

# Algebraic Geometry Buzzlist

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## 1 Algebraic Geometry

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## 1.1 General terms

### 1.1.1 Cartier divisor

Let  $\mathcal{K}_X$  be the *sheaf of total quotients* on  $X$ , and let  $\mathcal{O}_X^*$  be the sheaf of non-zero divisors on  $X$ . We have an exact sequence

$$1 \rightarrow \mathcal{O}_X^* \rightarrow \mathcal{K}_X \rightarrow \mathcal{K}_X / \mathcal{O}_X^* \rightarrow 1.$$

Then a **Cartier divisor** is a global section of the quotient sheaf at the right.

### 1.1.2 Categorical quotient

Let  $X$  be a scheme and  $G$  a group. A **categorical quotient** is a morphism  $\pi : X \rightarrow Y$  that satisfies the following two properties:

1. It is invariant, in the sense that  $\pi \circ \sigma = \pi \circ p_2$  where  $\sigma : G \times X \rightarrow X$  is the group action, and  $p_2 : G \times X \rightarrow X$  is the projection. That is, the following diagram should commute:

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ p_2 \downarrow & & \downarrow \pi \\ X & \xrightarrow{\pi} & Y \end{array}$$

2. The map  $\pi$  should be *universal*, in the following sense: If  $\pi' : X \rightarrow Z$  is any morphism satisfying the previous condition, it should uniquely factor through  $\pi$ . That is:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ \pi' \downarrow & \nearrow \exists! h & \\ Z & & \end{array}$$

Note: A categorical quotient need not be surjective.

### 1.1.3 Chow group

Let  $X$  be an algebraic variety. Let  $Z_r(X)$  be the group of  $r$ -dimension cycles on  $X$ , a *cycle* being a  $\mathbb{Z}$ -linear combination of  $r$ -dimensional subvarieties of  $X$ . If  $V \subset X$  is a subvariety of dimension  $r + 1$  and  $f : X \dashrightarrow \mathbb{A}^1$  is a rational function on  $X$ , then there is an integer  $\text{ord}_W(f)$  for each codimension one subvariety of  $V$ , the order of vanishing of  $f$ . For a given  $f$ , there will only be finitely many subvarieties  $W$  for which this number is non-zero. Thus we can define an element  $[\text{div}(f)]$  in  $Z_r(X)$  by  $\sum \text{ord}_W(f)[W]$ .

We say that two  $r$ -cycles  $U_1, U_2$  are *rationally equivalent* if there exist  $r + 1$ -dimensional subvarieties  $V_1, V_2$  together with rational functions  $f_1 : V_1 \dashrightarrow \mathbb{A}^1$ ,  $f_2 : V_2 \dashrightarrow \mathbb{A}^1$  such that  $U_1 - U_2 = \sum_i [\text{div}(f_i)]$ . The quotient group is called the **Chow group** of  $r$ -dimensional cycles on  $X$ , and denoted by  $A_r(X)$ .

### 1.1.4 Complete variety

Let  $X$  be an integral, **separated** scheme over a field  $k$ . Then  $X$  is **complete** if is **proper**.

Then  $\mathbb{P}^n$  is proper over any field, and  $\mathbb{A}^n$  is never proper.

### 1.1.5 Crepant resolution

A **crepant resolution** is a resolution of singularities  $f : X \rightarrow Y$  that does not change the **canonical bundle**, i.e. such that  $\omega_X \simeq f^*\omega_Y$ .

### 1.1.6 Dominant map

A rational map  $f : X \dashrightarrow Y$  is **dominant** if its image (or precisely: the image of one of its representatives) is dense in  $Y$ .

### 1.1.7 Étale map

A morphism of schemes of finite type  $f : X \rightarrow Y$  is **étale** if it is smooth of dimension zero. This is equivalent to  $f$  being flat and  $\Omega_{X/Y} = 0$ . This again is equivalent to  $f$  being flat and unramified.

### 1.1.8 Genus

The **geometric genus** of a smooth, algebraic variety, is defined as the number of sections of the **canonical sheaf**, that is, as  $H^0(V, \omega_X)$ . This is often denoted  $p_X$ .

### 1.1.9 Geometric quotient

Let  $X$  be an algebraic variety and  $G$  an algebraic group. Then a **geometric quotient** is a morphism of varieties  $\pi : X \rightarrow Y$  such that

1. For each  $y \in Y$ , the fiber  $\pi^{-1}(y)$  is an orbit of  $G$ .
2. The topology of  $Y$  is the quotient topology: a subset  $U$  of  $Y$  is open if and only if  $\pi^{-1}(U)$  is open.
3. For any open subset  $U \subset Y$ ,  $\pi^* : k[U] \rightarrow k[\pi^{-1}(U)]^G$  is an isomorphism of  $k$ -algebras.

The last condition may be rephrased as an isomorphism of structure sheaves:  $\mathcal{O}_Y \simeq (\pi_* \mathcal{O}_X)^G$ .

### 1.1.10 Hodge numbers

If  $X$  is a complex manifold, then the **Hodge numbers**  $h^{pq}$  of  $X$  are defined as the dimension of the cohomology groups  $H^q(X, \Omega_X^p)$ .

### 1.1.11 Intersection multiplicity (of curves on a surface)

Let  $C, D$  be two curves on a smooth surface  $X$  and  $P$  is a point on  $X$ , then the **intersection multiplicity**  $(C \cdot D)_P$  of  $C$  and  $D$  at  $P$  is defined to be the length of  $\mathcal{O}_{P,X} / (f, g)$ .

Example: let  $C, D$  be the curves  $C = \{y^2 = x^3\}$  and  $D = \{x = 0\}$ . Then  $\mathcal{O}_{0,A^2} / (y^2 - x^3, x) = k[x, y]_{(x,y)} / (x, y^2 - x) = k[y]_{(y)} / (y^2) = k \oplus y \cdot k$ , so the tangent line of the cusp meets it with multiplicity two.

### 1.1.12 Linear series

A **linear series** on a smooth curve  $C$  is the data  $(\mathcal{L}, V)$  of a line bundle on  $C$  and a vector subspace  $V \subseteq H^0(C, \mathcal{L})$ . We say that the linear series  $(\mathcal{L}, V)$  have *degree*  $\deg \mathcal{L}$  and *rank*  $\dim V - 1$ .

### 1.1.13 Log structure

A **prelog structure** on a scheme  $X$  is given by a pair  $(X, M)$ , where  $X$  is a scheme and  $M$  is a sheaf of monoids on  $X$  (on the **Étale site**) together with a morphisms  $\alpha : M \rightarrow \mathcal{O}_X$ . It is a **log structure** if the map  $\alpha : \alpha^{-1} \mathcal{O}_X^* \rightarrow \mathcal{O}_X^*$  is an isomorphism.

See [5].

#### 1.1.14 Néron-Severi group

Let  $X$  be a nonsingular projective variety of dimension  $\geq 2$ . Then we can define the subgroup  $\text{Cl}^\circ X$  of  $\text{Cl } X$ , the subgroup consisting of divisor classes algebraically equivalent to zero. Then  $\text{Cl } X / \text{Cl}^\circ X$  is a finitely-generated group. It is denoted by  $\text{NS}(X)$ .

