

# Findings

FM

## 1 Existence of a smoothing

Let  $\mathcal{K} = D_6 * D_6$  be the join of two 6-gons, and let  $A_{\mathcal{K}}$  be its Stanley-Reisner ring, and  $\mathbb{P}(\mathcal{K})$  its associated projective scheme.

**Lemma 1.1.** *The  $f$ -vector of  $\mathcal{K}$  is  $f = (1, 12, 48, 72, 36, 1)$ .*

The tangent and obstruction modules  $T^i(\mathbb{P}(\mathcal{K}))$  can be described as follows:

**Proposition 1.2.**

$$\dim_{\mathbb{C}} T^1(\mathcal{K}) = 12 + 72 = 84$$

$$\dim_{\mathbb{C}} T^2(\mathcal{K}) = 72$$

*Proof.* We use the results of [AC10]. A basis of  $T^1$  come in two orbits. They can be described as follows: let  $\text{lk}(\mathcal{K}, v_1)$  be the link of a vertex in  $\mathcal{K}$ . This is double pyramid over a hexagon, and this contributes with one dimension to  $T^1$ , by the results in [AC10]. There are 12 vertices.

For the other type, let  $v_i$  be one of the first six vertices and  $v_j$  be one of the last six. Then  $\text{lk}(\mathcal{K}, v_i v_j)$  is a square, which contributes 2 to  $T^1$ . In total, there are  $6 \times 6$  choices, hence  $2 \times 36 = 72$  dimensions to account for. In total  $\dim_{\mathbb{C}} T^1(\mathcal{K}) = 84$ .

For  $T^2$ , all contributions are of the same form as those for  $D_6$ , namely the crossing diagonals. There are three of them, so in total, we have  $\dim_{\mathbb{C}} T_2^2 = 6$ . However, each one of them can be multiplied by monomials to give something of degree 0. Thus there are in total  $3 \times 4 \times 6 = 72$  contributions to  $T^2$ .  $\square$

Here's an observation: Let  $\mathcal{G} = D_6 * \{v\}$ . Then  $\mathbb{P}(\mathcal{G})$  can be smoothed to a del Pezzo surface of degree 6. This follows because  $\mathcal{G}$  triangulates the associated polytope (a hexagon), by standard “Sturmfels theory”. It follows that  $D_6$  can be smoothed as well, because it is embedded as a complete intersection in  $\mathcal{G}$ .

**Proposition 1.3.** *There exists a smoothing of  $\mathbb{P}(\mathcal{K})$  to a complete intersection of two hyperplanes in a deformation of a toric variety.*

*Proof.* Now let  $\mathcal{G} = (D_6 * \{v\}) * (D_6 * \{w\})$  for two vertices  $v, w$ . Then using the above trick,  $\mathcal{G}$  can be deformed to the projective join  $T$  of two del Pezzo surfaces of degree 6. The ideal is just given by the sum of the ideal of each del Pezzo surface, in disjoint variables. Since  $\mathcal{K}$  is a complete intersection in  $\mathcal{G}$ , it follows that  $X_0 = \mathbb{P}(\mathcal{K})$  deforms as well, say to  $X_t$ . However,  $T$  has a singular locus of dimension 2. By Bertini, it follows that  $X_t$  have isolated singularities.

However: it is possible to computationally, by brute force, further deform  $T$  to a variety having singular locus of dimension 1. This implies that  $X_t$  deforms to something smooth.  $\square$

It would be nice to find a non-computational argument for the existence of the smoothing. Maybe a toric deformation? It would also be nice to know if the generic fiber of  $X_0$  over  $Def(X_0)$  is smooth, and if not, what is the smoothing component? From computational experiments, it looks like the total base space is way to complicated to approach directly.

## 2 Mirror symmetry

### 2.1 Notation and preliminaries

Since  $X_t$  above degenerates to a sphere, it is a Calabi-Yau-manifold, and in fact, the description of it is a complete intersection of two hyperplanes corresponds to a so-called nef partition of the polytope associated to  $T$  above. We recall the Batyrev-Borisov construction for mirrors of complete intersections in toric varieties. We follow the description in [DN10].

