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Fredrik Meyer

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1 Derived categories and derived functors

We first start with notation. Let $\mathbb{A}$ be an abelian category. Let $C(\mathbb{A})$ be the category of complexes with objects and maps from $\mathbb{A}$. We denote by $C^b(\mathbb{A})$ the category of bounded complexes. $C^+(\mathbb{A})$ is the category of bounded below complexes, and $C^-(\mathbb{A})$ is the category of bounded above complexes.

Definition 1.1. Two morphisms $f, g : A^\bullet \to B^\bullet$ are *homotopy equivalent* if there exists a set of maps $h^i : A^i \to B^{i-1}$ such that

$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i.$$

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The homotopy category $K(\mathbb{A})$ is the category with the same objects as $C(\mathbb{A})$, but where homotopic morphisms are identified.

This is a triangulated category. For this we need a shift operator: $A^\bullet \mapsto A^\bullet[1]$ (topologists like to write $A^\bullet[1] = \Sigma A^\bullet$). The differential is $d^i_{A[1]} = -d_A^{i+1}$.

The *cone of a morphisms* $j : A^\bullet \to B^\bullet$ is denoted by $C(j)$ and defined as follows: each term is $C(j)^i = A^{i+1} \oplus B^i$, and the maps are

$$d_{C(j)} = \begin{pmatrix} -d_A^{i+1} & 0 \\ j^{i+1} & d_B^i \end{pmatrix}.$$

Then a *distinguished triangle* is a sequence of objects in $K(\mathbb{A})$ and morphisms isomorphic to the sequence

$$A^\bullet \xrightarrow{j} B^\bullet \to C(j) \to A[1].$$

The derived category $D(\mathbb{A})$ is the category with objects the same as $C(\mathbb{A})$, and morphisms obtained by localizing $K(\mathbb{A})$ with respect to quasiisomorphisms (that is, we require quasiisomorphisms to be invertible). Formally, a morphisms $A^\bullet \to B^\bullet$i n the derived category is defined as a *roof*:

$$\begin{tikzcd} & C^\bullet \arrow[dl] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

The left arrow is a quasiisomorphism.

Knowing that $K(\mathbb{A})$ is triangulated, we deduce (somehow) that $D(\mathbb{A})$ is triangulated as well.

1.1 Two examples

1. Let X be a smooth point. Then $D(X)$ is easy to describe. All complexes of vector spaces are quasiisomorphic to the direct sum of their cohomologies. (more explicit??)

2. Let X be a smooth projective curve and $\mathcal{E} \in \mathsf{Coh}(()X)$. Then there is an exact sequence
$$0 \to \mathcal{T} \to \mathcal{E} \to \mathcal{E}_{tf} \to 0,$$
where the last term is torsionfree and the first term is a torsion sheaf. This sequence splits (why?). And one can show that
$$D^b(X) \ni \mathcal{E} = \oplus_{i\in I} \mathscr{H}^i(\mathcal{E})[-i] = \oplus_i (\mathscr{H}^i(\mathcal{E}_{tors}) \oplus \mathscr{H}^i(\mathcal{E})_{tf})$$
(I don't see why?)

1.2 Semiorthogonal decomposition for $D^b(X)$

Suppose we have a sequence of full triangulated subcategories of $D^b(X)$.

Definition 1.2. A colleection $T_1, \dots, T_n$ defines a *semiorthogonal decomposition* for $D^b(X)$ if

1. $\mathrm{Hom}_{D^b(X)}(T_i, T_j) = 0$ for $i > j$.

2. For all $F \in D^b(X)$, there exists a chain of morphisms
$$0 \to E_n \to E_{n-1} \to \dots \to E_0 = F,$$
such that $\mathrm{Cone}(E_i, E_{i-1}) \in T_i$.

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We write $D^b(X) = \langle T_1, \ldots, T_n \rangle$.

Example 1.3. Beilinson proved that $D^b(\mathbb{P}^n) = \langle \mathscr{O}_{\mathbb{P}^n}, \ldots \mathscr{O}_{\mathbb{P}^n}(n) \rangle$. ★

Example 1.4. If X is a curve (smooth projective over $\mathbb{C}$), then $D^b(X)$ has a nontrivial semiorthogonal decomposition if and only if the genus is zero. ★

Example 1.5. If X is smooth projective with $\omega_X = \mathscr{O}_X$, then $D^b(X)$ has no semi-orthogonal decomposition. ★

1.3 Derived functors