Smoothing a Calabi-Yau manifold

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Chapter 1

Preliminary definitions

We work over $\mathbb{C}$, but some theorems may be stated over a field k.

1.1 Stanley-Reisner basics

Given a simplicial complex $\mathcal{K}$, one can associate to it a projective scheme $\mathbb{P}(\mathcal{K})$ defined as follows. Let P be the polynomial ring with one variable for each vertex of $\mathcal{K}$. Then the *Stanley-Reisner ideal* $I_{\mathcal{K}}$ corresponding to $\mathcal{K}$ is generated by the monomials corresponding to *non-faces* of $\mathcal{K}$. Then we define the *Stanley-Reisner scheme* to be $\operatorname{Proj} P/I_{\mathcal{K}}$.

Example 1.1.1. Let $\mathcal{K}$ be the square, with vertices v_0, v_1, v_2, v_3. Then the Stanley-Reisner ideal is generated by v_0v_2 and v_1v_3. ★

Some of the topology of the simplicial complex is encoded in the scheme structure of $\mathbb{P}(\mathcal{K})$. In particular, the simplicial (co)homology groups of $\mathcal{K}$ can be computed as the sheaf cohomology of $\mathbb{P}(\mathcal{K})$

Lemma 1.1.2. *Let $(K;\mathbb{C})$ denote the singular cohomology groups of $\mathcal{K}$. Then there are isomorphisms $H^i(K;\mathbb{C}) = H^i(\mathbb{P}(\mathcal{K}), \mathscr{O}_{\mathbb{P}(\mathcal{K})})$ for all i.*

Proof. ———to come——— □

Corollary 1.1.3. *We have isomorphisms $H^i(K, \mathbb{C}) \simeq H^{2i}(\mathbb{P}(\mathcal{K}); \mathbb{C})$ of singular cohomology groups.*

Proof. Something about i-cells in even dimensions □

1.2 Calabi-Yau basics

Definition 1.2.1. A *Calabi-Yau variety* is a smooth projective variety satisfying the following two conditions:

1. $H^i(X, \mathscr{O}_X) = 0$ for $0 < i < \dim X$.
2. The canonical sheaf is trivial: $\omega_X \simeq \mathscr{O}_X$.

■

The classical example of a Calabi-Yau manifold is the quintic threefold in $\mathbb{P}^5$. Another example is the following:

Example 1.2.2. Let X be the double cover of $\mathbb{P}^3$ ramified along a smooth octic. The projection map is affine, so the conditions on $H^i(X, \mathscr{O}_X)$ are fulfilled. To see that the canonical sheaf is trivial, we use the adjunction formula, which says that $K_X = 2K_{\mathbb{P}^3}\big|_X + \deg R$, where R is the ramification divisor. Then, since $\omega_{\mathbb{P}^3} = \mathscr{O}_{\mathbb{P}^3}(-4)$, it follows that $K_X = 0$. ★

If $\mathcal{K}$ is a simplicial sphere, then a smoothing of $\mathbb{P}(\mathcal{K})$ will give a Calabi-Yau manifold.

—- ref: bayer-eisenbud graph curves.

The most basic invariants of Calabi-Yau manifolds are their *Hodge numbers* h^{pq}. In algebraic geometry these can be defined as the dimensions of the cohomology groups $H^q(X, \Omega^p_X)$. This definition is however not so transparent. On a complex manifold, it is true that $h^{pq} = h^{qp}$, but this is not obvious from our definition. Instead, let us define these groups in complex algebraic geometry terms.

The de Rham complex $(\Omega^\bullet, d)$ refines to a bigraded complex $(\Omega^{\bullet,\bullet}, d)$, where a differential form of bidegree (p, q) can be written as

$$\omega = \sum f_{IJ} dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z_1} \ldots \wedge d\overline{z_q}.$$

The differential d splits as $\partial + \overline{\partial}$, where $\partial : \Omega^{\bullet,\bullet} \to \Omega^{\bullet+1,\bullet}$, and $\overline{\partial} : \Omega^{\bullet,\bullet} \to \Omega^{\bullet,\bullet+1}$. The decomposition passes respects cohomology, so we can form the *Dolbeault cohomology groups* $H^{p,q}(X)$.

