

Smoothing a Calabi-Yau manifold

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Chapter 1

Preliminary definitions

We work over \mathbb{C} , but some theorems may be stated over a field k .

1.1 Stanley-Reisner basics

Given a simplicial complex \mathcal{K} , one can associate to it a projective scheme $\mathbb{P}(\mathcal{K})$ defined as follows. Let P be the polynomial ring with one variable for each vertex of \mathcal{K} . Then the *Stanley-Reisner ideal* $I_{\mathcal{K}}$ corresponding to \mathcal{K} is generated by the monomials corresponding to *non-faces* of \mathcal{K} . Then we define the *Stanley-Reisner scheme* to be $\text{Proj } P/I_{\mathcal{K}}$.

Example 1.1.1. Let \mathcal{K} be the square, with vertices v_0, v_1, v_2, v_3 . Then the Stanley-Reisner ideal is generated by v_0v_2 and v_1v_3 . ★

Some of the topology of the simplicial complex is encoded in the scheme structure of $\mathbb{P}(\mathcal{K})$. In particular, the simplicial (co)homology groups of \mathcal{K} can be computed as the sheaf cohomology of $\mathbb{P}(\mathcal{K})$

Lemma 1.1.2. *Let $(K; \mathbb{C})$ denote the singular cohomology groups of \mathcal{K} . Then there are isomorphisms $H^i(K; \mathbb{C}) = H^i(\mathbb{P}(\mathcal{K}), \mathcal{O}_{\mathbb{P}(\mathcal{K})})$ for all i .*

Proof. ———to come——— □

Corollary 1.1.3. *We have isomorphisms $H^i(K, \mathbb{C}) \simeq H^{2i}(\mathbb{P}(\mathcal{K}); \mathbb{C})$ of singular cohomology groups.*

Proof. Something about i -cells in even dimensions □

1.2 Calabi-Yau basics

Definition 1.2.1. A *Calabi-Yau variety* is a smooth projective variety satisfying the following two conditions:

1. $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$.
2. The canonical sheaf is trivial: $\omega_X \simeq \mathcal{O}_X$.

■

The classical example of a Calabi-Yau manifold is the quintic threefold in \mathbb{P}^5 . Another example is the following:

Example 1.2.2. Let X be the double cover of \mathbb{P}^3 ramified along a smooth octic. The projection map is affine, so the conditions on $H^i(X, \mathcal{O}_X)$ are fulfilled. To see that the canonical sheaf is trivial, we use the adjunction formula, which says that $K_X = 2K_{\mathbb{P}^3}|_X + R$, where R is the ramification divisor. In this case $R \sim 8H$, where H is a hyperplane in \mathbb{P}^3 . Then, since $K_{\mathbb{P}^3} = -4H$, it follows that $K_X = 0$. ★

If \mathcal{K} is a simplicial sphere, then a smoothing of $\mathbb{P}(\mathcal{K})$ will give a Calabi-Yau manifold.

— ref: bayer-eisenbud graph curves.

The most basic invariants of Calabi-Yau manifolds are their *Hodge numbers* h^{pq} . In algebraic geometry these can be defined as the dimensions of the cohomology groups $H^q(X, \Omega_X^p)$. This definition is however not so transparent. On a complex manifold, it is true that $h^{pq} = h^{qp}$, but this is not obvious from our definition. Instead, let us define these groups in complex algebraic geometry terms.

The de Rham complex (Ω^\bullet, d) refines to a bigraded complex $(\Omega^{\bullet,\bullet}, d)$, where a differential form of bidegree (p, q) can be written as

$$\omega = \sum f_{IJ} dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_q.$$

The differential d splits as $\partial + \bar{\partial}$, where $\partial : \Omega^{\bullet,\bullet} \rightarrow \Omega^{\bullet+1,\bullet}$, and $\bar{\partial} : \Omega^{\bullet,\bullet} \rightarrow \Omega^{\bullet,\bullet+1}$. The decomposition passes respects cohomology, so we can form the *Dolbeault cohomology groups* $H^{p,q}(X)$.

With this definition, applying complex conjugation shows that $H^{p,q} = \overline{H^{q,p}}$.

