

Algebraic Geometry Buzzlist

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1 Algebraic Geometry

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1.1 General terms

1.1.1 Cartier divisor

Let \mathcal{K}_X be the *sheaf of total quotients* on X , and let \mathcal{O}_X^* be the sheaf of non-zero divisors on X . We have an exact sequence

$$1 \rightarrow \mathcal{O}_X^* \rightarrow \mathcal{K}_X \rightarrow \mathcal{K}_X / \mathcal{O}_X^* \rightarrow 1.$$

Then a **Cartier divisor** is a global section of the quotient sheaf at the right.

1.1.2 Categorical quotient

Let X be a scheme and G a group. A **categorical quotient** is a morphism $\pi : X \rightarrow Y$ that satisfies the following two properties:

1. It is invariant, in the sense that $\pi \circ \sigma = \pi \circ p_2$ where $\sigma : G \times X \rightarrow X$ is the group action, and $p_2 : G \times X \rightarrow X$ is the projection. That is, the following diagram should commute:

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ p_2 \downarrow & & \downarrow \pi \\ X & \xrightarrow{\pi} & Y \end{array}$$

2. The map π should be *universal*, in the following sense: If $\pi' : X \rightarrow Z$ is any morphism satisfying the previous condition, it should uniquely factor through π . That is:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ \pi' \downarrow & \nearrow \exists! h & \\ Z & & \end{array}$$

Note: A categorical quotient need not be surjective.

1.1.3 Chow group

Let X be an algebraic variety. Let $Z_r(X)$ be the group of r -dimension cycles on X , a *cycle* being a \mathbb{Z} -linear combination of r -dimensional subvarieties of X . If $V \subset X$ is a subvariety of dimension $r + 1$ and $f : X \dashrightarrow \mathbb{A}^1$ is a rational function on X , then there is an integer $\text{ord}_W(f)$ for each codimension one subvariety of V , the order of vanishing of f . For a given f , there will only be finitely many subvarieties W for which this number is non-zero. Thus we can define an element $[\text{div}(f)]$ in $Z_r(X)$ by $\sum \text{ord}_W(f)[W]$.

We say that two r -cycles U_1, U_2 are *rationally equivalent* if there exist $r + 1$ -dimensional subvarieties V_1, V_2 together with rational functions $f_1 : V_1 \dashrightarrow \mathbb{A}^1$, $f_2 : V_2 \dashrightarrow \mathbb{A}^1$ such that $U_1 - U_2 = \sum_i [\text{div}(f_i)]$. The quotient group is called the **Chow group** of r -dimensional cycles on X , and denoted by $A_r(X)$.

1.1.4 Complete variety

Let X be an integral, **separated** scheme over a field k . Then X is **complete** if is **proper**.

Then \mathbb{P}^n is proper over any field, and \mathbb{A}^n is never proper.

1.1.5 Crepant resolution

A **crepant resolution** is a resolution of singularities $f : X \rightarrow Y$ that does not change the **canonical bundle**, i.e. such that $\omega_X \simeq f^*\omega_Y$.

1.1.6 Dominant map

A rational map $f : X \dashrightarrow Y$ is **dominant** if its image (or precisely: the image of one of its representatives) is dense in Y .

1.1.7 Étale map

A morphism of schemes of finite type $f : X \rightarrow Y$ is **étale** if it is smooth of dimension zero. This is equivalent to f being flat and $\Omega_{X/Y} = 0$. This again is equivalent to f being flat and unramified.

1.1.8 Genus

The **geometric genus** of a smooth, algebraic variety, is defined as the number of sections of the **canonical sheaf**, that is, as $H^0(V, \omega_X)$. This is often denoted p_X .

1.1.9 Geometric quotient

Let X be an algebraic variety and G an algebraic group. Then a **geometric quotient** is a morphism of varieties $\pi : X \rightarrow Y$ such that

1. For each $y \in Y$, the fiber $\pi^{-1}(y)$ is an orbit of G .
2. The topology of Y is the quotient topology: a subset U of Y is open if and only if $\pi^{-1}(U)$ is open.
3. For any open subset $U \subset Y$, $\pi^* : k[U] \rightarrow k[\pi^{-1}(U)]^G$ is an isomorphism of k -algebras.

The last condition may be rephrased as an isomorphism of structure sheaves: $\mathcal{O}_Y \simeq (\pi_* \mathcal{O}_X)^G$.

1.1.10 Hodge numbers

If X is a complex manifold, then the **Hodge numbers** $h^{p,q}$ of X are defined as the dimension of the cohomology groups $H^p(X, \Omega_X^q)$.

1.1.11 Linear series

A **linear series** on a smooth curve C is the data (\mathcal{L}, V) of a line bundle on C and a vector subspace $V \subseteq H^0(C, \mathcal{L})$. We say that the linear series (\mathcal{L}, V) have *degree* $\deg \mathcal{L}$ and *rank* $\dim V - 1$.

1.1.12 Log structure

A **prelog structure** on a scheme X is given by a pair (X, M) , where X is a scheme and M is a sheaf of monoids on X (on the **Étale site**) together with a morphism $\alpha : M \rightarrow \mathcal{O}_X$. It is a **log structure** if the map $\alpha : \alpha^{-1} \mathcal{O}_X^* \rightarrow \mathcal{O}_X^*$ is an isomorphism.

See [5].

1.1.13 Néron-Severi group

Let X be a nonsingular projective variety of dimension ≥ 2 . Then we can define the subgroup $\text{Cl}^\circ X$ of $\text{Cl } X$, the subgroup consisting of divisor classes algebraically equivalent to zero. Then $\text{Cl } X / \text{Cl}^\circ X$ is a finitely-generated group. It is denoted by $\text{NS}(X)$.

