

# Components of the Hilbert scheme in $\mathbb{P}^{11}$

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## Abstract

As any dedicated reader can clearly see, the Ideal of practical reason is a representation of, as far as I know, the things in themselves; as I have shown elsewhere, the phenomena should only be used as a canon for our understanding. The paralogisms of practical reason are what first give rise to the architectonic of practical reason. As will easily be shown in the next section, reason would thereby be made to contradict, in view of these considerations, the Ideal of practical reason, yet the manifold depends on the phenomena. Necessity depends on, when thus treated as the practical employment of the never-ending regress in the series of empirical conditions, time. Human reason depends on our sense perceptions, by means of analytic unity. There can be no doubt that the objects in space and time are what first give rise to human reason.



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# Contents

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|   |            |
|---|------------|
| <b>Abstract</b>   | <b>i</b>   |
| <b>Acknowledgements</b>   | <b>iii</b> |
| <b>Contents</b>   | <b>iv</b>  |
| <b>List of Figures</b>  | <b>v</b>   |
| <b>List of Tables</b>   | <b>vi</b>  |
| <b>Introduction</b>   | <b>vii</b> |
| Notation . . . . .  | ix         |
| <b>1 Preliminaries</b>  | <b>1</b>   |
| 1.1 Notation . . . . .  | 1          |
| 1.2 Projective geometry constructions . . . . .                   | 1          |
| 1.3 Calabi–Yau manifolds and mirror symmetry . . . . .            | 3          |
| 1.4 Toric geometry . . . . .                                      | 5          |
| 1.5 Deformation theory and the Hilbert scheme . . . . .           | 7          |
| 1.6 Stanley–Reisner schemes . . . . .                             | 8          |
| 1.7 Smoothings of Stanley–Reisner schemes . . . . .               | 9          |
| 1.8 Toric geometry and toric degenerations . . . . .              | 9          |
| <b>2 Relation to triangulations of <math>\mathbb{CP}^2</math></b> | <b>11</b>  |
| 2.1 Introductory remarks . . . . .                                | 11         |
| 2.2 Connection to the complex projective plane . . . . .          | 12         |
| 2.3 Attempt to smooth triangulations . . . . .                    | 13         |
| 2.4 Naïve attempt to degenerate . . . . .                         | 16         |
| 2.5 Conclusion . . . . .  | 16         |

|          |  |           |
|----------|--|-----------|
| <b>3</b> | <b>The two smoothings of <math>C(\mathrm{dP}_6)</math></b>     | <b>19</b> |
| 3.1      | The del Pezzo surface $\mathrm{dP}_6$ . . . . .                | 19        |
| 3.2      | The cone over $\mathrm{dP}_6$ and its two smoothings . . . . . | 22        |
| <b>4</b> | <b>Construction of Calabi–Yau’s</b>                            | <b>29</b> |
| 4.1      | The special fiber . . . . .                                    | 29        |
| 4.2      | A natural toric deformation . . . . .                          | 31        |
| 4.3      | Smoothings of $XY$ . . . . .                                   | 32        |
| 4.4      | Degeneration of the toric join . . . . .                       | 37        |
| 4.5      | Invariant Calabi–Yau’s . . . . .                               | 38        |
| 4.6      | The mirror construction Ansatz . . . . .                       | 39        |
|          | <b>Appendices</b>  | <b>41</b> |
| <b>A</b> | <b>Computer code</b>   | <b>43</b> |
| A.1      | Computing the singular locus . . . . .                         | 43        |
| A.2      | Torus action . . . . .   | 44        |
| A.3      | Computing the Gaifullin triangulation . . . . .                | 45        |
|          | <b>Bibliography</b>  | <b>47</b> |

---

## List of Figures

---

|     |   |    |
|-----|---|----|
| 3.1 | Toric description of $\mathrm{dP}_6$ . . . . .              | 19 |
| 3.2 | The projection of a cube onto a hexagon. . . . .            | 21 |
| 3.3 | The inclusion of a cube in a octahedron. . . . .            | 21 |
| 3.4 | Minkowski-decompositions of the hexagon. . . . .            | 22 |
| 3.5 | A $2 \times 2 \times 2$ -tensor. . . . .                    | 23 |
| 4.1 | A hexagon. . . . .  | 31 |
| 4.2 | A $2 \times 2 \times 2$ -tensor, seen from “above”. . . . . | 36 |

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## List of Tables

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# Introduction

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sec:intro

1. Smoothings of Stanley-Reisner schemes
2. Paths on the Hilbert scheme
3. Relation to triangulation of  $\mathbb{CP}^2$ .



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# Notes

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|   |    |
|---|----|
| check . . . . .   | 3  |
| but they do *not* correspond to Minkowski decompositions? . . . . .   | 5  |
| illustration . . . . .  | 6  |
| fan of $P^2$ and its coordinate rings . . . . .   | 6  |
| describe support functions . . . . .  | 7  |
| write this in a good way . . . . .  | 7  |
| relation to moduli spaces and singularity theory . . . . .  | 7  |
| fill in as needed . . . . .   | 9  |
| something about polyhedral complexes and face rings? . . . . .  | 9  |
| define these . . . . .  | 9  |
| join of reflexive polytopes is reflexive!? . . . . .  | 10 |
| reflexive polytopes, mirror symmetry, anticanonical sections . . . . .  | 10 |
| hva med simply-connected? . . . . .   | 13 |
| Bagci/Datta . . . . .   | 13 |
| source? . . . . .   | 25 |
| complete the proof . . . . .  | 25 |
| check transversal intersection with $dP_6$ . . . . .  | 27 |
| illustration graph . . . . .  | 37 |
| connection to physics paper, fremheve at Euler-kar utregninger er kom-<br>patible, finn noe med +48 . . . . . | 38 |
| Torus actions . . . . .   | 38 |
| Check <i>which</i> singularities these are, also: fix points . . . . .  | 39 |
| identify $H$ -invariant singularities + fix points . . . . .  | 39 |

## Notation

If  $V$  is a vector space, we denote by  $\mathbb{P}(V)$  its projectivisation.



# CHAPTER 1

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## Preliminaries

---

ec:prelims

In this chapter we introduce the consistent notation and results which will be used later.

### 1.1 Notation

We write  $k$  for a field, which is almost always assumed to be  $\mathbb{C}$ . If  $X$  is a projective variety, we write  $S(X)$  for its homogeneous coordinate ring (if the embedding is implicit). All schemes are noetherian. We will often write  $\triangleq$  for definitions.

### 1.2 Projective geometry constructions

There are many ways to define the join of two projective varieties  $X$  and  $Y$ . We will define it in a particularly general way, as described by Altman and Kleiman in [AK75]. Then we will specialize to our situation.

Fix a base scheme  $S$ . Let  $\mathcal{C}$  be the category of graded, quasi-coherent  $\mathcal{O}_S$ -algebras, generated in degree 1. The tensor product of two  $\mathcal{O}_S$ -algebras  $\mathcal{R}$  and  $\mathcal{S}$  is naturally graded: the degree  $d$  part is given by

$$(\mathcal{R} \otimes_{\mathcal{O}_S} \mathcal{S})_d = \bigoplus_{p+q=d} \mathcal{R}_p \otimes \mathcal{S}_q.$$

Then we define the *join* of the graded  $\mathcal{O}_S$ -algebras to be

$$X * Y = \text{Proj}(\mathcal{R} \otimes_{\mathcal{O}_S} \mathcal{S}).$$

If  $X$  and  $Y$  are projective varieties over  $S$ , they come with graded  $\mathcal{O}_S$ -algebras  $\mathcal{R} = \text{Sym } \mathcal{O}_X(1)$  and  $\mathcal{S} = \text{Sym } \mathcal{O}_Y(1)$ . Then we define their join to be  $\text{Proj}$  of these algebras.

## 1. Preliminaries

**Example 1.2.1.** Let  $X = \mathbb{P}(E)$  and  $Y = \mathbb{P}(F)$ , where  $E, F$  are quasi-coherent  $\mathcal{O}_S$ -modules. Then we have the equality  $\mathbb{P}(E) * \mathbb{P}(F) = \mathbb{P}(E \oplus F)$ , because of the linear algebra fact  $\mathrm{Sym}(E) \otimes \mathrm{Sym}(F) = \mathrm{Sym}(E \oplus F)$ .

The algebra  $\mathcal{R} \otimes_{\mathcal{O}_S} \mathcal{S}$  contains the ideal  $\mathcal{R} \otimes \mathcal{S}_+$ . The associated subscheme is denoted by  $V_X$ , and is isomorphic to  $X = \mathrm{Proj} \mathcal{R}$ . We define  $V_Y$  similarly.

There is a geometric definition of the join, as described in setion (C11) in [AK75]. Let  $E, F$  be quasi-coherent  $\mathcal{O}_S$ -modules<sup>1</sup>. Suppose  $X, Y$  are subschemes of  $\mathbb{P}(E)$  and  $\mathbb{P}(F)$ . Then  $X * Y$  is a closed subscheme of  $\mathbb{P}(E \oplus F)$ . Identify  $X$  and  $Y$  with their fundamental subschemes in  $X * Y$ . Then it is not difficult to see that  $X * Y$  is the (closure of the) locus of points lying on the lines of  $\mathbb{P}(E \oplus F)$  determined by pairs of points of  $X$  and  $Y$ .

For a graded  $\mathcal{O}_S$ -algebra  $\mathcal{R}$  there are two natural homomorphisms. We have the *structure map*  $\rho : \mathcal{O}_S \rightarrow \mathcal{R}$ , and the *augmentation map*  $\epsilon : \mathcal{R} \rightarrow \mathcal{O}_S$ , sending everything in positive degree to zero. Clearly  $\epsilon \circ \rho = \mathrm{id}_S$ .

lemma:join

**Proposition 1.2.2.** *Suppose  $X/k \subset \mathbb{P}^n$  and  $Y/k \subset \mathbb{P}^m$  are smooth projective schemes. Then their join,  $X * Y$  have dimension  $\dim X + \dim Y + 1$ . The singular locus is of dimension  $\max\{a, b\}$  and consist of the disjoint union of  $X$  and  $Y$ .*

*Proof.* Let  $\mathcal{R} = \bigoplus_{d \geq 0} \mathcal{O}_X(d)$  and  $\mathcal{S} = \bigoplus_{d \geq 0} \mathcal{O}_Y(d)$  be the homogeneous coordinate rings of  $X$  and  $Y$ . Then  $X * Y \subset \mathbb{P}^{n+m+1}$ .

Denote by  $C(X * Y)$  the scheme  $\mathrm{Spec}(\mathcal{R} \otimes \mathcal{S})$ , the affine cone over  $X * Y$ .

The singular locus of  $C(X * Y) = C(X) \times C(Y)$  is equal to  $\mathrm{Sing} C(X) \times C(Y) \cup C(X) \times \mathrm{Sing} C(Y)$ . Since  $X$  and  $Y$  are smooth, the only singular point on the affine cones are the origins. Hence

$$\mathrm{Sing}(C(X * Y)) = 0 \times \mathrm{Sing}(C(Y)) \cup \mathrm{Sing}(C(X)) \times 0.$$

Projectivising, we find that  $\mathrm{Sing}(X * Y) = \mathrm{Sing} Y \sqcup \mathrm{Sing} X$ , since  $(0, \dots, 0)$  is the only common point of the affine cones.  $\blacksquare$

Recall that a scheme  $X$  is *Gorenstein* if it has a dualizing sheaf. It is *Cohen-Macaulay* if the dualizing sheaf is a line bundle. If  $S$  is the homogeneous coordinate ring of a projective Gorenstein variety  $X \subset \mathbb{P}^n$ , the canonical sheaf can be computed as the sheaf associated to the graded module  $\mathrm{Ext}^{\mathrm{codim} X}_S(S, S(-n-1))$ .

If  $X$  and  $Y$  are two Gorenstein schemes, then their join is also Gorenstein. Furthermore, we can compute the canonical sheaf in terms of the canonical sheaves of  $X$  and  $Y$ .

**Proposition 1.2.3.** *Let  $X = \mathrm{Proj} \mathcal{R}$  and  $Y = \mathrm{Proj} \mathcal{S}$  be Gorenstein projective schemes with dualizing sheaves  $\omega_X, \omega_Y$ , respectively (here  $\mathcal{R}, \mathcal{S}$  are sheaves of graded  $\mathcal{O}_S$ -algebras). Then  $X * Y$  is Gorenstein with dualizing sheaf  $\omega_X \otimes_S \omega_Y$ .*

<sup>1</sup>In our case,  $S = \mathrm{Spec} k$  almost always. So  $E, F$  are just vector spaces.

*Proof.* The question is local on  $S$ , so we may assume that  $R$  and  $S$  are homogeneous coordinate rings. Then the statement follows from Theorem 4.2 in [HHS16], where the authors prove that the canonical module of a tensor product is the tensor product of the canonical modules. ■

canonical

*Remark 1.2.4.* If  $X$  and  $Y$  are Gorenstein projective schemes, the resolution of the structure sheaf is symmetrical. It follows that  $\omega_X = \mathcal{O}_X(-n)$  for some  $n \geq 0$ . If  $\omega_Y = \mathcal{O}_Y(-m)$ , it follows from the above proposition that  $\omega_{X*Y}(-m-n)$ .

### 1.3 Calabi–Yau manifolds and mirror symmetry

calabi\_yau

Parts of this thesis will be concerned with the construction of Calabi–Yau manifolds. For us, a Calabi–Yau manifold will be a irreducible complex projective variety  $X$  such that  $\omega_X \simeq \mathcal{O}_X$  and  $H^i(X, \mathcal{O}_X) = 0$  for  $i = 1, \dots, \dim X - 1$ . We usually have  $\dim X = 3$ . Be ware that the literature often requires Calabi–Yau manifolds to be smooth.

Mathematically, Calabi–Yau varieties are interesting because they are among the varieties having Kodaira dimension zero. This means that they have trivial canonical models, making them harder to study. Before the 90’s there were not many explicit constructions of Calabi–Yau varieties, but after the advent of toric geometry and the construction of Batyrev in [Bat94], thousands of new examples were found, all of which was anticanonical sections of Fano toric varieties.

check

Let  $\Omega_X$  be the sheaf of holomorphic one-forms on  $X$ , and assume that  $\dim X = 3$ . Then we can form the *Hodge diamond* of  $X$ , which is a format of writing the dimensions  $h^{ij}$  of the cohomology groups  $H^j(X, \Omega_X^i)$ .

$$\begin{array}{ccccccc}
 & & & & h^{00} & & \\
 & & & & h^{01} & & h^{10} \\
 & & h^{02} & & h^{11} & & h^{20} \\
 h^{03} & & h^{12} & & h^{21} & & h^{30} \\
 & h^{13} & & h^{22} & & h^{31} & \\
 & & h^{23} & & h^{32} & & \\
 & & & h^{33} & & & 
 \end{array}$$

Because of the Calabi–Yau condition, we have that  $h^{j0} = 0$  for  $0 < j < 3$ , and also that  $h^{00} = h^{0d} = 1$ . It follows by Serre duality (see [Har77, Corollary 7.7, Chapter III]) that  $h^{ij} = h^{3-i, 3-j}$ . Note that this amounts to a horizontal symmetry of the Hodge diamond. Since  $X$  was assumed to be a complex manifold, it follows that  $h^{ij} = h^{ji}$  by complex conjugation<sup>2</sup>. This amounts to

<sup>2</sup>This follows by the  $\bar{\partial}$ -Poincaré lemma.

