Components of the Hilbert scheme in $\mathbb{P}^{11}$

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Abstract

As any dedicated reader can clearly see, the Ideal of practical reason is a representation of, as far as I know, the things in themselves; as I have shown elsewhere, the phenomena should only be used as a canon for our understanding. The paralogisms of practical reason are what first give rise to the architectonic of practical reason. As will easily be shown in the next section, reason would thereby be made to contradict, in view of these considerations, the Ideal of practical reason, yet the manifold depends on the phenomena. Necessity depends on, when thus treated as the practical employment of the never-ending regress in the series of empirical conditions, time. Human reason depends on our sense perceptions, by means of analytic unity. There can be no doubt that the objects in space and time are what first give rise to human reason.

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Notation

If V is a vector space, we denote by $\mathbb{P}(V)$ its projectivisation.

CHAPTER 1

Preliminaries

ec:prelims

In this chapter we introduce the consistent notation and results which will be used later.

1.1 Notation

We write k for a field, which is almost always assumed to be $\mathbb{C}$. If X is a projective variety, we write $S(X)$ for its homogeneous coordinate ring (if the embedding is implicit). All schemes are noetherian.

1.2 Projective geometry constructions

There are many ways to define the join of two projective varieties X and Y. We will define it in a particularly general way, as described by Altman and Kleiman in [AK75]. Then we will specialize to our situation.

Fix a base scheme S. Let $\mathscr{C}$ be the category of graded, quasi-coherent $\mathscr{O}_S$-algebras, generated in degree 1. The tensor product of two $\mathscr{O}_S$-algebras $\mathscr{R}$ and $\mathscr{S}$ is naturally graded: the degree d part is given by

$$(\mathscr{R} \otimes_{\mathscr{O}_S} \mathscr{S})_d = \bigoplus_{p+q=d} \mathscr{R}_p \otimes \mathscr{S}_q.$$

Then we define the *join* of the graded $\mathscr{O}_S$-algebras to be

$$X * Y = \operatorname{Proj}(\mathscr{R} \otimes_{\mathscr{O}_S} \mathscr{S}).$$

If X and Y are projective varieties over S, they come with graded $\mathscr{O}_S$-algebras $\mathscr{R} = \operatorname{Sym} \mathscr{O}_X(1)$ and $\mathscr{S} = \operatorname{Sym} \mathscr{O}_Y(1)$. Then we define their join to be Proj of these algebras.

Example 1.2.1. Let $X = \mathbb{P}(E)$ and $Y = \mathbb{P}(F)$, where E, F are quasi-coherent $\mathscr{O}_S$-modules. Then we have the equality $\mathbb{P}(E) * \mathbb{P}(F) = \mathbb{P}(E \oplus F)$, because of the linear algebra fact $\operatorname{Sym}(E) \otimes \operatorname{Sym}(F) = \operatorname{Sym}(E \oplus F)$.

The algebra $\mathscr{R} \otimes_{\mathscr{O}_S} \mathscr{S}$ contains the ideal $\mathscr{R} \otimes \mathscr{S}_+$. The associated subscheme is denoted by V_X, and is isomorphic to $X = \operatorname{Proj} \mathscr{R}$. We define V_Y similarly.

There is a geometric definition of the join, as described in setion (C11) in [AK75]. Let E, F be quasi-coherent $\mathscr{O}_S$-modules[1]. Suppose X, Y are subschemes of $\mathbb{P}(E)$ and $\mathbb{P}(F)$. Then $X * Y$ is a closed subscheme of $\mathbb{P}(E \oplus F)$. Identify X and Y with their fundamental subschemes in $X * Y$. Then it is not difficult to see that $X * Y$ is the (closure of the) locus of points lying on the lines of $\mathbb{P}(E \oplus F)$ determined by pairs of points of X and Y.

For a graded $\mathscr{O}_S$-algebra $\mathscr{R}$ there are two natural homomorphisms. We have the *structure map* $\rho : \mathscr{O}_S \to \mathscr{R}$, and the *augmentation map* $\epsilon : \mathscr{R} \to \mathscr{O}_S$, sending everything in positive degree to zero. Clearly $\epsilon \circ \rho = \mathrm{id}_S$.

lemma:join **Proposition 1.2.2.** *Suppose $X/k \subset \mathbb{P}^n$ and $Y/k \subset \mathbb{P}^m$ are smooth projective schemes. Then their join, $X * Y$ have dimension $\dim X + \dim Y + 1$. The singular locus is of dimension $\max\{a, b\}$ and consist of the disjoint union of X and Y.*

Proof. Let $\mathscr{R} = \oplus_{d \geq 0} \mathscr{O}_X(d)$ and $\mathscr{S} = \oplus_{d \geq 0} \mathscr{O}_Y(d)$ be the homogeneous coordinate rings of X and Y. Then $X * Y \subset \mathbb{P}^{n+m+1}$.

Denote by $C(X * Y)$ the scheme $\operatorname{Spec}(\mathscr{R} \otimes \mathscr{S})$, the affine cone over $X * Y$.

The singular locus of $C(X * Y) = C(X) \times C(Y)$ is equal to $\operatorname{Sing} C(X) \times C(Y) \bigcup C(X) \times \operatorname{Sing} C(Y)$. Since X and Y are smooth, the only singular point on the affine cones are the origins. Hence

$$\operatorname{Sing}(C(X * Y)) = 0 \times \operatorname{Sing}(C(Y)) \cup \operatorname{Sing}(C(X)) \times 0.$$

Projectivising, we find that $\operatorname{Sing}(X * Y) = \operatorname{Sing} Y \sqcup \operatorname{Sing} X$, since $(0, \ldots, 0)$ is the only common point of the affine cones. ■

Recall that a scheme X is *Gorenstein* if it has a dualizing sheaf. It is *Cohen–Macaulay* if the dualizing sheaf is a line bundle. If S is the homogeneous coordinate ring of a projective Gorenstein variety $X \subset \mathbb{P}^n$, the canonical sheaf can be computed as the sheaf associated to the graded module $\operatorname{Ext}^{\operatorname{codim} X}(S, S(-n-1))$.

If X and Y are two Gorenstein schemes, then their join is also Gorenstein. Furthermore, we can compute the canonical sheaf in terms of the canonical sheaves of X and Y.

Proposition 1.2.3. *Let $X = \operatorname{Proj} \mathscr{R}$ and $Y = \operatorname{Proj} \mathscr{S}$ be Gorenstein projective schemes with dualizing sheaves ω_X, ω_Y, respectively (here $\mathscr{R}, \mathscr{S}$ are sheaves of graded $\mathscr{O}_S$-algebras). Then $X * Y$ is Gorenstein with dualizing sheaf $\omega_X \otimes_S \omega_Y$.*

Proof. The question is local on S, so we may assume that R and S are homogeneous coordinate rings. Then the statement follows from Theorem 4.2

[1] In our case, $S = \operatorname{Spec} k$ almost always. So E, F are just vector spaces.

in [HHS16], where the authors prove that the canonical module of a tensor product is the tensor product of the canonical modules. ■

:canonical

Remark 1.2.4. If X and Y are Gorenstein projective schemes, the resolution of the structure sheaf is symmetrical. It follows that $\omega_X = \mathscr{O}_X(-n)$ for some $n \geq 0$. If $\omega_Y = \mathscr{O}_Y(-m)$, it follows from the above proposition that $\omega_{X*Y}(-m-n)$.

1.3 Calabi–Yau manifolds and mirror symmetry

calabi_yau

Parts of this thesis will be concerned with the construction of Calabi–Yau manifolds. For us, a Calabi-Yau manifold will be a irreducible complex projective variety X such that $\omega_X \simeq \mathscr{O}_X$ and $H^i(X, \mathscr{O}_X) = 0$ for $i = 1, \dots, \dim X - 1$. We usually have $\dim X = 3$. Be ware that the literature often requires Calabi–Yau manifolds to be smooth.

Mathematically, Calabi–Yau varieties are interesting because they are among the varities having Kodaira dimension zero. This means that they have trivial canonical models, making them harder to study. Before the 90's there were not many explicit constructions of Calabi–Yau varieties, but after the advent of toric geometry and the construction of Batyrev in [Bat94], thousands of new examples were found, all of which was anticanonical sections of Fano toric varieties. check

Let Ω_X be the sheaf of holomorphic one-forms on X, and assume that $\dim X = 3$. Then we can form the *Hodge diamond* of X, which is a format of writing the dimensions h^{ij} of the cohomology groups $H^j(X, \Omega^i_X)$.

$$
\begin{array}{ccccccc}
 & & & h^{00} & & & \\
 & & h^{01} & & h^{10} & & \\
 & h^{02} & & h^{11} & & h^{20} & \\
h^{03} & & h^{12} & & h^{21} & & h^{30} \\
 & h^{13} & & h^{22} & & h^{31} & \\
 & & h^{23} & & h^{32} & & \\
 & & & h^{33} & & &
\end{array}
$$

Because of the Calabi–Yau condition, we have that $h^{j0} = 0$ for $0 < j < 3$, and also that $h^{00} = h^{0d} = 1$. It follows by Serre duality (see [Har77, Corollary 7.7, Chapter III]) that $h^{ij} = h^{3-i,3-j}$. Note that this amounts to a horizontal symmetry of the Hodge diamond. Since X was assumed to be a complex manifold, it follows that $h^{ij} = h^{ji}$ by complex conjugation[2]. This amounts to vertical symmetry of the Hodge diamond. It follows that for 3-dimensional

[2]This follows by the $\bar{\delta}$-Poincaré lemma.

Calabi–Yau varieties, the Hodge diamond simplifies to

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{11} & & 0 & \\ 1 & & h^{12} & & h^{12} & & 1 \\ & 0 & & h^{11} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}.$$

The *Hodge decomposition* theorem states that the singular cohomology groups decomposes as

$$H^k(X, \mathbb{C}) = \bigoplus_{i+j=k}^{\dim X} H^i(\Omega_X^j).$$

The *topological Euler characteristic* is defined as

$$\chi(X) = \sum_{k=0}^{2\dim X} \dim_{\mathbb{C}} H^k(X, \mathbb{C}).$$

For 3-dimensional Calabi–Yau varieties, it follows from the above discussion that $\chi(X)$ can be computed as $2(h^{11} - h^{12})$.

Example 1.3.1. The canonical example of a Calabi–Yau variety is the quintic in $\mathbb{P}^4$. Let $X = V(f)$ be defined by a general element in $H^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(5))$. Then X is smooth, and by adjunction formula we have $\omega_X = \omega_{\mathbb{P}^4}\big|_X \otimes \mathscr{O}_X(5) = \mathscr{O}_X(-5) \otimes \mathscr{O}_X(5) = \mathscr{O}_X$, so the canonical bundle is trivial. By the ideal sheaf sequence, we find that $H^i(X, \mathscr{O}_X) \simeq H^i(X, \mathscr{O}_{\mathbb{P}^4}(-5))$, which by [Har77, Theorem 5.1, Chapter III] implies the required vanishing of the structure sheaf cohomology groups.

The Euler characteristic can be computed as the degree of the top Chern class of X. If Y is a degree d hypersurface in $\mathbb{P}^n$, the following formula holds:

$$c_{n-1}(T_X) = h^{n-1}\left(\binom{n+1}{n-1} - d\binom{n+1}{n-2} + d^2\binom{n+1}{n-3} + \dots\right).$$

Putting $n = 4$ and $d = 5$, we find that $\chi(X) = -200$. To compute h^{11}, we see first from the conormal sequence that $H^1(\Omega_Y^1) \simeq H^1(\Omega_{\mathbb{P}^4}^1\big|_Y)$. Now it follows easily from the restricted Euler sequence that $h^{11} = 1$, and since $\chi(X) = 2(h^{11} - h^{12})$ it follows that $h^{12} = 101$.

In general it is very hard to compute the Hodge numbers of Calabi–Yau varieties, with the exception of hypersurfaces in four-dimensional toric varieties. Often the best one can hope for is the topological Euler characteristic $\chi(X)$, which is much easier to compute.

After the invention of string theory, Calabi–Yau varieties caught the attention of physicists. They discovered a duality between different Calabi–Yau 3-manifolds X and X° such that their Hodge numbers satisfy $h^{11}(X) = h^{12}(X^\circ)$ and $h^{12}(X) = h^{11}(X^\circ)$ (in particular $\chi(X) = -\chi(X^\circ)$). This a conjectural correspondence between the complex moduli space of X (which have tangent space $H^1(X, \Omega^2_X)$, and the Kähler moduli space of ω°, the Kähler class on X°. This correspondence is called *mirror symmetry*.

