

# Results so far

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February 3, 2015

## 1 Introduction

In the article [Gai09], it is proven that there exists a triangulation of  $\mathbb{CP}^2$  with 15 vertices and other nice properties [[expand]]. The triangulation is constructed by “glueing” cones of 3-spheres along a triangle. Call this triangulation for  $\mathcal{S}$ .

A smoothing of the associated Stanley-Reisner scheme  $\mathbb{P}(\mathcal{S})$  would have interesting properties. In particular, it would be a hyper-Kähler variety [[EXPLAIN]]

The Hilbert polynomial of  $\mathbb{P}(\mathcal{S})$  is  $(9/2)t^4 + (15/2)t^2 + 3$ . The simplicial complex  $\mathcal{S}$  have  $f$ -vector  $(15, 90, 240, 270, 108)$ . In particular, the degree of  $\mathbb{P}(\mathcal{S})$  is 108.

Here are a few computations:

**Lemma 1.1.** *The module of first-order deformations of  $\mathbb{P}(\mathcal{S})$  is 90-dimensional, i.e.  $\dim_k T^1(\mathbb{P}(\mathcal{S})) = 90$ .*

**Lemma 1.2.** *The obstruction module have  $\dim_k T^2(\mathbb{P}(\mathcal{K})) = 306$ .*

The link of  $\mathcal{S}$  at one of its vertices is a particularly simple 3-sphere, namely the join of the boundaries of two hexagons. Call this  $\mathcal{K}$ . Then  $\mathcal{K}$  have  $f$ -vector  $(1, 12, 48, 72, 36)$ . In particular the Stanley-Reisner scheme  $\mathbb{P}(\mathcal{K})$  have degree 36.

Consider now  $\mathcal{K} * \Delta^1$ . This a cone over a ball, so it is topologically a 5-dimensional ball. In fact, it is the join of two hexagons. So it is a 5-dimensional ball. Consider now the polytope  $P * P$  that is the convex hull

of the columns of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

This polytope is the join of two *actual* hexagons. Thus, by standard theory,

## 2 Deformations of the 5-dimensional toric variety

Recall that the 5-dimensional toric variety  $T$  had 2-dimensional singularities (actually two disjoint copies of  $dP_6$ ).

**Theorem 2.1.** *There exists a flat deformation of  $\mathbb{P}(\mathcal{K} * \Delta^1)$ ,  $\mathfrak{X} \rightarrow S$ , such that  $\mathfrak{X}_{t_1} = T$  for some  $t_1 \in S$  and such that the general fiber  $\tilde{T}$  have one-dimensional singularities.*

*Proof.* As per now, the proof is purely by computer. The technique is this: First, consider the monomial degeneration of  $T$  to the Stanley-Reisner ring  $A(\mathcal{K} * \Delta^1)$  (recall that  $\mathcal{K} = D_6 * D_6$ ). Choose deformation parameters  $t_i$  perturbing the equations “in the direction of  $T$ ”, meaning that we only choose parameters introducing terms already occurring in the equations of  $T$ .

Now, using the package **VersalDeformations**, it is possible to produce a flat family  $\mathfrak{X} \rightarrow S'$  with  $\mathfrak{X}_0 = \mathbb{P}(\mathcal{K} * \Delta^1)$ .

The base space is a union of toric varieties of dimensions 14, 13, 13, 12, respectively. Call the largest one for  $S$ . The equations are

$$\begin{vmatrix} t_1 & t_2 & t_2 & \cdots & t_6 \\ t_7 & t_8 & t_9 & \cdots & t_{12} \end{vmatrix} \leq 1 \quad \begin{vmatrix} t_{13} & t_{14} & t_{15} & \cdots & t_{18} \\ t_{19} & t_{20} & t_{21} & \cdots & t_{24} \end{vmatrix} \leq 1$$

By restriction, we get the claimed family  $\mathfrak{X} \rightarrow T$ . It is checked by setting  $t_i = i$  for  $i = 1, \dots, 12$  and  $t_i = 2i$  for  $i = 13, \dots, 24$  that the resulting fiber have one-dimensional singularities. The reason we don't set all  $t_i = 1$ , is that this point lies in the intersection of the components of  $S$ .  $\square$

