Algebraic groups and moduli theory

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Sammendrag

These are notes from the course Algebraic Geometry III.

1 Representation theory in general

Let V be a vector space. Briefly, a *representation* of any group G on V is just a group homomorphism $\rho : G \to \mathrm{GL}(V)$.

Example 1.1. The *trivial representation* is given by sending every $g \in G$ to the identity transformation. ★

Example 1.2. Suppose G is a finite group. Then there is an embedding $G \hookrightarrow S_n$, and every element of S_n can be represented by permutation matrices (that is, matrices M_g such that $Me_i = e_{g(i)}$ for all $g \in G$). This defines a representation of G in k^n. ★

Example 1.3. Suppose G acts on a (finite) set X. Let V be the vector space with basis identified with the elements of X. Then G acts on V by linearity: for each $g \in G$, $\rho(g)$ is the linear map sending e_x to e_{gx}. Such representations are called *permutation representations*. ★

A *morphism of representations* (ρ, V),(ρ', W) consists of commutative diagrams

$$\begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\rho'(g)"] \\ V \arrow[r, "\psi"'] & W \end{tikzcd}$$

for each $g \in G$. Thus, if ψ is invertible, this says that the linear operators $\rho(s), \rho'(s)$ are similar.

2 Algebraic groups

Algebraic groups are group objects in the category of affine varieties. More precisely:

Definition 2.1. Let A be a finitely generated k-algebra. An *affine algebraic group* is a quadruple $(A, \mu_A, \epsilon, \iota)$ where $\mu_A : A \to A \otimes_k A$ (the *coproduct*), $\epsilon : A \to k$ (the *coidentity*), $\iota : A \to A$ (the *coinverse*) are k-algebra homomorphisms, satisfying the following conditions:

1. Coassociativity. The following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\mu_A"] \arrow[d, "\mu_A"'] & A \otimes_k A \arrow[d, "\mathrm{id}_A \otimes \mu_A"] \\
A \otimes_k A \arrow[r, "\mu_A \otimes \mathrm{id}_A"'] & A \otimes_k A \otimes_k A
\end{tikzcd}$$

2. The following diagram commutes:

$$\begin{tikzcd}
 & & k \otimes_k A \arrow[dr, "\simeq"] & \\
A \arrow[r, "\mu"] & A \otimes_k A \arrow[ur, "\epsilon \otimes \mathrm{id}_A"] \arrow[dr, "\mathrm{id}_A \otimes \epsilon"'] & & A \\
 & & A \otimes_k k \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

   and is equal to the identity.

3. Inverse. The following diagram commutes:

$$\begin{tikzcd}
A \arrow[rr, "\epsilon"] \arrow[d, "\mu"] & & k \arrow[d] \\
A \otimes_k A \arrow[r, "\mathrm{id}_A \otimes \iota"'] & A \otimes_k A \arrow[r, "\cdot"'] & A
\end{tikzcd}$$

   Here the right arrow is the morphism making A a k-algebra. The last arrow in the lower sequence is multiplication in A.

■

Example 2.2. Let G be any group, and let $k[G]$ be its group ring. Let A be its k-linear dual, that is $A = \mathrm{Hom}_k(k[G], k)$. This is a priori just another vector space, but we can give it the structure of a k-algebra by defining multiplication as follows: let $\lambda : k[G] \to k$,$\gamma : k[G] \to k$ be k-linear maps. It is enough to say what should happen on a basis, and a basis is given by the elements g of G. Then, set $(\lambda \cdot \gamma)(g) = \lambda(g) \cdot \gamma(g)$.

Then set $\mu : A \to A \otimes A$ to be the dual of the multiplication map on $k[G]$. Explicitly, let $m : k[G] \otimes_k k[G] \to k[G]$ denoted the multiplication map. Let $\lambda : k[G] \to k$ be an element of A. Then we can form $m^*\lambda = \lambda \circ m$, which is an element of $(k[G] \otimes k[G])^\vee$. For finite-dimensional vector spaces, this is isomorphic to $A \otimes A$, which gives our multiplication map μ. The coidentity is given by sending $\lambda : k[G] \to k$ to $\lambda(1_G)$, where $1_G \in G \subseteq k[G]$.