The last chapter of this thesis is concerned about the construction of new examples of Calabi–Yau manifold. They are all complete intersections in toric varieties .

but they do *not* correspond to Minkowski decompoistions?

- Fano -> Calabi–Yau

1.4 Toric geometry

c_geometry

A toric variety is an irreducible and normal variety containing the torus $T = (\mathbb{C}^*)^n$ as a dense subset, such that the action of the torus on itself extends to an action on the variety.

We fix some notation that will be used throughout. Details and proofs can be found in [CLS11; Ful93]. Each toric variety comes with two dual lattices. The *lattice of 1-parameter subgroups* N and the *character lattice* M. A one-parameter subgroup is a morphism $\lambda : \mathbb{C}^* \to T$ that is a group homomorphism. The set of one-parameter subgroups is a lattice isomorphic to $\mathbb{Z}^n$, and we denote it by N. A *character* is a morphism $\chi : T \to \mathbb{C}^*$ that is a group homomorphism. The set of characters is a lattice M isomorphic to $\mathbb{Z}^n$ which is naturally dual to N.

Let V be a $\mathbb{R}$-vector space. Let $V^\vee$ be the dual vector space. A *convex polyhedral cone* is a subset σ of V of the form

$$\sigma = \{r_1 v_1 + \ldots + r_s v_s \mid r_i \geq 0 \text{ for all } i\},$$

where the v_i is a finite set of vectors in V and the r_i are real numbers. A *rational polyhedral cone* is a cone such that the vectors v_i can be taken to have rational coordinates.

The *dual cone* $\sigma^\vee$ lives in $V^\vee$, and is defined as the set of functionals that are positive on σ:

$$\sigma^\vee = \{u \in V^\vee \mid \langle u, v \rangle \geq 0, v \in \sigma\}.$$

Cones have two descriptions: either as the positive hull of a finite set of vectors (as above), or implicitly, as the intersection of finitely many half-spaces. If u_i generate $\sigma^\vee$, then it is true that

$$\sigma = \{v \in V \mid \langle u_i, v \rangle \geq 0 \text{ for all } i\}.$$

The vectors u_i are the inner normal vectors of the facets of σ.

A *(commutative) semigroup* is a set S with a commutative binary operation $S \times S \to S$, together with an identiy element $0 \in S$. Given a cone $\sigma \subset N$, we can form a semigroup $S = \sigma^\vee \cap M \subset M$. Given a semigroup S, we can form the *semigroup algebra* $\mathbb{C}[S]$: it is the algebra generated by the elements of S, with multiplicative structure inherited from S. We then define U_σ as $\operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$, and call it the *affine toric variety associated to* σ.

Example 1.4.1. Let $\sigma = \langle (1,0),(1,2) \rangle \subset \mathbb{R}^2$. Then $\sigma^\vee = \langle (2,-1),(1,0),(0,1) \rangle \subset \mathbb{R}^2$. Then $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[x,y,x^2/y]$, where we have identified x and y with the standard basis of $\mathbb{R}^2$. This ring is isomorphic to $\mathbb{C}[a,b,c]/(a^2 - bc)$.

illustration

General toric varities are described using collections of cones called *fans*. A set Σ of cones is called a *fan* if it closed under intersections and faces of cones: if $\sigma, \sigma' \in \Sigma$, then we should also have $\sigma \cap \sigma' \in \Sigma$, and if $\sigma' \subset \sigma$ is a face with $\sigma \in \Sigma$, then $\sigma' \in \Sigma$ also. Thus, given a fan Σ, we get a collection of affine toric varieties U_σ for each cone $\sigma \in \Sigma$. We have inclusions $U_{\sigma \cap \sigma'} \subset U_\sigma$, and using these inclusions one glue to get a separated toric variety. If the fan is *complete* (meaning that the union of its cones is equal to N), it follows that the corresponding toric variety is complete.

Example 1.4.2.

fan of P2 and its coordinate rings

Projective toric varieties can be constructed from lattice polytopes. We describe the procedure here. Let Δ be a lattice polytope in $M \simeq \mathbb{Z}^n$. Let $M' = M \oplus \mathbb{Z}$, and embed Δ in M' by sending v to $(v,1)$. Let $C(\Delta)$ be the cone over Δ in M'. Then $\mathbb{C}[C(\Delta) \cap M']$ is a $\mathbb{Z}$-graded algebra. We let X_Δ be the associated projective variety.

- Construction of CY varities in toric varities.

Divisors and Picard groups of toric varieties

Recall that a *Weil divisor* is a formal linear combination of codimension 1 subvarieties of a scheme X (satisfying condition "star" in Hartshorne). The group of Weil divisors modulo linear equivalence is the *class group* of X, and is denoted by $\operatorname{Cl}(X)$. The group of line bundles modulo isomorpism is the *Picard group* of X, and is denoted by $\operatorname{Pic}(X)$. The two groups coincide for smooth varieties. They are in general very hard to compute, but for toric varieties the computation is exceptionally easy, relying only on structure of the rays in the fan Σ defining the toric variety.

For completeness, we describe the divisors on toric varities. The description will be used in Chapter 3, where we work out the geometry of the two smoothings of the affine cone over the del Pezzo surface of degree 6.

Let X be a smooth toric variety, and let $\Sigma(1)$ denote the set of one-dimensional cones (called *rays*) in the fan Σ defining X. For each ray ρ, let $u_\rho \in N$ denote the primitive ray generator of ρ. Then one can show that the torus-invariant divisors on X are in one to one correspondence with the rays $\rho \in \Sigma(1)$. Furthermore, every divisor on X is linearly equivalent to a torus-invariant divisor. Using these two facts, one can prove the following:

There is an exact sequence:

$$0 \longrightarrow M \xrightarrow{C} \mathbb{Z}^{\Sigma(1)} \longrightarrow \operatorname{Pic}(X) \longrightarrow 0,$$

where the rows of the matrix C are the vectors u_ρ. See [CLS11], Chapter 4, for a proof.

There is also a description of the Cartier divisors on X in terms of *support functions* on N: a support function is a function $\varphi : |\Sigma| \to \mathbb{R}$ such that the restriction $\varphi\big|_\sigma$ of φ to each cone in Σ is linear. A support function is *integral with respect to* N if $\varphi(|\Sigma| \cap N) \subset \mathbb{Z}$. This means that for each cone σ, there is an $m_\sigma \in M$, such that $\varphi(v) = \langle v, m_\sigma \rangle$ if $v \in \sigma$.

The set of support functions is an abelian group under addition, and by Theorem 4.2.12 in [CLS11], there is an isomorphism between the group of integral support functions on Σ and the torus invariant Cartier divisors on X.

describe support functions

1.5 Deformation theory and the Hilbert scheme

Given a scheme X_0 over $\mathbb{C}$, a *family of deformations* of X_0 is a flat morphism $\pi : \mathscr{X} \to (S, 0)$ with S connected such that $\pi^{-1}(0) = X_0$. If S is the spectrum of an artinian $\mathbb{C}$-algebra, then π is an *infinitesimal deformation.* If $S = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$, then π is a *first order deformation.* An *embedded deformation* is a deformation such that the total space is contained in $S \times \mathbb{P}^M$ for some M. A *smoothing of* X_0 is a deformation such that the general fiber is smooth. A reference for deformation theory is the book by Hartshorne [Har10].

Let $\operatorname{Def}(X, S)$ denote the space of all deformations of the projective scheme $X \subset \mathbb{P}^n$ over the base scheme S, and let $\operatorname{EmbDef}(X, S)$ denote the space of all embedded deformations of X over S. There is a natural forgetful map $\operatorname{EmbDef}(X, S) \to \operatorname{Def}(X, S)$. The tangent space of $\operatorname{EmbDef}(X, S)$ at the point corresponding to X is given by $H^0(X, \mathcal{N}_{X/\mathbb{P}^n})$, and the tangent space of $\operatorname{Def}(X)$ is by definition given by $\operatorname{Def}(X) = \operatorname{Def}(k[\epsilon]) = T^1$

write this in a good way

- Normal sheaf classifies embedded deformations
- T^1 ()

relation to moduli spaces and singularity theory

1.6 Stanley–Reisner schemes

Stanley–Reisner schemes are certain degenerate projective schemes modelled on simplicial complexes. We first recall some notation. Let $[n]$ denote the set of numbers $\{0, \ldots, n\}$. The power set of $[n]$ is called the *n-simplex* and denoted by Δ_n. A *simplical complex* is a subset $\mathcal{K} \subseteq \Delta_n$ (for some n), such that if $f \in \mathcal{K}$ and $g \subseteq f$, then $g \in \mathcal{K}$. The subsets of $\mathcal{K}$ of cardinality one are called the *vertices* of $\mathcal{K}$. The subsets of $\mathcal{K}$ are called *faces*. The *dimension* of a face f is equal to $|f| - 1$. A good reference is Stanley's green book [Sta96].

Let k be a field, and let $P_{\mathcal{K}}$ be the polynomial ring over k with variables indexed by the vertices of $\mathcal{K}$. Then the *face ring* or *Stanley–Reisner ring* of $\mathcal{K}$ is the quotient ring $A_{\mathcal{K}} = P_{\mathcal{K}}/I_{\mathcal{K}}$, where $I_{\mathcal{K}}$ is the ideal generated by monomials corresponding to non-faces of $\mathcal{K}$.

Example 1.6.1. Let $\mathcal{K}$ be the triangle with vertices $\{v_1, v_2, v_3\}$. Its maximal faces are v_1v_2, v_2v_3 and v_1v_3. The Stanley–Reisner ring is $k[v_1, v_2, v_3]/(v_1v_2v_3)$.

The ideal $I_{\mathcal{K}}$ is graded since it is defined by monomials. This leads us to define the *Stanley–Reisner scheme* $\mathbb{P}(\mathcal{K})$ as $\operatorname{Proj} A_{\mathcal{K}}$.

The *join* of two simplicial complexes $\mathcal{K}$ and $\mathcal{K}'$ is defined as

$$\mathcal{K} * \mathcal{K}' \triangleq \{f \sqcup g \mid f \in \mathcal{K}, g \in \mathcal{K}'\},$$

where $\sqcup$ denotes the disjoint union. Note that $\mathbb{P}(\mathcal{K} * \mathcal{K}') = \mathbb{P}(\mathcal{K}) * \mathbb{P}(\mathcal{K}')$, where the second star means the join of two projective varieties.

If $f \subset \mathcal{K}$ is a face, the *link of f in* $\mathcal{K}$ is the simplicial complex defined by

$$\operatorname{lk}(f, \mathcal{K}) \triangleq \{g \in \mathcal{K} \mid f \cap g = \emptyset, f \cup g \in \mathcal{K}\}.$$

If $D_+(x_f) \subset \mathbb{P}(\mathcal{K})$ denotes the distinguished open set corresponding to the monomial x^f, we have that $D_+(x_f) = \mathbb{A}(\operatorname{lk}(f, \mathcal{K})) \times (k^*)^{\dim f}$.

Every simplicial complex has a *geometric realization*, which as a set is defined as follows:

$$|\mathcal{K}| \triangleq \{\alpha : [n] \to [0,1] \mid \operatorname{supp}(\alpha) \in \mathcal{K}, \sum_{i=1}^{n} \alpha(i) = 1\}.$$

This is an example of a piecewise linear manifold, or a PL-manifold for short. For more on PL-manifolds or combinatorial topology, we refer the reader to [Gla70; Spa66; Hud69].

We will need the following result of Christophersen ([AC10, Theorem 4.6]):

thm:t1dims **Theorem 1.6.2.** *If $\mathcal{K}$ is a simplicial manifold, and $\mathbf{c} = \mathbf{a} - \mathbf{b}$ (with disjoint supports a and b), then*

$$\dim_k T^1_{A_{\mathcal{K}}, \mathbf{c}} = \begin{cases} 1 & \text{if } a \in \mathcal{K} \text{ and } b \in \mathcal{B}(\operatorname{lk}(a, \mathcal{K})) \\ 0 & \text{otherwise.} \end{cases}$$

Here $\mathcal{B}(\mathcal{K})$ is defined as follows:

Definition 1.6.3. The set $\mathcal{B}(\mathcal{K})$ is the set of $b \subset \mathcal{K}$ with $|b| \geq 2$ such that

1. $\mathcal{K} = L * \partial b$, wehere $|L|$ is a $(n - |b| + 1)$-sphere, if $b \notin \mathcal{K}$.
2. $\mathcal{K} = L * \partial b \cup \partial L * \bar{b}$ where $|L|$ is a $(n - |b| + 1)$-ball, if $b \in \mathcal{K}$.

fill in as needed

1.7 Smoothings of Stanley–Reisner schemes

Because many properties of varieties are easier read off their degenerations, it is an interesting problem to study smoothings of Stanley–Reisner-schemes, which are highly singular. Below are the two main lemmas motivating the study of triangulations in deformation theory.

ma:srcohom **Lemma 1.7.1.** *If $\mathcal{K}$ is a simplicial complex, then $H^i(\mathcal{K}; k) \simeq H^i(\mathbb{P}(\mathcal{K}), \mathscr{O}_{\mathbb{P}(\mathcal{K})})$.*

The lemma is essentially due to Hochster, and is proved (in a different form) in Stanley's book [Sta96]. This is true essentially because the Čech complex computing the simplicial cohomology and the Čech complex computing sheaf cohomology look exactly the same.

