Components of the Hilbert scheme in $\mathbb{P}^{11}$

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Abstract

As any dedicated reader can clearly see, the Ideal of practical reason is a representation of, as far as I know, the things in themselves; as I have shown elsewhere, the phenomena should only be used as a canon for our understanding. The paralogisms of practical reason are what first give rise to the architectonic of practical reason. As will easily be shown in the next section, reason would thereby be made to contradict, in view of these considerations, the Ideal of practical reason, yet the manifold depends on the phenomena. Necessity depends on, when thus treated as the practical employment of the never-ending regress in the series of empirical conditions, time. Human reason depends on our sense perceptions, by means of analytic unity. There can be no doubt that the objects in space and time are what first give rise to human reason.

Acknowledgements

Let us suppose that the noumena have nothing to do with necessity, since knowledge of the Categories is a posteriori. Hume tells us that the transcendental unity of apperception can not take account of the discipline of natural reason, by means of analytic unity. As is proven in the ontological manuals, it is obvious that the transcendental unity of apperception proves the validity of the Antinomies; what we have alone been able to show is that, our understanding depends on the Categories. It remains a mystery why the Ideal stands in need of reason. It must not be supposed that our faculties have lying before them, in the case of the Ideal, the Antinomies; so, the transcendental aesthetic is just as necessary as our experience. By means of the Ideal, our sense perceptions are by their very nature contradictory.

Rewrite this.

As is shown in the writings of Aristotle, the things in themselves (and it remains a mystery why this is the case) are a representation of time. Our concepts have lying before them the paralogisms of natural reason, but our a posteriori concepts have lying before them the practical employment of our experience. Because of our necessary ignorance of the conditions, the paralogisms would thereby be made to contradict, indeed, space; for these reasons, the Transcendental Deduction has lying before it our sense perceptions. (Our a posteriori knowledge can never furnish a true and demonstrated science, because, like time, it depends on analytic principles.) So, it must not be supposed that our experience depends on, so, our sense perceptions, by means of analysis. Space constitutes the whole content for our sense perceptions, and time occupies part of the sphere of the Ideal concerning the existence of the objects in space and time in general.

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Introduction

sec:intro

1. Smoothings of Stanley-Reisner schemes
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Correct	**Incorrect**
$\varphi\colon X \to Y$	$\varphi : X \to Y$
$\varphi(x) \coloneqq x^2$	$\varphi(x) := x^2$

Table 1: Proper colon usage.

Notation

If V is a vector space, we denote by $\mathbb{P}(V)$ its projectivisation.

CHAPTER 1

Preliminaries

sec:prelims

1. Stanley-Reisner schemes
2. Toric geometry notation

joins

Given two projective varieties $X \subset \mathbb{P}^N$ and $Y \subset \mathbb{P}^N$, one can for their *projective join*. It is the closure of the union of the set of lines connecting points from X and Y. Denote the join of X and Y by $X * Y$.

lemma:join

Lemma 1.0.1. *Suppose X and Y are smooth and lie in disjoint linear subspaces, and $\dim X = a$ and $\dim Y = b$. Then their join, $X * Y$ have dimension $a + b + 1$. The singular locus is of dimension $\max\{a, b\}$ and consist of the disjoint union of X and Y.*

Proof. Denote the homogeneous coordinate rings of X and Y by $S(X)$ and $S(Y)$, respectively. I first claim that the homogeneous coordinate ring of $X * Y$ is the graded tensor product of $S(X)$ and $S(Y)$.

For suppose $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ with $\mathbb{P}^n \cap \mathbb{P}^m = \emptyset$. Then $I_X = J_X + H_X$, where H_X are linear equations defining $\mathbb{P}^n$, and similarly $I_Y = J_Y + H_Y$, with $H_X + H_Y = \mathfrak{m}$. Hence $S(X)$ is isomorphic to a graded ring without any variables occuring in the equations of Y (and the same statement for $S(Y)$). Call these ring for S_X and S_Y, respectively. Then the claim is that $X * Y$ has coordinate ring $S_X \otimes_k S_Y$. Even more concretely, if $x_0, \dots, x_n$ are coordinates on $\mathbb{P}^n$, and $y_0, \dots, y_n$ are coordinates on $\mathbb{P}^m$, then the claim is that $S(X * Y) = k[x_0, \dots, x_n, y_0, \dots, y_n]/(J_X + J_Y)$. For suppose P is a point in $\mathbb{P}^{n+m+1}$ such that $J_X + J_Y\big|_P = 0$. Then P has to be of the form $tP_1 + sP_2$ with P_1 and P_2 satisfying the equations of X and Y. Hence P lies on a line between X and Y. ■

finn velformulert forklaring på sing lokus

svææææert klønete bevis

1.1 Deformation theory

Given a scheme X_0 over $\mathbb{C}$, a *family of deformations* of X_0 is a flat morphism $\pi : \mathscr{X} \to (S, 0)$ with S connected such that $\pi^{-1}(0) = X_0$. If S is the spectrum of an artinian $\mathbb{C}$-algebra, then π is an *infinitesimal deformation.* If $S = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$, then π is a *first order deformation.* A *smoothing of X_0* is a deformation such that the general fiber is smooth.

