Notes mirror symmetry

Fredrik Meyer

October 13, 2014

1 The quintic

We want to completely understand the quintic in $\mathbb{P} = \mathbb{P}^4$ and its mirror. The quintic Calabi-Yau is defined to be the zero set of a general quintic in $H^0(\mathbb{P}, \mathscr{O}_X(5))$. Note that since $\omega_{\mathbb{P}} \simeq \mathscr{O}_{\mathbb{P}}(-5)$, this is also just a general section of the anticanonical bundle on $\mathbb{P}$.

Recall the defintion of Calabi-Yau:

Definition 1.1. An algebraic variety X is *Calabi-Yau* if $H^i(X, \mathscr{O}_X) = 0$ for $i \neq 0, n$ (where $n = \dim X$) and $\omega_X = \wedge^n \Omega_{X/k}$ is trivial, that is, $\omega_X \simeq \mathscr{O}_X$. ■

Denote the quintic by $Y \subset \mathbb{P}$. We want to show that Y is Calabi-Yau.

Proposition 1.2. *A general section of $H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(5))$ is Calabi-Yau. In addition $h^{11} = 1$ and $h^{12} = 101$.*

Proof. We have the ideal sheaf sequence

$$0 \to \mathscr{I} \to \mathscr{O}_{\mathbb{P}} \to i^* \mathscr{O}_Y \to 0,$$

where $i : Y \to \mathbb{P}^4$ is the inclusion. Note that $\mathscr{I} = \mathscr{O}_{\mathbb{P}}(-5)$. Thus we have from the long exact sequence of cohomology that

$$\cdots \to H^i(\mathbb{P}, \mathscr{I}) \to H^i(\mathbb{P}, \mathscr{O}_{\mathbb{P}}) \to H^i(Y, \mathscr{O}_Y) \to H^{i+1}(\mathbb{P}, \mathscr{I}) \to \cdots$$

Note that $H^{i+1}(\mathbb{P}, \mathscr{I}) = 0$ for $i \neq 3$ and 1 for $i = 3$. Also $H^i(\mathbb{P}, \mathscr{O}_{\mathbb{P}}) = 0$ unless $i = 0$ in which case it is 1. Thus we get that $H^i(Y, \mathscr{O}, Y)$ is k for $i = 0$, for $i = 1, 2$ it is 0, and for $i = 3$ it is k. For higher i it is zero by Grothendieck vanishing.

The adjunction formula relates the canonical bundles as follows: if $\omega_{\mathbb{P}}$ is the canonical bundle on $\mathbb{P}$, then $\omega_Y = i^*\omega_{\mathbb{P}} \otimes_{\mathscr{O}_{\mathbb{P}}} \det(\mathscr{I}/\mathscr{I}^2)^\vee$. The ideal sheaf is already a line bundle, so taking the determinant does not change anything. Now

$$\begin{aligned}(\mathscr{I}/\mathscr{I}^2)^\vee &= \mathrm{Hom}_Y(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_Y) \\ &= \mathrm{Hom}_X(\mathscr{I}, \mathscr{O}_Y) = \mathrm{Hom}_X(\mathscr{O}_Y(-5), \mathscr{O}_Y) = \mathscr{O}_Y(5).\end{aligned}$$

It follows that $\omega_Y = \mathscr{O}_Y(-5) \otimes \mathscr{O}_Y(5) = \mathscr{O}_Y$. Thus the canonical bundle is trivial and we conclude that Y is Calabi-Yau.