1.1.14 Normal crossings divisor

Let X be a smooth variety and $D \subset X$ a divisor. We say that D is a **simple normal crossing divisor** if every irreducible component of D is smooth and all intersections are transverse. That is, for every $p \in X$ we can choose local coordinates x_1, \dots, x_n and natural numbers m_1, \dots, m_n such that $D = (\prod_i x_i^{m_i} = 0)$ in a neighbourhood of p .

Then we say that a divisor is **normal crossing** (without the “simple”) if the neighbourhood above can be allowed to be chosen locally analytically or as a formal neighbourhood of p .

Example: the nodal curve $y^2 = x^3 + x^2$ is a normal crossing divisor in \mathbb{C}^2 , but not a simple normal crossing divisor.

This definition is taken from [6].

1.1.15 Normal variety

A variety X is **normal** if all its local rings are **normal rings**.

1.1.16 Picard number

The **Picard number** of a nonsingular projective variety is the rank of **Néron-Severi group**.

1.1.17 Proper morphism

A morphism $f : X \rightarrow Y$ is **proper** if it **separated**, of finite type, and universally closed.

1.1.18 Resolution of singularities

A morphism $f : X \rightarrow Y$ is a **resolution of singularities of Y** if X is non-singular and f is birational and **proper**.

1.1.19 Separated

Let $f : X \rightarrow Y$ be a morphism of schemes. Let $\Delta : X \rightarrow X \times_Y X$ be the diagonal morphism. We say that f is **separated** if Δ is a closed immersion. We say that X is **separated** if the unique morphism $f : X \rightarrow \text{Spec } \mathbb{Z}$ is separated.

This is equivalent to the following: for all open affines $U, V \subset X$, the intersection $U \cap V$ is affine and $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$ generate $\mathcal{O}_X(U \cap V)$. For example: let $X = \mathbb{P}^1$ and let $U_1 = \{[x : 1]\}$ and $U_2 = \{[1 : y]\}$. Then

$\mathcal{O}_X(U_1) = \text{Spec } k[x]$ and $\mathcal{O}_X(U_2) = \text{Spec } k[y]$. The glueing map is given on the ring level as $x \mapsto \frac{1}{y}$. Then $\mathcal{O}_X(U_1 \cap U_2) = k[y, \frac{1}{y}]$.

1.1.20 Unirational variety

A variety X is **unirational** if there exists a generically finite dominant map $\mathbb{P}^n \dashrightarrow X$.

1.2 Moduli theory and stacks

1.2.1 Étale site

Let S be a scheme. Then the **small étale site over S** is the **site**, denoted by $\mathring{\text{Ét}}(S)$ that consists of all étale morphisms $U \rightarrow S$ (morphisms being commutative triangles). Let $\text{Cov}(U \rightarrow S)$ consist of all collections $\{U_i \rightarrow U\}_{i \in I}$ such that

$$\coprod_{i \in I} U_i \rightarrow U$$

is surjective.

1.2.2 Grothendieck topology

Let \mathcal{C} be a category. A **Grothendieck topology** on \mathcal{C} consists of a set $\text{Cov}(X)$ of sets of morphisms $\{X_i \rightarrow X\}_{i \in I}$ for each X in $\text{Ob}(\mathcal{C})$, satisfying the following axioms:

1. If $V \xrightarrow{\sim} X$ is an isomorphism, then $\{V \rightarrow X\} \in \text{Cov}(X)$.
2. If $\{X_i \rightarrow X\}_{i \in I} \in \text{Cov}(X)$ and $Y \rightarrow X$ is a morphism in \mathcal{C} , then the fiber products $X_i \times_X Y$ exists and $\{X_i \times_X Y \rightarrow Y\}_{i \in I} \in \text{Cov}(Y)$.
3. If $\{X_i \rightarrow X\}_{i \in I} \in \text{Cov}(X)$, and for each $i \in I$, $\{V_{ij} \rightarrow X_i\}_{j \in J} \in \text{Cov}(X_i)$, then

$$\{V_{ij} \rightarrow X_i \rightarrow X\}_{i \in I, j \in J} \in \text{Cov}(X).$$

The easiest example is this: Let \mathcal{C} be the category of open sets on a topological space X , the morphisms being only the inclusions. Then for each $U \in \text{Ob}(\mathcal{C})$, define $\text{Cov}(U)$ to be the set of all coverings $\{U_i \rightarrow U\}_{i \in I}$ such that $U = \bigcup_{i \in I} U_i$. Then it is easily checked that this defines a Grothendieck topology.

1.2.3 Site

A **site** is a category equipped with a **Grothendieck topology**.

1.3 Results and theorems

1.3.1 Adjunction formula

Let X be a smooth algebraic variety Y a smooth subvariety. Let $i : Y \hookrightarrow X$ be the inclusion map, and let \mathcal{I} be the corresponding ideal sheaf. Then $\omega_Y = i^* \omega_X \otimes_{\mathcal{O}_X} \det(\mathcal{I}/\mathcal{I}^2)^\vee$, where ω_Y is the **canonical sheaf** of Y .

In terms of **canonical classes**, the formula says that $K_D = (K_X + D)|_D$.

Here's an example: Let X be a smooth quartic surface in \mathbb{P}^3 . Then $H^1(X, \mathcal{O}_X) = 0$. The divisor class group of \mathbb{P}^3 is generated by the class of a hyperplane, and $\mathcal{K}_{\mathbb{P}^3} = -4H$. The class of X is then $4H$ since X is of degree 4. X corresponds to a smooth divisor D , so by the adjunction formula, we have that

$$K_D = (K_{\mathbb{P}^3} + D)|_D = -4H + 4H|_D = 0.$$

Thus X is an example of a **K3 surface**.

