Results so far

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1 Introduction

In the article [Gaĭ09], it is proven that there exists a triangulation of $\mathbb{CP}^2$ with 15 vertices and other nice properties [[expand]]. The triangulation is constructed by "glueing" cones of 3-spheres along a triangle. Call this triangulation for $\mathcal{S}$.

A smoothing of the associated Stanley-Reisner scheme $\mathbb{P}(\mathcal{S})$ would have interesting properties. In particular, it would be an example of a hyper-Kähler variety. An irreducible hyper-Kähler manifold X is (for our purposes) a projective variety whose space of holomorphic two-forms is generated by an everywhere non-degenerate form. In cohomological terms, this means that $H^2(X, \mathscr{O}_X) \simeq \mathbb{C}$.

Now the cohomology of $\mathbb{CP}^2$ is given by $H^i(\mathbb{CP}^2, k) = k$ if i is even and as 0 if i is odd. Thus a smoothing of $\mathcal{K}$ will be a hyper-Kähler variety.

The Hilbert polynomial of $\mathbb{P}(\mathcal{S})$ is $\frac{9}{2}t^4 + \frac{15}{2}t^2 + 3$. The simplicial complex $\mathcal{S}$ have f-vector $(15, 90, 240, 270, 108)$. In particular, the degree of $\mathbb{P}(\mathcal{S})$ is 108.

Here are a few computations:

Lemma 1.1. *The module of first-order deformations of* $\mathbb{P}(\mathcal{S})$ *is* 90*-dimensional, i.e.* $\dim_k T^1(\mathbb{P}(\mathcal{S})) = 90$.

Lemma 1.2. *The obstruction module have* $\dim_k T^2(\mathbb{P}(\mathcal{K})) = 306$.

The link of $\mathcal{S}$ at three of its vertices is a particularly simple 3-sphere, namely the join of the boundaries of two hexagons. Call this $\mathcal{K}$. Then $\mathcal{K}$ have f-vector $(1, 12, 48, 72, 36)$. In particular the Stanley-Reisner scheme $\mathbb{P}(\mathcal{K})$ have degree 36.

Consider now $\mathcal{K} * \Delta^1$. This a cone over a ball, so it is topologically a 5-dimensional ball. In fact, it is the join of two hexagons. So it is a 5-dimensional ball. Consider now the polytope $P * P$ that is the convex hull of the columns of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

This polytope is the join of two *actual* hexagons in $\mathbb{R}^5$. Thus, by standard Sturmfels theory, there is a Gröebner degeneration of the associated toric variety (whose fan is the polar polytope of P).

2 Deformations of the 5-dimensional toric variety

The 5-dimensional toric variety T associated to P whose normal fan is spanned by P have 2-dimensional singularities, which are easily seen to be two disjoint copies of dP_6, the toric del Pezzo surface of degree 6.

Theorem 2.1. *There exists a flat deformation of $\mathbb{P}(\mathcal{K} * \Delta^1)$, $\mathfrak{X} \to S$, such that $\mathfrak{X}_{t_1} = T$ for some $t_1 \in S$ and such that the general fiber Y have one-dimensional singularities.*

Proof. As per now, the proof is purely by computer. The technique is this: First, consider the monomial degeneration of T to the Stanley-Reisner scheme $\mathbb{P}(\mathcal{K} * \Delta^1)$ (recall that $\mathcal{K} = D_6 * D_6$). Choose deformation parameters t_i perturbing the equations "in the direction of T", meaning that we only choose parameters introducing terms already occuring in the equations of T.

Now, using the package `VersalDeformations`, it is possible to produce a flat family $\mathfrak{X} \to S'$ with $\mathfrak{X}_0 = \mathbb{P}(\mathcal{K} * \Delta^1)$.

