

Results so far

Fredrik Meyer

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1 Introduction

In the article [Gai09], it is proven that there exists a triangulation of \mathbb{CP}^2 with 15 vertices and other nice properties [[expand]]. The triangulation is constructed by “glueing” cones of 3-spheres along a triangle. Call this triangulation for \mathcal{S} .

A smoothing of the associated Stanley-Reisner scheme $\mathbb{P}(\mathcal{S})$ would have interesting properties. In particular, it would be a hyper-Kähler variety [[EXPLAIN]]

The Hilbert polynomial of $\mathbb{P}(\mathcal{S})$ is $(9/2)t^4 + (15/2)t^2 + 3$. The simplicial complex \mathcal{S} have f -vector $(15, 90, 240, 270, 108)$. In particular, the degree of $\mathbb{P}(\mathcal{S})$ is 108.

Here are a few computations:

Lemma 1.1. *The module of first-order deformations of $\mathbb{P}(\mathcal{S})$ is 90-dimensional, i.e. $\dim_k T^1(\mathbb{P}(\mathcal{S})) = 90$.*

Lemma 1.2. *The obstruction module have $\dim_k T^2(\mathbb{P}(\mathcal{K})) = 306$.*

The link of \mathcal{S} at one of its vertices is a particularly simple 3-sphere, namely the join of the boundaries of two hexagons. Call this \mathcal{K} . Then \mathcal{K} have f -vector $(1, 12, 48, 72, 36)$. In particular the Stanley-Reisner scheme $\mathbb{P}(\mathcal{K})$ have degree 36.

Consider now $\mathcal{K} * \Delta^1$. This a cone over a ball, so it is topologically a 5-dimensional ball. In fact, it is the join of two hexagons. So it is a 5-dimensional ball. Consider now the polytope $P * P$ that is the convex hull

of the columns of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

This polytope is the join of two *actual* hexagons. Thus, by standard theory,

2 Deformations of the 5-dimensional toric variety

Recall that the 5-dimensional toric variety T had 2-dimensional singularities (actually two disjoint copies of dP_6).

Theorem 2.1. *There exists a flat deformation of $\mathbb{P}(\mathcal{K} * \Delta^1)$, $\mathfrak{X} \rightarrow S$, such that $\mathfrak{X}_{t_1} = T$ for some $t_1 \in S$ and such that the general fiber \tilde{T} have one-dimensional singularities.*

Proof. As per now, the proof is purely by computer. The technique is this: First, consider the monomial degeneration of T to the Stanley-Reisner ring $A(\mathcal{K} * \Delta^1)$ (recall that $\mathcal{K} = D_6 * D_6$). Choose deformation parameters t_i perturbing the equations “in the direction of T ”, meaning that we only choose parameters introducing terms already occurring in the equations of T .

Now, using the package **VersalDeformations**, it is possible to produce a flat family $\mathfrak{X} \rightarrow S'$ with $\mathfrak{X}_0 = \mathbb{P}(\mathcal{K} * \Delta^1)$.

The base space is a union of toric varieties of dimensions 14, 13, 13, 12, respectively. Call the largest one for S . The equations are

$$\begin{vmatrix} t_1 & t_2 & t_2 & \cdots & t_6 \\ t_7 & t_8 & t_9 & \cdots & t_{12} \end{vmatrix} \leq 1 \quad \begin{vmatrix} t_{13} & t_{14} & t_{15} & \cdots & t_{18} \\ t_{19} & t_{20} & t_{21} & \cdots & t_{24} \end{vmatrix} \leq 1$$

By restriction, we get the claimed family $\mathfrak{X} \rightarrow T$. It is checked by setting $t_i = i$ for $i = 1, \dots, 12$ and $t_i = 2i$ for $i = 13, \dots, 24$ that the resulting fiber have one-dimensional singularities. The reason we don't set all $t_i = 1$, is that this point lies in the intersection of the components of S . \square