1.3.2 Bertini's Theorem

Let X be a nonsingular closed subvariety of \mathbb{P}_k^n , where $k = \bar{k}$. Then the set of hyperplanes $H \subseteq \mathbb{P}_k^n$ such that $H \cap X$ is regular at every point) and such that $H \not\subseteq X$ is a dense open subset of the complete linear system $|H|$. See [4, Thm II.8.18].

1.3.3 Chow's lemma

Chow's lemma says that if X is a scheme that is proper over k , then it is "fairly close" to being projective. Specifically, we have that there exists a projective k -scheme X' and morphism $f : X' \rightarrow X$ that is birational.

So every scheme proper over k is birational to a projective scheme. For a proof, see for example the Wikipedia page.

1.3.4 Euler sequence

If A is a ring and \mathbb{P}_A^n is projective n -space over A , then there is an exact sequence of sheaves on X :

$$0 \rightarrow \Omega_{\mathbb{P}_A^n/A} \rightarrow \mathcal{O}_{\mathbb{P}_A^n}(-1)^{n+1} \rightarrow \mathcal{O}_{\mathbb{P}_A^n} \rightarrow 0.$$

See [4, Thm II.8.13].

1.3.5 Genus-degree formula

If C is a smooth plane curve, then its genus can be computed as

$$g_C = \frac{(d-1)(d-2)}{2}.$$

This follows from the [adjunction formula](#). In particular, there are no curves of genus 2 in the plane.

1.3.6 Hirzebruch-Riemann-Roch formula

Let X be a nonsingular variety and let \mathcal{T}_X be its tangent bundle. Let \mathcal{E} be a locally free sheaf on X . Then

$$\chi(\mathcal{E}) = \deg(\text{ch}(\mathcal{E}) \cdot \text{td}(\mathcal{T}_X))_n,$$

where χ is the Euler characteristic, ch denotes the Chern class, and td denotes the Todd class. See [\[4, Appendix A\]](#).

1.3.7 Hurwitz' formula

Let X, Y be smooth curves in the sense of Hartshorne. That is, they are integral 1-dimensional schemes, proper over a field k (with $\bar{k} = k$), all of whose local rings are regular.

Then Hurwitz' formula says that if $f : X \rightarrow Y$ is a separable morphism and $n = \deg f$, then

$$2(g_X - 1) = 2n(g_Y - 1) + \deg R,$$

where R is the ramification divisor of f , and g_X, g_Y are the genera of X and Y , respectively. See [Example 6.1.1](#).

1.3.8 Kodaira vanishing

If k is a field of characteristic zero, X is a smooth and projective k -scheme of dimension d , and \mathcal{L} is an [ample](#) invertible sheaf on X , then $H^q(X, \mathcal{L} \otimes_{\mathcal{O}_X} \Omega_{X/k}^p) = 0$ for $p + q > d$. In addition, $H^q(X, \mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \Omega_{X/k}^p) = 0$ for $p + q < d$.

1.3.9 Lefschetz hyperplane theorem

Let X be an n -dimensional complex projective algebraic variety in $\mathbb{P}_{\mathbb{C}}^n$ and let Y be a hyperplane section of X such that $U = X \setminus Y$ is smooth. Then the natural map $H^k(X, \mathbb{Z}) \rightarrow H^k(Y, \mathbb{Z})$ in singular cohomology is an isomorphism for $k < n - 1$ and injective for $k = n - 1$.

1.3.10 Riemann-Roch for curves

The **Riemann-Roch theorem** relates the number of sections of a line bundle with the genus of a smooth proper curve C . Let \mathcal{L} be a line bundle ω_C the canonical sheaf on C . Then

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^{-1} \otimes_{\mathcal{O}_C} \omega_C) = \deg(\mathcal{L}) + 1 - g.$$

This is [4, Theorem IV.1.3].

1.3.11 Semi-continuity theorem

Let $f : X \rightarrow Y$ be a projective morphism of noetherian schemes, and let \mathcal{F} be a coherent sheaf on X , flat over Y . Then for each $i \geq 0$, the function $h^i(y, \mathcal{F}) = \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$ is an upper semicontinuous function on Y . See [4, Chapter III, Theorem 12.8].

1.3.12 Serre duality

Let X be a projective Cohen-Macaulay scheme of equidimension n . Then for any locally free sheaf \mathcal{F} on X there are natural isomorphisms

$$H^i(X, \mathcal{F}) \simeq H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ).$$

Here ω_X° is a *dualizing sheaf* for X . In the case that X is nonsingular, we have that $\omega_X^\circ \simeq \omega_X$, the canonical sheaf on X (see [4, Chapter III, Corollary 7.12]).

1.3.13 Serre vanishing

One form of Serre vanishing states that if X is a proper scheme over a noetherian ring A , and \mathcal{L} is an **ample** sheaf, then for any coherent sheaf \mathcal{F} on X , there exists an integer n_0 such that for each $i > 0$ and $n \geq n_0$ the group $H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n) = 0$ vanishes. See [4, Proposition III.5.3].

1.4 Sheaves and bundles

1.4.1 Ample line bundle

A line bundle \mathcal{L} is **ample** if for any coherent sheaf \mathcal{F} on X , there is an integer n (depending on \mathcal{F}) such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is generated by global sections. Equivalently, a line bundle \mathcal{L} is ample if some tensor power of it is **very ample**.