#### 1.1.15 Normal crossings divisor

Let  $X$  be a smooth variety and  $D \subset X$  a divisor. We say that  $D$  is a **simple normal crossing divisor** if every irreducible component of  $D$  is smooth and all intersections are transverse. That is, for every  $p \in X$  we can choose local coordinates  $x_1, \dots, x_n$  and natural numbers  $m_1, \dots, m_n$  such that  $D = (\prod_i x_i^{m_i} = 0)$  in a neighbourhood of  $p$ .

Then we say that a divisor is **normal crossing** (without the “simple”) if the neighbourhood above can be chosen locally analytically or as a formal neighbourhood of  $p$ .

Example: the nodal curve  $y^2 = x^3 + x^2$  is a normal crossing divisor in  $\mathbb{C}^2$ , but not a simple normal crossing divisor.

This definition is taken from [6].

#### 1.1.16 Normal variety

A variety  $X$  is **normal** if all its local rings are **normal rings**.

#### 1.1.17 Picard number

The **Picard number** of a nonsingular projective variety is the rank of **Néron-Severi group**.

#### 1.1.18 Proper morphism

A morphism  $f : X \rightarrow Y$  is **proper** if it **separated**, of finite type, and universally closed.

#### 1.1.19 Resolution of singularities

A morphism  $f : X \rightarrow Y$  is a **resolution of singularities of  $Y$**  if  $X$  is non-singular and  $f$  is birational and **proper**.

### 1.1.20 Separated

Let  $f : X \rightarrow Y$  be a morphism of schemes. Let  $\Delta : X \rightarrow X \times_Y X$  be the diagonal morphism. We say that  $f$  is **separated** if  $\Delta$  is a closed immersion. We say that  $X$  is **separated** if the unique morphism  $f : X \rightarrow \operatorname{Spec} \mathbb{Z}$  is separated.

This is equivalent to the following: for all open affines  $U, V \subset X$ , the intersection  $U \cap V$  is affine and  $\mathcal{O}_X(U)$  and  $\mathcal{O}_X(V)$  generate  $\mathcal{O}_X(U \cap V)$ . For example: let  $X = \mathbb{P}^1$  and let  $U_1 = \{[x : 1]\}$  and  $U_2 = \{[1 : y]\}$ . Then  $\mathcal{O}_X(U_1) = \operatorname{Spec} k[x]$  and  $\mathcal{O}_X(U_2) = \operatorname{Spec} k[y]$ . The glueing map is given on the ring level as  $x \mapsto \frac{1}{y}$ . Then  $\mathcal{O}_X(U_1 \cap U_2) = k[y, \frac{1}{y}]$ .

### 1.1.21 Unirational variety

A variety  $X$  is **unirational** if there exists a generically finite dominant map  $\mathbb{P}^n \dashrightarrow X$ .

## 1.2 Moduli theory and stacks

### 1.2.1 Étale site

Let  $S$  be a scheme. Then the **small étale site over  $S$**  is the **site**, denoted by  $\acute{\text{E}}\text{t}(S)$  that consists of all étale morphisms  $U \rightarrow S$  (morphisms being commutative triangles). Let  $\operatorname{Cov}(U \rightarrow S)$  consist of all collections  $\{U_i \rightarrow U\}_{i \in I}$  such that

$$\coprod_{i \in I} U_i \rightarrow U$$

is surjective.

### 1.2.2 Grothendieck topology

Let  $\mathcal{C}$  be a category. A **Grothendieck topology** on  $\mathcal{C}$  consists of a set  $\operatorname{Cov}(X)$  of sets of morphisms  $\{X_i \rightarrow X\}_{i \in I}$  for each  $X$  in  $\operatorname{Ob}(\mathcal{C})$ , satisfying the following axioms:

1. If  $V \xrightarrow{\sim} X$  is an isomorphism, then  $\{V \rightarrow X\} \in \operatorname{Cov}(X)$ .
2. If  $\{X_i \rightarrow X\}_{i \in I} \in \operatorname{Cov}(X)$  and  $Y \rightarrow X$  is a morphism in  $\mathcal{C}$ , then the fiber products  $X_i \times_X Y$  exists and  $\{X_i \times_X Y \rightarrow Y\}_{i \in I} \in \operatorname{Cov}(Y)$ .
3. If  $\{X_i \rightarrow X\}_{i \in I} \in \operatorname{Cov}(X)$ , and for each  $i \in I$ ,  $\{V_{ij} \rightarrow X_i\}_{j \in J} \in \operatorname{Cov}(X_i)$ , then

$$\{V_{ij} \rightarrow X_i \rightarrow X\}_{i \in I, j \in J} \in \operatorname{Cov}(X).$$

The easiest example is this: Let  $\mathcal{C}$  be the category of open sets on a topological space  $X$ , the morphisms being only the inclusions. Then for each  $U \in \text{Ob}(\mathcal{C})$ , define  $\text{Cov}(U)$  to be the set of all coverings  $\{U_i \rightarrow U\}_{i \in I}$  such that  $U = \bigcup_{i \in I} U_i$ . Then it is easily checked that this defines a Grothendieck topology.

### 1.2.3 Site

A **site** is a category equipped with a **Grothendieck topology**.

## 1.3 Results and theorems

### 1.3.1 Adjunction formula

Let  $X$  be a smooth algebraic variety  $Y$  a smooth subvariety. Let  $i : Y \hookrightarrow X$  be the inclusion map, and let  $\mathcal{I}$  be the corresponding ideal sheaf. Then  $\omega_Y = i^* \omega_X \otimes_{\mathcal{O}_X} \det(\mathcal{I}/\mathcal{I}^2)^\vee$ , where  $\omega_Y$  is the **canonical sheaf** of  $Y$ .

In terms of **canonical classes**, the formula says that  $K_D = (K_X + D)|_D$ .

Here's an example: Let  $X$  be a smooth quartic surface in  $\mathbb{P}^3$ . Then  $H^1(X, \mathcal{O}_X) = 0$ . The divisor class group of  $\mathbb{P}^3$  is generated by the class of a hyperplane, and  $\mathcal{K}_{\mathbb{P}^3} = -4H$ . The class of  $X$  is then  $4H$  since  $X$  is of degree 4.  $X$  corresponds to a smooth divisor  $D$ , so by the adjunction formula, we have that

$$K_D = (K_{\mathbb{P}^3} + D)|_D = -4H + 4H|_D = 0.$$

Thus  $X$  is an example of a **K3 surface**.

### 1.3.2 Bertini's Theorem

Let  $X$  be a nonsingular closed subvariety of  $\mathbb{P}_k^n$ , where  $k = \bar{k}$ . Then the set of hyperplanes  $H \subseteq \mathbb{P}_k^n$  such that  $H \cap X$  is regular at every point) and such that  $H \not\subseteq X$  is a dense open subset of the complete linear system  $|H|$ . See [4, Thm II.8.18].

### 1.3.3 Chow's lemma

Chow's lemma says that if  $X$  is a scheme that is proper over  $k$ , then it is "fairly close" to being projective. Specifically, we have that there exists a projective  $k$ -scheme  $X'$  and morphism  $f : X' \rightarrow X$  that is birational.

