

# Representation theory

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## Abstract

These are notes from the course MAT4270 on representation theory, autumn 2015. The lectures were held by Sergey Neshyevy. The notes are mine, as well as any mistakes. The first half is about representations of finite groups, the second half about representations of compact Lie groups.

## 1 Introduction to representation theory

In the first part,  $G$  is always a *finite group* and  $V$  a finite-dimensional vector space over the complex numbers. Most results will hold over any algebraically closed field of characteristic zero.

### 1.1 Motivating example

In 1896 Dedekind made the following observation. Let  $\mathbb{C}[x_g \mid g \in G]$  be the free  $\mathbb{C}$ -algebra with basis the elements of  $G$ . Then one can form the following polynomial:

$$P_G(x_{\rho_1}, \dots, x_{\rho_n}) = \det \left( (x_{\rho_i g_j})_{i,j=1}^n \right).$$

Note that the matrix is just the multiplication table for the group  $G$  if we identify the variable  $x_g$  with the group element  $g$ . The problem is to decompose  $P_G$  into irreducible polynomials.

**Example 1.1.** Let  $G = S_3$ . Then  $G$  is generated by two elements  $r$  and  $s$  with the relation  $srs = r^2$ . If we write the elements of  $G$  as

$\{e, r, r^2, s, sr, sr^2\}$ , then the multiplication table looks like

$$\begin{pmatrix} e & r & r^2 & s & sr & sr^2 \\ r & r^2 & e & sr^2 & s & sr \\ r^2 & e & r & sr & sr^2 & s \\ s & sr & sr^2 & e & r & r^2 \\ sr & sr^2 & s & r^2 & e & r \\ sr^2 & s & sr & r & r^2 & e \end{pmatrix}.$$

The determinant is quite long, so I won't write it out. Suffice it to say it is a degree 6 polynomial with 146 terms. Using a computer algebra system such as **Macaulay2**, one can decompose it. It decomposes into three factors: one quadratic and two linear. They are the following (we change notation to avoid confusion about exponents):

$$x_e + x_r + x_{r^2} + x_s + x_{sr} + x_{sr^2}$$

and

$$x_e + x_r + x_{r^2} - x_s - x_{sr} - x_{sr^2}$$

and

$$x_e^2 - x_e x_r + x_{r^2} - x_e x_{r^2} - x_r x_{r^2} + x_{r^2}^2 - x_s^2 + x_s x_{sr} - x_{sr^2}^2 + x_s x_{sr^2} - x_{sr^2}^2 - x_{sr^2}^2.$$

We will see later that the first factor corresponds to the trivial representation, the second to the alternating representation, and the third is the so-called standard representation of  $G$ . ★

Frobenius proved the following theorem:

**Theorem 1.2.** *Let  $P_G = \prod_{i=1}^r P_i^{m_i}$  be the decomposition of  $P_G$  into irreducibles. Then*

1.  $m_i = \deg P_i$  for every  $i$ .
2.  $r$  is the number of conjugacy classes in  $G$ .

*In particular, since  $r = |G|$  if and only if  $G$  is abelian,  $P_G$  decomposes into linear factors if and only if  $G$  is abelian.*

In trying to prove the above theorem, Frobenius basically had to develop representation theory.

## 1.2 Representations

A **representation of  $G$  on  $V$**  is a homomorphism  $\pi : G \rightarrow \text{GL}(V)$ . We will denote a representation interchangeably by  $\pi, (\pi, V), V_\pi$  or  $V$ ,

depending upon the situation. We sometimes also say that  $V$  is a  **$G$ -module**.

Thus a representation is a *linear* action of  $G$  on  $V$ , so that the action of  $G$  on  $V$  always can be represented as some group of matrices. This matrix group is isomorphic to  $G$  if the action is **faithful**.

**Example 1.3.** The **trivial representation**  $\epsilon$  of any group  $G$  is given by letting  $V = \mathbb{C}$  and  $\epsilon(g) = e$  for any  $g \in G$ . ★

**Example 1.4.** We define the **left regular representation**. Let  $\mathbb{C}[G]$  be the space of functions on  $G$ . It is a finite-dimensional vector space under pointwise addition. Define  $\lambda : G \rightarrow \text{GL}(\mathbb{C}[G])$  by

$$g \mapsto (f \mapsto (h \mapsto f(g^{-1}h)))$$

This vector space have a basis consisting of the characteristic functions  $e_g$  for  $g \in G$ . On these elements one sees that the action is given by  $g \cdot e_h = e_{gh}$ .

Thus any group have at least one non-trivial representation. ★

**Example 1.5.** Similarly, one has the **permutation representation**. Let  $X$  be a  $G$ -set, that is, a set on which  $G$  act by permutations. Then one forms the space  $\mathbb{C}[X]$  of  $\mathbb{C}$ -valued functions on  $X$ . It has a basis  $e_x$  of characteristic functions, and the action of  $G$  on  $\mathbb{C}[X]$  is given by  $g e_x = e_{gx}$ . ★

As in any reasonable category, there is a notion of *isomorphism* of representations. An **equivalence** of representations  $(\pi, V)$  and  $(\theta, W)$  is given by a vector space isomorphism  $T : V \rightarrow W$  such that  $T(g \cdot v) = g \cdot T(v)$  for all  $g \in G$ .

A map  $T : V \rightarrow W$  satisfying the last condition above is called an **intertwiner** of  $V$  and  $W$ . The set of intertwiners is denoted by  $\text{Mor}(\pi, \theta)$  or  $\text{Hom}_G(V, W)$ .

A subspace  $W \subset V$  is **invariant** if  $g \cdot W \subset W$  for all  $g \in G$ . Letting  $\theta(g) = \pi(g)|_W$ , we get another representation of  $G$ , called a **subrepresentation** of  $G$ . We write  $\pi|_W$  for  $\theta$ .

If we have two representation  $(\pi, V)$  and  $(\pi', V')$ , we can form the **direct sum representation** by letting  $g \in G$  act on  $V \oplus V'$  componentwise. Note that  $\pi$  is a subrepresentation of  $V \oplus V'$ .

The following proposition has important consequences. Note that the proof works for any field of characteristic not dividing  $|G|$ .

**Proposition 1.6** (Mascke's theorem). *Let  $(\pi, V)$  be a representation of  $G$  and  $W \subset V$  an invariant subspace. Then there exists a complementary invariant subspace  $W^\perp$ . That is,  $W$  is also invariant and such that  $V = W \oplus W^\perp$ .*

*Proof.* We prove this by “averaging”, a process we will do again and again. Let  $P : V \rightarrow W$  be *any* projection from  $V$  to  $W$ . Then define

$$\Pi(v) = \frac{1}{|G|} \sum_{g \in G} gP(g^{-1}v).$$

This is a  $G$ -linear morphism, because

$$\begin{aligned} \Pi(h \cdot v) &= \frac{1}{|G|} \sum_{g \in G} gP(g^{-1}h \cdot v) \\ &= \frac{1}{|G|} \sum_{g \in G} gP((h^{-1}g) \cdot v) \\ &= \frac{h}{|G|} \sum_{g \in G} h^{-1}gP((h^{-1}g) \cdot v) = h \cdot \Pi(v). \end{aligned}$$

It is also surjective, since if  $v \in W$ , then clearly  $\Pi(v) = v$ . Hence  $\Pi$  is a  $G$ -linear surjection onto  $W$ . Then it is clear that  $\ker \Pi$  is a  $G$ -invariant subspace of  $V$  complementary to  $W$ .  $\square$

We say that a representation is **irreducible** if it has no proper invariant subspaces. A representation is **completely reducible** if it decomposes into a direct sum of irreducible representations.

**Example 1.7.** Let  $G = (\mathbb{R}, +)$  act on  $\mathbb{R}^2$  by the matrices

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

This leaves the  $x$ -axis fixed, so the representation is not irreducible. But it does not split into irreducible representations, since there is no complementary subspaces.  $\star$

**Example 1.8.** The same example with  $G = (\mathbb{F}_p, +)$  and  $V = \mathbb{F}_p^2$  works. Again, the space spanned by the vector  $(1, 0)^T$  is invariant, so  $V$  is not irreducible. The same computation as in the previous example shows that there are no complementary invariant subspaces.  $\star$

**Example 1.9.** Let  $G = \mathbb{Z}/2$  act on  $V = \mathbb{F}_p^2$  by  $(x, y) \mapsto (y, x)$ . Then the diagonal  $\Delta$  is an invariant subspace. Letting  $P : \mathbb{F}_p^2 \rightarrow \mathbb{F}_p^2$  be the map  $(x, y) \mapsto (x, x)$  be a projection onto  $\Delta$ , going through the proof, we see that

$$\Pi((x, y)) = \frac{1}{2} ((x, x) + (y, y)) = \frac{1}{2} (x + y, x + y)$$

is an intertwiner. The kernel is spanned by  $(1, -1)^T$  if  $p \neq 2$ , so we see that  $V$  is completely reducible if and only if  $p \neq 2$ .  $\star$

However, we don't worry, because we work over  $\mathbb{C}$ . Here things are nicer:

**Theorem 1.10.** *Any finitely-dimensional representation of a finite group  $G$  is completely reducible.*

*Proof.* The proof is a direct consequence of Maschke's theorem (1.6). Just inductively find invariant subspaces. Since the representation is finite-dimensional, the process must stop.  $\square$

We next prove Schur's lemma, which although it has a very simple proof, has important consequences.

**Proposition 1.11** (Schur's lemma). *Let  $(V, \pi)$  and  $(W, \theta)$  be irreducible representations. Then*

1. *If  $\pi \not\sim \theta$ , then  $\text{Hom}_G(V, W) = 0$ .*
2. *If  $\pi \sim \theta$ , then  $\text{Hom}_G(V, W)$  is 1-dimensional.*

*Proof.* Both the kernel and the image of a morphism of representations is a representation. But since  $V$  and  $W$  are irreducible, it follows that  $\ker \varphi = 0$  or  $\ker \varphi = V$  for any  $\varphi : V \rightarrow W$ . This proves i).

Now suppose  $\Pi : V \rightarrow W$  is an equivalence of representations. Let  $\varphi \in \text{Hom}_G(V, W)$  be non-zero. Then  $\varphi$  must be an equivalence as well, since the neither the kernel nor image can be non-zero. Consider  $T = \varphi^{-1} \circ \Pi : V \rightarrow V$ . This is a linear map, and since we work over  $\mathbb{C}$ , it has an eigenvalue  $\lambda$ . Now consider  $T - \lambda \cdot \text{id}_V$ . This has non-zero invariant kernel, hence  $T - \lambda \text{id}_V = 0$ .  $\square$

**Proposition 1.12.** *If  $G$  is abelian, then any irreducible representation of  $G$  is 1-dimensional.*

*Proof.* Each  $g \in G$  gives a map  $\pi(g) : V \rightarrow V \in \text{End}(V)$ . This is an intertwining map, because  $\pi(g)\pi(h) = \pi(gh) = \pi(hg) = \pi(h)\pi(g)$ .

If  $\pi$  is not trivial, by Schur's lemma, this map is just multiplication by some constant. But any 1-dimensional subspace of  $V$  is invariant under this action, so  $V$  must be 1-dimensional in order to be irreducible.  $\square$

Thus, up to equivalence, every irreducible representation of a finite abelian group is just a homomorphism  $\chi : G \rightarrow \mathbb{C}$ . But since  $G$  is finite, this actually maps into  $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ . Such maps are called **characters** of  $G$ .

Denote by  $\hat{G}$  the set of irreducible representations of  $G$ . We call  $\hat{G}$  the **Pontryagin dual** of  $G$ . It is an abelian group under pointwise multiplication.

**Remark.** *If  $G$  is non-abelian, there will always exist irreducible representations of dimension  $\geq 2$ . This follows because the regular representation  $G \rightarrow \text{GL}(\mathbb{C}[G])$  is injective.*

## 2 Density and orthogonality

### 2.1 Density theorems

Let  $(V, \pi)$  be an irreducible representation. The set of operators  $\pi(g)$  for  $g \in G$  is quite large. By definition, if  $x \neq 0$ , then

$$V = \text{Span}\{\pi(g)x \mid g \in G\}.$$

By Schur's lemma, we also know that

$$\text{End}(\pi) = \{T \in \text{End}(V) \mid T\pi(g) = \pi(g)T \forall g \in G\} = \mathbb{C} \cdot \text{id}_V.$$

**Remark.** Over any field,  $\text{End}(\pi)$  is a division algebra over that field. But if the field is algebraically closed, any division algebra is isomorphic to the field itself.

**Theorem 2.1** (The density theorem). Assume  $\pi_1, \dots, \pi_n$  are pairwise inequivalent irreducible representations, with  $V_i = V_{\pi_i}$ . Consider the direct sum representation  $V_1 \oplus \dots \oplus V_n$ .

Then

$$\text{Span } \pi(G) = \text{End}(V_1) \oplus \dots \oplus \text{End}(V_n).$$

*Proof.* Read the proof of Theorem 2.2 first. First break  $(V, \pi)$  into irreducible representations  $V_i$ . It follows directly from Schur's lemma that  $\pi(G)' = \mathbb{C}\text{id}_{V_1} \oplus \dots \oplus \mathbb{C}\text{id}_{V_n}$ .

Now, the elements of  $\mathbb{C}\text{id}_{V_1} \oplus \dots \oplus \mathbb{C}\text{id}_{V_n}$  are block diagonal scalar matrices. It is an easy exercise with matrices to see that the commutator of this set is just

$$\text{End}(V_1) \oplus \dots \oplus \text{End}(V_n).$$

This proves the theorem. □

First we need some notation. Let  $V$  be some vector space and  $X \subset \text{End}(V)$  a subset of the endomorphisms of  $V$ . Let

$$X' \triangleq \{T \in \text{End}(V) \mid TS = ST \text{ for all } S \in X\}.$$

We call  $X'$  the **commutant** of  $X$ . Thus if  $(V, \pi)$  is a representation, then by definition:

$$\text{End}(\pi) = \pi(G)'.$$

**Theorem 2.2.** For any finite-dimensional representation  $(V, \pi)$  we have

$$\pi(G)'' = \text{Span } \pi(G).$$

*Proof.* The inclusion  $\supseteq$  is clear, since  $\pi(G)'' = \text{End}(\pi)'$ , and we have  $\pi(g)S(v) = \pi(g)(Sv) = S(\pi(g)v)$  for all  $S \in \text{End}(\pi)$ .

For the other inclusion, let  $T \in \pi(G)''$ . Let  $x_1, \dots, x_n$  be a basis of  $V$ . Consider the representation  $\theta = \pi \oplus \dots \oplus \pi$  (n times) on  $W = \oplus_{i=1}^n V$ . Let  $S$  be the operator  $T \oplus \dots \oplus T$  on  $W$ .

Then  $S \in \theta(G)''$ , because

$$\begin{aligned} (S\theta)(v) &= S(\theta \cdot (v_1, \dots, v_n)) \\ &= S(\pi(v_1), \dots, \pi(v_n)) \\ &= (T\pi(v_1), \dots, T\pi(v_n)) \\ &= (\pi(Tv_1), \dots, \pi(Tv_n)) \\ &= \theta \cdot (Tv_1, \dots, Tv_n) \\ &= \theta Sv. \end{aligned}$$

Let  $x = (x_1, \dots, x_n) \in W$ . Then  $\text{Span } \theta(G)x$  is an invariant subspace of  $W$ . Then there exists an invariant complementary subspace. Let  $P : W \rightarrow \text{Span } \theta(G)x$  be the projection with kernel that invariant subspace, so that  $P \in \theta(G)'$  (that is,  $P$  is a  $G$ -morphism). Then

$$PSx = SPx = Sx$$

since  $Px \in \text{Span } \theta(G)x$ . Hence  $Sx \in \text{Span } \theta(G)x$  as well. This means that there exist  $\alpha_g \in \mathbb{C}$  such that

$$Sx = \sum_g \alpha_g \theta(g)x.$$

This is the same as saying that

$$Tx_i = \sum_g \alpha_g \pi(g)x_i,$$

hence  $T = \sum \alpha_g \pi(g)$ , so  $T \in \text{Span } \pi(G)$ .  $\square$

**Remark.** Note that any irreducible representation is contained in the regular representation. Indeed, let  $(V, \pi)$  be irreducible, and pick some non-zero  $x \in V$ . Then the map

$$\mathbb{C}[G] \xrightarrow{\cdot x} V$$

must be surjective, and by for example Schur's lemma it has a section. Thus the density theorem says that

$$|G|^2 = \sum_{\pi \in \widehat{G}} |V_\pi|^2.$$

In particular, there are only finitely many isomorphism classes of representations of  $G$ .



## 2.2 Orthogonality relations

Let  $(V, \pi)$  be a representation and let  $\rho \in V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ . The function  $\alpha_{\rho, x}^{\pi}$  on  $G$  defined by  $\alpha_{\rho, x}^{\pi}(g) = \rho(\pi(g)x)$  is called a **matrix coefficient** of  $\pi$ . If we fix a basis  $e_1, \dots, e_n$  of  $V$  and consider the dual basis  $e_1^*, \dots, e_n^*$  of  $V^*$ , then we write  $\alpha_{ij}^{\pi}$  instead of  $\alpha_{e_i^*, e_j}^{\pi}$ .

In the basis  $e_1, \dots, e_n$  we have

$$\pi(g) = (\alpha_{ij}^{\pi}(g))_{ij} \in GL(V).$$

The following theorem is usually called the **orthogonality relations**:

**Theorem 2.3.** *We have:*

1. *If  $\pi$  and  $\theta$  are inequivalent irreducible representations, then*

$$\frac{1}{|G|} \sum_{g \in G} \alpha_{f, x}^{\pi}(g) \alpha_{\rho, y}^{\theta}(g^{-1}) = 0$$

*for all  $f \in V_{\pi}^*$ ,  $\rho \in V_{\theta}^*$ ,  $x \in V_{\pi}$  and  $y \in V_{\theta}$ .*

2. *If  $\pi$  is irreducible, then*

$$\frac{1}{|G|} \sum_g \alpha_{f, x}^{\pi}(g) \alpha_{\rho, y}^{\pi}(g^{-1}) = \frac{f(y)\rho(x)}{\dim \pi}.$$

*Proof.* It is the averaging trick again. First observe that if  $T : V_{\theta} \rightarrow V_{\pi}$  is any linear operator, then

$$S = \frac{1}{|G|} \sum_{g \in G} \pi(g) T \theta(g^{-1})$$

is in  $\text{Hom}_G(\theta, \pi)$  by averaging. Thus, since  $\pi$  and  $\theta$  are inequivalent, it follows from Schur's lemma that this must be zero.

Now let  $T(v) = \rho(v)x$ . This is a linear operator. Then

$$\begin{aligned} 0 = f(Sy) &= f\left(\frac{1}{|G|} \sum_{g \in G} \pi(g) T \theta(g^{-1})(y)\right) \\ &= \frac{1}{|G|} \sum_{g \in G} f(\pi(g) \rho(\theta(g^{-1})y)x) \\ &= \frac{1}{|G|} \sum_{g \in G} f(\pi(g)x) \rho(\theta(g^{-1})y) \\ &= \frac{1}{|G|} \sum_{g \in G} \alpha_{f, x}^{\pi}(g) \alpha_{\rho, y}^{\theta}(g^{-1}) \end{aligned}$$

This proves part 1.

Now suppose  $\pi = \theta$ . Then  $S \in \text{End}(\pi) = \mathbb{C} \cdot \text{id}_V$ . Thus  $S = \alpha \cdot \text{id}_V$  for some  $\alpha \in \mathbb{C}$ . Taking traces on both sides, we get  $\alpha = \text{Tr } T / \dim \pi$  (note that  $T$  and  $S$  have the same trace).

Then

$$\frac{1}{|G|} \sum_g \alpha_{f,x}^\pi(g) \alpha_{\rho,y}^\pi(g^{-1}) = f(S(y)) = f(y) \frac{\text{Tr } T}{\dim \pi} = \frac{f(y) \rho(x)}{\dim \pi},$$

since  $\text{Tr } T = \rho(x)$ .  $\square$

Let  $\mathbb{C}[G]$  denote the vector space of functions  $G \rightarrow \mathbb{C}$ . Instead of writing expressions like  $\sum_g \alpha_g \pi(g)$  it is convenient to introduce the following notation. Define the **convolution** of two functions  $f_1, f_2 \in \mathbb{C}[G]$  by

$$(f_1 * f_2)(g) = \sum_{g=g_1 g_2} f_1(g_1) f_2(g_2) = \sum_{h \in G} f_1(h) f_2(h^{-1}g).$$

We have  $\delta_{g_1} * \delta_{g_2} = \delta_{g_1 g_2}$  (easy check!). The convolution product is associative and makes  $\mathbb{C}[G]$  into an algebra, called the **group algebra** of  $G$ .

Note that with this product,  $\mathbb{C}[G]$  could just as well have been defined as the associative algebra  $\mathbb{C}\langle x_g \mid g \in G \rangle$  module the relations defined by  $x_g x_h = x_{gh}$ .

Given a representation  $\pi : G \rightarrow \text{GL}(V)$  we can define an algebra homomorphism  $\mathbb{C}[G] \rightarrow \text{End}(V)$  by  $\pi(f) = \sum_{g \in G} f(g) \pi(g)$ . Thus  $V$  becomes a left  $\mathbb{C}[G]$ -module. Conversely, given any unital algebra homomorphism  $\pi : \mathbb{C}[G] \rightarrow \text{End}(V)$ , we get a representation by  $g \mapsto \pi(\delta_g)$ . Thus we have a correspondence between representations of  $G$  and left  $\mathbb{C}[G]$ -modules. Using these notions, the Density Theorem (Theorem 2.1) becomes  $\pi(\mathbb{C}[G]) = \oplus_i \text{End}(V_i)$ .

Formalizing Remark , we get:

**Theorem 2.4.** *Let  $\widehat{G}$  be the set of equivalence classes of irreducible representations of  $G$ . Then  $\widehat{G}$  is finite and  $\sum_{[\pi] \in \widehat{G}} (\dim \pi)^2 = |G|$ . Furthermore, we have*

$$\lambda \sim \bigoplus_{[\pi] \in \widehat{G}} \pi^{\oplus \dim \pi}.$$

Here  $\lambda$  is the left regular representation of  $G$ .

*Proof.* The first two statements are already proved in the Remark. The only thing to be proved is the part about the multiplicity of  $\pi$  in  $\lambda$ .

Note that as a representation,  $\mathbb{C}[G]$  is just the left regular representation of  $G$ . Note also that  $\text{End } V_i \simeq V^{\dim V}$ : let  $\{v_1, \dots, v_n\}$  be a basis of  $V$ . Then the map  $\text{End}(V_i) \ni \varphi \mapsto (\varphi(v_1), \dots, \varphi(v_n))$  is an isomorphism of representations.

Then the density theorem implies the statement.  $\square$

**Example 2.5.** Let  $G = S_3$ , the symmetric group on 3 elements. We have that  $|G| = 6$ . We have two one-dimensional representations: the trivial representation and the sign representation  $g \mapsto \text{sign}(g) \in \{\pm 1\}$ . Hence  $6 = 1^2 + 1^2 + n_1^2 + n_2^2 + \dots$ , where  $n_i$  is the dimension of the  $i$ th irreducible representation. This is only possible if  $n_1 = 2$  and  $n_i = 0$  for  $i > 1$ . Thus there are only three irreducible representations of  $S_3$ .

Let  $G = \langle \rho, \sigma \rangle$  with  $\rho^3 = e$ ,  $\sigma^2 = e$  and  $\sigma\rho\sigma = \rho^{-1}$ . The unique irreducible 2-dimensional representation of  $S_2$  can be realized as the action on  $\mathbb{C}^2$  given by

$$\rho \mapsto \begin{pmatrix} \sin\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \cos\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix}$$

and

$$\sigma \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The density theorem here says that

$$\mathbb{C}[G] = e\mathbb{C} \oplus \dots$$

[[sett inn dekomposisjon av  $\mathbb{C}[G]$  som algebra]]

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We will now motivate the name "orthogonality relations". First of all, note that if  $V_\pi$  is a representation of a finite group  $G$ , and  $\langle, \rangle'$  is any Hermitian scalar product on  $V_\pi$ , then

$$\langle v, w \rangle := \frac{1}{|G|} \sum_{g \in G} \langle \pi(g)v, \pi(g)w \rangle'$$

is another Hermitian scalar product on  $V_\pi$ , making the representation **unitary**, meaning that the inner product is invariant under the action of the group.

If  $\pi$  is any unitary representation, choose an orthonormal basis  $e_1, \dots, e_n \in V$ . Then the matrices  $\pi(g) = (a_{ij}^\pi(g))$  are unitary matrices. Since for unitary matrices  $U$  we have  $U^{-1} = U^*$  (conjugate transpose), we have  $\overline{a_{ij}^\pi(g)} = a_{ji}^\pi(g^{-1})$ .

There is a standard scalar product on  $\mathbb{C}[G]$  defined by

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

Then the orthogonality relations take the form:

**Theorem 2.6.** *For every irreducible representation  $\pi$  choose an invariant scalar product and an orthonormal basis  $\{e_1^\pi, \dots, e_n^{\dim \pi}\}$  and corresponding matrix coefficients  $a_{ij}^\pi$ . Then the functions  $a_{ij}^\pi$  are mutually orthogonal with*

$$(a_{ij}^\pi, a_{ij}^\pi) = \frac{1}{\dim \pi}.$$

*Proof.* This is just a reformulation of the Theorem.  $\square$

Let  $(V, \pi)$  be an irreducible representation of  $G$  and  $(W, \theta)$  any finite-dimensional representation. Denote by  $W(\pi)$  the space spanned by elements  $Tx$  for  $T \in \text{Mor}(\pi, \theta)$ . It is an invariant subspace of  $W$ , called the **isotypic component** of  $\theta$  corresponding to  $\pi$ .

**Lemma 2.7.** *The operator  $P : W \rightarrow W$  defined by*

$$P = \frac{\dim \pi}{|G|} \sum_{g \in G} \chi_\pi(g^{-1}) \theta(g),$$

where  $\chi_\pi(g) = \text{Tr } \pi(g)$ , is a projection onto  $W(\pi)$  along  $\sum_{[\pi'] \in \widehat{G} \setminus \{[\pi]\}} W(\pi')$ . In particular,

$$W = \bigoplus_{[\pi] \in \widehat{G}} W(\pi)$$

and

$$\theta|_{W(\pi)} \sim \pi^{\oplus n_\pi},$$

where  $n_\pi$  is the multiplicity of  $\pi$  in  $\theta$ .

*Proof.* For now, see Theorem 8, page 2.6 in [3].  $\square$

## 2.3 Decomposition of the regular representation

We want to decompose the left regular representation  $\lambda$  into irreducibles.

It is sometimes more convenient to work with  $\rho$ , the right regular representation. Note that the map  $\mathbb{C}[G] \rightarrow \mathbb{C}[G]$  defined by  $\delta_g \mapsto \delta_{g^{-1}}$  defines an equivalence (check!)<sup>1</sup>.

The following theorem was already proved using the density theorem, but we now get a more explicit proof.

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<sup>1</sup>Recall that the right regular representation is defined by  $(f \cdot h)(k) = f(hk)$

**Theorem 2.8.** *Every irreducible representation  $\pi$  of  $G$  has multiplicity  $\dim \pi$  in the regular representation, so*

$$\lambda \sim \rho \sim \bigoplus_{[\pi] \in \widehat{G}} \pi^{\oplus \dim \pi}.$$

*Proof.* Fix a basis  $v_1, \dots, v_n$  of  $V_\pi$ . In matrix form the identity  $\rho(gh) = \rho(g)\rho(h)$  takes the form

$$a_{ij}^\pi(gh) = \sum_k a_{ik}^\pi(g) a_{kj}^\pi(h).$$

Or equivalently,

$$\rho(h) a_{ij}^\pi = \sum_k a_{kj}^\pi(h) a_{ik}^\pi.$$

Therefore, for every  $i$ , the map

$$V_\pi \rightarrow \mathbb{C}[G], \quad e_j \mapsto a_{ij}^\pi \quad (1 \leq j \leq n)$$

is an intertwiner between  $\pi$  and  $\rho$ .

Assume now that  $V \subset \mathbb{C}[G]$  is a  $\rho$ -invariant subspace such that  $\rho|_V \sim \pi$ . Choose a basis  $f_1, \dots, f_n$  on  $V$  such that

$$\rho(h) f_j = \sum_k a_{kj}^\pi(h) f_k,$$

so

$$f_j(gh) = \sum_k a_{kj}^\pi(h) f_k(g).$$

Thus

$$f_j = \sum_k f_k(e) a_{kj}^\pi$$

by setting  $g = e$ .

We thus conclude that the isotypic component corresponding to  $\pi$  equals  $\text{Span}\{a_{ij}^\pi\}_{i,j=1}^n$ . The functions  $a_{ij}^\pi$  are linearly independent by the orthogonality relations, hence their span have dimension  $n^2$ , and the multiplicity of  $\pi$  in  $\rho$  is  $n$ .  $\square$

**Corollary 2.9.** *There are only finitely many equivalence classes of representations of  $G$ , and*

$$\sum_{[\pi] \in \widehat{G}} (\dim \pi)^2 = |G|.$$

### 3 Character theory

Let  $(\pi, V)$  be a representation.

The function  $\chi_\pi : G \rightarrow \mathbb{C}$  defined by  $\text{Tr } \pi(g)$  is called the **character** of  $\pi$ .

We say that a function  $f : G \rightarrow \mathbb{C}$  is **central** if  $f(hgh^{-1}) = f(g)$  for all  $h, g \in G$ . Note that characters are central, by standard properties of traces. Also note that  $\chi_\pi(e) = \dim \pi$ .

**Theorem 3.1.** *The characters  $\chi_\pi$  (for  $[\pi] \in \widehat{G}$ ) form an orthonormal basis for the space of central functions on  $G$ .*

*Proof.* We have

$$\chi_\pi(g) = \sum_{i=1}^{\dim \pi} a_{ii}^\pi(g),$$

and the orthogonality relations show that  $\{\chi_\pi \mid [\pi] \in \widehat{G}\}$  is an orthonormal system:

$$(\chi_\pi, \chi_\pi) = \sum_{ij}^{\dim \pi} (a_{ii}^\pi, a_{jj}^\pi) = \sum_{i=1}^{\dim \pi} \frac{1}{\dim \pi} = 1.$$

Thus we need to show that the characters *span* the space of central functions. Consider the projection  $P$  from  $\mathbb{C}[G]$  to the space of central functions defined by

$$f \mapsto (Pf)(g) = \frac{1}{|G|} \sum_{h \in G} f(hgh^{-1}).$$

It is obviously a projection. Now consider  $P(a_{ij})(g)$  (we skip the upper index  $\pi$ ). Then this is equal to

$$\frac{1}{|G|} \sum_{h \in G} a_{ij}(hgh^{-1}) = \frac{1}{|G|} \sum_{h \in G} \sum_k \sum_l a_{ik}(h) a_{kl}(g) a_{lj}(h^{-1}).$$

But after using that the orthogonality relations, this simplifies to

$$\frac{1}{\dim \pi} \delta_{ij} \chi_\pi(g).$$

But the  $a_{ij}^\pi$  constitute a basis of  $\mathbb{C}[G]$ , by the proof of Theorem 2.8.

Thus  $\text{image}(P) = \text{Span}\{\chi_\pi\}$ .  $\square$

The dimension of the space of central functions coincides with the number of conjugacy classes  $c(G)$  in  $G$ . We thus have:

**Corollary 3.2.** *Let  $c(G)$  be the number of conjugacy classes in  $G$ . We have*

$$|\widehat{G}| = c(G).$$

**Corollary 3.3.** *Two finite-dimensional representations  $\pi$  and  $\theta$  are equivalent if and only if  $\chi_\pi = \chi_\theta$ .*

*Proof.* You can read off the multiplicities from the character.  $\square$

**Corollary 3.4.** *For any  $[\pi] \in \widehat{G}$ , the multiplicity of  $\pi$  in the regular representation  $\lambda$  equals  $\dim \pi$ .*

*Proof.* We are interested in  $(\chi_\lambda, \chi_\pi)$ . But note that

$$\chi_\lambda(g) = \sum_{h \in G} \text{Tr}(\delta_h \mapsto \delta_{hg}) = \begin{cases} 0 & \text{if } g \neq e \\ |G| & \text{if } g = e. \end{cases}$$

Then  $(\chi_\lambda, \chi_\pi) = \frac{1}{|G|} \chi_\lambda(e) \cdot \overline{\chi_\pi(e)} = \dim \pi$ .  $\square$

## 4 The Frobenius determinant problem

Let  $G$  be a finite group and  $\mathbb{C}[x_g \mid g \in G]$  the algebra on generators indexed by  $g \in G$ . Then one can form the following determinant:

$$P_G = \det((x_{gh})_{gh}).$$

It is the determinant of the multiplication table of  $G$ . The problem is to decompose  $P_G$  into irreducible polynomials.

It is convenient to instead work with  $\widehat{P}_G = \det((x_{gh^{-1}})_{gh})$ . This has the same determinant as  $P_G$  up to multiplication by  $\pm 1$  since this just corresponds to permuting some of rows of the multiplication table.

Now here comes the *key* observation: the matrix  $(x_{gh^{-1}})$  is the matrix of the operator

$$\sum_{g \in G} x_g \lambda(g)$$

in the basis  $\delta_g \in \mathbb{C}[G]$ . Here you are supposed to view elements of  $\mathbb{C}[x_g \mid g \in G]$  as **functions** on  $G$ .

That this operator actually has matrix  $(x_{gh^{-1}})$  can be seen by applying the operator to basis elements:

$$\left( \sum_{g \in G} x_g \lambda(g) \right) (\delta_h) = \sum_{g \in G} x_g \delta_{gh}$$

by definition of the left regular action. Reparametrizing this last sum by setting  $g' = gh$  we get

$$\sum_{g \in G} x_g \delta_{gh} = \sum_{g' = gh} x_{gh^{-1}} \delta_g,$$

as wanted.

Now, the regular representation decomposes as

$$\lambda \sim \bigoplus_{[\pi] \in \widehat{G}} \pi^{\oplus \dim \pi}$$

Hence the determinant decomposes as well (since the operator is  $G$ -invariant) and we get

$$\widehat{P}_G = \prod_{[\pi] \in \widehat{G}} \det \left( \sum_{g \in G} x_g \pi(g) \right)^{\dim \pi}$$

The question is now of course if we can decompose further. The answer is no:



**Theorem 4.1** (Frobenius). *The polynomials  $\det\left(\sum_{g \in G} x_g \pi(g)\right)$  ( $[\pi] \in \widehat{G}$ ) are irreducible and no two of them are associates.*

We begin with a lemma:

**Lemma 4.2.** *The polynomial  $p = \det(x_{ij})$  is irreducible.*

*Proof.* Let  $X = (x_{ij})$ . Note that since  $p$  is linear in each variable,  $p$  is a sum of monomials with no square terms. Suppose for contradiction that  $\det X = fg$  and that any determinant of a  $(n-1) \times (n-1)$  matrix is irreducible (by induction).

Then one of  $f, g$  must be independent of  $x_{nn}$ , say  $f$ . Then  $g$  is linear in  $x_{nn}$ , hence can be written as  $g = x_{nn}a + b$ , where  $a, b$  are polynomials in which  $x_{nn}$  does not appear. Hence

$$\det X = fg = x_{nn}af + bf.$$

Now let  $Y$  be the square matrix obtained from  $X$  by removing the last row and column. By induction  $\det Y$  is irreducible, and have coefficient  $x_{nn}$  in the expansion of  $\det X$ . Hence  $af = \det Y$ . Hence either  $a = 1$  and  $f = \det Y$  or  $a = \det Y$  and  $f = 1$ . The last option would show that  $\det X$  is irreducible. So assume that  $a = 1$  and  $f = \det Y$ . Then  $\det X = x_{nn} \det Y + b \det Y$ .

