Exercises

Fredrik Meyer

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Contents

1 Algebra - Serge Lang

1.1 Chapter V

Exercise 1. Let $E = \mathbb{Q}(\alpha)$, where α is a root of the equation

$$\alpha^3 + \alpha^2 + \alpha + 2 = 0.$$

Express $(\alpha^2 + \alpha + 1)(\alpha^2 + \alpha)$ and $(\alpha - 1)^{-1}$ in the form

$$a\alpha^2 + b\alpha + c$$

with $a, b, c \in \mathbb{Q}$. ♠

Solution 1. Note that $\alpha^3 = -\alpha^2 - \alpha - 2$ and $\alpha^2 + \alpha + 1 = \frac{\alpha^3 - 1}{\alpha - 1}$ (it is the beginning of a geometric series). Then

$$\begin{aligned}
(\alpha^2 + \alpha + 1)(\alpha^2 + \alpha) &= \alpha(\alpha^2 + \alpha + 1)(\alpha + 1) \\
&= \alpha \left(\frac{\alpha^3 - 1}{\alpha - 1} \right) (\alpha + 1) \\
&= \alpha \left(\frac{-\alpha^2 - \alpha - 3}{\alpha - 1} \right) (\alpha + 1) \\
&= - \left(\frac{\alpha^3 + \alpha^2 + 3\alpha}{\alpha - 1} \right) (\alpha + 1) \\
&= - \left(\frac{-\alpha^2 - \alpha - 2 + \alpha^2 + 3\alpha}{\alpha - 1} \right) (\alpha + 1) \\
&= - \left(\frac{2\alpha - 2}{\alpha - 1} \right) (\alpha + 1) \\
&= -2\alpha - 2.
\end{aligned}$$

Alternatively, there's a much easier solution.

For $(\alpha - 1)^{-1}$, let

$$\frac{1}{\alpha - 1} = a\alpha^2 + b\alpha + c.$$

Then multiplaying on both sides and using that $1, \alpha, \alpha^2$ are linearly independent over $\mathbb{Q}$, we equate coefficients, and get

$$(\alpha - 1)^{-1} = -\frac{1}{5}\alpha^2 - \frac{2}{5}\alpha - \frac{3}{5}.$$

♡

Exercise 2. Let $E = F(\alpha)$ where α is algebraic over F of odd degree. Show that $E = F(\alpha^2)$. ♠

Solution 2. Cleary $F(\alpha^2) \subset F(\alpha)$. It will be enough to show $\alpha \in F(\alpha^2)$.

If $\alpha \notin F(\alpha^2)$, the extension $F(\alpha)/F(\alpha^2)$ must have degree 2 because α is a zero of $X^2 - \alpha^2$. Then $[F(\alpha) : F] = [F(\alpha) : F(\alpha^2)][F(\alpha^2) : F] = 2[F(\alpha^2) : F]$. ♡

Exercise 3. Let α, β be two elements which are algebraic over F. Let $f(X) = \mathrm{Irr}(\alpha, F, X)$ and $g(X) = \mathrm{Irr}(\beta, F, X)$. Suppose that $\deg f$ and $\deg g$ are relatively prime. Show that g is irreducible in the polynomial ring $F(\alpha)[X]$. ♠

Solution 3. Consider the tower of fields $F \subset F(\alpha) \subset F(\alpha, \beta)$. Then $[F(\alpha,\beta) : F] = \deg f \cdot n \leq \deg f \cdot \deg g$, by the same argument as in the proof of Proposition 1.2. Similarly $[F(\alpha,\beta) : F] = \deg g \cdot m \leq \deg f \cdot \deg g$. But then

$$\frac{\deg f}{\deg g} = \frac{m}{n}.$$

But $\deg f$ and $\deg g$ are relatively prime, and, and we must have $m \leq \deg f$ and $n \leq degg$. This forces $m = degf$ and $n = \deg g$, and hence $[F(\alpha,\beta) : F(\alpha)] = \deg g$ and $[F(\alpha,\beta) : F(\beta)] = \deg f$.

Now, if g were reducible in $F(\alpha)[X]$, we would have $[F(\alpha,\beta) : F] < \deg f \deg g$. But we don't. ♡

2 Algebraic Geometry - Hartshorne

2.1 Chapter I - Varieties

Exercise 4 (Exercise 1.1)**.** a) Let Y be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

c) Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

♠

Solution 4. a) We have $A(Y) = k[x, y]/(y - x^2)$. An isomorphism $A(Y) \to k[t]$ is given by $x \mapsto t$ and $y \mapsto t^2$.

b) We have $A(Z) = k[x,y]/(xy-1) \simeq k[x, \frac{1}{x}]$. So we must show that $k[x, \frac{1}{x}] \not\approx k[x]$. It can be computed that the first one has automorphisms given by $x \mapsto cx^n$ for c nonzero and $n \neq 0$. The second has as automorphisms $ax + b$ ($a \neq 0$). So the first one have an abelian automorphism group, the second has not.

c) What is special about $A(Y)$ and $A(Z)$? Staring at pictures, we see that any line in $\mathbb{A}^2$ intersects Y in at least one point, but in the case of Z, there exist two lines which do not intersect Z. We claim that this is the only two things that can happen.

First we claim that if we are in the second situation, that is, if there exist a pair of lines ℓ, ℓ' such that $W \cap \ell = W \cap \ell' = \emptyset$, then $W \simeq Z$.

A general quadric can be written as

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Suppose now $\ell \cap W = \emptyset$. This is equivalent to $I(f, \ell^\vee) = (1)$. Without loss of generality, we can assumme $\ell = \{x = 0\}$. Then

$$I(f, \ell) = (cy^2 + ey + f, x).$$

This generates $k[x,y]$ if and only if $c = e = 0$ and $f \neq 0$. Thus f must be of the form

$$ax^2 + bxy + dx + f = 0$$

with $f \neq 0$. But this can be written as

$$x(ax + by + d) + f = 0.$$

Put $y' = ax + by + d$. Then $I(W)$ takes the form $(xy' + f = 0)$, which is clearly isomorphic to Z after a linear change of coordinates. Note that the other line not meeting W is the line given by $y' = ax + by + d = 0$.

Assume now that we are in the other situation, namely that *every* line in $\mathbb{A}^2$ meets W. Now pick a tangent line ℓ of W. Without loss of generality, we can assume that ℓ is $\{y = 0\}$. This is a tangent line if and only if it meets W doubly, meaning that $I(W) + (\ell^\vee)$ takes the form (l^2, y) for some linear form l. We can also assume that $\ell \cap W = (0,0)$, so that $I(W) + (\ell^\vee) = (x^2, y)$. But this means that

$$\begin{aligned} I(W) + I(\ell) &= (ax^2 + bxy + cy^2 + dx + ey + f, y) \\ &= (ax^2 + dx + f, y) \end{aligned}$$

We want $ax^2 + dx + f = x^2$. This can happen only if $d = f = 0$ and $a \neq 0$. Thus the quadric takes the form

$$ax^2 + bxy + cy^2 + ey = 0.$$

Now we claim that there exist one line at each point of W that intersect W transversally in exactly one point. This is the case for Y. Consider the pencil of lines through $(0, 0)$ defined by $x = \lambda y$. We want to find λ such that the intersection is transversal and only one point. We have

$$(ax^2 + bxy + cy^2 + ey, x - \lambda y) = \left((a\lambda^2 + b\lambda + c)y^2 + ey, x - \lambda y\right).$$

This have exactly one solution if and only if $a\lambda^2 + b\lambda + c = 0$. This is solvable since $a \neq 0$ and since all lines intersect W. Thus choose λ as above. We can rotate this line such that it becomes $x = 0$. Then the equation takes the form

$$ax^2 + bxy + ey = 0.$$

We have still not arrived at $y = x^2$. Let now $y = \lambda x$ be a general line through the origin. We demand that this intersect W twice for every λ such that the line is not tangent. We get that the intersection is given by

$$ax^2 + b\lambda x + ex = x((a + \lambda b)x + e) = 0.$$

For this to have two solutions for every λ we must have $a + \lambda b \neq 0$ for all λ. But this requires $b = 0$. Thus the equation is

$$ax^2 + ey = 0$$

which is the conic we were looking for.

♡

Exercise 5 (Exercise 1.2, the twisted cubic curve)**.** Let $Y \subseteq \mathbb{A}^3$ be the set $\{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric equation* $x = t, y = t^2, z = t^3$. ♠

Solution 5. An affine variety is by definition a closed irreducible subset of $\mathbb{A}^3$. So we must find an irreducible ideal I such that $Z(I) = Y$ (forgive the abuse of notation).

I claim that $I(Y) = \langle x^2 - y, x^3 - z \rangle$. Clearly, every $P \in Y$ satisfies these equations. This shows the inclusion $Y \subset Z(I)$. Now suppose $P \in Z(I)$, that is, $f(P) = 0$ for all $f \in I$. In particular $(x^2 - y)(P) = 0$ and $(x^3 - z)(P) = 0$. Thus $y = x^2$ and $z = x^3$. So if $P = (a, b, c) \in k^3$, then $P = (a, a^2, a^3)$, so $P \in Y$. This shows that $Z(I) = Y$. If we can show that I is prime, then it follows that $I(Y) = I$ and that Y is a variety.

In fact, we claim that $k[x, y, z]/I \simeq k[t]$, implying that I is prime. The map φ is given by $x \mapsto t$, $y \mapsto t^2$, $z \mapsto t^3$. Then clearly $I \subseteq \ker \varphi$. We must show equality. So suppose $\varphi(f) = 0$.

First we claim that any $f \in k[x, y, z]$ can be written as $f = R(x) + S(x)y + T(x)z + i(x, y, z)$ where i is a polynomial in I. We prove this by induction on $\deg f$. If $\deg f = 1$, this is trivially true. The rest of the proof proceeds by tedious induction. ♡

2.2 Chapter II - Schemes

Exercise 6 (Exercise 1.2)**.** a) For any morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, show that for each point P, $(\ker \varphi)_P = \ker(\varphi_P)$ and $(\operatorname{im} \varphi)_P = \operatorname{im}(\varphi_P)$.

b) Show that φ is injective (resp. surjective) if and only if the induced map on the stalks φ_P is injective (resp. surjective) for all P.

c) Show that a sequence $\ldots \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \ldots$ of sheaves and morphisms is exact if and only if for each $P \in X$, the corresponding sequence of stalks is exact as a sequence of abelian groups.

♠

Solution 6. a) An element of $(\ker \varphi)_P$ is represented by a pair (U, f) with $f \in \mathscr{F}(U)$ satisfying $\varphi(U)(f) = 0$. We have $(U, f) \simeq (V, g)$ if there is a neighbourhood W of p contained in $U \cap V$ such that $f\big|_W = g\big|_W$ (then automatically $\varphi(W)(f) = 0$, since $\varphi(W) = \varphi(U)\big|_W$).

On the other hand, an element of $\ker \varphi_P$ is represented by a pair (U, f) satisfying the same conditions.

A similar argument works for $\operatorname{im} \varphi$. Alternatively, one can show that finite limits commute with direct limits.

b) Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is injective. Then by definition all $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ are injective, hence $(\ker \varphi)_P = \ker \varphi_P = 0$, hence φ_P is injective. Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is surjective. By definition, this means that $\operatorname{im} \varphi = \mathscr{G}$, hence $\mathscr{G}_P = (\operatorname{im} \varphi)_P = \operatorname{im} \varphi_P$, so the stalks are surjective.

c) Exactness means that $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$. Taking stalks, gives one implication. Assume that the stalks are exact. Then the same argument works.

♡

Exercise 7. a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. Show that the induced map $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ of associated sheaves is injective.

b) Use part a) to show that if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then $\operatorname{im} \varphi$ can be naturally identified with a subsheaf of $\mathscr{G}$, as mentioned in the text.

♠

Solution 7. a) From the universal property of the sheafification functor, we have a commutative square:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \mathscr{G} \arrow[d, "\theta"] \\ \mathscr{F}^+ \arrow[r, "\exists!"] & \mathscr{G}^+ \end{tikzcd}$$

The lower arrow follows from the universal property of sheafification applied to $\theta \circ \varphi$. Taking stalks induced the identity map on the vertical arrows, and since a map is injective if it is injective on stalks, the statement follows.

b) $\operatorname{im} \varphi$ is the sheafification of $(\operatorname{im} \varphi)_{pre}(U) = \{U \mapsto \varphi(U)\}$. We have $\operatorname{im} \varphi(U) \subset \mathscr{G}(U)$ for all U, hence $\operatorname{im} \varphi_P \subset \mathscr{G}_P$ for all P, hence $\operatorname{im} \varphi \to \mathscr{G}$ is injective.

♡

Exercise 8 (Exercise 1.14, Support)**.** Let $\mathscr{F}$ be a sheaf on X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. The *support of* s denoted $\operatorname{Supp}(s)$, is defined to be the set $\{P \in U \mid s_P \neq 0\}$ where s_P denotes the germ of s in the stalk s_P. Show that $\operatorname{Supp}(s)$ is a closed subset of U. We define the *support* of $\mathscr{F}$ by $\operatorname{Supp} \mathscr{F}$ to be $\{P \in X \mid \mathscr{F}_P \neq 0\}$. It need not be a closed subset. ♠

Solution 8. Showing that $\operatorname{Supp}(s)$ is closed is equivalent to showing that the complement is open. So let $P \in X \setminus \operatorname{Supp}(s)$. Then $s_P = 0$. But every germ is represented by a pair (s, U) (with $(s', U') \simeq (s, U)$ if $s|_W = s'|_W$ for some open $W \subset U \cap U'$). But since $s_P = 0$, there must be some neighbourhood U

such that s_P is represented by $s = 0$, hence $X \setminus \operatorname{Supp}(s)$ can be covered by those open U's.

To see that $\operatorname{Supp} \mathscr{F}$ need not be closed, let $X = \mathbb{A}^1_k$ with k an infinite field. Let $\mathbb{Z}$ be the constant sheaf and let $\mathscr{L}$ be the direct sum of infinitely many skyscraper sheaves, but not everyone. Let $\mathscr{F}/\mathscr{L}$ be the quotient. This has support on the infinitely many points chosen, which is not closed. ♡

Exercise 9 (Exercise 1.16, Flabby/flasque sheaves)**.** A sheaf $\mathscr{F}$ on a topological space X is *flasque* (flabby) if for every inclusion $U \subseteq V$, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.

a) Show that a constant sheaf on an irreducible topological space is flasque.

b) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ is flabby, then for any open set U, the sequence

$$0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$$

is exact.

c) Same as above, but suppose $\mathscr{F}'$ and $\mathscr{F}$ are flabby. Show that $\mathscr{F}''$ is flabby.

d) If $f : X \to Y$ is a continous map, and $\mathscr{F}$ is a flabby sheaf on X, then $f_* \mathscr{F}$ is flabby on Y.

e) Let $\mathscr{F}$ be any sheaf on X. We define a new sheaf $\mathscr{G}$, called the *sheaf of discontinous sections of* $\mathscr{F}$, as follows: For each open set $U \subset X$, $\mathscr{G}(U)$ is the set of maps $s : U \to \cup_{P \in U} \mathscr{F}_P$, such that for all $P \in U$, $s(P) \in \mathscr{F}_P$. Show that $\mathscr{G}$ is a flabby sheaf, and that there is a natural injective morphism from $\mathscr{F}$ to $\mathscr{G}$.

♠

Solution 9. a) Every open set in X is irreducible and dense, and dense sets are connected. Hence a constant sheaf is actually constant, and all the restriction maps are identities (except if one of them is the empty set).

b) The sheaf axiom for a sheaf $\mathscr{F}$ is equivalent to the following: for every covering $\{U_i\}$ of U, the following sequence is exact:

$$0 \to \mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \to \prod_{ij} \mathscr{F}(U_{ij}),$$

where $U_{ij} = U_i \cap U_j$. The first map sends a section s to the product of all its restrictions, and the second map sends $(s_i) \mapsto (s_i - s_j)_{ij \in I \times I}$.

