{p} \supset \mathfrak{a}$ . So suppose  $a \in \mathfrak{a} \subset B$ . Then  $a$  satisfies an equation

$$a^n + b_{n-1}a^{n-1} + \dots + b_1a + b_0 = 0$$

with  $b_i \in A$ . Since  $a \in \mathfrak{a}$ , we see that  $b_0 \in \mathfrak{q} = \mathfrak{p} \cap A$ . Hence

$$a^n + b_{n-1}a^{n-1} + \dots + b_1a = a(a^{n-1} + b_{n-1}a^{n-2} + \dots + b_1) \in \mathfrak{p}$$

since  $\mathfrak{q} \subset \mathfrak{p}$ . But  $\mathfrak{p}$  is prime so either  $a \in \mathfrak{p}$  and we are done, or  $a^{n-1}b_{n-1}a^{n-2} + \dots + b_1 \in \mathfrak{p}$ , and we can continue by induction.

Hence we are done. ♥

## 7.5 Chapter 7 - Noetherian rings

**Exercise 36** (Exercise 11). Let  $A$  be a ring such that  $A_{\mathfrak{p}}$  is Noetherian for each  $\mathfrak{p} \in \text{Spec } A$ . Is  $A$  necessarily noetherian? ♠

**Solution 36.** Consider the ring

$$A = \mathbb{Z}/2 \times \mathbb{Z}/2 \cdots$$

It is a countable product of noetherian rings. The primes are just the coordinate axes, and each localization is isomorphic to  $\mathbb{Z}/2$ . Thus each  $A_{\mathfrak{p}}$  is Noetherian, but  $A$  is not. ♥

**Exercise 37** (Exercise 15). Let  $A$  be a Noetherian local ring,  $\mathfrak{m}$  its maximal ideal and  $k$  its residue field and let  $M$  be a finitely generated  $A$ -module. Then the following are equivalent:

- i)  $M$  is free.
- ii)  $M$  is flat.
- iii) The mapping  $\mathfrak{m} \otimes M \rightarrow A \otimes M$  is injective.
- iv)  $\text{Tor}_1^A(k, M) = 0$ .

♠

**Solution 37.** The implication  $i) \Rightarrow ii)$  is trivial. One way is to compute  $\text{Tor}_1^A(M, N)$  for any  $A$ -module  $N$ . But a free resolution of  $M$  is just one-term, so  $\text{Tor}_1^A(M, N)$  is automatically zero.

The implication  $ii) \Rightarrow iii)$  follows by tensoring the inclusion  $\mathfrak{m} \hookrightarrow A$  with  $M$ .

The implication  $iii) \Rightarrow iv)$  follows from the Tor exact sequence

$$\text{Tor}_1^A(A, M) \rightarrow \text{Tor}_1^A(k, M) \rightarrow \mathfrak{m} \otimes M \rightarrow A \otimes M \rightarrow k \otimes M \rightarrow 0.$$



The leftmost term is zero since  $A$  is a free  $A$ -module, and by *iii*) and exactness we must as well have  $\text{Tor}_1^A(k, M)$ .

Now for *iv*  $\Rightarrow$  *i*). Choose element  $m_i \in M$  ( $0 \leq i \leq r$ ) such that they form a  $k$ -basis for  $M/\mathfrak{m}M$ . Choose a surjection  $f : A^r \rightarrow M$  and let  $E = \ker f$  be its kernel. Then we have an exact sequence

$$0 \rightarrow E \rightarrow A^r \rightarrow M \rightarrow 0.$$

of finitely-generated  $A$ -modules ( $E$  is finitely generated by Proposition 6.2). Tensor the sequence by  $k$ , and get

$$\text{Tor}_1^A(k, M) \rightarrow E/\mathfrak{m}E \rightarrow k^r \rightarrow M/\mathfrak{m}M \rightarrow 0.$$

The left-most term is zero by assumption. The last two spaces are  $k$ -vector spaces of the same dimension, and it follows that  $E/\mathfrak{m}E = 0$ . But then it follows that  $E$  is zero by Nakayama's lemma, hence  $M$  is free.  $\heartsuit$

**Exercise 38** (Exercise 16). Let  $A$  be a Noetherian ring,  $M$  a finitely-generated  $A$ -module. Then the following are equivalent:

- i)  $M$  is a flat  $A$ -module.
- ii)  $M_{\mathfrak{p}}$  is a free  $A_{\mathfrak{p}}$ -module for each  $\mathfrak{p} \in \text{Spec } A$ .
- iii)  $M_{\mathfrak{m}}$  is a free  $A_{\mathfrak{m}}$ -module for each maximal ideal  $\mathfrak{m}$ .

