Results so far

Fredrik Meyer

December 3, 2014

1 Deformations of the 5-dimensional toric variety

Recall that the 5-dimensional toric variety T had 2-dimensional singularities (actually two disjoint copies of dP_6).

Theorem 1.1. *There exists a flat deformation of $\mathbb{P}(\mathcal{K} * \Delta^1)$, $\mathfrak{X} \to S$, such that $\mathfrak{X}_{t_1} = T$ for some $t_1 \in S$ and such that the general fiber $\widetilde{T}$ have one-dimensional singularities.*

Proof. As per now, the proof is purely by computer. The technique is this: First, consider the monomial degeneration of T to the Stanley-Reisner ring $A(\mathcal{K} * \Delta^1)$ (recall that $\mathcal{K} = D_6 * D_6$). Choose deformation parameters t_i perturbing the equations "in the direction of T", meaning that we only choose parameters introducing terms already occuring in the equations of T.

Now, using the package `VersalDeformations`, it is possible to produce a flat family $\mathfrak{X} \to S'$ with $\mathfrak{X}_0 = \mathbb{P}(\mathcal{K} * \Delta^1)$.

The base space is a union of toric varieties of dimensions $14, 13, 13, 12$, respectively. Call the largest one for S. The equations are

$$\begin{vmatrix} t_1 & t_2 & t_2 & t_3 & \dots & t_{12} \\ t_{13} & t_{14} & t_{15} & t_{16} & \dots & t_{24} \end{vmatrix} \leq 1$$

By restriction, we get the claimed family $\mathfrak{X} \to T$. It is cheched by setting $t_i = i$ for $i = 1, \dots, 12$ and $t_i = 2i$ for $i = 13, \dots, 24$ that the resulting fiber have one-dimensional singularities. The reason we don't set all $t_i = 1$, is that this point lies in the intersection of the components of S. $\square$

Corollary 1.2. *The Stanley-Reisner scheme $\mathbb{P}(\mathcal{K})$ smooths to a smooth Calabi-Yau variety X.*

Proof. The scheme $\mathbb{P}(\mathcal{K})$ sits as a complete intersection in $\mathbb{P}(\mathcal{K} * \Delta^1)$. Complete intersections deform together with the ambient variety, so $\mathbb{P}(\mathcal{K} * \Delta^1)$ deforms to a general complete intersection in $\widetilde{T}$. Since $\widetilde{T}$ have curve singularities, it follows by two applications of Bertini's theorem [Har77, Theorem II.8.18], that X is smooth. $\square$

Now what we would really like to do is to compute the Hodge numbers $h^{ij} = \dim_k H^j(X, \Omega^i_X)$ of X.

We can however compute the Hodge numbers of $\widetilde{T}$. The hope is that there is some sort of Lefschetz theorem giving us the Hodge numbers of X.

Theorem 1.3. *We have $h^{11}(\widetilde{T}) = 1$ and $h^{12}(\widetilde{T}) = 13$.*

Proof. Again, this is purely computational. We use long exact sequences together with sheaf cohomology computations in `Macaulay2`.

Since the ideal of $\widetilde{T}$ is rather complicated, doing this naïvely does not work. The trick is to choose the right term order. Since we know that $\widetilde{T}$ has a nice degeneration, we would like to find a term order such that its initial ideal is precisely the Stanley-Reisner ideal.

The `Macaulay2` package `gfanInterface` provides an interface with `gfan`, which is a program that can compute weight vectors given polynomials with prescribed initial terms. The weight vector is

$$\omega = (1, 1, 4, 7, 7, 4, 1, 1, 4, 7, 7, 4, 1, 1).$$

With this term order, giving a very small Gröbner basis (18 elements), the computations are much faster than with the standard term order. We are able to compute resolutions of all the relevant modules within a few minutes in total.

We have an exact sequence of sheaves on $\widetilde{T}$:

$$0 \to \mathscr{T}_1 \hookrightarrow \mathcal{I}/\mathcal{I}^2 \xrightarrow{d} \Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}} \to \Omega^1_{\widetilde{T}} \to 0.$$

This sequence can be broken into two short exact sequences. The relevant one is this:

$$0 \to \operatorname{im} d \to \Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}} \to \Omega^1_{\widetilde{T}} \to 0. \tag{1}$$

We also have the restriced Euler sequence:

$$0 \to \Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}} \to \mathscr{O}_{\widetilde{T}}(-1)^{14} \to \mathscr{O}_{\widetilde{T}} \to 0. \tag{2}$$

We first compute h^{11}. From (1) we get a long exact sequence

$$\ldots \to H^1(\operatorname{im} d) \to H^1(\Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}}) \to H^1(\Omega^1_{\widetilde{T}}) \to H^2(\operatorname{im} d) \to \ldots$$

The cohomology of $H^1(\operatorname{im} d)$ and $H^2(\operatorname{im} d)$ was computed with `Macaulay2` to be both zero. Thus $H^1(\Omega^1_{\widetilde{T}}) \simeq H^1(\Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}})$. From the Euler sequence we get

$$\ldots \to H^0(\mathscr{O}_{\widetilde{T}}(-1)^{14}) \to H^0(\mathscr{O}_{\widetilde{T}}) \to H^1(\Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}}) \to H^1(\mathscr{O}_{\widetilde{T}}(-1)^{14}) \to \ldots$$

But the left and right terms are both zero. Hence $h^{11} = 1$. We now compute h^{12}.

From (1) we again get

$$0 \to H^2(\Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}}) \to H^2(\Omega^1_{\widetilde{T}}) \to H^3(\operatorname{im} d) \to H^3(\Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}}) \to \ldots,$$

where we have used that $H^2(\operatorname{im} d) = 0$. □

References

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.