Since the sequence of sheaves in the exercise is exact, for small enough U_i, the sequence $0 \to \mathscr{F}'(U_i) \to \mathscr{F}(U_i) \to \mathscr{F}''(U_i) \to 0$ is exact (for sheaves, exactness is a local property). Hence we can form the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{F}'(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{F}''(U) \arrow[d] & \\
0 \arrow[r] & \prod_i \mathscr{F}'(U_i) \arrow[r] \arrow[d, "f"] & \prod_i \mathscr{F}(U_i) \arrow[r] \arrow[d] & \prod_i \mathscr{F}''(U_i) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod_{ij} \mathscr{F}'(U_{ij}) \arrow[r] \arrow[d] & \prod_{ij} \mathscr{F}(U_{ij}) \arrow[r] & \prod_{ij} \mathscr{F}''(U_{ij}) \arrow[r] & 0 \\
 & \operatorname{coker} f & & &
\end{tikzcd}$$

Hm! If f was surjective, we could apply the snake lemma!! But f is not surjective. (...) All proofs I've found use Zorns lemma...

c) Use the same diagram. The middle column is surjective at the bottom, and by commutativity, the right column must be as well.

d) This is obvious, since $f_* \mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$.

e) It is clear that $\mathscr{G}$ is a sheaf. If $U \subset V$, let $s \in \mathscr{G}(U)$ be given. Then define $s' \in G(V)$ as follows: $s'(P) = s(P)$ if $P \in U$ and zero elsewhere. This element will be sent to s.

The injective morphism from $\mathscr{F}$ to $\mathscr{G}$ is defined as follows: send $s \in \mathscr{F}(U)$ to the function $s(P) = s_P$ in $\mathscr{G}(U)$.

♡

Exercise 10 (Exercise 2.19)**.** Let A be a ring. The following are equivalent:

1. $\operatorname{Spec} A$ is disconnected.

2. There exists nonzero elements $e_1, e_2 \in A$ such that $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1 + e_2 = 1$ (these are called *orthogonal idempotents*).

3. A is isomorphic to a direct product $A_1 \times A_2$ of two nonzero rings.

♠

Solution 10. $1 \Rightarrow 3$: Let U be a nonempty connected compontent of $X = \operatorname{Spec} A$. Let $V = X \backslash U$ be its complement, and let $i_1 : U \to X$ and $i_2 = V \to X$ be the natural inclusions on topological spaces. This can be extended to a map of schemes as well: we need to give a morphism $f^{\#} : \mathscr{O}_X \to f_* \mathscr{O}_U$. But $f_* \mathscr{O}_U(W) = \mathscr{O}_X(W \cap U)$, so $f_* \mathscr{O}_U = \mathscr{O}_X \big|_U$. Hence we just choose $f^{\$} : \mathscr{O}_X \to \mathscr{O}_U$ to be the natural map provided by the sheaf axioms.

We now have two morphisms $i_1 : U \to X$ and $i_2 : V \to X$ which are closed immersions, hence the induced ring morphisms $A \to A_1$ and $A \to A_2$ are surjective. Also, the universal property for products hold because the universal property for coproducts hold in the category of affine schemes. Hence $A \simeq A_1 \times A_2$. (a bit clumsy??)

$2 \Rightarrow 3$: Let $\pi_i : A \to A$ be given by multiplication by e_i and let A_i be its image. Then $\ker \pi_1 = A_2$, because if $e_1 f$ then $f = e_2 f$, so $f \in A_2$. The splitting maps are the natural inclusions.

$3 \Rightarrow 2$: If $A = A_1 \times A_2$, let $e_i = \pi_i(1)$.

$3 \Rightarrow 1$: Similar to the first argument, just opposite.

♡

Exercise 11 (Excercise 7.1)**.** Let $(X, \mathscr{O}_X)$ be a locally ringed space and let $f : \mathscr{L} \to \mathscr{M}$ be a surjective map of invertible sheaves on X. Show that f is an isomorphism. ♠

Solution 11. Since $\mathscr{L}, \mathscr{M}$ are invertible, we have isomorphisms $\mathscr{L}_x \approx \mathscr{O}_{X,x}$ and $\mathscr{M}_x \approx \mathscr{O}_{X,x}$ for each $x \in X$.

But $\operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{O}_{X,x}, \mathscr{O}_{X,x}) = \mathscr{O}_{X,x}$, that is, all homomorphisms are given by multiplication by some $h \in \mathscr{O}_{X,x}$. But since f was surjective, we conclude that h is outside $\mathfrak{m}_x$, the maximal ideal of $\mathscr{O}_{X,x}$. But then h is a unit, so f is an isomorphism. ♡

2.3 Chapter III - Cohomology

Exercise 12 (Exercise 2.1)**.** a) Let $X = \mathbb{A}^1_k$ be the affine line over an infinite field k. Let P, Q be distinct closed points on X and let $U = X - \{P, Q\}$. Show that $H^1(X, \mathbb{Z}_U) \neq 0$.

♠

Solution 12. a) We have an exact sequence

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to i_*(\mathbb{Z}\big|_Z) \to 0,$$

where $Z = P \cup Q$. The last sheaf is equal to the skyscraper sheaf $\mathbb{Z}_P \oplus \mathbb{Z}_Q$. Since $\mathbb{Z}$ is flabby, we have $H^1(X, \mathbb{Z}) = 0$. Hence the long exact sequence reads

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H^1(\mathbb{Z}_U) \to 0.$$

It follows that $H^1(\mathbb{Z}_U) = \mathbb{Z}^2$. In fact, this should please us, because if $k = \mathbb{C}$, we have that U is the complex plane minus two points, which is homotopic to the figure eight, which indeed have $H^1_{sing}(U, \mathbb{C}) = \mathbb{C}^2$.

♡

Exercise 13 (Exercise 4.3)**.** Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$ and let $U = X \backslash \{(0, 0)\}$. Use a suitable open cover of X by open affine subsets to show that $H^1(U, \mathscr{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^i y^j \mid i, j < 0\}$. In particular, it is infinitedimensional, and so U cannot be affine (not projective either). ♠

Solution 13. We can cover U by $U_1 = \mathbb{A}^2 \backslash \{x = 0\}$ and $U_2 = \mathbb{A}^2 \backslash \{y = 0\}$. We have $U_1 \cap U_2 = \mathbb{A}^2 \backslash \{xy = 0\}$. Also, $\mathscr{O}(U_1) = k[x, y, \frac{1}{x}]$ and $\mathscr{O}(U_2) = k[x, y, \frac{1}{y}]$ and $\mathscr{O}(U_1 \cap U_2) = k[x, y, \frac{1}{xy}]$. Then the Čech complex takes the form

$$0 \to k[x, y, \frac{1}{x}] \times k[x, y, \frac{1}{y}] \xrightarrow{d} k[x, y, \frac{1}{xy}] \to 0,$$

the differential being difference. Then $H^1(U, \mathscr{O}_U)$ can be computed as the homology at the second term. But nothing on the left side can hit anything of the form $x^i y^j$ with $i, j < 0$. Anything else is hit. Thus we have

$$H^1(U, \mathscr{O}_U) \simeq \{x^i y^j \mid i, j < 0\}$$

as k-vector spaces. ♡

Exercise 14 (Exercise 4.7)**.** Let X be the subscheme of $\mathbb{P}^2_k$ defined by a single homogeneous polynomial $f(x_0, x_1, x_2) = 0$ of degree d. Assume that $(1, 0, 0)$ is not on X. Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Čech complex

$$\Gamma(U, \mathscr{O}_X) \oplus \Gamma(V, \mathscr{O}_X) \to \Gamma(U \cap V, \mathscr{O}_X)$$

explicitly, and thus show that

$$\begin{aligned} \dim_k H^0(X, \mathscr{O}_X) &= 1 \\ \dim_k H^1(X, \mathscr{O}_X) &= \frac{1}{2}(d-1)(d-2). \end{aligned}$$

♠

Solution 14. X can be covered by just two open affines since $\mathbb{P}^2 \backslash (U \cup V) = \{(1:0:0)\}$, which was assumed not to lie on the curve.

The open affine subset $\Gamma(U, \mathscr{O}_X)$ can be identified with the polynomial ring $k[u,v]/\langle f(u,1,v)\rangle$, and $\Gamma(V, \mathscr{O}_X) = k[x,y]/f(x,y,1)$. The differential is then given by

$$(g(u,v), h(x,y)) \mapsto g(xy^{-1}, y^{-1}) - h(x,y) \in k[x,y,\frac{1}{y}].$$

We can assume that $f = x_0^d$, since what really matters is the degree, and we are just doing linear algebra.

We first calculate $H^0(X, \mathscr{O}_X)$. So suppose $g(xy^{-1}, y^{-1}) - h(x,y) = 0$ in $k[x,y,y^{-1}]/\langle f(x,y,1)\rangle$. By definition this means that

$$g(xy^{-1}, y^{-1}) - h(x,y) = f(x,y,1) \cdot \tilde{f}(x,y,\frac{1}{y})$$

for some polynomial $\tilde{f}$. Write $\tilde{f}$ as $\tilde{f}_0 + \tilde{f}_1$, where $\tilde{f}_0 = \sum_{j<0} a_{ij} x^i y^j$ and $\tilde{f}_1 \in k[x,y]$. Then we have the equality

$$g(xy^{-1}, y^{-1}) - h(x,y) = \sum_{j<0} a_{ij} x^{i+d} y^j + \sum_{j \geq 0} x^{i+d} y^j.$$

First of all, we see that the constant terms of g and h must be equal, because there are no constant terms on the right hand side. Secondly, $g(xy^{-1}, y^{-1})$ consists solely of terms with $j < 0$. Thus the non constant terms of $g(xy^{-1}, y^{-1})$ must be equal to the left term of the right hand side above. But both terms of the right hand side are zero modulo f, so the constant terms of $g(xy^{-1}, y^{-1})$ are also zero mod f. The same holds for $h(x,y)$. Thus $H^0(X, \mathscr{O}_X) = \{(c,c) \mid c \in k\} \simeq k$.

Now we compute $H^1(X, \mathscr{O}_X)$. Consider a monomial $x^i y^j$ in the target. If both $i, j \geq 0$, then it is hit by $(0, -x^i y^j)$. Likewise, if $j \geq i$, then $(x^i y^{j-i}, 0) \mapsto x^i x^{-j}$. Thus all monomials $x^i y^{-j}$ with $j \geq i$ is zero in the cokernel. Further, if $i \geq d$, then $x^i y^j$ is already zero! Thus, we can draw the non-zero monomials in the cokernel as points in the lattice $\mathbb{Z}^2$. This is a triangle of length $d-2$. Thus the dimension of $H^1(X, \mathscr{O}_X)$ is

$$1 + 2 + \ldots + d - 3 + d - 2 = \frac{1}{2}(d-2)(d-2+1) = \frac{1}{2}(d-2)(d-1).$$

$\heartsuit$

2.4 Chapter IV - Curves

Exercise 15 (Exercise 1.1)**.** Let X be a curve and $P \in X$ a point. Show that there exists a nonconstant rational function $f \in K(X)$ which is regular everywhere except at P. ♠

Solution 15. Let D be the divisor $D = nP$. The linear system

$$\{E = D + f \geq 0\}$$

consists of all divisors linearly equivalent to D. But these are classified by those f with $(f) \geq -nP$, i.e. those f with at most poles of order n at P.

By Riemann-Roch we have

$$l(D) - l(K - D) = \deg D + 1 - g = n + 1 - g.$$

If n is large enough, $K - D$ will have negative degree, so $l(K - D) = 0$. Thus for large n, we can get $l(D)$ as big as we want. ♡

Exercise 16 (Exercise 1.2)**.** Again let X be a curve and let $P_1, \cdots, P_r$ be points. Then there is a rational function $f \in K(X)$ having poles only at the P_i and regular elsewhere. ♠

Exercise 17. Again, this follows by Riemann-Roch. Let $D = n\sum_i P_i$. Then, as in the previous exercise, we get that the set

$$\{f \in K(X) \mid (f) + \sum nP_i \geq 0\}$$

is non-empty. But the condition is equivalent to $(f)_{P_i} \geq -nP_i$ and $(f)_P \geq 0$ for $P \neq P_i$. But this is exactly what is to be shown. ♠

Exercise 18 (Exercise 1.5)**.** For an effective divisor D on a curve X of genus g, show that $\dim|D| \leq \deg D$. Furthermore, equality holds if and only if $D = 0$ or $g = 0$. ♠

Solution 16. This is a simple consequence of Riemann-Roch. Note that the statement is equivalent to $l(D) - 1 \leq \deg D$. So by Riemann-Roch:

$$l(D) - 1 = \deg D - g + l(K - D).$$

So the statement is equivalent to $l(K-D) \leq g$. Now, by definition, $l(K) = g$, so we need only show that $l(K - D) \leq l(K)$ for any two divisors K, D. But by the identification

$$H^0(K - D) = \{f \in K(C) \mid (f) + K - D \leq 0\}$$

this is trivial since D is effective.

Now assume $l(K-D) = g$. If $D = 0$, then this is equal by definition. If not, then if f is such that $(f) + K \geq 0$, then $(f) + K - D \geq 0$. We must have $\deg f = 0$ and f must have at least $2g-2$ zeros and the same number of poles. .... ♡

2.5 Chapter V - Surfaces

Exercise 19 (Exercise 1.1)**.** Let C, D be any two divisors on a surface X, and let the corresponding invertible sheaves be $\mathscr{L}, \mathscr{M}$. Show that

$$C.D = \chi(\mathscr{O}_X) - \chi(\mathscr{L}^{-1}) - \chi(\mathscr{M}^{-1}) + \chi(\mathscr{L}^{-1} \otimes \mathscr{M}^{-1}).$$

♠

Solution 17. We have an exact sequence

$$0 \to \mathscr{L}^{-1} \to \mathscr{O}_X \to \mathscr{O}_C \to 0,$$

and similarly with $\mathscr{L}$ and C replaces by $\mathscr{M}$ and D, from which we obtain that $\chi(\mathscr{O}_C) = \chi(\mathscr{O}_X) - \chi(\mathcal{L}^{-1})$. We want to use the equality $C.D = \chi(\mathscr{O}_C) - \chi(\mathscr{L}(-D) \otimes \mathscr{O}_C)$ in the proof of Proposition 1.4.

Using what we've found, we have

$$C.D = -\chi(\mathscr{L}^{-1}) + \chi(\mathscr{O}_{C \cap D}).$$

Using the exact sequences

$$0 \to \mathscr{M}^{-1} \otimes \mathscr{O}_C \to \mathscr{O}_C \to \mathscr{O}_{C \cap D} \to 0$$

and

$$0 \to \mathscr{M}^{-1} \otimes \mathscr{L}^{-1} \to \mathscr{M}^{-1} \to \mathscr{M}^{-1} \otimes \mathscr{L}^{-1} \to 0,$$

together with additivty of Euler characteristics, we obtain the desired result. ♡

Exercise 20 (Exercise 1.4)**.** a) If a surface X of degree d contains a straight line $C = \mathbb{P}^1$, show that $C^2 = 2 - d$.

b) Assume that $char k = 0$ and show that for every $d \geq 1$, there exists a nonsingular surface X of degree d in $\mathbb{P}^3$ containing the line $x = y = 0$.

