

Exercises

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I solve and type exercises from different places (read *books*).

1 Algebraic Geometry - Hartshorne

1.1 Chapter I - Varieties

Exercise 1 (Exercise 1.1). a) Let Y be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k .

b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k .

c) Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f . Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?



Solution 1. a) We have $A(Y) = k[x, y]/(y - x^2)$. An isomorphism $A(Y) \rightarrow k[t]$ is given by $x \mapsto t$ and $y \mapsto t^2$.

b) We have $A(Z) = k[x, y]/(xy - 1) \simeq k[x, \frac{1}{x}]$. So we must show that $k[x, \frac{1}{x}] \not\simeq k[x]$. It can be computed that the first one has automorphisms given by $x \mapsto cx^n$ for c nonzero and $n \neq 0$. The second has as automorphisms $ax + b$ ($a \neq 0$). So the first one have an abelian automorphism group, the second has not.

c) What is special about $A(Y)$ and $A(Z)$? Staring at pictures, we see that any line in \mathbb{A}^2 intersects Y in at least one point, but in the case of Z , there exist two lines which do not intersect Z . We claim that this is the only two things that can happen.

First we claim that if we are in the second situation, that is, if there exist a pair of lines ℓ, ℓ' such that $W \cap \ell = W \cap \ell' = \emptyset$, then $W \simeq Z$.

A general quadric can be written as

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Suppose now $\ell \cap W = \emptyset$. This is equivalent to $I(f, \ell^\vee) = (1)$. Without loss of generality, we can assume $\ell = \{x = 0\}$. Then

$$I(f, \ell) = (cy^2 + ey + f, x).$$

This generates $k[x, y]$ if and only if $c = e = 0$ and $f \neq 0$. Thus f must be of the form

$$ax^2 + bxy + dx + f = 0$$

with $f \neq 0$. But this can be written as

$$x(ax + by + d) + f = 0.$$

Put $y' = ax + by + d$. Then $I(W)$ takes the form $(xy' + f = 0)$, which is clearly isomorphic to Z after a linear change of coordinates. Note that the other line not meeting W is the line given by $y' = ax + by + d = 0$.

Assume now that we are in the other situation, namely that *every* line in \mathbb{A}^2 meets W . Now pick a tangent line ℓ of W . Without loss of generality, we can assume that ℓ is $\{y = 0\}$. This is a tangent line if and only if it meets W doubly, meaning that $I(W) + (\ell^\vee)$ takes the form (l^2, y) for some linear form l . We can also assume that $\ell \cap W = (0, 0)$, so that $I(W) + (\ell^\vee) = (x^2, y)$. But this means that

$$\begin{aligned} I(W) + I(\ell) &= (ax^2 + bxy + cy^2 + dx + ey + f, y) \\ &= (ax^2 + dx + f, y) \end{aligned}$$

We want $ax^2 + dx + f = x^2$. This can happen only if $d = f = 0$ and $a \neq 0$. Thus the quadric takes the form

$$ax^2 + bxy + cy^2 + ey = 0.$$

Now we claim that there exist one line at each point of W that intersect W transversally in exactly one point. This is the case for Y . Consider the pencil of lines through $(0, 0)$ defined by $x = \lambda y$. We want to find λ such that the intersection is transversal and only one point. We have

$$(ax^2 + bxy + cy^2 + ey, x - \lambda y) = ((a\lambda^2 + b\lambda + c)y^2 + ey, x - \lambda y).$$

This have exactly one solution if and only if $a\lambda^2 + b\lambda + c = 0$. This is solvable since $a \neq 0$ and since all lines intersect W . Thus choose λ as above. We can rotate this line such that it becomes $x = 0$. Then the equation takes the form

$$ax^2 + bxy + ey = 0.$$

We have still not arrived at $y = x^2$. Let now $y = \lambda x$ be a general line through the origin. We demand that this intersect W twice for every λ such that the line is not tangent. We get that the intersection is given by

$$ax^2 + b\lambda x + ex = x((a + \lambda b)x + e) = 0.$$

For this to have two solutions for every λ we must have $a + \lambda b \neq 0$ for all λ . But this requires $b = 0$. Thus the equation is

$$ax^2 + ey = 0$$

which is the conic we were looking for.

