

Algebraic groups and moduli theory

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Abstract

These are notes from the course Algebraic Geometry III. We work over a field of characteristic zero. We start by doing some basic representation theory. Then we introduce algebraic groups. Then we study representations of algebraic groups. Finally we apply this to moduli problems.

1 Introduction and motivation

The construction of parameter spaces is central to algebraic geometry. Say you have some class of objects (e.g. isomorphism classes of elliptic curves, quadrics, planes in a vector space) that you want to correspond to the points of some space S . To do this, one takes the set of objects and divide out by an equivalence relation. This is very subtle, and the very first question one ask is if this question exists at all.

A good paramter space does not exist for curves of genus g for example. This is essentially because you want a family all of whose fibers are isomorphic to be the trivial family, but because of automorphisms, there are nontrivial families with isomorphic fibers. The problem was solved in the 60's by Mumford and Deligne (and their accomplices) by the introduction of *stacks*, which are generalizations of schemes. This allowed them to define $\overline{\mathfrak{M}}_g$, a compactification of the moduli space of genus g curves.

The introduction of stacks is somewhat unsatisfactory, however, in that we loose geometry (there are no points).

The quotient is often a quotient by a group action, and this is often an *algebraic* group action. So the situation is this: we have an action $G \curvearrowright X$ of an algebraic group G on some algebraic space X . Then we ask when the quotient X/G exists, and if it does, in what

sense? The naive approach of taking the points to be the orbits of G does not work as we will see in the examples below.

Let us first consider the affine case. That is, let $X = \operatorname{Spec} A$, where A is some finitely-generated k -algebra. What should the space X/G be? First of all, note that an action on X induce an action on A : if $f \in A$, then we see that $g \cdot f$ is the function defined by $(g \cdot f)(x) = f(g \cdot x)$ for all x . We want functions on X/G to correspond to functions that are constant on G -orbits. Thus we are led to consider A^G , namely the invariant subring of A .

Problem! Even if A is finitely generated, A^G is *usually* not!

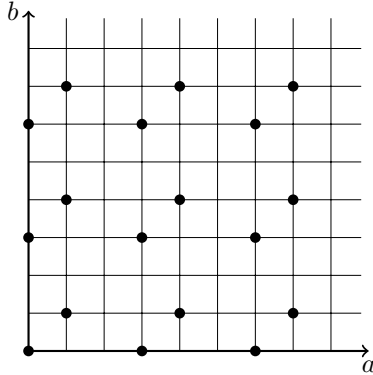
We end this section with a collection of examples of what can happen and what can go wrong when studying quotients. First an example where things are good.

Example 1.1. Consider the representation of \mathbb{Z}_n on $\operatorname{SL}_2(\mathbb{C})$ given by

$$\mathbb{Z}_n \ni [m] \mapsto \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix} \in \operatorname{SL}_2(\mathbb{C}),$$

where ζ_n is an n 'th root of unity. The action on the coordinate ring $\mathbb{C}[x, y]$ is given by $x \mapsto \zeta_n x$ and $y \mapsto \zeta_n^{-1} y$. Thus the action on a monomial is given by $x^a y^b \mapsto \zeta_n^{a-b} x^a y^b$. This monomial is invariant if and only if $a \equiv b \pmod{n}$.

Let us plot these monomials for $n = 3$:



As a semigroup, this is generated by the monomials x^n, xy and y^n , so that $\mathbb{C}[x, y]^{\mathbb{Z}_n} = \mathbb{C}[x^n, xy, y^n] = \mathbb{C}[u, v, w]/(uw - v^n)$. This is a toric surface singularity of type A_{n-1} .

The map $\mathbb{C}^2 \rightarrow \operatorname{Spec} \mathbb{C}[x, y]^{\mathbb{Z}_n}$ is of degree n , since the field extension $\mathbb{C}(x^n, xy, y^n) = \mathbb{C}(x^n, xy) \hookrightarrow \mathbb{C}(x, y)$ is of degree n . The only “bad” point is the origin, which have only one fiber. ★

Example 1.2. Let $G = \mathbb{C}^*$ and let $X = \mathbb{C}^n$. Consider the representation

$$\mathbb{C}^* \ni \lambda \mapsto \begin{pmatrix} \lambda & 0 & \\ 0 & \ddots & 0 \\ & 0 & \lambda \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}).$$

We have an induced action on the coordinate ring, sending x_i to λx_i . But the only invariants are the constants! Thus the “quotient” is just the constant map $\mathbb{C}^n \rightarrow \{p\}$. This is not what we want.

Let us analyze the orbits of G . If $x \in \mathbb{C}^n$ is non-zero, then the orbit $G \cdot x$ is the line spanned by x minus the origin. If $x = \vec{0}$, then the orbit is just the origin. Thus most orbits are not closed, and we notice that the origin is in the closure of all orbits.

This suggests a possible solution. If we instead consider $\mathbb{C}^n \setminus \{0\}$, the orbits are disjoint and closed in the subspace topology. ★

Example 1.3. Let again $G = \mathbb{C}^*$, but consider now the representation given by

$$\mathbb{C}^* \ni \lambda \mapsto \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix} \in \mathrm{GL}_4(\mathbb{C}).$$

The action extends to the coordinate ring $R = \mathbb{C}[x, y, z, w]$, and the invariants are $R^G = \mathbb{C}[xz, xw, yz, yw]$. Then $R^G \approx \mathbb{C}[t_1, t_2, t_3, t_4]/(t_1 t_4 - t_2 t_3)$. This is a hypersurface in \mathbb{C}^4 .

We have a commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^4 & & \\ \downarrow & \searrow \phi & \\ \mathrm{Spec} R^G & \hookrightarrow & \mathbb{C}^4 \end{array}$$

Here $\phi(x, y, z, w) = (xz, xw, yz, yw)$. Then we look at $\phi^{-1}(0)$. A small calculation reveals that this is the union of the xy -plane and the zw -plane, intersecting in the origin. The problem here is that the fiber over the origin is 2-dimensional, whereas all the other fibers are 1-dimensional. In particular the map is not flat. ★

Example 1.4. An even more serious problem is that R^G might not be finitely generated. It is difficult to construct examples where this happens, though. Masayoshi Nagata constructed a counterexample as follows: let \mathbb{G}_a^n act on \mathbb{A}^{2n} (in some specified way) and let $G = \mathbb{G}_a^{n-r}$ be a general linear subspace of codimension r . He proved

that if $r = 3$ and $n = 16$, then $R = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]^G$ is not finitely generated. See [3] for a readable account of the counterexample (and related problems). ★

It turns out that the “magic” property we want of an algebraic group is that it is “linearly reductive” - a technical property implying that R^G is finitely-generated.

The plan ahead is as follows. The next section will talk loosely about representation theory in general in order for the reader to get a feel for the subject. Section 3 will be about algebraic groups and some examples. Section 4 is longer and is about representations of algebraic groups. This is the technical section where linear reductivity is introduced.

2 Representation theory in general

Let V be a vector space. Briefly, a *representation* of any group G on V is just a group homomorphism $\rho : G \rightarrow \mathrm{GL}(V)$.

Example 2.1. The *trivial representation* is given by sending every $g \in G$ to the identity transformation. ★

Example 2.2. Suppose G is a finite group. Then there is an embedding $G \hookrightarrow S_n$, and every element of S_n can be represented by permutation matrices (that is, matrices M_g such that $M_e e_i = e_{g(i)}$ for all $g \in G$). This defines a representation of G in k^n . ★

Example 2.3. Suppose G acts on a (finite) set X . Let V be the vector space with basis identified with the elements of X . Then G acts on V by linearity: for each $g \in G$, $\rho(g)$ is the linear map sending e_x to e_{gx} . Such representations are called *permutation representations*. ★

A *morphism of representations* $(\rho, V), (\rho', W)$ consists of commutative diagrams

$$\begin{array}{ccc} V & \xrightarrow{\psi} & W \\ \rho(g) \downarrow & & \downarrow \rho'(g) \\ V & \xrightarrow{\psi} & W \end{array}$$

for each $g \in G$. Thus, if ψ is invertible, this says that the linear operators $\rho(s), \rho'(s)$ are similar.

3 Algebraic groups

Algebraic groups are group objects in the category of affine varieties. More precisely:

Definition 3.1. Let A be a finitely generated k -algebra. An *affine algebraic group* is a quadruple $(A, \mu_A, \epsilon, \iota)$ where $\mu_A : A \rightarrow A \otimes_k A$ (the *coproduct*), $\epsilon : A \rightarrow k$ (the *coidentity*), $\iota : A \rightarrow A$ (the *coinverse*) are k -algebra homomorphisms, satisfying the following conditions:

1. Coassociativity. The following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\mu_A} & A \otimes_k A \\ \mu_A \downarrow & & \downarrow \text{id}_A \otimes \mu_A \\ A \otimes_k A & \xrightarrow{\mu_A \otimes \text{id}_A} & A \otimes_k A \end{array}$$

2. The following diagram commutes:

$$\begin{array}{ccccc} & & k \otimes_k A & & \\ & \nearrow \epsilon \otimes \text{id}_A & & \searrow \simeq & \\ A & \xrightarrow{\mu} & A \otimes_k A & & A \\ & \searrow \text{id}_A \otimes \epsilon & & \nearrow \simeq & \\ & & A \otimes_k k & & \end{array}$$

and is equal to the identity.