Lemma 1.7.2. *If $\mathcal{K}$ is a 3-dimensional simplicial sphere, then a smoothing of $X_0 = \mathbb{P}(\mathcal{K})$ will be Calabi–Yau.*

Proof. Let $\pi : \mathscr{X} \to S$ be a smoothing. Since $\mathcal{K}$ is a sphere, it follows from 1.7.1 that $H^i(X_0, \mathscr{O}_{X_0}) = k$ for $i = 0, 3$, and zero for $i \neq 0, 3$. The triviality of the canonical bundle is proved in Theorem 6.1 in [BE91]. Since $H^1(\mathcal{K}; K) = H^2(\mathcal{K}; K) = 0$, it follows from the semicontinuity theorem (Theorem 12.8 in Chapter III in [Har77]) that $H^i(X_t, \mathscr{O}_{X_t}) = 0$ for all $t \in S$. Similarly, if ω_0 all nearby fibers must have trivial canonical bundle as well. ∎

something about polyhedral complexes and face rings?

1.8 Toric geometry and toric degenerations

There is a correspondence between certain degenerations of toric varieties and so-called unimodular triangulations.

define these

Let M be a lattice (by which we mean a free abelian group of finite rank). Let $\nabla \subset M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ be a lattice polytope, and let S_∇ be the semigroup in $M \times \mathbb{Z}$ generated by the elements $(u, 1) \in \nabla \cap M$. Then we define $\mathbb{P}(\nabla) = \operatorname{Proj} \mathbb{C}[S_\nabla]$, and call it the *toric variety associated to* ∇.

By Theorem 8.3 and Corollary 8.9 in [Stu96], there is a one-one correspondence between unimodular regular triangulations of ∇ and the square-free initial ideals of the toric ideal of $\mathbb{P}(\nabla)$.

ec:toricgeometry

join of reflexive polytopes is reflexive!?

imodular_triangs

Proposition 1.8.1. *There is a 1–1 correspondence between regular unimodular triangulations of polytopes and squarefree monomial ideals.*

By Theorem 8.3 and Corollary 8.9 in [Stu96]......

Thus, if we can find a polytope whose boundary has a regular unimodular triangulation, we know that the associated Stanley-Reisner ring has at least one deformation.

reflexive polytopes, mirror symmetry, anticanonical sections

CHAPTER 2

Relation to triangulations of $\mathbb{CP}^2$

cp2triangs

This chapter will not contain any new results of any signifance, but is rather a report on an idea which led to the deliberations in the later chapters.

We explain a connection between the topological space $\mathbb{CP}^2$ and hyper-Kähler manifolds.

2.1 Introductory remarks

Hyper-Kähler manifolds

_manifolds

Among the known families of manifolds, hyper-Kähler manifolds are among the most elusive. One often divides manifolds into three types: those with positive, negative or trivial canonical class. Of those with trivial canonical class, two prominent types stand out: Calab–Yau-manifolds and hyper-Kähler manifolds.

Definition 2.1.1. A *hyper-Kähler manifold* X is a simply connected compact Kähler manifold such that $H^0(X, \Omega_X^2)$ is generated by a non-degenerate $\sigma : TX \times TX \to \mathbb{C}$.

Remark 2.1.2. Since the two-form σ is non-degenerate, it follows that the canonical sheaf $\omega_X = \Omega^n_{X/\mathbb{C}}$ is trivial. The map $1 \mapsto \sigma^{n/2}$ gives an isomorphism $\mathscr{O}_X \to \omega_X$.

For example, in dimension two, K3 surfaces are hyper-Kähler (*and* Calab–Yau). Because of the non-degeneracy of the symplectic form $\sigma \in H^0(X, \Omega_X^2)$, hyper-Kähler manifolds only occur in even dimensions. Only a few explicit families of hyper-Kähler manifolds are known. Below we sketch the construction of one such family.

Let S be a K3-surface with symplectic form σ, and let $S^{(2)}$ be its symmetric square: $S \times S/\{(p,q) \sim (q,p)\}$. Let $\pi_i : S \times S \to S$ be the two projections ($i = 1, 2$). Then the 2-form $\pi_1^*\sigma + \pi_2^*\sigma$ is $\mathbb{Z}/2$-invariant, hence it decends to a 2-form τ on $S^{(2)}$.

The space $S^{(2)}$ is singular along the diagonal: locally it is isomorphic to $\mathbb{C} \times \mathbb{C}/(x \sim -x)$. The last factor is a quadric cone, so a single blowup along the diagonal will resolve the singularities. The form τ lifts to a non-degenerate form on $S^{[2]}$, and it can be shown that it is in fact a hyper-Kähler variety of dimension 4. The resulting space is denoted by $S^{[2]}$, and is called the *Hilbert square of* S, or the *Hilbert scheme of two points on* S. It parametrizes length two subschemes of S.

For more details on this construction, see Beauville's original paper [Bea83].

The variety of lines on a cubic fourfold

There is another construction of hyper-Kähler varieties that is interesting to us. Let X be a smooth cubic fourfold in $\mathbb{P}^5$. Let $F(X)$ denote the set of lines contained in X. It is the *Fano variety of lines on* X, and is a closed subset of the Grassmannian $\mathbb{G}(1, \mathbb{P}^5)$. One can can show that $F(X)$ is a hyper-Kähler variety of dimension 4.

In the article [BD85], Beauville and Donagi shows that $F(X)$ is deformation equivalent to $S^{[2]}$ for some K3 surface S. They also show that if X is a *pfaffian* hypersurface, then $F(X)$ is actually *isomorphic* to $S^{[2]}$ for some K3 surface S. Furthermore, the family $\{F(X)\}$ obtained this way is 19-dimensional, and is a hypersurface in the deformation space of $S^{[2]}$.

2.2 Connection to the complex projective plane

Let X be a topological space. Recall that the symmetric product $X^{(2)}$ is defined as follows:

$$X * Y = X \times Y/\{(x, y) \sim (y, x)\}.$$

If $X = S^2$, we have that $X^{(2)}$ is naturally isomorphic to $\mathbb{CP}^2$, which can be seen as follows: S^2 can be identified with $\mathbb{P}^1_{\mathbb{C}}$. Unordered pairs of points in $\mathbb{P}^1$ correspond to degree 2 polynomials up to scalar multiplication. Hence we have identifications

$$(S^2)^{(2)} = (\mathbb{P}^1)^{(2)} = \{(P, Q) \in \mathbb{P}^1 \times \mathbb{P}^1\}/\mathbb{Z}_2 = \mathbb{P}\left(H^0(\mathscr{O}_{\mathbb{P}^1}(2))\right) = \mathbb{CP}^2.$$

Stanley–Reisner degenerations of K3 surfaces correspond to triangulated spheres. Since the symmetric square of a sphere is $\mathbb{CP}^2$, a Stanley-Reisner degeneration of the symmetric square of a K3 surface should correspond to a triangulated $\mathbb{CP}^2$.

Proposition 2.2.1. *Suppose $\mathcal{K}$ is a triangulation of $\mathbb{CP}^2$ and $X_0 = \mathbb{P}(\mathcal{K})$ is its associated Stanley–Reisner-scheme. Then a smoothing X of X_0 will be a hyper-Kähler manifold.*

Proof. The dimensions of the groups $H^i(X, \mathscr{O}_X)$ are constant in flat families. Since $H^0(X, \Omega_X^2) = H^2(X, \mathscr{O}_X) = H^2(\mathcal{K}; \mathbb{C}) = \mathbb{C}$ (the first equality is complex conjugation), we have that $H^0(X, \Omega_X^2)$ is generated by a single 2-form. It is non-degenerate since $\omega_X \simeq \mathscr{O}_X$. It follows that X is hyper-Kähler. ■

hva med simply-connected?

2.3 Attempt to smooth triangulations

If $\mathcal{K}$ is a triangulation of $\mathbb{CP}^2$ and $\mathbb{P}(\mathcal{K})$ is the associated Stanley–Reisner-scheme, a smoothing of $\mathbb{P}(\mathcal{K})$ will give a hyper-Kähler variety. Using this idea, and the `Macaulay2` package `VersalDeformations` (by Nathan Ilten, see [Ilt12]), we tried to find potentially new hyper-Kähler varieties. Unfortunaly, it looks like all the triangulations we experimented with were non-smoothable.

In the next four subsections we describe four different triangulations of $\mathbb{CP}^2$, and also their deformation theory using the results of [AC10]. In all cases we conclude that the corresponding Stanley–Reisner scheme is probably not smoothable.

Before we go on to describe the triangulations, we recall some basic facts about combinatorial manifolds.

We can decompose $\mathbb{CP}^2$ into three four-dimensional closed balls B_j, whose pairwise intersections are solid tori Π_{ij}, and whose triple intersection is a two-dimensional torus T. The closed balls B_0 is defined as

$$B_0 = \{[x_0 : x_1 : x_2] \in \mathbb{CP}^2 \mid x_0\overline{x_0} \geq x_1\overline{x_1},\, x_0\overline{x_0} \geq x_2\overline{x_2}\},$$

and similarly for B_1 and B_2. This is sometimes called the *equilibrium decomposition* of the complex projective plane.

A triangulation of $\mathbb{CP}^2$ is *equilibrium* if the closed balls, the solid tori, and the torus T are subcomplexes of the triangulation. Several of the triangulation below are equilibrium.

1. Kühnels 9-vertex
2. The equilibrium triangulation (10 vertices)
3. Bagci/Datta
4. 15-vertex triang

The 15-vertex triangulation

A very interesting triangulation $\mathcal{T}$ of $\mathbb{CP}^2$ is found in [Gai09]. The author describes a triangulation of $\mathbb{CP}^2$ using 15 vertices. One reason it is interesting is that it the corresponding Stanley–Reisner scheme $\mathbb{P}(\mathcal{T})$ has the same Hilbert-polynomial as $F_1(X)$, the Fano variety of lines on a cubic hypersurface. This means that they live in the *same* Hilbert scheme, and one could naively hope that they live in the same component as well.

We will spend some time describing this tringulation, since parts of it inspired our the construction of the Calabi–Yau's in the last chapter. We cite the definition ad verbatim from [Gai09].

Definition 2.3.1. Let $V_4 \subset S_4$ be the Klein four group. The vertex set of $\mathcal{T}$ is defined as

$$V = (V_4 \backslash \{e\}) \sqcup (\{1,2,3,4\} \times \{1,2,3\})\,. \tag{2.1}$$

Thus the vertices of $\mathcal{T}$ are the permutations (12)(34),(13)(24) and (14)(23) and the pairs of integers (a,b) with $1 \leq a \leq 4$ and $1 \leq b \leq 3$. The maximal faces are spanned by the sets

$$\nu, (1,b_1), (2,b_2), (3,b_3), (4,b_4) \tag{2.2}$$

with $\nu \in V_4 \backslash \{e\}$ and $1 \leq b_a \leq 3$ $(a = 1,2,3,4)$ such that $b_{\nu(a)} \neq b_a$ for $a = 1,2,3,4$.

See Appendix A.3 for a SAGE [Dev17] script for computing the maximal facets of $\mathcal{T}$.

The triangulation $\mathcal{T}$ is the union over the cones over three 3-spheres S_j (the cone over S_j is the ball B_j in the definition of an equlibrium triangulation). Each S_j is a very simple 3-sphere. It is the join of two hexagons (recall that $S^1 * S^1 \approx S^3$).

We compute some deformation-theoretic invariants of $\mathbb{P}(\mathcal{T})$, the Stanley–Reisner scheme associated to $\mathcal{T}$.