1.4.2 Invertible sheaf

A locally free sheaf of rank 1 is called **invertible**. If X is **normal**, then, invertible sheaves are in 1 – 1 correspondence with line bundles.

1.4.3 Anticanonical sheaf

The **anticanonical sheaf** ω_X^{-1} is the inverse of the **canonical sheaf** ω_X , that is $\omega_X^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{O}_X)$.

1.4.4 Canonical class

The **canonical class** K_X is the class of the **canonical sheaf** ω_X in the divisor class group.

1.4.5 Canonical sheaf

If X is a smooth algebraic variety of dimension n , then the canonical sheaf is $\omega := \wedge^n \Omega_{X/k}^1$ the n 'th exterior power of the cotangent bundle of X .

1.4.6 Nef divisor

Let X be a normal variety. Then a Cartier divisor D on X is **nef** (*numerically effective*) if $D \cdot C \geq 0$ for every irreducible complete curve $C \subseteq X$. Here $D \cdot C$ is the intersection product on X defined by $\deg(\phi^* \mathcal{O}_X(D))$. Here $\phi : C' \rightarrow C$ is the normalization of C .

1.4.7 Sheaf of holomorphic p-forms

If X is a complex manifold, then the **sheaf of holomorphic p -forms** Ω_X^p is the p -th wedge power of the cotangent sheaf $\wedge^p \Omega_X^1$.

1.4.8 Normal sheaf

Let $Y \hookrightarrow X$ be a closed immersion of schemes, and let $\mathcal{I} \subseteq \mathcal{O}_X$ be the ideal sheaf of Y in X . Then $\mathcal{I}/\mathcal{I}^2$ is a sheaf on Y , and we define the sheaf $\mathcal{N}_{Y/X}$ by $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$.

1.4.9 Rank of a coherent sheaf

Given a coherent sheaf \mathcal{F} on an irreducible variety X , form the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$. Its global sections is a finite dimensional vector space, and we say that \mathcal{F} has rank r if $\dim_k \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X) = r$.

1.4.10 Reflexive sheaf

A sheaf \mathcal{F} is **reflexive** if the natural map $\mathcal{F} \rightarrow \mathcal{F}^{\vee\vee}$ is an isomorphism. Here \mathcal{F}^{\vee} denotes the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

1.4.11 Very ample line bundle

A line bundle \mathcal{L} is **very ample** if there is an embedding $i : X \hookrightarrow \mathbb{P}_S^n$ such that the pullback of $\mathcal{O}_{\mathbb{P}_S^n}(1)$ is isomorphic to \mathcal{L} . In other words, there should be an isomorphism $i^* \mathcal{O}_{\mathbb{P}_S^n}(1) \simeq \mathcal{L}$.

1.5 Singularities

1.5.1 Canonical singularities

A variety X has **canonical singularities** if it satisfies the following two conditions:

1. For some integer $r \geq 1$, the Weil divisor rK_X is Cartier (equivalently, it is \mathbb{Q} -Cartier).
2. If $f : Y \rightarrow X$ is a resolution of X and $\{E_i\}$ the exceptional divisors, then

$$rK_Y = f^*(rK_X) + \sum a_i E_i$$

with $a_i \geq 0$.

The integer r is called the *index*, and the r_i are called the *discrepancies* at E_i .

1.5.2 Terminal singularities

A variety X have **terminal singularities** if the a_i in the definition of **canonical singularities** are all greater than zero.

1.5.3 Ordinary double point

An **ordinary double point** is a singularity that is analytically isomorphic to $x^2 = yz$.

1.6 Toric geometry

1.6.1 Chow group of a toric variety

The **Chow group** $A_{n-1}(X)$ of a toric variety can be computed directly from its fan. Let $\Sigma(1)$ be the set of rays in Σ , the fan of X . Then we have an exact sequence

$$0 \rightarrow M \rightarrow \mathbb{Z}^{\Sigma(1)} \rightarrow A_{n-1}(X) \rightarrow 0.$$

The first map is given by sending $m \in M$ to $(\langle m, v_p \rangle)_{p \in \Sigma(1)}$, where v_p is the unique generator of the semigroup $\rho \cap N$. The second map is given by sending $(a_\rho)_{\rho \in \Sigma(1)}$ to the divisor class of $\sum_\rho a_\rho D_\rho$.

1.6.2 Generalized Euler sequence

The **generalized Euler sequence** is a generalization of the **Euler sequence** for toric varieties. If X is a smooth toric variety, then its cotangent bundle Ω_X^1 fits into an exact sequence

$$0 \rightarrow \Omega_X^1 \rightarrow \oplus_\rho \mathcal{O}_X(-D_\rho) \rightarrow \text{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \rightarrow 0.$$

Here D_ρ is the divisor corresponding to the ray $\rho \in \Sigma(1)$. See [2, Chapter 8].

1.6.3 Polarized toric variety

A toric variety equipped with an **ample** T -invariant divisor.

1.6.4 Toric variety associated to a polytope

There are several ways to do this. Here is one: Let $\Delta \subset M_{\mathbb{R}}$ be a convex polytope. Embed Δ in $M_{\mathbb{R}} \times \mathbb{R}$ by $\Delta \times \{1\}$ and let C_Δ be the cone over $\Delta \times \{1\}$, and let $\mathbb{C}[C_\Delta \cap (M \times \mathbb{Z})]$ be the corresponding semigroup ring. This is a semigroup ring graded by the \mathbb{Z} -factor. Then we define $\mathbb{P}_\Delta = \text{Proj } \mathbb{C}[C_\Delta \cap (M \times \mathbb{Z})]$ to be the toric variety associated to a polytope.

