

# Algebraic Geometry Buzzlist

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## 1 Algebraic Geometry

### 1.1 General properties

#### 1.1.1 Complete variety

Let  $X$  be an integral, **separated** scheme over a field  $k$ . Then  $X$  is **complete** if it is **proper**.

#### 1.1.2 Crepant resolution

A **crepant resolution** is a resolution of singularities  $f : X \rightarrow Y$  that does not change the **canonical bundle**, i.e. such that  $\omega_X \simeq f^*(\omega_Y)$ .

#### 1.1.3 Hodge numbers

If  $X$  is a complex manifold, then the **Hodge numbers**  $h^{p,q}$  of  $X$  are defined as the dimension of the cohomology groups  $H^p(X, \Omega_X^q)$ .

#### 1.1.4 Normal variety

A variety  $X$  is **normal** if all its local rings are **normal rings**.

#### 1.1.5 Proper morphism

A morphism  $f : X \rightarrow Y$  is **proper** if it **separated**, of finite type, and universally closed.

#### 1.1.6 Resolution of singularities

A morphism  $f : X \rightarrow Y$  is a **resolution of singularities of  $Y$**  if  $X$  is non-singular and  $f$  is birational and **proper**.

### 1.1.7 Separated morphism

Let  $f : X \rightarrow Y$  be a morphism of schemes. Let  $\Delta : X \rightarrow X \times_Y X$  be the diagonal morphism. We say that  $f$  is **separated** if  $\Delta$  is a closed immersion.

## 1.2 Results and theorems

### 1.2.1 Adjunction formula

Let  $X$  be a smooth algebraic variety  $Y$  a smooth subvariety. Let  $i : Y \hookrightarrow X$  be the inclusion map, and let  $\mathcal{I}$  be the corresponding ideal sheaf. Then  $\omega_Y = i^* \omega_X \otimes_{\mathcal{O}_X} \det(\mathcal{I}/\mathcal{I}^2)^\vee$ , where  $\omega_Y$  is the **canonical sheaf** of  $Y$ .

In terms of canonical classes, the formula says that  $K_D = (K_X + D)|_D$ .

### 1.2.2 Bertini's Theorem

Let  $X$  be a nonsingular closed subvariety of  $\mathbb{P}_k^n$ , where  $k = \bar{k}$ . Then the set of hyperplanes  $H \subseteq \mathbb{P}_k^n$  such that  $H \cap X$  is regular at every point) and such that  $H \not\subseteq X$  is a dense open subset of the complete linear system  $|H|$ . See [3, Thm II.8.18].

### 1.2.3 Euler sequence

If  $A$  is a ring and  $\mathbb{P}_A^n$  is projective  $n$ -space over  $A$ , then there is an exact sequence of sheaves on  $X$ :

$$0 \rightarrow \Omega_{\mathbb{P}_A^n/A} \rightarrow \mathcal{O}_{\mathbb{P}_A^n}(-1)^{n+1} \rightarrow \mathcal{O}_{\mathbb{P}_A^n} \rightarrow 0.$$

See [3, Thm II.8.13].

### 1.2.4 Kodaira vanishing

If  $k$  is a field of characteristic zero,  $X$  is a smooth and projective  $k$ -scheme of dimension  $d$ , and  $\mathcal{L}$  is an **ample** invertible sheaf on  $X$ , then  $H^q(X, \mathcal{L} \otimes_{\mathcal{O}_X} \Omega_{X/k}^p) = 0$  for  $p + q > d$ . In addition,  $H^q(X, \mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \Omega_{X/k}^p) = 0$  for  $p + q < d$ .

### 1.2.5 Lefschetz hyperplane theorem

Let  $X$  be an  $n$ -dimensional complex projective algebraic variety in  $\mathbb{P}_{\mathbb{C}}^n$  and let  $Y$  be a hyperplane section of  $X$  such that  $U = X \setminus Y$  is smooth. Then the natural map  $H^k(X, \mathbb{Z}) \rightarrow H^k(Y, \mathbb{Z})$  in singular cohomology is an isomorphism for  $k < n - 1$  and injective for  $k = n - 1$ .

### 1.2.6 Riemann-Roch for curves

The **Riemann-Roch theorem** relates the number of sections of a line bundle with the genus of a smooth curve  $C$ . Let  $\mathcal{L}$  be a line bundle  $\omega_C$  the canonical sheaf on  $C$ . Then

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^{-1} \otimes_{\mathcal{O}_C} \omega_C) = \deg(\mathcal{L}) + 1 - g.$$

This is [3, Theorem IV.1.3].

### 1.2.7 Semi-continuity theorem

Let  $f : X \rightarrow Y$  be a projective morphism of noetherian schemes, and let  $\mathcal{F}$  be a coherent sheaf on  $X$ , flat over  $Y$ . Then for each  $i \geq 0$ , the function  $h^i(y, \mathcal{F}) = \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$  is an upper semicontinuous function on  $Y$ . See [3, Chapter III, Theorem 12.8].