This leads to a contradiction: let  $M$  be the matrix  $(\delta_{i, n-i})$ . Then  $\det Y = 0$  but  $\det X \neq 0$ .  $\square$

*Proof of theorem:* Suppose that  $P_\pi = p_1 p_2$ . The polynomials  $p_1, p_2$  are homogeneous. Fix a basis  $e_1, \dots, e_n$  in  $V_\pi$ . Consider the corresponding matrix units  $m_{ij}$  defined by

$$m_{ij}e_k = \delta_{jk}e_i.$$

Since  $\pi(\mathbb{C}[G]) = \text{End}(V_\pi)$  by the density theorem, we can find  $a_{ij}(g) \in \mathbb{C}$  such that

$$m_{ij} = \sum_{g \in G} \alpha_{ij}(g) \pi(g).$$

Consider the ring map  $\mathbb{C}[x_g \mid g \in G] \rightarrow \mathbb{C}[x_{ij}]$  defined by

$$x_g \mapsto \sum_{ij} \alpha_{i,j}(g) x_{ij}.$$

Under this map the operator  $\widehat{P}_G$  becomes

$$\sum_{g \in G} x_g \pi(g) \mapsto \sum_{g \in G} \sum_{i,j} \alpha_{i,j}(g) x_{ij} \pi(g) = \sum_{i,j} m_{ij} x_{ij} = (x_{ij}).$$

Since the determinant is functorial with respect to ring maps, a decomposition  $P_G = p_1 p_2$  would induce a decomposition of  $\det(x_{ij})$ , which was just proved to be impossible.

We still have to check that no two of the determinants are equal (up to associates). But note that  $P_\pi|_{x_g=0, g \neq e} = x_e^{\dim \pi}$ . So we can recover the dimension of  $\pi$  from the determinant. Also note that if we fix  $g \in G$ , then

$$P_\pi|_{x_e=1, x_h=0 \text{ for } h \neq g} = \det(1 + x_g \pi(g)) = 1 + \text{Tr } \pi(g) x_g + h.o.t.$$

so we can recover the trace as well. But a representation is determined by its character, so if the polynomials are equal, then the representations are equivalent. The converse is similar.  $\square$

#### 4.1 Two constructions on representations

Recall that the tensor product of  $V$  and  $W$  have a basis  $v_i \otimes v_j$ . Hence  $\dim_{\mathbb{C}} V \otimes W = \dim_{\mathbb{C}} V \cdot \dim_{\mathbb{C}} W$ .

Assume  $(V, \pi)$  is a finite-dimensional representation and consider  $V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ . Then we define the **contragredient representation**  $\pi^c$  to be the representation of the dual space defined by

$$(\pi(g)f)(v) = f(\pi(g^{-1})v).$$

If  $v_1, \dots, v_n$  is a basis for  $V$  and  $\pi(g) = (a_{ij}(g))$ , then in this basis

$$\pi^c(g) = (a_{ji}(g^{-1}))_{ij}.$$

In particular, the character  $\chi_{\pi^c}(g) = \chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$ .

Also, we can define the **tensor product of two representations**  $\pi$  and  $\theta$  by  $(\pi \otimes \theta)(g)(v \otimes w) = \pi(g)v \otimes \theta(g)w$ .

Also:  $\text{Hom}(U, W \otimes V^*) = \text{Hom}(U \otimes V, W)$ .

Frobenius reciprocity:

$$\text{Mor}(\pi, \eta \otimes \theta^c) \simeq \text{Mor}(\pi \otimes \theta, \eta).$$

Denote by  $R(G)$  the abelian group generated by classes  $[\pi] \in \widehat{G}$  of finite-dimensional representations and relations  $[\pi] + [\pi'] = [\pi \oplus \pi']$ . Then it is an exercise to show that  $R(G)$  is a free abelian group with basis  $[\pi] \in \widehat{G}$ .

Via the tensor product we can form a product on  $R(G)$ , making it into a ring. See the Appendix for an explicit example.

## 5 Dimension of irreducible representations

Recall that

$$\sum_{[\pi] \in \widehat{G}} (\dim \pi)^2 = |G|.$$

In this lecture we will prove the following theorem:

**Theorem 5.1.** *The dimension of any irreducible representation divides the number  $|G/Z(G)|$ , where  $Z(G)$  is the center of  $G$ .*

Later we will strengthen this and show that  $Z(G)$  can be replaced by any normal abelian subgroup of  $G$ .

In order to prove this, we need to introduce some results from commutative algebra. Let  $R$  be a unital commutative ring and  $S \subset R$  a unital subring. Then we say that an element  $a \in R$  is **integral over  $S$**  if it satisfies a monic polynomial  $a^n + s_1 a^{n-1} + \dots + s_{n-1} = 0$  with coefficients in  $S$ .

We say that a complex number integral over  $\mathbb{Z}$  is an **algebraic integer**. An integral domain  $S$  is called **integrally closed** if any element in the fraction field of  $S$  integral over  $S$  is already in  $S$ . It is an easy exercise to see that  $\mathbb{Z}$  is integrally closed (and so is any UFD).

**Lemma 5.2.** *Let  $S \subset R$  be as above. Then  $a \in R$  is integral over  $S$  if and only if the subring  $S[a]$  is a finitely-generated  $S$ -module.*

*Proof.* This is Proposition 5.1 in Atiyah-MacDonald, [1]. □

It is an exercise to see that the set of elements of  $R$  integral over  $S$  actually form a ring. Alternatively, consult Corollary 5.3 in Atiyah-MacDonald.

**Proposition 5.3.** *Assume that  $\pi$  is an irreducible representation. Let  $g \in G$ . Let  $C(g)$  be the conjugacy class of  $G$ . Then*

1. *The number  $\chi_\pi(g)$  is an algebraic integer.*
2. *The number  $\frac{|C(g)|}{\dim \pi} \chi_\pi(g)$  is also an algebraic integer.*

*Proof.* i). Since  $G$  is finite, we have  $g^n = e$  for some  $n$ . Hence all eigenvalues of  $g$  are  $n$ th roots of unity which are algebraic integers. But  $\chi_\pi(g)$  is the sum of the eigenvalues, and the set of algebraic integers form a subring of  $\mathbb{C}$ , hence  $\chi_\pi(g)$  is an algebraic integer as well.

The second part is more difficult. Let  $p$  be the characteristic function of  $C(g)$ , that is the function defined as follows:

$$p(x) = \begin{cases} 1 & \text{if } x \in C(g) \\ 0 & \text{otherwise.} \end{cases}$$

Then  $p$  lies in the center of  $\mathbb{C}[G]$ . Consider the subring  $\mathbb{Z}[G] \subset \mathbb{C}[G]$  (the **group ring** of  $G$ ). Let  $R \subset \mathbb{Z}[G]$  be the subring of *central functions*. Then  $p \in R$ . As  $R$  is a finitely generated abelian group, any element of  $R$  is integral over  $\mathbb{Z} \cdot 1 \subset R$ .

It follows that  $\pi(p)$  is integral over  $\mathbb{Z} \cdot \text{id}_{V_\pi} \subset \text{End}(V_\pi)$ . Since  $p$  is central,  $\pi(p) \in \text{End}_G(V_\pi) = \mathbb{C} \cdot \text{id}_{V_\pi}$ . Hence  $\pi(p) = \alpha \cdot \text{id}_{V_\pi}$  for some complex number  $\alpha$ . This  $\alpha$  is an algebraic integer, since  $\pi(p)$  is integral over  $\mathbb{Z}$ .

But now

$$\begin{aligned} \alpha \cdot \dim \pi &= \text{Tr}(\alpha \cdot \text{id}_{V_\pi}) \\ &= \text{Tr} \pi(p) \\ &= \sum_{h \in C(g)} \text{Tr} \pi(h) \\ &= \sum_{h \in C(g)} \chi_\pi(h) \\ &= |C(g)| \chi_\pi(g). \end{aligned}$$

The last equality is because characters are constant on conjugacy classes. This proves the theorem.  $\square$

**Lemma 5.4.** *If  $\pi$  is irreducible, then  $\dim \pi \mid |G|$ .*

*Proof.* Recall that

$$(\chi_\pi, \chi_\pi) = 1.$$

This can be rewritten as

$$\frac{|G|}{\dim \pi} = \sum_{g \in G} \frac{\chi_\pi(g) \chi_\pi(g^{-1})}{\dim \pi}.$$

Now let  $C(G)$  denote the set of conjugacy classes in  $G$ .  $C$  is then a partition of  $G$ . Then the above sum can be rewritten as

$$\sum_{C \in C(G)} \sum_{h \in C} \frac{\chi_\pi(h) \chi_\pi(h^{-1})}{\dim \pi} = \sum_{C \in C(G)} \frac{|C| \chi_\pi(g_C)}{\dim \pi} \chi_\pi(g_C^{-1}),$$

where  $g_C$  is any representative from  $C$ . The right-hand-side is a sum of products that we know are algebraic integers. They form a ring, hence the sum is an algebraic integer as well.

But the left hand side is a fraction, so we must have  $|G|/\dim \pi \in \mathbb{Z}$ , which is equivalent to what was to be proven.  $\square$

Now we can prove Theorem 5.1. Recall that we want to show that  $\dim \pi$  divides  $|G/Z(G)|$ .

*Proof.* For every  $n \in \mathbb{N}$ , consider the representation of  $G^n = G \times \cdots \times G$  on  $V_\pi \otimes \cdots \otimes V_\pi$  defined by  $\pi_n(g_1, \dots, g_n) = \pi_1(g_1) \otimes \cdots \otimes \pi_n(g_n)$ . This is an irreducible representation, since any elementary tensor  $v_1 \otimes \cdots \otimes v_n$  can be sent to any other elementary tensor by the action of  $G^n$ .

Now consider the subgroup

$$Z_n = \{(g_1, \dots, g_n) \in Z(G)^n \mid \prod g_i = e\}.$$

It is isomorphic to  $Z(G)^{n-1}$ . If  $g \in Z(G)$ , then by Schur's lemma,  $\pi(g)$  is a scalar operator. So if  $(g_1, \dots, g_n) \in Z_n$ , then  $\pi(g_i) = \alpha_i \cdot 1$ , hence

$$\pi_n(g_1, \dots, g_n) = \alpha_1 \cdot \dots \cdot \alpha_n \text{id}_{V^{\otimes n}} = \text{id}_{V^{\otimes n}}$$

since  $g_1 \cdots g_n = e$ . Therefore  $Z_n \subset \ker \pi_n$ . It follows that  $\pi_n$  induces a representation of  $G^n/Z_n$ . Then by the previous lemma

$$(\dim \pi)^n \mid \frac{|G|^n}{|Z(G)|^{n-1}}$$

That is:

$$\left( \frac{|G|}{\dim \pi |Z(G)|} \right)^n \in \frac{1}{|Z(G)|} \mathbb{Z}$$

for any  $n$ . This is true for any  $n$ , hence

$$\mathbb{Z} \left[ \frac{|G|}{\dim \pi |Z(G)|} \right] \subset \frac{1}{|Z(G)|} \mathbb{Z}.$$

This implies that the left-hand-side is a finite-generated  $\mathbb{Z}$ -module, so by Lemma 5.2, the fraction  $|G|/\dim \pi |Z(G)|$  is an algebraic integer. Hence it is an integer. This proves that  $\dim \pi \mid |G|/|Z(G)|$ .  $\square$

## 6 Representation theory of $S_n$

It is easy to understand conjugacy classes in  $S_n$ . Every  $\sigma \in S_n$  defines a partition of  $[n] = \{1, \dots, n\}$  into orbits of  $\sigma$ . Let  $O_1, \dots, O_m$  be these orbits. Assume that they are decreasing in size. Let  $n_i = |O_i|$ . These numbers  $n_i$  sum to  $n$ .

We say that a **partition of  $n$**  is a decreasing sequence of integers  $n_1 \geq \dots \geq n_m \geq 1$  such that  $n = \sum n_i$ .

Thus every  $\sigma \in S_n$  gives a partition of  $n$ . Two elements of  $S_n$  are conjugate if and only if they define the same partition. To see this, note that a permutation is determined by its decomposition into disjoint cycles. If  $\sigma = \sigma_1 \dots \sigma_r$  is a product into disjoint cycles, then  $\rho \sigma \rho^{-1} = \rho \sigma_1 \rho^{-1} \dots \rho \sigma_r \rho^{-1}$  is also a decomposition into disjoint cycles. Hence conjugate elements define the same partition. For the converse, write the partition  $n = n_1 + n_2 \dots + n_m$  as

$$(1, 2, \dots, n_1)(n_1 + 1, \dots, n_1 + n_2) \dots$$

Then renaming the numbers correspond to conjugating.

Our goal is to produce an irreducible representation for every partition of  $n$ .

Recall that for any finite group  $G$ , we have

$$\mathbb{C}[G] \simeq \bigoplus_{[\pi] \in \widehat{G}} \text{End}(V_\pi),$$

the isomorphism sending  $g$  to the corresponding endomorphism of  $V = \bigoplus V_\pi$ . Now one can ask if it is possible to recover the space  $V_\pi$  from its matrix algebra? (essentially, yes)

We need a few definitions. An idempotent  $e$  in an algebra  $A$  is called **minimal** if  $e \neq 0$  and  $eAe = \mathbb{C}e$ .

**Lemma 6.1.** *Let  $V$  be a finite-dimensional vector space. Show that  $e \in \text{End}(V)$  is a minimal idempotent if and only if it is a projecting onto a 1-dimensional subspace  $\mathbb{C}v \subset V$ . Then we have an isomorphism*

$$\text{End}(V)e \simeq V$$

of  $\text{End}(V)$ -modules given by  $Te \mapsto Tv$ .

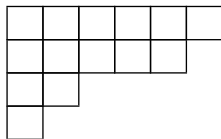
*Proof.* –TO COME–

□

Therefore finding irreducible representations of  $G$  is the same as finding minimal idempotents in the group algebra  $\mathbb{C}[G]$ : if  $e \in \mathbb{C}[G]$  is a minimal idempotent, then  $\mathbb{C}[G]e$  defines an irreducible representation ([WHY IS IT IRREDUCIBLE??]).

Thus, returning to  $S_n$ , we want to construct minimal idempotents in  $\mathbb{C}[S_n]$  for every partition.

It is convenient to present partitions as **Young diagrams**. Given a partition  $(n_1, \dots, n_m)$  of  $n$ , we draw a diagram having  $n_1$  boxes in the first row,  $n_2$  boxes in the second, and so on. For example, for the partition  $(6, 5, 2, 1)$  of 14, we draw the following diagram:



We can also fill these with numbers: a **Young tableau** (plural *tableaux*) is a Young diagram filled with numbers  $i \in [n]$  without repetitions. For example:

|   |   |
|---|---|
| 3 | 1 |
| 2 |   |

Two Young tableaux are said to be of the same **shape** if they arise from the same underlying Young diagram.

Note that the symmetric group act on the set of Young tableaux of the same shape  $\lambda$ .

Fix a Young tableau  $T$ . Let  $R(T) \subset S_n$  be the subgroup of elements of  $S_n$  permuting numbers in the *rows* of  $T$ . Let  $C(T)$  be the same group, but permuting columns instead. Define the following two elements of  $\mathbb{C}[S_n]$ :

$$a_T = \frac{1}{|R(T)|} \sum_{g \in R(T)} g$$

and

$$b_T = \frac{1}{|C(T)|} \sum_{g \in C(T)} \text{sgn}(g) \cdot g.$$

Then we define  $c_T = a_T b_T$ . The element  $c_T$  is called a **Young symmetrizer**. Our main theorem is this:

**Theorem 6.2.** 1. For any  $T$ , the element  $c_T$  is (up to a scalar) a minimal idempotent in  $\mathbb{C}[S_n]$ , so it defines an irreducible representation of  $S_n$ .  
2. If  $T_1, T_2$  are two Young tableaux, then the representations corresponding to them are equivalent if and only if  $T_1$  and  $T_2$  have the same shape.

The modules  $\mathbb{C}[S_n]c_T$  are called **Specht modules**.

*Proof.* TOO LONG FOR NOW. WILL PROB. COME LATER □

## 6.1 Tabloids

A bit more explicitly the Specht modules can be described as follows.

Fix a Young diagram  $\lambda$ . Introduce an equivalence relation on Young tableaux of shape  $\lambda$ :

$$T_1 \sim T_2$$

if  $T_1 = r(T_2)$  for some  $r \in R(T_2)$ . The equivalence class of a Young tableau  $T$  is called a **tabloid** (of shape  $\lambda$ ) and denoted by  $\{T\}$ .

Note that  $S_n$  act on the set of tabloids. Let  $(M_\lambda, S_n)$  be the corresponding permutation representation. Note also that action on the set of tabloids is transitive. Thus the basis of  $M_\lambda$  can be identified with  $S_n/R(T)$ <sup>2</sup>.

Thus we get isomorphisms of  $\mathbb{C}[S_n]$ -modules:

$$\mathbb{C}[S_n]a_T \simeq \mathbb{C}[S_n/R(T)] \simeq M_\lambda.$$

The first isomorphism is given by sending  $ga_T$  to  $\partial_{gR(T)}$  (the delta function), and the second is given by sending a coset  $\partial_{gR(T)}$  to  $\{g(T)\}$ .

Under these isomorphisms, the image of the submodule  $\mathbb{C}[S_n]b_Ta_T \subset \mathbb{C}[S_n]a_T$  is spanned by the elements  $b_{g(T)}\{g(T)\}$ . In other words,  $V_\lambda$  is spanned by  $e_{T'} := b_{T'}\{T'\}$  by all elements of shape  $\lambda$ . This gives us a description of  $V_\lambda \simeq V_T$  only in terms of  $\lambda$ .

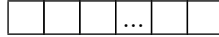
This is a spanning set. We would like a basis. We say that that tableau  $T$  is **standard** if the numbers in every row and column are increasing.

Here's a theorem that we will not prove:

**Theorem 6.3.** *For any Young diagram  $\lambda$  the elements  $e_T$  for all standard tableaux  $T$  of shape  $\lambda$  form a basis in  $V_\lambda$ .*

The complication is of course that  $ge_T = c_{g(T)}$  is a combination of standard tableaux!

**Example 6.4.** Let  $\lambda = n$ , be the trivial partition. Then the corresponding Young diagram looks like:



For any tableau  $T$  of shape  $\lambda$  we get  $C(T) = \{e\}$ , since all the columns are one-element sets. Also,  $R(T) = S_n$ . Hence

$$c_T = \frac{1}{n!} \sum_{g \in S_n} g.$$

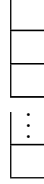
Then  $gc_T = c_Tg = c_T$ . Hence  $\mathbb{C}[S_n]c_T = \mathbb{C}c_T$ , so that  $\pi_T \sim \epsilon$ , the trivial representation. ★

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<sup>2</sup>Every  $\{T\}$  can be written as  $\sigma\{T_0\}$  for some fixed  $T_0$ . But the ambiguity lies in  $R(T)$ .



**Example 6.5.** Let  $\lambda = (1, \dots, 1)$  be the partition  $n = 1 + \dots + 1$ . Then the corresponding Young diagram looks like



Then  $C(T) = S_n$  and  $R(T) = \{e\}$ . Therefore

$$c_T = \frac{1}{n!} \sum_{g \in S_n} \text{sgn}(g) \cdot g.$$

So  $gc_T = c_Tg = \text{sgn}(g)c_T$ . Hence  $\mathbb{C}[S_n] = \mathbb{C}c_T$ , and  $\pi_T$  is the one-dimensional representation  $\text{sgn}$ . ★

## 6.2 Characters and the Hook length formula

Denote by  $\pi_\lambda$  the irreducible representation of the symmetric group corresponding to the Young diagram  $\lambda$ .

To describe the character of  $\pi_\lambda$ , it is convenient to index the conjugacy classes in  $S_n$  by the number of cycles.

Let  $I = (i_1, i_2, \dots)$  consist of non-negative integers such that

$$\sum_{k=1}^{\infty} ki_k = n.$$

Denote by  $c_I$  the conjugacy class in  $S_n$  consisting of elements  $\sigma$  which decompose into  $i_1$  orbits of length 1,  $i_2$  orbits of length two and so on.

This correspond to the partition  $ki_k + (k-1)i_{k-1} + \dots = n$ .

Let  $\chi_{\pi_\lambda}(C_I)$  denote the value of the character of  $\pi_\lambda$  on any representing of the conjugacy class of  $C_I$ . Then

**Theorem 6.6** (Frobenius character formula). *Assume  $\lambda = (\lambda_1, \dots, \lambda_r)$ . Take any number  $N \geq r$ . Then  $\chi_{\pi_\lambda}(C_I)$  is the coefficient of  $\prod_{i=1}^N x_i^{\lambda_i + N - i}$  in the polynomial*

$$\Delta(x) \prod_{k \geq 1} \left( \sum_{i=1}^N (x_i^k) \right)^{i_k}.$$

where  $\Delta(x) = \prod (x_i - x_j)$  is the Vandermonde determinant.

*Proof.* If time allows a proof will appear here. □

Now we quote the **hook length** formula. Let  $\lambda$  be a Young diagram and let  $(i, j)$  be the element in position row  $i$  and column  $j$ . Then the **hook length**  $h(i, j)$  is the number of boxes below and to the right of  $(i, j)$  including  $(i, j)$  itself.

Using the notation in the previous Theorem, let  $l_i = \lambda_i + N - i$  (we put  $\lambda_i = 0$  for  $i > r$ ). Then:

**Lemma 6.7.** *For any  $i$ , we have that*

$$\frac{l_i!}{\prod_{j=i+1}^n (l_i - l_j)} = \prod_{k=1}^{\lambda_i} h(i, k).$$

From this we get:

**Theorem 6.8.** *Let  $\lambda$  be partition and let  $\pi_\lambda$  be the corresponding irreducible representation of  $S_n$ . Then*

$$\dim \pi_\lambda = \frac{n!}{\prod_{(i,j) \in \lambda} h(i, j)}.$$

*This is also the number of standard Young tableaux of shape  $\lambda$ .*

Proof later.

**Example 6.9.** Let



be a Young diagram. Then  $\dim \pi_\lambda = \frac{3!}{1 \cdot 1 \cdot 1 \cdot 3} = 2$ .

★

## 7 Induced representations

Let  $G$  be a finite group and  $H \subset G$  a subgroup. Assume  $(\pi, V)$  is a representation of  $H$ . We want to construct a representation of  $G$  out of this.

We define the **induced representation**  $\text{Ind}_H^G \pi$  to be the  $\mathbb{C}[G]$ -module defined by

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V.$$

It is very useful in constructing representations of larger groups from smaller ones. The idea is that if we know the representations of some subgroup  $H$  of a group  $G$ , then in nice situations all representations of  $G$  can be found as subrepresentations of  $\text{Ind}_H^G \pi$  for various  $[\pi] \in \hat{H}$ .

**Remark.** *This is exactly the concept of extension of scalars from commutative algebra (see e.g. Atiyah-MacDonald page 28). Namely let  $A = \mathbb{C}[H]$  and  $B = \mathbb{C}[G]$  and let  $M = V$  be an  $A$ -module. Then we can form the  $A$ -module  $M_B = B \otimes_A M$ , which carries a natural  $B$ -module structure as well: the action of  $B$  on  $M_B$  is given by  $b(b' \otimes x) = bb' \otimes x$ .*

We have three equivalent descriptions of the induced representation, some of the descriptions being easier to work with than others.

**Theorem 7.1.** *Let  $V$  be a representation of  $H$ . The following representations are isomorphic.*

1.  $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$  where  $G$  acts by  $g \cdot (g' \otimes v) = gg' \otimes v$ .
2.  $\text{Hom}_H(\mathbb{C}[G], V) \simeq \{f : G \rightarrow V \mid f(hg) = hf(g) \forall h \in H, g \in G\}$ , where  $g$  acts by  $(g' \cdot f)(g) = f(gg')$ .
3.  $W = \bigoplus_{\sigma \in G/H} V_\sigma$ , where each  $V_\sigma$  is a copy of  $V$ . Let  $x_\sigma \in V_\sigma$ . Then the action of  $G$  is defined as follows: Let  $r_\sigma \in \sigma$  be a representative for the coset  $\sigma$ . Then  $g$  can be written as  $g = r_\sigma h$  for some  $h \in H$ . Then we define  $g \cdot x_\sigma = h \cdot x_{g\sigma}$ .

*Proof.* Start with 1). We want to show that this is the same as number 3). We will do this in a non-canonical way. The algebra  $\mathbb{C}[H]$  has a vector space basis  $\{e_{h_1}, \dots, e_{h_r}\}$  where  $r = |H|$ . By properties of cosets, this implies that  $\mathbb{C}[G]$  has a basis

$$\{e_{h_1}, \dots, e_{h_r}, e_{r_1 h_1}, \dots, e_{r_1 h_r}, \dots, e_{r_k h_1}, \dots, e_{r_k h_r}\}$$

where  $k = [G : H]$  and the  $r_i$ 's are a set of fixed representatives for the cosets  $G/H$ . This means that we can write

$$\mathbb{C}[G] = \bigoplus_{i=1}^k r_i \mathbb{C}[H].$$

This implies that  $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$  decomposes as

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V = \left( \bigoplus_{i=1}^k r_i \mathbb{C}[H] \right) \otimes_{\mathbb{C}[H]} V \simeq \bigoplus_{i=1}^k (r_i \mathbb{C}[H] \otimes_{\mathbb{C}[H]} V)$$

This is exactly the description in 3), because each  $r_i \mathbb{C}[H] \otimes_{\mathbb{C}[H]} V$  is isomorphic to  $V$  via the map  $r_i h \otimes v \mapsto h^{-1} v$  (we need the inverse in order for this to be a map of representations). The representation  $V$  is endowed with a left action of  $H$  (this is what a representation *is*), but it also has a right action, defined by  $v \cdot g := g^{-1} v$ . In this way,  $V$  becomes a left-right- $H$ -module.

One can check that the action in 3) and the natural  $G$ -action on  $r_i \mathbb{C}[H] \otimes_{\mathbb{C}[H]} V$  are the same.

The equivalence of 1) and 2) can be seen by standard isomorphisms from module theory. Namely, note that

$$\mathrm{Hom}_H(\mathbb{C}[G], V) \simeq \mathrm{Hom}_H(\mathbb{C}[G], \mathbb{C}) \otimes_{\mathbb{C}[H]} V.$$

We also have an identification of representations  $\mathbb{C}[G]$  with  $\mathrm{Hom}(\mathbb{C}[G], \mathbb{C})$  where we let  $\delta_g$  be a basis of  $\mathbb{C}[G]$ . Then the action of  $G$  on  $\mathbb{C}[G]$  by left-multiplication is identified with the contragradient action of  $G$  on  $\mathrm{Hom}(\mathbb{C}[G], \mathbb{C})$ . But all functions  $\mathbb{C}[G] \rightarrow \mathbb{C}$  are  $G$ -invariant (because  $f(gx) = gf(x) = f(x)$  since we are considering the trivial action on  $\mathbb{C}$ ). Hence  $\mathrm{Hom}_H(\mathbb{C}[G], \mathbb{C}) = \mathbb{C}[G]$ . So the two descriptions are equal.  $\square$

We say that the collection  $\{V_\sigma\} \subset W$  is called a **system of imprimitivity** of the representation  $V$ .

**Example 7.2.** Let  $\lambda_H$  be the regular representation of  $H$ . Then  $\lambda_H$  can be identified with  $\mathbb{C}[H]$ . In this case we have  $\mathrm{Ind}_H^G \lambda_H = \lambda_G$  since  $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}[H] \simeq \mathbb{C}[G]$ .  $\star$

Here are some basic properties of induction. Most of them follow from basic properties of tensor products.

1. If  $N \subset M \subset G$  are subgroups, then

$$\mathrm{Ind}_H^G \mathrm{Ind}_N^M \pi \sim \mathrm{Ind}_N^G \pi$$

2. If  $\pi$  is finitedimensional, then

$$(\mathrm{Ind}_H^G \pi)^\vee \simeq \mathrm{Ind}_H^G \pi^\vee.$$

Here  $\pi^\vee$  is the contragradient representation of  $\pi$ .

## 7.1 Frobenius reciprocity

Let  $\text{Rep}(G)$  denote the category of finite-dimensional  $G$ -representations where  $G$  is a finite group and let  $H \subset G$  be a subgroup of  $G$ . We have two functors in each direction: we have a functor  $\text{Res}_H^G$  sending a representation to its restriction to  $H$ , and we have a functor  $\text{Ind}_H^G$  in the other direction, sending a representation of  $H$  to the induced representation on  $G$ . The theorem of this section is that these two functors are adjoints to each other.

But this is a standard fact about tensor products: let  $S \subset R$  be rings (commutative or not) and let  $V$  be an  $S$ -module and  $W$  an  $R$ - $S$ -bimodule. In our case a representation  $W$  of  $G$  is both a  $\mathbb{C}[G]$ -module and a  $\mathbb{C}[H]$ -module by restriction. Then we always have

$$\text{Hom}_R(R \otimes_S V, W) \simeq \text{Hom}_S(V, W)$$

Note that as an  $S$ -module we have  $W = \text{Res}_H^G W$ .

If  $\chi$  is the character of  $\pi$  (a representation of  $H$ ), we denote by  $\text{Ind}_H^G \chi$  the character of the induced representation.

Note that it follows by Schur's lemma and the orthogonality relations that for any two representations, we have

$$(\chi_{\pi_1}, \chi_{\pi_2}) = \dim_{\mathbb{C}} \text{Mor}(\pi_1, \pi_2).$$

Therefore, in terms of characters, Frobenius reciprocity can be stated as

$$(\text{Ind}_H^G \chi_{\pi}, \chi_{\theta}) = (\chi_{\theta}, \text{Res}_H^G \chi_{\theta}).$$

Thus:

**Proposition 7.3.** *Let  $V$  be an irreducible representation of  $H$  and  $W$  an irreducible representation of  $G$ . Then the number of times that  $W$  occurs in  $\text{Ind}_H^G V$  is equal to the number of times that  $\text{Res}_H^G W$  occurs in  $V$ .*

*Proof.* Recall that if  $V$  is an irreducible representation and  $U$  is any representation, then  $(\chi_V, \chi_U)$  is equal to the number of times that  $V$  occurs in  $U$ . Now use the above remarks.  $\square$

## 7.2 Characters of induced representations

**Proposition 7.4.** *Assume  $G \supset H$  and that  $(\pi, V)$  is a finite-dimensional representation of  $H$  and  $\chi$  its character. Then the character  $\text{Ind}_H^G \chi$  of  $\text{Ind}_H^G \pi$  is given by*

$$(\text{Ind } \chi)(g) = \sum_{\bar{s} \in G/H, s^{-1}gs \in H} \chi(s^{-1}gs) = \frac{1}{|H|} \sum_{s \in G | s^{-1}gs \in H} \chi(s^{-1}gs),$$

where we extend  $\chi$  to  $G$  by zero.

*Proof.* Consider the imprimitivity system  $\{W_x\}_{x \in G/H}$  for the induced representation  $\theta = \text{Ind}_H^G \pi$ . Then

$$\begin{aligned}
\chi_\theta(g) &= \text{Tr}(g) \\
&= \sum_{x \in G/H, gx=x} \text{Tr}(\theta(g)|_{W_x}) \\
&= \sum_{\bar{s} \in G/H, s^{-1}gs \in H} \text{Tr}(\theta(g)|_{W_{\bar{s}}}) \\
&= \sum_{\bar{s} \in G/H, s^{-1}gs \in H} \text{Tr}(\theta(s^{-1})\theta(g)\theta(s)|_{W_{\bar{s}}}) \\
&= \sum_{\bar{s} \in G/H, s^{-1}gs \in H} \text{Tr}(\pi(s^{-1}gs)) \\
&= \sum_{\bar{s} \in G/H, s^{-1}gs \in H} \chi(s^{-1}gs).
\end{aligned}$$

The second equality follows by noting that  $\chi(s^{-1}gs)$  is constant on left cosets as  $s$  varies.  $\square$

This makes it easy to compute characters of induced representations. In practice one often induces from a normal subgroup containing few conjugacy classes, so that many of the terms in the sum are zero.

## 8 The Frobenius subgroup theorem

In this section we will prove a theorem of group theory, the proof of which relies heavily upon representation theory. To this day, no “elementary” proof is known.

**Theorem 8.1.** *Assume a finite group  $G$  acts transitively on a set  $X$  such that every  $g \neq e$  has at most one fixed point. Then there exists a unique normal subgroup  $K$  of  $G$  acting freely transitively on  $X$ .*

*Proof.* Put

$$K \triangleq \{e\} \cup \{g \in G \mid g \text{ has no fixed points}\}.$$

We have to show that  $K$  is a normal subgroup and that it acts freely transitively on  $X$ . Once this is done, we are finished, since  $K$  is clearly the largest possible subset of  $G$  satisfying the conclusion of the theorem.

To show that  $K$  acts freely transitively on  $X$ , it suffices to show that  $\#K = \#X$ . Let  $x \in X$ , and consider the stabilizer

$$H = G_x = \{g \in G \mid g \cdot x = x\}.$$

If  $g \notin H$ , by definition  $g \cdot x \neq x$ . Also note that

$$H \cap gHg^{-1} = G_x \cap G_{gx} = \{e\},$$

since otherwise  $h$  in the intersection would act trivially on both  $x$  and  $gx$ , but we assumed that every element acted with at most one fixpoint.

Hence we can write  $G$  as a disjoint union as follows:

$$G = K \coprod \left( \coprod_{y \in X} G_y \setminus \{e\} \right) = K \coprod \left( \coprod_{[g] \in G/H} gHg^{-1} \setminus \{e\} \right).$$

Therefore

$$\#G = \#K + \frac{\#G}{\#H}(\#H - 1)$$

so that  $\#K = \#G/\#H = \#X$ .

It remains to show that  $K$  is a normal subgroup. We will do this by showing that it is the kernel of some representation of  $G$ .

For an irreducible representation  $\pi$  of  $H$ , consider the function  $\widetilde{\chi}_\pi$  on  $G$  defined by

$$\begin{aligned} \widetilde{\chi}_\pi &= \text{Ind}_H^G \chi_\pi - (\dim \pi) \text{Ind}_H^G \chi_{\epsilon_H} + (\dim \pi) \chi_{\epsilon_G} \\ &= \text{Ind}_H^G \chi_\pi - (\dim \pi) \text{Ind}_H^G 1 + (\dim \pi) 1. \end{aligned}$$

Recall that in general, if  $f : H \rightarrow \mathbb{C}$  is any class function, we can induce a new class function on  $G$  by the formula

$$(\text{Ind}_H^G f)(h) = \sum_{[g] \in G/H, g^{-1}hg \in H} f(s^{-1}gs).$$

Here the sum chooses one representative from each coset. Now one sees easily that

$$\widetilde{\chi}_\pi(g) = \begin{cases} \dim \pi & \text{if } g \in K \\ \chi_\pi(h) & \text{if } g \text{ is conjugate to } h \in H \end{cases}$$

In the first case we get  $0 - 0 + \dim \pi$  in the definition, and in the second case we get  $\text{Ind}_H^G \chi_\pi(g) - \dim \pi + \dim \pi$ . That this completely describes  $\widetilde{\chi}_\pi(g)$  is clear because of our description of  $G$  as a disjoint union above.

