Exercises deformation theory

FM

1 Chapter 1.1

Exercise 1. Existence of the Hilbert scheme for curves in $\mathbb{P}^2$. Here a *curve* is a closed subscheme of $\mathbb{P}^2_k$ defined by any homogeneous polynomial of degree d in $S = k[x, y, z]$ (so curves are in 1-1 correspondence with points in $\mathbb{P}(S^d V)$ where V is a 3-dim k-vector space).

Write f has $a_0 x_0^d + \ldots + a_n z^d$ with $a_i \in k$ and $n = \binom{d+2}{2} - 1$. Consider $(a_0, \cdots, a_n)$ as a point in $\mathbb{P}^n_k$.

1. Curves in $\mathbb{P}^2$ of degree d are in 1-1 correspondence with points in $\mathbb{P}^n$ by this correspondence.
2. Define $\mathcal{C} \subseteq \mathbb{P}^2 \times \mathbb{P}^n$ by $a_0 x^d + \cdots + a_n + x^d = 0$. Show that the correspondence in a) is given by $a \in \mathbb{P}^n$ goes to the fiber $\mathcal{C}_a \subseteq \mathbb{P}^2$ over the point a. We call $\mathcal{C}$ the *tautological family.*
3. For any finitely generated k-algebra A, we define a *family of curves of degree d in $\mathbb{P}^2$ over A* to be a closed subscheme $X \subseteq \mathbb{P}^2_A$, flat over A, such that the fibers over closed points of $\operatorname{Spec} A$ are curves of degree d in $\mathbb{P}^2$. Show that the ideal $I_X \subseteq A[x, y, z]$ is generated by a single homogeneous polynomial f of degree d in $A[x, y, z]$.

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Solution 1. 1. Obvious.

2. Let $a \in \mathbb{P}^n$. Then $\mathcal{C}_a$ is precisely the subscheme $\subseteq \mathbb{P}^2$ cut out by $f = 0$.

$$\begin{tikzcd}
\mathcal{C}_a \arrow[r] \arrow[d] & \mathcal{C} \subseteq \mathbb{P}^2 \times \mathbb{P}^n \arrow[d] \\
\{a\} \arrow[r] & \mathbb{P}^2
\end{tikzcd}$$

3. By lifting, we can assume that $A = k[b_1, \cdots, b_l]$ for some l. Then the question is equivalent to: Suppose $I \subseteq k[b_1, \cdots, b_l] \otimes k[x, y, z]$ is such that $I \otimes_k A/\mathfrak{m} = \langle f \rangle$ for some $f \in k[x, y, z]$ for all $\mathfrak{m} \in \operatorname{Spec} A$. Suppose in addition that $A[x, y, z]/I$ is a flat A-module. Then $I = \langle \tilde{f} \rangle$ for some $\tilde{f} \in A[x, y, z]$ such that $\tilde{f} \otimes A/\mathfrak{m} = f$.

   This should follow from the equational criterium for flatness. In particular: in each fibre, $I \otimes A/\mathfrak{m}$ is generated by a single polynomial, and this lifts to a generator of I, together with the trivial relation. If I had more than one generator, there would be a relation that is trivial in all fibers. But then it must be trivial everywhere.

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