Algebraic groups and moduli theory

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Abstract

These are notes from the course Algebraic Geometry III. We work over a field of characteristic zero.

1 Representation theory in general

Let V be a vector space. Briefly, a *representation* of any group G on V is just a group homomorphism $\rho : G \to \mathrm{GL}(V)$.

Example 1.1. The *trivial representation* is given by sending every $g \in G$ to the identity transformation. ★

Example 1.2. Suppose G is a finite group. Then there is an embedding $G \hookrightarrow S_n$, and every element of S_n can be represented by permutation matrices (that is, matrices M_g such that $Me_i = e_{g(i)}$ for all $g \in G$). This defines a representation of G in k^n. ★

Example 1.3. Suppose G acts on a (finite) set X. Let V be the vector space with basis identified with the elements of X. Then G acts on V by linearity: for each $g \in G$, $\rho(g)$ is the linear map sending e_x to e_{gx}. Such representations are called *permutation representations.* ★

A *morphism of representations* (ρ, V),(ρ', W) consists of commutative diagrams

$$\begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\rho'(g)"] \\ V \arrow[r, "\psi"'] & W \end{tikzcd}$$

for each $g \in G$. Thus, if ψ is invertible, this says that the linear operators $\rho(s), \rho'(s)$ are similar.

2 Algebraic groups

Algebraic groups are group objects in the category of affine varieties. More precisely:

Definition 2.1. Let A be a finitely generated k-algebra. An *affine algebraic group* is a quadruple $(A, \mu_A, \epsilon, \iota)$ where $\mu_A : A \to A \otimes_k A$ (the *coproduct*), $\epsilon : A \to k$ (the *coidentity*), $\iota : A \to A$ (the *coinverse*) are k-algebra homomorphisms, satisfying the following conditions:

1. Coassociativity. The following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\mu_A"] \arrow[d, "\mu_A"'] & A \otimes_k A \arrow[d, "\mathrm{id}_A \otimes \mu_A"] \\
A \otimes_k A \arrow[r, "\mu_A \otimes \mathrm{id}_A"'] & A \otimes_k A \otimes_k A
\end{tikzcd}$$

2. The following diagram commutes:

$$\begin{tikzcd}
& & k \otimes_k A \arrow[dr, "\simeq"] & \\
A \arrow[r, "\mu"] & A \otimes_k A \arrow[ur, "\epsilon \otimes \mathrm{id}_A"] \arrow[dr, "\mathrm{id}_A \otimes \epsilon"'] & & A \\
& & A \otimes_k k \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

   and is equal to the identity.

3. Inverse. The following diagram commutes:

$$\begin{tikzcd}
A \arrow[rrr, "\epsilon"] \arrow[d, "\mu"] & & & k \arrow[d] \\
A \otimes_k A \arrow[r, "\mathrm{id}_A \otimes \iota"'] & A \otimes_k A \arrow[rr, "\cdot"'] & & A
\end{tikzcd}$$

   Here the right arrow is the morphism making A a k-algebra. The last arrow in the lower sequence is multiplication in A.

■

Example 2.2. Let G be any group, and let $k[G]$ be its group ring. Let A be its k-linear dual, that is $A = \mathrm{Hom}_k(k[G], k)$. This is a priori just another vector space, but we can give it the structure of a k-algebra by defining multiplication as follows: let $\lambda : k[G] \to k$,γ :

$k[G] \to k$ be k-linear maps. It is enough to say what should happen on a basis, and a basis is given by the elements g of G. Then, set $(\lambda \cdot \gamma)(g) = \lambda(g) \cdot \gamma(g)$.

Then set $\mu : A \to A \otimes A$ to be the dual of the multiplication map on $k[G]$. Explicitly, let $m : k[G] \otimes_k k[G] \to k[G]$ denoted the multiplication map. Let $\lambda : k[G] \to k$ be an element of A. Then we can form $m^*\lambda = \lambda \circ m$, which is an element of $(k[G] \otimes k[G])^\vee$. For finite-dimensional vector spaces, this is isomorphic to $A \otimes A$, which gives our multiplication map μ. The coidentity is given by sending $\lambda : k[G] \to k$ to $\lambda(1_G)$, where $1_G \in G \subseteq k[G]$.