3. Coinverse. The following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{\epsilon} & k & & \\ \downarrow \mu & & \downarrow & & \\ A \otimes_k A & \xrightarrow{\text{id}_A \otimes \iota} & A \otimes_k A & \xrightarrow{\cdot} & A \end{array}$$

Here the right arrow is the morphism making A a k -algebra. The last arrow in the lower sequence is multiplication in A . ■

Example 3.2. If $G = \text{GL}_n$, then $A = k[T_{ij}, \det T]$. Then μ_A is given by

$$T_{ij} \mapsto \sum_{h=1}^n T_{ih} \otimes T_{hj}.$$

The coinverse is given by the usual Cramer's rule. Also $\epsilon(T_{ij}) = \delta_{ij}$. ★

Example 3.3. A special case of the previous example is the one-dimensional torus $\mathbb{G}_m = \text{Spec } k[x, x^{-1}]$. Here $\mu_A(x) = x \otimes x$ and $\iota(x) = x^{-1}$. The coinverse is evaluation at 1, i.e. the map sending $x \mapsto 1$. ★

Example 3.4. If $G = \mathbb{G}_a = (\mathbb{A}^1, +) = \text{Spec } k[X]$, then $\mu_A(X) = X \otimes 1 + 1 \otimes X$. The coidentity is $\epsilon(X) = 0$, and the coinverse is $\iota(X) = -X$. ★

Example 3.5. Let $A = k[s]$ be the polynomial ring in one variable. This is the coordinate ring of \mathbb{A}_k^1 . We can define

$$\mu(s) = s \otimes 1 + 1 \otimes s.$$

Also, $\epsilon(s) = 0$, and $\iota(s) = -s$. ★

Definition 3.6. An *action* of an affine algebraic group $G = \text{Spec } A$ on an affine variety $X = \text{Spec } R$ is a morphism $G \times X \rightarrow X$ defined dually by a k -algebra morphism $\mu_R : R \rightarrow R \otimes_k A$ satisfying the following two conditions.

1. The following diagram is commutative:

$$\begin{array}{ccc} R & \xrightarrow{\mu_R} & R \otimes_k A \\ & \searrow \text{id}_R & \downarrow \text{id}_R \otimes \epsilon \\ & & R \simeq R \otimes_k k \end{array}$$

2. The diagram

$$\begin{array}{ccc} R & \xrightarrow{\mu_R} & R \otimes_k A \\ \mu_R \downarrow & & \downarrow \mu_R \otimes \text{id}_A \\ R \otimes_k A & \xrightarrow{\text{id}_R \otimes \mu_A} & R \otimes_k A \otimes_k A \end{array}$$

■

4 Representations of algebraic groups

Let $G = \text{Spec } A$ be an affine algebraic group over a field k .

Definition 4.1. An *algebraic representation* of G is a pair (V, μ_V) consisting of a k -vector space V and a k -linear map $\mu_V : V \rightarrow V \otimes_k A$ satisfying the following two conditions:

1. The diagram

$$\begin{array}{ccc}
 V & \xrightarrow{\mu_V} & V \otimes_k A \\
 & \searrow \text{id}_V & \downarrow \text{id}_V \otimes \epsilon \\
 & & V \simeq V \otimes_k k
 \end{array} \tag{1}$$

is commutative.

2. The diagram

$$\begin{array}{ccc}
 V & \xrightarrow{\mu_V} & V \otimes_k A \\
 \mu_V \downarrow & & \downarrow \mu_V \otimes \text{id}_A \\
 V \otimes_k A & \xrightarrow{\text{id}_V \otimes \mu_A} & V \otimes_k A \otimes_k A
 \end{array}$$

is commutative. Here μ_A is the coproduct in the coordinate ring of G . ■

Remark. In lieu of Definition 3.6, we see that any action of an algebraic group G on an affine variety $X = \text{Spec } R$ is a representation of G on the infinite-dimensional k -vector space $R = \Gamma(X, \mathcal{O}_X)$.

Remark. Mumford and Fogarty calls this a dual action of G on V , in their famous “Geometric Invariant Theory” [4].

We often drop the subscript from μ_V unless confusion may arise. The same comment applies to tensor products. They will always be over the ground field unless otherwise stated. We will sometimes refer to a representation (V, μ_V) sometimes as “a representation $\mu : V \rightarrow V \otimes A$ ” and sometimes as just “a representation V ”.

Definition 4.2. Let $\mu : V \rightarrow V \otimes A$ be a representation of $G = \text{Spec } A$. Then:

1. A vector $x \in V$ is said to be G -invariant if $\mu(x) = x \otimes 1$.
2. A subspace $U \subset V$ is called a *subrepresentation* if $\mu(U) \subseteq U \otimes A$. ■

Proposition 4.3. Every representation V of G is locally finite-dimensional. Precisely: every $x \in V$ is contained in a finite-dimensional subrepresentation of G .

Proof. Write $\mu(x)$ as a finite sum $\sum_i x_i \otimes f_i$ for $x_i \in V$ and linearly independent $f_i \in A$. This we can always do, by definition of tensor

product and bilinearity. Let U be the subspace of V spanned by the vectors x_i .

Now, by the commutativity of the diagram (1) it follows that

$$x = \sum_i \epsilon(f_i) x_i.$$

By the commutativity of the second diagram in the definition, it follows that

$$\sum_i \mu_V(x_i) \otimes f_i = \sum_i x_i \otimes \mu_A(f_i) \in U \otimes_k A \otimes_k A.$$

Because each term of the right-hand-side is contained in $U \otimes A \otimes A$, it follows that $\mu_V(x_i)$ is contained in U since the f_i are linearly independent.

Thus x is contained in the finite-dimensional representation $\mu_V|_U : U \rightarrow U \otimes A$. \square

We can classify representations of \mathbb{G}_m easily. They are all direct sums of “weight m ”-representations, that is, representations of the form

$$V \rightarrow V \otimes k[t, t^{-1}], v \mapsto v \otimes t^m.$$

Proposition 4.4. *Every representation V of \mathbb{G}_m is a direct sum $V = \oplus_{m \in \mathbb{Z}} V_{(m)}$, where each $V_{(m)}$ is a subrepresentation of weight m .*

Proof. For each $m \in \mathbb{Z}$, define

$$V_{(m)} = \{v \in V \mid \mu(v) = v \otimes t^m\}.$$

This is a subrepresentation of V : we must see that $\mu(V_{(m)}) \subset U \otimes A$, but this is true by construction. It is also clear that it has weight m . Next we show that $V = \oplus_{m \in \mathbb{Z}} V_{(m)}$. Write

$$\mu(v) = \sum_{m \in \mathbb{Z}} v_m \otimes t^m \in V \otimes k[t, t^{-1}].$$

Using the first condition in the definition of a representation, we get that $v = \sum_{m \in \mathbb{Z}} \epsilon(t^m) v_m$. It remains to check that each $v_m \in V_{(m)}$ (we can forget the scalars $\epsilon(t^m)$). But from definition ii), it follows that

$$\sum \mu(v_m) \otimes t^m = \sum v_m \otimes t^m \otimes t^m,$$

so that indeed $\mu(v_m) = v_m \otimes t^m$, as wanted. \square

Example 4.5. An action of \mathbb{G}_m on $X = \operatorname{Spec} R$ is equivalent to specifying a grading

$$R = \bigoplus_{m \in \mathbb{Z}} R_{(m)} \quad R_{(m)} R_{(n)} \subset R_{(m+n)}.$$

The invariants under this action are thus the homogeneous elements of weight zero, that is, the subring $R_{(0)}$. Moreover, we have a special operator. There is a linear endomorphism E of R that sends $f = \sum f_m \mapsto \sum m f_m$, and it is a derivation of R , called the Euler operator. We have $R^{\mathbb{G}_m} = \ker E$.

To see that E is a derivation, we must check that $E(fg) = fE(g) + gE(f)$. The operator is homogeneous, so it is enough to check on homogeneous elements. So let f_m, g_n be of degree m, n , respectively. Then

$$E(f_m g_n) = (m+n) f_m g_n = g_n (m f_m) + f_m (n g_n) = g_n E(f_m) + f_m E(g_n),$$

as wanted. ★

A character is a homomorphism $G \rightarrow \mathbb{C}^*$, so we have a corresponding notion of characters in this “dual” world:

Definition 4.6. Let $G = \operatorname{Spec} A$ be an affine algebraic group. A 1-dimensional character of G is an element $\chi \in A$ satisfying

$$\mu_A(\chi) = \chi \otimes \chi \quad \iota(\chi)\chi = 1.$$

■

Example 4.7. Let $G = \mathbb{G}_m = \operatorname{Spec} k[x, x^{-1}]$. Then since $\iota(x) = x^{-1}$ and $\mu_A(x) = x \otimes x$, we see that the elements x^n for $n \in \mathbb{Z}$ are characters. In fact they are all: ★

Lemma 4.8. *The characters of the general linear group $\operatorname{GL}(n) = \operatorname{Spec} k[x_{ij}, \det X]$ are precisely the integer powers of the determinant $(\det X)^n$ for $n \in \mathbb{Z}$.*

Proof. Characters are in particular invertible elements of the coordinate ring. The only invertible elements of $\Gamma(\operatorname{GL}(n), \mathcal{O}_{\operatorname{GL}(n)})$ are scalar multiples of integer powers of the determinant. But the only λ for which $\lambda(\det X)^n$ is a character is when $\lambda = 1$, by the first equation in the definition of character. □

Definition 4.9. Let χ be a character of an affine algebraic group G , and let V be a representation of G . A vector $v \in V$ satisfying

$$\mu_V(v) = v \otimes \chi$$

is called a *semi-invariant* of G with weight χ . The semi-invariants of V belonging to a given character χ form a subrepresentation $V_\chi \subset V$ of V . (easy to show) ■

We will often change the point of view depending upon the situation. Sometimes we think of a representation of an algebraic group as a k -linear map $V \rightarrow V \otimes_k A$ satisfying some axioms, and sometimes we think of a representation as a group G acting on a vector space V in the usual fashion.

Proposition 4.10. *Let $\mu : V \rightarrow V \otimes_k A$ be a representation of an algebraic group G . Let $g \in G(k)$ be a k -valued point and $\mathfrak{m}_g \subseteq A$ the corresponding maximal ideal. Denote by $\rho(g)$ the composition*

$$V \xrightarrow{\mu} V \otimes_k A \xrightarrow{\text{mod } \mathfrak{m}_g} V \otimes_k k \simeq V.$$

Then, if $A = \Gamma(G, \mathcal{O}_G)$ is an integral domain, a vector $v \in V$ such that $\rho(g)(v) = v$ for all $g \in G(k)$ is a G -invariant.