Let  $N \approx \mathbb{Z}^n$  be a lattice and let  $M = N^\vee$  be the dual lattice (the “equation lattice”). Let  $\Delta \subset N_\mathbb{R}$  be a reflexive polytope and  $\Delta^\circ$  its polar polytope (which is also reflexive). Let  $T := N \otimes_\mathbb{Z} \mathbb{C}^*$  be the  $n$ -dimensional torus. We denote by  $\mathbb{P}_\Delta$  the toric variety associated to the polytope  $\Delta$ . It can be described as the closure of the map  $T_N \rightarrow \mathbb{P}^{s-1}$  (here  $s$  is the number of vertices of lattice points of  $\Delta$ ) given by

$$t \mapsto (\chi^{m_1}(t), \dots, \chi^{m_s}(t)).$$

An equivalent (?) description more suitable for computations is the following: embed  $\Delta$  in  $M_\mathbb{R} \times \mathbb{R}$  as  $\Delta \times \{1\}$ . Let  $C$  be the cone over  $\Delta \times \{1\}$ , and consider

the semigroup ring  $\mathbb{C}[C \cap (M \times \mathbb{Z})]$ . This is a graded ring, so we define  $\mathbb{P}_\Delta$  to be  $\text{Proj } \mathbb{C}[C \cap (M \times \mathbb{Z})]$ .

Let the vertex set  $V$  of  $\Delta$  be partitioned into a disjoint union of subsets  $V_1 \sqcup V_2 \sqcup \dots \sqcup V_r$  with corresponding polytopes  $\Delta_i := \text{Conv}(V_i \cup \{0\})$ . We say that this partition is a *nef partition* if the Minkowski sum  $\Delta_1 + \dots + \Delta_r$  is also a reflexive polytope, which we will denote by  $\nabla^\circ \subset N_{\mathbb{R}}$  (so  $\nabla \subset M_{\mathbb{R}}$ ) (we say that the nef partition is *indecomposable* if no Minkowski sum of any proper subset of the partition gives a reflexive polytope).

The associated *Cayley polytope*  $P^* \subset N_{\mathbb{R}} \times \mathbb{R}^r$  of dimension  $n + r - 1$  is given by

$$P^* = \text{Conv} \left( \bigcup_{i=1}^r \Delta_i \times e_i \right),$$

where of course  $\{e_i\}_{i=1}^r$  is the standard basis of  $\mathbb{R}^r$ . The cone over the Cayley polytope is the *Cayley cone*  $C^* := \text{Cone}(P^*)$  with *dual Cayley cone*  $C \subset M_{\mathbb{R}} \times \mathbb{R}^r$ . The cone  $C^*$  is a reflexive Gorenstein cone of index  $r$ .

The intersections of  $P$  with affine subspaces given by intersections of hyperplanes  $x_i = 1, x_j = 0$  for  $i, j \in \{n+1, \dots, n+r\}$ , and  $i \neq j$  are polytopes  $\nabla'_1, \dots, \nabla'_r$ . Then  $\nabla_i = \pi(\nabla'_i)$ , the projection of  $\nabla'_i$  to  $N_{\mathbb{R}}$ . These polytopes correspond to the dual nef partition, such that  $\nabla = \text{Conv}(\nabla_1 \cup \dots \cup \nabla_r)$  and  $\Delta^\circ = \nabla_1 + \dots + \nabla_r$ .

To sum up, we have the following sets of polytopes:

$$\begin{array}{cc} N_{\mathbb{R}} & M_{\mathbb{R}} \\ \hline \Delta = \text{Conv}(\{\Delta_1, \dots, \Delta_r\}) & \Delta^\circ = \nabla_1 + \dots + \nabla_r \\ \nabla^\circ = \Delta_1 + \dots + \Delta_r & \nabla = \text{Conv}(\{\nabla_1, \dots, \nabla_r\}) \end{array}$$

This is the mirror duality of Batyrev-Borisov.

## 2.2 Applied to our case

Recall that the polytope associated to the del Pezzo surface of degree 6 is the polytope  $H$  depicted in 2.2.

Since we have a total of 7 points in  $H \cap M$ , the del Pezzo surface is naturally embedded in  $\mathbb{P}^6$ . Taking the projective join of two such, we get the (singular) toric variety  $T$  in  $\mathbb{P}^{11}$ . The polytope associated to this variety is  $H * H$ , the *join* of  $H$  with itself. This is obtained by placing each copy of  $H$  in disjoint hyperplanes and taking the convex hull. Explicitly, the join of two polytopes  $P_1 \subset M_{\mathbb{R}}$  and  $P_2 \subset M_{\mathbb{R}}$  can be defined in the following way: embed  $P_i$  in  $M_{\mathbb{R}} \times M_{\mathbb{R}} \times \mathbb{R}^2$  as  $P_i \times \{e_i\}$ . Then take the convex hull of  $P_1$  and  $P_2$ . In our case, we consider the convex hull of the columns of the matrix

