

Algebraic Geometry Buzzlist

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1 Algebraic Geometry

1.1 Results and theorems

1.1.1 Adjunction formula

Let X be a smooth algebraic variety Y a smooth subvariety. Let $i : Y \hookrightarrow X$ be the inclusion map, and let \mathcal{I} be the corresponding ideal sheaf. Then $\omega_Y = i^*\omega_X \otimes_{\mathcal{O}_X} \det(\mathcal{I}/\mathcal{I}^2)^\vee$, where ω_Y is the canonical sheaf of Y .

In terms of canonical classes, the formula says that $K_D = (K_X + D)|_D$.

1.1.2 Bertini's Theorem

Let X be a nonsingular closed subvariety of \mathbb{P}_k^n , where $k = \bar{k}$. Then the set of hyperplanes $H \subseteq \mathbb{P}_k^n$ such that $H \cap X$ is regular at every point) and such that $H \not\subseteq X$ is a dense open subset of the complete linear system $|H|$. See [1, Thm II.8.18].

1.1.3 Euler sequence

If A is a ring and \mathbb{P}_A^n is projective n -space over A , then there is an exact sequence of sheaves on X :

$$0 \rightarrow \Omega_{\mathbb{P}_A^n/A} \rightarrow \mathcal{O}_{\mathbb{P}_A^n}(-1)^{n+1} \rightarrow \mathcal{O}_{\mathbb{P}_A^n} \rightarrow 0.$$

See [1, Thm II.8.13].

1.1.4 Kodaira vanishing

If k is a field of characteristic zero, X is a smooth and projective k -scheme of dimension d , and \mathcal{L} is an ample invertible sheaf on X , then $H^q(X, \mathcal{L} \otimes_{\mathcal{O}_X} \Omega_{X/k}^p) = 0$ for $p + q > d$. In addition, $H^q(X, \mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \Omega_{X/k}^p) = 0$ for $p + q < d$.

1.1.5 Lefschetz hyperplane theorem

Let X be an n -dimensional complex projective algebraic variety in $\mathbb{P}_{\mathbb{C}}^n$ and let Y be a hyperplane section of X such that $U = X \setminus Y$ is smooth. Then the natural map $H^k(X, \mathbb{Z}) \rightarrow H^k(Y, \mathbb{Z})$ in singular cohomology is an isomorphism for $k < n - 1$ and injective for $k = n - 1$.

1.1.6 Riemann-Roch for curves

The **Riemann-Roch theorem** relates the number of sections of a line bundle with the genus of a smooth curve C . Let \mathcal{L} be a line bundle ω_C the canonical sheaf on C . Then

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^{-1} \otimes_{\mathcal{O}_C} \omega_C) = \deg(\mathcal{L}) + 1 - g.$$

This is [1, Theorem IV.1.3].

1.1.7 Serre vanishing

One form of Serre vanishing states that if X is a proper scheme over a noetherian ring A , and \mathcal{L} is an ample sheaf, then for any coherent sheaf \mathcal{F} on X , there exists an integer n_0 such that for each $i > 0$ and $n \geq n_0$ the group $H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n) = 0$ vanishes. See [1, Proposition III.5.3].

1.2 Sheaves and bundles

1.2.1 Ample line bundle

A line bundle \mathcal{L} is **ample** if for any coherent sheaf \mathcal{F} on X , there is an integer n (depending on \mathcal{F}) such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is generated by global sections. Equivalently, a line bundle \mathcal{L} is ample if some tensor power of it is very ample.

1.2.2 Very ample line bundle

A line bundle \mathcal{L} is **very ample** if there is an embedding $i : X \hookrightarrow \mathbb{P}_{\mathbb{S}}^n$ such that the pullback of $\mathcal{O}_{\mathbb{P}_{\mathbb{S}}^n}(1)$ is isomorphic to \mathcal{L} . In other words, there should be an isomorphism $i^* \mathcal{O}_{\mathbb{P}_{\mathbb{S}}^n}(1) \simeq \mathcal{L}$.

1.2.3 Anticanonical sheaf

The **anticanonical sheaf** ω_X^{-1} is the inverse of the canonical sheaf ω_X , that is $\omega_X^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{O}_X)$.