Using this decomposition, we can calculate  $(\widetilde{\chi}_\pi, \widetilde{\chi}_\pi)$ :

$$(\widetilde{\chi}_\pi, \widetilde{\chi}_\pi) = \frac{1}{|G|} \sum_{g \in G} |\widetilde{\chi}_\pi(g)|^2 = \frac{1}{|G|} \frac{|G|}{|H|} \dim \pi^2 + \frac{1}{|G|} \frac{|G|}{|H|} \sum_{h \in H \setminus \{e\}} |\chi_\pi(h)|^2$$

But this last expression is equal to  $(\chi_\pi, \chi_\pi) = 1$ .

As  $\widetilde{\chi}_\pi$  is a linear combination of characters with integer coefficients, we must have  $\widetilde{\chi}_\pi = \pm \chi_{\widetilde{\pi}}$  for some irreducible representation  $\widetilde{\pi}$ .<sup>3</sup> But  $\widetilde{\chi}_\pi(e) = \dim \pi > 0$ , so that actually  $\widetilde{\chi}_\pi = \chi_{\widetilde{\pi}}$ .

Note that as  $\chi_{\widetilde{\pi}}|_H = \chi_\pi$ , we have  $\text{Res}_H^G \widetilde{\pi} \sim \pi$ .

Now we claim that

$$K = \bigcap_{[\pi] \in \hat{H}} \ker \widetilde{\pi}.$$

This implies that  $K$  is a normal subgroup. This would finishing the proof.

First we look at  $\subseteq$ : Let  $g \in K$ . Then  $\chi_{\widetilde{\pi}}(g) = \dim \pi = \dim \widetilde{\pi}$ . But  $\widetilde{\pi}(g)^n = 1$  for some  $n \geq 1$ . Hence all the eigenvalues of  $\widetilde{\pi}(g)$  are roots of unity. The matrix  $\widetilde{\pi}(g)$  is diagonalizable, hence the sum of  $\dim \widetilde{\pi}$  roots of unity is equal to  $\dim \widetilde{\pi}$ . This is only possible if all the eigenvalues are 1 by the triangle inequality, i.e. that  $\widetilde{\pi}(g) = \text{id}$ . Thus  $g \in \ker \widetilde{\pi}$ .

Now look at  $\supseteq$ . Let  $g \in G \setminus K$ . Then  $g$  is conjugate to an element  $h \in H \setminus \{e\}$ . Then we can find some irreducible representation  $[\pi] \in \hat{H}$  such that  $\pi(h) \neq 1$ . Hence  $\widetilde{\pi}(g) \neq 1$ .

This finishes the proof.  $\square$

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<sup>3</sup>To see this, note that the norm of  $\widetilde{\chi}_\pi$  is a sum of squares of integers, summing to 1.



**Example 8.2.** Let  $k$  be a finite field and let  $G$  be the translation group of  $\mathbb{A}^1$ , or in other words the “ax+b”-group. It is isomorphic to the matrix group

$$G \simeq \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in k^\times, b \in k \right\}.$$

Then  $G$  satisfies the conditions of the theorem (we say that  $G$  is a **Frobenius group**). Then the group  $K$  in the theorem is the group of translations:

$$K = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in k \right\}.$$

★

## 9 The theorems of Artin and Brauer and fields of definition

**Theorem 9.1** (Artin). *Let  $G$  be a finite group and  $\mathcal{C}$  a collection of subgroups of  $G$ . Then the following two conditions are equivalent:*

1. *Any character of  $G$  is a rational combination of characters of representations induced from subgroups in  $\mathcal{C}$ .*
2. *Any  $g \in G$  is conjugate to an element in a subgroup in  $\mathcal{C}$ .*

*Proof.* The proof is “surprisingly simple”.

We first prove  $1 \Rightarrow 2$ . Recall that the characters span the space of central functions on  $G$ .

At the same time, the characters induced from subgroups in  $\mathcal{C}$  are zero on elements that are not conjugate to elements of subgroups in  $\mathcal{C}$ , by Proposition 7.4. But any non-identity  $g \in G$  have a character with non-zero value at  $g$  (take for example the induced regular representation). Thus  $g$  must meet some conjugacy class in  $\mathcal{C}$ .

Conversely, denote by  $R_{\mathbb{C}}(G)$  the space of central functions on  $G$  with the usual scalar product.

Consider the map

$$T : \bigoplus_{H \in \mathcal{C}} R_{\mathbb{C}}(H) \rightarrow R_{\mathbb{C}}(G)$$

$$T((f)_{H \in \mathcal{C}}) = \sum_{H \in \mathcal{C}} \text{Ind}_H^G f_H.$$

Recall that by Frobenius reciprocity, the functor  $\text{Res}_H^G$  is adjoint to  $\text{Ind}_H^G$ . Therefore the adjoint map of  $T$  is given by

$$T^* : R_{\mathbb{C}}(G) \rightarrow \bigoplus_{H \in \mathcal{C}} R_{\mathbb{C}}(H)$$

sending  $f$  to  $(\text{Res}_H^G f)_{H \in \mathcal{C}}$ .

Now suppose  $\text{Res}_H^G f = 0$  for all  $H \in \mathcal{C}$  and some  $f \in R_{\mathbb{C}}(G)$ . But as above, since any  $g \in G$  is conjugate to an element from some  $H$ , it follows that  $f$  is zero on all conjugacy classes, hence  $f$  is zero. So the map  $T^*$  is injective, hence  $T$  is surjective.<sup>4</sup>

This means that we can find a basis in  $R_{\mathbb{C}}(G)$  consisting of characters  $\chi_1, \dots, \chi_n$  of representations induced from subgroups in  $H \in \mathcal{C}$ . The characters  $\chi'_1, \dots, \chi'_n$  of irreducible representations of  $G$  form a basis in  $R_{\mathbb{C}}(G)$  by Prop .... Therefore the transition matrix  $A$  from  $\{\chi_1, \dots, \chi_n\}$  to  $\{\chi'_1, \dots, \chi'_n\}$  has integer entries. Hence  $A^{-1}$  has rational entries.

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<sup>4</sup>Quote Sergey: “we didn’t do any real work”.

Therefore any character induced from an irreducible representation can be written as a  $\mathbb{Q}$ -linear combination of the  $\chi_i$ 's. Therefore this holds for all characters.  $\square$

We can take  $\mathcal{C}$  to be the collection of cyclic subgroups of  $G$ .

**Corollary 9.2.** *Any character can be written as a  $\mathbb{Q}$ -linear combination with rational coefficients of characters induced from cyclic subgroups.*

*Proof.* Any  $g \in G$  is contained in some cyclic subgroup, say  $\langle g \rangle$ .  $\square$

The following is more difficult to prove (we won't):

**Theorem 9.3** (Brauer). *For any finite group  $G$ , the characters of  $G$  can be written as a linear combination with integer coefficients of characters  $\text{Ind}_H^G \chi_\pi$  of one-dimensional representations  $\pi$  of subgroups  $H \subset G$ .*

*In fact we can let  $H$  range through products of cyclic and  $p$ -subgroups.*

For a proof, see for example [3].

## 9.1 Field of definition

Let  $\pi : G \rightarrow \text{GL}(V)$  be a representation over the complex numbers. Let  $K \subset \mathbb{C}$ . We say that  $\pi$  is **defined over**  $K$  if there exists a vector space  $V'$  over  $K$  and a representation  $\pi' : G \rightarrow \text{GL}(V')$  such that  $\pi$  is equivalent to  $\pi'_\mathbb{C}$ , where

$$\pi'_\mathbb{C}(g) = 1 \otimes \pi(g) : \mathbb{C} \otimes_K V' \rightarrow \mathbb{C} \otimes_K V'.$$

**Example 9.4.** Let  $G = \mathbb{Z}/n\mathbb{Z}$ . Representations of  $G$  are characters, all of the form  $e^{\frac{2\pi i}{n}}$ . Hence any representation of  $G$  is defined over  $\mathbb{Q}(\zeta_n)$ .  $\star$

**Example 9.5.** If  $G = S_n$  (a much more complicated group), every representation is in fact defined over  $\mathbb{Q}$ .  $\star$

As long as the field  $K$  is of characteristic zero, many of the results are still true with the same proofs. The main difference is that if  $\pi$  is an irreducible representation, then  $\text{End}(\pi)$  forms a *division algebra* over  $K$ , instead of being just equal to  $K$  itself. There are not many division algebras of characteristic zero, so this is not too bad.

Many results remain true, however. Maschke's theorem (that is, proving complete reducibility) goes through for any field of characteristic zero (or for any group of order not dividing the characteristic). We also have a weaker form of Schur's lemma: if  $\pi$  and  $\pi'$  are

non-equivalent irreducible representations, then  $\text{Mor}(\pi, \pi') = 0$  and  $(\chi_{\pi'}, \chi_{\pi}) = 0$ .

Note that  $\chi_{\pi' \mathbb{C}} = \chi_{\pi'}$ . This implies that disjoint irreducibles stay disjoint when complexifying. What can happen, of course, is that irreducibles split into disjoint representations.

If  $G$  is a finite group, then the **exponent of  $G$**  is the least common multiple of orders of all elements in  $G$ .

**Theorem 9.6** (Brauer). *Let  $G$  be a finite group and let  $m$  be the exponent of  $G$ . Then any complex representation of  $G$  is defined over  $\mathbb{Q}(\zeta_m)$ , where  $\zeta_m$  is a primitive  $m$ -th root of unity.*

*Proof.* Let  $\pi_1, \dots, \pi_k$  be representatives of the equivalence classes of irreducible representations of  $G$  over  $\mathbb{Q}(\zeta_m)$ .

Note that any one-dimensional representation of a subgroup of  $G$  is defined over  $\mathbb{Q}(\zeta_m)$ . Hence the induction of any such representation is defined over  $\mathbb{Q}(\zeta_m)$ .

Now let  $\pi$  be any finite-dimensional complex representation representation of  $G$ . It follows by Brauer's Theorem 9.3 that

$$\chi_{\pi} = \sum_{i=1}^k m_i \chi_{\pi_i \mathbb{C}} = \sum_{i=1}^k m_i \chi_{\pi_i}$$

for some  $m_i \in \mathbb{Z}$ , since each  $\text{Ind}_H^G \chi$  is a  $\mathbb{Z}$ -linear combination of the  $\pi_i$ 's.

Assume in addition that  $\pi$  is irreducible. Then

$$\begin{aligned} 1 = (\chi_{\pi}, \chi_{\pi}) &= \sum_{i,j} (\chi_{\pi_i}, \chi_{\pi_j}) \\ &= \sum_i m_i^2 (\chi_{\pi_i}, \chi_{\pi_i}). \end{aligned}$$

This is only possible if  $\chi_{\pi} = \chi_{\pi_i}$  for some  $i$ , hence  $\pi \sim (\pi_i)_{\mathbb{C}}$ . So  $\pi$  is defined over  $\mathbb{Q}(\zeta_m)$ . Thus the theorem is true for irreducible representations, and so true for every representation.  $\square$

## 10 Mackey's irreducibility criterion

When is an induced representation irreducible? In practice, this is almost never the case.

**Example 10.1.** Let  $H \subset G$  be a subgroup. Then  $\text{Ind}_H^G \epsilon_H = \mathbb{C}[G/H]$ , which is never irreducible. ★

We need a unpleasant lemma.

Let  $G$  be a finite group and let  $H, K$  be two subgroups and suppose  $\pi$  is a representation of  $H$ . Define a subgroup  $H_g \subset H$  by

$$H_g = gHg^{-1} \cap K, \quad g \in G.$$

Then, for any  $g \in G$ , define a representation  $\pi^g$  of  $H_g$  by

$$\pi^g(h) \triangleq \pi(g^{-1}hg).$$

**Lemma 10.2.** *We have*

$$\text{Res}_K^G \text{Ind}_H^G \pi \sim \bigoplus_{\bar{g} \in K \backslash G/H} \text{Ind}_{H_g}^K \pi^g,$$

where  $K \backslash G/H = G/\sim$ , where  $g_1 \sim g_2$  if  $g_1 = kg_2h$  with  $k \in K$  and  $h \in H$ .

*Proof.* To come. □

**Theorem 10.3** (Mackey). *Assume  $\pi$  is an irreducible representation of  $H \subset G$ . Then  $\text{Ind}_H^G \pi$  is irreducible if and only if the representations  $\text{Res}_{H_g}^H \pi$  and  $\pi^g = \pi(g^{-1} - g)$  are disjoint for all  $g \in G \backslash H$ .*

*Proof.* With the lemma in hand, the proof is quite short.

Let  $\chi = \chi_\pi$ . Then we want to compute  $(\text{Ind}_H^G \chi_\pi, \text{Ind}_H^G \chi_\pi)$ . By Frobenius reciprocity, this is the same as  $(\text{Res}_H^G \text{Ind}_H^G \chi, \chi)$ . Hence, by the lemma, this is (using Frobenius reciprocity again)

$$\sum_{\bar{g} \in H \backslash G/H} (\text{Ind}_{H_g}^H \chi_{\pi^g}, \chi) = \sum_{\bar{g} \in H \backslash G/H} (\chi_{\pi^g}, \text{Res}_{H_g}^H \chi).$$

Thus  $\pi$  is irreducible if and only if  $(\chi_{\pi^g}, \chi|_{H_g}) = 0$  for all  $g \notin H$ . □

## 11 Induction from normal subgroups

We study the following question: let  $G$  be a finite group and  $H \subset G$  a normal subgroup. Can we describe the representations of  $G$  in terms of those of  $H$  and those of  $G/H$ ?

Assume  $(W, \theta)$  is an irreducible representation of  $G$ . Let  $(V, \pi)$  be an irreducible representation contained in  $\text{Res}_H^G \theta$ . Then, for every  $g \in G$ ,  $\text{Res}_H^G \theta$  also contains the representation  $\pi^g$  on  $H$  defined by

$$\pi^g(h) = \pi(g^{-1}hg).$$

Indeed, if  $T : V \rightarrow W$  intertwines  $\pi$  with  $\text{Res } \theta$  (meaning that  $T\pi(g) = \theta(g)T$  for all  $g \in H$ ), then  $\theta(g)T$  intertwines  $\pi^g$  with  $\theta$ :

$$\theta(g)T\pi^g(h) = \theta(g)T\pi(g^{-1}hg) = \theta(g)\theta(g^{-1}hg)T = \theta(h)\theta(g)T.$$

Furthermore, if we consider the isotypic components of  $W(\pi^g)$  of  $\text{Res } \theta$  corresponding to  $\pi^g$ , then  $\theta(g_1)W(\pi^{g_2}) = W(\pi^{g_1g_2})$ .

Thus the space

$$\sum_{g \in G} W(\pi^g)$$

is invariant under  $G$ , hence it coincides with  $W$ , since  $W$  was assumed to be irreducible.

In other words,  $\text{Res } \theta$  decomposes into copies of  $\pi^g$ ,  $g \in G$ .

We can define an action of  $G$  on  $\widehat{H}$  by

$$[\pi] \mapsto [\pi^g].$$

What we have shown, is that any irreducible representation  $W_\theta$  of  $G$  defines a  $G$ -orbit  $S \subset \widehat{H}$  and that

$$W = \bigoplus_{x \in S} W(x), \quad \theta(g)W(x)W(gx).$$

For  $x \in \widehat{H}$ , denote by  $\mathcal{I}(x) \subset G$ , the stabilizer of  $x$  in  $G$ . It is called the **inertia subgroup** of  $x$ . Note that the action of  $H$  on  $\widehat{H}$  is trivial, so that we always have  $H \subset \mathcal{I}(x)$ .

Then

$$\theta \sim \text{Ind}_{\mathcal{I}(x)}^G \left( \text{Res}_{\mathcal{I}(x)}^G \theta \Big|_{W(x)} \right).$$

(why??)

What can be say about  $\text{Res}_{\mathcal{I}(x)}^G \theta$ ? In other words, assume we take  $[\pi] \in \widehat{H}$ . Then

$$\mathcal{I} = \mathcal{I}([\pi]) = \{g \in G \mid \pi^g \sim \pi\}.$$

Can we describe all irreducible representations  $\eta$  of  $\mathcal{I}$  such that  $\text{Res}_H^{\mathcal{I}} \eta$  is isotypic to  $\pi$ ?

It can be shown that all such representations of  $\mathcal{I}$  can be described in terms of *projective* irreducible representations of  $\mathcal{I}/H$ .

In general, a **projective representation** of a group  $K$  on a vector space  $V$  is a homomorphism  $K \rightarrow \mathrm{PGL}(V) \triangleq \mathrm{GL}(V)/\mathbb{C}^* \cdot \mathrm{id}_V$ .

Why do projective representations appear? We have that  $\pi^g \sim \pi$  for all  $g \in \mathcal{I}$ . Thus there exists an invertible  $T_g \in \mathrm{GL}(V_\pi)$  such that

$$\pi(g^{-1}hg) = T_g^{-1}\pi(h)T_g$$

for all  $h \in H$ . By irreducibility,  $T_g$  is unique up to a scalar factor. This implies that  $T_{g_1g_2} = c(g_1, g_2)T_{g_1}T_{g_2}$  for some  $c(g_1, g_2) \in \mathbb{C}^*$ . There the operators  $T_g$  define a projective representation of  $\mathcal{I}$  on  $V$ .

In some cases projective representations do not appear. We will make the following assumptions:  $H$  is abelian, so  $\pi$  is a character of  $H$ . Also,  $\pi : H \rightarrow \mathrm{GL}(\mathbb{C})$  extends to a homomorphism  $\tilde{\pi} : \mathcal{I} \rightarrow \mathbb{C}^*$ .

We want to describe all irreducible representations of  $\mathcal{I}$  such that  $\mathrm{Res}_H^{\mathcal{I}} \sigma$  is isotypic to  $\pi$ .

Assume  $\sigma$  is such a representation. Then consider  $\tilde{\sigma} = \tilde{\pi} \otimes \sigma$  (recall that by assumption,  $\pi$  is a character, so conjugation makes sense). Then  $\tilde{\sigma}$  is a representation on the same space given by  $\sigma(g) = \tilde{\pi}(g)^{-1}\sigma(g)$ . Then  $\tilde{\sigma}$  is irreducible and  $H \subset \ker \tilde{\sigma}$ . So  $\sigma$  is an irreducible representation of  $\mathcal{I}([\pi])/H$ .

Conversely, if  $\tilde{\sigma}$  is an irreducible representation of  $\mathcal{I}/H$ , then we can define  $\sigma$  by  $\sigma = \tilde{\pi} \otimes \tilde{\sigma}$ .

We have almost proved:

**Theorem 11.1.** *Let  $G$  be a finite group and  $H \subset G$  a normal abelian subgroup. Assume that every character  $\chi \in \hat{H}$  extends to a homomorphism  $\tilde{\chi} : \mathcal{I}(\chi) \rightarrow \mathbb{C}^*$ , where*

$$\mathcal{I}(\chi) = \{g \in G \mid \chi^g = \chi\}.$$

*Then the irreducible representations of  $G$  can be described as follows: for every  $G$ -orbit in  $\hat{H}$ , fix a representative  $\chi$ . Then for every  $[\sigma] \in \widehat{\mathcal{I}(\chi)/H}$ , define  $\mathrm{Ind}_{\mathcal{I}(\chi)}^G(\tilde{\chi} \otimes \sigma)$ .*

*Therefore, as a set,  $\hat{G}$  can be identified with*

$$\bigsqcup_{[\chi] \in \hat{H}/G} \widehat{\mathcal{I}(\chi)/H},$$

*where the  $[\chi]$  ranges over representatives of  $G$ -orbits in  $\hat{H}$ .*

**Example 11.2.** Consider a finite field  $\mathbb{F}_p$  with  $p$  prime. Consider the corresponding  $ax + b$  group:

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_p^*, b \in \mathbb{F}_p \right\}.$$

Let  $H$  be the subgroup  $H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_p \right\} \simeq \mathbb{Z}/p$ . Then  $\widehat{H} = \mathbb{Z}/p$  consists of characters  $\chi_0, \dots, \chi_{p-1}$  where

$$\chi_k(b) = e^{\frac{2\pi i}{p} kb}.$$

We have

$$\chi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \chi(a^{-1} -).$$

The stabilizer of  $\chi_0$  is  $G$ . The stabilizers of  $\chi_k$  for  $k = 1, \dots, p-1$  are trivial in  $G/H$ , and  $G$  acts transitively on  $\{\chi_1, \dots, \chi_{p-1}\}$ .

The assumptions of the theorem are satisfied, as  $\chi_0$  extends to  $G$  by  $\tilde{\chi}_0(g) = 1$ .

Consider the representatives of the two  $G$ -orbits on  $\tilde{H}$ :

$\underline{\chi_0}$ : Here  $\mathcal{I}(\chi_0) = G$ . We have  $G/H \simeq \mathbb{F}_p^\times$ . This leads to  $|\mathbb{F}_p^\times| = p-1$  one-dimensional representations. Namely,

$$G \ni \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \chi(a)$$

for  $\chi \in \mathbb{F}_p^\times \simeq \mathbb{Z}/(p-1)$ .

$\underline{\chi_1}$ : Here  $(\chi_1) = H$ . Then we get one irreducible  $p-1$ -dimensional representation  $\text{Ind}_H^G \chi_1$ . ★

## 11.1 Divisibility

**Theorem 11.3.** *Let  $G$  be a finite group,  $H \subset G$  a normal abelian subgroup, and  $\pi$  an irreducible representation of  $G$ . Then  $\dim \pi$  divides  $|G|/|H|$ .*

*Proof.* Recall that we already proved this for  $H = Z(G)$ . For general  $H$ , we know that  $\pi \simeq \text{Ind}_I^G \sigma$  for some  $I \supset H$  and  $\text{Res}_H^I \sigma$  is isotypic to  $\chi \in \widehat{H}$ .

Consider the groups  $\tilde{I} = \sigma(I)$  and  $\tilde{H} = \sigma(H) = \{\chi(h) \cdot 1 \mid h \in H\}$ .

Then  $\tilde{H} \subset Z(\tilde{I})$  and the representation of  $\tilde{I}$  on  $V_\sigma$  is irreducible.

Hence

$$\dim \sigma \mid |\tilde{I}/\tilde{H}| = |I/H \cdot \ker \sigma|,$$

so

$$\dim \sigma \mid |I/H|.$$

Hence

$$\dim \pi = |G/I| \dim \sigma \mid |G/I| |G/H| = |G/H|.$$

□



## 12 Part II - Compact groups

When one wants to develop a representation theory for infinite groups, one has to restrict the scope somewhat. Infinite groups are too wild in general, and they can have quite bad representation theory. For example,  $\mathbb{Z}$  acting on  $\mathbb{C}^2$  by  $(x, y) \mapsto (x + ny, y)$  has the  $y$ -axis as an irreducible subspace, but the representation does not split, because there is no complementary invariant subspace. Thus the property of complete reducibility fails.

We want to at least restrict to groups with a topology. But what made wonders for the theory of finite groups was the presence of an averaging operator. If we restrict to *compact* groups, we will see that we can mimic this operator in this setting as well.

Recall that a **topological group** is a group  $G$  equipped with a topology such that the group law and the inverse are continuous maps.

Here is a key fact:

**Proposition 12.1** (Haar, von Neumann, André Weil). *For any locally compact group  $G$  there exists a nonzero left-invariant Radon measure on  $G$  which is unique up to scalar. We call such a measure a **Haar measure**.*

Recall that a **Radon measure**  $\mu$  on a locally compact space is a Borel measure which is outer regular, inner regular on open subsets and  $\mu(K) < \infty$  for all compacts  $K$ .

To be given a Radon measure on a locally compact space  $X$  is equivalent to be given a linear functional  $F : C_c(X) \rightarrow \mathbb{C}$  such that  $F(f) \geq 0$  if  $f \geq 0$  (here  $C_c(X)$  is the space of compactly supported functions on  $X$ ). Thus we can think of a Radon measure as equivalent to having a way to integrate functions.

If  $G$  is compact, we usually normalize so that  $\mu(G) = 1$ . This makes the Haar measure unique.

**Example 12.2.** Let  $G = (\mathbb{R}, +)$ . The Haar measure is the usual Lebesgue measure. ★

There are also right-invariant measures. If the left- and right-invariant Haar measures coincide, then we say that the group  $G$  is **unimodular**.

**Lemma 12.3.** *Any compact group is unimodular.*

*Proof.* Let  $\mu_l$  be the left-invariant measure. Then the measure  $\mu'_l(U) = \mu_l(Ug)$  is left-invariant as well. Hence  $\mu'_l = \alpha\mu_l$  for some  $\alpha \in \mathbb{R}$ . But  $1 = \mu'_l(G) = \alpha\mu_l(G) = \alpha$ . Hence  $\mu'_l = \mu$ . Thus  $\mu_l(Ug) = \mu'_l(U) = \mu_l(U)$ , so  $\mu_l$  is right-invariant as well. □

Many of the results for finite groups carry over to compact groups: the proofs are the same except expressions like  $\frac{1}{|G|}$  are replaced by  $\int_G f d\mu$ .

A **representation** of a compact group  $G$  is a continuous homomorphism  $\pi : G \rightarrow \text{GL}(V)$ .

Here are some results that holds for compact groups as well (with the same proofs):

- **Maschke's theorem:** Any finite-dimensional representation of  $G$  is completely reducible.
- Any finite-dimensional representation is unitarizable.
- The orthogonality relations. In particular, two finite-dimensional representations of a  $G$  are equivalent if and only if the characters coincide.

The notion of regular representation is however more complicated. We are forced to (briefly) introduce infinite-dimensional spaces.

A **unitary** representation of a topological group  $G$  on a Hilbert space  $\mathcal{H}$  is a homomorphism  $\pi : G \rightarrow U(\mathcal{H})$ , continuous in the **strong operator topology**, meaning the following: if  $g_i \rightarrow g$  in  $G$ , then  $\pi(g_i)\zeta \rightarrow \pi(g)\zeta$  for all  $\zeta \in \mathcal{H}$ . This can be rephrased as saying that the map  $G \rightarrow \mathcal{H}$  given by  $g \mapsto \pi(g)\zeta$  is continuous for all  $\zeta \in \mathcal{H}$ .

**Example 12.4.** Let  $G$  be a locally compact group with fixed Haar measure. Consider  $L^2(G)$  (the vector space of square integrable functions on  $G$ ) with respect to the given measure. Define

$$\begin{aligned} G &\longrightarrow U(L^2(G)) \\ g &\longmapsto (\lambda(g)f)(h) = f(g^{-1}h). \end{aligned}$$

Then  $\lambda$  is continuous in the above sense.

We call  $\lambda$  the **regular representation of the compact group**  $G$ . ★

**Theorem 12.5** (Peter-Weyl). *Let  $G$  be a compact group and  $\pi : G \rightarrow U(\mathcal{H})$  a unitary representation. Then  $\pi$  decomposes into a direct sum (in the sense of Hilbert spaces) of finite-dimensional irreducible representations.*

*Proof.* The key part is to show that there exists a nonzero invariant finite-dimensional subspace  $K \subset \mathcal{H}$ .

Then one uses transfinite induction and Zorn's lemma to prove the theorem.

To find  $K$ , take any nonzero compact operator  $A \geq 0$ . Then define

$$B = \int_G \pi(g)A\pi(g^{-1})dg$$

Then  $B$  is nonzero and  $G$ -invariant.

Now apply the spectral theorem for compact self-adjoint operators to see that  $B$  has an eigenvalue  $\alpha > 0$  and the corresponding eigenspace  $K$  is finite-dimensional.  $\square$

**Corollary 12.6.** *For any compact group  $G$ , the intersection of all kernels of irreducible representations on  $G$  is trivial.*

*Proof.* The regular representation decomposes into a sum of irreducibles, and is injective.  $\square$

Denote by  $\mathbb{C}[G]$  the linear span of matrix coefficients of finite dimensional representations of  $G$ . This is an algebra under pointwise multiplication: the product of two matrix coefficients (from representations  $\pi_1, \pi_2$ , say) is a matrix coefficient in  $\pi_1 \otimes \pi_2$ . This algebra is closed under complex conjugation, since the matrix coefficient of the dual representation is given by  $\overline{\pi(g^{-1})_{ji}}$ . Thus  $\mathbb{C}[G]$  is a  $C^*$ -algebra.

**Corollary 12.7.** *The algebra  $\mathbb{C}[G]$  is norm-dense in  $C(G)$ .*

*Proof.* Recall the Stone-Weierstrass theorem: if  $A$  is a subalgebra of  $C(G)$  that separates points, then  $A$  is norm-dense in  $C(G)$ . So let  $g_1, g_2 \in G$ . By Theorem 12.5, the regular representation decomposes into irreducibles. Suppose there is no  $\pi$  such that  $\pi(g_1) \neq \pi(g_2)$ . Then  $\pi(g_1 g_2^{-1}) = e$  for all  $\pi$ . But the regular representation is injective.  $\square$

The algebra  $\mathbb{C}[G]$  is called the **algebra of regular/polynomial functions on  $G$** .

**Theorem 12.8** (Peter-Weyl). *For every compact group  $G$ , we have*

$$\lambda \sim \bigoplus_{[\pi] \in \widehat{G}} \pi^{\dim \pi},$$

where  $\lambda : G \rightarrow U(L^2(G))$  is the regular representation.

*Proof.* For any unitary irreducible representation  $\pi$ , choose an orthonormal basis  $\{e_1, \dots, e_{\dim \pi}\}$ , and consider the corresponding matrix coefficients  $a_{ij}^\pi(g) = (\pi(g)e_i, e_j)$ . Then, by the orthogonality relations, we have isometric embeddings  $V_\pi \hookrightarrow L^2(G)$  given by  $e_j \mapsto (\dim \pi)^{\frac{1}{2}} a_{ij}^\pi$  for all  $j$ . This gives an isometric embedding of  $\bigoplus_{[\pi] \in \widehat{G}} \pi^{\oplus \dim \pi}$  into  $\lambda$ .

As  $\mathbb{C}[G]$  is normdense in  $C(G)$ , hence in  $L^2(G)$ , this embedding is unitary.  $\square$

Just as in the case of finite groups, it is shown that the characters of finite-dimensional representations form an orthonormal basis in the space

$$\{f \in L^2(G) \mid f(g - g^{-1}) = f(-)\forall g \in G\} \subseteq L^2(G).$$

In fact, for any compact group  $G$ , the characters of finite-dimensional representations of  $G$  span a *dense* subspace in the space of continuous central functions on  $G$  (equivalently:  $g$  and  $h$  are conjugate if and only if  $\chi_\pi(g) = \chi_\pi(h)$  for all  $[\pi] \in \widehat{G}$ ).

**Theorem 12.9.** *Suppose  $G$  is a compact subgroup of the unitary group  $U(n) \subseteq \mathbb{R}^{n \times n}$ . Then  $\mathbb{C}[G]$  is the unital algebra generated by the matrix coefficients  $a_{ij}$  and  $\det^{-1}$ .*

*Proof.* Since the matrix coefficients of  $[a_{ij}(g)]^{-1}$  can be expressed as polynomials in  $a_{ij}$  and  $\det^{-1}$ . The algebra generated by these is closed under complex conjugation and separates points, hence it is dense in  $C(G)$ .

It follows by the orthogonality relations that  $A = \mathbb{C}[G]$ .  $\square$

Equivalently, let  $\pi$  be the standard representation of  $G$  (it has one since it is a matrix group). Then the theorem says that any irreducible representation of  $G$  occurs as a subrepresentation of  $(\det^{-1})^{\otimes k} \otimes \pi^{\otimes l}$  for some  $k, l \geq 0$ .

## 13 Representation theory of compact Lie groups

A **Lie group** is a smooth manifold  $G$  equipped with smooth maps  $\mu : G \times G \rightarrow G$  and  $\iota : G \rightarrow G$  making  $(G, \mu, \iota)$  into a group.

Equivalently, a Lie group is a group object in the category of smooth manifolds.

**Example 13.1.** Let  $V$  be a finite-dimensional vector space. Then  $\mathrm{GL}(V)$  is a Lie group. The smooth structure is inherited from  $\mathrm{End}(V) \simeq \mathbb{R}^N$ , since  $\mathrm{GL}(V)$  is an open subset of  $\mathrm{End}(V)$ . ★

Before giving further examples, we will develop some general results.

### 13.1 The exponential map

We recall some results from manifold theory.

Let  $M^n$  be a smooth manifold of dimension  $n$  and let  $p \in M$ . The tangent space  $T_p M$  can be equivalently defined as

- Equivalence classes of smooth curves passing through  $p$ : let  $\gamma_i : (a_i, b_i) \rightarrow M$ , with  $a_i < 0 < b_i$  and  $\gamma_i(0) = p$  be two such curves ( $i = 1, 2$ ). Then  $\gamma_1 \sim \gamma_2$  if and only if

$$(x \circ \gamma_1)'(0) = (x \circ \gamma_2)'(0)$$

for some/any coordinate system  $(x, U)$  around  $p$ . We denote the equivalence class by  $\gamma'(0)$ , or

- the more algebraic definition: the space of derivations of  $C^\infty(M)$  at  $p$ . This is by definition a linear map  $L : C^\infty(M) \rightarrow \mathbb{C}$  such that  $L(fg) = f(p)L(g) + g(p)L(f)$ .

The relation between these two definitions is that the derivation  $L$  defined by  $\gamma'(0)$  is

$$L(f) = \frac{\partial}{\partial f}(f \circ \gamma) \Big|_{t=0}.$$

In local coordinates  $(x, U)$ , any tangent vector can be written as

$$\sum_{i=1}^n a^i(p) \frac{\partial}{\partial x^i} \Big|_p.$$

Recall that a smooth vector field is a smooth section of the tangent bundle. The space of vector fields can be identified with the space of derivations of  $C^\infty(M)$ , i.e. as linear maps  $L : C^\infty(M) \rightarrow C^\infty(M)$  satisfying the Leibniz rule.

Given a vector field  $X$ , an **integral curve of  $X$**  is a smooth curve  $\gamma : (a, b) \rightarrow M$  such that  $\gamma'(t) = X_{\gamma(t)}$  for all  $t \in (a, b)$ . It is a standard fact of manifold theory that for any  $p \in M$ , there exists a unique maximal integral curve passing through  $p$ .

Recall also: if  $f : M \rightarrow N$  is a smooth map, then the differential at  $p$  is the map  $d_p f : T_p M \rightarrow T_{f(p)} N$  given by  $\gamma'(0) \mapsto (f \circ \gamma)'(0)$ .

Now let  $G$  be a Lie group. For each  $g \in G$ , we have diffeomorphism  $l_g : G \rightarrow G$  defined by left-multiplication:  $h \mapsto gh$ . This induces a map on vector fields:

$$(l_g)_*(X)_p = (dl_g)(X_{l_g^{-1}(p)})$$

We say that a vector field  $X$  on a Lie group  $G$  is **left-invariant** if  $(l_g)_*(X) = X$  for all  $g \in G$ . Such a vector field is determined by its value at  $e \in G$ :  $X_g = (d_e l_g)(X_e)$ .

Thus we get a one to one correspondence between left-invariant vector fields and  $v \in T_e G$ .

**Proposition 13.2.** *Let  $G$  be a Lie group,  $X \in T_e G$  and  $\alpha_X : (a, b) \rightarrow G$  be the maximal integral curve of the corresponding left-invariant vector field such that  $\alpha_X(0) = e$ .*

*Then  $(a, b) = \mathbb{R}$  and  $\alpha_X(s+t) = \alpha_X(s)\alpha_X(t)$  for all  $s, t \in \mathbb{R}$ .*

*Proof.* Consider the two curves  $\gamma_1 : (a, b) \rightarrow G$  and  $\gamma_2 : (a, b) \rightarrow G$  defined by  $t \mapsto \alpha_X(s)\alpha_X(t)$  and  $t \mapsto \alpha_X(s+t)$ , respectively.

Then  $\gamma_1(0) = \alpha_X(s) = \gamma_2(0)$ .