Lemma 2.3.2. *We have that* $\dim_{\mathbb{C}} = T^1_{A(\mathcal{T}),0} = 90$ *and* $\dim_{\mathbb{C}} = T^2_{A(\mathcal{T}),0} = 306$. *The normal sheaf has* $\mathcal{N}_{\mathbb{P}(\mathcal{T})/\mathbb{P}^{14}}$ *has* 300 *global sections.*

The proof is a computation in `Macaulay2`. We remark that since $\mathbb{P}(\mathcal{T})$ is not Cohen–Macaulay, some standard comparison theorems does not hold. In our case we only have an inclusion $T^i_{A(\mathcal{T})} \hookrightarrow T^1_{\mathbb{P}(\mathcal{T})}$ (see the article of Kleppe [Kle79] and Theorem 3.9).

Kühnel's 9-vertex triangulation

2.4 Naïve attempt to degenerate

On the other hand, degenerating the ideal of $F_1(X) \subset \mathbb{P}^N$ to a square-free monomial ideal should give a triangulation of $\mathbb{CP}^2$. Since $F_1(X)$ sits inside $\mathbb{G}(1,5)$, and there are many known degenerations of $\mathbb{G}(1,5)$, we hoped that maybe $F_1(X)$ would generate inside $\mathbb{G}(1,5)$. Unfortunaly, we did not succed, mainly because we could not see any structure in the ideal of $F_1(X)$.

1. Conversely, a smoothing of a triangulation of $\mathbb{CP}^2$ would give a potentially new hyper-Kähler family.

2. Results: computed $F_1(X)$ for some hypersurfaces, both pfaffian and non-pfaffian.

3. Also looked at deformations of a specific 15-vertex triangulation of $\mathbb{CP}^2$. It deforms to the union of three toric varieties (which are cones over joins of del Pezzo surfaces). Probably not smoothable. T^2 is very big.

CHAPTER 3

The two smoothings of $C(\mathrm{dP}_6)$

smoothings

In this chapter we study the toric singularity that is the cone over the del Pezzo surface of degree 6. it has two topologically different smoothings, which we haven't seen studied in some detail before.

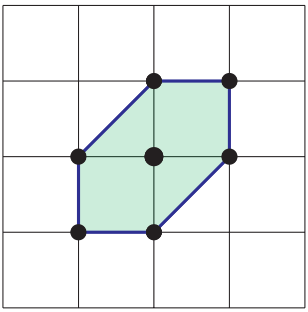
3.1 The del Pezzo surface dP_6

smoothings

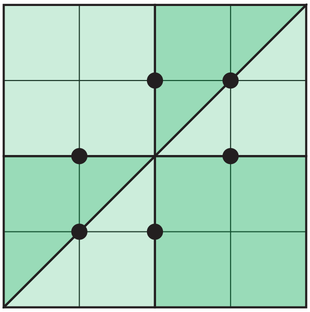
We start this chapter by talking about the del Pezzo surface in some generality.

Denote by dP_6 the blow-up of $\mathbb{P}^2$ in three non-collinear points. These points can be chosen to be the coordinate points $(1:0:0), (0:1:0)$ and $(0:0:1)$. Since the coordinate points are invariant under the natural torus action on $\mathbb{P}^2$, it follows that the dP_6 is a toric variety.

As a toric variety, it can be described as toric variety of the planar hexagon depicted in Figure 3.1(a). The normal fan is in Figure 3.1(b)



(a) The hexagon corresponding to dP_6.



(b) The fan over the polar polytope.

Figure 3.1: Toric description of dP_6.

We will spend some time describing the different embeddings of dP_6. Different embeddings give rise to different smoothings of the affine cones.

Embedding in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Since dP_6 is the blowup of $\mathbb{P}^2$ in three points, we can blow them up separately. Let x_0, x_1, x_2 be coordinates of $\mathbb{P}^2$. Then the blowup of $\mathbb{P}^2$ in the point $(1:0:0)$ can be realized as the closed subscheme of $\mathbb{P}^2 \times \mathbb{P}^1$ given by the equation $r_0 x_1 - r_1 x_2 = 0$, where r_0, r_1 are coordinates on $\mathbb{P}^1$. We can repeat this procedure on the two other points $(0:1:0)$ and $(0:0:1)$ to obtain similar equations. Collecting these, we see that dP_6 is given by the matrix equation

$$M\vec{x} = \begin{pmatrix} 0 & r_0 & -r_1 \\ s_1 & 0 & -s_0 \\ -t_0 & t_1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0.$$

in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Since $\vec{x}$ is non-zero, it follows that we must have $\det M = 0$. It is not difficult to see that M cannot have rank 1 or lower, because that would force some of the $\mathbb{P}^1$-coordinates to be zero. Consider the projection forgetting the $\mathbb{P}^2$-factor:

$$\pi : \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Consider the hypersurface $\det M = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Any solution to this equation gives a unique solution to the equation $M\vec{x} = 0$, meaning that the restriction of π to dP_6 is an isomorphism onto the hypersurface $\det M = r_0 s_0 t_0 - r_1 s_1 t_1 = 0$.

It is also interesting to see how this embedding arises from a toric perspective using polytopes. Since $\mathbb{P}^1$ is the toric variety associated with the interval $[-1,1] \subset \mathbb{R}$, it follows that $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the toric variety associated with the cube $\Delta = [-1,1]^3 \subset M_{\mathbb{R}} = \mathbb{R}^3$. The inclusion of dP_6 in M induces a surjection of coordinate rings $\mathbb{C}[M] \to \mathbb{C}[\mathrm{dP}_6]$. This correspond to the fact that there is a lattice projection of the cube onto the hexagon. See Figure 3.2.

Embedding in $\mathbb{P}^2 \times \mathbb{P}^2$

On the other hand, blowups can also be realized as closures of graphs of rational maps. Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the Cremona transformation given by $(x_0 : x_1 : x_2) \mapsto \left(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right)$. Then, in coordinates $(a_0 : a_1 : a_2) \times (b_0 : b_1 : b_2)$
eq:dp6_inp2p2 on $\mathbb{P}^2 \times \mathbb{P}^2$, the equations $a_0 b_0 = a_1 b_1 = a_2 b_2$ hold . These are the equations of the blowup along the indeterminacy locus of the rational map φ. The indeterminacy locus is exactly the three coordinate points. Hence dP_6 can also be realized as the intersection of two $(1,1)$-divisors in $\mathbb{P}^2 \times \mathbb{P}^2$.

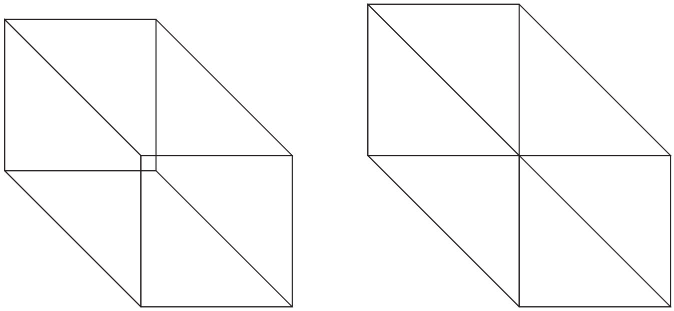


Figure 3.2: The projection of a cube onto a hexagon.

fig:cube_project

There is also in this case a description in terms of polytopes. The polytope associated with $\mathbb{P}^2 \times \mathbb{P}^2$ is $\Delta^2 \times \Delta^2$, the product of two two-simplices. Also in this case, there is a projection onto a hexagon in $\mathbb{R}^2$.

visualize some-how?

Hence, using the Segre embedding, dP_6 lives naturally in both $\left(\mathbb{P}^1\right)^3 \hookrightarrow \mathbb{P}^7$ and $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$.

Remark 3.1.1. Intersecting $\mathbb{P}^2 \times \mathbb{P}^2$ with a single $(1,1)$-divisor gives us the projective space bundle corresponding to the tangent bundle of $\mathbb{P}^2$, which we denote by $\mathcal{T}(\mathbb{P}^2)$. This follows from the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_{\mathbb{P}^2}(1)^3 \to \mathcal{T}_{\mathbb{P}^2} \to 0.$$

Since $\mathbb{P}(\mathscr{O}_{\mathbb{P}^2}(1)^3) = \mathbb{P}^2$, $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ can be realized as the subset of $\mathbb{P}^2 \times \mathbb{P}^2$ such that $a_0b_0 + a_1b_1 + a_2b_2 = 0$.

can the topology of this space be studied using e.g. chern classes?

3.2 The cone over dP_6 and its two smoothings

The singularity $Z = C(\mathrm{dP}_6)$ is one of the most studied singularities with an obstructed deformation space, In the paper [Alt97], Klaus Altmann describe a method to study the versal deformations of isolated affine Gorenstein toric singularities using only the combinatorial data of the toric variety. He shows that different components of the base space correspond to different ways of writing the defining polytope as a Minkowski sum of other polytopes.

See the illustation in [[TO COME]].

illustration of dP_6 as a minkowski sum of different things

Let $A = A(Z)$ denote the affine coordinate ring of $C(\mathrm{dP}_6)$. It has a natural $\mathbb{Z}$-grading. From Altmann's article, or by using `Macaulay2`, ones computes that $\dim T^1(A) = 3$, and that $\dim T^2(A) = 2$. The versal base space decomposes into a union of a line and a plane.

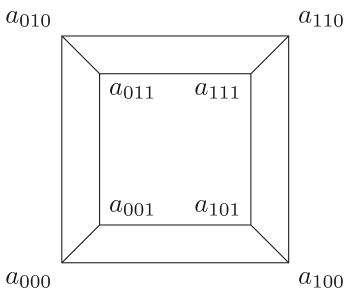


Figure 3.3: A $2 \times 2 \times 2$-tensor. fig:p1p1p1

For well-behaved singularities, one can often describe all of its deformations by writing up a "format" of the equations. For example, for codimension three Gorenstein projective schemes, there is a structure theorem for the whole resolution, involving Pfaffians. For codimension 4, there is no such result, though there have been some research in this direction, see for example [Rei15], where Miles Reid discusses a very general structure theorem for codimension 4 projective schemes.

It is worthwhile to note that both smoothings of Z arise by "sweeping out the cone": if X is a projective variety in $\mathbb{P}^n$, and Y is equal to $X \cap H$, where H is a section of $\mathscr{O}_{\mathbb{P}^n}(1)$, then the affine cone over Y deforms to a general hyperplane section of the affine cone over X. See the introduction of [Ste03] for more details.

Using the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ and substituting from the linear equations in the description Section 3.1, we can write the equations of dP_6 inside $\mathbb{P}^6$ as

$$\begin{vmatrix} y & x_1 & x_2 \\ x_3 & y & x_4 \\ x_5 & x_6 & y \end{vmatrix} \leq 1, \tag{3.1}$$

where ≤ 1, means taking all 2×2-minors.

On the other hand, dP_6 can be realized as a subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as well. The equations can be described as follows: draw a cube, and let each vertex correspond to a variable. Then the equations of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding are given by taking all "minors" along all sides of the cube together with the three long diagonals. See Figure 3.3. To get dP_6, one identifies two opposite corners. Thus in total there are $8 - 1 = 7$ variables, just as above.

The first smoothing is obtained by deforming the equations of dP_6 as a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$. It can be described by perturbing two of the entries of the matrix below:

$$\begin{vmatrix} y & x_1 & x_2 \\ x_4 & y+t_1 & x_3 \\ x_5 & x_6 & y+t_2 \end{vmatrix} \leq 1. \tag{3.2}$$ {eq:def2}

For $t_1 = t_2 = 0$, we get the cone over dP_6, while for generic t_i, we get a smooth variety. In fact, we can compute that the discriminant locus (the set of points in $\mathbb{A}^2_{t_1,t_2}$ with singular fiber) are the t_1-axis, the t_2-axis and the line $t_1 = t_2$. Notice that the total space is equal to the cone over $\mathbb{P}^2 \times \mathbb{P}^2$.

Call (any) smooth fiber X_2.