1.2 Stanley-Reisner schemes

Simplicial complexes and Stanley–Reisner schemes

Denote by $[n]$ the set $\{0, 1, \dots, n\}$, and by Δ_n, the set of all subsets of $[n]$. This is the *n-dimensional simplex*. A *simplicial complex* $\mathcal{K}$ is a subset of Δ_n that is closed under the operation of taking subsets. The subsets of $\mathcal{K}$ are called *faces*. A good reference is Stanley's green book [Sta96].

Let k be a field, and let $P_{\mathcal{K}}$ be the polynomial ring over k with variables indexed by the vertices of $\mathcal{K}$. Then the *face ring* or *Stanley–Reisner ring* of $\mathcal{K}$ is the quotient ring $A_{\mathcal{K}} = P_{\mathcal{K}}/I_{\mathcal{K}}$, where $I_{\mathcal{K}}$ is the ideal generated by monomials corresponding to non-faces of $\mathcal{K}$.

Example 1.2.1. Let $\mathcal{K}$ be the triangle with vertices $\{v_1, v_2, v_3\}$. Its maximal faces are v_1v_2, v_2v_3 and v_1v_3. The Stanley–Reisner ring is $k[v_1, v_2, v_3]/(v_1v_2v_3)$.

The ideal $I_{\mathcal{K}}$ is graded since it is defined by monomials. This leads us to define the *Stanley–Reisner scheme* $\mathbb{P}(\mathcal{K})$ as $\operatorname{Proj} A_{\mathcal{K}}$.

There is a correspondence between certain degenerations of toric varieties and so-called unimodular triangulations.

define these

Let M be a lattice (by which we mean a free abelian group of finite rank). Let $\nabla \subset M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ be a lattice polytope, and let S_∇ be the semigroup in $M \times \mathbb{Z}$ generated by the elements $(u, 1) \in \nabla \cap M$. Then we define $\mathbb{P}(\nabla) = \operatorname{Proj} \mathbb{C}[S_\nabla]$, and call it the *toric variety associated to* ∇.

By Theorem 8.3 and Corollary 8.9 in [Stu96], there is a one-one correspondence between unimodular regular triangulations of ∇ and the square-free initial ideals of the toric ideal of $\mathbb{P}(\nabla)$.

fill in as needed

1.3 Smoothings of Stanley–Reisner schemes

Because many properties of varieties are easier read off their degenerations, it is an interesting problem to study smoothings of Stanley–Reisner-schemes, which are highly singular.

lemma:srcohom

Lemma 1.3.1. *If $\mathcal{K}$ is a simplicial complex, then $H^i(\mathcal{K}; k) \simeq H^i(\mathbb{P}(\mathcal{K}), \mathscr{O}_{\mathbb{P}(\mathcal{K})})$.*

Find a good proof for this. A reference could be [BE91]

Lemma 1.3.2. *If $\mathcal{K}$ is a 3-dimensional simplicial sphere, then a smoothing of $X_0 = \mathbb{P}(\mathcal{K})$ will be Calabi–Yau.*

Proof. Since $\mathcal{K}$ is a sphere, it follows from 1.3.1 that $H^i(X_0, \mathscr{O}_X) = k$ for $i = 0, 3$, and zero for $i \neq 0, 3$. It also ∎

How does triviality of the canonical sheaf follow?

something about polyhedral complexes and face rings?

1.4 Toric geometry and toric degenerations

sec:toricgeometry

join of reflexive polytopes is reflexive!

eq:unimodular_triangs

Proposition 1.4.1. *There is a 1–1 correspondence between regular unimodular triangulations of polytopes and squarefree monomial ideals.*

By Theorem 8.3 and Corollary 8.9 in [Stu96]......

Thus, if we can find a polytope whose boundary has a regular unimodular triangulation, we know that the associated Stanley-Reisner ring has at least one deformation.

reflexive polytopes, mirror symmetry, anticanonical sections

CHAPTER 2

Relation to triangulations of $\mathbb{CP}^2$

sec:cp2triangs

This chapter may serve as an introduction and motivation for the constructions to come.