♠

Solution 18. a) This follows from the adjunction formulas. We have that $K_X = (d-4)H$ by the adjunction formula and also that $-2 = C^2 + C.K$ by Prop 1.5. Thus

$$C^2 = -2 - C.K = -2 - C.((d-4)H) = -2 - d + 4 = 2 - d$$

since $C.H$ is the degree of C which is 1.

b) Let X be the surface defined by $f = x^d + y^d - xz^{d-1} - yw^{d-1}$. Then f is non-singular and contains $x = y = 0$. We check nonsingularity by the Jacobian criterium. The ideal generated by the partial derivates is

$$Jac(f) = \langle dx^{d-1} - z^{d-1}, dy^{d-1} - w^{d-1}, xz^{d-2}, yw^{d-2}\rangle$$

Takin radicals gets rid of some of the powers:

$$\sqrt{Jac(f)} = \langle dx^{d-1} - z^{d-1}, dy^{d-1} - w^{d-1}, xz, yw\rangle$$

So suppose $(x : y : z : w)$ is a singular point. Then $xz = 0$. So, say, $x = 0$. This implies $z = 0$ (by the first equation), which implies $y(y^{d-1} - w^{d-1}) = 0$ by the equation of f. If $y = 0$, we must have $w = 0$, so all coordinates are zero! Contradiction. So assume $y^{d-1} - w^{d-1} = 0$. Then by the second equation, we must have $dy^{d-1} - w^{d-1} = (d-1)y^{d-1} = 0$, so $y = 0$ now as well. So $w = 0$ again! So either way, we don't get a point in projective space.

So suppoze $z = 0$. Then $x = 0$ by the first equation (and we repeat the arguments). ♡

Exercise 21 (Exercise 1.5)**.** a) If X is a surface of degree d in $\mathbb{P}^3$, then $K^2 = d(d-4)^2$.

b) If X is a product of nonsingular curves C, C' og genus g, g', then $K^2 = 8(g-1)(g'-1)$.

♠

Solution 19. a) By the adjunction formula, we have $\omega_X = \omega_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}}(d) \otimes \mathscr{O}_X = \mathscr{O}_X(d-4)$. Thus, taking classes, we find that

$$K_X = (d-4)H,$$

where H is the pullback of the hyperplane class in $\mathbb{P}^3$. Hence $K^2 = (d-4)^2H^2$. Now, H^2 is the number of points in the intersection of $H \cap X$ and $H' \cap X$ where H, H' are generic hyperplanes. But $H \cap X$ is a plane curve of degree d. Intersecting this with the line $H \cap H'$ gives d points. Hence $K^2 = d(d-4)^2$.

b). ♡

3 Categories for the working mathematician - Saunders MacLane

3.1 Chapter 1.1

Exercise 22 (1.1)**.** Show that each of the following constructions can be regarded as a functor: The field of quotiens of an integral domain; the Lie algebra of a Lie group. ♠

Solution 20. The first is a functor $\mathsf{Rings} \to \mathsf{Fields}$, since "obviously" the assignment $R \mapsto K(R)$ is functorial, since a morphism is determined by what it does to elements of R.

The second is a functor from the category of Lie groups to the category of Lie algebras. It sends a Lie group G to its tangent space T_eG at the identity. ♡

Exercise 23 (1.2)**.** Show that functors $1 \to C$, $2 \to C$ and $3 \to C$ correspond respectively to objects, arrows and composable pairs of arrows of C. ♠

Solution 21. A functor $1 \to C$ need not satisfy any compatability conditions, so the only data is an object in C.

A functor $F : 2 \to C$ correspond to a choice of two objects $F(1), F(2)$ in C such that there is an arrow $F(1) \to F(2)$.

A functor $F : 3 \to C$ correspond to a choice of three objects in C and arrows between them such that that the corresponding triangle commutes. ♡

Exercise 24 (1.4)**.** Prove that there is no functor $\mathsf{Grp} \to \mathsf{Ab}$ sending a group to its center. ♠

Solution 22. We take the hint and consider maps $S_2 \to S_3 \to S_2$. Let the first map be given by sending a permutation $(a_1 a_2)$ to $(a_1 a_2 3)$[1] Let the map $S_3 \to S_2$ be given by sending σ to its signature ± 1. Then the composite is the identity.

Denote the purported functor by Z.

But the center of S_3 is trivial. This forces $Z(f \circ g)$ to be the constant function $g \mapsto e$. However, as $f \circ g = \mathrm{id}_{S_2}$, this sshould be the identity, which it is not. ♡

Exercise 25 (1.5)**.** Find two different functors $T : \mathsf{Grp} \to \mathsf{Grp}$ with object function $T(G) = G$ for every group G. ♠

[1] Notation: $(a_1 \dots a_n)$ denotes the permutation sending i to a_i.

Solution 23. Here are two different functors: let the first one be the identity functor, sending each morphism $G \to H$ to itself.

Let the other be the "trivial" functor, sending each morphism $G \to H$ to the trivial morphism $\epsilon : G \to H$, $g \mapsto e \in H$.

It is trivial that both of these are functors. ♡

4 Calculus on Manifolds - Spivak

4.1 Functions on Euclidean Space

Exercise 26 (Exercise 1.1)**.** Prove that $|x| \leq \sum_{i=1}^{n} |x^i|$. ♠

Solution 24. By induction, one can prove that $\sqrt{\sum_i a_i} \leq \sum_i \sqrt{a_i}$. The claim then follows trivially. ♡

Exercise 27 (Exercise 1.7)**.** A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is *norm preserving* if $|T(x)| = |x|$ for all $x \in \mathbb{R}^n$. It is inner product preserving if $\langle Tx, Ty \rangle = \langle x, y \rangle y$.

a) Prove that T is norm preserving if and only if it is inner product preserving.

b) Prove that such a linear transformation is $1-1$ and T^{-1} is of the same sort.

♠

Solution 25. a). The direction $\Leftarrow$ is trivial. For the other direction, choose a basis $\{x_1, \dots, x_n\}$ of $\mathbb{R}^n$ such that $x = x_1$ and $y = \sum a_i x_i$. Then $\langle Tx, a_i x_i \rangle = a_i \langle Tx, x_i \rangle = 0$ if $i \neq 1$ and a_1 else. Then since $T(0) = 0$ it follows that

$$\langle Tx, Ty \rangle = \langle Tx, T(a_1 x_1) \rangle = a_1 \langle Tx_1, Tx_1 \rangle = a_1 |Tx_1|^2 = a_1 |x_1|^2 = a_1 \langle x_1, x_1 \rangle.$$

b). Suppose $T(x) = 0$. Then $0 = \langle Tx, Tx \rangle = \langle x, x \rangle$, but this happens if and only if $x = 0$. Also $\langle T^{-1}y, T^{-1}y \rangle = \langle T^{-1}T(x), T^{-1}T(x) \rangle = \langle TT^{-1}(x), TT^{-1}(x) \rangle = \langle T^{-1}x, T^{-1}x \rangle = \langle y, y \rangle$. ♡

5 Commutative Algebra - Eisenbud

5.1 Chapter 16 - Modules of Differentials

Exercise 28 (Exercise 16.1)**.** Show that if $b \in S$ is an idempotent ($b^2 = b$), and $d : S \to M$ is any derivation, then $db = 0$. ♠

Solution 26. This is trivial. $db = d(b^2) = 2db$. If $2 = 0$, then the statement is automatically true. If not, then $db = 0$ by subtraction. ♡

6 Complex Geometry - Daniel Huybrechts

6.1 1 Local Theory

Exercise 29 (1.1.1)**.** Show that every holomorphic map $f : \mathbb{C} \to \mathbb{H} = \{z \mid \Im z > 0\}$ is constant. ♠

Solution 27. Note that $\mathbb{H}$ is holomorphic to the open unit disc via the Möbius transformation $z \mapsto i/z$. Then it follows from Liouville's theorem that any such map must be constant. ♡

Exercise 30 (1.1.2)**.** Show that the real and imaginary part u and v, respectively, of a holomorphic function $f = u + iv$ are harmonic, i.e.

$$\sum_i \frac{\partial^2 u}{\partial x_i^2} + \sum_i \frac{\partial^2 u}{\partial y_i^2} = 0,$$

and similarly for v. ♠

Solution 28. It will be clear from the solution that without loss of generality, we can do this for the one-dimensional case. Then this follows trivially from the Cauchy-Riemann equations:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\frac{\partial u}{\partial y} \\ &= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \\ &= 0. \end{aligned}$$

♡

7 Deformation Theory - Hartshorne

7.1 Chapter I.3 - The T^i functors

Exercise 31 (Exercise 3.1)**.** Let $B = k[x, y](xy)$. Show that $T^1(B/k, M) = M \otimes k$ and $T^2(B/k, M) = 0$ for any B-module M. ♠

Solution 29. Since B is defined by a principal ideal in $P = k[x, y]$, it follows that $L_2 = 0$ in the cotangent complex. Thus $T^2(B/k, M)$ is automatically zero.

We have that $L_1 = B$ and $L_0 = Bdx \oplus Bdy$ with d_1 being $f \mapsto (fy, fx)$. Applying $\mathrm{Hom}(-, M)$, we get $\mathrm{Hom}(L_0, M) = M \oplus M$ and $\mathrm{Hom}(L_1, M) = M$.

We have $\mathrm{Hom}(B \oplus B, M) \simeq M \oplus M$ by $\phi \mapsto (\phi(1,0), \phi(0,1)$. We have a diagram

$$\begin{tikzcd}
\mathrm{Hom}(B \oplus B, M) \arrow[r, "\psi^*"] \arrow[d, "\simeq"'] & \mathrm{Hom}(B, M) \arrow[d, "\simeq"] \\
M \oplus M \arrow[r] & M
\end{tikzcd}$$

Under these isomorphisms, it is easy to see that the bottom map is given by

$$(\phi(1,0), \phi(0,1)) \mapsto y\phi(1,0) + x\phi(0,1).$$

Thus since T^1 is the cokernel of this map, we must have $T^1(B/k, M) = M \otimes k$. ♡

Exercise 32 (Exercise 3.3)**.** Let $B = k[x,y]/(x^2, xy, y^2)$. Show that $T^0(B/k, B) = k^4$, $T^1(B/k, B) = k^4$ and $T^2(B/k, B) = k$. ♠

Solution 30. Let's compute L_2 first. For that we need part of a resolution of I. We have in fact

$$0 \to \mathrm{im} \begin{pmatrix} -y & 0 \\ x & -y \\ 0 & x \end{pmatrix} \to P(-2)^3 \to I \to 0.$$

The Koszul relations are given by

$$\mathrm{im} \begin{pmatrix} -y^2 & -xy & 0 \\ 0 & x^2 & -y^2 \\ x^2 & 0 & xy \end{pmatrix}.$$

Let's compute Q/F_0 (relations modulo Koszul relations). Since Q is generated in degree 3, and F_0 is of degree 4, we have $\dim_k(Q/F_0)_3 = 2$. Let's consider degree 4. As a k-vector space Q_4 is spanned by the four elements

$$\begin{pmatrix} -y^2 \\ xy \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -y^2 \\ xy \end{pmatrix}, \begin{pmatrix} -yx \\ x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -yx \\ x^2 \end{pmatrix}.$$

The two in the middle are already Koszul relations, so that $(Q/F_0)_4$ have dimension ≤ 2. But we also have

$$\begin{pmatrix} -y^2 \\ xy \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ yx \\ -x^2 \end{pmatrix} + \begin{pmatrix} -y^2 \\ 0 \\ x^2 \end{pmatrix}.$$

Thus $\dim_k (Q/F_0)_4 = 1$, since the second term above is a Koszul relation. Similarly we find that $\dim_k (Q/F_0)_5 = 0$. Hence, L_2 is the 3-dimensional k-vector space spanned by Q_3 and one more relation. L_1 is $F \otimes B = B^3$, and L_0 is $B \oplus B$, spanned by dx, dy.

Taking duals, we get that $L_2 = \operatorname{Hom}(Q/F_0, B)$. This set can be identified with

$$\begin{aligned} \operatorname{Hom}(Q/F_0, B) &= \{\varphi : Q \to B \mid \varphi\big|_{F_0} = 0\} \\ &= \{\varphi : Q \to P \mid \operatorname{im} f\big|_{F_0} \subseteq I\} \end{aligned}$$

Thus, since $I = \mathfrak{m}^2$, we must have that φ sends the two generators of Q to something of degree 1 (degree 0 is not ok, since then F_0 would be sent outside I). Thus $\operatorname{Hom}(Q/F_0, B)$ is $2 \times 2 = 4$-dimensional, spanned by

$$\operatorname{im} \begin{pmatrix} y & x & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}.$$

But d_2 is the dual of the inclusion $Q \to F$ from the exact sequence above. The dual is given by transposing, and we are left with one column - in conclusion, $T^2(B/k, B)$ is one-dimensional.

The Jacobian of I is given by

$$\begin{pmatrix} 2x & y & 0 \\ 0 & x & 2y \end{pmatrix},$$

and it is easily seen that the kernel of $\operatorname{Jac} \otimes B$ is given by $\mathfrak{m} \oplus \mathfrak{m} \oplus \mathfrak{m} \subset B^3$. The two relations kill off two dimensions, so $\dim_k T^1(B/k, B) = \dim_k \mathfrak{m}^{\oplus 3} - 2 = 6 - 2 = 4$.

Also $T^0(B/k, B)$ is B^2 modulo the image of the Jacobian. The constants are left untouched, so $\dim_k T^0(B/k, B) = 2 + 2 + 2 - 3 = 3$. A basis is given by $(1, 0), (0, 1)$ and (x, y). (thus Hartshorne is wrong?) ♡

8 Geometry of differential forms - Morita

8.1 Chapter 1 - Manifolds

Exercise 33 (1.1)**.** For a natural number m, define a map $f_m : \mathbb{C} \to \mathbb{C}$ by $z \mapsto z^m$. Let $z = x + iy$, and consider f_m as a function of x and y. Compute the Jacobian matrix of f_m. ♠

Solution 31. We consider f_m as a map $\mathbb{R}^2 \to \mathbb{R}^2$. Write $f_m = r_m + ij_m$. Then we want compute $\frac{\partial r_m}{\partial x}, \frac{\partial r_m}{\partial y}$ and $\frac{\partial j_m}{\partial x}$ and $\frac{\partial j_m}{\partial y}$. Now

$$\frac{\partial}{\partial x}(x+iy)^m = m(x+iy)^{m-1}$$

and

$$\frac{\partial}{\partial y}(x+iy)^m = im(x+iy)^{m-1}.$$

Now taking real parts ($\Re$) and taking imaginary parts ($\Im$) are continous operations $\mathbb{C} \to \mathbb{R}$, so they commute with limits and hence derivatives. Hence

$$\frac{\partial r_m}{\partial x} = \Re\left(m(x+iy)^{m-1}\right) = m\Re(z^{m-1})$$

and

$$\frac{\partial r_m}{\partial y} = \Re\left(mi(x+iy)^{m-1}\right) = -m\Im(z^{m-1})$$

since in general $\Re(iz) = -\Im(z)$. Similarly

$$\frac{\partial j_m}{\partial x} = \Im\left(m(x+iy)^{m-1}\right) = m\Im(z^{m-1})$$

and

$$\frac{\partial j_m}{\partial x} = \Im\left(im(x+iy)^{m-1}\right) = \Re(z^{m-1}).$$

Hence the Jacobian matrix is

$$\begin{bmatrix} m\Re(z^{m-1}) & -m\Im(z^{m-1}) \\ m\Im(z^{m-1}) & m\Re(z^{m-1}) \end{bmatrix},$$

with determinant $m^2|z|^2$. ♡

Exercise 34 (1.2)**.** Prove that the set of all the orthogonal matrices of order 2, denoted $O(2)$ becomes a C^∞ manifold in a natural way. ♠

Solution 32. Let a, b, c, d be coordinates on $M_2(\mathbb{R})$. Then the equations of $O(2)$ are given by

$$\begin{aligned} a^2 + b^2 - 1 &= 0 \\ ac + bd &= 0 \\ c^2 + d^2 - 1 &= 0 \end{aligned}$$

Hence the Jacobian is

$$\begin{bmatrix} 2a & 2b & 0 & 0 \\ c & d & a & b \\ 0 & 0 & 2c & 2d \end{bmatrix}.$$

We want this to have maximal rank. The determinant of the first minor (choosing the first three columns) is $4c(ad - bc)$. Hence this has maximal rank if and only if $c \neq 0$. So suppose $c = 0$. Then $d^2 = \pm 1$, hence $d = \pm 1$, hence $b = 0$, hence $a^2 = 1$, hence $a = \pm 1$. Hence the only two matrices in $O(2)$ with $c = 0$ are the identity matrix I and $-I$. But by inspection, both these values are regular, hence $O(2)$ is a C^∞ submanifold of $M_2(\mathbb{R})$ by the inverse function theorem.