1.2.4 Canonical divisor

The **canonical divisor** K_X is the class of the canonical sheaf ω_X in the divisor class group.

1.2.5 Canonical sheaf

If X is a smooth algebraic variety of dimension n , then the canonical sheaf is $\omega := \wedge^n \Omega_{X/k}^1$ the n 'th exterior power of the cotangent bundle of X .

1.3 Toric geometry

1.3.1 Polarized toric variety

A toric variety equipped with an ample T -invariant divisor.

1.4 Types of varieties

1.4.1 Calabi-Yau variety

In algebraic geometry, a **Calabi-Yau** variety is a smooth, proper variety X over a field k such that the canonical sheaf is trivial, that is, $\omega_X \simeq \mathcal{O}_X$, and such that $H^j(X, \mathcal{O}_X) = 0$ for $1 \leq j \leq n - 1$.

1.4.2 Fano variety

A variety X is **Fano** if the anticanonical sheaf ω_X^{-1} is ample.

1.4.3 K3 surface

A **K3 surface** is a complex algebraic surface X such that the canonical sheaf is trivial, $\omega_X \simeq \mathcal{O}_X$, and such that $H^1(X, \mathcal{O}_X) = 0$. These conditions completely determine the Hodge numbers of X .

2 Commutative algebra

2.1 Modules

2.1.1 Depth

Let R be a noetherian ring, and M a finitely-generated R -module and I an ideal of R such that $IM \neq M$. Then the I -depth of M is (see Ext):

$$\inf\{i \mid \text{Ext}_R^i(R/I, M) \neq 0\}.$$

This is also the length of a maximal M -sequence in I .

2.2 Rings

2.2.1 Cohen-Macaulay ring

A local Cohen-Macaulay ring (CM-ring for short) is a commutative noetherian local ring with Krull dimension equal to its depth. A ring is Cohen-Macaulay if its localization at all prime ideals are Cohen-Macaulay.

2.2.2 Depth of a ring

The depth of a ring R is its depth as a module over itself.

2.2.3 Gorenstein ring

A commutative ring R is Gorenstein if each localization at a prime ideal is a Gorenstein local ring. A Gorenstein local ring is a local ring with finite injective dimension as an R -module. This is equivalent to the following: $\text{Ext}_R^i(k, R) = 0$ for $i \neq n$ and $\text{Ext}_R^n(k, R) \simeq k$ (here $k = R/\mathfrak{m}$ and n is the Krull dimension of R).

3 Convex geometry

3.1 Cones

3.1.1 Simplicial cone

A cone C generated by $\{v_1, \dots, v_k\} \subseteq N_{\mathbb{R}}$ is simplicial if the v_i are linearly independent.

3.2 Polyhedra

3.2.1 Dual (polar) polyhedron

If Δ is a polyhedron, its dual Δ° is defined by

$$\Delta^\circ = \{x \in N_{\mathbb{R}} \mid \langle x, y \rangle \geq -1 \forall y \in \Delta\}.$$

3.2.2 Reflexive polytope

A polytope Δ is reflexive if the following two conditions hold:

1. All facets Γ of Δ are supported by affine hyperplanes of the form $\{m \in M_{\mathbb{R}} \mid \langle m, v_{\Gamma} \rangle\}$ for some $v_{\Gamma} \in N$.
2. The only interior point of Δ is 0, that is: $\text{Int}(\Delta) \cap M = \{0\}$.

4 Homological algebra

4.1 Derived functors

4.1.1 Ext

Let R be a ring and M, N be R -modules. Then $\text{Ext}_R^i(M, N)$ is the right-derived functors of the $\text{Hom}(M, -)$ -functor. In particular, $\text{Ext}_R^i(M, N)$ can be computed as follows: choose a projective resolution C_{\bullet} of N over R . Then apply the left-exact functor $\text{Hom}_R(M, -)$ to the resolution and take homology. Then $\text{Ext}_R^i(M, N) = h^i(C_{\bullet})$.

4.1.2 Tor

Let R be a ring and M, N be R -modules. Then $\text{Tor}_R^i(M, N)$ is the right-derived functors of the $- \otimes_R N$ -functor. In particular $\text{Tor}_R^i(M, N)$ can be computed by taking a projective resolution of M , tensoring with N , and then taking homology.

References

- [1] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.