Smoothing a Calabi-Yau manifold

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Chapter 1

Preliminary definitions

We work over $\mathbb{C}$, but some theorems may be stated over a field k.

1.1 Stanley-Reisner basics

Given a simplicial complex $\mathcal{K}$, one can associate to it a projective scheme $\mathbb{P}(\mathcal{K})$ defined as follows. Let P be the polynomial ring with one variable for each vertex of $\mathcal{K}$. Then the *Stanley-Reisner ideal* $I_{\mathcal{K}}$ corresponding to $\mathcal{K}$ is generated by the monomials corresponding to *non-faces* of $\mathcal{K}$. Then we define the *Stanley-Reisner scheme* to be $\operatorname{Proj} P/I_{\mathcal{K}}$.

Example 1.1.1. Let $\mathcal{K}$ be the square, with vertices v_0, v_1, v_2, v_3. Then the Stanley-Reisner ideal is generated by v_0v_2 and v_1v_3. ★

Some of the topology of the simplicial complex is encoded in the scheme structure of $\mathbb{P}(\mathcal{K})$. In particular, the simplicial (co)homology groups of $\mathcal{K}$ can be computed as the sheaf cohomology of $\mathbb{P}(\mathcal{K})$

Lemma 1.1.2. *Let* $(K;\mathbb{C})$ *denote the singular cohomology groups of* $\mathcal{K}$. *Then there are isomorphisms* $H^i(K;\mathbb{C}) = H^i(\mathbb{P}(\mathcal{K}), \mathscr{O}_{\mathbb{P}(\mathcal{K})})$ *for all* i.

Proof. ———to come———— □

Corollary 1.1.3. *We have isomorphisms* $H^i(K,\mathbb{C}) \simeq H^{2i}(\mathbb{P}(\mathcal{K});\mathbb{C})$ *of singular cohomology groups. <- WRONG*

Find correct statement: H^{2n} should be number of facets.

Proof. Something about i-cells in even dimensions □

1.2 Calabi-Yau basics

Definition 1.2.1. A *Calabi-Yau variety* is a smooth projective variety satisfying the following two conditions:

1. $H^i(X, \mathscr{O}_X) = 0$ for $0 < i < \dim X$.

2. The canonical sheaf is trivial: $\omega_X \simeq \mathscr{O}_X$.

■

The classical example of a Calabi-Yau manifold is the quintic threefold in $\mathbb{P}^5$. Another example is the following:

Example 1.2.2. Let X be the double cover of $\mathbb{P}^3$ ramified along a smooth octic. The projection map is affine, so the conditions on $H^i(X, \mathscr{O}_X)$ are fulfilled. To see that the canonical sheaf is trivial, we use the adjunction formula, which says that $K_X = 2K_{\mathbb{P}^3}\big|_X + R$, where R is the ramification divisor. In this case R $8H$, where H is a hyperplane in $\mathbb{P}^3$. Then, since $K_{\mathbb{P}^3} = -4H$, it follows that $K_X = 0$. ★

If $\mathcal{K}$ is a simplicial sphere, then a smoothing of $\mathbb{P}(\mathcal{K})$ will give a Calabi-Yau manifold.

—- ref: bayer-eisenbud graph curves.

The most basic invariants of Calabi-Yau manifolds are their *Hodge numbers* h^{pq}. In algebraic geometry these can be defined as the dimensions of the cohomology groups $H^q(X, \Omega_X^p)$. This definition is however not so transparent. On a complex manifold, it is true that $h^{pq} = h^{qp}$, but this is not obvious from our definition. Instead, let us define these groups in complex algebraic geometry terms.

The de Rham complex $(\Omega^\bullet, d)$ refines to a bigraded complex $(\Omega^{\bullet,\bullet}, d)$, where a differential form of bidegree (p, q) can be written as

$$\omega = \sum f_{IJ} dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z_1} \ldots \wedge d\overline{z_q}.$$

The differential d splits as $\partial + \overline{\partial}$, where $\partial : \Omega^{\bullet,\bullet} \to \Omega^{\bullet+1,\bullet}$, and $\overline{\partial} : \Omega^{\bullet,\bullet} \to \Omega^{\bullet,\bullet+1}$. The decomposition passes respects cohomology, so we can form the *Dolbeault cohomology groups* $H^{p,q}(X)$.

