

Exercises

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I solve and type exercises from different places (read *books*).

1 Algebraic Geometry - Hartshorne

1.1 Chapter I - Varieties

Exercise 1 (Exercise 1.1). a) Let Y be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k .

b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k .

c) Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f . Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?



Solution 1. a) We have $A(Y) = k[x, y]/(y - x^2)$. An isomorphism $A(Y) \rightarrow k[t]$ is given by $x \mapsto t$ and $y \mapsto t^2$.

b) We have $A(Z) = k[x, y]/(xy - 1) \simeq k[x, \frac{1}{x}]$. So we must show that $k[x, \frac{1}{x}] \not\simeq k[x]$. It can be computed that the first one has automorphisms given by $x \mapsto cx^n$ for c nonzero and $n \neq 0$. The second has as automorphisms $ax + b$ ($a \neq 0$). So the first one have an abelian automorphism group, the second has not.

c) What is special about $A(Y)$ and $A(Z)$? Staring at pictures, we see that any line in \mathbb{A}^2 intersects Y in at least one point, but in the case of Z , there exist two lines which do not intersect Z . We claim that this is the only two things that can happen.

First we claim that if we are in the second situation, that is, if there exist a pair of lines ℓ, ℓ' such that $W \cap \ell = W \cap \ell' = \emptyset$, then $W \simeq Z$.

A general quadric can be written as

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Suppose now $\ell \cap W = \emptyset$. This is equivalent to $I(f, \ell^\vee) = (1)$. Without loss of generality, we can assume $\ell = \{x = 0\}$. Then

$$I(f, \ell) = (cy^2 + ey + f, x).$$

This generates $k[x, y]$ if and only if $c = e = 0$ and $f \neq 0$. Thus f must be of the form

$$ax^2 + bxy + dx + f = 0$$

with $f \neq 0$. But this can be written as

$$x(ax + by + d) + f = 0.$$

Put $y' = ax + by + d$. Then $I(W)$ takes the form $(xy' + f = 0)$, which is clearly isomorphic to Z after a linear change of coordinates. Note that the other line not meeting W is the line given by $y' = ax + by + d = 0$.

Assume now that we are in the other situation, namely that *every* line in \mathbb{A}^2 meets W . Now pick a tangent line ℓ of W . Without loss of generality, we can assume that ℓ is $\{y = 0\}$. This is a tangent line if and only if it meets W doubly, meaning that $I(W) + (\ell^\vee)$ takes the form (l^2, y) for some linear form l . We can also assume that $\ell \cap W = (0, 0)$, so that $I(W) + (\ell^\vee) = (x^2, y)$. But this means that

$$\begin{aligned} I(W) + I(\ell) &= (ax^2 + bxy + cy^2 + dx + ey + f, y) \\ &= (ax^2 + dx + f, y) \end{aligned}$$

We want $ax^2 + dx + f = x^2$. This can happen only if $d = f = 0$ and $a \neq 0$. Thus the quadric takes the form

$$ax^2 + bxy + cy^2 + ey = 0.$$

Now we claim that there exist one line at each point of W that intersect W transversally in exactly one point. This is the case for Y . Consider the pencil of lines through $(0, 0)$ defined by $x = \lambda y$. We want to find λ such that the intersection is transversal and only one point. We have

$$(ax^2 + bxy + cy^2 + ey, x - \lambda y) = ((a\lambda^2 + b\lambda + c)y^2 + ey, x - \lambda y).$$

This have exactly one solution if and only if $a\lambda^2 + b\lambda + c = 0$. This is solvable since $a \neq 0$ and since all lines intersect W . Thus choose λ as above. We can rotate this line such that it becomes $x = 0$. Then the equation takes the form

$$ax^2 + bxy + ey = 0.$$

We have still not arrived at $y = x^2$. Let now $y = \lambda x$ be a general line through the origin. We demand that this intersect W twice for every λ such that the line is not tangent. We get that the intersection is given by

$$ax^2 + b\lambda x + ex = x((a + \lambda b)x + e) = 0.$$

For this to have two solutions for every λ we must have $a + \lambda b \neq 0$ for all λ . But this requires $b = 0$. Thus the equation is

$$ax^2 + ey = 0$$

which is the conic we were looking for.

♡

Exercise 2 (Exercise 1.2, the twisted cubic curve). Let $Y \subseteq \mathbb{A}^3$ be the set $\{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k . We say that Y is given by the *parametric equation* $x = t, y = t^2, z = t^3$. ♠

Solution 2. An affine variety is by definition a closed irreducible subset of \mathbb{A}^3 . So we must find an irreducible ideal I such that $Z(I) = Y$ (forgive the abuse of notation).

I claim that $I(Y) = \langle x^2 - y, x^3 - z \rangle$. Clearly, every $P \in Y$ satisfies these equations. This shows the inclusion $Y \subset Z(I)$. Now suppose $P \in Z(I)$, that is, $f(P) = 0$ for all $f \in I$. In particular $(x^2 - y)(P) = 0$ and $(x^3 - z)(P) = 0$. Thus $y = x^2$ and $z = x^3$. So if $P = (a, b, c) \in k^3$, then $P = (a, a^2, a^3)$, so $P \in Y$. This shows that $Z(I) = Y$. If we can show that I is prime, then it follows that $I(Y) = I$ and that Y is a variety.

In fact, we claim that $k[x, y, z]/I \simeq k[t]$, implying that I is prime. The map φ is given by $x \mapsto t, y \mapsto t^2, z \mapsto t^3$. Then clearly $I \subseteq \ker \varphi$. We must show equality. So suppose $\varphi(f) = 0$.

First we claim that any $f \in k[x, y, z]$ can be written as $f = R(x) + S(x)y + T(x)z + i(x, y, z)$ where i is a polynomial in I . We prove this by induction on $\deg f$. If $\deg f = 1$, this is trivially true. The rest of the proof proceeds by tedious induction. ♡

1.2 Chapter II - Schemes

Exercise 3 (Exercise 1.16, Flabby/flasque sheaves). A sheaf \mathcal{F} on a topological space X is *flasque* (flabby) if for every inclusion $U \subseteq V$, the restriction map $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is surjective.

- a) Show that a constant sheaf on an irreducible topological space is flasque.
- b) If $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is an exact sequence of sheaves, and if \mathcal{F}' is flabby, then for any open set U , the sequence

$$0 \rightarrow \mathcal{F}'(U) \rightarrow \mathcal{F}(U) \rightarrow \mathcal{F}''(U) \rightarrow 0$$

is exact.

- c) Same as above, but suppose \mathcal{F}' and \mathcal{F} are flabby. Show that \mathcal{F}'' is flabby.
- d) If $f : X \rightarrow Y$ is a continuous map, and \mathcal{F} is a flabby sheaf on X , then $f_* \mathcal{F}$ is flabby on Y .
- e) Let \mathcal{F} be any sheaf on X . We define a new sheaf \mathcal{G} , called the *sheaf of discontinuous sections of \mathcal{F}* , as follows: For each open set $U \subset X$, $\mathcal{G}(U)$ is the set of maps $s : U \rightarrow \cup_{P \in U} \mathcal{F}_P$, such that for all $P \in U$, $s(P) \in \mathcal{F}_P$. Show that \mathcal{G} is a flabby sheaf, and that there is a natural injective morphism from \mathcal{F} to \mathcal{G} .