Proof. We need to check that $\mu(v) = v \otimes 1$. First, since G is the spectrum of a finitely generated k -algebra, we can write A as $k[y_1, \dots, y_m]/I$ for some prime ideal I . Then the same trick as in the proof of Proposition 4.3 works. Write $\mu(v) = \sum v_i \otimes f_i$ with $f_i \in A$ for all i . Since the composition is the identity, we have that $f_i \equiv 1 \pmod{\mathfrak{m}_g}$ for all $g \in G$. This implies that $f_i - 1$ is contained in the Jacobson radical of A . But A is an integral domain, so $f_i - 1 = 0$. □

Thus, in a sense, the two notions of G -invariance coincides.

4.1 Algebraic groups and their Lie spaces

In the spirit of Grothendieck (or maybe the spirit of Newton¹), we will consider infinitesimal neighbourhoods of the identity $e \in G = \text{Spec } A$.

Definition 4.11. Let R be a k -algebra and M an R -module. An M -valued derivation is a k -linear map $D : R \rightarrow M$ satisfying the Leibniz rule $D(xy) = xD(y) + yD(x)$ for $x, y \in R$. ■

The set of M -valued derivations is also an R -module, denoted by $\text{Der}_k(R, M)$. This is used to define tangent spaces in algebraic geometry as follows: Let $p \in \text{Spec } A = X$ be a closed point. Then we have a local ring $\mathcal{O}_{X,p}$ and a quotient map $\mathcal{O}_{X,p} \rightarrow \mathcal{O}_{X,p}/\mathfrak{m}_p \simeq k$. Then the k -module $(\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$ is called the *Zariski tangent space* of $p \in X$. In fact:

¹Or Leibniz, or Lie...

Proposition 4.12. *We have isomorphisms of $\mathcal{O}_{X,p}$ -modules:*

$$\mathrm{Der}_k(\mathcal{O}_{X,p}, k) \simeq (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee \simeq \mathrm{Hom}_{k\text{-alg}}(\mathcal{O}_{X,p}, k[\epsilon])$$

Proof. We first prove the existence of the first isomorphism.

Send $D \in \mathrm{Der}_k(\mathcal{O}_{X,p}, k)$ to $D|_{\mathfrak{m}_p}$. This is well-defined if D vanishes on \mathfrak{m}_p^2 . But if $m_1, m_2 \in \mathfrak{m}_p$, then

$$D(m_1 m_2) = m_1 D(m_2) + m_2 D(m_1) = 0 \in \mathcal{O}_{X,p}/\mathfrak{m}_p.$$

So the map is well-defined.

We have a map in the opposite direction as well. Let $\ell : \mathfrak{m}_p/\mathfrak{m}_p^2 \rightarrow k$ be a linear functional on the k -vector space $\mathfrak{m}_p/\mathfrak{m}_p^2$. Define D_ℓ to be the k -linear map $D_\ell : \mathcal{O}_{X,p} \rightarrow k$ given by

$$D_\ell(f) = \begin{cases} 0 & \text{if } f \in k \\ \ell(f) & \text{if } f \in \mathfrak{m}_p \\ 0 & \text{if } f \in \mathfrak{m}_p^2. \end{cases}$$

I now claim that this is a derivation. Write $f, g \in \mathcal{O}_{X,p}$ as $c + m, c' + m'$ where c is outside the maximal ideal and $m \in \mathfrak{m}_p$. Then

$$D_\ell(fg) = D_\ell(cc' + cm' + c'm + m'm') = cD_\ell(m') + c'D_\ell(m),$$

by definition of D_ℓ .

It is easy to check that these two maps are inverse to each other.

We give an isomorphism between the first and the third module. We begin first by studying the elements of $\mathrm{Hom}_{k\text{-alg}}(\mathcal{O}_{X,p}, k[\epsilon])$. As a k -vector space, $k[\epsilon] = k \oplus \epsilon k$. Thus every $f \in \mathrm{Hom}_{k\text{-alg}}(\mathcal{O}_{X,p}, k[\epsilon])$ can be written as $f = \lambda(f) + \epsilon \delta(f)$. But we have a diagram

$$\begin{array}{ccc} & k & \\ \swarrow & & \searrow \\ \mathcal{O}_{X,p} & \xrightarrow{f} & k[\epsilon] \\ \searrow & & \swarrow \\ & k & \end{array}$$

Thus $\lambda(f)(x) = x \pmod{\mathfrak{m}_p} = x(p)$. Thus the “constant part” of f is determined by default. Furthermore, since f is an algebra morphism, we must have that

$$\begin{aligned} f(xy) &= (xy)(p) + \epsilon \delta(f)(xy) \\ f(x)f(y) &= (x(p) + \epsilon \delta(f)(x))(y(p) + \epsilon \delta(f)(y)) \\ &= x(p)y(p) + \epsilon (x(p)\delta(f)(y) + y(p)\delta(f)(x)). \end{aligned}$$

Thus we see that the function $\delta(f) : \mathcal{O}_{X,p} \rightarrow k$ is a derivation. We thus have a k -linear map $\mathrm{Hom}_{k\text{-alg}}(\mathcal{O}_{X,p}, k[\epsilon]) \rightarrow \mathrm{Der}_k(\mathcal{O}_{X,p}, k)$ by $f \mapsto \delta(f)$. The inverse map is given by the function $D \mapsto (x \mapsto x(p) + \epsilon D(x))$. \square

Remark. *This smells of some kind of Taylor expansion. In fact, this can be used to compute tangent spaces of Hilbert schemes. Let $X \subset \mathbb{P}^n$ be a projective variety, and $[X] \in \mathrm{Hilb}_{P(t)}$ the corresponding point in the Hilbert scheme. By the above, the set of morphisms $\mathrm{Spec} k[\epsilon] \rightarrow \mathrm{Hilb}_{P(t)}$ are in one to one correspondence with the tangent space of $\mathrm{Hilb}_{P(t)}$ at $[X]$. But by the universal property of the Hilbert scheme, this set is in one-one correspondence with the set of flat families $\mathfrak{X} \rightarrow \mathrm{Spec} k[\epsilon]$. But these are classified by global sections of $\mathcal{N}_{X/\mathbb{P}^n}$, the normal bundle.*

In fact, the module of derivations is what is called a *corepresentable functor*. There is an R -module $\Omega_{R/k}$ such that $\mathrm{Der}_k(R, M) = \mathrm{Hom}_R(\Omega_{R/k}, M)$, functorially in M . This is the module of *Kähler differentials*.

Example 4.13. We compute the Zariski tangent space of SL_n at the identity element. Note that $\mathrm{SL}_n = \mathrm{Spec} k[x_{ij}]/(\det X - 1) = \mathrm{Spec} R$. Let $I = (\det X - 1)$, the principal ideal generated by $\det X - 1$. There is an exact sequence

$$0 \rightarrow \mathrm{Der}_k(R, k) \rightarrow \mathrm{Der}_k(k[x_{ij}], k) \rightarrow \mathrm{Hom}_k(I/I^2, k)$$

of R -modules. The first map is just the pullback of the projection $k[x_{ij}] \rightarrow R$. The second map send $D \in \mathrm{Der}_k(k[x_{ij}], k)$ to the restriction $D|_{I/I^2}$. This does not a priori make sense. But since we are considering homomorphisms to $k = R/\mathfrak{m}_e$, it is easy to see that any D vanishes on I^2 . Thus the Zariski tangent space of SL_n at the identity can be computed as the kernel of the map to the right. The middle term is spanned as a k -vector space by $\frac{\partial}{\partial x_{ij}} \Big|_{X=I_n}$, so that we can identify it with $\mathrm{End}(V)$.

Since I is principal, it follows that $\mathrm{Hom}_k(I/I^2, k) \simeq \mathrm{Hom}_k(A, k) \simeq k$. Doing all the identifications, the right map is then given by $D \mapsto D(\det M)|_{I_n}$. A computation gives that $D(\det M)|_{I_n} = \mathrm{tr} M$, so that the kernel is given by the matrices with trace zero. \star

A derivation is a k -linear map vanishing at \mathfrak{m}_p^2 . The generalization of this are *local distributions*:

Definition 4.14. Let $\mathfrak{m}_p \in \mathrm{Spec} X$. A *local distribution with support* $\mathfrak{m}_p \in X$ is a k -linear map $\alpha : R \rightarrow k$ with the property that $\alpha(\mathfrak{m}_p^N) = 0$ for some $N \in \mathbb{N}$.

The minimal N such that $\alpha(\mathfrak{m}_p^{N+1}) = 0$ is called the *degree* of the distribution. Thus the distributions of degree 1 are the derivations (up to isomorphism). ■

We can identify the set of local distributions of degree $\leq d$ with k -module $(R/\mathfrak{m}_p^{d+1})^\vee$. The surjections $R/\mathfrak{m}_p^{d+1} \rightarrow R/\mathfrak{m}_p^d$ induce injections $(R/\mathfrak{m}_p^d)^\vee \hookrightarrow (R/\mathfrak{m}_p^{d+1})^\vee$. We can thus identify the set of distributions supported at p with $\varinjlim (R/\mathfrak{m}_p^i)^\vee = \bigcup_i (R/\mathfrak{m}_p^i)^\vee$.

Lemma 4.15. *For a k -linear map $\alpha : R \rightarrow k$, the following are equivalent:*

1. α is a derivation of $X = \operatorname{Spec} R$ at the point $p \in X$.
2. α is a local distribution supported at $p \in X$ of degree 1 satisfying $\alpha(1) = 0$.

4.1.1 The distribution algebra

If $G = \operatorname{Spec} A$ is an affine algebraic group with coordinate ring A , then denote by $\mathcal{H}(G)$ the vector space of distributions $\alpha : A \rightarrow k$ supported at $e \in G$. The Zariski tangent space at $e \in G$ is called the *Lie space* of G and is denoted by $\mathfrak{g} \subset \mathcal{H}(G)$.