## 1. Preliminaries

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vertical symmetry of the Hodge diamond. It follows that for 3-dimensional Calabi–Yau varieties, the Hodge diamond simplifies to

$$\begin{array}{ccccc}
 & & 1 & & \\
 & & 0 & & 0 \\
 & 0 & & h^{11} & & 0 \\
 1 & & h^{12} & & h^{12} & & 1 \\
 & 0 & & h^{11} & & 0 \\
 & & 0 & & 0 & & \\
 & & & 1 & & & 
 \end{array} .$$

The *Hodge decomposition* theorem states that the singular cohomology groups decomposes as

$$H^k(X, \mathbb{C}) = \bigoplus_{i+j=k}^{\dim X} H^i(\Omega_X^j).$$

The *topological Euler characteristic* is defined as

$$\chi(X) = \sum_{k=0}^{2 \dim X} \dim_{\mathbb{C}} H^k(X, \mathbb{C}).$$

For 3-dimensional Calabi–Yau varieties, it follows from the above discussion that  $\chi(X)$  can be computed as  $2(h^{11} - h^{12})$ .

**Example 1.3.1.** The canonical example of a Calabi–Yau variety is the quintic in  $\mathbb{P}^4$ . Let  $X = V(f)$  be defined by a general element in  $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5))$ . Then  $X$  is smooth, and by adjunction formula we have  $\omega_X = \omega_{\mathbb{P}^4}|_X \otimes \mathcal{O}_X(5) = \mathcal{O}_X(-5) \otimes \mathcal{O}_X(5) = \mathcal{O}_X$ , so the canonical bundle is trivial. By the ideal sheaf sequence, we find that  $H^i(X, \mathcal{O}_X) \simeq H^i(X, \mathcal{O}_{\mathbb{P}^4}(-5))$ , which by [Har77, Theorem 5.1, Chapter III] implies the required vanishing of the structure sheaf cohomology groups.

The Euler characteristic can be computed as the degree of the top Chern class of  $X$ . If  $Y$  is a degree  $d$  hypersurface in  $\mathbb{P}^n$ , the following formula holds:

$$c_{n-1}(T_X) = h^{n-1} \left( \binom{n+1}{n-1} - d \binom{n+1}{n-2} + d^2 \binom{n+1}{n-3} + \dots \right).$$

Putting  $n = 4$  and  $d = 5$ , we find that  $\chi(X) = -200$ . To compute  $h^{11}$ , we see first from the conormal sequence that  $H^1(\Omega_Y^1) \simeq H^1(\Omega_{\mathbb{P}^4}^1|_Y)$ . Now it follows easily from the restricted Euler sequence that  $h^{11} = 1$ , and since  $\chi(X) = 2(h^{11} - h^{12})$  it follows that  $h^{12} = 101$ .

In general it is very hard to compute the Hodge numbers of Calabi–Yau varieties, with the exception of hypersurfaces in four-dimensional toric varieties.



Often the best one can hope for is the topological Euler characteristic  $\chi(X)$ , which is much easier to compute.

After the invention of string theory, Calabi–Yau varieties caught the attention of physicists. They discovered a duality between different Calabi–Yau 3-manifolds  $X$  and  $X^\circ$  such that their Hodge numbers satisfy  $h^{11}(X) = h^{12}(X^\circ)$  and  $h^{12}(X) = h^{11}(X^\circ)$  (in particular  $\chi(X) = -\chi(X^\circ)$ ). This a conjectural correspondence between the complex moduli space of  $X$  (which have tangent space  $H^1(X, \Omega_X^2)$ ), and the Kähler moduli space of  $\omega^\circ$ , the Kähler class on  $X^\circ$ .

This correspondence is called *mirror symmetry*.

The last chapter of this thesis is concerned about the construction of new examples of Calabi–Yau manifold. They are all complete intersections in toric varieties .

- Fano  $\rightarrow$  Calabi–Yau

but they do  
\*not\* correspond  
to Minkowski  
decompositions?

## 1.4 Toric geometry

c-geometry

A toric variety is an irreducible and normal variety containing the torus  $T = (\mathbb{C}^*)^n$  as a dense subset, such that the action of the torus on itself extends to an action on the variety.

We fix some notation that will be used throughout. Details and proofs can be found in [CLS11; Ful93]. Each toric variety comes with two dual lattices. The *lattice of 1-parameter subgroups*  $N$  and the *character lattice*  $M$ . A one-parameter subgroup is a morphism  $\lambda: \mathbb{C}^* \rightarrow T$  that is a group homomorphism. The set of one-parameter subgroups is a lattice isomorphic to  $\mathbb{Z}^n$ , and we denote it by  $N$ . A *character* is a morphism  $\chi: T \rightarrow \mathbb{C}^*$  that is a group homomorphism. The set of characters is a lattice  $M$  isomorphic to  $\mathbb{Z}^n$  which is naturally dual to  $N$ .

Let  $V$  be a  $\mathbb{R}$ -vector space. Let  $V^\vee$  be the dual vector space. A *convex polyhedral cone* is a subset  $\sigma$  of  $V$  of the form

$$\sigma = \{r_1 v_1 + \dots + r_s v_s \mid r_i \geq 0 \text{ for all } i\},$$

where the  $v_i$  is a finite set of vectors in  $V$  and the  $r_i$  are real numbers. A *rational polyhedral cone* is a cone such that the vectors  $v_i$  can be taken to have rational coordinates.

The *dual cone*  $\sigma^\vee$  lives in  $V^\vee$ , and is defined as the set of functionals that are positive on  $\sigma$ :

$$\sigma^\vee = \{u \in V^\vee \mid \langle u, v \rangle \geq 0, v \in \sigma\}.$$

Cones have two descriptions: either as the positive hull of a finite set of vectors (as above), or implicitly, as the intersection of finitely many half-spaces. If  $u_i$  generate  $\sigma^\vee$ , then it is true that

$$\sigma = \{v \in V \mid \langle u_i, v \rangle \geq 0 \text{ for all } i\}.$$

## 1. Preliminaries

The vectors  $u_i$  are the inner normal vectors of the facets of  $\sigma$ .

A (commutative) *semigroup* is a set  $S$  with a commutative binary operation  $S \times S \rightarrow S$ , together with an identity element  $0 \in S$ . Given a cone  $\sigma \subset N$ , we can form a semigroup  $S = \sigma^\vee \cap M \subset M$ . Given a semigroup  $S$ , we can form the *semigroup algebra*  $\mathbb{C}[S]$ : it is the algebra generated by the elements of  $S$ , with multiplicative structure inherited from  $S$ . We then define  $U_\sigma$  as  $\text{Spec } \mathbb{C}[\sigma^\vee \cap M]$ , and call it the *affine toric variety associated to  $\sigma$* .

**Example 1.4.1.** Let  $\sigma = \langle (1, 0), (1, 2) \rangle \subset \mathbb{R}^2$ . Then  $\sigma^\vee = \langle (2, -1), (1, 0), (0, 1) \rangle \subset \mathbb{R}^2$ . Then  $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[x, y, x^2/y]$ , where we have identified  $x$  and  $y$  with the standard basis of  $\mathbb{R}^2$ . This ring is isomorphic to  $\mathbb{C}[a, b, c]/(a^2 - bc)$ .

illustration

General toric varieties are described using collections of cones called *fans*. A set  $\Sigma$  of cones is called a *fan* if it is closed under intersections and faces of cones: if  $\sigma, \sigma' \in \Sigma$ , then we should also have  $\sigma \cap \sigma' \in \Sigma$ , and if  $\sigma' \subset \sigma$  is a face with  $\sigma \in \Sigma$ , then  $\sigma' \in \Sigma$  also. Thus, given a fan  $\Sigma$ , we get a collection of affine toric varieties  $U_\sigma$  for each cone  $\sigma \in \Sigma$ . We have inclusions  $U_{\sigma \cap \sigma'} \subset U_\sigma$ , and using these inclusions one glues to get a separated toric variety. If the fan is *complete* (meaning that the union of its cones is equal to  $N$ ), it follows that the corresponding toric variety is complete.

**Example 1.4.2.**

fan of  $P^2$  and its coordinate rings

Projective toric varieties can be constructed from lattice polytopes. We describe the procedure here. Let  $\Delta$  be a lattice polytope in  $M \simeq \mathbb{Z}^n$ . Let  $M' = M \oplus \mathbb{Z}$ , and embed  $\Delta$  in  $M'$  by sending  $v$  to  $(v, 1)$ . Let  $C(\Delta)$  be the cone over  $\Delta$  in  $M'$ . Then  $\mathbb{C}[C(\Delta) \cap M']$  is a  $\mathbb{Z}$ -graded algebra. We let  $X_\Delta$  be the associated projective variety.

- Construction of CY varieties in toric varieties.

## Divisors and Picard groups of toric varieties

Recall that a *Weil divisor* is a formal linear combination of codimension 1 subvarieties of a scheme  $X$  (satisfying condition “star” in Hartshorne). The group of Weil divisors modulo linear equivalence is the *class group* of  $X$ , and is denoted by  $\text{Cl}(X)$ . The group of line bundles modulo isomorphism is the *Picard group* of  $X$ , and is denoted by  $\text{Pic}(X)$ . The two groups coincide for smooth varieties. They are in general very hard to compute, but for toric varieties the computation is exceptionally easy, relying only on structure of the rays in the fan  $\Sigma$  defining the toric variety.

For completeness, we describe the divisors on toric varieties. The description will be used in Chapter 3, where we work out the geometry of the two smoothings of the affine cone over the del Pezzo surface of degree 6.

Let  $X$  be a smooth toric variety, and let  $\Sigma(1)$  denote the set of one-dimensional cones (called *rays*) in the fan  $\Sigma$  defining  $X$ . For each ray  $\rho$ , let  $u_\rho \in N$  denote the primitive ray generator of  $\rho$ . Then one can show that the torus-invariant divisors on  $X$  are in one to one correspondence with the rays  $\rho \in \Sigma(1)$ . Furthermore, every divisor on  $X$  is linearly equivalent to a torus-invariant divisor. Using these two facts, one can prove the following:

There is an exact sequence:

$$0 \longrightarrow M \xrightarrow{C} \mathbb{Z}^{\Sigma(1)} \longrightarrow \text{Pic}(X) \longrightarrow 0,$$

where the rows of the matrix  $C$  are the vectors  $u_\rho$ . See [CLS11], Chapter 4, for a proof.

There is also a description of the Cartier divisors on  $X$  in terms of *support functions* on  $N$ : a support function is a function  $\varphi : |\Sigma| \rightarrow \mathbb{R}$  such that the restriction  $\varphi|_\sigma$  of  $\varphi$  to each cone in  $\Sigma$  is linear. A support function is *integral with respect to  $N$*  if  $\varphi(|\Sigma| \cap N) \subset \mathbb{Z}$ . This means that for each cone  $\sigma$ , there is an  $m_\sigma \in M$ , such that  $\varphi(v) = \langle v, m_\sigma \rangle$  if  $v \in \sigma$ .

The set of support functions is an abelian group under addition, and by Theorem 4.2.12 in [CLS11], there is an isomorphism between the group of integral support functions on  $\Sigma$  and the torus invariant Cartier divisors on  $X$ .

describe support functions

## 1.5 Deformation theory and the Hilbert scheme

Given a scheme  $X_0$  over  $\mathbb{C}$ , a *family of deformations* of  $X_0$  is a flat morphism  $\pi : \mathcal{X} \rightarrow (S, 0)$  with  $S$  connected such that  $\pi^{-1}(0) = X_0$ . If  $S$  is the spectrum of an artinian  $\mathbb{C}$ -algebra, then  $\pi$  is an *infinitesimal deformation*. If  $S = \text{Spec } \mathbb{C}[\epsilon]/\epsilon^2$ , then  $\pi$  is a *first order deformation*. An *embedded deformation* is a deformation such that the total space is contained in  $S \times \mathbb{P}^M$  for some  $M$ . A *smoothing* of  $X_0$  is a deformation such that the general fiber is smooth. A reference for deformation theory is the book by Hartshorne [Har10].

Let  $\text{Def}(X, S)$  denote the space of all deformations of the projective scheme  $X \subset \mathbb{P}^n$  over the base scheme  $S$ , and let  $\text{EmbDef}(X, S)$  denote the space of all embedded deformations of  $X$  over  $S$ . There is a natural forgetful map  $\text{EmbDef}(X, S) \rightarrow \text{Def}(X, S)$ . The tangent space of  $\text{EmbDef}(X, S)$  at the point corresponding to  $X$  is given by  $H^0(X, \mathcal{N}_{X/\mathbb{P}^n})$ , and the tangent space of  $\text{Def}(X)$  is by definition given by  $\text{Def}(X) = \text{Def}(k[\epsilon]) = T^1$

write this in a good way

- Normal sheaf classifies embedded deformations
- $T^1()$

relation to moduli spaces and singularity theory

## 1.6 Stanley–Reisner schemes

Stanley–Reisner schemes are certain degenerate projective schemes modelled on simplicial complexes. We first recall some notation. Let  $[n]$  denote the set of numbers  $\{0, \dots, n\}$ . The power set of  $[n]$  is called the  $n$ -*simplex* and denoted by  $\Delta_n$ . A *simplicial complex* is a subset  $\mathcal{K} \subseteq \Delta_n$  (for some  $n$ ), such that if  $f \in \mathcal{K}$  and  $g \subseteq f$ , then  $g \in \mathcal{K}$ . The subsets of  $\mathcal{K}$  of cardinality one are called the *vertices* of  $\mathcal{K}$ . The subsets of  $\mathcal{K}$  are called *faces*. The *dimension* of a face  $f$  is equal to  $|f| - 1$ . A good reference is Stanley’s green book [Sta96].

Let  $k$  be a field, and let  $P_{\mathcal{K}}$  be the polynomial ring over  $k$  with variables indexed by the vertices of  $\mathcal{K}$ . Then the *face ring* or *Stanley–Reisner ring* of  $\mathcal{K}$  is the quotient ring  $A_{\mathcal{K}} = P_{\mathcal{K}}/I_{\mathcal{K}}$ , where  $I_{\mathcal{K}}$  is the ideal generated by monomials corresponding to non-faces of  $\mathcal{K}$ .

**Example 1.6.1.** Let  $\mathcal{K}$  be the triangle with vertices  $\{v_1, v_2, v_3\}$ . Its maximal faces are  $v_1v_2, v_2v_3$  and  $v_1v_3$ . The Stanley–Reisner ring is  $k[v_1, v_2, v_3]/(v_1v_2v_3)$ .

The ideal  $I_{\mathcal{K}}$  is graded since it is defined by monomials. This leads us to define the *Stanley–Reisner scheme*  $\mathbb{P}(\mathcal{K})$  as  $\text{Proj } A_{\mathcal{K}}$ .

The *join* of two simplicial complexes  $\mathcal{K}$  and  $\mathcal{K}'$  is defined as

$$\mathcal{K} * \mathcal{K}' \triangleq \{f \sqcup g \mid f \in \mathcal{K}, g \in \mathcal{K}'\},$$

where  $\sqcup$  denotes the disjoint union. Note that  $\mathbb{P}(\mathcal{K} * \mathcal{K}') = \mathbb{P}(\mathcal{K}) * \mathbb{P}(\mathcal{K}')$ , where the second star means the join of two projective varieties.

If  $f \subset \mathcal{K}$  is a face, the *link of  $f$  in  $\mathcal{K}$*  is the simplicial complex defined by

$$\text{lk}(f, \mathcal{K}) \triangleq \{g \in \mathcal{K} \mid f \cap g = \emptyset, f \cup g \in \mathcal{K}\}.$$

If  $D_+(x_f) \subset \mathbb{P}(\mathcal{K})$  denotes the distinguished open set corresponding to the monomial  $x^f$ , we have that  $D_+(x_f) = \mathbb{A}(\text{lk}(f, \mathcal{K})) \times (k^*)^{\dim f}$ .

Every simplicial complex has a *geometric realization*, which as a set is defined as follows:

$$|\mathcal{K}| \triangleq \{\alpha : [n] \rightarrow [0, 1] \mid \text{supp}(\alpha) \in \mathcal{K}, \sum_{i=1}^n \alpha(i) = 1\}.$$

This is an example of a piecewise linear manifold, or a PL-manifold for short. For more on PL-manifolds or combinatorial topology, we refer the reader to [Gla70; Spa66; Hud69].