Solution 3. a) Every open set in X is irreducible and dense, and dense sets are connected. Hence a constant sheaf is actually constant, and all the restriction maps are identities (except if one of them is the empty set).

- b) The sheaf axiom for a sheaf \mathcal{F} is equivalent to the following: for every covering $\{U_i\}$ of U , the following sequence is exact:

$$0 \rightarrow \mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightarrow \prod_{ij} \mathcal{F}(U_{ij}),$$

where $U_{ij} = U_i \cap U_j$. The first map sends a section s to the product of all its restrictions, and the second map sends $(s_i) \mapsto (s_i - s_j)_{ij \in I \times I}$.

Since the sequence of sheaves in the exercise is exact, for small enough U_i , the sequence $0 \rightarrow \mathcal{F}'(U_i) \rightarrow \mathcal{F}(U_i) \rightarrow \mathcal{F}''(U_i) \rightarrow 0$ is exact (for sheaves, exactness is a local property). Hence we can form the following diagram,

where the columns are exact by the sheaf property, and the rows are exact by what was just said:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}'(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{F}''(U) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_i \mathcal{F}'(U_i) & \longrightarrow & \prod_i \mathcal{F}(U_i) & \longrightarrow & \prod_i \mathcal{F}''(U_i) \longrightarrow 0 \\
& & \downarrow f & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_{ij} \mathcal{F}'(U_{ij}) & \longrightarrow & \prod_{ij} \mathcal{F}(U_{ij}) & \longrightarrow & \prod_{ij} \mathcal{F}''(U_{ij}) \longrightarrow 0 \\
& & \downarrow & & & & \\
& & \text{coker } f & & & &
\end{array}$$

Now, $\text{coker } f = 0$ since \mathcal{F}' is flabby! Hence apply the snake lemma to see that the top right map is surjective as well!

- c) Use the same diagram. The middle column is surjective at the bottom, and by commutativity, the right column must be as well.
- d) This is obvious, since $f_* \mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$.
- e) It is clear that \mathcal{G} is a sheaf. If $U \subset V$, let $s \in \mathcal{G}(U)$ be given. Then define $s' \in \mathcal{G}(V)$ as follows: $s'(P) = s(P)$ if $P \in U$ and zero elsewhere. This element will be sent to s .

The injective morphism from \mathcal{F} to \mathcal{G} is defined as follows: send $s \in \mathcal{F}(U)$ to the function $s(P) = s_P$ in $\mathcal{G}(U)$.

♡

Exercise 4 (Exercise 2.19). Let A be a ring. The following are equivalent:

1. $\text{Spec } A$ is disconnected.
2. There exists nonzero elements $e_1, e_2 \in A$ such that $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1 + e_2 = 1$ (these are called *orthogonal idempotents*).
3. A is isomorphic to a direct product $A_1 \times A_2$ of two nonzero rings.

♠

Proof. $1 \Rightarrow 3$: Let U be a nonempty connected component of $X = \operatorname{Spec} A$. Let $V = X \setminus U$ be its complement, and let $i_1 : U \rightarrow X$ and $i_2 : V \rightarrow X$ be the natural inclusions on topological spaces. This can be extended to a map of schemes as well: we need to give a morphism $f^\# : \mathcal{O}_X \rightarrow f_* \mathcal{O}_U$. But $f_* \mathcal{O}_U(W) = \mathcal{O}_X(W \cap U)$, so $f_* \mathcal{O}_U = \mathcal{O}_X|_U$. Hence we just choose $f^\# : \mathcal{O}_X \rightarrow \mathcal{O}_U$ to be the natural map provided by the sheaf axioms.

We now have two morphisms $i_1 : U \rightarrow X$ and $i_2 : V \rightarrow X$ which are closed immersions, hence the induced ring morphisms $A \rightarrow A_1$ and $A \rightarrow A_2$ are surjective. Also, the universal property for products hold because the universal property for coproducts hold in the category of affine schemes. Hence $A \simeq A_1 \times A_2$. (a bit clumsy??)

$2 \Rightarrow 3$: Let $\pi_i : A \rightarrow A_i$ be given by multiplication by e_i and let A_i be its image. Then $\ker \pi_1 = A_2$, because if $e_1 f$ then $f = e_2 f$, so $f \in A_2$. The splitting maps are the natural inclusions.

$3 \Rightarrow 2$: If $A = A_1 \times A_2$, let $e_i = \pi_i(1)$.

$3 \Rightarrow 1$: Similar to the first argument, just opposite. \square

Exercise 5 (Exercise 7.1). Let (X, \mathcal{O}_X) be a locally ringed space and let $f : \mathcal{L} \rightarrow \mathcal{M}$ be a surjective map of invertible sheaves on X . Show that f is an isomorphism. \spadesuit

Solution 4. Since \mathcal{L}, \mathcal{M} are invertible, we have isomorphisms $\mathcal{L}_x \approx \mathcal{O}_{X,x}$ and $\mathcal{M}_x \approx \mathcal{O}_{X,x}$ for each $x \in X$.

But $\operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{X,x}) = \mathcal{O}_{X,x}$, that is, all homomorphisms are given by multiplication by some $h \in \mathcal{O}_{X,x}$. But since f was surjective, we conclude that h is outside \mathfrak{m}_x , the maximal ideal of $\mathcal{O}_{X,x}$. But then h is a unit, so f is an isomorphism. \heartsuit

1.3 Chapter III - Cohomology

Exercise 6 (Exercise 4.3). Let $X = \mathbb{A}_k^2 = \operatorname{Spec} k[x, y]$ and let $U = X \setminus \{(0, 0)\}$. Use a suitable open cover of X by open affine subsets to show that $H^1(U, \mathcal{O}_U)$ is isomorphic to the k -vector space spanned by $\{x^i y^j \mid i, j < 0\}$. In particular, it is infinite-dimensional, and so U cannot be affine (not projective either). \spadesuit