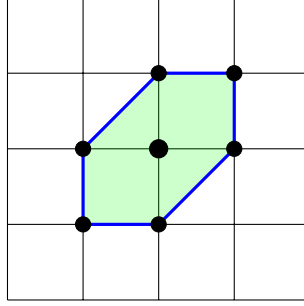


Figure 1: The hexagon, a.k.a the polytope associated to a degree 6 del Pezzo surface.

$$P' = \text{Conv} \left( \begin{bmatrix} 0 & 1 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \right)$$

Note however that  $P'$  is not a reflexive polytope anymore. The origin is a vertex, and there are no interior points. This is solved by considering  $P := 2 \cdot P' - (1, 1, 1, 1, 1)$ . This polytope is now reflexive. Since a polytope and any of its multiples have the same normal fan, we can just as well consider  $P$  as the polytope associated to  $T$ .

To get the same notation as in the previous section, we from now on denote  $P$  by  $\Delta$ . Now  $\Delta$  is the convex hull of the columns of the matrix:

$$\Delta = \text{Conv} \left( \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & -2 & -2 & -1 & -1 \end{bmatrix} \right)$$

Using the program PALP, which is implemented in SAGE, one finds that if one takes  $\Delta_1$  (resp.  $\Delta_2$ ) to be convex hull of the first six columns (resp. six last) and the origin, then  $\Delta_1 + \Delta_2$  is reflexive, which we will, as above, denote by  $\nabla^\circ$ .

**Lemma 2.1.** *The  $f$ -vector of  $\nabla^\circ$  is  $f = (1, 48, 156, 194, 108, 24, 1)$ .*

The associated Cayley polytope  $P^*$  is as above the convex hull of  $\Delta_1 \times \{e_1\}$  and  $\Delta_2 \times \{e_2\}$  in  $N_{\mathbb{R}} \times \mathbb{R}^2$ . Explicitly, it is given by the same matrix as  $\Delta$ , but with  $(1, 0)$  after the first six columns and  $(0, 1)$  after the last six.

The Cayley cone  $C^*$  is just the 7-dimensional cone supported on  $P^*$ . To find  $P$ , one just takes the convex hull of the ray generators of  $C = (C^*)^*$ . As above, one find  $\nabla_1, \nabla_2$  by intersecting with hyperplanes, and then projecting back to  $M_{\mathbb{R}}$ .

**Lemma 2.2.**  *$\nabla_1$  and  $\nabla_2$  are isomorphic lattice polytopes, differing by a translation in the lattice. In particular, they have the same normal fan.*

A calculation in PALP gives the following:

**Lemma 2.3.** *The Hodge numbers associated to this nef partition are  $h^{12} = h^{22} = 19$  (and consequently, they are the same for the dual nef partition).*

Here's a summary of the data of all the polytopes found so far:

Polytope	Dimension	Lattice points	Interior points	Reflexive	f-vector
$\Delta$	5	15	1	Yes	(12, 48, 74, 48, 12)
$\Delta^\circ$	5	87	1	Yes	(12, 48, 74, 48, 12)
$\Delta_1$	3	8	0	No (2)	(7, 12, 7)
$\Delta_2$	3	8	0	No (2)	(7, 12, 7)
$\nabla$	5	27	1	Yes	(24, 108, 194, 156, 48)
$\nabla^\circ$	5	63	1	Yes	(48, 156, 194, 108, 24)
$\nabla_1$	5	14	0	No (2)	(12, 48, 74, 48, 12)
$\nabla_2$	5	14	0	No (2)	(12, 48, 74, 48, 12)

By “No (2)” is ment that  $2 \cdot P$  is reflexive.

### 3 More - details to arrive

- Check all details. In which world do  $P$  live?
- Find explicit description of the nef hypersurfaces.
- Understand concretely the mirror candidate.
- Find subgroup  $G$  of  $\text{Aut } \mathcal{K}$  acting on  $X_t$ . Then find subgroup  $H$  of  $\mathbb{C}^{*n}$  acting on the generic element of the family. Mirror candidate is also  $X_t/H$ . To do this, need to know what kind of singularities  $X_t$  has. The group should permute them as well.
- Iflg Jan, “obvious that  $\text{Aut } \mathcal{K}$  acts on generic fiber  $Y$ .”

## References

- [AC10] Klaus Altmann and Jan Arthur Christophersen. Deforming Stanley-Reisner schemes. *Math. Ann.*, 348(3):513–537, 2010.
- [DN10] Charles F. Doran and Andrey Y. Novoseltsev. Closed form expressions for Hodge numbers of complete intersection Calabi-Yau threefolds in toric varieties. In *Mirror symmetry and tropical geometry*, volume 527 of *Contemp. Math.*, pages 1–14. Amer. Math. Soc., Providence, RI, 2010.