Definition 4.16. Let $\alpha, \beta \in \mathcal{H}(G)$. Then we define the *convolution product* $\alpha \star \beta$ to be the composition

$$A \xrightarrow{\mu_A} A \otimes_k A \xrightarrow{\alpha \otimes \beta} k \otimes_k k \simeq k.$$

■

Lemma 4.17. *The convolution product $\alpha \star \beta$ is again a local distribution supported at the identity with*

$$\deg \alpha \star \beta \leq \deg \alpha + \deg \beta.$$

Proof. The fact that $(e, e) \mapsto e$ under $m : G \times G \rightarrow G$, is equivalent to

$$\mu_A^{-1}(A \otimes \mathfrak{m} + \mathfrak{m} \otimes A) = \mathfrak{m},$$

from which it follows that $A \otimes \mathfrak{m} + \mathfrak{m} \otimes A = \mu_A(\mathfrak{m})$. Since μ_A is a ring homomorphism, we have

$$\mu_A(\mathfrak{m}^{a+b+1}) \subset \sum_{i+j=a+b+1} \mathfrak{m}^i \otimes \mathfrak{m}^j.$$

Taking $a = \deg \alpha$ and $\beta = \deg \beta$ proves the lemma. □

Note that the map $\epsilon : A \rightarrow k$ corresponding to $e \in G$ is a distribution of degree zero.

Lemma 4.18. *The structure map $\epsilon : A \rightarrow k$ from Definition 3.1 (“evaluation at the identity”) is an identity element for the convolution product.*

Proof. This follows from the following diagram:

$$A \xrightarrow{\mu_A} A \otimes_k A \xrightarrow{\epsilon \otimes \text{id}} k \otimes A \xrightarrow{\text{id} \otimes \alpha} k \otimes k \simeq k.$$

The composition is equal to $\epsilon \star \alpha$, but by Part 2 of Definition 3.1, it is also equal to α . \square

It follows from coassociativity that \star is associative (easy diagram chase). This makes $\mathcal{H}(G)$ into an associative algebra over k , called the *distribution algebra* of the algebraic group G .

Proposition 4.19. *The distribution algebra $\mathcal{H}(G)$ is a Lie algebra with bracket*

$$[\alpha, \beta] = \alpha \star \beta - \beta \star \alpha.$$

Proof. Same proof as in the matrix case. \square

Example 4.20. Let $G = \mathbb{G}_m$, the multiplicative group. The derivations are just scalar multiples of $E = \frac{\partial}{\partial t} \Big|_e$.

I claim that the distribution algebra $\mathcal{H}(G)$ is generated by E as an algebra. That is, distributions of degree n are precisely scalar multiples of $E \star E \cdots \star E$. Since $k[t, t^{-1}]_d = \langle t^d \rangle_k$, it is clear that the set of distributions of degree exactly n is one-dimensional. Hence they are multiples of $E^{\star n}$. \star

Now let $\mu_V : V \rightarrow V \otimes A$ be an algebraic representation, with associated representation $\rho : G \rightarrow \text{GL}(V)$. Let $\alpha \in \mathcal{H}(G)$ be a local distribution. Then composition

$$V \xrightarrow{\mu_V} V \otimes_k A \xrightarrow{\text{id}_V \otimes \alpha} V \otimes k \simeq V$$

is an endomorphism of V , which we denote by $\tilde{\rho}(\alpha) \in \text{End}(V)$. Clearly the map $\alpha \mapsto \tilde{\rho}(\alpha)$ is linear in α . It is also associative, so we get a ring homomorphism

$$\tilde{\rho} : \mathcal{H}(G) \rightarrow \text{End } V. \quad (2)$$

Thus V is a (non-commutative) $\mathcal{H}(G)$ -module (if $\alpha : A \rightarrow k$ is a distribution and $v \in V$ a vector, then $\alpha \cdot v$ is $\tilde{\rho}(\alpha)(v)$).

Consider the action of G on itself by conjugation: $G \times G \rightarrow G$, $(g, h) \mapsto ghg^{-1}$. Since $e \in G$ is fixed under conjugation, this induces an action on each quotient A/\mathfrak{m}^n , and on its dual space. It follows that $\mathcal{H}(G)$ becomes a linear representation of G .

Given a representation $\rho : G \rightarrow \mathrm{GL}(V)$, the space $\mathrm{End} V$ also becomes a linear representation by mapping $g \in G$ to $(T \mapsto \rho(g)T\rho(g)^{-1}) \in \mathrm{GL}(\mathrm{End}(V))$.

Lemma 4.21. *With respect to these actions, we have that the map*

$$\tilde{\rho} : \mathcal{H}(G) \rightarrow \mathrm{End}(V)$$

is a homomorphism of G -representations.

Proof. We must show that the map $\tilde{\rho}$ is G -equivariant. [[HOW TO DO THIS??] \square

In particular, the Lie space $\mathfrak{g} \subset \mathcal{H}(G)$ is a subrepresentation of $\mathcal{H}(G)$. This is called the *adjoint representation* and is denoted by $\mathrm{Ad} : G \rightarrow \mathrm{GL}(\mathfrak{g})$.

4.1.2 The Casimir operator

Let $\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow k$ be an inner product on the Lie algebra \mathfrak{g} of G . That is, κ is a symmetric and nondegenerate bilinear form on \mathfrak{g} . We will assume that κ is invariant under the adjoint representation $\mathrm{Ad} : G \rightarrow \mathrm{GL}(\mathfrak{g})$. This just means that for $X, Y \in \mathfrak{g}$, we have $\kappa(g \cdot X, g \cdot Y) = \kappa(X, Y)$.

Definition 4.22 (The Casimir element). Let κ be as above. Let $X_1, \dots, X_N \in \mathfrak{g}$ be a basis for \mathfrak{g} and let $X'_1, \dots, X'_N \in \mathfrak{g}$ be its dual basis with respect to κ . Then the distribution

$$\Omega := X_1 \star X'_1 + \dots + X_N \star X'_N \in \mathcal{H}(G)$$

is called the *Casimir element* over G with respect to κ . \blacksquare

Proposition 4.23. *The Casimir element Ω is independent of choice of basis.*

Proof. Suppose $\{Y_1, \dots, Y_N\}$ be another basis and let $\{Y'_1, \dots, Y'_N\}$ be its dual basis. Then

$$Y_i = \sum_{j=1}^N a_{ij} X_j$$

and

$$Y'_i = \sum_{j=1}^N a'_{ij} X'_j,$$

for some matrices $A = (a_{ij})$ and $A' = (a'_{ij})$ satisfying $A^T A' = I$. To see this, compute $\kappa(Y_i, Y'_j)$.

Then:

$$\begin{aligned} \sum_{i=1}^N Y_i \star Y'_i &= \sum_{i=1}^N \left(\sum_{j=1}^N a_{ij} X_j \right) \star \left(\sum_{k=1}^N a'_{ik} X'_k \right) \\ &= \sum_{j,k}^N \left(\sum_{i=1}^N a_{ij} a'_{ik} \right) X_j \star X'_k \\ &= \sum_{j,k} \delta_{jk} X_j \star X'_k = \Omega. \end{aligned}$$

This proves the statement. \square

Remark. *This can also be seen as follows: the non-degenerate form κ gives a canonical isomorphism of \mathfrak{g} with \mathfrak{g}^* . This isomorphism is an element of $\text{Hom}(\mathfrak{g}, \mathfrak{g}^*) \simeq \text{Hom}(\mathfrak{g}, k) \otimes_k \mathfrak{g}^* \simeq \mathfrak{g}^* \otimes \mathfrak{g}^* \simeq (\mathfrak{g} \otimes \mathfrak{g})^*$, and writing the isomorphism in one basis gives exactly the Casimir element.*

Recall that $\text{Ad}(g) : \text{GL}(\mathfrak{g}) \rightarrow \text{GL}(\mathfrak{g})$ is the endomorphism of $\text{GL}(\mathfrak{g})$ given by $v \mapsto dc_g(v)$ where c_g is conjugation by g . Since the inner product κ is assumed to be G -invariant, the sets

$$\{\text{Ad}(g)(X_1), \dots, \text{Ad}(g)(X_n)\}$$

and

$$\{\text{Ad}(g)(X'_1), \dots, \text{Ad}(g)(X'_n)\}$$

are again dual bases. We deduce from this and the previous proposition that:

Corollary 4.24. *The Casimir element $\Omega \in \mathcal{H}(G)$ is invariant under the action of G on the distribution algebra $\mathcal{H}(G)$.*

Now let $\rho : G \rightarrow \text{GL}(V)$ be any representation of G . This is an $\mathcal{H}(G)$ -module via $\tilde{\rho} : \mathcal{H}(G) \rightarrow \text{End } V$ from Equation 2. In particular, the Casimir element determined an endomorphism of V , $\tilde{\rho}(\Omega) : V \rightarrow V$, called the *Casimir operator*.

By the Corollary and Lemma 4.21 $\tilde{\rho}(\Omega)$ is invariant under the conjugation action of G on $\text{End } V$. Moreover, since \mathfrak{g} kills the G -invariant $V^G \subset V$, so does the Casimir operator $[[\text{HVA MENES HER??}]]$. We conclude:

Corollary 4.25. *The Casimir operator is a G -endomorphism of each representation V of G and*

$$V^G \subset \ker(\tilde{\rho}(\Omega)).$$

This result will be crucial when proving Hilbert's finiteness theorem.