Solution 5. We can cover U by $U_1 = \mathbb{A}^2 \setminus \{x = 0\}$ and $U_2 = \mathbb{A}^2 \setminus \{y = 0\}$. We have $U_1 \cap U_2 = \mathbb{A}^2 \setminus \{xy = 0\}$. Also, $\mathcal{O}(U_1) = k[x, y, \frac{1}{x}]$ and $\mathcal{O}(U_2) = k[x, y, \frac{1}{y}]$ and $\mathcal{O}(U_1 \cap U_2) = k[x, y, \frac{1}{xy}]$. Then the Čech complex takes the form

$$0 \rightarrow k[x, y, \frac{1}{x}] \times k[x, y, \frac{1}{y}] \xrightarrow{d} k[x, y, \frac{1}{xy}] \rightarrow 0,$$

the differential being difference. Then $H^1(U, \mathcal{O}_U)$ can be computed as the homology at the second term. But nothing on the left side can hit anything of the form $x^i y^j$ with $i, j < 0$. Anything else is hit. Thus we have

$$H^1(U, \mathcal{O}_U) \simeq \{x^i y^j \mid i, j < 0\}$$

as k -vector spaces. ♡

Exercise 7 (Exercise 4.7). Let X be the subscheme of \mathbb{P}_k^2 defined by a single homogeneous polynomial $f(x_0, x_1, x_2) = 0$ of degree d . Assume that $(1, 0, 0)$ is not on X . Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Čech complex

$$\Gamma(U, \mathcal{O}_X) \oplus \Gamma(V, \mathcal{O}_X) \rightarrow \Gamma(U \cap V, \mathcal{O}_X)$$

explicitly, and thus show that

$$\begin{aligned} \dim_k H^0(X, \mathcal{O}_X) &= 1 \\ \dim_k H^1(X, \mathcal{O}_X) &= \frac{1}{2}(d-1)(d-2). \end{aligned}$$

♠

Solution 6. X can be covered by just two open affines since $\mathbb{P}^2 \setminus (U \cup V) = \{(1 : 0 : 0)\}$, which was assumed not to lie on the curve.

The open affine subset $\Gamma(U, \mathcal{O}_X)$ can be identified with the polynomial ring $k[u, v]/\langle f(u, 1, v) \rangle$, and $\Gamma(V, \mathcal{O}_X) = k[x, y]/f(x, y, 1)$. The differential is then given by

$$(g(u, v), h(x, y)) \mapsto g(xy^{-1}, y^{-1}) - h(x, y) \in k[x, y, \frac{1}{y}].$$

We can assume that $f = x_0^d$, since what really matters is the degree, and we are just doing linear algebra.

We first calculate $H^0(X, \mathcal{O}_X)$. So suppose $g(xy^{-1}, y^{-1}) - h(x, y) = 0$ in $k[x, y, y^{-1}]/\langle f(x, y, 1) \rangle$. By definition this means that

$$g(xy^{-1}, y^{-1}) - h(x, y) = f(x, y, 1) \cdot \tilde{f}(x, y, \frac{1}{y})$$

for some polynomial \tilde{f} . Write \tilde{f} as $\tilde{f}_0 + \tilde{f}_1$, where $\tilde{f}_0 = \sum_{j < 0} a_{ij} x^i y^j$ and $\tilde{f}_1 \in k[x, y]$. Then we have the equality

$$g(xy^{-1}, y^{-1}) - h(x, y) = \sum_{j < 0} a_{ij} x^{i+d} y^j + \sum_{j \geq 0} x^{i+d} y^j.$$

First of all, we see that the constant terms of g and h must be equal, because there are no constant terms on the right hand side. Secondly, $g(xy^{-1}, y^{-1})$ consists solely of terms with $j < 0$. Thus the non constant terms of $g(xy^{-1}, y^{-1})$ must be equal to the left term of the right hand side above. But both terms of the right hand side are zero modulo f , so the constant terms of $g(xy^{-1}, y^{-1})$ are also zero mod f . The same holds for $h(x, y)$. Thus $H^0(X, \mathcal{O}_X) = \{(c, c) \mid c \in k\} \simeq k$.

Now we compute $H^1(X, \mathcal{O}_X)$. Consider a monomial $x^i y^j$ in the target. If both $i, j \geq 0$, then it is hit by $(0, -x^i y^j)$. Likewise, if $j \geq i$, then $(x^i y^{j-i}, 0) \mapsto x^i x^{-j}$. Thus all monomials $x^i y^{-j}$ with $j \geq i$ is zero in the cokernel. Further, if $i \geq d$, then $x^i y^j$ is already zero! Thus, we can draw the non-zero monomials in the cokernel as points in the lattice \mathbb{Z}^2 . This is a triangle of length $d - 2$. Thus the dimension of $H^1(X, \mathcal{O}_X)$ is

$$1 + 2 + \dots + d - 3 + d - 2 = \frac{1}{2}(d - 2)(d - 2 + 1) = \frac{1}{2}(d - 2)(d - 1).$$

♡

1.4 Chapter IV - Curves

Exercise 8 (Exercise 1.1). Let X be a curve and $P \in X$ a point. Show that there exists a nonconstant rational function $f \in K(X)$ which is regular everywhere except at P . ♠

Solution 7. Let D be the divisor $D = nP$. The linear system

$$\{E = D + f \geq 0\}$$

consists of all divisors linearly equivalent to D . But these are classified by those f with $(f) \geq -nP$, i.e. those f with at most poles of order n at P .

By Riemann-Roch we have

$$l(D) - l(K - D) = \deg D + 1 - g = n + 1 - g.$$

If n is large enough, $K - D$ will have negative degree, so $l(K - D) = 0$. Thus for large n , we can get $l(D)$ as big as we want.

♡

2 Commutative Algebra - Eisenbud

2.1 Chapter 16 - Modules of Differentials

Exercise 9 (Exercise 16.1). Show that if $b \in S$ is an idempotent ($b^2 = b$), and $d : S \rightarrow M$ is any derivation, then $db = 0$. ♠

Solution 8. This is trivial. $db = d(b^2) = 2db$. If $2 = 0$, then the statement is automatically true. If not, then $db = 0$ by subtraction. \heartsuit

3 Deformation Theory - Hartshorne

3.1 Chapter I.3 - The T^i functors

Exercise 10 (Exercise 3.1). Let $B = k[x, y](xy)$. Show that $T^1(B/k, M) = M \otimes k$ and $T^2(B/k, M) = 0$ for any B -module M . \spadesuit

Solution 9. Since B is defined by a principal ideal in $P = k[x, y]$, it follows that $L_2 = 0$ in the cotangent complex. Thus $T^2(B/k, M)$ is automatically zero.

We have that $L_1 = B$ and $L_0 = Bdx \oplus Bdy$ with d_1 being $f \mapsto (fy, fx)$. Applying $\text{Hom}(-, M)$, we get $\text{Hom}(L_0, M) = M \oplus M$ and $\text{Hom}(L_1, M) = M$.