Corollary 2.2. *The Stanley-Reisner scheme $\mathbb{P}(\mathcal{K})$ smooths to a smooth Calabi-Yau variety X .*

Proof. The scheme $\mathbb{P}(\mathcal{K})$ sits as a complete intersection in $\mathbb{P}(\mathcal{K} * \Delta^1)$. Complete intersections deform together with the ambient variety, so $\mathbb{P}(\mathcal{K} * \Delta^1)$ deforms to a general complete intersection in \tilde{T} . Since \tilde{T} have curve singularities, it follows by two applications of Bertini's theorem [Har77, Theorem II.8.18], that X is smooth. \square

Now what we would really like to do is to compute the Hodge numbers $h^{ij} = \dim_k H^j(X, \Omega_X^i)$ of X .

We can however compute the Hodge numbers of \tilde{T} . The hope is that there is some sort of Lefschetz theorem giving us the Hodge numbers of X .

Theorem 2.3. *We have $h^{11}(\tilde{T}) = 1$ and $h^{12}(\tilde{T}) = 12$.*

Proof. Again, this is purely computational. We use long exact sequences together with sheaf cohomology computations in `Macaulay2`.

Since the ideal of \tilde{T} is rather complicated, doing this naïvely does not work. The trick is to choose the right term order. Since we know that \tilde{T} has a nice degeneration, we would like to find a term order such that its initial ideal is precisely the Stanley-Reisner ideal.

The `Macaulay2` package `gfanInterface` provides an interface with `gfan`, which is a program that can compute weight vectors given polynomials with prescribed initial terms. The weight vector is

$$\omega = (1, 1, 4, 7, 7, 4, 1, 1, 4, 7, 7, 4, 1, 1).$$

With this term order, giving a very small Gröbner basis (18 elements), the computations are much faster than with the standard term order. We are able to compute resolutions of all the relevant modules within a few minutes in total.

We have an exact sequence of sheaves on \tilde{T} :

$$0 \rightarrow \mathcal{T}_1 \hookrightarrow \mathcal{I}/\mathcal{I}^2 \xrightarrow{d} \Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_{\tilde{T}} \rightarrow \Omega_{\tilde{T}}^1 \rightarrow 0.$$

This sequence can be broken into two short exact sequences. The relevant one is this:

$$0 \rightarrow \operatorname{im} d \rightarrow \Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_{\tilde{T}} \rightarrow \Omega_{\tilde{T}}^1 \rightarrow 0. \quad (1)$$

We also have the restricted Euler sequence:

$$0 \rightarrow \Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_{\tilde{T}} \rightarrow \mathcal{O}_{\tilde{T}}(-1)^{14} \rightarrow \mathcal{O}_{\tilde{T}} \rightarrow 0. \quad (2)$$

We first compute h^{11} . From (1) we get a long exact sequence

$$\dots \rightarrow H^1(\mathrm{im} d) \rightarrow H^1(\Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_{\tilde{T}}) \rightarrow H^1(\Omega_{\tilde{T}}^1) \rightarrow H^2(\mathrm{im} d) \rightarrow \dots$$

The cohomology of $H^1(\mathrm{im} d)$ and $H^2(\mathrm{im} d)$ was computed with `Macaulay2` to be both zero. Thus $H^1(\Omega_{\tilde{T}}^1) \simeq H^1(\Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_{\tilde{T}})$. From the Euler sequence we get

$$\dots \rightarrow H^0(\mathcal{O}_{\tilde{T}}(-1)^{14}) \rightarrow H^0(\mathcal{O}_{\tilde{T}}) \rightarrow H^1(\Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_{\tilde{T}}) \rightarrow H^1(\mathcal{O}_{\tilde{T}}(-1)^{14}) \rightarrow \dots$$

But the left and right terms are both zero. Hence $h^{11} = 1$. We now compute h^{12} .