♡

Exercise 2 (Exercise 1.2, the twisted cubic curve). Let $Y \subseteq \mathbb{A}^3$ be the set $\{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k . We say that Y is given by the *parametric equation* $x = t, y = t^2, z = t^3$. ♠

Solution 2. An affine variety is by definition a closed irreducible subset of \mathbb{A}^3 . So we must find an irreducible ideal I such that $Z(I) = Y$ (forgive the abuse of notation).

I claim that $I(Y) = \langle x^2 - y, x^3 - z \rangle$. Clearly, every $P \in Y$ satisfies these equations. This shows the inclusion $Y \subset Z(I)$. Now suppose $P \in Z(I)$, that is, $f(P) = 0$ for all $f \in I$. In particular $(x^2 - y)(P) = 0$ and $(x^3 - z)(P) = 0$. Thus $y = x^2$ and $z = x^3$. So if $P = (a, b, c) \in k^3$, then $P = (a, a^2, a^3)$, so $P \in Y$. This shows that $Z(I) = Y$. If we can show that I is prime, then it follows that $I(Y) = I$ and that Y is a variety.

In fact, we claim that $k[x, y, z]/I \simeq k[t]$, implying that I is prime. The map φ is given by $x \mapsto t, y \mapsto t^2, z \mapsto t^3$. Then clearly $I \subseteq \ker \varphi$. We must show equality. So suppose $\varphi(f) = 0$.

First we claim that any $f \in k[x, y, z]$ can be written as $f = R(x) + S(x)y + T(x)z + i(x, y, z)$ where i is a polynomial in I . We prove this by induction on $\deg f$. If $\deg f = 1$, this is trivially true. The rest of the proof proceeds by tedious induction. ♡

1.2 Chapter II - Schemes

Exercise 3 (Exercise 7.1). Let (X, \mathcal{O}_X) be a locally ringed space and let $f : \mathcal{L} \rightarrow \mathcal{M}$ be a surjective map of invertible sheaves on X . Show that f is an isomorphism. ♠

Solution 3. Since \mathcal{L}, \mathcal{M} are invertible, we have isomorphisms $\mathcal{L}_x \approx \mathcal{O}_{X,x}$ and $\mathcal{M}_x \approx \mathcal{O}_{X,x}$ for each $x \in X$.

But $\text{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{X,x}) = \mathcal{O}_{X,x}$, that is, all homomorphisms are given by multiplication by some $h \in \mathcal{O}_{X,x}$. But since f was surjective, we conclude that h is outside \mathfrak{m}_x , the maximal ideal of $\mathcal{O}_{X,x}$. But then h is a unit, so f is an isomorphism. ♥

1.3 Chapter III - Cohomology

Exercise 4 (Exercise 4.3). Let $X = \mathbb{A}_k^2 = \text{Spec } k[x, y]$ and let $U = X \setminus \{(0, 0)\}$. Use a suitable open cover of X by open affine subsets to show that $H^1(U, \mathcal{O}_U)$ is isomorphic to the k -vector space spanned by $\{x^i y^j \mid i, j < 0\}$. In particular, it is infinite-dimensional, and so U cannot be affine (not projective either). ♠

Solution 4. We can cover U by $U_1 = \mathbb{A}^2 \setminus \{x = 0\}$ and $U_2 = \mathbb{A}^2 \setminus \{y = 0\}$. We have $U_1 \cap U_2 = \mathbb{A}^2 \setminus \{xy = 0\}$. Also, $\mathcal{O}(U_1) = k[x, y, \frac{1}{x}]$ and $\mathcal{O}(U_2) = k[x, y, \frac{1}{y}]$ and $\mathcal{O}(U_1 \cap U_2) = k[x, y, \frac{1}{xy}]$. Then the Čech complex takes the form

$$0 \rightarrow k[x, y, \frac{1}{x}] \times k[x, y, \frac{1}{y}] \xrightarrow{d} k[x, y, \frac{1}{xy}] \rightarrow 0,$$