4.2 Linear reductivity

Definition 4.26. An algebraic group G is said to be *linearly reductive* if, for every epimorphism $\varphi : V \rightarrow W$ of G -representations, the induced map of G -invariants $\varphi^G : V^G \rightarrow W^G$ is surjective. ■

Proposition 4.27. *Every finite group G is linearly reductive.*

Proof. Let $\varphi : V \rightarrow W$ be the given epimorphism of representations. Let $R : V \rightarrow V^G \subset V$ be given by $v \mapsto \sum_{g \in G} g \cdot v$. Let $w \in W^G$. Then it is an easy calculation to check that $\varphi(R(v)) = R(\varphi(v))$, from which it follows that $\varphi(R(v)) = w$ (note that $R|_{W^G} = \text{id}_{W^G}$). □

The homomorphism R above is called the *Reynolds operator*.

Proposition 4.28. *The following are equivalent:*

- i) G is linearly reductive.
- ii) For every epimorphism $V \rightarrow W$ of finite dimensional G -representations, the induced map $V^G \rightarrow W^G$ is surjective.
- iii) If V is any finite-dimensional representation and $U \subseteq V$ is a proper subrepresentation and $\bar{v} \in V/U$ is G -invariant, then the coset $v + U$ (for any lifting of \bar{v}) contains a non-trivial G -invariant vector.

Proof. $i) \Rightarrow ii)$ is trivial. For $ii) \Rightarrow iii)$, apply $ii)$ to the quotient map $V \rightarrow V/U$. Then $V^G \rightarrow (V/U)^G$ is surjective. This implies that for every nonzero $\bar{v} \in (V/U)^G$, there exists a G -invariant $v \in \pi^{-1}(v) = U + \bar{v}$.

$iii) \Rightarrow i)$ is hardest. Suppose $\phi : V \rightarrow W$ is an epimorphism of representations (not necessarily finite-dimensional). Suppose $\phi(v) = w \in W^G$ for some $v \in V$.

By Proposition 4.3 there exists a finite-dimensional subrepresentation $V_0 \subseteq V$ containing v . Now $v \in V_0$ is G -invariant modulo $U_0 := V_0 \cap \ker \phi$ (since $V/\ker \phi \simeq W$ as G -representations), so by $iii)$, there exists a G -invariant vector $v' \in V_0$ such that $v' - v \in U_0$. But $\phi(v') = w$, so $\phi^G : V^G \rightarrow W^G$ is surjective. □

Lemma 4.29. *Direct products of linearly reductive groups are linearly reductive. If $H \subset G$ is a normal subgroup and G is linearly reductive, then so is G/H . Moreover, if both H and G/H are linearly reductive, then so is G .*

Proof. Suppose given an endomorphism of representation of $G \times H$: $V \rightarrow W$. In particular, they are representations of G, H separately, by the rule $g \cdot v = (g, e) \cdot v$. In particular, if an element $w \in W$ is $G \times H$ -invariant, it is also G, H -invariant. Thus by assumption, there is an G, H -invariant $v \in V$ mapping to w . But if something is G, H -invariant, it is also $G \times H$ -invariant, since G, H commute in $G \times H$.

Similarly, every G/H -representation gives a G -representation, by the rule $g \cdot v = \bar{g} \cdot v$, where \bar{g} denotes the class of g in G/H . Now if $w \in W^{G/H}$ is G/H -invariant, then it is by definition G -invariant, and by linear reductivity of G , the map is surjective.

Finally, if both H and G/H are linearly reductive, suppose $\phi : V \rightarrow W$ is a surjection of G -representations. This is also a surjection of H -representations, and since H was linearly reductive, we get that $V^H \rightarrow W^H$ is surjective. It follows that the map ϕ and the vector spaces V, W splits as $(\phi^H, \phi') : V^H \oplus V' \rightarrow W^H \oplus W'$, where H acts trivially on the second factor. This implies that G/H acts on V', W' , and it follows that $V' \rightarrow W'$ is surjective. \square

Proposition 4.30. *Every algebraic torus $(\mathbb{G}_m)^N$ is linearly reductive.*

Proof. By the lemma, it suffices to prove this for $N = 1$. We use Proposition 4.28 iii). By Proposition 4.4, we can write a representation V and a subrepresentation U as

$$V = \bigoplus_{m \in \mathbb{Z}} V_{(m)} \quad \text{and} \quad U = \bigoplus_{m \in \mathbb{Z}} U_{(m)}.$$

Here $U_{(m)} \subset V_{(m)}$. An element $v \in V/U$ is \mathbb{G}_m -invariant if any lifting of v to V lies in $U_{(m)}$ for $m \neq 0$. Thus $v_{(0)}$ is \mathbb{G}_m -invariant and lies in the coset $v + U$. \square

The classical example of a group that is not linearly reductive is the affine line \mathbb{A}^1 under addition:

Example 4.31. Consider the 2-dimensional representation given by

$$\mathbb{G}_a \rightarrow \mathrm{GL}_2, \quad t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

This is a representation by the rules of matrix multiplication. Algebraically, this as follows: let x, y be a basis for V . Then we define a k -linear map $V \rightarrow V \otimes_k k[t]$ by $x \mapsto x \otimes 1$ and $y \mapsto x \otimes t + y \otimes 1$. This extends to a representation of $k[V] = k[x, y]$ in the obvious

way. Then we can define an epimorphism of representations by sending $k[x, y] \rightarrow k[x, y]/(x) \simeq k[y]$. Taking invariants, we get that $k[x, y]^{\mathbb{G}_a} = k[x]$ but $k[y]^{\mathbb{G}_a} = k[y]$, but the map sends x to 0, so is not surjective. ★

The main aim of this section is to prove that SL_n is linearly reductive. From this it will follow that also GL_n is linearly reductive, because it fits into an exact sequence

$$1 \rightarrow k^* \rightarrow \mathrm{GL}_n \rightarrow \mathrm{SL}_n \rightarrow 1$$

because, by Lemma 4.29, the middle term must also be linearly subsection.

Let U be a finite-dimensional vector space. The Lie algebra of $\mathrm{GL}(U)$ is canonically isomorphic to $\mathrm{End}(U)$, and we have a nondegenerate inner product on $\mathrm{End}(U)$ given by the trace:

$$\kappa : \mathrm{End}(U) \times \mathrm{End}(U) \rightarrow k, \quad (f, g) \mapsto \mathrm{tr}(fg).$$

Write $\kappa(f) := \kappa(f, f)$.

Lemma 4.32. *The inner product κ is invariant under the adjoint action of $\mathrm{GL}(U)$. In other words:*

$$\kappa(\alpha f \alpha^{-1}) = \kappa(f)$$

is satisfied for all $f \in \mathrm{End}(U)$ and $\alpha \in \mathrm{GL}(U)$.

Proof. The trace is just the natural map $\mathrm{End}(U) \simeq U \otimes U^* \rightarrow k$ given by $(v, \lambda) \mapsto \lambda(v)$. Under this identification, the adjoint action of $\mathrm{GL}(U)$ on $\mathrm{End}(U)$ is given by

$$g \cdot (v \otimes \lambda) = (gv \otimes (w \mapsto \lambda(g^{-1}w))).$$

Then we see that the trace map is given by

$$g \cdot (v \otimes w) \mapsto \lambda(g^{-1}gv)\lambda(v).$$

□

The Lie algebra $\mathfrak{sl}(U)$ of $\mathrm{SL}(U)$ is the sub Lie-algebra of $\mathfrak{g}(U)$ given by the traceless matrices.

Lemma 4.33. *The inner product κ is non-degenerate when restricted to $\mathfrak{sl}(U)$, and invariant under the adjoint action.*

Proof. κ is non-degenerate on $\text{End}(U)$ and $\mathfrak{sl}(U)$ is the orthogonal complement of the identity element I_U . Thus every element $v \in \text{End}(U)$ can be written uniquely as $s + j$ with $s \in \mathfrak{sl}(U)$ and j a multiple of I_U . Suppose $\kappa(s, s')$ is zero for all $s' \in \mathfrak{sl}(U)$. Since κ is non-degenerate on $\text{End } U$, this implies that s is a multiple of the identity element. But s was assumed to be traceless, so s is zero. \square

Example 4.34. The Casimir element of $\text{SL}(2)$ is

$$\Omega = e \star f + f \star e + \frac{1}{2} h \star h$$

where e, f, h are

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is so because e, f are dual with respect to κ , and h have dual $\frac{1}{2}h$ since $\text{tr}(h^2) = 2$. ★

Proposition 4.35. *For a representation $\rho : \text{SL}(n) \rightarrow \text{GL}(V)$, the following are equivalent:*

1. *The representation is trivial (that is, $\rho(g) = \text{id}_V$ for all g).*
2. $\tilde{\rho}(\Omega) = 0$.
3. $\text{tr } \rho(\Omega) = 0$.

Proof. i) \Rightarrow ii) trivially by Corollary 4.25 once one notices that the representation is trivial if and only if $V^G = V$. ii) implies iii) even more trivially. So we have to show that iii) \Rightarrow i).

Let $T \subset \text{SL}(n)$ be the torus subgroup of diagonal matrices and $\mathfrak{h} \subset \mathfrak{sl}(n)$ its Lie algebra. Then we can decompose V as

$$V = \bigoplus_{\chi \in X(T)} V_{\chi}.$$

Each $\chi : T \rightarrow \mathbb{G}_m$ corresponds to a linear form $\bar{\chi} : \mathfrak{h} \rightarrow k$ with integer coefficients². To see this, note that a homomorphism $T \rightarrow \mathbb{G}_m$ is given by $(t_1, \dots, t_n) \mapsto t^{\sum t_i \lambda_i}$ for some λ_i .

For each $h \in \mathfrak{h}$ we have:

$$\text{tr } \tilde{\rho}(h)^2 = \sum_{\chi \neq 1} (\dim V_{\chi}) \bar{\chi}(h)^2 \in \mathbb{Z}.$$

²Note that since $\text{char } k = 0$, the integer are naturally embedded in k .