With this definition, applying complex conjugation shows that $H^{p,q} = \overline{H^{q,p}}$.

Lemma 1.2.3. *We have natural isomorphisms* $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$.

Proof. Use that the de Rham complex is flabby □

For more details on this and other details from complex geometry, see [Voi07].

The “Hodge diamond” is ...

Example 1.2.4. Let X be a smooth quintic in $\mathbb{P}^4$. We will compute its Hodge numbers. Let us first compute $H^{1,1}(X)$. We have the following exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{P}^4}\big|_X \to \Omega_X^1 \to 0$$

Since $\mathcal{I}/\mathcal{I}^2 \simeq \mathscr{O}_X(-5)$, it follows from the long exact sequence of cohomology that $H^1(X, \Omega_X^1) \simeq H^1(X, \Omega_{PP^4}^1\big|_X)$. .... ★

1.3 Deformation theory

Deformation theory is the study how varieties (or other algebraic structures like line bundles, vector bundles, ...) vary in families.

There is a lot of technical machinery available for the deformation theorist, but for us just a few vector spaces will be of importance.

Definition 1.3.1. Let X be a scheme over k. Then a *deformation of X over S* is a flat morphism $\mathfrak{X} \to S$ together with an isomorphism $X \simeq \mathfrak{X} \times_S 0$ for a closed point $0 \in S$:

$$\begin{tikzcd}
X \simeq X_0 \arrow[r, hook] \arrow[d] & \mathfrak{X} \arrow[d] \\
0 \arrow[r] & S
\end{tikzcd}$$

■

Recall that a morphism $f : X \to Y$ is *flat* if the associated morphism $f^\$: \mathscr{O}_Y \to f_* \mathscr{O}_X$ of $\mathscr{O}_Y$-modules is a flat morphism.

Chapter 2

Two topologically distinct smoothings

Denote by dP_6 the del Pezzo surface of degree 6 embedded in $\mathbb{P}^6$. This can be realized as the blow-up of $\mathbb{P}^2$ in three points not lying on a line. Let X denote the affine cone over dP_6. Then it has long been known that X has two smoothing components, and we show here that they are topologically distinct.

Recall that a *del Pezzo* surface is a surface such that the anti-canonical bundle is ample. The degree is the degree given by the anticanonical embedding. It is a classical result that every del Pezzo surface is obtained either by blowing up $\mathbb{P}^2$ in $r = 0, \ldots, 6$ points in suitable positions, or as the 2-uple embedding of a quadric surface in $\mathbb{P}^3$.

2.1 Different embeddings of dP_6

We first obtain the equations of dP_6 directly from the description of it as blow-up. Let x_0, x_1, x_2 be coordinates of $\mathbb{P}^2$. Recall that the blowup of $\mathbb{P}^2$ in the point $(1:0:0)$ can be realized as the closed subscheme of $\mathbb{P}^2 \times \mathbb{P}^1$ given by the equation $r_0x_1 - r_1x_2 = 0$, where r_0, r_1 are coordinates on $\mathbb{P}^1$. We can repeat this process on the points $(0:1:0)$ and $(0:0:1)$ to obtain similar equations.

Collecting these, we see that dP_6 is given by the matrix equation

$$M\vec{x} = \begin{pmatrix} 0 & r_0 & -r_1 \\ s_1 & 0 & -s_0 \\ -t_0 & t_1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0.$$

in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Here r_i, s_i and t_i $(i = 0, 1)$ are of course coordinates on $\mathbb{P}^1$.

We can do more than this however.