So every scheme proper over  $k$  is birational to a projective scheme. For a proof, see for example the Wikipedia page.

### 1.3.4 Euler sequence

If  $A$  is a ring and  $\mathbb{P}_A^n$  is projective  $n$ -space over  $A$ , then there is an exact sequence of sheaves on  $X$ :

$$0 \rightarrow \Omega_{\mathbb{P}_A^n/A} \rightarrow \mathcal{O}_{\mathbb{P}_A^n}(-1)^{n+1} \rightarrow \mathcal{O}_{\mathbb{P}_A^n} \rightarrow 0.$$

See [4, Thm II.8.13].

### 1.3.5 Genus-degree formula

If  $C$  is a smooth plane curve, then its genus can be computed as

$$g_C = \frac{(d-1)(d-2)}{2}.$$

This follows from the [adjunction formula](#). In particular, there are no curves of genus 2 in the plane.

### 1.3.6 Hirzebruch-Riemann-Roch formula

Let  $X$  be a nonsingular variety and let  $\mathcal{T}_X$  be its tangent bundle. Let  $\mathcal{E}$  be a locally free sheaf on  $X$ . Then

$$\chi(\mathcal{E}) = \deg(\text{ch}(\mathcal{E}) \cdot \text{td}(\mathcal{T}_X))_n,$$

where  $\chi$  is the Euler characteristic,  $\text{ch}$  denotes the Chern class, and  $\text{td}$  denotes the Todd class. See [4, Appendix A].

### 1.3.7 Hurwitz' formula

Let  $X, Y$  be smooth curves in the sense of Hartshorne. That is, they are integral 1-dimensional schemes, proper over a field  $k$  (with  $\bar{k} = k$ ), all of whose local rings are regular.

Then Hurwitz' formula says that if  $f : X \rightarrow Y$  is a separable morphism and  $n = \deg f$ , then

$$2(g_X - 1) = 2n(g_Y - 1) + \deg R,$$

where  $R$  is the ramification divisor of  $f$ , and  $g_X, g_Y$  are the genera of  $X$  and  $Y$ , respectively. See Example [7.1.1](#).



### 1.3.8 Kodaira vanishing

If  $k$  is a field of characteristic zero,  $X$  is a smooth and projective  $k$ -scheme of dimension  $d$ , and  $\mathcal{L}$  is an **ample** invertible sheaf on  $X$ , then  $H^q(X, \mathcal{L} \otimes_{\mathcal{O}_X} \Omega_{X/k}^p) = 0$  for  $p + q > d$ . In addition,  $H^q(X, \mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \Omega_{X/k}^p) = 0$  for  $p + q < d$ .

### 1.3.9 Lefschetz hyperplane theorem

Let  $X$  be an  $n$ -dimensional complex projective algebraic variety in  $\mathbb{P}_{\mathbb{C}}^n$  and let  $Y$  be a hyperplane section of  $X$  such that  $U = X \setminus Y$  is smooth. Then the natural map  $H^k(X, \mathbb{Z}) \rightarrow H^k(Y, \mathbb{Z})$  in singular cohomology is an isomorphism for  $k < n - 1$  and injective for  $k = n - 1$ .

### 1.3.10 Riemann-Roch for curves

The **Riemann-Roch theorem** relates the number of sections of a line bundle with the genus of a smooth proper curve  $C$ . Let  $\mathcal{L}$  be a line bundle  $\omega_C$  the canonical sheaf on  $C$ . Then

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^{-1} \otimes_{\mathcal{O}_C} \omega_C) = \deg(\mathcal{L}) + 1 - g.$$

This is [4, Theorem IV.1.3].

### 1.3.11 Semi-continuity theorem

Let  $f : X \rightarrow Y$  be a projective morphism of noetherian schemes, and let  $\mathcal{F}$  be a coherent sheaf on  $X$ , flat over  $Y$ . Then for each  $i \geq 0$ , the function  $h^i(y, \mathcal{F}) = \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$  is an upper semicontinuous function on  $Y$ . See [4, Chapter III, Theorem 12.8].

### 1.3.12 Serre duality

Let  $X$  be a projective Cohen-Macaulay scheme of equidimension  $n$ . Then for any locally free sheaf  $\mathcal{F}$  on  $X$  there are natural isomorphisms

$$H^i(X, \mathcal{F}) \simeq H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ).$$

Here  $\omega_X^\circ$  is a *dualizing sheaf* for  $X$ . In the case that  $X$  is nonsingular, we have that  $\omega_X^\circ \simeq \omega_X$ , the canonical sheaf on  $X$  (see [4, Chapter III, Corollary 7.12]).

### 1.3.13 Serre vanishing

One form of Serre vanishing states that if  $X$  is a proper scheme over a noetherian ring  $A$ , and  $\mathcal{L}$  is an **ample** sheaf, then for any coherent sheaf  $\mathcal{F}$  on  $X$ , there exists an integer  $n_0$  such that for each  $i > 0$  and  $n \geq n_0$  the group  $H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n) = 0$  vanishes. See [4, Proposition III.5.3].

### 1.3.14 Weil conjectures

The Weil conjectures is a theorem relating the properties of a variety over finite fields with its properties over fields over characteristic zero.

Specifically, let

$$\zeta(X, s) = \exp \left( \sum_{m=1}^{\infty} \frac{N_m}{m} q^{-sm} \right)$$

be the zeta function of  $X$  (with respect to  $q$ ).  $N_m$  is the number of points of  $X$  over  $\mathbb{F}_{q^n}$ . Then the Weil conjectures are the following four statements:

1. The zeta function  $\zeta(X, s)$  is a rational function of  $T = q^{-s}$ :

$$\zeta(X, T) = \prod_{i=1}^{2n} P_i(T)^{(-1)^{i+1}},$$

where the  $P_i$ 's are integral polynomials. Furthermore,  $P_0(T) = 1 - T$  and  $P_{2n}(T) = 1 - q^n T$ . For  $1 \leq i \leq 2n - 1$ ,  $P_i(T)$  factors as  $P_j(T) = \prod (1 - \alpha_{ij} T)$  over  $\mathbb{C}$ .

2. There is a functional equation. Let  $E$  be the topological Euler characteristic of  $X$ . Then

$$\zeta(X, q^{-n} T^{-1}) = \pm q^{\frac{nE}{2}} T^E \zeta(X, T).$$

3. A ‘‘Riemann hypothesis’’:  $|\alpha_{ij}| = q^{i/2}$  for all  $1 \leq i \leq 2n - 1$  and all  $j$ . This implies that the zeroes of  $P_k(T)$  all lie on the critical line  $\Re(z) = k/2$ .
4. If  $X$  is a good reduction modulo  $p$ , then the degree of  $P_i$  is equal to the  $i$ 'th Betti number of  $X$ , seen as a complex variety.

## 1.4 Sheaves and bundles

### 1.4.1 Ample line bundle

A line bundle  $\mathcal{L}$  is **ample** if for any coherent sheaf  $\mathcal{F}$  on  $X$ , there is an integer  $n$  (depending on  $\mathcal{F}$ ) such that  $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$  is generated by global sections. Equivalently, a line bundle  $\mathcal{L}$  is ample if some tensor power of it is **very ample**.