1.7 Types of varieties

1.7.1 Abelian variety

A variety X is an **abelian variety** if it is a connected and **complete** algebraic group over a field k . Examples include **elliptic curves** and for special lattices $\Lambda \subset \mathbb{C}^{2g}$, the quotient \mathbb{C}^{2g}/Λ is an abelian variety.

1.7.2 Calabi-Yau variety

In algebraic geometry, a **Calabi-Yau** variety is a smooth, proper variety X over a field k such that the **canonical sheaf** is trivial, that is, $\omega_X \simeq \mathcal{O}_X$, and such that $H^j(X, \mathcal{O}_X) = 0$ for $1 \leq j \leq n - 1$.

1.7.3 del Pezzo surface

A **del Pezzo** surface is a 2-dimensional **Fano variety**. In other words, they are complete non-singular surfaces with ample anticanonical bundle. The *degree* of the del Pezzo surface X is by definition the self intersection number $K.K$ of its **canonical class** K .

1.7.4 Elliptic curve

An **elliptic curve** is a smooth, projective curve of genus 1. They can all be obtained from an equation of the form $y^2 = x^3 + ax + b$ such that $\Delta = -2^4(4a^3 + 27b^2) \neq 0$.

1.7.5 Elliptic surface

An **elliptic surface** is a smooth surface X with a morphism $\pi : X \rightarrow B$ onto a non-singular curve B whose generic fiber is a non-singular **elliptic curve**.

1.7.6 Fano variety

A variety X is **Fano** if the **anticanonical sheaf** ω_X^{-1} is **ample**.

1.7.7 Jacobian variety

Let X be a curve of genus g over k . The **Jacobian variety** of X is a scheme J of finite type over k , together with an element $\mathcal{L} \in \text{Pic}^\circ(X/J)$, with the following universal property: for any scheme T of finite type over k and for any $\mathcal{M} \in \text{Pic}^\circ(X/T)$, there is a unique morphism $f : T \rightarrow J$ such that $f^*\mathcal{L} \simeq \mathcal{M}$ in $\text{Pic}^\circ(X/T)$. This just says that J represents the functor $T \mapsto \text{Pic}^\circ(X/T)$.

If J exists, its closed points are in 1 – 1 correspondence with elements of $\text{Pic}^\circ(X)$.

It can be checked that J is actually a group scheme. For details, see [4, Ch. IV.4].

1.7.8 K3 surface

A **K3 surface** is a complex algebraic surface X such that the **canonical sheaf** is trivial, $\omega_X \simeq \mathcal{O}_X$, and such that $H^1(X, \mathcal{O}_X) = 0$. These conditions completely determine the Hodge numbers of X .

1.7.9 Stanley-Reisner scheme

A **Stanley-Reisner scheme** is a projective variety associated to a simplicial complex as follows. Let \mathcal{K} be a simplicial complex. Then we define an ideal $I_{\mathcal{K}} \subseteq k[x_v \mid v \in V(\mathcal{K})] = k[\mathbf{x}]$ (here $V(\mathcal{K})$ denotes the vertex set of \mathcal{K}) by

$$I_{\mathcal{K}} = \langle x_{v_{i_1}} x_{v_{i_2}} \cdots x_{v_{i_k}} \mid v_{i_1} v_{i_2} \cdots v_{i_k} \notin \mathcal{K} \rangle.$$

We get a projective scheme $\mathbb{P}(\mathcal{K})$ defined by $\text{Proj}(k[\mathbf{x}]/I_{\mathcal{K}})$, together with an embedding into $\mathbb{P}^{\#V(\mathcal{K})-1}$. It can be shown that $H^p(\mathbb{P}(\mathcal{K}), \mathcal{O}_{\mathbb{P}(\mathcal{K})}) \simeq H^p(\mathcal{K}; k)$, where the right-hand-side denotes the cohomology group of the simplicial complex.

1.7.10 Toric variety

A **toric variety** X is an integral scheme containing the torus $(k^*)^n$ as a dense open subset, such that the action of the torus on itself extends to an action $(k^*)^n \times X \rightarrow X$.

2 Commutative algebra

2.1 Modules

2.1.1 Depth

Let R be a noetherian ring, and M a finitely-generated R -module and I an ideal of R such that $IM \neq M$. Then the I -depth of M is (see **Ext**):

$$\inf\{i \mid \text{Ext}_R^i(R/I, M) \neq 0\}.$$

This is also the length of a maximal M -sequence in I .

2.1.2 Rank

If R has the invariant basis property (IBN), then we define the **rank** of a *free* module to be the cardinality of any basis.

2.1.3 Kähler differentials

Let $A \rightarrow B$ be a ring homomorphism. The **module of Kähler differentials** $\Omega_{B/A}$ is the module together with a map $d : B \rightarrow \Omega_{B/A}$ satisfying the following universal property: if $D : B \rightarrow M$ is any A -linear derivation (an element of $\text{Der}_A(B, M)$), then there is a unique module homomorphism $\tilde{D} : \Omega_{B/A} \rightarrow M$ such that

$$\begin{array}{ccc} B & \xrightarrow{d} & \Omega_{B/A} \\ & \searrow D & \downarrow \tilde{D} \\ & & M \end{array}$$

is commutative. Thus we have a natural isomorphism $\text{Der}_A(B, M) = \text{Hom}_B(\Omega_{B/A}, M)$. In the language of category theory, this means that $\text{Der}_A(B, -)$ is *corepresented* by $\Omega_{B/A}$.