Lemma 3.2.1. *Let $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ be the projective bundle associated to the tangent sheaf on $\mathbb{P}^2$. Then the smoothing X_2 is isomorphic to $M \setminus \mathrm{dP}_6$.*

Proof. First homogenize the equations (3.2) with respect to y_1. Call the homogenized variety N. Put $y_0' = y_0$, $y_1' = y_0 - ty_1$ and $y_2' = y_0 - t_2 y_1$. Then we have the relation

$$h = t_2 y_1' - t_1 y_2' - (t_1 - t_2) y_0' = 0.$$

Hence we see that $N = \mathbb{P}^2 \times \mathbb{P}^2 \cap (h = 0)$. We can pull back the coordinates y_i' to $\mathbb{P}^2 \times \mathbb{P}^2$. Let $\mathbb{P}^2 \times \mathbb{P}^2$ have coordinates x_0, x_1, x_2 and y_0, y_1, y_2. Then h pulls back to the equation

$$(x_0, x_1, x_2) \cdot (-t_1 y_2, (t_1 - t_2) y_0, t_2 y_1) = 0$$

in $\mathbb{P}^2 \times \mathbb{P}^2$. As long as $t_1 \neq t_2$ and $t_1, t_2 \neq 0$, we can do a change of coordinates in $\mathbb{P}^2_{y_0 y_1 y_2}$, so that h transforms to

$$(x_0, x_1, x_2) \cdot (y_0, y_1, y_2) = 0.$$

Hence we see that M is isomorphic to the total space of the Grassmannian of lines in $\mathbb{P}^2$ (each point in one of the $\mathbb{P}^2$'s give a line in the other $\mathbb{P}^2$). This is in turn isomorphic to $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$, since each tangent vector through a point determines a line through it.

Now, what have we gained by homogenizing? The divisor at infinity is $y_1 = 0$, which is a dP_6 again. In our new coordinates this is equivalent to $y_1' = y_2' = y_0'$. Hence in the coordinates of $\mathbb{P}^2 \times \mathbb{P}^2$, dP_6 is given by the two equations $x_1 y_0 - x_2 y_1 = x_1 y_0 - x_0 y_2 = 0$. ∎

The other smoothing is the obtained by replacing one of the corners of the cube in Figure 3.3 with $a_{000}' = a_{000} + t$, obtained a one-parameter smoothing. The total space is now that affine cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Call this smoothing X_1.

Lemma 3.2.2. *The smoothing X_1 is isomorpic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathrm{dP}_6$.*

Proof. Homogenize, notice what is gained, then subtract. ∎

source?

Observe that $\mathcal{T}(\mathbb{P}^2)$ is homotopy equivalent to $\mathbb{P}^1 \times \mathbb{P}^2$. It follows that its Euler characteristic, which is invariant under homotopy, is equal to $2 \times 3 = 6$.

This information let us calculate the Euler characteristics of the smoothings. Note that $\chi(\mathbb{P}^1) = 2$ and $\chi(\mathcal{T}(\mathbb{P}^2)) = 6$. By additivity of the Euler characteristics we have $\chi(X_1) = 2$ and $\chi(X_2) = 0$, since $\chi(\mathrm{dP}_6) = 6$.

It follows that the two smoothing components correspond to topologically different smoothings. This can explain the obstructedness of the deformations of X_0 in Chapter 4.

Lemma 3.2.3. *The cohomology ring of* $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ *is* $\mathbb{Z}[x, y]/(x^3, y^2+c_1y+c_2)$, *where* x *and* y *have degree* 2*. In particular, the cohomology of* M *is given by* $(1, 0, 2, 0, 2, 0, 1)$.

Proof. The first claim follows from the Leray-Hirch theorem. See [BT82, page 270]. The next claim follows since x and y both have degree 2. ■

We can use what we know about the topology of these spaces to compute homology groups of the two affine smoothings.

We first need a prelimenary lemma from toric geometry, which we state and prove in general form, because we could not find a proper reference.

Lemma 3.2.4. *Let* $Y \overset{i}{\hookrightarrow} X$ *be an closed immersion of smooth toric varieties, corresponding to a map of fans* $\Sigma_1 \xrightarrow{A} \Sigma_2$. *Let* M_1 *and* M_2 *be the corresponding character lattices. Then we have a commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & M_2 \arrow[r] \arrow[d, "A^T"] & \mathbb{Z}^{\Sigma_2(1)} \arrow[r] \arrow[d, "C^T"] & \mathrm{Pic}(X) \arrow[r] \arrow[d, "i^*"] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & \mathbb{Z}^{\Sigma_1(1)} \arrow[r] & \mathrm{Pic}(Y) \arrow[r] & 0
\end{tikzcd}$$

Where in addition $i^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ *is the map of Picard groups induced by the closed embedding.*

Proof. The vertical rows are well-known. See for example Theorem 4.1.3 in [CLS11].

The matrix C^T is defined as follows: each primitive ray generator of cones in $\Sigma_1(1)$ thought of as lying in N_2 via the embedding A has a unique description as a linear combination of rays in $\Sigma_2(1)$ lying in the same minimal cone. Let the columns of C be the coefficients of this linear combination. Then, by definition, the first square commutes.

It follows that there is an induced map of Picard groups. We must show that the induced map is exactly the one induced by the closed embedding. To see this, we use Proposition 6.2.7 in [CLS11], where a description of this map is given in terms of convex functions on N. ■

This lemma let us compute explicitly what the induced map of Picard groups is using only the toric data.

Theorem 3.2.5. *The two affine smoothings are topologically different. The homology groups are:*

Group	0	1	2	3	4	5	6	Euler-characteristic
$H^i(X_1,\mathbb{Z})$	1	0	2	1	0	0	0	2
$H^i(X_2,\mathbb{Z})$	1	0	1	2	0	0	0	0

Proof. The singular cohomology of $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is given by $(1,0,3,0,3,0,1)$, which can be computed by the Künneth formula. The cohomology of dP_6 is given by $(1,0,4,0,1)$.

We will use the Lefschetz duality theorem [Spa66], which in this case says that $H_q(M \backslash dP_6, \mathbb{Z}) \simeq H^{6-q}(M, dP_6, \mathbb{Z})$. Then the long exact sequence of the pair (M, dP_6) immediately gives $h_0(X_1,\mathbb{Z}) = 1$. Similarly, we see that $h_5(X_1,\mathbb{Z}) = h_6(X_1,\mathbb{Z}) = 0$, since the map $H^1(M,\mathbb{Z}) \to H^1(D,\mathbb{Z})$ is an isomorphism.

The other groups depend upon the explicit form of the maps $H^2(M,\mathbb{Z}) \to H^2(D,\mathbb{Z})$ and $H^4(M,\mathbb{Z}) \to H^4(D,\mathbb{Z})$.

By Poincaré duality ((reference)), the induced map corresponds to intersecting the divisors on M with dP_6. Computing, we get that map is given by the following matrix: Find reference

$$H^2(M,\mathbb{Z}) \simeq H_4(M,\mathbb{Z}) \simeq \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \mathbb{Z}^4 \simeq H_2(dP_6,\mathbb{Z}) \simeq H^2(dP_6,\mathbb{Z}).$$

This is an injective map, and it follows from the long-exact sequence and the Lefschetz theorem that $H_3(X_1) \simeq H^3(M, dP_6) \simeq \mathbb{Z}$, and also that $H_4(X_1) = 0$.

Similarly, the map $H^4(M) \to H^4(dP_6)$ is computed to be given by $(a,b,c) \mapsto a+b+c$, since the three $\mathbb{P}^1$'s intersect dP_6 in a single point. This map has two-dimensional kernel, and we conclude that $H_2(X_1) \simeq H^4(M, \mathrm{dP}_6) = \mathbb{Z}^2$, and that $H^1(X_1) = 0$.

The computations for X_2 are similar but more involved. We first note that the Picard group of $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ is generated by the pullbacks F, G of the generators of $\mathrm{Pic}(\mathbb{P}^2_{x_0x_1x_2} \times \mathbb{P}^2_{y_0y_1y_2})$. Say F is represented by $V(x_0)$ and G is represented by $V(y_0)$. check transversal intersection with dP6

Again we compute the intersections of F and G with dP_6. Intersecting with F is computed by decomposing the ideal $(x_0, x_1y_0 - x_2y_1, x_1y_0 - x_0y_2)$ in $k[x_0,x_1,x_2,y_0,y_1,y_2]$ and saturating by (x_0,x_1,x_2) and (y_0,y_1,y_2). This can either be done by hand or by using `Macaulay2`. Either way, we find that $F\big|_{dP_6} = E_3 + L_{23} + E_2 = H$, using the notation from earlier this chapter.

Similarly $G\big|_{dP_6} = L_{23} + L_{12} + E_2 = 2H - E_1 - E_2 - E_3$. Hence the map on cohomology is given by the matrix

$$H^2(M,\mathbb{Z}) \simeq H_4(M,\mathbb{Z}) \simeq \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}} \mathbb{Z}^4 \simeq H_2(\mathrm{dP}_6,\mathbb{Z}) \simeq H^2(\mathrm{dP}_6,\mathbb{Z}).$$

This is an injective map, and as above, we conclude that $H_3(X_2) \simeq H^3(M,\mathrm{dP}_6) \simeq \mathbb{Z}^2$, and also that $H_4(X_1) = 0$. ■

Remark 3.2.6. In fact, the Andreotti-Frankel theorem [AF59] states the following: if V is any smooth affine variety of complex dimension n, then it has the homotopy type of a CW complex of real dimension n. Thus it should be no surprise that $H^j(X_i) = 0$ for $j > 3$.

CHAPTER 4

Construction of Calabi–Yau's

structions

In this chapter I describe the construction of three topologically different smoothings of a singular Calabi-Yau manifold. They (should) correspond to different components of the Hilbert scheme of threefolds with Hilbert polynomial $p(t) = 6t^3 + 6$ in $\mathbb{P}^{11}$.

We first describe a degenerate Calabi–Yau in the form of a Stanley-Reisner scheme $\mathbb{P}(\mathcal{K})$, which has a quite large symmetry group. There is a natural deformation to a X_0, which is a hypersurface inside toric variety, with isolated singularities.

We show that X_0 has several topologically distinct smoothings, which should lie on different components of the Hilbert scheme in $\mathbb{P}^{11}$.

4.1 The mirror construction Ansatz

ion_ansatz

In many cases of interest, given a construction of a Calabi–Yau manifold, the following Ansatz produces a mirror.

Let $\mathcal{K}$ be a simplicial complex, with associated Stanley–Reisner scheme X_0. Let G be the automorphism group of $\mathcal{K}$. Then G induces an action on $T^1_{X_0}$ in the following way: each element of $T^1_{X_0}$ can be represented by a $\phi \in \operatorname{Hom}(I/I^2, A)$, and then $g \cdot \phi$ is given by $(g \cdot \phi)(f) = g \cdot \phi(g^{-1} \cdot f)$, the contragredient action.

There is an action of $T_n = (\mathbb{C}^*)^{n+1}/\mathbb{C}^*$ on $\mathbb{P}^n$, and since I_{X_0} is generated by monomials, the action restricts to an action on X_0 as well.

Given a smoothing family with general fiber X and special fiber X_0, we can consider a subfamily with only isolated singularities on which G act. Let $H \subset T_n$ be the subgroup of the torus acting on this family. Then the mirror candidate to X is given by a crepant resolution of $Y_t = X_t/H$.

Though it is often overlooked (or stated in another language) in the literature, even the mirror construction of the famous quintic arises this way. Briefly, the quintic Calabi–Yau is given by the zero locus of a general quintic in $H^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(5))$. The special quintic given by $f = x_0x_1x_2x_3x_4$ is the Stanley–Reisner scheme corresponding to the 3-simplex. The automorphism group is S_5, and an invariant 1-parameter family is given by $f_t = \sum_{i=0}^{4} x_i^5 + tx_0x_1x_2x_3x_4$. There is a $H = (\mathbb{Z}/5)^5/\mathbb{Z}^5$-action on $X_t = Z(f_t)$ given by coordinate-wise multiplication by fifth roots of unity. The general element of the family X_t is smooth, so the only singularities of the quotient $Y_t = X_t/H$ comes from points with non-trivial stabilizer. These can be resolved by methods from toric geometry. For details, see the first chapter of Ingrid Fausk's thesis [Fau12].

4.2 The special fiber

Let E_6 be the hexagon as a simplicial complex. The associated Stanley–Reisner scheme $\mathbb{P}(E_6)$ is a degenerated elliptic curve in $\mathbb{P}^5$. If $\mathbb{P}^5$ have coordinates $x_0, \ldots, x_5$, the equations of E_6 are $x_ix_{i+2} = x_ix_i + 3 = 0$, where i is taken modulo 6. This gives a total of 9 quadratic generators.

Lemma 4.2.1. *The Hilbert polynomial of* $\mathbb{P}(E_6)$ *is* $h(t) = 6t$.

Proof. We want to count the dimension of $S_t = S_{E_6}(t)$. Any monomial in S_k has support on the simplicial complex E_6, so its support is either a vertex or an edge. In the first case, the monomial has the form x_i^t, so there are six of these.