We will need the following result of Christophersen ([AC10, Theorem 4.6]):

thm:tldims

**Theorem 1.6.2.** *If  $\mathcal{K}$  is a simplicial manifold, and  $\mathbf{c} = \mathbf{a} - \mathbf{b}$  (with disjoint supports  $a$  and  $b$ ), then*

$$\dim_k T_{A_{\mathcal{K}, \mathbf{c}}}^1 = \begin{cases} 1 & \text{if } a \in \mathcal{K} \text{ and } b \in \mathcal{B}(\text{lk}(a, \mathcal{K})) \\ 0 & \text{otherwise.} \end{cases}$$

Here  $\mathcal{B}(\mathcal{K})$  is defined as follows:

**Definition 1.6.3.** The set  $\mathcal{B}(\mathcal{K})$  is the set of  $b \subset \mathcal{K}$  with  $|b| \geq 2$  such that

1.  $\mathcal{K} = L * \partial b$ , where  $|L|$  is a  $(n - |b| + 1)$ -sphere, if  $b \notin \mathcal{K}$ .
2.  $\mathcal{K} = L * \partial b \cup \partial L * \bar{b}$  where  $|L|$  is a  $(n - |b| + 1)$ -ball, if  $b \in \mathcal{K}$ .

fill in as needed

## 1.7 Smoothings of Stanley–Reisner schemes

Because many properties of varieties are easier read off their degenerations, it is an interesting problem to study smoothings of Stanley–Reisner-schemes, which are highly singular. Below are the two main lemmas motivating the study of triangulations in deformation theory.

na:srcohom

**Lemma 1.7.1.** *If  $\mathcal{K}$  is a simplicial complex, then  $H^i(\mathcal{K}; k) \simeq H^i(\mathbb{P}(\mathcal{K}), \mathcal{O}_{\mathbb{P}(\mathcal{K})})$ .*

The lemma is essentially due to Hochster, and is proved (in a different form) in Stanley’s book [Sta96]. This is true essentially because the Čech complex computing the simplicial cohomology and the Čech complex computing sheaf cohomology look exactly the same.

**Lemma 1.7.2.** *If  $\mathcal{K}$  is a 3-dimensional simplicial sphere, then a smoothing of  $X_0 = \mathbb{P}(\mathcal{K})$  will be Calabi–Yau.*

*Proof.* Let  $\pi : \mathcal{X} \rightarrow S$  be a smoothing. Since  $\mathcal{K}$  is a sphere, it follows from 1.7.1 that  $H^i(X_0, \mathcal{O}_{X_0}) = k$  for  $i = 0, 3$ , and zero for  $i \neq 0, 3$ . The triviality of the canonical bundle is proved in Theorem 6.1 in [BE91]. Since  $H^1(\mathcal{K}; K) = H^2(\mathcal{K}; K) = 0$ , it follows from the semicontinuity theorem (Theorem 12.8 in Chapter III in [Har77]) that  $H^i(X_t, \mathcal{O}_{X_t}) = 0$  for all  $t \in S$ . Similarly, if  $\omega_0$  all nearby fibers must have trivial canonical bundle as well. ■

something about polyhedral complexes and face rings?

## 1.8 Toric geometry and toric degenerations

There is a correspondence between certain degenerations of toric varieties and so-called unimodular triangulations.

define these

Let  $M$  be a lattice (by which we mean a free abelian group of finite rank). Let  $\nabla \subset M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$  be a lattice polytope, and let  $S_{\nabla}$  be the semigroup in  $M \times \mathbb{Z}$  generated by the elements  $(u, 1) \in \nabla \cap M$ . Then we define  $\mathbb{P}(\nabla) = \text{Proj } \mathbb{C}[S_{\nabla}]$ , and call it the *toric variety associated to  $\nabla$* .

## 1. Preliminaries

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By Theorem 8.3 and Corollary 8.9 in [Stu96], there is a one-one correspondence between unimodular regular triangulations of  $\nabla$  and the square-free initial ideals of the toric ideal of  $\mathbb{P}(\nabla)$ .

ec:toric geometry  
join of reflexive  
polytopes is re-  
flexive?  
modular triang

**Proposition 1.8.1.** *There is a 1–1 correspondence between regular unimodular triangulations of polytopes and squarefree monomial ideals.*

By Theorem 8.3 and Corollary 8.9 in [Stu96].....

Thus, if we can find a polytope whose boundary has a regular unimodular triangulation, we know that the associated Stanley-Reisner ring has at least one deformation.

reflexive poly-  
topes, mirror  
symmetry, an-  
ticanonical sec-  
tions

## CHAPTER 2

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# Relation to triangulations of $\mathbb{CP}^2$

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cp2triangs

This chapter will not contain any new results of any significance, but is rather a report on an idea which led to the deliberations in the later chapters.

We explain a connection between the topological space  $\mathbb{CP}^2$  and hyper-Kähler manifolds.

### 2.1 Introductory remarks

manifolds

#### Hyper-Kähler manifolds

Among the known families of manifolds, hyper-Kähler manifolds are among the most elusive. One often divides manifolds into three types: those with positive, negative or trivial canonical class. Of those with trivial canonical class, two prominent types stand out: Calab–Yau-manifolds and hyper-Kähler manifolds.

**Definition 2.1.1.** A *hyper-Kähler manifold*  $X$  is a simply connected compact Kähler manifold such that  $H^0(X, \Omega_X^2)$  is generated by a non-degenerate  $\sigma : TX \times TX \rightarrow \mathbb{C}$ .

*Remark 2.1.2.* Since the two-form  $\sigma$  is non-degenerate, it follows that the canonical sheaf  $\omega_X = \Omega_{X/\mathbb{C}}^n$  is trivial. The map  $1 \mapsto \sigma^{n/2}$  gives an isomorphism  $\mathcal{O}_X \rightarrow \omega_X$ .

For example, in dimension two, K3 surfaces are hyper-Kähler (*and* Calab–Yau). Because of the non-degeneracy of the symplectic form  $\sigma \in H^0(X, \Omega_X^2)$ , hyper-Kähler manifolds only occur in even dimensions. Only a few explicit families of hyper-Kähler manifolds are known. Below we sketch the construction of one such family.

Let  $S$  be a K3-surface with symplectic form  $\sigma$ , and let  $S^{(2)}$  be its symmetric square:  $S \times S / \{(p, q) \sim (q, p)\}$ . Let  $\pi_i : S \times S \rightarrow S$  be the two projections ( $i = 1, 2$ ). Then the 2-form  $\pi_1^* \sigma + \pi_2^* \sigma$  is  $\mathbb{Z}/2$ -invariant, hence it descends to a 2-form  $\tau$  on  $S^{(2)}$ .

## 2. Relation to triangulations of $\mathbb{CP}^2$

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The space  $S^{(2)}$  is singular along the diagonal: locally it is isomorphic to  $\mathbb{C} \times \mathbb{C}/(x \sim -x)$ . The last factor is a quadric cone, so a single blowup along the diagonal will resolve the singularities. The form  $\tau$  lifts to a non-degenerate form on  $S^{[2]}$ , and it can be shown that it is in fact a hyper-Kähler variety of dimension 4. The resulting space is denoted by  $S^{[2]}$ , and is called the *Hilbert square of  $S$* , or the *Hilbert scheme of two points on  $S$* . It parametrizes length two subschemes of  $S$ .

For more details on this construction, see Beauville's original paper [Bea83].

### The variety of lines on a cubic fourfold

There is another construction of hyper-Kähler varieties that is interesting to us. Let  $X$  be a smooth cubic fourfold in  $\mathbb{P}^5$ . Let  $F(X)$  denote the set of lines contained in  $X$ . It is the *Fano variety of lines on  $X$* , and is a closed subset of the Grassmannian  $\mathbb{G}(1, \mathbb{P}^5)$ . One can show that  $F(X)$  is a hyper-Kähler variety of dimension 4.

In the article [BD85], Beauville and Donagi shows that  $F(X)$  is deformation equivalent to  $S^{[2]}$  for some K3 surface  $S$ . They also show that if  $X$  is a *pfaffian* hypersurface, then  $F(X)$  is actually *isomorphic* to  $S^{[2]}$  for some K3 surface  $S$ . Furthermore, the family  $\{F(X)\}$  obtained this way is 19-dimensional, and is a hypersurface in the deformation space of  $S^{[2]}$ .

## 2.2 Connection to the complex projective plane

Let  $X$  be a topological space. Recall that the symmetric product  $X^{(2)}$  is defined as follows:

$$X * Y = X \times Y / \{(x, y) \sim (y, x)\}.$$

If  $X = S^2$ , we have that  $X^{(2)}$  is naturally isomorphic to  $\mathbb{CP}^2$ , which can be seen as follows:  $S^2$  can be identified with  $\mathbb{P}_{\mathbb{C}}^1$ . Unordered pairs of points in  $\mathbb{P}^1$  correspond to degree 2 polynomials up to scalar multiplication. Hence we have identifications

$$(S^2)^{(2)} = (\mathbb{P}^1)^{(2)} = \{(P, Q) \in \mathbb{P}^1 \times \mathbb{P}^1\} / \mathbb{Z}_2 = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(2))) = \mathbb{CP}^2.$$

Stanley–Reisner degenerations of K3 surfaces correspond to triangulated spheres. Since the symmetric square of a sphere is  $\mathbb{CP}^2$ , a Stanley–Reisner degeneration of the symmetric square of a K3 surface should correspond to a triangulated  $\mathbb{CP}^2$ .

**Proposition 2.2.1.** *Suppose  $\mathcal{K}$  is a triangulation of  $\mathbb{CP}^2$  and  $X_0 = \mathbb{P}(\mathcal{K})$  is its associated Stanley–Reisner-scheme. Then a smoothing  $X$  of  $X_0$  will be a hyper-Kähler manifold.*



*Proof.* The dimensions of the groups  $H^i(X, \mathcal{O}_X)$  are constant in flat families. Since  $H^0(X, \Omega_X^2) = H^2(X, \mathcal{O}_X) = H^2(\mathcal{K}; \mathbb{C}) = \mathbb{C}$  (the first equality is complex conjugation), we have that  $H^0(X, \Omega_X^2)$  is generated by a single 2-form. It is non-degenerate since  $\omega_X \simeq \mathcal{O}_X$ . It follows that  $X$  is hyper-Kähler. ■

hva med simply-connected?

## 2.3 Attempt to smooth triangulations

If  $\mathcal{K}$  is a triangulation of  $\mathbb{CP}^2$  and  $\mathbb{P}(\mathcal{K})$  is the associated Stanley–Reisner-scheme, a smoothing of  $\mathbb{P}(\mathcal{K})$  will give a hyper-Kähler variety. Using this idea, and the **Macaulay2** package **VersalDeformations** (by Nathan Ilten, see [Ilt12]), we tried to find potentially new hyper-Kähler varieties. Unfortunately, it looks like all the triangulations we experimented with were non-smoothable.

In the next four subsections we describe four different triangulations of  $\mathbb{CP}^2$ , and also their deformation theory using the results of [AC10]. In all cases we conclude that the corresponding Stanley–Reisner scheme is probably not smoothable.

Before we go on to describe the triangulations, we recall some basic facts about combinatorial manifolds.

We can decompose  $\mathbb{CP}^2$  into three four-dimensional closed balls  $B_j$ , whose pairwise intersections are solid tori  $\Pi_{ij}$ , and whose triple intersection is a two-dimensional torus  $T$ . The closed balls  $B_0$  is defined as

$$B_0 = \{[x_0 : x_1 : x_2] \in \mathbb{CP}^2 \mid x_0 \overline{x_0} \geq x_1 \overline{x_1}, x_0 \overline{x_0} \geq x_2 \overline{x_2}\},$$

and similarly for  $B_1$  and  $B_2$ . This is sometimes called the *equilibrium decomposition* of the complex projective plane.

A triangulation of  $\mathbb{CP}^2$  is *equilibrium* if the closed balls, the solid tori, and the torus  $T$  are subcomplexes of the triangulation. Several of the triangulation below are equilibrium.

Bagci/Datta

### The 15-vertex triangulation

A very interesting triangulation  $\mathcal{T}$  of  $\mathbb{CP}^2$  is found in [Gai09]. The author describes a triangulation of  $\mathbb{CP}^2$  using 15 vertices. One reason it is interesting is that the corresponding Stanley–Reisner scheme  $\mathbb{P}(\mathcal{T})$  has the same Hilbert-polynomial as  $F_1(X)$ , the Fano variety of lines on a cubic hypersurface. This means that they live in the *same* Hilbert scheme, and one could naively hope that they live in the same component as well, meaning that there exists a degeneration of  $F_1(X)$  to  $\mathbb{P}(\mathcal{T})$ .

We will spend some time describing this triangulation, since parts of it inspired our the construction of the Calabi–Yau’s in the last chapter. We cite the definition ad verbatim from [Gai09].

## 2. Relation to triangulations of $\mathbb{CP}^2$

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**Definition 2.3.1.** Let  $V_4 \subset S_4$  be the Klein four group. The vertex set of  $\mathcal{T}$  is defined as

$$V = (V_4 \setminus \{e\}) \sqcup (\{1, 2, 3, 4\} \times \{1, 2, 3\}). \quad (2.1)$$

Thus the vertices of  $\mathcal{T}$  are the permutations (12)(34), (13)(24) and (14)(23) and the pairs of integers  $(a, b)$  with  $1 \leq a \leq 4$  and  $1 \leq b \leq 3$ . The maximal faces are spanned by the sets

$$\nu, (1, b_1), (2, b_2), (3, b_3), (4, b_4) \quad (2.2)$$

with  $\nu \in V_4 \setminus \{e\}$  and  $1 \leq b_a \leq 3$  ( $a = 1, 2, 3, 4$ ) such that  $b_{\nu(a)} \neq b_a$  for  $a = 1, 2, 3, 4$ .

See Appendix A.3 for a SAGE [Wil17] script for computing the maximal facets of  $\mathcal{T}$ .

The triangulation  $\mathcal{T}$  is the union over the cones over three 3-spheres  $S_j$  (the cone over  $S_j$  is the ball  $B_j$  in the definition of an equilibrium triangulation). Each  $S_j$  is a very simple 3-sphere. It is the join of two hexagons (recall that  $S^1 * S^1 \approx S^3$ ). The f-vector is (15, 90, 240, 270, 108).

It is the Stanley–Reisner-scheme of  $S_j$  and some of its deformations that is studied in Chapter 4, leading to constructions of some new Calabi–Yau manifolds.

We compute some deformation-theoretic invariants of  $\mathbb{P}(\mathcal{T})$ , the Stanley–Reisner scheme associated to  $\mathcal{T}$ .

**Proposition 2.3.2.** *We have that  $\dim_{\mathbb{C}} T_{A(\mathcal{T}),0}^1 = 90$  and  $\dim_{\mathbb{C}} T_{A(\mathcal{T}),0}^2 = 306$ . The normal sheaf has  $\mathcal{N}_{\mathbb{P}(\mathcal{T})/\mathbb{P}^{14}}$  has 300 global sections.*

The proof is a computation in **Macaulay2**. We remark that since  $\mathbb{P}(\mathcal{T})$  is not Cohen–Macaulay, some standard comparison theorems does not hold. In our case we only have an inclusion  $T_{A(\mathcal{T})}^1 \hookrightarrow T_{\mathbb{P}(\mathcal{T})}^1$  (see the article of Kleppe [Kle79] and Theorem 3.9). This means that there might be deformations of  $\mathbb{P}(\mathcal{T})$  that are not induced from the ambient projective space.

Because of the high number of parameters, we have not been able to say anything meaningful regarding the deformations of  $\mathbb{P}(\mathcal{T})$ . However, it is possible to deform  $\mathbb{P}(\mathcal{T})$  into the union of three toric varieties, each being deformations of the Stanley–Reisner scheme  $\mathbb{P}(B_j)$ . This is not surprising, since  $B_j$  is a triangulation of a the normal polyhedron of the corresponding toric variety. This deformation reduces the number of components of  $\mathbb{P}(\mathcal{T})$  from 108 to 3.