### 1.4.2 Invertible sheaf

A locally free sheaf of rank 1 is called **invertible**. If  $X$  is **normal**, then, invertible sheaves are in 1 – 1 correspondence with line bundles.

### 1.4.3 Anticanonical sheaf

The **anticanonical sheaf**  $\omega_X^{-1}$  is the inverse of the **canonical sheaf**  $\omega_X$ , that is  $\omega_X^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{O}_X)$ .

### 1.4.4 Canonical class

The **canonical class**  $K_X$  is the class of the **canonical sheaf**  $\omega_X$  in the divisor class group.

### 1.4.5 Canonical sheaf

If  $X$  is a smooth algebraic variety of dimension  $n$ , then the canonical sheaf is  $\omega := \wedge^n \Omega_{X/k}^1$  the  $n$ 'th exterior power of the cotangent bundle of  $X$ .

### 1.4.6 Nef divisor

Let  $X$  be a normal variety. Then a Cartier divisor  $D$  on  $X$  is **nef** (*numerically effective*) if  $D \cdot C \geq 0$  for every irreducible complete curve  $C \subseteq X$ . Here  $D \cdot C$  is the intersection product on  $X$  defined by  $\deg(\phi^* \mathcal{O}_X(D))$ . Here  $\phi : C' \rightarrow C$  is the normalization of  $C$ .

### 1.4.7 Sheaf of holomorphic p-forms

If  $X$  is a complex manifold, then the **sheaf of holomorphic  $p$ -forms**  $\Omega_X^p$  is the  $p$ -th wedge power of the cotangent sheaf  $\wedge^p \Omega_X^1$ .

#### 1.4.8 Normal sheaf

Let  $Y \hookrightarrow X$  be a closed immersion of schemes, and let  $\mathcal{I} \subseteq \mathcal{O}_X$  be the ideal sheaf of  $Y$  in  $X$ . Then  $\mathcal{I}/\mathcal{I}^2$  is a sheaf on  $Y$ , and we define the sheaf  $\mathcal{N}_{Y/X}$  by  $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ .

#### 1.4.9 Rank of a coherent sheaf

Given a coherent sheaf  $\mathcal{F}$  on an irreducible variety  $X$ , form the sheaf  $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ . Its global sections is a finite dimensional vector space, and we say that  $\mathcal{F}$  has rank  $r$  if  $\dim_k \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X) = r$ .

#### 1.4.10 Reflexive sheaf

A sheaf  $\mathcal{F}$  is **reflexive** if the natural map  $\mathcal{F} \rightarrow \mathcal{F}^{\vee\vee}$  is an isomorphism. Here  $\mathcal{F}^{\vee}$  denotes the sheaf  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ .

#### 1.4.11 Very ample line bundle

A line bundle  $\mathcal{L}$  is **very ample** if there is an embedding  $i : X \hookrightarrow \mathbb{P}_S^n$  such that the pullback of  $\mathcal{O}_{\mathbb{P}_S^n}(1)$  is isomorphic to  $\mathcal{L}$ . In other words, there should be an isomorphism  $i^* \mathcal{O}_{\mathbb{P}_S^n}(1) \simeq \mathcal{L}$ .

### 1.5 Singularities

#### 1.5.1 Canonical singularities

A variety  $X$  has **canonical singularities** if it satisfies the following two conditions:

1. For some integer  $r \geq 1$ , the Weil divisor  $rK_X$  is Cartier (equivalently, it is  $\mathbb{Q}$ -Cartier).
2. If  $f : Y \rightarrow X$  is a resolution of  $X$  and  $\{E_i\}$  the exceptional divisors, then

$$rK_Y = f^*(rK_X) + \sum a_i E_i$$

with  $a_i \geq 0$ .

The integer  $r$  is called the *index*, and the  $r_i$  are called the *discrepancies* at  $E_i$ .

### 1.5.2 Terminal singularities

A variety  $X$  have **terminal singularities** if the  $a_i$  in the definition of **canonical singularities** are all greater than zero.

### 1.5.3 Ordinary double point

An **ordinary double point** is a singularity that is analytically isomorphic to  $x^2 = yz$ .

## 1.6 Toric geometry

### 1.6.1 Chow group of a toric variety

The **Chow group**  $A_{n-1}(X)$  of a toric variety can be computed directly from its fan. Let  $\Sigma(1)$  be the set of rays in  $\Sigma$ , the fan of  $X$ . Then we have an exact sequence

$$0 \rightarrow M \rightarrow \mathbb{Z}^{\Sigma(1)} \rightarrow A_{n-1}(X) \rightarrow 0.$$

The first map is given by sending  $m \in M$  to  $(\langle m, v_\rho \rangle)_{\rho \in \Sigma(1)}$ , where  $v_\rho$  is the unique generator of the semigroup  $\rho \cap N$ . The second map is given by sending  $(a_\rho)_{\rho \in \Sigma(1)}$  to the divisor class of  $\sum_\rho a_\rho D_\rho$ .

### 1.6.2 Generalized Euler sequence

The **generalized Euler sequence** is a generalization of the **Euler sequence** for toric varieties. If  $X$  is a smooth toric variety, then its cotangent bundle  $\Omega_X^1$  fits into an exact sequence

$$0 \rightarrow \Omega_X^1 \rightarrow \oplus_\rho \mathcal{O}_X(-D_\rho) \rightarrow \text{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \rightarrow 0.$$

Here  $D_\rho$  is the divisor corresponding to the ray  $\rho \in \Sigma(1)$ . See [2, Chapter 8].

### 1.6.3 Polarized toric variety

A toric variety equipped with an **ample**  $T$ -invariant divisor.

### 1.6.4 Toric variety associated to a polytope

There are several ways to do this. Here is one: Let  $\Delta \subset M_{\mathbb{R}}$  be a convex polytope. Embed  $\Delta$  in  $M_{\mathbb{R}} \times \mathbb{R}$  by  $\Delta \times \{1\}$  and let  $C_\Delta$  be the cone over  $\Delta \times \{1\}$ , and let  $\mathbb{C}[C_\Delta \cap (M \times \mathbb{Z})]$  be the corresponding semigroup ring. This is a semigroup ring graded by the  $\mathbb{Z}$ -factor. Then we define  $\mathbb{P}_\Delta = \text{Proj } \mathbb{C}[C_\Delta \cap (M \times \mathbb{Z})]$  to be the toric variety associated to a polytope.

## 1.7 Types of varieties

### 1.7.1 Abelian variety

A variety  $X$  is an **abelian variety** if it is a connected and **complete** algebraic group over a field  $k$ . Examples include **elliptic curves** and for special lattices  $\Lambda \subset \mathbb{C}^{2g}$ , the quotient  $\mathbb{C}^{2g}/\Lambda$  is an abelian variety.

### 1.7.2 Calabi-Yau variety

In algebraic geometry, a **Calabi-Yau** variety is a smooth, proper variety  $X$  over a field  $k$  such that the **canonical sheaf** is trivial, that is,  $\omega_X \simeq \mathcal{O}_X$ , and such that  $H^j(X, \mathcal{O}_X) = 0$  for  $1 \leq j \leq n - 1$ .