Both curves are integral curves for  $X$ . This is clear for  $\gamma_2$ , while for  $\gamma_1$ , we have that

$$\begin{aligned} \gamma_1'(t) &= (l_{\alpha_X(s)} \circ \alpha_X)'(t) = d_{\alpha_X(t)} l_{\alpha_X(s)}(\alpha_X'(t)) \\ &= (d_{\alpha_X(t)} l_{\alpha_X(s)})(\widetilde{X_{\alpha_X(t)}}) \\ &= \widetilde{X_{\alpha_X(s)\alpha_X(t)}} = \widetilde{X_{\gamma_1(t)}}. \end{aligned}$$

Hence  $\gamma_1 = \gamma_2$  and their domain can be extended to all of  $\mathbb{R}$ . □

A **1-parameter subgroup of  $G$**  is a smooth map  $\gamma : \mathbb{R} \rightarrow G$  that is a group homomorphism.

**Corollary 13.3.** *The map  $X \mapsto \alpha_X$  is a bijection between  $T_e G$  and one-parameter subgroups of  $G$ .*

*Proof.* Assume  $\gamma : \mathbb{R} \rightarrow G$  is a one-parameter subgroup of  $G$ . Put  $X = \gamma'(0) \in T_e G$ . Then  $\gamma = \alpha_X$ :

$$\gamma'(t) = (d_e \gamma(t))(\gamma'(0))$$

as  $\gamma(t+s) = \gamma(t)\gamma(s)$ . □

**Remark.** In general  $\gamma(\mathbb{R})$  is not a closed subgroup of  $G$ . Take for example  $\mathbb{R} \rightarrow S^1 \times S^1$  given by  $t \mapsto (e^{it}, e^{i\theta t})$  for some irrational  $\theta$ . Then the image is dense in  $S^1 \times S^1$ .

We define the **exponential map** as  $\exp : T_e G \rightarrow G$  given by  $X \mapsto \alpha_X(1)$ .

Note that for any  $s \in \mathbb{R}$ :

$$\frac{d}{dt} \alpha_X(st) = s\alpha'_X(st) = sX_{\alpha_X(st)}.$$

Thus  $t \mapsto \alpha_X(st)$  is the integral curve of the vector field corresponding to  $sX$ , hence  $\alpha_X(st) = \alpha_{sX}(t)$ .

Therefore  $\exp(tX) = \alpha_{tX}(1) = \alpha_X(t)$ . So by the corollary, all 1-parameter subgroups of  $G$  have the form  $t \mapsto \exp(tX)$ .

We have

$$\left. \frac{d}{dt} \exp(tX) \right|_{t=0} = X.$$

In other words, if we identify  $T_0(T_e G)$  with  $T_e G$ , we conclude that  $d_0 \exp = \text{id}_{T_e G}$ . Thus  $\exp$  gives us a diffeomorphism between a neighbourhood of  $0 \in T_e G$  and a neighbourhood of the identity  $e \in G$ .

That  $\exp$  is smooth is a standard result on integral curves. See for example [4].

**Theorem 13.4.** *The exponential map is the unique smooth map  $\exp : T_e G \rightarrow G$  such that  $\exp((s+t)X) = \exp(sX)\exp(tX)$  and*

$$\left. \frac{\partial}{\partial t} \exp(tX) \right|_{t=0} = X$$

In general, the exponential map is neither injective nor surjective, even for connected groups.

**Example 13.5.** Let  $G = \text{GL}(V)$ . As  $\text{GL}(V)$  is a open subset of  $\text{End}(V)$ , the tangent space at every  $M \in \text{GL}(V)$  is identified with  $\text{End}(V)$  for all  $M \in \text{GL}(V)$ . Then  $\exp : \text{End}(V) \rightarrow \text{GL}(V)$  is given by

$$X \mapsto e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

★

A **homomorphism of Lie groups** is a homomorphism  $\pi : G \rightarrow H$  which is also smooth.

**Theorem 13.6.** Assume that  $\pi : G \rightarrow H$  is a Lie group homomorphism. Then the diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi} & H \\ \exp \uparrow & & \uparrow \exp \\ T_e G & \xrightarrow{d_e \pi} & T_e H \end{array}$$

is commutative.

*Proof.* See [4]. □

**Example 13.7.** Let  $G = \mathrm{GL}(V)$ . Consider the map  $\det : \mathrm{GL}(V) \rightarrow K^\times = \mathrm{GL}(K)$ . Then  $d_e \det = \mathrm{Trace}$ . Then the theorem says that  $\det(e^A) = e^{\mathrm{trace}(A)}$ . ★

### 13.2 The Lie algebra of a Lie group

Recall that for vector field  $X, Y$  on a manifold  $M$ , there is a bracket operation  $[X, Y]$ . If  $L_X, L_Y$  are the corresponding derivations, then  $[X, Y]$  is defined as the vector field corresponding to the derivation  $L_{[X, Y]} = L_X L_Y - L_Y L_X$ .<sup>5</sup> If  $X$  and  $Y$  are given in local coordinates as

$$X = \sum f^i \frac{\partial}{\partial x^i}, \quad Y = \sum g^i \frac{\partial}{\partial x^i},$$

respectively, then

$$[X, Y] = \sum_{i,j} \left( f^i \frac{\partial g^j}{\partial x^i} - g^i \frac{\partial f^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

The definition of a Lie algebra over a field  $k$  is then the following: a **Lie algebra** is a vector space  $\mathfrak{g}$  over  $k$  equipped with a bilinear form  $[-, -]$  called the **Lie bracket** that is skew-symmetric and satisfies the **Jacobi identity**:

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

The Jacobi identity can be thought of as a "linearization of associativity".

We want to compute the commutator more explicitly. Let  $\gamma_X$  be a curve on  $G$  with  $\gamma_X(0) = e$  and  $\gamma'_X(0) = X$ . Then for any  $g \in G$ , the curve  $g\gamma_X$  satisfies  $(g\gamma_X)(0) = g$  and  $(g\gamma_X)'(0) = (d_e l_g)(X)$ .

Hence we have

$$L_X(f)(g) = \left. \frac{d}{dt} f(g\gamma_X(t)) \right|_{t=0}$$

---

<sup>5</sup>It is not immediately obvious that this is a derivation, but it is easily checked.



Then

$$\begin{aligned}(L_X L_Y)(f)(g) &= \frac{d}{dt} L_Y(f)(g\gamma_X(t)) \Big|_{t=0} \\ &= \frac{\partial^2}{\partial s \partial t} f(g\gamma_X(t)\gamma_Y(s)) \Big|_{s=t=0}\end{aligned}$$

Hence

$$\begin{aligned}(d_e f)([X, Y]) &= (L_X L_Y - L_Y L_X)(f)(e) \\ &= \frac{\partial^2}{\partial s \partial t} (f(\gamma_X(t)\gamma_Y(s)) - f(\gamma_Y(s)\gamma_X(t))) \Big|_{s=t=0} \quad (1)\end{aligned}$$

This is reasonable explicit, but can be made more so by working with coordinates. Choose local coordinates  $(x, U)$  around the identity  $e \in G$ . We can identify the codomain of  $x : U \rightarrow \mathbb{R}^n$  with  $\mathfrak{g} = T_e G$ . We require that  $x(e) = 0$  and  $d_e x = \text{id}$  (we can for example choose  $x = \exp^{-1}$ ).

Now consider the map  $m : V \times V \rightarrow \mathfrak{g}$  given by  $(X, Y) \mapsto x(x^{-1}(X)x^{-1}(Y))$ . This is well-defined for a small enough neighbourhood  $U$  of  $0 \in \mathfrak{g}$ .

Applying equation (1) to  $f = \ell \circ x$  (where  $\ell$  is a linear functional on  $\mathfrak{g}$ ) we get

$$\begin{aligned}\ell([X, Y]) &= \frac{\partial^2}{\partial s \partial t} \ell(x(x^{-1}(tX)x^{-1}(sY)) - x(x^{-1}(sY)x^{-1}(tX))) \Big|_{t=s=0} \\ &= \ell \left( \frac{\partial^2}{\partial s \partial t} (m(tX, sY) - m(sY, tX)) \Big|_{t=s=0} \right)\end{aligned}$$

Therefore

$$[X, Y] = \frac{\partial^2}{\partial s \partial t} (m(tX, sY) - m(sY, tX)) \Big|_{t=s=0}$$

This can further be written as follows: as  $m(0, 0) = 0$ ,  $m(X, 0) = X$  and  $m(0, Y) = Y$ , the Taylor expansion of  $m$  at  $(0, 0)$  has the form

$$m(X, Y) = X + Y + B(X, Y) + h.o.t$$

where  $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  is a bilinear map. The reason it is bilinear is the following: the term  $B(X, Y)$  consist by definition of terms

$$\frac{\partial^2 m}{\partial x_i \partial x_j} + 2 \frac{\partial^2 m}{\partial x_i \partial x_j} + \frac{\partial^2 m}{\partial y_i \partial y_j}.$$

But the first and the second must vanish for the following reason:  
computing the first-order derivative at zero:

$$\begin{aligned}\frac{\partial m_j}{\partial x_i} &= \lim_{h \rightarrow 0} \frac{m_j(x_1, \dots, x_j + h, \dots, x_n, 0, \dots, 0) - m_j(x_1, \dots, x_n, 0, \dots, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x_1, \dots, x_j + h, \dots, x_n)_j - (x_1, \dots, x_n)_j}{h} = \delta_{ij},\end{aligned}$$

since  $m(X, 0) = X$ . Thus the first order derivative is a linear function.  
Differentiating one more time, we get zero.

Hence in this description, we have that

$$[X, Y] = B(X, Y) - B(Y, X).$$

**Example 13.8.** Let  $G = \text{GL}(V)$  for  $V$  a finite-dimensional real or complex vector space. Choose coordinates by setting  $x(A) = A - I$ . Then

$$m(X, Y) = (1 + X)(1 + Y) - 1 = XY + X + Y.$$

Hence in this case  $[X, Y] = XY - YX$ . ★

The space  $\text{End } V = T_e \text{GL}(V)$  with this bracket is denoted by  $\mathfrak{gl}(V)$ .

With this done, here's a theorem:

**Theorem 13.9.** *If  $\pi : G \rightarrow H$  is a Lie group homomorphism, then  $\pi_* = d_e \pi : \mathfrak{g} \rightarrow \mathfrak{h}$  is a Lie algebra homomorphism. That is, the map  $\pi_*$  preserves the bracket:*

$$\pi_*([X, Y]) = [\pi_*(X), \pi_*(Y)].$$

*Proof.* Recall first that for all  $X \in \mathfrak{g}$  we have  $\pi(\exp(tX)) = \exp(t\pi_*(X))$ . First off, we have, as above, that

$$L_X(f)(g) = \left. \frac{d}{dt}(f(g \exp(tX))) \right|_{t=0}$$

and one also verifies that

$$L_{\pi_*(X)}(f)(\pi(g)) = L_X(f \circ \pi)(g).$$

This follows more or less directly from the definitions.

[[[[ rest of the proof here ]]]]

□

## 14 Lie subgroups

By definition, a **closed Lie subgroup** of a Lie group  $G$  is a subgroup  $H$  which is also a closed submanifold.

Then  $H$  is itself a Lie group. We can identify  $\mathfrak{h} = T_e H$  with a subspace of  $\mathfrak{g} = T_e G$ . As the embedding  $H \hookrightarrow G$  is a Lie group homomorphism, this gives an identification of  $\mathfrak{h}$  with a Lie subalgebra of  $\mathfrak{g}$ .

In fact, the condition that  $H$  is a submanifold is redundant:

**Theorem 14.1.** *If  $G$  is a Lie group and  $H$  is a closed subgroup, then  $H$  is a closed submanifold.*

*Proof.* This will be a bit tricky.

Consider the subspace  $V \subset \mathfrak{g}$  consisting of  $X \in \mathfrak{g}$  such that there exist  $X_n \in \mathfrak{g}$  and  $\alpha_n \in \mathbb{R}$  such that  $\exp(X_n) \in H$  and  $\alpha_n X_n \rightarrow X$  as  $n \rightarrow \infty$  and also  $|\alpha_n| \rightarrow \infty$  as  $n \rightarrow \infty$ .

Note that  $V$  is non-empty since  $0 \in V$ .

**Claim 1:** If  $X \in V$ , then  $\exp(tX) \in H$  for all  $t \in \mathbb{R}$ .

Take  $t \in \mathbb{R}$ . As  $|\alpha_n| \rightarrow \infty$ , we can find  $m_n \in \mathbb{Z}$  such that

$$\frac{m_n}{\alpha_n} \rightarrow t$$

as  $t \rightarrow \infty$  (take for example  $m_n = [\alpha_n t]$ ). Then

$$\begin{aligned} H \ni \exp(X_n)^{m_n} &= \exp(m_n X_n) \\ &= \exp\left(\frac{m_n}{\alpha_n} \alpha_n X_n\right) \\ &\rightarrow \exp(tX). \end{aligned}$$

This implies that  $\exp(tX) \in H$ , since  $H$  is closed.

**Claim 2:** The set  $V \subset \mathfrak{g}$  is a sub-vector space of  $\mathfrak{g}$ .

By Claim 1,  $V$  is closed under multiplication by scalars. Therefore, we have to show that if  $X, Y \in V$ , then  $X + Y \in V$ .

Consider the map  $\gamma(t) = \exp(tX) \exp(tY)$ . Then  $\gamma'(0) = X + Y$ . For small  $|t|$ , we can write  $\gamma(t) = \exp(f(t))$ , where  $f$  is a smooth  $\mathfrak{g}$ -valued function, since  $\exp$  is a local diffeomorphism at  $e$ .

Then  $f'(0) = X + Y$ , so  $\frac{f(t)}{t} \rightarrow X + Y$  as  $t \rightarrow 0$ . In particular,  $nf(\frac{1}{n}) \rightarrow X + Y$  as  $n \rightarrow \infty$ . By definition of  $\gamma$ , we have  $\exp(f(\frac{1}{n})) \in H$ . Now by definition of  $V$ , letting  $\alpha_n = n$  and  $X_n = f(\frac{1}{n})$ , we conclude that  $X + Y \in V$ .

**Claim 3:**  $H$  is a closed submanifold with  $T_e H = V$ .

Let  $V'$  be a complementary subspace to  $V$  in  $\mathfrak{g}$ , i.e. a subspace such that  $\mathfrak{g} = V \oplus V'$ .

Consider the map  $\varphi : V \times V' \rightarrow G$  defined by

$$\varphi(X, Y) = \exp(X) \exp(Y).$$

Then  $d_{(0,0)}\varphi = \text{id}_{\mathfrak{g}}$ , so we can find a neighbourhood  $\Omega$  of 0 in  $V$  and  $\Omega'$  of 0 in  $V'$  such that  $\varphi$  is a diffeomorphism of  $\Omega \times \Omega'$  onto  $\varphi(\Omega \times \Omega')$ .

We will show that for  $n$  large enough, we have the equality

$$\varphi\left(\Omega \times \frac{1}{n}\Omega'\right) \cap H = \exp(\Omega \times \{0\}).$$

Suppose this is not true. Then we can find  $(X_n, X'_n) \in \Omega \times \Omega'$  such that  $\exp(X_n)\exp(\frac{X'_n}{n}) \in H$  and  $X'_n \neq 0$ .

Take any limit point  $X'$  of the sequence  $\frac{X'_n}{\|X'_n\|}$ . Then  $X' \neq 0$  and  $X' \in V'$  and  $\exp(\frac{X'_n}{n}) \in H$  (since the left term is in  $H$ ). Then also

$$\frac{n}{\|X'_n\|} \cdot \frac{X'_n}{n} \rightarrow X',$$

Then by definition of  $V$ , we must have  $X' \in V$ . This is a contradiction, since  $V'$  is closed.

Hence for some  $n$ , we have  $\varphi\left(\Omega \times \frac{1}{n}\Omega'\right) \cap H = \exp(\Omega \times \{0\})$ . Thus, letting  $U = \varphi\left(\Omega \times \frac{1}{n}\Omega'\right)$ , we can take as coordinate system  $x = \varphi^{-1}$ , such that

$$U \cap H = \{g \in H \mid \text{the coordinates } x(g) \text{ in } V' \text{ are zero}\}.$$

Thus  $H$  satisfies the definition of a closed submanifold near the identity  $e$ . For arbitrary  $h \in H$ , we can take the neighbourhood  $hU$  of  $h \in G$  and coordinates  $x_h(-) = x(h^{-1}-)$ .  $\square$

Here's a surprising consequence:

**Theorem 14.2.** *If  $\pi : G \rightarrow H$  is a continuous homomorphism of Lie group, then  $\pi$  is smooth.*

In particular, for Lie groups, there is no difference between finite-dimensional representations of locally compact groups and Lie groups.

*Proof.* Consider the graph  $\Gamma$  of  $\pi$ :

$$\gamma = \{(g, \pi(g)) \mid g \in G\} \subset G \times H.$$

Since  $\pi$  is continuous,  $\Gamma$  is a closed subgroup of the product. Hence by the theorem,  $\Gamma$  is also a closed submanifold.

Consider the projection  $\pi : G \times H \rightarrow G$ , and let  $q = p_1|_{\Gamma}$ . Then  $q$  is a smooth isomorphism between  $\Gamma$  and  $G$ .

Since  $q$  is a Lie group homomorphism, it has constant rank (proof: for all  $x \in \Gamma$ , we have  $q = l_{q(x)^{-1}} \circ q \circ l_x$ ).

Now, by the constant rank theorem, a constant rank bijection is a diffeomorphism. Thus  $q^{-1}$  is smooth.

But  $\pi$  is the composition of  $q^{-1}$  with  $p_2 : G \times H \rightarrow H$ , so  $\pi$  is smooth.  $\square$

**Proposition 14.3.** *If  $\pi : G \rightarrow H$  is a Lie group homomorphism, then  $\ker \pi$  is a closed Lie subgroup of  $G$  with Lie algebra  $\ker(\pi_* : \mathfrak{g} \rightarrow \mathfrak{h})$ .*

*Proof.* Clearly  $\ker \pi$  is closed, hence it is a closed subgroup. Hence it is a closed Lie subgroup by Theorem 14.1. So we need to show the second statement (let  $\text{LieAlg}(G)$  for the moment denote the Lie algebra of a Lie group):

$$\begin{aligned} X \in \text{LieAlg}(\ker \pi) &\Leftrightarrow \exp(tX) \in \ker \pi \forall t \\ &\Leftrightarrow \pi(\exp tX) = e \forall t \\ &\Leftrightarrow \exp(t\pi_* X) = e \forall t \\ &\Leftrightarrow \pi_*(X) = 0, \end{aligned}$$

so  $X \in \ker \pi_*$ . □

**Proposition 14.4.** *Let  $\pi : G \rightarrow \text{GL}(V)$  be a representation of a Lie group  $G$  on a finite-dimensional real or complex vector space  $V$ . Take any  $v \in V$ . Then*

$$H = \{g \in G \mid \pi(g)v = v\}$$

*is a closed Lie subgroup of  $G$  with Lie algebra*

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \pi_*(X)v = 0\}.$$

*Proof.* That  $H$  is closed Lie subgroup is immediate.

Suppose  $X \in \mathfrak{h}$ . Then  $\exp(tX) \in H$  for all  $t$ . But this means that  $\pi(\exp(tX))v = \exp(t\pi_*(X))v = v$  for all  $t$ . But differentiating both sides give  $\pi_*(X)v = 0$ . □

## 15 Classical Lie groups

This section will give examples of Lie groups and their Lie algebras explicitly in terms of matrices. The **classical Lie groups** are groups of linear transformations preserving some symmetric/skew-symmetric/Hermitian form on a real/complex/quaternionic space, as well as their special versions.

### 15.1 The general linear group

Let  $V$  be a real or complex vector space of dimension  $n$ . Then  $\mathrm{GL}(V)$  is the set of invertible linear operators  $V \rightarrow V$ . If a basis for  $V$  is given, then  $\mathrm{GL}(V)$  is realized as the set of invertible  $n \times n$  matrices with determinant non-zero.

If  $V$  is a real vector space, then the dimension of  $\mathrm{GL}(V)$  is  $n^2$ , and if  $V$  is complex, then the dimension is  $2n^2$ .

The Lie algebra of  $\mathrm{GL}(V)$ , denoted by  $\mathfrak{gl}(V)$  is identified with  $\mathrm{End}(V)$ , or, if a basis is given, with  $\mathrm{Mat}_{n,n}(V)$ , since  $\mathrm{GL}(V)$  is an open subset in  $\mathrm{End}(V)$ .

### 15.2 The special linear group

Again, let  $V$  be a real or a complex vector space. The **special linear group**  $\mathrm{SL}(V)$  is by definition the subset of  $\mathrm{GL}(V)$  consisting of operators of determinant 1. Since this is a closed condition, it follows from Theorem 14.1 that  $\mathrm{SL}(V)$  is actually a closed Lie subgroup of  $\mathrm{GL}(V)$ . Note that  $\dim \mathrm{SL}(V) = n^2 - 1$  if  $V$  is a real vector space.

I claim that the Lie algebra of  $\mathrm{SL}(V)$  is the space of matrices in  $\mathrm{End}(V)$  of trace zero. To prove this, let  $\{A_t\}$  be an arc in  $\mathrm{SL}(V)$  such that  $A_0 = I_n$  and  $\frac{d}{dt}\big|_{t=0} A_t = X$ . Then  $A_t e_1 \wedge \dots \wedge A_t e_n = e_1 \wedge \dots \wedge e_n$  for all  $t$  since the determinant of  $A_t$  is 1 for all  $t$ . Thus, taking derivatives of both sides, we get

$$\begin{aligned} 0 &= \frac{d}{dt}\bigg|_{t=0} A_t e_1 \wedge \dots \wedge A_t e_n \\ &= \sum_{i=1}^n e_1 \wedge \dots \wedge X e_i \wedge \dots \wedge e_n \\ &= \mathrm{Tr}(X) e_1 \wedge \dots \wedge e_n. \end{aligned}$$

In the second line we used the product rule for the derivative (which works just as well with exterior products (in fact, with any bilinear operator)). Hence  $\mathrm{Tr}(X) = 0$ . Counting dimensions, we see that the space of traceless matrices have the same dimension as  $\mathrm{SL}(V)$ , so that we can identify  $\mathfrak{sl}(V)$  with the space of traceless matrices.

### 15.3 Groups defined by preservation of a bilinear form

Let  $V$  be as above and consider a bilinear  $B : V \otimes V \rightarrow \mathbb{R}$  (or  $\mathbb{C}$ ) form on  $V$ . Let

$$G = \{g \in \mathrm{GL}(V) \mid B(gv, gw) = B(v, w) \text{ for all } v, w \in V\}.$$

Then  $G$  is a subgroup of  $\mathrm{GL}(V)$ . We shall verify that it is indeed a Lie subgroup.

Let  $W$  be the space of bilinear forms on  $V$ . Note that  $W \simeq V^* \otimes V^*$ . Define a representation  $\pi$  of  $\mathrm{GL}(V)$  on  $W$  by

$$(\pi(g)B')(v, w) = B'(g^{-1}v, g^{-1}w).$$

Then  $G$  is exactly the set  $\{g \in \mathrm{GL}(V) \mid \pi(g)B = B\}$ , so that from Proposition 14.4, it follows that  $G$  is a Lie group. with Lie algebra

$$\mathfrak{g} = \{X \in \mathfrak{gl}(V) \mid \pi_*(X)B = 0\}.$$

We want this a little more explicit. So note that we can write

$$\pi_*(X) = \left. \frac{d}{dt} \pi(\exp(tX)) \right|_{t=0}$$

This is a simple calculation using the functoriality of  $\pi_*$  and the definition of the exponential map. Hence we get

$$\begin{aligned} (\pi_*(X)B)(v, w) &= \left. \frac{d}{dt} (\pi(\exp(tX))B)(v, w) \right|_{t=0} \\ &= \left. \frac{d}{dt} B(e^{-tX}v, e^{-tX}w) \right|_{t=0} \\ &= -B(Xv, w) - B(v, Xw). \end{aligned}$$

We used the chain rule in the last line.

Thus

$$\mathfrak{g} = \{X \in \mathfrak{gl}(V) \mid B(Xv, w) + B(v, Xw) = 0\}.$$

The dimension of  $G$  will of course depend upon the choice of  $B$ .

### 15.4 The orthogonal group $O(n)$

Let  $V = \mathbb{R}^n$  and let  $B$  be the usual scalar product

$$(v, w) = \sum_{i=1}^n v_i w_i.$$

Then the group  $G$  for  $B$  as above is called the **orthogonal group** and denoted by  $O(n)$ ,  $O_n(\mathbb{R})$  or  $O(n; \mathbb{R})$ .

Note that we have

$$(Xv, w) + (v, Xw) = (Xv, w) + (X^T v, w) = ((X + X^T)v, w).$$

This inner product should be zero for all  $v, w$ . Hence we have:

$$\mathfrak{o}(n) = \{X \in \mathfrak{gl}_n(\mathbb{R}) \mid X + X^T = 0\},$$

by the previous section. Thus the Lie algebra consists of all skew-symmetric matrices. We see that  $O(n)$  have dimension  $1 + 2 + \dots + (n-1) = n(n-1)/2$ . The **special orthogonal group**  $SO(n)$  is the subgroup of  $O(n)$  whose determinants are 1. In other words:  $SO(n) = O(n) \cap \mathrm{SL}_n(\mathbb{R})$ . This is the connected component of the identity of  $O_n(\mathbb{R})$ . Then the Lie algebra is given by

$$\mathfrak{sl}_n(\mathbb{R}) = \{X \in \mathfrak{gl}_n(\mathbb{R}) \mid X^T = -X, \mathrm{Tr} X = 0\},$$

which is equal to  $\mathfrak{o}(n)$  since the condition  $X^T = -X$  implies that  $\mathrm{Tr} X = 0$ .

We can do the same for  $V = \mathbb{C}^n$  with the same bilinear form. The corresponding groups are denoted by  $O_n(\mathbb{C})$  and  $SO_n(\mathbb{C})$ . These are however not compact anymore, since  $B \otimes \mathbb{C}$  is not positive definite.

## 15.5 Indefinite orthogonal groups $O(k, l)$

Let  $V = \mathbb{R}^{k+l}$  for  $k, l \geq 1$  and consider the bilinear form

$$\begin{aligned} B(v, w) &= \sum_{i=1}^k v_i w_i - \sum_{i=k+1}^{k+l} v_i w_i \\ &= (Av, w) \end{aligned}$$

where

$$A = \begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix}.$$

The group of transformations preserving this form is then by definition the **indefinite orthogonal group**  $O(k, l)$ . As above, we see that its Lie algebra is

$$\mathfrak{o}(k, l) = \{X \in \mathfrak{gl}_{k+l}(\mathbb{R}) \mid AX + X^T A = 0\}.$$

This can be made even more explicit. Let  $X \in \mathfrak{o}(k, l)$ . Write

$$X = \begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$



where  $P$  is a  $k \times k$  matrix,  $Q$  is a  $l \times k$  matrix,  $R$  is a  $k \times l$  matrix, and  $S$  is a  $l \times l$  matrix. Then one can check that the condition for  $X$  to be in  $\mathfrak{o}(k, l)$  is equivalent to  $P = -P^T$ ,  $S = -S^T$  and  $R = Q$ . Thus the dimension of  $\mathfrak{o}(k, l)$  is

$$\frac{k(k-1)}{2} + \frac{l(l-1)}{2} + kl = \frac{n(n-1)}{2},$$

where  $n = k + l$ .

We also have its special version:  $SO(k, l) := O(k, l) \cap SL_{k+l}(\mathbb{R})$ .

## 15.6 The unitary group $U(n)$

Let  $V = \mathbb{C}^n$ , and consider the Hermitian form

$$\langle v, w \rangle = \sum_{i=1}^n v_i \overline{w_i}.$$

The **unitary group**  $U(n)$  is the group of  $\mathbb{C}$ -linear transformations preserving this form. The proof in the case of  $O(n)$  goes through here as well, so that we see that

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid X = -X^*\},$$

where the star means the conjugate transpose of  $X$ . We also have the **special unitary group**  $SU(n) = U(n) \cap SL_n(\mathbb{C})$  with Lie algebra

$$\mathfrak{su}(n) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid X = -X^*, \operatorname{Tr} X = 0\}.$$

The dimension of  $U(n)$  is  $n^2$ . To see this, note that matrices in the Lie algebra consist of purely complex entries along the diagonal, plus free entries above the diagonal (the entries below the diagonal are minus conjugate those above). Thus the total freedom is  $n + 2 \cdot \frac{n(n-1)}{2} = n^2$ .

## 15.7 The symplectic Lie groups $Sp(2n, \mathbb{R})$

Let  $V = \mathbb{R}^{2n}$  or  $V = \mathbb{C}^{2n}$  and consider

$$\begin{aligned} (v, w) &= \sum_{i=1}^{2n} (v_i w_{i+n} - v_{i+n} w_i) \\ &= (Jv, w) \end{aligned}$$

where

$$J = \begin{pmatrix} 0 & -I_n \\ -I_n & 0 \end{pmatrix}.$$

The group of transformations preserving this form are the **symplectic groups**  $Sp(2n, \mathbb{R})$  or  $Sp(2n, \mathbb{C})$ . The Lie algebra is

$$\mathfrak{sp}(2n; k) = \{X \in \mathfrak{gl}_n(k) \mid JX + X^T J = 0\}.$$

In the same way as with the indefinite orthogonal group, one can compute the Lie algebra more explicitly. Namely, write  $X$  as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Here  $A, B, C$  and  $D$  are  $n \times n$  matrices. Here the condition on  $X$  to be in the Lie algebra is equivalent to  $D = -A^T$  and  $B$  and  $C$  being skew-symmetric matrices. Hence the dimension is  $n^2 + 2 \cdot \frac{n^2 - n}{2} = 2n^2 - n$ .

### 15.8 The compact symplectic group $Sp(n)$

This is the group defined by  $Sp(n) = Sp(2n; \mathbb{C}) \cap U(2n)$  with Lie algebra

$$\mathfrak{sp}(n) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid X + X^* = 0, JX + X^T J = 0\},$$

where  $J$  is as above. It can be shown that  $Sp(n)$  is the subgroup of  $GL_n(\mathbb{H})$  preserving  $\sum_{i=1}^n v_i \bar{w}_i$ , where  $\mathbb{H}$  are the quaternions.

## 16 Simply connected Lie groups

There is a close correspondence between the category of Lie algebras and the category of Lie groups. In particular, it is true that any Lie subalgebra correspond to a Lie subgroup with that Lie algebra.

To show this, we need to remind ourselves of some definitions from manifold theory, however.

### 16.1 Some facts about distributions

Let  $M$  be a smooth manifold and let  $D \subset TM$  be a subbundle of the tangent bundle of  $M$ . Then an immersed submanifold  $N \subset M$  is called an **integral manifold** if  $T_p N = D_p$  for all  $p \in N$ .

Here's the so-called *Frobenius integrability condition*:

**Theorem 16.1.** *Suppose  $D$  is integrable. Then for all  $p_0 \in M$ , there exists a coordinate chart  $(x, U)$  around  $p_0$  such that*

$$D_p = \text{Span} \left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^k} \Big|_p \right\}$$

for all  $p \in U$ .

The proof is surprisingly short, and can be found in for example Spivak's book [4].

This implies that there locally exists a unique integral manifold passing through a given point:

$$N = \{p \in U \mid x^j(p) = x^j(p_0) \text{ for } j = k+1, \dots, n\}.$$

Uniqueness means that if  $N'$  is another integral manifold passing through  $p$ , then  $N \cap N'$  is open in  $N$  and  $N'$ .

This implies that every point lies in a unique maximal connected integral manifold, called a **leaf of the foliation defined by  $D$** .

### 16.2 Back to Lie subgroups

**Theorem 16.2.** *Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ . Let  $\mathfrak{h} \subset \mathfrak{g}$  be a Lie subalgebra.*

*Then there exists a unique Lie subgroup  $H \subset G$  with Lie algebra  $\mathfrak{h}$ .*

*Proof.* Consider the distribution  $D$  on  $G$  defined by  $D_g = (d_e l_g)(\mathfrak{h})$ . As  $\mathfrak{h}$  is a Lie subalgebra of  $\mathfrak{g}$ , it is integrable.

Since left translations map  $D$  into itself, they map leaves into leaves of the corresponding foliation.

It follows that if  $H$  is the leaf passing through  $e$ , then  $H = hH$  for all  $h \in H$ . Therefore  $H \subset G$  is a subgroup. It is a Lie subgroup because it is locally closed.

Now, if  $K \subset G$  is another connected Lie subgroup with Lie algebra  $\mathfrak{h}$ , then  $K$  would be an integral manifold of  $D$  passing through  $e$ . Thus  $K$  is an open subgroup. But  $K$  is closed as well, since

$$K = H \setminus \left( \bigcup_{h \notin K} hK \right).$$

As  $H$  is connected,  $H \subset K$ . □

**Proposition 16.3.** *Assume  $\pi : G \rightarrow H$  is a Lie group homomorphism with  $G$  connected. Then  $\pi$  is completely determined by  $\pi_* : \mathfrak{g} \rightarrow \mathfrak{h}$ .*

*Proof.* Consider the subgroup  $K$  generated by  $\exp \mathfrak{g} \subset G$ . Since  $\exp \mathfrak{g}$  contains an open neighbourhood of  $e$ ,  $K$  is an open subgroup of  $G$ . But this implies that  $K$  is closed, hence  $K = G$ .

Thus, the statement is proved since  $\pi(\exp X) = \exp \pi_* X$ , and every element of  $G$  is a product of terms  $\exp X_j$  with  $X_j \in \mathfrak{g}$  because the image of  $\exp \mathfrak{g}$  generates  $G$ . □

**Remark.** *This proves that the functor  $\text{LieAlg} : \text{LieGrps} \rightarrow \text{LieAlgs}$  sending a Lie group to its Lie algebra is faithful (for connected Lie groups in the domain).*

*It is not full, however. Take for example  $G = S^1$  and  $H = \mathbb{R}$ . Then  $\mathfrak{g} = \mathfrak{h} = \mathbb{R}$ , so  $\text{Hom}_{\text{LieAlg}}(\mathfrak{g}, \mathfrak{h}) = \mathbb{R}$ , but  $\text{Hom}_{\text{LieGrp}}(S^1, \mathbb{R}) = \{0\}$ <sup>6</sup>.*

Recall that a map  $p : N \rightarrow M$  is a **covering map** if  $p$  is continuous and surjective and a local homeomorphism such that each point  $p \in M$  has a neighbourhood  $U$  such that the pre-image decomposes as

$$p^{-1}(U) = \bigsqcup_{i \in I} U_i$$

with  $p|_{U_i} : U_i \rightarrow U$  a homeomorphism.

**Example 16.4.** Suppose that  $\pi : G \rightarrow H$  is a Lie group homomorphism with  $H$  connected, such that  $\pi_* : \mathfrak{g} \rightarrow \mathfrak{h}$  is an isomorphism. Then  $\pi$  is a covering map:

We can find  $U \supset e \in G$  such that  $\pi|_U : U \rightarrow \pi(U)$  is a diffeomorphism. Choose a neighbourhood  $V$  of  $e \in G$  such that  $VV^{-1} \subset U$ .