Lemma 2.1.1. *We can also realize dP_6 embedded in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with equation $r_0 s_0 t_0 = r_1 s_1 t_1$.*

Proof. Note that the matrix cannot have rank 1 or lower. Now consider the projection onto the last three factors:

$$\pi : \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

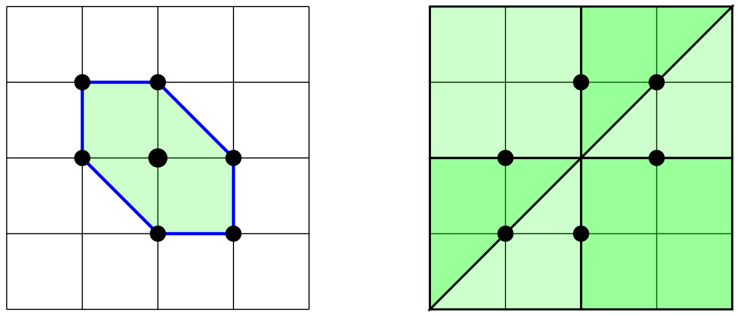
Each point P in the product on the right-hand side gives a matrix M_P of rank 2. Thus there is a line of solutions, which correspond exactly to a point in $\mathbb{P}^2$.

Hence the restriction of π to dP_6 is an isomorphism onto the hypersurface given by $\det M = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. $\square$

Another way to realize blow-ups is this: let $\mathfrak{d}$ be the linear system of quadrics with assigned basepoints $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$ in $\mathbb{P}^2$. We can choose a basis given by x_0x_1, x_0x_2 and x_1x_2. This gives a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. The closure of the graph of this map is a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$ defined by two bilinear equations. Each of the projections correspond to the blowup.

Explicitly, if we let y_0, y_1, y_2 be coordinates on the other $\mathbb{P}^2$, then the equations are $x_1y_0 - x_2y_1 = x_1y_0 - x_0y_2 = 0$.

We also have a natural embedding in $\mathbb{P}^6$ as follows. Denote by E_1, E_2, E_3 the exceptional divisors on the blowup. Let L be a line in $\mathbb{P}^2$. Then the divisor $\pi^*3L - E_1 - E_2 - E_3$ is ample, and gives an embedding in $\mathbb{P}^6$ (see [Har77, Chapter V, Theorem 4.6]). A basis for the corresponding linear system is given by all monomials in $\Gamma(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(3))$ except x^3, y^3 and z^3.



(a) The hexagon. (b) The fan of dP_6.

Figure 2.1: Toric description of dP_6.

The equations can be arranged in a particularly symmetric form: let $y, x_1, \ldots, x_1$ be coordinates on $\mathbb{P}^6$. Then the equations of dP_6 are the 2×2 minors of the matrix

$$\begin{pmatrix} x_1 & y & x_6 \\ x_2 & x_3 & y \\ y & x_4 & x_5 \end{pmatrix}.$$

This gives 9 equations, which can be compactly written as $x_i x_{i+2} - y x_i = 0$ and $x_i x_{i+3} - y^2 = 0$, for $i = 1, \ldots, 6$ (where i is taken modulo 6). Note that the equations have a visible D_6-symmetry, where D_6 denotes the dihedral group.

2.1.1 As a toric variety

There is a nice combinatorial description of dP_6 as a toric variety associated to a polytope. Namely, let P denote the hexagon in Figure 2.1a. Then the normal fan of this polytope defines a fan in $N_{\mathbb{R}}$, defining a toric variety.

The polytope is reflexive, implying that the normal fan of P is the face fan over the same polytope. See Figure 2.1b. From standard toric geometry, it is clear that dP_6 is the blowup of $\mathbb{P}^2$ in the three torus-fixed points.

2.2 Divisors and topology

Consider the exponential sequence

$$0 \to \mathbb{Z} \to \mathscr{O}_{dP_6} \to \mathscr{O}^*_{dP_6} \to 0.$$

Since dP_6 is a rational surface, it follows from the long-exact sequence that $H^2(dP_6, \mathbb{Z}) \simeq \mathrm{Pic}(dP_6)$. From the description of dP_6 as a blowup, we know from [Har77, Chapter V], that the Picard group is spanned by the three exceptional divisors, together with the class of the pullback of a hyperplane. Denote these by E_i and H, respectively. Then, since the E_i are exceptional divisors, we have $E_i \cdot E_j = -\delta_{ij}$, and also $H^2 = 1$, since we can compute intersections downstairs.