For example: let $G = C_n$ be the cyclic group of order n. Then $k[G] = k[t]/(t^n - 1)$, and since this is finite-dimensional over k, we can find an isomorphism $k[G] \approx A$. Unwinding definitions, we see that [????] (I dont see this) ★

Example 2.3. Let $A = k[s]$ be the polynomial ring in one variable. This is the coordinate ring of $\mathbb{A}^1_k$. We can define

$$\mu(s) = s \otimes 1 + 1 \otimes s.$$

Also, $\epsilon(s) = 0$, and $\iota(s) = -s$. ★

Definition 2.4. An *action* of an affine algebraic group $G = \operatorname{Spec} A$ on an affine variety $X = \operatorname{Spec} R$ is a morphism $G \times X \to X$ defined dually by a k-algebra morphism $\mu_R : R \to R \otimes_k A$ satisfying the following two conditions.

1. The following diagram is commutative:

$$\begin{tikzcd} R \arrow[r, "\mu_R"] \arrow[dr, "\mathrm{id}_R"'] & R \otimes_k A \arrow[d, "\mathrm{id}_R \otimes \epsilon"] \\ & R \simeq R \otimes_k k \end{tikzcd}$$

2. The diagram

$$\begin{tikzcd} R \arrow[r, "\mu_R"] \arrow[d, "\mu_R"'] & R \otimes_k A \arrow[d, "\mu_R \otimes \mathrm{id}_A"] \\ R \otimes_k A \arrow[r, "\mathrm{id}_R \otimes \mu_A"'] & R \otimes_k A \otimes_k A \end{tikzcd}$$

■

3 Representations of algebraic groups

Let $G = \operatorname{Spec} A$ be an affine algebraic group over a field k.

Definition 3.1. An *algebraic representation of* G is a pair (V, μ_V) consisting of a k-vector space V and a k-linear map $\mu_V : V \to V \otimes_k A$ satisfying the following two conditions:

1. The diagram

$$\begin{tikzcd} V \arrow[r, "\mu_V"] \arrow[dr, "\mathrm{id}_V"'] & V \otimes_k A \arrow[d, "\mathrm{id}_V \otimes \epsilon"] \\ & V \simeq V \otimes_k k \end{tikzcd} \tag{1}$$

   is commutative.

2. The diagram

$$\begin{tikzcd} V \arrow[r, "\mu_V"] \arrow[d, "\mu_V"'] & V \otimes_k A \arrow[d, "\mu_V \otimes \mathrm{id}_A"] \\ V \otimes_k A \arrow[r, "\mathrm{id}_V \otimes \mu_A"'] & V \otimes_k A \otimes_k A \end{tikzcd}$$

   is commutative. Here μ_A is the coproduct in the coordinate ring of G.

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Remark. *In lieu of Definition 2.4, we see that any action of an algebraic group G on an affine variety $X = \operatorname{Spec} R$ is a representation of G on the infinite-dimensional k-vector space $R = \Gamma(X, \mathscr{O}_X)$.*

Remark. *Mumford calls this a* dual action *of G on V, in his 1965 book "Geometric Invariant Theory".*

We often drop the subcript from μ_V unless confusion may arise. The same comment applies to tensor products. They will always be over the ground field unless otherwise stated. We will sometimes refer to a representation (V, μ_V) sometimes as "a representation $\mu : V \to V \otimes A$" and sometimes as just "a representation V".

Definition 3.2. Let $\mu : V \to V \otimes A$ be a representation of $G = \operatorname{Spec} A$. Then:

1. A vector $x \in V$ is said to be *G-invariant* if $\mu(x) = x \otimes 1$.
2. A subspace $U \subset V$ is called a *subrepresentation* if $\mu(U) \subseteq U \otimes A$.

■

Proposition 3.3. *Every representation V of G is locally finite-dimensional. Precisely: every $x \in V$ is contained in a finite-dimensional subrepresentation of G.*

Bevis. Write $\mu(x)$ as a finite sum $\sum_i x_i \otimes f_i$ for $x_i \in V$ and linearly independent $f_i \in A$. This we can always do, by definition of tensor product and bilinearity. Let U be the subspace of V spanned by the vectors x_i.

Now, by the commutativity of the diagram (1) it follows that

$$x = \sum_i \epsilon(f_i) x_i.$$

By the commutativity of the second diagram in the definition, it follows that

$$\sum_i \mu_V(x_i) \otimes f_i = \sum_i x_i \otimes \mu_A(f_i) \in U \otimes A_k \otimes_k A.$$

Because each term of the right-hand-side is contained in $U \otimes A \otimes A$, it follows that $\mu_V(x_i)$ is contained in U since the f_i are linearly independent.