**Corollary 2.2.** *The Stanley-Reisner scheme  $\mathbb{P}(\mathcal{K})$  smooths to a smooth Calabi-Yau variety  $X$ .*

*Proof.* The scheme  $\mathbb{P}(\mathcal{K})$  sits as a complete intersection in  $\mathbb{P}(\mathcal{K} * \Delta^1)$ . Complete intersections deform together with the ambient variety, so  $\mathbb{P}(\mathcal{K} * \Delta^1)$  deforms to a general complete intersection in  $\tilde{T}$ . Since  $\tilde{T}$  have curve singularities, it follows by two applications of Bertini's theorem [Har77, Theorem II.8.18], that  $X$  is smooth.  $\square$

Now what we would really like to do is to compute the Hodge numbers  $h^{ij} = \dim_k H^j(X, \Omega_X^i)$  of  $X$ .

We can however compute the Hodge numbers of  $\tilde{T}$ . The hope is that there is some sort of Lefschetz theorem giving us the Hodge numbers of  $X$ .

**Theorem 2.3.** *We have  $h^{11}(\tilde{T}) = 1$  and  $h^{12}(\tilde{T}) = 12$ .*

*Proof.* Again, this is purely computational. We use long exact sequences together with sheaf cohomology computations in `Macaulay2`.

Since the ideal of  $\tilde{T}$  is rather complicated, doing this naïvely does not work. The trick is to choose the right term order. Since we know that  $\tilde{T}$  has a nice degeneration, we would like to find a term order such that its initial ideal is precisely the Stanley-Reisner ideal.

The `Macaulay2` package `gfanInterface` provides an interface with `gfan`, which is a program that can compute weight vectors given polynomials with prescribed initial terms. The weight vector is

$$\omega = (1, 1, 4, 7, 7, 4, 1, 1, 4, 7, 7, 4, 1, 1).$$

With this term order, giving a very small Gröbner basis (18 elements), the computations are much faster than with the standard term order. We are able to compute resolutions of all the relevant modules within a few minutes in total.

We have an exact sequence of sheaves on  $\tilde{T}$ :

$$0 \rightarrow \mathcal{T}_1 \hookrightarrow \mathcal{I}/\mathcal{I}^2 \xrightarrow{d} \Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_{\tilde{T}} \rightarrow \Omega_{\tilde{T}}^1 \rightarrow 0.$$

This sequence can be broken into two short exact sequences. The relevant one is this:

$$0 \rightarrow \operatorname{im} d \rightarrow \Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_{\tilde{T}} \rightarrow \Omega_{\tilde{T}}^1 \rightarrow 0. \quad (1)$$

We also have the restricted Euler sequence:

$$0 \rightarrow \Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_{\tilde{T}} \rightarrow \mathcal{O}_{\tilde{T}}(-1)^{14} \rightarrow \mathcal{O}_{\tilde{T}} \rightarrow 0. \quad (2)$$

We first compute  $h^{11}$ . From (1) we get a long exact sequence

$$\dots \rightarrow H^1(\text{im } d) \rightarrow H^1(\Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_{\tilde{T}}) \rightarrow H^1(\Omega_{\tilde{T}}^1) \rightarrow H^2(\text{im } d) \rightarrow \dots$$

The cohomology of  $H^1(\text{im } d)$  and  $H^2(\text{im } d)$  was computed with `Macaulay2` to be both zero. Thus  $H^1(\Omega_{\tilde{T}}^1) \simeq H^1(\Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_{\tilde{T}})$ . From the Euler sequence we get

$$\dots \rightarrow H^0(\mathcal{O}_{\tilde{T}}(-1)^{14}) \rightarrow H^0(\mathcal{O}_{\tilde{T}}) \rightarrow H^1(\Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_{\tilde{T}}) \rightarrow H^1(\mathcal{O}_{\tilde{T}}(-1)^{14}) \rightarrow \dots$$

But the left and right terms are both zero. Hence  $h^{11} = 1$ . We now compute  $h^{12}$ .

