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1 Derived categories and derived functors

We first start with notation. Let \mathbb{A} be an abelian category. Let $C(\mathbb{A})$ be the category of complexes with objects and maps from \mathbb{A} . We denote by $C^b(\mathbb{A})$ the category of bounded complexes. $C^+(\mathbb{A})$ is the category of bounded below complexes, and $C^-(\mathbb{A})$ is the category of bounded above complexes.

Definition 1.1. Two morphisms $f, g: A^{\bullet} \to B^{\bullet}$ are homotopy equivalent if there exists a set of maps $h^i: A^i \to B^{i-1}$ such that

$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i.$$

The homotopy category $K(\mathbb{A})$ is the category with the same objects as $C(\mathbb{A})$, but where homotopic morphisms are identified.

This is a triangulated category. For this we need a shift operator: $A^{\bullet} \mapsto A^{\bullet}[1]$ (topologists like to write $A^{\bullet}[1] = \Sigma A^{\bullet}$). The differential is $d^{i}_{A[1]} = -d^{i+1}_{A}$.

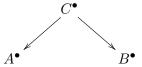
The cone of a morphisms $j: A^{\bullet} \to B^{\bullet}$ is denoted by C(j) and defined as follows: each term is $C(j)^i = A^{i+1} \oplus B^i$, and the maps are

$$d_{C(j)} = \begin{pmatrix} -d_A^{i+1} & 0\\ j^{i+1} & d_B^i \end{pmatrix}.$$

Then a distinguised triangle is a sequence of objects in $K(\mathbb{A})$ and morphisms isomorphic to the sequence

$$A^{\bullet} \xrightarrow{j} B^{\bullet} \to C(j) \to A[1].$$

The derived category $D(\mathbb{A})$ is the category with objects the same as $C(\mathbb{A})$, and morphisms obtained by localizing $K(\mathbb{A})$ with respect to quasiisomorphisms (that is, we require quasiisomorphisms to be invertible). Formally, a morphisms $A^{\bullet} \to B^{\bullet}$ in the derived category is defined as a *roof*:



The left arrow is a quasiisomorphism.

Knowing that $K(\mathbb{A})$ is triangulated, we deduce (somehow) that $D(\mathbb{A})$ is triangulated as well.

1.1 Two examples

- 1. Let X be a smooth point. Then D(X) is easy to describe. All complexes of vector spaces are quasiisomorphic to the direct sum of their cohomologies. (more explicit??)
- 2. Let X be a smooth projective curve and $\mathcal{E} \in \mathsf{Coh}(()X)$. Then there is an exact sequence

$$0 \to \mathcal{T} \to \mathcal{E} \to \mathcal{E}_{tf} \to 0$$
,

where the last term is torsionfree and the first term is a torsion sheaf. This sequence splits (why?). And one can show that

$$D^b(X) \ni \mathcal{E} = \bigoplus_{i \in I} \mathscr{H}^i(\mathcal{E})[-i] = \bigoplus_i (\mathscr{H}^i(\mathcal{E}_{tors}) \oplus \mathscr{H}^i(\mathcal{E})_{tf})$$

(I don't see why?)

1.2 Semiorthogonal decomposition for $D^b(X)$

Suppose we have a sequence of full triangulated subcategories of $D^b(X)$.

Definition 1.2. A collection T_1, \ldots, T_n defines a semiorthogonal decomposition for $D^b(X)$ if

- 1. $\operatorname{Hom}_{D^b(X)}(T_i, T_j) = 0 \text{ for } i > j.$
- 2. For all $F \in D^b(X)$, there exists a chain of morphisms

$$0 \to E_n \to E_{n-1} \to \dots \to E_0 = F,$$

such that $Cone(E_i, E_{i-1}) \in T_i$.

We write $D^b(X) = \langle T_1, \dots, T_n \rangle$.

Example 1.3. Beilinson proved that $D^b(\mathbb{P}^n) = \langle \mathscr{O}_{\mathbb{P}^n}, \dots \mathscr{O}_{\mathbb{P}^n}(n) \rangle$.

Example 1.4. If X is a curve (smooth projective over \mathbb{C}), then $D^b(X)$ has a nontrivial semiorthogonal decomposition if and only if the genus is zero. \bigstar

Example 1.5. If X is smooth projective with $\omega_X = \mathscr{O}_X$, then $D^b(X)$ has no semi-orthogonal decomposition.

1.3 Derived functors