So flatness is the same as being locally free.  $\spadesuit$

**Solution 38.** The implications  $i \Rightarrow ii$ ) and  $ii) \Rightarrow iii$ ) follows trivially from the previous exercise. We prove  $iii) \Rightarrow i$ ).

Applying the Tor functor commutes with localization, hence we have  $\text{Tor}_1^A(M, N)_{\mathfrak{m}} = \text{Tor}_1^{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) = 0$  for all  $\mathfrak{m}$ . But being zero is a local property, so it follows that  $\text{Tor}_1^A(M, N) = 0$  for all  $A$ -modules  $N$ . Hence  $M$  is flat.  $\heartsuit$

## 8 The K-book - Charles Weibel

### 8.0.1 Chapter 1.1 - Free modules, GL $_n$ and stably free modules

**Exercise 39** (Semisimple rings). A nonzero  $R$ -module  $M$  is called *simple* if it has no submodules other than 0 and  $M$ , and semisimple if it is a direct sum of simple modules. A ring  $R$  is called *semisimple* if it is a semisimple

$R$ -module. If  $R$  is semisimple, show that  $R$  is a direct sum of a *finite* (say  $n$ ) number of simple modules.

Then use the Jordan-Hölder theorem to show that every stably free module is free. ♠

**Solution 39.** Suppose  $R$  is semisimple, that is  $R = \oplus M_i$  with a priori infinitely many  $M_i$ . But write  $1 = \sum a_i m_i$  with  $m_i \in M_i$  and  $a_i \in R$ . This is a finite sum, and since for any  $r \in R$ , we have  $r = \sum a_i r m_i$ , only finitely many  $M_i$  need occur in the decomposition  $R = \oplus M_i$ .

To see that any stably free module is free if  $R$  is semisimple, suppose  $M \oplus R^n \simeq R^m$ . Then we can write  $M \oplus M_i^n \simeq \oplus M_j^m$  as above, and note that the image of a simple module must be simple, hence the  $M_j$  on the right must be mapped to copies of themselves isomorphically. Hence we can cancel  $M_i$ -terms on both sides until we arrive at  $R^k \simeq M$  for some  $k$ . ♡

**Exercise 40.** Consider the following conditions on a ring  $R$ :

- i)  $R$  satisfies the invariant basis property (IBP).
- ii) For all  $m, n$ , if  $R^m \simeq R^n \oplus P$ , then  $m \leq n$ .
- iii) For all  $n$ , if  $R^n \simeq R^n \oplus P$ , then  $P = 0$ .

Show that  $\text{iii} \Rightarrow \text{ii} \Rightarrow \text{i}$ . ♠

**Solution 40.** Suppose  $R^m \simeq R^n \oplus P$  and suppose  $n > m$ . Then we can write  $R^m \simeq R^m \oplus (R^{n-m} \oplus P)$ . Then from *iii* we must have  $R^{n-m} \oplus P = 0$ , but this is impossible.

Now suppose  $R^n \simeq R^m$ . Then for  $P = 0$ , we have  $R^m \simeq R^n \oplus P$ , hence  $m \leq n$ . But the opposite argument works as well, hence  $m = n$ . ♡

**Exercise 41.** Show that *iii*) in the previous exercise and the following matrix conditions are equivalent:

- a) For all  $n$ , every surjection  $f : R^n \rightarrow R^m$  is an isomorphism.
- b) For all  $n$  and  $f, g \in M_n(R)$ , if  $fg = 1_n$ , then  $gf = 1_n$  and  $g \in \text{GL}_n(R)$ .