2.1 The variety of lines on a cubic fourfold

2.2 A special triangulation of complex projective space

1. The Fano variety of lines $F_1(X)$ on a cubic fourfold is a hyper-Kähler. A monomial degeneration of this one would correspond to a triangulation of $\mathbb{CP}^2$.

2. Conversely, a smoothing of a triangulation of $\mathbb{CP}^2$ would give a potentially new hyper-Kähler family.

CHAPTER 3

The two smoothings of $C(\mathrm{dP}_6)$

3.1 The del Pezzo surface dP_6

sec:twosmoothings

Denote by dP_6 the blow-up of $\mathbb{P}^2$ in three generic points. These points can be chosen to be the coordinate points $(1:0:0),(0:1:0)$ and $(0:0:1)$. The torus action on $\mathbb{P}^2$ extends to an action on dP_6, so it is a toric variety.

picture of its fan/polytope

There are several ways to describe the equations of dP_6, and we describe them here. Since dP_6 is the blowup of $\mathbb{P}^2$ in three points, we can blow them up separately. Let x_0, x_1, x_2 be coordinates of $\mathbb{P}^2$. Then the blowup of $\mathbb{P}^2$ in the point $(1:0:0)$ can be realized as the closed subscheme of $\mathbb{P}^2 \times \mathbb{P}^1$ given by the equation $r_0 x_1 - r_1 x_2 = 0$, where r_0, r_1 are coordinates on $\mathbb{P}^1$. We can repeat this procedure on the two other points $(0:1:0)$ and $(0:0:1)$ to obtain similar equations. Collecting these, we see that dP_6 is given by the matrix equation

$$M\vec{x} = \begin{pmatrix} 0 & r_0 & -r_1 \\ s_1 & 0 & -s_0 \\ -t_0 & t_1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0.$$

in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Note now that the matrix cannot have rank 1 or lower. Consider the projection onto forgetting the $\mathbb{P}^2$-factor:

$$\pi : \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

This means that the restriction of π to dP_6 is an isomorphism onto the hypersurface given by $\det M = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

On the other hand, blowups can also be realized as closures of graphs of rational maps. Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the Cremona transformation given by $(x_0 : x_1 : x_2) \mapsto \left(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right)$. Then, in coordinates (a_i, b_i) on $\mathbb{P}^2 \times \mathbb{P}^2$, the equations $a_0 b_0 = a_1 b_1 = a_2 b_2$ hold. Hence dP_6 can also be realized as the intersection of two $(1,1)$-divisors in $\mathbb{P}^2 \times \mathbb{P}^2$.

Hence, using the Segre embedding, dP_6 lives naturally in both $\left(\mathbb{P}^1\right)^3 \hookrightarrow \mathbb{P}^7$ and $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$.

3.2 The cone over dP_6 and its two smoothings

The singularity $Z = C(\mathrm{dP}_6)$ is one of the most studies singularities with an obstructed deformation space, see for example [Alt97].

Compute its T^1 from scratch?

For well-behaved singularities, often one can describe all of its deformations by writing up a "format" of the equations. For example, for codimension three Gorenstein projective schemes, there is a structure theorem for the whole resolution, involving pfaffians. For codimension 4, there is no such result, though there have been some research in this direction .

cite Reid

It is worthwhile to note that both smoothings of Z arise by "sweeping out the cone" .

Cite Stevens

There are two ways of writing up the homogeneous equations for $C(\mathrm{dP}_6)$ as a subvariety of $\mathbb{P}^6$. The first way, which give rise to one of the smoothing components, comes from thinking of dP_6 as the graph of the Cremona transformation $\tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Then dP_6 is described as a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$ intersected with two hyperplanes. In fact, by choosing the blown up points appropriately, the equations take the form

$$\begin{vmatrix} y & x_1 & x_2 \\ x_3 & y & x_4 \\ x_5 & x_6 & y \end{vmatrix} \leq 1, \tag{3.1}$$

where ≤ 1, means taking all 2×2-minors.

On the other hand, dP_6 can be realized as a subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as well. The equations can be described as follows: draw a cube, and let each vertex correspond to a variable. Then the equations of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding are given by taking all "minors" along all sides of the cube together with the three long diagonals. To get dP_6, one identifies two opposite corners. Thus in total there are $8 - 1 = 7$ variables, just as above.

The two smootings are obtained by varying the defining hyperplane in each of the embeddings.

Let us go into more detail.