Lemma 1.2.3. *We have natural isomorphisms $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$.*

Proof. Use that the de Rham complex is flabby □

For more details on this and other details from complex geometry, see [Voi07].

The “Hodge diamond” is ...

Example 1.2.4. Let X be a smooth quintic in \mathbb{P}^4 . We will compute its Hodge numbers. Let us first compute $H^{1,1}(X)$. We have the following exact sequence

$$0 \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{\mathbb{P}^4}|_X \rightarrow \Omega_X^1 \rightarrow 0$$

Since $\mathcal{I}/\mathcal{I}^2 \simeq \mathcal{O}_X(-5)$, it follows from the long exact sequence of cohomology that $H^1(X, \Omega_X^1) \simeq H^1(X, \Omega_{\mathbb{P}^4}^1|_X)$ ★

1.3 Deformation theory

Deformation theory is the study how varieties (or other algebraic structures like line bundles, vector bundles, ...) vary in families.

There is a lot of technical machinery available for the deformation theorist, but for us just a few vector spaces will be of importance.

Definition 1.3.1. Let X be a scheme over k . Then a *deformation of X over S* is a flat morphism $\mathfrak{X} \rightarrow S$ together with an isomorphism $X \simeq \mathfrak{X} \times_S 0$ for a closed point $0 \in S$:

$$\begin{array}{ccc} X \simeq X_0 & \hookrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & S \end{array}$$

■

Recall that a morphism $f : X \rightarrow Y$ is *flat* if the associated morphism $f^{\$} : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ of \mathcal{O}_Y -modules is a flat morphism.

Chapter 2

Two topologically distinct smoothings

Denote by dP_6 the del Pezzo surface of degree 6 embedded in \mathbb{P}^6 . This can be realized as the blow-up of \mathbb{P}^2 in three points not lying on a line. Let X denote the affine cone over dP_6 . Then it has long been known that X has two smoothing components, and we show here that they are topologically distinct.

Recall that a *del Pezzo* surface is a surface such that the anti-canonical bundle is ample. The degree is the degree given by the anticanonical embedding. It is a classical result that every del Pezzo surface is obtained either by blowing up \mathbb{P}^2 in $r = 0, \dots, 6$ points in suitable positions, or as the 2-uple embedding of a quadric surface in \mathbb{P}^3 .

2.1 Different embeddings of dP_6

We first obtain the equations of dP_6 directly from the description of it as blow-up. Let x_0, x_1, x_2 be coordinates of \mathbb{P}^2 . Recall that the blowup of \mathbb{P}^2 in the point $(1 : 0 : 0)$ can be realized as the closed subscheme of $\mathbb{P}^2 \times \mathbb{P}^1$ given by the equation $r_0x_1 - r_1x_2 = 0$, where r_0, r_1 are coordinates on \mathbb{P}^1 . We can repeat this process on the points $(0 : 1 : 0)$ and $(0 : 0 : 1)$ to obtain similar equations.

Collecting these, we see that dP_6 is given by the matrix equation

$$M\vec{x} = \begin{pmatrix} 0 & r_0 & -r_1 \\ s_1 & 0 & -s_0 \\ -t_0 & t_1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0.$$

in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Here r_i, s_i and t_i ($i = 0, 1$) are of course coordinates on \mathbb{P}^1 .

We can do more than this however.

Lemma 2.1.1. *We can also realize dP_6 embedded in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with equation $r_0 s_0 t_0 = r_1 s_1 t_1$.*

Proof. Note that the matrix cannot have rank 1 or lower. Now consider the projection onto the last three factors:

$$\pi : \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Each point P in the product on the right-hand side gives a matrix M_P of rank 2. Thus there is a line of solutions, which correspond exactly to a point in \mathbb{P}^2 .

Hence the restriction of π to dP_6 is an isomorphism onto the hypersurface given by $\det M = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. \square

Another way to realize blow-ups is this: let \mathfrak{d} be the linear system of quadrics with assigned basepoints $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$ in \mathbb{P}^2 . We can choose a basis given by $x_0 x_1$, $x_0 x_2$ and $x_1 x_2$. This gives a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. The closure of the graph of this map is a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$ defined by two bilinear equations. Each of the projections correspond to the blowup.

