Triangulations of $\mathbb{CP}^2$ and deformations

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1 Approach 1

Let $X \subseteq \mathbb{P}^5$ we a smooth pfaffian cubic hypersurface, and let $F_1(X)$ be its Fano variety of lines. The latter is a reducible subvariety of the Grassmannian $\mathbb{G}(2,6)$.

If we assume that X is smooth, then it is known [[ref]] that $F_1(X)$ is deformation equivalent to the Hilbert scheme $S^{[2]}$ of pairs of points on S, where S is a K3 surface of degree 14 in $\mathbb{P}^8$. By standard theory, a Stanley-Reisner degeneration of a K3 surface is a triangulated sphere, or just $\mathbb{P}^1_{\mathbb{C}}$. Thus, a Stanley-Reisner degeneration of $S^{[2]}$ should give a triangulation of $\mathbb{P}^1_{\mathbb{C}} * \mathbb{P}^1_{\mathbb{C}} \approx \mathbb{P}^2_{\mathbb{C}}$, the complex projective plane, as topological spaces.

Since $F_1(X)$ is embedded in $\mathbb{P}^{14}$ as a closed subscheme of the Grassmannian, finding such a triangulation is equivalent to finding a square-free initial ideal of the ideal of this embedding.

Since smooth hypersurfaces have too computationally consuming equations, we start naïvely with a singular hypersurface $X = V(f) = V(x_0x_2x_4 - x_1x_3x_5)$. This is both a toric variety and a pfaffian hypersurface. However, because of the form of f, $F_1(X)$ is reducible:

Proposition 1.1. *If $X = V(x_0x_2x_4 - x_1x_3x_5)$, the variety of lines $F_1(X)$ is reducible. It decomposes into 15 components: 9 of them are copies of $\mathbb{G}(2,4)$, and the 6 other are toric varieties, all of them isomorphic.*

Proof. The Grassmannian components are easy to see: if one of the even numbered coordinates are zero, then one of the odd must be too. Thus X contains 3×3 special copies of $\mathbb{P}^3$. Lines in $\mathbb{P}^3$ are parametrized by Grassmannian $\mathbb{G}(2,4)$. This have dimension 4.

The other components are not that easy: Consider the rational map $\psi : \mathbb{P}^2_{abc} \times \mathbb{P}^2_{xyz} \dashrightarrow X$ given by sending $(a:b:c) \times (x:y:z)$ to $(ax : bx : by : cy : cz : az)$. Then this map is well-defined outside the points $(1:0:$

0) $\times$ (0 : 1 : 0), (0 : 1 : 0) $\times$ (0 : 0 : 1) and (0 : 0 : 1) $\times$ (1 : 0 : 0). Resolving the locuf of indeterminacy give an isomorphism of a desingularization of X with the blowup of $\mathbb{P}^2 \times \mathbb{P}^2$ in 3 points. There are six ways of choosing the map ψ, which gives the six different toric components of X.

Since the map is of bidegree $(1, 1)$, we have a well-defined induced map $(\mathbb{P}^2)^\wedge \times \mathbb{P}^2 \dashrightarrow F_1(X)$ given by

$$\ell \times p \mapsto \ell_p := \big[\psi(x,p), \psi(y,p)\big]_{x,y \in \ell},$$

where ℓ_p denotes the line connecting $\psi(x,p)$ and $\psi(y,p)$ in $\mathbb{P}^5$. This gives one of the components in $F_1(X)$.

That these are all components can either be checked on a computer, or by degree considerations. [[unsure of the latter]] □

The equations of $F_1(X)$ inside $\mathbb{P}^{14}$ are rather messy. The embedding is of degree 108 and there are 70 equations (15 quadrics corresponding to the inclusion in $\mathbb{G}(2, 6)$ and 55 cubics, of which 6 are monomials).

However, the Hilbert polynomial is $h(t) = \frac{9}{2}t^4 + \frac{15}{2}i^2 + 3$, giving a topological Euler characteristic of 3, which agrees with the Euler characteristic of (the topological space) $\mathbb{CP}^2$.

Thus, it is at least not totally unreasonable to predict that a Stanley-Reisner degeneration of $F_1(X)$ would correspond to a triangulation of $\mathbb{CP}^2$.

2 Bad er good news

The article [[ref]] presents a triangulation T of $\mathbb{CP}^2$ having 15 vertices, *and* the same f-vector as the triangulation we're looking for. This gives us a Stanley-Reisner scheme X_T living in the same Hilbert scheme as $F_1(X)$. [[see the reference (for now) for a description]]

Now, the Hilbert scheme is a terribly lonely place, so it could be that this new triangulation lives in a completely different component than the one $F_1(X)$ lives in. However, if they do - we could hope for one particularly nice situation: $F_1(X)$ degenerates to X_T, and we would have another "algebro-geometric" explanation of the existence of T.

One way to find a degeneration is to try to deform X_T. First: $\mathcal{C}\mathbb{P}^2$ have a cell structure consisting of three 4-cells, where their intersections are 3 different filled tori. The triple intersection is a 2-torus. Here's a surprising result:

Proposition 2.1. *There exists a deformation of X_T, the Stanley-Reisner scheme of the triangulation of T, to the union of three toric varieties. Their* *pairwise intersections are Stanley-Reisner schemes of affine dimension* 4*, corresponding to the product of two tori. [[... FILL IN THIS...]]*

The question is now of course if this union of toric varieties is smoothable, and if so, does it smooth to $F_1(X)$?

Here is some information about the equations of the deformation: its degree is of course 108. The ideal is generated by 37 elements, a lot less than the 70 of $F_1(X)$. There are 15 quadrics (as with $F_1(X)$) and 22 cubics. The natural question now is of course: what can we say about the number of defining equations for deformation equivalent objects?

Remark. *The quadric equations have four terms, whereas the Grassmannian terms in $F_1(X)$ have three terms. Unless there is some change of coordinate to decrease the number of terms, there is no way (?) this can deform to the Grassmannian.*