Note that  $\pi$  is open: if  $U$  is a small open set in  $G$ , containing  $g$ , then  $g^{-1}U$  is small and contains  $e$ . Thus  $\pi(g^{-1}U)$  is open since  $\pi$  is a homeomorphism close to the identity. Hence  $\pi(U) = \pi(g)\pi(g^{-1}U)$  is open since  $\pi(g)$  is a diffeomorphism. Also  $\text{image}(G)$  is closed, since it

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<sup>6</sup>Proof: Any such homomorphism must have compact image. But the image must also be discrete, since otherwise it would generate  $\mathbb{R}$ . Hence the image is a finite subgroup of  $\mathbb{R}$ . But the only finite subgroup of  $\mathbb{R}$  is  $\{0\}$ .

is equal to  $H \setminus \cup_{U \subset G} \pi(U)$ . Thus, since  $H$  is connected,  $\pi$  is surjective. Then for all  $g \in G$ , we have

$$\pi^{-1}(\pi(g)\pi(V)) = \bigsqcup_{h \in \ker \pi} hgV.$$

We have to check that  $h_1g \cap h_2gV = \emptyset$  for  $h_1, h_2 \in \ker \pi$  and  $h_1 \neq h_2$ .

Suppose  $h_1gv_1 = h_2gv_2$  for some  $v_1, v_2 \in V$ . This is equivalent to  $g^{-1}h_2^{-1}h_1g = v_2v_1^{-1} \in VV^{-1} \subset U$ . The left hand side is in the kernel of  $\pi$ , but  $\pi$  is a homeomorphism on  $U \ni e$ , hence  $g^{-1}h_2^{-1}h_1g = e$ , hence  $h_1 = h_2$ .  $\star$

If a topological space is nice enough (for example a connected manifold), then there exists a unique universal covering space  $\pi : \widetilde{M} \rightarrow M$  such that if  $p : P \rightarrow N$  is any covering space and  $f : M \rightarrow N$  is a continous map, then there exists a unique map  $\tilde{f} : \widetilde{M} \rightarrow P$  such that  $p \circ \tilde{f} = f \circ \pi$ .

Furthermore, a covering map  $\lambda : Q \rightarrow M$  is universal if and only if  $Q$  is simply connected.

We can define  $\widetilde{M}$  as the set of equivalence classes  $[\gamma]$  of curves  $\gamma : [0, 1] \rightarrow M$  such that  $\gamma_1 \sim \gamma_2$  if  $\gamma_1(1) = \gamma_2(1)$  and  $\gamma_1$  and  $\gamma_2$  are homotopic within the class of curves ending at  $\gamma_1(1) = \gamma_2(1)$ . This set has a natural topology. See e.g. Munkres [2].

**Proposition 16.5.** *Let  $G$  be a connected Lie group and  $\pi : \widetilde{G} \rightarrow G$  a universal covering space. Then  $\widetilde{G}$  has a natural structure of a Lie group so that  $\pi$  becomes a Lie group homomorphism.*

*Proof.* Suppose  $\bar{e} \in \pi^{-1}(e)$ . Then using that  $\pi \times \pi : \widetilde{G} \times \widetilde{G} \rightarrow G \times G$  is a universal cover, there exist a unique lift of the product map  $m : G \times G \rightarrow G$  to a map  $\tilde{m} : \widetilde{G} \times \widetilde{G} \rightarrow \widetilde{G}$  such that  $\tilde{m}(\bar{e}, \bar{e}) = \bar{e}$ .

Then  $\widetilde{G}$  is a Lie group with inverse given by the lift of the inverse such that  $\tilde{e} \mapsto \tilde{e}$ .  $\square$

**Theorem 16.6.** *Let  $G, H$  be Lie groups with  $G$  simply connected. Then any Lie algebra homomorphism  $\rho : \mathfrak{g} \rightarrow \mathfrak{h}$  integrates to a Lie group homomorphism  $\pi : G \rightarrow H$ .*

*Proof.* Consider the Lie subalgebra

$$\mathfrak{p} = \{(X, \rho(X)) \mid X \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{h}.$$

Let  $P \subset G \times H$  be the connected Lie subgroup with Lie algebra  $\mathfrak{p}$ . Consider the projection  $p_1 : G \times H \rightarrow G$  and let  $q = p_1|_P$ . Then  $q_* : \mathfrak{p} \rightarrow \mathfrak{g}$  is an isomorphism.

Hence  $q : P \rightarrow G$  is a covering map. As  $G$  is already simply-connected,  $q$  must be an isomorphism. Then we can define  $\pi$  as the composition of  $q^{-1} : G \rightarrow P \subset G \times H$  with the projection  $p_2 : G \times H \rightarrow H$ .  $\square$

**Corollary 16.7.** *Two simply connected Lie groups are isomorphic if and only if  $\mathfrak{g} \simeq \mathfrak{h}$ .*

Here is a theorem we will only state, known as “**Lie’s third fundamental theorem**”, which is due to Cartan.

**Theorem 16.8.** *For any finite-dimensional real Lie algebra  $\mathfrak{g}$ , there exists a simply connected Lie group  $G$  with Lie algebra  $\mathfrak{g}$ .*

Thus the category of simply connected Lie groups is equivalent to the category of Lie algebras.

One possible proof is based on **Ado’s theorem**, which we also will not prove:

**Theorem 16.9.** *Any finite-dimensional real Lie algebra is isomorphic to a Lie subalgebra of  $\mathfrak{gl}_n(\mathbb{R})$ .*

Given this, to prove Lie’s theorem, we can take  $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$ . Then let  $G$  be the Lie subgroup of  $GL_n(\mathbb{R})$  corresponding to  $\mathfrak{g}$ , and then pass to the universal covering.

**Remark.** *It is not true that every Lie group can be embedded in  $GL_n(\mathbb{R})$ . Here’s an example: let  $G$  be the universal cover of  $SL_2(\mathbb{R})$ .*

*One can show that  $SL_2(\mathbb{R})$  is a fiber bundle with fiber  $\mathbb{R}$  over  $\mathbb{R}^2 \setminus \{0\}$ , hence its fundamental group is  $\mathbb{Z}$ .*

*Complexifying, we get the corresponding group  $SL_2(\mathbb{C})$ , which in the same manner can be shown to have trivial fundamental group.*

*Now suppose we have a representation  $p : G \rightarrow GL_n(\mathbb{R})$  for some  $n$ . By inclusion, we have a representation  $G \xrightarrow{p} GL_n(\mathbb{R}) \rightarrow GL_n(\mathbb{C})$ .*

*This is the same map as the composition*

$$G \xrightarrow{\pi} SL_2(\mathbb{R}) \rightarrow SL_2(\mathbb{C}) \rightarrow GL_n(\mathbb{C})$$

*where  $\pi$  is the covering space map. Since the map on tangent spaces is the same. But  $\pi$  is not injective, hence  $p$  cannot be injective.*

If  $G$  is connected but not simply-connected, then  $G \simeq \tilde{G}/\Gamma$  for some discrete normal subgroup  $\Gamma \subset \tilde{G}$ . It is easy to show that any such subgroup must be contained in the center of  $G$ .

**Example 16.10.** The groups  $SU(n)$ ,  $Sp(n)$  are simply connected.

The group  $SO(n; \mathbb{R})$  is connected, but not simply connected. We have  $\pi_1(SO(2)) = \pi_1(S^1) = \mathbb{Z}$ . For  $n > 2$ , we have  $\pi_1(SO(n)) = \mathbb{Z}/2$ . The universal covers are the “spin groups”.

We will prove (almost) this later.

★

## 17 The adjoint representation

Let  $G$  be a Lie group. For any  $g \in G$  consider the automorphism  $h \mapsto ghg^{-1}$  of  $G$ . Thus we have a map  $G \rightarrow \text{Aut}(G)$ . Its differential at the identity is denoted by  $\text{Ad}$ , and we get a homomorphism  $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$ , called the **adjoint representation**.

The differential of  $\text{Ad}$  at  $e \in G$  gives a Lie algebra homomorphism

$$\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}),$$

also called the **adjoint representation**.

We can compute this map more explicitly. Let  $X, Y \in \mathfrak{g}$ . Then by definition

$$(\text{ad } X)(Y) = \left. \frac{d}{dt} (\text{Ad } \exp(tX))(Y) \right|_{t=0}.$$

Let  $f \in C^\infty(G)$ . Then

$$\begin{aligned} (d_e f)(\text{ad}(X)(Y)) &= \left. \frac{d}{dt} (d_e f)(\text{Ad } \exp(tX)(Y)) \right|_{t=0} \\ &= \left. \frac{\partial^2}{\partial t \partial s} f(\exp(tX) \exp(sY) \exp(-tX)) \right|_{s=t=0} \\ &= \left. \frac{\partial^2}{\partial s \partial t} f(\exp(tX) \exp(sY)) - f(\exp(sY) \exp(tX)) \right|_{s=t=0} \\ &= (d_e f)([X, Y]). \end{aligned}$$

Therefore,  $(\text{ad } X)(Y) = [X, Y]$ .

Of course the adjoint map  $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  defined by  $X \cdot Y = [X, Y]$  is well-defined for any Lie algebra  $\mathfrak{g}$ . Note that the identity

$$\text{ad}[X, Y] = [\text{ad}(X), \text{ad}(Y)] = \text{ad}(X) \text{ad}(Y) - \text{ad}(Y) \text{ad}(X)$$

is exactly the Jacobi identity.

The kernel of  $\text{ad}$  is exactly the **center of  $\mathfrak{g}$** :

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \forall Y \in \mathfrak{g}\}.$$

**Proposition 17.1.** *Let  $G$  be a Lie group and let  $X, Y \in \mathfrak{g}$ . Then the following two conditions are equivalent:*

1.  $[X, Y] = 0$ .
2.  $\exp(tX) \exp(sY) = \exp(sY) \exp(tX)$  for all  $s, t \in \mathbb{R}$ .

*Furthermore, if these conditions are satisfied, then  $\exp(X + Y) = \exp(X) \exp(Y)$ .*

*Proof.* This is Lemma 13 on page 157 in [4]. □

**Proposition 17.2.** *Any connected abelian Lie group  $G$  is isomorphic to  $\mathbb{R}^k \times \mathbb{T}^l$  for some natural numbers  $k, l \geq 0$ .*

*Proof.* Consider  $\mathfrak{g}$  as a Lie group under addition. Then  $\exp : \mathfrak{g} \rightarrow G$  is a Lie group homomorphism by Proposition 17.1.

By Example 16.4,  $\exp$  is a covering map. Let  $\Gamma$  be its kernel, which must be a discrete subgroup of  $\mathfrak{g}$ .

But for any discrete subgroup  $\Gamma$  of  $\mathfrak{g} \simeq \mathbb{R}^n$ , there is a basis  $\{e_1, \dots, e_k\}$  such that  $\Gamma$  is equal to  $\mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_k$ . Hence

$$\mathfrak{g}/\Gamma \simeq \mathbb{R}^n/(\mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_k) \simeq \mathbb{T}^k \times \mathbb{R}^{n-k}.$$

□



## 18 Structure and representation theory of $SU(2)$

Recall that the Lie algebra of  $SU(2)$  is

$$\mathfrak{su}(2) = \{X \in \mathfrak{gl}_2(\mathbb{C}) \mid X^* + X = 0, \quad \text{Tr}(X) = 0\}.$$

Recall also that

$$\mathfrak{sl}_2(\mathbb{C}) = \{X \in \mathfrak{gl}_2(\mathbb{C}) \mid \text{Tr}(X) = 0\}.$$

The latter is a complex vector space, and in fact we have  $\mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}_2(\mathbb{C})$ . Note that one inclusion is obvious. For the other, note that any  $X \in \mathfrak{sl}_2(\mathbb{C})$  can be written as

$$X = \frac{X - X^*}{2} + i \frac{X + X^*}{2i},$$

namely as a sum of two matrices in  $\mathfrak{su}(2)$ .

There is a standard  $\mathbb{C}$ -basis for  $\mathfrak{sl}_2(\mathbb{C})$  given by the following three matrices:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A basis for  $\mathfrak{su}(2)$  over  $\mathbb{R}$  is  $E - F$ ,  $i(E + F)$  and  $iH$ . Note that  $\mathbb{R}iH \subset \mathfrak{su}(2)$  is the Lie algebra of the torus

$$T = \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \mid z \in S^1 \right\}.$$

The Lie algebra structure on  $\mathfrak{sl}_2(\mathbb{C})$  is defined by the rules

$$[H, E] = 2E \quad [H, F] = -2F \quad [E, F] = H.$$

**Lemma 18.1.** *The group  $SU(2)$  is simply-connected.*

*Proof.* One checks that

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1 \right\}$$

This is clearly homeomorphic to  $S^3$ , which is simply-connected.  $\square$

Therefore, from Theorem 16.6, we have a one to one correspondence between Lie group homomorphisms  $SU(2) \rightarrow GL(V)$  and Lie algebra homomorphisms  $\mathfrak{su}(2) \rightarrow \mathfrak{gl}(V)$ .

Recall that if  $\mathfrak{g}$  is Lie algebra, then a **representation** of  $\mathfrak{g}$  on  $V$  is the same thing as a Lie algebra homomorphism  $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$ . We also say that  $V$  is a  **$\mathfrak{g}$ -module**. We say that a subspace  $W \subset V$  is **invariant** if  $X \cdot W \subset W$  for all  $X \in \mathfrak{g}$ . A representation  $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$  is **irreducible** (or the  $\mathfrak{g}$ -module  $V$  is **simple**) if there are no proper invariant subspaces.

It is an exercise to see that if  $G$  is connected Lie group, then these notions coincide with those of representation of  $G$  (as opposed to representations of  $\mathfrak{g}$ ).

We now want to classify irreducible representations of  $SU(2)$  on *complex* finite-dimensional vector spaces  $V$ . Equivalently, we want to classify irreducible representations of  $\mathfrak{su}(2)$  on complex finite-dimensional vectors spaces  $V$ .

Now, since  $\mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{sl}_2(\mathbb{C})$ , any Lie algebra homomorphism  $\mathfrak{su}(2) \rightarrow \mathfrak{gl}(V)$  (where we consider  $V$  as a real vector space) extends uniquely to a Lie algebra homomorphism  $\mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{gl}(V)$  of complex vector spaces.

So classifying  $\mathfrak{su}(2)$ -modules is the same as classifying  $\mathfrak{sl}_2(\mathbb{C})$ -modules.

Assume now  $V$  is a finite-dimensional  $\mathfrak{sl}_2(\mathbb{C})$ -module. Then this exponentiates to a representation of  $SU(2)$ :

$$\begin{array}{ccc}
 \mathfrak{su}(2) \hookrightarrow \mathfrak{sl}_2(\mathbb{C}) & & SU(2) \hookrightarrow SL_2(\mathbb{C}) \\
 \searrow \downarrow & \xrightarrow{\exp} & \searrow \downarrow \\
 & \mathfrak{gl}(V) & GL(V)
 \end{array}$$

By restriction, the torus  $T \subset SU(2)$  act on  $V$  as well, and by complete reducibility,  $V$  decomposes as a sum of  $T$ -invariant subspaces. Since  $\{tiH \mid t \in \mathbb{R}\}$  exponentiates to the torus  $\mathbb{T}$ ,  $V$  decomposes in as a sum of  $H$ -invariant subspaces as well.

**Lemma 18.2.** *The eigenvalues of  $H$  are integers.*

*Proof.* Suppose  $H \cdot v = \lambda \cdot v$ . Then  $\exp(2\pi i H) \cdot v = Iv = v$ , since  $\exp(2\pi i H) = I$ . But then  $v = \exp(2\pi i \lambda)v$ . This is only true if  $\lambda \in \mathbb{Z}$ .  $\square$

It is common to index the eigenvalues of  $H$  on  $V$  by half-integers. Thus for  $s \in \frac{1}{2}\mathbb{Z}$ , we put

$$V(s) = \{\xi \in V \mid H\xi = 2s\xi\}.$$

Elements of  $V(s)$  are called vectors of **weight**  $s$ . Thus we write

$$V = \bigoplus_{s \in \frac{1}{2}\mathbb{Z}} V(s).$$

Now recall the commutation relations in  $\mathfrak{sl}_2(\mathbb{C})$ :  $[H, E] = 2E$  and  $[H, F] = -2F$ .

**Lemma 18.3.** *We have*

$$EV(s) \subset V(s+1)$$

and

$$FV(s) \subset V(s-1).$$

*In particular, if  $v \neq 0$  is a vector of highest weight, then  $Ev = 0$ .*

*Proof.* Let  $v \in EV(s)$ . Then  $v = Ew$  for  $w \in V(s)$ . Then

$$\begin{aligned} Hv &= HEw \\ &= HEw - EHw + EHw \\ &= [H, E]w + EHw \\ &= 2Ew + 2sEw \\ &= 2v + 2sv \\ &= 2(v+1)v. \end{aligned}$$

So  $v \in V(s+1)$ . The calculation with  $F$  is similar.  $\square$

These two observations are enough to prove quite a lot about representations of  $\mathfrak{sl}_2(\mathbb{C})$ .

**Lemma 18.4.** *Suppose  $V$  is a finite-dimensional  $\mathfrak{sl}_2(\mathbb{C})$ -module and  $\xi \in V(s) \setminus \{0\}$  satisfies  $E\xi = 0$  (that is,  $\xi$  is of highest weight). Then*

1.  $s \geq 0$ .
2.  $F^{2s+1}\xi = 0$  and  $F^k\xi \neq 0$  for  $k = 0, \dots, 2s$ .
3.  $EF^k\xi = (2s - k + 1)kF^{k-1}\xi$  for all  $k \geq 0$ .

*Proof.* We start by proving number 3 by induction. The base case  $k = 0$  gives  $E\xi = 0$ , which is clear by assumption.

Note that  $EF = [E, F] + FE = H + FE$ . Note also that since  $\xi \in V(s)$ , we have  $F^k\xi \in V(s-k)$  by the previous lemma. Then:

$$\begin{aligned} EF^{k+1}\xi &= EFF^k\xi \\ &= (H + FE)F^k\xi \\ &= H(F^k\xi) + F(EF^k\xi) \\ &= 2(s-k)F^k\xi + (2s-k+1)kF^{k-1}\xi \\ &= (2s-k+2sk-k^2)F^k\xi. \end{aligned}$$

Now note that this is what we want, because

$$(2s - (k + 1) + 1)(k + 1) = (2s - k)(k + 1) = 2sk + 2s - k^2 - k.$$

Thus number 3) is proved.

Now let  $n$  be the smallest number such that  $F^n \xi = 0$  (the **smallest weight**). Then  $F^n \xi = 0$ . Then it follows from 3) that

$$0 = EF^n \xi = (2s - n + 1)nF^{n-1} \xi.$$

Hence  $2s - n + 1 = 0$ , so  $n = 2s + 1$ . This proves 2). This also proves 1) because  $n \geq 1$ .  $\square$

In view of this lemma, for any  $s \in \frac{1}{2}\mathbb{Z}_+ = \{0, \frac{1}{2}, 1, \dots\}$ , define an  $\mathfrak{sl}_2(\mathbb{C})$ -module  $V_s$  as follows: it is a vector space with basis elements denoted by

$$\{\xi_{-s}^s, \xi_{-s+1}^s, \dots, \xi_s^s\},$$

and where  $H, F$  and  $E$  act as follows:

$$H\xi_t^s = 2t\xi_t^s \quad F\xi_t^s = \xi_{t-1}^s \quad E\xi_{s-k}^s = (2s - k + 1)k\xi_{s-k+1}^s.$$

Then it is an (easy) exercise to see that this defines an  $\mathfrak{sl}_2(\mathbb{C})$ -module. It is also easy to see that  $V_s$  is irreducible, since, starting from any basis vector  $\xi_t^s$ , we can use  $F$  and  $E$  to raise and lower the indices to generate all of  $V_s$ .

The number  $s$  in  $V_s$  is called the **highest weight** or **spin** of  $V_s$ .

The lemma tells us that if  $V$  is a finite-dimensional  $\mathfrak{sl}_2(\mathbb{C})$ -module and  $\xi \in V(s)$ ,  $\xi \neq 0$  and  $E\xi = 0$  (that is,  $\xi$  is a highest weight vector), then we get an embedding  $V_s \hookrightarrow V$  given by  $\xi_{s-k}^s \mapsto F^k \xi$ .

**Theorem 18.5.** *The  $\mathfrak{sl}_2(\mathbb{C})$ -modules  $V_s$  ( $s \in \frac{1}{2}\mathbb{Z}$ ) are irreducible, pairwise non-isomorphic, and any finite-dimensional irreducible  $\mathfrak{sl}_2(\mathbb{C})$ -module is isomorphic to some  $V_s$ .*

*Proof.* The first statement is clear, since all the  $V_s$  have different dimension. Now, if  $V$  is an irreducible  $\mathfrak{sl}_2(\mathbb{C})$ -module, the observation above tells that we have an embedding  $V_s \hookrightarrow V$ .

Since  $\mathfrak{sl}_2(\mathbb{C})$  is semisimple, it follows from Weyl's theorem ([ref??]), that every invariant subspace has a complementary invariant subspace. But  $V$  was irreducible, so  $V_s = V$ .  $\square$

**Example 18.6.** Let  $s = 0$ . Then  $\dim V_0 = 1$ , and this corresponds to the trivial representation of  $SU(2)$ .  $\star$

**Example 18.7.** Let  $s = \frac{1}{2}$ . Then  $\dim V_{\frac{1}{2}} = 2$ , and one sees that  $H, F$  and  $E$  act precisely as the canonical representation of  $\mathfrak{sl}_2(\mathbb{C})$  on  $\mathbb{C}^2$ .  $\star$

**Example 18.8.** Let  $s = 1$ . Then  $\dim V_1 = 3$ , and  $V_1$  is the adjoint representation of  $\mathfrak{sl}_2(\mathbb{C})$  with highest weight vector  $\xi_1^1 = E$ .  $\star$

## 19 Tori

A **torus** is a Lie group isomorphic to  $\mathbb{T}^n = (S^1)^n$ .

Recall that by Proposition 17.2, any compact connected abelian Lie group is a torus.

**Proposition 19.1.** *Let  $G$  be a compact connected Lie group. Then there exists a maximal torus in  $G$ . Furthermore, we have a one to one correspondence between the maximal tori in  $G$  and the maximal abelian Lie subalgebras of  $\mathfrak{g}$ .*

*Proof.* As  $\mathfrak{g}$  is finite-dimensional, maximal abelian subalgebras exist. Let  $\mathfrak{a}$  be such a maximal abelian subalgebra of  $\mathfrak{g}$ . Then  $\exp \mathfrak{a}$  is a connected abelian Lie subgroup of  $G$ .

Then  $H = \overline{\exp \mathfrak{a}}$  is a compact connected abelian subgroup of  $G$ . Hence  $H$  is a torus. Then  $\mathfrak{h}$  is abelian and  $\mathfrak{h} \supset \mathfrak{a}$ , hence  $\mathfrak{h} = \mathfrak{a}$ . Hence  $H = \exp \mathfrak{a}$ , since the inclusion  $\mathfrak{a} \subset H$  must be a covering map.

On the other hand, if  $T \subset G$  is a maximal torus, then  $\mathfrak{t} \subset \mathfrak{g}$  is an abelian Lie algebra, hence it is contained in a maximal abelian Lie algebra  $\mathfrak{a}$ . Then  $\exp \mathfrak{a}$  is a torus containing  $T$ , hence  $T = \exp \mathfrak{a}$ , and  $\mathfrak{t} = \mathfrak{a}$ .  $\square$

Our goal in this section is to show that any two maximal tori are conjugate. This is not so difficult in concrete cases:

**Example 19.2.** Let  $G = U(n)$ . Then

$$T = \left\{ \begin{pmatrix} z & & \\ & \ddots & \\ & & z \end{pmatrix} \mid z \in \mathbb{C} \right\} \cap SU(n) \simeq \mathbb{T}^n$$

is a torus.

Suppose  $H \subset G$  is a torus. As  $H$  is compact abelian,  $\mathbb{C}^n$  decomposes into a direct sum of 1-dimensional  $H$ -invariant subspaces. Hence there exist  $g \in U(n)$  such that  $gHg^{-1} \subset T$ , hence there exist  $g \in U(n)$  such that  $gHg^{-1} \subset T$ .

This shows that  $T$  is maximal and any maximal torus is conjugate to  $T$ .  $\star$

We need some preparations to prove the general case.

If  $T$  is a torus, we say that  $g \in T$  is a **topological generator** if

$$\overline{\{g^n \mid n \in \mathbb{Z}\}} = T.$$

**Theorem 19.3** (Kronecker). *Consider  $T = \mathbb{T}^n$ . Then*

$$g = (e^{2\pi i t_1}, \dots, e^{2\pi i t_n})$$

*is a topological generator if and only if the numbers  $1, t_1, \dots, t_n$  are linearly independent over  $\mathbb{Q}$ .*

*Proof.* Consider  $H = \overline{\{g^n \mid n \in \mathbb{Z}\}}$ . If  $H \neq T$ , then  $T/H$  is a non-trivial compact abelian group, hence  $T/H$  has a nontrivial character. By composition,  $T$  has a non-trivial character  $\chi$  such that  $\chi|_H = 1$  (constantly the identity).

Note that  $G/H$  is trivial if and only if for every character  $\chi$ , the implication  $\chi(g) = 1 \Rightarrow \chi = 1$  holds.

We have, for any character,

$$\chi(e^{2\pi i t_1}, \dots, e^{2\pi i t_n}) = e^{2\pi i m_1 t_1} \dots e^{2\pi i m_n t_n}$$

for some  $m_1, \dots, m_n \in \mathbb{Z}$ . So suppose  $\chi(g) = 1$ . Then the  $m_i$ 's constitute an equation of linear dependence among the  $t_i$ . But if these are linearly independent, then  $\chi$  is trivial.  $\square$

**Corollary 19.4.** *For any torus  $T$ , the set of topological generators is dense in  $T$ .*

**Proposition 19.5.** *Let  $G$  be a compact connected Lie group. Let  $T \subset G$  be a maximal torus. Then*

1.  $N(T) \triangleq \{g \in G \mid gTg^{-1} = T\}$  is a closed subgroup of  $G$ .
2.  $N(T)^\circ = T$ .
3. The group  $W(G, T) \triangleq N(T)/T$  is finite. It is called the **Weyl group of  $G$** .

*Proof.* It is obvious that  $N(T)$  is a closed subgroup, since the definition is a closed condition.

For 2., consider the homomorphism

$$\begin{aligned} \alpha : N(T) &\rightarrow \text{Aut}(T) \\ g &\mapsto \alpha(g)(h) = ghg^{-1}. \end{aligned}$$

The automorphism group of the torus  $T$  is  $\text{GL}_n(\mathbb{Z}) \subset \text{GL}_n(\mathbb{R})$ .

The group  $\text{GL}_n(\mathbb{Z})$  is discrete, so  $\alpha$  is locally constant. Thus  $N(T)^\circ \subset \ker \alpha$ . Thus  $gh = hg$  for all  $g \in N(T)^\circ$  and all  $h \in T$ . Let  $X$  be an element of the Lie algebra of  $N(T)$ . Then  $\exp(tX)$  commutes with  $T$  for all  $t \in \mathbb{R}$ . It follows that the closure of the group generated by  $\exp(tX)$  is a compact connected abelian Lie group, hence must be a torus.

But since  $T$  is maximal, this torus coincides with  $T$ , so  $\exp(tX) \in T$  for all  $t \in \mathbb{R}$ .

Thus the Lie algebra of  $N(T)$  coincides with the Lie algebra of  $T$ . Hence  $N(T)^\circ = T$ .

For 3., note that by 2., we have

$$N(T)/T = N(T)/N(T)^\circ,$$

which is finite because  $N(T)$  is compact.  $\square$

**Example 19.6.** Let  $G = U(n)$ . And let  $T$  consist of the unitary diagonal matrices. It is a maximal torus.

Then  $N(T)$  consist of all the matrices  $g \in U(n)$  such that every row and column contains exactly one nonzero coefficient.

Thus  $N(T)/T \simeq S_n$ .  $\star$

## 19.1 Digression on Riemannian geometry

Recall that a Riemannian manifold is a manifold  $M$  such that  $T_p M$  is equipped with a scalar product depending smoothly on  $p$ . Thus, if  $X, Y \in \mathfrak{X}(M)$  are two vector fields, then the map  $p \mapsto (X_p, Y_p)$  is a smooth map. Equivalently, a scalar product defines a smooth section of  $S^2(T^*M)$ .

Given a piecewise  $C^1$ -curve, we define its **length** by

$$L(\gamma) \triangleq \int_a^b \|\gamma'(t)\| dt.$$

Then we can define a metric on  $M$  (assuming  $M$  is connected), by

$$d(a, b) = \inf\{L(\gamma) \mid \gamma : [0, 1] \rightarrow M \text{ piecewise } C^1, \gamma(0) = a, \gamma(1) = b\}.$$

It is not difficult to see that the topology on  $M$  coincides with the topology defined by this metric  $d$ .

A map  $\gamma : [a, b] \rightarrow M$  is called a **geodesic** if for any  $t_0 \in (a, b)$ , there exists a  $\delta > 0$  such that  $d(\gamma(s), \gamma(t)) = |s - t|$  for  $s, t \in (t_0 - \delta, t_0 + \delta)$ .

We also allow linear change of variables, so  $t \mapsto \gamma(at)$  is also called a geodesic for  $\alpha > 0$ .

Here is a theorem we wont prove.

**Theorem 19.7.** *Let  $M$  be a Riemannian manifold. Then:*

1. *Any geodesic is smooth.*
2. *For any  $p \in M$  and  $X \in T_p M$ ,  $X \neq 0$ , there is a unique maximal geodesic  $\gamma$  such that  $\gamma(0) = p$  and  $\gamma'(0) = X$ .*
3. *(Hopf-Rinow theorem) If  $M$  is compact and connected, then any maximal geodesic is defined on the whole line  $\mathbb{R}$ , and for any  $p, q \in M$ , there exist a geodesic passing through  $p$  and  $q$ .*

## 19.2 Returning to tori

From now on, assume  $G$  is a compact connected Lie group. Consider the adjoint representation  $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$ . Since  $G$  is compact, there exist an  $\text{Ad } G$ -invariant scalar product (see Lemma 20.3). Then  $\text{Ad}: G \rightarrow O(\mathfrak{g}, \langle -, - \rangle)$ . Recall that the Lie algebra of  $O(\mathfrak{g})$  consist of  $T \in \mathfrak{gl}(\mathfrak{g})$  such that

$$\langle Tu, v \rangle + \langle u, Tv \rangle = 0$$

for all  $u, v \in \mathfrak{g}$ .

Now we use left- or right-translations to define a scalar product on  $T_g G$  for any  $g \in G$ . Thus  $G$  becomes a Riemannian manifold such that both left and right translations preserve the Riemannian structure.

**Theorem 19.8.** *The geodesics in  $G$  are translations (left or right) of one-parameter subgroups of  $G$ .*

*Proof.* The result is true for tori, since they are of the form  $\mathbb{R}^n/\mathbb{Z}^n$ , so their geodesics are lines.

In general, it suffices to consider geodesics passing through  $e$ . Let  $\gamma: \mathbb{R} \rightarrow G$  be a geodesic with  $\gamma(0) = e$ . Put  $X = \gamma'(0) \in \mathfrak{g}$ . For any  $g \in G$ , the map  $t \mapsto g\gamma(t)g^{-1}$  is also a geodesic (because the scalar product is invariant under left- and right-translations). Its derivative at 0 is  $\text{Ad}(g)(X)$ .

Let  $T$  be a maximal torus containing  $\exp(tX)$  for  $t \in \mathbb{R}$ . Then  $\text{Ad}(g)(X) = X$  for all  $g \in T$ .

It follows that  $\gamma(t) = g\gamma(t)g^{-1}$  for all  $t \in \mathbb{R}$ , since by point 2. in Theorem 19.7, geodesics are unique. In particular,  $\gamma(t) \in N(T)^\circ = T$ .

Thus  $\text{Im } \gamma \subset T$ . Hence  $\gamma$  is a geodesic in  $T$ , where the result is true. Hence  $\gamma(t) = \exp(tX)$ , since the theorem is true for  $T$ .  $\square$

The next result is completely fundamental for everything that follows.

**Theorem 19.9** (Cartan). *Let  $G$  be a compact connected Lie group and  $T \subset G$  a maximal torus. Then for all  $g \in G$  there exists  $h_g \in G$  such that  $h_g g h_g^{-1} \in T$ .*

*Proof.* Let  $t_0 \in T$  be a topological generator of  $T$ . Let  $H_0 \in \mathfrak{t}$  be such that  $\exp(H_0) = t_0$ .

Choose a geodesic  $\gamma$  passing through  $e$  and  $g$  with  $\gamma(0) = e$ . Let  $X = \gamma'(0)$ . Then  $\gamma(t) = \exp(tX)$  by the previous theorem.

Consider the function

$$G \ni h \mapsto \langle \text{Ad}(h)(X), H_0 \rangle \in \mathbb{R}$$

This is a function from a compact space to the real line, so it has a maximum, achieved at, say,  $h \in G$ .



**Claim:**  $[(\text{Ad } h)(X), H_0] = 0$ .

Assuming the claim, it follows that  $\exp(t(\text{Ad } h)(X)) = h \exp(tX) h^{-1}$  commutes with  $\exp(H_0) = t_0$  for  $t \in \mathbb{R}$ . As  $t_0$  is a topological generator of  $T$ , it follows that  $h \exp(tX) h^{-1}$  commutes with  $T$ .

As  $T$  is a maximal torus, it follows that  $h \exp(tX) h^{-1} \in T$  for all  $t \in \mathbb{R}$ . In particular,  $hgh^{-1} \in T$ .

Thus it remains to prove the claim. Take  $Y \in \mathfrak{g}$ . Then the function

$$t \mapsto \langle \text{Ad}(\exp(tY)h)(X), H_0 \rangle$$

attains its maximum at  $t = 0$ . Therefore its derivative at 0 is zero. So

$$\begin{aligned} 0 &= \langle \text{ad}(Y)(\text{Ad } h)(X), H_0 \rangle \\ &= -\langle [\text{Ad}(h)X, Y], H_0 \rangle \\ &= \langle Y, [\text{Ad}(h)(X), H_0] \rangle. \end{aligned}$$

This is true for all  $Y \in \mathfrak{g}$ , hence  $[\text{Ad}(h)(X), H_0] = 0$ .  $\square$

**Corollary 19.10.** *Any two maximal tori in  $G$  are conjugate.*

*Proof.* Let  $T_1$  and  $T_2$  be two maximal tori. Let  $t_1 \in T_1$  be a topological generator. Then there exists  $h \in G$  such that  $ht_1h^{-1} \in T_2$ . But then  $hT_1h^{-1} \subset T_2$ , since  $t_1$  was a topological generator. So  $T_1 \subset h^{-1}T_2h$ . But  $T_1$  was maximal, so  $T_1 = T_2$ .  $\square$

**Corollary 19.11.** *Let  $G$  be a compact connected Lie group. Then any element of  $G$  is contained in a maximal torus. In particular,  $\exp: \mathfrak{g} \rightarrow G$  is surjective.*

*Proof.* Let  $g \in G$ , and let  $T$  be a maximal torus. Then  $hgh^{-1} \in T$  for some  $h \in G$ . But then  $g \in h^{-1}Th$ , which is maximal.

To see the "in particular", let  $g \in G$ . Then  $g$  is contained in a maximal torus. Then the geodesic starting at  $e$  and ending at  $g$  must have the form  $\exp(tX)$ .  $\square$

**Corollary 19.12.** *Any maximal torus  $T$  in  $G$  is a maximal abelian subgroup of  $G$ .*

*Proof.* Assume  $g$  commutes with  $T$ . Consider

$$H = C(g)^\circ = \{h \in G \mid hg = gh\}^\circ,$$

the centralizer of  $g$  in  $G$ . This is a compact connected Lie group. Since  $T$  is connected,  $T \subset H$ . As  $g$  is contained in a maximal torus in  $G$ , we have  $g \in H$ .