the differential being difference. Then $H^1(U, \mathcal{O}_U)$ can be computed as the homology at the second term. But nothing on the left side can hit anything of the form $x^i y^j$ with $i, j < 0$. Anything else is hit. Thus we have

$$H^1(U, \mathcal{O}_U) \simeq \{x^i y^j \mid i, j < 0\}$$

as k -vector spaces. ♥

Exercise 5 (Exercise 4.7). Let X be the subscheme of \mathbb{P}_k^2 defined by a single homogeneous polynomial $f(x_0, x_1, x_2) = 0$ of degree d . Assume that $(1, 0, 0)$ is not on X . Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Čech complex

$$\Gamma(U, \mathcal{O}_X) \oplus \Gamma(V, \mathcal{O}_X) \rightarrow \Gamma(U \cap V, \mathcal{O}_X)$$

explicitly, and thus show that

$$\begin{aligned}\dim_k H^0(X, \mathcal{O}_X) &= 1 \\ \dim_k H^1(X, \mathcal{O}_X) &= \frac{1}{2}(d-1)(d-2).\end{aligned}$$

♠

Solution 5. X can be covered by just two open affines since $\mathbb{P}^2 \setminus (U \cup V) = \{(1 : 0 : 0)\}$, which was assumed not to lie on the curve.

The open affine subset $\Gamma(U, \mathcal{O}_X)$ can be identified with the polynomial ring $k[u, v]/\langle f(u, 1, v) \rangle$, and $\Gamma(V, \mathcal{O}_X) = k[x, y]/f(x, y, 1)$. The differential is then given by

$$(g(u, v), h(x, y)) \mapsto g(xy^{-1}, y^{-1}) - h(x, y) \in k[x, y, \frac{1}{y}].$$

We can assume that $f = x_0^d$, since what really matters is the degree, and we are just doing linear algebra.

We first calculate $H^0(X, \mathcal{O}_X)$. So suppose $g(xy^{-1}, y^{-1}) - h(x, y) = 0$ in $k[x, y, y^{-1}]/\langle f(x, y, 1) \rangle$. By definition this means that

$$g(xy^{-1}, y^{-1}) - h(x, y) = f(x, y, 1) \cdot \tilde{f}(x, y, \frac{1}{y})$$

for some polynomial \tilde{f} . Write \tilde{f} as $\tilde{f}_0 + \tilde{f}_1$, where $\tilde{f}_0 = \sum_{j < 0} a_{ij} x^i y^j$ and $\tilde{f}_1 \in k[x, y]$. Then we have the equality

$$g(xy^{-1}, y^{-1}) - h(x, y) = \sum_{j < 0} a_{ij} x^{i+d} y^j + \sum_{j \geq 0} x^{i+d} y^j.$$

First of all, we see that the constant terms of g and h must be equal, because there are no constant terms on the right hand side. Secondly, $g(xy^{-1}, y^{-1})$ consists solely of terms with $j < 0$. Thus the non constant terms of $g(xy^{-1}, y^{-1})$ must be equal to the left term of the right hand side above. But both terms of the right hand side are zero modulo f , so the constant terms of $g(xy^{-1}, y^{-1})$ are also zero mod f . The same holds for $h(x, y)$. Thus $H^0(X, \mathcal{O}_X) = \{(c, c) \mid c \in k\} \simeq k$.

Now we compute $H^1(X, \mathcal{O}_X)$. Consider a monomial $x^i y^j$ in the target. If both $i, j \geq 0$, then it is hit by $(0, -x^i y^j)$. Likewise, if $j \geq i$, then $(x^i y^{j-i}, 0) \mapsto x^i x^{-j}$. Thus all monomials $x^i y^{-j}$ with $j \geq i$ is zero in the cokernel. Further, if $i \geq d$, then $x^i y^j$ is already zero! Thus, we can draw

the non-zero monomials in the cokernel as points in the lattice \mathbb{Z}^2 . This is a triangle of length $d - 2$. Thus the dimension of $H^1(X, \mathcal{O}_X)$ is

$$1 + 2 + \dots + d - 3 + d - 2 = \frac{1}{2}(d - 2)(d - 2 + 1) = \frac{1}{2}(d - 2)(d - 1).$$