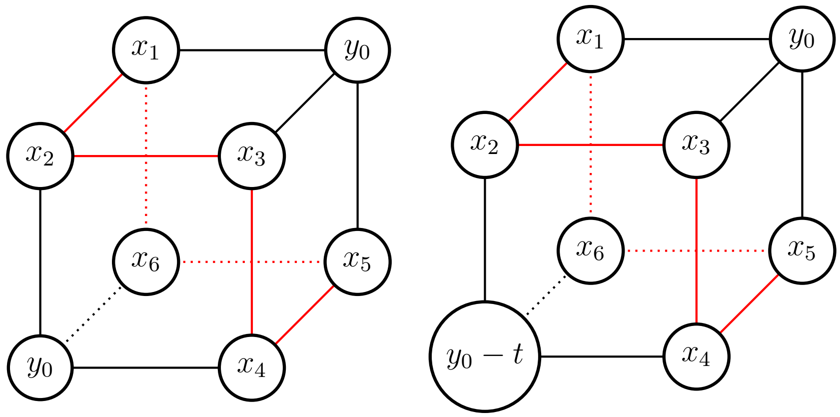
Consider the embedding of dP_6 in $\mathbb{P}^2 \times \mathbb{P}^2$ given by the equations $x_1y_0 - x_2y_1 = x_1y_0 - x_0y_2 = 0$. This is the closure of the graph of a Cremona transformation of $\mathbb{P}^2$. The three exception divisors in dP_6 are given by $\pi_1(P_i)$, where P_i are the three (torus-invariant) blown up points. There are three more interesting lines in dP_6: let L_{ij} be the line through P_i and P_j in $\mathbb{P}^2$. Also denote the proper transform of L_{ij} by L_{ij}. These are elements of the Picard group, and hence is a linear combination of the E_i and H. Since L_{ij} goes through P_i and P_j once each, we must have $L_{ij} \cdot E_i = 1$. Similarly, $L_{ij} \cdot H = 1$. Since $E_i \cdot E_j = -\delta_{ij}$, it follows that $L_{ij} = H - E_i - E_j$.

Make hexagon figure

2.3 The affine cone and its two smoothings

Let X denote the affine cone over dP_6. It is an affine variety with an isolated singularity at the origin. One can compute that it has two smoothing components: the union of a plane and a line. They both come from different ways of perturbing the equations of dP_6.

Look at Figure 2.2a. One can read off the equations of dP_6 by taking minors along "faces" and long diagonals of this square. This correspond to a hyperplane cut of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in the Segre embedding. Then the one-dimensional component of the versal deformation of X is obtained by perturbing one of the y_0-corners as in Figure 2.2b.



(a) Equations dP_6.

(b) Deforming $C(dP_6)$.

Figure 2.2: Forms of equations.

It is clear the corresponding deformation is smooth, since it is a hyperplane cut of cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ outside the origin. Call this smoothing X_1.

Lemma 2.3.1. *The smoothing X_1 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \backslash dP_6$.*

Proof. Specialize to some $t \neq 0$. Then we can homogenize the equations with respect to y_1 to obtain a projective variety in 8 variables. However, in this form, $y_0 - ty_1$ and y_0 are linearly independent, hence by a change of variables, we see that this variety is in fact isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding. See Figure 2.2b.

What we gained by homogenizing is exactly the projective variety given by setting $y_1 = 0$. But then we get back the equations of dP_6 in $\mathbb{P}^6$. $\square$

The second smoothing is obtained by deforming the equations of dP_6 as a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$. Namely, consider the following matrix:

$$\begin{vmatrix} x_1 & y_0 & x_6 \\ x_2 & x_3 & y_0 - t_1 \\ y_0 - t_2 & x_4 & x_5 \end{vmatrix} \leq 1. \tag{2.1}$$

For $t_1 = t_2 = 0$, we get the cone over dP_6, while for generic t_i, we get a smooth variety. In fact, we can compute that the discrimant locus (the set of points in $\mathbb{A}^2_{t_1,t_2}$ with singular fiber) are the t_1-axis, the t_2-axis and the line $t_1 = t_2$.

