Results

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1 Construction of a smooth Calabi-Yau

Let X_0 be the Stanley-Reisner scheme corresponding to the join of two hexagons. Call this simplicial complex for $\mathcal{K}$. Then consider the join of $\mathcal{K}$ with Δ^1. This corresponds to adding two free variables. The resulting Stanley-Reisner scheme Y_0 corresponds to a 5-dimensional ball.

By standard "Sturmfels theory", Y_0 deforms to a toric variety Y, whose associated polytope is the join of two hexagons. Since X_0 sits inside Y_0 as a linear section, it deforms as well to a generic hyperplane section in Y. Y sits inside $\mathbb{P}^{13}$, and its ideal sheaf is the sum of the ideal sheaf of two del Pezzo surfaces in $\mathbb{P}^6$, anti-canonically embedded. It follows that the singular locus of Y is 2-dimensional, consisting of two disjoint copies of a del Pezzo surface.

Hence the intersection of Y with two generic hyperplanes is a 3-dimensional variety with isolated singularities. Since the del Pezzos are of degree 6, there should in total be 12 singularities, looking locally like cones over del Pezzos. In the following we will try to describe their topology.

There exists a deformation of Y, reducing its singular locus to a normal crossing cycle of dimension 1. This implies that there exists a smoothing of X_0, yielding a smooth Calabi-Yau. One of our main tasks will be to try to compute some of its invariants.

1.1 Recipe for finding the smoothing

This is a general heuristic for finding smoothings of Stanley-Reisner schemes, which worked well in my master's thesis, where I studied degenerations of the Grassmannian $\mathbb{G}(3,6)$.

The main ingredient will be the package `Macaulay2` package `VersalDeformations` by Nathan Ilten. It comes with a method that takes as input an ideal, a basis for the first-order deformations, and a basis for the obstruction space. Then the algorithm tries to successively lift the equations to higher order. One can optionally only give the algorithm a subset of a basis of the first-order deformations.

[[rediscover this]]

1.2 Dimension of some cohomology groups

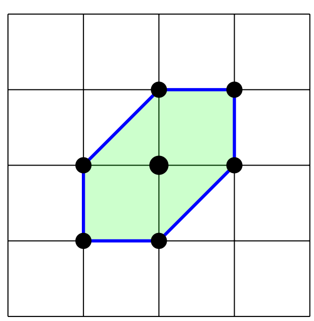
Group	0	1	2	3	4	5	Euler-characteristic
$H^i(X, \Omega_{\mathbb{P}^{12}} \otimes \mathscr{O}_X)$	0	1	0	167	0	0	-168
$H^i(Y, I_Y/I_Y^2)$	0	36	0	12	2	0	-46
$H^i(Y, \Omega_Y)$	0	1	12	2	0	0	-9

2 The singular locus of Y

By computing in each chart and taking closures, it can be computed that the singular locus of Y is of dimension 1, and consists of the union of projective lines. [[do this explcitily]]

3 Descriptions of dP_6

Recall that dP_6 is the toric variety whose associated polytope is the hexagon:



This induces an embedding into $\mathbb{P}^6$, by standard toric geometry. Let $\mathbb{P}^6$ have coordinates $y_0, x_1..x_6$ (corresponding to the center and the vertices, respectively). Then the ideal of dP_6 inside $\mathbb{P}^6$ is given by the 2×2-minors of the matrix

$$\begin{vmatrix} x_1 & y_0 & x_6 \\ x_2 & x_3 & y_0 \\ y_0 & x_4 & x_5 \end{vmatrix} \leq 1. \tag{1}$$

The $\mathbb{Z}_6$-symmetry is visible by permuting columns and rows.

Note that this representation of the ideal gives us an embedding of dP_6 into $\mathbb{P}^2 \times \mathbb{P}^2$ as a section of $\mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1) \oplus \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1)$ (namely as the zeros of $t_{12} - t_{23} = t_{23} - t_{31}$ (where t_{ij} are the natural coordinates on the product).

There are several other ways to veiw dP_6.