♡

1.4 Chapter IV - Curves

Exercise 6 (Exercise 1.1). Let X be a curve and $P \in X$ a point. Show that there exists a nonconstant rational function $f \in K(X)$ which is regular everywhere except at P . ♠

Solution 6. Let D be the divisor $D = nP$. The linear system

$$\{E = D + f \geq 0\}$$

consists of all divisors linearly equivalent to D . But these are classified by those f with $(f) \geq -nP$, i.e. those f with at most poles of order n at P .

By Riemann-Roch we have

$$l(D) - l(K - D) = \deg D + 1 - g = n + 1 - g.$$

If n is large enough, $K - D$ will have negative degree, so $l(K - D) = 0$. Thus for large n , we can get $l(D)$ as big as we want.

♡

2 Commutative Algebra - Eisenbud

2.1 Chapter 16 - Modules of Differentials

Exercise 7 (Exercise 16.1). Show that if $b \in S$ is an idempotent ($b^2 = b$), and $d : S \rightarrow M$ is any derivation, then $db = 0$. ♠

Solution 7. This is trivial. $db = d(b^2) = 2db$. If $2 = 0$, then the statement is automatically true. If not, then $db = 0$ by subtraction. ♡

3 Deformation Theory - Hartshorne

3.1 Chapter I.3 - The T^i functors

Exercise 8 (Exercise 3.1). Let $B = k[x, y](xy)$. Show that $T^1(B/k, M) = M \otimes k$ and $T^2(B/k, M) = 0$ for any B -module M . ♠

Solution 8. Since B is defined by a principal ideal in $P = k[x, y]$, it follows that $L_2 = 0$ in the cotangent complex. Thus $T^2(B/k, M)$ is automatically zero.

We have that $L_1 = B$ and $L_0 = Bdx \oplus Bdy$ with d_1 being $f \mapsto (fy, fx)$. Applying $\text{Hom}(-, M)$, we get $\text{Hom}(L_0, M) = M \oplus M$ and $\text{Hom}(L_1, M) = M$.

We have $\text{Hom}(B \oplus B, M) \simeq M \oplus M$ by $\phi \mapsto (\phi(1, 0), \phi(0, 1))$. We have a diagram

$$\begin{array}{ccc} \text{Hom}(B \oplus B, M) & \xrightarrow{\psi^*} & \text{Hom}(B, M) \\ \simeq \downarrow & & \downarrow \simeq \\ M \oplus M & \longrightarrow & M \end{array}$$

Under these isomorphisms, it is easy to see that the bottom map is given by

$$(\phi(1, 0), \phi(0, 1)) \mapsto y\phi(1, 0) + x\phi(0, 1).$$

Thus since T^1 is the cokernel of this map, we must have $T^1(B/k, M) = M \otimes k$. \heartsuit

Exercise 9 (Exercise 3.3). Let $B = k[x, y]/(x^2, xy, y^2)$. Show that $T^0(B/k, B) = k^4$, $T^1(B/k, B) = k^4$ and $T^2(B/k, B) = k$. \spadesuit

Solution 9. Let's compute L_2 first. For that we need part of a resolution of I . We have in fact

$$0 \rightarrow \text{im} \begin{pmatrix} -y & 0 \\ x & -y \\ 0 & x \end{pmatrix} \rightarrow P(-2)^3 \rightarrow I \rightarrow 0.$$

The Koszul relations are given by

$$\text{im} \begin{pmatrix} -y^2 & -xy & 0 \\ 0 & x^2 & -y^2 \\ x^2 & 0 & xy \end{pmatrix}.$$

Let's compute Q/F_0 (relations modulo Koszul relations). Since Q is generated in degree 3, and F_0 is of degree 4, we have $\dim_k(Q/F_0)_3 = 2$. Let's consider degree 4. As a k -vector space Q_4 is spanned by the four elements

$$\begin{pmatrix} -y^2 \\ xy \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -y^2 \\ xy \end{pmatrix}, \begin{pmatrix} -yx \\ x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -yx \\ x^2 \end{pmatrix}.$$