Thus if $\text{tr } \tilde{\rho}(h) = 0$ for all $h \in \mathfrak{h}$, we must have $\dim V_\chi = 0$ for all $\chi \neq 1$. In other words, V is the trivial representation of the torus T , or equivalently, $\rho(t)(v) = v$ for all $t \in T \subseteq \text{SL}(n)$. The same argument works for any subgroup conjugate to T , so that V is the trivial representation for the subset of all diagonalizable matrices. But these are dense in $\text{SL}(n)$, so V is trivial as a representation of $\text{SL}(n)$ as well.

Thus the proposition is proved if we can show that $\text{tr } \tilde{\rho}(\Omega) = 0$ implies that $\text{tr } \tilde{\rho}(h) = 0$ for all $h \in \mathfrak{h}$.

We will do this for $\text{SL}(2)$. In this case \mathfrak{h} is spanned by the single matrix h from Example 4.30. The Casimir operator [SKJØNNER IKKE!!!!] \square

4.3 Finite generation

Let G be an algebraic group acting on a polynomial ring S .

Theorem 4.36 (Hilbert). *If G is linearly reductive, then the ring of invariants S^G is finitely generated as a k -algebra.*

Proof. The ring S^G inherits the grading from S , i.e. we have a direct sum composition

$$S^G = \bigoplus_{e=0}^{\infty} S^G \cap S_e.$$

Let $S_+^G = \bigoplus_{e=1}^{\infty} S^G \cap S_e$, and denote by J the ideal generated by S_+^G . By Hilbert's basis theorem, J is generated by finitely many polynomials $f_1, \dots, f_N \in S_+^G$. In other words, we have a surjective S -module map

$$\bigoplus_{i=1}^N S \xrightarrow{(f_1, \dots, f_N)} J \rightarrow 0.$$

Now we claim that in fact S^G is generated as a k -algebra by f_1, \dots, f_N . Let $h \in S^G$, homogeneous. We want to show that $h \in k[f_1, \dots, f_N]$ by induction on $\deg h$. If $\deg h = 0$, this is obvious. If $\deg h > 0$, h belongs to J , also to J^G since $h \in S^G$. View J as a representation of G . The surjective module map above is a map of G -representations, so by linear reductivity, the induced map

$$\bigoplus_{i=1}^N S^G \xrightarrow{(f_1, \dots, f_N)} J^G \rightarrow 0$$

is also surjective. Therefore there exists invariant polynomials $h'_1, \dots, h'_N \in S^G$ such that $h = \sum_{i=1}^N h'_i f_i$. But all of the f_i have $\deg f_i > 0$, we

must have $\deg h'_i < \deg h$. By the inductive hypothesis, we have $h'_i \in k[f_1, \dots, f_N]$, and thus $h \in k[f_1, \dots, f_N]$ also. \square

Suppose now that G acts on a finitely generated k -algebra R . It follows easily from Hilbert's theorem that also R^G is finitely-generated when G is reductive.

Lemma 4.37. *Suppose that an algebraic group G acts on a finitely-generated k -algebra R . Then there exists generators r_1, \dots, r_N of R whose k -linear span $\langle r_1, \dots, r_N \rangle \subset R$ is a G -invariant vector subspace.*

Proof. Let $s_1, \dots, s_M \in R$ be any set of generators. Now each s_i is contained in a finite-dimensional subrepresentation V_i of R . Now extend s_1, \dots, s_M to a basis r_1, \dots, r_N of $\sum_{i=1}^M V_i \subset R$. \square

To paraphrase Mukai, this says that “an affine algebraic variety acted on by an algebraic group can always be equivariantly embedded in an affine space \mathbb{A}^N on which G acts linearly”.

Theorem 4.38. *If a linearly reductive group G acts on a finitely generated k -algebra R , then the invariant ring R^G is finitely generated.*

Proof. Start by choosing generators r_1, \dots, r_N of R as in the lemma. Then we have a surjection $S = k[x_1, \dots, x_N] \rightarrow R$. Taking invariants and using Hilbert's theorem, it follows immediately that R^G is finite-generated. \square

5 The construction of quotient varieties

In this chapter we will actually construct some objects we will call quotients.

5.1 Affine quotients

We begin with an example.

Example 5.1. Let $G = \mathbb{C}^*$ act on k^2 by $t \cdot (x, y) = (tx, t^{-1}y)$. We have three types of orbits.

1. The origin $(0, 0)$ is an orbit consisting of one point.
2. The axes $x = 0, y = 0$ are two other orbits.
3. All other orbits are hyperbolas $xy = a$ with $a \neq 0$.

The action on the coordinate ring is given by $x \mapsto \lambda^{-1}x$ and $y \mapsto \lambda x$. Thus the only invariant is the monomial xy . We thus get a projection map $\phi : \mathbb{A}^2 \rightarrow \mathbb{A}^1$ given by $(x, y) = xy$.

This “quotient map” separates the third type of orbits, but cannot distinguish between the orbits 1. and 2, all of which map to zero. ★

Theorem 5.2. *Suppose that a linearly reductive group G acts on an affine variety X . Given two orbits $O, O' \subset X$, the following conditions are equivalent:*

- i) The closures of O and O' have a common point, $\overline{O} \cap \overline{O'} \neq \emptyset$.*
- ii) O and O' fail to be separated by G -invariants $k[X]^G$.*

Proof. $i \Rightarrow ii$ is clear: any G -invariant function is automatically constant on orbits, and by continuity constant on orbit closures. If the orbit closures intersect, then any G -invariant function is constant on $\overline{O} \cup \overline{O'}$ also.

We prove the converse of $ii \Rightarrow i$. Namely, if $oO \cap \overline{O'} = \emptyset$, then O and O' are separated by $k[X]^G$.

The condition is equivalent to $I(O) + I(O') = k[X]$. Since all three terms are G -invariant, we have a map of G -representations

$$I(O) \oplus I(O') \xrightarrow{+} k[X].$$

Taking G -invariants, we get a surjection $I(O)^G \oplus I(O')^G \rightarrow k[X]^G$ since G is linearly reductive. Thus there exist functions f, f' such that $f + f' = 1$ and $f(O) = 0, f'(O') = 0$. But $f(O') = 1 - f'(O') = 1$, and $f'(O) = 1 - f(O) = 1$, so f and f' separate O and O' . □

Corollary 5.3. *If G is a linearly reductive group acting on an affine variety X , then distinct closed G -orbits are separated by the G -invariants $k[X]^G$.*

Proof. Distinct closed orbits are automatically disjoint. □

Corollary 5.4. *Given an orbit, the set S of orbits with closure meeting this orbit contains exactly one closed orbit. Moreover, this orbit is contained in the closure of every other orbit in the same set.*

Proof. The set of such orbits can contain at most one closed set by the previous corollary. Let O be an orbit with minimal dimension of S . We claim that O is closed. Suppose not. Then $\overline{O} \setminus O$ is a nonempty union of G -orbits in S because $\overline{\overline{O} \setminus O} \cap \overline{O} \neq \emptyset$, and is of smaller dimension, contradiction.

For the second part... □

Consider again the map $\phi : X \rightarrow \mathbb{A}^n$, $x \mapsto (f_1(x), \dots, f_n(x))$, where the f_i are the generators of R^G .

Proposition 5.5. *The image $\phi(X) \subset \mathbb{A}^n$ is closed.*

Proof. Consider the homomorphism of R -modules

$$\pi : R \oplus \dots \oplus R \rightarrow R, \quad (b_1, \dots, b_n) \mapsto \sum b_i(f_i - a_i).$$

This is a homomorphism of G -representations. The induced map has image $\mathfrak{m}_a \subset R^G$, corresponding to the point (a_1, \dots, a_n) . This implies that π cannot be surjective. Therefore its image is contained in some maximal ideal $\mathfrak{m} \subset R$ (a priori not a unique ideal). But $\mathfrak{m} \cap R^G$ is maximal in R^G and therefore coincides with \mathfrak{m}_a .

Thus $a \in \phi(X)$ is the image of the point corresponding to \mathfrak{m} . \square

Definition 5.6. We denote the affine variety $\text{Spec } R^G$ by $X//G$. The inclusion $R^G \subset R$ determines a morphism of affine varieties $\Phi : X \rightarrow X//G$, which we call the *affine quotient map*. \blacksquare

We sum this up in a theorem:

Theorem 5.7. *If G is a linearly reductive group acting on an affine variety X , then the affine quotient map*

$$\Phi : X \rightarrow X//G = \text{Spec}(k[X]^G)$$

is surjective and gives a one-to-one correspondence between points of $X//G$ and closure-equivalence classes of G -orbits in X .

In addition: if $Z \subset X$ is closed and G -invariant, then $\Phi(Z)$ is closed.

Proof. Only the last claim needs proof.

Let $\mathfrak{a} \subset R$ be the ideal of Z . This is G -invariant, so G acts on R/\mathfrak{a} . By linear reductivity, we get a surjective map $R^G \rightarrow (R/\mathfrak{a})^G$. The kernel is $R^G \cap \mathfrak{a}$ and so

$$R^G/(R^G \cap \mathfrak{a}) \simeq (R/\mathfrak{a})^G.$$

The left hand side is the coordinate ring of the closure of $\Phi(X)$ in $X//G$. The right hand side is the coordinate ring of $Z//G$. Thus, applying $\text{Spec}(-)$, we get an isomorphism

$$Z//G \simeq \overline{\Phi(Z)}.$$

But as we have seen, the affine quotient map $Z \rightarrow Z//G$ is surjective, so we must have $\Phi(Z) = \overline{\Phi(Z)}$. \square

5.2 Stability

Definition 5.8. Suppose that a linearly reductive group G acts on an affine variety X . A point $x \in X$ is said to be *stable* for the action of G if the following two conditions hold:

1. The orbit $Gx \subset X$ is closed.
2. The stabilizer subgroup $\text{Stab}(x) = \{g \in G \mid gx = x\}$ is finite.

We denote the set of all stable point by $X^s \subset X$. ■

The reason we don't want unstable points is that if the stabilizers have positive dimension, the dimension of the quotient will be too low.