3.1 As $\mathbb{P}^2$ blown up in 3 points

Consider the monoidal transformation $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ given by $(u : v : w) \mapsto (uv : uw : vw)$. This is a birational involution with three points of indeterminacy: $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$ and $P_3 = (0 : 0 : 1)$. We blow up $\mathbb{P}^2$ in these three points to get a scheme $\widetilde{X}$ and a morphism $\pi : \widetilde{X} \to \mathbb{P}^2$. Then dP_6 is $\widetilde{X}$.

Remark. *Note that the involution $\widetilde{\varphi}$ lifts to an involution $\widetilde{\varphi} : \widetilde{X} \to \widetilde{X}$. We can realize $\widetilde{X}$ as the closure of the graph of φ:*

$$\widetilde{X} = \{(u : v : w) \times (a : b : c) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid vb = wa = uc\}.$$

Then the involution is $\widetilde{\varphi}(u : v : w, a : b : c) = (a : b : c, u : v : w)$.

It can be shown that the automorphism group of dP_6 is $(\mathbb{C}^*)^2 \rtimes (S_2 \times S_3)$ ([DOLGACHEV]). The lifted involution $\widetilde{\varphi}$ generates the S_2 part. The $(\mathbb{C}^*)^2$-part is inherited from the corresponding action on $\mathbb{P}^2$ and it can be computed to be given by

$$(t_1, t_2) \cdot ((u : v : w) \times (a : b : c)) = (t_1 u, t_2 v, t_1^{-1} t_2^{-1} w) \times (t_1 t_2 a : t_2^{-1} b : t_1^{-1} c).$$

The S_3 part comes from permuting the three points P_1, P_2 and P_3. If $\sigma \in S_3$ is a permutation of the variables u, v, w, then the corresponding action on $\widetilde{X}$ is given by $\sigma(P \times Q) = \sigma P \times \sigma^{-1} Q$. For example, the cyclic part is generated by

$$(u : v : w) \times (a : b : c) \mapsto (w : u : v) \times (b : c : a).$$

3.2 A natural embedding in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Let D be the divisor $D = 2(u) + 2(v) + 2(w)$ in $Div(\mathbb{P}^2)$ and consider the linear system $|D|$. Let

$$f_1 = \frac{uv}{w^2} \qquad f_2 = \frac{uw}{v^2} \qquad f_3 = \frac{vw}{u^2}$$

be three sections. Together they define a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The base locus consist exactly of the three points P_1, P_2, P_3 above. So again we can blow up to resolve the locus of indeterminacy to get a map $\widetilde{X} \to (\mathbb{P}^1)^3$.

If t_i, s_i are coordinates on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, 2, 3$, then the equation of the image is given by $t_1 t_2 t_3 = s_1 s_2 s_3$.

4 Deformations of dP_6

Since dP_6 is smooth, the only singularity of its affine cone, $C(dP_6)$, is the origin. One can compute that $T^1(C(dP_6)) = 3$, and that the versal base space splits into two components: a line and a plane intersecting transversely.

4.1 The first smoothing of the affine cone

We attempt to give explicit descriptions of the two affine smoothing components of $C(dP_6)$.

One of the components is given by:

$$\begin{vmatrix} x_1 & y_0 & x_6 \\ x_2 & x_3 & y_0 - t_1 \\ y_0 - t_2 & x_4 & x_5 \end{vmatrix} \leq 1.$$

That is, as the 2×2-minors of the above matrix. This time we see that the affine cone $C(dP_6)$ embeds naturally in the affine cone over $C(\mathbb{P}^2 \times \mathbb{P}^2)$, again as the intersection of two hyperplanes, but with some coefficients added.

It can be computed that the locus of points in $\mathbb{A}^2$ with singular fibers have ideal generated by $st(s + t) = s^2 t + t^2 s$, namely the union of the axes and a line.

4.2 The other smoothing of the affine cone

The other smoothing is derived from another way of writing the equations of dP_6. See Figure 1. One obtains the equations for this "$2 \times 2 \times 2$-tensor" by taking 2×2-minors along the faces and along long diagonals.

