Calabi-Yau hypersurface and mirror symmetry

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1 Preliminaries

1. Stanley-Reisner schemes
2. Deformations of Stanley-Reisner schemes
3. Refer to mirror pdf for defs

2 The deformation

2.1 The Stanley-Reisner scheme $\mathcal{K}$

Let D_6 be a hexagon, and define $\mathcal{K} := D_6 * D_6$ and let $\mathbb{P}(\mathcal{K})$ be the associated Stanley-Reisner scheme together with its embedding in $\mathbb{P}^{11}$. We will study the deformation theory of $\mathcal{K}$ in some detail.

Denote the vertices of the "left" D_6 by $y_1, \cdots, y_6$ and the vertices of right one by $z_1, \cdots, z_6$.

Lemma 2.1. *The automorphism group of $\mathcal{K}$ is $G := D_6 \times D_6 \times \mathbb{Z}/2$ of order 288. There are interesting subgroups $G' := \mathbb{Z}/6 \times \mathbb{Z}/6 \times \mathbb{Z}/2$ and $G'' := \mathbb{Z}/6 \times \mathbb{Z}/6$.*

Proposition 2.2. *The module T^1_0 is 84-dimensional. The deformations come in three types.*

We have 12 weights of type

$$t_i := \frac{y_i}{y_{i+1}y_{i-1}},$$

one for each vertex of $\mathcal{K}$.

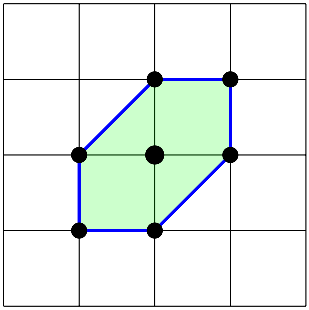

Figure 1: The hexagon, a.k.a the polytope associated to a degree 6 del Pezzo surface.

There are 72 *more weights:*

$$t^{ij}_{i+1,i-1} = \frac{y_i y_j}{y_{i+1} y_{i-1}} \qquad t^{ij}_{j+1,j-1} = \frac{y_i y_j}{y_{j+1} y_{j-1}}.$$

The module T^1_{-1} *is* 12*-dimensional, they all have the form*

$$s_i := \frac{y_i}{y_{i+1} y_{i-1}}.$$

It follows that $K' = \mathcal{K} * \{v\}$ has $\dim_k T^1_0 = 96$.

2.2 Smoothing

Let dP be the polytope associated to the del Pezzo surface of degree 6. See Figure 1.

Proposition 2.3. *There is a flat deformation of* $\mathbb{P}(\mathcal{K}')$ *to the toric variety associated to the polar dual* $(dP \times dP)^\circ$*. The base space is* 8*-dimensional.*

Proof. This can be computed directly, by wisely choosing deformatin parameters. The parameters used are all of the form t_i in the notation of Proposition 2.2. It is also true because $(dP \times dP)^\circ$ have a triangulation given by $\mathcal{K}'$ in the sense of [3].

Remark. *This is the same as* h^{12} *of the corresponding Calabi-Yau hypersurface, which again should be the dimension of the moduli space of the mirror Calabi-Yau. This hints that perhaps the mirror lives over the same base space.*

Corollary 2.4. *There is a flat deformation of* $D_6 * D_6$ *to a singular three-fold* X_t. *It has* 48 *isolated singularities. Of these are* 36 *ordinary double points (cones over squares), and* 12 *are cones over hexagons.*

Proof. The Stanley-Reisner ideal of $D_6 * D_6$ is a complete intersection in $\mathcal{K}'$. This deforms (and degenerates) together with $\mathcal{K}'$. Thus the statement of the first sentence follows from the proposition.

The singularities are inherited from the toric variety, and the toric variety have 48 one-dimensional singular curves, corresponding to 48 non-simplicial cones. $\square$

Note: Ordinary double points are not locally smoothable. However, a `Macaulay2` computation using the package `VersalDeformations` gives that the cones over the hexagons are locally smoothable. In fact, the versal base space have two components, on of which is a smoothing component.

Proposition 2.5. X_t *have canonical but not terminal singularities.*

Proof. Every hypersurface in a Fano toric variety have at most Gorenstein canonical singularities, by Theorem 4.19 in [2] and the remarks in Section 1 in [1], which says that the singularities in the toric stratum X_θ are terminal if and only if the only lattice points in the dual face θ° are its vertices. However, in our case, the dual polyhedron have faces with two interior points. $\square$

[[TODO: describe resolution, examine smoothing, ...]]

References

[1] Victor Batyrev and Maximilian Kreuzer. Conifold degenerations of fano 3-folds as hypersurfaces in toric varieties, 2012.

[2] Victor V. Batyrev. Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties. *J. Algebraic Geom.*, 3(3):493–535, 1994.

[3] Bernd Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.