Summary

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January 14, 2016

1 Dimension of some cohomology groups

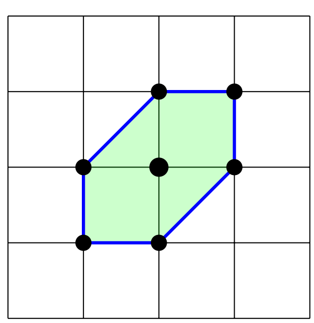
Group	0	1	2	3	4	5	Euler-characteristic
$H^i(X, \Omega_{\mathbb{P}^{12}} \otimes \mathscr{O}_X)$	0	1	0	167	0	0	-168
$H^i(Y, I_Y/I_Y^2)$	0	36	0	12	2	0	-46
$H^i(Y, \Omega_Y)$	0	1	12	2	0	0	-9

2 The singular locus of Y

By computing in each chart and taking closures, it can be computed that the singular locus of Y is of dimension 1, and consists of the union of projective lines. [[do this explcitily]]

3 Descriptions of dP_6

Recall that dP_6 is the toric variety whose associated polytope is the hexagon:



This induces an embedding into $\mathbb{P}^6$, by standard toric geometry. Let $\mathbb{P}^6$ have coordinates $y_0, x_1..x_6$ (corresponding to the center and the vertices, respectively). Then the ideal of dP_6 inside $\mathbb{P}^6$ is given by the 2×2-minors of the matrix

$$\begin{vmatrix} x_1 & y_0 & x_6 \\ x_2 & x_3 & y_0 \\ y_0 & x_4 & x_5 \end{vmatrix} \leq 1. \tag{1}$$

The $\mathbb{Z}_6$-symmetry is visible by permuting columns and rows.

Note that this representation of the ideal gives us an embedding of dP_6 into $\mathbb{P}^2 \times \mathbb{P}^2$ as a section of $\mathscr{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1,1) \oplus \mathscr{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1,1)$ (namely as the zeros of $t_{12} - t_{23} = t_{23} - t_{31}$ (where t_{ij} are the natural coordinates on the product).

There are several other ways to veiw dP_6.

3.1 As $\mathbb{P}^2$ blown up in 3 points

Consider the monoidal transformation $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ given by $(u : v : w) \mapsto (uv : uw : vw)$. This is a birational involution with three points of indeterminacy: $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$ and $P_3 = (0 : 0 : 1)$. We blow up $\mathbb{P}^2$ in these three points to get a scheme $\widetilde{X}$ and a morphism $\pi : \widetilde{X} \to \mathbb{P}^2$. Then dP_6 is $\widetilde{X}$.

Remark. *Note that the involution $\widetilde{\varphi}$ lifts to an involution $\widetilde{\varphi} : \widetilde{X} \to \widetilde{X}$. We can realize $\widetilde{X}$ as the closure of the graph of φ:*

$$\widetilde{X} = \{(u : v : w) \times (a : b : c) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid vb = wa = uc\}.$$

Then the involution is $\widetilde{\varphi}(u : v : w, a : b : c) = (a : b : c, u : v : w)$.

It can be shown that the automorphism group of dP_6 is $(\mathbb{C}^*)^2 \rtimes (S_2 \times S_3)$ ([DOLGACHEV]). The lifted involution $\widetilde{\varphi}$ generates the S_2 part. The $(\mathbb{C}^*)^2$-part is inherited from the corresponding action on $\mathbb{P}^2$ and it can be computed to be given by

$$(t_1, t_2) \cdot ((u : v : w) \times (a : b : c)) = (t_1 u, t_2 v, t_1^{-1} t_2^{-1} w) \times (t_1 t_2 a : t_2^{-1} b : t_1^{-1} c).$$

The S_3 part comes from permuting the three points P_1, P_2 and P_3. If $\sigma \in S_3$ is a permutation of the variables u, v, w, then the corresponding action on $\widetilde{X}$ is given by $\sigma(P \times Q) = \sigma P \times \sigma^{-1} Q$. For example, the cyclic part is generated by

$$(u : v : w) \times (a : b : c) \mapsto (w : u : v) \times (b : c : a).$$

3.2 A natural embedding in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Let D be the divisor $D = 2(u) + 2(v) + 2(w)$ in $Div(\mathbb{P}^2)$ and consider the linear system $|D|$. Let

$$f_1 = \frac{uv}{w^2} \qquad f_2 = \frac{uw}{v^2} \qquad f_3 = \frac{vw}{u^2}$$

be three sections. Together they define a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The base locus consist exactly of the three points P_1, P_2, P_3 above. So again we can blow up to resolve the locus of indeterminacy to get a map $\widetilde{X} \to (\mathbb{P}^1)^3$.

If t_i, s_i are coordinates on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, 2, 3$, then the equation of the image is given by $t_1 t_2 t_3 = s_1 s_2 s_3$.

4 Deformations of dP_6

Since dP_6 is smooth, the only singularity of its affine cone, $C(dP_6)$, is the origin. One can compute that $T^1(C(dP_6)) = 3$, and that the versal base space splits into two components: a line and a plane intersecting transversely.

4.1 The first smoothing of the affine cone

We attempt to give explicit descriptions of the two affine smoothing components of $C(dP_6)$.