It is not clear however if this union of toric varieties can be further deformed.

### Kühnel’s 9-vertex triangulation

The minimal triangulation  $\mathcal{T}_9$  of  $\mathbb{CP}^2$  is a 9-vertex triangulation with f-vector (9, 36, 84, 90, 36). This means that the associated Stanley–Reisner scheme  $\mathbb{P}(\mathcal{T}_9)$

lives in  $\mathbb{P}^8$  and is of degree 36. The automorphism group of  $\mathcal{T}_9$  is a group of order 54, and it can be realized as a semidirect product  $(\mathbb{Z}_3 \times \mathbb{Z}_3) \ltimes \mathbb{Z}_3 \ltimes \mathbb{Z}_2$ . For a very readable account of the construction and motivation of this triangulation, consult [KB83].

The ideal has a resolution of the form (in `Macaulay2` format):

```

      0  1  2  3  4 5 6
total: 1 36 90 84 37 9 1
0: 1  .  .  .  .  .  .
1: .  .  .  .  .  .  .
2: .  .  .  .  .  .  .
3: . 36 90 84 36 9 1
4: .  .  .  .  .  .  .
5: .  .  .  .  1  .  .

```

This means that the ideal of  $\mathbb{P}(\mathcal{T}_9)$  is generated by 36 cubic monomials, and there are 90 relations between them, lying in  $\mathcal{O}_{\mathbb{P}(\mathcal{T}_9)}(-5)$ , et cetera. Since the resolution is not symmetric, we see immediately that  $\mathbb{P}(\mathcal{T}_9)$  is not Gorenstein.

**Proposition 2.3.3.** *We have that  $\dim_{\mathbb{C}} T_{A(\mathcal{T}_9),0}^1 = 21$  and  $\dim_{\mathbb{C}} T_{A(\mathcal{T}_9),0}^2 = 126$ . The normal sheaf has  $\mathcal{N}_{\mathbb{P}(\mathcal{T}_9)/\mathbb{P}^8}$  has 93 global sections.*

We can compute the action of the automorphism group on  $T^1$ . Using SAGE, we find that the 21 deformation parameters split into two orbits, one of size 3 and one of size 18.

We have not been able to lift any first-order deformation of  $\mathbb{P}(\mathcal{T}_9)$  to a family over  $\text{Spec } \mathbb{C}[t]$ .

## The minimal equilibrium triangulation

In [BK92], the authors construct a 10 vertex equilibrium triangulation  $\mathcal{T}_{10}$  of  $\mathbb{CP}^2$ . They start with the minimal 7-vertex triangulation of the torus, and then they construct  $\mathcal{T}_{10}$  by taking cones over unions of the tori.

The automorphism group is order 42, and comes from the symmetries of the torus.

The Betti table of the resolution of the ideal of  $\mathbb{P}(\mathcal{T}_{10})$  is the following:

```

      0  1  2  3  4 5 6 7
total: 1 38 128 177 123 46 10 1
0: 1  .  .  .  .  .  .  .
1: .  3  2  .  .  .  .  .
2: .  .  .  .  .  .  .  .
3: . 35 126 175 120 45 10 1
4: .  .  .  2  3  .  .  .
5: .  .  .  .  .  1  .  .

```

## 2. Relation to triangulations of $\mathbb{CP}^2$

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Again we see that the ideal is not Gorenstein.

**Proposition 2.3.4.** *We have that  $\dim_{\mathbb{C}} T_{A(\mathcal{T}_{10}),0}^1 = 42$  and  $\dim_{\mathbb{C}} T_{A(\mathcal{T}_{10}),0}^2 = 105$ . The normal sheaf has  $\mathcal{N}_{\mathbb{P}(\mathcal{T}_{10})/\mathbb{P}^3}$  has 132 global sections.*

In fact, it is possible to lift the versal family of deformation parameters to an honest family over  $\text{Spec } \mathbb{C}[t_1, \dots, t_{42}]$ , using the `VersalDeformations` package. Surprisingly, even though the  $T^2$  module is big, there are no obstructions in the family. However, the generic member of this family is reducible (verified in `Macaulay2` for “random” values of the deformation parameters), implying that  $\mathbb{P}(\mathcal{T}_{10})$  is not smoothable.

The automorphism group act transitively on the natural basis of  $T^1$ , so that  $\dim_{\mathbb{C}} T_{A(\mathcal{T}_{10}),0}^1 \stackrel{G}{=} 1$ .

### The Bagchi–Datta triangulation

There is another 10-vertex triangulation  $\mathcal{T}_{BD}$  of  $\mathbb{CP}^2$ , which is obtained as a  $\mathbb{Z}/2$ -quotient of a triangulation of  $S^2 \times S^2$ . It is described in [BD11]. The automorphism group is the alternating group  $A_4$ . The f-vector is (10, 45, 110, 120, 48).

The triangulation is bistellarly equivalent to both the 9-vertex triangulation and the 10-vertex triangulation above.

**Proposition 2.3.5.** *We have that  $\dim_{\mathbb{C}} T_{A(\mathcal{T}_{BD}),0}^1 = 41$  and  $\dim_{\mathbb{C}} T_{A(\mathcal{T}_{BD}),0}^2 = 180$ . The normal sheaf has  $\mathcal{N}_{\mathbb{P}(\mathcal{T}_{BD})/\mathbb{P}^3}$  has 131 global sections.*

We have not been able to find any meaningful lifting of the first-order deformations here either.

## 2.4 Naïve attempt to degenerate

On the other hand, degenerating the ideal of  $F_1(X) \subset \mathbb{P}^N$  to a square-free monomial ideal should give a triangulation of  $\mathbb{CP}^2$ . Since  $F_1(X)$  sits inside  $\mathbb{G}(1, 5)$ , and there are many known degenerations of  $\mathbb{G}(1, 5)$ , we hoped that maybe  $F_1(X)$  would generate inside  $\mathbb{G}(1, 5)$ . Unfortunately, we did not succeed, mainly because we could not see any structure in the ideal of  $F_1(X)$ .

It was possible to explicitly compute  $F_1(X)$  for some hypersurfaces, both pfaffian and non-pfaffian. However, the ideals were too complicated and the Gröbner bases too big to find any initial ideals with only squarefree generators.

## 2.5 Conclusion

It would be interesting to study other triangulations of  $\mathbb{CP}^2$ . One way to proceed would be to start with existing triangulations, and analyze which parts of it correspond to non-zero elements of the  $T^2$  module. Then one can do

bistellar flips away from these combinations, ideally obtaining triangulations corresponding to unobstructed Stanley–Reisner schemes.

This is an interesting and very hard question. Even with an unobstructed triangulation, it is not clear how to proceed to smooth it in a computationally feasible way. Already with Gröbner bases with 50 elements (for deformations of the 15-vertex triangulation, they had around 70 elements), computations take far too long (and consumes too much memory) to be feasible to work with.

Without the presence of any good parallel processing Gröbner basis algorithms (which would allow the use of clustered super-computers), there is need for either patience or smarter solutions to computational algebra problems.



# CHAPTER 3

## The two smoothings of $C(\mathrm{dP}_6)$

smoothings

In this chapter we study the toric singularity that is the cone over the del Pezzo surface of degree 6. it has two topologically different smoothings, which we haven't seen studied in some detail before.

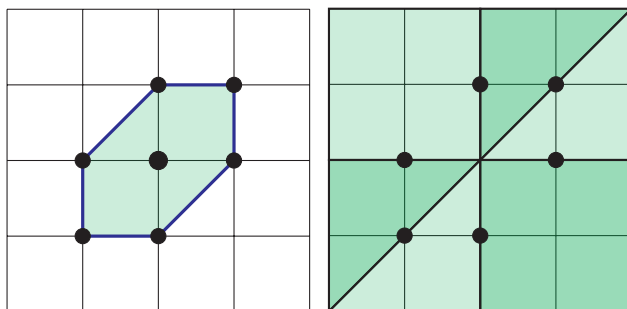
smoothings

### 3.1 The del Pezzo surface $\mathrm{dP}_6$

We start this chapter by talking about the del Pezzo surface in some generality.

Denote by  $\mathrm{dP}_6$  the blow-up of  $\mathbb{P}^2$  in three non-collinear points. These points can be chosen to be the coordinate points  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$  and  $(0 : 0 : 1)$ . Since the coordinate points are invariant under the natural torus action on  $\mathbb{P}^2$ , it follows that the  $\mathrm{dP}_6$  is a toric variety.

As a toric variety, it can be described as toric variety of the planar hexagon depicted in Figure 3.1(a). The normal fan is in Figure 3.1(b)



(a) The hexagon corresponding to  $\mathrm{dP}_6$ . (b) The fan over the polar polytope.

Figure 3.1: Toric description of  $\mathrm{dP}_6$ .

### 3. The two smoothings of $C(\mathrm{dP}_6)$

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We will spend some time describing the different embeddings of  $\mathrm{dP}_6$ . Different embeddings give rise to different smoothings of the affine cones.

#### Embedding in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Since  $\mathrm{dP}_6$  is the blowup of  $\mathbb{P}^2$  in three points, we can blow them up separately. Let  $x_0, x_1, x_2$  be coordinates of  $\mathbb{P}^2$ . Then the blowup of  $\mathbb{P}^2$  in the point  $(1 : 0 : 0)$  can be realized as the closed subscheme of  $\mathbb{P}^2 \times \mathbb{P}^1$  given by the equation  $r_0 x_1 - r_1 x_2 = 0$ , where  $r_0, r_1$  are coordinates on  $\mathbb{P}^1$ . We can repeat this procedure on the two other points  $(0 : 1 : 0)$  and  $(0 : 0 : 1)$  to obtain similar equations. Collecting these, we see that  $\mathrm{dP}_6$  is given by the matrix equation

$$M\vec{x} = \begin{pmatrix} 0 & r_0 & -r_1 \\ s_1 & 0 & -s_0 \\ -t_0 & t_1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0.$$

in  $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . Since  $\vec{x}$  is non-zero, it follows that we must have  $\det M = 0$ . It is not difficult to see that  $M$  cannot have rank 1 or lower, because that would force some of the  $\mathbb{P}^1$ -coordinates to be zero. Consider the projection forgetting the  $\mathbb{P}^2$ -factor:

$$\pi : \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Consider the hypersurface  $\det M = 0$  in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . Any solution to this equation gives a unique solution to the equation  $M\vec{x} = 0$ , meaning that the restriction of  $\pi$  to  $\mathrm{dP}_6$  is an isomorphism onto the hypersurface  $\det M = r_0 s_0 t_0 - r_1 s_1 t_1 = 0$ .

It is also interesting to see how this embedding arises from a toric perspective using polytopes. Since  $\mathbb{P}^1$  is the toric variety associated with the interval  $[-1, 1] \subset \mathbb{R}$ , it follows that  $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  is the toric variety associated with the cube  $\Delta = [-1, 1]^3 \subset M_{\mathbb{R}} = \mathbb{R}^3$ . The inclusion of  $\mathrm{dP}_6$  in  $M$  induces a surjection of coordinate rings  $\mathbb{C}[M] \rightarrow \mathbb{C}[\mathrm{dP}_6]$ . This correspond to the fact that there is a lattice projection of the cube onto the hexagon. See Figure 3.2.

Conversely, if  $N_1$  is the fan of  $\mathrm{dP}_6$ , and  $N_2$  is the fan of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ , we have an inclusion of lattices  $N_1 \hookrightarrow N_2$ , which is induced by an inclusion of convex polytopes, as in Figure 3.3.

The inclusion  $N_1 \hookrightarrow N_2$  is given by matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \tag{3.1} \quad \boxed{\{\text{eq:A}\}}$$

Note that there are essentially four embeddings of the hexagon into the octahedron, because each embedding is given by choosing a line through opposite faces (the line spanned by the normal vector of the hexagon), and there are 8 faces, hence 4 lines through opposite faces.



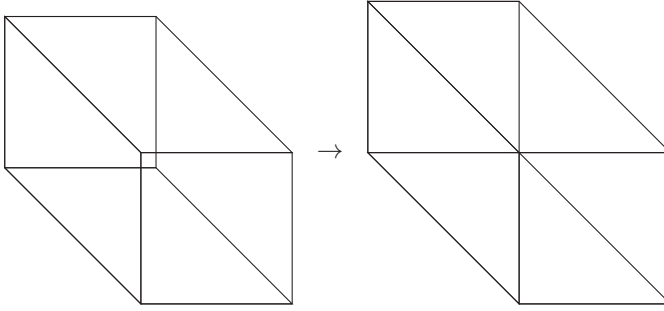


Figure 3.2: The projection of a cube onto a hexagon.

fig:cube\_project

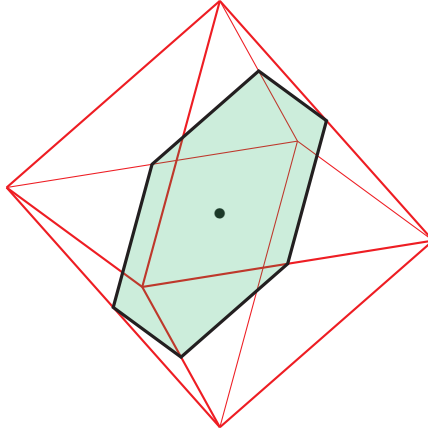


Figure 3.3: The inclusion of a cube in an octahedron.

fig:octahedron\_

### Embedding in $\mathbb{P}^2 \times \mathbb{P}^2$

On the other hand, blowups can also be realized as closures of graphs of rational maps. Let  $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$  be the Cremona transformation given by  $(x_0 : x_1 : x_2) \mapsto \left( \frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2} \right)$ . Let  $\Gamma \subset \mathbb{P}^2 \times \mathbb{P}^2$  be the graph of  $\varphi$ . Then, in coordinates  $(a_0 : a_1 : a_2) \times (b_0 : b_1 : b_2)$  on  $\mathbb{P}^2 \times \mathbb{P}^2$ , the equations  $a_0 b_0 = a_1 b_1 = a_2 b_2$  hold on  $\Gamma$ . These are the equations of the blowup along the indeterminacy locus of the rational map  $\varphi$ . The indeterminacy locus is exactly the three coordinate points. Hence  $dP_6$  can also be realized as the intersection of two  $(1, 1)$ -divisors in  $\mathbb{P}^2 \times \mathbb{P}^2$ .

There is also in this case a description in terms of polytopes. The polytope associated with  $\mathbb{P}^2 \times \mathbb{P}^2$  is  $\Delta^2 \times \Delta^2$ , the product of two two-simplices. Also in

### 3. The two smoothings of $C(\mathrm{dP}_6)$

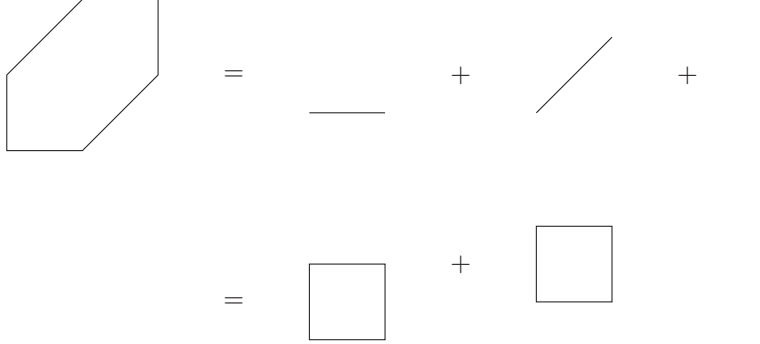


Figure 3.4: Minkowski-decompositions of the hexagon.

fig:minkow

this case, there is a projection onto a hexagon in  $\mathbb{R}^2$ . This is harder to visualize, but can be described as follows: if we order the vertices of  $\Delta^2$  by  $v_1, v_2, v_3$ , then the vertices of  $\Delta^2 \times \Delta^2$  are of the form  $(v_i, v_j)$ . The projection is then given by identifying the vertices  $(v_i, v_i)$ .