### 1.7.3 del Pezzo surface

A **del Pezzo** surface is a 2-dimensional **Fano variety**. In other words, they are complete non-singular surfaces with ample anticanonical bundle. The *degree* of the del Pezzo surface  $X$  is by definition the self intersection number  $K.K$  of its **canonical class**  $K$ .

### 1.7.4 Elliptic curve

An **elliptic curve** is a smooth, projective curve of genus 1. They can all be obtained from an equation of the form  $y^2 = x^3 + ax + b$  such that  $\Delta = -2^4(4a^3 + 27b^2) \neq 0$ .

### 1.7.5 Elliptic surface

An **elliptic surface** is a smooth surface  $X$  with a morphism  $\pi : X \rightarrow B$  onto a non-singular curve  $B$  whose generic fiber is a non-singular **elliptic curve**.

### 1.7.6 Fano variety

A variety  $X$  is **Fano** if the **anticanonical sheaf**  $\omega_X^{-1}$  is **ample**.

### 1.7.7 Jacobian variety

Let  $X$  be a curve of genus  $g$  over  $k$ . The **Jacobian variety** of  $X$  is a scheme  $J$  of finite type over  $k$ , together with an element  $\mathcal{L} \in \text{Pic}^\circ(X/J)$ , with the following universal property: for any scheme  $T$  of finite type over  $k$  and for any  $\mathcal{M} \in \text{Pic}^\circ(X/T)$ , there is a unique morphism  $f : T \rightarrow J$  such

that  $f^*\mathcal{L} \simeq \mathcal{M}$  in  $\text{Pic}^\circ(X/T)$ . This just says that  $J$  represents the functor  $T \mapsto \text{Pic}^\circ(X/T)$ .

If  $J$  exists, its closed points are in 1 – 1 correspondence with elements of  $\text{Pic}^\circ(X)$ .

It can be checked that  $J$  is actually a group scheme. For details, see [4, Ch. IV.4].

### 1.7.8 K3 surface

A **K3 surface** is a complex algebraic surface  $X$  such that the **canonical sheaf** is trivial,  $\omega_X \simeq \mathcal{O}_X$ , and such that  $H^1(X, \mathcal{O}_X) = 0$ . These conditions completely determine the Hodge numbers of  $X$ .

### 1.7.9 Stanley-Reisner scheme

A **Stanley-Reisner scheme** is a projective variety associated to a simplicial complex as follows. Let  $\mathcal{K}$  be a simplicial complex. Then we define an ideal  $I_{\mathcal{K}} \subseteq k[x_v \mid v \in V(\mathcal{K})] = k[\mathbf{x}]$  (here  $V(\mathcal{K})$  denotes the vertex set of  $\mathcal{K}$ ) by

$$I_{\mathcal{K}} = \langle x_{v_{i_1}} x_{v_{i_2}} \cdots x_{v_{i_k}} \mid v_{i_1} v_{i_2} \cdots v_{i_k} \notin \mathcal{K} \rangle.$$

We get a projective scheme  $\mathbb{P}(\mathcal{K})$  defined by  $\text{Proj}(k[\mathbf{x}]/I_{\mathcal{K}})$ , together with an embedding into  $\mathbb{P}^{\#V(\mathcal{K})-1}$ . It can be shown that  $H^p(\mathbb{P}(\mathcal{K}), \mathcal{O}_{\mathbb{P}(\mathcal{K})}) \simeq H^p(\mathcal{K}; k)$ , where the right-hand-side denotes the cohomology group of the simplicial complex.

### 1.7.10 Toric variety

A **toric variety**  $X$  is an integral scheme containing the torus  $(k^*)^n$  as a dense open subset, such that the action of the torus on itself extends to an action  $(k^*)^n \times X \rightarrow X$ .

## 2 Category theory

### 2.1 Basic concepts

#### 2.1.1 Adjoints pair

Let  $\mathcal{C}, \mathcal{C}'$  be categories. Let  $F : \mathcal{C} \rightarrow \mathcal{C}'$  and  $F' : \mathcal{C}' \rightarrow \mathcal{C}$  be functors. We call  $(F, F')$  an **adjoint pair**, or that  $F$  is **left adjoint** to  $F'$  (or  $F'$  right adjoint) if for each  $A \in \mathcal{C}$  and  $A' \in \mathcal{C}'$ , we have a *natural* bijection

$$\text{Hom}_{\mathcal{C}'}(F(A), A') \simeq \text{Hom}_{\mathcal{C}}(A, F'(A')).$$

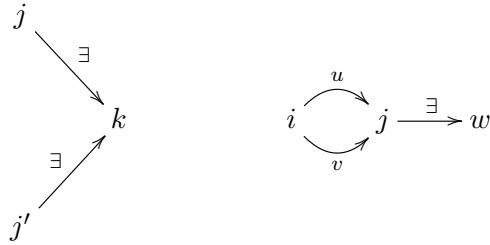
The naturality condition assures us that adjoints are unique up to isomorphism.

## 2.2 Limits

### 2.2.1 Direct limit

### 2.2.2 Filtered category

A category  $J$  is **filtered** when it satisfies the following three conditions: 1) it is non-empty. 2) For every two objects  $j, j' \in \mathbf{ob}(J)$ , there exists an object  $k \in \mathbf{ob}(J)$  and two arrows  $f : j \rightarrow k$  and  $f : j' \rightarrow k$ . 3) For every two parallel arrows  $u, v : i \rightarrow j$  there exists an object  $w \in \mathbf{ob}(J)$  and an arrow  $w : j \rightarrow k$  such that  $wu = wv$ .



## 3 Commutative algebra

### 3.1 Modules

#### 3.1.1 Depth

Let  $R$  be a noetherian ring, and  $M$  a finitely-generated  $R$ -module and  $I$  an ideal of  $R$  such that  $IM \neq M$ . Then the  $I$ -depth of  $M$  is (see **Ext**):

$$\inf\{i \mid \text{Ext}_R^i(R/I, M) \neq 0\}.$$

This is also the length of a maximal  **$M$ -sequence** in  $I$ .

#### 3.1.2 $M$ -sequence

Let  $M$  be an  $A$ -module and  $x \in A$ . We say that  $x$  is  **$M$ -regular** if multiplication by  $x$  is injective on  $M$ . We say that a sequence of elements  $a_1, \dots, a_r$  is an  **$M$ -sequence** if

- $a_1$  is  $M$ -regular,  $a_2$  is  $M/a_1M$ -regular,  $a_3$  is  $M/(a_1, a_2)M$ -regular, and so on.



- $M/\sum_i a_i M \neq 0$ .

The length of a maximal  $M$ -sequence is the **depth** of  $M$ .

### 3.1.3 Rank

If  $R$  has the invariant basis property (IBN), then we define the **rank** of a *free* module to be the cardinality of any basis.

### 3.1.4 Stably free module

A module  $M$  is **stably free** (of rank  $n - m$ ) if  $P \oplus R^m \simeq R^n$  for some  $m$  and  $n$ .