From (1) we again get

$$0 \rightarrow H^2(\Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_{\tilde{T}}) \rightarrow H^2(\Omega_{\tilde{T}}^1) \rightarrow H^3(\text{im } d) \rightarrow H^3(\Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_{\tilde{T}}) \rightarrow \dots,$$

where we have used that  $H^2(\text{im } d) = 0$ . But from the Euler sequence we get that the right term is also zero. Thus  $H^2(\Omega_{\tilde{T}}^1) \simeq H^3(\text{im } d)$ . This last group can be computed in `Macaulay2` to be 12-dimensional.  $\square$

By `Macaulay2` computations we find that

$$h^i(\widetilde{\text{im } d}) = \begin{cases} 12 & i = 3 \\ 2 & i = 4 \\ 0 & \text{else,} \end{cases}$$

and

$$h^i(\widetilde{\text{im } d}(-1)) = \begin{cases} 0 & i = 0, 1, 2 \\ 24 & i = 3 \\ 12 & i = 4 \\ 18 & i = 5, \end{cases}$$

and

$$h^i(\widetilde{\text{im } d}(-2)) = \begin{cases} 0 & i = 0, 1, 2 \\ 36 & i = 3 \\ 24 & i = 4 \\ 218 & i = 5, \end{cases}$$

In the same manner we find that:

**Proposition 2.4.** *We have  $H^3(\Omega_Y^1) = 2$ . The other Hodge groups  $H^i(\Omega_Y^1) = 0$  (for  $i = 0, 4, 5$ ).*

By twisting all the exact sequences above, we can also calculate:

**Proposition 2.5.** *We have*

$$h^i(\Omega_Y^1(-1)) = \begin{cases} 0 & i = 0 \\ 0 & i = 1 \\ 24 & i = 2 \\ 12 & i = 3, \end{cases}$$

and also  $h^4(\Omega_Y^1(-1)) - h^5(\Omega_Y^1(-1)) = 4$ .

Similarly:

$$h^i(\Omega_Y^1(-2)) = \begin{cases} 0 & i = 0 \\ 0 & i = 1 \\ 36 & i = 2 \\ 24 & i = 3, \end{cases}$$

and  $h^4(\Omega_Y^1(-2)) - h^5(\Omega_Y^1(-2)) = 23$ .

**Remark.** *The reader may wonder we just didn't ask Macaulay2 to compute the cohomology sheaf  $\Omega_{\tilde{T}}^1$  directly, e.g. by the command `HH^i(cotangentSheaf Proj A)`. The reason is that Macaulay2's algorithms actually compute the sheaf, and not just the dimension, and this is too computationally intensive.*

**Remark (Question).** *A computation reveals that  $H^i(\mathcal{I}/\mathcal{I}^2) \simeq H^i(\text{im } d)$  for  $i \geq 2$ . This could be because the singularities are of dimension 1. Is there a theoretical result to this effect?*

We can compute it Macaulay2 that:

**Lemma 2.6.** *The first cotangent module of  $\tilde{T}$  has  $\dim_k T^1(\tilde{T}/k) = 26$ .*

We have that  $h^0(\mathbb{P}(\mathcal{K} * \Delta^1), \mathcal{T}) = 14$  by Theorem 5.2 in [AC10]. Thus these numbers fit in the narrative that we should have  $T_{X_0}^1 = h^1(\mathcal{T}_{X_t}) + h^0(\mathcal{T}_{X_0})$ . (IS THERE ANY HEURISTIC FOR THIS??)

### 3 Computing the Hodge numbers of X

Since  $X$  is a complete intersection of two hyperplanes in  $Y$ , we have an exact sequence

$$0 \rightarrow \mathcal{O}_Y(-2) \rightarrow \mathcal{O}_Y(-1)^2 \rightarrow \mathcal{I}_{X/Y} \rightarrow 0,$$

where  $\mathcal{I}_{X/Y}$  is the ideal sheaf of  $X$  in  $Y$ . We also have the sequence

$$0 \rightarrow \mathcal{I}_{X/Y} \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_X \rightarrow 0. \quad (3)$$

This allows us to compute the Hodge numbers of  $X$ :