Then show that commutative rings satisfy b), hence *iii*). ♠

**Solution 41.** First we see that  $\text{iii} \Rightarrow \text{a)$ . Suppose  $f : R^n \rightarrow R^m$  is a surjection. Then we have an exact sequence

$$0 \rightarrow K \rightarrow R^n \xrightarrow{f} R^m \rightarrow 0.$$

Since  $R^n$  is free, the sequence splits and we have  $R^n \simeq R^n \oplus K$ , but then by assumption  $K = 0$ . Hence  $f$  is an isomorphism.

Now suppose  $a$ ), that is, every surjection is an isomorphism. Next suppose  $R^n \simeq R^n \oplus P$ . Compose with the surjection  $R^n \oplus P \rightarrow R^n$  to get a surjective map  $R^n \rightarrow R^n$ . The kernel of this is  $P$ . But every surjection  $R^n \rightarrow R^n$  is an isomorphism, hence  $P = 0$ .

Now suppose  $iii$ ). The condition  $fg = 1_n$  implies that  $g$  is injective and that we have a splitting  $R^n \simeq R^n \oplus \text{cokerg}$ . But then  $\text{cokerg} = 0$  by  $iii$ ), hence  $g$  is an isomorphism. To show that  $gf = 1_n$ , suppose  $gf(x) = y$ . Applying  $f$  to both sides give  $f(x) = f(y)$ . So we must show that  $f$  is injective. So suppose that  $f(x) = 0$ . Since  $g$  was surjective, we can write  $x = g(y)$  for some  $y$ . Then  $f(g(y)) = x = 0$ . Hence  $f$  is injective. So we have proven  $b$ ).

Now suppose  $b$ ). Then  $b$  implies  $a$ , hence  $iii$ ).

♡

## 8.1 Chapter 1.2 - Projective modules

**Exercise 42** (Radical ideals). Let  $I$  be a radical ideal in  $R$ . If  $P_1, P_2$  are finitely generated projective  $R$ -modules such that  $P_1/IP_1 \simeq P_2/IP_2$ , show that  $P_1 \simeq P_2$ .

♠

**Solution 42.**

♡

## 9 Representation Theory - Fulton, Harris

### 9.1 Representations of Finite Groups

**Exercise 43** (Exercise 1.1). Verify that the relation

$$\langle g \cdot v^*, g \cdot v \rangle = \langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle$$

is satisfied when we define

$$\rho^*(g) = \rho(g^{-1})^t : V^* \rightarrow V^*,$$

that is,  $(\rho^*g)(v^*)(w) = \langle (\rho^*g)(v^*), w \rangle = \langle v^*, (\rho g^{-1})(w) \rangle$ .

♠

**Solution 43.** This is a matter of calculation.

$$\langle gv^*, gv \rangle = \langle v^*, (\rho g^{-1})(gv) \rangle = \langle v^*, v \rangle.$$

So the definition is ok.

♡

**Exercise 44** (Exercise 1.2). Verify that in general the vector space of  $G$ -linear maps between two representations  $V$  and  $W$  of  $G$  is just the subspace  $\text{Hom}(V, W)^G$  of elements of  $\text{Hom}(V, W)$  fixed under the action of  $G$ . This subspace is often denoted  $\text{Hom}_G(V, W)$ . ♠

**Solution 44.** A map  $\varphi : V \rightarrow W$  is  $G$ -linear when  $\varphi(gv) = g\varphi(v)$ . The action of  $G$  on  $\varphi$  is given by  $g\varphi(v) = g\varphi(g^{-1}v)$ . But by  $G$ -linearity, this is

$$\varphi(gv) = gg^{-1}\varphi(gv) = gg^{-1}\varphi(v) = \varphi(v).$$

Hence a map is  $G$ -linear if and only if it is fixed by the action of  $G$ . ♡

**Exercise 45** (Exercise 1.3). Let  $\rho : G \rightarrow \text{GL}(V)$  be any representation of the finite group  $G$  on an  $n$ -dimensional vector space  $V$  and suppose that for any  $g \in G$ , the determinant of  $\rho(g)$  is 1. Show that the spaces  $\wedge^k V$  and  $\wedge^{n-k} V^*$  are isomorphic as representations of  $G$ . ♠

**Solution 45.** This is (again) just a matter of writing out the definitions. First we define the isomorphism, and then we check that it is actually an isomorphism of representations.