### 3.1.5 Kähler differentials

Let  $A \rightarrow B$  be a ring homomorphism. The **module of Kähler differentials**  $\Omega_{B/A}$  is the module together with a map  $d : B \rightarrow \Omega_{B/A}$  satisfying the following universal property: if  $D : B \rightarrow M$  is any  $A$ -linear derivation (an element of  $\text{Der}_A(B, M)$ ), then there is a unique module homomorphism  $\tilde{D} : \Omega_{B/A} \rightarrow M$  such that

$$\begin{array}{ccc} B & \xrightarrow{d} & \Omega_{B/A} \\ & \searrow D & \downarrow \tilde{D} \\ & & M \end{array}$$

is commutative. Thus we have a natural isomorphism  $\text{Der}_A(B, M) = \text{Hom}_B(\Omega_{B/A}, M)$ . In the language of category theory, this means that  $\text{Der}_A(B, -)$  is *corepresented* by  $\Omega_{B/A}$ .

A concrete construction of  $\Omega_{B/A}$  is given as follows. Let  $M$  be the free  $B$ -module generated by all symbols  $df$ , where  $f \in B$ . Let  $N$  be the submodule generated by  $da$  if  $a \in A$ ,  $d(f + g) - df - dg$  and the Leibniz rule  $d(fg) - f dg - g df$ . Then  $M/N \simeq \Omega_{B/A}$  as  $B$ -modules.

## 3.2 Results and theorems

### 3.2.1 The conormal sequence

The **conormal sequence** is a sequence relating Kähler differentials in different rings. Specifically, if  $A \rightarrow B \rightarrow 0$  is a surjection of rings with kernel  $I$ , then we have an exact sequence of  $B$ -modules:

$$I/I^2 \xrightarrow{d} B \otimes_A \Omega_{B/A} \xrightarrow{D\pi} \Omega_{T/R} \rightarrow 0$$

The map  $d$  sends  $f \mapsto 1 \otimes df$ , and  $D\pi$  sends  $c \otimes db \mapsto cdb$ . For proof, see [3, Chapter 16].

### 3.2.2 Determinant of an exact sequence

Suppose we have an exact sequence of free  $R$ -modules:

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0,$$

of ranks  $l, m$  and  $n$ , respectively. Then there is a natural isomorphism  $\wedge^m M \simeq \wedge^l L \otimes_R \wedge^n N$ . This is used in proving the **adjunction formula**.

### 3.2.3 The Unmixedness Theorem

Let  $R$  be a ring. If  $I = \langle x_1, \dots, x_n \rangle$  is an ideal generated by  $n$  elements such that  $\text{codim } I = n$ , then all minimal primes of  $I$  have codimension  $n$ . If in addition  $R$  is **Cohen-Macaulay**, then every associated prime of  $I$  is minimal over  $I$ . See the discussion after [3, Corollary 18.14] for more details.

## 3.3 Rings

### 3.3.1 Cohen-Macaulay ring

A local Cohen-Macaulay ring (CM-ring for short) is a commutative noetherian local ring with Krull dimension equal to its depth. A ring is Cohen-Macaulay if its localization at all prime ideals are Cohen-Macaulay.

### 3.3.2 Depth of a ring

The depth of a ring  $R$  is its depth as a module over itself.

### 3.3.3 Gorenstein ring

A commutative ring  $R$  is Gorenstein if each localization at a prime ideal is a Gorenstein local ring. A Gorenstein local ring is a local ring with finite injective dimension as an  $R$ -module. This is equivalent to the following:  $\text{Ext}_R^i(k, R) = 0$  for  $i \neq n$  and  $\text{Ext}_R^n(k, R) \simeq k$  (here  $k = R/\mathfrak{m}$  and  $n$  is the Krull dimension of  $R$ ).

### 3.3.4 Invariant basis property

A ring  $R$  satisfies the **invariant basis property** (IBP) if  $R^n \not\simeq R^{n+t}$   $R$ -modules for any  $t \neq 0$ . Any commutative ring satisfies the IBP.

### 3.3.5 Normal ring

An integral domain  $R$  is **normal** if all its localizations at prime ideals  $\mathfrak{p} \in \text{Spec } R$  are integrally closed domains.

## 4 Convex geometry

### 4.1 Cones

#### 4.1.1 Gorenstein cone

A strongly convex cone  $C \subset M_{\mathbb{R}}$  is **Gorenstein** if there exists a point  $n \in N$  in the dual lattice such that  $\langle v, n \rangle = 1$  for all generators of the semigroup  $C \cap M$ .

#### 4.1.2 Reflexive Gorenstein cone

A cone  $C$  is **reflexive** if both  $C$  and its dual  $C^{\vee}$  are **Gorenstein cones**. See for example [1].

#### 4.1.3 Simplicial cone

A cone  $C$  generated by  $\{v_1, \dots, v_k\} \subseteq N_{\mathbb{R}}$  is **simplicial** if the  $v_i$  are linearly independent.

### 4.2 Polytopes

#### 4.2.1 Dual (polar) polytope

If  $\Delta$  is a polyhedron, its dual  $\Delta^{\circ}$  is defined by

$$\Delta^{\circ} = \{x \in N_{\mathbb{R}} \mid \langle x, y \rangle \geq -1 \forall y \in \Delta\}.$$

#### 4.2.2 Gorenstein polytope of index $r$

A lattice polytope  $P \subset \mathbb{R}^{d+r-1}$  is called a **Gorenstein polytope of index  $r$**  if  $rP$  contains a single interior lattice point  $p$  and  $rP - p$  is a **reflexive polytope**.

### 4.2.3 Nef partition

Let  $\Delta \subset M_{\mathbb{R}}$  be a  $d$ -dimensional **reflexive polytope**, and let  $m = \text{int}(\Delta) \cap M$ . A Minkowski sum decomposition  $\Delta = \Delta_1 + \dots + \Delta_r$  where  $\Delta_1, \dots, \Delta_r$  are lattice polytopes is called a **nef partition of  $\Delta$  of length  $r$**  if there are lattice points  $p_i \in \Delta_i$  for all  $i$  such that  $p_1 + \dots + p_r = m$ . The nef partition is called *centered* if  $p_i = 0$  for all  $i$ .

This is equivalent to the toric divisor  $D_j = \mathcal{O}(\Delta_j) = \sum_{\rho \in \Delta_j} D_\rho$  being a Cartier divisor generated by its global sections. See [1, Chapter 4.3].

### 4.2.4 Reflexive polytope

A polytope  $\Delta$  is **reflexive** if the following two conditions hold:

1. All facets  $\Gamma$  of  $\Delta$  are supported by affine hyperplanes of the form  $\{m \in M_{\mathbb{R}} \mid \langle m, v_\Gamma \rangle = -1\}$  for some  $v_\Gamma \in N$ .
2. The only interior point of  $\Delta$  is 0, that is:  $\text{Int}(\Delta) \cap M = \{0\}$ .

It can be proved that a polytope  $\Delta$  is reflexive if and only if the associated toric variety  $\mathbb{P}_\Delta$  is **Fano**.