In fact, since the Jacobian have rank 3, it follows that $O(2)$ is one-dimensional. In fact it is diffeomorphic to two disjoint copies of S^1. ♡

Exercise 35 (1.3)**.** Show that the 1-dimensional complex projective space $\mathbb{C}P^1$ is diffeomorphic to S^2. ♠

Solution 33. We define a map $\mathbb{C}P^1 \to S^2$ locally. That is, both spaces are covered by two charts, and we check that the maps agree on intersections. It will be clear from the construction that it is injective and surjective and C^∞, and since the spaces are compact, there exists an inverse.

$\mathbb{C}P^1$ is covered by two charts $i_1 : (x, y) \mapsto [x + iy, 1]$ and $i_2 : (x, y) \mapsto [1, x + iy]$. Similarly, S^2 is covered by two charts, given by stereographic projection. The formulas are $j_1^{-1}(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$ and $j_2^{-1}(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$ with inverses $j_1(a, b) = \left(\frac{2a}{1+a^2+b^2}, \frac{2b}{1+a^2+b^2}, \frac{-1+a^2+b^2}{1+a^2+b^2}\right)$ and $j_2(a, b) = \left(\frac{2a}{1+a^2+b^2}, \frac{2b}{1+a^2+b^2}, \frac{1-a^2-b^2}{1+a^2+b^2}\right)$.

Define a map $\mathbb{C}P^1 \to S^2$ by the identity map on the chart corresponding to i_1 and on the other chart, let the map be given by $\varphi : (a, b) \mapsto (a, -b)$. Then one can check that $j_2 \circ \varphi \circ i_2^{-1} \circ i_1 = j_1$.

Hence the diagram

$$\begin{tikzcd}
& \mathbb{C}P^1 \arrow[r] & S^2 & \\
\mathbb{R}^2 \arrow[ur, "i_1"] \arrow[rr] \arrow[dr, "i_2^{-1}\circ i_1"'] & & \mathbb{R}^2 \arrow[u, "j_1"] \arrow[dr, "j_2^{-1}\circ j_1"] & \\
& \mathbb{R}^2 \arrow[uu, "i_2"'] \arrow[rr, "\varphi"] & & \mathbb{R}^2 \arrow[uul, "j_2"']
\end{tikzcd}$$

And this defines a map $\mathbb{C}P^1 \to S^2$.

This map globalizes to

$$\mathbb{C}P^1 \ni [x+iy, a+ib] \mapsto \left(\frac{2ax+2by}{a^2+b^2+x^2+y^2}, \frac{2ay-2bx}{a^2+b^2+x^2+y^2}, \frac{a^2+b^2-x^2-y^2}{a^2+b^2+x^2+y^2}\right) \in S^2$$

♡

9 Foundations of Algebraic Geometry - Vakil

9.1 Chapter 21 - Differentials

Exercise 36 (Excercise 21.2.F)**.** a) Suppose K/k is a separable algebraic field extension. Show that $\Omega_{K/k} = 0$.

b) Suppose $char k = p$ and and let $K = k(t^p)$ and $L = k(t)$. Compute $\Omega_{L/K}$.

♠

Solution 34.

1. Choose a nonzero $\alpha \in K$. By definition it satisfies a polynomial equation $f(\alpha) = 0$ such that $f'(\alpha) \neq 0$. Then a computation gives that
$$0 = d(f\alpha) = f'(\alpha)d\alpha.$$
This implies $d\alpha = 0$, so $\Omega_{K/k} = 0$.

2. Write $L = K[x]/(x^p - t^p)$. Then $\Omega_{L/K} = Ldx/df$. But $df = d(x^p - t^p) = px^{p-1}dx = 0$. Hence $\Omega_{L/K}$ is just the free L-module generated by t.

♡

10 Introduction to Differential Geometry - Spivak

10.1 Chapter 1 - Manifolds

Exercise 37 (Exercise 3)**.** a) Every manifold is locally compact.

b) Every manifold is locally pathwise connected, and a connected manifold is pathwise connected.

c) A connected manifold is arcwise connected. (*Arcwise connected* means that any two points can be connected by a $1-1$ path.)

♠

Solution 35. a) Indeed, let $x : \mathbb{R}^n \to U \subset M$ be a homeomorphism of an open subset of M with $\mathbb{R}^n$. Then the image of $[0,1]^n$ is compact in M.

b) The first part follows in the same way, since M is locally homeomorphic to $\mathbb{R}^n$. Now assume that M is connected. Fix $q \in M$. Let V be the set of all points in M such that there is a path from q to p. Clearly V is non-empty, by the first part of the exercise.

   V is also open: For let $p \in V$. Choose a neighbourhood U around p homeomorphic to $\mathbb{R}^n$. By composing paths, any point in U can be reached as well. Hence V is open.

   We show that V is closed: let $\{p_i\}$ be a convergent sequence of points $p_i \in M$ with all $p_i \in V$. We want to show that the limit point is contained in V. Choose a compact neighbourhood around $\lim p_i = p$, which we can assume to be $\approx [0,1]^n$. Then $p \in [0,1]^n$, and this is path connected. Hence V is closed.

c) For $n > 2$, one can always homotope away from the points of non-injectivity. For $n = 2$, one can do "Reidemeister" moves.

♡

Exercise 38 (Exercise 4)**.** A space X is called locally connected if for each $x \in X$, it is the case that every neoughbourhood of x contains a connected neighbourhood.

a) Connectedness does not imply local connectedness.

b) An open subset of a locally connected space is locally connected.

c) X is locally connected if and only if components of open sets are open, so every neigbourhood of a point in a locally connected space contains an *open* connected neighbourhood.

d) A locally connected space is homeomorphic to the disjoint union of its components.

e) Every manifold is locally connected, and consequently homeomorphic to the disjoint union of its components, which are open submanifolds.

♠

Solution 36. a) Consider the topologist's since curve. Every neighbourhood of 0 is diconnected.

b) This is "trivial". Let U be the said open subset. The open subsets of U are intersections $U \cap V$ where V is open in X. Hence local connectedness is trivially inherited.

c) Suppose X is locally connected. Let $U \subset X$ be an open set, and let $U = \cup_i U_i$ be its decomposition into its components. We want to show that each U_i is open. So let $x \in U_i$. Then U_i contains a connected neighbourhood containing x, by definition, hence U_i is open.

Conversely, assume components of open sets are open. Let $x \in X$, and let U be an open neighbourhood of x. Then U_i as above is connected and can be chosen to contain x, hence x is locally connected.

d) This is trivial, since the components are open.

e) Pathwise local connectedness implies local connectedness.

♡

Exercise 39 (Exercise 15)**.** a) Show that $\mathbb{P}^1$ is homeomorphic to S^1. (in fact, diffeomorphic)

b) Show that $\mathbb{P}^n \backslash \mathbb{P}^{n-1}$ is homeomorphic to the interior $D^n = \{x \in \mathbb{R}^n \mid d(x, 0) < 1\}$.

♠

Solution 37. a) Both $\mathbb{P}^1$ and S^1 can be covered by two open subsets homeomorphic to $\mathbb{R}$, and it can be checked that in both cases, the transition mapping is given by $x \mapsto \frac{1}{x}$, hence they are glued in the same way, hence they must be diffeomorphic.S

b) By using homogeneous coordinates, we see that $\mathbb{P}^n \backslash \mathbb{P}^{n-1}$ is homeomorphic to $\mathbb{R}^n$ which is again homeomorphic to the interior of a disc.

♡

10.2 Chapter 3 - The tangent bundle

Exercise 40. Show that in the definition of an equivalence it suffices to assume that the map $E_1 \to E_2$ is continous. ♠

Solution 38. Okay, so the assumptions are now that there is a continous map $f : E_1 \to E_2$ and a commuting diagram

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[dr, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ & B \end{tikzcd}$$

taking each fibre $\pi_1^{-1}(p)$ isomorphically onto $\pi_2^{-1}(p)$. We want to show that f is a homeomorphism.

First we show that f is bijective by defining a set-theoretic inverse $g : E_2 \to E_1$. Suppose $q \in E_2$. Then $q \in \pi_2^{-1}(\pi_2(q))$. But $f\big|_{\pi_1^{-1}(\pi_2(q))}$ is a vector space isomorphism, so we define

$$g(q) = f\big|_{\pi_1^{-1}(\pi_2(q))}^{-1}(q).$$

The next step is to show that g is continous. This actually follows from the local triviality condition. For each open U on B on which E_1, E_2 both are trivial. Then there are isomorphisms $E_1\big|_U \simeq U \times \mathbb{R}^n$ and $E_2\big|_U \simeq U \times \mathbb{R}^n$, then we have a commutative diagram

$$\begin{tikzcd} U \times \mathbb{R}^n \arrow[r, "f'"] \arrow[dr, "\pi_1"] & U \times \mathbb{R}^n \arrow[d, "\pi_2"] \\ & U \end{tikzcd}$$

The commuting condition implies that f' is the identity on the first factor, and thus f' is *linear* map on the second factor. Linear maps are always continous. ♡

Exercise 41. Show that in the definition of bundle map, continuity of $f : B_1 \to B_2$ follows automatically from continuity of $\widetilde{f} : E_1 \to E_2$. ♠

Solution 39. Again, this is local triviality. Choose a trivialization so that the map is given by $U \times \mathbb{R}^n \to V \times R^m$. This is continous by assmuption. Choose open $W \subset V \subset B_2$. The map π_2 is continous (of course), and by continuity of $\widetilde{f}$ and openness of π_1, we have $f^{-1}(W) = \pi_1(\widetilde{f}^{-1}(\pi_2^{-1}(W)))$ which is open. ♡

Exercise 42. For a bundle map $(\tilde{f}, f)$, with $f : B_1 \to B_2$, let K_p be the kernel of the map $\tilde{f}\big|_{\pi_1^{-1}(p)}$ from $\pi_1^{-1}(p)$ to $\pi_2^{-1}(f(p))$.

1. If $p \mapsto \dim K_p$ is continous, then $\ker \tilde{f}$, the union of all K_p is a bundle over B_1.

2.

♠

Solution 40. 1. The condition says that $\dim \ker K_p$ is constant on connected components, so we may as well assume that $\dim \ker K_p$ is constant and B_1 is connected.

First choose a small open set $U \subset B_2$ such that E_1, E_2 are both trivial over $V = f^{-1}(U)$ and E_2, respectively.

Then the map $E_1\big|_V \to E_2\big|_U$ takes the form $(v, p) \mapsto (f(v), f_2(v, p))$, since this is a bundle map (follows from commutativity).

Then

$$\ker \tilde{f}\Big|_V \simeq \{(v, p) \in V \times \mathbb{R}^n \mid f_2(v, p) = 0\}.$$

Since $\tilde{f}_2$ is linear in the second variable, we can write $\ker \tilde{f}\Big|_V$ as the kernel of matrix with functional entries. If the dimension of the kernel is q, then there is some $q \times q$-minor that doesn't vanish. ♡

10.3 Chapter 4 - Tensors

Exercise 43 (Exc 1)**.** Let $f : M^n \to N^m$ and suppose $(x, U), (y, V)$ are coordinate systems around p and $f(p)$, respectively.

a) If $g : N \to \mathbb{R}$, then

$$\frac{\partial(g \circ f)}{\partial x^i}(p) = \sum_{j=1}^{m} \frac{\partial g}{\partial y^j}(f(p)) \frac{\partial(y^j \circ f)}{\partial x^i}(p).$$

b) Show that

$$f_* \left(\frac{\partial}{\partial x^i}\bigg|_p \right) = \sum_{j=1}^{m} \frac{\partial(y^j \circ f)}{\partial x^i}(p) \frac{\partial}{\partial y^j}\bigg|_{f(p)}.$$

c) Show that

$$(f^* dy^j)(p) = \sum_{i=1}^{n} \frac{\partial(y^j \circ f)}{\partial x^i}(p) dx^i(p).$$

♠

Solution 41. a)

This is of course the chain rule together with deciphering of the definitions. By definition

$$\frac{\partial(g \circ f)}{\partial x^i}(p) = D_i(g \circ f \circ x^{-1}).$$

Which is equal to

$$D_i(g \circ f \circ x^{-1}) = D_i(g \circ y^{-1} \circ y \circ f \circ x^{-1})$$

which by the chain rule is

$$\sum_{j=1}^{n} D_j(g \circ y^{-1})(y \circ f(p)) \cdot D_i(y \circ f \circ x^{-1})^j(x(p))$$

which by definition is equal to what we want.

b)

This is easier. We know that $f_*(\frac{\partial}{\partial x^i})$ is a certain linear combination of $\frac{\partial}{\partial y^j}$'s. So we just have to verify the coefficients.

We use that $\frac{\partial}{\partial y^j} y^i = \delta_i^j$. That is, the coefficients are determined by the m coordinate functions y^i.

Thus each coefficient is given by $(f_* \frac{\partial}{\partial x^i})(y^j) \overset{def}{=} \frac{\partial(y^j \circ f)}{\partial x^i}$.

c)

Solved similarly as above. We know that $f^* dy^j$ is a certain linear combination of dx^i's. To get the coefficients, we use that $dx_j(\frac{\partial}{\partial x^i}) = \delta_i^j$. Then

$$(f^* dy^j)_p(\frac{\partial}{\partial x^i}) \overset{def}{=} dy^j_{f(p)} \left(f_* \frac{\partial}{\partial x^i}\bigg|_p \right).$$

By the above, this is

$$dy^j_{f(p)} \left(\sum_{j=1}^{m} \frac{\partial(y^j \circ f)}{\partial x^i}(p) \frac{\partial}{\partial y^j}\bigg|_{f(p)} \right)$$

The only surviving term in the sum is $\frac{(y^j \circ f)}{x^i}(p)$. This is exactly the coefficient we want.

♡

10.4 Chapter 5 - Vector fields and differential equations

Exercise 44 (Exercise 1)**.**
- If $\alpha : M \to N$ is C^∞, then $\alpha_* : TM \to TN$ is C^∞.
- If $\alpha : M \to N$ is a diffeomorphism and X is a vector field on M,then $\alpha_* M$ is a C^∞ vector field on N.
- If $\alpha : \mathbb{R} \to \mathbb{R}$ is $\alpha(t) = t^3$, then there is a C^∞ vector field on $\mathbb{R}$ such that $\alpha_* X$ is not C^∞.

♠

Solution 42.
- This is clear, since the map from tangent bundles is just the Jacobian matrix. Explicitly, we can WLOG assume $M = \mathbb{R}^n$ and $N = \mathbb{R}^m$ with $TM = \mathbb{R}^n \times \mathbb{R}^n$ and $TN = \mathbb{R}^m \times \mathbb{R}^m$. Then a_* is given by $(v, p) \mapsto (d\alpha(p)(v), \alpha(p))$, where $d\alpha(p)$ is the Jacobian matrix evaluated at p.
- The same strategy. We assume $M = \mathbb{R}^n$ and $N = \mathbb{R}^m$. The difference between α_* and $\alpha_* X$ is that the former is a map between the tangent bundles and the latter is a section of N.