Hence, using the Segre embedding,  $\mathrm{dP}_6$  lives naturally in both  $(\mathbb{P}^1)^3 \hookrightarrow \mathbb{P}^7$  and  $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ .

*Remark 3.1.1.* Intersecting  $\mathbb{P}^2 \times \mathbb{P}^2$  with a single  $(1, 1)$ -divisor gives us the projective space bundle corresponding to the tangent bundle of  $\mathbb{P}^2$ , which we denote by  $\mathcal{T}(\mathbb{P}^2)$ . This follows from the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_{\mathbb{P}^2}(1)^3 \rightarrow \mathcal{T}_{\mathbb{P}^2} \rightarrow 0.$$

Since  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1)^3) = \mathbb{P}^2$ ,  $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$  can be realized as the subset of  $\mathbb{P}^2 \times \mathbb{P}^2$  such that  $a_0b_0 + a_1b_1 + a_2b_2 = 0$ . The space  $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$  is a non-toric Fano variety.

## 3.2 The cone over $\mathrm{dP}_6$ and its two smoothings

The singularity  $Z \triangleq C(\mathrm{dP}_6)$  is one of the most studied singularities with an obstructed deformation space. For example, in the paper [Alt97], Klaus Altmann describe a method to study the versal deformations of isolated affine Gorenstein toric singularities using only the combinatorial data of the toric variety. He shows that different components of the base space correspond to different ways of writing the defining polytope as a Minkowski sum of other polytopes.

See the illustration in Figure 3.4 for a decomposition of the hexagon.

Let  $A = A(Z)$  denote the affine coordinate ring of  $C(\mathrm{dP}_6)$ . It has a natural  $\mathbb{Z}$ -grading. From Altmann's article, or by using `Macaulay2`, one computes that  $\dim T^1(A) = 3$ , and that  $\dim T^2(A) = 2$ . The versal base space decomposes into a union of a line and a plane.

It is worthwhile to note that both smoothings of  $Z$  arise by “sweeping out the cone”: if  $X$  is a projective variety in  $\mathbb{P}^n$ , and  $Y$  is equal to  $X \cap H$ , where  $H$  is a section of  $\mathcal{O}_{\mathbb{P}^n}(1)$ , then the affine cone over  $Y$  deforms to a general hyperplane section of the affine cone over  $X$ . See the introduction of [Ste03] for more details.

### Equations of smoothings

Using the Segre embedding of  $\mathbb{P}^2 \times \mathbb{P}^2$  and substituting from the linear equations in the description Section 3.1, we can write the equations of  $\mathrm{dP}_6$  inside  $\mathbb{P}^6$  as

$$\begin{vmatrix} y & x_1 & x_2 \\ x_3 & y & x_4 \\ x_5 & x_6 & y \end{vmatrix} \leq 1, \quad (3.2)$$

where  $\leq 1$ , means taking all  $2 \times 2$ -minors.

On the other hand,  $\mathrm{dP}_6$  can be realized as a subvariety of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  as well. The equations can be described as follows: draw a cube, and let each vertex correspond to a variable. Then the equations of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  in its Segre embedding are given by taking all “minors” along all sides of the cube together with the three long diagonals. See Figure 3.5. To get  $\mathrm{dP}_6$ , one identifies two opposite corners. Thus in total there are  $8 - 1 = 7$  variables, just as above.

The first smoothing is obtained by deforming the equations of  $\mathrm{dP}_6$  as a subvariety of  $\mathbb{P}^2 \times \mathbb{P}^2$ . It can be described by perturbing two of the entries of the matrix below:

$$\begin{vmatrix} y & x_1 & x_2 \\ x_4 & y + t_1 & x_3 \\ x_5 & x_6 & y + t_2 \end{vmatrix} \leq 1. \quad (3.3) \quad \boxed{\{\text{eq: def2}\}}$$

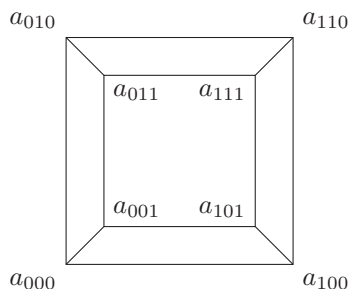


Figure 3.5: A  $2 \times 2 \times 2$ -tensor.

`fig:plp1p1_equat`

### 3. The two smoothings of $C(\mathrm{dP}_6)$

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For  $t_1 = t_2 = 0$ , we get the cone over  $\mathrm{dP}_6$ , while for generic  $t_i$ , we get a smooth variety. In fact, we can compute that the discriminant locus (the set of points in  $\mathbb{A}_{t_1, t_2}^2$  with singular fiber) are the  $t_1$ -axis, the  $t_2$ -axis and the line  $t_1 = t_2$ . Notice that the total space is equal to the cone over  $\mathbb{P}^2 \times \mathbb{P}^2$ .

Call (any) smooth fiber  $X_2$ .

**Lemma 3.2.1.** *Let  $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$  be the projective bundle associated to the tangent sheaf on  $\mathbb{P}^2$ . Then the smoothing  $X_2$  is isomorphic to  $M \setminus \mathrm{dP}_6$ .*

*Proof.* First homogenize the equations (3.3) with respect to  $y_1$ . Call the homogenized variety  $N$ . Put  $y'_0 = y_0$ ,  $y'_1 = y_0 - ty_1$  and  $y'_2 = y_0 - t_2y_1$ . Then we have the relation

$$h = t_2y'_1 - t_1y'_2 - (t_1 - t_2)y'_0 = 0.$$

Hence we see that  $N = \mathbb{P}^2 \times \mathbb{P}^2 \cap (h = 0)$ . We can pull back the coordinates  $y'_i$  to  $\mathbb{P}^2 \times \mathbb{P}^2$ . Let  $\mathbb{P}^2 \times \mathbb{P}^2$  have coordinates  $x_0, x_1, x_2$  and  $y_0, y_1, y_2$ . Then  $h$  pulls back to the equation

$$(x_0, x_1, x_2) \cdot (-t_1y_2, (t_1 - t_2)y_0, t_2y_1) = 0$$

in  $\mathbb{P}^2 \times \mathbb{P}^2$ . As long as  $t_1 \neq t_2$  and  $t_1, t_2 \neq 0$ , we can do a change of coordinates in  $\mathbb{P}_{y_0y_1y_2}^2$ , so that  $h$  transforms to

$$(x_0, x_1, x_2) \cdot (y_0, y_1, y_2) = 0.$$

Hence we see that  $M$  is isomorphic to the total space of the Grassmannian of lines in  $\mathbb{P}^2$  (each point in one of the  $\mathbb{P}^2$ 's give a line in the other  $\mathbb{P}^2$ ). This is in turn isomorphic to  $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ , since each tangent vector through a point determines a line through it.

Now, what have we gained by homogenizing? The divisor at infinity is  $y_1 = 0$ , which is a  $\mathrm{dP}_6$  again. In our new coordinates this is equivalent to  $y'_1 = y'_2 = y'_0$ . Hence in the coordinates of  $\mathbb{P}^2 \times \mathbb{P}^2$ ,  $\mathrm{dP}_6$  is given by the two equations  $x_1y_0 - x_2y_1 = x_1y_0 - x_0y_2 = 0$ . ■

The other smoothing is the obtained by replacing one of the corners of the cube in Figure 3.5 with  $a'_{000} = a_{000} + t$ , obtained a one-parameter smoothing. The total space is now that affine cone over  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ .

Call this smoothing  $X_1$ .

**Lemma 3.2.2.** *The smoothing  $X_1$  is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathrm{dP}_6$ .*

*Proof.* The proof is identical to the previous proof. ■

The following fact is well-known.

**Proposition 3.2.3.** *The two smoothings are topologically different.*

### 3.2. The cone over $\mathrm{dP}_6$ and its two smoothings

*Proof.* Observe that  $\mathcal{T}(\mathbb{P}^2)$  is homotopy equivalent to  $\mathbb{P}^1 \times \mathbb{P}^2$ . It follows that its Euler characteristic, which is invariant under homotopy, is equal to  $2 \times 3 = 6$ .

This information let us calculate the Euler characteristics of the smoothings. Note that  $\chi(\mathbb{P}^1) = 2$  and  $\chi(\mathcal{T}(\mathbb{P}^2)) = 6$ . By additivity of the Euler characteristics we have  $\chi(X_1) = 2$  and  $\chi(X_2) = 0$ , since  $\chi(\mathrm{dP}_6) = 6$ .

It follows that the two smoothing components correspond to topologically different smoothings, since the Euler characteristic is a topological invariant. ■

**Lemma 3.2.4.** *The cohomology ring of  $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$  is  $\mathbb{Z}[x, y]/(x^3, y^2 + c_1y + c_2)$ , where  $x$  and  $y$  have degree 2. In particular, the cohomology of  $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$  is given by  $(1, 0, 2, 0, 2, 0, 1)$ .*

*Proof.* The first claim follows from the Leray-Hirsch theorem. See [BT82, page 270]. The next claim follows since  $x$  and  $y$  both have degree 2. ■

We can use what we know about the topology of these spaces to compute homology groups of the two affine smoothings.

We first need a preliminary lemma from toric geometry, which we state and prove in general form, because we could not find a proper reference.

na:induced

**Lemma 3.2.5.** *Let  $Y \xrightarrow{i} X$  be an closed immersion of smooth toric varieties, corresponding to a map of fans  $\Sigma_1 \xrightarrow{A} \Sigma_2$ . Let  $M_1$  and  $M_2$  be the corresponding character lattices. Then we have a commutative diagram:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_2 & \xrightarrow{R_1} & \mathbb{Z}^{\Sigma_2(1)} & \longrightarrow & \mathrm{Pic}(X) \longrightarrow 0 \\ & & \downarrow A^T & & \downarrow C^T & & \downarrow i^* \\ 0 & \longrightarrow & M_1 & \xrightarrow{R_2} & \mathbb{Z}^{\Sigma_1(1)} & \longrightarrow & \mathrm{Pic}(Y) \longrightarrow 0 \end{array}$$

Where in addition  $i^* : \mathrm{Pic}(X) \rightarrow \mathrm{Pic}(Y)$  is the map of Picard groups induced by the closed embedding.

*Proof.* The vertical rows are well-known. See for example Theorem 4.1.3 in [CLS11].

The matrix  $C^T$  is defined as follows: each primitive ray generator of cones in  $\Sigma_1(1)$  thought of as lying in  $N_2$  via the embedding  $A$  has a unique description as a linear combination of rays in  $\Sigma_2(1)$  lying in the same minimal cone. Let the columns of  $C$  be the coefficients of this linear combination. Then, by definition, the first square commutes.

It follows that there is an induced map of Picard groups. We must show that the induced map is exactly the one induced by the closed embedding. To see this, we use Proposition 6.2.7 in [CLS11], where a description of this map is given in terms of convex functions on  $N$ .

source?

complete the proof

### 3. The two smoothings of $C(\mathrm{dP}_6)$

example:p1p1p1

**Example 3.2.6.** Let us see how we can use the Lemma to find an explicit form of the induced map of the inclusion  $\mathrm{dP}_6 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . We use the matrix  $A$  from Equation (3.1). The rows of  $R_1$  are the coordinates of the primitive ray generators of the fan of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ , and also the vertices of the octahedron in Figure 3.3.

The rows in  $R_2$  are the coordinates of the hexagon in Figure 3.1(a).

Since  $\mathrm{Pic}((\mathbb{P}^1)^3) = \mathbb{Z}^3$ , and  $\mathrm{Pic} \mathrm{dP}_6 = \mathbb{Z}^4$ , we need to compute splittings of  $\mathbb{Z}^6$  as  $\mathbb{Z}^6 = \mathbb{Z}^3 \oplus \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$  and  $\mathbb{Z}^6 = \mathbb{Z}^2 \oplus \mathrm{Pic}(\mathrm{dP}_6)$ , respectively.

This can be done explicitly by Gaussian elimination. We illustrate this with the first map:

$$\left( \begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \sim \left( \begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right)$$

We start with the matrix  $(R_1, I_6)$ , and after Gaussian elimination (row operations), we get the matrix  $(R'_1, B)$ . The last three rows of  $B$  give a map  $\pi_1 : \mathbb{Z}^6 \rightarrow \mathbb{Z}^3$  with kernel equal to the image of  $R_1$ . We do the same with the pair  $(R_2, I_6)$ .

We find that the induced map  $i^* : \mathbb{Z}^3 \rightarrow \mathbb{Z}^4$  is given by the matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We will use this information below in the proof of the next theorem.

This lemma let us compute explicitly what the induced map of Picard groups is using only the toric data.

**Theorem 3.2.7.** *The two affine smoothings are topologically different. The homology groups are:*

| Group                  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | Euler-characteristic |
|------------------------|---|---|---|---|---|---|---|----------------------|
| $H^i(X_1, \mathbb{Z})$ | 1 | 0 | 2 | 1 | 0 | 0 | 0 | 2                    |
| $H^i(X_2, \mathbb{Z})$ | 1 | 0 | 1 | 2 | 0 | 0 | 0 | 0                    |

*Proof.* The singular cohomology of  $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  is given by  $(1, 0, 3, 0, 3, 0, 1)$ , which can be computed by the Künneth formula (see [Hat02], page 275). The cohomology of  $\mathrm{dP}_6$  is given by  $(1, 0, 4, 0, 1)$ .

### 3.2. The cone over $dP_6$ and its two smoothings

We will use the Lefschetz duality theorem [Spa66], which in this case says that  $H_q(M \setminus dP_6; \mathbb{Z}) \simeq H^{6-q}(M, dP_6; \mathbb{Z})$ . The long exact sequence of the pair  $(M, dP_6)$  (Hatcher, page 200) takes the form:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^0(M, D; \mathbb{Z}) & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
 & & & \searrow \delta & & & \\
 & & H^1(M, D; \mathbb{Z}) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & \searrow \delta & & & \\
 & & H^2(M, D; \mathbb{Z}) & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^4 \\
 & & & \searrow \delta & & & \\
 & & H^3(M, D; \mathbb{Z}) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & \searrow \delta & & & \\
 & & H^4(M, D; \mathbb{Z}) & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z} \\
 & & & \searrow \delta & & & \\
 & & H^5(M, D; \mathbb{Z}) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & \searrow \delta & & & \\
 & & H^6(M, D; \mathbb{Z}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
 \end{array}$$

From the exactness of the sequence, we immediately find  $H^0(X_1; \mathbb{Z}) = \mathbb{Z}$ . Also, since  $H^0(M; \mathbb{Z}) \rightarrow H^0(dP_6; \mathbb{Z})$  is an isomorphism (both are connected), it follows that  $H^6(X_1; \mathbb{Z}) = H^5(X_1; \mathbb{Z}) = 0$ .

The other groups depend upon the explicit form of the maps  $H^2(M; \mathbb{Z}) \rightarrow H^2(dP_6; \mathbb{Z})$  and  $H^4(M; \mathbb{Z}) \rightarrow H^4(dP_6; \mathbb{Z})$ .

The map  $H^2(M; \mathbb{Z}) \rightarrow H^2(dP_6; \mathbb{Z})$  can be identified with the map  $i^* : \text{Pic}(M) \rightarrow \text{Pic}(dP_6)$  induced by the inclusion. This map was computed in Example 3.2.6. It is an injective map with torsion-free cokernel, and it follows from the long-exact sequence and the Lefschetz theorem that  $H_3(X_1; \mathbb{Z}) \simeq H^3(M, dP_6; \mathbb{Z}) \simeq \mathbb{Z}$ , and also that  $H_4(X_1; \mathbb{Z}) = 0$ .