**Theorem 3.1.** *There exists a non-singular Calabi-Yau with  $X$  with  $\chi(\Omega_X^1) = 36$ . Hence the topological Euler characteristic is  $-72$ .*

*Proof.* Since  $X$  is a complete intersection in  $Y$ , we have  $\mathcal{I}_{X/Y}/\mathcal{I}_{X/Y}^2 \simeq \mathcal{O}_X(-1)^2$  as  $\mathcal{O}_X$ -modules. Hence we have an exact sequence

$$0 \rightarrow \mathcal{O}_X(-1)^2 \rightarrow \Omega_Y^1 \otimes \mathcal{O}_X \rightarrow \Omega_X^1 \rightarrow 0.$$

The first sheaf have cohomology only in  $H^3(\mathcal{O}_X(-1)) = H^0(\mathcal{O}_X(1)) = 12$ , which can be computed from its Stanley-Reisner degeneration. Hence the Euler characteristics are related by  $\chi(\Omega_X^1) = \chi(\Omega_Y^1|_X) + 24$ .

Now tensor the exact sequence (3) with  $\Omega_Y^1$  to get

$$0 \rightarrow \mathcal{I}_{X/Y} \otimes \Omega_Y^1 \rightarrow \Omega_Y^1 \rightarrow \Omega_Y^1 \otimes \mathcal{O}_X \rightarrow 0.$$

Tensoring with  $\Omega_Y^1$  is exact because the singularities of  $Y$  lie outside  $X$  (recall that the sheaf on the right is extended by zero outside  $X$ ). Do the same with the  $\mathcal{O}_Y$ -resolution of  $\mathcal{I}_{X/Y}$  to get

$$0 \rightarrow \Omega_Y^1(-2) \rightarrow \Omega_Y(-1)^2 \rightarrow \mathcal{I}_{X/Y} \otimes \Omega_Y \rightarrow 0.$$

Taking Euler characteristics, we find that  $\chi(\mathcal{I}_{X/Y} \otimes \Omega_Y) = -3$ . Since  $\chi(\Omega_Y^1)$  was computed to be 9, it follows from the first exact sequence that  $\chi(\Omega_X^1) = 36$ .

The topological Euler characteristic is double the Euler characteristic of the tangent sheaf, hence  $\chi(X) = -24$ .  $\square$

To actually find the Hodge numbers, we need a few lemmas.

**Lemma 3.2.** *Let  $\mathcal{N}_{X/\mathbb{P}^{13}}$  be the normal sheaf of  $X$  in  $\mathbb{P}^{13}$ . Then  $h^3(\mathcal{I}_X/\mathcal{I}_X^2) = h^0(\mathcal{N}_{X/\mathbb{P}^{13}})$ .*

*Proof.* By Serre duality  $h^{3-i}(\mathcal{I}_X/\mathcal{I}_X^2) = h^i((\mathcal{I}_X/\mathcal{I}_X^2)^\vee \otimes \omega)$ , where  $\omega$  is the dualizing sheaf. But  $X$  is Calabi-Yau, so  $\omega \simeq \mathcal{O}_X$ . The dual of  $\mathcal{I}_X/\mathcal{I}_X^2$  is by definition the normal bundle.  $\square$

Consider the Euler sequence

$$0 \rightarrow \Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X \rightarrow \mathcal{O}_X(-1)^{14} \rightarrow \mathcal{O}_X \rightarrow 0.$$

Since  $X$  is a deformation of a Stanley-Reisner sphere, we know the cohomology of  $\mathcal{O}_X$ . So we can extract the cohomology of  $\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X$ .