The two in the middle are already Koszul relations, so that $(Q/F_0)_4$ have dimension ≤ 2 . But we also have

$$\begin{pmatrix} -y^2 \\ xy \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ yx \\ -x^2 \end{pmatrix} + \begin{pmatrix} -y^2 \\ 0 \\ x^2 \end{pmatrix}.$$

Thus $\dim_k(Q/F_0)_4 = 1$, since the second term above is a Koszul relation. Similarly we find that $\dim_k(Q/F_0)_5 = 0$. Hence, L_2 is the 3-dimensional k -vector space spanned by Q_3 and one more relation. L_1 is $F \otimes B = B^3$, and L_0 is $B \oplus B$, spanned by dx, dy .

Taking duals, we get that $L_2 = \text{Hom}(Q/F_0, B)$. This set can be identified with

$$\begin{aligned} \text{Hom}(Q/F_0, B) &= \{\varphi : Q \rightarrow B \mid \varphi|_{F_0} = 0\} \\ &= \{\varphi : Q \rightarrow P \mid \text{im } \varphi|_{F_0} \subseteq I\} \end{aligned}$$

Thus, since $I = \mathfrak{m}^2$, we must have that φ sends the two generators of Q to something of degree 1 (degree 0 is not ok, since then F_0 would be sent outside I). Thus $\text{Hom}(Q/F_0, B)$ is $2 \times 2 = 4$ -dimensional, spanned by

$$\text{im} \begin{pmatrix} y & x & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}.$$

But d_2 is the dual of the inclusion $Q \rightarrow F$ from the exact sequence above. The dual is given by transposing, and we are left with one column - in conclusion, $T^2(B/k, B)$ is one-dimensional.

The Jacobian of I is given by

$$\begin{pmatrix} 2x & y & 0 \\ 0 & x & 2y \end{pmatrix},$$

and it is easily seen that the kernel of $\text{Jac} \otimes B$ is given by $\mathfrak{m} \oplus \mathfrak{m} \oplus \mathfrak{m} \subset B^3$. The two relations kill off two dimensions, so $\dim_k T^1(B/k, B) = \dim_k \mathfrak{m}^{\oplus 3} - 2 = 6 - 2 = 4$.

Also $T^0(B/k, B)$ is B^2 modulo the image of the Jacobian. The constants are left untouched, so $\dim_k T^0(B/k, B) = 2 + 2 + 2 - 3 = 3$. A basis is given by $(1, 0)$, $(0, 1)$ and (x, y) . (thus Hartshorne is wrong?) \heartsuit

4 Introduction to Commutative Algebra - Atiyah-MacDonald

4.1 Chapter 1 - Rings and ideals

Exercise 10. Let x be a nilpotent element of a ring A . Show that $1 + x$ is a unit of A . Deduce that the sum of a nilpotent element and a unit is a unit. ♠

Solution 10. Suppose $x^{n+1} = 0$ and that $x^n \neq 0$. Consider

$$s = 1 - x + x^2 - x^3 + \dots + x^n$$

Then

$$sx = x - x^2 + x^3 - x^4 + \dots - x^n$$

since $x^{n+1} = 0$. But then $s + sx = 1$, so that $s(1 + x) = 1$. Hence $1 + x$ is a unit. To prove that the sum of any unit and any nilpotent is a unit, note that if u is any unit, then $u^{-1}x$ is still nilpotent. So since $u + x = u(1 + u^{-1}x)$ and product of units are units, the claim follows. ♥

Exercise 11 (Exercise 11). A ring A is *Boolean* if $x^2 = x$ every $x \in A$. In a Boolean ring A , show that

- i) $2x = 0$ for all $x \in A$.
- ii) Every prime ideal \mathfrak{p} is maximal, and A/\mathfrak{p} is a field with two elements.
- iii) Every finitely generated ideal in A is principal.

♠

Solution 11. i) We have $4x = 4x^2 = (2x)^2 = 2x$, hence $2x = 0$.

