A study of a variety

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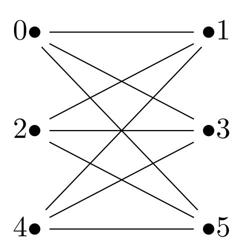
1 The variety X

Let $X \subseteq \mathbb{P}^5$ be defined by the ideal $I = (x_0x_2x_4 - x_1x_3x_5)$. It is a singular 4-fold, and the singular locus is a union of 9 lines. One sees easily that the ideal of the singular locus of X is

$$I(\operatorname{Sing} X) = (x_2x_4, x_0x_4, x_0x_2, x_3x_5, x_1x_5, x_1x_3)$$

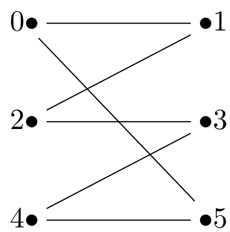
If one draws the the 2-skeleton of the associated simplicial complex, one gets the complete bipartite graph $K_{3,3}$:



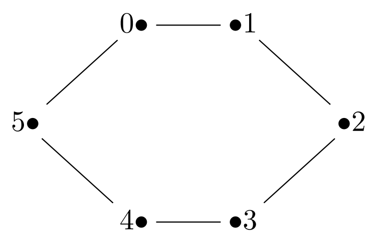
There are no 3-cycles, so the associated simplicial complex cannot be 2-dimensional. In particular, we can identify the singular locus from this drawing: The numbers $0 - 5$ correspond to the "standard points" of $\mathbb{P}^5$, i.e. the points $(1:0:0:0:0:0), \ldots, (0:0:0:0:0:1)$. So one of the lines in $\operatorname{Sing} X$ is the line [01], that is, all points of the form $(\lambda : \mu : 0 : 0 : 0 : 0)$. One sees that each line in $\operatorname{Sing} X$ meet intersect four other lines, in two points.

Note that the generators of the ideal of the singular locus give rise to a birational map $\mathbb{P}^5 \dashrightarrow X$, and we could blow up X on the singular locus, to try to desingularize X. However, there may be other ways to remove the singularities.

In particular, consider the graph below:



This is a cycle, so it may better be drawn as



This give rise to a rational map $\varphi : \mathbb{P}^5 \dashrightarrow X$ given by

$$(z_{01} : z_{12} : \cdots : z_{50}) \mapsto (x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_0x_5)$$

Its base locus, B, lies inside $\operatorname{Sing} X$. The inverse images of the lines $[01], [12], \ldots, [50]$, are $\mathbb{P}^3$'s. In particular, they have codimension 2 in $\mathbb{P}^5$. However, this map is dominant, and we want a map from a projective space of the same dimension. So the first drawing is actually better: It gives rive to a map $\psi : \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow X$ defined by

$$(x_0 : x_2 : x_4, x_1 : x_3 : x_5) \mapsto (x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_0x_5)$$

the same formula. Now the inverse images are better: Let $[ij]$ be the line between the exceptional points P_i, P_j. Then its inverse image is ....

Line	Inverse image	$\dim \psi^{-1}([\text{line}])$
[01]	$(\lambda x_0^2 : \lambda^2 : \mu x_0^2, 0 : 1 : 0)$	2

Proposition 1.1. *The variety X is naturally isomorphic to the projective spectrum of the semi group ring*

$$S = k[x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_0].$$

That is, $X = \operatorname{Proj}(S)$. This gives a coordinate-independent description of X.

Proof. The kernel of the homomorphism

$$k[y_1, \cdots, y_6] \to k[x_0, \cdots, x_6]$$

sending y_i to $x_i x_{i+1}$ (modulo 6), is precisely $I = (x_0x_2x_4 - x_1x_3x_5)$. $\square$

Observation: In the same manner, we can identify $\mathbb{P}^2 \times \mathbb{P}^2$ with

$$\mathrm{Proj}(k[x_0x_1, x_0x_3, x_0x_5, \cdots, x_4x_1, x_4x_3, x_4x_5])$$

This mean that we have a rational morphism $\psi : \mathbb{P}^2 \to \mathbb{P}^2 \dashrightarrow X$.

Proposition 1.2. *The variety X is a Pfaffian hypersurface. In particular, it is given as det radical of the determinant of the* 6×6*-matrix*

$$\begin{pmatrix} 0 & x_0 & 0 & 0 & 0 & -x_5 \\ -x_0 & 0 & x_3 & 0 & 0 & 0 \\ 0 & -x_3 & 0 & x_4 & 0 & 0 \\ 0 & 0 & -x_4 & 0 & x_1 & 0 \\ 0 & 0 & 0 & -x_1 & 0 & x_2 \\ x_5 & 0 & 0 & 0 & -x_2 & 0 \end{pmatrix}$$

2 The Fano variety $F_1(X)$

The variety of lines contained in X is the Fano variety $F_1(X)$, lying inside the Grassmannian $\mathbb{G}(2,6)$. It is reducible and consists of 15 irreducible components. Nine of them are copies of $\mathbb{G}(2,4)$, and the six others are varieties defined by binomials, i.e. toric varieties.

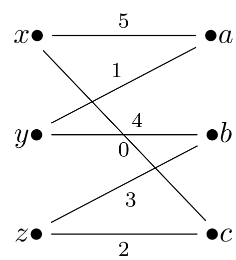
Using the map from the previous section and noting that it is linear, we get a map into the Fano variety:

$$\begin{aligned} \Xi : (\mathbb{P}^2)^\vee \times \mathbb{P}^2 &\dashrightarrow F_1(X) \subseteq \mathbb{G}(2,6) \\ \ell \times p &\mapsto \ell_p := [\psi(x,p), \psi(y,p)]_{x,y \in \ell} \end{aligned}$$

Here ψ is the map below:

$$\begin{aligned} \psi : \mathbb{P}^2 \times \mathbb{P}^2 &\dashrightarrow X \\ (x,y,z) \times (a,b,c) &\mapsto (ay, by, bz, cz, cx, ax) \end{aligned}$$

The map can be visualized with the following figure:



To get the Plücker coordinates of the map, one forms the 2×6-matrix having as rows the value of ψ at two generic points, and then one takes maximal minors. These are the Plücker coordinates. Since the image of ψ lies inside X, the image of Ξ lies inside $F_1(X)$.

The nine other components are isomorphic to $\mathbb{G}(2, 4)$. They occur when one odd and one even coordinate is zero.

3 Connection to algebraic statistics

The ideal I is an example of an ideal arising from permutation matrices. For details, see [1, page 56] and [2, page 148].

References

[1] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[2] Bernd Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.