In the other case, it has the form $x_i^a x_{i+1}^b$, with $a + b = t$ and $a, b \neq 0$. Counting, there are $6(t-1)$ of these monomials. In total, the dimension is $6 + 6(t-1) = 6t$. ∎

Remark 4.2.2. Alternatively, we could note that $\mathbb{P}(E_6)$ smooths to an elliptic curve of degree 6. Since Hilbert polynomials are constant in flat families, it follows from Riemann–Roch that $h(t) = \deg \mathscr{O}_{\mathbb{P}(E_6)}(6t) - 1 + 1 = 6t$.

Note that the Hilbert polynomial only differ from the Hilbert function for $t = 0$, since $h(0) = 0$, while $\dim_{\mathbb{C}} S_0 = 1$.

We now introduce the central fiber in the discussions onward. Let $\mathcal{K}$ be the simplicial complex $E_6 * E_6$. It is a triangulation of the 3-sphere. The maximal faces are unions of maximal faces from each factor.

Denote the vertices of the left E_6 with $x_1, \ldots, x_6$, and the vertices of the right E_6 with $z_1, \ldots, z_6$. Then the maximal faces of $\mathcal{K}$ are of the form $x_ix_{i+i}z_jz_{j+1}$, where $i, j \in \mathbb{Z}_6$. The number of i-faces follows are easy to compute:

Lemma 4.2.3. *The f-vector of* $\mathcal{K}$ *is* $(1, 12, 48, 36)$.

Proof. There are 12 vertices, and $6 \times 6 = 36$ maximal facets. Since $\mathcal{K}$ is a 3-sphere, it follows that $12 - f_1 + 36 = \chi(S^3) = 0$ that $f_1 = 48$.[1] ∎

Lemma 4.2.4. *The Hilbert polynomial of* $\mathbb{P}(\mathcal{K})$ *is* $h(t) = 6t^3 + 6$.

[1] Here we used that in a cell complex, the Euler characteristic is also the alternating sum of the number of cells in each dimension. This is Theorem 2.44 in [Hat02].

Proof. The homogeneous coordinate ring $S = \oplus_{t\geq 0} S_t$ of $\mathbb{P}(\mathcal{K})$ is the graded tensor product of $\mathbb{P}(E_6)$ with itself. It follows from the previous lemma that

$$\dim S_t = \sum_{i+j=k, ij\neq 0} 36ij + 12k,$$

where the last term is a correction term because $h(t) \neq 1$. It is now a routine computation using formulas for sums of squares to verify the claim. ■

Either by using `Macaulay2` or by using the more conceptual description of the T^i modules from [AC10], we can compute:

Lemma 4.2.5. *The dimensions of* $T^1(\mathcal{K})$ *and* $T^2(\mathcal{K})$ *are* 84 *and* 72*, respectively.*

Proof. We will prove this using the techniques and notation from [AC10]. Our goal is to compute the degree zero part of $T^1_{A_{\mathcal{K}}}$. We will do this using Theorem 1.6.2.

First notice that all links of vertices of $\mathcal{K} = E_6 * E_6$ are double suspensions over hexagons (they are denoted by ΣE_6 in Christophersen's article).

According to Table 1 in Christophersen's article, double suspensions over hexagons contribute with one dimension to $T^1_{A_{\mathcal{K}}}$, namely in degree $x_i^2/x_{i-1}x_{i+1}$ (if $\mathbf{a} = x_i^2$). In total there are $6 + 6 = 12$ contributions of this form.

Taking the link at the vertex $x_i z_j$ produces a square with vertices $x_{i+1}, z_{j+1}, x_{i-1}, z_{j-1}$ (in that order). According to Table 1 in Christophersen's article, these links contribute with dimension 2 to $T^1_{A_{\mathcal{K}}}$. The contributions have degrees $x_i z_j/x_{i+1}x_{i-1}$ and $x_i z_j/z_{j+1}z_{j-1}$. There are $2 \cdot 6 \cdot 6 = 72$ contributions of this form.

Thus, in total, $T^1_{A_{\mathcal{K}}}$ have $\mathbb{C}$-dimension 84.

We now compute $T^2_{A_{\mathcal{K}}}$. The contributions come from choosing $\mathbf{a} = x_i^2$ and $\mathbf{a} = x_i x_{i+1}$, respectively. If $|a| = 1$ (as in the first case), the results from Christophersen's article imply that $L_b := \cap_{b' \subset b} \mathrm{lk}(b', \mathrm{lk}(x_i, \mathcal{K}))$ must have more than one connected component (the contribution comes from $\widetilde{H}^0(L_b, \mathbb{C})$). This is the case if b consist of two opposite vertices in the suspended circle. In total there are $2 \cdot 6 \cdot 3 = 36$ contributions of this form.

If $|a| = 2$, the contributing links are hexagons, and in this case the contributions come from b such that $L_b = \emptyset$. Again consisthoosing b to consist of opposite vertices of the hexagon, we find three pairs b with $L_b = \emptyset$ for each hexagon. Thus in total there are $2 \cdot 6 \cdot 3 = 36$ contributions of this form.

In sum, $T^2_{A_{\mathcal{K}}}$ is $36 + 36$-dimensional. ■

The automorphism group of $\mathcal{K}$ is $D_6 \times D_6 \times \mathbb{Z}_2$, and of order $12 \cdot 12 \cdot 2 = 288$. It is not difficult to see that the induced action on $T^1_{X_0}$ have two orbits under $\mathrm{Aut}(\mathcal{K})$, corresponding to deformations of the form $x_i x_{i-2} + t x_{i+1} z_j$ and $x_{i-1}x_{i+1} + t x_i^2$, respectively.

4.3 A natural toric deformation

Consider Figure 4.1. It is the 2-dimensional polytope associated to the del Pezzo surface of degree 6. The fan over this polytope correspond to a unimodular regular triangulation of the polytope, and it follows by Proposition 1.8.1, that dP_6 degenerates to the Stanley–Reisner scheme $\mathbb{P}(E_6 * \{pt\})$, where $\{pt\}$ correspond to the origin. This is an embedded deformation inside $\mathbb{P}^6$.

Concretely, the equations of dP_6 are given by $x_i x_{i+2} - y x_{i+1} = x_i x_{i+3} - y^2 = 0$. The degeneration to $\mathbb{P}(E_6 * \{pt\})$ is given by setting the second terms to zero.

Now form the join of two copies of dP_6, to get a new variety $Y \subset \mathbb{P}^{13}$. By Proposition 1.2.2, this is a $2 + 2 + 1 = 5$-dimensional toric variety with singular locus consisting of two copies of dP_6. Since the coordinate ring is just the tensor product of two copies of $S(\mathrm{dP}_6)$, it follows that Y degenerates to $\mathbb{P}(E_6 * \{pt\} * E_6 * \{pt\}) = \mathbb{P}(\mathcal{K} * \Delta^1)$.

The following holds:

Proposition 4.3.1. *There is a deformation of the Stanley–Reisner scheme X_0 to an irreducible Calabi–Yau variety $X_Y \subset Y$ with isolated singularities. There are twelve of them, and they are locally isomorphic to cones over del Pezzo surfaces. More precisely: let (U, p_i) be the germ of X_0 at p_i. Then $(U, p_i) \simeq (C(\mathrm{dP}_6), 0)$.*

Proof. Since X_0 is a complete intersection side $\mathbb{P}(\mathcal{K} * \Delta^1)$, it follows that X_0 deforms to a complete intersection inside any deformation of $\mathbb{P}(\mathcal{K} * \Delta^1)$. We explained above that $\mathbb{P}(\mathcal{K} * \Delta^1)$ deforms to the join Y of two del Pezzo surfaces, and it follows that X_0 deforms to Y intersected with two generic hyperplanes.

Since Y has singular locus of dimension 2 and degree $6 + 6 = 12$, it follows by Bertini's theorem [Har77, Chapter II, Theorem 8.18] that X_0 has twelve isolated singularities p_i.

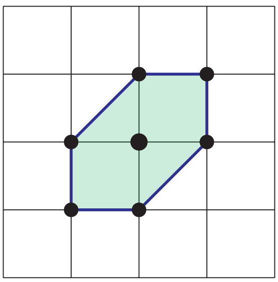


Figure 4.1: A hexagon. fig:hexago

To see how the singularities look locally, we argue as follows. Locally, Y looks like $\mathbb{A}^2_{a_1,a_2} \times C(\mathrm{dP}_6)_{x_i}$, where the subscripts refer to the coordinates. This is the ideal of Y consists of two sets of equations, each defining a smooth toric variety, and smooth toric varieties are isomorphic to $\mathbb{A}^d$ in affine charts.

The claim now follows from two applications of Theorem 3.1.5 in [Bat94], which says that the singularities on Σ-regular toric hypersurfaces are inherited from the ambient toric variety. ■

Since the cone over dP_6 deforms in two topologically different ways, we might expect that X_Y does this too. This is indeed true.

4.4 Smoothings of X_Y

By embedding dP_6 in different spaces, we obtain different smoothings of subvarieties of the join of these spaces.

The block matrix construction

We are inspired by the construction in Rødland's thesis [Rød00].

Let E be a 3-dimensional vector space. Let $\{e_1, e_2, e_3\}$ be a basis for E. Then we can form the vector space $V = (E \otimes E) \oplus (E \otimes E)$, which has dimension 18. Let $\mathbb{P}^{17} = \mathbb{P}(V)$. Choose coordinates $x_1, \ldots, x_{18}$ on $\mathbb{P}^{17}$.

Thinking of $E \otimes E$ as 3×3-matrices, we can think of the elements of $\mathbb{P}^{17}$ as pairs of 3×3-matrices up to scalar, not both zero. Concretely, two pairs of matrices (A', B') and (A, B) are equivalent if $(A', B') = (\lambda A, \lambda B)$ for some $\lambda \in \mathbb{C}^*$.

There is a natural rational map $\pi : \mathbb{P}^{17} \dashrightarrow \mathbb{P}^8 \times \mathbb{P}^8$, which is the identity on coordinates, given by dividing out by the antidiagonal $\mathbb{C}^*$-action: $\lambda' \cdot (A, B) = (\lambda', \lambda'^{-1} B)$.

Denote by V_1 and V_2 the subspaces $x_1 = \ldots = x_9 = 0$ and $x_{10} = \ldots = x_{18} = 0$, respectively. Blow up $\mathbb{P}^{17}$ in $V_1 \cup V_2$, to get $\widetilde{\mathbb{P}^{17}}$. The spaces V_i are exactly the indeterminacy locus of π, so π extends to a map $\pi : \widetilde{\mathbb{P}^{17}} \to \mathbb{P}^8 \times \mathbb{P}^8$. Denote by π_1 and π_2 the two natural projections to $\mathbb{P}^8$. Then it is true that $\widetilde{\mathbb{P}^{17}} = \mathbb{P}_{\mathbb{P}^8 \times \mathbb{P}^8}(\pi_1^* \mathscr{O}_{\mathbb{P}^8}(1) \oplus \pi_2^* \mathscr{O}_{\mathbb{P}^8}(1)) = \mathbb{P}(\mathscr{O}_{\mathbb{P}^8 \times \mathbb{P}^8} \oplus \mathscr{O}_{\mathbb{P}^8 \times \mathbb{P}^8}(1, -1))$. This is explained further in Section C7 in [AK75].

Let M be the closure of the set of pairs (A, B) where $\operatorname{rank} A = \operatorname{rank} B = 1$.

prop:m **Proposition 4.4.1.** *The variety M is the join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, and has singular locus $\mathbb{P}^2 \times \mathbb{P}^2 \subset V_i$ of dimension 4.*

The canonical sheaf is $\omega_M = \mathscr{O}_M(-6)$, so that M is a Fano toric variety.

Proof. If $\mathbb{P}^{17}$ have coordinates $x_1, \ldots, x_{18}$, let M_1 and M_2 be the matrices

$$M_1 = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} \\ x_{16} & x_{16} & x_{17} \end{pmatrix}.$$

Then M is defined by the zeroes of the 2×2-minors of M_1 and M_2. Then it is clear that M is the projective join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8 \subset \mathbb{P}^{17}$, since the variable sets are disjoint.

The variety M is 9-dimensional: the affine cone over M, $C(M)$, is equal to $C(\mathbb{P}^2 \times \mathbb{P}^2) \times C(\mathbb{P}^2 \times \mathbb{P}^2)$. This variety has dimension $5 + 5 = 10$, hence its projectivization M is 9-dimensional.

The singular locus of M consists of the pairs $(0, B)$, and $(A, 0)$, where $\operatorname{rank} A = \operatorname{rank} B = 1$, hence $\dim \operatorname{Sing} M = 4$. See also Proposition 1.2.2.