- ii) Consider A/\mathfrak{p} . This is an integral domain in which $x^2 = x$ for all $x \in A/\mathfrak{p}$. But then $x^2 - x = x(x - 1) = 0$. Hence either $x = 0$ or $x = 1$, hence A/\mathfrak{p} can have only two elements. Thus it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ which is a field, hence \mathfrak{p} is maximal.
- iii) Let $I = (a_1, \dots, a_r)$. Every ideal is contained in a maximal ideal \mathfrak{m} . Consider the image of I in A/\mathfrak{m} .
- iv) By induction we can assume that I is generated by two elements, say $I = (a_1, a_2)$. Then I claim that $I = (a_1 + a_2)$. Clearly $(a_1 + a_2) \subseteq (a_1, a_2)$. The other direction will follow if we can see that $a_1 a_2 = 0$ (or

So suppose $M \otimes_A N = 0$. Then $(M \otimes_A N)_k = 0$. But this is isomorphic to $M_k \otimes_A N_k$ since $k \otimes_A k = k$. But $M_k \otimes_A N_k \simeq M_k \otimes_k N_k$, as k -modules, since everything in \mathfrak{m} acts trivially on M_k . But these are vector spaces over a field, now we must have $M_k = 0$ or $N_k = 0$, and by Nakayama we are done. \heartsuit

Exercise 16 (Exercise 24). If M is an A -module, the following are equivalent:

- i) M is flat.
- ii) $\text{Tor}_n^A(M, N) = 0$ for all $n > 0$ and A -modules N .
- iii) $\text{Tor}_1^A(M, N) = 0$ for all A -modules N .

Solution 16. To compute $\text{Tor}_n^A(M, N)$, one takes an A -resolution of N and tensor it with M and take homology. But M is flat, so the sequence stays exact, so the homology is zero. This shows $i) \Rightarrow ii)$.

The implication $ii) \Rightarrow iii)$ is trivial.

Now let

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

be any exact sequence of A -modules. Then by properties of the Tor functor, we have an exact sequence

$$\text{Tor}_1(M, N'') \rightarrow N' \otimes M \rightarrow N \otimes M \rightarrow N'' \otimes M \rightarrow 0.$$

But $\text{Tor}_1(M, N'') = 0$, so the sequence is short exact. Hence M is flat. \heartsuit

Exercise 17 (Exercise 25). Let

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

be an exact sequence with N'' flat. Then N' is flat if and only if N is flat. \spadesuit

Solution 17. We have from the Tor exact sequence

$$0 \rightarrow \text{Tor}_1(N', M) \rightarrow \text{Tor}_1(N, M) \rightarrow 0$$

since $\text{Tor}_2(N'', M) = \text{Tor}_1(N'', M) = 0$. The statement follows. \heartsuit

\spadesuit

4.3 Chapter III - Rings and modules of fractions

Exercise 18 (Exercise 1). Let S be a multiplicatively closed subset of a ring A , and let M be a finitely-generated A -module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$. ♠

Solution 18. Suppose there exists such s . Let $m/s' \in S^{-1}M$. This is zero if and only if there exists $s \in M$ such that $s(s'm) = 0$. But $ss'm = s'sm = s'0 = 0$. So $m = 0$ in $S^{-1}M$. (note that we did not use finite generation)

Now let m_1, \dots, m_r be a set of generators for M and suppose that $S^{-1}M = 0$. Then for each i ($i = 1, \dots, r$), there exists s_i such that $s_i m_i = 0$. Since every element of M is an A -linear combination of the m_i , it follows that the product $s_1 s_2 \cdots s_r$ makes $sM = 0$. ♥

4.4 Chapter 5 - Integral dependence and valuations

Exercise 19 (Exercise 1). Let $f : A \rightarrow B$ be an integral morphism of rings. Show that $f^* : \text{Spec } B \rightarrow \text{Spec } A$ is a closed mapping. ♠

Proof. The map f^* is by definition given by $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p}) = \mathfrak{p} \cap A$. A closed subset of $\text{Spec } B$ is by definition

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec } B \mid \mathfrak{p} \supset \mathfrak{a}\}$$

for some ideal $\mathfrak{a} \subset B$.

