Algebraiske grupper og moduliteori

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1 Representation theory in general

2 Algebraic groups

Definition 2.1. Let A be a finitely generated k-algebra. An affine algebraic group is a quadruple $(A, \mu_A, \epsilon, \iota)$ where $\mu_A : A \to A \otimes_k A$ (the coproduct), $\epsilon : A \to k$ (the coidentity), $\iota : A \to A$ (the coinverse) are k-algebra homomorphisms, satisfying the following conditions:

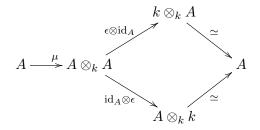
1. Coassociativity. The following diagram commutes:

$$A \xrightarrow{\mu_A} A \otimes_k A$$

$$\downarrow^{\operatorname{id}_A \otimes \mu_A}$$

$$A \otimes_k A \xrightarrow{\mu_A \otimes \operatorname{id}_A} A \otimes_k A \otimes_k A$$

2. The following diagram commutes:



and is equal to the identity.

3. Inverse. The following diagram commutes:

Here the right arrow is the morphism making A a k-algebra. The last arrow in the lower sequence is multiplication in A.

Definition 2.2. An action of an affine algebraic group $G = \operatorname{Spec} A$ on an affine variety $X = \operatorname{Spec} R$ is a morphism $G \times X \to X$ defined dually by a k-algebra morphism $\mu_R : R \to R \otimes_k A$ satisfying the following two conditions.

1. The following diagram is commutative:

$$R \xrightarrow{\mu_R} R \otimes_k A$$

$$\downarrow^{\mathrm{id}_R} \qquad \downarrow^{\mathrm{id}_R \otimes \epsilon}$$

$$R \simeq R \otimes_k k$$

2. The diagram

$$R \xrightarrow{\mu_R} R \otimes_k A$$

$$\downarrow^{\mu_R \otimes \mathrm{id}_A}$$

$$R \otimes_k A \xrightarrow{\mathrm{id}_R \otimes \mu_A} R \otimes_k A \otimes_k A$$

3 Representations of algebraic groups

Let $G = \operatorname{Spec} A$ be an affine algebraic group over a field k.

Definition 3.1. An algebraic representation of G is a pair (V, μ_V) consisting of a k-vector space V and a k-linear map $\mu_V : V \to V \otimes_k A$ satisfying the following two conditions:

1. The diagram

$$V \xrightarrow{\mu_{V}} V \otimes_{k} A$$

$$\downarrow^{\mathrm{id}_{V}} \qquad \downarrow^{\mathrm{id}_{V} \otimes \epsilon}$$

$$V \simeq V \otimes_{k} k$$

$$(1)$$

is commutative.

2. The diagram

$$V \xrightarrow{\mu_{V}} V \otimes_{k} A$$

$$\downarrow^{\mu_{V} \otimes \mathrm{id}_{A}} V \otimes_{k} A \otimes_{k} A$$

$$V \otimes_{k} A \xrightarrow{\mathrm{id}_{V} \otimes \mu_{A}} V \otimes_{k} A \otimes_{k} A$$

is commutative. Here μ_A is the coproduct in the coordinate ring of G.

Remark. In lieu of Definition 1.2, we see that any action of an algebraic group G on an affine variety $X = \operatorname{Spec} R$ is a representation of G on the infinite-dimensional k-vector space $R = \Gamma(X, \mathcal{O}_X)$.

We often drop the subcript from μ_V unless confusion may arise. The same comment applies to tensor products. They will always be over the ground field unless otherwise stated. We will sometimes refer to a representation (V, μ_V) sometimes as "a representation $\mu: V \to V \otimes A$ " and sometimes as just "a representation V".

Definition 3.2. Let $\mu:V\to V\otimes A$ be a representation of $G=\operatorname{Spec} A.$ Then:

- 1. A vector $x \in V$ is said to be G-invariant if $\mu(x) = x \otimes 1$.
- 2. A subspace $U \subset V$ is called a subrepresentation if $\mu(U) \subseteq U \otimes A$.

Proposition 3.3. Every representation V of G is locally finite-dimensional. Precisely: every $x \in V$ is contained in a finite-dimensional subrepresentation of G.

Bevis. Write $\mu(x)$ as a finite sum $\sum_i x_i \otimes f_i$ for $x_i \in V$ and linearly independent $f_i \in A$. This we can always do, by definition of tensor product and bilinearity. Let U be the subspace of V spanned by the vectors x_i .

Now, by the commutativity of the diagram (1) it follows that

$$x = \sum_{i} \epsilon(f_i) x_i.$$

By the commutativity of the second diagram in the definition, it follows that

$$\sum_{i} \mu_{V}(x_{i}) \otimes f_{i} = \sum_{i} x_{i} \otimes \mu_{A}(f_{i}) \in U \otimes A_{k} \otimes_{k} A.$$

Because each term of the right-hand-side is contained in $U \otimes A \otimes A$, it follows that $\mu_V(x_i)$ is contained in U since the f_i are linearly independent.

Thus x is contained in the finite-dimensional representation $\mu_V|_U:U\to U\otimes A.$

Repr. of \mathbb{G}_m .

Euler operator.

Characters.