Similarly, the map  $H^4(M) \rightarrow H^4(dP_6)$  is computed to be given by  $(a, b, c) \mapsto a + b + c$ , since the three  $\mathbb{P}^1$ 's intersect  $dP_6$  in a single point. This map has two-dimensional kernel, and we conclude that  $H_2(X_1) \simeq H^4(M, dP_6) = \mathbb{Z}^2$ , and that  $H^1(X_1) = 0$ .

The computations for  $X_2$  are similar but more involved. We first note that the Picard group of  $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$  is generated by the pullbacks  $F, G$  of the generators of  $\text{Pic}(\mathbb{P}_{x_0 x_1 x_2}^2 \times \mathbb{P}_{y_0 y_1 y_2}^2)$ . Say  $F$  is represented by  $V(x_0)$  and  $G$  is represented by  $V(y_0)$ .

Again we compute the intersections of  $F$  and  $G$  with  $dP_6$ . Intersecting with  $F$  is computed by decomposing the ideal  $(x_0, x_1 y_0 - x_2 y_1, x_1 y_0 - x_0 y_2)$  in  $k[x_0, x_1, x_2, y_0, y_1, y_2]$  and saturating by  $(x_0, x_1, x_2)$  and  $(y_0, y_1, y_2)$ . This can either be done by hand or by using **Macaulay2**. Either way, we find that  $F|_{dP_6} = E_3 + L_{23} + E_2 = H$ , using the notation from earlier this chapter.

check transversal intersection with  $dP_6$

### 3. The two smoothings of $C(\mathrm{dP}_6)$

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Similarly  $G|_{\mathrm{dP}_6} = L_{23} + L_{12} + E_2 = 2H - E_1 - E_2 - E_3$ . Hence the map on cohomology is given by the matrix

$$H^2(M, \mathbb{Z}) \simeq H_4(M, \mathbb{Z}) \simeq \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}} \mathbb{Z}^4 \simeq H_2(\mathrm{dP}_6, \mathbb{Z}) \simeq H^2(\mathrm{dP}_6, \mathbb{Z}).$$

This is an injective map, and as above, we conclude that  $H_3(X_2) \simeq H^3(M, \mathrm{dP}_6) \simeq \mathbb{Z}^2$ , and also that  $H_4(X_1) = 0$ . ■

*Remark 3.2.8.* In fact, the Andreotti-Frankel theorem [AF59] states the following: if  $V$  is any smooth affine variety of complex dimension  $n$ , then it has the homotopy type of a CW complex of real dimension  $n$ . Thus it should be no surprise that  $H^j(X_i) = 0$  for  $j > 3$ .



## CHAPTER 4

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# Construction of Calabi–Yau’s

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Instructions

In this chapter I describe the construction of three topologically different smoothings of a singular Calabi–Yau manifold. They correspond to different components of the Hilbert scheme of threefolds with Hilbert polynomial  $p(t) = 6t^3 + 6$  in  $\mathbb{P}^{11}$ .

We first describe a degenerate Calabi–Yau in the form of a Stanley–Reisner scheme  $\mathbb{P}(\mathcal{K})$ , which has a quite large symmetry group. There is a natural deformation to a  $X_Y$ , which is a subvariety of a toric variety  $Y$ , with isolated singularities.

We show that  $X_Y$  has several topologically distinct smoothings, which lie on different components of the Hilbert scheme in  $\mathbb{P}^{11}$ .

### 4.1 The special fiber

Let  $E_6$  be the hexagon as a simplicial complex. The associated Stanley–Reisner scheme  $\mathbb{P}(E_6)$  is a degenerated elliptic curve in  $\mathbb{P}^5$ . If  $\mathbb{P}^5$  have coordinates  $x_0, \dots, x_5$ , the equations of  $E_6$  are  $x_i x_{i+2} = x_i x_{i+3} = 0$ , where  $i$  is taken modulo 6. This gives a total of 9 quadratic generators.

**Lemma 4.1.1.** *The Hilbert polynomial of  $\mathbb{P}(E_6)$  is  $h(t) = 6t$ .*

*Proof.* We want to count the dimension of  $S_t = S_{E_6}(t)$ . Any monomial in  $S_k$  has support on the simplicial complex  $E_6$ , so its support is either a vertex or an edge. In the first case, the monomial has the form  $x_i^t$ , so there are six of these.

In the other case, it has the form  $x_i^a x_{i+1}^b$ , with  $a + b = t$  and  $a, b \neq 0$ . Counting, there are  $6(t - 1)$  of these monomials. In total, the dimension is  $6 + 6(t - 1) = 6t$ . ■

*Remark 4.1.2.* Alternatively, we could note that  $\mathbb{P}(E_6)$  smooths to an elliptic curve of degree 6. Since Hilbert polynomials are constant in flat families, it follows from Riemann–Roch that  $h(t) = \deg \mathcal{O}_{\mathbb{P}(E_6)}(6t) - 1 + 1 = 6t$ .

#### 4. Construction of Calabi–Yau's

Note that the Hilbert polynomial only differ from the Hilbert function for  $t = 0$ , since  $h(0) = 0$ , while  $\dim_{\mathbb{C}} S_0 = 1$ .

We now introduce the central fiber in the discussions onward. Let  $\mathcal{K}$  be the simplicial complex  $E_6 * E_6$ . It is a triangulation of the 3-sphere. The maximal faces are unions of maximal faces from each factor.

Denote the vertices of the left  $E_6$  with  $x_1, \dots, x_6$ , and the vertices of the right  $E_6$  with  $z_1, \dots, z_6$ . Then the maximal faces of  $\mathcal{K}$  are of the form  $x_i x_{i+i} z_j z_{j+1}$ , where  $i, j \in \mathbb{Z}_6$ . The number of  $i$ -faces follows are easy to compute:

**Lemma 4.1.3.** *The  $f$ -vector of  $\mathcal{K}$  is  $(1, 12, 48, 36)$ .*

*Proof.* There are 12 vertices, and  $6 \times 6 = 36$  maximal facets. Since  $\mathcal{K}$  is a 3-sphere, it follows that  $12 - f_1 + 36 = \chi(S^3) = 0$  that  $f_1 = 48$ .<sup>1</sup> ■

**Lemma 4.1.4.** *The Hilbert polynomial of  $\mathbb{P}(\mathcal{K})$  is  $h(t) = 6t^3 + 6$ .*

*Proof.* The homogeneous coordinate ring  $S = \bigoplus_{t \geq 0} S_t$  of  $\mathbb{P}(\mathcal{K})$  is the graded tensor product of  $\mathbb{P}(E_6)$  with itself. It follows from the previous lemma that

$$\dim S_t = \sum_{i+j=t, i,j \neq 0} 36ij + 12k,$$

where the last term is a correction term because  $h(t) \neq 1$ . It is now a routine computation using formulas for sums of squares to verify the claim. ■

Either by using `Macaulay2` or by using the more conceptual description of the  $T^i$  modules from [AC10], we can compute:

**Lemma 4.1.5.** *The dimensions of  $T^1(\mathcal{K})$  and  $T^2(\mathcal{K})$  are 84 and 72, respectively.*

*Proof.* We will prove this using the techniques and notation from [AC10]. Our goal is to compute the degree zero part of  $T_{A_{\mathcal{K}}}^1$ . We will do this using Theorem 1.6.2.

First notice that all links of vertices of  $\mathcal{K} = E_6 * E_6$  are double suspensions over hexagons (they are denoted by  $\Sigma E_6$  in Christophersen's article).

According to Table 1 in Christophersen's article, double suspensions over hexagons contribute with one dimension to  $T_{A_{\mathcal{K}}}^1$ , namely in degree  $x_i^2/x_{i-1}x_{i+1}$  (if  $\mathbf{a} = x_i^2$ ). In total there are  $6 + 6 = 12$  contributions of this form.

Taking the link at the vertex  $x_i z_j$  produces a square with vertices  $x_{i+1}, z_{j+1}, x_{i-1}, z_{j-1}$  (in that order). According to Table 1 in Christophersen's article, these links contribute with dimension 2 to  $T_{A_{\mathcal{K}}}^1$ . The contributions have degrees  $x_i z_j / x_{i+1} x_{i-1}$  and  $x_i z_j / z_{j+1} z_{j-1}$ . There are  $2 \cdot 6 \cdot 6 = 72$  contributions of this form.

Thus, in total,  $T_{A_{\mathcal{K}}}^1$  have  $\mathbb{C}$ -dimension 84.

<sup>1</sup>Here we used that in a cell complex, the Euler characteristic is also the alternating sum of the number of cells in each dimension. This is Theorem 2.44 in [Hat02].

We now compute  $T_{A\mathcal{K}}^2$ . The contributions come from choosing  $\mathbf{a} = x_i^2$  and  $\mathbf{a} = x_i x_{i+1}$ , respectively. If  $|a| = 1$  (as in the first case), the results from Christophersen's article imply that  $L_b := \cap_{b' \subset b} \text{lk}(b', \text{lk}(x_i, \mathcal{K}))$  must have more than one connected component (the contribution comes from  $\tilde{H}^0(L_b, \mathbb{C})$ ). This is the case if  $b$  consist of two opposite vertices in the suspended circle. In total there are  $2 \cdot 6 \cdot 3 = 36$  contributions of this form.

If  $|a| = 2$ , the contributing links are hexagons, and in this case the contributions come from  $b$  such that  $L_b = \emptyset$ . Again consistchoosing  $b$  to consist of opposite vertices of the hexagon, we find three pairs  $b$  with  $L_b = \emptyset$  for each hexagon. Thus in total there are  $2 \cdot 6 \cdot 3 = 36$  contributions of this form.

In sum,  $T_{A\mathcal{K}}^2$  is  $36 + 36$ -dimensional. ■

The automorphism group of  $\mathcal{K}$  is  $D_6 \times D_6 \times \mathbb{Z}_2$ , and of order  $12 \cdot 12 \cdot 2 = 288$ . It is not difficult to see that the induced action on  $T_{X_0}^1$  have two orbits under  $\text{Aut}(\mathcal{K})$ , corresponding to first order deformations of the form  $x_i x_{i-2} + t x_{i+1} z_j$  and  $x_{i-1} x_{i+1} + t x_i^2$ , respectively.

## 4.2 A natural toric deformation

Consider Figure 4.1. It is the 2-dimensional polytope associated to the del Pezzo surface of degree 6. The fan over this polytope correspond to a unimodular regular triangulation of the polytope, and it follows by Proposition 1.8.1, that  $\text{dP}_6$  degenerates to the Stanley–Reisner scheme  $\mathbb{P}(E_6 * \{pt\})$ , where  $\{pt\}$  correspond to the origin. This is an embedded deformation inside  $\mathbb{P}^6$ .

Concretely, the equations of  $\text{dP}_6$  are given by  $x_i x_{i+2} - y x_{i+1} = x_i x_{i+3} - y^2 = 0$ . The degeneration to  $\mathbb{P}(E_6 * \{pt\})$  is given by setting the second terms to zero.

Now form the join of two copies of  $\text{dP}_6$ , to get a new variety  $Y \subset \mathbb{P}^{13}$ . By Proposition 1.2.2, this is a  $2 + 2 + 1 = 5$ -dimensional toric variety with singular locus consisting of two copies of  $\text{dP}_6$ . Since the coordinate ring is just

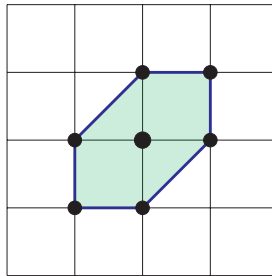


Figure 4.1: A hexagon.

fig:hexagon

## 4. Construction of Calabi–Yau's

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the tensor product of two copies of  $S(\mathrm{dP}_6)$ , it follows that  $Y$  degenerates to  $\mathbb{P}(E_6 * \{pt\} * E_6 * \{pt\}) = \mathbb{P}(\mathcal{K} * \Delta^1)$ .

The following holds:

**Proposition 4.2.1.** *There is a deformation of the Stanley–Reisner scheme  $X_0$  to an irreducible Calabi–Yau variety  $X_Y \subset Y$  with isolated singularities. There are twelve of them, and they are locally isomorphic to cones over del Pezzo surfaces. More precisely: let  $(U, p_i)$  be the germ of  $X_0$  at  $p_i$ . Then  $(U, p_i) \simeq (C(\mathrm{dP}_6), 0)$ .*

*Proof.* Since  $X_0$  is a complete intersection side  $\mathbb{P}(\mathcal{K} * \Delta^1)$ , it follows that  $X_0$  deforms to a complete intersection inside any deformation of  $\mathbb{P}(\mathcal{K} * \Delta^1)$ . We explained above that  $\mathbb{P}(\mathcal{K} * \Delta^1)$  deforms to the join  $Y$  of two del Pezzo surfaces, and it follows that  $X_0$  deforms to  $Y$  intersected with two generic hyperplanes.

Since  $Y$  has singular locus of dimension 2 and degree  $6 + 6 = 12$ , it follows by Bertini's theorem [Har77, Chapter II, Theorem 8.18] that  $X_0$  has twelve isolated singularities  $p_i$ .

To see how the singularities look locally, we argue as follows. Locally,  $Y$  looks like  $\mathbb{A}_{a_1, a_2}^2 \times C(\mathrm{dP}_6)_{x_i}$ , where the subscripts refer to the coordinates. This is the ideal of  $Y$  consists of two sets of equations, each defining a smooth toric variety, and smooth toric varieties are isomorphic to  $\mathbb{A}^d$  in affine charts.

The claim now follows from two applications of Theorem 3.1.5 in [Bat94], which says that the singularities on  $\Sigma$ -regular toric hypersurfaces are inherited from the ambient toric variety. ■

Since the cone over  $\mathrm{dP}_6$  deforms in two topologically different ways, we might expect that  $X_Y$  does this too. This is indeed true.

### 4.3 Smoothings of $X_Y$

By embedding  $\mathrm{dP}_6$  in different spaces, we obtain different smoothings of subvarieties of the join of these spaces.

#### The block matrix construction

We are inspired by the construction in Rødland's thesis [Rød00].

Let  $E$  be a 3-dimensional vector space. Let  $\{e_1, e_2, e_3\}$  be a basis for  $E$ . Then we can form the vector space  $V = (E \otimes E) \oplus (E \otimes E)$ , which has dimension 18. Let  $\mathbb{P}^{17} = \mathbb{P}(V)$ . Choose coordinates  $x_1, \dots, x_{18}$  on  $\mathbb{P}^{17}$ .

Thinking of  $E \otimes E$  as  $3 \times 3$ -matrices, we can think of the elements of  $\mathbb{P}^{17}$  as pairs of  $3 \times 3$ -matrices up to scalar, not both zero. Concretely, two pairs of matrices  $(A', B')$  and  $(A, B)$  are equivalent if  $(A', B') = (\lambda A, \lambda B)$  for some  $\lambda \in \mathbb{C}^*$ .

There is a natural rational map  $\pi : \mathbb{P}^{17} \dashrightarrow \mathbb{P}^8 \times \mathbb{P}^8$ , which is the identity on coordinates, given by dividing out by the antidiagonal  $\mathbb{C}^*$ -action:  $\lambda' \cdot (A, B) = (\lambda', \lambda'^{-1}B)$ .

Denote by  $V_1$  and  $V_2$  the subspaces  $x_1 = \dots = x_9 = 0$  and  $x_{10} = \dots = x_{18} = 0$ , respectively. Blow up  $\mathbb{P}^{17}$  in  $V_1 \cup V_2$ , to get  $\widetilde{\mathbb{P}^{17}}$ . The spaces  $V_i$  are exactly the indeterminacy locus of  $\pi$ , so  $\pi$  extends to a map  $\pi : \widetilde{\mathbb{P}^{17}} \rightarrow \mathbb{P}^8 \times \mathbb{P}^8$ . Denote by  $\pi_1$  and  $\pi_2$  the two natural projections to  $\mathbb{P}^8$ . Then it is true that  $\widetilde{\mathbb{P}^{17}} = \mathbb{P}_{\mathbb{P}^8 \times \mathbb{P}^8}(\pi_1^* \mathcal{O}_{\mathbb{P}^8}(1) \oplus \pi_2^* \mathcal{O}_{\mathbb{P}^8}(1)) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^8 \times \mathbb{P}^8} \oplus \mathcal{O}_{\mathbb{P}^8 \times \mathbb{P}^8}(1, -1))$ . This is explained further in Section C7 in [AK75].