Call (any) smooth fiber X_2.

Lemma 2.3.2. *Let $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ be the projective bundle associated to the tangent sheaf on $\mathbb{P}^2$. Then the smoothing X_2 is isomorphic to $M \backslash dP_6$.*

Proof. The technique is the same as in the previous proof. First homogenize the equations (2.1) with respect to y_1. Call the homogenized variety M. Put $y_0' = y_0$, $y_1' = y_0 - ty_1$ and $y_2' = y_0 - t_2 y_1$. Then we have the relation

$$h = t_2 y_1' - t_1 y_2' - (t_1 - t_2) y_0' = 0.$$

Hence we see that $M = \mathbb{P}^2 \times \mathbb{P}^2 \cap (h = 0)$. We can pull back the coordinates y_i' to $\mathbb{P}^2 \times \mathbb{P}^2$. Let $\mathbb{P}^2 \times \mathbb{P}^2$ have coordinates x_0, x_1, x_2 and y_0, y_1, y_2. Then h pulls back to the equation

$$(x_0, x_1, x_2) \cdot (-t_1 y_2, (t_1 - t_2) y_0, t_2 y_1) = 0$$

in $\mathbb{P}^2 \times \mathbb{P}^2$. As long as $t_1 \neq t_2$ and $t_1, t_2 \neq 0$, we can do a change of coordinates in $\mathbb{P}^2_{y_0 y_1 y_2}$, so that h transforms to

$$(x_0, x_1, x_2) \cdot (y_0, y_1, y_2) = 0.$$

Hence we see that M is isomorphic to the total space of the Grassmannian of lines in $\mathbb{P}^2$ (each point in one of the $\mathbb{P}^2$'s give a line in the other $\mathbb{P}^2$). This is in turn isomorphic to $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$, since each tangent vector through a point determines a line through it.

Now, what have we gained by homogenizing? The divisor at infinity is $y_1 = 0$, which is a dP_6 again. In our new coordinates this is equivalent to $y_1' = y_2' = y_0'$. Hence in the coordinates of $\mathbb{P}^2 \times \mathbb{P}^2$, the dP_6 is given by the two equations $x_1 y_0 - x_2 y_1 = x_1 y_0 - x_0 y_2 = 0$. $\square$

Lemma 2.3.3. *The cohomology ring of $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ is $\mathbb{Z}[x, y]/(x^3, y^2 + c_1 y + c_2)$, where x and y have degree 2. In particular, the cohomology of M is given by $(1, 0, 2, 0, 2, 0, 1)$.*

Proof. The first claim follows from the Leray-Hirch theorem. See [BT82, page 270]. The next claim follows since x and y both have degree 2. $\square$

We can use what we know about the topology of these spaces to compute homology groups of the two affine smoothings.

Theorem 2.3.4. *The two affine smoothings are topologically different. The homology groups are:*

Group	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*Euler-characteristic*
$H^i(X_1, \mathbb{Z})$	*1*	*0*	*2*	*1*	*0*	*0*	*0*	*2*
$H^i(X_2, \mathbb{Z})$	*1*	*0*	*1*	*2*	*0*	*0*	*0*	*0*

Proof. The singular cohomology of $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is given by $(1, 0, 3, 0, 3, 0, 1)$, which can be computed by the Künneth formula. The cohomology of dP_6 is given by $(1, 0, 4, 0, 1)$.

We will use the Lefschetz duality theorem [Spa66], which in this case says that $H_q(M \backslash dP_6, \mathbb{Z}) \simeq H^{6-q}(M, dP_6, \mathbb{Z})$. Then the long exact sequence of the pair (M, dP_6) immediately gives $h_0(X_1, \mathbb{Z}) = 1$. Similarly, we see that $h_5(X_1, \mathbb{Z}) = h_6(X_1, \mathbb{Z}) = 0$, since the map $H^1(M, \mathbb{Z}) \to H^1(D, \mathbb{Z})$ is an isomorphism.