One of the components is given by:

$$\begin{vmatrix} x_1 & y_0 & x_6 \\ x_2 & x_3 & y_0 - t_1 \\ y_0 - t_2 & x_4 & x_5 \end{vmatrix} \leq 1.$$

That is, as the 2×2-minors of the above matrix. This time we see that the affine cone $C(dP_6)$ embeds naturally in the affine cone over $C(\mathbb{P}^2 \times \mathbb{P}^2)$, again as the intersection of two hyperplanes, but with some coefficients added.

It can be computed that the locus of points in $\mathbb{A}^2$ with singular fibers have ideal generated by $st(s+t) = s^2 t + t^2 s$, namely the union of the axes and a line.

4.2 The other smoothing of the affine cone

The other smoothing is derived from another way of writing the equations of dP_6. See Figure 1. One obtains the equations for this "$2 \times 2 \times 2$-tensor" by taking 2×2-minors along the faces and along long diagonals.

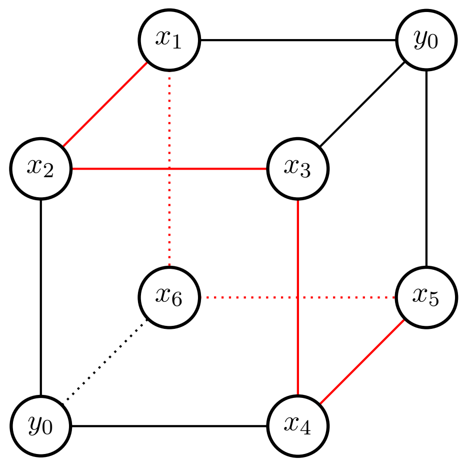


Figure 1: Equations of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

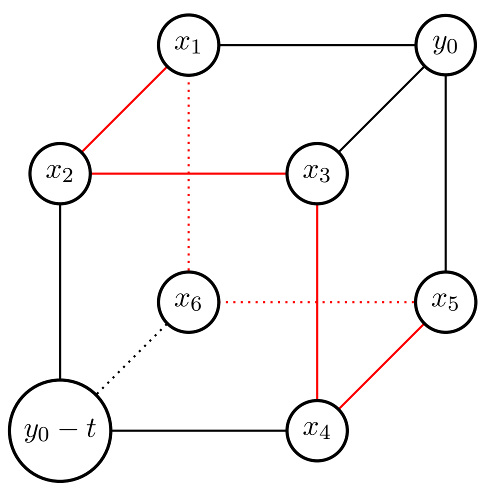


Figure 2: Deforming $C(dP_6)$.

It is clear that the one-dimensional component is a smoothing of $C(dP_6)$, since it can be obtained as a generic hyperplane in $C(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$.

5 Topology of the smootings of the affine cones

Since the singularities of our Calabi-Yau look locally like affine cones over dP_6, and the smoothings are given by locally smoothing these cones, we would like to compute their topology.

5.1 The first smoothing of $C(dP_6)$

Recall that one of the smoothing components of $C(dP_6)$ is given by the equations of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding in $\mathbb{P}^7$, replacing one of the corners by $y_0 + t$.

The total family lies in $\mathbb{A}^8$, and we can take its projective closure in $\mathbb{P}^8$ by homogenizing the equations (treating the variable s_1 as a constant of degree 0). Thus we get a family $\mathscr{X} \to \mathbb{A}^1$, where $\mathscr{X}_s$ is a projective variety for each $s \in \mathbb{A}^1$. For $s = 0$, we get the projective cone over dP_6, and for $s \neq 0$, we get something isomorphic (after a linear change of coordinates) to $(\mathbb{P}^1)^3$. By inspection, we see that what is gained in the projective closure is exactly dP_6 (for $s \neq 0$). Hence the smoothing of $C(dP_6)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \backslash dP_6$.

Let $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \backslash dP_6$ and let $D = dP_6$. There is the so-called "homology Gysin sequence", which will help us compute much of the homology of M:

$$\ldots \to H_{k+1}(M) \to H_{k_1}(D) \to H_k(M \backslash D) \to H_k(M) \to H_{k-2}(D) \to H_{k-1}(M \backslash D) \to \ldots$$

Since $\mathbb{P}^1 \simeq S^2$, we can use the Künneth formula to compute the homology of $(\mathbb{P}^1)^3$:

$$H^i(M) = \begin{cases} 1 & i = 0 \\ 0 & i = 1 \\ 3 & i = 2 \\ 0 & i = 3 \\ 3 & i = 4 \\ 0 & i = 5 \\ 1 & i = 6 \end{cases}$$

The homology of the del Pezzo is given by

$$H^i(D) = \begin{cases} 1 & i = 0 \\ 0 & i = 1 \\ 4 & i = 2 \\ 0 & i = 3 \\ 1 & i = 4 \end{cases}$$

Writing up the long exact sequence (and being happy for all the zeroes), we find almost all the homology of $M \backslash D$:

$$H^i(M \backslash D) = \begin{cases} 1 & i = 0 \\ 0 & i = 1 \\ 2 & i = 2 \\ ? & i = 3 \\ ?' & i = 4 \\ 0 & i = 5, 6. \end{cases}$$

The two questions marks are not independent, however. Since $\chi(M) = 8$[1] and $\chi(dP_6) = 6$, we know that $\chi(M \backslash D) = 2$. In fact, $?' = ? - 1$. ((there may be some codimension argument to the effect that $H_4(M) = H_4(M \backslash D)$ ...))

5.2 The second smoothing

5.2.1 Hei hei

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[1]Heuristic: Euler-characteristic is some kind of volume. And $\mathbb{P}^1$ has Euler characteristic two, and volume should be multiplicative.