Proposition 5.9. *Let $Z \subset X$ be the set*

$$\{x \in X \mid \dim \text{Stab}(x) > 0\}.$$

Then X^s is the complement in X of $\Phi^{-1}(\Phi(Z))$.

Proof. Suppose $\Phi(x) \in \Phi(Z)$. If $x \in Z$, then $z \notin X^s$. If $x \notin Z$, then the fiber $\Phi^{-1}(\Phi(x))$ contains at least two G -orbits (the orbit(s) containing Z and the orbit(s) containing x), and by Corollary 5.4 the orbit Gx cannot be closed, so the first condition is violated. This shows $\Phi^{-1}(\Phi(Z)) \subset X \setminus X^s$. The converse is similar [...]. □

Remark. *We have $X^s = X$ if and only if all points have finite stabilizer.*

Being stable is an open condition. Equivalently, Z is a closed set:

Proof. Consider the map

$$s : G \times X \rightarrow X \times X, \quad (g, x) \mapsto (gx, x).$$

Then let \tilde{Z} be $s^{-1}(\Delta)$, where Δ is the diagonal. Then consider the projection

$$\pi : \tilde{Z} \rightarrow X, \quad (g, x) \mapsto x.$$

The fiber $\pi^{-1}(x)$ is canonically isomorphic to $\text{Stab}(x)$. Now it is well-known that the dimension of the fibers is an upper semicontinuous function, so it follows that the positive-dimensional locus Z is a closed set. See for example Chapter 11 in Vakil. Chapter I.6.3 in Shafarevich, or EGA IV 13.1.3. □

It follows at once that...

Proposition 5.10. ... the stable set $X^s \subset X$ and its image $\Phi(X^s) \subset X//G$ are open sets.

Theorem 5.11. Suppose that a linearly reductive group G acts on an affine variety X , and suppose that $x \in X$ is a stable point for this action. Then for any $y \in X \setminus Gx$, there exists an invariant function $f \in k[X]^G$ such that $f(x) \neq f(y)$.

Proof. We apply Theorem 5.2: suppose there does not exist a function separating x and y . Then by the theorem, the closures \overline{Gx} and \overline{Gy} intersect in a common point. Since x is stable Gx is closed, and this implies $Gx \subset \overline{Gy}$. Now, $\dim Gx = \dim G$ since x has finite stabilizer, and so $\dim Gx \geq \dim \overline{Gy}$, hence $Gx = \overline{Gy}$, contrary to $y \in X \setminus Gx$. \square

For future reference, here are a few definitions.

Definition 5.12. Let X be a scheme and G an (affine) algebraic group, with G acting on X . Then a *categorical quotient* is a morphism $\pi : X \rightarrow Y$ such that

1. π is G -equivariant with G acting trivially on Y . In other words: $\pi(gx) = \pi(x)$ for all g, x . Or:

$$\begin{array}{ccc} G \times X & \xrightarrow{\mu} & X \times Y \\ \pi_2 \downarrow & & \downarrow \pi \\ X & \xrightarrow{\pi} & Y \end{array}$$

should commute.

2. $\pi : X \rightarrow Y$ is universal with respect to 1. In other words: if $\phi : X \rightarrow Z$ is any other such map ($\phi(gx) = \phi(x)$), then there should exist a unique $\gamma : Y \rightarrow Z$ making $\gamma \circ \pi = \phi$:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Z \\ \pi \downarrow & \nearrow \exists! \gamma & \\ Y & & \end{array}$$

■

Clearly a categorical quotient is unique if it exists. In fact, $\text{Spec } R^G$ is an example. But as we have seen, categorical quotients can be quite bad on some occasions: if $G = \mathbb{C}^*$ acts by scaling lines through the origin in \mathbb{A}^2 , then, as we have seen, $R^G = k$, so the categorical quotient is just a point.

Definition 5.13. A *good categorical quotient* is a map $\pi : X \rightarrow Y$ such that:

1. If $U \subset Y$ is open, then $\mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(\pi^{-1}(U))$ induces an isomorphism $\mathcal{O}_Y(U) \xrightarrow{\sim} \mathcal{O}_X(\pi^{-1}(U))^G$.
2. If $W \subset X$ is closed and G -invariant, then $\pi(W)$ is closed in Y .
3. If W_1, W_2 are closed, disjoint and G -invariant, then $\pi(W_1) \cap \pi(W_2) = \emptyset$.

■

It is an **exercise** to see that good implies categorical. Also, the following is true:

Proposition 5.14. *For a good categorical quotient $\pi : X \rightarrow Y$:*

1. π is surjective.
2. If $U \subset X//G$ is open, then the restriction

$$\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \rightarrow U$$

is also a good categorical quotient.

3. $\pi(x) = \pi(y) \Leftrightarrow \overline{Gx} \cap \overline{Gy} \neq \emptyset$.

Proposition 5.15. *Let $X \rightarrow X//G$ be a good categorical quotient. Then the following are equivalent:*

1. All orbits are closed.
2. If $x, y \in X$ have $\pi(x) = \pi(y)$ then $Gx = Gy$.
3. π induces a bijection between G -orbits and closed points of $X//G$.
4. The image of $G \times X \rightarrow X \times X$, $(g, x) \mapsto (gx, x)$ is isomorphic to $X \times_{X//G} X$.

Definition 5.16. If any of the equivalent conditions of the proposition holds, then we say that $X \rightarrow Y$ is a *geometric quotient*, and we write it as X/G . ■

5.3 Moduli of hypersurfaces in \mathbb{P}^n

Let us try to make the theory concrete by some examples. We will study hypersurfaces in \mathbb{P}^n modulo the natural $\mathrm{GL}(n+1)$ -action.

5.3.1 Classical invariants and discriminants

A form of degree d (an element of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$) in coordinates x_0, \dots, x_n can be written

$$f = \sum_{|I|=d} a_I x^I,$$

in “multiindex” notation. Note that there are $\binom{n+d}{d}$ values of I . Denote by $V_{n,d}$ the vector space of polynomials of degree d (synonymous with $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$), and by $\mathbb{V}_{n,d}$ its associated affine space.

The group $\mathrm{GL}(n+1)$ acts on $\mathbb{V}_{n,d}$ on the right by $f(x) \mapsto f(gx)$.

Remark. *This is really a right action. Recall that a group action is a right action if $f \cdot (gh) = (f \cdot g) \cdot h$.*

In fact, we have

$$(f \cdot (gh))(x) = f(ghx) = (f \cdot g)(hx) = ((f \cdot g) \cdot h)(x),$$

so this is really a right action.

Explicitly, this can be written as follows: Let F denote the “generic” d -form, that is a form with the coefficients replaced by variables (that would be forms on $\mathbb{P}^n \times \mathbb{A}^{\binom{n+d}{d}}$).

Then a matrix $g = (g_{ij}) \in \mathrm{GL}(n+1)$ transform F to

$$F = \sum_{|I|=d} a_I \left(\sum g_{0j} x_j \right)^{i_0} \left(\sum g_{1j} x_j \right)^{i_1} \dots \left(\sum a_{nj} x_j \right)^{i_n}.$$

After rearranging, this can be written as

$$F = \sum_{|I|=d} \xi_I(g) x^I,$$

where the $\xi_I(g)$ are polynomials in g_{ij} and a_I , linear in the a_I . Thus the assignment $a_I \mapsto \xi_I(g)$ is [representation ??].

Lets do a very concrete example.

Example 5.17 (Pairs of points on the Riemann sphere, $d = 2$, $n = 2$). A form of degree 2 on \mathbb{P}^1 is given by a polynomial

$$F = a_{20}x_0^2 + a_{11}x_0x_1 + a_{02}x_1^2.$$

A generic element of $\mathrm{GL}(2)$ looks like

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

This g acts on x by sending it to $g_{11}x_0 + g_{12}x_1$. Thus it can be computed that the action on F is given by

$$\begin{aligned} F \cdot g &= (a_{20}g_{11}^2 + a_{11}g_{11}g_{21} + a_{02}g_{21}^2)x_0^2 \\ &+ (a_{20}g_{11}g_{12} + a_{11}g_{12}g_{21} + a_{11}g_{11}g_{22} + 2a_{02}g_{21}g_{22})x_0x_1 \\ &+ (a_{20}g_{12}^2 + a_{11}g_{12}g_{22} + a_{02}g_{22}^2)x_1^2. \end{aligned}$$

We see that the $\xi_I(g)$ are linear combinations of polynomials in g_{ij} , so we can write the transformation in matrix form:

$$\xi(g) = \begin{pmatrix} g_{11}^2 & 2g_{11}g_{12} & g_{12}^2 \\ g_{11}g_{21} & g_{12}g_{21} + g_{11}g_{22} & g_{12}g_{22} \\ g_{21}^2 & 2g_{21}g_{22} & g_{22}^2 \end{pmatrix}.$$

This is a representation of $GL(2)$ on $V_{1,2}$ (a map $\rho : GL(2) \rightarrow GL(3)$). It would be a fun exercise to describe the its irreducible components (an educated guess would be that it splits into two irreducible representations, one given by the forms with a double root, and one given by the forms with distinct roots).