From (1) we again get

$$0 \rightarrow H^2(\Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_{\tilde{T}}) \rightarrow H^2(\Omega_{\tilde{T}}^1) \rightarrow H^3(\mathrm{im} d) \rightarrow H^3(\Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_{\tilde{T}}) \rightarrow \dots,$$

where we have used that $H^2(\mathrm{im} d) = 0$. But from the Euler sequence we get that the right term is also zero. Thus $H^2(\Omega_{\tilde{T}}^1) \simeq H^3(\mathrm{im} d)$. This last group can be computed in `Macaulay2` to be 12-dimensional. \square

By `Macaulay2` computations we find that

$$h^i(\widetilde{\mathrm{im} d}) = \begin{cases} 12 & i = 3 \\ 2 & i = 4 \\ 0 & \text{else,} \end{cases}$$

and

$$h^i(\widetilde{\mathrm{im} d(-1)}) = \begin{cases} 0 & i = 0, 1, 2 \\ 24 & i = 3 \\ 12 & i = 4 \\ 18 & i = 5, \end{cases}$$

and

$$h^i(\widetilde{\mathrm{im} d(-2)}) = \begin{cases} 0 & i = 0, 1, 2 \\ 36 & i = 3 \\ 24 & i = 4 \\ 218 & i = 5, \end{cases}$$

In the same manner we find that:

Proposition 2.4. *We have $H^3(\Omega_Y^1) = 2$. The other Hodge groups $H^i(\Omega_Y^1) = 0$ (for $i = 0, 4, 5$).*

By twisting all the exact sequences above, we can also calculate:

Proposition 2.5. *We have*

$$h^i(\Omega_Y^1(-1)) = \begin{cases} 0 & i = 0 \\ 0 & i = 1 \\ 24 & i = 2 \\ 12 & i = 3, \end{cases}$$

and also $h^4(\Omega_Y^1(-1)) - h^5(\Omega_Y^1(-1)) = 4$.

Similarly:

$$h^i(\Omega_Y^1(-2)) = \begin{cases} 0 & i = 0 \\ 0 & i = 1 \\ 36 & i = 2 \\ 24 & i = 3, \end{cases}$$

and $h^4(\Omega_Y^1(-2)) - h^5(\Omega_Y^1(-2)) = 23$.

Remark. *The reader may wonder we just didn't ask `Macaulay2` to compute the cohomology sheaf $\Omega_{\tilde{T}}^1$ directly, e.g. by the command `HH^i(cotangentSheaf Proj A)`. The reason is that `Macaulay2`'s algorithms actually compute the sheaf, and not just the dimension, and this is too computationally intensive.*

Remark (Question). *A computation reveals that $H^i(\mathcal{I}/\mathcal{I}^2) \simeq H^i(\text{im } d)$ for $i \geq 2$. This could be because the singularities are of dimension 1. Is there a theoretical result to this effect?*

We can compute it `Macaulay2` that:

Lemma 2.6. *The first cotangent module of \tilde{T} has $\dim_k T^1(\tilde{T}/k) = 26$.*

We have that $h^0(\mathbb{P}(\mathcal{K} * \Delta^1), \mathcal{T}) = 14$ by Theorem 5.2 in [AC10]. Thus these numbers fit in the narrative that we should have $T_{X_0}^1 = h^1(\mathcal{T}_{X_t}) + h^0(\mathcal{T}_{X_0})$. (IS THERE ANY HEURISTIC FOR THIS??)