With this definition, applying complex conjugation shows that $H^{p,q} = \overline{H^{q,p}}$.

Lemma 1.2.3. *We have natural isomorphisms $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$.*

Proof. Use that the de Rham complex is flabby $\square$

For more details on this and other details from complex geometry, see [3].

The “Hodge diamond” is ...

Example 1.2.4. Let X be a smooth quintic in $\mathbb{P}^4$. We will compute its Hodge numbers. Let us first compute $H^{1,1}(X)$. We have the following exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{P}^4}\big|_X \to \Omega_X^1 \to 0$$

Since $\mathcal{I}/\mathcal{I}^2 \simeq \mathscr{O}_X(-5)$, it follows from the long exact sequence of cohomology that $H^1(X, \Omega_X^1) \simeq H^1(X, \Omega_{PP^4}^1\big|_X)$. .... ★

1.3 Deformation theory

Deformation theory is the study how varieties (or other algebraic structures like line bundles, vector bundles, ...) vary in families.

There is a lot of technical machinery available for the deformation theorist, but for us just a few vector spaces will be of importance.

Definition 1.3.1. Let X be a scheme over k. Then a *deformation of X over S* is a flat morphism $\mathfrak{X} \to S$ together with an isomorphism $X \simeq \mathfrak{X} \times_S 0$ for a closed point $0 \in S$:

$$\begin{tikzcd} X \simeq X_0 \arrow[r, hook] \arrow[d] & \mathfrak{X} \arrow[d] \\ 0 \arrow[r] & S \end{tikzcd}$$

■

Recall that a morphism $f : X \to Y$ is *flat* if the associated morphism $f^\$: \mathscr{O}_Y \to f_* \mathscr{O}_X$ of $\mathscr{O}_Y$-modules is a flat morphism.

Chapter 2

Two topologically distinct smoothings

Denote by dP_6 the del Pezzo surface of degree 6 embedded in $\mathbb{P}^6$. This can be realized as the blow-up of $\mathbb{P}^2$ in three points not lying on a line. Let X denote the affine cone over dP_6. Then it has long been known that X has two smoothing components, and we show here that they are topologically distinct.

Recall that a *del Pezzo* surface is a surface such that the anti-canonical bundle is ample. The degree is the degree given by the anticanonical embedding. It is a classical result that every del Pezzo surface is obtained either by blowing up $\mathbb{P}^2$ in $r = 0, \ldots, 6$ points in suitable positions, or as the 2-uple embedding of a quadric surface in $\mathbb{P}^3$.

2.1 Different embeddings of dP_6

We first obtain the equations of dP_6 directly from the description of it as blow-up. Let x_0, x_1, x_2 be coordinates of $\mathbb{P}^2$. Recall that the blowup of $\mathbb{P}^2$ in the point $(1 : 0 : 0)$ can be realized as the closed subscheme of $\mathbb{P}^2 \times \mathbb{P}^1$ given by the equation $r_0 x_1 - r_1 x_2 = 0$, where r_0, r_1 are coordinates on $\mathbb{P}^1$. We can repeat this process on the points $(0 : 1 : 0)$ and $(0 : 0 : 1)$ to obtain similar equations. Collecting these, we see that dP_6 is given by the matrix equation

$$M\vec{x} = \begin{pmatrix} 0 & r_0 & -r_1 \\ s_1 & 0 & -s_0 \\ -t_0 & t_1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0.$$

in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Here r_i, s_i and t_i ($i = 0, 1$) are of course coordinates on $\mathbb{P}^1$.

We can do more than this however.

Lemma 2.1.1. *We can also realize dP_6 embedded in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with equation $r_0 s_0 t_0 = r_1 s_1 t_1$.*

Proof. Note that the matrix cannot have rank 1 or lower.