### 1.2.8 Serre vanishing

One form of Serre vanishing states that if  $X$  is a proper scheme over a noetherian ring  $A$ , and  $\mathcal{L}$  is an **ample** sheaf, then for any coherent sheaf  $\mathcal{F}$  on  $X$ , there exists an integer  $n_0$  such that for each  $i > 0$  and  $n \geq n_0$  the group  $H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n) = 0$  vanishes. See [3, Proposition III.5.3].

## 1.3 Sheaves and bundles

### 1.3.1 Ample line bundle

A line bundle  $\mathcal{L}$  is **ample** if for any coherent sheaf  $\mathcal{F}$  on  $X$ , there is an integer  $n$  (depending on  $\mathcal{F}$ ) such that  $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$  is generated by global sections. Equivalently, a line bundle  $\mathcal{L}$  is ample if some tensor power of it is **very ample**.

### 1.3.2 Invertible sheaf

A locally free sheaf of rank 1 is called **invertible**. If  $X$  is **normal**, then, invertible sheaves are in 1 – 1 correspondence with line bundles.

### 1.3.3 Anticanonical sheaf

The **anticanonical sheaf**  $\omega_X^{-1}$  is the inverse of the **canonical sheaf**  $\omega_X$ , that is  $\omega_X^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{O}_X)$ .

### 1.3.4 Canonical divisor

The **canonical divisor**  $K_X$  is the class of the **canonical sheaf**  $\omega_X$  in the divisor class group.

### 1.3.5 Canonical sheaf

If  $X$  is a smooth algebraic variety of dimension  $n$ , then the canonical sheaf is  $\omega := \wedge^n \Omega_{X/k}^1$  the  $n$ 'th exterior power of the cotangent bundle of  $X$ .

### 1.3.6 Sheaf of holomorphic p-forms

If  $X$  is a complex manifold, then the **sheaf of holomorphic  $p$ -forms**  $\Omega_X^p$  is the  $p$ -th wedge power of the cotangent sheaf  $\wedge^p \Omega_X^1$ .

### 1.3.7 Normal sheaf

Let  $Y \hookrightarrow X$  be a closed immersion of schemes, and let  $\mathcal{I} \subseteq \mathcal{O}_X$  be the ideal sheaf of  $Y$  in  $X$ . Then  $\mathcal{I}/\mathcal{I}^2$  is a sheaf on  $Y$ , and we define the sheaf  $\mathcal{N}_{Y/X}$  by  $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ .

### 1.3.8 Reflexive sheaf

A sheaf  $\mathcal{F}$  is **reflexive** if the natural map  $\mathcal{F} \rightarrow \mathcal{F}^{\vee\vee}$  is an isomorphism. Here  $\mathcal{F}^{\vee}$  denotes the sheaf  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ .

### 1.3.9 Very ample line bundle

A line bundle  $\mathcal{L}$  is **very ample** if there is an embedding  $i : X \hookrightarrow \mathbb{P}_S^n$  such that the pullback of  $\mathcal{O}_{\mathbb{P}_S^n}(1)$  is isomorphic to  $\mathcal{L}$ . In other words, there should be an isomorphism  $i^* \mathcal{O}_{\mathbb{P}_S^n}(1) \simeq \mathcal{L}$ .

## 1.4 Toric geometry

### 1.4.1 Polarized toric variety

A toric variety equipped with an **ample**  $T$ -invariant divisor.

## 1.5 Types of varieties

### 1.5.1 Calabi-Yau variety

In algebraic geometry, a **Calabi-Yau** variety is a smooth, proper variety  $X$  over a field  $k$  such that the **canonical sheaf** is trivial, that is,  $\omega_X \simeq \mathcal{O}_X$ , and such that  $H^j(X, \mathcal{O}_X) = 0$  for  $1 \leq j \leq n - 1$ .

### 1.5.2 Fano variety

A variety  $X$  is **Fano** if the **anticanonical sheaf**  $\omega_X^{-1}$  is **ample**.

### 1.5.3 K3 surface

A **K3 surface** is a complex algebraic surface  $X$  such that the **canonical sheaf** is trivial,  $\omega_X \simeq \mathcal{O}_X$ , and such that  $H^1(X, \mathcal{O}_X) = 0$ . These conditions completely determine the Hodge numbers of  $X$ .

## 2 Commutative algebra

### 2.1 Modules

#### 2.1.1 Depth

Let  $R$  be a noetherian ring, and  $M$  a finitely-generated  $R$ -module and  $I$  an ideal of  $R$  such that  $IM \neq M$ . Then the  $I$ -depth of  $M$  is (see **Ext**):

$$\inf\{i \mid \text{Ext}_R^i(R/I, M) \neq 0\}.$$

This is also the length of a maximal  $M$ -sequence in  $I$ .

### 2.2 Results and theorems

#### 2.2.1 The Unmixedness Theorem

Let  $R$  be a ring. If  $I = \langle x_1, \dots, x_n \rangle$  is an ideal generated by  $n$  elements such that  $\text{codim } I = n$ , then all minimal primes of  $I$  have codimension  $n$ . If in addition  $R$  is **Cohen-Macaulay**, then every associated prime of  $I$  is minimal over  $I$ . See the discussion after [2, Corollary 18.14] for more details.