$$\bigwedge^k V \rightarrow \bigwedge^{n-k} V^* \\ v_1 \wedge \cdots \wedge v_k \mapsto (w_1 \wedge \cdots \wedge w_{n-k} \mapsto v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_{n-k})$$

Being a map of representations is equivalent to  $g^{-1}\varphi(gv) = \varphi(v)$ , so we just need to check that all the  $g$ 's disappear from the left hand side.

$$\begin{aligned} g^{-1}\varphi(gv) &= g^{-1}(w_1 \cdots w_{n-k} \mapsto gv_1 \cdots gv_k w_1 \cdots w_{n-k}) \\ &= (gv_1 \cdots gv_k gw_1 \cdots gw_{n-k}) \\ &= \det \rho(g) v_1 \wedge \cdots \wedge w_{n-k}. \end{aligned}$$

Hence  $\varphi$  is a map of representations if and only if  $\det \rho(g) = 1$  for all  $g \in G$ .

(it is an isomorphism because it has zero kernel: because what would the kernel be? Every subspace is the same, and this is a basis free description)

♡

**Exercise 46** (Exercise 1.4). The permutation representation  $R$  of  $G$  acting on a finite set  $X$  have two descriptions: one is given by letting  $V$  be the vector space with basis  $\{e_x \mid x \in X\}$  and letting  $g$  act on  $V$  by  $ge_x = e_{gx}$ .

Alternatively  $R$  is the set of functions  $f : X \rightarrow \mathbb{C}$  with action  $(g\alpha)(h) = \alpha(g^{-1}h)$ .

- a) Show that these two descriptions agree by identifying  $e_x$  with the characteristic function which takes the value 1 on  $x$  and 0 elsewhere.
- b) The space of functions on  $G$  can also be made into a  $G$ -module by the rule  $(g\alpha)(h) = \alpha(hg)$ . Show that this is an isomorphic representation.



**Solution 46.** a). Clearly the vector space dimensions agree (since the characteristic functions are a basis). So we need to check that this is a map of representations. Denote the characteristic function by  $\chi_x$ . Then  $\varphi(ge_x)(h) = \varphi(e_{gx})(h) = \chi_{gx}(h)$ . Similarly  $g\varphi(e_x)(h) = g\chi_x(h) = \chi_x(g^{-1}h)$ . The first function is 1 if  $gx = h$ , and the second function is 1 if  $g^{-1}h = x$ , and these are equivalent.

b). Send  $\alpha$  to the function  $g \mapsto \alpha(g^{-1})$ . Call this assignment  $\psi$ . We need to check that  $\psi(g\alpha) = g\psi(\alpha)$ .

First the left hand side. We have:  $\psi(g\alpha)(h) = \psi(h \mapsto \alpha(g^{-1}h))(h) = \alpha(g^{-1}h^{-1})$ .

And similarly:  $g\psi(\alpha)(h) = g(h \mapsto \alpha(h^{-1}))(h) = g\alpha(h^{-1}) = \alpha(g^{-1}h^{-1})$ .

And these are equal.



**Exercise 47** (Exercise 1.10).  $G = S_3$ . Verify that with  $\sigma = (12)$ ,  $\tau = (123)$ , the standard representation has a basis  $\alpha = (\omega, 1, \omega^2)$ ,  $\beta = (1, \omega, \omega^2)$ , with

$$\tau\alpha = \omega\alpha, \quad \tau\beta = \omega^2\beta, \quad \sigma\alpha = \beta, \quad \sigma\beta = \alpha.$$



**Solution 47.** The standard representation  $V$  is the subspace  $\{x_1 + x_2 + x_3 = 0\}$  of  $\mathbb{C}^3$ . Since  $1 + \omega + \omega^2 = 0$ , and  $\alpha \cdot \beta = 3\omega \neq 0$ , these two span  $V$ .

The identities are easy.



**Exercise 48** (Exercise 1.11). Use this approach to find the decomposition of the representations  $\text{Sym}^2 V$  and  $\text{Sym}^3 V$ .