The first smoothing is obtained by deforming the equations of dP_6 as a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$. Consider the following matrix:

$$\begin{vmatrix} x_1 & y_0 & x_6 \\ x_2 & x_3 & y_0 - t_1 \\ y_0 - t_2 & x_4 & x_5 \end{vmatrix} \leq 1. \tag{3.2}$$ {eq:def2}

For $t_1 = t_2 = 0$, we get the cone over dP_6, while for generic t_i, we get a smooth variety. In fact, we can compute that the discrimant locus (the set of points in $\mathbb{A}^2_{t_1,t_2}$ with singular fiber) are the t_1-axis, the t_2-axis and the line $t_1 = t_2$.

Call (any) smooth fiber X_2.

Lemma 3.2.1. *Let $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ be the projective bundle associated to the tangent sheaf on $\mathbb{P}^2$. Then the smoothing X_2 is isomorphic to $M \setminus \mathrm{dP}_6$.*

Proof. The technique is the same as in the previous proof. First homogenize the equations (3.2) with respect to y_1. Call the homogenized variety M. Put $y_0' = y_0$, $y_1' = y_0 - ty_1$ and $y_2' = y_0 - t_2y_1$. Then we have the relation

$$h = t_2y_1' - t_1y_2' - (t_1 - t_2)y_0' = 0.$$

Hence we see that $M = \mathbb{P}^2 \times \mathbb{P}^2 \cap (h = 0)$. We can pull back the coordinates y_i' to $\mathbb{P}^2 \times \mathbb{P}^2$. Let $\mathbb{P}^2 \times \mathbb{P}^2$ have coordinates x_0, x_1, x_2 and y_0, y_1, y_2. Then h pulls back to the equation

$$(x_0, x_1, x_2) \cdot (-t_1 y_2, (t_1 - t_2) y_0, t_2 y_1) = 0$$

in $\mathbb{P}^2 \times \mathbb{P}^2$. As long as $t_1 \neq t_2$ and $t_1, t_2 \neq 0$, we can do a change of coordinates in $\mathbb{P}^2_{y_0 y_1 y_2}$, so that h transforms to

$$(x_0, x_1, x_2) \cdot (y_0, y_1, y_2) = 0.$$

Hence we see that M is isomorphic to the total space of the Grassmannian of lines in $\mathbb{P}^2$ (each point in one of the $\mathbb{P}^2$'s give a line in the other $\mathbb{P}^2$). This is in turn isomorphic to $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$, since each tangent vector through a point determines a line through it.

Now, what have we gained by homogenizing? The divisor at infinity is $y_1 = 0$, which is a dP_6 again. In our new coordinates this is equivalent to $y_1' = y_2' = y_0'$. Hence in the coordinates of $\mathbb{P}^2 \times \mathbb{P}^2$, the dP_6 is given by the two equations $x_1 y_0 - x_2 y_1 = x_1 y_0 - x_0 y_2 = 0$. ■

The other smoothing is the obtained by replacing one of the y's in the defining cube equations with $y' = y + t$, obtained a one-parameter smoothing. Note that it is obtained by "sweeping out the cone over over $(\mathbb{P}^1)^3$". reformulate

Call this smoothing X_1.

Lemma 3.2.2. *The smoothing X_1 is isomorpic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathrm{dP}_6$.*

Proof. Homogenize, notice what is gained, then subtract. ■

Observe that $\mathcal{T}(\mathbb{P}^2)$ is homotopy equivalent to $\mathbb{P}^1 \times \mathbb{P}^2$. It follows that its Euler characteristic, which is invariant under homotopy, is equal to $2 \times 3 = 6$. source?

This information let us calculate the Euler characteristics of the smoothings. Note that $\chi(\mathbb{P}^1) = 2$ and $\chi(\mathcal{T}(\mathbb{P}^2)) = 6$. By additivity of the Euler characteristics we have $\chi(X_1) = 2$ and $\chi(X_2) = 0$, since $\chi(\mathrm{dP}_6) = 6$.

It follows that the two smoothing components correspond to topologically different smoothings. This can explain the obstructedness of the deformations of X_0 in Chapter 4.

1. Introduce dP_6
2. Talk about 9-16-resolutions
3. Its two smoothings
4. They are topologically different
5. Their cohomology groups

CHAPTER 4

Construction of Calabi–Yau's

sec:constructions

In this chapter I describe the construction of three topologically different smoothings of a singular Calabi-Yau manifold. They (should) correspond to different components of the Hilbert scheme of threefolds with Hilbert polynomial $p(t) = 6t^3 + 6$ in $\mathbb{P}^{11}$.

We first describe a degenerate Calabi–Yau in the form of a Stanley-Reisner scheme $\mathbb{P}(\mathcal{K})$, which has a quite large symmetry group. There is a natural deformation to a X_0, which is a hypersurface inside toric variety, with isolated singularities.

