Calculations

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August 29, 2016

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1 Computations on dP6

1.1 Finding equations of deformations

Consider the del Pezzo surface dP_6 of degree 6 embedded in $\mathbb{P}^6$. Its ideal is defined as follows:

```
restart
S = QQ[x_1..x_6,y_0]
I = ideal(x_1*x_3-x_2*y_0,
      x_2*x_4-x_3*y_0,
  x_3*x_5-x_4*y_0,
  x_4*x_6-x_5*y_0,
  x_5*x_1-x_6*y_0,
  x_6*x_2-x_1*y_0,
  x_1*x_4-y_0^2,
  x_2*x_5-y_0^2,
  x_3*x_6-y_0^2)
```

We compute the two deformations of its affine cone using the package `VersalDeformations`.

```
(F,R,G,C) = versalDeformation(gens I);
decompose ideal transpose mingens ideal G
```

The output are four lists of matrices entries in $\mathbb{Q}[\mathbf{x}] \otimes \mathbb{Q}[t_1, t_2, t_3]$. The list F consists of the equations of the family, and the list R of the relations. The list G gives equations for the base space. We have that F_0 is the matrix of generators of I, and that $F_i R_i \equiv 0 \pmod{t^{i+1}}$.

The decomposition of `ideal G` is the following:

```
i9 : decompose ideal transpose mingens ideal G

o9 = {ideal(t  - t ), ideal (t  - t , t )}
             1    3            2    3   1
```

Thus the base space splits into two components meeting transversely at the origin, of dimension 2 and 1, respectively. By doing a change of variables we can get rid of the linear terms:

```
T = QQ[x_1..x_6,y_0,t_1,t_2,t_3,s_1,s_2,s_3];
gsub = sub(sub(ideal mingens ideal G,T), {t_2 => s_2+s_3-s_1,
    t_1 => s_3, t_3 => s_3-s_1})
fsub = transpose  sub(sub(sum F,T), {t_2 => s_2+s_3-s_1, t_1 =>
    s_3, t_3 => s_3-s_1})
```

Now the equations are easier:

```
i13 : decompose gsub

o13 = {ideal(s ), ideal (s , s )}
              1          3   2
```

We can get equations for each of these families by setting $s_1 = 0$ and $s_3 = s_2 = 0$, respectively:

```
i25 : fsub1 = sub(fsub, s_1 => 0)

o25 = {-2} | x_1x_3-x_2y_0                         |
      {-2} | x_2x_4-x_3y_0+x_3s_2+x_3s_3           |
      {-2} | x_3x_5-x_4y_0+x_4s_3                  |
      {-2} | x_4x_6-x_5y_0                         |
      {-2} | x_1x_5-x_6y_0+x_6s_2+x_6s_3           |
```

```
      {-2} | x_2x_6-x_1y_0+x_1s_3                         |
      {-2} | x_1x_4-y_0^2+y_0s_2+y_0s_3                   |
      {-2} | x_2x_5-y_0^2+y_0s_2+2y_0s_3-s_2s_3-s_3^2     |
      {-2} | x_3x_6-y_0^2+y_0s_3                          |

              9       1
o25 : Matrix T  <--- T
```

And:

```
i26 : fsub2 = sub(fsub, {s_3 => 0, s_2 => 0})

o26 = {-2} | x_1x_3-x_2y_0        |
      {-2} | x_2x_4-x_3y_0-x_3s_1 |
      {-2} | x_3x_5-x_4y_0        |
      {-2} | x_4x_6-x_5y_0-x_5s_1 |
      {-2} | x_1x_5-x_6y_0        |
      {-2} | x_2x_6-x_1y_0-x_1s_1 |
      {-2} | x_1x_4-y_0^2-y_0s_1  |
      {-2} | x_2x_5-y_0^2-y_0s_1  |
      {-2} | x_3x_6-y_0^2-y_0s_1  |

              9       1
o26 : Matrix T  <--- T
```

1.2 Intersecting with two special hyperplanes

Consider dP_6 defined as above. Then consider the two hyperplanes

$$h_1 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

and

$$h_2 = x_1 - x_2 + x_3 - x_4 + x_5 - x_6.$$

We can compute the intersection with dP_6 in `Macaulay2` as follows:

```
h1 = x_1+x_2+x_3+x_4+x_5+x_6
h2 = x_1-x_2+x_3-x_4+x_5-x_6

SS = S[r]/(r^2+3)
apply(decompose(sub(I + h1 + h2,SS)), j -> ideal mingens sub(j,
    y_0 => 1))
```

The reason we create a new ring is that we need $\sqrt{-3}$ in order for the ideals to decompose. The results is the following list of ideals:

1. $(x_5 - 1, x_4 + x_6 + 1, x_3 + x_6 + 1, x_2 - 1, x_1 - x_6, r - 2x_6 - 1)$.
2. $(x_5 - 1, x_4 + x_6 + 1, x_3 + x_6 + 1, x_2 - 1, x_1 - x_6, r + 2x_6 + 1)$.
3. $(x_6 - 1, x_4 - x_5, x_3 - 1, x_2 + x_5 + 1, x_1 + x_5 + 1, r - 2x_5 - 1)$.
4. $(x_6 - 1, x_4 - x_5, x_3 - 1, x_2 + x_5 + 1, x_1 + x_5 + 1, r + 2x_5 + 1)$.
5. $(x_5 - x_6, x_4 - 1, x_3 + x_6 + 1, x_2 + x_6 + 1, x_1 - 1, r - 2x_6 - 1)$.
6. $(x_5 - x_6, x_4 - 1, x_3 + x_6 + 1, x_2 + x_6 + 1, x_1 - 1, r + 2x_6 + 1)$.

From this we can read off the coordinates in $\mathbb{P}^6$:

Table 1: Table of singular points.

x_1	x_2	x_3	x_4	x_5	x_6	y_0
$\frac{-1+\sqrt{3}}{2}$	1	$\frac{-1-\sqrt{-3}}{2}$	$\frac{-1-\sqrt{-3}}{2}$	1	$\frac{-1+\sqrt{-3}}{2}$	1
$\frac{-1+\sqrt{-3}}{2}$	1	$\frac{-1+\sqrt{-3}}{2}$	$\frac{-1+\sqrt{-3}}{2}$	1	$\frac{-1-\sqrt{-3}}{2}$	1
$\frac{-1-\sqrt{-3}}{2}$	$\frac{-1-\sqrt{-3}}{2}$	1	$\frac{-1+\sqrt{-3}}{2}$	$\frac{-1+\sqrt{-3}}{2}$	1	1
$\frac{-1+\sqrt{-3}}{2}$	$\frac{-1+\sqrt{-3}}{2}$	1	$\frac{-1-\sqrt{-3}}{2}$	$\frac{-1-\sqrt{-3}}{2}$	1	1
1	$\frac{-1-\sqrt{-3}}{2}$	$\frac{-1-\sqrt{-3}}{2}$	1	$\frac{-1+\sqrt{-3}}{2}$	$\frac{-1+\sqrt{-3}}{2}$	1
1	$\frac{-1+\sqrt{-3}}{2}$	$\frac{-1+\sqrt{-3}}{2}$	1	$\frac{-1-\sqrt{-3}}{2}$	$\frac{-1-\sqrt{-3}}{2}$	1

Note that the hexagonal group D_6 act transitively on the set of singular points.

2 Toric mirrors

Let I and I' be the ideals of dP_6 with a disjoint set of variables. Let $J = I + I'$. Then $Y_0 = \mathrm{Proj}(S/J)$ is a toric variety whose associated polytope P has vertices

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 2 & 2 \\ -2 & 2 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{1}$$

The polytope P is reflexive with 87 lattice points. We can use the Batyrev-Borisov construction to find a toric mirror candidate to Y_0.

The construction actually give two such families. Here's how: start with a reflexive polytope $P \subset M_{\mathbb{R}}$. A *nef partition* induces a decomposition V_1, V_2 of the vertex set of $P^{\vee} \subset N_{\mathbb{R}}$, such that $P^{\vee} = \mathrm{conv}(V_1, V_2)$. Let $Q_i = \mathrm{conv}(V_i)$. Then one forms $Q := Q_1 + Q_2 \subset N_{\mathbb{R}}$. Similarly, we have a dual nef partition, so that we can write $P = P_1 + P_2$.

A nef partition correspond to a complete intersection in X_P, and the dual nef partition correspond to a complete intersection in X_Q.

We can find (using SAGE for example) the polytopes P_1 and P_2. They are given as follows:

$$P_1 = \mathrm{conv}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{2}$$

and

$$P_2 = \mathrm{conv}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

There is a formula for the Q_i given the P_i:

$$Q_i = \{u \in N_{\mathbb{R}} \mid \langle u, m\rangle \geq -1 \text{ for all } \; m \in P_i, \langle u, m\rangle \geq 0 \,\forall\, m \in P_j, j \neq i\}$$

Then we find that Q_1 is given by

$$Q_1 = \operatorname{conv}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ -1 & -1 & -1 & 0 & -1 & -1 & -1 \end{pmatrix} \tag{4}$$

and

$$Q_2 = \operatorname{conv}\begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \tag{5}$$

The polytope Q has 48 vertices:

$$\left(\begin{array}{cccccccccccccccccccccccccccccccccccccccccccccccc} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 1 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

The polar polytope $Q^{\vee}$ is given by

$$\begin{pmatrix} -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

One can use the program $\texttt{nef}_x$ $(writtenbySkarkeetal)tocomputetheHodgenumberscorrespondingt$
P_2 and $Q_1 + Q_2$. It turns out that they are $(19, 19)$.

Note that J can be degenerated to a more singular toric variety by letting .....