Now consider the projection onto the last three factors:

$$\pi : \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

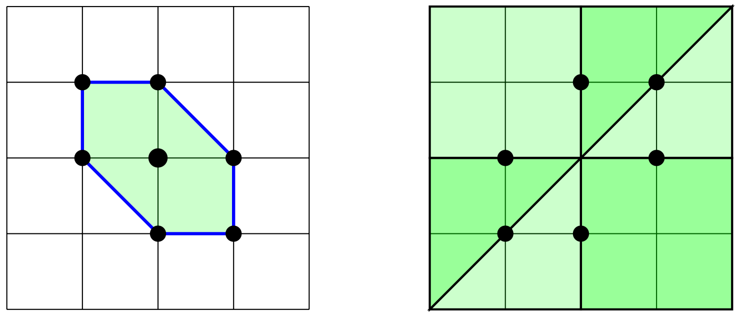
Each point P in the product on the right-hand side gives a matrix M_P of rank 2. Thus there is a line of solutions, which correspond exactly to a point in $\mathbb{P}^2$.

Hence the restriction of π to dP_6 is an isomorphism onto the hypersurface given by $\det M = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. $\square$

Another way to realize blow-ups is this: let $\mathfrak{d}$ be the linear system of quadrics with assigned basepoints $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$ in $\mathbb{P}^2$. We can choose a basis given by x_0x_1, x_0x_2 and x_1x_2. This gives a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. The closure of the graph of this map is a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$ defined by two bilinear equations. Each of the projections correspond to the blowup.

Explicitly, if we let y_0, y_1, y_2 be coordinates on the other $\mathbb{P}^2$, then the equations are $x_1y_0 - x_1y_1 = x_1y_1 - x_2y_2 = 0$.

We also have a natural embedding in $\mathbb{P}^6$ as follows. Denote by E_1, E_2, E_3 the exceptional divisors on the blowup. Let L be a line in $\mathbb{P}^2$. Then the divisor $\pi^*3L - E_1 - E_2 - E_3$ is ample, and gives an embedding in $\mathbb{P}^6$ (see [2, Chapter V, Theorem 4.6]). A basis for the corresponding linear system is given by all monomials in $\Gamma(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(3))$ except x^3, y^3 and z^3.



(a) The hexagon. (b) The fan of dP_6.

Figure 2.1: Toric description of dP_6.

The equations can be arranged in a particularly symmetric form: let $y, x_1, \ldots, x_1$ be coordinates on $\mathbb{P}^6$. Then the equations of dP_6 are the 2×2 minors of the matrix

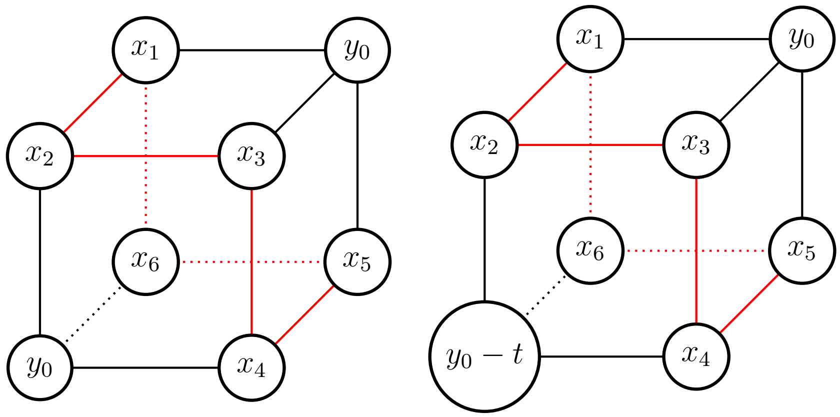
$$\begin{pmatrix} x_1 & y & x_6 \\ x_2 & x_3 & y \\ y & x_4 & x_5 \end{pmatrix}.$$

This gives 9 equations, which can be compactly written as $x_i x_{i+2} - y x_i = 0$ and $x_i x_{i+3} - y^2 = 0$, for $i = 1, \ldots, 6$ (where i is taken modulo 6). Note that the equations have a visible D_6-symmetry, where D_6 denotes the dihedral group.

2.1.1 As a toric variety

There is a nice combinatorial description of dP_6 as a toric variety associated to a polytope. Namely, let P denote the hexagon in Figure 2.1a. Then the normal fan of this polytope defines a fan in $N_{\mathbb{R}}$, defining a toric variety.

The polytope is reflexive, implying that the normal fan of P is the face fan over the same polytope. See Figure 2.1b. From standard toric geometry, it is clear that dP_6 is the blowup of $\mathbb{P}^2$ in the three torus-fixed points.



(a) Equations dP_6.