The base space is a union of toric varieties of dimensions $14, 13, 13, 12$, respectively. Call the largest one for S. The equations for S are

$$\begin{vmatrix} t_1 & t_2 & t_2 & \cdots & t_6 \\ t_7 & t_8 & t_9 & \cdots & t_{12} \end{vmatrix} \leq 1 \qquad \begin{vmatrix} t_{13} & t_{14} & t_{15} & \cdots & t_{18} \\ t_{19} & t_{20} & t_{21} & \cdots & t_{24} \end{vmatrix} \leq 1$$

By restriction, we get the claimed family $\mathfrak{X} \to T$. It is checked by setting $t_i = i$ for $i = 1, \ldots, 12$ and $t_i = 2i$ for $i = 13, \ldots, 24$ that the resulting fiber have one-dimensional singularities. The reason we don't set all $t_i = 1$, is that this point lies in the intersection of the components of S.

To check for singularities in `Macaulay2`, one has to reduce the number of equations. One way to do this is to find the singularities locally, setting each variable equal to 1. In 12 out of 14 cases, this allows us to reduce the number of variables enough so that the singularities in each chart can be computed. There were two charts where this method didn't work, but since the complement of all the other charts constitute a $\mathbb{P}^1$, the singular locus there can at most be isolated. □

Corollary 2.2. *The Stanley-Reisner scheme $\mathbb{P}(\mathcal{K})$ smooths to a smooth Calabi-Yau variety X.*

Proof. The scheme $\mathbb{P}(\mathcal{K})$ sits as a complete intersection in $\mathbb{P}(\mathcal{K} * \Delta^1)$. Complete intersections deform together with the ambient variety, so $\mathbb{P}(\mathcal{K} * \Delta^1)$ deforms to a general complete intersection in $\widetilde{T}$. Since $\widetilde{T}$ have curve singularities, it follows by two applications of Bertini's theorem [Har77, Theorem II.8.18], that X is smooth. □

Now what we would really like to do is to compute the Hodge numbers $h^{ij} = \dim_k H^j(X, \Omega^i_X)$ of X.

We can however compute the Hodge numbers of Y. The hope is that there is some sort of Lefschetz theorem giving us the Hodge numbers of X.

Theorem 2.3. *We have $h^{11}(Y) = 1$ and $h^{12}(Y) = 12$ and $h^{13} = 2$. The other Hodge groups $H^i(\Omega^1_Y) = 0$ ($i = 0, 4, 5$). In particular $\chi(\Omega^1_Y) = 9$. 0* $\to \mathscr{T}_1^Y \hookrightarrow \mathcal{I}/\mathcal{I}^2 \xrightarrow{d} \Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_Y \to \Omega^1_Y \to 0$.Here $\mathscr{T}_1^Y$ is by definition the kernel of the map d. This sequence can be broken into two short exact sequences. The relevant one is this:

$$0 \to \operatorname{im} d \to \Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}} \to \Omega^1_{\widetilde{T}} \to 0. \tag{1}$$

We also have the restriced Euler sequence:

$$0 \to \Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}} \to \mathscr{O}_{\widetilde{T}}(-1)^{14} \to \mathscr{O}_{\widetilde{T}} \to 0. \tag{2}$$

We first compute h^{11}. From (1) we get a long exact sequence

$$\ldots \to H^1(\operatorname{im} d) \to H^1(\Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}}) \to H^1(\Omega^1_{\widetilde{T}}) \to H^2(\operatorname{im} d) \to \ldots$$

The cohomology of $H^1(\operatorname{im} d)$ and $H^2(\operatorname{im} d)$ was computed with `Macaulay2` to be both zero. Thus $H^1(\Omega^1_{\widetilde{T}}) \simeq H^1(\Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}})$. From the Euler sequence we get

$$\ldots \to H^0(\mathscr{O}_{\widetilde{T}}(-1)^{14}) \to H^0(\mathscr{O}_{\widetilde{T}}) \to H^1(\Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}}) \to H^1(\mathscr{O}_{\widetilde{T}}(-1)^{14}) \to \ldots$$

But the left and right terms are both zero. Hence $h^{11} = 1$. We now compute h^{12}.