**Solution 48.** The elements  $\{\alpha^2, \alpha\beta, \beta^2\}$  are a basis of  $\text{Sym}^2 V$ , and the eigenvalues are  $\omega^2, 1$  and  $\omega$ , respectively. Thus  $\langle \alpha\beta \rangle$  span a representation isomorphic to  $U$ , the trivial representation, and  $\langle \alpha^2, \beta^2 \rangle$  span a representation isomorphic to  $V$ , the standard representation. Hence  $\text{Sym}^2 V = U \oplus V$ .

The elements  $\{\alpha^3, \alpha^2\beta, \alpha\beta^2, \beta^3\}$  are a basis of  $\text{Sym}^3 V$ . The eigenvalues are  $1, \omega, \omega^2$  and  $1$ , respectively. Looking at the action of  $\sigma = (12)$ , we see that  $U \simeq \langle \alpha^3 + \beta^3 \rangle$ , and  $U' \simeq \langle \alpha^3 - \beta^3 \rangle$ . The remaining  $\langle \alpha^2\beta, \alpha\beta^2 \rangle$  span a representation isomorphic to  $V$ . Hence  $\text{Sym}^3 V = U \oplus U' \oplus V$ .



**Exercise 49** (Exercise 2.2). For  $\text{Sym}^2 V$ , verify that

$$\chi_{\text{Sym}^2 V}(g) = \frac{1}{2} [\chi_V(g)^2 + \chi_V(g^2)].$$

Note that this is compatible with the decomposition  $V \otimes V = \text{Sym}^2 V \oplus \wedge^2 V$ . ♠

**Solution 49.** The eigenvalues of  $g$  acting on  $\text{Sym}^2 V$  are  $\{\lambda_i \lambda_j\}$ . Hence

$$\begin{aligned} \chi_{\text{Sym}^2 V}(g) &= \sum_{i \leq j} \lambda_i \lambda_j \\ &= \sum_{i < j} \lambda_i \lambda_j + \sum_i \lambda_i^2 \\ &= \frac{1}{2} (\chi_V(g)^2 - \chi_V(g^2)) + \chi_V(g^2) \\ &= \frac{1}{2} (\chi_V(g)^2 + \chi_V(g^2)). \end{aligned}$$

♡

**Exercise 50** (Exercise 2.5, The original fixed point formula). If  $V$  is a permutation representation associated to the action of a group  $G$  on a finite set  $X$ , show that  $\chi_V(g)$  is the number of elements fixed by  $g$ . ♠

**Solution 50.** This is easy. The matrix associated to  $g$  is a permutation matrix with a 1 in row  $j$  if element number  $i$  is sent to  $j$ . Then number of fixed points is the number of ones on the diagonal, and this is  $\chi_V(g)$ . ♡

**Exercise 51** (Exercise 2.34). Let  $V, W$  be irreducible representations of  $G$  and  $L_0 : V \rightarrow W$  any linear mapping. Define  $L : V \rightarrow W$  by

$$L(v) = \frac{1}{|G|} \sum_g g^{-1} L_0(gv).$$

Show that  $L = 0$  if  $V$  and  $W$  are not isomorphic, and that  $L$  is multiplication by  $\text{tr}(L_0)/\dim(V)$  if  $V = W$ . ♠

**Solution 51.** We want to apply Schur's lemma. We check that  $L$  is a  $G$ -module homomorphism. We have

$$\begin{aligned} L(hv) &= \frac{1}{|G|} \sum_g g^{-1} L_0(ghv) \\ &= \frac{1}{|G|} \sum_{gh} hgh^{-1} L_0(ghv) \\ &= \frac{1}{|G|} \sum_{g'} hg'^{-1} L_0(g'v) \end{aligned}$$

Hence  $L$  is a  $G$ -module homomorphism. Hence by Schur's lemma,  $L$  is either the zero map or an isomorphism. In particular, if they are not isomorphic,  $L = 0$ . Now suppose  $V = W$ . ♥

## 9.2 Chapter 7 - Lie groups

**Exercise 52** (Exercise 7.11). a) Show that any discrete normal subgroup  $H$  of a connected Lie group  $G$  is in the center  $Z(G)$ .