As  $T$  is a maximal torus in  $G$ , it is a maximal torus in  $H$ . Hence there exists  $h \in H$  such that  $hgh^{-1} \in T$ . But  $g \in Z(H)$ , hence  $g \in T$ .  $\square$

**Remark.** *This implies that the map*

$$\begin{aligned}\alpha : W(G, T) &\rightarrow \text{Aut}(T) \\ g &\mapsto \alpha(gT)(h) = ghg^{-1}\end{aligned}$$

*has trivial kernel. Hence  $W(G, T) \hookrightarrow \text{Aut}(T)$ .*

## 20 Roots

First we recall some notation from the representation theory of  $SU(2)$ . There we had a maximal torus, called  $T$ , which was given by

$$T = \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \mid z \in S^1 \subset \mathbb{C} \right\}.$$

The Lie algebra of  $SU(2)$  is contained in the Lie algebra of  $SL(2, \mathbb{C})$ . We write  $\mathfrak{su}(2) \subset \mathfrak{sl}_2(\mathbb{C})$ . The Lie algebra is spanned by  $E, F, H$ , satisfying  $[H, E] = 2E$ ,  $[H, F] = -2F$  and  $[E, F] = H$ .

### 20.1 Roots

Let  $G$  be a compact connected Lie group. Fix a maximal torus  $T$ . Consider  $\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ , the complexification of the Lie algebra of  $G$ . This is a complex Lie algebra, where we define  $[c \otimes X, d \otimes Y] = cd \otimes [X, Y]$ .

We can identify  $\mathfrak{g}$  with the elements of the form  $1 \otimes X$  for  $X \in \mathfrak{g}$ . Sometimes we write elements of  $\mathfrak{g}_{\mathbb{C}}$  as  $X + iY$ .

The adjoint representation  $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$  extends to a representation  $G \rightarrow \text{GL}(\mathfrak{g}_{\mathbb{C}})$ , which we will also denote by  $\text{Ad}$ .

Consider  $\text{Ad}|_T$ . As  $T$  is a compact abelian group, we have a decomposition of  $\text{Ad}|_T$  into isotypic components corresponding to characters  $\chi \in \widehat{T}$ . Thus we can write

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\chi \in \widehat{T}} \mathfrak{g}_{\mathbb{C}}(\chi)$$

where

$$\mathfrak{g}_{\mathbb{C}}(\chi) = \{X \in \mathfrak{g}_{\mathbb{C}} \mid (\text{Ad } g)X = \chi(g)X \text{ for all } g \in T\}$$

is the eigenspace of  $X$ .

We view  $\mathbb{T} = S^1$  as a subgroup of  $\mathbb{C}^* = \text{GL}_1(\mathbb{C})$ . In this picture, the Lie algebra of  $\mathbb{T}$  is  $i\mathbb{R} \subset \mathbb{C}$ .

Let  $\mathfrak{t}$  be the Lie algebra of  $T$ . Any character  $\chi \in \widehat{T}$  is determined by its differential  $\chi_* : \mathfrak{t} \rightarrow i\mathbb{R}$ , so we can think of  $\chi_* \in i\mathfrak{t}^*$ , where  $\mathfrak{t}^* = \text{Hom}_{\mathbb{R}}(\mathfrak{t}, \mathbb{R})$ .

Then the set of differentials  $\chi_*$  ( $\chi \in \widehat{T}$ ) form a subset  $X^*(T) \subset i\mathfrak{t}^*$ . It is a *lattice* in  $i\mathfrak{t}^*$  (here by *lattice* we mean a discrete subgroup of maximal rank), called the **character lattice**.

Indeed, if  $T = \mathbb{R}^n / \mathbb{Z}^n$ , then  $\mathfrak{t} = \mathbb{R}^n$ , so  $\mathfrak{t}^* = \mathbb{R}^n$ , and an element  $(it_1, \dots, it_n) \in i\mathfrak{t}^*$  defines a character  $\mathbb{R}^n \rightarrow \mathbb{T}$  by

$$(a_1, \dots, a_n) \mapsto e^{ia_1 t_1} \cdots e^{ia_n t_n},$$

which factors through  $\mathbb{R}^n/\mathbb{Z}^n$  if and only if all the  $a_i \in 2\pi\mathbb{Z}$ . Then  $X^*(T) = 2\pi i\mathbb{Z}^n \subset i\mathbb{R}^n$ .

Consider now the complexification  $\mathfrak{t}_{\mathbb{C}}$  of the Lie algebra of  $T$ . We denote this complexification by  $\mathfrak{h}$ . It is called a **Cartan subalgebra** of  $\mathfrak{g}_{\mathbb{C}}$ . It is a “maximal toral subalgebra”.

Functionals  $\mathfrak{t} \rightarrow \mathbb{R}$  can be extended by complex linearity to  $\mathfrak{h}$ , so  $i\mathfrak{t}^* \subset \mathfrak{h}^* = \text{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ . We then have

$$\begin{aligned}\mathfrak{g}_{\mathbb{C}}(\chi) &= \{X \in \mathfrak{g}_{\mathbb{C}} \mid (\text{Ad } g)X = \chi(g)X, \forall g \in T\} \\ &= \{X \in \mathfrak{g}_{\mathbb{C}} \mid [Y, X] = \chi_*(Y)X, \forall Y \in \mathfrak{t}\} \\ &= \{X \in \mathfrak{g}_{\mathbb{C}} \mid [Y, X] = \chi_*(Y)X, \forall Y \in \mathfrak{h}\}.\end{aligned}$$

For  $\alpha \in \mathfrak{h}^*$ , denote

$$\mathfrak{g}_{\alpha} = \{X \in \mathfrak{g}_{\mathbb{C}} \mid [H, X] = \alpha(H)X, \forall H \in \mathfrak{h}\}.$$

In our earlier notation, this says that  $\mathfrak{g}_{\alpha} = \mathfrak{g}_{\mathbb{C}}(\alpha)$ , where we think of  $\alpha \in i\mathfrak{t}^* \subset \mathfrak{h}^*$ .

**Lemma 20.1.** *We have that  $\mathfrak{g}_0 = \mathfrak{h}$ .*

*Proof.* This follows because  $\mathfrak{t}$  is a maximal abelian Lie subalgebra of  $\mathfrak{g}$ , so that

$$\mathfrak{t} = \{X \in \mathfrak{g} \mid [H, X] = 0 \quad \forall H \in \mathfrak{t}\}.$$

Now complexify both sides. □

The elements  $\alpha \in X^*(T) \setminus \{0\}$  such that  $\mathfrak{g}_{\alpha} \neq 0$  are called the **roots** of  $G$  or  $\mathfrak{g}_{\mathbb{C}}$ . We denote the set of roots by  $\Delta$ .

To remember the position of  $\mathfrak{g}$  inside  $\mathfrak{g}_{\mathbb{C}}$ , it is convenient to introduce an involution on  $\mathfrak{g}_{\mathbb{C}}$ , defined by

$$(X + iY)^* = -X + iY,$$

so that  $\mathfrak{g} = \{X \in \mathfrak{g}_{\mathbb{C}} \mid X = -X^*\}$ .<sup>7</sup>

This induces an involution  $\alpha \mapsto \bar{\alpha}$  in  $\mathfrak{h}^*$  as well. Namely, define

$$\bar{\alpha}(H) \triangleq \overline{\alpha(H^*)}.$$

This is actually an involution, because

$$\overline{\bar{\alpha}}(H) = \overline{\overline{\alpha(H^*)}} = \overline{\alpha(H^{**})} = \alpha(H).$$

In fact, this definition ensures that  $\bar{\alpha}$  is *complex linear* as well: we have that  $\bar{\alpha}(zH) = z\bar{\alpha}(H)$  (this is an easy but somewhat tedious exercise).

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<sup>7</sup>Note that I have chosen to write  $X^*$  for the involution to distinguish it from taking duals.

This lets us recognize  $i\mathfrak{t}^*$  inside  $\mathfrak{h}^*$  as well. Namely,

$$\begin{aligned} i\mathfrak{t}^* &= \{\alpha \in \mathfrak{h}^* \mid \alpha(\mathfrak{t}) \subset i\mathbb{R}\} \\ &= \{\alpha \in \mathfrak{h}^* \mid \bar{\alpha} = \alpha\}, \end{aligned}$$

as those functionals fixed by the involution.

## 20.2 Some lemmas

The rest of this lecture will consist of several small but fundamental results.

**Lemma 20.2.** *We have a decomposition*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}.$$

*Also:*

1.  $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subset \mathfrak{g}_{\alpha+\beta}$ .
2.  $\mathfrak{g}_{\alpha}^* = \mathfrak{g}_{-\alpha}$ . Here  $\mathfrak{g}_{\alpha}^* = \{X^* \mid X \in \mathfrak{g}_{\alpha}\}$ .

*Proof.* The decomposition of  $\mathfrak{g}_{\mathbb{C}}$  is just that of writing  $\mathfrak{g}_{\mathbb{C}}$  into simultaneous eigenspaces for the action of  $\mathfrak{h}$ , where  $\mathfrak{h}$  has zero eigenvalues, as proved above.

We have that

$$\mathfrak{g}_{\alpha} = \{X \in \mathfrak{g}_{\mathbb{C}} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}.$$

Let  $Y \in \mathfrak{g}_{\beta}$ . Then we look at  $[H, [X, Y]]$ . By the Jacobi identity, this is equal to

$$\begin{aligned} [H, [X, Y]] &= -[Y, [H, X]] - [X, [Y, H]] \\ &= -\alpha(H)[Y, X] + [X, [H, Y]] \\ &= \alpha(H)[X, Y] + \beta(H)[X, Y] \\ &= (\alpha(H) + \beta(H))[X, Y]. \end{aligned}$$

Hence  $[X, Y] \in \mathfrak{g}_{\alpha+\beta}$ .

For ii), note that  $\alpha(\mathfrak{t}) \subset i\mathbb{R}$ . Hence for  $H \in \mathfrak{t}$ , we have

$$\bar{\alpha}(H) = \overline{\alpha(H^*)} = -\overline{\alpha(H)} = \alpha(H).$$

Hence we have

$$\begin{aligned} \mathfrak{g}_{\alpha} &= \{X \in \mathfrak{g}_{\mathbb{C}} \mid [H, X] = -\alpha(H)X \quad \forall H \in \mathfrak{h}\} \\ &= \{X \in \mathfrak{g} \mid [H, X] = -\alpha(H)X \quad \forall H \in i\mathfrak{t}^*\} \otimes_{\mathbb{R}} \mathbb{C} \\ &= \{X \in \mathfrak{g} \mid [H, X^*] = \alpha(H)X^* \quad \forall H \in i\mathfrak{t}^*\} \otimes_{\mathbb{R}} \mathbb{C}. \end{aligned}$$

In the last equality, we used the involution on both sides to get

$$[H^*, X^*] = -[H, X] = \alpha(H)X = \alpha(H)X^*,$$

since  $X$  is real (which gives  $X^* = -X$ ) and  $H^* = H$ . □

### 20.3 An Ad-invariant inner product

To proceed, it is convenient to introduce an Ad-invariant scalar product on  $\mathfrak{g}$ . These exist:

**Lemma 20.3.** *Let  $G$  be compact Lie group. Then there is an Ad-invariant inner product  $(-, -)$  on  $\mathfrak{g}$ , in the sense that*

$$(g \cdot X, g \cdot Y) = (X, Y)$$

for  $X, Y \in \mathfrak{g}$ , where  $g \cdot X$  denotes the adjoint action  $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$ . This condition further implies that

$$([X, Y], Z) + (Y, [X, Z]) = 0.$$

*Proof.* Choose any scalar product  $\langle -, - \rangle$  on  $\mathfrak{g}$  and a normalized Haar measure  $dg$  on  $G$ . Then

$$(X, Y) = \int_G (g \cdot X, g \cdot Y) dg$$

is an Ad-invariant scalar product on  $\mathfrak{g}$ . To see the other equality, let  $\gamma : \mathbb{R} \rightarrow G$  be an arc with  $\gamma(0) = e \in G$  and  $\gamma'(0) = Z \in \mathfrak{g}$ . Then

$$\begin{aligned} 0 &= \frac{d}{dt} (\gamma(t) \cdot X, \gamma(t) \cdot Y) \Big|_{t=0} \\ &= \lim_{t \rightarrow 0} \frac{1}{t} ((\gamma(t) \cdot X, \gamma(t) \cdot Y) - (X, \gamma(t) \cdot Y) + (X, \gamma(t) \cdot Y) - (X, Y)) \\ &= \lim_{t \rightarrow 0} \left( \frac{1}{t} (\gamma(t) \cdot X - X, Y) + \left( X, \frac{1}{t} (\gamma(t) \cdot Y - Y) \right) \right) \\ &= ([Z, X], Y) + (X, [Z, Y]) \end{aligned}$$

Renaming  $X, Y, Z$  appropriately gives the required identity.  $\square$

It will be convenient to require that  $(-, -)$  is *negative* definite on  $\mathfrak{g}$ . We can extend this form to  $\mathfrak{g}_{\mathbb{C}}$  be complex linearity. It will no longer be definite (it doesn't make sense), but it will still be non-degenerate (easy exercise).

**Lemma 20.4.** *We have*

$$(\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}) = 0$$

if  $\alpha \neq -\beta$ .

*Proof.* This follows from the invariance of Ad. Namely let  $X_{\alpha} \in \mathfrak{g}_{\alpha}$  and  $X_{\beta} \in \mathfrak{g}_{\beta}$ . Then:

$$\begin{aligned} 0 &= ([H, X_{\alpha}], X_{\beta}) + (X_{\alpha}, [H, X_{\beta}]) \\ &= (\alpha(H)X_{\alpha}, X_{\beta}) + (X_{\alpha}, \beta(H)X_{\beta}) \\ &= (\alpha(H) + \beta(H))(X_{\alpha}, X_{\beta}) \end{aligned}$$

for all  $H \in \mathfrak{h}$ .  $\square$

Hence the eigenspaces  $\mathfrak{g}_\alpha$  and  $\mathfrak{g}_\beta$  are orthogonal.

By duality we can define a symmetric bilinear form on  $\mathfrak{h}^*$  as well: let  $\alpha \in \mathfrak{h}^*$ . Define its **dual element**  $h_\alpha$  by the equality

$$(h_\alpha, H) = \alpha(H)$$

for all  $H \in \mathfrak{h}$ . To see that this is well-defined, choose an orthogonal basis  $\{e_i\}$  for  $\mathfrak{h}$  and let  $\{e_i^*\}$  be the dual basis (with respect to the form  $(-, -)$ ). Then the vector  $h_\alpha$  is just the transpose of the vector  $\alpha$ .

We define

$$(h_\alpha, h_\beta) \triangleq (\alpha, \beta).$$

**Lemma 20.5.** *We have*

$$h_\alpha^* = h_{\bar{\alpha}}$$

for every  $\alpha \in \mathfrak{h}^*$ .

*Proof.* First off, note that by  $(X^*, Y) = \overline{(X, Y^*)}$ . Then

$$(h_\alpha^*, H) = \overline{(h_\alpha, H^*)} = \overline{\alpha(H^*)} = \overline{\overline{\alpha(H)}} = \bar{\alpha}(H) = (h_{\bar{\alpha}}, H).$$

Here we used the definition of  $\bar{\alpha}$  as  $\bar{\alpha}(H) = \overline{\alpha(H^*)}$ .  $\square$

It follows that if  $\alpha \in \mathfrak{h}$  satisfies  $\bar{\alpha} = \alpha$ , then  $h_\alpha^* = h_\alpha$ , which implies that  $h_\alpha \in i\mathfrak{t}$ . Since  $(-, -)$  was negative definite on  $\mathfrak{g}$ , it follows that  $(-, -)$  is positive definite on  $i\mathfrak{t}$ .

Now let  $\alpha \in \Delta$  be a root. Then define

$$H_\alpha = \frac{2h_\alpha}{(\alpha, \alpha)} \in i\mathfrak{t} \subset \mathfrak{h}.$$

**Lemma 20.6.** *Let  $\alpha \in \Delta$ , and  $X \in \mathfrak{g}_\alpha$ . Then*

$$[X, X^*] = (X, X^*)h_\alpha.$$

*Proof.* We know already from Lemma (20.2) that  $[X, X^*] \in \mathfrak{h}$ . Let  $\beta \in \mathfrak{h}^*$ . Then

$$\begin{aligned} \beta([X, X^*]) &= (h_\beta, [X, X^*]) \\ &= -(h_\beta, [X^*, X]) \\ &= ([X^*, h_\beta], X) \\ &= (\alpha(h_\beta)X^*, X) = (X^*, X)\beta(h_\alpha). \end{aligned}$$

This holds for all  $\beta \in \mathfrak{h}^*$ , which implies the statement.  $\square$

Choose a non-zero  $E_\alpha \in \mathfrak{g}_\alpha$ , and normalize it so that  $(E_\alpha, E_\alpha^*) = \frac{2}{(\alpha, \alpha)}$ . Then

$$[E_\alpha, E_\alpha^*] = H_\alpha.$$

Put  $F_\alpha = E_\alpha^*$ . We thus get

$$[H_\alpha, E_\alpha] = 2E_\alpha \quad [H_\alpha, F_\alpha] = -2F_\alpha \quad [E_\alpha, F_\alpha] = H_\alpha.$$

This defines a sub-Lie-algebra isomorphic to  $\mathfrak{sl}_2(\mathbb{C})$ . Thus for every  $\alpha \in \Delta$  we get an inclusion of Lie algebras

$$i_\alpha : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{g}_\mathbb{C}.$$

**Lemma 20.7.** *a) The spaces  $\mathfrak{g}_\alpha$  are one-dimensional.*

*b) Suppose  $\alpha \in \Delta$  and  $c \in \mathbb{R}^*$  and  $c\alpha \in \Delta$ . Then  $c = \pm 1$ .*

*Proof.* We can suppose  $\alpha$  to be minimal, in the sense that  $c\alpha \notin \Delta$  if  $c \in (0, 1)$ . Now consider the space

$$V = \mathbb{C}F_\alpha \oplus \mathfrak{h} \oplus \left( \bigoplus_{c>0, c\alpha \in \Delta} \mathfrak{g}_{c\alpha} \right).$$

This is a finite-dimensional  $\mathfrak{sl}_2(\mathbb{C})$ -module. But we know from 18.5 that their spectra are symmetric about the origin, counted with multiplicities. Hence the big term must be isomorphic to the left term, which is one-dimensional, and we must have  $c = 1$ .  $\square$

**Remark.** *The isomorphism  $\mathfrak{h}^* \simeq \mathfrak{h}$  depends upon the choice of invariant form, but the elements  $H_\alpha$  ( $\alpha \in \Delta$ ) does not depend upon this choice:  $H_\alpha$  is the unique element of the one-dimensional space  $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$  such that  $\alpha(H_\alpha) = 2$ . Then  $E_\alpha \in \mathfrak{g}_\alpha$  is defined up to scalar of modulus 1, such that  $[E_\alpha, E_\alpha^*] = H_\alpha$ .*



## 21 Root systems

In this lecture, we study root systems in the abstract.

Let  $V$  be a finite-dimensional Euclidean vector space (that is, a real vector space equipped with a scalar product). A finite subset  $\Delta \subset V \setminus \{0\}$  is a **root system** if:

1. For every  $\alpha \in \Delta$ , the reflection  $s_\alpha$  with respect to the hyperplane  $\alpha^\perp$  leaves  $\Delta$  invariant.

Note that

$$s_\alpha(v) = v - \frac{2(\alpha, v)}{(\alpha, \alpha)} \alpha.$$

2. The number

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)}$$

is an integer for all  $\alpha, \beta \in \Delta$ .

A root system is called **reduced**, if in addition the following condition is satisfied:

3. If  $\alpha, c\alpha \in \Delta$ , where  $c \in \mathbb{R}^\times$ , then we must have  $c = \pm 1$ .

**Remark.** Number 3. is often assumed to be part of the definition of a root system. It is also often assumed that  $\Delta$  span the whole space  $V$ , but for us it will be important not to assume this.

**Example 21.1.** Here's an example of a root system. It is denoted by  $A_2$  (and is associated with  $SU(3)$ ). It is a regular hexagon.

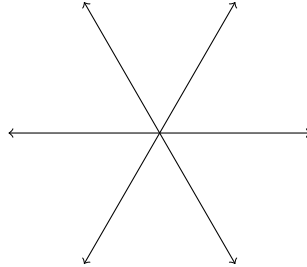


Figure 1: The root system  $A_2$ .

To check the conditions, we can choose the vertices  $\alpha_i$  to be given by  $\alpha_i = (\cos \theta_i, \sin \theta_i)$  with  $\theta_i = \frac{\pi}{3}i$  for  $i = 0, \dots, 5$ . Then we have that

$$(\cos \theta_i, \sin \theta_i) \cdot (\cos \theta_j, \sin \theta_j) = \cos(\theta_i - \theta_j) = \pm \frac{1}{2} \text{ or } \pm 1,$$

since  $\theta_i - \theta_j$  is a multiple of  $\pi/3$ . In either case, the second condition is fulfilled. ★

We say that two root systems  $(V_1, \Delta_1)$  and  $(V_2, \Delta_2)$  are **isomorphic** if there exists a linear isomorphism  $T : V_1 \rightarrow V_2$  such that  $T(\Delta_1) = \Delta_2$  and  $Ts_\alpha = s_{T(\alpha)}T$  for all  $\alpha \in \Delta_1$ . Note that we do not require the isomorphism to preserve the scalar product.

**Theorem 21.2.** *Let  $G$  be a compact connected Lie group and  $T \subset G$  a maximal torus. Then  $(i\mathfrak{t}^*, \Delta)$  is a reduced root system. Up to isomorphism, this root system does not depend upon the choice of  $T$  and the choice of an invariant form on  $\mathfrak{g}$ .*

*Proof.* Take  $\alpha \in \Delta$ . Consider the corresponding embedding

$$i_\alpha : \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}_\mathbb{C}.$$

Since  $SU(2)$  is simply-connected, it defines a homomorphism  $SU(2) \rightarrow G$  (because  $i_\alpha(\mathfrak{su}(2)) \subset \mathfrak{g}$  since  $i_\alpha(X)^* = i_\alpha(X^*)$ ).

Let  $w_\alpha$  be the image of

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SU(2)$$

under this homomorphism. Note that  $w_\alpha^2 = -w_\alpha$ , hence  $w_\alpha^{-1} = -w_\alpha$ . Then  $(\text{Ad } w_\alpha)(H_\alpha) = -H_\alpha$ ,  $\text{Ad}(w_\alpha)(E_\alpha) = -E_\alpha$  and  $\text{Ad}(w_\alpha)(F_\alpha) = -F_\alpha$ .

On the other hand, if we take  $H \in \ker \alpha \subset \mathfrak{h}$ , then  $H$  commutes with  $i_\alpha(\mathfrak{sl}_2(\mathbb{C}))$ . Hence  $(\text{Ad } w_\alpha)(H) = H$ . In particular, since  $\mathfrak{h} = \ker \alpha \oplus \mathbb{C}H_\alpha$ , we see that  $(\text{Ad } w_\alpha)(\mathfrak{h}) = \mathfrak{h}$ , hence  $w_\alpha$  normalizes the maximal torus  $T$ .

It follows that  $\text{Ad } w_\alpha$  maps  $\mathfrak{g}_\beta$  into  $\mathfrak{g}_{\beta \circ \text{Ad } w_\alpha}$  (check!)

Thus the map  $\beta \mapsto \beta \circ \text{Ad } w_\alpha$  maps  $\Delta$  into itself.

Therefore, to prove the first condition, it suffices to show that that  $\beta \circ \text{Ad } w_\alpha = s_\alpha(\beta)$  for  $\beta \in V$ . Equivalently, since  $h_{\beta \circ \text{Ad } w_\alpha} = (\text{Ad } w_\alpha)(h_\beta)$ , we have to show that  $(\text{Ad } w_\alpha)(h_\beta) = h_{s_\alpha(\beta)}$ .

If  $\beta = \alpha$ , then this is true because  $s_\alpha(\alpha) = -\alpha$  and  $(\text{Ad } w_\alpha)(h_\alpha) = -h_\alpha$ . If  $(\beta, \alpha) = 0$ , then it is true as well, because then  $s_\alpha(\beta) = \beta$  and  $(\text{Ad } w_\alpha)(h_\beta) = h_\beta$ , since  $h_\beta \in \ker \alpha$ . This proves i).

To prove the second part we have to show that  $\beta(H_\alpha) \in \mathbb{Z}$  for all  $\alpha, \beta \in \Delta$ . We will show a bit more:  $\beta(H_\alpha) \in \mathbb{Z}$  for all  $\beta \in X^*(T)$ . Using the map  $i_\alpha$ , any differential of a character in  $X^*(T)$  restricts to a differential of a character of  $SU(2)$ . But these are all integer-valued, hence we are done.  $\square$

Note that although the reflections  $s_\alpha$  are defined using a chosen inner product, they are independent of this choice, because  $(\alpha, \beta) = \beta(H_\alpha)$ , and  $H_\alpha$  is independent of any choices (because  $h_\alpha$  is the unique element such that  $[X, X^*] = (X, X^*)h_\alpha$ ).

## 21.1 Simple roots, Weyl chambers, and the Weyl group

Let  $(V, \Delta)$  be a root system and assume that  $\text{Span} \Delta = V$ . We say that a subset  $\Pi \subset \Delta$  is a **system of simple roots** if (1)  $\Pi$  is a basis of  $V$  and (2) that if  $\beta = \sum_{\alpha \in \Pi} c_\alpha \alpha \in \Delta$ , then either all  $c_\alpha \geq 0$  or all  $c_\alpha \leq 0$ .

Reflections  $s_\alpha$  defined by simple roots are called **simple reflections**.

If we fix a set of simple roots  $\Pi$ , then the roots  $\beta = \sum_{\alpha \in \Pi} c_\alpha \alpha \in \Delta$  with positive  $c_\alpha$  are called **positive roots**. We denote the set of positive roots by  $\Delta_+$ . Thus  $\Delta = \Delta_+ \cup -\Delta_+$ .

It is a fact that every root system possesses systems of simple roots. In order to construct  $\Pi$ , we introduce the notion of a **Weyl chamber**: consider the set

$$V \setminus \bigcup_{\alpha \in \Delta} \alpha^\perp.$$

The connected components of this set are called the (open) *Weyl chambers* of  $\Delta$ . Let  $C$  be such a Weyl chamber.

By definition,  $(\alpha, \beta) \neq 0$  for all  $\alpha \in \Delta$  and  $\beta \in C$ . Hence for any  $\alpha \in \Delta$  we have either  $(\alpha, \beta) > 0$  for all  $\beta \in C$  or  $(\alpha, \beta) < 0$  for all  $\beta \in C$ . In the first case, we say that  $\alpha$  is  **$C$ -positive**. Let  $\Pi(C) \subset \Delta$  be the set of  $C$ -positive roots  $\alpha$  such that it is impossible to write  $\alpha$  as  $\beta + \gamma$  for  $C$ -positive roots  $\beta, \gamma \in \Delta$ .

**Proposition 21.3.** *The set  $\Pi(C)$  is a system of simple roots. Furthermore, the assignment  $C \mapsto \Pi(C)$  is a bijection between Weyl chambers and systems of simple roots.*

*Proof.* By definition of  $\Pi(C)$ , any  $C$ -positive root  $\beta$  has the form  $\sum_{\alpha \in \Pi(C)} c_\alpha \alpha$  for  $c_\alpha \in \mathbb{Z}_+$  (why: because  $\Pi(C)$  is exactly the set of  $C$ -positives that are such sums with only one term).

It follows that  $\text{Span} \Pi(C) = V$ , since any cone of the same dimension as  $V$  span  $V$ .

Therefore we have to show that  $\Pi(C)$  is a basis. Observe first that if  $\alpha, \beta \in \Pi(C)$ , then  $\alpha - \beta \notin \Delta$ . Indeed, otherwise, either  $\alpha - \beta$  or  $\beta - \alpha$  is  $C$ -positive. But then  $\alpha = \beta + (\alpha - \beta)$  or  $\beta = \alpha + (\beta - \alpha)$  gives a contradiction to the definition of  $\Pi(C)$ .

Observe next that if  $\alpha, \beta \in \Pi(C)$ ,  $\alpha \neq \beta$ , then  $(\alpha, \beta) \leq 0$ . Indeed, assume that  $(\alpha, \beta) > 0$ . We may assume that  $\|\alpha\| \geq \|\beta\|$ . Then

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} < \frac{2\|\alpha\|\|\beta\|}{\|\alpha\|^2} = \frac{2\|\beta\|}{\|\alpha\|} \leq 2.$$

But the left term is a positive integer, hence it must be equal to 1. But then  $s_\alpha(\beta) = \beta - \alpha \in \Delta$ , contradicting the previous observation.

Assume now that  $\sum_{\alpha \in \Pi(C)} c_\alpha \alpha = 0$  for some  $c_\alpha \in \mathbb{R}$ . Consider the sets

$$A = \{\alpha \in \Pi(C) \mid c_\alpha \geq 0\}, \quad B = \Pi(C) \setminus A,$$

and let  $v = \sum_{\alpha \in A} c_\alpha \alpha = \sum_{\alpha \in B} (-c_\alpha) \alpha$ . Then

$$(v, v) = \sum_{\alpha \in A, \beta \in B} c_\alpha (-c_\beta) \cdot (\alpha, \beta) \leq 0,$$

Since the coefficients are positive by definition of  $A$  and  $B$ , and  $(\alpha, \beta) \leq 0$  by the above observation. Hence  $v = 0$ .

Now take any  $\beta \in C$ . Then

$$0 = (v, \beta) = \sum_{\alpha \in A} c_\alpha (\alpha, \beta) = \sum_{\alpha \in B} (-c_\alpha) (\alpha, \beta)$$

The coefficients in the last equality are strictly negative and the dot products are strictly positive (since the  $\alpha$ s are  $C$ -positive). Hence we must conclude that  $B = \emptyset$ . Likewise we conclude that  $c_\alpha = 0$  for all  $\alpha \in A$ . Thus  $\Pi(C)$  is a basis, and  $\Pi(C)$  is a set of simple roots.

Now we have to show that the assignment is a bijection.

Note that for any Weyl chamber  $C$ , we have

$$\begin{aligned} C &= \{\beta \in V \mid (\alpha, \beta) > 0 \text{ for any } C\text{-positive root } \alpha\} \\ &= \{\beta \in V \mid (\alpha, \beta) > 0 \quad \forall \quad \alpha \in \Pi(C)\}. \end{aligned}$$

Thus we see that we can recover  $C$  from  $\Pi(C)$ , so  $C \mapsto \Pi(C)$  is injective.

Assume now that  $\Pi$  is a system of simple roots. As  $\Pi$  is a basis, there exists a  $\beta_0 \in V$  such that  $(\alpha, \beta_0) = 1$  for all  $\alpha \in \Pi$ . Then  $(\alpha, \beta_0) \neq 0$  for all  $\alpha \in \Delta$ , hence  $\beta_0$  lies in a uniquely defined Weyl chamber  $C$ .

Then we claim that  $\Pi \subset \Pi(C)$ . Indeed, if  $\alpha \in \Pi$  is equal to  $\alpha = \beta + \gamma$  for some  $C$ -positive  $\beta, \gamma$ , then  $\beta$  and  $\gamma$  are positive with respect to  $\Pi$ , as  $(\beta, \beta_0) > 0$  and  $(\gamma, \beta_0) > 0$ .

As  $\beta, \gamma$  cannot be proportional to  $\alpha$  ( $\Delta$  is reduced), decomposing  $\beta$  and  $\gamma$  as

$$\beta = \sum_{\delta \in \Pi} c'_\delta \delta$$

and

$$\gamma = \sum_{\delta \in \Pi} c''_\delta \delta$$

with  $c'_\delta, c''_\delta \geq 0$ , we have

$$\alpha = \sum_{\delta \in \Pi} (c'_\delta + c''_\delta) \delta,$$

contradicting the assumption that  $\Pi$  is a basis. But  $\#\Pi = \#\Pi(C)$ , so they must be equal.  $\square$

Given a Weyl chamber  $C$ , we say that a hyperplane  $L \subset V$  is a **wall** of  $C$  if  $L \cap \overline{C}$  has nonempty interior in  $L$  (in the subspace topology).

Here are some facts that were given as an exercise:

- For all  $\alpha \in \Delta$ , the hyperplane  $\alpha^\perp$  is a wall of some Weyl chamber.
- For any Weyl chamber  $C$ , the walls of  $C$  are the hyperplanes  $\alpha^\perp$  with  $\alpha \in \Pi(C)$ .
- This implies that any  $\alpha \in \Delta$  lies in  $\Pi(C)$  for some Weyl chamber  $C$ .

By definition, the **Weyl group**  $W$  of  $(V, \Delta)$  is the subgroup of  $O(V)$  generated by the reflections  $s_\alpha$  for  $\alpha \in \Delta$ .

Since the reflections preserve the root system (which is a finite set), the Weyl group is itself finite. Note also that since the Weyl group act on the set of hyperplanes  $\alpha^\perp$  for  $\alpha \in \Delta$ , it also act on the set  $C$  of Weyl chambers.

**Theorem 21.4.** *We have:*

1.  $W$  acts freely and transitively on the set of Weyl chambers.
2. For any Weyl chamber  $C$ ,  $W$  is generated by the reflections  $s_\alpha$  for  $\alpha \in \Pi(C)$ .

*Half-proof:* Let  $C$  be a Weyl chamber and let  $C'$  be an adjacent Weyl chamber, meaning that there exists a hyperplane  $L$  such that  $L \cap \overline{C} \cap \overline{C'}$  has nonempty interior in  $L$ .

Then  $L = \alpha^\perp$  for some  $\alpha \in \Pi(C)$  and  $C' = s_\alpha(C)$ . Then  $s_\alpha(\Pi(C)) = \Pi(C')$ .

It follows that for any  $\beta' = s_\alpha(\beta) \in \Pi(C')$  (with  $\beta \in \Pi(C)$ ), we have

$$s_{\beta'} = s_{s_\alpha(\beta)} = s_\alpha s_\beta s_\alpha,$$

since for any orthogonal transformation we have  $s_{T\alpha} = Ts_\alpha T^{-1}$ .

Thus reflections  $s_{\beta'}$  for  $\beta' \in \Pi(C')$  lie in the group generated by  $s_\alpha$ ,  $\alpha \in \Pi(C)$ .

Now, for any Weyl chamber  $C''$  we can find a sequence of adjacent Weyl chambers  $C = C_0, \dots, C_n = C''$ . Using the above argument and induction on  $n$ , we conclude that there exists  $w \in W$  such that  $w(C) = C''$  and the reflections  $s_{\beta''}$  for  $\beta'' \in \Pi(C'')$  lie in the group generated by  $s_\alpha$  for  $\alpha \in \Pi(C)$ .

As any root in  $\Delta$  lies in some  $\Pi(C'')$ , we see that  $W$  is generated by  $s_\alpha$ ,  $\alpha \in \Delta$ .

This proves that the action of  $W$  on the set of Weyl chambers is transitive. We will not prove nor need the fact that the action is free.

For root spaces coming from compact groups, this will follow from the proof of the next result.  $\square$

**Example 21.5.** Again consider the root system  $A_2$ . One computes that the Weyl chambers are the interiors of the rays generated by  $(\cos(\frac{\pi}{6} + \frac{\pi}{3}k), \sin(\frac{\pi}{6} + \frac{\pi}{3}k))$  and  $(\cos(\frac{\pi}{6} + \frac{\pi}{3}(k+1)), \sin(\frac{\pi}{6} + \frac{\pi}{3}(k+1)))$  for  $k = 0, \dots, 5$ .