For example: let $G = C_n$ be the cyclic group of order n. Then $k[G] = k[t]/(t^n - 1)$, and since this is finite-dimensional over k, we can find an isomorphism $k[G] \approx A$. Unwinding definitions, we see that [????] (I dont see this) ★

Example 2.3. Let $A = k[s]$ be the polynomial ring in one variable. This is the coordinate ring of $\mathbb{A}^1_k$. We can define

$$\mu(s) = s \otimes 1 + 1 \otimes s.$$

Also, $\epsilon(s) = 0$, and $\iota(s) = -s$. ★

Definition 2.4. An *action* of an affine algebraic group $G = \operatorname{Spec} A$ on an affine variety $X = \operatorname{Spec} R$ is a morphism $G \times X \to X$ defined dually by a k-algebra morphism $\mu_R : R \to R \otimes_k A$ satisfying the following two conditions.

1. The following diagram is commutative:

$$\begin{tikzcd} R \arrow[r, "\mu_R"] \arrow[dr, "\mathrm{id}_R"'] & R \otimes_k A \arrow[d, "\mathrm{id}_R \otimes \epsilon"] \\ & R \simeq R \otimes_k k \end{tikzcd}$$

2. The diagram

$$\begin{tikzcd} R \arrow[r, "\mu_R"] \arrow[d, "\mu_R"'] & R \otimes_k A \arrow[d, "\mu_R \otimes \mathrm{id}_A"] \\ R \otimes_k A \arrow[r, "\mathrm{id}_R \otimes \mu_A"'] & R \otimes_k A \otimes_k A \end{tikzcd}$$

■

3 Representations of algebraic groups

Let $G = \operatorname{Spec} A$ be an affine algebraic group over a field k.

Definition 3.1. An *algebraic representation of* G is a pair (V, μ_V) consisting of a k-vector space V and a k-linear map $\mu_V : V \to V \otimes_k A$ satisfying the following two conditions:

1. The diagram

$$\begin{tikzcd} V \arrow[r, "\mu_V"] \arrow[dr, "\mathrm{id}_V"'] & V \otimes_k A \arrow[d, "\mathrm{id}_V \otimes \epsilon"] \\ & V \simeq V \otimes_k k \end{tikzcd} \tag{1}$$

   is commutative.

2. The diagram

$$\begin{tikzcd} V \arrow[r, "\mu_V"] \arrow[d, "\mu_V"'] & V \otimes_k A \arrow[d, "\mu_V \otimes \mathrm{id}_A"] \\ V \otimes_k A \arrow[r, "\mathrm{id}_V \otimes \mu_A"'] & V \otimes_k A \otimes_k A \end{tikzcd}$$

   is commutative. Here μ_A is the coproduct in the coordinate ring of G.

■

Remark. *In lieu of Definition 2.4, we see that any action of an algebraic group G on an affine variety $X = \operatorname{Spec} R$ is a representation of G on the infinite-dimensional k-vector space $R = \Gamma(X, \mathscr{O}_X)$.*

Remark. *Mumford calls this a* dual action of G on V, *in his 1965 book "Geometric Invariant Theory".*

We often drop the subcript from μ_V unless confusion may arise. The same comment applies to tensor products. They will always be over the ground field unless otherwise stated. We will sometimes refer to a representation (V, μ_V) sometimes as "a representation $\mu : V \to V \otimes A$" and sometimes as just "a representation V".

Definition 3.2. Let $\mu : V \to V \otimes A$ be a representation of $G = \operatorname{Spec} A$. Then:

1. A vector $x \in V$ is said to be *G-invariant* if $\mu(x) = x \otimes 1$.
2. A subspace $U \subset V$ is called a *subrepresentation* if $\mu(U) \subseteq U \otimes A$.

■

Proposition 3.3. *Every representation V of G is locally finite-dimensional. Precisely: every $x \in V$ is contained in a finite-dimensional subrepresentation of G.*

Proof. Write $\mu(x)$ as a finite sum $\sum_i x_i \otimes f_i$ for $x_i \in V$ and linearly independent $f_i \in A$. This we can always do, by definition of tensor product and bilinearity. Let U be the subspace of V spanned by the vectors x_i.

Now, by the commutativity of the diagram (1) it follows that

$$x = \sum_i \epsilon(f_i) x_i.$$

By the commutativity of the second diagram in the definition, it follows that

$$\sum_i \mu_V(x_i) \otimes f_i = \sum_i x_i \otimes \mu_A(f_i) \in U \otimes A_k \otimes_k A.$$

Because each term of the right-hand-side is contained in $U \otimes A \otimes A$, it follows that $\mu_V(x_i)$ is contained in U since the f_i are linearly independent.

Thus x is contained in the finite-dimensional representation $\mu_V\big|_U : U \to U \otimes A$. $\square$

We can classify representations of $\mathbb{G}_m$ easily. They are all direct sums of "weight m"-representations, that is, representations of the form

$$V \to V \otimes k[t, t^{-1}], v \mapsto v \otimes t^m.$$

Proposition 3.4. *Every representation V of $\mathbb{G}_m$ is a direct sum $V = \oplus_{m \in \mathbb{Z}} V_{(m)}$, where each $V_{(m)}$ is a subrepresentation of weight m.*

Proof. For each $m \in \mathbb{Z}$, define

$$V_{(m)} = \{v \in V \mid \mu(v) = v \otimes t^m\}.$$

This is a subrepresentation of V: we must see that $\mu(V_{(m)}) \subset U \otimes A$, but this is true by construction. It is also clear that is has weight m. Next we show that $V = \oplus_{m \in \mathbb{Z}} V_{(m)}$. Write

$$\mu(v) = \sum_{m \in \mathbb{Z}} v_m \otimes t^m \in V \otimes k[t, t^{-1}].$$

Using the first condition in the definition of a representation, we get that $v = \sum_{m \in \mathbb{Z}} \epsilon(t^m) v_m$. It remains to check that each

$v_m \in V_{(m)}$ (we can forget the scalars $\epsilon(t^m)$). But from definition ii), it follows that

$$\sum \mu(v_m) \otimes t^m = \sum v_m \otimes t^m \otimes t^m,$$

so that indeed $\mu(v_m) = v_m \otimes t^m$, as wanted. □

Example 3.5. An action of $\mathbb{G}_m$ on $X = \operatorname{Spec} R$ is equivalent to specifying a grading

$$R = \oplus_{m\in\mathbb{Z}} R_{(m)} \qquad R_{(m)}R_{(n)} \subset R_{(m+n)}.$$

The invariants under this action are thus the homogeneous elements of weight zero, that is, the subring $R_{(0)}$. Moreover, we have a special operator. There is a linear endomorphism E of R that sends $f = \sum f_m \mapsto \sum m f_m$, and it is a derivation of R, called the Euler operator. We have $R^{\mathbb{G}_m} = \ker E$.

To see that E is a derivation, we must check that $E(fg) = fE(g) + gE(f)$. The operator is homogeneous, so it is enough to check on homogeneous elements. So let f_m, g_n be of degree m, n, respectively. Then

$$E(f_m g_n) = (m+n) f_m g_n = g_n(m f_m) + f_m(n g_n) = g_n E(f_m) + f_m E(g_m),$$

as wanted. ★

A character in regular representation theory is a homomorphism $G \to \mathbb{C}^*$, so do we have a corresponding notion of characters in this "dual" world:

Definition 3.6. Let $G = \operatorname{Spec} A$ be an affine algebraic group. A 1-dimensional character of G is a function $\chi \in A$ satisfying

$$\mu_A(\chi) = \chi \otimes \chi \qquad \iota(\chi)\chi = 1.$$

■

Lemma 3.7. *The characters of the general linear group* $\mathrm{GL}(n) = \operatorname{Spec} k[x_{ij}, \det X]$ *are precisely the integer powers of the determinant* $(\det X)^n$ *for* $n \in Z$*.*

Definition 3.8. Let χ be a character of an affine algebraic group G, and let V be a representation of G. A vector $v \in V$ satisfying

$$\mu_V(v) = v \otimes \chi$$

is called a *semi-invariant* of G with weight χ. The semi-invariants of V belonging to a given character χ form a subrepresentation $V_\chi \subset V$ of V. ■

We will often change the point of view depending upon the situation. Sometimes we think of a representation of an algebraic group as a k-linear map $V \to V \otimes_k A$ satisfying some axioms, and sometimes we think of a representation as a group G acting on a vector space V in the usual fashion.

Proposition 3.9. *Let $\mu : V \to V \otimes_k A$ be a representation of an algebraic group G. Let $g \in G(k)$ be a k-valued point and $\mathfrak{m}_g \subseteq A$ the corresponding maximal ideal. Denote by $\rho(g)$ the composition*

$$V \xrightarrow{\mu} V \otimes_k A \xrightarrow{\text{mod } \mathfrak{m}_g} V \otimes_k k \simeq V.$$

Then, if $A = \Gamma(G, \mathscr{O}_G)$ is an integral domain, a vector $v \in V$ such that $\rho(g)(v) = v$ for all $g \in G(k)$ is a G-invariant.

Proof. We need to check that $\mu(v) = v \otimes 1$. First, since G is the spectrum of a finitely generated k-algebra, we can write A as $k[y_1, \cdots, y_m]/I$ for some prime ideal I. Then the same trick as in the proof of Proposition 3.3 works. Write $\mu(v) = \sum v_i \otimes f_i$ with $f_i \in A$ for all i. Since the composition is the identity, we have that $f_i \equiv 1 \pmod{\mathfrak{m}_g}$ for all $g \in G$. This implies that $f_i - 1$ is contained in the Jacobson radical of A. But A is an integral domain, so $f_i - 1 = 0$. □

Thus, in a sense, the two notions of G-invariance coincides.

3.1 Algebraic groups and their Lie spaces

[[something about local study]]

Definition 3.10. Let R be a k-algebra and M an R-module. An *M-valued derivation* is a k-linear map $D : R \to M$ satisfying the Leibniz rule $D(xy) = xD(y) + yD(x)$ for $x, y \in R$. ■

The set of M-valued derivations is also an R-module, denoted by $\mathrm{Der}_k(R, M)$. This is used to define tangent spaces in algebraic geometry as follows: Let $p \in \operatorname{Spec} A = X$ be a closed point. Then we have a local ring $\mathscr{O}_{X,p}$ and a quotient map $\mathscr{O}_{X,p} \to k$ with kernel $\mathfrak{m}_p$. Then the k-module $(\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$ is called the *Zariski tangent space* of $p \in X$. In fact:

Proposition 3.11. *We have an isomorphism of A-modules:*

$$\mathrm{Der}_k(\mathscr{O}_{X,p}, k) \simeq (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$$

Proof. sketch send D to $D\big|_{\mathfrak{m}_p}$. □

[[Also $\mathrm{Hom}_k(k[\epsilon], \mathscr{O}_{X,p})$.

3.2 Linear reductivity

Definition 3.12. An algebraic group G is said to be *linearly reductive* if, for every epimorphism $\varphi : V \to W$ of G-representations, the induced map of G-invariants $\varphi^G : V^G \to W^G$ is surjective. ■

Proposition 3.13. *Every finite group G is linearly reductive.*

Proof. Let $\varphi : V \to W$ be the given epimorphism of representations. Let $R : V \to V^G \subset V$ be given by $v \mapsto \sum_{g \in G} g \cdot v$. Let $w \in W^G$. Then it is an easy calculation to check that $\varphi(R(v)) = R(\varphi(v))$, from which it follows that $\varphi(R(v)) = w$ (note that $R\big|_{W^G} = \mathrm{id}_{W^G}$). □

The homomorphism R above is called the *Reynolds operator.*

Proposition 3.14. *The following are equivalent:*

i) *G is linearly reductive.*

ii) *For every epimorphism $V \to W$ of* finite-dimensional *G-representations, the induced map $V^G \to W^G$ is surjective.*

iii) *If V is any finite-dimensional representation and $U \subseteq$ is a proper subrepresentation and $\bar{v} \in V/U$ is G-invariant, then the coset $v + U$ (for any lifting of $\bar{v}$) contains a non-trivial G-invariant vector.*

Proof. *i)* $\Rightarrow$ *ii)* is trivial. For *ii)* $\Rightarrow$ *iii)*, apply *ii)* to the quotient map $V \to V/U$. Then $V^G \to (V/U)^G$ is surjective. This implies that for every nonzero $\bar{v} \in (V/U)^G$, there exists a G-invariant $v \in \pi^{-1}(v) = U + \bar{v}$.

iii) $\Rightarrow$ *i)* is hardest. Suppose $\phi : V \to W$ is an epimorphism of representations (not necessarily finite-dimensional). Suppose $\phi(v) = w \in W^G$ for some $v \in V$.