  Then, chasing through the definitions, if the X is given by $p \mapsto (f(p), p)$, the section $\alpha_* X(q)$ is given by $q \mapsto \big(d(\alpha^{-1}(q))(f(\alpha^{-1}(q))), q\big)$.

  But this is clearly C^∞ since α^{-1} is.
- Every vector field on $\mathbb{R}$ have the form $X = f\frac{\partial}{\partial t}$ for some function f. Let $f = 1$. That is, X is the vector field $s \mapsto (1, s)$. Then by the above, $\alpha_* X(s) = ((3s^2)\big|_{s^{\frac{1}{3}}} \cdot 1, s) = (3s^{\frac{2}{3}}, s)$ which is not C^∞.

♡

Exercise 45 (Exercise 10a)**.** Prove that

$$L_X(f\omega) = \omega L_X f + f L_X w.$$

♠

Solution 43. Two things are needed for this: 1) Note that $\varphi_h^*(f\omega)(p) = f(\varphi_h(p))\varphi_h^*\omega(p)$. 2) The standard proof of the product rule. The rest is routine. ♡

11 Introduction to Commutative Algebra - Atiyah-MacDonald

11.1 Chapter 1 - Rings and ideals

Exercise 46. Let x be a nilpotent element of a ring A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit. ♠

Solution 44. Suppose $x^{n+1} = 0$ and that $x^n \neq 0$. Consider

$$s = 1 - x + x^2 - x^3 + \ldots + x^n$$

Then

$$sx = x - x^2 + x^3 - x^4 + \ldots - x^n$$

since $x^{n+1} = 0$. But then $s + sx = 1$, so that $s(1 + x) = 1$. Hence $1 + x$ is a unit. To prove that the sum of any unit and any nilpotent is a unit, note that if u is any unit, then $u^{-1}x$ is still nilpotent. So since $u+x = u(1+u^{-1}x)$ and product of units are units, the claim follows. ♡

Exercise 47 (Exercise 11)**.** A ring A is *Boolean* if $x^2 = x$ every $x \in A$. In a Boolean ring A, show that

i) $2x = 0$ for all $x \in A$.

ii) Every prime ideal $\mathfrak{p}$ is maximal, and $A/\mathfrak{p}$ is a field with two elements.

iii) Every finitely generated ideal in A is principal.

♠

Solution 45. i) We have $4x = 4x^2 = (2x)^2 = 2x$, hence $2x = 0$.

ii) Consider $A/\mathfrak{p}$. This is an integral domain in which $x^2 = x$ for all $x \in A/\mathfrak{p}$. But then $x^2 - x = x(x - 1) = 0$. Hence either $x = 0$ or $x = 1$, hence $A/\mathfrak{p}$ can have only two elements. Thus it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ which is a field, hence $\mathfrak{p}$ is maximal.

iii) Let $I = (a_1, \cdots, a_r)$. Every ideal is contained in a maximal ideal $\mathfrak{m}$. Consider the image of I in $A/\mathfrak{m}$.

iv) By induction we can assume that I is generated by two elements, say $I = (a_1, a_2)$. Then I claim that $I = (a_1 + a_2)$. Cleary $(a_1 + a_2) \subseteq (a_1, a_2)$. The other direction will follow if we can see that $a_1a_2 = 0$ (or they can be assumed to satisfy this), because $a_1a_2 + a_1 \in (a_1 + a_2)$. [[[[[[[[[[????]]]]]]]]]]

♡

Exercise 48 (Exercise 12)**.** A local ring contains no nontrivial idempotents. ♠

Solution 46. Suppose $x \neq 0, 1$ and that $x^2 = x$. Then $x^2 - x = x(x-1) = 0$. Both x and $x - 1$ cannot be contained in $\mathfrak{m}$ since they generate A. Hence one of the is unit. Hence either $x = 0$ or $x = 1$, contradiction. ♡

Exercise 49 (Exercise 15, The prime spectrum of a ring)**.** Let A be a ring and let X be the set of prime ideals of A. For each subset E of A, let $V(E)$ denote the set of prime ideals of A which contain E. Prove that

1. If $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$[2].

2. $V(0) = X$ and $V(1) = \emptyset$.

3. If $(E_i)_{i \in I}$ is a family of subsets of A, then

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i).$$

4. $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for all ideals $\mathfrak{a}, \mathfrak{b}$ of A.

These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topolgoy is called the *Zariski topology.* The topological space X is called the *prime spectrum of* A and denoted $\operatorname{Spec} A$. ♠

Solution 47. We do these one by one.

1. Clearly $\mathfrak{p} \supset \langle E \rangle \supset E$, where the brackets denote the ideal generated by E. Hence $V(\mathfrak{a}) \subset V(E)$. But if $\mathfrak{p} \supset E$, we must have $\mathfrak{p} \supset \mathfrak{a}$ since $\langle \mathfrak{p} \rangle = \mathfrak{p}$. Thus the first equality is established.

   Since $r(\mathfrak{a}) \subset \mathfrak{a}$, we have $V(\mathfrak{a}) \subset V(r(\mathfrak{a}))$. Suppose $\mathfrak{p} \supset r(\mathfrak{a})$ and suppose $a \in \mathfrak{a}$. We want to show $a \in \mathfrak{p}$. We know that $a^n \in r(\mathfrak{a})$ for some n, hence $a^n \in \mathfrak{p}$. But $\mathfrak{p}$ is a prime ideal, so $a \in \mathfrak{p}$ also. Hence equality is established.

2. Every ideal contains the zero ideal and (1) is not a prime ideal.

[2]Here $r(\mathfrak{a})$ denotes the radical of $\mathfrak{a}$

3. Suppose $\mathfrak{p} \supset \cup E_i$. Then $\mathfrak{p} \supset E_i$ for all i, so $\mathfrak{p} \in \cap V(E_i)$. Thus this is just a formal consequence of the contravariant nature of $V(-)$.

4. Since $\mathfrak{ab} \subset \mathfrak{a} \cap \mathfrak{b}$, we automatically have $V(\mathfrak{a} \cap \mathfrak{b}) \subset V(\mathfrak{ab})$. So suppose $\mathfrak{p} \supset \mathfrak{ab}$ and let $a \in \mathfrak{a} \cap \mathfrak{b}$. Then $a^2 \in \mathfrak{ab} \subset \mathfrak{p}$, but then $a \in \mathfrak{p}$ since $\mathfrak{p}$ is prime.

   Now suppose $\mathfrak{p} \supset \mathfrak{a}$ or $\mathfrak{p} \supset \mathfrak{b}$. Then if $a \in \mathfrak{a} \cap \mathfrak{b}$, we have $a \in \mathfrak{p}$, so $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subset V(\mathfrak{a} \cap \mathfrak{b})$. Now suppose $\mathfrak{p} \supset \mathfrak{a} \cap \mathfrak{b}$. Then by Proposition 1.11, we have $\mathfrak{p} \supset \mathfrak{a}$ or $\mathfrak{p} \supset \mathfrak{b}$.

♡

Exercise 50 (Exercise 17)**.** For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \operatorname{Spec} A$. The sets X_f are open. Show that they form a basis for the Zariski topology, and that

1. $X_f \cap X_g = X_{fg}$.
2. $X_f = \emptyset \Leftrightarrow f$ is nilpotent.
3. $X_f = X \Leftrightarrow f$ is a unit.
4. $X_f = X_g \Leftrightarrow r((f)) = r((g))$.
5. X is quasi-compact.
6. More generally, each X_f is quasi-compact.
7. An open subset of X is quasi-compact if and only if it is a finite union of the sets X_f.

The sets X_f are called *basic open sets* of $X = \operatorname{Spec} A$. ♠

Solution 48. We need to show that the sets X_f forms a basis for the Zariski topology on X. This means that each open in X can be written as a union of the X_f. An open in X have the form

$$U(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \not\supset \mathfrak{a}\}.$$

The sets X_f have the form

$$X_f = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}.$$

Let $\{f_i\}_{i \in I}$ generate $\mathfrak{a}$. I claim that $\bigcup X_{f_i} = U(\mathfrak{a})$. Let $\mathfrak{p}$ be an element of the left hand side. This means by definition that $f_i \notin \mathfrak{p}$ for some i. But f_i is an element of $\mathfrak{a}$, so $\mathfrak{a} \not\subset \mathfrak{p}$, hence $\mathfrak{p} \in U(\mathfrak{a})$.

Conversely, suppose $\mathfrak{p} \not\supset \mathfrak{a}$. Then some generator f_i of $\mathfrak{a}$ is not contained in $\mathfrak{p}$. Hence $\mathfrak{p} \in X_{f_i}$.

1. We have
$$X_f \cap X_g = \{\mathfrak{p} \mid f, g \notin \mathfrak{p}\} = \{\mathfrak{p} \mid fg \notin \mathfrak{p}\},$$
since $\mathfrak{p}$ is a prime ideal: for suppose $f, g \notin \mathfrak{p}$, then $fg \notin \mathfrak{p}$ also, because if $fg \in \mathfrak{p}$, primality implies either f or $gin\mathfrak{p}$. Conversely, suppose $fg \notin \mathfrak{p}$. Then neither f, g can be in $\mathfrak{p}$ by defintion of ideals.

2. Suppose X_f is empty. Then there are no prime ideals with $f \notin \mathfrak{p}$. But that means that f is contained in every prime ideal, hence f is nilpotent.

3. Suppose $X_f = X$. Then for all prime ideals, $f \notin \mathfrak{p}$, hence f generates the unit ideal, hence f is a unit. For if f did not generate the unit ideal, f would be contained in some maximal ideal $\mathfrak{m}$, and maximal ideals are prime.

4. Suppose $X_f = X_g$. By definition, this means that for every prime $\mathfrak{p}$ with $f \notin \mathfrak{p}$, we have $g \notin \mathfrak{p}$ (and conversely). The contrapositive of this is $g \in \mathfrak{p} \Leftrightarrow f \in \mathfrak{p}$. Hence we have
$$r((f)) = \bigcap_{\mathfrak{p} \supset (f)} \mathfrak{p} = \bigcap_{\mathfrak{p} \ni f} \mathfrak{p} = \bigcap_{\mathfrak{p} \ni g} \mathfrak{p} = r((g)).$$

5. Let $\{X_f\}_{f \in I}$ be a covering of X by basic opens, that is, $X = \bigcup_{f \in I} X_f$. This means that for every $\mathfrak{p} \in X$, there is some $f \in I$ with $f \notin \mathfrak{p}$. I claim that the f_i generate the unit ideal: for if not, $\langle f_i \rangle$ would be contained in some prime ideal, but by the above, this is not the case. Hence there is an equation of the form $1 = \sum g_i f_i$ with $g_i \in A$, which is a *finite* sum. Hence these finitely many f_i suffice.

6. ...

♡

11.2 Chapter 2 - Modules

Exercise 51 (Excercise 1)**.** Show that $\mathbb{Z}/m \otimes_Z \mathbb{Z}/n = 0$ if m, n are coprime.

♠

Solution 49. Write $1 = am + bn$. Then
$$\begin{aligned} 1 \otimes 1 = (am + bn) \otimes 1 &= am \otimes 1 + bn \otimes 1 \\ &= 0 + bn \otimes 1 = 1 \otimes bn = 1 \otimes 0 = 0. \end{aligned}$$
And we are done. ♡

Exercise 52 (Exercise 2)**.** Let A be a ring, $\mathfrak{a}$ an ideal, and M an A-module. Then $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$. ♠

Solution 50. Start with

$$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0.$$

Tensoring with M gives

$$\mathfrak{a} \otimes M \to M \to A/\mathfrak{a} \otimes_A M \to 0.$$

But $\mathfrak{a} \otimes_A M \simeq \mathfrak{a}M$, so that the sequence reads $A/\mathfrak{a} \otimes M \simeq M/\mathfrak{a}M$. ♡

Exercise 53 (Exercise 3)**.** Let A be a local ring, M, N finitely generated A-modules. Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$. ♠

Solution 51. First a counterexample if A is not a local ring. Let $A = k[x]$ and $M = k[x]/(x-1)$ and $N = k[x]/(x)$. We can write $1 = -(x-1)+x$. Then $M \otimes_A N = 0$ by the same method as in Exercise 1 ($1 \otimes 1 = (-x+1+x) \otimes 1 = x \otimes 1 = 1 \otimes x = 0$).

Let $M_k := M \otimes k = M/\mathfrak{m}M$. By Nakayama's lemma, $M_k = 0 \Rightarrow M = 0$.

So suppose $M \otimes_A N = 0$. Then $(M \otimes_A N)_k = 0$. But this is isomorphic to $M_k \otimes_A N_k$ since $k \otimes_A k = k$. But $M_k \otimes_A N_k \simeq M_k \otimes_k N_k$, as k-modules, since everything in $\mathfrak{m}$ acts trivially on M_k. But these are vector spaces over a field, now we must have $M_k = 0$ or $N_k = 0$, and by Nakayama we are done. ♡

Exercise 54 (Exercise 4)**.** Let M_i ($i \in I$) be any family of A-modules, and let M be their direct sum. Then M is flat if and only if each M_i is flat. ♠

Solution 52. Let

$$0 \to N' \to N \to N'' \to 0$$

be any exact sequence. Then tensoring with M gives

$$0 \to N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M \to 0.$$

We only need to check that the left map is injective. But we have $N' \otimes_A M \simeq \bigoplus_i N' \otimes_A M_i$ and $N \otimes_A M \simeq \bigotimes_i N \otimes_A M_i$, and thus the left map is just the direct sum of all the maps

$$0 \to N' \otimes_A M_i \to N \otimes_A M,$$

which is injective if and only if each M_i is flat. ♡

Exercise 55 (Exercise 5)**.** Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is flat A-algebra. ♠

Solution 53. We have $A[x] = \bigoplus_{i=0}^{\infty} x^i A$ as an A-module. Now use Exercise 4. ♡

Exercise 56 (Exercise 24)**.** If M is an A-module, the following are equivalent:

i) M is flat.

ii) $\mathrm{Tor}_n^A(M,N) = 0$ for all $n > 0$ and A-modules N.

iii) $\mathrm{Tor}_1^A(M,N) = 0$ for all A-modules N.

Solution 54. To compute $\mathrm{Tor}_A^n(M,N)$, one takes an A-resolution of N and tensor it with M and take homology. But M is flat, so the sequence stays exact, so the homology is zero. This shows $i) \Rightarrow ii)$.

The implication $ii) \Rightarrow iii)$ is trivial.

Now let

$$0 \to N' \to N \to N'' \to 0$$

be any exact sequence of A-modules. Then by properties of the Tor functor, we have an exact sequence

$$\mathrm{Tor}_1(M,N'') \to N' \otimes M \to N \otimes M \to N'' \otimes M \to 0.$$

But $\mathrm{Tor}_1(M,N'') = 0$, so the sequence is short exact. Hence M is flat. ♡

Exercise 57 (Exercise 25)**.** Let

$$0 \to N' \to N \to N'' \to 0$$

be an exact sequence with N'' flat. Then N' is flat if and only if N is flat. ♠

Solution 55. We have from the Tor exact sequence

$$0 \to \mathrm{Tor}_1(N',M) \to Tor_1(N,M) \to 0$$

since $\mathrm{Tor}_2(N'',M) = \mathrm{Tor}_1(N'',M) = 0$. The statement follows. ♡

♠

11.3 Chapter III - Rings and modules of fractions

Exercise 58 (Exercise 1)**.** Let S be a multiplicatively closed subset of a ring A, and let M be a finitely-generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$. ♠

Solution 56. Suppose there exists such s. Let $m/s' \in S^{-1}M$. This is zero if and only if there exists $s \in M$ such that $s(s'm) = 0$. But $ss'm = s'sm = s'0 = 0$. So $m = 0$ in $S^{-1}M$. (note that we did not use finite generation)

Now let $m_1, \ldots, m_r$ be a set of generators for M and suppose that $S^{-1}M = 0$. Then for each i ($i = 1, \ldots, r$), there exists s_i such that $s_i m_i = 0$. Since every element of M is an A-linear combination of the m_i, it follows that the product $s_1 s_2 \cdots s_r$ makes $sM = 0$. ♡

Exercise 59. A multiplicatively closed subset S of a ring A is said to be *saturated* if

$$xy \in S \Leftrightarrow x \in S \text{ and } y \in S.$$

Prove that

i) S is saturated $\Leftrightarrow$ $A \backslash S$ is a union of prime ideals.

ii) If S is any multiplicatively closed subset of A, there is a unique smallest saturated multiplicatively closed subset $\overline{S}$ containing S, and $\overline{S}$ is the complement in A of the union of the prime ideals which do not meet S.