(b) Deforming $C(dP_6)$.

Figure 2.2: Forms of equations.

2.2 Divisors and topology

2.3 The affine cone and its two smoothings

Let X denote the affine cone over dP_6. It is an affine variety with an isolated singularity at the origin. One can compute that it has two smoothing components: the union of a plane and a line. They both come from different ways of perturbing the equations of dP_6.

Look at Figure 2.2a. One can read off the equations of dP_6 by taking minors along "faces" and long diagonals of this square. This correspond to a hyperplane cut of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in the Segre embedding. Then the one-dimensional component of the versal deformation of X is obtained by perturbing one of the y_0-corners as in Figure 2.2b.

It is clear the corresponding deformation is smooth, since it is a hyperplane cut of cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ outside the origin. Call this smoothing X_1.

Lemma 2.3.1. *The smoothing X_1 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \backslash dP_6$.*

Proof. Specialize to some $t \neq 0$. Then we can homogenize the equations with respect to y_1 to obtain a projective variety in 8 variables. However, in this form, $y_0 - ty_1$ and y_0 are linearly independent, hence by a change of variables, we see that this variety is in fact isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding.

What we gained by homogenizing is exactly the projective variety given by setting $y_1 = 0$. But then we get back the equations of dP_6 in $\mathbb{P}^6$. $\square$

The second smoothing is obtained by deforming the equations of dP_6 as a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$. Namely, consider the following matrix:

$$\begin{vmatrix} x_1 & y_0 & x_6 \\ x_2 & x_3 & y_0 - t_1 \\ y_0 - t_2 & x_4 & x_5 \end{vmatrix} \leq 1. \tag{2.1}$$

For $t_1 = t_2 = 0$, we get the cone over dP_6, while for generic t_i, we get a smooth variety. In fact, we can compute that the discrimant locus (the set of points in $\mathbb{A}^2_{t_1,t_2}$ with singular fiber) are the t_1-axis, the t_2-axis and the line $t_1 = t_2$.

Call (any) smooth fiber X_2.

Lemma 2.3.2. *The variety X_2 is isomorphic to $(\mathbb{P}^2 \times \mathbb{P}^2) \cap H \backslash dP_6$, where H is a hyperplane in $\mathbb{P}^8$.*

Proof. The technique is the same as in the previous proof. First homogenize the equations (2.1) with respect to y_1. If we let t_{ij} $(i, j = 0, 1, 2)$ be coordinates of $\mathbb{P}^8$, then the homogenized variety is $\mathbb{P}^2 \times \mathbb{P}^2 \cap \{h = 0\}$, where h is the hyperplane $(t(t_{32} - t_{11}) = s(t_{23} - t_{11}$.

Again, what we gained by homogenizing is given by intersecting with $y_1 = 0$. But this is exactly dP_6 again, in the form $\mathbb{P}^2 \times \mathbb{P}^2 \cap \{t_{23} = t_{32} = 0\}$. $\square$

We can use what we know about the topology of these spaces to compute homology groups of the two affine smoothings.

Theorem 2.3.3. *The two affine smoothings are topologically different. The homology groups are:*

Group	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*Euler-characteristic*
$H^i(X_1, \mathbb{Z})$	*1*	*0*	*2*	*1*	*0*	*0*	*0*	*2*
$H^i(X_2, \mathbb{Z})$	*1*	*0*	*1*	*2*	*0*	*0*	*0*	*0*

Proof. Long exact sequence of a pair + Lefschetz duality $\square$

Remark. *In fact, the Andreotti-Frankel theorem [1] states the following: if V is any smooth affine variety of complex dimension n, then it has the homotopy type of a CW complex of dimension n.*

Chapter 3

A smooth Calabi-Yau

Consider the hexagon E_6. The join $E_6 * E_6$ is a 3-dimensional sphere, and so a smoothing of the corresponding Stanley-Reisner scheme would correspond to a smooth Calabi-Yau manifold. In this chapter I prove that there does indeed exist a smoothing, and I describe some of its properties.

Description, singularities, etc.

Bibliography

[1] Aldo Andreotti and Theodore Frankel. The Lefschetz theorem on hyperplane sections. *Ann. of Math. (2)*, 69:713–717, 1959.

[2] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[3] Claire Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps.