Algebraic Geometry Buzzlist

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1 Algebraic Geometry

1.1 General properties

1.1.1 Complete variety

Let X be an integral, separated scheme over a field k. Then X is **complete** if is proper.

1.1.2 Crepant resolution

A **crepant resolution** is a resolution of singularities $f : X \to Y$ that does not change the canonical bundle, i.e. such that $\omega_X \simeq f^*(\omega_Y)$.

1.1.3 Dominant map

A rational map $f : X \dashrightarrow Y$ is **dominant** if its image (or precisely: the image of one of its representatives) is dense in Y.

1.1.4 Étale map

A morphism of schemes of finite type $f : X \to Y$ is **étale** if it is smooth of dimension zero. This is equivalent to f being flat and $\Omega_{X/Y} = 0$. This again is equivalent to f being flat and unramified.

1.1.5 Genus

The **geometric genus** of a smooth, algebraic variety, is defined as the number of sections of the canonical sheaf, that is, as $H^0(V, \omega_X)$. This is often denoted p_X.

1.1.6 Hodge numbers

If X is a complex manifold, then the **Hodge numbers** h^{pg} of X are defined as the dimension of the cohomology groups $H^p(X, \Omega_X^q)$.

1.1.7 Linear series

A **linear series** on a smooth curve C is the data $(\mathcal{L}, V)$ of a line bundle on C and a vector subspace $V \subseteq H^0(C, \mathcal{L})$. We say that the linear series $(\mathcal{L}, V)$ have *degree* $\deg \mathcal{L}$ and *rank* $\dim V - 1$.

1.1.8 Normal variety

A variety X is **normal** if all its local rings are normal rings.

1.1.9 Proper morphism

A morphism $f : X \to Y$ is **proper** if it separated, of finite type, and universally closed.

1.1.10 Resolution of singularities

A morphism $f : X \to Y$ is a **resolution of singularities of** Y if X is non-singular and f is birational and proper.

1.1.11 Separated morphism

Let $f : X \to Y$ be a morphism of schemes. Let $\Delta : X \to X \times_Y X$ be the diagonal morphism. We say that f is **separated** if Δ is a closed immersion.

1.2 Moduli theory and stacks

1.2.1 Étale site

Let S be a scheme. Then the **small étale site over** S is the site, denoted by $\mathsf{\acute{E}t}(S)$ that consists of all étale morphisms $U \to S$ (morphisms being commutative triangles). Let $\mathrm{Cov}(U \to S)$ consist of all collections $\{U_i \to U\}_{i \in I}$ such that

$$\coprod_{i \in I} U_i \to U$$

is surjective.

1.2.2 Grothendieck topology

Let $\mathcal{C}$ be a category. A **Grothendieck topology** on $\mathcal{C}$ consists of a set $\mathrm{Cov}(X)$ of sets of morphisms $\{X_i \to X\}_{i \in I}$ for each X in $\mathrm{Ob}(\mathcal{C})$, satisfying the following axioms:

1. If $V \xrightarrow{\approx} X$ is an isomorphism, then $\{V \to X\} \in \mathrm{Cov}(X)$.
2. If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is a morphism in $\mathcal{C}$, then the fiber products $X_i \times_X Y$ exists and $\{X_i \times_X Y \to Y\}_{i \in I} \in \mathrm{Cov}(Y)$.
3. If $\{X_i \in X\}_{i \in I} \in \mathrm{Cov}(X)$, and for each $i \in I$, $\{V_{ij} \to X_i\}_{j \in J} \in \mathrm{Cov}(X_i)$, then

$$\{V_{ij} \to X_i \to X\}_{i \in I, j \in J} \in \mathrm{Cov}(X).$$

The easiest example is this: Let $\mathcal{C}$ be the category of open sets on a topological space X, the morphisms being only the inclusions. Then for each $U \in \mathrm{Ob}(\mathcal{C})$, define $\mathrm{Cov}(U)$ to be the set of all coverings $\{U_i \to U\}_{i \in I}$ such that $U = \bigcup_{i \in I} U_i$. Then it is easily checked that this defines a Grothendieck topology.

1.2.3 Site

A **site** is a category equipped with a Grothendieck topology.

1.3 Results and theorems

1.3.1 Adjunction formula

Let X be a smooth algebraic variety Y a smooth subvariety. Let $i : Y \hookrightarrow X$ be the inclusion map, and let $\mathcal{I}$ be the corresponding ideal sheaf. Then $\omega_Y = i^* \omega_X \otimes_{\mathscr{O}_X} \det(\mathcal{I}/\mathcal{I}^2)^\vee$, where ω_Y is the canonical sheaf of Y.

In terms of canonical classes, the formula says that $K_D = (K_X + D)\big|_D$.