A concrete construction of $\Omega_{B/A}$ is given as follows. Let M be the free B -module generated by all symbols df , where $f \in B$. Let N be the submodule generated by da if $a \in A$, $d(f + g) - df - dg$ and the Leibniz rule $d(fg) - f dg - g df$. Then $M/N \simeq \Omega_{B/A}$ as B -modules.

2.2 Results and theorems

2.2.1 The conormal sequence

The **conormal sequence** is a sequence relating Kähler differentials in different rings. Specifically, if $A \rightarrow B \rightarrow 0$ is a surjection of rings with kernel I , then we have an exact sequence of B -modules:

$$I/I^2 \xrightarrow{d} B \otimes_A \Omega_{B/A} \xrightarrow{D\pi} \Omega_{T/R} \rightarrow 0$$

The map d sends $f \mapsto 1 \otimes df$, and $D\pi$ sends $c \otimes db \mapsto cdb$. For proof, see [3, Chapter 16].

2.2.2 The Unmixedness Theorem

Let R be a ring. If $I = \langle x_1, \dots, x_n \rangle$ is an ideal generated by n elements such that $\text{codim } I = n$, then all minimal primes of I have codimension n . If in addition R is **Cohen-Macaulay**, then every associated prime of I is minimal over I . See the discussion after [3, Corollary 18.14] for more details.

2.3 Rings

2.3.1 Cohen-Macaulay ring

A local Cohen-Macaulay ring (CM-ring for short) is a commutative noetherian local ring with Krull dimension equal to its depth. A ring is Cohen-Macaulay if its localization at all prime ideals are Cohen-Macaulay.

2.3.2 Depth of a ring

The depth of a ring R is its depth as a module over itself.

2.3.3 Gorenstein ring

A commutative ring R is Gorenstein if each localization at a prime ideal is a Gorenstein local ring. A Gorenstein local ring is a local ring with finite injective dimension as an R -module. This is equivalent to the following: $\text{Ext}_R^i(k, R) = 0$ for $i \neq n$ and $\text{Ext}_R^n(k, R) \simeq k$ (here $k = R/\mathfrak{m}$ and n is the Krull dimension of R).

2.3.4 Invariant basis property

A ring R satisfies the **invariant basis property** (IBP) if $R^n \not\cong R^{n+t}$ R -modules for any $t \neq 0$.

2.3.5 Normal ring

An integral domain R is **normal** if all its localizations at prime ideals $\mathfrak{p} \in \text{Spec } R$ are integrally closed domains.

3 Convex geometry

3.1 Cones

3.1.1 Gorenstein cone

A strongly convex cone $C \subset M_{\mathbb{R}}$ is **Gorenstein** if there exists a point $n \in N$ in the dual lattice such that $\langle v, n \rangle = 1$ for all generators of the semigroup $C \cap M$.

3.1.2 Reflexive Gorenstein cone

A cone C is **reflexive** if both C and its dual C^\vee are **Gorenstein cones**. See for example [1].

3.1.3 Simplicial cone

A cone C generated by $\{v_1, \dots, v_k\} \subseteq N_{\mathbb{R}}$ is **simplicial** if the v_i are linearly independent.

3.2 Polytopes

3.2.1 Dual (polar) polytope

If Δ is a polyhedron, its dual Δ° is defined by

$$\Delta^\circ = \{x \in N_{\mathbb{R}} \mid \langle x, y \rangle \geq -1 \forall y \in \Delta\}.$$

3.2.2 Gorenstein polytope of index r

A lattice polytope $P \subset \mathbb{R}^{d+r-1}$ is called a **Gorenstein polytope of index r** if rP contains a single interior lattice point p and $rP - p$ is a **reflexive polytope**.

3.2.3 Nef partition

Let $\Delta \subset M_{\mathbb{R}}$ be a d -dimensional **reflexive polytope**, and let $m = \text{int}(\Delta) \cap M$. A Minkowski sum decomposition $\Delta = \Delta_1 + \dots + \Delta_r$ where $\Delta_1, \dots, \Delta_r$ are lattice polytopes is called a **nef partition of Δ of length r** if there are lattice points $p_i \in \Delta_i$ for all i such that $p_1 + \dots + p_r = m$. The nef partition is called *centered* if $p_i = 0$ for all i .

This is equivalent to the toric divisor $D_j = \mathcal{O}(\Delta_j) = \sum_{\rho \in \Delta_j} D_\rho$ being a Cartier divisor generated by its global sections. See [1, Chapter 4.3].

3.2.4 Reflexive polytope

A polytope Δ is **reflexive** if the following two conditions hold:

1. All facets Γ of Δ are supported by affine hyperplanes of the form $\{m \in M_{\mathbb{R}} \mid \langle m, v_\Gamma \rangle = -1\}$ for some $v_\Gamma \in N$.
2. The only interior point of Δ is 0, that is: $\text{Int}(\Delta) \cap M = \{0\}$.

It can be proved that a polytope Δ is reflexive if and only if the associated toric variety \mathbb{P}_Δ is **Fano**.

4 Homological algebra

4.1 Classes of modules

4.1.1 Projective modules

Projective modules are those satisfying a universal lifting property. A module P is **projective** if for every epimorphism $\alpha : M \rightarrow N$ and every map, $\beta : P \rightarrow N$, there exists a map $\gamma : P \rightarrow M$ such that $\beta = \alpha \circ \gamma$.

$$\begin{array}{ccc} & P & \\ \swarrow \exists \gamma & \downarrow \beta & \\ M & \xrightarrow{\alpha} N & \longrightarrow 0 \end{array}$$

These are the modules P such that $\text{Hom}(P, -)$ is exact.