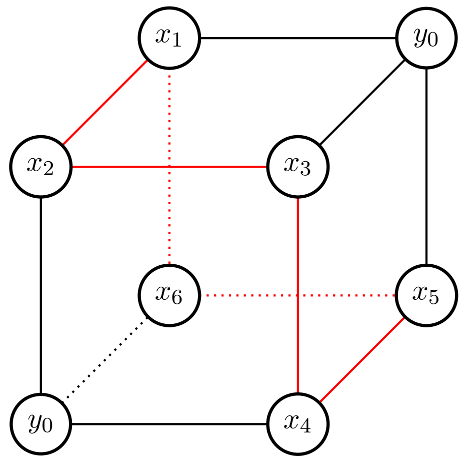


Figure 1: Equations of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

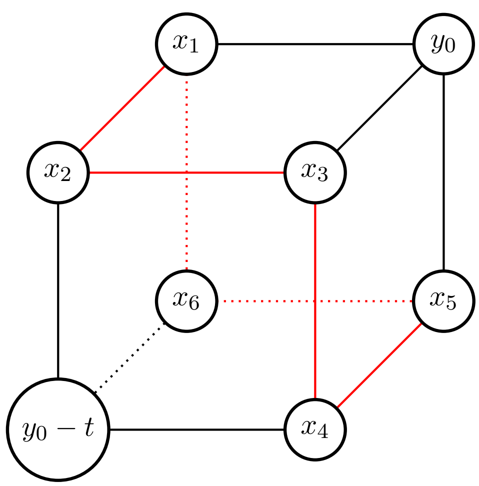


Figure 2: Deforming $C(dP_6)$.

It is clear that the one-dimensional component is a smoothing of $C(dP_6)$, since it can be obtained as a generic hyperplane in $C(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$.

5 Topology of the smootings of the affine cones

Since the singularities of our Calabi-Yau look locally like affine cones over dP_6, and the smoothings are given by locally smoothing these cones, we would like to compute their topology.

5.1 The first smoothing of $C(dP_6)$

Recall that one of the smoothing components of $C(dP_6)$ is given by the equations of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding in $\mathbb{P}^7$, replacing one of the corners by $y_0 + t$.

The total family lies in $\mathbb{A}^8$, and we can take its projective closure in $\mathbb{P}^8$ by homogenizing the equations (treating the variable s_1 as a constant of degree 0). Thus we get a family $\mathscr{X} \to \mathbb{A}^1$, where $\mathscr{X}_s$ is a projective variety for each $s \in \mathbb{A}^1$. For $s = 0$, we get the projective cone over dP_6, and for $s \neq 0$, we get something isomorphic (after a linear change of coordinates) to $(\mathbb{P}^1)^3$. By inspection, we see that what is gained in the projective closure is exactly dP_6 (for $s \neq 0$). Hence the smoothing of $C(dP_6)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \backslash dP_6$.

Let $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and let $D = dP_6$. There is the so-called "homology Gysin sequence" (from [1]), which will help us compute much of the homology of M:

$$\ldots \to H_{k+1}(M) \to H_{k-1}(D) \to H_k(M \backslash D) \to H_k(M) \to H_{k-2}(D) \to H_{k-1}(M \backslash D) \to \ldots$$

Since $\mathbb{P}^1 \simeq S^2$, we can use the Künneth formula to compute the homology of $(\mathbb{P}^1)^3$:

$$H_i(M) = \begin{cases} 1 & i = 0 \\ 0 & i = 1 \\ 3 & i = 2 \\ 0 & i = 3 \\ 3 & i = 4 \\ 0 & i = 5 \\ 1 & i = 6 \end{cases}$$

The homology of the del Pezzo is given by

$$H_i(D) = \begin{cases} 1 & i = 0 \\ 0 & i = 1 \\ 4 & i = 2 \\ 0 & i = 3 \\ 1 & i = 4 \end{cases}$$

We have that $H^6(M\backslash D) = 0$, since $M\backslash D$ is non-compact. It follows by the long-exact sequence that $H^5(M\backslash D) = 0$ also. Writing up the long exact sequence, we find some of the homology

$$H_i(M\backslash D) = \begin{cases} 1 & i = 0 \\ ? & i = 1 \\ ?' & i = 2 \\ ?'' & i = 3 \\ ?''' & i = 4 \\ 0 & i = 5, 6. \end{cases}$$

Since $\chi(M) = 8$[1] and $\chi(dP_6) = 6$, we know that $\chi(M\backslash D) = 2$.