We have $\text{Hom}(B \oplus B, M) \simeq M \oplus M$ by $\phi \mapsto (\phi(1, 0), \phi(0, 1))$. We have a diagram

$$\begin{array}{ccc} \text{Hom}(B \oplus B, M) & \xrightarrow{\psi^*} & \text{Hom}(B, M) \\ \simeq \downarrow & & \downarrow \simeq \\ M \oplus M & \longrightarrow & M \end{array}$$

Under these isomorphisms, it is easy to see that the bottom map is given by

$$(\phi(1, 0), \phi(0, 1)) \mapsto y\phi(1, 0) + x\phi(0, 1).$$

Thus since T^1 is the cokernel of this map, we must have $T^1(B/k, M) = M \otimes k$. \heartsuit

Exercise 11 (Exercise 3.3). Let $B = k[x, y]/(x^2, xy, y^2)$. Show that $T^0(B/k, B) = k^4$, $T^1(B/k, B) = k^4$ and $T^2(B/k, B) = k$. \spadesuit

Solution 10. Let's compute L_2 first. For that we need part of a resolution of I . We have in fact

$$0 \rightarrow \text{im} \begin{pmatrix} -y & 0 \\ x & -y \\ 0 & x \end{pmatrix} \rightarrow P(-2)^3 \rightarrow I \rightarrow 0.$$

The Koszul relations are given by

$$\text{im} \begin{pmatrix} -y^2 & -xy & 0 \\ 0 & x^2 & -y^2 \\ x^2 & 0 & xy \end{pmatrix}.$$

Let's compute Q/F_0 (relations modulo Koszul relations). Since Q is generated in degree 3, and F_0 is of degree 4, we have $\dim_k(Q/F_0)_3 = 2$. Let's consider degree 4. As a k -vector space Q_4 is spanned by the four elements

$$\begin{pmatrix} -y^2 \\ xy \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -y^2 \\ xy \end{pmatrix}, \begin{pmatrix} -yx \\ x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -yx \\ x^2 \end{pmatrix}.$$

The two in the middle are already Koszul relations, so that $(Q/F_0)_4$ have dimension ≤ 2 . But we also have

$$\begin{pmatrix} -y^2 \\ xy \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ yx \\ -x^2 \end{pmatrix} + \begin{pmatrix} -y^2 \\ 0 \\ x^2 \end{pmatrix}.$$

Thus $\dim_k(Q/F_0)_4 = 1$, since the second term above is a Koszul relation. Similarly we find that $\dim_k(Q/F_0)_5 = 0$. Hence, L_2 is the 3-dimensional k -vector space spanned by Q_3 and one more relation. L_1 is $F \otimes B = B^3$, and L_0 is $B \oplus B$, spanned by dx, dy .

Taking duals, we get that $L_2 = \text{Hom}(Q/F_0, B)$. This set can be identified with

$$\begin{aligned} \text{Hom}(Q/F_0, B) &= \{\varphi : Q \rightarrow B \mid \varphi|_{F_0} = 0\} \\ &= \{\varphi : Q \rightarrow P \mid \text{im } \varphi|_{F_0} \subseteq I\} \end{aligned}$$

Thus, since $I = \mathfrak{m}^2$, we must have that φ sends the two generators of Q to something of degree 1 (degree 0 is not ok, since then F_0 would be sent outside I). Thus $\text{Hom}(Q/F_0, B)$ is $2 \times 2 = 4$ -dimensional, spanned by

$$\text{im} \begin{pmatrix} y & x & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}.$$

But d_2 is the dual of the inclusion $Q \rightarrow F$ from the exact sequence above. The dual is given by transposing, and we are left with one column - in conclusion, $T^2(B/k, B)$ is one-dimensional.

The Jacobian of I is given by

$$\begin{pmatrix} 2x & y & 0 \\ 0 & x & 2y \end{pmatrix},$$

and it is easily seen that the kernel of $\text{Jac} \otimes B$ is given by $\mathfrak{m} \oplus \mathfrak{m} \oplus \mathfrak{m} \subset B^3$. The two relations kill off two dimensions, so $\dim_k T^1(B/k, B) = \dim_k \mathfrak{m}^{\oplus 3} - 2 = 6 - 2 = 4$.

Also $T^0(B/k, B)$ is B^2 modulo the image of the Jacobian. The constants are left untouched, so $\dim_k T^0(B/k, B) = 2 + 2 + 2 - 3 = 3$. A basis is given by $(1, 0)$, $(0, 1)$ and (x, y) . (thus Hartshorne is wrong?) \heartsuit

4 Introduction to Commutative Algebra - Atiyah-MacDonald

4.1 Chapter 1 - Rings and ideals

Exercise 12. Let x be a nilpotent element of a ring A . Show that $1 + x$ is a unit of A . Deduce that the sum of a nilpotent element and a unit is a unit. \spadesuit

Solution 11. Suppose $x^{n+1} = 0$ and that $x^n \neq 0$. Consider

$$s = 1 - x + x^2 - x^3 + \dots + x^n$$

Then

$$sx = x - x^2 + x^3 - x^4 + \dots - x^n$$

since $x^{n+1} = 0$. But then $s + sx = 1$, so that $s(1 + x) = 1$. Hence $1 + x$ is a unit. To prove that the sum of any unit and any nilpotent is a unit, note that if u is any unit, then $u^{-1}x$ is still nilpotent. So since $u + x = u(1 + u^{-1}x)$ and product of units are units, the claim follows. \heartsuit

Exercise 13 (Exercise 11). A ring A is *Boolean* if $x^2 = x$ every $x \in A$. In a Boolean ring A , show that

- i) $2x = 0$ for all $x \in A$.
- ii) Every prime ideal \mathfrak{p} is maximal, and A/\mathfrak{p} is a field with two elements.

iii) Every finitely generated ideal in A is principal.



Solution 12. i) We have $4x = 4x^2 = (2x)^2 = 2x$, hence $2x = 0$.

ii) Consider A/\mathfrak{p} . This is an integral domain in which $x^2 = x$ for all $x \in A/\mathfrak{p}$. But then $x^2 - x = x(x - 1) = 0$. Hence either $x = 0$ or $x = 1$, hence A/\mathfrak{p} can have only two elements. Thus it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ which is a field, hence \mathfrak{p} is maximal.

iii) Let $I = (a_1, \dots, a_r)$. Every ideal is contained in a maximal ideal \mathfrak{m} . Consider the image of I in A/\mathfrak{m} .

iv) By induction we can assume that I is generated by two elements, say $I = (a_1, a_2)$. Then I claim that $I = (a_1 + a_2)$. Clearly $(a_1 + a_2) \subseteq (a_1, a_2)$. The other direction will follow if we can see that $a_1 a_2 = 0$ (or they can be assumed to satisfy this), because $a_1 a_2 + a_1 \in (a_1 + a_2)$.
 [((((((((([?])))))))))]



Exercise 14 (Exercise 12). A local ring contains no nontrivial idempotents.



Solution 13. Suppose $x \neq 0, 1$ and that $x^2 = x$. Then $x^2 - x = x(x - 1) = 0$. Both x and $x - 1$ cannot be contained in \mathfrak{m} since they generate A . Hence one of the is unit. Hence either $x = 0$ or $x = 1$, contradiction.



Exercise 15 (Exercise 15, The prime spectrum of a ring). Let A be a ring and let X be the set of prime ideals of A . For each subset E of A , let $V(E)$ denote the set of prime ideals of A which contain E . Prove that

1. If \mathfrak{a} is the ideal generated by E , then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))^1$.
2. $V(0) = X$ and $V(1) = \emptyset$.
3. If $(E_i)_{i \in I}$ is a family of subsets of A , then

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i).$$

4. $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for all ideals $\mathfrak{a}, \mathfrak{b}$ of A .

