Toolbox

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1 Techniques

1.1 Compute the ideal of an affine toric variety

Suppose given an affine toric X_σ defined by a full-dimensional rational polyhedral convex cone in $N_\mathbb{R} \simeq \mathbb{R}^d$. Then the coordinate ring of X_σ is given by the semigroup algebra $k[S_\sigma]$, where $S_\sigma = \sigma^\vee \cap M$.

Here $\sigma^\vee$ is the dual cone and M is the dual lattice $N^\vee$. There is a canonical k-algebra basis for $k[S_\sigma]$ given by the *Hilbert basis* of the semigroup $\sigma^\vee \cap M$. This gives us a presentation $k[\mathbf{x}] \to k[S_\sigma]$.

Thus there are three steps in computing the toric ideal:

1. First compute the dual cone $\sigma^\vee$.
2. Compute a Hilbert basis $\{m_1, \cdots, m_r\}$ of $\sigma^\vee \cap M$.
3. Compute the kernel of the map

$$\begin{aligned} k[x_1, \cdots, x_r] &\to k[S_\sigma] \\ x_i &\mapsto m_i. \end{aligned}$$

Here is a `Macaulay2` session that starts with a cone $\sigma \subseteq \mathbb{N}_\mathbb{R}$, and prints the corresponding toric ideal.

```
M = matrix{{3,1},{1,2}}
C = posHull M
Cd = dualCone C
hB = transpose matrix apply(hilbertBasis Cd, a -> entries a_0)
I  =  toricGroebner(hB, QQ[vars(0..#hB-1)])
```

1.2 Computing ideals of rational secant varieties

Let $X \subset \mathbb{P}^n$ be a projective variety. Then consider

$$\mathrm{Sec}(X) = \overline{\bigcup_{p,q \in X} \overline{pq}}.$$

Here $\overline{pq}$ denotes the line through p and q. The overline indicates closure in the Zariski topology.

We do as an example the rational normal curve C of degree $10 \in \mathbb{P}^{10}$. It has a parametrization given by

$$\mathbb{P}^1 \ni (a:b) \mapsto (a^{10} : a^9 b : \ldots : ab^9 : b^{10}) \in \mathbb{P}^{10}.$$

Then we have a parametrization of $\mathrm{Sec}(C)$ given by

$$(s:t) \times (a:b) \times (a':b') \mapsto (\ldots : sa^i b^{10-i} + ta^i b^{10-i} : \ldots).$$

On the ring level, we get the dual map

$$\begin{aligned} \varphi : k[x_0..x_{10}] &\to k[s,t,a,b,a',b'] \\ x_i &\mapsto sa^i b^{10-i} + ta^i b^{10-i}. \end{aligned}$$

The kernel of this map is the ideal of $\mathrm{Sec}(C)$. This can be computed in `Macaulay2` as follows:

```
R2 = QQ[a,b,a1,b1,s,t]
S  = QQ[x_0..x_10]
aa = matrix{toList apply(0..10, i -> a^(10-i)*b^i)}
a2 = matrix{toList apply(0..10, i -> a1^(10-i)*b1^i)}
phi= map(R2, S', s*(aa) + t*(a2))
I = ker phi
```

The ideal have 84 cubic generators, and the Betti table looks like

```
             0  1   2   3   4   5   6  7
o15 = total: 1 84 378 756 840 540 189 28
          0: 1  .   .   .   .   .   .  .
          1: .  .   .   .   .   .   .  .
          2: . 84 378 756 840 540 189 28
```

This means that all relations in the ideal are linear. It has dimension 4, so that $\mathrm{Sec}(C)$ have dimension 3 as a subset of $\mathbb{P}^n$.