By the way, I didn't compute this by hand. Here is how to do it in Macaulay2.

```
R = QQ[x_0,x_1, a,b,c, g_11,g_12,g_21,g_22]
h = map(R,R,{g_11*x_0+g_12*x_1, g_21*x_0+g_22*x_1,
              a,b,c,g_11,g_12,g_21,g_22})
F = a*x_0^2+b*x_0*x_1+c*x_1^2
trans = h F
(M,C) = coefficients(trans,
  Monomials => {x_0^2,x_0*x_1, x_1^2}, Variables => {x_0,x_1})
(V,G0) = coefficients(C_0_0,
  Monomials => {a,b,c}, Variables => {a,b,c})
(V,G1) = coefficients(C_0_1,
  Monomials => {a,b,c}, Variables => {a,b,c})
(V,G2) = coefficients(C_0_2,
  Monomials => {a,b,c}, Variables => {a,b,c})
G = G0 | G1 | G2
```

The last computation, G , is the matrix $\xi(g)$ above. To get \LaTeX code, type `tex G` and copy. ★

Example 5.18 (Conics in the plane, $d = 2$, $n = 3$). To get a sense of how complicated things get with a slight increase in dimension,

this is how $\xi(g)$ looks in the case of conics in \mathbb{P}^2 .

$$\begin{pmatrix} g_{11}^2 & 2g_{11}g_{12} & 2g_{11}g_{13} & g_{12}^2 & 2g_{12}g_{13} & g_{13}^2 \\ g_{11}g_{21} & g_{12}g_{21} + g_{11}g_{22} & g_{13}g_{21} + g_{11}g_{23} & g_{12}g_{22} & g_{13}g_{22} + g_{12}g_{23} & g_{13}g_{23} \\ g_{11}g_{31} & g_{12}g_{31} + g_{11}g_{32} & g_{13}g_{31} + g_{11}g_{33} & g_{12}g_{32} & g_{13}g_{32} + g_{12}g_{33} & g_{13}g_{33} \\ g_{21}^2 & 2g_{21}g_{22} & 2g_{21}g_{23} & g_{22}^2 & 2g_{22}g_{23} & g_{23}^2 \\ g_{21}g_{31} & g_{22}g_{31} + g_{21}g_{32} & g_{23}g_{31} + g_{21}g_{33} & g_{22}g_{32} & g_{23}g_{32} + g_{22}g_{33} & g_{23}g_{33} \\ g_{31}^2 & 2g_{31}g_{32} & 2g_{31}g_{33} & g_{32}^2 & 2g_{32}g_{33} & g_{33}^2 \end{pmatrix}$$

There is some symmetry in the equations, but for modern mathematicians, these equations are very scary. At least to handle with bare hands. ★

Definition 5.19. If a homogeneous polynomial F satisfies $F(xg) = F(x)$ for all $g \in \mathrm{SL}(n+1)$ (in other words, if it is $\mathrm{SL}(n+1)$ -invariant for the right action of $\mathrm{SL}(n+1)$), then we call it a *classical invariant*. ■

Let $H_{nd} := \mathbb{P}V_{nd}$. This is the Hilbert scheme of hypersurfaces of degree d in \mathbb{P}^n .

Proposition 5.20. For a homogeneous polynomial $F \in \bigoplus_i H^i(H_{n,d}, \mathcal{O}_{H_{n,d}}(i)) = k[\mathbb{V}_{n,d}]$, the following are equivalent:

1. F is a classical invariant.
2. The subvariety $F = 0$ in $H_{n,d}$ is $\mathrm{GL}(n+1)$ -invariant.

Proof. The easiest direction is $i \Rightarrow ii$). That the subvariety $F = 0$ is $\mathrm{GL}(n+1)$ -invariant, means that $F \cdot g = \lambda F$ for some $\lambda \neq 0$ for each $g \in \mathrm{GL}(n+1)$. Now, each $g \in \mathrm{GL}(n+1)$ can be written as the product of a diagonal matrix and an element of $\mathrm{SL}(n+1)$ (the diagonal matrix is $\sqrt[n]{\det g}I$). Write this as $g = \lambda h$. Then

$$F(gx) = F(\lambda hx) = F(hx) \cdot \lambda = F(x) \cdot \lambda = F(\lambda x) = \lambda^n F(x).$$

Thus $F = 0$ is $\mathrm{GL}(n+1)$ -invariant.

For the other direction: the ideal $\langle F \rangle$ is $\mathrm{GL}(n+1)$ -invariant. Thus the one-dimensional k -vector space spanned by F is a one-dimensional representation of $\mathrm{GL}(n+1)$. But this is a character of $\mathrm{GL}(n+1)$, and by Lemma 4.8, we know that F must have the form $\lambda \det X$, and this is clearly $\mathrm{SL}(n+1)$ -invariant. [[Is this correct?? Sound too good to be true]] □

We will define the *discriminant*, but we need some discussion first. Let $X \subseteq \mathbb{P}^n$ be the hypersurface given by $f \in H^0(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(d))$. Then recall that $p \in \mathbb{P}^n$ is a singular point of X if it is a solution of

$$\frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_1} = \dots = \frac{\partial f}{\partial x_n} = 0.$$

If these have no solutions in \mathbb{P}^n , then we say that X is *smooth*.

Denote by $H_{n,d}^{sing} \subset H_{n,d}$ the subset of singular hypersurfaces $X \subset \mathbb{P}^n$. Consider the set

$$Z := \{(p, X) \mid p \in \mathbb{P}^n \text{ is a singular point on } X \subset \mathbb{P}^n\}.$$

This is an algebraic variety given by the $n+1$ equations

$$\frac{\partial}{\partial x_0} \sum_I a_I x^I = \dots = \frac{\partial}{\partial x_n} \sum_I a_I x^I = 0.$$

In particular, Z is closed. Let $\psi : Z \rightarrow \mathbb{P}^n$ and $\varphi : Z \rightarrow H_{n,d}$ be the projections. By definition, the image of φ is $H_{n,d}^{sing}$. On the other hand $\varphi^{-1}(p)$ is the set of all hypersurfaces that are singular at p , and in $H_{n,d}$, this is defined by $n+1$ linear equations (the equation above is linear in the a_I). Thus

$$\begin{aligned} \dim Z &= \dim \mathbb{P}^n + \dim \phi^{-1}(p) \\ &\geq n + \dim H_{n,d} - (n+1) \\ &= \dim H_{n,d} - 1. \end{aligned}$$

In fact, equality holds. The reason is this: first of all, the image of φ is a proper subset of $H_{n,d}$, since there exist nonsingular hypersurfaces in any dimension³. Also, $\dim Z = \dim H_{n,d}^{sing}$, because 1), Z surjects onto $H_{n,d}^{sing}$, and 2) because of the following fact: if $f : X \rightarrow Y$ is a dominant morphism of integral schemes of finite type over k , then the dimension of the fibers $f^{-1}(y)$ (for $y \in f(X)$), satisfy $\dim f^{-1}(y) \geq \dim X - \dim Y$. In our case $f^{-1}([X]) = \text{Sing } X$, so if we can prove that there exists hypersurfaces with isolated singularities, then we get $\dim H_{n,d}^{sing} \geq \dim Z$. This is true, consider for example $x_1^k + \dots + x_n^k = 0$ (x_0 is left out)⁴.

Thus $H_{n,d}^{sing}$ is itself a hypersurface, so it is defined by a unique (up to scalar) homogeneous polynomial.

Definition 5.21. The defining equation of $H_{n,d}^{sing}$ is called the *discriminant* of forms of degree d on \mathbb{P}^n . ■

³Take the Fermat hypersurface $x_0^l + \dots + x_n^l = 0$

⁴See for example [2], page 95, Exercise 3.22. See also [1], Theorem 11.12.

Clearly $H_{n,d}^{sing}$ is invariant under $\mathrm{GL}(n+1)$, so:

Corollary 5.22. *The discriminant D is a classical invariant.*

Example 5.23. We continue on Example 5.17. The hypersurface X is singular if and only if

$$\begin{aligned}\frac{\partial F}{\partial x_0} &= 2a_{20}x_0 + a_{11}x_1 = 0 \\ \frac{\partial F}{\partial x_1} &= a_{11}x_0 + 2a_{02}x_1 = 0\end{aligned}$$

But this is a linear equation, having a non-trivial solution if and only if the determinant $4a_{20}a_{02} - a_{11}^2$ is zero.

This should remind everyone of the abc-formula from early childhood. ★

6 Moduli theory

The problem of moduli theory is *classification*. This can be made precise in various ways, depending upon the sophistication of the reader.

Let V be an n -dimensional vector space. Suppose you want to classify k -planes in V . Up to isomorphism, this is trivial, since all vector spaces are determined once the integer k is given. In that sense, the moduli space is just the point $\{k\}$. However, let us instead look at the whole set

$$\mathbb{G}(k, n) = \{L \mid L \text{ is a } k\text{-plane in } V\}.$$

So far, this is just a set. What does it mean to give a k -plane? It is certainly sufficient to give a spanning set, i.e. k linearly independent vectors in V . These are elements of an open subset U of \mathbb{A}^{kn} . But this is redundant: we have an action of $G = \mathrm{GL}_d$ on \mathbb{A}^{kn} . Two elements $v, v' \in \mathbb{A}^{kn}$ represent the same k -plane if and only if $v = gv'$ for $g \in \mathrm{GL}_d$.

We are thus led to ask for the existence of U/GL_d . It turns out that this object exists. It is a projective variety called the Grassmannian $\mathbb{G}(k, n)$.

6.1 Some GIT

[5]

Suppose we are in the situation of an reductive algebraic group G acting on a projective variety X . Assume further that G is a subgroup of $\mathrm{SL}(n+1, \mathbb{C})$.

The construction of GIT quotient: the group acts on each $H^0(X, \mathcal{O}_X(r))$. Then we simply define X/G to be

$$\mathrm{Proj} \left(\bigoplus_r H^0(X, \mathcal{O}_X(r))^G \right).$$

Lemma 6.1. $\bigoplus_r H^0(X, \mathcal{O}_X(r))^G$ is finitely generated.

Lemma 6.2. Hilbert's basis theorem + ...

Definition 6.3. A point $x \in X$ is *semistable* if there exists an $s \in H^0(X, \mathcal{O}_x(r))^G$ with $r > 0$ such that $s(x) \neq 0$. ■

Point which are not semistable are *unstable*.

Definition 6.4. A semistable point $x \in X$ is *stable* if $\bigoplus_r H^0(X, \mathcal{O}_x(r))^G$ separates orbits near x and the stabiliser of x is finite. ■

By “separates orbits near x ” we mean the following: given any $v \in T_x X \setminus T_x(G.x)$ there is another element in $H^0(X, \mathcal{O}_x(r))^G$ whose derivative with respect to v is nonzero.

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