From (1) we again get

$$0 \to H^2(\Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}}) \to H^2(\Omega^1_{\widetilde{T}}) \to H^3(\operatorname{im} d) \to H^3(\Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_{\widetilde{T}}) \to \ldots,$$

where we have used that $H^2(\operatorname{im} d) = 0$. But from the Euler sequence we get that the right term is also zero. Thus $H^2(\Omega^1_{\widetilde{T}}) \simeq H^3(\operatorname{im} d)$. This last group can be computed in `Macaulay2` to be 12-dimensional.

By `Macaulay2` computations we find that

$$h^i(\widetilde{\operatorname{im} d}) = \begin{cases} 12 & i = 3 \\ 2 & i = 4 \\ 0 & \text{else,} \end{cases}$$

and

$$h^i(\widetilde{\operatorname{im} d}(-1)) = \begin{cases} 0 & i = 0,1,2 \\ 24 & i = 3 \\ 12 & i = 4 \\ 18 & i = 5, \end{cases}$$

and

$$h^i(\widetilde{\operatorname{im} d}(-2)) = \begin{cases} 0 & i = 0,1,2 \\ 36 & i = 3 \\ 24 & i = 4 \\ 218 & i = 5, \end{cases}$$

By twisting all the exaxt sequences above, we can also calculate:

$$h^i(\Omega^1_Y(-1)) = \begin{cases} 0 & i = 0 \\ 0 & i = 1 \\ 24 & i = 2 \\ 12 & i = 3, \end{cases}$$

and also $h^4(\Omega^1_Y(-1)) - h^5(\Omega^1_Y(-1)) = 4$.

Similarly:

$$h^i(\Omega^1_Y(-2)) = \begin{cases} 0 & i=0 \\ 0 & i=1 \\ 36 & i=2 \\ 24 & i=3, \end{cases}$$

and $h^4(\Omega^1_Y(-2)) - h^5(\Omega^1_Y(-2)) = 23$.

Remark. *The reader may wonder we just didn't ask* `Macaulay2` *to compute the cohomology sheaf* Ω^1_Y *directly, e.g. by the command* `HH^i(cotangentSheaf Y)`. *The reason is that* `Macaulay2`*'s algorithims actually compute the* sheaf, *and not just the dimension, and this is too computationally intensive.*

Remark (Question)**.** *A computation reveals that* $H^i(\mathcal{I}/\mathcal{I}^2) \simeq H^i(\operatorname{im} d)$ *for* $i \geq 2$. *This could be because the singularities are of dimension* 1. *Is there a theoretical result to this effect?*

We can compute it `Macaulay2` that:

Lemma 2.4. *The first cotangent module of* Y *has* $\dim_{\mathbb{C}} T^1(Y/\mathbb{C}) = 26$. *The second cotangent module has* $\mathbb{C}$*-dimension* 24.

Remark. *All computations are done over* $\mathbb{Q}$, *but by standard base change theorems (just note that the extension* $\mathbb{Q} \hookrightarrow \mathbb{C}$ *is flat), the results are still true over* $\mathbb{C}$. *The same applies for the cohomology groups, by Proposition 9.3 in Hartshorne [Har77], Chapter III.*

We have that $h^0(\mathbb{P}(\mathcal{K} * \Delta^1), \Theta_{\mathbb{P}(\mathcal{K}*\Delta^1)}) = 14$ by Theorem 5.2 in [AC10]. Thus these numbers fit in the narrative that we should have $T^1_{X_0} = h^1(\mathcal{T}_{X_t}) + h^0(\mathcal{T}_{X_0})$. (IS THERE ANY HEURISTIC FOR THIS??)

3 Computing the Hodge numbers of X

Since X is a complete intersection of two hyperplanes in Y, we have an exact sequence

$$0 \to \mathscr{O}_Y(-2) \to \mathscr{O}_Y(-1)^2 \to \mathscr{I}_{X/Y} \to 0,$$

where $\mathscr{I}_{X/Y}$ is the ideal sheaf of X in Y. We also have the sequence

$$0 \to \mathscr{I}_{X/Y} \to \mathscr{O}_Y \to i^* \mathscr{O}_X \to 0, \tag{3}$$

where $i : X \to Y$ is the inclusion.