By Remark 1.2.4, it follows that $\omega_M = \mathscr{O}_M(-6)$, since

$$\omega_{\mathbb{P}^2 \times \mathbb{P}^2} = \mathscr{O}_{\mathbb{P}^8}(-3)\big|_{\mathbb{P}^2 \times \mathbb{P}^2}.$$

■

Here comes our first construction. Let X_1 be the intersection of M with a generic $\mathbb{P}^{11}$. Then the following is true.

prop:x1 **Proposition 4.4.2.** *X_1 is a smooth Calabi–Yau variety with $\chi(X_1) = -72$.*

Proof. The singularities of M are of dimension 4. By Bertini's theorem, intersecting M with a codimension 6 hyperplane gives a smooth variety X_1.

To see that X_1 is Calabi–Yau, we use the adjunction formula, which in this case takes the form

$$\omega_{X_1} = \omega_M \otimes \wedge^6 \mathscr{O}_X(1)^{\oplus 6} = \mathscr{O}_X(-6) \otimes \mathscr{O}_X(6) = \mathscr{O}_X,$$

showing that the canonical sheaf is trivial.

To find the topological Euler characteristic, we compute in `Macaulay2`. Since computing the whole cotangent sheaf of X_1 is impossible with current computer technology[2], we make use of standard exact sequences. Let $\mathscr{I}$ be the ideal sheaf of M in $\mathbb{P}^{17}$. First off, we have the exact sequence

$$0 \to \mathscr{I}/\mathscr{I}^2\big|_{X_1} \to \Omega^1_{\mathbb{P}}\big|_{X_1} \to \Omega^1_M\big|_{X_1} \to 0.$$

The restriction to X_1 is exact since $\mathscr{I}/\mathscr{I}^2$ is locally free on the smooth locus.

[2] An external computer has been trying to compute this sheaf for a few months now without terminating.

The `Macaulay2` command `eulers` computes the Euler characteristics of generic linear sections of a sheaf $\mathscr{F}$ (behind the scene, this is equivalent to computing the Koszul resolution of the relative ideal sheaf $\mathscr{I}_{X_1/M}$). Using this command, we find that $\chi(\mathscr{I}/\mathscr{I}^2\big|_{X_1}) = -180$. Using the exact sequence

$$0 \to \Omega^1_{\mathbb{P}}\big|_{X_1} \to \mathscr{O}_{X_1}(-1)^{18} \to \mathscr{O}_{X_1} \to 0,$$

we find that the Euler characteristic of $\Omega^1_{\mathbb{P}}\big|_{X_1}$ is $-216 = 12 \cdot 18$. It follows from the first exact sequence that $\Omega^1_M\big|_{X_1}$ has Euler characteristic -36.

Since X_1 is a complete intersection in M, the conormal sequence is

$$0 \to \mathscr{O}_{X_1}(-1)^6 \to \Omega_M\big|_{X_1} \to \Omega^1_{X_1} \to 0.$$

Hence $\chi(\Omega^1_X) = -36 + 72 = 36$.

It follows that the topological Euler characteristic is $\chi_{X_1} = -2\chi(\Omega^1_{X_1}) = -72$. ■

Remark 4.4.3. We can give explicit equations for a flat family with special fiber X_Y and general fiber X_1. Let $y_0 - h_1$ and $y_1 - h_2$ be the generic linear forms in $\mathbb{P}^{13}$ defining X_Y as a subscheme of Y. Let g_i (for $i = 1, \ldots, 6$) be generic linear forms in $\mathbb{P}^{17}$. Then such a flat family is defined by the 2×2-minors of the two matrices below:

$$A_1 = \begin{pmatrix} h_1 + tg_1 & x_2 & x_3 \\ x_4 & h_1 + tg_2 & x_6 \\ x_7 & x_8 & h_1 + tg_3 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} h_2 + tg_4 & x_{11} & x_{12} \\ x_{13} & h_2 + tg_5 & x_{15} \\ x_{16} & x_{16} & h_2 + tg_6 \end{pmatrix}.$$

For $t = 0$, we get X_Y.

Remark 4.4.4. Since X_1 avoids $V_1 \cup V_2$, the inverse image $\pi^{-1}(X_1) \subset \widetilde{\mathbb{P}^{17}}$ is isomorphic to X_1. Thus we can realize X_1 as a subvariety of a *smooth* variety. Unfortunaly, X_1 is cut out by non-ample divisors in $\widetilde{\mathbb{P}^{17}}$, making computations there just as hard.

Remark 4.4.5. I have not been able to compute the Hodge nubmers of X_1. However, counting parameters, we can give a conjectural size of $H^1(X, \mathcal{T}_{X_1})$, which is the space of complex structures on X_1:

X_1 lies in a family parametrized by planes containing twelve 3×3 block matrices (spanning the $\mathbb{P}^{11}$), giving $12 \cdot 18 = 216$ parameters. These matrices correspond to elements of $V = (E \otimes E)^{\oplus 2}$. There is an action of the group $\prod_{i=1}^4 \mathrm{GL}(E)$ on V, reducing the amount of parameters by $4 \dim \mathrm{GL}(V) = 36$. Furthermore, rotation in $\mathbb{P}^{11}$ reduces the dimension by $H^0(\mathbb{P}^{11}, \mathcal{T}) = 12^2 - 1 = 143$. In total, we have $216 - 36 - 143 = 37$ complex parameters. Since the Euler characteristic is -72, heuristically, the Hodge numbers should be $(1, 37)$.

This is very interesting, for the following reason: in the article [Kap15], the author produces list of Calabi–Yau manifolds with $\operatorname{Pic} X = \mathbb{Z}$, among which there are some that are only conjectured to exist. There is exactly one variety in that list of degree 36, and Euler-characteristic -72, and it has Hodge numbers $(1, 37)$. It is conjectured to exist based on the conjecture that to every differential equation of “Calabi–Yau type”, there should exist a Calabi–Yau variety having that equation as its Picard–Fuchs-equation (see [ES06]).

The three-tensor construction

The construction in the previous section used the embedding of dP_6 in $\mathbb{P}^2 \times \mathbb{P}^2$ to deform X_Y. There is also the embedding of dP_6 in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to exploit. The construction is similar.

Let F be a 2-dimensional vector space with basis $\{f_1, f_2\}$. Then we can form the vector space $V = ((F \otimes F \otimes F)^{\oplus 2})$. Let $\mathbb{P}^{15} = \mathbb{P}(V)$. Choose coordinates $a_{ijk} = (f_i \otimes f_j \otimes f_j, 0)$ and $b_{ijk} = (0, f_i \otimes f_j \otimes f_k)$ $(i, j, k = 0, 1)$ for $\mathbb{P}^{15}$.

The elements of $\mathbb{P}^{15}$ are pairs (A, B) of $2 \times 2 \times 2$ tensors, not both zero. There is also in this case a natural map $\pi : \mathbb{P}^{15} \to \mathbb{P}^7 \times \mathbb{P}^7$, given by dividing out by the antidiagonal $\mathbb{C}^*$-action. Just as above, let V_1 and V_2 be the subspaces $A = 0$ and $B = 0$, respectively. Let $\widetilde{\mathbb{P}^{15}}$ be the blowup of $\mathbb{P}^{15}$ in $V_1 \cup V_2$. The V_i's are exactly the indeterminacy locus of π, so π extends to a morphism $\pi : \widetilde{\mathbb{P}^{15}} \to \mathbb{P}^8 \times \mathbb{P}^8$, which is a $\mathbb{P}^1$-bundle. Also in this case it is true that $\widetilde{\mathbb{P}^{15}} = \mathbb{P}\left(\mathscr{O}_{\mathbb{P}^7 \times \mathbb{P}^7} \oplus \mathscr{O}_{\mathbb{P}^7 \times \mathbb{P}^7}(1, -1)\right)$.

Let N be the closure of set of pairs (A, B) where both A and B have tensor rank 1[3].

Proposition 4.4.6. *The variety N is the join of two copies of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$, and has singular locus $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset V_i$ of dimension 3.*

The canonical sheaf is $\omega_N = \mathscr{O}_N(-4)$, so that N is a Fano toric variety.

Proof. A pure $2 \times 2 \times 2$-tensor can be visualized as a cube with vertices a_{ijk}. See the diagram in Figure 4.2 on the facing page.

The equations of the set of rank 1 tensors in $\mathbb{P}(F \otimes F \otimes F)$ are obtained as the “minors” along the 6 sides of the cube, together with the minors along with the 3 long diagonals, giving a total of 9 binomial equations. We write this symbolically as $[a_{ijk}] \leq 1$.

Hence the equations for N are given by $[a_{ijk}] \leq 1$, together with $[b_{ijk}] \leq 1$. Since these are equations in a disjoint set of variables, it is clear that $N = (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)^{*2}$.

The claim about the singular locus and the canonical sheaf follow as in the proof of Proposition 4.4.1. ■

[3] An element of $F^{\otimes 3}$ have rank 1 if it is a pure tensor. It has rank $\leq k$ if it can be written as a sum of k pure tensors.

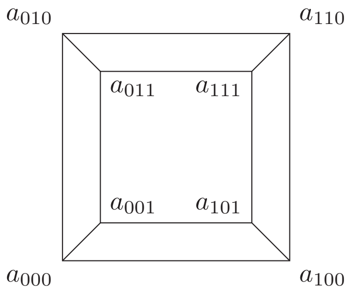


Figure 4.2: A $2 \times 2 \times 2$-tensor, seen from "above". fig:222tensor

Let X_2 be the intersection of N with a general $\mathbb{P}^{11}$.

Proposition 4.4.7. *The topological Euler characteristic of X_2 is -48.*

Proof. The proof is identical to the proof of Proposition 4.4.2. ∎

Remark 4.4.8. A heuristic moduli count works also in this case. We argue slightly differently in this case.

X_2 lies in a plane spanned by 12-plane in $\mathbb{P}((F \otimes F \otimes F)^{\oplus 2})$. Such a plane are parametrized by $\mathbb{G}(12, 16)$, the Grassmannian of 12-planes in k^{16}. This space is $12 \cdot (16 - 12) = 48$-dimensional. There is an action of the group $\prod_{i=1}^{6} \mathrm{GL}(F)$ on $(F \otimes F \otimes F)^{\oplus 2}$. However, the diagonal subgroup $\mathbb{C}^*$ acts trivially, so we really have an action of $\prod_{i=1}^{6} \mathrm{GL}(F)/\mathbb{C}^*$ on $\mathbb{P}^{15}$. This group is $4 \cdot 6 - 1 = 23$-dimensional. Thus in total we have $48 - 23 = 25$ moduli parameters.

Since we know the Euler characteristic, we predict the Hodge numbers to be $(1, 25)$.

The mixed smoothing

In the above cases, we formed the join of equal varieties. We mix things up: let $V = (E \otimes E) \oplus (F \otimes F \otimes F)$. Then let $\mathbb{P}^{16} = \mathbb{P}(V)$.

Now let W be the set of "mixed" rank 1 tensors. In a way similar to above, we find that W is a singular Fano toric variety of dimension 8. The singular locus is of dimension 4, so a 5-fold complete intersection is again a smooth Calabi-Yau variety X_3.

Proposition 4.4.9. *The Euler characteristic of X_3 is -60.*

Proof. The proof is identical to the proofs above. ∎

Remark 4.4.10. The heuristic above gives $(h^{11}, h^{12}) = (1, 31)$.

4.5 Degeneration of the toric join

The coordinate ring of the toric variety Y can be described as $R = S \otimes_k S$, where S is the coordinate ring of the del Pezzo surface of degree 6 in its anticanonical embedding. S has a special element y_0 corresponding to the origin of the defining polytope. Concretely, R can be described as follows: it is a quotient of the polynomial ring $\mathbb{C}[x_1, \ldots, x_6, y_0, z_1, \ldots, z_6, y_1]$. We then mod out by the ideal $I_1 + I_2$, where I_i $(i = 1, 2)$ is the ideal of the del Pezzo surface in the variables (x_i, y_0 and z_i, y_1, respectively). Then $Y = \operatorname{Proj}(\mathbb{C}[x_1, \ldots, x_6, y_0, z_1, \ldots, z_6, y_1]/(I_1 + I_2)$.