¹Here $r(\mathfrak{a})$ denotes the radical of \mathfrak{a}

These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the *Zariski topology*. The topological space X is called the *prime spectrum of A* and denoted $\text{Spec } A$. ♠

Solution 14. We do these one by one.

1. Clearly $\mathfrak{p} \supset \langle E \rangle \supset E$, where the brackets denote the ideal generated by E . Hence $V(\mathfrak{a}) \subset V(E)$. But if $\mathfrak{p} \supset E$, we must have $\mathfrak{p} \supset \mathfrak{a}$ since $\langle \mathfrak{p} \rangle = \mathfrak{p}$. Thus the first equality is established.

Since $r(\mathfrak{a}) \subset \mathfrak{a}$, we have $V(\mathfrak{a}) \subset V(r(\mathfrak{a}))$. Suppose $\mathfrak{p} \supset r(\mathfrak{a})$ and suppose $a \in \mathfrak{a}$. We want to show $a \in \mathfrak{p}$. We know that $a^n \in r(\mathfrak{a})$ for some n , hence $a^n \in \mathfrak{p}$. But \mathfrak{p} is a prime ideal, so $a \in \mathfrak{p}$ also. Hence equality is established.

2. Every ideal contains the zero ideal and (1) is not a prime ideal.
3. Suppose $\mathfrak{p} \supset \cup E_i$. Then $\mathfrak{p} \supset E_i$ for all i , so $\mathfrak{p} \in \cap V(E_i)$. Thus this is just a formal consequence of the contravariant nature of $V(-)$.
4. Since $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$, we automatically have $V(\mathfrak{a} \cap \mathfrak{b}) \subset V(\mathfrak{a}\mathfrak{b})$. So suppose $\mathfrak{p} \supset \mathfrak{a}\mathfrak{b}$ and let $a \in \mathfrak{a} \cap \mathfrak{b}$. Then $a^2 \in \mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$, but then $a \in \mathfrak{p}$ since \mathfrak{p} is prime.

Now suppose $\mathfrak{p} \supset \mathfrak{a}$ or $\mathfrak{p} \supset \mathfrak{b}$. Then if $a \in \mathfrak{a} \cap \mathfrak{b}$, we have $a \in \mathfrak{p}$, so $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subset V(\mathfrak{a} \cap \mathfrak{b})$. Now suppose $\mathfrak{p} \supset \mathfrak{a} \cap \mathfrak{b}$. Then by Proposition 1.11, we have $\mathfrak{p} \supset \mathfrak{a}$ or $\mathfrak{p} \supset \mathfrak{b}$.

♡

Exercise 16 (Exercise 17). For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \text{Spec } A$. The sets X_f are open. Show that they form a basis for the Zariski topology, and that

1. $X_f \cap X_g = X_{fg}$.
2. $X_f = \emptyset \Leftrightarrow f$ is nilpotent.
3. $X_f = X \Leftrightarrow f$ is a unit.
4. $X_f = X_g \Leftrightarrow r((f)) = r((g))$.
5. X is quasi-compact.
6. More generally, each X_f is quasi-compact.

7. An open subset of X is quasi-compact if and only if it is a finite union of the sets X_f .

The sets X_f are called *basic open sets* of $X = \text{Spec } A$. ♠

Solution 15. We need to show that the sets X_f forms a basis for the Zariski topology on X . This means that each open in X can be written as a union of the X_f . An open in X have the form

$$U(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec } A \mid \mathfrak{p} \not\supset \mathfrak{a}\}.$$

The sets X_f have the form

$$X_f = \{\mathfrak{p} \in \text{Spec } A \mid f \notin \mathfrak{p}\}.$$

Let $\{f_i\}_{i \in I}$ generate \mathfrak{a} . I claim that $\bigcup X_{f_i} = U(\mathfrak{a})$. Let \mathfrak{p} be an element of the left hand side. This means by definition that $f_i \notin \mathfrak{p}$ for some i . But f_i is an element of \mathfrak{a} , so $\mathfrak{a} \not\subset \mathfrak{p}$, hence $\mathfrak{p} \in U(\mathfrak{a})$.

Conversely, suppose $\mathfrak{p} \not\supset \mathfrak{a}$. Then some generator f_i of \mathfrak{a} is not contained in \mathfrak{p} . Hence $\mathfrak{p} \in X_{f_i}$.

1. We have

$$X_f \cap X_g = \{\mathfrak{p} \mid f, g \notin \mathfrak{p}\} = \{\mathfrak{p} \mid fg \notin \mathfrak{p}\},$$

since \mathfrak{p} is a prime ideal: for suppose $f, g \notin \mathfrak{p}$, then $fg \notin \mathfrak{p}$ also, because if $fg \in \mathfrak{p}$, primality implies either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Conversely, suppose $fg \notin \mathfrak{p}$. Then neither f, g can be in \mathfrak{p} by definition of ideals.

2. Suppose X_f is empty. Then there are no prime ideals with $f \notin \mathfrak{p}$. But that means that f is contained in every prime ideal, hence f is nilpotent.
3. Suppose $X_f = X$. Then for all prime ideals, $f \notin \mathfrak{p}$, hence f generates the unit ideal, hence f is a unit. For if f did not generate the unit ideal, f would be contained in some maximal ideal \mathfrak{m} , and maximal ideals are prime.
4. Suppose $X_f = X_g$. By definition, this means that for every prime \mathfrak{p} with $f \notin \mathfrak{p}$, we have $g \notin \mathfrak{p}$ (and conversely). The contrapositive of this is $g \in \mathfrak{p} \Leftrightarrow f \in \mathfrak{p}$. Hence we have

$$r((f)) = \bigcap_{\mathfrak{p} \supset (f)} \mathfrak{p} = \bigcap_{\mathfrak{p} \ni f} \mathfrak{p} = \bigcap_{\mathfrak{p} \ni g} \mathfrak{p} = r((g)).$$

5. Let $\{X_f\}_{f \in I}$ be a covering of X by basic opens, that is, $X = \bigcup_{f \in I} X_f$. This means that for every $\mathfrak{p} \in X$, there is some $f \in I$ with $f \notin \mathfrak{p}$. I claim that the f_i generate the unit ideal: for if not, $\langle f_i \rangle$ would be contained in some prime ideal, but by the above, this is not the case. Hence there is an equation of the form $1 = \sum g_i f_i$ with $g_i \in A$, which is a *finite* sum. Hence these finitely many f_i suffice.
6. ...

♡

4.2 Chapter 2 - Modules

Exercise 17 (Exercise 1). Show that $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n = 0$ if m, n are coprime.

♠

Solution 16. Write $1 = am + bn$. Then

$$\begin{aligned} 1 \otimes 1 &= (am + bn) \otimes 1 = am \otimes 1 + bn \otimes 1 \\ &= 0 + bn \otimes 1 = 1 \otimes bn = 1 \otimes 0 = 0. \end{aligned}$$

And we are done.