Theorem 3.1. *There exists a non-singular Calabi-Yau with X with $\chi(\Omega^1_X) = 36$.*

Proof. Since X is a complete intersection in Y, we have $\mathscr{I}_{X/Y}/\mathscr{I}^2_{X/Y} \simeq \mathscr{O}_X(-1)^2$ as $\mathscr{O}_X$-modules. Hence we have an exact sequence

$$0 \to \mathscr{O}_X(-1)^2 \to \Omega^1_Y \otimes \mathscr{O}_X \to \Omega^1_X \to 0.$$

The first sheaf have cohomology only in $H^3(\mathscr{O}_X(-1)) = H^0(\mathscr{O}_X(1)) = 12$, which can be computed from its Stanley-Reisner degeneration. Hence the Euler characteristics are related by $\chi(\Omega^1_X) = \chi(\Omega^1_Y\big|_X) + 24$.

Now tensor the exact sequence (3) with Ω^1_Y to get

$$0 \to \mathscr{I}_{X/Y} \otimes \Omega^1_Y \to \Omega^1_Y \to \Omega^1_Y \otimes \mathscr{O}_X \to 0.$$

Tensoring with Ω^1_Y is exact because the singularities of Y lie outside X (recall that the sheaf on the right is extended by zero outside X). Do the same with the $\mathscr{O}_Y$-resolution of $\mathscr{I}_{X/Y}$ to get

$$0 \to \Omega^1_Y(-2) \to \Omega_Y(-1)^2 \to \mathscr{I}_{X/Y} \otimes \Omega^1_Y \to 0.$$

Taking Euler characteristics, we find that $\chi(\mathscr{I}_{X/Y} \otimes \Omega_Y) = -3$. Since $\chi(\Omega^1_Y)$ was computed to be 9, it follows from the first exact sequence that $\chi(\Omega^1_X) = 36$. $\square$

Remark. *The standard toric Batyrev-Borisov construction used on the toric variety T gives a Calabi-Yau X' with $\chi(\Omega^1_{X'}) = -36$ with Hodge numbers $(44, 8)$, so we would* really *want our Calabi-Yau to have Hodge numbers $(8, 44)$. In that case, it would be an example of an* extremal transition*, in the sense of Morrison.*

To actually find the Hodge numbers, we need a few lemmas.

Lemma 3.2. *Let $\mathcal{N}_{X/\mathbb{P}^{13}}$ be the normal sheaf of X in $\mathbb{P}^{13}$. Then $h^3(\mathcal{I}_X/\mathcal{I}^2_X) = h^0(\mathcal{N}_{X/\mathbb{P}^{13}})$.*

Proof. By Serre duality $h^{3-i}(\mathcal{I}_X/\mathcal{I}^2_X) = h^i((\mathcal{I}_X/\mathcal{I}^2_X)^\vee \otimes \omega)$, where ω is the dualizing sheaf. But X is Calabi-Yau, so $\omega \simeq \mathscr{O}_X$. The dual of $\mathcal{I}_X/\mathcal{I}^2_X$ is by definition the normal bundle. $\square$

Consider the Euler sequence

$$0 \to \Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X \to \mathscr{O}_X(-1)^{14} \to \mathscr{O}_X \to 0.$$

Since X is a deformation of a Stanley-Reisner sphere, we know the cohomology of $\mathscr{O}_X$. So we can extract the cohomology of $\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X$.

Lemma 3.3. *We have*

$$H^i(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1 \\ 0 & \text{if } i = 2 \\ 167 & \text{if } i = 3. \end{cases}$$

Proof. The full long exact sequence is:

$$\begin{aligned} 0 \to H^0(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) &\to H^0(\mathscr{O}_X(-1)^{14}) \to H^0(\mathscr{O}_X) \to \\ H^1(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) \to H^1(\mathscr{O}_X(-1)^{14}) &\to H^1(\mathscr{O}_X) \to \\ H^2(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) \to H^2(\mathscr{O}_X(-1)^{14}) &\to H^2(\mathscr{O}_X) \to \\ H^3(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) \to H^3(\mathscr{O}_X(-1)^{14}) &\to H^3(\mathscr{O}_X) \to 0 \end{aligned}$$