b) If  $Z(G)$  is discrete, then  $G/Z(G)$  have trivial center. ♠

**Solution 52.** a) We must show that for any  $z \in H$  and any  $g \in G$ , we have  $gz = zg \Leftrightarrow z = gzg^{-1}$ , that is, that  $z$  is fixed by conjugation by all elements of  $g$ . Since  $H$  is normal, we must have  $gzg^{-1} = z'$  for some  $z' \in H$ . Conjugation is a continuous mapping, hence if  $g'$  is close to  $g$ ,  $g'zg'^{-1}$  is close to  $z'$ . But by normality this must still be an element of  $H$ , but  $H$  is discrete, for  $g'$  close enough to  $g$  the only element of  $H$  that can be hit is  $z$ . Hence the mapping  $z \mapsto gzg^{-1}$  is locally constant.  $G$  is connected, so the mapping must be constant, and since  $geg^{-1} = e$ , the mapping must be the identity mapping, hence  $gzg^{-1} = z$  for all  $g \in G$  and  $z \in H$ .

b) Suppose  $a \in Z(G/Z(G))$ . Let  $\pi : G \rightarrow G/Z(G)$  be the quotient map. Let  $Z = \pi^{-1}(Z(G/Z(G)))$ . We want to show that if  $a$  is a representative for  $a$ , then  $a$  lies in  $Z(G)$ . Lying in  $Z(G/Z(G))$  means that  $[a, b] \in Z(G)$  for all  $a \in Z$  and  $b \in G$ . But this implies that the map  $[\cdot, \cdot] : Z \times G \rightarrow G$  lands in  $Z(G)$ , which is discrete, hence the map must be constant, hence by definition,  $a$  lies in  $Z(G)$ . ♥

**Exercise 53** (Exercise 7.12). If  $\varphi : H \rightarrow G$  is a covering of connected Lie groups, show that  $Z(G)$  is discrete if and only if  $Z(H)$  is discrete, and then  $H/Z(H) = G/Z(G)$ . Therefore, if  $Z(G)$  is discrete, the adjoint form of  $G$  exists and is  $G/Z(G)$ . ♠

**Solution 53.** Suppose  $Z(G)$  is discrete, and let  $h \in Z(H)$ . Since  $\varphi$  is a covering, the image  $\varphi(h)$  lies in  $Z(G)$ . Thus, since  $Z(G)$  is discrete, we can find a small neighbourhood around  $\varphi(h)$  such that  $\varphi(h) \cap Z(G) = \{\varphi(h)\}$ . By shrinking the neighbourhood if necessary, it can be shrunk so that  $\varphi$  is a diffeomorphism around  $h$ , hence  $Z(H)$  is discrete as well.

Suppose  $Z(H)$  is discrete. Then the image of any  $h \in Z(H)$  lies in  $Z(G)$  and  $\varphi$  is a local diffeomorphism.

Now for the other part. We note that we have a diagram:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \ker \varphi|_{Z(H)} & \longrightarrow & \ker \varphi & \longrightarrow & \ker \bar{\varphi} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & Z(H) & \longrightarrow & H & \longrightarrow & H/Z(H) \longrightarrow 1 \\
 & & \downarrow & & \downarrow \varphi & & \downarrow \\
 1 & \longrightarrow & Z(G) & \longrightarrow & G & \longrightarrow & G/Z(G) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

The vertical lower maps are all surjective since  $\varphi$  is a covering map and by the proof above. By the previous exercise, we find that  $\ker \varphi|_{Z(H)} = \ker \varphi$ , hence  $H/Z(H) \simeq G/Z(G)$  by the snake lemma (which holds here, see mathoverflow 53124). ♡

### 9.3 Lecture 8 - Lie Algebras and Lie groups

**Exercise 54** (Exercise 8.1). Let  $G$  be a connected Lie group, and  $U \subset G$  any neighbourhood of the identity. Show that  $U$  generates  $G$ . ♠

**Solution 54.** The subgroup generated by  $U$  can be written

$$H = \bigcup_{n \in \mathbb{Z}} U^n,$$

hence  $H$  is an open subgroup. But since  $G \setminus H = \bigcup_{h \notin H} hH$ , we see that any open subgroup is also closed, hence  $H$  is both open and closed, hence  $H = G$ . (this solution was given by Theo Bühler at math.stackexchange). ♡