3 Computing the Hodge numbers of X

Since X is a complete intersection of two hyperplanes in Y , we have an exact sequence

$$0 \rightarrow \mathcal{O}_Y(-2) \rightarrow \mathcal{O}_Y(-1)^2 \rightarrow \mathcal{I}_{X/Y} \rightarrow 0,$$

where $\mathcal{I}_{X/Y}$ is the ideal sheaf of X in Y . We also have the sequence

$$0 \rightarrow \mathcal{I}_{X/Y} \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_X \rightarrow 0. \quad (3)$$

This allows us to compute the Hodge numbers of X :

Theorem 3.1. *There exists a non-singular Calabi-Yau with X with $\chi(\Omega_X^1) = -12$. Hence the topological Euler characteristic is -24 .*

Proof. Since X is a complete intersection in Y , we have $\mathcal{I}_{X/Y}/\mathcal{I}_{X/Y}^2 \simeq \mathcal{O}_X(-1)^2$ as \mathcal{O}_X -modules. Hence we have an exact sequence

$$0 \rightarrow \mathcal{O}_X(-1)^2 \rightarrow \Omega_Y^1 \otimes \mathcal{O}_X \rightarrow \Omega_X^1 \rightarrow 0.$$

The first sheaf have no cohomology, which can be computed from its Stanley-Reisner degeneration. Hence the Euler characteristics of the two sheaves on the right must be equal.

Now tensor the exact sequence (3) with Ω_Y^1 to get

$$0 \rightarrow \mathcal{I}_{X/Y} \otimes \Omega_Y^1 \rightarrow \Omega_Y^1 \rightarrow \Omega_Y^1 \otimes \mathcal{O}_X \rightarrow 0.$$

Tensoring with Ω_Y^1 is exact because it is torsionfree (??). Do the same with the \mathcal{O}_Y -resolution of $\mathcal{I}_{X/Y}$ to get

$$0 \rightarrow \Omega_Y^1(-2) \rightarrow \Omega_Y(-1)^2 \rightarrow \mathcal{I}_{X/Y} \otimes \Omega_Y \rightarrow 0.$$

Taking Euler characteristics, we find that $\chi(\mathcal{I}_{X/Y} \otimes \Omega_Y) = -3$. Since $\chi(\Omega_Y^1)$ was computed to be 9, it follows from the first exact sequence that $\chi(\Omega_X^1) = 12$.

The topological Euler characteristic is double the Euler characteristic of the tangent sheaf, hence $\chi(X) = -24$. \square

To actually find the Hodge numbers, we need a few lemmas.

Lemma 3.2. *Let $\mathcal{N}_{X/\mathbb{P}^{13}}$ be the normal sheaf of X in \mathbb{P}^{13} . Then $h^3(\mathcal{I}_X/\mathcal{I}_X^2) = h^0(\mathcal{N}_{X/\mathbb{P}^{13}})$.*

Proof. By Serre duality $h^{3-i}(\mathcal{I}_X/\mathcal{I}_X^2) = h^i((\mathcal{I}_X/\mathcal{I}_X^2)^\vee \otimes \omega)$, where ω is the dualizing sheaf. But X is Calabi-Yau, so $\omega \simeq \mathcal{O}_X$. The dual of $\mathcal{I}_X/\mathcal{I}_X^2$ is by definition the normal bundle. \square

Consider the Euler sequence

$$0 \rightarrow \Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X \rightarrow \mathcal{O}_X(-1)^{14} \rightarrow \mathcal{O}_X \rightarrow 0.$$

Since X is a deformation of a Stanley-Reisner sphere, we know the cohomology of \mathcal{O}_X . So we can extract the cohomology of $\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X$.

Lemma 3.3. *We have*

$$H^i(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1 \\ 0 & \text{if } i = 2 \\ 167 & \text{if } i = 3. \end{cases}$$

Proof. The full long exact sequence is:

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) & \rightarrow & H^0(\mathcal{O}_X(-1)^{14}) & \rightarrow & H^0(\mathcal{O}_X) \rightarrow \\ H^1(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) & \rightarrow & H^1(\mathcal{O}_X(-1)^{14}) & \rightarrow & H^1(\mathcal{O}_X) & \rightarrow & \\ H^2(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) & \rightarrow & H^2(\mathcal{O}_X(-1)^{14}) & \rightarrow & H^2(\mathcal{O}_X) & \rightarrow & \\ H^3(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) & \rightarrow & H^3(\mathcal{O}_X(-1)^{14}) & \rightarrow & H^3(\mathcal{O}_X) & \rightarrow & 0 \end{array}$$