Explicitly, if we let y_0, y_1, y_2 be coordinates on the other \mathbb{P}^2 , then the equations are $x_1 y_0 - x_0 y_1 = x_1 y_1 - x_2 y_2 = 0$.

We also have a natural embedding in \mathbb{P}^6 as follows. Denote by E_1, E_2, E_3 the exceptional divisors on the blowup. Let L be a line in \mathbb{P}^2 . Then the divisor $\pi^* 3L - E_1 - E_2 - E_3$ is ample, and gives an embedding in \mathbb{P}^6 (see [Har77, Chapter V, Theorem 4.6]). A basis for the corresponding linear system is given by all monomials in $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ except x^3, y^3 and z^3 .

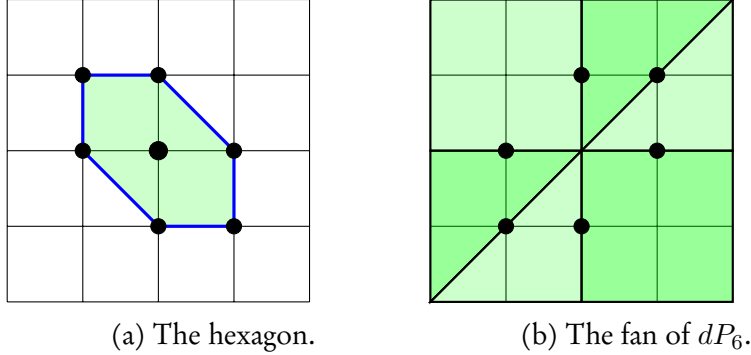


Figure 2.1: Toric description of dP_6 .

The equations can be arranged in a particularly symmetric form: let y, x_1, \dots, x_1 be coordinates on \mathbb{P}^6 . Then the equations of dP_6 are the 2×2 minors of the matrix

$$\begin{pmatrix} x_1 & y & x_6 \\ x_2 & x_3 & y \\ y & x_4 & x_5 \end{pmatrix}.$$

This gives 9 equations, which can be compactly written as $x_i x_{i+2} - y x_i = 0$ and $x_i x_{i+3} - y^2 = 0$, for $i = 1, \dots, 6$ (where i is taken modulo 6). Note that the equations have a visible D_6 -symmetry, where D_6 denotes the dihedral group.

2.1.1 As a toric variety

There is a nice combinatorial description of dP_6 as a toric variety associated to a polytope. Namely, let P denote the hexagon in Figure 2.1a. Then the normal fan of this polytope defines a fan in $N_{\mathbb{R}}$, defining a toric variety.

The polytope is reflexive, implying that the normal fan of P is the face fan over the same polytope. See Figure 2.1b. From standard toric geometry, it is clear that dP_6 is the blowup of \mathbb{P}^2 in the three torus-fixed points.

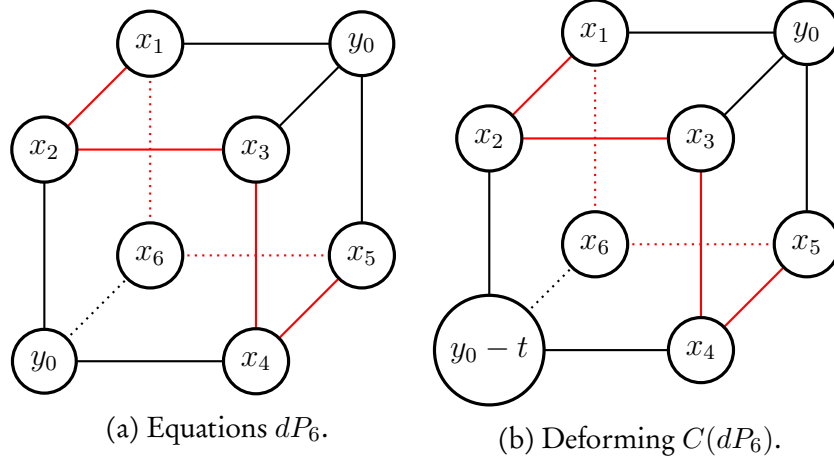


Figure 2.2: Forms of equations.