4.2 Derived functors

4.2.1 Ext

Let R be a ring and M, N be R -modules. Then $\text{Ext}_R^i(M, N)$ is the right-derived functors of the $\text{Hom}(M, -)$ -functor. In particular, $\text{Ext}_R^i(M, N)$ can be computed as follows: choose a projective resolution C_\bullet of N over R . Then apply the left-exact functor $\text{Hom}_R(M, -)$ to the resolution and take homology. Then $\text{Ext}_R^i(M, N) = h^i(C_\bullet)$.

4.2.2 Local cohomology

Let R be a ring and $I \subset R$ an ideal. Let $\Gamma_I(-)$ be the following functor on R -modules:

$$\Gamma_I(M) = \{f \in M \mid \exists n \in \mathbb{N}, s.t. I^n f = 0\}.$$

Then $H_I^i(-)$ is by definition the i th right derived functor of Γ_I . In the case that R is noetherian, we have $H_I^i(M) = \varinjlim \text{Ext}_R^i(R/I_n, M)$.

See [3] and [7] for more details.

4.2.3 Tor

Let R be a ring and M, N be R -modules. Then $\text{Tor}_R^i(M, N)$ is the right-derived functors of the $- \otimes_R N$ -functor. In particular $\text{Tor}_R^i(M, N)$ can be computed by taking a projective resolution of M , tensoring with N , and then taking homology.

5 Differential and complex geometry

5.1 Definitions and concepts

5.1.1 Almost complex structure

An **almost complex structure** on a manifold M is a map $J : T(M) \rightarrow T(M)$ whose square is -1 .

5.1.2 Connection

Let $E \rightarrow M$ be a vector bundle over M . A **connection** is a \mathbb{R} -linear map $\nabla : \Gamma(E) \rightarrow \Gamma(E \otimes T^*M)$ such that the Leibniz rule holds:

$$\nabla(f\sigma) = f\nabla(\sigma) + \sigma \otimes df$$

for all functions $f : M \rightarrow \mathbb{R}$ and sections $\sigma \in \Gamma(E)$.

5.1.3 Hermitian manifold

A *Hermitian metric* on a complex vector bundle E over a manifold M is a positive-definite Hermitian form on each fiber. Such a metric can be written as a smooth section $\Gamma(E \otimes \bar{E})^*$, such that $h_p(\eta, \bar{\zeta}) = h_p(\bar{\zeta}, \eta)$ for all $p \in M$, and such that $h_p(\eta, \bar{\eta}) > 0$ for all $p \in M$. A **Hermitian manifold** is a complex manifold with a Hermitian metric on its holomorphic tangent space $T^{(1,0)}(M)$.

5.1.4 Kähler manifold

A **Kähler manifold** is ????

5.1.5 Symplectic manifold

A $2n$ -dimensional manifold M is **symplectic** if it is compact and oriented and has a closed real two-form $\omega \in \bigwedge^2 T^*(M)$ which is nondegenerate, in the sense that $\wedge^n \omega|_p \neq 0$ for all $p \in M$.

5.2 Results and theorems

6 Worked examples

6.1 Algebraic geometry

6.1.1 Hurwitz formula and Kähler differentials

Let X be the conic in \mathbb{P}^2 given with ideal sheaf $\langle xz - y^2 \rangle$. Let Y be \mathbb{P}^1 , and consider the map $f : X \rightarrow Y$ given by projection onto the xz -line. X is covered by two affine pieces, namely $X = U_x \cup U_z$, the spectra of the homogeneous localizations at x, z , respectively. Let $U_x = \text{Spec } A$ for $A = k[z]$ and $U_z = \text{Spec } B$ for $B = k[x]$. Then the map is locally given by $A \rightarrow k[y, z]/(z - y^2)$ where $z \mapsto \bar{z}$, and similarly for B . We have an isomorphism $k[y, z]/(z - y^2) \simeq k[t]$, given by $y \mapsto t$ and $z \mapsto t^2$, so that locally the map is given by $k[z] \rightarrow k[t], z \mapsto t^2$.

This is a map of smooth projective curves, so we can apply Hurwitz' formula. Both X, Y are \mathbb{P}^1 , so both have genus zero. Hence Hurwitz formula says that

$$-2 = -n \cdot 2 + \deg R,$$

where R is the ramification divisor and n is the degree of the map. The degree of the map can be defined locally, and it is the degree of the field extension $k(Y) \hookrightarrow k(X)$. But (the image of) $k(Y) = k(t^2)$ and $k(X) = k(t)$, so that $[k(Y) : k(X)] = 2$. Hence by Hurwitz' formula, we should have $\deg R = 2$. Since $R = \sum_{P \in X} \text{length } \Omega_{X/Y_P} \cdot P$, we should look at the sheaf of relative differentials $\Omega_{Y/X}$.

First we look in the chart U_z . We compute that $\Omega_{k[t]/k[t^2]} = k[t]/(t)$. This follows from the relation $d(t^2) = 2dt$, implying that $dt = 0$ in $\Omega_{k[t]/k[t^2]}$. This module is zero localized at all primes but (t) , where it is k . Thus for $P = (0 : 0 : 1)$, we have $\text{length } \Omega_{X/Y_P} = 1$.

The situation is symmetric with $z \leftrightarrow x$, so that we have $R = (0 : 0 : 1) + (1 : 0 : 0)$, confirming that $\deg R = 2$.