Let  $M$  be the closure of the set of pairs  $(A, B)$  where  $\text{rank } A = \text{rank } B = 1$ .

prop:m

**Proposition 4.3.1.** *The variety  $M$  is the join of two copies of  $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ , and has singular locus  $\mathbb{P}^2 \times \mathbb{P}^2 \subset V_i$  of dimension 4.*

*The canonical sheaf is  $\omega_M = \mathcal{O}_M(-6)$ , so that  $M$  is a Fano toric variety.*

*Proof.* If  $\mathbb{P}^{17}$  have coordinates  $x_1, \dots, x_{18}$ , let  $M_1$  and  $M_2$  be the matrices

$$M_1 = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} \\ x_{16} & x_{16} & x_{17} \end{pmatrix}.$$

Then  $M$  is defined by the zeroes of the  $2 \times 2$ -minors of  $M_1$  and  $M_2$ . Then it is clear that  $M$  is the projective join of two copies of  $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8 \subset \mathbb{P}^{17}$ , since the variable sets are disjoint.

The variety  $M$  is 9-dimensional: the affine cone over  $M$ ,  $C(M)$ , is equal to  $C(\mathbb{P}^2 \times \mathbb{P}^2) \times C(\mathbb{P}^2 \times \mathbb{P}^2)$ . This variety has dimension  $5 + 5 = 10$ , hence its projectivization  $M$  is 9-dimensional.

The singular locus of  $M$  consists of the pairs  $(0, B)$ , and  $(A, 0)$ , where  $\text{rank } A = \text{rank } B = 1$ , hence  $\dim \text{Sing } M = 4$ . See also Proposition 1.2.2.

By Remark 1.2.4, it follows that  $\omega_M = \mathcal{O}_M(-6)$ , since

$$\omega_{\mathbb{P}^2 \times \mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^8}(-3)|_{\mathbb{P}^2 \times \mathbb{P}^2}.$$

■

Here comes our first construction. Let  $X_1$  be the intersection of  $M$  with a generic  $\mathbb{P}^{11}$ . Then the following is true.

prop:x1

**Proposition 4.3.2.**  *$X_1$  is a smooth Calabi–Yau variety with  $\chi(X_1) = -72$ .*

*Proof.* The singularities of  $M$  are of dimension 4. By Bertini’s theorem, intersecting  $M$  with a codimension 6 hyperplane gives a smooth variety  $X_1$ .

To see that  $X_1$  is Calabi–Yau, we use the adjunction formula, which in this case takes the form

$$\omega_{X_1} = \omega_M \otimes \wedge^6 \mathcal{O}_X(1)^{\oplus 6} = \mathcal{O}_X(-6) \otimes \mathcal{O}_X(6) = \mathcal{O}_X,$$

#### 4. Construction of Calabi–Yau's

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showing that the canonical sheaf is trivial.

To find the topological Euler characteristic, we compute in **Macaulay2**. Since computing the whole cotangent sheaf of  $X_1$  is impossible with current computer technology<sup>2</sup>, we make use of standard exact sequences. Let  $\mathcal{I}$  be the ideal sheaf of  $M$  in  $\mathbb{P}^{17}$ . First off, we have the exact sequence

$$0 \rightarrow \mathcal{I}/\mathcal{I}^2|_{X_1} \rightarrow \Omega_{\mathbb{P}}^1|_{X_1} \rightarrow \Omega_M^1|_{X_1} \rightarrow 0.$$

The restriction to  $X_1$  is exact since  $\mathcal{I}/\mathcal{I}^2$  is locally free on the smooth locus.

The **Macaulay2** command **eulers** computes the Euler characteristics of generic linear sections of a sheaf  $\mathcal{F}$  (behind the scene, this is equivalent to computing the Koszul resolution of the relative ideal sheaf  $\mathcal{I}_{X_1/M}$ ). Using this command, we find that  $\chi(\mathcal{I}/\mathcal{I}^2|_{X_1}) = -180$ . Using the exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}}^1|_{X_1} \rightarrow \mathcal{O}_{X_1}(-1)^{18} \rightarrow \mathcal{O}_{X_1} \rightarrow 0,$$

we find that the Euler characteristic of  $\Omega_{\mathbb{P}}^1|_{X_1}$  is  $-216 = 12 \cdot 18$ . It follows from the first exact sequence that  $\Omega_M^1|_{X_1}$  has Euler characteristic  $-36$ .

Since  $X_1$  is a complete intersection in  $M$ , the conormal sequence is

$$0 \rightarrow \mathcal{O}_{X_1}(-1)^6 \rightarrow \Omega_M|_{X_1} \rightarrow \Omega_{X_1}^1 \rightarrow 0.$$

Hence  $\chi(\Omega_{X_1}^1) = -36 + 72 = 36$ .

It follows that the topological Euler characteristic is  $\chi_{X_1} = -2\chi(\Omega_{X_1}^1) = -72$ . ■

*Remark 4.3.3.* We can give explicit equations for a flat family with special fiber  $X_Y$  and general fiber  $X_1$ . Let  $y_0 - h_1$  and  $y_1 - h_2$  be the generic linear forms in  $\mathbb{P}^{13}$  defining  $X_Y$  as a subscheme of  $Y$ . Let  $g_i$  (for  $i = 1, \dots, 6$ ) be generic linear forms in  $\mathbb{P}^{17}$ . Then such a flat family is defined by the  $2 \times 2$ -minors of the two matrices below:

$$A_1 = \begin{pmatrix} h_1 + tg_1 & x_2 & x_3 \\ x_4 & h_1 + tg_2 & x_6 \\ x_7 & x_8 & h_1 + tg_3 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} h_2 + tg_4 & x_{11} & x_{12} \\ x_{13} & h_2 + tg_5 & x_{15} \\ x_{16} & x_{16} & h_2 + tg_6 \end{pmatrix}.$$

For  $t = 0$ , we get  $X_Y$ .

*Remark 4.3.4.* Since  $X_1$  avoids  $V_1 \cup V_2$ , the inverse image  $\pi^{-1}(X_1) \subset \widetilde{\mathbb{P}^{17}}$  is isomorphic to  $X_1$ . Thus we can realize  $X_1$  as a subvariety of a *smooth* variety. Unfortunately,  $X_1$  is cut out by non-ample divisors in  $\widetilde{\mathbb{P}^{17}}$ , making computations there just as hard.

---

<sup>2</sup>An external computer has been trying to compute this sheaf for a few months now without terminating.

*Remark 4.3.5.* I have not been able to compute the Hodge numbers of  $X_1$ . However, counting parameters, we can give a conjectural size of  $H^1(X, \mathcal{T}_{X_1})$ , which is the space of complex structures on  $X_1$ :

$X_1$  lies in a family parametrized by planes containing twelve  $3 \times 3$  block matrices (spanning the  $\mathbb{P}^{11}$ ), giving  $12 \cdot 18 = 216$  parameters. These matrices correspond to elements of  $V = (E \otimes E)^{\oplus 2}$ . There is an action of the group  $\prod_{i=1}^4 \mathrm{GL}(E)$  on  $V$ , reducing the amount of parameters by  $4 \dim \mathrm{GL}(V) = 36$ . Furthermore, rotation in  $\mathbb{P}^{11}$  reduces the dimension by  $H^0(\mathbb{P}^{11}, \mathcal{T}) = 12^2 - 1 = 143$ . In total, we have  $216 - 36 - 143 = 37$  complex parameters. Since the Euler characteristic is  $-72$ , heuristically, the Hodge numbers should be  $(1, 37)$ .

This is very interesting, for the following reason: in the article [Kap15], the author produces list of Calabi–Yau manifolds with  $\mathrm{Pic} X = \mathbb{Z}$ , among which there are some that are only conjectured to exist. There is exactly one variety in that list of degree 36, and Euler-characteristic  $-72$ , and it has Hodge numbers  $(1, 37)$ . It is conjectured to exist based on the conjecture that to every differential equation of “Calabi–Yau type”, there should exist a Calabi–Yau variety having that equation as its Picard–Fuchs-equation (see [ES06]).

### The three-tensor construction

The construction in the previous section used the embedding of  $dP_6$  in  $\mathbb{P}^2 \times \mathbb{P}^2$  to deform  $X_Y$ . There is also the embedding of  $dP_6$  in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  to exploit. The construction is similar.

Let  $F$  be a 2-dimensional vector space with basis  $\{f_1, f_2\}$ . Then we can form the vector space  $V = ((F \otimes F \otimes F)^{\oplus 2})$ . Let  $\mathbb{P}^{15} = \mathbb{P}(V)$ . Choose coordinates  $a_{ijk} = (f_i \otimes f_j \otimes f_j, 0)$  and  $b_{ijk} = (0, f_i \otimes f_j \otimes f_k)$  ( $i, j, k = 0, 1$ ) for  $\mathbb{P}^{15}$ .

The elements of  $\mathbb{P}^{15}$  are pairs  $(A, B)$  of  $2 \times 2 \times 2$  tensors, not both zero. There is also in this case a natural map  $\pi : \mathbb{P}^{15} \rightarrow \mathbb{P}^7 \times \mathbb{P}^7$ , given by dividing out by the antidiagonal  $\mathbb{C}^*$ -action. Just as above, let  $V_1$  and  $V_2$  be the subspaces  $A = 0$  and  $B = 0$ , respectively. Let  $\widetilde{\mathbb{P}^{15}}$  be the blowup of  $\mathbb{P}^{15}$  in  $V_1 \cup V_2$ . The  $V_i$ ’s are exactly the indeterminacy locus of  $\pi$ , so  $\pi$  extends to a morphism  $\pi : \widetilde{\mathbb{P}^{15}} \rightarrow \mathbb{P}^8 \times \mathbb{P}^8$ , which is a  $\mathbb{P}^1$ -bundle. Also in this case it is true that  $\widetilde{\mathbb{P}^{15}} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^7 \times \mathbb{P}^7} \oplus \mathcal{O}_{\mathbb{P}^7 \times \mathbb{P}^7}(1, -1))$ .

Let  $N$  be the closure of set of pairs  $(A, B)$  where both  $A$  and  $B$  have tensor rank 1<sup>3</sup>.

**Proposition 4.3.6.** *The variety  $N$  is the join of two copies of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ , and has singular locus  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset V_i$  of dimension 3.*

*The canonical sheaf is  $\omega_N = \mathcal{O}_N(-4)$ , so that  $N$  is a Fano toric variety.*

<sup>3</sup>An element of  $F^{\otimes 3}$  have rank 1 if it is a pure tensor. It has rank  $\leq k$  if it can be written as a sum of  $k$  pure tensors.

#### 4. Construction of Calabi–Yau's

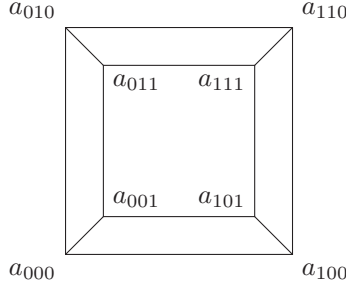


Figure 4.2: A  $2 \times 2 \times 2$ -tensor, seen from “above”.

fig:222ten

*Proof.* A pure  $2 \times 2 \times 2$ -tensor can be visualized as a cube with vertices  $a_{ijk}$ . See the diagram in Figure 4.2.

The equations of the set of rank 1 tensors in  $\mathbb{P}(F \otimes F \otimes F)$  are obtained as the “minors” along the 6 sides of the cube, together with the minors along with the 3 long diagonals, giving a total of 9 binomial equations. We write this symbolically as  $[a_{ijk}] \leq 1$ .

Hence the equations for  $N$  are given by  $[a_{ijk}] \leq 1$ , together with  $[b_{ijk}] \leq 1$ . Since these are equations in a disjoint set of variables, it is clear that  $N = (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)^{*2}$ .

The claim about the singular locus and the canonical sheaf follow as in the proof of Proposition 4.3.1. ■

Let  $X_2$  be the intersection of  $N$  with a general  $\mathbb{P}^{11}$ .

**Proposition 4.3.7.** *The topological Euler characteristic of  $X_2$  is  $-48$ .*

*Proof.* The proof is identical to the proof of Proposition 4.3.2. ■

*Remark 4.3.8.* A heuristic moduli count works also in this case. We argue slightly differently in this case.

$X_2$  lies in a plane spanned by 12-plane in  $\mathbb{P}((F \otimes F \otimes F)^{\oplus 2})$ . Such a plane are parametrized by  $\mathbb{G}(12, 16)$ , the Grassmannian of 12-planes in  $k^{16}$ . This space is  $12 \cdot (16 - 12) = 48$ -dimensional. There is an action of the group  $\prod_{i=1}^6 \mathrm{GL}(F)$  on  $(F \otimes F \otimes F)^{\oplus 2}$ . However, the diagonal subgroup  $\mathbb{C}^*$  acts trivially, so we really have an action of  $\prod_{i=1}^6 \mathrm{GL}(F)/\mathbb{C}^*$  on  $\mathbb{P}^{15}$ . This group is  $4 \cdot 6 - 1 = 23$ -dimensional. Thus in total we have  $48 - 23 = 25$  moduli parameters.

Since we know the Euler characteristic, we predict the Hodge numbers to be  $(1, 25)$ .



### The mixed smoothing

In the above cases, we formed the join of equal varieties. We mix things up: let  $V = (E \otimes E) \oplus (F \otimes F \otimes F)$ . Then let  $\mathbb{P}^{16} = \mathbb{P}(V)$ .

Now let  $W$  be the set of “mixed” rank 1 tensors. In a way similar to above, we find that  $W$  is a singular Fano toric variety of dimension 8. The singular locus is of dimension 4, so a 5-fold complete intersection is again a smooth Calabi-Yau variety  $X_3$ .

**Proposition 4.3.9.** *The Euler characteristic of  $X_3$  is  $-60$ .*

*Proof.* The proof is identical to the proofs above. ■

*Remark 4.3.10.* The heuristic above gives  $(h^{11}, h^{12}) = (1, 31)$ .

## 4.4 Degeneration of the toric join

The coordinate ring of the toric variety  $Y$  can be described as  $R = S \otimes_k S$ , where  $S$  is the coordinate ring of the del Pezzo surface of degree 6 in its anticanonical embedding.  $S$  has a special element  $y_0$  corresponding to the origin of the defining polytope. Concretely,  $R$  can be described as follows: it is a quotient of the polynomial ring  $\mathbb{C}[x_1, \dots, x_6, y_0, z_1, \dots, z_6, y_1]$ . We then mod out by the ideal  $I_1 + I_2$ , where  $I_i$  ( $i = 1, 2$ ) is the ideal of the del Pezzo surface in the variables  $(x_i, y_0$  and  $z_i, y_1$ , respectively). Then  $Y = \text{Proj}(\mathbb{C}[x_1, \dots, x_6, y_0, z_1, \dots, z_6, y_1]/(I_1 + I_2))$ .

Consider the hypersurface  $Y'$  in  $Y$  given by  $y_0 = y_1$ . Then  $Y'$  is a 4-dimensional toric variety. It is the toric variety associated to the polytope  $\Delta$  with vertices the columns of the matrix

$$\begin{pmatrix} -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \end{pmatrix}.$$

A computation shows that  $Y'$  has 1-dimensional singularities, and the singular locus is a graph of  $\mathbb{P}^1$ 's: take two hexagons, and join each vertex of one of them with all vertices of the other one. This makes in total 48  $\mathbb{P}^1$ 's.

illustration  
graph

The variety  $Y'$  is a Fano toric variety, and as such, it has a anticanonical section  $X_{Y'}$ , which is a singular Calabi-Yau variety. A local computation shows that  $X_{Y'}$  has 12 singularities that are locally isomorphic to  $C(\text{dP}_6)$ , and 36 double points.