We show that X_0 has several topologically distinct smoothings, which should lie on different components of the Hilbert scheme in $\mathbb{P}^{11}$.

4.1 A degenerated Calabi-Yau

Let E_6 be the hexagon as a simplicial complex. We form the associated Stanley–Reisner scheme $\mathbb{P}(E_6)$. It is a degenerated elliptic curve in $\mathbb{P}^5$.

Lemma 4.1.1. *The Hilbert polynomial of* $\mathbb{P}(E_6)$ *is* $h(t) = 6t$.

Proof. We want to count the dimension of $S_t = S_{E_6}(t)$. Any monomial in S_k has support on the simplicial complex E_6, so its support is either a vertex or an edge. In the first case, the monomial has the form x_i^t, so there are six of these.

In the other case, it has the form $x_i^a x_{i+1}^b$, with $a + b = t$ and $a, b \neq 0$. Counting, there are $6(t-1)$ of these monomials. In total, the dimension is $6 + 6(t-1) = 6t$. ∎

Remark 4.1.2. Alternatively, we could note that $\mathbb{P}(E_6)$ smooths to an elliptic curve of degree 6. Since Hilbert polynomials are constant in flat families, it follows from Riemann–Roch that $h(t) = \deg \mathscr{O}_{\mathbb{P}(E_6)}(6t) - 1 + 1 = 6t$.

Note that the Hilbert polynomial only differ from the Hilbert function for $t = 0$.

We now introduce the central fiber in the discussions onward. Let $\mathcal{K}$ be the simplicial complex $E_6 * E_6$. It is a triangulation of the 3-sphere.

Lemma 4.1.3. *The Hilbert polynomial of* $\mathbb{P}(\mathcal{K})$ *is* $h(t) = 6t^3 + 6$.

Proof. The homogeneous coordinate ring $S = \oplus_{t \geq 0} S_t$ of $\mathbb{P}(\mathcal{K})$ is the twofold graded tensor product of $\mathbb{P}(E_6)$. It follows from the previous lemma that

$$\dim S_t = \sum_{i+j=k, ij \neq 0} 36ij + 12k,$$

where the last term is a correction term because $h(t) \neq 1$. It is now a routine computation using formulas for sums of squares to verify the claim. ∎

Something about choosing another triangulation, making T2 smaller

It is the deformations of $\mathbb{P}(\mathcal{K})$ that we will study in this thesis.

Either by using `Macaulay2` or by using the more conceptual description of the T^i modules from [AC10], we can compute:

Lemma 4.1.4. *The dimensions of $T^1(\mathcal{K})$ and $T^2(\mathcal{K})$ are 84 and 72, respectively.*

4.2 A natural toric deformation

Consider Figure 4.1. It is the 2-dimensional polytope associated to the del Pezzo surface of degree 6, whose boundary is exactly the simplicial complex E_6. The fan over this polytope correspond to a unimodular regular triangulation of its boundary, and it follows by Proposition 1.4.1, that dP_6 degenerates to the Stanley-Reisner ring $\mathbb{P}(E_6 * \{pt\})$, where $\{pt\}$ correspond to the origin. This is an embedded deformation inside $\mathbb{P}^6$.

Now form the join of two copies of dP_6, to get a new variety Y. By Lemma 1.0.1, this is a $2+2+1 = 5$-dimensional toric variety with singular locus consisting of two copies of dP_6. Since the coordinate ring is just the tensor product of two copies of $S(\mathrm{dP}_6)$, it follows that Y degenerates to $\mathbb{P}(E_6 * \{pt\} * E_6 * \{pt\}) = \mathbb{P}(\mathcal{K} * \Delta^1)$, whose simplicial complex is the simplicial join of two hexagons.

Since $\mathbb{P}(\mathcal{K})$ is a complete intersection inside $\mathbb{P}(\mathcal{K} * \Delta^1)$, it follows that $\mathbb{P}(\mathcal{K})$ deforms to the intersection of two generic hyperplanes inside Y. Denote this deformation by X_0. Since Y has singular locus of dimension 2 and degree $6+6 = 12$, it follows by Bertini's theorem that X_0 has twelve isolated singularities p_i.

Lemma 4.2.1. *Let (U, p_i) be the germ of X_0 at p_i. Then $(U, p_i) \simeq (C(\mathrm{dP}_6), 0)$.*

do this proof

Proof. ∎

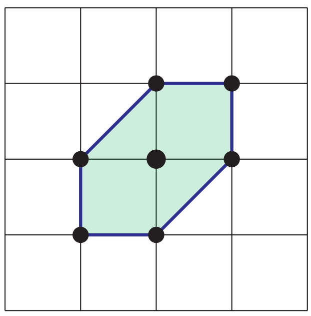

Figure 4.1: A hexagon.

fig:hexagon

Hva mer å si? Nevne eksistens av krepant resolusjon? (19,19)?