## 10 Twenty-Four Hours of Local Cohomology

### 10.1 Lecture 1 - Basic notions

**Exercise 55.** Let  $k$  be a finite field.

1. For every point  $p \in k^n$ , construct a polynomial  $f \in k[x_1, \dots, x_n]$  such that  $f(p) = 1$  and  $f(q) = 0$  for all points  $q \in k^n \setminus \{p\}$ .
2. Given a function  $q : k^n \rightarrow k$ , show that there is a polynomial  $f \in k[x_1, \dots, x_n]$  with  $f(p) = g(p)$  for all  $p \in k^n$ .
3. Prove that any subset of  $k^n$  is the zero set of a single polynomial.



**Solution 55.** Suppose that  $\text{char}(k) = p$ . 1. Suppose  $p = (p_1, \dots, p_n)$ . If  $n = 1$ , then  $p = (p_1)$ . Then the polynomial  $f(x) = \prod_{q \neq p} (x - q)$  is zero on all of  $k$ . On the other hand,  $f(p)$  is the product of all non-zero elements in  $k$ . But this must be one, as one sees by grouping inverses together.

This generalizes to  $k^n$  by taking products over products.

2.



## 11 Riemannian geometry - Do Carmo

### 11.1 Chapter 0 - Differentiable manifolds

**Exercise 56** (Exercise 2). Prove that the tangent bundle of a differentiable manifold  $M$  is orientable.



**Solution 56.** Locally the tangent bundle is given by  $\mathbb{R}^n \times \mathbb{R}^n$ , and if  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a transition function between two charts, then the induced transition function on the tangent bundle is given by  $f \times df$ . Hence the differential of the transition map is given by a block diagonal matrix with  $df$  appearing twice. Hence the determinant is  $(\det df)^2 > 0$ , hence  $TM$  is orientable.



**Exercise 57** (Exercise 4). Show that the projective plane  $\mathbb{P}^2(\mathbb{R})$  is non-orientable.



**Solution 57.** From the hint, we see that it is enough to find an open subset of  $\mathbb{P}^2(\mathbb{R})$  that is non-orientable.



**Exercise 58** (Exercise 5 - Embedding of  $P^2(\mathbb{R})$  in  $\mathbb{R}^4$ ). Let  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be given by

$$F(x, y, z) = (x^2 - y^2, xy, xz, yz).$$

Let  $S^2 \subset \mathbb{R}^3$  be the unit sphere. Observe that the restriction  $\varphi = F|_{S^2}$  is such that  $\varphi(p) = \varphi(-p)$ , and consider the mapping  $\tilde{\varphi} : P^2(\mathbb{R}) \rightarrow \mathbb{R}^4$  given by

$$\tilde{\varphi}([p]) = \varphi(p).$$

Prove that a)  $\tilde{\varphi}$  is an immersion and b) that  $\tilde{\varphi}$  is injective. This implies, together with the compactness of  $P^2(\mathbb{R})$  that  $\tilde{\varphi}$  is an embedding. ♠

**Solution 58.** Since  $S^2$  is locally diffeomorphic to  $P(\mathbb{R}^2)$ , it is enough to check that  $\varphi$  is an immersion. We do this on charts. One chart of  $S^2$  is given by

$$(x, y) \mapsto \left( \frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right).$$

In this chart (forgetting the scaling, since by the chain rule, that will only contribute by multiplication by a scalar), the Jacobian look like

$$\begin{pmatrix} 8x & 4y & 6x + 2y^2 - 2 & 4xy \\ -8y & 4x & 4xy & 2x^2 + 6y^2 - 2 \end{pmatrix}.$$

The first minor (the first two columns) is only zero if  $x = y = 0$ , and in that case, the last minor is non-zero. Hence (at least in this chart), the mapping is an immersion.

For b), note the  $xy = ab$  and  $xz = bc$  together imply  $y/z = b/c$  which implies  $yc = bz$ , hence  $y = bc/z$ . Hence  $bc^2 = bz^2$ , hence  $c = \pm z$ . Inserting this into  $xz = ac$  gives  $x = \pm a$ , and similarly  $y = \pm b$ , hence  $\tilde{\varphi}$  is injective. ♡