Hyper-Kähler manifolds

Fredrik Meyer

April 23, 2016

Abstract

These are notes from the "summer school" at IMPA, Warzaw, held by Kieran O'Grady.

1 Lecture 1 - Introduction

We will first motivate the definition of hyper-Kähler by looking at K3 surfaces.

By definition, a K3 surface is compact Kähler 2-dimensional complex manifold that is simply connected and has trivial canonical bundle ($K_X \simeq \mathscr{O}_X$).

Example 1.1. Let $X \subset \mathbb{P}^3$ be a smooth quartic surface. Then by Lefschetz:

$$\pi_(X,*) \xrightarrow{\sim} \pi_1(\mathbb{P}^3,*) = \{1\},$$

so X is simply-connected. By adjunction, we have:

$$K_x \simeq (K_{\mathbb{P}^3}\big|_X) \otimes \mathcal{N}_{X/\mathbb{P}^3} = \mathscr{O}_X(-4) \otimes \mathscr{O}_X(4) = \mathscr{O}_X\,.$$

★

We list some of the know results about K3's:

1. Any two K3's are deformation equivalent (Kodaira).
2. There is a Hodge-theoretic description of the Kähler cone of a K3 (after having chosen one Kähler class).
3. There is a *Global Torelli Theorem* (Shafarevich and Piateski-Shapiro). Namely, a Hodge structure on $H^2(K3;\mathbb{C})$ and a lattice structure on $H^2(K3,\mathbb{Z})$ determines X up to isomorphism.

We now state the definition of a hyper-Kähler manifold:

A hyper-Kähler manifold X is a compact Kähler manifold X, simply connected, such that $H^0(X, \Omega^2, X) = \mathbb{C}\sigma$, where σ is *symplectic*, meanig that $T_xX \times T_xX \xrightarrow{\sigma(x)} \mathbb{C}$ is non-degenerate for all $x \in X$.

Remark. *A HK must have even dimension: the skew-symmetric form σ can be represented by a $n \times n$-matrix A that is skew-symmetric with non-zero determinant. Skew-symmety of A means that $A^T = -A$. Hence* $\det A = \det A^T = (-1)^n \det A$. *This forces n to be even if we want* $\det A \neq 0$.

In dimension 2, K3's are hyper-Kähler.

To motivate hyper-Käher manifolds, we state the Beaville-Bogomolov decomposition theorem:

Theorem 1.2. *Let Z be a compact Kähler manifold with $c_1(Z) = 0$ (what does this mean?) in $H^2(X, \mathbb{Q})$. Then there exists a finite étale cover $\widetilde{Z} \to Z$ such that*

$$\widetilde{Z} = \mathbb{C}^d/\Lambda \times \prod_i X_i \times \prod_j Y_i$$

where the first factor is a compact torus. The second factor is a product of hyper-Kähler manifolds, and the second is a product of Calabi-Yau manifolds (that is, manifolds with $K_{Y_i} \simeq \mathscr{O}_{Y_i}$ and $h^0(\Omega^p_{Y_i}) = 0$ for all $0 < p < \dim Y_i$).

1.1 First example of a HK

Now we define some higher-dimensional examples of HK's. Terminology: when we say "HK variety", we shall mean a *projective* HK manifold.

Let S be a K3 surface. Then let $S^{(2)}$ be the symmetric square of S (that is, $S \times S/(p,q) \sim (q,p)$). This space comes equipped with two projections, π_1, π_2, and the form $\pi_1^*\sigma + \pi_2^*\sigma \in H^0(\Omega^2_{S\times S})$ is τ-invariant, hence it descends to a holomorphic 2-form on $S^{(2)}$.

But the symmetric square is singular along the (image of the) diagonal, where two points come together. In fact, one can see that it locally looks like $\mathbb{C} \times \mathbb{C}/x \sim -x$. The last factor is a quadric cone, hence a single blowup along the diagonal will resolve the singularities in this case.

Let D be the diagonal. We have a diagram

$$\begin{tikzcd} Bl_D(S^2) \arrow[r, "\widetilde{\rho}"] \arrow[d] & S^{[2]} \arrow[d, "\gamma"] \\ S^2 \arrow[r, "\rho"] & S^{(2)} \end{tikzcd}$$

The top right space $S^{[2]}$ can be thought of in two ways: first, notice that the involution on S^2 act on the blowup as well. Hence we can take the quotient of the blowup. This is one description of $S^{[2]}$. Secondly, one can think of $S^{[2]}$ as the blowup of $S^{(2)}$ along the points $2P$ (the image of the diagonal).