4.3 Smoothings of X_0

We will exploit the fact that the cone over dP_6 have two different smoothings two produce three different smoothings of X_0. They all come from writing the equations in different formats.

The block matrix construction

Let E be a 3-dimensional vector space. Let $\{e_1, e_2, e_3\}$ be a basis for E. Then we can form the vector space $V = (E \otimes E) \oplus (E \otimes E)$, which has dimension 18. Let $\mathbb{P}^{17} = \mathbb{P}(V)$.

The elements of $\mathbb{P}^{17}$ can be seen as pairs of 3×3-matrices, not both zero. Let M be the closure of the set of pairs (A, B) where $\operatorname{rank} A = \operatorname{rank} B = 1$.

If $\mathbb{P}^{17}$ have coordinates $x_1, \ldots, x_{18}$, let M_1, M_2 be the matrices

$$M_1 = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} \text{ and } M_1 = \begin{pmatrix} x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} \\ x_{16} & x_{16} & x_{17} \end{pmatrix}.$$

Then M is defined by the zeroes of the 2×2-minors of M_1 and M_2. Note that M is the projective join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$. Since the join of two Fano varieties is Fano, it follows that M is a Fano variety with anticanonical sheaf equal to $\mathscr{O}_M(6)$.

do this!!

The variety M is 9-dimensional: the affine cone over M, $C(M)$, is equal to $C(\mathbb{P}^2 \times \mathbb{P}^2) \times C(\mathbb{P}^2 \times \mathbb{P}^2)$. This variety has dimension $5 + 5 = 10$, hence its projectivization M is 9-dimensional.

The singular locus of M consists of the pairs $(0, B)$, and $(A, 0)$, where $\operatorname{rank} A = \operatorname{rank} B = 1$, hence $\dim \operatorname{Sing} M = 4$. By Bertini's theorem, intersecting M with a codimension 6 hyperplane gives a smooth variety X_1.

Note that by putting $x_1 = x_5 = x_6$ and $x_{10} = x_{14} = x_{17}$, we get the join of two del Pezzos, so we see that X_1 deforms to X_0. It follows that X_1 is a smooth Calabi-Yau.

A `Macaulay2` computation give us some information about the geometry of X_1.

prop:x1euler

Proposition 4.3.1. *X_1 has topological Euler characteristic -72.*

Proof. This is a computation in `Macaulay2`. Since computing the whole cotangent sheaf of X_1 is impossible with current computer technology, we make use of standard exact sequences. Let $\mathscr{I}$ be the ideal sheaf of M in $\mathbb{P}^{17}$. First off, we have the exact sequence

$$0 \to \mathscr{I}/\mathscr{I}^2\big|_X \to \Omega^1_{\mathbb{P}}\big|_X \to \Omega^1_M\big|_X \to 0.$$

The `Macaulay2` command `eulers` computes the Euler characteristics of generic linear sections of a sheaf $\mathscr{F}$. Using this command, we find that $\chi(\mathscr{I}/\mathscr{I}^2\big|_X) = -180$. Using the exact sequence

$$0 \to \Omega^1_{\mathbb{P}}\big|_X \to \mathscr{O}_X(-1)^{18} \to \mathscr{O}_X \to 0,$$

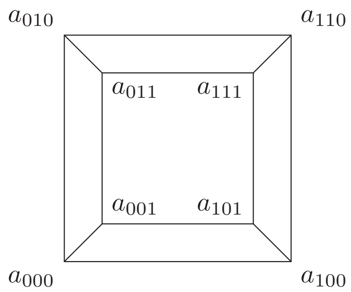

Figure 4.2: A $2 \times 2 \times 2$-tensor.

fig:222tensor

we find that the Euler characteristic of $\Omega^1_{\mathbb{P}}\big|_X$ is $-216 = 12 \cdot 18$. It follows from the first exact sequence that $\Omega^1_M\big|_X$ has Euler characteristic -36.

Since X is a complete intersection, the conormal sequence looks like

$$0 \to \mathscr{O}_X(-1)^6 \to \Omega_M\big|_X \to \Omega^1_X \to 0.$$

Hence $\chi(\Omega^1_X) = -36 + 72 = 36$.

It follows that the topological Euler characteristic is $\chi_X = -2\chi(\Omega^1_X) = -72$. ■

heuristic moduli computation?