By Proposition 3.3 there exists a finite-dimensional subrepresentation $V_0 \subseteq V$ containing v. Now $v \in V_0$ is G-invariant modulo $U_0 := V_0 \cap \ker \phi$ (since $V/\ker\phi \simeq W$ as G-representations), so by *iii)*, there exists a G-invariant vector $v' \in V_0$ such that $v' - v \in U_0$. But $\phi(v') = w$, so $\phi^G : V^G \to W^G$ is surjective. □

Lemma 3.15. *Direct products of linearly reductive groups are linearly reductive. If $H \subset G$ is a normal subgroup and G is linearly reductive, then so is G/H. Moreover, if both H and G/H are linearly redutive, then so is G.*

Proof. Suppose given an endomorphism of representation of $G \times H$: $V \to W$. In particular, they are representations of G, H separately, by the rule $g \cdot v = (g, e) \cdot v$. In particular, if an element $w \in W$

is $G \times H$-invariant, it is also G, H-invariant. Thus by assumption, there is an G, H-invariant $v \in V$ mapping to w. But if something is G, H-invariant, it is also $G \times H$-invariant, since G, H commute in $G \times H$.

Similarly, every G/H-representation gives a G-representation, by the rule $g \cdot v = \bar{g} \cdot v$, where $\bar{g}$ denotes the class of g in G/H. Now if $w \in W^{G/H}$ is G/H-invariant, then it is by definition G-invariant, and by linearly reductivity of G, the map is surjective.

Finally, if both H and G/H are linearly reductive, suppose $\phi : V \to W$ is a surjection of G-representations. This is also a surjection of H-representations, and since H was linearly reductive, we get that $V^H \to W^H$ is surjective. It follows that the map ϕ and the vector spaces V, W splits as $(\phi^H, \phi') : V^H \oplus V' \to W^H \oplus W'$, where H acts trivially on the second factor. This implies that G/H acts on V', W', and it follows that $V' \to W'$ is surjective. □

Proposition 3.16. *Every algebraic torus $(\mathbb{G}_m)^N$ is linearly reductive.*

Proof. By the lemma, it suffices to prove this for $N = 1$. We use Proposition 3.14 iii). By Proposition 3.4, we can write a representation V and a subrepresentation U as

$$V = \bigoplus_{m \in \mathbb{Z}} V_{(m)} \qquad \text{and} \qquad U = \bigoplus_{m \in \mathbb{Z}} U_{(m)}.$$

Here $U_{(m)} \subset V_{(m)}$. An element $v \in V/U$ is $\mathbb{G}_m$-invariant if any lifting of v to V lies in $U_{(m)}$ for $m \neq 0$. Thus $v_{(0)}$ is $\mathbb{G}_m$-invariant and lies in the coset $v + U$. □

The classical example of a group that is not linearly reductive is the affine line $\mathbb{A}^1$ under addition:

Example 3.17. Consider the 2-dimensional representation given by

$$\mathbb{G}_a \to \mathrm{GL}_2, \qquad t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

This is a representation by the rules of matrix multiplication. Algebraically, this as follows: let x, y be a basis for V. Then we define a k-linear map $V \to V \otimes_k k[t]$ by $x \mapsto x \otimes 1$ and $y \mapsto x \otimes t + y \otimes 1$. This extends to a representation of $k[V] = k[x, y]$ in the obvious way. Then we can define an epimorphism of representations by sending $k[x, y] \to k[x, y]/(x) \simeq k[y]$. Taking invariants, we get that $k[x, y]^{\mathbb{G}_a} = k[x]$ but $k[y]^{\mathbb{G}_a} = k[y]$, but the map sends x to 0, so is not surjective. ★