♡

Exercise 18 (Exercise 2). Let A be a ring, \mathfrak{a} an ideal, and M an A -module. Then $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$.

♠

Solution 17. Start with

$$0 \rightarrow \mathfrak{a} \rightarrow A \rightarrow A/\mathfrak{a} \rightarrow 0.$$

Tensoring with M gives

$$\mathfrak{a} \otimes M \rightarrow M \rightarrow A/\mathfrak{a} \otimes_A M \rightarrow 0.$$

But $\mathfrak{a} \otimes_A M \simeq \mathfrak{a}M$, so that the sequence reads $A/\mathfrak{a} \otimes M \simeq M/\mathfrak{a}M$.

♡

Exercise 19 (Exercise 3). Let A be a local ring, M, N finitely generated A -modules. Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$.

♠

Solution 18. First a counterexample if A is not a local ring. Let $A = k[x]$ and $M = k[x]/(x-1)$ and $N = k[x]/(x)$. We can write $1 = -(x-1) + x$. Then $M \otimes_A N = 0$ by the same method as in Exercise 1 ($1 \otimes 1 = (-x+1+x) \otimes 1 = x \otimes 1 = 1 \otimes x = 0$).

Let $M_k := M \otimes k = M/\mathfrak{m}M$. By Nakayama's lemma, $M_k = 0 \Rightarrow M = 0$.

So suppose $M \otimes_A N = 0$. Then $(M \otimes_A N)_k = 0$. But this is isomorphic to $M_k \otimes_A N_k$ since $k \otimes_A k = k$. But $M_k \otimes_A N_k \simeq M_k \otimes_k N_k$, as k -modules, since everything in \mathfrak{m} acts trivially on M_k . But these are vector spaces over a field, now we must have $M_k = 0$ or $N_k = 0$, and by Nakayama we are done. \heartsuit

Exercise 20 (Exercise 4). Let M_i ($i \in I$) be any family of A -modules, and let M be their direct sum. Then M is flat if and only if each M_i is flat. \spadesuit

Solution 19. Let

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

be any exact sequence. Then tensoring with M gives

$$0 \rightarrow N' \otimes_A M \rightarrow N \otimes_A M \rightarrow N'' \otimes_A M \rightarrow 0.$$

We only need to check that the left map is injective. But we have $N' \otimes_A M \simeq \bigoplus_i N' \otimes_A M_i$ and $N \otimes_A M \simeq \bigoplus_i N \otimes_A M_i$, and thus the left map is just the direct sum of all the maps

$$0 \rightarrow N' \otimes_A M_i \rightarrow N \otimes_A M_i,$$

which is injective if and only if each M_i is flat. \heartsuit

Exercise 21 (Exercise 5). Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A . Prove that $A[x]$ is flat A -algebra. \spadesuit

Solution 20. We have $A[x] = \bigoplus_{i=0}^{\infty} x^i A$ as an A -module. Now use Exercise 4. \heartsuit

Exercise 22 (Exercise 24). If M is an A -module, the following are equivalent:

- i) M is flat.
- ii) $\text{Tor}_n^A(M, N) = 0$ for all $n > 0$ and A -modules N .
- iii) $\text{Tor}_1^A(M, N) = 0$ for all A -modules N .

Solution 21. To compute $\text{Tor}_A^n(M, N)$, one takes an A -resolution of N and tensor it with M and take homology. But M is flat, so the sequence stays exact, so the homology is zero. This shows $i) \Rightarrow ii)$.

The implication $ii) \Rightarrow iii)$ is trivial.

Now let

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

be any exact sequence of A -modules. Then by properties of the Tor functor, we have an exact sequence

$$\mathrm{Tor}_1(M, N'') \rightarrow N' \otimes M \rightarrow N \otimes M \rightarrow N'' \otimes M \rightarrow 0.$$

But $\mathrm{Tor}_1(M, N'') = 0$, so the sequence is short exact. Hence M is flat. ♡

Exercise 23 (Exercise 25). Let

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

be an exact sequence with N'' flat. Then N' is flat if and only if N is flat. ♠

Solution 22. We have from the Tor exact sequence

$$0 \rightarrow \mathrm{Tor}_1(N', M) \rightarrow \mathrm{Tor}_1(N, M) \rightarrow 0$$

since $\mathrm{Tor}_2(N'', M) = \mathrm{Tor}_1(N'', M) = 0$. The statement follows. ♡

♠

4.3 Chapter III - Rings and modules of fractions

Exercise 24 (Exercise 1). Let S be a multiplicatively closed subset of a ring A , and let M be a finitely-generated A -module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$. ♠

Solution 23. Suppose there exists such s . Let $m/s' \in S^{-1}M$. This is zero if and only if there exists $s \in M$ such that $s(s'm) = 0$. But $ss'm = s'sm = s'0 = 0$. So $m = 0$ in $S^{-1}M$. (note that we did not use finite generation)

Now let m_1, \dots, m_r be a set of generators for M and suppose that $S^{-1}M = 0$. Then for each i ($i = 1, \dots, r$), there exists s_i such that $s_i m_i = 0$. Since every element of M is an A -linear combination of the m_i , it follows that the product $s_1 s_2 \cdots s_r$ makes $sM = 0$. ♡

4.4 Chapter 5 - Integral dependence and valuations

Exercise 25 (Exercise 1). Let $f : A \rightarrow B$ be an integral morphism of rings. Show that $f^* : \mathrm{Spec} B \rightarrow \mathrm{Spec} A$ is a closed mapping. ♠

Solution 24. The map f^* is by definition given by $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p}) = \mathfrak{p} \cap A$. A closed subset of $\mathrm{Spec} B$ is by definition

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec} B \mid \mathfrak{p} \supset \mathfrak{a}\}$$

for some ideal $\mathfrak{a} \subset B$.

Then the image of $V(\mathfrak{a})$ is the set

$$f^*(V(\mathfrak{a})) = \{\mathfrak{p} \cap A \mid \mathfrak{p} \in \operatorname{Spec} B, \mathfrak{p} \supset \mathfrak{a}\}$$

I claim that this is equal to

$$V(\mathfrak{a} \cap A) = \{\mathfrak{q} \in \operatorname{Spec} A \mid \mathfrak{q} \supset \mathfrak{a} \cap A\},$$

which clearly is a closed subset of $\operatorname{Spec} A$.

One direction is obvious: let $\mathfrak{p} \cap A$ be an element of $f^*(V(\mathfrak{a}))$. This is a point of $\operatorname{Spec} A$, and clearly $\mathfrak{p} \cap A \supset \mathfrak{a} \cap A$ since $\mathfrak{p} \supset \mathfrak{a}$.

