Components of the Hilbert scheme in $\mathbb{P}^{11}$

Fredrik Meyer

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Abstract

After a preliminary chapter introducing concepts and definitions, we explore a connection between triangulations of $\mathbb{CP}^2$ and degenerations of hyper-Kähler manifolds. Then we study the topology of the two smoothings of the affine cone over dP_6, and prove that they are topologically different by computing their singular cohomology groups. In the last chapter we construct new examples of smooth Calabi–Yau manifolds, and their respective mirrors.

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General theme: smoothings of SR-schemes in order to construct new manifolds (or to put known manifolds in nice families). Use orbifold construction to construct mirrors. Relation to other work?

1. Smoothings of Stanley-Reisner schemes
2. Paths on the Hilbert scheme
3. Relation to triangulation of $\mathbb{CP}^2$.

The source code of the thesis and all computer computations are available on github

Notes

Notation

If V is a vector space, we denote by $\mathbb{P}(V)$ its projectivisation. We write k for a field, which is almost always assumed to be $\mathbb{C}$. If X is a projective variety, we write $S(X)$ for its homogeneous coordinate ring (if the embedding is implicit). If X is a scheme over k, we write X/k. We will write $h^i(X, \mathscr{F})$ for $\dim_k H^i(X, \mathscr{F})$. All schemes are noetherian. We will often write $\triangleq$ for definitions (instead of ":=", common in computer science literature).

CHAPTER 1

Preliminaries

ec:prelims

In this chapter we introduce the consistent notation and results which will be used later. Some of the material in this chapter plays the role of motivation more than preliminary results.

1.1 The join of projective varieties

There are many ways to define the join of two projective varieties X and Y. We will define it in a particularly general way, as described by Altman and Kleiman in [AK75]. Then we will specialize to our situation.

Fix a base scheme S. Let $\mathscr{C}$ be the category of graded, quasi-coherent $\mathscr{O}_S$-algebras, generated in degree 1. The tensor product of two $\mathscr{O}_S$-algebras $\mathscr{R}$ and $\mathscr{S}$ is naturally graded: the degree d part is given by

$$(\mathscr{R} \otimes_{\mathscr{O}_S} \mathscr{S})_d = \bigoplus_{p+q=d} \mathscr{R}_p \otimes \mathscr{S}_q.$$

Let $X = \operatorname{Proj} \mathscr{R}$ and $Y = \operatorname{Proj} \mathscr{S}$. Then we define the *join* of the graded $\mathscr{O}_S$-algebras to be

$$X * Y \triangleq \operatorname{Proj}(\mathscr{R} \otimes_{\mathscr{O}_S} \mathscr{S}).$$

If X and Y are projective varieties over S, they come with graded $\mathscr{O}_S$-algebras $\mathscr{R} = \operatorname{Sym}_S \mathscr{O}_X(1)$ and $\mathscr{S} = \operatorname{Sym}_S \mathscr{O}_Y(1)$. Then we define the join of X and Y to be join of these algebras.

The join construction is a contravariant functor in two variables from the category of graded $\mathscr{O}_S$-algebras and surjective maps to the category of projective varieties.

Example 1.1.1. Let $X = \mathbb{P}(E)$ and $Y = \mathbb{P}(F)$, where E, F are quasi-coherent $\mathscr{O}_S$-modules. Then we have the equality $\mathbb{P}(E) * \mathbb{P}(F) = \mathbb{P}(E \oplus F)$, because of the linear algebra fact $\operatorname{Sym}(E) \otimes \operatorname{Sym}(F) = \operatorname{Sym}(E \oplus F)$. ♡

The algebra $\mathscr{R}\otimes_{\mathscr{O}_S}\mathscr{S}$ contains the ideal $\mathscr{R}\otimes\mathscr{S}_+$. The associated subscheme is denoted by V_X, and is isomorphic to $X = \operatorname{Proj}\mathscr{R}$. We define V_Y similarly. We call V_X and V_Y the *fundamental subschemes* of $X * Y$.

There is a geometric definition of the join, as described in setion (C11) in [AK75]. Let E, F be quasi-coherent $\mathscr{O}_S$-modules[1]. Suppose X, Y are subschemes of $\mathbb{P}(E)$ and $\mathbb{P}(F)$. Then $X * Y$ is a closed subscheme of $\mathbb{P}(E \oplus F)$. Identify X and Y with their fundamental subschemes in $X * Y$. Then it is not difficult to see that $X * Y$ is the (closure of the) locus of points lying on the lines of $\mathbb{P}(E \oplus F)$ determined by pairs of points of X and Y.

For a graded $\mathscr{O}_S$-algebra $\mathscr{R}$ there are two natural homomorphisms. We have the *structure map* $\rho : \mathscr{O}_S \to \mathscr{R}$, and the *augmentation map* $\epsilon : \mathscr{R} \to \mathscr{O}_S$, sending everything in positive degree to zero. Clearly $\epsilon \circ \rho = \mathrm{id}_S$.

lemma:join

Proposition 1.1.2. *Suppose $X/k \subset \mathbb{P}^n$ and $Y/k \subset \mathbb{P}^m$ are smooth projective schemes. Then their join, $X * Y$ have dimension* $\dim X + \dim Y + 1$. *The singular locus is of dimension* $\max\{a, b\}$ *and consist of the disjoint union of X and Y.*

Proof. Let $\mathscr{R} = \bigoplus_{d\geq 0}\mathscr{O}_X(d)$ and $\mathscr{S} = \bigoplus_{d\geq 0}\mathscr{O}_Y(d)$ be the homogeneous coordinate rings of X and Y. Then $X * Y \subset \mathbb{P}^{n+m+1}$.

Denote by $C(X * Y)$ the scheme $\operatorname{Spec}(\mathscr{R}\otimes\mathscr{S})$, the affine cone over $X * Y$.

The singular locus of $C(X * Y) = C(X) \times C(Y)$ is equal to $\operatorname{Sing} C(X) \times C(Y) \bigcup C(X) \times \operatorname{Sing} C(Y)$. Since X and Y are smooth, the only singular point on the affine cones are the origins. Hence

$$\operatorname{Sing}\left(C(X * Y)\right) = 0 \times \operatorname{Sing}(C(Y)) \bigcup \operatorname{Sing}(C(X)) \times 0.$$

Projectivising, we find that $\operatorname{Sing}(X * Y) = \operatorname{Sing} Y \sqcup \operatorname{Sing} X$, since $(0, \ldots, 0)$ is the only common point of the affine cones. ■

Sjekk definisjoner!!

Recall that a scheme X is *Gorenstein* if it has a dualizing sheaf. It is *Cohen–Macaulay* if the dualizing sheaf is a line bundle. If S is the homogeneous coordinate ring of a projective Gorenstein variety $X \subset \mathbb{P}^n$, the canonical sheaf can be computed as the sheaf associated to the graded module $\operatorname{Ext}^{\operatorname{codim} X}(S, S(-n-1))$.

sjekk definisjoner!!

If X and Y are two Gorenstein schemes, then their join is also Gorenstein. Furthermore, we can compute the canonical sheaf in terms of the canonical sheaves of X and Y.

Proposition 1.1.3. *Let $X = \operatorname{Proj}\mathscr{R}$ and $Y = \operatorname{Proj}\mathscr{S}$ be Gorenstein projective schemes with dualizing sheaves ω_X, ω_Y, respectively (here $\mathscr{R}, \mathscr{S}$ are sheaves of graded $\mathscr{O}_S$-algebras). Then $X * Y$ is Gorenstein with dualizing sheaf $\omega_X \otimes_S \omega_Y$.*

[1] In our case, $S = \operatorname{Spec} k$ almost always. So E, F are just vector spaces.

Proof. The question is local on S, so we may assume that R and S are homogeneous coordinate rings. Then the statement follows from Theorem 4.2 in [HHS16], where the authors prove that the canonical module of a tensor product is the tensor product of the canonical modules. ■

:canonical

Remark 1.1.4. If X and Y are Gorenstein projective schemes, the resolution of the structure sheaf is symmetrical. It follows that $\omega_X = \mathscr{O}_X(-n)$ for some $n \geq 0$. If $\omega_Y = \mathscr{O}_Y(-m)$, it follows from the above proposition that $\omega_{X*Y}(-m-n)$.

1.2 Calabi–Yau manifolds and mirror symmetry

calabi_yau

Parts of this thesis will be concerned with the construction of new Calabi–Yau manifolds. For us, a Calabi-Yau manifold will be a irreducible complex projective variety X such that $\omega_X \simeq \mathscr{O}_X$ and $H^i(X, \mathscr{O}_X) = 0$ for $i = 1, \dots, \dim X - 1$. We will always have $\dim X = 3$. Be ware that the literature often requires Calabi–Yau manifolds to be smooth, or to have only certain kinds of singularities.

Mathematically, Calabi–Yau varieties are interesting because they are among the varities having Kodaira dimension zero. This means that they have trivial canonical models, making them harder to study. Before the 90's there were only sporadic constructions of Calabi–Yau varieties, but after the advent of toric geometry and the construction of Batyrev in [Bat94], thousands of new examples were found, all of which was anticanonical sections of Fano toric varieties. We will explain this construction in the next section. do that

Let Ω^1_X be the sheaf of holomorphic one-forms on X, and assume that $\dim X = 3$. Then we can form the *Hodge diamond* of X, which is a format of writing the dimensions h^{ij} of the cohomology groups $H^j(X, \Omega^i_X)$. Here Ω^i_X is by definition the wedge product $\wedge^i \Omega^1_X$.

$$\begin{array}{ccccccc}
 & & & h^{00} & & & \\
 & & h^{01} & & h^{10} & & \\
 & h^{02} & & h^{11} & & h^{20} & \\
h^{03} & & h^{12} & & h^{21} & & h^{30} \\
 & h^{13} & & h^{22} & & h^{31} & \\
 & & h^{23} & & h^{32} & & \\
 & & & h^{33} & & &
\end{array}$$

Because of the Calabi–Yau condition, we have that $h^{j0} = 0$ for $0 < j < 3$, and also that $h^{00} = h^{0d} = 1$. It follows by Serre duality (see [Har77, Corollary 7.7, Chapter III]) that $h^{ij} = h^{3-i,3-j}$. Note that this amounts to a horizontal symmetry of the Hodge diamond. Since X was assumed to be a complex manifold, it follows that $h^{ij} = h^{ji}$ by complex conjugation[2]. This amounts to

[2] This follows from the $\bar{\delta}$-Poincaré lemma.

vertical symmetry of the Hodge diamond. It follows that for 3-dimensional Calabi–Yau varieties, the Hodge diamond simplifies to

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{11} & & 0 & \\ 1 & & h^{12} & & h^{12} & & 1 \\ & 0 & & h^{11} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}.$$

The *Hodge decomposition* theorem [Voi02, page 142] states that the singular cohomology groups decomposes as

$$H^k(X,\mathbb{C}) = \bigoplus_{i+j=k}^{\dim X} H^i(X,\Omega_X^j).$$

The *topological Euler characteristic* is defined as

$$\chi(X) = \sum_{k=0}^{2\dim X} \dim_{\mathbb{C}} H^k(X,\mathbb{C}).$$

For 3-dimensional Calabi–Yau varieties, it follows from the above discussion that $\chi(X)$ can be computed as $2(h^{11} - h^{12})$.

Example 1.2.1. The canonical example of a Calabi–Yau variety is the quintic in $\mathbb{P}^4$. Let $X = V(f)$ be defined by a general element in $H^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(5))$. Then X is smooth, and by adjunction formula we have $\omega_X = \omega_{\mathbb{P}^4}\big|_X \otimes \mathscr{O}_X(5) = \mathscr{O}_X(-5) \otimes \mathscr{O}_X(5) = \mathscr{O}_X$, so the canonical bundle is trivial. By the ideal sheaf sequence, we find that $H^i(X,\mathscr{O}_X) \simeq H^i(X,\mathscr{O}_{\mathbb{P}^4}(-5))$, which by [Har77, Theorem 5.1, Chapter III] implies the required vanishing of the structure sheaf cohomology groups.

The Euler characteristic can be computed as the degree of the top Chern class of X. If Y is a degree d hypersurface in $\mathbb{P}^n$, the following formula holds:

$$c_{n-1}(T_X) = h^{n-1}\left(\binom{n+1}{n-1} - d\binom{n+1}{n-2} + d^2\binom{n+1}{n-3} + \dots\right).$$

Putting $n = 4$ and $d = 5$, we find that $\chi(X) = -200$. To compute h^{11}, we see first from the conormal sequence that $H^1(\Omega^1_Y) \simeq H^1(\Omega^1_{\mathbb{P}^4}\big|_Y)$. Now it follows easily from the restricted Euler sequence that $h^{11} = 1$, and since $\chi(X) = 2(h^{11} - h^{12})$ it follows that $h^{12} = 101$. ♡

In general it is very hard to compute the Hodge numbers of Calabi–Yau varieties, with the exception of hypersurfaces in four-dimensional toric varieties. Often the best one can hope for is the topological Euler characteristic $\chi(X)$, which is much easier to compute.

There is a connection between Calabi–Yau manifolds and Fano manifolds. In particular, the following is true:

calsection **Proposition 1.2.2.** *Let $Y \subset \mathbb{P}^N$ be an n-dimensional Fano variety with $\omega_Y = \mathscr{O}_Y(-k)$. Then a general section X of $\mathscr{O}_Y(1)^{\oplus k}$ is an $n-k$-dimensional Calabi–Yau variety.*

Proof. This follows from the adjunction formula, which says that

$$\omega_X = \omega_Y \otimes \wedge^k (\mathcal{I}_X/\mathcal{I}_X^2)^\vee.$$

A general section of $\mathscr{O}_Y(1)^{\oplus k}$ is a complete intersection, and the normal bundle is then equal to $\mathscr{O}_X(1)^{\oplus k}$. It is then true that $\wedge^k \mathscr{O}_X(1)^{\oplus k} = \mathscr{O}_X(k)$, from which the triviality of the canonical bundle follows.

Recall that the structure cohomology groups $H^i(Y, \mathscr{O}_Y) = 0$ for $i > 0$ when Y is a Fano variety. That the cohomology groups $H^i(X, \mathscr{O}_X)$ are zero for $i = 1, \ldots, n-k$ can be ∎

prove this (use technique of [CK99])

The last chapter of this thesis is concerned about the construction of new examples of Calabi–Yau manifold. They are complete intersections in different (singular) toric varieties.

Mirror symmetry

After the invention of string theory in the late 60's, Calabi–Yau varieties caught the attention of theoretical physicists. They predict that space-time is really 10-dimensional, and locally looks like $\mathbb{R}^4 \times X$, where M is a Calabi–Yau manifold, of complex dimension 3. They discovered a duality between different Calabi–Yau 3-manifolds X and X° such that their Hodge numbers satisfy $h^{11}(X) = h^{12}(X^\circ)$ and $h^{12}(X) = h^{11}(X^\circ)$ (in particular $\chi(X) = -\chi(X^\circ)$). This a conjectural correspondence between the complex moduli space of X (which have tangent space $H^1(X, \Omega_X^2)$, and the Kähler moduli space of ω°, the Kähler class on X°. These two manifolds, though mathematically different, are thought to give rise to equivalent physical theories. This correspondence is called *mirror symmetry*.

In the 90's, Candelas et al. constructed the mirror of the general quintic [Can+91]. They calculate certain Hodge theoretic invariants on the mirror, and use them to count *rational curves* on the quintic. This greatly surprised the mathematical community, and led to the invention of Gromov–Witten-invariants and homological mirror symmetry.

Mirror symmetry is a fascinating and notoriously technical topic. There are several good introductions, depending upon taste and technical proficiency. Two of the most comprehensive introductions are [CK99; Hor+03].

Constructing mirrors of Calabi–Yau manifolds have become a small industry in the mathematics community. In the last chapter of this thesis, I propose mirror candidates for two of my Calabi–Yau constructions.

flytt ansatz-avsnittet hit. Forklar med eksempler.

1.3 Toric geometry

:toric_geometry

A toric variety is an irreducible and normal variety containing the torus $T = (\mathbb{C}^*)^n$ as a dense subset, such that the action of the torus on itself extends to an action on the variety.

We fix some notation that will be used throughout. Details and proofs can be found in [CLS11; Ful93]. Each toric variety comes with two dual lattices. The *lattice of 1-parameter subgroups* N and the *character lattice* M. A one-parameter subgroup is a morphism $\lambda : \mathbb{C}^* \to T$ that is a group homomorphism. The set of one-parameter subgroups is a lattice isomorphic to $\mathbb{Z}^n$, and we denote it by N. A *character* is a morphism $\chi : T \to \mathbb{C}^*$ that is a group homomorphism. The set of characters is a lattice M isomorphic to $\mathbb{Z}^n$ which is naturally dual to N.

Let V be a $\mathbb{R}$-vector space. Let $V^\vee$ be the dual vector space. A *convex polyhedral cone* is a subset σ of V of the form

$$\sigma = \{r_1 v_1 + \ldots + r_s v_s \mid r_i \geq 0 \text{ for all } i\},$$

where the v_i is a finite set of vectors in V and the r_i are real numbers. A *rational polyhedral cone* is a cone such that the vectors v_i can be taken to have rational coordinates.

The *dual cone* $\sigma^\vee$ lives in $V^\vee$, and is defined as the set of functionals that are positive on σ:

$$\sigma^\vee = \{u \in V^\vee \mid \langle u, v\rangle \geq 0, v \in \sigma\}.$$

Cones have two descriptions: either as the positive hull of a finite set of vectors (as above), or implicitly, as the intersection of finitely many half-spaces. If u_i generate $\sigma^\vee$, then it is true that

$$\sigma = \sigma^{\vee\vee} = \{v \in V \mid \langle u_i, v\rangle \geq 0 \text{ for all } i\}.$$

The vectors u_i are the inner normal vectors of the facets of σ.

A *(commutative) semigroup* is a set S with a commutative binary operation $S \times S \to S$, together with an identity element $0 \in S$. Given a cone $\sigma \subset N$, we can form a semigroup $S \triangleq \sigma^\vee \cap M \subset M$. Given a semigroup S, we can form the *semigroup algebra* $\mathbb{C}[S]$: it is the algebra generated by the elements

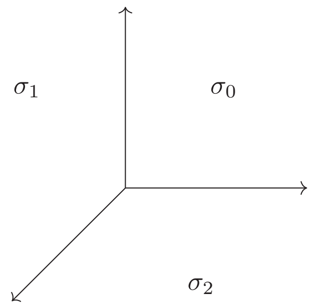


Figure 1.1: The fan corresponding to the toric variety $\mathbb{P}^2$. fig:p2fan

of S, with multiplicative structure inherited from S. We then define U_σ as $\operatorname{Spec}\mathbb{C}[\sigma^\vee \cap M]$, and call it the *affine toric variety associated to* σ.

Example 1.3.1. Let $\sigma = \langle (1,0),(1,2)\rangle \subset \mathbb{R}^2$. Then

$$\sigma^\vee = \langle (2,-1),(1,0),(0,1)\rangle \subset \mathbb{R}^2.$$

Then $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[x,y,x^2/y]$, where we have identified x and y with the standard basis of $\mathbb{R}^2$. This ring is isomorphic to $\mathbb{C}[a,b,c]/(a^2-bc)$, which is a quadric cone. ♡

General toric varities are described using collections of cones called *fans*. A set Σ of cones is called a *fan* if it closed under intersections and faces of cones: if $\sigma, \sigma' \in \Sigma$, then we also have $\sigma \cap \sigma' \in \Sigma$, and if $\sigma' \subset \sigma$ is a face with $\sigma \in \Sigma$, then $\sigma' \in \Sigma$ also. Thus, given a fan Σ, we get a collection of affine toric varieties U_σ for each cone $\sigma \in \Sigma$. We have inclusions $U_{\sigma\cap\sigma'} \subset U_\sigma$, and using these inclusions we glue to get a separated toric variety. If the fan is *complete* (meaning that the union of its cones is equal to N), the corresponding toric variety is complete.

Example 1.3.2. Consider Figure 1.1. This is the fan corresponding to the toric variety $\mathbb{P}^2$. The dual cones $\sigma_i^\vee$ give rise to algebras $\mathbb{C}[x,y]$, $\mathbb{C}[\frac{1}{x},\frac{y}{x}]$ and $\mathbb{C}[\frac{x}{y},\frac{1}{y}]$. These glue to form $\mathbb{P}^2$. More complicated fans give rise to exponents in the monomial generators. ♡

Projective toric varieties can be constructed from lattice polytopes. We describe the procedure here. Let Δ be a lattice polytope in $M \simeq \mathbb{Z}^n$. Let $M' = M \oplus \mathbb{Z}$, and embed Δ in M' by sending v to $(v,1)$. Let $C(\Delta)$ be the cone over Δ in M'. Then $\mathbb{C}[C(\Delta)\cap M']$ is a $\mathbb{Z}$-graded algebra. We let X_Δ be the associated projective variety.

Divisors and Picard groups of toric varieties

Recall that a *Weil divisor* is a formal linear combination of codimension 1 subvarieties of a scheme X (satisfying the "star" condition in Hartshorne). The group of Weil divisors modulo linear equivalence is the *class group* of X, and is denoted by $\mathrm{Cl}(X)$. The group of line bundles modulo isomorpism is the *Picard group* of X, and is denoted by $\mathrm{Pic}(X)$. The two groups coincide for smooth varieties. They are in general very hard to compute, but for toric varieties the computation is exceptionally easy, relying only on structure of the rays in the fan Σ defining the toric variety.

For completeness, we describe the divisors on toric varities. The description will be used in Chapter 3, where we work out the geometry of the two smoothings of the affine cone over the del Pezzo surface of degree 6.

Let X be a smooth toric variety, and let $\Sigma(1)$ denote the set of one-dimensional cones (called *rays*) in the fan Σ defining X. For each ray ρ, let $u_\rho \in N$ denote the primitive ray generator of ρ. Then one can show that the torus-invariant divisors on X are in one to one correspondence with the rays $\rho \in \Sigma(1)$. Furthermore, every divisor on X is linearly equivalent to a torus-invariant divisor. Using these two facts, one can prove the following:

There is an exact sequence:

$$0 \longrightarrow M \xrightarrow{C} \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Pic}(X) \longrightarrow 0,$$

where the rows of the matrix C are the vectors u_ρ. See [CLS11], Chapter 4, for a proof.

There is also a description of the Cartier divisors on X in terms of *support functions* on N: a support function is a function $\varphi : |\Sigma| \to \mathbb{R}$ such that the restriction $\varphi\big|_\sigma$ of φ to each cone in Σ is linear. A support function is *integral with respect to* N if $\varphi(|\Sigma| \cap N) \subset \mathbb{Z}$. This means that for each cone σ, there is an $m_\sigma \in M$, such that $\varphi(v) = \langle v, m_\sigma \rangle$ if $v \in \sigma$.

The set of support functions is an abelian group under addition, and by Theorem 4.2.12 in [CLS11], there is an isomorphism between the group of integral support functions on Σ and the torus invariant Cartier divisors on X.

describe support functions

1.4 Deformation theory and the Hilbert scheme

Deformation theory is the infinetesimal study of algebro-geometric objects in families. Examples of such objects can be families of schemes, families of projective schemes (respecting the embedding), families of vector bundles, and so on.

In this section I will review some notation and motivation from deformation theory. Although results from deformation theory are not central in thesis, many of the methods and objects have roots from or connections with deformation theory. A reference for deformation theory is the book by Hartshorne [Har10]. For a leisurely popular account, the article [Maz04] is a nice read.

Given a scheme X_0 over $\mathbb{C}$, a *family of deformations* of X_0 is a flat morphism $\pi : \mathscr{X} \to (S, 0)$ with S connected such that $\pi^{-1}(0) = X_0$. If S is the spectrum of an artinian $\mathbb{C}$-algebra, then π is an *infinitesimal deformation.* If $S = \operatorname{Spec}\mathbb{C}[\epsilon]/\epsilon^2$, then π is a *first order deformation.* An *embedded deformation* of an embedded scheme $X_0 \subset \mathbb{P}^n$ is a deformation $\pi : \mathscr{X} \to (S, 0)$ with $\mathscr{X} \subset \mathbb{P}^n \times S$ such that π is the restriction of the projection $\pi : \mathbb{P}^n \times S \to S$. A deformation is *trivial* if it is isomorphic to the projection $X_0 \times S \to S$.

A *smoothing of* X_0 is a deformation such that the general fiber is smooth.

The set of embedded deformations and the set of abstract deformations have interpretations in terms of "familiar" objects (see [Har10] for proofs):

Proposition 1.4.1. *The set of all first order embedded deformations of a projective scheme X is in one–one correspondence with the group $H^0(X, \mathcal{N}_{X/\mathbb{P}^n})$, where $\mathcal{N}_{X/\mathbb{P}^n}$ is the normal sheaf of X in $\mathbb{P}^n$.*[3]

Proposition 1.4.2. *The set of all first order deformations of a smooth scheme X is in one–one correspondence with the group $H^1(X, \mathcal{T}_X)$.*

Remark 1.4.3. The intuition behind this result is the following. From the normal sequence

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}}\big|_X \to \mathcal{N}_{X/\mathbb{P}^n} \to 0,$$

we get a surjection (for $n \geq 2$):

$$H^0(X, \mathcal{T}_{\mathbb{P}}\big|_X) \to H^0(X, \mathcal{N}_{X/\mathbb{P}^n}) \to H^1(X, \mathcal{T}_X) \to 0.$$

The interpretation is that abstract deformations correspond to embedded deformations modulo infinetesimal automorphisms of $\mathbb{P}^n$.

If we denote by $\operatorname{Def}(X)$ (resp. $\operatorname{EmbDef}(X)$) denote the "space" of all (resp. embedded) deformations of a scheme X, then then the above proposition tells us that $H^1(X, \mathcal{T}_X)$ (resp. $H^0(X, \mathcal{N}_{X/\mathbb{P}^n})$) is the tangent space of the point $[X]$ in $\operatorname{Def}(X)$ (resp. $\operatorname{EmbDef}(X)$).

There is a complex, called the *cotangent complex*, associated to commutative rings R and modules M, that measures various deformation theoretic aspects of $\operatorname{Spec} R$. These are modules $T^i(B/k, M)$ for $i \geq 0$. Only the first three will be relevant for us, and we will present some ad hoc definitions.

Let B be an A-algebra, where A is a commutative ring. Let R be a polynomial ring surjecting onto B and I the kernel. Let F be a free module surjecting onto I, and let Q be its kernel. Then we have an exact sequence

$$0 \to Q \to F \xrightarrow{j} I \to 0.$$

[3] Recall that this is by definition $\mathcal{H}om(\mathcal{I}/\mathcal{I}^2, \mathscr{O}_X)$, where $\mathcal{I}$ is the ideal sheaf of X.

There is a "Koszul" submodule F_0 of F generated by the elements $aj(b) - bj(a)$. Note that $j(F_0) = 0$, implying that $F_0 \subset Q$. Let $L_2 \overset{\Delta}{=} Q/F_0$. Let $L_1 = F \otimes_R B$, and let $L_0 = \Omega^1_{R/A} \otimes_R B$. These are the first few terms of the cotangent complex:

$$L_\bullet : L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \xrightarrow{0} 0.$$

The map d_2 is induced by the inclusion $Q \to F$. The map d_1 is the composition of $j : F \to I$ and the derivation $R \to \Omega^1_{R/A}$.

For any R-module M, we now define $T^i(B/A, M)$ as the homology $H^i(\mathrm{Hom}_B(L_\bullet, M))$. There are many things to be checked, but the details are all in [Har10].

We list a few of the important properties of the T^i-functors here:

- We have an equality $T^0(B/A, M) = \mathrm{Der}_A(B, M)$. If $M = B$, this is the *tangent module* of B over A.
- If $A = k[x_1, \ldots, x_n]$ and $B = A/I$, then we have an exact sequence

  $$\mathrm{Hom}(\Omega^1_{A/k}, M) \to \mathrm{Hom}(I/I^2, M) \to T^1(B/k, M) \to 0. \qquad (1.1)$$ {eq:t1seq}

  This gives us a way to compute $T^1(B/k, M)$ which is amenable to computer algebra software. Algorithms for computing $T^i(B/k, B)$ for $i = 0, 1, 2$ are implemented in the `Macaulay2` package `VersalDeformations` by Nathan Ilten [Ilt12].
- The module $T^1(B/k, B)$ classifies first order deformations of the affine scheme $\mathrm{Spec}\, B$. It is a finite-dimensional k-vector space if $\mathrm{Spec}\, B$ has only isolated singularities. Both $T^1(B/k, B)$ and $T^2(B/k, B)$ are zero if B is smooth.
- The module $T^2(B/k, B)$ contains "obstructions" for lifting deformations to larger artinian rings.
- If B and M are graded, then $T^i(B/A, M)$ are graded as well.

If X is a projective variety and A its homogeneous coordinate ring, let U_X denote $\mathrm{Spec}\, A$. Then the deformation theory of the affine cone and X itself is closely related. This is studied for example in Schlessinger's article [Sch73], from which the following useful result can be proved:

prop:t1h1 **Proposition 1.4.4.** *Let X/k be a smooth projective Calabi–Yau variety, and let A be its homogeneous coordinate ring. Then we have an isomorphism*

$$T^1(A/k, A)_0 \simeq H^1(X, \mathcal{T}_X),$$

where $\mathcal{T}_X$ is the tangent sheaf of X.

Proof. This is a combination of Theorem 2.5 and Corollary 2.6 in [DFF15], using the fact that X is Calabi–Yau and smooth. ■

This result makes computing Hodge numbers of projective smooth Calabi–Yau's relatively easy.

We include a somewhat lengthy example of how to compute the T^i modules for a relatively simple ring.

Example 1.4.5. Let $B = k[x,y]/(x^2, xy, y^2)$ be the coordinate ring of the double point in $\mathbb{A}^2$. We want to compute $T^i(B/k, B)$ for $i = 0, 1, 2$.

We have that $T^0(B/k, B) = \mathrm{Der}_k(B, B)$, and this can be identified with the kernel of the map (see Proposition 3.10 in [Har10])

$$\mathrm{Hom}(\Omega_{k[x,y]/k}, B) \xrightarrow{\varphi} \mathrm{Hom}(I/I^2, B).$$

The map φ can be identified with the transpose of the Jacobian matrix of I. The module to the left is free, generated by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Up to scalars, φ is given by

$$\varphi = \begin{pmatrix} x & 0 \\ y & x \\ 0 & y \end{pmatrix}.$$

Thus $T^0(B/k, B)$ is equal to the set of $(f, g) \in R^2$ annihilated by the ideal $\mathfrak{m} = (x, y)$ in B. But since R is $k[x,y]/\mathfrak{m}^2$, this is equal to $\mathfrak{m} \oplus \mathfrak{m}$. Thus $\dim_k \mathrm{Der}_k(B, B) = 4$, corresponding to the fact that a fat point can move by moving its support and also by moving its "tangent arrow".

We can use the exact sequence (1.1) to compute $T^1(B/k, B)$. We see that $T^1(B/k, B)$ is the cokernel of φ. We must first identify $\mathrm{Hom}(I/I^2, B)$.

To compute this module, we start with a free resolution of I over $P = k[x, y]$:

$$0 \to P^2 \xrightarrow{d_1 = \begin{pmatrix} y & 0 \\ -x & y \\ 0 & -x \end{pmatrix}} P^3 \xrightarrow{d_0 = \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}} I \to 0.$$

It is then true that $\mathrm{Hom}(I/I^2, B)$ can be identified with $\ker(d_1^\vee \otimes B)$. An easy argument shows that this is $\mathfrak{m} \oplus \mathfrak{m} \oplus \mathfrak{m}$.

But the image of φ is a two-dimensional subset of $\mathrm{Hom}_B(I/I^2, B)$. Hence $\dim_k T^1(B/k, B) = 6 - 2 = 4$.

The computation of $T^2(B/k, B)$ is usually the hardest. We can identify $T^2(B/k, B)$ with $\mathrm{Hom}_B(Q/F_0, B)/\mathrm{image}\,(d_1 \otimes B)^\vee$, where F_0 is the module of Koszul relations and $Q = \mathrm{image} d_1$. Let us first compute $\mathrm{Hom}_B(Q/F_0, B)$.

We start with finding a presentation for Q/F_0. The module F_0 is the submodule of $F = P^3$ generated by the columns of the matrix

$$\psi = \begin{pmatrix} y^2 & xy & 0 \\ 0 & -x^2 & y^2 \\ -x^2 & 0 & xy \end{pmatrix}.$$

The image of d_1 is isomorphic to R^2. Using this isomorphism, Q/F_0 fits into an exact sequence

$$R^3 \xrightarrow{M=\begin{pmatrix} x & y & 0 \\ 0 & x & y \end{pmatrix}} R^2 \to Q/F_0 \to 0.$$

Applying $\mathrm{Hom}_B(-, B)$ is left-exact, so we get an exact sequence:

$$0 \to \mathrm{Hom}_B(Q/F_0, B) \to B^2 \xrightarrow{\begin{pmatrix} x & 0 \\ y & x \\ 0 & y \end{pmatrix}} B^3$$

It follows that $\mathrm{Hom}_B(Q/F_0, B) = \mathfrak{m} \oplus \mathfrak{m}$. The image of $d_1^\vee \otimes B$ kills off three of the four generators, so that $T^2(B/k, B)$ is a $4 - 3 = 1$-dimensional vector space over k. This reflects the fact that the fat point correspond to a singular point in its Hilbert scheme.

As we can see, already for this small example, there is a lot of computation involved. Especially the computation of a free resolution is resource demanding when the ideal have more generators. Therefore computer algebra software is essential when doing experiments in deformation theory. ♡

A few words about Hilbert schemes

The *Hilbert scheme* $\mathscr{H}_{P(t)}$ parametrizes projective schemes with a given Hilbert polynomial $P(t)$. The proof of its existence is non-trivial, and given first by Grothendieck in [Gro95]. The proof was later simplified by Mumford [Mum66]. However, it is often just as easy to work with the functorial description of the Hilbert scheme - namely with the functor it represents rather than the scheme itself.

The functor that the Hilbert scheme represents is the following: $h_{P(t)}(S)$ is the set set of all flat families $\mathscr{X} \subset S \times \mathbb{P}^n \to S$ where the fibers have Hilbert polynomial $P(t)$. With this definition, it is not difficult to show for example that the tangent space of $\mathscr{H}_{P(t)}$ at a point corresponding to a scheme X is given by $H^0(X, \mathscr{N}_{X/\mathbb{P}^n})$, where $\mathscr{N}_{X/\mathbb{P}^n}$ is the normal sheaf of X. Thus for a "generic" scheme, the dimension of the component on the Hilbert scheme on which it lies, is given by $h^0(X, \mathscr{N}_{X/\mathbb{P}^n})$.

Note that two different points on $\mathscr{H}_{P(t)}$ might represent isomorphic schemes. Two schemes are different if they occupy different points in $\mathbb{P}^n$. Allowing deformations outside $\mathbb{P}^n$ corresponds to applying the forgetful functor $\mathrm{Def}(X, S) \to \mathrm{Def}(X)$.

1.5 Stanley–Reisner schemes

Stanley–Reisner schemes are certain degenerate projective schemes modelled on simplicial complexes. We first recall some facts about simplicial complexes.

Let $[n]$ denote the set of numbers $\{0, \ldots, n\}$. The power set of $[n]$ is called the *n-simplex* and is denoted by Δ_n.

Definition 1.5.1. A *simplical complex* is a subset $\mathcal{K} \subseteq \Delta_n$ (for some n), such that if $f \in \mathcal{K}$ and $g \subseteq f$, then $g \in \mathcal{K}$. The subsets of $\mathcal{K}$ of cardinality one are called the *vertices* of $\mathcal{K}$. The subsets of codimension one are called facets of $\mathcal{K}$. The subsets of $\mathcal{K}$ are called *faces*. The *dimension* of a face f is equal to $|f| - 1$.

The *join* of two simplicial complexes $\mathcal{K}$ and $\mathcal{K}'$ is defined as

$$\mathcal{K} * \mathcal{K}' \triangleq \{f \sqcup g \mid f \in \mathcal{K}, g \in \mathcal{K}'\},$$

where $\sqcup$ denotes the disjoint union.

A good reference for this and the following is Stanley's green book [Sta96].

If $f \subset \mathcal{K}$ is a face, the *link of f in $\mathcal{K}$* is the simplicial complex defined by

$$\mathrm{lk}(f, \mathcal{K}) \triangleq \{g \in \mathcal{K} \mid f \cap g = \emptyset, f \cup g \in \mathcal{K}\}.$$

If $D_+(x_f) \subset \mathbb{P}(\mathcal{K})$ denotes the distinguished open set corresponding to the monomial x^f, we have that $D_+(x_f) = \mathbb{A}(\mathrm{lk}(f, \mathcal{K})) \times (k^*)^{\dim f}$.

Every simplicial complex has a *geometric realization*, which as a set is defined as follows:

$$|\mathcal{K}| \triangleq \{\alpha : [n] \to [0,1] \mid \mathrm{supp}(\alpha) \in \mathcal{K}, \sum_{i=1}^{n} \alpha(i) = 1\}.$$

This is an example of a piecewise linear manifold, or a PL-manifold for short. For more on PL-manifolds or combinatorial topology, we refer the reader to [Gla70; Spa66; Hud69].

Motivated by this, we single out a class of simplicial complexes:

Definition 1.5.2. A simplicial complex $\mathcal{K}$ is called a *manifold* if the geometric realization of every link $\mathrm{lk}(\mathcal{K}, v)$ (v is a vertex) is homeomorphic to a sphere.

Let k be a field, and let $P_\mathcal{K}$ be the polynomial ring over k with variables indexed by the vertices of $\mathcal{K}$. Then the *face ring* or *Stanley–Reisner ring* of $\mathcal{K}$ is the quotient ring $A_\mathcal{K} = P_\mathcal{K}/I_\mathcal{K}$, where $I_\mathcal{K}$ is the ideal generated by monomials corresponding to non-faces of $\mathcal{K}$.

Example 1.5.3. Let $\mathcal{K}$ be the triangle with vertices $\{v_1, v_2, v_3\}$. Its maximal faces are v_1v_2, v_2v_3 and v_1v_3. The Stanley–Reisner ring is $k[v_1, v_2, v_3]/(v_1v_2v_3)$. ♡

The ideal $I_\mathcal{K}$ is graded since it is defined by monomials. This leads us to define the *Stanley–Reisner scheme* $\mathbb{P}(\mathcal{K})$ as $\operatorname{Proj} A_\mathcal{K}$.

Note that $\mathbb{P}(\mathcal{K} * \mathcal{K}') = \mathbb{P}(\mathcal{K}) * \mathbb{P}(\mathcal{K}')$, where the second star means the join of two projective varieties.

The following result of Christophersen ([AC10, Theorem 4.6]) is useful. It expresses the deformation theory of Stanley–Reisner schemes purely in terms of their combinatorial data.

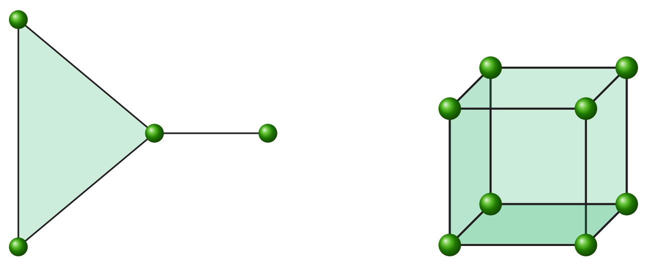
thm:t1dims **Theorem 1.5.4.** *If $\mathcal{K}$ is a simplicial manifold, and $\mathbf{c} = \mathbf{a} - \mathbf{b}$ (with disjoint supports a and b), then*

$$\dim_k T^1_{A_\mathcal{K},\mathbf{c}} = \begin{cases} 1 & \textit{if } a \in \mathcal{K} \textit{ and } b \in \mathcal{B}(\operatorname{lk}(a, \mathcal{K})) \\ 0 & \textit{otherwise.} \end{cases}$$

Here $\mathcal{B}(\mathcal{K})$ is defined as follows:

Definition 1.5.5. The set $\mathcal{B}(\mathcal{K})$ is the set of $b \subset \mathcal{K}$ with $|b| \geq 2$ such that

1. $\mathcal{K} = L * \partial b$, wehere $|L|$ is a $(n - |b| + 1)$-sphere, if $b \notin \mathcal{K}$.
2. $\mathcal{K} = L * \partial b \cup \partial L * \bar{b}$ where $|L|$ is a $(n - |b| + 1)$-ball, if $b \in \mathcal{K}$.



(a) A non-manifold simplicial complex. The link at the two right vertices are not spheres.

(b) A simplicial manifold.

Figure 1.2: Two examples of simplicial complexes.

1.6 Smoothings of Stanley–Reisner schemes

Because many properties of varieties are easier read off their degenerations, it is an interesting problem to study smoothings of Stanley–Reisner-schemes, which are highly singular. Below are the two main lemmas motivating the study of triangulations in deformation theory.

ma:srcohom **Lemma 1.6.1.** *If $\mathcal{K}$ is a simplicial complex, then $H^i(\mathcal{K}; k) \simeq H^i(\mathbb{P}(\mathcal{K}), \mathscr{O}_{\mathbb{P}(\mathcal{K})})$.*

The lemma is essentially due to Hochster, and is proved (in a different form) in Stanley's book [Sta96]. This is true essentially because the Čech complex computing the simplicial cohomology and the Čech complex computing sheaf cohomology look exactly the same.

Lemma 1.6.2. *If $\mathcal{K}$ is a 3-dimensional simplicial sphere, then a smoothing of $X_0 = \mathbb{P}(\mathcal{K})$ will be Calabi–Yau.*

Proof. Let $\pi : \mathscr{X} \to S$ be a smoothing. Since $\mathcal{K}$ is a sphere, it follows from 1.6.1 that $H^i(X_0, \mathscr{O}_{X_0}) = k$ for $i = 0, 3$, and zero for $i \neq 0, 3$. The triviality of the canonical bundle is proved in Theorem 6.1 in [BE91]. Since $H^1(\mathcal{K}; K) = H^2(\mathcal{K}; K) = 0$, it follows from the semicontinuity theorem (Theorem 12.8 in Chapter III in [Har77]) that $H^i(X_t, \mathscr{O}_{X_t}) = 0$ for all $t \in S$. Similarly, if ω_0 all nearby fibers must have trivial canonical bundle as well. ■

1.7 Toric geometry and toric degenerations

There is a correspondence between certain degenerations of toric varieties and so-called unimodular triangulations. define these

Let M be a lattice (by which we mean a free abelian group of finite rank). Let $\nabla \subset M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ be a lattice polytope, and let S_∇ be the semigroup in $M \times \mathbb{Z}$ generated by the elements $(u, 1) \in \nabla \cap M$. Then we define $\mathbb{P}(\nabla) = \operatorname{Proj} \mathbb{C}[S_\nabla]$, and call it the *toric variety associated to* ∇.

By Theorem 8.3 and Corollary 8.9 in [Stu96], there is a one-one correspondence between unimodular regular triangulations of ∇ and the square-free initial ideals of the toric ideal of $\mathbb{P}(\nabla)$. check this correspondence

ar_triangs **Proposition 1.7.1.** *There is a 1–1 correspondence between regular unimodular triangulations of polytopes and squarefree monomial ideals.*

By Theorem 8.3 and Corollary 8.9 in [Stu96]......

Thus, if we can find a polytope whose boundary has a regular unimodular triangulation, we know that the associated Stanley-Reisner ring has at least one deformation.

CHAPTER 2

Relation to triangulations of $\mathbb{CP}^2$

cp2triangs

This chapter will not contain any new results of any signifance, but is rather a report on an idea which led to the deliberations in the next chapters.

We explain an interesting connection between the topological space $\mathbb{CP}^2$ and degenerations of hyper-Kähler manifolds.

2.1 Hyper-Kähler manifolds

_manifolds

One often divides manifolds into three types: those with positive, negative or trivial canonical class. Of those with trivial canonical class, two prominent types stand out: Calab–Yau-manifolds and hyper-Kähler manifolds.

Definition 2.1.1. A *hyper-Kähler manifold* X is a simply connected compact Kähler[1] complex manifold such that $H^0(X, \Omega^2_X)$ is generated by a non-degenerate $\sigma : TX \times TX \to \mathbb{C}$.

Remark 2.1.2. Since the two-form σ is non-degenerate, it follows that the canonical sheaf $\omega_X = \Omega^n_{X/\mathbb{C}}$ is trivial. The map $1 \mapsto \sigma^{n/2}$ gives an isomorphism $\mathscr{O}_X \to \omega_X$.

Remark 2.1.3. In dimension 2, there is no difference between Calabi–Yau varieties and hyper-Kähler manifolds. These are the K3 surfaces[2].

Because of the non-degeneracy of the symplectic form $\sigma \in H^0(X, \Omega^2_X)$, hyper-Kähler manifolds only occur in even dimensions: the determinant of the skew-symmetric form is equal to $\det \sigma = (-1)^n \det \sigma$, implying $(-1)^n = 1$, so that n has to be even.

Only a few explicit families of hyper-Kähler manifolds are known. Below we sketch the construction of two such families.

[1] Recall that a complex manifold is *Kähler* if it is equipped with a Hermitian metric h whose associated two-form σ is closed. The two-form σ is defined by $\sigma(u, v) = \Re h(iu, v)$.

[2] K3 surfaces are named after Kummer, Kähler and Kodaira.

The Hilbert square $S^{[2]}$

Let S be a K3-surface with symplectic form σ, and let $S^{(2)}$ be its symmetric square: $S \times S/\{(p,q) \sim (q,p)\}$. Let $\pi_i : S \times S \to S$ be the two projections ($i = 1,2$). Then the 2-form $\pi_1^*\sigma + \pi_2^*\sigma$ is $\mathbb{Z}/2$-invariant, hence it decends to a 2-form τ on $S^{(2)}$.

The space $S^{(2)}$ is singular along the diagonal: locally it is isomorphic to $\mathbb{C}^2 \times \mathbb{C}^2/(x \sim -x)$. The last factor is a quadric cone, so a single blowup along the diagonal will resolve the singularities. The form τ lifts to a non-degenerate form on the blowup $\mathrm{Bl}_\Delta S^{(2)}$, which we denote by $S^{[2]}$. It can be shown that it is in fact a hyper-Kähler variety of dimension 4. The resulting space is denoted by $S^{[2]}$, and is called the *Hilbert square of* S, or the *Hilbert scheme of two points on* S. It parametrizes length two subschemes of S.

For more details on this construction, see Beauville's original paper [Bea83].

Lines on hypersurfaces

There is another construction of hyper-Kähler varieties that is interesting to us. Let X be a smooth cubic fourfold in $\mathbb{P}^5$. Let $F(X)$ denote the set of lines contained in X. It is the *Fano variety of lines on* X, and is a closed subset of the Grassmannian $\mathbb{G}(1,\mathbb{P}^5)$. One can can show that $F(X)$ is a hyper-Kähler variety of dimension 4.

In the article [BD85], Beauville and Donagi shows that $F(X)$ is deformation equivalent to $S^{[2]}$ for some K3 surface S. They also show that if X is a *pfaffian* hypersurface, then $F(X)$ is actually *isomorphic* to $S^{[2]}$ for some K3 surface S. Furthermore, the family $\{F(X)\}$ obtained this way is 19-dimensional, and is a hypersurface in the deformation space of $S^{[2]}$.

2.2 Connection to the complex projective plane

Let X be a topological space. Recall that the symmetric product $X^{(2)}$ is defined as follows:

$$X^{(2)} = X * X = X \times X/\{(x,y) \sim (y,x)\}.$$

If $X = S^2$, we have that $X^{(2)}$ is naturally isomorphic to $\mathbb{CP}^2$, which can be seen as follows: S^2 can be identified with $\mathbb{P}^1_{\mathbb{C}}$. Unordered pairs of points in $\mathbb{P}^1$ correspond to degree 2 polynomials up to scalar multiplication. Hence we have identifications

$$(S^2)^{(2)} = (\mathbb{P}^1)^{(2)} = \{(P,Q) \in \mathbb{P}^1 \times \mathbb{P}^1\}/\mathbb{Z}_2 = \mathbb{P}\left(H^0(\mathscr{O}_{\mathbb{P}^1}(2))\right) = \mathbb{CP}^2.$$

Here is an observation.

Lemma 2.2.1. *If $\mathcal{K}$ is a simplicial complex that is a* manifold*, isomorphic to S^2, then a smoothing of $\mathcal{K}$ is K3 surface.*

Proof. See [BE91]. ■

Stanley–Reisner degenerations of K3 surfaces correspond to triangulated 2-spheres. Since the symmetric square of a sphere is $\mathbb{CP}^2$, a Stanley-Reisner degeneration of the symmetric square of a K3 surface should correspond to a triangulated $\mathbb{CP}^2$.

Proposition 2.2.2. *Suppose $\mathcal{K}$ is a triangulation of $\mathbb{CP}^2$ and $X_0 = \mathbb{P}(\mathcal{K})$ is its associated Stanley–Reisner-scheme. Then a smoothing X of X_0 will be a hyper-Kähler manifold.*

Proof. The dimensions of the groups $H^i(X, \mathscr{O}_X)$ are constant in flat families. Since $H^0(X, \Omega^2_X) = H^2(X, \mathscr{O}_X) = H^2(\mathcal{K}; \mathbb{C}) = \mathbb{C}$ (the first equality is complex conjugation), we have that $H^0(X, \Omega^2_X)$ is generated by a single 2-form. It is non-degenerate since $\omega_X \simeq \mathscr{O}_X$. It follows that X is hyper-Kähler. ■

hva med simply-connected?

2.3 Attempt to smooth triangulations

If $\mathcal{K}$ is a triangulation of $\mathbb{CP}^2$ and $\mathbb{P}(\mathcal{K})$ is the associated Stanley–Reisner-scheme, a smoothing of $\mathbb{P}(\mathcal{K})$ will give a hyper-Kähler variety. Using this idea, and the `Macaulay2` package `VersalDeformations` (by Nathan Ilten, see [Ilt12]), we tried to find potentially new hyper-Kähler varieties. Unfortunaly, it looks like all the triangulations we experimented with were non-smoothable.

In the next four subsections we describe four different triangulations of $\mathbb{CP}^2$, and also their deformation theory using the results of [AC10]. In all cases we conclude that the corresponding Stanley–Reisner scheme is probably not smoothable.

Before we go on to describe the triangulations, we recall some basic facts about combinatorial manifolds.

We can decompose $\mathbb{CP}^2$ into three four-dimensional closed balls B_j, whose pairwise intersections are solid tori Π_{ij}, and whose triple intersection is a two-dimensional torus T. The closed balls B_0 is defined as

$$B_0 = \left\{ [x_0 : x_1 : x_2] \in \mathbb{CP}^2 \mid x_0\overline{x_0} \geq x_1\overline{x_1},\, x_0\overline{x_0} \geq x_2\overline{x_2} \right\},$$

and similarly for B_1 and B_2. This is sometimes called the *equilibrium decomposition* of the complex projective plane.

A triangulation of $\mathbb{CP}^2$ is *equilibrium* if the closed balls, the solid tori, and the torus T are subcomplexes of the triangulation. Several of the triangulation below are equilibrium.

The 15-vertex triangulation

A very interesting triangulation $\mathcal{T}$ of $\mathbb{CP}^2$ is found in [Gai09]. The author describes a triangulation of $\mathbb{CP}^2$ using 15 vertices. One reason it is interesting is that it the corresponding Stanley–Reisner scheme $\mathbb{P}(\mathcal{T})$ has the same Hilbert-polynomial as $F_1(X)$, the Fano variety of lines on a cubic hypersurface. This means that they live in the *same* Hilbert scheme, and one could naively hope that they live in the same component as well, meaning that there exists a degeneration of $F_1(X)$ to $\mathbb{P}(\mathcal{T})$.

We will spend some time describing this tringulation, since parts of it inspired our the construction of the Calabi–Yau's in the last chapter. We cite the definition ad verbatim from [Gai09].

Definition 2.3.1. Let $V_4 \subset S_4$ be the Klein four group. The vertex set of $\mathcal{T}$ is defined as

$$V = (V_4 \backslash \{e\}) \sqcup (\{1,2,3,4\} \times \{1,2,3\}) \,. \tag{2.1}$$

Thus the vertices of $\mathcal{T}$ are the permutations (12)(34),(13)(24) and (14)(23) and the pairs of integers (a,b) with $1 \leq a \leq 4$ and $1 \leq b \leq 3$. The maximal faces are spanned by the sets

$$\nu, (1,b_1), (2,b_2), (3,b_3), (4,b_4) \tag{2.2}$$

with $\nu \in V_4 \backslash \{e\}$ and $1 \leq b_a \leq 3$ $(a = 1,2,3,4)$ such that $b_{\nu(a)} \neq b_a$ for $a = 1,2,3,4$.

See Appendix A.4 for a SAGE [Wil17] script for computing the maximal facets of $\mathcal{T}$.

The triangulation $\mathcal{T}$ is the union over the cones over three 3-spheres S_j (the cone over S_j is the ball B_j in the definition of an equlibrium triangulation). Each S_j is a very simple 3-sphere. It is the join of two hexagons (recall that $S^1 * S^1 \approx S^3$). The f-vector is $(15, 90, 240, 270, 108)$.

It is the Stanley–Reisner-scheme of S_j and some if its deformations that is studied in Chapter 4, leading to constructions of some new Calabi–Yau manifolds.

We compute some deformation-theoretic invariants of $\mathbb{P}(\mathcal{T})$, the Stanley–Reisner scheme associated to $\mathcal{T}$.

Proposition 2.3.2. *We have that* $\dim_{\mathbb{C}} T^1_{A(\mathcal{T}),0} = 90$ *and* $\dim_{\mathbb{C}} T^2_{A(\mathcal{T}),0} = 306$. *The normal sheaf has* $\mathcal{N}_{\mathbb{P}(\mathcal{T})/\mathbb{P}^{14}}$ *has* 300 *global sections.*

The proof is a computation in `Macaulay2`. We remark that since $\mathbb{P}(\mathcal{T})$ is not Cohen–Macaulay, some standard comparison theorems does not hold. In our case we only have an inclusion $T^1_{A(\mathcal{T})} \hookrightarrow T^1_{\mathbb{P}(\mathcal{T})}$ (see the article of Kleppe [Kle79] and Theorem 3.9). This means that there might be deformations of $\mathbb{P}(\mathcal{T})$ that are not induced from the ambient projective space.

Because of the high number of parameters, we have not been able to say anything meaningful regarding the deformations of $\mathbb{P}(\mathcal{T})$. However, it is possible to deform $\mathbb{P}(\mathcal{T})$ into the union of three toric varieties, each being deformations of the Stanley–Reisner scheme $\mathbb{P}(B_j)$. This is not surprising, since B_j is a triangulation of a the normal polyhedron of the corresponding toric variety. This deformation reduces the number of components of $\mathbb{P}(\mathcal{T})$ from 108 to 3.

It is not clear however if this union of toric varieties can be further deformed.

Kühnel's 9-vertex triangulation

The minimal triangulation $\mathcal{T}_9$ of $\mathbb{CP}^2$ is a 9-vertex triangulation with f-vector $(9, 36, 84, 90, 36)$. This means that the associated Stanley–Reisner scheme $\mathbb{P}(\mathcal{T}_9)$ lives in $\mathbb{P}^8$ and is of degree 36. The automorphism group of $\mathcal{T}_9$ is a group of order 54, and it can be realized as a semidirect product $(\mathbb{Z}_3 \times \mathbb{Z}_3) \ltimes \mathbb{Z}_3 \ltimes \mathbb{Z}_2$. For a very readable account of the construction and motivation of this triangulation, consult [KB83].

The ideal has a resolution of the form (in `Macaulay2` format):

```
        0  1  2  3  4 5 6
total: 1 36 90 84 37 9 1
    0: 1  .  .  .  . . .
    1: .  .  .  .  . . .
    2: .  .  .  .  . . .
    3: . 36 90 84 36 9 1
    4: .  .  .  .  . . .
    5: .  .  .  .  1 . .
```

This means that the ideal of $\mathbb{P}(\mathcal{T}_9)$ is generated by 36 cubic monomials, and there are 90 relations between them, lying in $\mathcal{O}_{\mathbb{P}(\mathcal{T}_9)}(-5)$, et cetera. Since the resolution is not symmetric, we see immediately that $\mathbb{P}(\mathcal{T}_9)$ is not Gorenstein.

Proposition 2.3.3. *We have that* $\dim_{\mathbb{C}} T^1_{A(\mathcal{T}_9),0} = 21$ *and* $\dim_{\mathbb{C}} = T^2_{A(\mathcal{T}_9),0} = 126$. *The normal sheaf has* $\mathcal{N}_{\mathbb{P}(\mathcal{T}_9)/\mathbb{P}^8}$ *has* 93 *global sections.*

We can compute the action of the automorphism group on T^1. Using SAGE, we find that the 21 deformation parameters split into two orbits, one of size 3 and one of size 18.

We have not been able to lift any first-order deformation of $\mathbb{P}(\mathcal{T}_9)$ to a family over $\operatorname{Spec} \mathbb{C}[t]$.

The minimal equilibrium triangulation

In [BK92], the authors construct a 10 vertex equilibrium triangulation $\mathcal{T}_{10}$ of $\mathbb{CP}^2$. They start with the minimal 7-vertex triangulation of the torus, and then they construct $\mathcal{T}_{10}$ by taking cones over unions of the tori.

The automorphism group is order 42, and comes from the symmetries of the torus.

The Betti table of the resolution of the ideal of $\mathbb{P}(\mathcal{T}_{10})$ is the following:

```
        0  1   2   3   4  5  6 7
total: 1 38 128 177 123 46 10 1
    0: 1  .   .   .   .  .  . .
    1: .  3   2   .   .  .  . .
    2: .  .   .   .   .  .  . .
    3: . 35 126 175 120 45 10 1
    4: .  .   .   2   3  .  . .
    5: .  .   .   .   .  1  . .
```

Again we see that the ideal is not Gorenstein.

Proposition 2.3.4. *We have that* $\dim_{\mathbb{C}} T^1_{A(\mathcal{T}_{10}),0} = 42$ *and* $\dim_{\mathbb{C}} = T^2_{A(\mathcal{T}_{10}),0} = 105$*. The normal sheaf has* $\mathcal{N}_{\mathbb{P}(\mathcal{T}_{10})/\mathbb{P}^8}$ *has* 132 *global sections.*

In fact, it is possible to lift the versal family of deformation parameters to an honest family over $\operatorname{Spec} \mathbb{C}[t_1, \cdots, t_{42}]$, using the `VersalDeformations` package. Surprisingly, even though the T^2 module is big, there are no obstructions in the family. However, the generic member of this family is reducible (verified in `Macaulay2` for "random" values of the deformation parameters), implying that $\mathbb{P}(\mathcal{T}_{10})$ is not smoothable.

The automorphism group act transitively on the natural basis of T^1, so that $\dim_{\mathbb{C}} {T^1_{A(\mathcal{T}_{10}),0}}^G = 1$.

The Bagchi–Datta triangulation

There is another 10-vertex triangulation $\mathcal{T}_{BD}$ of $\mathbb{CP}^2$, which is obtained as a $\mathbb{Z}/2$-quotient of a triangulation of $S^2 \times S^2$. It is described in [BD11]. The automorphism group is the alternating group A_4. The f-vector is $(10, 45, 110, 120, 48)$.

The triangulation is bistellarly equivalent to both the 9-vertex triangulation and the 10-vertex triangulation above.

Proposition 2.3.5. *We have that* $\dim_{\mathbb{C}} T^1_{A(\mathcal{T}_{BD}),0} = 41$ *and* $\dim_{\mathbb{C}} = T^2_{A(\mathcal{T}_{BD}),0} = 180$*. The normal sheaf has* $\mathcal{N}_{\mathbb{P}(\mathcal{T}_{BD})/\mathbb{P}^8}$ *has* 131 *global sections.*

We have not been able to find any meaningful lifting of the first-order deformations here either.

2.4 Naïve attempt to degenerate

On the other hand, degenerating the ideal of $F_1(X) \subset \mathbb{P}^N$ to a square-free monomial ideal should give a triangulation of $\mathbb{CP}^2$. Since $F_1(X)$ sits inside

$\mathbb{G}(1,5)$, and there are many known degenerations of $\mathbb{G}(1,5)$, we hoped that maybe $F_1(X)$ would generate inside $\mathbb{G}(1,5)$. Unfortunaly, we did not succed, mainly because we could not see any structure in the ideal of $F_1(X)$.

It was possible to explicitly compute $F_1(X)$ for some hypersurfaces, both pfaffian and non-pfaffian. However, the ideals were too complicated and the Gröbner bases too big to find any initial ideals with only squarefree generators.

2.5 Conclusion

It would be interesting to study other triangulations of $\mathbb{CP}^2$. One way to proceed would be to start with existing triangulations, and analyze which parts of it correspond to non-zero elements of the T^2 module. Then one can do bistellar flips away from these combinations, ideally obtaining triangulations corresponding to unobstructed Stanley–Reisner schemes.

This is an interesting and very hard question. Even with an unobstructed triangulation, it is not clear how to proceed to smooth it in a computationally feasible way. Already with Gröbner bases with 50 elements (for deformations of the 15-vertex triangulation, they had around 70 elements), computations take far too long (and consumes too much memory) to be feasible to work with.

Without the presence of any good parallell processing Gröbner basis algorithms (which would allow the use of clustered super–computers), there is need for either patience or smarter solutions to computational algebra problems.

CHAPTER 3

The two smoothings of $C(\mathrm{dP}_6)$

smoothings

In this chapter we study the toric singularity that is the cone over the del Pezzo surface of degree 6. it has two topologically different smoothings, which we haven't seen studied in some detail before.

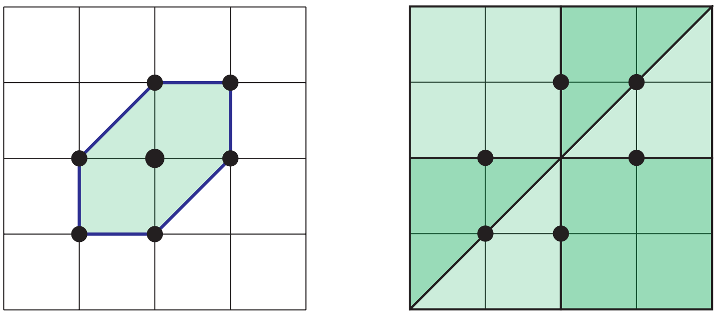
3.1 The del Pezzo surface dP_6

smoothings

We start this chapter by talking about the del Pezzo surface in some generality.

Denote by dP_6 the blow-up of $\mathbb{P}^2$ in three non-collinear points. These points can be chosen to be the coordinate points $(1:0:0), (0:1:0)$ and $(0:0:1)$. Since the coordinate points are invariant under the natural torus action on $\mathbb{P}^2$, it follows that the dP_6 is a toric variety.

As a toric variety, it can be described as toric variety of the planar hexagon depicted in Figure 3.1(a). The normal fan is in Figure 3.1(b)



(a) The hexagon corresponding to dP_6.

(b) The fan over the polar polytope.

Figure 3.1: Toric description of dP_6.

We will spend some time describing the different embeddings of dP_6. Different embeddings give rise to different smoothings of the affine cones.

Embedding in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Since dP_6 is the blowup of $\mathbb{P}^2$ in three points, we can blow them up separately. Let x_0, x_1, x_2 be coordinates of $\mathbb{P}^2$. Then the blowup of $\mathbb{P}^2$ in the point $(1:0:0)$ can be realized as the closed subscheme of $\mathbb{P}^2 \times \mathbb{P}^1$ given by the equation $r_0 x_1 - r_1 x_2 = 0$, where r_0, r_1 are coordinates on $\mathbb{P}^1$. We can repeat this procedure on the two other points $(0:1:0)$ and $(0:0:1)$ to obtain similar equations. Collecting these, we see that dP_6 is given by the matrix equation

$$M\vec{x} = \begin{pmatrix} 0 & r_0 & -r_1 \\ s_1 & 0 & -s_0 \\ -t_0 & t_1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0.$$

in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Since $\vec{x}$ is non-zero, it follows that we must have $\det M = 0$. It is not difficult to see that M cannot have rank 1 or lower, because that would force some of the $\mathbb{P}^1$-coordinates to be zero. Consider the projection forgetting the $\mathbb{P}^2$-factor:

$$\pi : \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Consider the hypersurface $\det M = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Any solution to this equation gives a unique solution to the equation $M\vec{x} = 0$, meaning that the restriction of π to dP_6 is an isomorphism onto the hypersurface $\det M = r_0 s_0 t_0 - r_1 s_1 t_1 = 0$.

It is also interesting to see how this embedding arises from a toric perspective using polytopes. Since $\mathbb{P}^1$ is the toric variety associated with the interval $[-1,1] \subset \mathbb{R}$, it follows that $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the toric variety associated with the cube $\Delta = [-1,1]^3 \subset M_{\mathbb{R}} = \mathbb{R}^3$. The inclusion of dP_6 in M induces a surjection of coordinate rings $\mathbb{C}[M] \to \mathbb{C}[\mathrm{dP}_6]$. This correspond to the fact that there is a lattice projection of the cube onto the hexagon. See Figure 3.2.

Conversely, if N_1 is the fan of dP_6, and N_2 is the fan of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, we have an inclusion of lattices $N_1 \hookrightarrow N_2$, which is induced by an inclusion of convex polytopes, as in Figure 3.3.

The inclusion $N_1 \hookrightarrow N_2$ is given by matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \tag{3.1}$$

{eq:A}

Note that there are essentialy four embeddings of the hexagon into the octahedron, because each embedding is given by choosing a line through opposite faces (the line spanned by the normal vector of the hexagon), and there are 8 faces, hence 4 lines through opposite faces.

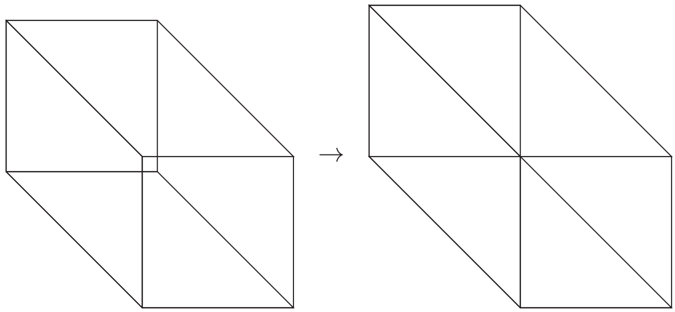


Figure 3.2: The projection of a cube onto a hexagon. fig:cube_project

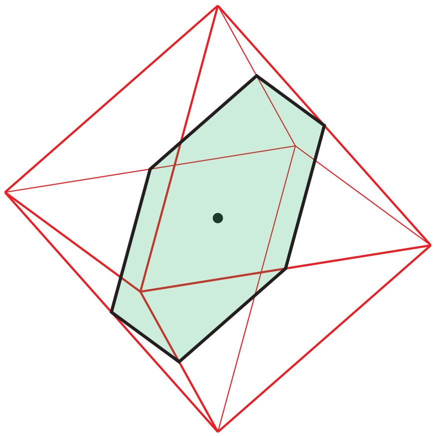


Figure 3.3: The inclusion of a cube in a octahedron. fig:octahedron_

Embedding in $\mathbb{P}^2 \times \mathbb{P}^2$

On the other hand, blowups can also be realized as closures of graphs of rational maps. Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the Cremona transformation given by $(x_0 : x_1 : x_2) \mapsto \left(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right)$. Let $\Gamma \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the graph of φ. Then, in coordinates $(a_0 : a_1 : a_2) \times (b_0 : b_1 : b_2)$ on $\mathbb{P}^2 \times \mathbb{P}^2$, the equations $a_0 b_0 = a_1 b_1 = a_2 b_2$ hold on Γ. These are the equations of the blowup along the indeterminacy locus of the rational map φ. The indeterminacy locus is exactly the three coordinate points. Hence dP_6 can also be realized as the intersection of two $(1,1)$-divisors in $\mathbb{P}^2 \times \mathbb{P}^2$. dp6_inp2p2

There is also in this case a description in terms of polytopes. The polytope associated with $\mathbb{P}^2 \times \mathbb{P}^2$ is $\Delta^2 \times \Delta^2$, the product of two two-simplices. Also in

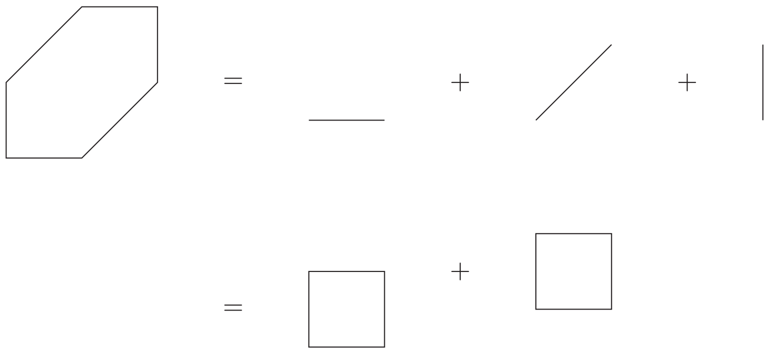


Figure 3.4: Minkowski-decompositions of the hexagon. fig:minkow

this case, there is a projection onto a hexagon in $\mathbb{R}^2$. This is harder to visualize, but can be described as follows: if we order the vertices of Δ^2 by v_1, v_2, v_3, then the vertices of $\Delta^2 \times \Delta^2$ are of the form (v_i, v_j). The projection is then given by identifying the vertices (v_i, v_i).

Hence, using the Segre embedding, dP_6 lives naturally in both $\left(\mathbb{P}^1\right)^3 \hookrightarrow \mathbb{P}^7$ and $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$.

Remark 3.1.1. Intersecting $\mathbb{P}^2 \times \mathbb{P}^2$ with a single $(1,1)$-divisor gives us the projective space bundle corresponding to the tangent bundle of $\mathbb{P}^2$, which we denote by $\mathcal{T}(\mathbb{P}^2)$. This follows from the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_{\mathbb{P}^2}(1)^3 \to \mathcal{T}_{\mathbb{P}^2} \to 0.$$

Since $\mathbb{P}(\mathscr{O}_{\mathbb{P}^2}(1)^3) = \mathbb{P}^2$, $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ can be realized as the subset of $\mathbb{P}^2 \times \mathbb{P}^2$ such that $a_0 b_0 + a_1 b_1 + a_2 b_2 = 0$. The space $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ is a non-toric Fano variety.

3.2 The cone over dP_6 and its two smoothings

The singularity $Z \stackrel{\Delta}{=} C(\mathrm{dP}_6)$ is one of the most studied singularities with an obstructed deformation space, For example, in the paper [Alt97], Klaus Altmann describe a method to study the versal deformations of isolated affine Gorenstein toric singularities using only the combinatorial data of the toric variety. He shows that different components of the base space correspond to different ways of writing the defining polytope as a Minkowski sum of other polytopes.

See the illustation in Figure 3.4 for a decomposition of the hexagon.

Let $A = A(Z)$ denote the affine coordinate ring of $C(\mathrm{dP}_6)$. It has a natural $\mathbb{Z}$-grading. From Altmann's article, or by using `Macaulay2`, ones computes that $\dim T^1(A) = 3$, and that $\dim T^2(A) = 2$. The versal base space decomposes into a union of a line and a plane. Both components are smoothing components, which are topologically different. We will prove this by computing their singular cohomology groups below.

It is worthwhile to note that both smoothings of Z arise by "sweeping out the cone": if X is a projective variety in $\mathbb{P}^n$, and Y is equal to $X \cap H$, where H is a section of $\mathscr{O}_{\mathbb{P}^n}(1)$, then the affine cone over Y deforms to a general hyperplane section of the affine cone over X. See the introduction of [Ste03] for more details.

Equations of smoothings

Using the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ and substituting from the linear equations in the description Section 3.1, we can write the equations of dP_6 inside $\mathbb{P}^6$ as

$$\begin{vmatrix} y & x_1 & x_2 \\ x_3 & y & x_4 \\ x_5 & x_6 & y \end{vmatrix} \leq 1, \tag{3.2}$$

where ≤ 1, means taking all 2×2-minors.

On the other hand, dP_6 can be realized as a subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as well. The equations can be described as follows: draw a cube, and let each vertex correspond to a variable. Then the equations of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding are given by taking all "minors" along all sides of the cube together with the three long diagonals. See Figure 3.5, in which we look at the cube from the front face. To get dP_6, one identifies two opposite corners, corresponding to the equation $a_{000} = a_{111}$ inside $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Thus in total there are $8 - 1 = 7$ variables, just as above.

The first smoothing is obtained by deforming the equations of dP_6 as a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$. It can be described by perturbing two of the entries of the matrix below:

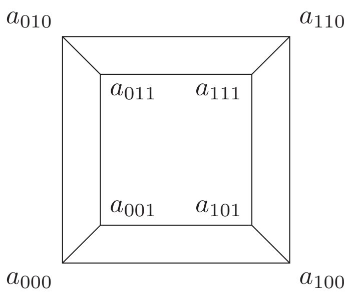


Figure 3.5: A $2 \times 2 \times 2$-tensor.

fig:p1p1p1_equa

$$\begin{vmatrix} y & x_1 & x_2 \\ x_4 & y+t_1 & x_3 \\ x_5 & x_6 & y+t_2 \end{vmatrix} \leq 1. \qquad (3.3)$$ {eq:def2}

For $t_1 = t_2 = 0$, we get the cone over dP_6, while for generic t_i, we get a smooth variety. In fact, we can compute that the discrimant locus (the set of points in $\mathbb{A}^2_{t_1,t_2}$ with singular fiber) are the t_1-axis, the t_2-axis and the line $t_1 = t_2$. Notice that the total space is equal to the cone over $\mathbb{P}^2 \times \mathbb{P}^2$.

Call (any) smooth fiber X_2.

Lemma 3.2.1. *Let $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ be the projective space bundle associated to the tangent sheaf on $\mathbb{P}^2$. Then the smoothing X_2 is isomorphic to $M \setminus \mathrm{dP}_6$.*

Proof. First homogenize the equations (3.3) with respect to y_1. Call the homogenized variety N. Put $y_0' = y_0$, $y_1' = y_0 - ty_1$ and $y_2' = y_0 - t_2 y_1$. Then we have the relation

$$h = t_2 y_1' - t_1 y_2' - (t_1 - t_2) y_0' = 0.$$

Hence we see that $N = \mathbb{P}^2 \times \mathbb{P}^2 \cap (h = 0)$. We can pull back the coordinates y_i' to $\mathbb{P}^2 \times \mathbb{P}^2$. Let $\mathbb{P}^2 \times \mathbb{P}^2$ have coordinates x_0, x_1, x_2 and y_0, y_1, y_2. Then h pulls back to the equation

$$(x_0, x_1, x_2) \cdot (-t_1 y_2, (t_1 - t_2) y_0, t_2 y_1) = 0$$

in $\mathbb{P}^2 \times \mathbb{P}^2$. As long as $t_1 \neq t_2$ and $t_1, t_2 \neq 0$, we can do a change of coordinates in $\mathbb{P}^2_{y_0 y_1 y_2}$, so that h transforms to

$$(x_0, x_1, x_2) \cdot (y_0, y_1, y_2) = 0.$$

Hence we see that M is isomorphic to the total space of the Grassmannian of lines in $\mathbb{P}^2$ (each point in one of the $\mathbb{P}^2$'s give a line in the other $\mathbb{P}^2$). This is in turn isomorphic to $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$, since each tangent vector through a point determines a line through it.

Now, what have we gained by homogenizing? The divisor at infinity is $y_1 = 0$, which is a dP_6 again. In our new coordinates this is equivalent to $y_1' = y_2' = y_0'$. Hence in the coordinates of $\mathbb{P}^2 \times \mathbb{P}^2$, dP_6 is given by the two equations $x_1 y_0 - x_2 y_1 = x_1 y_0 - x_0 y_2 = 0$. ∎

The other smoothing is the obtained by replacing one of the corners of the cube in Figure 3.5 with $a_{000}' = a_{000} + t$, obtained a one-parameter smoothing. The total space is now that affine cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Call this smoothing X_1.

Lemma 3.2.2. *The smoothing X_1 is isomorpic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathrm{dP}_6$.*

Proof. The proof is identical to the previous proof. ∎

The following fact is well-known, and follows from the above two lemmas.

Proposition 3.2.3. *The two smoothings are topologically different.*

Proof. The Euler characteristic of $\mathbb{P}(\mathscr{T}_{\mathbb{P}^2})$ is 6, which follows from the next lemma.

This let us calculate the Euler characteristics of the smoothings. Note that $\chi(\mathbb{P}^1) = 2$. By the Künneth formula, the Euler characteristic is multiplicative on products, so that $\chi(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 8$. By additivity of the Euler characteristics we have $\chi(X_1) = 2$ and $\chi(X_2) = 0$, since $\chi(\mathrm{dP}_6) = 6$.

It follows that the two smoothing components correspond to topologically different smoothings, since the Euler characteristic is a topological invariant. ∎

Lemma 3.2.4. *The cohomology ring of $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ is $\mathbb{Z}[x,y]/(x^3, y^2+3y+3)$, where x and y have degree 2. In particular, the cohomology of $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ is given by $(1,0,2,0,2,0,1)$.*

Proof. The first claim follows from the Leray-Hirch theorem. See [BT82, page 270]. The next claim follows since x and y both have degree 2. ∎

We can use what we know about the topology of these spaces to compute homology groups of the two affine smoothings.

We first need a prelimenary lemma from toric geometry, which we state and prove in general form, because we could not find a proper reference.

ma:induced **Lemma 3.2.5.** *Let $Y \overset{i}{\hookrightarrow} X$ be an closed immersion of smooth toric varieties, corresponding to a map of fans $\Sigma_1 \xrightarrow{A} \Sigma_2$. Let M_1 and M_2 be the corresponding character lattices. Then we have a commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & M_2 \arrow[r, "R_1"] \arrow[d, "A^T"] & \mathbb{Z}^{\Sigma_2(1)} \arrow[r] \arrow[d, "C^T"] & \mathrm{Pic}(X) \arrow[r] \arrow[d, "i^*"] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "R_2"] & \mathbb{Z}^{\Sigma_1(1)} \arrow[r] & \mathrm{Pic}(Y) \arrow[r] & 0
\end{tikzcd}$$

Where in addition $i^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is the map of Picard groups induced by the closed embedding.

Proof. The vertical rows are well-known. See for example Theorem 4.1.3 in [CLS11].

The matrix C^T is defined as follows: each primitive ray generator of cones in $\Sigma_1(1)$ thought of as lying in N_2 via the embedding A has a unique description as a linear combination of rays in $\Sigma_2(1)$ lying in the same minimal cone. Let the columns of C be the coefficients of this linear combination. Then, by definition, the first square commutes.

It follows that there is an induced map of Picard groups. We must show that the induced map is exactly the one induced by the closed embedding. To see this, we use Proposition 6.2.7 in [CLS11], where a description of this map is given in terms of convex functions on N.

complete the proof

■

example:p1p1p1

Example 3.2.6. Let us see how we can use the Lemma to find an explicit form of the induced map of the inclusion $\mathrm{dP}_6 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We use the matrix A from Equation (3.1). The rows of R_1 are the coordinates of the primitive ray generators of the fan of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and also the vertices of the octahedron in Figure 3.3.

The rows in R_2 are the coordinates of the hexagon in Figure 3.1(a).

Since $\mathrm{Pic}((\mathbb{P}^1)^3) = \mathbb{Z}^3$, and $\mathrm{Pic}\,\mathrm{dP}_6 = \mathbb{Z}^4$, we need to compute splittings of $\mathbb{Z}^6$ as $\mathbb{Z}^6 = \mathbb{Z}^3 \oplus \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ and $\mathbb{Z}^6 = \mathbb{Z}^2 \oplus \mathrm{Pic}(\mathrm{dP}_6)$, respectively.

This can be done explicitly by Gaussian elimination. We illustrate this with the first map:

$$\left(\begin{array}{ccc|cccccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|cccccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

We start with the matrix (R_1, I_6), and after Gaussian elimination (row operations), we get the matrix (R_1', B). The last three rows of B give a map $\pi_1 : \mathbb{Z}^6 \to \mathbb{Z}^3$ with kernel equal to the image of R_1. We do the same with the pair (R_2, I_6).

We find that the induced map $i^* : \mathbb{Z}^3 \to \mathbb{Z}^4$ is given by the matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We will use this information below in the proof of the next theorem. ♡

example:p2p2

Example 3.2.7. Now we consider the embedding $\mathrm{dP}_6 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$. On the level of coordinate rings, it is induced by a projection of polytopes $\Delta^2 \times \Delta^2 \to H$ (where H denotes the hexagon in Figure 4.1).

The anticanonical polytope of $\mathbb{P}^2$ is the convex hull of the points $v_1 = (-1, 2)$, $v_2 = (-1, -1)$ and $v_3 = (2, -1)$. Then the anticanonical polytope of $\mathbb{P}^2 \times \mathbb{P}^2$ is the convex hull of the 9 vertices $v_{ij} \triangleq v_i \times v_j \in \mathbb{R}^4$.

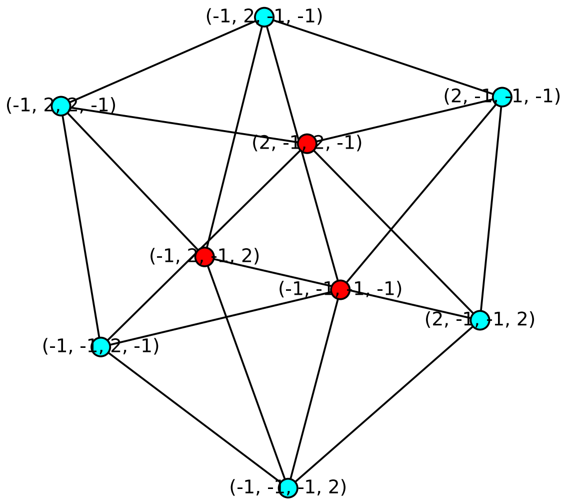


Figure 3.6: The edge graph of $\Delta \times \Delta$. The red vertices are the diagonal vertices v_{ii}.

fig:edgegraphDD

We want a projection sending the vertices v_{ii} $(i = 1, 2, 3)$ to the origin in $\mathbb{R}^2$. In Figure 3.6, we have visualized the edge graph of $\Delta^2 \times \Delta^2$. The three vertices that are sent to zero are marked in red.

make nicer figure

By demanding that $v_{12} \mapsto (1, 0)$ and $v_{23} \mapsto (0, 1)$, together with $v_{ii} \mapsto (0, 0)$, we get a system of 8 linear equations, corresponding to a unique map $\mathbb{R}^4 \to \mathbb{R}^2$ with the required properties. We get:

$$A'^T = \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

The image generates a sublattice $\frac{1}{3}\mathbb{Z}^2 \subset \mathbb{Z}^2$. Replace A' by $A \triangleq 3A'$, and consider only the sublattice.

The images of the rays of the fan of dP_6 under A^T are exacty the 6 rays of the fan of $\mathbb{P}^2 \times \mathbb{P}^2$. This means that the map C^T in the Lemma is the identity matrix I_6, and we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^4 \arrow[r, "R_1"] \arrow[d, "A^T"] & \mathbb{Z}^6 \arrow[r] \arrow[d, "I_6"] & \mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2) = \mathbb{Z}^2 \arrow[r] \arrow[d, "i^*"] & 0 \\
0 \arrow[r] & \mathbb{Z}^2 \arrow[r, "R_2"] & \mathbb{Z}^6 \arrow[r] & \mathrm{Pic}(\mathrm{dP}_6) = \mathbb{Z}^4 \arrow[r] & 0
\end{tikzcd}$$

It now follows from the snake lemma that i^* is injective with zero cokernel. $\heartsuit$

This lemma let us compute explicitly what the induced map of Picard groups is using only the toric data.

Theorem 3.2.8. *The two affine smoothings are topologically different. The homology groups are:*

Group	1	1	2	3	4	5	6	*Euler characteristic*
$H^i(X_1, \mathbb{Z})$	1	0	2	1	0	0	0	*2*
$H^i(X_2, \mathbb{Z})$	1	0	1	2	0	0	0	*0*

Proof. The singular cohomology of $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is given by $(1, 0, 3, 0, 3, 0, 1)$, which can be computed by the Künneth formula (see [Hat02], page 275). The cohomology of dP_6 is given by $(1, 0, 4, 0, 1)$.

We will use the Lefschetz duality theorem [Spa66], which in this case says that $H_q(M \backslash dP_6; \mathbb{Z}) \simeq H^{6-q}(M, dP_6; \mathbb{Z})$. The long exact sequence of the pair (M, dP_6) ([Hat02], page 200) takes the form:

$$\begin{tikzcd}
0 \arrow[r] & H^0(M,D;\mathbb{Z}) \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[dll, "\delta"] \\
 & H^1(M,D;\mathbb{Z}) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, "\delta"] \\
 & H^2(M,D;\mathbb{Z}) \arrow[r] & \mathbb{Z}^3 \arrow[r] & \mathbb{Z}^4 \arrow[dll, "\delta"] \\
 & H^3(M,D;\mathbb{Z}) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, "\delta"] \\
 & H^4(M,D;\mathbb{Z}) \arrow[r] & \mathbb{Z}^3 \arrow[r] & \mathbb{Z} \arrow[dll, "\delta"] \\
 & H^5(M,D;\mathbb{Z}) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, "\delta"] \\
 & H^6(M,D;\mathbb{Z}) \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

From the exactness of the sequence, we immediately find $H^0(X_1; \mathbb{Z}) = \mathbb{Z}$. Also, since $H^0(M; \mathbb{Z}) \to H^0(\mathrm{dP}_6; \mathbb{Z})$ is an isomorpism (both are connected), it follows that $H^6(X_1; \mathbb{Z}) = H^5(X_1; \mathbb{Z}) = 0$.

The other groups depend upon the explicit form of the maps $H^2(M; \mathbb{Z}) \to H^2(\mathrm{dP}_6; \mathbb{Z})$ and $H^4(M; \mathbb{Z}) \to H^4(\mathrm{dP}_6, \mathbb{Z})$.

The map $H^2(M; \mathbb{Z}) \to H^2(\mathrm{dP}_6; \mathbb{Z})$ can be identified with the map $i^* \colon \mathrm{Pic}(M) \to \mathrm{Pic}(\mathrm{dP}_6)$ induced by the inclusion. This map was computed in Example 3.2.6. It is an injective map with torsion-free cokernel, and it follows from the long-exact sequence and the Lefschetz theorem that $H_3(X_1; \mathbb{Z}) \simeq H^3(M, dP_6; \mathbb{Z}) \simeq \mathbb{Z}$, and also that $H_4(X_1; \mathbb{Z}) = 0$.

To compute the map $H^4(M; \mathbb{Z}) \to H^4(\mathrm{dP}_6; \mathbb{Z})$, note that $H^4(M; \mathbb{Z})$ is Poincaré dual to $H_2(M; \mathbb{Z})$, and this group is generated by $\mathbb{P}^1 \times \{pt\} \times \{pt\}$

(and permutations)[1]. Also, $H^4(\mathrm{dP}_6;\mathbb{Z}) \simeq H_0(\mathrm{dP}_6,\mathbb{Z}) = \mathbb{Z}$. In this description, pullback corresponds to intersection, and one sees that the map is given by $(a,b,c) \mapsto a+b+c$, since the three $\mathbb{P}^1$'s intersect dP_6 in a single point each. This map has two-dimensional kernel, and we conclude that $H_2(X_1;\mathbb{Z}) \simeq H^4(M,\mathrm{dP}_6;\mathbb{Z}) = \mathbb{Z}^2$, and that $H^1(X_1;\mathbb{Z}) = 0$.

The computations for X_2 are similar. We first note that the Picard group of $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ is generated by the pullbacks F, G of the generators of $\mathrm{Pic}(\mathbb{P}^2_{x_0x_1x_2} \times \mathbb{P}^2_{y_0y_1y_2})$. Say F is represented by $V(x_0)$ and G is represented by $V(y_0)$.

Again we compute the intersections of F and G with dP_6. Intersecting with F is computed by decomposing the ideal $(x_0, x_1y_0 - x_2y_1, x_1y_0 - x_0y_2)$ in $k[x_0,x_1,x_2,y_0,y_1,y_2]$ and saturating by (x_0,x_1,x_2) and (y_0,y_1,y_2). This can either be done by hand or by using `Macaulay2`. Either way, we find that $F\big|_{dP_6} = E_3 + L_{23} + E_2 = H$, using the notation from earlier this chapter. Similarly $G\big|_{dP_6} = L_{23} + L_{12} + E_2 = 2H - E_1 - E_2 - E_3$. Hence the map on cohomology is given by the matrix

$$H^2(M;\mathbb{Z}) \simeq H_4(M;\mathbb{Z}) \simeq \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}} \mathbb{Z}^4 \simeq H_2(\mathrm{dP}_6;\mathbb{Z}) \simeq H^2(\mathrm{dP}_6;\mathbb{Z}).$$

This is an injective map, and as above, we conclude that $H_3(X_2;\mathbb{Z}) \simeq H^3(M,\mathrm{dP}_6;\mathbb{Z}) \simeq \mathbb{Z}^2$, and also that $H_4(X_1;\mathbb{Z}) = 0$. ■

Remark 3.2.9. In fact, the Andreotti-Frankel theorem [AF59] states the following: if V is any smooth affine variety of complex dimension n, then it has the homotopy type of a CW complex of real dimension n. Thus it comes for free that $H^j(X_i,\mathbb{Z}) = 0$ for $j > 3$.

[1] I thank the math.stackexchange user *nefertiti* for this argument.

CHAPTER 4

Construction of Calabi–Yau's

structions

In this chapter I describe the construction of three topologically different smoothings of a singular Calabi-Yau manifold. They correspond to different components of the Hilbert scheme of threefolds in $\mathbb{P}^{11}$ with Hilbert polynomial $p(t) = 6t^3 + 6$.

We first describe a degenerate Calabi–Yau in the form of a Stanley–Reisner scheme $\mathbb{P}(\mathcal{K})$, which has a quite large symmetry group. There is a natural deformation to a X_Y, which is a subvariety of a toric variety Y, with isolated singularities.

We show that X_Y has several topologically distinct smoothings, which lie on different components of the Hilbert scheme in $\mathbb{P}^{11}$.

Furthermore, in the last section, we propose mirror candidates for two of the constructions, based on orbifolding.

4.1 The special fiber

Let E_6 be the hexagon as a simplicial complex. The associated Stanley–Reisner scheme $\mathbb{P}(E_6)$ is a degenerated elliptic curve in $\mathbb{P}^5$. If $\mathbb{P}^5$ have coordinates $x_0, \dots, x_5$, the equations of $\mathbb{P}(E_6)$ are $x_i x_{i+2} = x_i x_{i+3} = 0$, where i is taken modulo 6. This gives a total of 9 quadratic generators.

Lemma 4.1.1. *The Hilbert polynomial of* $\mathbb{P}(E_6)$ *is* $h(t) = 6t$.

Proof. We want to count the dimension of $S_t = S_{E_6}(t)$. Any monomial in S_k has support on the simplicial complex E_6, so its support is either a vertex or an edge. In the first case, the monomial has the form x_i^t, so there are six of these.

In the other case, it has the form $x_i^a x_{i+1}^b$, with $a + b = t$ and $a, b \neq 0$. Counting, there are $6(t-1)$ of these monomials. In total, the dimension is $6 + 6(t-1) = 6t$. ■

Remark 4.1.2. Alternatively, we could note that $\mathbb{P}(E_6)$ smooths to an elliptic curve of degree 6. Since Hilbert polynomials are constant in flat families, it follows from Riemann–Roch that $h(t) = \deg \mathscr{O}_{\mathbb{P}(E_6)}(6t) - 1 + 1 = 6t$.

Note that the Hilbert polynomial only differ from the Hilbert function for $t = 0$, since $h(0) = 0$, while $\dim_{\mathbb{C}} S_0 = 1$.

We now introduce the central fiber in the discussions onward. Let $\mathcal{K}$ be the simplicial complex $E_6 * E_6$. It is a triangulation of the 3-sphere. The maximal faces are unions of maximal faces from each factor.

Denote the vertices of the left E_6 with $x_1, \dots, x_6$, and the vertices of the right E_6 with $z_1, \dots, z_6$. Then the maximal faces of $\mathcal{K}$ are of the form $x_i x_{i+i} z_j z_{j+1}$, where $i, j \in \mathbb{Z}_6$. The number of i-faces follows are easy to compute:

Lemma 4.1.3. *The f-vector of $\mathcal{K}$ is* $(1, 12, 48, 36)$.

Proof. There are 12 vertices, and $6 \times 6 = 36$ maximal facets. Since $\mathcal{K}$ is a 3-sphere, it follows that $12 - f_1 + 36 = \chi(S^3) = 0$ that $f_1 = 48$.[1] ■

Lemma 4.1.4. *The Hilbert polynomial of* $\mathbb{P}(\mathcal{K})$ *is* $h(t) = 6t^3 + 6$.

Proof. The homogeneous coordinate ring $S = \oplus_{t \geq 0} S_t$ of $\mathbb{P}(\mathcal{K})$ is the graded tensor product of $\mathbb{P}(E_6)$ with itself. It follows from the previous lemma that

$$\dim S_t = \sum_{i+j=k, ij \neq 0} 36ij + 12k,$$

where the last term is a correction term because $h(t) \neq 1$. It is now a routine computation using formulas for sums of squares to verify the claim. ■

Corollary 4.1.5. *Any smoothing of* $\mathbb{P}(\mathcal{K})$ *satisfy* $|H| = 12$, $c_2 \cdot H = 72$, *and* $H^3 = 36$.

Proof. All these invariants can be read off from the Hilbert polynomial. ■

Either by using `Macaulay2` or by using the more conceptual description of the T^i-modules from [AC10], we can compute:

Proposition 4.1.6. *The dimensions of* $T^1(A_{\mathcal{K}})_0$ *and* $T^2(A_{\mathcal{K}})_0$ *are* 84 *and* 72, *respectively.*

[1] Here we used that in a cell complex, the Euler characteristic is also the alternating sum of the number of cells in each dimension. This is Theorem 2.44 in [Hat02].

Proof. We will prove this using the techniques and notation from [AC10]. Our goal is to compute the degree zero part of $T^1_{A_\mathcal{K}}$. We will do this using Theorem 1.5.4.

First notice that all links of vertices of $\mathcal{K} = E_6 * E_6$ are double suspensions over hexagons (they are denoted by ΣE_6 in Christophersen's article).

According to Table 1 in Christophersen's article, double suspensions over hexagons contribute with one dimension to $T^1_{A_\mathcal{K}}$, namely in degree $x_i^2/x_{i-1}x_{i+1}$ (if $\mathbf{a} = x_i^2$). In total there are $6 + 6 = 12$ contributions of this form.

Taking the link at the vertex $x_i z_j$ produces a square with vertices x_{i+1},z_{j+1},x_{i-1},z_{j-1} (in that order). According to Table 1 in Christophersen's article, these links contribute with dimension 2 to $T^1_{A_\mathcal{K}}$. The contributions have degrees $x_i z_j/x_{i+1}x_{i-1}$ and $x_i z_j/z_{j+1}z_{j-1}$. There are $2 \cdot 6 \cdot 6 = 72$ contributions of this form.

Thus, in total, $T^1_{A_\mathcal{K}}$ have $\mathbb{C}$-dimension 84.

We now compute $T^2_{A_\mathcal{K}}$. The contributions come from choosing $\mathbf{a} = x_i^2$ and $\mathbf{a} = x_i x_{i+1}$, respectively. If $|a| = 1$ (as in the first case), the results from Christophersen's article imply that $L_b := \cap_{b' \subset b} \mathrm{lk}(b', \mathrm{lk}(x_i, \mathcal{K}))$ must have more than one connected component (the contribution comes from $\widetilde{H}^0(L_b, \mathbb{C})$). This is the case if b consist of two opposite vertices in the suspended circle. In total there are $2 \cdot 6 \cdot 3 = 36$ contributions of this form.

If $|a| = 2$, the contributing links are hexagons, and in this case the contributions come from b such that $L_b = \emptyset$. Again consisthoosing b to consist of opposite vertices of the hexagon, we find three pairs b with $L_b = \emptyset$ for each hexagon. Thus in total there are $2 \cdot 6 \cdot 3 = 36$ contributions of this form.

In sum, $T^2_{A_\mathcal{K}}$ is $36 + 36$-dimensional. ∎

The automorphism group of $\mathcal{K}$ is $D_6 \times D_6 \times \mathbb{Z}_2$, and of order $12 \cdot 12 \cdot 2 = 288$. It is not difficult to see that the induced action on $T^1_{X_0}$ have two orbits under $\mathrm{Aut}(\mathcal{K})$, corresponding to first order deformations of the form $x_i x_{i-2} + t x_{i+1} z_j$ and $x_{i-1}x_{i+1} + t x_i^2$, respectively.

4.2 A natural toric deformation

Consider Figure 4.1. It is the 2-dimensional polytope associated to the del Pezzo surface of degree 6. The fan over this polytope correspond to a unimodular regular triangulation of the polytope, and it follows by Proposition 1.7.1, that dP_6 degenerates to the Stanley–Reisner scheme $\mathbb{P}(E_6 * \{pt\})$, where $\{pt\}$ correspond to the origin. This is an embedded deformation inside $\mathbb{P}^6$.

Concretely, the equations of dP_6 are given by $x_i x_{i+2} - y x_{i+1} = x_i x_{i+3} - y^2 = 0$. The degeneration to $\mathbb{P}(E_6 * \{pt\})$ is given by setting the second terms to zero.

Now form the join of two copies of dP_6, to get a new variety $Y \subset \mathbb{P}^{13}$. By Proposition 1.1.2, this is a $2 + 2 + 1 = 5$-dimensional toric variety with singular locus consisting of two copies of dP_6. Since the coordinate ring is just the tensor product of two copies of $S(\mathrm{dP}_6)$, it follows that Y degenerates to $\mathbb{P}(E_6 * \{pt\} * E_6 * \{pt\}) = \mathbb{P}(\mathcal{K} * \Delta^1)$.

The following holds:

Proposition 4.2.1. *There is a deformation of the Stanley–Reisner scheme X_0 to an irreducible Calabi–Yau variety $X_Y \subset Y$ with isolated singularities. There are twelve of them, and they are locally isomorphic to cones over del Pezzo surfaces. More precisely: let (U, p_i) be the germ of X_0 at p_i. Then $(U, p_i) \simeq (C(\mathrm{dP}_6), 0)$.*

Proof. Since X_0 is a complete intersection side $\mathbb{P}(\mathcal{K} * \Delta^1)$, it follows that X_0 deforms to a complete intersection inside any deformation of $\mathbb{P}(\mathcal{K} * \Delta^1)$. We explained above that $\mathbb{P}(\mathcal{K} * \Delta^1)$ deforms to the join Y of two del Pezzo surfaces, and it follows that X_0 deforms to Y intersected with two generic hyperplanes.

Since Y has singular locus of dimension 2 and degree $6 + 6 = 12$, it follows by Bertini's theorem [Har77, Chapter II, Theorem 8.18] that X_0 has twelve isolated singularities p_i.

To see how the singularities look locally, we argue as follows. Locally, Y looks like $\mathbb{A}^2_{a_1,a_2} \times C(\mathrm{dP}_6)_{x_i}$, where the subscripts refer to the coordinates. This is the ideal of Y consists of two sets of equations, each defining a smooth toric variety, and smooth toric varieties are isomorphic to $\mathbb{A}^d$ in affine charts.

The claim now follows from two applications of Theorem 3.1.5 in [Bat94], which says that the singularities on Σ-regular toric hypersurfaces are inherited from the ambient toric variety. ∎

Since the cone over dP_6 deforms in two topologically different ways, we might expect that X_Y does this too. This is indeed true.

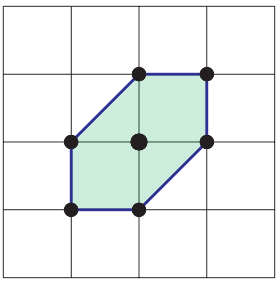


Figure 4.1: A hexagon.

fig:hexago

4.3 Smoothings of X_Y

By embedding dP_6 in different spaces, we obtain different smoothings of subvarieties of the join of these spaces.

The block matrix construction

We are inspired by the construction in Rødland's thesis [Rød00][2].

Let E be a 3-dimensional vector space. Let $\{e_1, e_2, e_3\}$ be a basis for E. Then we can form the vector space $V = (E \otimes E) \oplus (E \otimes E)$, which has dimension 18. Let $\mathbb{P}^{17} = \mathbb{P}(V)$. Choose coordinates $x_1, \ldots, x_{18}$ on $\mathbb{P}^{17}$.

Thinking of $E \otimes E$ as 3×3-matrices, we can think of the elements of $\mathbb{P}^{17}$ as pairs of 3×3-matrices up to scalar, not both zero. Concretely, two pairs of matrices (A', B') and (A, B) are equivalent if $(A', B') = (\lambda A, \lambda B)$ for some $\lambda \in \mathbb{C}^*$.

We can also interpret $\mathbb{P}^{17}$ as the geometric join of $\mathbb{P}(E \otimes E)$ with itself. This is the set of all lines connecting pairs of 3×3-matrices.

There is a natural rational map $\pi : \mathbb{P}^{17} \dashrightarrow \mathbb{P}^8 \times \mathbb{P}^8$, which is the identity on coordinates, given by dividing out by the antidiagonal $\mathbb{C}^*$-action: $\lambda' \cdot (A, B) = (\lambda', \lambda'^{-1} B)$.

Denote by V_1 and V_2 the subspaces $x_1 = \ldots = x_9 = 0$ and $x_{10} = \ldots = x_{18} = 0$, respectively. Blow up $\mathbb{P}^{17}$ in $V_1 \cup V_2$, to get $\widetilde{\mathbb{P}^{17}}$. The spaces V_i are exactly the indeterminacy locus of π, so π extends to a map $\pi : \widetilde{\mathbb{P}^{17}} \to \mathbb{P}^8 \times \mathbb{P}^8$. Denote by π_1 and π_2 the two natural projections to $\mathbb{P}^8$. Then it is true that $\widetilde{\mathbb{P}^{17}} = \mathbb{P}_{\mathbb{P}^8 \times \mathbb{P}^8}(\pi_1^* \mathscr{O}_{\mathbb{P}^8}(1) \oplus \pi_2^* \mathscr{O}_{\mathbb{P}^8}(1)) = \mathbb{P}(\mathscr{O}_{\mathbb{P}^8 \times \mathbb{P}^8} \oplus \mathscr{O}_{\mathbb{P}^8 \times \mathbb{P}^8}(1, -1))$. This is explained further in Section C7 in [AK75].

Let M be the closure of the set of pairs (A, B) where $\operatorname{rank} A = \operatorname{rank} B = 1$.

prop:m

Proposition 4.3.1. *The variety M is the join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, and has singular locus $\mathbb{P}^2 \times \mathbb{P}^2 \subset V_i$ of dimension 4.*

The canonical sheaf is $\omega_M = \mathscr{O}_M(-6)$, so that M is a Fano toric variety.

Proof. If $\mathbb{P}^{17}$ have coordinates $x_1, \ldots, x_{18}$, let M_1 and M_2 be the matrices

$$M_1 = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} \\ x_{16} & x_{16} & x_{17} \end{pmatrix}.$$

Then M is defined by the zeroes of the 2×2-minors of M_1 and M_2. Then it is clear that M is the projective join of two copies of $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8 \subset \mathbb{P}^{17}$, since the sets of variables are disjoint.

[2] Rødland's construction is a linear subvariety of $\mathbb{P}(E \wedge E)$, where E is 7-dimensional.

The variety M is 9-dimensional: the affine cone over M, $C(M)$, is equal to $C(\mathbb{P}^2 \times \mathbb{P}^2) \times C(\mathbb{P}^2 \times \mathbb{P}^2)$. This variety has dimension $5+5=10$, hence its projectivization M is 9-dimensional.

The singular locus of M consists of the pairs $(0, B)$, and $(A, 0)$, where $\operatorname{rank} A = \operatorname{rank} B = 1$, hence $\dim \operatorname{Sing} M = \dim \left(\mathbb{P}^2 \times \mathbb{P}^2\right) = 4$. See also Proposition 1.1.2.

By Remark 1.1.4, it follows that $\omega_M = \mathscr{O}_M(-6)$, since

$$\omega_{\mathbb{P}^2 \times \mathbb{P}^2} = \mathscr{O}_{\mathbb{P}^8}(-3)\big|_{\mathbb{P}^2 \times \mathbb{P}^2}.$$

■

Here comes our first construction. Let X_1 be the intersection of M with a generic $\mathbb{P}^{11}$. Then the following is true.

prop:x1 **Proposition 4.3.2.** *X_1 is a smooth Calabi–Yau variety with $\chi(X_1) = -72$.*

Proof. The singularities of M are of dimension 4. By Bertini's theorem, intersecting M with a codimension 6 hyperplane gives a smooth variety X_1.