The other groups depend upon the explicit form of the maps $H^2(M, \mathbb{Z}) \to H^2(D, \mathbb{Z})$ and $H^4(M, \mathbb{Z}) \to H^4(D, \mathbb{Z})$.

By Poincaré duality ((reference)), the induced map corresponds to intersecting the divisors on M with dP_6. Computing, we get that map is given by the following matrix: Find reference

$$H^2(M, \mathbb{Z}) \simeq H_4(M, \mathbb{Z}) \simeq \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \mathbb{Z}^4 \simeq H_2(dP_6, \mathbb{Z}) \simeq H^2(dP_6, \mathbb{Z}).$$

This is an injective map, and it follows from the long-exact sequence and the Lefschetz theorem that $H_3(X_1) \simeq H^3(M, dP_6) \simeq \mathbb{Z}$, and also that $H_4(X_1) = 0$.

Similarly, the map $H^4(M) \to H^4(dP_6)$ is computed to be given by $(a, b, c) \mapsto a + b + c$, since the three $\mathbb{P}^1$'s intersect dP_6 in a single point. This map has two-dimensional kernel, and we conclude that $H_2(X_1) \simeq H^4(M, dP_6) = \mathbb{Z}^2$, and that $H^1(X_1) = 0$.

The computations for X_2 are similar but more involved. We first note that the Picard group of $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ is generated by the pullbacks F, G of the generators of $\mathrm{Pic}(\mathbb{P}^2_{x_0x_1x_2} \times \mathbb{P}^2_{y_0y_1y_2})$. Say F is represented by $V(x_0)$ and G is represented by $V(y_0)$.

check transversal intersection with dP6

Again we compute the intersections of F and G with dP_6. Intersecting with F is computed by decomposing the ideal $(x_0, x_1y_0 - x_2y_1, x_1y_0 - x_0y_2)$ in $k[x_0, x_1, x_2, y_0, y_1, y_2]$ and saturating by (x_0, x_1, x_2) and (y_0, y_1, y_2). This can either be done by hand or by using `Macaulay2`. Either way, we find that $F\big|_{dP_6} = E_3 + L_{23} + E_2 = H$, using the notation from earlier this chapter. Similarly $G\big|_{dP_6} = L_{23} + L_{12} + E_2 = 2H - E_1 - E_2 - E_3$. Hence the map on cohomology is given by the matrix

$$H^2(M, \mathbb{Z}) \simeq H_4(M, \mathbb{Z}) \simeq \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}} \mathbb{Z}^4 \simeq H_2(dP_6, \mathbb{Z}) \simeq H^2(dP_6, \mathbb{Z}).$$

This is an injective map, and as above, we conclude that $H_3(X_2) \simeq H^3(M, dP_6) \simeq \mathbb{Z}^2$, and also that $H_4(X_1) = 0$.

$\square$

Remark. *In fact, the Andreotti-Frankel theorem [AF59] states the following: if V is any smooth affine variety of complex dimension n, then it has the homotopy type of a CW complex of dimension n.*

Chapter 3

A smooth Calabi-Yau

Consider the hexagon E_6. The join $E_6 * E_6$ is a 3-dimensional sphere, and so a smoothing of the corresponding Stanley-Reisner scheme would correspond to a smooth Calabi-Yau manifold. In this chapter I prove that there does indeed exist a smoothing, and I describe some of its properties.

Description, singularities, etc.

3.1 Isolated singularities

find better section title

The smoothing process is done by deforming the ambient space. First, note that $\mathcal{K} = E_6 * \mathcal{E}_6 * \Delta^0 * \Delta^0$, is a 5-dimensional simplicial sphere, which is the join of two hexagons with interior. The Stanley-Reisner ring of $\mathcal{K}$ is the tensor product $k[E_6 * \Delta^0] \otimes_k k[E_6 * \Delta^0]$. Each of the factors deform to the affine cone over a del Pezzo surface of degree 6, the same on as in Chapter 2.