## 5 Homological algebra

### 5.1 Classes of modules

#### 5.1.1 Projective modules

Projective modules are those satisfying a universal lifting property. A module  $P$  is **projective** if for every epimorphism  $\alpha : M \rightarrow N$  and every map,  $\beta : P \rightarrow N$ , there exists a map  $\gamma : P \rightarrow M$  such that  $\beta = \alpha \circ \gamma$ .

$$\begin{array}{ccc} & P & \\ \swarrow \exists \gamma & \downarrow \beta & \\ M & \xrightarrow{\alpha} N & \longrightarrow 0 \end{array}$$

These are the modules  $P$  such that  $\text{Hom}(P, -)$  is exact.

### 5.2 Derived functors

#### 5.2.1 Ext

Let  $R$  be a ring and  $M, N$  be  $R$ -modules. Then  $\text{Ext}_R^i(M, N)$  is the right-derived functors of the  $\text{Hom}(M, -)$ -functor. In particular,  $\text{Ext}_R^i(M, N)$  can

be computed as follows: choose a projective resolution  $C$  of  $N$  over  $R$ . Then apply the left-exact functor  $\text{Hom}_R(M, -)$  to the resolution and take homology. Then  $\text{Ext}_R^i(M, N) = h^i(C)$ .

### 5.2.2 Local cohomology

Let  $R$  be a ring and  $I \subset R$  an ideal. Let  $\Gamma_I(-)$  be the following functor on  $R$ -modules:

$$\Gamma_I(M) = \{f \in M \mid \exists n \in \mathbb{N}, \text{ s.t. } I^n f = 0\}.$$

Then  $H_I^i(-)$  is by definition the  $i$ th right derived functor of  $\Gamma_I$ . In the case that  $R$  is noetherian, we have  $H_I^i(M) = \varinjlim \text{Ext}_R^i(R/I_n, M)$ .

See [3] and [7] for more details.

### 5.2.3 Tor

Let  $R$  be a ring and  $M, N$  be  $R$ -modules. Then  $\text{Tor}_R^i(M, N)$  is the right-derived functors of the  $- \otimes_R N$ -functor. In particular  $\text{Tor}_R^i(M, N)$  can be computed by taking a projective resolution of  $M$ , tensoring with  $N$ , and then taking homology.

## 6 Differential and complex geometry

### 6.1 Definitions and concepts

#### 6.1.1 Almost complex structure

An **almost complex structure** on a manifold  $M$  is a map  $J : T(M) \rightarrow T(M)$  whose square is  $-1$ .

#### 6.1.2 Connection

Let  $E \rightarrow M$  be a vector bundle over  $M$ . A **connection** is a  $\mathbb{R}$ -linear map  $\nabla : \Gamma(E) \rightarrow \Gamma(E \otimes T^*M)$  such that the Leibniz rule holds:

$$\nabla(f\sigma) = f\nabla(\sigma) + \sigma \otimes df$$

for all functions  $f : M \rightarrow \mathbb{R}$  and sections  $\sigma \in \Gamma(E)$ .

### 6.1.3 Hermitian manifold

A *Hermitian metric* on a complex vector bundle  $E$  over a manifold  $M$  is a positive-definite Hermitian form on each fiber. Such a metric can be written as a smooth section  $\Gamma(E \otimes \bar{E})^*$ , such that  $h_p(\eta, \bar{\zeta}) = h_p(\bar{\zeta}, \eta)$  for all  $p \in M$ , and such that  $h_p(\eta, \bar{\eta}) > 0$  for all  $p \in M$ . A **Hermitian manifold** is a complex manifold with a Hermitian metric on its holomorphic tangent space  $T^{(1,0)}(M)$ .

### 6.1.4 Kähler manifold

A **Kähler manifold** is a Hermitian manifold (that is, a complex manifold equipped with a Hermitian metric at every complex tangent space) such that its associated Hermitian form is closed.

### 6.1.5 Morse function

A **Morse function**  $f : M \rightarrow \mathbb{R}$  on a manifold  $M$  is a smooth function whose Hessian matrix is no-where singular. The set of Morse functions forms a dense open set on  $C^\infty(M)$  in the  $C^2$ -topology.

The Morse lemma states that a Morse function can be written as

$$f(x) = f(b) - x_1^2 - x_2^2 - \dots - x_\alpha^2 + x_{\alpha+1}^2 + \dots + x_n^2$$

such that  $f(x) = 0$ , in a neighbourhood of a point  $x \in M$ . The number  $\alpha$  is called the **index** of  $f$  at  $b$ .

Let  $M^a = f^{-1}((-\infty, a])$ . The first of the two fundamental theorems of Morse theory says the following: suppose  $f$  is a Morse function and  $f^{-1}([a, b])$  is compact, and that there are no critical values of  $f$  in  $[a, b]$ , then  $M^a$  is diffeomorphic to  $M^b$  and  $M^b$  deformation retracts onto  $M^a$ .

The other theorem says the following: let  $f$  be a Morse function and let  $p$  be a critical point of  $f$  of index  $\gamma$ , and that  $f(p) = q$ . Suppose also that  $f^{-1}([q - \epsilon, q + \epsilon])$  is compact and contains no other critical points. Then  $M^{q+\epsilon}$  is homotopy equivalent to  $M^{q-\epsilon}$  with a  $\gamma$ -cell attached.

Thus Morse functions are nice for studying the topology of manifolds.

### 6.1.6 Symplectic manifold

A  $2n$ -dimensional manifold  $M$  is **symplectic** if it is compact and oriented and has a closed real two-form  $\omega \in \bigwedge^2 T^*(M)$  which is nondegenerate, in the sense that  $\wedge^n \omega|_p \neq 0$  for all  $p \in M$ .

## 6.2 Results and theorems

## 7 Worked examples

### 7.1 Algebraic geometry

#### 7.1.1 Hurwitz formula and Kähler differentials

Let  $X$  be the conic in  $\mathbb{P}^2$  given with ideal sheaf  $\langle xz - y^2 \rangle$ . Let  $Y$  be  $\mathbb{P}^1$ , and consider the map  $f : X \rightarrow Y$  given by projection onto the  $xz$ -line.  $X$  is covered by two affine pieces, namely  $X = U_x \cup U_z$ , the spectra of the homogeneous localizations at  $x, z$ , respectively. Let  $U_x = \text{Spec } A$  for  $A = k[z]$  and  $U_z = \text{Spec } B$  for  $B = k[x]$ . Then the map is locally given by  $A \rightarrow k[y, z]/(z - y^2)$  where  $z \mapsto \bar{z}$ , and similarly for  $B$ . We have an isomorphism  $k[y, z]/(z - y^2) \simeq k[t]$ , given by  $y \mapsto t$  and  $z \mapsto t^2$ , so that locally the map is given by  $k[z] \rightarrow k[t], z \mapsto t^2$ .

This is a map of smooth projective curves, so we can apply Hurwitz' formula. Both  $X, Y$  are  $\mathbb{P}^1$ , so both have genus zero. Hence Hurwitz formula says that

$$-2 = -n \cdot 2 + \deg R,$$

where  $R$  is the ramification divisor and  $n$  is the degree of the map. The degree of the map can be defined locally, and it is the degree of the field extension  $k(Y) \hookrightarrow k(X)$ . But (the image of)  $k(Y) = k(t^2)$  and  $k(X) = k(t)$ , so that  $[k(Y) : k(X)] = 2$ . Hence by Hurwitz' formula, we should have  $\deg R = 2$ . Since  $R = \sum_{P \in X} \text{length } \Omega_{X/Y_P} \cdot P$ , we should look at the sheaf of relative differentials  $\Omega_{Y/X}$ .