Here's an example: Let X be a smooth quartic surface in $\mathbb{P}^3$. Then $H^1(X, \mathscr{O}_X) = 0$. The divisor class group of $\mathbb{P}^3$ is generated by the class of a hyperplane, and $\mathcal{K}_{\mathbb{P}^3} = -4H$. The class of X is then $4H$ since X is of degree 4. X corresponds to a smooth divisor D, so by the adjunction formula, we have that

$$K_D = (K_{\mathbb{P}^3} + D)\big|_D = -4H + 4H\big|_D = 0.$$

Thus X is an example of a K3 surface.

1.3.2 Bertini's Theorem

Let X be a nonsingular closed subvariety of $\mathbb{P}^n_k$, where $k = \bar{k}$. Then the set of of hyperplanes $H \subseteq \mathbb{P}^n_k$ such that $H \cap X$ is regular at every point) and such that $H \not\subseteq X$ is a dense open subset of the complete linear system $|H|$. See [3, Thm II.8.18].

1.3.3 Euler sequence

If A is a ring and $\mathbb{P}^n_A$ is projective n-space over A, then there is an exact sequence of sheaves on X:

$$0 \to \Omega_{\mathbb{P}^n_A/A} \to \mathscr{O}_{\mathbb{P}^n_A}(-1)^{n+1} \to \mathscr{O}_{\mathbb{P}^n_A} \to 0.$$

See [3, Thm II.8.13].

1.3.4 Kodaira vanishing

If k is a field of characteristic zero, X is a smooth and projective k-scheme of dimension d, and $\mathcal{L}$ is an ample invertible sheaf on X, then $H^q(X, \mathcal{L} \otimes_{\mathscr{O}_X} \Omega^p_{X/k}) = 0$ for $p + q > d$. In addition, $H^q(X, \mathcal{L}^{-1} \otimes_{\mathscr{O}_X} \Omega^p_{X/k}) = 0$ for $p + q < d$.

1.3.5 Lefschetz hyperplane theorem

Let X be an n-dimensional complex projective algebraic variety in $\mathbb{P}^n_{\mathbb{C}}$ and let Y be a hyperplane section of X such that $U = X \backslash Y$ is smooth. Then the natural map $H^k(X, \mathbb{Z}) \to H^k(Y, \mathbb{Z})$ in singular cohomology is an isomorphism for $k < n - 1$ and injective for $k = n - 1$.

1.3.6 Riemann-Roch for curves

The **Riemann-Roch theorem** relates the number of sections of a line bundle with the genus of a smooth curve C. Let $\mathcal{L}$ be a line bundle ω_C the canonical sheaf on C. Then

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^{-1} \otimes_{\mathscr{O}_C} \omega_C) = \deg(\mathcal{L}) + 1 - g.$$

This is [3, Theorem IV.1.3].

1.3.7 Semi-continuity theorem

Let $f : X \to Y$ be a projective morphism of noetherian schemes, and let $\mathscr{F}$ be a coherent sheaf on X, flat over Y. Then for each $i \geq 0$, the function $h^i(y, \mathscr{F}) = \dim_{k(y)} H^i(X_y, \mathscr{F}_y)$ is an upper semicontinuous function on Y. See [3, Chapter III, Theorem 12.8].

1.3.8 Serre vanishing

One form of Serre vanishing states that if X is a proper scheme over a noetherian ring A, and $\mathcal{L}$ is an ample sheaf, then for any coherent sheaf $\mathscr{F}$ on X, there exists an integer n_0 such that for each $i > 0$ and $n \geq n_0$ the group $H^i(X, \mathscr{F} \otimes_{\mathscr{O}_X} \mathcal{L}^n) = 0$ vanishes. See [3, Proposition III.5.3].

1.4 Sheaves and bundles

1.4.1 Ample line bundle

A line bundle $\mathcal{L}$ is **ample** if for any coherent sheaf $\mathscr{F}$ on X, there is an integer n (depending on $\mathscr{F}$) such that $\mathscr{F} \otimes_{\mathscr{O}_X} \mathcal{L}^{\otimes n}$ is generated by global sections. Equivalently, a line bundle $\mathcal{L}$ is ample if some tensor power of it is very ample.

1.4.2 Invertible sheaf

A locally free sheaf of rank 1 is called **invertible**. If X is normal, then, invertible sheaves are in $1-1$ correspondence with line bundles.

1.4.3 Anticanonical sheaf

The **anticanonical sheaf** ω_X^{-1} is the inverse of the canonical sheaf ω_X, that is $\omega_X^{-1} = \mathscr{H}\mathrm{om}_{\mathscr{O}_X}(\omega_X, \mathscr{O}_X)$.

1.4.4 Canonical class

The **canonical class** K_X is the class of the canonical sheaf ω_X in the divisor class group.

1.4.5 Canonical sheaf

If X is a smooth algebraic variety of dimension n, then the canonical sheaf is $\omega := \wedge^n \Omega^1_{X/k}$ the n'th exterior power of the cotangent bundle of X.

1.4.6 Sheaf of holomorphic p-forms

If X is a complex manifold, then the **sheaf of of holomorphic p-forms** Ω_X^p is the p-th wedge power of the cotangent sheaf $\wedge^p \Omega_X^1$.

1.4.7 Normal sheaf

Let $Y \hookrightarrow X$ be a closed immersion of schemes, and let $\mathcal{I} \subseteq \mathscr{O}_X$ be the ideal sheaf of Y in X. Then $\mathcal{I}/\mathcal{I}^2$ is a sheaf on Y, and we define the sheaf $\mathcal{N}_{Y/X}$ by $\mathscr{H}\mathrm{om}_{\mathscr{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathscr{O}_Y)$.

1.4.8 Reflexive sheaf

A sheaf $\mathscr{F}$ is **reflexive** if the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorpism. Here $\mathscr{F}^{\vee}$ denotes the sheaf $\mathscr{H}\mathrm{om}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$.

1.4.9 Very ample line bundle

A line bundle $\mathcal{L}$ is **very ample** if there is an embedding $i : X \hookrightarrow \mathbb{P}_S^n$ such that the pullback of $\mathscr{O}_{\mathbb{P}_S^n}(1)$ is isomorphic to $\mathcal{L}$. In other words, there should be an isomorphism $i^* \mathscr{O}_{\mathbb{P}_S^n}(1) \simeq \mathcal{L}$.

1.5 Toric geometry

1.5.1 Polarized toric variety

A toric variety equipped with an ample T-invariant divisor.

1.5.2 Toric variety associated to a polytope

There are several ways to do this. Here is one: Let $\Delta \subset M_{\mathbb{R}}$ be a convex polytope. Embed Δ in $M_R \times \mathbb{R}$ by $\Delta \times \{1\}$ and let C_Δ be the cone over $\Delta \times \{1\}$, and let $\mathbb{C}[C_\Delta \cap (M \times \mathbb{Z})]$ be the corresponding semigroup ring. This is a semigroup ring graded by the $\mathbb{Z}$-factor. Then we define $\mathbb{P}_\Delta = \mathrm{Proj}\,\mathbb{C}[C_\Delta \cap (M \times \mathbb{Z})]$ to be the toric variety associated to a polytope.

1.6 Types of varieties

1.6.1 Abelian variety

A variety X is an **abelian variety** if it is a connected and complete algebraic group over a field k. Examples include elliptic curves and for special lattices $\Lambda \subset \mathbb{C}^{2g}$, the quotient $\mathbb{C}^{2g}/\Lambda$ is an abelian variety.

1.6.2 Calabi-Yau variety

In algebraic geometry, a **Calabi-Yau** variety is a smooth, proper variety X over a field k such that the canonical sheaf is trivial, that is, $\omega_X \simeq \mathscr{O}_X$, and such that $H^j(X, \mathscr{O}_X) = 0$ for $1 \leq j \leq n-1$.

1.6.3 del Pezzo surface

A **del Pezzo** surface is a 2-dimensional Fano variety. In other words, they are complete non-singular surfaces with ample anticanonical bundle. The *degree* of the del Pezzo surface X is by definition the self intersection number $K.K$ of its canonical class K.

1.6.4 Elliptic curve

An **elliptic curve** is a smooth, projective curve of genus 1. They can all be obtained from an equation of the form $y^2 = x^3 + ax + b$ such that $\Delta = -2^4(4a^3 + 27b^2) \neq 0$.

1.6.5 Fano variety

A variety X is **Fano** if the anticanonical sheaf ω_X^{-1} is ample.

1.6.6 K3 surface

A **K3 surface** is a complex algebraic surface X such that the canonical sheaf is trivial, $\omega_X \simeq \mathscr{O}_X$, and such that $H^1(X, \mathscr{O}_X) = 0$. These conditions completely determine the Hodge numbers of X.

2 Commutative algebra

2.1 Modules

2.1.1 Depth

Let R be a noetherian ring, and M a finitely-generated R-module and I an ideal of R such that $IM \neq M$. Then the I-depth of M is (see Ext):

$$\inf\{i \mid \mathrm{Ext}^i_R(R/I, M) \neq 0\}.$$

This is also the length of a maximal M-sequence in I.

2.2 Results and theorems

2.2.1 The Unmixedness Theorem

Let R be a ring. If $I = \langle x_1, \cdots, x_n \rangle$ is an ideal generated by n elements such that $\operatorname{codim} I = n$, then all minimal primes of I have codimension n. If in addition R is Cohen-Macaulay, then every associated prime of I is minimal over I. See the discussion after [2, Corollary 18.14] for more details.

2.3 Rings

2.3.1 Cohen-Macaulay ring

A local Cohen-Macaulay ring (CM-ring for short) is a commutative noetherian local ring with Krull dimension equal to its depth. A ring is Cohen-Macaulay if its localization at all prime ideals are Cohen-Macaulay.

2.3.2 Depth of a ring

The depth of a ring R is is its depth as a module over itself.

2.3.3 Gorenstein ring

A commutative ring R is Gorenstein if each localization at a prime ideal is a Gorenstein local ring. A Gorenstein local ring is a local ring with finite injective dimension as an R-module. This is equivalent to the following: $\operatorname{Ext}^i_R(k, R) = 0$ for $i \neq n$ and $\operatorname{Ext}^n_R(k, R) \simeq k$ (here $k = R/\mathfrak{m}$ and n is the Krull dimension of R).

2.3.4 Normal ring

An integral domain R is **normal** if all its localizations at prime ideals $\mathfrak{p} \in \operatorname{Spec} R$ are integrally closed domains.

3 Convex geometry

3.1 Cones

3.1.1 Gorenstein cone

A strongly convex cone $C \subset M_{\mathbb{R}}$ is **Gorenstein** if there exists a point $n \in N$ in the dual lattice such that $\langle v, n \rangle = 1$ for all generators of the semigroup $C \cap M$.

3.1.2 Reflexive Gorenstein cone

A cone C is **reflexive** if both C and its dual $C^\vee$ are Gorenstein cones. See for example [1].

3.1.3 Simplicial cone

A cone C generated by $\{v_1, \cdots, v_k\} \subseteq N_\mathbb{R}$ is **simplicial** if the v_i are linearly independent.

3.2 Polytopes

3.2.1 Dual (polar) polytope

If Δ is a polyhedron, its dual Δ° is defined by

$$\Delta^\circ = \{x \in N_\mathbb{R} \mid \langle x, y\rangle \geq -1 \, \forall \, y \in \Delta\}\,.$$

3.2.2 Gorenstein polytope of index r

A lattice polytope $P \subset \mathbb{R}^{d+r-1}$ is called a **Gorenstein polytope of index** r if rP contains a single interior lattice point p and $rP - p$ is a reflexive polytope.

3.2.3 Nef partition

Let $\Delta \subset M_\mathbb{R}$ be a d-dimensional reflexive polytope, and let $m = \mathrm{int}(\Delta) \cap M$. A Minkowski sum decomposition $\Delta = \Delta_1 + \ldots + \Delta_r$ where $\Delta_1, \ldots, \Delta_r$ are lattice polytopes is called a **nef partition of Δ of length** r if there are lattice points $p_i \in \Delta_i$ for all i such that $p_1 + \cdots + p_r = m$. The nef partition is called *centered* if $p_i = 0$ for all i.

This is equivalent to the toric divisor $D_j = \mathscr{O}(\Delta_i) = \sum_{\rho \in \Delta_i} D_\rho$ being a Cartier divisor generated by its global sections. See [1, Chapter 4.3].

3.2.4 Reflexive polytope

A polytope Δ is **reflexive** if the following two conditions hold:

1. All facets Γ of Δ are supported by affine hyperplanes of the form $\{m \in M_\mathbb{R} \mid \langle m, v_\Gamma\rangle\}$ for some $v_\Gamma \in N$.

2. The only interior point of Δ is 0, that is: $\mathrm{Int}(\Delta) \cap M = \{0\}$.

4 Homological algebra

4.1 Derived functors

4.1.1 Ext

Let R be a ring and M, N be R-modules. Then $\mathrm{Ext}^i_R(M, N)$ is the right-derived functors of the $\mathrm{Hom}(M, -)$-functor. In particular, $\mathrm{Ext}^i_R(M, N)$ can be computed as follows: choose a projective resolution $C_.$ of N over R. Then apply the left-exact functor $\mathrm{Hom}_R(M, -)$ to the resolution and take homology. Then $\mathrm{Ext}^i_R(M, N) = h^i(C_.)$.

4.1.2 Local cohomology

Let R be a ring and $I \subset R$ an ideal. Let $\Gamma_I(-)$ be the following functor on R-modules:

$$\Gamma_I(M) = \{f \in M \mid \exists n \in \mathbb{N}, s.t. I^n f = 0\}.$$

Then $H^i_I(-)$ is by definition the ith right derived functor of Γ_I. In the case that R is noetherian, we have $H^i_I(M) = \varinjlim \mathrm{Ext}^i_R(R/I_n, M)$.

See [2] and [4] for more details.

4.1.3 Tor

Let R be a ring and M, N be R-modules. Then $\mathrm{Tor}^i_R(M, N)$ is the right-derived functors of the $- \otimes_R N$-functor. In particular $\mathrm{Tor}^i_R(M, N)$ can be computed by taking a projective resolution of M, tensoring with N, and then taking homology.

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