Inserting the dimensions we know, we get:

$$\begin{aligned} 0 \to H^0(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) & \qquad 0 \to 1 \to \\ H^1(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) \to 0 & \qquad \to 0 \to \\ H^2(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) \to 0 & \qquad \to 0 \to \\ H^3(\Omega_{\mathbb{P}^{13}} \otimes \mathscr{O}_X) \to 168 & \qquad \to 1 \to 0 \end{aligned}$$

Hence we conclude. □

Since X is smooth, the conormal sequence is exact, so we have

$$0 \to \mathcal{I}_X/\mathcal{I}_X^2 \to \Omega^1_{\mathbb{P}^{13}} \otimes \mathscr{O}_X \to \Omega^1_X \to 0.$$

Lemma 3.4. *We have*

$$h^{21}(X) = h^0(\mathcal{N}_{X/\mathbb{P}^{13}}) - 167.$$

where χ denotes the Euler characteristic.

Proof. Write up the long exact sequence coming from the conormal sequence of X and use Lemma 3.2. □

Lemma 3.5. *There is an exact sequence*

$$0 \to \mathscr{O}_X(1)^2 \to \mathcal{N}_{X/\mathbb{P}^{13}} \to \mathcal{N}_{Y/\mathbb{P}^{13}}\big|_X \to 0$$

Proof. First note that there is an exact sequence of conormal sheaves:

$$0 \to \mathcal{I}_Y/\mathcal{I}_Y^2\big|_X \to \mathcal{I}_X/\mathcal{I}_X^2 \to \mathscr{O}_X(-1)^2 \to 0.$$

The last term is $\mathcal{N}_{X/Y}$, since X is a complete intersection in Y. Dualizing is exact even though Y is not smooth, because by the long exact sequence of $\mathscr{E}xt$ sheaves, we must have $\mathscr{E}xt^1(\mathscr{O}_X(-1), \mathscr{O}_X) = 0$. But this is true, since both of these are locally free. $\square$

Proposition 3.6. *We have*

$$h^{21}(X) = 39.$$

Hence $h^{11}(X) = 3$.

Proof. By lemma 3.4, we need to compute $h^0(\mathcal{N}_{X/\mathbb{P}^{13}})$. By the previous lemma, we have $h^0(\mathcal{N}_{X/\mathbb{P}^{13}}) = \mathcal{N}_{Y/\mathbb{P}^{13}}\big|_X + 24$.

We have an exact sequence

$$0 \to \mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}} \to \mathcal{N}_{Y/\mathbb{P}^{13}} \to \mathcal{N}_{Y/\mathbb{P}^{13}}\big|_X \to 0.$$

And also an exact sequence:

$$0 \to \mathcal{N}_{Y/\mathbb{P}^{13}}(-2) \to \mathcal{N}_{Y/\mathbb{P}^{13}}(-1)^{\oplus 2} \to \mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}} \to 0.$$

The cohomology of $\mathcal{N}_{Y/\mathbb{P}^{13}}(-i)$ is possible to compute in `Macaulay2`, and it follows from the long exact sequence that $h^0(\mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}}) = 36$, and $h^1(\mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}}) = 0$. Hence it follows from the same computation that $h^0(\mathcal{N}_{Y/\mathbb{P}^{13}}\big|_X) = 182$ and that $h^0(\mathcal{N}_{X/\mathbb{P}^{13}}) = 206$. We conclude that $h^{21} = 206 - 167 = 39$. $\square$

References

[AC10] Klaus Altmann and Jan Arthur Christophersen. Deforming Stanley-Reisner schemes. *Math. Ann.*, 348(3):513–537, 2010.

[Gaĭ09] A. A. Gaĭfullin. A minimal triangulation of the complex projective plane that admits a chessboard coloring of four-dimensional simplices. *Tr. Mat. Inst. Steklova*, 266(Geometriya, Topologiya i Matematicheskaya Fizika. II):33–53, 2009.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.