Now $S^{[2]}$ is smooth projective and "$\pi_1^*\sigma + \pi_2^*\sigma$" is symplectic.

Then $H^2(S^{[2]})$ is spanned by this symplectic 2-form. We can show that $S^{[2]}$ is simply-connected as well:

$$\pi_1(S^{[2]}\backslash D, *) \twoheadrightarrow \pi_1(S^{[2]}, *).$$

The first group is $\mathbb{Z}/2$, generated by a loop around D (we abuse notation: D denotes the image of the diagonal in $S^{[2]}$), and it is $\mathbb{Z}/2$ because $S^2\backslash D$ is a double cover of $S^{[2]}\backslash D$.

We call $S^{[2]}$ the *Hilbert square* of S. It is a HK variety of dimension 4. It can also be realized as the Hilbert scheme parametrizing length 2 subschemes of S.

In general,

$$Y^{[n]} := \{Z \subset Y \mid l(Z) = n\},$$

is smooth, irreducible of dimension $2n$ (if Y is a surface).

Beauville shows that if S is a K3, then $S^{[n]}$ is always a HK variety of dimension $2n$. Hence we have examples of hyper-Kählers in each even dimension!

If $n \geq 2$, then $b_2(S^{[2]}) = 23$, which is $22 + 1$, the last divisor coming from the blowup.

1.2 Second example of a HK

The analogue of $S^{[2]}$ with S replaced by A, an abelian surface.

The space $A^{[2]}$ has a holomorphic symplectic form, but is far from being simply connected. But consider the maps

$$A^{[2]} \xrightarrow{\gamma_2} A^{(2)} \xrightarrow{s_2} A.$$

The first map sends a points z to the sum $\sum_{p\in A} l(\mathscr{O}_{Z,P}\, P$. The second map sends $P + Q$ to $(p) + (q) \in A$, where the parenthesis means that we actually consider the sum in the *group* A.

Then the composition $s_2 \circ \gamma_2$ is a locally trivial fibration in the étale/analytic topology. It follows that the cohomology group $H^1(A) = \mathbb{C}^4 \hookrightarrow H^1(A^{[2]})$.

Now look at the fiber $s_2 \circ \gamma_2^{-1}(0)$. This is a smooth Kummer surface $\approx A/\langle -1 \rangle$. These are K3 surfaces!

In general we look at the sequence

$$A^{[n+1]} \xrightarrow{\gamma_{n+1}} A^{(n+1)} \xrightarrow{s_{n+1}} A$$

defined analogously, and we define the *generalized Kummer* to be $Kum^{[n]}(A) := (s_{n+1} \circ \gamma_{n+1})^{-1}(0)$.

Beauville proved that $K^{[n]}$ is a HK variety of dimension $2n$. For $n \geq 2$, we have $b_2(K^{[n]}(A)) = 7 = b_2(A) + 1$, so we do actually have two topologically distinct families.

1.3 Third example, lines on a cubic 4-fould

Let $Y \subset \mathbb{P}^n$ be some algebraic variety. Let $X = F(Y)$ be the set of lines contained in Y. It is a closed subset of the Grassmannian $\mathbb{G}(1, \mathbb{P}^n)$, which we can think of as embedded via Plücker in $\mathbb{P}^{\binom{n-1}{2}-1}$.

Theorem 1.3. *Let $Y \subset \mathbb{P}^n$ be a smooth cubic hypersurface. Then $X = F(Y)$ is a smooth connected variety of dimension $2n-6$ and $K_X \simeq \mathscr{O}_X(5-n)$.*

Since we are interested in HK varieties, we put $n = 5$.

Remark. *All HK's have trivial canonical bundle: the $n/2$th power of the symplectic form gives a trivialization of $K_X = \Omega^n_X$.*

We want to look at a incidence correspondence $\mathcal{I} \subset Y \times X$:

$$\begin{tikzcd} & \mathcal{I} = \{(y, L) \in Y \times X \mid y \in L\} \arrow[dl, "\rho"'] \arrow[dr, "\pi"] & \\ Y & & X \end{tikzcd}$$

The fibers of ρ are $\mathbb{P}^1$s (how to see this?). In general we get a map

$$H^{n-1}(Y) \xrightarrow{c} H^{n-3}(X)$$

given by $\alpha \mapsto \pi_*(\rho^*\alpha)$. So for $n = 5$, we get a map $H^4(Y) \to H^2(X)$.

Beauville and Donagi showed that if $Y \subset \mathbb{P}^5$ is a smooth cubic hypersurface, then $X = F(Y)$ is a HK variety of type $K3^{[2]}$. Moreover, the restriction of c to the primitive cohomology

$$H^4(Y)_0 := \{\alpha \in H^4(Y) \mid \alpha - c_1(\mathscr{O}_Y(1)) = 0\}$$

is an isomorphism to $H^2(X)_0$ of Hodge structures.

We get an isomorphism $H^{p,q}(Y)_0 \simeq H^{p-1,q-1}(X)_0)$, and there exists a bilinear symmetric form $\langle , \rangle$ on $H^2(X)_0$ such that

$$\langle c(\alpha), c(\beta) \rangle = -\int_Y \alpha \wedge \beta$$

for all $\alpha, \beta \in H^4(Y)_0$.

One consequence: if $Y \subset \mathbb{P}^5$ is very general, then $H^2(X)_0$ has no nonzero integral $(1,1)$-classes. [explanation follows]

Main point: if Y is very general, then $F(Y) \not\simeq S^{[2]}$ (not isomorphic) (S is K3).

Beauville proved however that if Y is a pfaffian cubic

$$\{(t_0 : \dots : t_5) \in \mathbb{P}^5 \mid Pf(t_0 A_0 + \dots t_5 A_5)\},$$

where the A_i are skewsymmetric matrices, then $F(Y) \simeq S^{[2]}$.

2 Lecture 2 - some of the main general results

The general theory is developed by Bogomolov, Fujiki, Beauville, Verlotsky, Huybrechts and others. Today's lecture have three main ingredients:

1. The deformation theory if HK's are unobstructed.
2. BBF (Bogomolov-Beauville-Fujiki) quadratic form on $H^2(HK)$.
3. Twistor families of HK's. This leads to the deepest results.

2.1 Deformations of a HK X

Let σ be the given holomorphic symplectic form on X. Contraction with σ defines an isomorphism $L_\sigma : \Theta_X \to \Omega^1_X$. Hence we get isomorphism

$$H^i(L_\sigma) : H^i(X, \Theta_X) \xrightarrow{\sim} H^i(X, \Omega^1_X).$$

In particular, $H^0(X, \Theta_X) = H^0(X, \Omega^1_X) = 0$, since X is simply-connected (to see this: by the Hodge decomposition we have $0 = H^1(X; \mathbb{C}) = H^0(X, \Omega^1_X) \oplus H^1(\mathscr{O}_X))$.

This implies (by general results?) that there exists a universal deformation space of X:

$$\begin{tikzcd} X \simeq X_0 \arrow[r, hook] \arrow[d] & \mathscr{X} \arrow[d, "\pi"] \\ 0 \arrow[r] & B \end{tikzcd}$$

Then we have $T_0 B = H^1(X, \Theta_X) = H^1(X, \Omega^1_X)$.

Theorem 2.1 (Bogomolov)**.** *If X is HK, then X is unobstructed. That is, there exists a universal deformation space B of X with B smooth.*

Corollary 2.2.

$$\dim Def(X) = dim B = b_2(X) - 2.$$

Proof. Since

$$Def(X)$$

is smooth, we have

$$\dim Def(X) = h^1(X, \Theta_X) = h^1(X, \Omega^1_X) = b_2(X) - h^{2,0} - h^{0,2} = b_2(X) - 2.$$

□

Remark. *This is because the Kodaira-Spencer map $T_0 B \xrightarrow{\kappa} H^1(X, \Theta_X)$ is an isomorphism if $B = Def(X)$.*

Example 2.3. Let $X = S^{[n]}$ and $n \geq 2$. Then $b_2(X) = 23$, hence $\dim Def(X) = 21$. If X is K3, then $dim Def(X) = 20$. ★

2.2 The BBF quadratic form

We will study the *local period map*. Again, let $\pi : \mathscr{X} \to B$ be a family of HK's, with π a proper submersion. If $b \in B$, we write $X_b := \pi^{-1}(b)$.

By choosing B small enough, we can assume that the local system $R^2\pi_*\mathbb{Z}$ is trivial: so we can identify $H^2(X_0, \mathbb{Z}) \simeq H^2(X_{b_2}, \mathbb{Z}) = \Lambda$ for all $b_2 \in B$ and for a fixed finitely generated torsion free abelian group Λ of rank r (i.e $R^2\pi_*\mathbb{Z} \simeq B \times \Lambda$).

For $b \in B$, we get isomorphisms $p_b : H^2(X_0, \mathbb{Z}) \to \Lambda$. Extension of scalars gives a map $p_b : H^2(X_b, \mathbb{C}) \to \Lambda_{\mathbb{C}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$. Note that $H^{2,0}(X_b)$ is contained in the source.

We are now ready to define the period map:

$$B \xrightarrow{\mathcal{P}_\pi} \mathbb{P}(\Lambda_\mathbb{C})$$

$$b \longmapsto p_b(H^{2,0}(X))$$

This makes sense, because $H^{2,0}(X)$ is one-dimensional, hence spans a line in Λ_C. We want to compute $d\mathcal{P}_\pi(0) = d\mathcal{P}_\pi$. [1]

Let $v \in T_0B$. A priori $d\mathcal{P}_\pi(v) \in \mathrm{Hom}(H^{2,0}(X), F^1H^2(X)/H^{2,0}(X))$. This follows from Griffiths transversality. Recall $F^1H^2(X) = H^{2,0}(X) \oplus H^{1,1}(X)$. This last Hom group is equal to $\mathrm{Hom}(H^{2,0}(X), H^{1,1}(X))$. Griffiths in this case tells us that $\langle d\mathcal{P}_\pi(v), \sigma\rangle = L_sigma(\kappa(v))$.

We see that (?) the differential of the period map is injective with image $\mathrm{Hom}(H^{2,0}(X), H^{2,1}(X))$. Hence we conclude that the image of the period map is a local analytic hypersurface in $\mathbb{P}(\Lambda_\mathbb{C})$.

We now state the theorem of the existence of the *Bogomolov-Beauville-Fujiki quadratic form*:

Theorem 2.4. *There exists a quadratic form (with an associated bilinear form $(,)_X$)*

$$q_X : H^2(X) \to \mathbb{C}$$

which is integral, indivisible[2]*, and a $c_X \in \mathbb{Q}_+$ such that*

$$\int_X \alpha^{2n} = c_X \frac{(2n)!}{n!2^n} q_X \alpha^n \tag{1}$$

for all $\alpha \in H^2(X)$.

Why the factor $\frac{(2n)!}{n!2^n}$? It has to with (1) is equivalent to the assertion

$$\int_X \alpha_1 \wedge \cdots \wedge \alpha_{2n} = c_X \sum_{\tau \in S_n} (\alpha_{\tau(1)}, \alpha_{\tau(2)})_X \cdot \ldots \cdot (\alpha_{\tau(2n-1)}, \alpha_{\tau(2n)})_X,$$

where we sum over permutations "without stupid repetitions" (interpret this).

Now let X be of type $K3^{[n]}$. If S is K3, then $H^2(S^{[n]}, \mathbb{Z}) = im\mu_n \oplus \mathbb{Z}\zeta_n$, where μ_n is the composition

$$H^2(S, \mathbb{Z}) \to H^2(S^{(n)}, \mathbb{Z}) \to H^2(S^{[n]}, \mathbb{Z}).$$

[1] The rest of this section will be quite sketchy, mostly because I didn't understand so much.

[2] What does this mean?

The first map sends a curve C to the $\{Z \in S^{[n]} \mid Z \cap C \neq \emptyset\}$. The ζ_n is the reduced image of the diagonal: one can see that the class of the diagonal in $H^2(S^{[n]}, \mathbb{Z})$ is divisible by two. This is analogous to the fact that the map $Bl_D S^2 \to S^{[2]}$ is $2:1$-ramified along D. However, for general n, the construction of $S^{[n]}$ by a sequence of blowups and blowdowns is quite complicated. But for dimension reasons, we only need to look at points of $S^{(n)}$ where two points come together.

In this case the BBF quadratic form takes the form (no pun...): On $im\mu_n$ it is defined by $(\mu_n(\alpha), \mu_n(\beta)) = \int_S \alpha \wedge \beta$, and on the other summand we have $q(\zeta_n) = -2(n-1)$. And $c_X = 1$.

This is useful, for it lets us compute for example the self-intersection of the diagonal:

$$\Delta_n \cdot \ldots \cdot \Delta_n = \frac{(2n)!}{n!2^n}(-8(n-1))^n.$$

We sum up some of the things we know in a table:

X	$\dim X$	$b_2(X)$	c_X	$H^2(X;\mathbb{Z})$
$K3^{[n]}$	2n	23	1	$U^3 \oplus E_8^2 \oplus (-2(n-1))$
$Kum^{[n]}$	2n	7	n+1	$U^3 \oplus (-2(n-1))$

(he goes on to prove the theorem, but I drew a pig instead)

3 Lecture 3 - more results

Recall from last lecture that we had the BBF quadratic form q_X on $H^2(X)$. We denote the associated bilinear form by $(,)_X$. The Fujiki constant is denoted $c_X \in \mathbb{Q}_+$.

They key property is this:

$$\int_X \alpha_1 \wedge \ldots \wedge \alpha_{2n} = c_X \sum_{\tau \in S_n} (\alpha_{\tau(1)}, \alpha_{\tau(2)})_X \cdot \ldots \cdot (\alpha_{\tau(2n-1)}, \alpha_{\tau(2n)})_X$$

And also the property $(\sigma, \overline{\sigma}) > 0$, which determines σ uniquely.

From the properties we can deduce the following list of consequences:

1. $H^{pq}(X) \perp H^{p',q'}$ unless $p' = 2-p$ and $q' = 2-q$. (don't ask me why)
2. If $\omega \in H^n(X, \mathbb{R})$ is a Kähler class (n??), then $(\omega, \omega)_X > 0$.
3. The signature of $q_X\big|_{H^2(X,\mathbb{R})}$ is $(3, b_2(X) - 3)$.

4. If X and Y are HK's and $f : X \dashrightarrow Y$ is birational, then there exists $I_X \subset X$, $I_Y \subset Y$ of codimension $geq2$ such that

$$(X \backslash I_X) \xrightarrow{\sim} (Y \backslash I_Y)$$

is an isomorphism. Hence f induces an isomorphism on H^2 (with integral coefficients). Also $(f^*\alpha, f^*\beta)_X = (\alpha, \beta)_Y$. That is, f^* is an isometry.

3.1 Revisiting the period map

Again, let $\mathfrak{X} \to B$ be a family, and assume that we have a trivialization $R^2\pi_*\mathbb{Z} = B_*\Lambda$, for a fixed lattice Λ. Hence for every point $b \in B$, we have an isomorphism $f_b : H^2(X_b, \mathbb{Z}) \to \Lambda = \mathbb{Z}^r$.

Then the period map $\mathcal{P} : B \to \mathbb{P}(\Lambda_{\mathbb{C}})$ is defined by sending b to $f_b(H^{2,0}(X_b))$.

Now, given $\alpha \in \Lambda_{\mathbb{R}}$, we can ask for which $b \in B$, $f_b^{-1}(\alpha) \in H^{1,1}(X_b, \mathbb{R})$? Answer: this happens if and only if $\mathcal{P}_B \in \alpha^\perp$ (then: explanation which I don't understand).

Now assume that $\alpha \in H^{1,1}_{\mathbb{Z}}(X)$ and that $image(\mathcal{P}_{\mathcal{B}}) \subset \alpha^\perp$.

...

3.2 Back to smooth cubic hypersurfaces

Back to $Y \subset \mathbb{P}^5$, a smooth cubic hypersurface. let $X = F_1(Y)$ be the variety of lines on X. Then X is HK of type $K3^{[n]}$, and is endowed with the Plücker line bundle $\mathscr{O}_X(1)$.

Point: not all small deformations of $F(Y)$ are isomorphic to $F(Y')$, but all small deformations of $F(Y)$ which keeps $c_1(\mathscr{O}_X(1))$ of type $(1,1)$ *are* isomorphic to $F(Y')$ (for some Y')).

3.3 Kähler cone

The Kähler cone is the set

$$K_X = \{\alpha \in H^{1,1}_{\mathbb{R}}(X) \mid \ \alpha \text{ is a Kähler class}\}.$$

The set of $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ such that $q_X(\alpha) > 0$ has two connected components. A choice of Kähler class determines one of these, which we call the *positive cone.*

Huybrechts showed that for *very general* $b \in B$, the Kähler cone is equal to the positive cone. Bayer-Hasset-Tschukul (?) gave a numerical description of $Ample(X)$ for X projective of type $K3^{[n]}$.

Huybrechts showed that if X, Y are birational HK's, then they are in fact deformation equivalent. (!!!)

3.4 More examples

Now let S be a projective abelian surface. Let $\mathscr{M}$ be the moduli space of $\mathscr{O}_S(1)$-stable shaves on S with fixed rank, c_1 and c_2 (Chern classes). Then it turns out that $\mathscr{M}$ is smooth and has a regular symplectic form!

Let $[\mathscr{F}] \in \mathscr{M}$. And say $0 \neq \omega \in H^0(K_S)$ is the canonical form on S. Then $\Theta_{[\mathscr{F}]}\mathscr{M} = H^1(\operatorname{End}\mathscr{F}))$ (the tangent space).

We want to define a pairing

$$\Theta_{[\mathscr{F}]}\mathscr{M} \times \Theta_{[\mathscr{F}]}\mathscr{M} \to \mathbb{C}$$

by $(\alpha, \beta) \mapsto \int_S \alpha \wedge \operatorname{Tr}(\alpha \cup \beta)$. But what is $\operatorname{Tr}(\alpha \cup \beta)$? $\alpha \cup \beta \in H^{0,2}(S, \operatorname{End}\mathscr{F})$. And we have a trace map from here to $H^{0,2}(S)$. (???)

Now let S be K3. Then we can consider $\mathscr{M}_v^{st}$, constisting of stable sheaves $\mathscr{F}$ with Mukai vector v (the Mukai vector is tuple containing rank and Chern classes). If the moduli space is projective (which happens sometimes), then it turns out that it is a HK variety of type $K3^{[n]}$.

If S is an abelian surface, then we have a map $\mathscr{M}_v^{st}(S) \to S^\wedge \times S$ given by $\mathscr{F} \mapsto (\det \mathscr{F}, \sum c_2(\mathscr{F})$. Then it turns out that $f^{-1}(0,0)$ is a HK variety of type $Kum^{[n]}$!

But most choices of v gives noncompact moduli spaces! One can try to compactify, but the compactification is usually singular at the boundary, outside the stable locus.

However, there exists in a few cases a symplectic desingularization of $\mathfrak{M}_v(S)$. In the case when $\dim \mathfrak{M}_v(S) = 10$, O'Grady have shown that we get a new deformation class of HK's! If $S = K3$, then these have $b_2 = 24$, and if S is abelian, we instead look at the fiber at 0 as before, giving a 6-dimensional HK with $b_2(fiber) = 8$.

4 More constructrions of HK's

In this last lecture, we will look at more constructions of HK's. Below is a table of all known deformation classes of HK's:

X	$\dim X$	$b_2(X)$	c_X	$H^2(X;\mathbb{Z})$
$K3^{[n]}$	2n	23	1	$U^3 \oplus E_8^2 \oplus (-2(n-1))$
$Kum^{[n]}$	2n	7	n+1	$U^3 \oplus (-2(n-1))$
OG6	6	8	4	$U^3 \oplus (-2)^2$
OG10	10	24	1	$U^3 \oplus E_8^2 \oplus A_2$

Today we will mention some other constructions (but no new deformation classes).

The holy grail of the theory is to describe all deformation classes of HK's. First, let us look at how Kodaira proved that $K3$s form a single deformation class.

There are a few steps. First: Let X_0 be a K3. Then by Noethers formula: $2 = \chi(\mathscr{O}_X) = (K^2 + c_2)/12$. It follows that $c_2 = 24$. This is the topological Euler characteristic. Hence $b_2 = 22$. The signature of $\langle , \rangle$ on $H^2(X_0, \mathbb{R})$ is $(3, 19)$. We also know that $H^2(X_0, \mathbb{Z})$ is a unimodular lattice (by Poincaré duality, appearantly). Since $K_X = 0$, the intersection form is even. By lattice theory, we conclude that $H^2(X_0, \mathbb{Z}) = U^3 \oplus E_8^2 = \Lambda$.

Hence all K3's have the same lattice.

Second: one calculates the period map $B \to \mathbb{P}(\Lambda_{\mathbb{C}}) = \mathbb{P}^{21}$. The image is contained in a smooth quadric. Then one proves that there exists a $b \in B$ such that $H^{1,1}_{\mathbb{Z}}(X_b) = \mathbb{Z}\alpha_b$ with $\alpha_b^2 = 2$.

Third: Let $X = X_b$. Then let $\alpha_b = c_1(L)$, where L is a holomorphic line bundle. By Hirzebruch-Riemann-Roch, we have

$$\chi(L) = \chi(\mathscr{O}_X) + \frac{1}{2}c_1(L)^2 = 2 + \frac{2}{2} = 3.$$

And by Serre duality:

$$3 = h^0(L) - h^1(L) + h^2(L) = h^0(L) - h^1(L) + h^2(L^{-1}).$$

Hence $h^0(L) + h^0(L^{-1}) = 3 + h^1(L)$. So we can assume L has at least three sections. Then one analyses the map $\varphi_L : X \dashrightarrow |L|^\vee$. By some argument, φ_L is regular with 2-dimensional image. It follows that φ_L is a $2:1$ map ramified over a sextic.

Conclusion: there is a deformation of X_0 to X, a double cover ramified over a sextic. But the deformation space of such double covers is irreducible! Hence all HK's are deformation equivalent.

Remark. *Now he goes on to explain a program of copying Kodaira's profo for a certain type of HK's ("numerical $K3^{[n]}$s"). Then he states a theorem of his:*

Theorem 4.1. *There is a rational map $\varphi : X \dashrightarrow \mathbb{P}^5$. One of the following holds:*

1. *φ is regular, 2:1 onto an EPW sextic. These are parametrized by a connected family, and all are of type $K3^{[n]}$.*
2. *φ is birational onto its image.*

The conjecture *is that the second option never occurs.*

4.1 EPW sextics and their double covers

EPW stands for Eisenbud-Popescu-Walter. Let $LG(\Lambda^3 V_6)$ be the set of Lagrangian subspaces of $\Lambda^3 V_6$ (here V_6 is a 6-dimensional vector space), and A one such. We say that A is Lagrangian if $\alpha, \beta \in A$ implies that $\alpha \wedge \beta = 0$

Then we define $Y_A \subset \mathbb{P}(V_6)$, to be the set of $[v] \in \mathbb{P}(V_6)$ such that the map $A \ni \alpha \mapsto v \wedge \alpha \in \Lambda^4 V_6$ has nontrivial kernel.

Then in the generic case, Y_A is a sextic hypersurface (so a 4-fold). The singular locus of Y_A is a smooth surface, and one can show that if $p \in \operatorname{Sing}(Y_A)$, then $(Y_A, p) \simeq (\mathbb{C}^2 \times C, 0$, where C is a cone. There is a smooth double cover of Y_A ramified along the singular locus. We denote this cover by X_A.

Then:

Theorem 4.2. *X_A is a HK of type $K3^{[n]}$. Varying A, we get a locally complete family of projective HK's.*

4.2 Other explicit examples of HK's

Here comes some more examples.

Let $\sigma \in \Lambda^3 V_{10}^\vee$ be generic. Then let

$$Y_\sigma = \{v \in \mathbb{G}(6, V_{10}) \mid \sigma\big|_{\wedge^3 v} = 0\}.$$

Then Y_σ is a HK of type

4.2.1 Ranestad-Iliev

Start with $Y \subset \mathbb{P}^5$, a cubic hypersurface. Then consider the set $VSP(Y, 10)$ ("variety of sums of powers"), consisting of $\Lambda \in |\mathscr{O}_{\mathbb{P}^5}(3)|$ such that $Y \in \Lambda$ and Λ is 10-secant to Veronese of Y.

Then $VSP(Y, 10)$ is HK of type $K3^{[n]}$. But now the polarization has degree 38.

Another one

Let $Y \subset \mathbb{P}^5$ be a generic cubic hypersurface. Then look at $M_3(Y)$, the moduli space of twisted cubics contained in Y. This is of dimension 10. There exists a map $M_3(Y) \to X(Y)$, where $X(Y)$ is hyper-Kähler of type $K3^{[n]}$ again.

4.3 More types?

Note that all the examples found are of type $K3^{[n]}$. It would be a nice challenge to find examples of type $Kum^{[n]}$.