Inserting the dimensions we know, we get:

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) & & 0 & \rightarrow & 1 \rightarrow \\ H^1(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) & \rightarrow & 0 & & \rightarrow & 0 & \rightarrow \\ H^2(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) & \rightarrow & 0 & & \rightarrow & 0 & \rightarrow \\ H^3(\Omega_{\mathbb{P}^{13}} \otimes \mathcal{O}_X) & \rightarrow & 168 & & \rightarrow & 1 & \rightarrow 0 \end{array}$$

Hence we conclude. □

Since X is smooth, the conormal sequence is exact, so we have

$$0 \rightarrow \mathcal{I}_X / \mathcal{I}_X^2 \rightarrow \Omega_{\mathbb{P}^{13}}^1 \otimes \mathcal{O}_X \rightarrow \Omega_X^1 \rightarrow 0.$$

Lemma 3.4. *We have*

$$h^{21}(X) = h^0(\mathcal{N}_{X/\mathbb{P}^{13}}) - 167.$$

where χ denotes the Euler characteristic.

Proof. Write up the long exact sequence coming from the conormal sequence of X and use Lemma 3.2. □

Lemma 3.5. *There is an exact sequence*

$$0 \rightarrow \mathcal{O}_X(1)^2 \rightarrow \mathcal{N}_{X/\mathbb{P}^{13}} \rightarrow \mathcal{N}_{Y/\mathbb{P}^{13}}|_X \rightarrow 0$$

Proof. First note that there is an exact sequence of conormal sheaves:

$$0 \rightarrow \mathcal{I}_Y/\mathcal{I}_Y^2|_X \rightarrow \mathcal{I}_X/\mathcal{I}_X^2 \rightarrow \mathcal{O}_X(-1)^2 \rightarrow 0.$$

The last term is $\mathcal{N}_{X/Y}$, since X is a complete intersection in Y . Dualizing is exact even though Y is not smooth, because by the long exact sequence of $\mathcal{E}xt$ sheaves, we must have $\mathcal{E}xt^1(\mathcal{O}_X(-1), \mathcal{O}_X) = 0$. But this is true, since both of these are locally free. \square

Proposition 3.6. *We have*

$$h^{21}(X) = 39.$$

Hence $h^{11}(X) = 27$.

Proof. By lemma 3.4, we need to compute $h^0(\mathcal{N}_{X/\mathbb{P}^{13}})$. By the previous lemma, we have $h^0(\mathcal{N}_{X/\mathbb{P}^{13}}) = \mathcal{N}_{Y/\mathbb{P}^{13}}|_X + 24$.

We have an exact sequence

$$0 \rightarrow \mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}} \rightarrow \mathcal{N}_{Y/\mathbb{P}^{13}} \rightarrow \mathcal{N}_{Y/\mathbb{P}^{13}}|_X \rightarrow 0.$$

And also an exact sequence:

$$0 \rightarrow \mathcal{N}_{Y/\mathbb{P}^{13}}(-2) \rightarrow \mathcal{N}_{Y/\mathbb{P}^{13}}(-1)^{\oplus 2} \rightarrow \mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}} \rightarrow 0.$$

The cohomology of $\mathcal{N}_{Y/\mathbb{P}^{13}}(-i)$ is possible to compute in `Macaulay2`, and it follows from the long exact sequence that $h^0(\mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}}) = 36$, and $h^1(\mathcal{I}_{X/Y} \otimes \mathcal{N}_{Y/\mathbb{P}^{13}}) = 0$. Hence it follows from the same computation that $h^0(\mathcal{N}_{Y/\mathbb{P}^{13}}|_X) = 182$ and that $h^0(\mathcal{N}_{X/\mathbb{P}^{13}}) = 206$. We conclude that $h^{21} = 206 - 167 = 39$. \square

References

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