2.2 Divisors and topology

Consider the exponential sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_{dP_6} \rightarrow \mathcal{O}_{dP_6}^* \rightarrow 0.$$

Since dP_6 is a rational surface, it follows from the long-exact sequence that $H^2(dP_6, \mathbb{Z}) \simeq \text{Pic}(dP_6)$. From the description of dP_6 as a blowup, we know from [Har77, Chapter V], that the Picard group is spanned by the three exceptional divisors, together with the class of the pullback of a hyperplane.

2.3 The affine cone and its two smoothings

Let X denote the affine cone over dP_6 . It is an affine variety with an isolated singularity at the origin. One can compute that it has two smoothing components: the union of a plane and a line. They both come from different ways of perturbing the equations of dP_6 .

Look at Figure 2.2a. One can read off the equations of dP_6 by taking minors along “faces” and long diagonals of this square. This correspond to a hyperplane cut of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in the Segre embedding. Then the one-dimensional component of the versal deformation of X is obtained by perturbing one of the

y_0 -corners as in Figure 2.2b.

It is clear the corresponding deformation is smooth, since it is a hyperplane cut of cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ outside the origin. Call this smoothing X_1 .

Lemma 2.3.1. *The smoothing X_1 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus dP_6$.*

Proof. Specialize to some $t \neq 0$. Then we can homogenize the equations with respect to y_1 to obtain a projective variety in 8 variables. However, in this form, $y_0 - ty_1$ and y_0 are linearly independent, hence by a change of variables, we see that this variety is in fact isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding. See Figure 2.2b.

What we gained by homogenizing is exactly the projective variety given by setting $y_1 = 0$. But then we get back the equations of dP_6 in \mathbb{P}^6 . \square

The second smoothing is obtained by deforming the equations of dP_6 as a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$. Namely, consider the following matrix:

$$\begin{vmatrix} x_1 & y_0 & x_6 \\ x_2 & x_3 & y_0 - t_1 \\ y_0 - t_2 & x_4 & x_5 \end{vmatrix} \leq 1. \quad (2.1)$$

For $t_1 = t_2 = 0$, we get the cone over dP_6 , while for generic t_i , we get a smooth variety. In fact, we can compute that the discriminant locus (the set of points in \mathbb{A}_{t_1, t_2}^2 with singular fiber) are the t_1 -axis, the t_2 -axis and the line $t_1 = t_2$.

Call (any) smooth fiber X_2 .

Lemma 2.3.2. *Let $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ be the projective bundle associated to the tangent sheaf on \mathbb{P}^2 . Then the smoothing X_2 is isomorphic to $M \setminus dP_6$.*

Proof. The technique is the same as in the previous proof. First homogenize the equations (2.1) with respect to y_1 . Call the homogenized variety M . Put $y'_0 = y_0$, $y'_1 = y_0 - ty_1$ and $y'_2 = y_0 - t_2y_1$. Then we have the relation

$$h = t_2y'_1 - t_1y'_2 - (t_1 - t_2)y'_0 = 0.$$

Hence we see that $M = \mathbb{P}^2 \times \mathbb{P}^2 \cap (h = 0)$. We can pull back the coordinates y'_i to $\mathbb{P}^2 \times \mathbb{P}^2$. Let $\mathbb{P}^2 \times \mathbb{P}^2$ have coordinates x_0, x_1, x_2 and y_0, y_1, y_2 . Then h pulls back to the equation

$$(x_0, x_1, x_2) \cdot (-t_1 y_2, (t_1 - t_2) y_0, t_2 y_1) = 0$$

in $\mathbb{P}^2 \times \mathbb{P}^2$. As long as $t_1 \neq t_2$ and $t_1, t_2 \neq 0$, we can do a change of coordinates in $\mathbb{P}^2_{y_0 y_1 y_2}$, so that h transforms to

$$(x_0, x_1, x_2) \cdot (y_0, y_1, y_2) = 0.$$

Hence we see that M is isomorphic to the total space of the Grassmannian of lines in \mathbb{P}^2 (each point in one of the \mathbb{P}^2 's give a line in the other \mathbb{P}^2). This is in turn isomorphic to $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$, since each tangent vector through a point determines a line through it.