If $S = 1 + \mathfrak{a}$, where $\mathfrak{a}$ is an ideal of A, find $\overline{S}$. ♠

Solution 57. i) Suppose S is saturated. Then let C be the set of all prime ideals not meeting S. Then by Proposition 3.11, the prime ideals of $S^{-1}A$ are precisely those in C, and they are all extensions of prime ideals of A. Thus let $a \in A \backslash S$ and consider $i : A \to S^{-1}A$. Let $\overline{a}$ be the image of a in $S^{-1}A$. Then $\overline{a}$ is contained in some prime ideal $\mathfrak{p} \subset S^{-1}A$. Let $\mathfrak{p}'$ be the inverse image of $\mathfrak{p}$ under i. Then $a \in \mathfrak{p}'$ and $\mathfrak{p}' \cap S = \emptyset$. Hence every $a \in A \backslash S$ is contained in some prime ideal, hence $A \backslash S$ is a union of prime ideals. (note: we didn't use that S was saturated)

Now assume $A \backslash S$ is a union of prime ideals, say $A \backslash S = \bigcup_{i \in I} \mathfrak{p}_i$. Then we have the following chain of equivalences:

$$\begin{aligned} xy \in S &\Leftrightarrow xy \notin A \backslash S \\ &\Leftrightarrow \exists \text{ no } i \in I \text{ with } xy \in \mathfrak{p}_i \\ &\Leftrightarrow \forall i \in I, xy \notin \mathfrak{p}_i \\ &\Leftrightarrow \forall i \in I, (x \notin \mathfrak{p}_i \text{ and } y \notin \mathfrak{p}_i) \\ &\Leftrightarrow x \in S \text{ and } y \in S. \end{aligned}$$

ii) Clearly $\overline{S}$ as defined in the exercise contains S, and it is clear from the previous paragraph that it is maximal.

Now let $S = 1 + \mathfrak{a}$. ♡

11.4 Chapter 5 - Integral dependence and valuations

Exercise 60 (Exercise 1)**.** Let $f : A \to B$ be an integral morphism of rings. Show that $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$ is a closed mapping. ♠

Solution 58. The map f^* is by definition given by $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p}) = \mathfrak{p} \cap A$. A closed subset of $\operatorname{Spec} B$ is by definition

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} B \mid \mathfrak{p} \supset \mathfrak{a}\}$$

for some ideal $\mathfrak{a} \subset B$.

Then the image of $V(\mathfrak{a})$ is the set

$$f^*(V(\mathfrak{a})) = \{\mathfrak{p} \cap A \mid \mathfrak{p} \in \operatorname{Spec} B, \quad \mathfrak{p} \supset \mathfrak{a}\}$$

I claim that this is equal to

$$V(\mathfrak{a} \cap A) = \{\mathfrak{q} \in \operatorname{Spec} A \mid \mathfrak{q} \supset \mathfrak{a} \cap A\},$$

which clearly is a closed subset of $\operatorname{Spec} A$.

One direction is obvious: let $\mathfrak{p} \cap A$ be an element of $f^*(V(\mathfrak{a}))$. This is a point of $\operatorname{Spec} A$, and clearly $\mathfrak{p} \cap A \supset \mathfrak{a} \cap A$ since $\mathfrak{p} \supset \mathfrak{a}$.

The other direction needs the going up Theorem 5.10. Suppose $\mathfrak{q} \in V(\mathfrak{a} \cap A)$. Then by Going Up, there exists $\mathfrak{p} \in \operatorname{Spec} B$ with $\mathfrak{p} \cap A = \mathfrak{q}$. But we need to check that $\mathfrak{p} \supset \mathfrak{a}$. That is, we need to prove the assertion that if $\mathfrak{q} = \mathfrak{p} \cap A$ and $\mathfrak{q} \supset \mathfrak{a} \cap A$, then $\mathfrak{p} \supset \mathfrak{a}$. So suppose $a \in \mathfrak{a} \subset B$. Then a satisfies an equation

$$a^n + b_{n-1}a^{n-1} + \ldots + b_1 a + b_0 = 0$$

with $b_i \in A$. Since $a \in \mathfrak{a}$, we see that $b_0 \in \mathfrak{q} = \mathfrak{p} \cap A$. Hence

$$a^n + b_{n-1}a^{n-1} + \ldots + b_1 a = a(a^{n-1} + b_{n-1}a^{n-2} + \ldots + b_1) \in \mathfrak{p}$$

since $\mathfrak{q} \subset \mathfrak{p}$. But $\mathfrak{p}$ is prime so either $a \in \mathfrak{p}$ and we are done, or $a^{n-1}b_{n-1}a^{n-2} + \ldots + b_1 \in \mathfrak{p}$, and we can continue by induction.

Hence we are done. ♡

11.5 Chapter 7 - Noetherian rings

Exercise 61 (Exercise 11)**.** Let A be a ring such that $A_{\mathfrak{p}}$ is Noetherian for each $\mathfrak{p} \in \operatorname{Spec} A$. Is A necessarily noetherian? ♠

Solution 59. Consider the ring

$$A = \mathbb{Z}/2 \times \mathbb{Z}/2 \cdots .$$

It is a countable product of noetherian rings. The primes are just the coordinate axes, and each localization is isomorphic to $\mathbb{Z}/2$. Thus each $A_{\mathfrak{p}}$ is Noetherian, but A is not. ♡

Exercise 62 (Exercise 15)**.** Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal and k its residue field and let M be a finitely generated A-module. Then the following are equivalent:

i) M is free.

ii) M is flat.

iii) The mapping $\mathfrak{m} \otimes M \to A \otimes M$ is injeective.

iv) $\operatorname{Tor}_1^A(k, M) = 0$.

♠

Solution 60. The implication $i) \Rightarrow ii)$ is trivial. One way is to compute $\operatorname{Tor}_1^A(M, N)$ for any A-module N. But a free resolution of M is just one-term, so $\operatorname{Tor}_1^A(M, N)$ is automatically zero.

The implication $ii \Rightarrow iii)$ follows by tensoring the incusion $\mathfrak{m} \hookrightarrow A$ with M.

The implication $iii) \Rightarrow iv)$ follows from the Tor exact sequence

$$\operatorname{Tor}_1^A(A, M) \to \operatorname{Tor}_1^A(k, M) \to \mathfrak{m} \otimes M \to A \otimes M \to k \otimes M \to 0.$$

The leftmost term is zero since A is a free A-module, and by $iii)$ and exactness we must as well have $\operatorname{Tor}_1^a(k, M)$.

Now for $iv \Rightarrow i)$. Choose element $m_i \in M$ $(0 \leq i \leq r)$ such that they form a k-basis for $M/\mathfrak{m}M$. Choose a surjection $f : A^r \to M$ and let $E = \ker f$ be its kernel. Then we have an exact sequence

$$0 \to E \to A^r \to M \to 0.$$

of finitely-generated A-modules (E is finitely generated by Proposition 6.2). Tensor the sequence by k, and get

$$\mathrm{Tor}_1^A(k, M) \to E/\mathfrak{m}E \to k^r \to M/\mathfrak{m}M \to 0.$$

The left-most term is zero by assumption. The last two spaces are k-vector spaces of the same dimension, and it follows that $E/\mathfrak{m}E = 0$. But then it follows that E is zero by Nakayama's lemma, hence M is free. ♡

Exercise 63 (Exercise 16)**.** Let A be a Noetherian ring, M a finitely-generated A-module. Then the following are equivalent:

i) M is a flat A-module.

ii) $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module for each $\mathfrak{p} \in \operatorname{Spec} A$.

iii) $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m}$.

So flatness is the same as being locally free. ♠

Solution 61. The implications $i \Rightarrow ii)$ and $ii) \Rightarrow iii)$ follows trivially from the previous exercise. We prove $iii) \Rightarrow i)$.

Applying the Tor functor commutes with localization, hence we have $\mathrm{Tor}_1^A(M, N)_{\mathfrak{m}} = \mathrm{Tor}_1^{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) = 0$ for all $\mathfrak{m}$. But being zero is a local property, so it follows that $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N. Hence M is flat. ♡

12 Lie groups, Lie algebras and Representations - Hall

12.1 Matrix Lie groups

Exercise 64. Let a be an irrational number and let G be the following subgroup of $\mathrm{GL}(2; \mathbb{C})$:

$$G = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

Show that

$$\overline{G} = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{is} \end{pmatrix} \mid t, s \in \mathbb{R} \right\}.$$

♠

Solution 62. It suffices to show that every pair (e^{it}, e^{is}) can be arbitrarily apprixmated by a pair (e^{it}, e^{iat}).

Let $t' = 2\pi k + t$. The set

$$\{e^{iat'} \mid k \in \mathbb{Z}\}$$

is dense in S^1, hence for some k we come very close to the pair (e^{it}, e^{iat}). Hence the closure is the torus. ♡

13 Linear representations of finite groups - Serre

13.1 Chapter 1 - Representations and characters

[no exercises]

13.2 Chapter 2 - The character of a representation

Exercise 65 (Exercise 2.1)**.** Let χ, χ' be the characters of two representations. Prove the formulas:

$$(\chi + \chi')^2_\sigma = \chi^2_\sigma + \chi'^2_\sigma + \chi\chi'$$

and

$$(\chi + \chi')^2_\alpha = \chi^2_\alpha + \chi'^2_\alpha + \chi\chi'$$

Here χ_σ, χ_α are the characters of the symmetric and the alternating representation, respectively. ♠

Solution 63. The left hand side is the character of $\mathrm{Sym}^2(V \oplus W)$, and a calculation shows that this decomposes as $\mathrm{Sym}^2 V \oplus \mathrm{Sym}^2 W \oplus (V \otimes W)$. Hence the result follows by Proposition 2.

For the alternating representation we have a similar decomposition. ♡

Exercise 66 (Exercise 2.2)**.** Let X be a finite set on which G acts and let ρ be the corresponding permutation representation and χ_X the character of ρ. Let $s \in G$. Show that $\chi_X(s)$ is the number of elements of X fixed by s. ♠

Solution 64. We have by definition that $\chi_X(s)$ is the trace of a matrix representing $\rho(s)$, but $\rho(s)$ is a permutation matrix. All basis elements corresponding to elements moved by s will be moved off the diagonal, so the remaning elements, the elements fixed by s, are on the diagonal, and are all 1's. Done. ♡

Exercise 67 (Excercise 2.3)**.** Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation with character χ and let V' be the dual of V. Show that there exists a unique linear representation $\rho' : G \to \mathrm{GL}(V')$ such that

$$\langle \rho_s x, \rho' x' \rangle = \langle x, x' \rangle$$

for $s \in G$, $x \in V$, $x' \in V'$. Here $\langle , \rangle$ is the natural pairing between V and V', given by $\langle x, x' \rangle = x'(x)$.

This is the **contragradient** or **dual** representation of ρ. Its character is $\overline{\chi}$. ♠

Solution 65. We first define the representation ρ', and then show that is unique. So let $\rho'(s)$ be defined by

$$(\rho'(s)f)(v) = f(g^{-1}v).$$

Then (we omit the ρ from the notation, it being clear which action is referred to):

$$\langle g \cdot x, g \cdot f \rangle = (g \cdot f)(g \cdot x) = f(g^{-1}gx) = f(x) = \langle x, f \rangle.$$

Now let $\sigma : G \to \mathrm{GL}(V')$ be another representation of V' satisfying this identity, and suppose $\sigma \neq \rho'$. Then there is some $s \in G$ such that $\sigma(s) \neq \rho'(s) \in \mathrm{GL}(V')$. This means there is some $f \in GL(V')$ such that $\sigma(s)(f) \neq \rho'(s)(f) \in V'$. This means that there is some $v \in V$ such that $\sigma(s)(f)(v) \neq \rho'(s)(f)(v) \in \mathbb{C}$. Writing this more compactly, this means

$$(\sigma_s f)(v) \neq (\rho'_s f)(v).$$

But then (letting $w = s^{-1} \cdot v$)

$$\begin{aligned} f(w) =& \langle w, f \rangle = \langle s \cdot w, \sigma_s f \rangle = (\sigma_s f)(v) \neq (\rho'_s f)(v) = \langle v, \rho'_s f \rangle \\ &= \langle s \cdot w, (\rho'_s)(f) \rangle = \langle w, f \rangle = f(w) \end{aligned}$$

which is a contradiction. ♡

Exercise 68. Let $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ be two linear representations with characters χ_1, χ_2. Let $W = \mathrm{Hom}(V_1, V_2)$.

For $s \in G$ and $f \in W$, let $\rho_s f = \rho_{2,s} \circ f \circ \rho_{1,s}^{-1}$ so $\rho_s f \in W$. Show that this defines a linear representation $\rho : G \to \mathrm{GL}(W)$ and that its character is $\chi_1^* \cdot \chi_2$. This representation is isomorphic to $\rho'_1 \otimes \rho_2$. ♠

Solution 66. It is easy but notationally challenging to check that this defines a representation.

To prove the statement about characters, we first show that

$$\mathrm{Hom}(V_1, V_2) \simeq V_1^* \otimes V_2$$

as representations. Then the statement will follow by properties of characters.

The map is given by (from right to left) by $f \otimes v_2 \mapsto (v_1 \mapsto f(v_1)v_2)$. We check that this map intertwines the action of G on the tensor product with that on the Hom-set, that is calling the isomorphism for φ, that $g \cdot \varphi(f \otimes v_2) = \varphi(g \cdot (f \otimes v_2))$. But

$$g \cdot \varphi(f \otimes v_2) = g \cdot (v_1 \mapsto f(v_1)v_2) = v_1 \mapsto f(g^{-1}v_1)gv_2$$

Note that the notation is ambigious here. With $gf(g^{-1}v)$, we really means $(\rho_2(g)f)(\rho_1(g)v)$, but who wants to write that much? On the other hand we have

$$\varphi(g \cdot (f \otimes v_2)) = \varphi(gf \otimes gv_2) = \varphi((v_1 \mapsto f(g^{-1}v_1)) \otimes gv_2) = v_1 \mapsto f(g^{-1}v_1)gv_2.$$

But these are equal! ♡

Exercise 69 (Exercise 2.5)**.** Let ρ be a linear representation with character χ. Show that the number of times that ρ contains the unit representation is equal to $(\chi \mid 1) = \frac{1}{g} \sum_{s \in G} \chi(s)$. ♠

Solution 67. The character χ_ρ decomposes as $\sum a_i \chi_{\pi_i}$ where π_i are its irreducible components with multiplicities a_i. Since the unit representation have character $\chi_\epsilon(g) = 1$ for all g, the result follows trivially. ♡

Exercise 70 (Exercise 2.6)**.** Let X be a finite set on which G acts and let ρ be the corresponding permutation representation and χ its character.

a) The set Gx of images under G of an element $x \in X$ is the *orbit* of x. Let c be the number of distinct orbits. Show that c is the number of times that ρ contains the unit representation. Deduce that $c = (\chi \mid 1)$. In particular, if G is transitive, ρ can be decomposed into $1 \oplus \theta$ where θ does not contain the unit representation. If ψ is the character of θ, we have $\chi = 1 + \psi$ and $(\psi \mid 1) = 0$.

b) Let G act diagonally on $X \times X$. That is, by $s(x, y) = (sx, sy)$. Show that the character of the corresponding representation is equal to χ^2.

c) Suppose that G acts transitively on X and that X has at least two elements. We say that G is doubly transitive if for all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$, there exists $s \in G$ with $x' = sx$ and $y' = sy$. Prove the equivalence of the following properties:

   i) G is doubly transitive.
   ii) The action of G on $X \times X$ have two orbits, the diagonal and the complement.
   iii) $(\chi^2 \mid 1) = 2$.
   iv) The representation θ defined in a) is irreducible.