Let  $C$  be the Weyl chamber generated by the first two rays. Then one computes that there are three  $C$ -positive roots:  $\alpha_1, \alpha_2, \alpha_3$  (the first one of angle zero, next of angle  $60^\circ$ , and last of angle  $120^\circ$ ). But  $\alpha_2 = \alpha_1 + \alpha_3$ , so we discard it. Thus the  $C$ -simple roots of  $A_2$  are  $\alpha_1$  and  $\alpha_3$ .

We see that the Weyl group is the group  $D_6$  of symmetries preserving the hexagon.  $\star$

## 21.2 Root systems of compact Lie groups

Assume now that  $G$  is a compact connected Lie group and  $T \subset G$  a maximal torus. Consider the corresponding root system  $\Delta$  on  $V = i\mathfrak{t}^* \subset \mathfrak{h}^*$ .

Let  $V_0 = \text{Span}_{\mathbb{R}} \Delta$ . Then for any  $\alpha \in \Delta$ , the reflection  $s_\alpha$  act trivially on  $V_0^\perp$ . It follows that the group generated by  $s_\alpha$  ( $\alpha \in \Delta$ ) can be identified with the Weyl group  $W$  generated by  $s_\alpha|_{V_0}$ .

We have a homomorphism

$$r: N(T)/T \rightarrow O(V)$$

defined by  $r(gT)(\alpha) = \alpha \circ (\text{Ad } g^{-1})$ .

**Theorem 21.6.** *The homomorphism  $r$  defines an isomorphism*

$$N(T)/T \simeq W,$$

*so that the two definitions of the Weyl group are equivalent.*

*Proof.* The map  $r$  leaves  $\Delta$  invariant. Also,  $s_\alpha(\beta) = \beta \circ \text{Ad } w_\alpha$ , so that  $W \subset \text{image}(r)$ .

Thus it suffices to prove that the action of  $N(T)/T$  on the set of Weyl chambers is free.

[[proof to come]]  $\square$

## 22 Cartan matrices and Dynkin diagrams

Let  $(V, \Delta)$  be a reduced root system with  $\text{Span } \Delta = V$ . Fix a system  $\Pi$  of simple roots, say  $\Pi = \{\alpha_1, \dots, \alpha_r\}$ . The  $r \times r$  matrix

$$A = (a_{ij})_{i,j=1\dots r}, \quad a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

is called the **Cartan matrix** of  $(V, \Delta)$ .

Since any two systems of simple roots are related by an orthogonal transformation, the matrix  $A$  does not depend on any choices up to permutation of indices.

It is also clear that isomorphic root systems have the same Cartan matrix (up to permutation of indices).

The Cartan matrices determine the root system:

**Proposition 22.1.** *Assume  $(V_1, \Delta_1)$  and  $(V_2, \Delta_2)$  are reduced root systems with the same Cartan matrix and  $\text{Span } \Delta_i = V_i$ . Then they are isomorphic.*

*Proof.* The assumption means that there exist systems of simple roots  $\Pi_1 \subset \Delta_1$  and  $\Pi_2 \subset \Delta_2$  and a linear isomorphism  $T : V_1 \xrightarrow{\sim} V_2$  such that  $T\Pi_1 = \Pi_2$  and such that  $Ts_\alpha = s_{T\alpha}T$  for all  $\alpha \in \Pi_1$ . To see that the last condition is true, note that

$$Ts_\alpha = Tv - \frac{2(\alpha, v)}{(\alpha, \alpha)}T\alpha$$

and

$$s_{T\alpha}T(v) = Tv - \frac{2(T\alpha, Tv)}{(T\alpha, T\alpha)}T\alpha,$$

and since the Cartan matrices are equal, we can assume that these two are equal (perhaps up to reordering of some indices).

As the Weyl groups  $W_1, W_2$  of  $\Delta_1, \Delta_2$  are generated by simple reflections, we see that

$$\pi(\omega) = T\omega T^{-1}$$

is an isomorphism  $\pi : W_1 \rightarrow W_2$ .

Recall next that  $\Delta_1 = \cup_{\omega \in W_1} \omega\Pi_1$  (and same for  $\Delta_2$ ). Hence  $T\Delta_1 = \Delta_2$  as  $T\omega(\alpha) = \pi(\omega)T\alpha$  for all  $\alpha \in \Pi_1$ .

Finally, for any root  $\beta = w\alpha$  ( $w \in W_1, \alpha \in \Pi_1$ ), we have

$$Ts_\beta = Tw s_\alpha w^{-1} = \pi(w) s_{T\alpha} \pi(w)^{-1} T = s_{\pi(w)T\alpha} T = s_{Tw\alpha} T = s_{T\beta} T.$$

□

Instead of the Cartan matrix, we can also represented the root system by its **Dynkin diagram**: it is a (semidirected) graph defined as follows:

- Its **vertices** correspond to the set of simple roots  $\alpha_1, \dots, \alpha_r$ .
- We connect  $\alpha_i$  with  $\alpha_j$  by  $a_{ij}a_{ji}$  edges for  $i \neq j$ .
- If  $|a_{ij}| < |a_{ji}|$  (i.e.  $\|\alpha_i\| > \|\alpha_j\|$ ), then the edge between  $\alpha_i$  and  $\alpha_j$  is directed  $\alpha_i \rightarrow \alpha_j$ .

The Dynkian diagram contains the same information as the Cartan matrix: first of all  $a_{ii} = 2$  for all  $i$ . Since the entries are integers, the condition  $a_{ij}a_{ji} = n$  ( $n$  is the number of edges), means that at least one of  $a_{ij}$  and  $a_{ji}$  must be equal to  $-1$ . Which one is equal to  $-1$  is determined by the direction of the edge between them.

**Remark.** *Note that*

$$a_{ij}a_{ji} = \frac{4(\alpha_i, \alpha_j)^2}{\|\alpha_i\|\|\alpha_j\|} < 4,$$

by Schwarz' inequality. Therefore, in the Dynkian diagram, the only possible number of edges between two vertices is 0, 1, 2, 3. Note also that if  $\theta_{ij}$  is the angle between  $\alpha_i$  and  $\alpha_j$ , then  $a_{ij}a_{ji} = 4\cos^2\theta_{ij}$ . Hence there are only finitely many possible angles:  $90^\circ, 60^\circ, 45^\circ$  and  $30^\circ$ .

We say that a root system  $(V, \Delta)$  is **indecomposable** if there are no root systems  $(V_1, \Delta_1), (V_2, \Delta_2)$  with  $(V, \Delta) \simeq (V_1 \oplus V_2, \Delta_1 \times \{0\} \cup \{0\} \times \Delta_2)$ .

**Lemma 22.2.** *Show that a root system is indecomposable if and only if its Dynkin diagram is connected.*

*Proof.* If a root system is decomposable, then there are edges with  $a_{ij}a_{ji} = 0$ , so some  $\alpha_i$  is not connected any  $\alpha_j$ .

On the other hand, if the Dynkin diagram is disconnected, then some  $\alpha_i$  is orthogonal to all  $\alpha_j$ . This induces a decomposition  $V \simeq V' \oplus \mathbb{C}\alpha_i$ .  $\square$

**Theorem 22.3.** *All indecomposable, reduced root systems (apart from*



$(\mathbb{R}, \emptyset)$ , are given by the following diagrams.

$$\begin{array}{llll}
A_l & \begin{array}{c} \circ - \circ - \cdots - \circ - \circ \\ \alpha_1 \quad \alpha_2 \qquad \alpha_{l-1} \quad \alpha_l \end{array} & l \geq 1 & \mathfrak{su}(l+1) \\
B_l & \begin{array}{c} \circ - \circ - \cdots - \circ \Rightarrow \circ \\ \alpha_1 \quad \alpha_2 \qquad \alpha_{l-1} \quad \alpha_l \end{array} & l \geq 2 & \mathfrak{so}(l+1) \\
C_l & \begin{array}{c} \circ - \circ - \cdots - \circ \Leftarrow \circ \\ \alpha_1 \quad \alpha_2 \qquad \alpha_{l-1} \quad \alpha_l \end{array} & l \geq 3 & \mathfrak{sp}(2l) \\
D_l & \begin{array}{c} \qquad \qquad \qquad \circ_{\alpha_l} \\ \qquad \qquad \qquad | \\ \circ - \circ - \cdots - \circ - \circ \\ \alpha_1 \quad \alpha_2 \qquad \alpha_{l-2} \quad \alpha_{l-1} \end{array} & l \geq 4 & \mathfrak{so}(2l) \\
E_6 & \begin{array}{c} \qquad \qquad \qquad \circ_{\alpha_4} \\ \qquad \qquad \qquad | \\ \circ - \circ - \circ - \circ - \cdots - \circ \\ \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_5 \qquad \alpha_l \end{array} & l = 6, 7, 8 & \\
F_4 & \begin{array}{c} \circ - \circ \Rightarrow \circ - \circ \\ \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \end{array} & & \\
G_2 & \begin{array}{c} \circ \Rightarrow \circ \\ \alpha_1 \quad \alpha_2 \end{array} & & 
\end{array}$$

To read the details how it is obtained is maybe not “the most rewarding experience”.

The proof classifies Cartan matrices and implies the following result:

**Proposition 22.4.** *An integer  $r \times r$  matrix  $A = (a_{ij})_{i,j=1}^r$  is a Cartan matrix of a reduced root system if and only if*

1.  $a_{ii} = 2$ .
2.  $a_{ij} < 0$ ,  $i \neq j$ .
3. *There exists a diagonal matrix  $D$  with positive entries such that  $DA$  is symmetric and positive definite.*

The classification is based on a series of elementary observation about configurations of vectors in Euclidean spaces. For example, we can prove the following:

**Lemma 22.5.** *Any Dynkin diagram has no loops apart from multiple edges, that is, there are no different vertices  $\alpha_1, \dots, \alpha_n$  ( $n \geq 3$ ) with  $\alpha_i$  and  $\alpha_{i+1}$  connected for  $i, \dots, n-1$ , as well as  $\alpha_n$  and  $\alpha_i$  connected.*

*Proof.* Consider  $v_i = \frac{\alpha_i}{\|\alpha_i\|}$ . Note that  $4(v_i, v_j)^2 = a_{ij}a_{ji} \in \{0, 1, 2, 3\}$ . Recall also that  $(v_i, v_j) \leq 0$  for all  $i \neq j$ . Hence either  $(v_i, v_j) = 0$  or  $(v_i, v_j) \leq -\frac{1}{2}$  for  $i \neq j$ . Now consider  $v = \sum_i v_i$ . Then

$$(v, v) = n + \sum_{i \neq j} (v_i, v_j) \leq n + 2 \left( \sum_{i=1}^{n-1} (v_i, v_i + 1) + (v_n, v_1) \right) \leq 0$$

Hence  $v = 0$ . But this contradicts the linear independence of the  $\alpha_i$ .  $\square$

So the only type of allowed diagrams are trees.

**Example 22.6.** Let  $G = SU(n)$ . A maximal torus  $T$  is given by the set of diagonal matrices in  $SU(n)$ . It is isomorphic to  $\mathbb{T}^{n-1}$ .

We have the following identifications:

$$\begin{aligned}\mathfrak{g} &= \mathfrak{su}(n) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid X + X^* = 0, \text{Tr } X = 0\} \\ \mathfrak{g}_{\mathbb{C}} &= \mathfrak{sl}_n(\mathbb{C}) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid \text{Tr } X = 0\} \\ \mathfrak{h} &= \{\text{diagonal matrices with trace } 0\}.\end{aligned}$$

Define  $\epsilon_i \in \mathfrak{h}^*$  by  $\epsilon_i(H) = H_{ii}$ . Then  $\epsilon_1 + \dots + \epsilon_n = 0$ .  
Then

$$\Delta = \{\alpha_{ij} = \epsilon_i - \epsilon_j \mid i \neq j\}$$

with root spaces  $\mathfrak{g}_{\alpha_{ij}} = \mathbb{C}E_{ij}$ , because  $[H, E_{ij}] = (H_{ii} - H_{jj})E_{ij}$ .

We need an Ad-invariant inner product. One such is given by  $(X, Y) = \text{Tr}(XY)$ . It is Ad-invariant since  $\text{Ad}(g)(X) = gXg^{-1}$ . For  $X \in \mathfrak{su}(n)$ , we have  $(X, X) = \text{Tr}(X^2) = \text{Tr}(-X * X) < 0$  for  $X \neq 0$ . Thus  $(-, -)$  is negative definite on  $\mathfrak{su}(n)$ .

Then  $h_{\alpha_{ij}} = H_{\alpha_{ij}} = E_{ii} - E_{jj}$  (as defined above Lemma 20.6). As a system of simple roots, we can take

$$\alpha_1 = \epsilon_1 - \epsilon_2, \dots, \alpha_i = \epsilon_i - \epsilon_{i+1}, \dots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n.$$

This is obviously a system of simple roots. We have

$$(\alpha_i, \alpha_k) = (h_{\alpha_i}, h_{\alpha_k}) = \text{Tr}((E_{ii} - E_{i+1, i+1})(E_{kk} - E_{k+1, k+1})) = \begin{cases} 2 & i = k \\ -1 & i = k \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the Dynkin diagram is precisely  $A_{n-1}$ . ★

## 23 Semisimple Lie groups

An **ideal**  $\mathfrak{a}$  of a Lie algebra  $\mathfrak{g}$  is by definition a subalgebra of  $\mathfrak{g}$  such that  $[X, Y] \in \mathfrak{a}$  for all  $X \in \mathfrak{g}$  and  $Y \in \mathfrak{a}$ .

We say that a Lie algebra is **semisimple** if it has no non-zero abelian ideals. A Lie group is **semisimple** if its Lie algebra is semisimple.

**Lemma 23.1.** *Suppose  $G$  is a connected Lie group and  $H \subset G$  a connected Lie subgroup. Then show that  $\mathfrak{h} \subset \mathfrak{g}$  is an ideal if and only if  $H$  is a normal subgroup.*

*Proof.* Suppose  $\mathfrak{h}$  is an ideal. Then, the condition is that  $\text{ad}(X)$  actually lands in  $\mathfrak{h}$ . Since  $\text{ad}$  is the derivative of  $\text{Ad}$ , this means that  $\exp(tX)\mathfrak{h}\exp(-tX) \in H$  for all  $t \in \mathbb{R}$  and  $X \in \mathfrak{g}$ . But the image of  $\exp$  generates  $G$ , so this must be true for all of  $G$ . Thus  $H$  is a normal subgroup.

The converse is similar. □

It follows that a connected Lie group is semisimple if and only if any normal abelian subgroup of  $G$  is discrete.

In particular, if  $G$  is a compact connected Lie group, then  $G$  is semisimple if and only if any normal abelian subgroup is finite.

Recall that if  $T \subset G$  is a maximal torus, then  $N(T)^\circ = T$ . The same argument shows that if a torus  $T_1$  is a normal subgroup, then  $T_1 \subset Z(G)$  (normality implies that we have a homomorphism  $G \rightarrow \text{Aut}(T_1) \simeq \text{GL}_k(\mathbb{Z})$ ).

Hence, if  $A \subset G$  is a normal abelian subgroup, then  $(\overline{A})^\circ$  is a torus normalized by  $G$ , so  $(\overline{A})^\circ \subset Z(G)$ . Therefore  $G$  is semisimple if and only if  $Z(G)$  is finite.

Recall next that the Lie algebra of  $Z(G)$  is

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \forall Y \in \mathfrak{g}\}.$$

Thus  $G$  is semisimple if and only if  $\mathfrak{z}(\mathfrak{g}) = 0$ .

Now fix a maximal torus  $T$ , and consider the corresponding root decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{h}_{\alpha}.$$

We have  $\mathfrak{z}(\mathfrak{g})_{\mathbb{C}} = \mathfrak{z}(\mathfrak{g}_{\mathbb{C}})$ . Assume  $X = H + \sum_{\alpha \in \Delta} X_{\alpha} \in \mathfrak{z}(\mathfrak{g})$ . Therefore, for any  $H' \in \mathfrak{h}$ , we have

$$0 = [X, H'] = \sum_{\alpha \in \Delta} \alpha(H')X_{\alpha}.$$

Hence  $X_{\alpha} = 0$  for all  $\alpha \in \Delta$ . In other words,  $\mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) \subset \mathfrak{h}$ .

Now assume  $H \in \mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) \subset \mathfrak{h}$ . We have  $0 = [H, X] = \alpha(H)X$  for  $X \in \mathfrak{g}_{\alpha}$ . Hence  $H \in \ker \alpha$ . Thus

$$\mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) = \bigcap_{\alpha \in \Delta} \ker \alpha.$$

Thus  $\mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) = 0$  if and only if  $\bigcap_{\alpha \in \Delta} \ker \alpha = 0$  if and only if  $\text{Span}_{\mathbb{C}} \Delta = \mathfrak{h}^*$ .

Thus, we have proved the following proposition:

**Proposition 23.2.** *If  $G$  is any compact connected Lie group, the following are equivalent:*

- $G$  is semisimple.
- $Z(G)$  is finite.
- $\mathfrak{z}(\mathfrak{g}) = 0$ .
- $\text{Span}_{\mathbb{C}} \Delta = \mathfrak{h}^*$ .
- $\text{Span}_{\mathbb{R}} \Delta = i\mathfrak{t}^* \subset \mathfrak{h}^*$ .

**Example 23.3.** The center of  $SU(n)$  are the scalar matrices of norm one. This is isomorphic to  $\mathbb{Z}/n$ . ★

Now let  $G$  be a semisimple compact connected Lie group. Fix a maximal torus  $T \subset G$  and consider the corresponding root system  $\Delta$  inside  $\Delta \subset i\mathfrak{t}^*$ .

As a system of simple roots  $\Pi \subset \Delta$  form a basis in  $i\mathfrak{t}^*$ , the additive subgroup  $Q \subset i\mathfrak{t}^*$  generated by  $\Delta$  is a lattice, called the **root lattice**. Define

$$P = \left\{ \beta \in i\mathfrak{t}^* \mid \beta(H_{\alpha}) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z} \forall \alpha \in \Delta \right\}.$$

We see that  $P$  is clearly a subgroup of  $i\mathfrak{t}^*$  with  $Q \subset P$ .  $P$  is also a lattice: indeed, if  $\beta \in P$  is close to zero, then  $\beta(H_{\alpha}) = 0$  (for all  $\alpha \in \Delta$ ), so  $\beta = 0$  since  $\text{Span} \Delta = i\mathfrak{t}^*$ .

Hence  $P$  is a lattice. It is called the **weight lattice**. In particular, the quotient  $P/Q$  is finite abelian.

**Example 23.4.** Consider the root system  $A_1$  corresponding to  $SU(2)$ . Then  $i\mathfrak{t}^* = \mathbb{R}$ . We have that  $\Delta = \{-1, 1\}$ . Also,  $P = \frac{1}{2}\mathbb{Z}$ . Hence  $P/Q = \mathbb{Z}/2$ . ★

Recall that the character lattice  $X^*(T) \subset i\mathfrak{t}^*$  satisfies  $Q \subset X^*(T) \subset P$ . ((WHY??))

**Proposition 23.5.** *Assume that  $G$  is a semisimple compact connected Lie group. Then its universal cover  $\tilde{G}$  is still compact.*

*Proof.* Let  $\pi : \tilde{G} \rightarrow G$  be the covering map.

We will need two facts which we will not prove from algebraic topology. The first is that  $\ker \pi = \pi_1(G)$ . The second is if  $G$  is a compact manifold, then  $\pi_1(G)$  is finitely generated as a group (this follows from the van Kampen theorem).

As  $\ker \pi$  is a normal discrete subgroup of  $\tilde{G}$ , we have  $\ker \pi \subset Z(\tilde{G})$ . Hence  $\ker \pi \simeq \mathbb{Z}^n \otimes \Gamma$  where  $\Gamma$  is a finite abelian group. We want to show that  $n = 0$ . This will imply the statement since then the covering is finite. So assume that  $n > 0$ .

Then we can find a subgroup  $H \subset \ker \pi$  such that  $\ker \pi/H$  is finite, but arbitrarily large. Then consider  $G_1 = \tilde{G}/H$ . We have a homomorphism  $\pi_1 : G_1 \rightarrow G$  with  $\ker \pi_1 = \ker \pi/H$ , hence  $G_1$  is compact, but  $\ker \pi_1$  can be chosen arbitrarily large. This will lead to a contradiction.

Fix a maximal torus  $T \subset G$ . We identify  $\mathfrak{g}_1$  with  $\mathfrak{g}$  using  $(\pi_1)_*$ . Consider the maximal torus  $\pi_1 = \exp_{G_1} \mathfrak{t} \subset G_1$ .

Consider the root and weight lattices  $Q, P \subset i\mathfrak{t}^* = i\mathfrak{t}_1^*$ . We have  $Q \subset X^*(T) \subset X^*(T_1) \subset P$ . Hence

$$|X^*(T_1)/X^*(T)| \leq |P/Q|.$$

One can show that  $\ker \pi_1 = X^*(T_1)/X^*(T)$ . Hence

$$|\ker \pi_1| = |X^*(T_1)/X^*(T)| \leq |P/Q|.$$

Since the left-hand-side can be arbitrarily large, this is a contradiction, since  $P/Q$  is finite.  $\square$

Let  $\text{Ad}(G)$  be the Lie group  $G/Z(G)$ . Then  $\text{Ad}(G)$  is semisimple, because .....

**Proposition 23.6.** *Assume  $G$  is a compact connected Lie group. Then  $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ .*

**Proposition 23.7.** *Any compact connected Lie group  $G$  can be written as  $(\mathbb{T}^n \times H)/\Gamma$ , where  $H$  is a compact simply connected semisimple Lie group and  $\Gamma$  is a finite subgroup of  $\mathbb{T}^n \times Z(H)$ .*

*Proof.* The group  $G/Z(G)$  is semisimple with Lie algebra  $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \simeq [\mathfrak{g}, \mathfrak{g}]$ .

Consider the universal cover  $H$  of  $G/Z(G)$ . Then  $H$  is compact and semisimple and simply-connected. The identification of the Lie algebra of  $H$  with  $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$  defines a homomorphism  $H \xrightarrow{\pi} G$ . Then we get a homomorphism  $\pi' : Z(G)^\circ \times H \rightarrow G$  given by  $(g, h) \mapsto g\pi(h)$ . At the Lie algebra level, this is an isomorphism.

Hence  $\pi'$  is surjective and  $\Gamma = \ker \pi'$  is finite.  $\square$

## 24 Classification of simply connected semisimple Lie groups

Let  $G$  be a compact connected Lie group. Fix a maximal torus  $T \subset G$ . Consider a finite-dimensional representation  $\pi : G \rightarrow \mathrm{GL}(V)$ . We can decompose  $\pi|_T$  into isotypic components. Thus, for  $\lambda \in X^*(T)$ , define

$$V(\lambda) = \{v \in V \mid Hv = \lambda(H)v \forall H \in \mathfrak{h}\}.$$

The elements of  $V(\lambda)$  are called **vectors of weight**  $\lambda$ . Put

$$P(V) = \{\lambda \in \mathfrak{h}^* \mid V(\lambda) \neq 0\} \subset X^*(T).$$

Then

$$V = \bigoplus_{\lambda \in P(V)} V(\lambda).$$

If  $X \in \mathfrak{g}_\alpha$ , then  $XV(\lambda) = V(\lambda + \alpha)$ . Indeed, for  $H \in \mathfrak{h}$ ,  $v \in V(\lambda)$ , we have

$$\begin{aligned} HXv &= (HX - XH)v + XHv \\ &= [H, X]v + XHv \\ &= \alpha(H)Xv + \lambda(H)Xv \\ &= (\alpha(H) + \lambda(H))Xv. \end{aligned}$$

**Proposition 24.1.** *Let  $\lambda \in P(V)$  and  $\alpha \in \Delta$ . Then*

1. *The set  $\{\lambda + k\alpha \in P(V) \mid k \in \mathbb{Z}\}$  have the form*

$$\lambda - q\alpha, \lambda - (q-1)\alpha, \dots, \lambda + p\alpha$$

*for some  $p, q \geq 0$  such that  $p - q = \lambda(H_\alpha)$ .*

2.  $E_\alpha^{p+q}V(\lambda - q\alpha) = V(\lambda + p\alpha)$ .

*Proof.* Consider the copy of  $\mathfrak{sl}_2(\mathbb{C}) \subset \mathfrak{g}_\mathbb{C}$  spanned by  $\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha$  and  $H_\alpha$ . Consider  $V$  as a  $\mathfrak{sl}_2(\mathbb{C})$ -module. From the representation theory of  $\mathfrak{sl}_2(\mathbb{C})$ , we know that the set

$$\{\lambda(H_\alpha) + 2k \mid \lambda + k\alpha \in P(V)\}$$

has the form

$$\{-n, -n+2, \dots, n-2, n\}$$

for some integer  $n = 2s$ . This means that

$$\{\lambda + k\alpha \mid \lambda + k\alpha \in P(V)\} = \{\lambda - q\alpha, \dots, \lambda + p\alpha\}.$$

Hence  $-n = \lambda(H_\alpha) - 2q$  and  $n = \lambda(H_\alpha) + 2p$ . Adding these equations give  $p - q = \lambda(H_\alpha)$ .

The second property follows since  $E$  raises the indices and that  $E_{-s}^{2s}\xi_{-s} = \xi_s$ , up to scalar.  $\square$

**Corollary 24.2.** *If  $\lambda, \lambda + \alpha \in P(V)$  for some  $\alpha \in \Delta$ , then  $E_\alpha V(\lambda) \neq 0$ .*

In particular, applying this to the adjoint representation, we conclude that if  $\alpha, \beta$  and  $\alpha + \beta \in \Delta$ , then  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ : we know from earlier that the left-hand-side is contained in the right-hand-side. By the above corollary, it is nonzero, and both sides are one-dimensional.

**Proposition 24.3.** *Fix a system of simple roots  $\Pi \subset \Delta$ . Then every  $\alpha \in \Delta_+$  can be written as  $\alpha_{i_1} + \dots + \alpha_{i_k}$  such that  $\alpha_{i_j} \in \Pi$  and  $\alpha_{i_1} + \dots + \alpha_{i_l} \in \Delta$  for all  $l = 1, \dots, k$ .*

*Proof.* FFF □

**Corollary 24.4.** *The space  $\mathfrak{g}_\mathbb{C}$  is generated by  $\mathfrak{h}$  and  $E_\alpha, F_\alpha$  for  $\alpha \in \Pi$ .*

*Proof.* If  $\lambda = \alpha_{i_1} + \dots + \alpha_{i_k} \in \Delta_+$  as in the proposition, then  $\mathfrak{g}_\lambda = [[E_{\alpha_1}, E_{\alpha_2}], E_{\alpha_3}], \dots, E_{\alpha_k}] \otimes \mathbb{C}$ . □

**Theorem 24.5.** *Two compact simply connected semisimple Lie groups are isomorphic if and only if their root systems are isomorphic.*

*Proof.* Fix a maximal torus  $T$  and a system of simple roots  $\Pi = \{\alpha_1, \dots, \alpha_r\}$ . Write  $H_i, E_i, F_i$  for  $H_{\alpha_i}, E_{\alpha_i}, F_{\alpha_i}$ . Let  $A$  denote the Cartan matrix  $A = (a_{ij})_{i,j=1}^r$ , with  $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ .

Let  $\mathfrak{a}$  be the universal Lie algebra with generators  $\widehat{H}_i, \widehat{E}_i, \widehat{F}_i$  for  $i = 1, \dots, r$  subject to the relations

$$\begin{aligned} [\widehat{H}_i, \widehat{H}_i] &= 0 & [\widehat{E}_i, \widehat{F}_i] &= \widehat{H}_i \\ [\widehat{H}_i, \widehat{E}_j] &= a_{ij} \widehat{E}_j & [\widehat{E}_i, \widehat{F}_j] &= 0 \ (i \neq j) \\ [\widehat{H}_i, \widehat{F}_j] &= a_{ij} \widehat{F}_j \end{aligned}$$

Then we get a Lie algebra homomorphism  $\pi : \mathfrak{a} \rightarrow \mathfrak{g}_\mathbb{C}$ . We need to describe  $\ker \pi$ . Denote by  $\widehat{\mathfrak{h}} = \text{Span}\{\widehat{H}_i\}_{i=1, \dots, r}$ , and by  $\mathfrak{a}_+ \subset \mathfrak{a}$  the subalgebra generated by  $\widehat{E}_i$  ( $i = 1, \dots, r$ ), and by  $\mathfrak{a}_- \subset \mathfrak{a}$  the subalgebra generated by  $\widehat{F}_i$  ( $i = 1, \dots, r$ ).

Then it is not difficult to see that  $\mathfrak{a} = \mathfrak{a}_- \oplus \widehat{\mathfrak{h}} \oplus \mathfrak{a}_+$  as a vector space.

Define  $\widehat{\alpha}_i \in \widehat{\mathfrak{h}}^*$  by  $\widehat{\alpha}_i(\widehat{H}_j) = a_{ji}$ .

Then we can define root spaces

$$\mathfrak{a}_\lambda = \{\widehat{X} \in \mathfrak{a} \mid [\widehat{H}, \widehat{X}] = \lambda(\widehat{H})\widehat{X} \quad \forall \quad \widehat{H} \in \widehat{\mathfrak{h}}\}.$$

Then  $\mathfrak{a}_+$  and  $\mathfrak{a}_-$  decomposes into positive and negative root spaces, respectively. That is:

$$\mathfrak{a}_+ = \bigoplus_{\lambda = n_1 \widehat{\alpha}_1 + \dots + n_r \widehat{\alpha}_r, \quad n_i \in \mathbb{Z}_+} \mathfrak{a}_\lambda \quad \text{and} \quad \mathfrak{a}_- = \bigoplus_{\lambda = n_1 \widehat{\alpha}_1 + \dots + n_r \widehat{\alpha}_r, \quad n_i \in \mathbb{Z}_-} \mathfrak{a}_\lambda$$

Now, if  $\mathfrak{p} \subset \mathfrak{a}$  is an ideal<sup>8</sup>, then  $\mathfrak{p} = \bigoplus_{\lambda} (\mathfrak{a}_{\lambda} \cap \mathfrak{p})$ . It follows that the sum of all ideals  $\mathfrak{p}$  in  $\mathfrak{a}$  such that  $\mathfrak{p} \cap \widehat{\mathfrak{h}} = 0$  still is an ideal  $\mathfrak{m}$  with the same property, that is,  $\mathfrak{m} \cap \widehat{\mathfrak{h}} = 0$ .

The claim is now that  $\ker(\pi : \mathfrak{a} \rightarrow \mathfrak{g}_{\mathbb{C}}) = \mathfrak{m}$ . Thus  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{a}/\mathfrak{m}$ .

As  $\pi : \mathfrak{a} \rightarrow \mathfrak{g}_{\mathbb{C}}$  gives an isomorphism  $\widehat{\mathfrak{h}} \simeq \mathfrak{h}$ , we have that  $\ker \pi \cap \widehat{\mathfrak{h}} = 0$ . Hence  $\ker \pi \subset \mathfrak{m}$ .

Consider the ideal  $\mathfrak{p} = \mathfrak{m}/\ker \pi$  in  $\mathfrak{g}_{\mathbb{C}}$ . Then  $\mathfrak{p} \cap \mathfrak{h} = 0$ . Then

$$\mathfrak{p} = \bigoplus_{\alpha \in \Delta} (\mathfrak{p} \cap \mathfrak{g}_{\alpha}).$$

If  $\mathfrak{p} \neq 0$ , then  $\mathfrak{p} \cap \mathfrak{g}_{\alpha} \neq 0$  for some  $\alpha \in \Delta$ , so  $E_{\alpha} \in \mathfrak{p}$ . But then  $H_{\alpha} = [E_{\alpha}, F_{\alpha}] \in \mathfrak{p} \cap \mathfrak{h}$ , which is a contradiction. Hence  $\ker \pi = \mathfrak{m}$ .

Therefore  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{a}/\mathfrak{m}$ , so  $\mathfrak{g}_{\mathbb{C}}$  is described entirely in terms of the Cartan matrix.

Recall also from earlier that we can recover  $\mathfrak{g}$  from  $\mathfrak{g}_{\mathbb{C}}$  as the subspace

$$\{X \in \mathfrak{g}_{\mathbb{C}} \mid X^{\star} = -X\},$$

where  $X^{\star}$  was the involution introduced earlier. The involution is completely determined by what it does on generators, so we can recover  $\mathfrak{g}$ .

Finally, we conclude that if  $G_1, G_2$  are two simply-connected compact Lie groups with isomorphic root systems, then their Cartan matrices coincide, hence  $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ , and therefore  $G_1 \simeq G_2$ .  $\square$

In addition to the obvious relations in  $\mathfrak{g}$ , we have the so-called **Serre relations**:

**Proposition 24.6** (Fact). *The relations*

$$(\operatorname{ad} E_j)^{1-a_{ji}}(E_i) = 0 \quad (\operatorname{ad} F_j)^{1-a_{ji}}(F_i) = 0 \quad i \neq j$$

*together with relations used in the proof are relations in  $\mathfrak{g}_{\mathbb{C}}$ .*

As mentioned earlier, conversely, any reduced root system  $(V, \Delta)$  with  $\operatorname{Span} \Delta = V$  arises from a (simply-connected) compact semisimple Lie group. This can be shown in different ways: one way is as in the proof above. Define  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{a}/\mathfrak{m}$  as above and integrate. The problem in this approach is to show that  $\mathfrak{g}_{\mathbb{C}}$  is finite-dimensional (and gives rise to a finite-dimensional Lie group).

The other approach is to use the classification of reduced root systems and construct each Lie group corresponding to each root system. Those missing are the 5 exceptional ones.

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<sup>8</sup>Keep in mind that being an ideal is the same as being an  $\operatorname{ad} \widehat{\mathfrak{h}}$ -invariant subspace



## 25 Representation theory of compact connected Lie groups

Let  $G$  be a compact connected Lie group. Fix a maximal torus  $T \subset G$ . Fix a system of simple roots  $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset \Delta$ . Then for any finite-dimensional representation  $V_\pi$  of  $G$ , we have a weight decomposition

$$V = \bigoplus_{\lambda \in P(V)} V(\lambda).$$

where  $V(\lambda) = \{v \in V \mid Hv = \lambda(H)v \ \forall \ H \in \mathfrak{h}\} \neq 0$ , and  $P(V) \subset X^*(T) \subset i\mathfrak{t}^* \subset \mathfrak{h}^*$ .

We saw in the last lecture that  $E_\alpha V(\lambda) \subset V(\lambda + \alpha)$  and  $F_\alpha V(\lambda) \subset V(\lambda - \alpha)$ .

We call a nonzero vector  $v \in V(\lambda)$  a **highest weight vector** (of weight  $\lambda$ ) if  $E_\alpha v = 0$  for all  $\alpha \in \Delta_+$ . Equivalently, if  $E_i v = 0$  for all  $i = 1, \dots, r$ .

If in addition,  $v$  generated  $V$  as a  $\mathfrak{g}_\mathbb{C}$ -module, then  $V$  is called a **highest weight module**.

Clearly, if  $V$  is an irreducible representation, then it is a highest weight module: the highest weight is the largest element in  $P(V)$  with respect to the partial order

$$\mu \geq \eta \Leftrightarrow \eta = \mu - \alpha_{i_1} - \dots - \alpha_{i_k}.$$

Thus any irreducible finite-dimensional representation of  $G$  has a uniquely assigned weight  $\lambda \in X^*(T)$ .

Recalling the representation theory of  $\mathfrak{sl}_2(\mathbb{C})$ , we also know that  $\lambda(H_i) \geq 0$  for all  $i = 1, \dots, r$ .