Now, what have we gained by homogenizing? The divisor at infinity is $y_1 = 0$, which is a dP_6 again. In our new coordinates this is equivalent to $y'_1 = y'_2 = y'_0$. Hence in the coordinates of $\mathbb{P}^2 \times \mathbb{P}^2$, the dP_6 is given by the two equations $x_1 y_0 - x_2 y_1 = x_1 y_0 - x_0 y_2 = 0$. \square

Lemma 2.3.3. *The cohomology ring of $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ is $\mathbb{Z}[x, y]/(x^3, y^2 + c_1(E)y + c_2(E))$.*

Proof. This is a consequence of the Leray-Hirsch theorem. See [BT82] for a proof. \square

We can use what we know about the topology of these spaces to compute homology groups of the two affine smoothings.

Theorem 2.3.4. *The two affine smoothings are topologically different. The homology groups are:*

Group	0	1	2	3	4	5	6	Euler-characteristic
$H^i(X_1, \mathbb{Z})$	1	0	2	1	0	0	0	2
$H^i(X_2, \mathbb{Z})$	1	0	1	2	0	0	0	0

Proof. The singular cohomology of $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is given by $(1, 0, 3, 0, 3, 0, 1)$, which can be computed by the Künneth formula. The cohomology of dP_6 is given by $(1, 0, 4, 0, 1)$.

We will use the Lefschetz duality theorem [Spa66], which in this case says that $H_q(M \setminus dP_6, \mathbb{Z}) \simeq H^{6-q}(M, dP_6, \mathbb{Z})$. Then the long exact sequence of the pair (M, dP_6) immediately gives $h_0(X_1, \mathbb{Z}) = 1$. Similarly, we see that $h_5(X_1, \mathbb{Z}) = h_6(X_1, \mathbb{Z}) = 0$, since the map $H^1(M, \mathbb{Z}) \rightarrow H^1(D, \mathbb{Z})$ is an isomorphism.

The other groups depend upon the explicit form of the maps $H^2(M, \mathbb{Z}) \rightarrow H^2(D, \mathbb{Z})$ and $H^4(M, \mathbb{Z}) \rightarrow H^4(D, \mathbb{Z})$.

By Poincaré duality ((reference)), the induced map corresponds to intersecting the divisors on M with dP_6 . Computing, we get that map is given by the following matrix:

$$H^2(M, \mathbb{Z}) \simeq H_4(M, \mathbb{Z}) \simeq \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \mathbb{Z}^4 \simeq H_2(dP_6, \mathbb{Z}) \simeq H^2(dP_6, \mathbb{Z}).$$

This is an injective map, and it follows from the long-exact sequence and the Lefschetz theorem that $H_3(X_1) \simeq H^3(M, dP_6) \simeq \mathbb{Z}$, and also that $H_4(X_1) = 0$.

Similarly, the map $H^4(M) \rightarrow H^4(dP_6)$ is computed to be given by $(a, b, c) \mapsto a + b + c$, since the three \mathbb{P}^1 's intersect dP_6 in a single point. This map has two-dimensional kernel, and we conclude that $H_2(X_1) \simeq H^4(M, dP_6) = \mathbb{Z}^2$, and that $H^1(X_1) = 0$.

The computations for X_2 are similar but more involved. We first

□

Remark. In fact, the Andreotti-Frankel theorem [AF59] states the following: if V is any smooth affine variety of complex dimension n , then it has the homotopy type of a CW complex of dimension n .

Chapter 3

A smooth Calabi-Yau

Consider the hexagon E_6 . The join $E_6 * E_6$ is a 3-dimensional sphere, and so a smoothing of the corresponding Stanley-Reisner scheme would correspond to a smooth Calabi-Yau manifold. In this chapter I prove that there does indeed exist a smoothing, and I describe some of its properties.

Description, singularities, etc.

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