All of it

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Chapter 1

Preliminary definitions

We work over $\mathbb{C}$.

1.1 Stanley-Reisner basics

Given a simplicial complex $\mathcal{K}$, one can associate to it a projective scheme $\mathbb{P}(\mathcal{K})$ defined as follows. Let P be the polynomial ring with one variable for each vertex of $\mathcal{K}$. Then the *Stanley-Reisner ideal* $I_{\mathcal{K}}$ corresponding to $\mathcal{K}$ is generated by the monomials corresponding to *non-faces* of $\mathcal{K}$. Then we define the *Stanley-Reisner scheme* to be $\operatorname{Proj} P/I_{\mathcal{K}}$.

Example 1.1.1. Let $\mathcal{K}$ be the square, with vertices v_0, v_1, v_2, v_3. Then the Stanley-Reisner ideal is generated by v_0v_2 and v_1v_3. ★

Some of the topology of the simplicial complex is encoded in the scheme structure of $\mathbb{P}(\mathcal{K})$. In particular, the simplicial (co)homology groups of $\mathcal{K}$ can be computed as the sheaf cohomology of $\mathbb{P}(\mathcal{K})$

Lemma 1.1.2. *Let $H^i(K;\mathbb{C})$ denote the singular cohomology groups of $\mathcal{K}$. Then there are isomorphisms $H^i(K;\mathbb{C}) = H^i(\mathbb{P}(\mathcal{K}), \mathscr{O}_{\mathbb{P}(\mathcal{K})})$ for all i.*

Proof. ———to come——— □

1.2 Calabi-Yau basics

Definition 1.2.1. A *Calabi-Yau variety* is a smooth projective variety satisfying the following two conditions:

1. $H^i(X, \mathscr{O}_X) = 0$ for $0 < i < \dim X$.

2. The canonical sheaf is trivial: $\omega_X \simeq \mathscr{O}_X$.

■

The classical example of a Calabi-Yau manifold is the quintic threefold in $\mathbb{P}^5$. Another example is the following:

Example 1.2.2. Let X be the double cover of $\mathbb{P}^3$ ramified along a smooth octic. The projection map is affine, so the conditions on $H^i(X, \mathscr{O}_X)$ are fulfilled. To see that the canonical sheaf is trivial, we use the adjunction formula, which says that $K_X = 2K_{\mathbb{P}^3}\big|_X + \deg R$, where R is the ramification divisor. Then, since $\omega_{\mathbb{P}^3} = \mathscr{O}_{\mathbb{P}^3}(-4)$, it follows that $K_X = 0$. ★

If $\mathcal{K}$ is a simplicial sphere, then a smoothing of $\mathbb{P}(\mathcal{K})$ will give a Calabi-Yau manifold.

—- ref: bayer-eisenbud graph curves.

1.3 Deformation theory

Chapter 2

Two topologically distinct smoothings

Denote by dP_6 the del Pezzo surface of degree 6 embedded in $\mathbb{P}^6$. This can be realized as the blow-up of $\mathbb{P}^2$ in three points not lying on a line. Let X denote the affine cone over dP_6. Then it has long been known that X has two smoothing components, and we show here that they are topologically distinct.

Recall that a *del Pezzo* surface is a surface such that the anti-canonical bundle is ample. The degree is the degree given by the anticanonical embedding. It is a classical result that every del Pezzo surface is obtained either by blowing up $\mathbb{P}^2$ in $r = 0, \dots, 6$ points in suitable positions, or as the 2-uple embedding of a quadric surface in $\mathbb{P}^3$.

2.1 Different embeddings of dP_6

We first obtain the equations of dP_6 directly from the description of it as blow-up. Let x_0, x_1, x_2 be coordinates of $\mathbb{P}^2$. Recall that the blowup of $\mathbb{P}^2$ in the point $(1:0:0)$ can be realized as the closed subscheme of $\mathbb{P}^2 \times \mathbb{P}^1$ given by the equation $r_0x_1 - r_1x_2 = 0$, where r_0, r_1 are coordinates on $\mathbb{P}^1$. We can repeat this process on the points $(0:1:0)$ and $(0:0:1)$ to obtain similar equations. Collecting these, we see that dP_6 is given by the matrix equation

$$M\vec{x} = \begin{pmatrix} 0 & r_0 & -r_1 \\ s_1 & 0 & -s_0 \\ -t_0 & t_1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0.$$

in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Here r_i, s_i and t_i $(i = 0, 1)$ are of course coordinates on $\mathbb{P}^1$.