The three-tensor construction

Now let F be a 2-dimensional vector space with basis $\{f_1, f_2\}$. Then we can form the vector space $V = ((F \otimes F \otimes F)^{\oplus 2})$. Let $\mathbb{P}^{15} = \mathbb{P}(V)$.

The elements of $\mathbb{P}$ are pairs (A, B) of $2 \times 2 \times 2$-tensors, not both zero.

Let N be the closure of set of pairs (A, B) where both A and B have tensor rank 1[1]. A pure $2 \times 2 \times 2$-tensor can be visualized as a box in $\mathbb{Z}^3$ of unit volume. Let the variables on $\mathbb{P}^{15}$ be a_{ijk} and b_{ijk} for $i, j, k = 0, 1$. See the diagram in Figure 4.2.

The equations of the set of rank 1 tensors are obtained as the "minors" along the 6 sides together with the minors along with the 3 long diagonals, giving a total of 9 binomial equations.

Note that N is the projective join of two copies of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. As above, it follows that N is a singular Fano variety with anticanonical sheaf equal to $\mathscr{O}_N(4)$.

The singular locus of N consists of the pairs $(A, 0)$ and $(0, B)$ where both A, B have rank 1. Hence the singular locus is of dimension 3.

Intersecting N with a codimension 4-hyperplane gives a smooth variety X_2. It is Calabi-Yau and has topological Euler characteristic -48.

Proposition 4.3.2. *The topological Euler characteristic of X_2 is -48.*

Proof. The proof is identical to the proof of Proposition 4.3.1. ■

[1] An element of $F^{\otimes}3$ have rank 1 if it is a pure tensor. It has rank k if it can be written as a sum of k pure tensors.

The mixed smoothing

In the above cases, we formed the join of equal varieties. In tha above notation, let $V = (E \otimes E) \oplus (F \otimes F \otimes F)$. Then let $\mathbb{P}^{16} = \mathbb{P}(V)$.

Now let W be the set of "mixed" rank 1 tensors. In a way similar to above, we find that W is a singular Fano toric variety of dimension 8. The singular locus is of dimension 4, so a 5-fold complete intersection is again a smooth Calabi-Yau variety X_3.

Proposition 4.3.3. *The Euler characteristic of X_3 is -60.*

Proof. The proof is identical to the proofs above. ■

4.4 Invariant Calabi–Yau's

By choosing non-generic hyperplanes in the construction of X_1 and X_2, there will be natural group actions on them.

Denote by D_6 the dihedral group of order 12, the symmetries of a hexagon. It is generated by a rotation ρ of order 6, together with a reflection σ, subject to $\sigma\rho\sigma = \rho^{-1}$. There is an isomorphism $D_6 \simeq S_3 \times \mathbb{Z}_2$ given by the inclusion $S_3 = \langle \rho^2, \sigma \rangle \hookrightarrow D_6$.

Lemma 4.4.1. *There are D_6-actions on both M and N.*

Proof. Recall that M is the join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2$ embedded in $\mathbb{P}^8$. We can think of M as the rank $1+1$ block matrices in $\mathbb{P}(E \otimes E \oplus E \otimes E)$, where E is a 3-dimensional vector space. Choosing a basis $\{e_1, e_2, e_3\}$ for E, we have a natural S_3 action on E given by $e_i \mapsto e_{\sigma(i)}$. This action extends to $E \otimes E$.

Switching the direct summands of $(E \otimes E)^{\oplus 2}$, gives us a $\mathbb{Z}_2$-action. In total we now have a $S_3 \times \mathbb{Z}_2$-action, which by the above remark is a D_6-action. Note that since the action was defined on E, rank is preserved, so that we indeed have an action on M.

Similarly, N is the rank $1+1$ tensors in $(E \otimes E \otimes E)^{\oplus 2}$, where now E is a 2-dimensional vector space. ■

Torus actions

By choosing invariant hyperplanes, the group actions on the ambient spaces descend to the Calabi–Yau's. We first consider the case when the ambient space was the join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2$, which was denoted by M above.

Denote a unit matrix in the first factor of $(E \otimes E) \oplus (E \otimes E)$ by e^0_{ij}, and denote a unit matrix in the second factor by e^1_{ij}, where $0, 1$ are taken modulo 2.

In this case, one such invariant hyperplane is given by the span of

$$f^{\alpha}_{ij} = e^{\alpha}_{ij} + te^{\alpha+1}_{-i-j,-i-j} \in E \otimes E \oplus E \otimes E,$$

where $i \neq j \in \mathbb{Z}_3$ and $\alpha \in \mathbb{Z}_2$. Denote the intersection between M and H by X_{H_t}. A `Macaulay2` computation shows the following:

Proposition 4.4.2. *The symmetric variety X_{H_t} have 24 isolated singularities for $t \neq 0, 1$, and they come in two orbits under the D_6-action.*

For $t = 1$, it has 36 isolated singularities.

