Dictionary

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1 Algebraic Geometry

1.1 General properties

1.1.1 Fano variety

A variety X is Fano if the anticanonical sheaf ω_X^{-1} is ample.

1.2 Results and theorems

1.2.1 Bertini's Theorem

Let X be a nonsingular closed subvariety of $\mathbb{P}_k^n$, where $k = \bar{k}$. Then the set of of hyperplanes $H \subseteq \mathbb{P}_k^n$ such that $H \cap X$ is regular at every point) and such that $H \not\subseteq X$ is a dense open subset of the complete linear system $|H|$. See [1, Thm II.8.18].

1.2.2 Euler sequence

If A is a ring and $\mathbb{P}_A^n$ is projective n-space over A, then there is an exact sequence of sheaves on X:

$$0 \to \Omega_{\mathbb{P}_A^n/A} \to \mathscr{O}_{\mathbb{P}_A^n}(-1)^{n+1} \to \mathscr{O}_{\mathbb{P}_A^n} \to 0.$$

See [1, Thm II.8.13].

1.2.3 Kodaira vanishing

If k is a field of characteristic zero, X is a smooth and projective k-scheme of dimension d, and $\mathcal{L}$ is an ample invertible sheaf on X, then $H^q(X, \mathcal{L} \otimes_{\mathscr{O}_X} \Omega_{X/k}^p) = 0$ for $p + q > d$. In addition, $H^q(X, \mathcal{L}^{-1} \otimes_{\mathscr{O}_X} \Omega_{X/k}^p) = 0$ for $p + q < d$.

1.2.4 Riemann-Roch for curves

The **Riemann-Roch theorem** relates the number of sections of a line bundle with the genus of a smooth curve C. Let $\mathcal{L}$ be a line bundle ω_C the canonical sheaf on C. Then

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^{-1} \otimes_{\mathscr{O}_C} \omega_C) = \deg(\mathcal{L}) + 1 - g.$$

This is [1, Theorem IV.1.3].

1.2.5 Serre vanishing

One form of Serre vanishing states that if X is a proper scheme over a noetherian ring A, and $\mathcal{L}$ is an ample sheaf, then for any coherent sheaf $\mathcal{F}$ on X, there exists an integer n_0 such that for each $i > 0$ and $n \geq n_0$ the group $H^i(X, \mathcal{F} \otimes_{\mathscr{O}_X} \mathcal{L}^n) = 0$ vanishes. See [1, Proposition III.5.3].

1.3 Sheaves and bundles

1.3.1 Ample line bundle

A line bundle $\mathcal{L}$ is **ample** if for any coherent sheaf $\mathcal{F}$ on X, there is an integer n (depending on $\mathcal{F}$) such that $\mathcal{F} \otimes_{\mathscr{O}_X} \mathcal{L}^{\otimes n}$ is generated by global sections. Equivalently, a line bundle $\mathcal{L}$ is ample if some tensor power of it is very ample.

1.3.2 Very ample line bundle

A line bundle $\mathcal{L}$ is **very ample** if there is an embedding $i : X \hookrightarrow \mathbb{P}^n_S$ such that the pullback of $\mathscr{O}_{\mathbb{P}^n_S}(1)$ is isomorphic to $\mathcal{L}$. In other words, there should be an isomorphism $i^* \mathscr{O}_{\mathbb{P}^n_S}(1) \simeq \mathcal{L}$.

1.3.3 Anticanonical sheaf

The **anticanonical sheaf** ω_X^{-1} is the inverse of the canonical sheaf ω_X, that is $\omega_X^{-1} = \mathscr{H}om_{\mathscr{O}_X}(\omega_X, \mathscr{O}_X)$.

1.3.4 Canonical divisor

The **canonical divisor** K_X is the class of the canonical sheaf ω_X in the divisor class group.

1.3.5 Canonical sheaf

If X is a smooth algebraic variety of dimension n, then the canonical sheaf is $\omega := \wedge^n \Omega^1_{X/k}$ the n'th exterior power of the cotangent bundle of X.

1.4 Toric geometry

1.4.1 Polarized toric variety

A toric variety equipped with an ample T-invariant divisor.

2 Commutative algebra

2.1 Modules

2.1.1 Depth

Let R be a noetherian ring, and M a finitely-generated R-module and I an ideal of R such that $IM \neq M$. Then the I-depth of M is (see Ext):

$$\inf\{i \mid \mathrm{Ext}^i_R(R/I, M) \neq 0\}.$$

This is also the length of a maximal M-sequence in I.

2.2 Rings

2.2.1 Cohen-Macaulay ring

A local Cohen-Macaulay ring (CM-ring for short) is a commutative noetherian local ring with Krull dimension equal to its depth. A ring is Cohen-Macaulay if its localization at all prime ideals are Cohen-Macaulay.

2.2.2 Depth of a ring

The depth of a ring R is is its depth as a module over itself.

2.2.3 Gorenstein ring

A commutative ring R is Gorenstein if each localization at a prime ideal is a Gorenstein local ring. A Gorenstein local ring is a local ring with finite injective dimension as an R-module. This is equivalent to the following: $\mathrm{Ext}^i_R(k, R) = 0$ for $i \neq n$ and $\mathrm{Ext}^n_R(k, R) \simeq k$ (here $k = R/\mathfrak{m}$ and n is the Krull dimension of R).

3 Convex geometry

3.1 Cones

3.1.1 Simplicial cone

A cone C generated by $\{v_1, \cdots, v_k\} \subseteq N_{\mathbb{R}}$ is simplicial if the v_i are linearly independent.

3.2 Polyhedra

3.2.1 Dual (polar) polyhedron

If Δ is a polyhedron, its dual Δ° is defined by

$$\Delta^\circ = \{x \in N_{\mathbb{R}} \mid \langle x, y \rangle \geq -1 \,\forall\, y \in \Delta\}\,.$$

3.2.2 Reflexive polytope

A polytope Δ is reflexive if the following two conditions hold:

1. All facets Γ of Δ are supported by affine hyperplanes of the form $\{m \in M_{\mathbb{R}} \mid \langle m, v_\Gamma \rangle\}$ for some $v_\Gamma \in N$.
2. The only interior point of Δ is 0, that is: $\mathrm{Int}(\Delta) \cap M = \{0\}$.

4 Homological algebra

4.1 Derived functors

4.1.1 Ext

Let R be a ring and M, N be R-modules. Then $\mathrm{Ext}^i_R(M, N)$ is the right-derived functors of the $\mathrm{Hom}(M, -)$-functor. In particular, $\mathrm{Ext}^i_R(M, N)$ can be computed as follows: choose a projective resolution $C_.$ of N over R. Then apply the left-exact functor $\mathrm{Hom}_R(M, -)$ to the resolution and take homology. Then $\mathrm{Ext}^i_R(M, N) = h^i(C_.)$.

4.1.2 Tor

Let R be a ring and M, N be R-modules. Then $\mathrm{Tor}^i_R(M, N)$ is the right-derived functors of the $- \otimes_R N$-functor. In particular $\mathrm{Tor}^i_R(M, N)$ can be computed by taking a projective resolution of M, tensoring with N, and then taking homology.

References

[1] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.