Then the image of $V(\mathfrak{a})$ is the set

$$f^*(V(\mathfrak{a})) = \{\mathfrak{p} \cap A \mid \mathfrak{p} \in \text{Spec } B, \mathfrak{p} \supset \mathfrak{a}\}$$

..... ???

□

4.5 Chapter 7 - Noetherian rings

Exercise 20 (Exercise 11). Let A be a ring such that $A_{\mathfrak{p}}$ is Noetherian for each $\mathfrak{p} \in \text{Spec } A$. Is A necessarily noetherian? ♠

Solution 19. Consider the ring

$$A = \mathbb{Z}/2 \times \mathbb{Z}/2 \cdots .$$

It is a countable product of noetherian rings. The primes are just the coordinate axes, and each localization is isomorphic to $\mathbb{Z}/2$. Thus each $A_{\mathfrak{p}}$ is Noetherian, but A is not. ♥

Exercise 21 (Exercise 15). Let A be a Noetherian local ring, \mathfrak{m} its maximal ideal and k its residue field and let M be a finitely generated A -module. Then the following are equivalent:

- i) M is free.
- ii) M is flat.
- iii) The mapping $\mathfrak{m} \otimes M \rightarrow A \otimes M$ is injective.
- iv) $\mathrm{Tor}_1^A(k, M) = 0$.



Solution 20. The implication $i) \Rightarrow ii)$ is trivial. One way is to compute $\mathrm{Tor}_1^A(M, N)$ for any A -module N . But a free resolution of M is just one-term, so $\mathrm{Tor}_1^A(M, N)$ is automatically zero.

The implication $ii) \Rightarrow iii)$ follows by tensoring the inclusion $\mathfrak{m} \hookrightarrow A$ with M .

The implication $iii) \Rightarrow iv)$ follows from the Tor exact sequence

$$\mathrm{Tor}_1^A(A, M) \rightarrow \mathrm{Tor}_1^A(k, M) \rightarrow \mathfrak{m} \otimes M \rightarrow A \otimes M \rightarrow k \otimes M \rightarrow 0.$$

The leftmost term is zero since A is a free A -module, and by $iii)$ and exactness we must as well have $\mathrm{Tor}_1^A(k, M) = 0$.

Now for $iv) \Rightarrow i)$. Choose element $m_i \in M$ ($0 \leq i \leq r$) such that they form a k -basis for $M/\mathfrak{m}M$. Choose a surjection $f : A^r \rightarrow M$ and let $E = \ker f$ be its kernel. Then we have an exact sequence

$$0 \rightarrow E \rightarrow A^r \rightarrow M \rightarrow 0.$$

of finitely-generated A -modules (E is finitely generated by Proposition 6.2). Tensor the sequence by k , and get

$$\mathrm{Tor}_1^A(k, M) \rightarrow E/\mathfrak{m}E \rightarrow k^r \rightarrow M/\mathfrak{m}M \rightarrow 0.$$

The left-most term is zero by assumption. The last two spaces are k -vector spaces of the same dimension, and it follows that $E/\mathfrak{m}E = 0$. But then it follows that E is zero by Nakayama's lemma, hence M is free. \heartsuit

Exercise 22 (Exercise 16). Let A be a Noetherian ring, M a finitely-generated A -module. Then the following are equivalent:

- i) M is a flat A -module.

ii) $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module for each $\mathfrak{p} \in \operatorname{Spec} A$.

iii) $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$ -module for each maximal ideal \mathfrak{m} .

So flatness is the same as being locally free. ♠

Solution 21. The implications $i \Rightarrow ii$) and $ii) \Rightarrow iii)$ follows trivially from the previous exercise. We prove $iii) \Rightarrow i$).

Applying the Tor functor commutes with localization, hence we have $\operatorname{Tor}_1^A(M, N)_{\mathfrak{m}} = \operatorname{Tor}_1^{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) = 0$ for all \mathfrak{m} . But being zero is a local property, so it follows that $\operatorname{Tor}_1^A(M, N) = 0$ for all A -modules N . Hence M is flat. ♡