♠

Solution 68. a) Decompose $X = \cup X_i$ where X_i are the disjoint orbits. Then ρ decomposes as $\rho = \oplus \rho_i$. Thus it suffices to show that if G act transitively on X, the corresponding permutation representation contains exactly one copy of the unit representation.

It is clear that it contains one copy, namely the subspace spanned by the element $(1, 1, \ldots, 1)$.

Now, two representations are isomorphic if and only if their characters are equal. Any representation isomorphic to the unit representation must therefore act as the identity. So let $v \in V_\rho$ be any vector. Then its coordinates are permutes by G. If the subspace spanned by v is invariant and if G act as the identity on it and since G acts transitively on the coordinates, all its coordinates must be equal. Thus the unit representation occurs exactly once in ρ.

Thus $c = (\chi \mid 1)$. The rest of the claims follows immediately.

b) Just note that the corresponding permutation representation is just the tensor product $\rho \otimes \rho$.

c) i and ii are clearly equivalent. The implication ii $\Rightarrow$ iii is the content of a) and b). The implication iii *Rightarrow* ii is a and b also.

Now suppose that θ is irreducible. We will show iii. It is easy to see that $(\chi^2, 1) = (\chi, \chi)$. This in turn is equal to $(1 + \psi, 1 + \psi)$ which is equal to $(1, 1) + 2(1, \psi) + (\psi, \psi)$. The first term is 1. The second is zero, since θ does not contain the unit representation. The third term is 1 by assumption. Hence $(\chi^2, 1) = 2$.

Clearly we can go the other way as well.

♡

13.3 Chapter 3

Exercise 71 (Exercise 3.1)**.** Show directly using Schur's lemma, that each irreducible representation of an abelian group, finite or not, has degree 1. ♠

Solution 69. Let A be an abelian group, and let V be an irreducible representation of A. Then each $g \in A$ gives a nonzero G-morphism $V \to V$ by left-multiplication. Here we use that A is abelian.

But by Schur's lemma, every such morphism is given by multiplication by some $\alpha(g) \in \mathbb{C}\backslash\{0\}$.

Let $v \neq 0 \in V$ and let $W \subset V$ be the subspace spanned by v. Then W is G-invariant: $g \cdot v = \alpha(g)v \in W$. Hence $V = W$ since W was irreducible. ♡

Exercise 72. Let ρ be an irreducible representation of G of degree n and character χ. Let $Z(G)$ be the center of G and let c be its order.

a) Show that ρ_s is a homothety for each $s \in Z(G)$. Deduce that $|\chi(s)| = n$ for all $s \in C$.

♠

14 The K-book - Charles Weibel

14.0.1 Chapter 1.1 - Free modules, GL_n and stably free modules

Exercise 73 (Semisimple rings)**.** A nonzero R-module M is called *simple* if it has no submodules other than 0 and M, and semisimple if it is a direct sum of simple modules. A ring R is called *semisimple* if it is a semisimple R-module. If R is semisimple, show that R is a direct sum of a *finite* (say n) number of simple modules.

Then use the Jordan-Hölder theorem to show that every stably free module is free. ♠

Solution 70. Suppose R is semisimple, that is $R = \oplus M_i$ with a priori infinitely many M_i. But write $1 = \sum a_i m_i$ with $m_i \in M_i$ and $a_i \in R$. This is a finite sum, and since for any $r \in R$, we have $r = \sum a_i r m_i$, only finitely many M_i need occur in the decomposition $R = \oplus M_i$.

To see that any stably free module is free if R is semisimple, suppose $M \oplus R^n \simeq R^m$. Then we can write $M \oplus M_i^n \simeq \oplus M_j^m$ as above, and note that the image of a simple module must be simple, hence the M_j on the right must be mapped to copies of themselves isomorphically. Hence we can cancel M_i-terms on both sides until we arrive at $R^k \simeq M$ for some k. ♡

Exercise 74. Consider the following conditions on a ring R:

i) R satisfies the invariant basis property (IBP).

ii) For all m, n, if $R^m \simeq R^n \oplus P$, then $m \leq n$.

iii) For all n, if $R^n \simeq R^n \oplus P$, then $P = 0$.

Show that iii $\Rightarrow$ ii $\Rightarrow$ i. ♠

Solution 71. Suppose $R^m \simeq R^n \oplus P$ and suppose $n > m$. Then we can write $R^m \simeq R^m \oplus (R^{n-m} \oplus P)$. Then from *iii* we must have $R^{n-m} \oplus P = 0$, but this is impossible.

Now suppose $R^n \simeq R^m$. Then for $P = 0$, we have $R^m \simeq R^n \oplus P$, hence $m \leq n$. But the opposite argument works as well, hence $m = n$. ♡

Exercise 75. Show that iii) in the previous exercise and the following matrix conditions are equivalent:

a) For all n, every surjection $f : R^n \to R^m$ is an isomorphism.

b) For all n and $f, g \in M_n(R)$, if $fg = 1_n$, then $gf = 1_n$ and $g \in \mathrm{GL}_n(R)$.

Then show that commutative rings satisfy b), hence iii). ♠

Solution 72. First we see that *iii* $\Rightarrow$ *a*). Suppose $f : R^n \to R^m$ is a surjection. Then we have an exact sequence

$$0 \to K \to R^n \xrightarrow{f} R^n \to 0.$$

Since R^n is free, the sequence splits and we have $R^n \simeq R^n \oplus K$, but then by assumption $K = 0$. Hence f is an isomorphism.

Now suppose *a*), that is, every surjection is an isomorphism. Next suppose $R^n \simeq R^n \oplus P$. Compose with the surjection $R^n \oplus P \to R^n$ to get a surjective map $R^n \to R^n$. The kernel of this is P. But every surjection $R^n \to R^n$ is an isomorphism, hence $P = 0$.

Now suppose *iii*). The condition $fg = 1_n$ implies that g is injective and that we have a splitting $R^n \simeq R^n \oplus \operatorname{coker} g$. But then $\operatorname{coker} g = 0$ by *iii*), hence g is an isomorphism. To show that $gf = 1_n$, suppose $gf(x) = y$. Applying f to both sides give $f(x) = f(y)$. So we must show that f is injective. So suppose that $f(x) = 0$. Since g was surjective, we can write $x = g(y)$ for some y. Then $f(g(x)) = x = 0$. Hence f is injective. So we have proven b).

Now suppose b). Then b implies a, hence iii). ♡

14.1 Chapter 1.2 - Projective modules

Exercise 76 (Radical ideals)**.** Let I be a radical ideal in R. If P_1, P_2 are finitely generated projective R-modules such that $P_1/IP_1 \simeq P_2/IP_2$, show that $P_1 \simeq P_2$. ♠

Solution 73. ♡

15 Representation Theory - Fulton, Harris

15.1 Representations of Finite Groups

Exercise 77 (Exercise 1.1)**.** Verify that the relation

$$\langle g \cdot v^*, g \cdot v \rangle = \langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle$$

is satisfied when we define

$$\rho^*(g) = \rho(g^{-1})^t : V^* \to V^*,$$

that is, $(\rho^* g)(v^*)(w) = \langle (\rho^* g)(v^*), w \rangle = \langle v^*, (\rho g^{-1})(w) \rangle$. ♠

Solution 74. This is a matter of calculation.

$$\langle gv^*, gv \rangle = \langle v^*, (\rho g^{-1})(gv) \rangle = \langle v^*, v \rangle.$$

So the definition is ok. ♡

Exercise 78 (Exercise 1.2)**.** Verify that in general the vector space of G-linear maps between two representations V and W of G is just the subspace $\mathrm{Hom}(V, W)^G$ of elements of $\mathrm{Hom}(V, W)$ fixed under the action of G. This subspace is often denoted $\mathrm{Hom}_G(V, W)$. ♠

Solution 75. A map $\varphi : V \to W$ is G-linear when $\varphi(gv) = g\varphi(v)$. The action of G on φ is given by $g\varphi(v) = g\varphi(g^{-1}v)$. But by G-linearity, this is

$$\varphi(gv) = gg^{-1}\varphi(gv) = gg^{-1}\varphi(v) = \varphi(v).$$

Hence a map is G-linear if and only if it is fixed by the action of G. ♡

Exercise 79 (Exercise 1.3)**.** Let $\rho : G \to \mathrm{GL}(V)$ be any representation of the finite group G on an n-dimensional vector space V and suppose that for any $g \in G$, the determinant if $\rho(g)$ is 1. Show that the spaces $\wedge^k V$ and $\wedge^{n-k} V^*$ are isomorphic as representations of G. ♠

Solution 76. This is (again) just a matter of writing out the definitions. First we define the isomorphism, and then we check that it is actually an isomorphism of representations.

$$\bigwedge^{k} V \to \bigwedge^{n-k} V^*$$
$$v_1 \wedge \cdots \wedge v_k \mapsto (w_1 \wedge \cdots \wedge w_{n-k} \mapsto v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_{n-k})$$

Being a map of representations is equivalent to $g^{-1}\varphi(gv) = \varphi(v)$, so we just need to check that all the g's disappear from the left hand side.

$$\begin{aligned} g^{-1}\varphi(gv) &= g^{-1}(w_1 \cdots w_{n-k} \mapsto gv_1 \cdots gv_k w_1 \cdots w_{n-k}) \\ &= (gv_1 \cdots gv_k gw_1 \cdots gw_{n-k}) \\ &= \det \rho(g) v_1 \wedge \cdots \wedge w_{n-k}. \end{aligned}$$

Hence φ is a map of representations if and only if $\det \rho(g) = 1$ for all $g \in G$.

(it is an isomorphism because it has zero kernel: because what would the kernel be? Every subspace is the same, and this is a basis free description)

♡

Exercise 80 (Exercise 1.4)**.** The permutation representation R of G acting on a finite set X have two descriptions: one is given by letting V be the vector space with basis $\{e_x \mid x \in X\}$ and letting g act on V by $ge_x = e_{gx}$.

Alternatively R is the set of functions $f : X \to \mathbb{C}$ with action $(g\alpha)(h) = \alpha(g^{-1}h)$.

a) Show that these two decriptions agree by identifying e_x with the characteristic function which takes the value 1 on x and 0 elsewhere.

b) The space of functions on G can also be made into a G-module by the rule $(g\alpha)(h) = \alpha(hg)$. Show that this is an isomorphic representation.

♠

Solution 77. a). Clearly the vector space dimensions agree (since the characteristic functions are a basis). So we need to check that this is a map of representations. Denote the characteristic function by χ_x. Then $\varphi(ge_x)(h) = \varphi(e_{gx})(h) = \chi_{gx}(h)$. Similarly $g\varphi(e_x)(h) = g\chi_x(h) = \chi_x(g^{-1}h)$, The first function is 1 if $gx = h$, and the second function is 1 if $g^{-1}h = x$, and these are equivalent.

b). Send α to the function $g \mapsto \alpha(g^{-1})$. Call this assignment ψ. We need to check that $\psi(g\alpha) = g\psi(\alpha)$.

First the left hand side. We have: $\psi(g\alpha)(h) = \psi(h \mapsto \alpha(g^{-1}h))(h) = \alpha(g^{-1}h^{-1})$.

And similarly: $g\psi(\alpha)(h) = g(h \mapsto \alpha(h^{-1}))(h) = g\alpha(h^{-1}) = \alpha(g^{-1}h^{-1})$.

And these are equal. ♡

Exercise 81 (Exercise 1.10)**.** $G = S_3$. Verify that with $\sigma = (12)$, $\tau = (123)$, the standard representation has a basis $\alpha = (\omega, 1, \omega^2)$, $\beta = (1, \omega, \omega^2)$, with

$$\tau\alpha = \omega\alpha, \qquad \tau\beta = \omega^2\beta, \qquad \sigma\alpha = \beta, \qquad \sigma\beta = \alpha.$$

♠

Solution 78. The standard representation V is the subspace $\{x_1+x_2+x_3 = 0\}$ of $\mathbb{C}^3$. Since $1 + \omega + \omega^2 = 0$, and $\alpha \cdot \beta = 3\omega \neq 0$, these two span V.

The identities are easy. ♡

Exercise 82 (Exercise 1.11)**.** Use this approach to find the decomposition of the representations $\operatorname{Sym}^2 V$ and $\operatorname{Sym}^3 V$. ♠

Solution 79. The elements $\{\alpha^2, \alpha\beta, \beta^2\}$ are a basis of $\operatorname{Sym}^2 V$, and the eigenvalues are $\omega^2, 1$ and ω, respectively. Thus $\langle\alpha\beta\rangle$ span a representation isomorphic to U, the trivial representation, and $\langle\alpha^2, \beta^2\rangle$ span a representation isomorphic to V, the standard representation. Hence $\operatorname{Sym}^2 V = U \oplus V$.

The elements $\{\alpha^3, \alpha^2\beta, \alpha\beta^2, \beta^3\}$ are a basis of $\operatorname{Sym}^3 V$. The eigenvalues are $1, \omega, \omega^2$ and 1, respectively. Looking at the action of $\sigma = (12)$, we see that $U \simeq \langle\alpha^3 + \beta^3\rangle$, and $U' \simeq \langle\alpha^3 - \beta^3\rangle$. The remaining $\langle\alpha^2\beta, \alpha\beta^2\rangle$ span a representation isomorphic to V. Hence $\operatorname{Sym}^3 V = U \oplus U' \oplus V$. ♡

Exercise 83 (Exercise 2.2)**.** For $\operatorname{Sym}^2 V$, verify that

$$\chi_{\operatorname{Sym}^2 V}(g) = \frac{1}{2}\left[\chi_V(g)^2 + \chi_V(g^2)\right].$$

Note that this is compatible with the decomposition $V \otimes V = \operatorname{Sym}^2 V \oplus \wedge^2 V$. ♠

Solution 80. The eigenvalues of g acting on $\operatorname{Sym}^2 V$ are $\{\lambda_i\lambda_j\}$. Hence

$$\begin{aligned}
\chi_{\operatorname{Sym}^2 V}(g) &= \sum_{i\leq j} \lambda_i\lambda_j \\
&= \sum_{i<j} \lambda_i\lambda_j + \sum_i \lambda_i^2 \\
&= \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right) + \chi_V(g^2) \\
&= \frac{1}{2}\left(\chi_V(g)^2 + \chi_V(g^2)\right).
\end{aligned}$$

♡

Exercise 84 (Exercise 2.5, The original fixed point formula)**.** If V is a permutation representation associated to the action of a group G on a finite set X, show that $\chi_V(g)$ is the number of elements fixed by g. ♠

Solution 81. This is easy. The matrix associated to g is a permutation matrix with a 1 in row j if element number i is sent to j. Then number of fixed points is the number of ones on the diagonal, and this is $\chi_V(g)$. ♡

Exercise 85 (Exercise 2.7)**.** Consider the standard representation V of S_3 and its nth tensor power $V^{\otimes n}$. Decompose it using character theory. ♠

Solution 82. We know that there are three irreducible representations of S_3. Hence $V^{\otimes n} \simeq U^a \oplus {U'}^b \oplus V^c$ for some numbers a, b, c. Here U is the trivial representation, U' is the alternating representation and V the standard representation.