We can do more than this however.

Lemma 2.1.1. *We can also realize dP_6 embedded in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with equation $r_0 s_0 t_0 = r_1 s_1 t_1$.*

Proof. Note that the matrix cannot have rank 1 or lower.

Now consider the projection onto the last three factors:

$$\pi : \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Each point P in the product on the right-hand side gives a matrix M_P of rank 2. Thus there is a line of solutions, which correspond exactly to a point in $\mathbb{P}^2$.

Hence the restriction of π to dP_6 is an isomorphism onto the hypersurface given by $\det M = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. □

Another way to realize blow-ups is this: let $\mathfrak{d}$ be the linear system of quadrics with assigned basepoints $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$ in $\mathbb{P}^2$. We can choose a basis given by $x_0 x_1, x_0 x_2$ and $x_1 x_2$. This gives a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. The closure of the graph of this map is a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$ defined by two bilinear equations. Each of the projections correspond to the blowup.

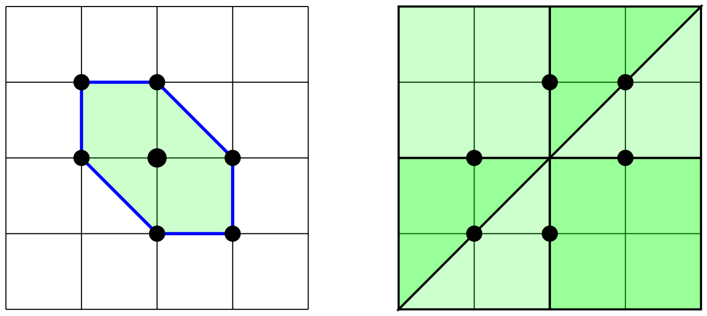
Explicitly, if we let y_0, y_1, y_2 be coordinates on the other $\mathbb{P}^2$, then the equations are $x_1 y_0 - x_1 y_1 = x_1 y_1 - x_2 y_2 = 0$.

We also have a natural embedding in $\mathbb{P}^6$ as follows. Denote by E_1, E_2, E_3 the exceptional divisors on the blowup. Let L be a line in $\mathbb{P}^2$. Then the divisor $\pi^* 3L - E_1 - E_2 - E_3$ is ample, and gives an embedding in $\mathbb{P}^6$ (see [1, Chapter V, Theorem 4.6]). A basis for the corresponding linear system is given by all monomials in $\Gamma(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(3))$ except x^3, y^3 and z^3.

The equations can be arranged in a particularly symmetric form: let $y, x_1, \ldots, x_1$ be coordinates on $\mathbb{P}^6$. Then the equations of dP_6 are the 2×2 minors of the matrix

$$\begin{pmatrix} x_1 & y & x_6 \\ x_2 & x_3 & y \\ y & x_4 & x_5 \end{pmatrix}.$$

This gives 9 equations, which can be compactly written as $x_i x_{i+2} - y x_i = 0$ and $x_i x_{i+3} - y^2 = 0$, for $i = 1, \ldots, 6$ (where i is taken modulo 6). Note that the equations have a visible D_6-symmetry, where D_6 denotes the dihedral group.



(a) The hexagon. (b) The fan of dP_6.

2.1.1 As a toric variety

There is a nice combinatorial description of dP_6 as a toric variety associated to a polytope. Namely, let P denote the hexagon in Figure 2.1a. Then the normal fan of this polytope defines a fan in $N_{\mathbb{R}}$, defining a toric variety.

The polytope is reflexive, implying that the normal fan of P is the face fan over the same polytope. See Figure 2.1b. From standard toric geometry, it is clear that dP_6 is the blowup of $\mathbb{P}^2$ in the three torus-fixed points.

2.2 The affine cone

2.3 The two smoothings

Chapter 3

A smooth Calabi-Yau

Consider the hexagon E_6. The join $E_6 * E_6$ is a 3-dimensional sphere, and so a smoothing of the corresponding Stanley-Reisner scheme would correspond to a smooth Calabi-Yau manifold. In this chapter I prove that there does indeed exist a smoothing, and I describe some of its properties.

Bibliography

[1] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.