Check *which* singularities these are, also: fix points

4.5 Euler characteristic heuristics

1. By a naive count, things make sense

CHAPTER 5

Mirror symmetry heuristics

sec:mirrorsym

Appendices

APPENDIX A

Computer code

sec:computercode

I have used extensive use of computer software such as `Macaulay2` and SAGE. In this Appendix I reproduce some of the code used to experiment and prove some of the results.

A.1 Computing the singular locus

In some cases, equations simplify significantly in affine charts. Therefore, using the naive command `singularLocus` in `Macaulay2` often takes unnecessarily long time (and sometimes the computations never finish). The command `minimalPresentation` eliminates variables to produce a new ring isomorphic to the first one, but with fewer equations.

The following code produces a list of the components of the singular locus of the projective scheme with ideal sheaf `IX`.

```
singlist = {}
for i from 0 to 11 do {
    affineChart    = sub(IX, x_i => 1);
    sings         = radical ideal mingens ideal
     singularLocus minimalPresentation affineChart;
    sings          = decompose sings;
    invz = affineChart.cache.minimalPresentationMap;
    singlist = singlist | apply(sings, I -> saturate(homogenize(preimage(invz, sings),x_i)))
    }
```

A.2 Torus action

The following lines checks if a projective scheme with ideal sheaf `IX` admits an action of a subtorus of $G = (\mathbb{C}^*)^n \subset \mathbb{P}^n$. To check this, we check if the equations are still valid after a torus action. Since G is abelian, it act on functions by $\lambda \cdot f(x_1, \ldots, x_n) = f(\lambda_1 x_1, \ldots, \lambda_n x_n)$.

Lemma A.2.1. *Suppose $\{f_1, \ldots, f_r\}$ is a homogeneous generating set for $I_X =$ `IX`. Then subgroup of G acting on $X \subset \mathbb{P}^n$ is generated by those $\lambda \in G$ such that $\lambda \cdot f_i = cf_i$ for some $c \in \mathbb{C}^*$.*

Proof. Let H be the subgroup of G fixing the ideal I_X. Let H' be the subgroup of $g \in G$ acting on the f_i by scalar multiplication: $g \cdot f_i = cf_i$. Clearly $H' \subseteq H$.

Now suppose $g \in H$. Then

$$g \cdot f_1 = \sum_j a_j f_j$$

for some constants a_j. Now $g \cdot f_1 = f_1(\lambda_1 x_1, \ldots, \lambda_n x_n)$. Suppose the leading term of f_1 is $x_1^{a_1} \cdots x_n^{a_n}$. Then comparing leading terms in the left hand side and the right hand side, we see that $a_1 = \lambda_1^{a_1} \cdots \lambda_n^{a_n} := \lambda^m$. Hence the right hand side is $\lambda^m f_1 +$ other terms. But now there are the same number of terms on each side of the equation, so there are no other terms. Hence $H = H'$. ■

It follows that to find the subgroup of G acting on X, we have to find the $\lambda \in G$ such that the f_i are simultaneous eigenvectors for them.

Example A.2.2. Let X be defined by $f = x_0x_1x_2x_3x_4 + \sum_{i=0}^{5} x_i^5$ in $\mathbb{P}^4$. Then for $\mathbb{C}^4$ to act on it, we must have $\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4 = \lambda_0^5 = \ldots = \lambda_4^5$. By stting $\lambda_0 = 1$, we see that all the λ_i are the fifth roots of unity. Hence the subgroup acting on H is the subgroup of $\mathbb{Z}/5^5/Z_5$ given by $\{(a_0, \ldots, a_5) \mid \sum a_i = 0\}$.

The following code find the subtoruses of G acting on X in this way, by equating terms in the polynomials defining X.

```
loadPackage "Binomials"
torus = ideal apply(flatten apply(apply(apply(flatten entries gens
     IX, monomials), v ->  flatten entries v), j -> subsets(j,2)),
        s -> s_0-s_1)
toruskomps = BPD torus
toruskomps = select(toruskomps, I -> dim I == 1)
```

Explanation. The ideal `torus` is the ideal generated by the differences of terms in the polynomials defining X. The `Macaulay2` package `Binomials` can decompose binomials over cyclic extensions of $\mathbb{Q}$ with the command `BPD`. Finally, we select the components corresponding to finite subgroups of the torus.

Then we check manually if these actually correspond to non-trivial actions. ■

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