## 2.3 Rings

### 2.3.1 Cohen-Macaulay ring

A local Cohen-Macaulay ring (CM-ring for short) is a commutative noetherian local ring with Krull dimension equal to its depth. A ring is Cohen-Macaulay if its localization at all prime ideals are Cohen-Macaulay.

### 2.3.2 Depth of a ring

The depth of a ring  $R$  is its depth as a module over itself.

### 2.3.3 Gorenstein ring

A commutative ring  $R$  is Gorenstein if each localization at a prime ideal is a Gorenstein local ring. A Gorenstein local ring is a local ring with finite injective dimension as an  $R$ -module. This is equivalent to the following:  $\text{Ext}_R^i(k, R) = 0$  for  $i \neq n$  and  $\text{Ext}_R^n(k, R) \simeq k$  (here  $k = R/\mathfrak{m}$  and  $n$  is the Krull dimension of  $R$ ).

### 2.3.4 Normal ring

An integral domain  $R$  is **normal** if all its localizations at prime ideals  $\mathfrak{p} \in \text{Spec } R$  are integrally closed domains.

## 3 Convex geometry

### 3.1 Cones

#### 3.1.1 Gorenstein cone

A strongly convex cone  $C \subset M_{\mathbb{R}}$  is **Gorenstein** if there exists a point  $n \in N$  in the dual lattice such that  $\langle v, n \rangle = 1$  for all generators of the semigroup  $C \cap M$ .

#### 3.1.2 Reflexive Gorenstein cone

A cone  $C$  is **reflexive** if both  $C$  and its dual  $C^\vee$  are **Gorenstein cones**. See for example [1].

### 3.1.3 Simplicial cone

A cone  $C$  generated by  $\{v_1, \dots, v_k\} \subseteq N_{\mathbb{R}}$  is **simplicial** if the  $v_i$  are linearly independent.

## 3.2 Polytopes

### 3.2.1 Dual (polar) polytope

If  $\Delta$  is a polyhedron, its dual  $\Delta^\circ$  is defined by

$$\Delta^\circ = \{x \in N_{\mathbb{R}} \mid \langle x, y \rangle \geq -1 \forall y \in \Delta\}.$$

### 3.2.2 Gorenstein polytope of index $r$

A lattice polytope  $P \subset \mathbb{R}^{d+r-1}$  is called a **Gorenstein polytope of index  $r$**  if  $rP$  contains a single interior lattice point  $p$  and  $rP - p$  is a **reflexive polytope**.

### 3.2.3 Nef partition

Let  $\Delta \subset M_{\mathbb{R}}$  be a  $d$ -dimensional **reflexive polytope**, and let  $m = \text{int}(\Delta) \cap M$ . A Minkowski sum decomposition  $\Delta = \Delta_1 + \dots + \Delta_r$  where  $\Delta_1, \dots, \Delta_r$  are lattice polytopes is called a **nef partition of  $\Delta$  of length  $r$**  if there are lattice points  $p_i \in \Delta_i$  for all  $i$  such that  $p_1 + \dots + p_r = m$ . The nef partition is called *centered* if  $p_i = 0$  for all  $i$ .

This is equivalent to the toric divisor  $D_j = \mathcal{O}(\Delta_j) = \sum_{\rho \in \Delta_j} D_\rho$  being a Cartier divisor generated by its global sections. See [1, Chapter 4.3].

### 3.2.4 Reflexive polytope

A polytope  $\Delta$  is **reflexive** if the following two conditions hold:

1. All facets  $\Gamma$  of  $\Delta$  are supported by affine hyperplanes of the form  $\{m \in M_{\mathbb{R}} \mid \langle m, v_\Gamma \rangle = 0\}$  for some  $v_\Gamma \in N$ .
2. The only interior point of  $\Delta$  is 0, that is:  $\text{Int}(\Delta) \cap M = \{0\}$ .

## 4 Homological algebra

### 4.1 Derived functors

#### 4.1.1 Ext

Let  $R$  be a ring and  $M, N$  be  $R$ -modules. Then  $\mathrm{Ext}_R^i(M, N)$  is the right-derived functors of the  $\mathrm{Hom}(M, -)$ -functor. In particular,  $\mathrm{Ext}_R^i(M, N)$  can be computed as follows: choose a projective resolution  $C_\bullet$  of  $N$  over  $R$ . Then apply the left-exact functor  $\mathrm{Hom}_R(M, -)$  to the resolution and take homology. Then  $\mathrm{Ext}_R^i(M, N) = h^i(C_\bullet)$ .

#### 4.1.2 Tor

Let  $R$  be a ring and  $M, N$  be  $R$ -modules. Then  $\mathrm{Tor}_R^i(M, N)$  is the right-derived functors of the  $- \otimes_R N$ -functor. In particular  $\mathrm{Tor}_R^i(M, N)$  can be computed by taking a projective resolution of  $M$ , tensoring with  $N$ , and then taking homology.

## References

- [1] David A. Cox and Sheldon Katz. *Mirror symmetry and algebraic geometry*, volume 68 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.
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- [3] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.