The other direction needs the going up Theorem 5.10. Suppose $\mathfrak{q} \in V(\mathfrak{a} \cap A)$. Then by Going Up, there exists $\mathfrak{p} \in \operatorname{Spec} B$ with $\mathfrak{p} \cap A = \mathfrak{q}$. But we need to check that $\mathfrak{p} \supset \mathfrak{a}$. That is, we need to prove the assertion that if $\mathfrak{q} = \mathfrak{p} \cap A$ and $\mathfrak{q} \supset \mathfrak{a} \cap A$, then $\mathfrak{p} \supset \mathfrak{a}$. So suppose $a \in \mathfrak{a} \subset B$. Then a satisfies an equation

$$a^n + b_{n-1}a^{n-1} + \dots + b_1a + b_0 = 0$$

with $b_i \in A$. Since $a \in \mathfrak{a}$, we see that $b_0 \in \mathfrak{q} = \mathfrak{p} \cap A$. Hence

$$a^n + b_{n-1}a^{n-1} + \dots + b_1a = a(a^{n-1} + b_{n-1}a^{n-2} + \dots + b_1) \in \mathfrak{p}$$

since $\mathfrak{q} \subset \mathfrak{p}$. But \mathfrak{p} is prime so either $a \in \mathfrak{p}$ and we are done, or $a^{n-1}b_{n-1}a^{n-2} + \dots + b_1 \in \mathfrak{p}$, and we can continue by induction.

Hence we are done. ♥

4.5 Chapter 7 - Noetherian rings

Exercise 26 (Exercise 11). Let A be a ring such that $A_{\mathfrak{p}}$ is Noetherian for each $\mathfrak{p} \in \operatorname{Spec} A$. Is A necessarily noetherian? ♠

Solution 25. Consider the ring

$$A = \mathbb{Z}/2 \times \mathbb{Z}/2 \cdots.$$

It is a countable product of noetherian rings. The primes are just the coordinate axes, and each localization is isomorphic to $\mathbb{Z}/2$. Thus each $A_{\mathfrak{p}}$ is Noetherian, but A is not. ♥

Exercise 27 (Exercise 15). Let A be a Noetherian local ring, \mathfrak{m} its maximal ideal and k its residue field and let M be a finitely generated A -module. Then the following are equivalent:

- i) M is free.
- ii) M is flat.
- iii) The mapping $\mathfrak{m} \otimes M \rightarrow A \otimes M$ is injective.
- iv) $\mathrm{Tor}_1^A(k, M) = 0$.



Solution 26. The implication $i) \Rightarrow ii)$ is trivial. One way is to compute $\mathrm{Tor}_1^A(M, N)$ for any A -module N . But a free resolution of M is just one-term, so $\mathrm{Tor}_1^A(M, N)$ is automatically zero.

The implication $ii) \Rightarrow iii)$ follows by tensoring the inclusion $\mathfrak{m} \hookrightarrow A$ with M .

The implication $iii) \Rightarrow iv)$ follows from the Tor exact sequence

$$\mathrm{Tor}_1^A(A, M) \rightarrow \mathrm{Tor}_1^A(k, M) \rightarrow \mathfrak{m} \otimes M \rightarrow A \otimes M \rightarrow k \otimes M \rightarrow 0.$$

The leftmost term is zero since A is a free A -module, and by $iii)$ and exactness we must as well have $\mathrm{Tor}_1^A(k, M) = 0$.

Now for $iv) \Rightarrow i)$. Choose element $m_i \in M$ ($0 \leq i \leq r$) such that they form a k -basis for $M/\mathfrak{m}M$. Choose a surjection $f : A^r \rightarrow M$ and let $E = \ker f$ be its kernel. Then we have an exact sequence

$$0 \rightarrow E \rightarrow A^r \rightarrow M \rightarrow 0.$$

of finitely-generated A -modules (E is finitely generated by Proposition 6.2). Tensor the sequence by k , and get

$$\mathrm{Tor}_1^A(k, M) \rightarrow E/\mathfrak{m}E \rightarrow k^r \rightarrow M/\mathfrak{m}M \rightarrow 0.$$

The left-most term is zero by assumption. The last two spaces are k -vector spaces of the same dimension, and it follows that $E/\mathfrak{m}E = 0$. But then it follows that E is zero by Nakayama's lemma, hence M is free. \heartsuit

Exercise 28 (Exercise 16). Let A be a Noetherian ring, M a finitely-generated A -module. Then the following are equivalent:

- i) M is a flat A -module.
- ii) $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module for each $\mathfrak{p} \in \mathrm{Spec} A$.
- iii) $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$ -module for each maximal ideal \mathfrak{m} .

So flatness is the same as being locally free. ♠

Solution 27. The implications $i \Rightarrow ii$ and $ii \Rightarrow iii$ follows trivially from the previous exercise. We prove $iii \Rightarrow i$.

Applying the Tor functor commutes with localization, hence we have $\text{Tor}_1^A(M, N)_{\mathfrak{m}} = \text{Tor}_1^{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) = 0$ for all \mathfrak{m} . But being zero is a local property, so it follows that $\text{Tor}_1^A(M, N) = 0$ for all A -modules N . Hence M is flat. ♡

5 Representation Theory - Fulton, Harris

5.1 Representations of Finite Groups

Exercise 29 (Exercise 1.1). Verify that the relation

$$\langle g \cdot v^*, g \cdot v \rangle = \langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle$$

is satisfied when we define

$$\rho^*(g) = \rho(g^{-1})^t : V^* \rightarrow V^*,$$

that is, $(\rho^*g)(v^*)(w) = \langle (\rho^*g)(v^*), w \rangle = \langle v^*, (\rho g^{-1})(w) \rangle$. ♠

Solution 28. This is a matter of calculation.

$$\langle gv^*, gv \rangle = \langle v^*, (\rho g^{-1})(gv) \rangle = \langle v^*, v \rangle.$$

So the definition is ok. ♡

Exercise 30 (Exercise 1.2). Verify that in general the vector space of G -linear maps between two representations V and W of G is just the subspace $\text{Hom}(V, W)^G$ of elements of $\text{Hom}(V, W)$ fixed under the action of G . This subspace is often denoted $\text{Hom}_G(V, W)$. ♠

Solution 29. A map $\varphi : V \rightarrow W$ is G -linear when $\varphi(gv) = g\varphi(v)$. The action of G on φ is given by $g\varphi(v) = g\varphi(g^{-1}v)$. But by G -linearity, this is

$$\varphi(gv) = gg^{-1}\varphi(gv) = gg^{-1}\varphi(v) = \varphi(v).$$

Hence a map is G -linear if and only if it is fixed by the action of G . ♡

Exercise 31 (Exercise 1.3). Let $\rho : G \rightarrow \text{GL}(V)$ be any representation of the finite group G on an n -dimensional vector space V and suppose that for any $g \in G$, the determinant of $\rho(g)$ is 1. Show that the spaces $\wedge^k V$ and $\wedge^{n-k} V^*$ are isomorphic as representations of G . ♠

Solution 30. This is (again) just a matter of writing out the definitions. First we define the isomorphism, and then we check that it is actually an isomorphism of representations.

$$\bigwedge^k V \rightarrow \bigwedge^{n-k} V^*$$

$$v_1 \wedge \cdots \wedge v_k \mapsto (w_1 \wedge \cdots \wedge w_{n-k} \mapsto v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_{n-k})$$

Being a map of representations is equivalent to $g^{-1}\varphi(gv) = \varphi(v)$, so we just need to check that all the g 's disappear from the left hand side.

$$\begin{aligned} g^{-1}\varphi(gv) &= g^{-1}(w_1 \cdots w_{n-k} \mapsto gv_1 \cdots gv_k w_1 \cdots w_{n-k}) \\ &= (gv_1 \cdots gv_k gw_1 \cdots gw_{n-k}) \\ &= \det \rho(g) v_1 \wedge \cdots \wedge w_{n-k}. \end{aligned}$$

Hence φ is a map of representations if and only if $\det \rho(g) = 1$ for all $g \in G$.