It remain to compute the Hodge numbers. We start with $h^{11} = \dim_k H^1(Y, \Omega_Y)$. We have the conormal sequence of sheaves on Y:

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_{\mathbb{P}} \otimes \mathscr{O}_Y \to \Omega_Y \to 0,$$

which gives us the long exact sequence:

$$\cdots \to H^i(\mathscr{I}/\mathscr{I}^2) \to H^i(\Omega_{\mathbb{P}} \otimes \mathscr{O}_Y) \to H^i(\Omega_Y) \to H^{i+1}(\mathscr{I}/\mathscr{I}^2) \to \cdots$$

We first compute the cohomology of $\mathscr{I}/\mathscr{I}^2$. We use the short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}}(-10) \to \mathscr{O}_{\mathbb{P}}(-5) \to \mathscr{I}/\mathscr{I}^2 \to 0. \tag{1}$$

we have $H^i(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(-10)) = 0$ for $i = 0, 1, 2, 3$, and for $i = 4$ we have $H^4(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(-10)) = H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(5)) = k^{126}$. Similarly $H^i(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(-5)) = 0$ for $i = 0, 1, 2, 3$ and $H^4(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(-5)) = H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}) = k$. We conclude that $h^i(Y, \mathscr{I}/\mathscr{I}^2) = 0$ for $i = 0, 1, 2$ and 125 for $i = 3$.

In particular $H^1(\Omega_Y) \simeq H^1(\Omega_{\mathbb{P}} \otimes \mathscr{O}_Y)$. We have the Euler sequence:

$$0 \to \Omega_{\mathbb{P}} \to \mathscr{O}_{\mathbb{P}}(-1)^{\oplus 5} \to \mathscr{O}_{\mathbb{P}} \to 0$$

Now $\mathscr{O}_Y = \mathscr{O}_{\mathbb{P}}/\mathscr{I}$ is a flat $\mathscr{O}_P P$-module since $\mathscr{I}$ is principal and generated by a non-zero divisor. Thus we can tensor the Euler sequence with $\mathscr{O}_Y$ and get

$$0 \to \Omega_{\mathbb{P}} \otimes \mathscr{O}_Y \to \mathscr{O}_Y(-1)^5 \to \mathscr{O}_Y \to 0,$$

from which it easily follows that $H^1(Y, \Omega_{\mathbb{P}} \otimes \mathscr{O}_Y) \simeq H^0(\mathscr{O}_Y) = k$. We conclude that $h^{11} = 1$.

Now we compute $h^{12} = \dim_k H^1(Y, \Omega^2)$. This is equal to $H^2(Y, \Omega_Y)$ by Serre duality. Again we use the conormal sequence. From the Euler sequence we get that $H^2(Y, \Omega_{\mathbb{P}} \otimes \mathscr{O}_Y) = 0$. We also get that $h^3(Y, \Omega_{\mathbb{P}} \otimes \mathscr{O}_Y) = 24$. NOW $H^3(\Omega_Y) = 0$ (WHY??), and it follows from the above computations that $h^{12} = 125 - 24 = 101$. $\square$

The moduli space of all quintics is 101-dimensional, and we can see that as follows: the space of all quintic polynomials is $h^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}}(5)) = 125$-dimensional. But $\dim \mathrm{PGL}(n+1) = 25 - 1 = 24$, so that the space of quintics up to automorpisms is $125 - 24 = 101$-dimensional. Note that this is the same as $h^{12} = \dim_k H^2(Y, \Omega_Y)$, and this is no coincidence.

The mirror quintic should have its Hodge numbers switched, so we are looking for a 1-dimensional family. The family $|H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(5))|$ have a subfamily invariant under S_5, given by

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + \psi x_0 x_1 x_2 x_3 x_4 = 0 \tag{2}$$

for $\psi \in k$. Now let

$$G = \{(a_0, \cdots, a_4) \in \mathbb{Z}_5^5 \mid \sum a_i \equiv 0 \pmod 5\}/\mathbb{Z}_5$$

act on $\mathbb{P}^4$ by multiplication:

$$g \cdot (x_0 : \cdots : x_4) = (\zeta^{a_1} x_0 : \cdots : \zeta^{a_4} x_4),$$

where ζ is a fifth root of unity. The family of (2) is invariant under G, so that we get a family of hypersurfaces in $\mathbb{P}^4/G$. This can be shown to give the "correct" Calabi-Yau. [[see proofs]]

- Complete computation of the quintic and its mirror. h^{11},h^{12}.
- Picard Fuchs etc?

2 Batyrev and Batyrev-Borisov

The Batyrev construction gives a huge number of mirror candidates using toric geometry.