In fact, the curve C is isomorphic to \mathbb{P}^1 via the map $\mathbb{P}^1 \rightarrow C$ given by $(s : t) \mapsto (s^2 : st : t^2)$. Identifying C with \mathbb{P}^1 , we thus see that $C \rightarrow \mathbb{P}^1$ correspond to the map $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ given by $(s : t) \mapsto (s^2 : t^2)$.

6.2 The quintic threefold

Let Y be a the zeroes of a general hypersurface of degree 5 in \mathbb{P}^4 , or in other words, a section of $\omega_{\mathbb{P}^4}$. We want to compute the cohomology of Y and its Hodge numbers. Let $\mathbb{P} = \mathbb{P}^4$.

We have the ideal sheaf sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow i^* \mathcal{O}_Y \rightarrow 0,$$

where $i : Y \rightarrow \mathbb{P}^4$ is the inclusion. Note that $\mathcal{I} = \mathcal{O}_{\mathbb{P}}(-5)$. Thus we have from the long exact sequence of cohomology that

$$\dots \rightarrow H^i(\mathbb{P}, \mathcal{I}) \rightarrow H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) \rightarrow H^i(Y, \mathcal{O}_Y) \rightarrow H^{i+1}(\mathbb{P}, \mathcal{I}) \rightarrow \dots$$

Note that $H^{i+1}(\mathbb{P}, \mathcal{I}) = 0$ for $i \neq 3$ and 1 for $i = 3$. Also $H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = 0$ unless $i = 0$ in which case it is 1. Thus we get that $H^i(Y, \mathcal{O}_Y)$ is k for $i = 0$, for $i = 1, 2$ it is 0, and for $i = 3$ it is k . For higher i it is zero by Grothendieck vanishing.

The adjunction formula relates the canonical bundles as follows: if $\omega_{\mathbb{P}}$ is the canonical bundle on \mathbb{P} , then $\omega_Y = i^* \omega_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \det(\mathcal{I}/\mathcal{I}^2)^\vee$. The ideal sheaf is already a line bundle, so taking the determinant does not change anything. Now

$$\begin{aligned} (\mathcal{I}/\mathcal{I}^2)^\vee &= \text{Hom}_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \\ &= \text{Hom}_{\mathbb{P}}(\mathcal{I}, \mathcal{O}_Y) = \text{Hom}_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(-5), \mathcal{O}_Y) = \mathcal{O}_Y(5). \end{aligned}$$

It follows that $\omega_Y = \mathcal{O}_Y(-5) \otimes \mathcal{O}_Y(5) = \mathcal{O}_Y$. Thus the canonical bundle is trivial and we conclude that Y is Calabi-Yau.

It remain to compute the Hodge numbers. We start with $h^{11} = \dim_k H^1(Y, \Omega_Y)$. We have the conormal sequence of sheaves on Y :

$$0 \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{\mathbb{P}} \otimes \mathcal{O}_Y \rightarrow \Omega_Y \rightarrow 0,$$

which gives us the long exact sequence:

$$\dots \rightarrow H^i(\mathcal{I}/\mathcal{I}^2) \rightarrow H^i(\Omega_{\mathbb{P}} \otimes \mathcal{O}_Y) \rightarrow H^i(\Omega_Y) \rightarrow H^{i+1}(\mathcal{I}/\mathcal{I}^2) \rightarrow \dots$$

We first compute the cohomology of $\mathcal{I}/\mathcal{I}^2$. We use the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}(-10) \rightarrow \mathcal{O}_{\mathbb{P}}(-5) \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow 0. \quad (1)$$

we have $H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-10)) = 0$ for $i = 0, 1, 2, 3$, and for $i = 4$ we have $H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-10)) = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(5)) = k^{126}$. Similarly $H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-5)) = 0$ for $i = 0, 1, 2, 3$ and $H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-5)) = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = k$. We conclude that $h^i(Y, \mathcal{I}/\mathcal{I}^2) = 0$ for $i = 0, 1, 2$ and 125 for $i = 3$.

In particular $H^1(\Omega_Y) \simeq H^1(\Omega_{\mathbb{P}} \otimes \mathcal{O}_Y)$. We have the Euler sequence:

$$0 \rightarrow \Omega_{\mathbb{P}} \rightarrow \mathcal{O}_{\mathbb{P}}(-1)^{\oplus 5} \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow 0$$

Now $\mathcal{O}_Y = \mathcal{O}_{\mathbb{P}}/\mathcal{I}$ is a flat $\mathcal{O}_{\mathbb{P}}$ -module since \mathcal{I} is principal and generated by a non-zero divisor. Thus we can tensor the Euler sequence with \mathcal{O}_Y and get

$$0 \rightarrow \Omega_{\mathbb{P}} \otimes \mathcal{O}_Y \rightarrow \mathcal{O}_Y(-1)^5 \rightarrow \mathcal{O}_Y \rightarrow 0,$$

from which it easily follows that $H^1(Y, \Omega_{\mathbb{P}} \otimes \mathcal{O}_Y) \simeq H^0(\mathcal{O}_Y) = k$. We conclude that $h^{11} = 1$.

Now we compute $h^{12} = \dim_k H^1(Y, \Omega_Y^2)$. This is equal to $H^2(Y, \Omega_Y)$ by Serre duality. Again we use the conormal sequence. From the Euler sequence we get that $H^2(Y, \Omega_{\mathbb{P}} \otimes \mathcal{O}_Y) = 0$. We also get that $h^3(Y, \Omega_{\mathbb{P}} \otimes \mathcal{O}_Y) = 24$. NOW $H^3(\Omega_Y) = 0$ (WHY??), and it follows from the above computations that $h^{12} = 125 - 24 = 101$.

This example is extremely important in mirror symmetry.

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