A weight  $\lambda \in X^*(T)$  such that  $\lambda(H_i) \geq 0$  for all  $i = 1, \dots, r$  is called **dominant**.

**Theorem 25.1.** *For any compact connected Lie group with maximal torus  $T$  and a fixed system of simple roots, we have:*

1. *For every dominant weight  $\lambda \in X^*(T)$ , there exists an irreducible finite-dimensional representation  $\pi_\lambda$  of  $G$  of highest weight  $\lambda$ .*
2. *Any finite-dimensional irreducible representation of  $G$  is equivalent to  $\pi_\lambda$  for a uniquely defined  $\lambda$ .*

We will prove this only for the simplyconnected semisimple case. Simultaneously, we will prove that  $X^*(T) = P$ .

*Proof.* Here is a sketch of the proof.

We know that  $\text{Span}_\mathbb{C} \Delta = \mathfrak{h}^*$  and  $Q \subset X^*(T) \subset P$ .

Recall that

$$P = \left\{ \beta \in i\mathfrak{t}^* \mid \beta(H_\alpha) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z} \ \forall \ \alpha \in \Delta \right\}.$$

and that  $Q = \text{Span}_{\mathbb{Z}} \Delta$ . The set of dominant weights in  $P$ , that is, the set

$$\{\lambda \in P \mid (\lambda, \alpha_i) \geq 0 \forall i\}$$

is denoted by  $P_+$ . The elements of  $P_+$  are called **dominant integral weights**. Note that  $P_+ = P \cap \overline{C}$ .

The goal is to show that for any  $\lambda \in P_+$ , there exist a unique finite-dimensional irreducible highest weight  $\mathfrak{g}_{\mathbb{C}}$ -module of weight  $\lambda$ .

This will prove the theorem in the semisimple simplyconnected case, as well that  $X^*(T) = P$ .

Denote by  $U\mathfrak{g}$  the universal enveloping algebra of  $\mathfrak{g}$ . The Poincaré-Birkhoff-Witt-theorem says that if  $X_1, \dots, X_n$  is a basis of  $\mathfrak{g}$ , then  $X_1^{a_1} \cdots X_n^{a_n}$  with  $a_1, \dots, a_n \geq 0$  is a basis of  $U\mathfrak{g}$ .

This implies that the map  $\mathfrak{g} \rightarrow U\mathfrak{g}$  is injective, so we can consider  $\mathfrak{g}$  as a subspace of  $U\mathfrak{g}$ . If  $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$  are Lie algebras and  $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$  as a vector space, then the homomorphisms  $U\mathfrak{a} \rightarrow U\mathfrak{b}$  and  $U\mathfrak{b} \rightarrow U\mathfrak{g}$  induce a linear isomorphism  $U\mathfrak{a} \otimes_U \mathfrak{b} \rightarrow U\mathfrak{g}$ .

Returning to semisimple simplyconnected Lie groups, we have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}.$$

Consider the following two Lie subalgebras in  $\mathfrak{g}_{\mathbb{C}}$ :

$$\mathfrak{a}_{-} = \bigoplus_{\alpha \in \Delta_{-}} \mathfrak{g}_{\alpha} \quad \mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_{+}} \mathfrak{g}_{\alpha}.$$

The module  $\mathfrak{b}$  is called a **Borel subalgebra**. Then  $U(\mathfrak{g}_{\mathbb{C}}) = U\mathfrak{a}_{-} \otimes_{\mathbb{C}} U\mathfrak{b}$ .

For any  $\lambda \in \mathfrak{h}^*$ , we have a 1-dimensional representation  $\rho_{\lambda}$ :  $\rho_{\lambda} : U\mathfrak{b} \rightarrow \mathbb{C}$  defined by  $\rho_{\lambda}(H) = \lambda(H)$ , and  $\rho_{\lambda}(\mathfrak{g}_{\alpha}) = 0$ . Denote this representation by  $\mathbb{C}_{\lambda}$ .

Then we define a  $U\mathfrak{g}_{\mathbb{C}}$ -module by induction:

$$L_{\lambda} = U\mathfrak{g}_{\mathbb{C}} \otimes_{U\mathfrak{b}} \mathbb{C}_{\lambda}.$$

Note that, as vector spaces,  $L_{\lambda} \simeq U\mathfrak{a}_{-}$ . Put  $v_{\lambda} = 1 \otimes 1 \in L_{\lambda}$ . Then  $v_{\lambda}$  generates  $L_{\lambda}$  and  $Hv_{\lambda} = \lambda(H)v_{\lambda}$  and  $E_{\alpha}v_{\lambda} = 0$ . Thus  $L_{\lambda}$  is a highest weight module with highest weight vector  $\lambda$ . It is called a **Verma module**.

It has the following universal property: if  $V$  is a highest weight module with highest weight vector  $w \in V(\lambda)$ , then there exists a unique morphism  $L_{\lambda} \rightarrow V$  such that  $v_{\lambda} \mapsto w$ . Indeed,  $\mathbb{C}_{\lambda} \rightarrow V$  ( $1 \mapsto w$ ) is a morphism of  $\mathfrak{b}$ -modules. Then we define  $L_{\lambda} \rightarrow V$  to be the composition

$$L_{\lambda} = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U\mathfrak{b}} \mathbb{C}_{\lambda} \rightarrow U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U\mathfrak{b}} V \rightarrow V.$$

We have a weight decomposition

$$L_\lambda = \bigoplus_{\mu \leq \lambda} L_\lambda(\mu).$$

We can see that  $\dim L_\lambda(\mu)$  is equal to the number of ways of writing  $\lambda - \mu$  as  $\sum_{j=1}^m a_j \beta_j$  for  $a_j \in \mathbb{Z}_+$ . This number is called **Kostants partition function** of  $\lambda - \mu$ . In particular,  $\dim L_\lambda(\mu)$  is always finite.

For any submodule  $M \subset L_\lambda$  we have

$$M = \bigoplus_{\mu \leq \lambda} (M \cap L_\lambda(\mu)).$$

It follows that the sum of all submodules  $M_\lambda$  such that  $v_\lambda \notin M$  still has the same property.

Now define  $V_\lambda = L_\lambda / M_\lambda$ .

Denote by  $\xi_\lambda$  the image of  $v_\lambda$  in  $V_\lambda$ .

**Lemma 25.2.** *For any  $\lambda \in \mathfrak{h}^*$ ,  $V_\lambda$  is a simple module. Furthermore, any simple highest weight module is isomorphic to  $V_\lambda$  for a uniquely defined  $\lambda \in \mathfrak{h}^*$ .*

*Proof inside proof:* If there is a proper nonzero submodule  $M$  of  $V_\lambda$ , then it lifts to proper inclusions  $M_\lambda \subset M \subset L_\lambda$ . We cannot have  $v_\lambda \in M$ , for then  $M = L_\lambda$  (since  $v_\lambda$  generates  $L_\lambda$ ). Hence  $v_\lambda \notin M$ , so  $M_\lambda = M$  by maximality of  $M_\lambda$ . Thus  $V_\lambda$  is simple.

Now suppose that  $V$  is a simple highest weight module with highest weight vector  $v \in V(\lambda)$ . Then we have a morphism  $\pi : L_\lambda \rightarrow V$  given by  $v_\lambda \mapsto w$ . But  $L_\lambda$  is simple, so this must be an isomorphism.

The highest weight is uniquely determined, as all other weights of  $V$  are lower than  $\lambda$ .  $\square$

As any finite-dimensional simple module is a highest weight module, to finish the classification of such modules (thus finite-dimensional representations of  $G$ ), it remains to understand when  $V_\lambda$  is finite-dimensional.

**Proposition 25.3.** *The module  $V_\lambda$  is finite-dimensional if and only if  $\lambda \in P_+$ .*

*Proof.* This is long and tricky. The hardest part.  $\square$

Once the above proposition is proved, we see that the  $V_\lambda$  for  $\lambda \in P_+$  are all irreducible representations of  $\mathfrak{g}$ .  $\square$

**Proposition 25.4.** *Any finite-dimensional highest weight module is simple, hence isomorphic to  $V_\lambda$  for some  $\lambda \in P_+$ .*

## 26 Classical approach to representation theory of $SU(n)$

Recall that the standard choice of a maximal torus is

$$T = \left\{ \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \mid z_i \in \mathbb{T}, \prod z_i = 1 \right\}.$$

The Lie algebra is

$$\mathfrak{su}(n) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid \text{Tr } X = 0, X + X^* = 0\}.$$

The complexification of  $\mathfrak{su}(n)$  is identified with  $\mathfrak{sl}_n(\mathbb{C})$ , which consist of complex matrices with trace zero.

The Cartan subalgebra is

$$\mathfrak{h} = \left\{ H = \begin{pmatrix} H_{11} & & \\ & \ddots & \\ & & H_{nn} \end{pmatrix} \mid \sum_i H_{ii} = 0 \right\}.$$

Define  $L_i \in \mathfrak{h}^*$  by  $L_i(H) = H_{ii}$ . Then  $L_1 + \dots + L_n = 0$ , and

$$\Delta = \{L_i - L_j \mid i \neq j\}.$$

As a system of simple roots we can take  $\Pi$  to be  $\alpha_i = L_i - L_{i+1}$  for  $i = 1, \dots, n-1$ . The root spaces are  $\mathfrak{g}_{L_i - L_j} = \mathbb{C}E_{ij}$ , where  $E_{ij}$  is the matrix  $(\delta_{ij})$ . As an Ad-invariant inner product on  $\mathfrak{h}$ , we may take  $(H, H') = \text{Tr}(HH')$ .

Recall that the isomorphism  $\mathfrak{h}^* \simeq \mathfrak{h}$ ,  $\alpha \mapsto h_\alpha$  is defined by  $(H, h_\alpha) = \alpha(H)$  for all  $H$ . For  $\alpha = L_i$  we get  $H_{ii} = \text{Tr}(Hh_{L_i})$  for  $i = 1, \dots, n$ . By inspection we see that then we must have  $h_{L_i} = E_{ii} - \frac{1}{n}I$ . Then the form on  $\mathfrak{h}^*$  is given by

$$(L_i, L_j) = (h_{L_i}, h_{L_j}) = \text{Tr}\left((E_{ii} - \frac{1}{n}I)(E_{jj} - \frac{1}{n}I)\right) = \delta_{ij} - \frac{1}{n}.$$

Hence  $(\alpha_i, \alpha_i) = (L_i - L_{i+1}, L_i - L_{i+1}) = 1 - \frac{1}{n} + \frac{2}{n} + 1 - \frac{1}{n} = 2$ . Thus

$$H_{\alpha_i} = \alpha_i^\vee = \frac{2h_{\alpha_i}}{(\alpha_i, \alpha_i)} = E_{ii} - E_{i+1, i+1},$$

which we can identify with  $\alpha_i$ .

The **fundamental weights**  $\omega_i$  are defined by  $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ . From this we see that

$$\omega_i = L_1 + \dots + L_i, \quad i = 1, \dots, n-1.$$

It follows that  $P = \mathbb{Z}\omega_1 + \dots + \mathbb{Z}\omega_{n-1} = \mathbb{Z}L_1 + \dots + \mathbb{Z}L_{n-1}$  and that  $P_+ = \mathbb{Z}_+\omega_1 + \dots + \mathbb{Z}_+\omega_{n-1}$ .

Put  $V = \mathbb{C}^n$ . Consider the obvious representation  $\pi$  of  $G = SU(n)$  on  $V$ . It is obviously irreducible.

The vector  $e_1 = (1, \dots, 0)$  is a highest weight vector,  $E_{ij}e_1 = 0$  for  $i < j$  of weight  $\omega_1 = L_1$ , as  $He_1 = H_{11}e_1$ . Thus  $\pi = \pi_{\omega_1} = \pi_{L_1}$ .

The representations corresponding to the fundamental weights are called **fundamental**. Since  $G$  is a compact subgroup of  $SL(V)$ , we know that every irreducible representation of  $G$  is a subrepresentation of  $\pi_{\omega_1}^{\otimes k}$  for some  $k \geq 0$ . How do we explicitly realize  $\pi_{a_1 L_1 + \dots + a_n L_n}$  ( $a_1 \in \mathbb{Z}_+$ ,  $a_1 \geq a_2 \geq \dots \geq a_{n-1}$ ) as a subrepresentation of  $\pi_{\omega_1}^{\otimes k}$ ?

Fix  $k \geq 2$ . Consider the representation  $\sigma_k$  of  $S_k$  on  $V^{\otimes k}$  defined by

$$\sigma_k(\tau)(v_1 \otimes \dots \otimes v_k) = v_{\tau^{-1}(1)} \otimes \dots \otimes v_{\tau^{-1}(k)}.$$

**Theorem 26.1** (Schur-Weyl duality). *The subalgebras  $\text{Span } \pi^{\otimes k}(SU(n))$  and  $\text{Span } \sigma_k(S_k)$  of  $\text{End}(V^{\otimes k})$  are commutants of each other.*

[[insert proof]]

Let  $\rho_i : S_k \rightarrow GL(V_i)$  be the different irreducible representations contained in  $\sigma_k$ . We then have a decomposition into isotypic components:

$$V^{\otimes k} = \bigoplus_{i=1}^m V_i \otimes W_i,$$

with  $\sigma_k(\tau) = \bigoplus_{i=1}^m \rho_i(\tau) \otimes 1$ .

Then  $\sigma_k(\mathbb{C}[S_k]) = \bigoplus 1 \otimes \text{End}(W_i)$ . It follows that we get representations  $\tilde{\rho}_i : SU(n) \rightarrow GL(W_i)$  such that .....

Therefore we get a 1-1 correspondence between the irreducible representations of  $S_k$  contained in  $\sigma_k$  and the irreducible representations of  $SU(n)$  contained in  $\pi_{\omega_1}^{\otimes k}$ .

**Schur functor:** Let  $\lambda$  be a Young diagram and consider the corresponding element  $e_\lambda = e_{T_\lambda}$  defining the Specht module. Put  $S^\lambda(V) = \sigma_k(e_\lambda)V^{\otimes k}$ , or in other words:

$$S^\lambda(V) = c_{T_\lambda} \mathbb{C}[S_n] \otimes_{S_n} V^{\otimes n}.$$

....

**Remark.** *To complete the classification of irreducible representations of  $SU(n)$  without relying on Verma modules and such, it would remain to show that two irreducible representations with highest weight are equivalent, that is,  $S^\lambda(V) = S^{\lambda'}(V)$  if  $\lambda_i = \lambda'_i + a$  for some  $a \in \mathbb{Z}$ .*

## A Exercises

### A.1 Lecture 1

**Exercise 1.** Suppose  $V_1 \oplus \dots \oplus V_r \sim W_1 \oplus \dots \oplus W_s$  as representations and that the  $V_i, W_i$  are irreducible. Show that  $n = m$  and there exists  $\rho \in S_r$  such that  $V_i \sim W_{\rho(i)}$  for all  $i$ . ♠

**Solution 1.** First note a lemma: If  $V \oplus W \sim V' \oplus W'$  and  $V \sim V'$ , then  $W \sim W'$ . This follows from the five-lemma.

Now consider the composition  $V_1 \hookrightarrow V_1 \oplus \dots \oplus V_r \rightarrow W_1 \oplus \dots \oplus W_s \rightarrow W_i$ . This has to be non-zero for at least one  $i$ . Hence  $V_1 \sim W_i$  for some  $i$ . By rearranging, we are in the situation of the lemma above. Hence, inductively, if  $r \leq s$ , we find  $0 \sim W_{r+1} \oplus \dots \oplus W_s$ , which is impossible. Similarly for  $s \leq r$ . Hence  $r = s$  and we conclude. ♥

**Exercise 2.** Let  $\hat{G}$  denote the Pontryagin dual of  $G$ . Then, for any finite abelian group  $G$ , we have  $G \simeq \hat{\hat{G}}$ . ♠

**Solution 2.** We first do this when  $G = \mathbb{Z}/n$  for some  $n \geq 0$ . Let  $\varphi : \mathbb{Z}/n \rightarrow \mathbb{C}^*$  be a character. Then  $\phi(1) = z$  for some  $z$ , but  $\phi(n) = \phi(0) = z^n = 1$ , so  $z$  must be a  $n$ 'th root of unity.

Realize  $\mathbb{Z}/n$  as the  $n$ th roots of unity. Then we can define a homomorphism  $\widehat{\mathbb{Z}/n} \rightarrow \mathbb{Z}/n$  by  $\varphi \mapsto \varphi(1)$ . Similarly, we can define an inverse map by sending  $e^{2\pi i/n}$  to the character  $m \mapsto e^{2\pi i m/n}$ . These are inverses.

Now, every finite abelian group is a product of these groups. So it remains to show (by induction) that if  $G, H$  are two finite abelian groups, then  $\widehat{G \times H} \simeq \hat{G} \times \hat{H}$ . The inclusion maps  $G \rightarrow G \times H$  and  $H \rightarrow G \times H$  induces a map  $\hat{G} \times \hat{H} \rightarrow \widehat{G \times H}$  by  $(\varphi_1, \varphi_2) \mapsto ((g, h) \mapsto \varphi_1(g)\varphi_2(h))$ .

It is easy to see that this map is injective. To see that it is surjective, let  $\varphi : G \times H \rightarrow \mathbb{C}$  be a character. Write  $\varphi$  as  $gh \mapsto \varphi(gh)$ . Then we can define characters on  $G, H$  by  $g \mapsto \varphi(g \cdot 1)$  which maps to  $\varphi$ . Ok. ♥

### A.2 Exercises 3

**Exercise 3.** Show that the  $S$  defined in the proof of Theorem (2.1) lies in  $\Theta(G)''$ . ♠

**Solution 3.** This is, I think, definition-hunting. DETAILS COME LATER ♥

**Exercise 4.** 1. Check that indeed

$$\langle v, w \rangle := \frac{1}{|G|} \sum_{g \in G} \langle \pi(g)v, \pi(g)w \rangle'$$

is an invariant Hermitian scalar product.

2. Show that if  $\pi$  is irreducible, then an invariant scalar product is unique up to a factor.



**Solution 4.** i). Invariant means that  $\langle \pi(h)v, \pi(h)w \rangle = \langle v, w \rangle$ . This is clear from the definitions, since if  $g \in G$  ranges over all of  $G$ , then so does  $gh$ .

The only (slightly) nontrivial thing to check is that  $\langle, \rangle$  is positive definite. But this is so.

ii) If  $\langle, \rangle$  is invariant, then it is an element of  $\text{Hom}_G(V \otimes V, \mathbb{C})$ . This is canonically isomorphic to  $\text{Hom}_G(V \otimes, V^*)$ . If we can show that  $V$  and  $V^*$  are isomorphic as representations, then we are done by Schur's lemma.

But we are given an inner product  $\langle, \rangle$ . We can define a map  $V \rightarrow V^*$  by sending  $v \in V$  to the function  $w \mapsto \langle v, w \rangle$ . This is clearly a linear isomorphism, and it is also a map of representations. Recall that the action of  $G$  on  $V^*$  is defined by  $\pi^*(g)\varphi(v) = \varphi(g^{-1}v)$ . Then in the diagram

$$\begin{array}{ccc} V & \longrightarrow & V^* \\ \downarrow \cdot g & & \downarrow \cdot g \\ V & \longrightarrow & V^* \end{array}$$

we want  $\langle gv, w \rangle$  to be equal to  $\langle v, g^{-1}w \rangle$ . But this is true since  $\langle, \rangle$  is invariant:

$$\langle gv, w \rangle = \langle gvgg^{-1}w \rangle = \langle v, g^{-1}w \rangle.$$

Hence  $V$  and  $V^*$  are isomorphic as representations and then  $\text{Hom}_G(V, V^*)$  (space of bilinear forms) is one-dimensional. ♡

**Exercise 5.** Let  $(V, \pi)$  be an irreducible representation and  $(W, \theta)$  a finite-dimensional representation. Let  $W(\pi)$  be the isotypic component corresponding to  $\pi$ .

1. Show that the operator

$$P = \frac{\dim \pi}{|G|} \sum_{g \in G} \chi_\pi(g^{-1}) \theta(g)$$

on  $W$  is the projection onto  $W(\pi)$  along  $\sum_{\pi' \neq \pi} W(\pi')$ . In particular:

$$W = \bigoplus_{[\pi] \in \hat{G}} W(\pi).$$

2. Show that  $\theta|_{W(\pi)} \sim \pi^{n_\pi}$  where  $n_\pi = \dim \text{Mor}(\pi, \theta)$ .

**Solution 5.** dddd



### A.3 Lecture 4

- Exercise 6.**
1.  $\pi^{cc} = \pi$ .
  2.  $(\pi \otimes \theta)^c = \pi^c \otimes \theta^c$ .
  3.  $\pi$  is irreducible if and only if  $\pi^c$  is.



### A.4 Exercises 5

**Exercise 7** (Exercise 1). Show that the ring  $\mathbb{Z}$  is integrally closed. ♠

**Solution 6.** Let  $x = p/q \in \mathbb{Q}$  and suppose that we have an equation of integral dependence:

$$x^n + b_1 x^{n-1} + \dots + b_n = 0$$

with  $b_i \in \mathbb{Z}$ . We can suppose that  $p$  and  $q$  have no common factor. Multiply by  $q^n$  on both sides to get

$$p^n + b_1 q p^{n-1} + \dots + b_n q^n = 0.$$

Looking modulo  $q$  we see that  $q \mid p^n$ , which to avoid contradiction, must imply that  $q = \pm 1$ , hence  $x \in \mathbb{Z}$ . ♡

**Exercise 8** (Exercise 2). Suppose  $R$  is a unital commutative ring and  $S \subset R$  a subring with  $1 \in S$ . Show that the set of elements of  $R$  integral over  $S$  form a subring of  $R$ . ♠

**Solution 7.** Note that  $x$  is integral over  $S$  if and only if  $S[x]$  is a finitely generated  $S$ -module. Hence if  $x$  and  $y$  are integral over  $S$ , it follows that  $S[x] \otimes_S S[y] = S[x, y]$  is a finitely generated  $S$ -module. Hence  $x \pm y$  and  $xy$  are integral over  $S$ . ♡

### A.5 Exercises 11

### A.6 Lecture 13

**Exercise 9.** Let  $G = \mathrm{GL}_n(\mathbb{C})$ . Show that  $\exp : \mathfrak{g} \rightarrow G$  is surjective. Show also that it is not open. ♠

**Solution 8.** Note that for matrix groups, the exponential is given by

$$\exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \dots + \frac{1}{n!}A^n + \dots$$



Also note that

$$\exp(P^{-1}AP) = P^{-1}\exp(A)P$$

for invertible matrices  $P \in \mathrm{GL}_n(\mathbb{C}) \subset \mathfrak{g} = \mathrm{End}(\mathbb{C}^n)$ . Thus, if  $A$  is diagonalizable with eigenvalues  $\lambda_i$ , the matrix  $P^{-1}\mathrm{diag}(\log \lambda_1, \dots, \log \lambda_n)P$  (with  $P$  diagonalizing  $A$ ) is mapped to  $A$ . Here  $\log$  is of course the complex logarithm (which is multi-valued, so choose one value for each).

Now if  $A$  is not diagonalizable, the matrix  $A + \epsilon I$  is invertible and diagonalizable for generic  $\epsilon$ . Thus we can find a sequence  $A_n$  with all  $A_n$  diagonalizable and  $A_n \rightarrow A$  with corresponding  $B_n$  with  $\exp(B_n) = A_n$ . I claim that the  $B_n$  can be chosen such that  $\{B_n\}$  is a convergent sequence. This is because the eigenvalues of  $A_n$  are continuous with respect to  $\epsilon$ , and as  $\log : \mathbb{C}^* \rightarrow \mathbb{C}$  is continuous, we can choose the logs of close eigenvalues to be close. Thus we get a convergent sequence  $B_n \rightarrow B$ , and by continuity of  $\exp$ ,  $\exp(B) = A$ .

Now we explain why  $\exp$  is not open. [[[HOW???\]]] ♡

## A.7 Exercises 13

**Exercise 10** (Exercise 1). Let  $G$  be a group that is also a compact space (that is, a compact Hausdorff topological space) such that the multiplication map  $\mu : G \times G \rightarrow G$  is continuous. Show that  $G$  is a compact group, that is, that the inverse map  $\iota : G \rightarrow G$  is continuous as well. ♠

**Solution 9.** Let  $U \subset G$ . Then

$$\begin{aligned}\iota^{-1}(U) &= \{g \in G \mid \iota(g) \in U\} \\ &= \{g \in G \mid g^{-1} \in U\} \\ &= \{g \in G \mid e \in gU\}\end{aligned}$$

[[[[[[[ something about bijection between compact spaces is a homeomorphism ]]]]]]] ♡

**Exercise 11.** 1. Find a Haar measure on  $\mathrm{GL}_2(\mathbb{R})$  and check that this group is unimodular<sup>9</sup>.

2. Find left- and right-invariant Haar measures on the  $ax + b$  group over  $\mathbb{R}$ , and check that this group is not unimodular. ♠

**Solution 10.** For the first part, we start with the usual measure on  $\mathbb{R}^4$ , and see what happens when we translate by an element of  $\mathrm{GL}_2(\mathbb{R})$ .

We assume the Haar measure has the following form, for some yet unknown function  $f$ .

$$\mu(S) = \int_S f(A) |da_{11} da_{12} da_{21} da_{22}|.$$

<sup>9</sup>Recall that this means that the left and right invariant Haar measures coincide

Now consider the same formula on the set  $gS$ . Then the change of variable formula tells us that

$$\mu(gS) = \int_S f(gS) |\det D(g)| |da_{11} da_{12} da_{21} da_{22}|.$$

where  $\det D(g)$  is the Jacobian matrix of the map  $A \mapsto gA$ . A computation shows that this map is row-equivalent to a block diagonal matrix with two copies of  $g$ . Thus  $\det D(g) = (\det g)^2$ . Hence a left-invariant Haar measure on  $\mathrm{GL}_2(\mathbb{R})$  is given by

$$\mu(S) = \int_S \frac{1}{|\det A|^2} dA.$$

The same computation with  $g \mapsto Ag$  gives the same result. Hence  $\mathrm{GL}_2(\mathbb{R})$  is unimodular.

ii). For the second part, note that the  $ax+b$ -group is topologically  $\mathbb{R} \times \mathbb{R}^*$ , so that we can use the measure  $dx dy$  here. Then a similar computation gives that a left-invariant Haar measure is given by

$$\mu(S) = \int_S \frac{1}{a^2} da db$$

and a right-invariant Haar measure is given by

$$\mu(S) = \int_S \frac{1}{a} da db.$$

♡

## A.8 Exercises 16

**Exercise 12** (Exc 2). Let  $G$  be a topological group and  $\Gamma \subset G$  a discrete normal subgroup. Show that  $\Gamma \subset Z(G)$ . ♠

**Solution 11.** Since  $\Gamma \subset G$  is normal, we have that  $h^{-1}gh \in \Gamma$  for all  $g \in G$ . So we get a continuous map  $c : G \rightarrow \Gamma$  given by conjugation. But a continuous map from a connected topological space to a discrete space must be constant (let  $\{g_1, g_2\} \subset c(G)$ , which is open (in the discrete topology), hence  $c^{-1}(\{g_1, g_2\})$  is open, which is only possible if the inverse image is  $\emptyset$  or all of  $G$ ).

Since  $c$  is constant it must be constantly equal to  $e$ , since  $c(e) = e$ . ♡

## A.9 Exercises 18

**Exercise 13** (Exc 2). Let  $\pi : G \rightarrow \mathrm{GL}(V)$  and  $\eta : G \rightarrow \mathrm{GL}(W)$  be two finite-dimensional representations of a Lie group  $G$ . Show that

$$(\pi \otimes \eta)_*(X) = \pi_*(X) \otimes 1 + 1 \otimes \eta_*(X)$$

for all  $X \in \mathfrak{g}$ . ♠

**Solution 12.** We have that  $\text{GL}(V \otimes W)$  is an open subset of  $\text{End}(V \otimes W)$ , so that we may use the additive structure when computing the derivative.

Thus, let  $\gamma : \mathbb{R} \rightarrow \text{GL}(V \otimes W)$  be a continuous curve with  $\gamma(0) = e$  and  $\gamma'(0) = X$ . Then

$$\begin{aligned}\pi_*(\pi \otimes \eta)(X) &= \frac{\partial}{\partial t}(\pi(\gamma(t)) \otimes \eta(\gamma(t))) \big|_{t=0} \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (\pi(\gamma(t)) \otimes \eta(\gamma(t)) - \pi(\gamma(0)) \otimes \eta(\gamma(0))) \\ &= 1 \otimes X + X \otimes 1\end{aligned}$$

by the usual proof of the product rule. ♥

## A.10 Exercises 19

**Exercise 14** (Exercise 3). Let  $G$  be a closed subgroup of  $U(n)$ . Verify that the formula

$$\langle X, Y \rangle = -\text{Tr}(XY)$$

defines an Ad-invariant scalar product on  $\mathfrak{g} \subset \mathfrak{u}(n)$ . ♠

**Solution 13.** We first verify that the formula defines a scalar product. It is clearly symmetric and bilinear. We only need to check that it is positive definite. Note that every  $X \in \mathfrak{g}$  satisfies  $X = -X^*$ . Hence  $\langle X, X \rangle = -\text{Tr}(XX) = -\text{Tr}(-X^*X) = \text{Tr}(XX^*)$ . An easy computation shows that the  $i$ th diagonal entry of  $XX^*$  is the sum of the moduli of the elements in the  $i$ th row of  $X$ . Hence  $\text{tr}(XX^*)$  is the sum of the lengths of all the entries in  $X$  (with certain repetitions). But this is zero if and only if  $X = 0$ . Hence  $\langle X, Y \rangle$  is positive definite.

Recall the adjoint action of  $G$  on  $\mathfrak{g}$ , given by  $g \cdot X = gXg^{-1}$ . Being Ad-invariant means that  $\langle g \cdot X, g \cdot Y \rangle = \langle X, Y \rangle$  for all  $X, Y \in \mathfrak{g}$ . But the action on the Lie algebra is given by conjugating, and we have

$$\langle g \cdot X, g \cdot Y \rangle = \langle gXg^{-1}, gYg^{-1} \rangle = -\text{Tr}(gXg^{-1}gYg^{-1}) = -\text{Tr}(gXYg^{-1}) = -\text{tr}(XY).$$

And we are done. ♥

## B Some worked examples

### B.1 The representation ring of $S_3$

We want to explicitly compute the representation ring of  $S_3$ .

First we find all irreducible representations. The first one is the trivial representation  $\epsilon : S_3 \rightarrow \mathbb{C}$ . We also have the *sign* representation given by  $g \mapsto \text{sgn}(g) \in \mathbb{C}$ . Both of these are one-dimensional representations.

$S_3$  have a natural action on  $\mathbb{R}^3$  given by permuting the basis vectors. But it acts trivially on the subspace spanned by  $e_1 + e_2 + e_3$ , hence  $\mathbb{R}^3$  decomposes as  $\epsilon \oplus V$ , where  $V$  is some 2-dimensional representation. It is irreducible: if not, any  $v \in V$  would be sent to a scalar multiple of itself, but this is not the case. This representation is called the **standard representation** of  $S_3$ . We have also now found all representations of  $S_3$ , since  $1^2 + 1^2 + 2^2 = 6$ .

Thus the representation ring  $R(S_3)$  is  $\mathbb{Z}[A, S]$  modulo some relations to be found. Here  $A$  is the alternating representation and  $S$  is the standard representation. The trivial representation is 1. It is easy to see that  $A \otimes A \sim \epsilon$ , so  $A^2 = 1 \in R(S_3)$ . The representation  $S \otimes S$  is 4-dimensional. To compute how it decomposes, we use character theory.

Recall that characters are class functions on  $G$ , that is, they only depend on the conjugacy classes of  $G$ . So we write a character table:

|            | $e$ | $\tau$ | $\sigma$ |
|------------|-----|--------|----------|
| $\epsilon$ | 1   | 1      | 1        |
| $A$        | 1   | 1      | -1       |
| $S$        | 2   | -1     | 0        |

To compute the character of the standard representation, we first note that it is, as a vector space given by  $\mathbb{R}^3/(1,1,1)$ . A basis is then given by the images of  $e_1, e_2$ . Let  $\tau$  be the transposition (123), sending  $e_i$  to  $e_{i+1}$ . Let  $\sigma$  be reflection fixing  $e_1$  and exchanging  $e_2$  and  $e_3$ . In this basis that means  $e_2$  is sent to  $-e_1 - e_2$ . Writing up the corresponding matrices lets us find the value of the character.

Now one can compute by hand (or use a result on characters), that the character of  $S \otimes S$  is given by  $\chi_S \cdot \chi_S$ , so that its entry in the character table is (4, 1, 0). If  $S \otimes S = V_1 \oplus V_2$ , then  $\chi_S = \chi_{V_1} + \chi_{V_2}$ . Using also that the characters are linearly independent, we see that the only option is  $S \otimes S \sim \epsilon \oplus A \oplus S$ .

Hence  $S^2 = 1 + A + S$  in the representation ring. Similarly, we find that  $AS = S$  in the representation ring. All in all

$$R(S_3) = \mathbb{Z}[A, S]/(A^2 - 1, S^2 - 1 - A - S, AS - S).$$

## B.2 Explicit Specht modules

Again, we work with  $G = S_3$ . Consider the following Young-diagram:

$$T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$$

We want to use the theorem from Lecture 5 (?) to find the standard representation of  $S_3$ . The elements of  $S_3$  are generated by  $\rho = (123)$  and  $s = (23)$  with the relations  $\rho^3 = e$  and  $s\rho s = \rho^2$ .

First off, the elements permuting the rows is the subgroup consisting of  $e$  and  $(12) = s\rho^2$ . The elements permuting the columns is the subgroup generated by  $(13) = s\rho$ . Then

$$a_T = \frac{1}{2}(e + (12))$$

and

$$b_T = \frac{1}{2}(e + (13)).$$

Thus

$$c_T = \frac{1}{4}(e - (13) + (12) - (132)).$$

This gives us as in the lecture a map

$$\mathbb{C}[S_3] \rightarrow V \subseteq \mathbb{C}[S_3]$$

given by  $g \mapsto gc_T$ , whose image is supposed to be an irreducible representation of  $S_3$ . Let's find this. One computes the action of  $c_T$  on the basis elements of  $\mathbb{C}[S_3]$  by direct computation:

$$\begin{aligned} c_T &= \frac{1}{4}(e - s\rho + s\rho^2 - \rho^2) \\ \rho c_T &= \frac{1}{4}(\rho - s + s\rho - e) \\ \rho^2 c_T &= \frac{1}{4}(\rho^2 - s\rho^2 + s - \rho) \\ s c_T &= \frac{1}{4}(s - \rho + \rho^2 - s\rho^2) \\ s\rho c_T &= \frac{1}{4}(s\rho - e + \rho - s) \\ s\rho^2 c_T &= \frac{1}{4}(s\rho^2 - \rho^2 + e - s\rho) \end{aligned}$$

We claim that  $c_T$  and  $\rho c_T$  span the image. For we have  $\rho^2 c_T = -c_T - \rho c_T$ . And  $s c_T = \rho^2 c_T$ . And  $s\rho c_T = \rho c_T$ . And  $s\rho^2 c_T = c_T$ .

Thus we have some 2-dimensional representation of  $S_3$  with basis  $c_T, \rho c_T$ . We compute the matrices of  $S_3$  with respect to this basis:

$$\begin{aligned} e &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \rho &= \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} & \rho^2 &= \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \\ s &= \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} & s\rho &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & s\rho^2 &= \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

As expected, the character of this representation is exactly the character of the standard representation from the previous example (as is seen by computing traces). This is no coincidence.

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