It follows that $\mathbb{P}(\mathcal{K})$ deforms to a toric variety Y_0, whose ideal in $\mathbb{P}^{13}$ is the sum of the ideals of the del Pezzo surface in $\mathbb{P}^6$ in a disjoint set of variables. Then it is not hard to see that the singular locus of Y_0 consists of two disjoint copies of dP_6.

Lemma 3.1.1. *Let X_0 be the intersection of Y_0 with two general hyperplanes in $\mathbb{P}^{13}$. Then X_0 is a singular Calabi-Yau variety with 12 isolated singularities.*

Proof. Away from the singular locus of Y_0, the intersection is smooth by Bertini.

The singular locus of Y_0 is equal to $\mathrm{dP}_6 \sqcup \mathrm{dP}_6$, hence is of dimension 2 and degree 12. Two general hyperplanes will intersect the singular locus in 12 points.

Then one form of Bertini's theorem [Har95, page 216], says that all singular points on X comes from those of Y. $\square$

We can determine the types of the singularities of X_0.

Lemma 3.1.2. *Let (U, p_i) be the germ of X_0 at p_i. Then $(U, p_i) \simeq (C(\mathrm{dP}_6), 0)$.*

Proof. In each chart, X_0 looks like $\mathbb{A}^2 \times C(dP_6)$. Let $\mathbb{A}^2$ have coordinates x_2, x_6 and $C(dP_6)$ have coordinates $z_1, \ldots, z_6, y_1$. Then X_0 is the zero set of $I(f, g)$, where f, g are polynomials that are linear in the z_i, y_1 and degree 4 in x_2 and x_6 (in fact, the Newton polyhedron of $f(x_2, x_6, 0, \ldots, 0)$ is a hexagon).

Let $p_i = (a_1, a_2, 0, \ldots, 0)$ be a singular point. By a change of variables, we can translate p_i to the origin. Then $f = x_2 u_1 + l(z_1, \ldots, z_6, y_1)$ and $g = x_6 u_2 + l'(z_1, \ldots, z_6, y_1)$, where u_1, u_2 are units around the origin, and l, l' are linear forms in $k[z_1, \ldots, z_6, y_1]$.

Hence for a small enough U (such that u_1, u_2 are units restricted to U), we can do another change of variables, letting $x_2' = x_2 u_1$, and $x_6' = x_6 u_2$. This allows us to eliminate x_2, x_6 locally from the equations, and we are left with $C(\mathrm{dP}_6)$. $\square$

3.2 Smoothing X_0

There is a smoothing of X_0 to a smooth Calabi-Yau X with Euler characteristic -72.

Lemma 3.2.1. *The toric variety Y_0 deforms to a variety Y with singularities of dimension ≤ 1.*

Proof. Consider the family Y_t defined by

$$\left|\begin{matrix} z_1 & z_2 & y_1 - t(2x_6 - x_2) \\ y_1 - (x_6 - x_2) & z_3 & z_4 \\ z_6 & y_1 & z_5 \end{matrix}\right| \leq 1. \quad \left|\begin{matrix} x_1 & x_2 & y_0 - t(2z_6 - z_2) \\ y_0 - t(z_6 - z_2) & x_3 & x_4 \\ x_6 & y_0 & x_5 \end{matrix}\right| \leq 1.$$

Then Y_0 is the central fiber, and one can check with computer help that the singular locus is a chain of $\mathbb{P}^1$'s. $\square$

Since X_0 is a complete intersection in Y_0, it deforms as well. By Bertini's theorem $X = X_1$ is smooth. In fact, with the help of a computer we can say a little more:

Proposition 3.2.2. *The smoothing X of X_0 has topological Euler characteristic -72.*

Proof. Since $\chi(X) = 2\chi(\mathcal{T}_X) = -2\chi(\Omega^1_X)$, we will show that $\chi(\Omega^1_X) = 36$. $\square$

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Todo list