First we look in the chart  $U_z$ . We compute that  $\Omega_{k[t]/k[t^2]} = k[t]/(t)$ . This follows from the relation  $d(t^2) = 2dt$ , implying that  $dt = 0$  in  $\Omega_{k[t]/k[t^2]}$ . This module is zero localized at all primes but  $(t)$ , where it is  $k$ . Thus for  $P = (0 : 0 : 1)$ , we have  $\text{length } \Omega_{X/Y_P} = 1$ .

The situation is symmetric with  $z \leftrightarrow x$ , so that we have  $R = (0 : 0 : 1) + (1 : 0 : 0)$ , confirming that  $\deg R = 2$ .

In fact, the curve  $C$  is isomorphic to  $\mathbb{P}^1$  via the map  $\mathbb{P}^1 \rightarrow C$  given by  $(s : t) \mapsto (s^2 : st : t^2)$ . Identifying  $C$  with  $\mathbb{P}^1$ , we thus see that  $C \rightarrow \mathbb{P}^1$  correspond to the map  $\mathbb{P}^1 \rightarrow \mathbb{P}^1$  given by  $(s : t) \mapsto (s^2 : t^2)$ .

### 7.2 The quintic threefold

Let  $Y$  be a the zeroes of a general hypersurface of degree 5 in  $\mathbb{P}^4$ , or in other words, a section of  $\omega_{\mathbb{P}^4}^\vee$ . We want to compute the cohomology of  $Y$  and its Hodge numbers. Let  $\mathbb{P} = \mathbb{P}^4$ .

We have the ideal sheaf sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow i^* \mathcal{O}_Y \rightarrow 0,$$

where  $i : Y \rightarrow \mathbb{P}^4$  is the inclusion. Note that  $\mathcal{I} = \mathcal{O}_{\mathbb{P}}(-5)$ . Thus we have from the long exact sequence of cohomology that

$$\dots \rightarrow H^i(\mathbb{P}, \mathcal{I}) \rightarrow H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) \rightarrow H^i(Y, \mathcal{O}_Y) \rightarrow H^{i+1}(\mathbb{P}, \mathcal{I}) \rightarrow \dots$$

Note that  $H^{i+1}(\mathbb{P}, \mathcal{I}) = 0$  for  $i \neq 3$  and 1 for  $i = 3$ . Also  $H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = 0$  unless  $i = 0$  in which case it is 1. Thus we get that  $H^i(Y, \mathcal{O}_Y)$  is  $k$  for  $i = 0$ , for  $i = 1, 2$  it is 0, and for  $i = 3$  it is  $k$ . For higher  $i$  it is zero by Grothendieck vanishing.

The adjunction formula relates the canonical bundles as follows: if  $\omega_{\mathbb{P}}$  is the canonical bundle on  $\mathbb{P}$ , then  $\omega_Y = i^* \omega_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \det(\mathcal{I}/\mathcal{I}^2)^\vee$ . The ideal sheaf is already a line bundle, so taking the determinant does not change anything. Now

$$\begin{aligned} (\mathcal{I}/\mathcal{I}^2)^\vee &= \text{Hom}_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \\ &= \text{Hom}_{\mathbb{P}}(\mathcal{I}, \mathcal{O}_Y) = \text{Hom}_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(-5), \mathcal{O}_Y) = \mathcal{O}_Y(5). \end{aligned}$$

It follows that  $\omega_Y = \mathcal{O}_Y(-5) \otimes \mathcal{O}_Y(5) = \mathcal{O}_Y$ . Thus the canonical bundle is trivial and we conclude that  $Y$  is Calabi-Yau.

It remain to compute the Hodge numbers. We start with  $h^{11} = \dim_k H^1(Y, \Omega_Y)$ . We have the conormal sequence of sheaves on  $Y$ :

$$0 \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{\mathbb{P}} \otimes \mathcal{O}_Y \rightarrow \Omega_Y \rightarrow 0,$$

which gives us the long exact sequence:

$$\dots \rightarrow H^i(\mathcal{I}/\mathcal{I}^2) \rightarrow H^i(\Omega_{\mathbb{P}} \otimes \mathcal{O}_Y) \rightarrow H^i(\Omega_Y) \rightarrow H^{i+1}(\mathcal{I}/\mathcal{I}^2) \rightarrow \dots$$

We first compute the cohomology of  $\mathcal{I}/\mathcal{I}^2$ . We use the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}(-10) \rightarrow \mathcal{O}_{\mathbb{P}}(-5) \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow 0. \quad (1)$$

we have  $H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-10)) = 0$  for  $i = 0, 1, 2, 3$ , and for  $i = 4$  we have  $H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-10)) = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(5)) = k^{126}$ . Similarly  $H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-5)) = 0$  for  $i = 0, 1, 2, 3$  and  $H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-5)) = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = k$ . We conclude that  $h^i(Y, \mathcal{I}/\mathcal{I}^2) = 0$  for  $i = 0, 1, 2$  and 125 for  $i = 3$ .

In particular  $H^1(\Omega_Y) \simeq H^1(\Omega_{\mathbb{P}} \otimes \mathcal{O}_Y)$ . We have the Euler sequence:

$$0 \rightarrow \Omega_{\mathbb{P}} \rightarrow \mathcal{O}_{\mathbb{P}}(-1)^{\oplus 5} \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow 0$$



Now  $\mathcal{O}_Y = \mathcal{O}_{\mathbb{P}}/\mathcal{I}$  is a flat  $\mathcal{O}_{\mathbb{P}}$ -module since  $\mathcal{I}$  is principal and generated by a non-zero divisor. Thus we can tensor the Euler sequence with  $\mathcal{O}_Y$  and get

$$0 \rightarrow \Omega_{\mathbb{P}} \otimes \mathcal{O}_Y \rightarrow \mathcal{O}_Y(-1)^5 \rightarrow \mathcal{O}_Y \rightarrow 0,$$

from which it easily follows that  $H^1(Y, \Omega_{\mathbb{P}} \otimes \mathcal{O}_Y) \simeq H^0(\mathcal{O}_Y) = k$ . We conclude that  $h^{11} = 1$ .

Now we compute  $h^{12} = \dim_k H^1(Y, \Omega_Y^2)$ . This is equal to  $H^2(Y, \Omega_Y)$  by Serre duality. Again we use the conormal sequence. From the Euler sequence we get that  $H^2(Y, \Omega_{\mathbb{P}} \otimes \mathcal{O}_Y) = 0$ . We also get that  $h^3(Y, \Omega_{\mathbb{P}} \otimes \mathcal{O}_Y) = 24$ . NOW  $H^3(\Omega_Y) = 0$  (WHY??), and it follows from the above computations that  $h^{12} = 125 - 24 = 101$ .

This example is extremely important in mirror symmetry.

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