**Lemma 3.3.** *We have*

$$H^i(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1 \\ 0 & \text{if } i = 2 \\ 167 & \text{if } i = 3. \end{cases}$$

*Proof.* The full long exact sequence is:

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) & \rightarrow & H^0(\mathcal{O}_X(-1)^{14}) & \rightarrow & H^0(\mathcal{O}_X) \rightarrow \\ & & H^1(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) & \rightarrow & H^1(\mathcal{O}_X(-1)^{14}) & \rightarrow & H^1(\mathcal{O}_X) \rightarrow \\ & & H^2(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) & \rightarrow & H^2(\mathcal{O}_X(-1)^{14}) & \rightarrow & H^2(\mathcal{O}_X) \rightarrow \\ & & H^3(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) & \rightarrow & H^3(\mathcal{O}_X(-1)^{14}) & \rightarrow & H^3(\mathcal{O}_X) \rightarrow 0 \end{array}$$

Inserting the dimensions we know, we get:

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) & & & 0 & \rightarrow 1 \rightarrow \\ & & H^1(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) \rightarrow 0 & & & \rightarrow 0 & \rightarrow \\ & & H^2(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) \rightarrow 0 & & & \rightarrow 0 & \rightarrow \\ & & H^3(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) \rightarrow 167 & & & \rightarrow 1 & \rightarrow 0 \end{array}$$

Hence we conclude. □

Since  $X$  is smooth, the conormal sequence is exact, so we have

$$0 \rightarrow \mathcal{I}_X/\mathcal{I}_X^2 \rightarrow \Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_X \rightarrow \Omega_X^1 \rightarrow 0.$$

**Lemma 3.4.** *We have*

$$h^{21}(X) = h^0(\mathcal{N}_{X/\mathbb{P}^{13}}) - 167.$$

where  $\chi$  denotes the Euler characteristic.

*Proof.* Write up the long exact sequence coming from the conormal sequence of  $X$  and use Lemma 3.2. □

**Lemma 3.5.** *There is an exact sequence*

$$0 \rightarrow \mathcal{O}_X(1)^2 \rightarrow \mathcal{N}_{X/\mathbb{P}^{13}} \rightarrow \mathcal{N}_{Y/\mathbb{P}^{13}}|_X \rightarrow 0$$

*Proof.* First note that there is an exact sequence of conormal sheaves:

$$0 \rightarrow \mathcal{I}_Y/\mathcal{I}_Y^2|_X \rightarrow \mathcal{I}_X/\mathcal{I}_X^2 \rightarrow \mathcal{O}_X(-1)^2 \rightarrow 0.$$

The last term is  $\mathcal{N}_{X/Y}$ , since  $X$  is a complete intersection in  $Y$ . Dualizing is exact even though  $Y$  is not smooth, because by the long exact sequence of  $\mathcal{E}xt$  sheaves, we must have  $\mathcal{E}xt^1(\mathcal{O}_X(-1), \mathcal{O}_X) = 0$ . But this is true, since both of these are locally free.  $\square$

**Proposition 3.6.** *We have*

$$h^{21}(X) = 39.$$

Hence  $h^{11}(X) = 27$ .

*Proof.* By lemma 3.4, we need to compute  $h^0(\mathcal{N}_{X/\mathbb{P}^{13}})$ . By the previous lemma, we have  $h^0(\mathcal{N}_{X/\mathbb{P}^{13}}) = \mathcal{N}_{Y/\mathbb{P}^{13}}|_X + 24$ .

We have an exact sequence

$$0 \rightarrow \mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}} \rightarrow \mathcal{N}_{Y/\mathbb{P}^{13}} \rightarrow \mathcal{N}_{Y/\mathbb{P}^{13}}|_X \rightarrow 0.$$

And also an exact sequence:

$$0 \rightarrow \mathcal{N}_{Y/\mathbb{P}^{13}}(-2) \rightarrow \mathcal{N}_{Y/\mathbb{P}^{13}}(-1)^{\oplus 2} \rightarrow \mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}} \rightarrow 0.$$

The cohomology of  $\mathcal{N}_{Y/\mathbb{P}^{13}}(-i)$  is possible to compute in `Macaulay2`, and it follows from the long exact sequence that  $h^0(\mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}}) = 36$ , and  $h^1(\mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}}) = 0$ . Hence it follows from the same computation that  $h^0(\mathcal{N}_{Y/\mathbb{P}^{13}}|_X) = 182$  and that  $h^0(\mathcal{N}_{X/\mathbb{P}^{13}}) = 206$ . We conclude that  $h^{21} = 206 - 167 = 39$ .  $\square$

## References

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