Proposition 5.1. *Let $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and let $D = \{x_0x_1x_2 = y_0y_1y_2\}$. Let $U = M\backslash U$. Then:*

$$H^1(U, \mathbb{Q}) = 0 \text{ and} \qquad H^3(U, \mathbb{Q}) = \mathbb{Q}$$
$$H^2(U, \mathbb{Q}) = 2 \text{ and} \qquad H^4(U, \mathbb{Q}) = 0.$$

Proof. We use Lefschetz duality (see [3, Chapter 6, p. 297]), which says that in this case, we have that $H_1(M\backslash D) \simeq \overline{H}^5(M, D)$. The long exact sequence of a pair is in this case:

$$0 \to H^4(M, D) \to H^4(M) \xrightarrow{j^*} H^4(D) \to H^5(M, D) \to 0.$$

We have that $H^4(M) = \mathbb{Q}^3$, generated by the duals of the three divisors on M, call the divisors D_1, D_2, D_3. $H^4(D)$ is generated by the fundamental class. Over a field, cohomology is dual to homology (see [2, Chapter 3, p. 195]), so we look at the dual map

$$j_* : H_4(D) \to H_4(M).$$

[1]Heuristic: Euler-characteristic is some kind of volume. And $\mathbb{P}^1$ has Euler characteristic two, and volume should be multiplicative.

The equation $x_0x_1x_2 = y_0y_1y_2$ for D can be deformed by adding a variable t, and letting $t \to 0$: we write $x_0x_1x_2 - ty_0y_1y_2 = 0$. Then for $t = 0$, we get a divisor in M equivalent to $D_1 + D_2 + D_3$. Thus the map j_* is just given by $D \mapsto D_1 + D_2 + D_3$, hence the dual map is given by $D_i^* \mapsto D^*$. This is a surjective map, and we conclude that $H_1(U) = 0$. □

Thus we know all the homology groups of the complement. In particular, the Euler characteristic is correct.

5.2 The second smoothing

Proposition 5.2. *The second smoothing M_2 is isomorphic to $(\mathbb{P}^2 \times \mathbb{P}^2) \cap H \backslash dP_6$, where H is a hyperplane section.*

Proof. Compare the equations of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ with the equations of a generic fiber of the second smoothing:

$$\begin{vmatrix} x_1 & y_0 & x_6 \\ x_2 & x_3 & y_0 + t \\ y_0 + s & x_4 & x_5 \end{vmatrix} \leq 1 \qquad\qquad \begin{vmatrix} x_1 & y_0 & x_6 \\ x_2 & x_3 & y_1 \\ y_2 & x_4 & x_5 \end{vmatrix} \leq 1.$$

Homogenize the first equations with respect to the variable y_1. Then, in the coordinates of $\mathbb{P}^8$, the homogenized variety corresponds to the variety given as $(\mathbb{P}^2 \times \mathbb{P}^2) \cap \{(t(t_{32} - t_{11}) = s(t_{23} - t_{11})\}$.

Now we check what we gained by homogenizing. $\mathbb{P}^8 \backslash \mathbb{A}^8$ is given by the divisor $\{y_1 = 0\}$, but this is exactly equal to dP_6. □

Using the same technique as with the other deformation, we can compute the homology groups of the complement. They are:

$$(h_0, h_1, \ldots, h_6) = (1, 0, 1, 2, 0, 0, 0),$$

giving a topological Euler characteristic of 0. Comparing with the h-vector of the other deformation, it appears that the other deformation is a degeneration of this one (contracting a 3-cell to a 2-cell).

References

[1] Alexandru Dimca. *Singularities and topology of hypersurfaces*. Universitext. Springer-Verlag, New York, 1992.

[2] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[3] Edwin H. Spanier. *Algebraic topology.* McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.