We also know that the characters are orthogonal, so to find a we have to compute $\langle \chi_U, \chi_{V^{\otimes n}}\rangle$. This we do by using that $\chi_{V^{\otimes n}} = \chi_V^n$. Then we see that $a = \frac{1}{3}(2^{n-1} + (-1)^n)$ since

$$\langle \chi_U, \chi_{V^{\otimes n}}\rangle = \frac{1}{6}\left(2^n + 0 + (-1)^n\right).$$

Similarly for b, c. ♡

Exercise 86 (Exercise 2.34)**.** Let V, W be irreducible representations of G and $L_0 : V \to W$ any linear mapping. Define $L : V \to W$ by

$$L(v) = \frac{1}{|G|}\sum_g g^{-1} L_0(gv).$$

Show that $L = 0$ if V and W are not isomorphic, and that L is multiplication by $\operatorname{Tr}(L_0)/\dim(V)$ if $V = W$. ♠

Solution 83. We want to apply Schur's lemma. We check that L is a G-module homomorphism. We have

$$\begin{aligned} L(hv) &= \frac{1}{|G|}\sum_g g^{-1}L_0(ghv) \\ &= \frac{1}{|G|}\sum_{gh} hgh^{-1}L_0(ghv) \\ &= \frac{1}{|G|}\sum_{g'} hg'^{-1}L_0(g'v) \end{aligned}$$

Hence L is a G-module homomorphism. Hence by Schur's lemma, L is either the zero map or an isomorphism. In particular, if they are not isomorphic, $L = 0$. Now suppose $V = W$. ♡

15.2 Chapter 7 - Lie groups

Exercise 87 (Exercise 7.11)**.** a) Show that any discrete normal subgroup H of a connected Lie group G is in the center $Z(G)$.

b) If $Z(G)$ is discrete, then $G/Z(G)$ have trivial center.

♠

Solution 84. a) We must show that for any $z \in H$ and any $g \in G$, we have $gz = zg \Leftrightarrow z = gzg^{-1}$, that is, that z is fixed by conjugation by all elements of g. Since H is normal, we must have $gzg^{-1} = z'$ for some $z' \in H$. Conjugation is a continous mapping, hence if g' is close to g, $g'zg'^{-1}$ is close to z'. But by normality this must still be an element of H, but H is discrete, for for g' close enough to g the only element of H that can be hit is is z. Hence the mapping $z \mapsto gzg^{-1}$ is locally constant. G is connected, so the mapping must be constant, and since $geg^{-1} = e$, the mapping must be the identity mapping, hence $gzg^{-1} = z$ for all $g \in G$ and $z \in H$.

b) Suppose $a \in Z(G/Z(G))$. Let $\pi : G \to G/Z(G)$ be the quotient map. Let $Z = \pi^{-1}(Z(G/Z(G))$. We want to show that if a is a representative for a, then a lies in $Z(G)$. Lying in $Z(G/Z(G))$ means that $[a, b] \in Z(G)$ for all $a \in Z$ and $b \in G$. But this implies that the map $[,] : Z \times G \to G$ lands in $Z(G)$, which is discrete, hence the map must be constant, hence by definition, a lies in $Z(G)$.

♡

Exercise 88 (Exercise 7.12)**.** If $\varphi : H \to G$ is a covering of connected Lie groups, show that $Z(G)$ is discrete if and only if $Z(H)$ is discrete, and then $H/Z(H) = G/Z(G)$. Therefore, if $Z(G)$ is discrete, the adjoint form of G exists and is $G/Z(G)$. ♠

Solution 85. Suppose $Z(G)$ is discrete, and let $h \in Z(H)$. Since φ is a covering, the image $\varphi(h)$ lies in $Z(G)$. Thus, since $Z(G)$ is discrete, we can find a small neighbourhood around $\varphi(h)$ such that $\varphi(h) \cap Z(G) = \{\varphi(h)\}$. By shrinking the neighbourhood if necessary, it can be shrunk so that φ is a diffeomorphism around h, hence $Z(H)$ is discrete as well.

Suppose $Z(H)$ is discrete. Then the image of any $h \in Z(H)$ lies in $Z(G)$ and φ is a local diffeomorphism.

Now for the other part. We note that we have a diagram:

$$\begin{tikzcd}
1 \arrow[r] & \ker \varphi\big|_{Z(H)} \arrow[r] \arrow[d] & \ker \varphi \arrow[r] \arrow[d] & \ker \bar{\varphi} \arrow[d] & \\
1 \arrow[r] & Z(H) \arrow[r] \arrow[d] & H \arrow[r] \arrow[d, "\varphi"] & H/Z(H) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Z(G) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & G/Z(G) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

The vertical lower maps are all surjective since φ is a covering map and by the proof above. By the previous exercise, we find that $\ker \varphi\big|_{Z(H)} = \ker \varphi$, hence $H/Z(H) \simeq G/Z(G)$ by the snake lemma (which holds here, see mathoverflow 53124). ♡

15.3 Lecture 8 - Lie Algebras and Lie groups

Exercise 89 (Exercise 8.1)**.** Let G be a connected Lie group, and $U \subset G$ any neighbourhood of the identity. Show that U generates G. ♠

Solution 86. The subgroup generated by U can be written

$$H = \bigcup_{n \in \mathbb{Z}} U^n,$$

hence H is an open subgroup. But since $G \backslash H = \bigcup_{h \notin H} hH$, we see that any open subgroup is also closed, hence H is both open and closed, hence $H = G$. (this solution was given by Theo Bühler at math.stackexchange). ♡

16 Twenty-Four Hours of Local Cohomology

16.1 Lecture 1 - Basic notions

Exercise 90 (Exercise 1.6)**.** Let k be a finite field.

1. For every point $p \in k^n$, construct a polynomial $f \in k[x_1, \dots, x_n]$ such that $f(p) = 1$ and $f(q) = 0$ for all points $q \in k^n \backslash \{p\}$.

2. Given a function $g : k^n \to k$, show that there is a polynomial $f \in k[x_1, \dots, x_n]$ with $f(p) = g(p)$ for all $p \in k^n$.

3. Prove that any subset of k^n is the zero set of a single polynomial.

♠

Solution 87. Suppose that $char(k) = p$. 1. Suppose $p = (p_1, \dots, p_n)$. If $n = 1$, then $p = (p_1)$. Then the polynomial $f(x) = \prod_{q \neq p}(x - q)$ is zero on all of k. On the other hand, $f(p)$ is the product of all non-zero elements in k. But this must be one, as one sees by grouping inverses together.

This generalizes to k^n by taking products over products.

2. This follows from the previous exercise. Let f_p be as in 1. Then form the polynomial

$$f = \sum_{p \in k^n} g(p) f_p.$$

Then $f(p) = g(p)$ for all p.

3. Let S be any subset. Let q be a function that is 1 outside S and is 0 on S. Then the result follows from 2. «««< HEAD ♡

Oppgave 1 (Exercise 1.11)**.** Prove that if K is not algebraically closed, then any algebraic set in K^n is the zero set of a single polynomial. ■

Løsning 1. Suppose $Z = V(F_1, \dots, F_r)$ is an algebraic set. We first show that for any n, there is a polynomial whose only zero is the origin.

The case $n = 1$ is clear, and the case $n = 2$ can be gotten this way: choose an irreducible polynomial $f(x)$ of degree greater than one (here we use that K is not algebraically closed). Then form the homogenization of $f(x)$ with respect to y. Call this new polynomial $G(x, y)$. Then $G(x, y)$ have only one zero at the origin.

Now assume we have such a polynomial H for $n = k - 1$. Then form $G(H(x_1, \dots, x_{n-1}), x_n)$. Done.

Let $n = r$. Let $F(x_1, \dots, x_n) = H(f_1, \dots, f_r)$. Done.

(I must admit I cheated on this one. I had to Google the first part) ♡

Oppgave 2 (Exc 1.14)**.** If a principal ideal domain is not a field, prove that it has dimension one. ■

Løsning 2. Suppose we have a minimal strict chain of *three* prime ideals. That is, suppose the dimension of R is two or more. $(0) \subset \mathfrak{p}_1 \subset \mathfrak{p}_2$. But $\mathfrak{p}_1 = (a)$ for some $a \in R$ and $\mathfrak{p}_2 = (b)$. But since $\mathfrak{p}_1 \subset \mathfrak{p}_2$, we must have $b = ar$ for some $r \in R$. But $\mathfrak{p}_2$ is a prime ideal, hence either $a \in \mathfrak{p}_2$ or $r \in \mathfrak{p}_2$. The latter cannot happen, because in that case, the inclusion would not be minimal. Neither the first, because then we would have $\mathfrak{p}_2 = \mathfrak{p}_1$. ♡

Exercise 91. What is the dimension of $\mathbb{Z}[x]$? ♠

Solution 88. The answer is 2, because of

$$(0) \subset (p) \subset (p, x).$$

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Oppgave 3 (Exercise 1.11)**.** Prove that if K is not algebraically closed, then any algebraic set in K^n is the zero set of a single polynomial. ■

Løsning 3. Suppose $Z = V(F_1, \ldots, F_r)$ is an algebraic set. We first show that for any n, there is a polynomial whose only zero is the origin.

The case $n = 1$ is clear, and the case $n = 2$ can be gotten this way: choose an irreducible polynomial $f(x)$ of degree greater than one (here we use that K is not algebraically closed). Then form the homogenization of $f(x)$ with respect to y. Call this new polynomial $G(x, y)$. Then $G(x, y)$ have only one zero at the origin.

Now assume we have such a polynomial H for $n = k - 1$. Then form $G(H(x_1, \ldots, x_{n-1}), x_n)$. Done.

Let $n = r$. Let $F(x_1, \ldots, x_n) = H(f_1, \ldots, f_r)$. Done.

(I must admit I cheated on this one. I had to Google the first part) ♡

Oppgave 4 (Exc 1.14)**.** If a principal ideal domain is not a field, prove that it has dimension one. ■

Løsning 4. Suppose we have a minimal strict chain of *three* prime ideals. That is, suppose the dimension of R is two or more. $(0) \subset \mathfrak{p}_1 \subset \mathfrak{p}_2$. But $\mathfrak{p}_1 = (a)$ for some $a \in R$ and $\mathfrak{p}_2 = (b)$. But since $\mathfrak{p}_1 \subset \mathfrak{p}_2$, we must have $b = ar$ for some $r \in R$. But $\mathfrak{p}_2$ is a prime ideal, hence either $a \in \mathfrak{p}_2$ or $r \in \mathfrak{p}_2$. The latter cannot happen, because in that case, the inclusion would not be minimal. Neither the first, because then we would have $\mathfrak{p}_2 = \mathfrak{p}_1$. ♡

Exercise 92. What is the dimension of $\mathbb{Z}[x]$? ♠

Solution 89. The answer is 2, because of

$$(0) \subset (p) \subset (p, x).$$

♡

16.2 Lecture 7 - Local cohomology

Exercise 93 (Exercise 7.2)**.** Check that the $\mathfrak{a}$-torsion functor is left-exact. ♠

Solution 90. Suppose $\varphi : M \hookrightarrow N$ is an injection of R-modules. Then clearly the induced morphism $\Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(N)$ is injective as well since $\Gamma_{\mathfrak{a}}(M)$ (N) is a submodule of M (N). ♡

17 Riemannian geometry - Do Carmo

17.1 Chapter 0 - Differentiable manifolds

Exercise 94 (Excercise 2)**.** Prove that the tangent bundle of a differentiable manifold M is orientable. ♠

Solution 91. Locally the tangent bundle is given by $\mathbb{R}^n \times \mathbb{R}^n$, and if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a transition function between two charts, then the induced transition function on the tangent bundle is given by $f \times df$. Hence the differential of the transition map is given by a block diagonal matrix with df appearing twice. Hence the determinant is $(\det df)^2 > 0$, hence TM is orientable. ♡

Exercise 95 (Exercise 4)**.** Show that the projective plane $\mathbb{P}^2(\mathbb{R})$ is non-orientable. ♠

Solution 92. From the hint, we see that it is enough to find an open subset of $\mathbb{P}^2(\mathbb{R})$ that is non-orientable. ♡

Exercise 96 (Exercise 5 - Embedding of $P^2(\mathbb{R})$ in $\mathbb{R}^4$)**.** Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$$F(x, y, z) = (x^2 - y^2, xy, xz, yz).$$

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. Observe that the restriction $\varphi = F \mid S^2$ is such that $\varphi(p) = \varphi(-p)$, and consider the mapping $\tilde{\varphi} : P^2(\mathbb{R}) \to \mathbb{R}^4$ given by

$$\tilde{\varphi}([p]) = \varphi(p).$$

Prove that a) $\tilde{\varphi}$ is an immersion and b) that $\tilde{\varphi}$ is injective. This implies, together with the compactness of $P^2(\mathbb{R})$ that $\tilde{\varphi}$ is an embedding. ♠

Solution 93. Since S^2 is locally diffeomorphic to $P(\mathbb{R}^2)$, it is enough to check that φ is an immersion. We do this on charts. One chart of S^2 is given by

$$(x,y) \mapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right).$$

In this chart (forgetting the scaling, since by the chain rule, that will only contribute by multiplication by a scalar), the Jacobian look like

$$\begin{pmatrix} 8x & 4y & 6x+2y^2-2 & 4xy \\ -8y & 4x & 4xy & 2x^2+6y^2-2 \end{pmatrix}.$$

The first minor (the first two columns) is only zero if $x = y = 0$, and in that case, the last minor is non-zero. Hence (at least in this chart), the mapping is an immersion.

For b), note the $xy = ab$ and $xz = bc$ together imply $y/z = b/c$ which implies $yc = bz$, hence $y = bc/z$. Hence $bc^2 = bz^2$, hence $c = \pm z$. Inserting this into $xz = ac$ gives $x = \pm a$, and similarly $y = \pm b$, hence $\tilde{\varphi}$ is injective. ♡

17.2 Chapter 2 - Affine and Riemannian connections

Exercise 97 (Exercise 8)**.** Consider the upper half plane

$$\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \mid y > 0\},$$

with the metric given by $g_{11} = g_{22} = \frac{1}{y^2}$ and $g_{12} = g_{21} = 0$.

a) Show that the Christoffel symbols of the Riemannian connection are $\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0$, $\Gamma^2_{11} = \frac{1}{y}$, $\Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}$.

b) Let $v_0 = (0,1)$ be a tangent vector at the point $(0,1)$ of $\mathbb{R}^2_+$. Let $v(t)$ be the parallel transport of v_0 along the curve $x = t, y = 1$. Show that $v(t)$ makes an angle t with the direction of the y-axis, measured in the clockwise sense.

♠

Solution 94. a)

This part is easy but tedious, using the fact that

$$\Gamma^m_{ij} = \frac{1}{2} \sum_k \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right) g^{km}$$

where (g^{km}) is the inverse matrix of (g_{ij}), it being

$$\begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}.$$

b)

The parallel transport satisfies the equation

$$0 = \frac{dv^k}{dt} + \sum_{ij} \Gamma_{ij} v^j \frac{dx_i}{dt}$$

for $k = 1, 2$, on page 53. In this case, using the values of Γ_{ij}^k, and the fact that $y = 1$ along the curve $v(t)$, these equations simplify to

$$\begin{cases} 0 & = \frac{da}{dt} - b(t) \\ 0 & = \frac{db}{dt} + a(t). \end{cases}$$

Now, since v_0 was a unit vector, and paralell transport is an isometry, the image of v_0 must lie on the unit circle on each point of the curve. Thus we can write $(a(t), b(t)) = (\sin\theta(t), \cos\theta(t))$ for some function $\theta(t)$. Using the chain rule the equations transform to

$$\begin{cases} 0 & = -\sin\theta(t)\theta'(t) - \sin\theta(t) = 0 \\ 0 & = \cos\theta(t)\theta'(t) + \cos\theta(t) = 0. \end{cases}$$

This implies that $\theta'(t) = -1$, hence $\theta(t) = \pi/2 - t$. ♡