Consider the hypersurface Y' in Y given by $y_0 = y_1$. Then Y' is a 4-dimensional toric variety. It is the toric variety associated to the polytope Δ with vertices the columns of the matrix

$$\begin{pmatrix} -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \end{pmatrix}.$$

A computation shows that Y' has 1-dimensional singularities, and the singular locus is a graph of $\mathbb{P}^1$'s: take two hexagons, and join each vertex of one of them with all vertices of the other one. This makes in total 48 $\mathbb{P}^1$'s.

illustration graph

The variety Y' is a Fano toric variety, and as such, it has a anticanonical section $X_{Y'}$ which is a singular Calabi–Yau variety. A local computation shows that $X_{Y'}$ has 12 singularities that are locally isomorphic to $C(\mathrm{dP}_6)$, and 36 double points.

Since Y' is a four-fold, it follows that $X_{Y'}$ has a maximal projective crepant resolution of singularities (a *MPCP-desingularization*) (see for example the comment on page 55 in [CK99]), which we denote by $\widetilde{X_{Y'}}$.

A computation using PALP [KS04] shows that $\widetilde{X_{Y'}}$ has Hodge numbers $(44, 8)$ and Euler-characteristic 72.

Remark 4.5.1. The variety $X_{Y'}$ has also been described elsewhere. The polar polytope Δ° is equal to the product of two hexagons, and it follows that $\mathbb{P}_{\Delta^\circ}$ is equal to the product of two del Pezzo surfaces. An anticanonical hypersurface in $\mathrm{dP}_6 \times \mathrm{dP}_6$ has Euler characteristic -72 (see Theorem 3.1 in [Hüb92]).

In the article [BCD10], Braun et al. study this hypersurface and a group action on it. They also describe a resolution of singularities of X_{00}.

Remark 4.5.2. There is a heuristic surgical reason for the Euler characteristic being $+72$. Our $X_{Y'}$ deforms to X_1, which has Euler characteristic -72. This is obtained by smoothing 36 double points and 12 cones over del Pezzo surfaces. By the inclusion-exclusion principle, it follows that a resolution of the singularities of $X_{Y'}$ have Euler characteristic $\chi(X_1) + 2 \cdot 36 + 6 \cdot 12 = 72$.

connection to physics paper, fremheve at Euler-kar utregninger er kompatible, finn noe med +48

4.6 Invariant Calabi–Yau's

In this section, I will explain natural group actions on the X_i´s constructed above.

Denote by D_6 the dihedral group of order 12, the symmetries of a hexagon. It is generated by a rotation ρ of order 6, together with a reflection σ, subject to $\sigma\rho\sigma = \rho^{-1}$. There is an isomorphism $D_6 \simeq S_3 \times \mathbb{Z}_2$: S_3 is identified with $\langle \rho^2, \sigma \rangle$ and $\mathbb{Z}_2$ is identied with ρ^3.

Lemma 4.6.1. *There are D_6-actions on both M and N.*

Proof. Recall that M is the join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2$ embedded in $\mathbb{P}^8$. We can think of M as block matrices of rank $1+1$ in $\mathbb{P}(E \otimes E \oplus E \otimes E)$, where E is a 3-dimensional vector space. Choosing a basis $\{e_1, e_2, e_3\}$ for E, we have a natural S_3 action on E given by $e_i \mapsto e_{\sigma(i)}$. This action extends to $E \otimes E$ by $v \oplus w \mapsto g \cdot v \oplus g \cdot w$.

Switching the direct summands of $(E \otimes E)^{\oplus 2}$, gives us a $\mathbb{Z}_2$-action. In total we now have a $S_3 \times \mathbb{Z}_2$-action, which by the above remark is a D_6-action. Note that since the action was defined on E, the rank of elements of $E \otimes E$ is preserved, so that we indeed have an action on M.

Similarly, N is the rank $1+1$ tensors in $(E \otimes E \otimes E)^{\oplus 2}$, where now E is a 2-dimensional vector space. ∎

Torus actions

By choosing invariant hyperplanes, the group actions on the ambient spaces descend to the Calabi–Yau's. We first consider the case when the ambient space was the join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2$, which was denoted by M above.

Denote a unit matrix in the first factor of $(E \otimes E) \oplus (E \otimes E)$ by e^0_{ij}, and denote a unit matrix in the second factor by e^1_{ij}, where $0, 1$ are taken modulo 2.

In this case, one such invariant hyperplane is given by the span of

$$f^\alpha_{ij} = e^\alpha_{ij} + te^{\alpha+1}_{-i-j,-i-j} \in E \otimes E \oplus E \otimes E,$$

where $i \neq j \in \mathbb{Z}_3$ and $\alpha \in \mathbb{Z}_2$. Denote the intersection between M and H by X_{H_t}. Then the following is true:

Proposition 4.6.2. *Both the finite group D_6 and the group $\mathbb{Z}_3$ act on X_{H_t}. The symmetric variety X_{H_t} have 24 isolated singularities for $t \neq 0, 1$, and they come in two orbits under the D_6-action.*

For $t = 1$, it has 36 isolated singularities.

Check *which* singularities these are, also: fix points

There is also a torus action on E, defined by $e_i \mapsto \omega^i e_i$, where ω is a third root of unity. This a $\mathbb{Z}/3$-action, which extends to an action on $E \otimes E \oplus E \otimes E$. Let $H = \mathbb{Z}/3$. Then note that X_{H_t} is $\mathbb{Z}/3$-invariant as well.

identify H-invariant singularities + fix points

Appendices

APPENDIX A

Computer code

mputercode

Extensive use of computer software such as `Macaulay2` and SAGE has been invaluable during my work. In this Appendix I collect some computer code for reproducing some of my calculations.

A.1 Computing the singular locus

In some cases, equations simplify significantly in affine charts. Therefore, using the naive command `singularLocus` in `Macaulay2` often takes unnecessarily long time (and sometimes the computations never finish), as it computes the minors of a very large Jacobian matrix. Restricting to each affine chart, we can use the command `minimalPresentation` to eliminate variables to produce a new ring isomorphic to the first one, but with fewer equations.

The following code produces a list of the components of the singular locus of the projective scheme with ideal I.

```
fastSingularities = I -> (
    R := ring I;
    n := numgens R;breaklines=true,
    gensR := gens R;
    singlist := {};
    for i from 0 to (n-1) do {
        affineChart := I + ideal(gensR_i - 1);
        sing        := radical ideal mingens ideal singularLocus minimalPresentation affineChart;
        inv         := affineChart.cache.minimalPresentationMap;
        singlist = singlist | {(homogenize(preimage(inv,sing), gensR_i))};
        };
    saturate intersect(singlist)
    )
```

The method works by computing the singular locus in each affine chart, taking the radical, and then pulling back to the homogeneous coordinate ring.

Finally, we get a list of singular loci in each affine chart. We return the (saturation of) the intersection of the singular loci of the affine charts.

It is especially fast when computing the singular locus of toric varieties with low-dimensional singularities.

The following code finds the singular locus of the projectice cone $\overline{(}C(\mathbb{P}^2 \times \mathbb{P}^2)) \subset \mathbb{P}^9$.

```
R = QQ[x_0..x_8,x_9]
M = genericMatrix(R,3,3)
I = minors(2,M)
time fastSingularities I
time radical ideal singularLocus I
```

Our function performs significantly faster. On a modern Mac, the times are 1.14 seconds versus 4.31 seconds, respectively.

Here is a more involved example. Let Y' be four-dimensional toric variety from Chapter 4. It is defined by the 2×2-minors two matrices. In `Macaulay2` we can define it as follows:

```
S = QQ[x_1..x_6,z_1..z_6,y]
M1 = matrix{{y,x_1,x_2},{x_4,y,x_3},{x_5,x_6,y}}
M2 = matrix{{y,z_1,z_2},{z_4,y,z_3},{z_5,z_6,y}}
J = minors(2,M1) + minors(2,M2)
```

Here the performance difference is even more impressive. Our function computes the singular locus in 7.29 seconds, but the built-in function `singularLocus` used more than 22 minutes (at which point I interrupted the computation).

A.2 Torus action

The following lines checks if a projective scheme with ideal sheaf `IX` admits an action of a subtorus of $G = (\mathbb{C}^*)^n \subset \mathbb{P}^n$. To check this, we check if the equations are still valid after a torus action. Since G is abelian, it act on functions by $\lambda \cdot f(x_1, \ldots, x_n) = f(\lambda_1 x_1, \ldots, \lambda_n x_n)$.

Lemma A.2.1. *Suppose $\{f_1, \ldots, f_r\}$ is a homogeneous generating set for $I_X =$ `IX`. Then subgroup of G acting on $X \subset \mathbb{P}^n$ is generated by those $\lambda \in G$ such that $\lambda \cdot f_i = c f_i$ for some $c \in \mathbb{C}^*$.*

Proof. Let H be the subgroup of G fixing the ideal I_X. Let H' be the subgroup of $g \in G$ acting on the f_i by scalar multiplication: $g \cdot f_i = c f_i$. Clearly $H' \subseteq H$. Now suppose $g \in H$. Then

$$g \cdot f_1 = \sum_j a_j f_j$$

for some constants a_j. Now $g \cdot f_1 = f_1(\lambda_1 x_1, \ldots, \lambda_n x_n)$. Suppose the leading term of f_1 is $x_1^{a_1} \cdots x_n^{a_n}$. Then comparing leading terms in the left hand side and the right hand side, we see that $a_1 = \lambda_1^{a_1} \cdots \lambda_n^{a_n} := \lambda^m$. Hence the right hand side is $\lambda^m f_1 +$ other terms. But now there are the same number of terms on each side of the equation, so there are no other terms. Hence $H = H'$. ■

It follows that to find the subgroup of G acting on X, we have to find the $\lambda \in G$ such that the f_i are simultaneous eigenvectors for them.

Example A.2.2. Let X be defined by $f = x_0x_1x_2x_3x_4 + \sum_{i=0}^{5} x_i^5$ in $\mathbb{P}^4$. Then for $\mathbb{C}^4$ to act on it, we must have $\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4 = \lambda_0^5 = \ldots = \lambda_4^5$. By stting $\lambda_0 = 1$, we see that all the λ_i are the fifth roots of unity. Hence the subgroup acting on H is the subgroup of $\mathbb{Z}/5^5/Z_5$ given by $\{(a_0, \ldots, a_5) \mid \sum a_i = 0\}$.

The following code find the subtoruses of G acting on X in this way, by equating terms in the polynomials defining X.

```
loadPackage "Binomials"
torus = ideal apply(flatten apply(apply(apply(flatten entries gens
    IX, monomials), v ->  flatten entries v), j -> subsets(j,2)),
      s -> s_0-s_1)
toruskomps = BPD torus
toruskomps = select(toruskomps, I -> dim I == 1)
```

Explanation. The ideal `torus` is the ideal generated by the differences of terms in the polynomials defining X. The `Macaulay2` package `Binomials` can decompose binomials over cyclic extensions of $\mathbb{Q}$ with the command `BPD`. Finally, we select the components corresponding to finite subgroups of the torus.

Then we check manually if these actually correspond to non-trivial actions. ■

A.3 Computing the Gaifullin triangulation

_gaifullin

Below is a short SAGE script computing the 15 vertex triangulation of $\mathbb{CP}^2$ as described in [Gai09]. The last line returns a `SimplicialComplex` object in SAGE.

```
#Defines the Klein 4 group.
V4 = PermutationGroup([Permutation("(1,2)(3,4)"),Permutation("
    (1,3)(2,4)")])

def isValidFace(F):
    '''
    Assumes the first vertex is a permutation.
    Then checks if F satisfies the condition in the
    definition of T.
    '''
```

```
    g = F[0]
    for v in (1,2,3,4):
        if (F[g(v)][1] == F[v][1]):
            return False
    return True


# Makes a list of all possible maximal faces of the correct form
candidates = [(g,(1,a1),(2,a2),(3,a3),(4,a4)) for g in V4.list()[1:] for a1 in (1,2,3) for a2 in (1,2,3) for a3 in (1,2,3) for a4 in (1,2,3)]

# Filters out the faces not fullfilling the condition
maximalFacets = filter(lambda F: isValidFace(F), candidates)

# Renames the vertices
S = SimplicialComplex(maximalFacets)
vertexSet = S.vertices()
D = dict([(F,i) for i,F in enumerate(vertexSet)])
renamedMaximalFacets = [[D[v] for v in F] for F in maximalFacets]
SS = SimplicialComplex(renamedMaximalFacets)
```

To get the Stanley–Reisner ideal, one can write:

```
list(SS.stanley_reisner_ring().defining_ideal().gens())
```

The returned value is a list of the monomials generating the Stanley–Reisner ideal of $\mathcal{T}$. This can then be copied into `Macaulay2` for further computation.

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