(it is an isomorphism because it has zero kernel: because what would the kernel be? Every subspace is the same, and this is a basis free description)

♡

Exercise 32 (Exercise 1.4). The permutation representation R of G acting on a finite set X have two descriptions: one is given by letting V be the vector space with basis $\{e_x \mid x \in X\}$ and letting g act on V by $ge_x = e_{gx}$.

Alternatively R is the set of functions $f : X \rightarrow \mathbb{C}$ with action $(g\alpha)(h) = \alpha(g^{-1}h)$.

- a) Show that these two descriptions agree by identifying e_x with the characteristic function which takes the value 1 on x and 0 elsewhere.
- b) The space of functions on G can also be made into a G -module by the rule $(g\alpha)(h) = \alpha(hg)$. Show that this is an isomorphic representation.

♠

Solution 31. a). Clearly the vector space dimensions agree (since the characteristic functions are a basis). So we need to check that this is a map of representations. Denote the characteristic function by χ_x . Then $\varphi(ge_x)(h) = \varphi(e_{gx})(h) = \chi_{gx}(h)$. Similarly $g\varphi(e_x)(h) = g\chi_x(h) = \chi_x(g^{-1}h)$. The first function is 1 if $gx = h$, and the second function is 1 if $g^{-1}h = x$, and these are equivalent.

b). Send α to the function $g \mapsto \alpha(g^{-1})$. Call this assignment ψ . We need to check that $\psi(g\alpha) = g\psi(\alpha)$.

First the left hand side. We have: $\psi(g\alpha)(h) = \psi(h \mapsto \alpha(g^{-1}h))(h) = \alpha(g^{-1}h^{-1})$.

And similarly: $g\psi(\alpha)(h) = g(h \mapsto \alpha(h^{-1}))(h) = g\alpha(h^{-1}) = \alpha(g^{-1}h^{-1})$.

And these are equal. ♥

Exercise 33 (Exercise 1.10). $G = S_3$. Verify that with $\sigma = (12)$, $\tau = (123)$, the standard representation has a basis $\alpha = (\omega, 1, \omega^2)$, $\beta = (1, \omega, \omega^2)$, with

$$\tau\alpha = \omega\alpha, \quad \tau\beta = \omega^2\beta, \quad \sigma\alpha = \beta, \quad \sigma\beta = \alpha.$$

♠

Solution 32. The standard representation V is the subspace $\{x_1 + x_2 + x_3 = 0\}$ of \mathbb{C}^3 . Since $1 + \omega + \omega^2 = 0$, and $\alpha \cdot \beta = 3\omega \neq 0$, these two span V .

The identities are easy. ♥

Exercise 34 (Exercise 1.11). Use this approach to find the decomposition of the representations $\text{Sym}^2 V$ and $\text{Sym}^3 V$. ♠

Solution 33. The elements $\{\alpha^2, \alpha\beta, \beta^2\}$ are a basis of $\text{Sym}^2 V$, and the eigenvalues are $\omega^2, 1$ and ω , respectively. Thus $\langle \alpha\beta \rangle$ span a representation isomorphic to U , the trivial representation, and $\langle \alpha^2, \beta^2 \rangle$ span a representation isomorphic to V , the standard representation. Hence $\text{Sym}^2 V = U \oplus V$.

The elements $\{\alpha^3, \alpha^2\beta, \alpha\beta^2, \beta^3\}$ are a basis of $\text{Sym}^3 V$. The eigenvalues are $1, \omega, \omega^2$ and 1 , respectively. Looking at the action of $\sigma = (12)$, we see that $U \simeq \langle \alpha^3 + \beta^3 \rangle$, and $U' \simeq \langle \alpha^3 - \beta^3 \rangle$. The remaining $\langle \alpha^2\beta, \alpha\beta^2 \rangle$ span a representation isomorphic to V . Hence $\text{Sym}^3 V = U \oplus U' \oplus V$. ♥

Exercise 35 (Exercise 2.2). For $\text{Sym}^2 V$, verify that

$$\chi_{\text{Sym}^2 V}(g) = \frac{1}{2} [\chi_V(g)^2 + \chi_V(g^2)].$$

Note that this is compatible with the decomposition $V \otimes V = \text{Sym}^2 V \oplus \wedge^2 V$. ♠

Proof. The eigenvalues of g acting on $\text{Sym}^2 V$ are $\{\lambda_i \lambda_j\}$. Hence

$$\begin{aligned}\chi_{\text{Sym}^2 V}(g) &= \sum_{i \leq j} \lambda_i \lambda_j \\ &= \sum_{i < j} \lambda_i \lambda_j + \sum_i \lambda_i^2 \\ &= \frac{1}{2} (\chi_V(g)^2 - \chi_V(g^2)) + \chi_V(g^2) \\ &= \frac{1}{2} (\chi_V(g)^2 + \chi_V(g^2)).\end{aligned}$$

□

Exercise 36 (Exercise 2.5, The original fixed point formula). If V is a permutation representation associated to the action of a group G on a finite set X , show that $\chi_V(g)$ is the number of elements fixed by g . ♠

Solution 34. This is easy. The matrix associated to g is a permutation matrix with a 1 in row j if element number i is sent to j . Then number of fixed points is the number of ones on the diagonal, and this is $\chi_V(g)$. ♡

Exercise 37 (Exercise 2.34). Let V, W be irreducible representations of G and $L_0 : V \rightarrow W$ any linear mapping. Define $L : V \rightarrow W$ by

$$L(v) = \frac{1}{|G|} \sum_g g^{-1} L_0(gv).$$

Show that $L = 0$ if V and W are not isomorphic, and that L is multiplication by $\text{tr}(L_0)/\dim(V)$ if $V = W$. ♠

Solution 35. We want to apply Schur's lemma. We check that L is a G -module homomorphism. We have

$$\begin{aligned}L(hv) &= \frac{1}{|G|} \sum_g g^{-1} L_0(ghv) \\ &= \frac{1}{|G|} \sum_{gh} hgh^{-1} L_0(ghv) \\ &= \frac{1}{|G|} \sum_{g'} hg'^{-1} L_0(g'v)\end{aligned}$$

Hence L is a G -module homomorphism. Hence by Schur's lemma, L is either the zero map or an isomorphism. In particular, if they are not isomorphic, $L = 0$. Now suppose $V = W$. ♡