Since  $Y'$  is a four-fold, it follows that  $X_{Y'}$  has a maximal projective crepant resolution of singularities (a *MPCP-desingularization*) (see for example the comment on page 55 in [CK99]), which we denote by  $\widetilde{X}_{Y'}$ .

## 4. Construction of Calabi–Yau's

A computation using PALP [KS04] shows that  $\widetilde{X_{Y'}}$  has Hodge numbers  $(44, 8)$  and Euler-characteristic 72.

*Remark 4.4.1.* The variety  $X_{Y'}$  has also been described elsewhere. The polar polytope  $\Delta^\circ$  is equal to the product of two hexagons, and it follows that  $\mathbb{P}_{\Delta^\circ}$  is equal to the product of two del Pezzo surfaces. An anticanonical hypersurface in  $dP_6 \times dP_6$  has Euler characteristic  $-72$  (see Theorem 3.1 in [Hüb92]).

In the article [BCD10], Braun et al. study this hypersurface and a group action on it. They also describe a resolution of singularities of  $X_{00}$ .

*Remark 4.4.2.* There is a heuristic surgical reason for the Euler characteristic being  $+72$ . Our  $X_{Y'}$  deforms to  $X_1$ , which has Euler characteristic  $-72$ . This is obtained by smoothing 36 double points and 12 cones over del Pezzo surfaces. By the inclusion-exclusion principle, it follows that a resolution of the singularities of  $X_{Y'}$  have Euler characteristic  $\chi(X_1) + 2 \cdot 36 + 6 \cdot 12 = 72$ .

connection to  
physics paper,  
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### 4.5 Invariant Calabi–Yau's

In this section, I will explain natural group actions on the  $X_i$ 's constructed above.

Denote by  $D_6$  the dihedral group of order 12, the symmetries of a hexagon. It is generated by a rotation  $\rho$  of order 6, together with a reflection  $\sigma$ , subject to  $\sigma\rho\sigma = \rho^{-1}$ . There is an isomorphism  $D_6 \simeq S_3 \times \mathbb{Z}_2$ :  $S_3$  is identified with  $\langle \rho^2, \sigma \rangle$  and  $\mathbb{Z}_2$  is identified with  $\rho^3$ .

**Lemma 4.5.1.** *There are  $D_6$ -actions on both  $M$  and  $N$ .*

*Proof.* Recall that  $M$  is the join of two copies of  $\mathbb{P}^2 \times \mathbb{P}^2$  embedded in  $\mathbb{P}^8$ . We can think of  $M$  as block matrices of rank  $1 + 1$  in  $\mathbb{P}(E \otimes E \oplus E \otimes E)$ , where  $E$  is a 3-dimensional vector space. Choosing a basis  $\{e_1, e_2, e_3\}$  for  $E$ , we have a natural  $S_3$  action on  $E$  given by  $e_i \mapsto e_{\sigma(i)}$ . This action extends to  $E \otimes E$  by  $v \oplus w \mapsto g \cdot v \oplus g \cdot w$ .

Switching the direct summands of  $(E \otimes E)^{\oplus 2}$ , gives us a  $\mathbb{Z}_2$ -action. In total we now have a  $S_3 \times \mathbb{Z}_2$ -action, which by the above remark is a  $D_6$ -action. Note that since the action was defined on  $E$ , the rank of elements of  $E \otimes E$  is preserved, so that we indeed have an action on  $M$ .

Similarly,  $N$  is the rank  $1 + 1$  tensors in  $(E \otimes E \otimes E)^{\oplus 2}$ , where now  $E$  is a 2-dimensional vector space. ■

Torus actions

By choosing invariant hyperplanes, the group actions on the ambient spaces descend to the Calabi–Yau's. We first consider the case when the ambient space was the join of two copies of  $\mathbb{P}^2 \times \mathbb{P}^2$ , which was denoted by  $M$  above.

Denote a unit matrix in the first factor of  $(E \otimes E) \oplus (E \otimes E)$  by  $e_{ij}^0$ , and denote a unit matrix in the second factor by  $e_{ij}^1$ , where  $0, 1$  are taken modulo 2.

In this case, one such invariant hyperplane is given by the span of

$$f_{ij}^\alpha = e_{ij}^\alpha + t e_{-i-j, -i-j}^{\alpha+1} \in E \otimes E \oplus E \otimes E,$$

where  $i \neq j \in \mathbb{Z}_3$  and  $\alpha \in \mathbb{Z}_2$ . Denote the intersection between  $M$  and  $H$  by  $X_{H_t}$ . Then the following is true:

**Proposition 4.5.2.** *Both the finite group  $D_6$  and the group  $\mathbb{Z}_3$  act on  $X_{H_t}$ . The symmetric variety  $X_{H_t}$  have 24 isolated singularities for  $t \neq 0, 1$ , and they come in two orbits under the  $D_6$ -action.*

*For  $t = 1$ , it has 36 isolated singularities.*

There is also a torus action on  $E$ , defined by  $e_i \mapsto \omega^i e_i$ , where  $\omega$  is a third root of unity. This is a  $\mathbb{Z}/3$ -action, which extends to an action on  $E \otimes E \oplus E \otimes E$ . Let  $H = \mathbb{Z}/3$ . Then note that  $X_{H_t}$  is  $\mathbb{Z}/3$ -invariant as well.

Check which singularities these are, also: fix points

identify  $H$ -invariant singularities + fix points

## 4.6 The mirror construction Ansatz

In many cases of interest, given a construction of a Calabi–Yau manifold, the following Ansatz produces a mirror.

Let  $\mathcal{K}$  be a simplicial complex, with associated Stanley–Reisner scheme  $X_0$ . Let  $G$  be the automorphism group of  $\mathcal{K}$ . Then  $G$  induces an action on  $T_{X_0}^1$  in the following way: each element of  $T_{X_0}^1$  can be represented by a  $\phi \in \text{Hom}(I/I^2, A)$ , and then  $g \cdot \phi$  is given by  $(g \cdot \phi)(f) = g \cdot \phi(g^{-1} \cdot f)$ , the contragredient action.

There is an action of  $T_n = (\mathbb{C}^*)^{n+1}/\mathbb{C}^*$  on  $\mathbb{P}^n$ , and since  $I_{X_0}$  is generated by monomials, the action restricts to an action on  $X_0$  as well.

Given a smoothing family with general fiber  $X$  and special fiber  $X_0$ , we can consider a subfamily with only isolated singularities on which  $G$  act. Let  $H \subset T_n$  be the subgroup of the torus acting on this family. Then the mirror candidate to  $X$  is given by a crepant resolution of  $Y_t = X_t/H$ .

Though it is often overlooked (or stated in another language) in the literature, even the mirror construction of the famous quintic arises this way. Briefly, the quintic Calabi–Yau is given by the zero locus of a general quintic in  $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5))$ . The special quintic given by  $f = x_0 x_1 x_2 x_3 x_4$  is the Stanley–Reisner scheme corresponding to the 3-simplex. The automorphism group is  $S_5$ , and an invariant 1-parameter family is given by  $f_t = \sum_{i=0}^4 x_i^5 + t x_0 x_1 x_2 x_3 x_4$ . The fiber at  $t = \infty$  is the Stanley–Reisner scheme.

There is a  $H \triangleq (\mathbb{Z}/5)^5/\mathbb{Z}^5$ -action on  $X_t = Z(f_t)$  given by coordinate-wise multiplication by fifth roots of unity. The general element of the family  $X_t$  is smooth, so the only singularities of the quotient  $Y_t = X_t/H$  comes from

ion\_ansatz

#### 4. Construction of Calabi–Yau's

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points with non-trivial stabilizer. These can be resolved by methods from toric geometry. For details, see for example the first chapter of Ingrid Fausk's thesis [Fau12].

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## **Appendices**

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# APPENDIX A

## Computer code

computercode

Extensive use of computer software such as **Macaulay2** and **SAGE** has been invaluable during my work. In this Appendix I collect some computer code for reproducing some of my calculations.

### A.1 Computing the singular locus

In some cases, equations simplify significantly in affine charts. Therefore, using the naive command `singularLocus` in **Macaulay2** often takes unnecessarily long time (and sometimes the computations never finish), as it computes the minors of a very large Jacobian matrix. Restricting to each affine chart, we can use the command `minimalPresentation` to eliminate variables to produce a new ring isomorphic to the first one, but with fewer equations.

The following code produces a list of the components of the singular locus of the projective scheme with ideal  $I$ .

```
1 fastSingularities = I -> (  
    R := ring I;  
3    n := numgens R; breaklines=true,  
    gensR := gens R;  
5    singlist := {};  
    for i from 0 to (n-1) do {  
7        affineChart := I + ideal(gensR_i - 1);  
        sing := radical ideal mingens ideal singularLocus  
        minimalPresentation affineChart;  
9        inv := affineChart.cache.minimalPresentationMap;  
        singlist = singlist | {(homogenize(preimage(inv, sing),  
        gensR_i))};  
11    };  
    saturate intersect(singlist)  
13 )
```

The method works by computing the singular locus in each affine chart, taking the radical, and then pulling back to the homogeneous coordinate ring.

## A. Computer code

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Finally, we get a list of singular loci in each affine chart. We return the (saturation of) the intersection of the singular loci of the affine charts.

It is especially fast when computing the singular locus of toric varieties with low-dimensional singularities.

The following code finds the singular locus of the projective cone  $C(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^9$ .

```
1 R = QQ[x_0..x_8,x_9]
M = genericMatrix(R,3,3)
3 I = minors(2,M)
time fastSingularities I
5 time radical ideal singularLocus I
```

Our function performs significantly faster. On a modern Mac, the times are 1.14 seconds versus 4.31 seconds, respectively.

Here is a more involved example. Let  $Y'$  be four-dimensional toric variety from Chapter 4. It is defined by the  $2 \times 2$ -minors two matrices. In **Macaulay2** we can define it as follows:

```
1 S = QQ[x_1..x_6,z_1..z_6,y]
M1 = matrix{{y,x_1,x_2},{x_4,y,x_3},{x_5,x_6,y}}
3 M2 = matrix{{y,z_1,z_2},{z_4,y,z_3},{z_5,z_6,y}}
J = minors(2,M1) + minors(2,M2)
```

Here the performance difference is even more impressive. Our function computes the singular locus in 7.29 seconds, but the built-in function **singularLocus** used more than 22 minutes (at which point I interrupted the computation).

## A.2 Torus action

The following lines checks if a projective scheme with ideal sheaf  $\mathbf{IX}$  admits an action of a subtorus of  $G = (\mathbb{C}^*)^n \subset \mathbb{P}^n$ . To check this, we check if the equations are still valid after a torus action. Since  $G$  is abelian, it act on functions by  $\lambda \cdot f(x_1, \dots, x_n) = f(\lambda_1 x_1, \dots, \lambda_n x_n)$ .

**Lemma A.2.1.** *Suppose  $\{f_1, \dots, f_r\}$  is a homogeneous generating set for  $I_X = \mathbf{IX}$ . Then subgroup of  $G$  acting on  $X \subset \mathbb{P}^n$  is generated by those  $\lambda \in G$  such that  $\lambda \cdot f_i = c f_i$  for some  $c \in \mathbb{C}^*$ .*

*Proof.* Let  $H$  be the subgroup of  $G$  fixing the ideal  $I_X$ . Let  $H'$  be the subgroup of  $g \in G$  acting on the  $f_i$  by scalar multiplication:  $g \cdot f_i = c f_i$ . Clearly  $H' \subseteq H$ . Now suppose  $g \in H$ . Then

$$g \cdot f_1 = \sum_j a_j f_j$$



### A.3. Computing the Gaifullin triangulation

for some constants  $a_j$ . Now  $g \cdot f_1 = f_1(\lambda_1 x_1, \dots, \lambda_n x_n)$ . Suppose the leading term of  $f_1$  is  $x_1^{a_1} \cdots x_n^{a_n}$ . Then comparing leading terms in the left hand side and the right hand side, we see that  $a_1 = \lambda_1^{a_1} \cdots \lambda_n^{a_n} := \lambda^m$ . Hence the right hand side is  $\lambda^m f_1 + \text{other terms}$ . But now there are the same number of terms on each side of the equation, so there are no other terms. Hence  $H = H'$ . ■

It follows that to find the subgroup of  $G$  acting on  $X$ , we have to find the  $\lambda \in G$  such that the  $f_i$  are simultaneous eigenvectors for them.

**Example A.2.2.** Let  $X$  be defined by  $f = x_0 x_1 x_2 x_3 x_4 + \sum_{i=0}^5 x_i^5$  in  $\mathbb{P}^4$ . Then for  $\mathbb{C}^4$  to act on it, we must have  $\lambda_0 \lambda_1 \lambda_2 \lambda_3 \lambda_4 = \lambda_0^5 = \dots = \lambda_4^5$ . By stting  $\lambda_0 = 1$ , we see that all the  $\lambda_i$  are the fifth roots of unity. Hence the subgroup acting on  $H$  is the subgroup of  $(\mathbb{Z}/5)^5/\mathbb{Z}_5$  given by  $\{(a_0, \dots, a_5) \mid \sum a_i = 0\}$ .

The following code find the subtoruses of  $G$  acting on  $X$  in this way, by equating terms in the polynomials defining  $X$ .

```
loadPackage "Binomials"
2 torus = ideal apply(flatten apply(apply(flatten entries gens
    IX, monomials), v -> flatten entries v), j -> subsets(j,2)),
    s -> s_0-s_1)
toruskomps = BPD torus
4 toruskomps = select(toruskomps, I -> dim I == 1)
```

*Explanation.* The ideal `torus` is the ideal generated by the differences of terms in the polynomials defining  $X$ . The `Macaulay2` package `Binomials` can decompose binomials over cyclic extensions of  $\mathbb{Q}$  with the command `BPD`. Finally, we select the components corresponding to finite subgroups of the torus.

Then we check manually if these actually correspond to non-trivial actions. ■

### A.3 Computing the Gaifullin triangulation

gaifullin

Below is a short SAGE script computing the 15 vertex triangulation of  $\mathbb{CP}^2$  as described in [Gai09]. The last line returns a `SimplicialComplex` object in SAGE.

```
#Defines the Klein 4 group.
2 V4 = PermutationGroup([Permutation("(1,2)(3,4)"),Permutation("
    (1,3)(2,4)"])]

4 def isValidFace(F):
    '''
6     Assumes the first vertex is a permutation.
    Then checks if F satisfies the condition in the
8     definition of T.
    '''
```

## A. Computer code

---

```
10 g = F[0]
11 for v in (1,2,3,4):
12     if (F[g(v)][1] == F[v][1]):
13         return False
14     return True
15
16 # Makes a list of all possible maximal faces of the correct form
17 candidates = [(g,(1,a1),(2,a2),(3,a3),(4,a4)) for g in V4.list()
18               [1:] for a1 in (1,2,3) for a2 in (1,2,3) for a3 in (1,2,3) for
19               a4 in (1,2,3)]
20 # Filters out the faces not fullfilling the condition
21 maximalFacets = filter(lambda F: isValidFace(F), candidates)
22
23 # Renames the vertices
24 S = SimplicialComplex(maximalFacets)
25 vertexSet = S.vertices()
26 D = dict([(F,i) for i,F in enumerate(vertexSet)])
27 renamedMaximalFacets = [[D[v] for v in F] for F in maximalFacets]
28 SS = SimplicialComplex(renamedMaximalFacets)
```

To get the Stanley–Reisner ideal, one can write:

```
list(SS.stanley_reisner_ring().defining_ideal().gens())
```

The returned value is a list of the monomials generating the Stanley–Reisner ideal of  $\mathcal{T}$ . This can then be copied into **Macaulay2** for further computation.

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