Thus x is contained in the finite-dimensional representation $\mu_V\big|_U : U \to U \otimes A$. $\square$

We can classify representations of $\mathbb{G}_m$ easily. They are all direct sums of "weight m"-representations, that is, representations of the form

$$V \to V \otimes k[t, t^{-1}], v \mapsto v \otimes t^m.$$

Proposition 3.4. *Every representation V of $\mathbb{G}_m$ is a direct sum $V = \oplus_{m \in \mathbb{Z}} V_{(m)}$, where each $V_{(m)}$ is a subrepresentation of weight m.*

Bevis. For each $m \in \mathbb{Z}$, define

$$V_{(m)} = \{v \in V \mid \mu(v) = v \otimes t^m\}.$$

This is a subrepresentation of V: we must see that $\mu(V_{(m)}) \subset U \otimes A$, but this is true by construction. It is also clear that is has weight m. Next we show that $V = \oplus_{m \in \mathbb{Z}} V_{(m)}$. Write

$$\mu(v) = \sum_{m \in \mathbb{Z}} v_m \otimes t^m \in V \otimes k[t, t^{-1}].$$

Using the first condition in the definition of a representation, we get that $v = \sum_{m \in \mathbb{Z}} \epsilon(t^m) v_m$. It remains to check that each $v_m \in V_{(m)}$ (we can forget the scalars $\epsilon(t^m)$). But from definition ii), it follows that

$$\sum \mu(v_m) \otimes t^m = \sum v_m \otimes t^m \otimes t^m,$$

so that indeed $\mu(v_m) = v_m \otimes t^m$, as wanted. $\square$

Example 3.5. An action of $\mathbb{G}_m$ on $X = \operatorname{Spec} R$ is equivalent to specifying a grading

$$R = \oplus_{m \in \mathbb{Z}} R_{(m)} \qquad R_{(m)} R_{(n)} \subset R_{(m+n)}.$$

The invariants under this action are thus the homogeneous elements of weight zero, that is, the subring $R_{(0)}$. Moreover, we have a special operator. There is a linear endomorphism E of R that sends $f = \sum f_m \mapsto \sum m f_m$, and it is a derivation of R, called the Euler operator. We have $R^{\mathbb{G}_m} = \ker E$.

To see that E is a derivation, we must check that $E(fg) = fE(g) + gE(f)$. The operator is homogeneous, so it is enough to check on homogeneous elements. So let f_m, g_n be of degree m, n, respectively. Then

$$E(f_m g_n) = (m+n) f_m g_n = g_n(m f_m) + f_m(n g_n) = g_n E(f_m) + f_m E(g_m),$$

as wanted. ★

A character in regular representation theory is a homomorphism $G \to \mathbb{C}^*$, so do we have a corresponding notion of characters in this "dual" world:

Definition 3.6. Let $G = \operatorname{Spec} A$ be an affine algebraic group. A 1-dimensional character of G is a function $\chi \in A$ satisfying

$$\mu_A(\chi) = \chi \otimes \chi \qquad \iota(\chi)\chi = 1.$$

■

Lemma 3.7. *The characters of the general linear group* $\mathrm{GL}(n) = \operatorname{Spec} k[x_{ij}, \det X]$ *are precisely the integer powers of the determinant* $(\det X)^n$ *for* $n \in Z$*.*

Definition 3.8. Let χ be a character of an affine algebraic group G, and let V be a representation of G. A vector $v \in V$ satisfying

$$\mu_V(v) = v \otimes \chi$$

is called a *semi-invariant* of G with weight χ. The semi-invariants of V belonging to a given character χ form a subrepresentation $V_\chi \subset V$ of V. ■

3.1 Linear reductivity

Definition 3.9. An algebraic group G is said to be *linearly reductive* if, for every epimorphism $\varphi : V \to W$ of G-representations, the induced map of G-invariants $\varphi^G : V^G \to W^G$ is surjective. ■

For the following proposition, assume that *chark* does not divide $|G|$.

Proposition 3.10. *Every finite group G is linearly reductive.*

Bevis. Let $\varphi : V \to W$ be the given epimorphism of representations. Let $R : V \to V^G \subset V$ be given by $v \mapsto \sum_{g \in G} g \cdot v$. Let $w \in W^G$. Then it is an easy calculation to check that $\varphi(R(v)) = R(\varphi(v))$, from which it follows that $\varphi(R(v)) = w$ (note that $R\big|_{W^G} = \mathrm{id}_{W^G}$). $\square$

The homomorphism R above is called the *Reynolds operator*.