That X_1 has trivial canonical bundle, follows from Proposition 1.2.2.

To find the topological Euler characteristic, we compute in `Macaulay2`. Computing the whole cotangent sheaf of X_1 is infeasible with current computer technology[3], we make use of standard exact sequences. Let $\mathscr{I}$ be the ideal sheaf of M in $\mathbb{P}^{17}$. First off, we have the exact sequence

$$0 \to \mathscr{I}/\mathscr{I}^2\big|_{X_1} \to \Omega^1_{\mathbb{P}^{17}}\big|_{X_1} \to \Omega^1_M\big|_{X_1} \to 0.$$

The restriction to X_1 is exact since $\mathscr{I}/\mathscr{I}^2$ is locally free on the smooth locus.

The `Macaulay2` command `eulers` computes the Euler characteristics of generic linear sections of a sheaf $\mathscr{F}$ (behind the scene, this is equivalent to computing the Koszul resolution of the relative ideal sheaf $\mathscr{I}_{X_1/M}$). Using this command, we find that $\chi(\mathscr{I}/\mathscr{I}^2\big|_{X_1}) = -180$. Using the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^{17}}\big|_{X_1} \to \mathscr{O}_{X_1}(-1)^{18} \to \mathscr{O}_{X_1} \to 0,$$

we find that the Euler characteristic of $\Omega^1_{\mathbb{P}^{17}}\big|_{X_1}$ is $-216 = 12 \cdot 18$. It follows from the first exact sequence that $\Omega^1_M\big|_{X_1}$ has Euler characteristic -36.

Since X_1 is a complete intersection in M, the conormal sequence is

$$0 \to \mathscr{O}_{X_1}(-1)^6 \to \Omega_M\big|_{X_1} \to \Omega^1_{X_1} \to 0.$$

[3]An external computer has been trying to compute this sheaf for several months now without terminating.

Hence $\chi(\Omega^1_X) = -36 + 72 = 36$.

It follows that the topological Euler characteristic is $\chi(X_1) \triangleq \chi(\mathcal{T}_{X_1}) = -2\chi(\Omega^1_{X_1}) = -72$. ■

remark_X1 *Remark* 4.3.3. We can give explicit equations for a flat family with special fiber X_Y and general fiber X_1. Let $y = h_1(x_1, \ldots, x_{12})$ and $y_1 = h_2(x_1, \ldots, x_{12})$ be the generic linear forms in $\mathbb{P}^{13}$ defining X_Y as a subscheme of Y. Let g_i (for $i = 1, \ldots, 6$) be generic linear forms in $\mathbb{P}^{11}$. Then such a flat family is defined by the 2×2-minors of the two matrices below:

$$A_1 = \begin{pmatrix} h_1 + tg_1 & x_2 & x_3 \\ x_4 & h_1 + tg_2 & x_6 \\ x_7 & x_8 & h_1 + tg_3 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} h_2 + tg_4 & x_{11} & x_{12} \\ x_{13} & h_2 + tg_5 & x_{15} \\ x_{16} & x_{16} & h_2 + tg_6 \end{pmatrix}.$$

For $t = 0$, we get X_Y. Note that the subscheme defined by the minors of

$$A_1 = \begin{pmatrix} y_1 & x_2 & x_3 \\ x_4 & y_1 & x_6 \\ x_7 & x_8 & y_1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} y_2 & x_{11} & x_{12} \\ x_{13} & y_2 & x_{15} \\ x_{16} & x_{16} & y_2 \end{pmatrix}$$

is the join of dP_6 with itself. Since $X_Y \subset \mathrm{dP}_6 * \mathrm{dP}_6$, we see that X_1 lies in a deformation of $\mathrm{dP}_6 * \mathrm{dP}_6$.

Remark 4.3.4. Since X_1 avoids the fundamental subscheme $V_1 \cup V_2$, the inverse image $\pi^{-1}(X_1) \subset \widetilde{\mathbb{P}^{17}}$ is isomorphic to X_1. Thus we can realize X_1 as a subvariety of a *smooth* variety. Unfortunaly, X_1 is cut out by non-ample divisors in $\widetilde{\mathbb{P}^{17}}$.

Remark 4.3.5. I have not been able to rigorously compute the Hodge nubmers of X_1. However, over several finite fields I have computed the dimension of the degree zero part of the T^1 module in `Macaulay2`. By Proposition 1.4.4, we have that $(T^1(S_{X_1}/k, S_{X_1}))_0 = H^1(X_1, \Omega^2_X)$. This module can be computed in `Macaulay2`.

After about a week of computation on a modern desktop computer, the answer turns out to be $\dim_{\mathbb{F}_p}(T^1(S_{X_1}/\mathbb{F}_p, S_{X_1}))_0 = 39$ for several large primes p.

This is plausible because of the following heuristic moduli count: X_1 is parametrized by the Grassmannian $\mathbb{G}r(12, (E \otimes E)^{\oplus 2})$, which has dimension $(18 - 12) \cdot 12 = 72$. Each E-factor is acted upon by $\mathrm{GL}(E)$. There are four of these factors, so we have an action of $\prod^4 \mathrm{GL}(E)$ on the Grassmannian. There is a torus subgroup $(\mathbb{C}^a st)^4$ acting by $(v \otimes w, r \otimes s) \mapsto (t_1 t_2 v \otimes w, t_3 t_4 r \otimes s)$ on $\mathbb{P}^{17}$. Elements of $(\mathbb{C}^*)^4$ satisfying $t_1 t_2 = t_3 t_4$ act trivially, forming a isotropy subgroup K. Hence we have an action of the quotient group $G \triangleq \left(\prod_{i=1}^4 \mathrm{GL}(4)\right)/K$ on the Grassmannian. This quotient group hasa dimension $9 \cdot 4 - 3 = 33$.

We form the quotient $\mathbb{G}(12, (E \otimes E)^{\oplus 2})/G$, which have dimension $72 - 33 = 39$.

If this is true, then X_1 has Hodge numbers $h^{11} = 3$ and $h^{12} = 39$. It is not clear which other divisors there are besides the hyperplane divisor.

The three-tensor construction

The construction in the previous section used the embedding of dP_6 in $\mathbb{P}^2 \times \mathbb{P}^2$ to deform X_Y. There is also the embedding of dP_6 in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to exploit. The construction is similar.

Let F be a 2-dimensional vector space with basis $\{f_1, f_2\}$. Then we can form the vector space $V = (F \otimes F \otimes F)^{\oplus 2}$. Let $\mathbb{P}^{15} = \mathbb{P}(V)$. Choose coordinates $a_{ijk} = (f_i \otimes f_j \otimes f_j, 0)$ and $b_{ijk} = (0, f_i \otimes f_j \otimes f_k)$ $(i, j, k = 0, 1)$ for $\mathbb{P}^{15}$.

The elements of $\mathbb{P}^{15}$ are pairs (A, B) of $2 \times 2 \times 2$ tensors, not both zero. There is also in this case a natural map $\pi : \mathbb{P}^{15} \to \mathbb{P}^7 \times \mathbb{P}^7$, given by dividing out by the antidiagonal $\mathbb{C}^*$-action. Just as above, let V_1 and V_2 be the subspaces $A = 0$ and $B = 0$, respectively. Let $\widetilde{\mathbb{P}^{15}}$ be the blowup of $\mathbb{P}^{15}$ in $V_1 \cup V_2$. The V_i's are exactly the indeterminacy locus of π, so π extends to a morphism $\pi : \widetilde{\mathbb{P}^{15}} \to \mathbb{P}^8 \times \mathbb{P}^8$, which is a $\mathbb{P}^1$-bundle. Also in this case it is true that $\widetilde{\mathbb{P}^{15}} = \mathbb{P}\left(\mathscr{O}_{\mathbb{P}^7 \times \mathbb{P}^7} \oplus \mathscr{O}_{\mathbb{P}^7 \times \mathbb{P}^7}(1, -1)\right)$.

Let N be the closure of set of pairs (A, B) where both A and B have tensor rank 1[4].

Proposition 4.3.6. *The variety N is the join of two copies of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$, and has singular locus $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset V_i$ of dimension 3.*

The canonical sheaf is $\omega_N = \mathscr{O}_N(-4)$, so that N is a Fano toric variety.

Proof. A pure $2 \times 2 \times 2$-tensor can be visualized as a cube with vertices a_{ijk}. See the diagram in Figure 4.2 on the facing page.

The equations of the set of rank 1 tensors in $\mathbb{P}(F \otimes F \otimes F)$ are obtained as the "minors" along the 6 sides of the cube, together with the minors along with the 3 long diagonals, giving a total of 9 binomial equations. We write this symbolically as $[a_{ijk}] \leq 1$.

Hence the equations for N are given by $[a_{ijk}] \leq 1$, together with $[b_{ijk}] \leq 1$. Since these are equations in a disjoint set of variables, it is clear that $N = (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)^{*2}$.

The claim about the singular locus and the canonical sheaf follow as in the proof of Proposition 4.3.1. ■

Let X_2 be the intersection of N with a general $\mathbb{P}^{11}$.

[4] An element of $F^{\otimes 3}$ have rank 1 if it is a pure tensor. It has rank $\leq k$ if it can be written as a sum of k pure tensors.

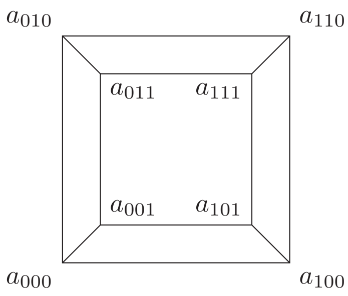


Figure 4.2: A $2 \times 2 \times 2$-tensor, seen from "above". fig:222tensor

Proposition 4.3.7. *The topological Euler characteristic of X_2 is -48.*

Proof. The proof is identical to the proof of Proposition 4.3.2. ∎

Remark 4.3.8. A heuristic moduli count works also in this case.

X_2 lies in a 12-plane in $\mathbb{P}((F \otimes F \otimes F)^{\oplus 2})$. Such planes are parametrized by $\mathbb{G}(12, 16)$, the Grassmannian of 12-planes in k^{16}. This space is $12 \cdot (16 - 12) = 48$-dimensional. There is an action of the group $\prod_{i=1}^{6} \mathrm{GL}(F)$ on $(F \otimes F \otimes F)^{\oplus 2}$. There is also in this case a torus subgroup acting trivially. Namely, the elements satisfying $t_1 t_2 t_3 = t_4 t_5 t_6$. Call this subgroup K. Thus we really have an action of the group $\left(\prod_{i=1}^{6} \mathrm{GL}(F)\right)/K$, which have dimension $6 \cdot 4 - 5 = 19$. Thus in total we have $48 - 19 = 29$ moduli parameters.

Since we know the Euler characteristic, we predict the Hodge numbers to be $(h^{11}, h^{12}) = (5, 29)$.

The mixed smoothing

In the above cases, we formed the join of equal varieties. We mix things up: let $V = (E \otimes E) \oplus (F \otimes F \otimes F)$. Then let $\mathbb{P}^{16} = \mathbb{P}(V)$.

Now let W be the set of "mixed" rank 1 tensors. In a way similar to above, we find that W is a singular Fano toric variety of dimension 8. The singular locus is of dimension 4, so a 5-fold complete intersection is again a smooth Calabi-Yau variety X_3.

Proposition 4.3.9. *The Euler characteristic of X_3 is -60.*

Proof. The proof is identical to the proofs above. ∎

Remark 4.3.10. We again give a heuristic moduli count. The Grassmannian in this case is 60-dimensional. The group acting on it is $\prod^2 \mathrm{GL}(E) \times \prod^3 \mathrm{GL}(F)$.

Here the trivially acting torus subgroup will be those satisfying $t_1t_2 = t_3t_4t_5$. It follows the the parameter space is $60 - (18 + 12 - 4) = 34$-dimensional.

Hence we predict the Hodge numbers to be $(h^{11}, h^{12}) = (4, 34)$.

4.4 Degeneration of the toric join

sec:degenjoin

Consider the construction of X_1 from above, and the explicit equations from Remark 4.3.3. Putting $t = 0$ and $h_1 = h_2$, gives a degeneration of X_Y to another, more singular, variety $X_{Y'}$. Explicitly, it is given by the 2×2-minors of the following two matrices, where h is a generic hypersurface in the variables.

$$A_1 = \begin{pmatrix} h & x_2 & x_3 \\ x_4 & h & x_6 \\ x_7 & x_8 & h \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} h & x_{11} & x_{12} \\ x_{13} & h & x_{15} \\ x_{16} & x_{16} & h \end{pmatrix}.$$

We can realize $X_{Y'}$ as a hypersurface in a toric variety as follows. Introduce a new variable y, and consider the variety defined by the 2×2 minors of

$$A_1 = \begin{pmatrix} y & x_2 & x_3 \\ x_4 & y & x_6 \\ x_7 & x_8 & y \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} y & x_{11} & x_{12} \\ x_{13} & y & x_{15} \\ x_{16} & x_{16} & y \end{pmatrix}.$$

This is a 4-dimensional toric variety. It is the toric variety associated to the polytope Δ with vertices the columns of the matrix

$$\begin{pmatrix} -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \end{pmatrix}.$$

A computation shows that Y' has 1-dimensional singularities, and the singular locus is a graph of $\mathbb{P}^1$'s: take two hexagons, and join each vertex of one of them with all vertices of the other one. This makes in total 48 $\mathbb{P}^1$'s.

The variety Y' is a Fano toric variety, and as such, it has a anticanonical section $X_{Y'}$ which is a singular Calabi–Yau variety. A local computation shows that $X_{Y'}$ has 12 singularities that are locally isomorphic to $C(\mathrm{dP}_6)$, and 36 double points. This can also be seen torically: the cones in the fan of Y' corresponding to the singular locus comes in two types. The first type is a cone over a hexagon, and the other type is the cone over a square. These correspond to (algebro-geometrically) cones over dP_6 and double points, respectively.

Since Y' is a four-fold, it follows that $X_{Y'}$ has a maximal projective crepant resolution of singularities (a *MPCP-desingularization*) (see for example the comment on page 55 in [CK99]), which we denote by $\widetilde{X_{Y'}}$.

A computation using PALP [KS04] shows that $\widetilde{X_{Y'}}$ has Hodge numbers $(44, 8)$ and Euler-characteristic 72.

Remark 4.4.1. There is a heuristic surgical reason for the Euler characteristic being $+72$. Our $X_{Y'}$ deforms to X_1, which has Euler characteristic -72. This is obtained by smoothing 36 double points and 12 cones over del Pezzo surfaces. By the inclusion-exclusion principle, it follows that a small resolution of the singularities of $X_{Y'}$ have Euler characteristic $\chi(X_1) + 2 \cdot 36 + 6 \cdot 12 = 72$.

Remark 4.4.2. The variety $X_{Y'}$ has also been described elsewhere. The polar polytope Δ° is equal to the product of two hexagons, and it follows that $\mathbb{P}_{\Delta^\circ}$ is equal to the product of two del Pezzo surfaces. An anticanonical hypersurface in $\mathrm{dP}_6 \times \mathrm{dP}_6$ has Euler characteristic -72 (see Theorem 3.1 in [Hüb92]).

In the article [BCD10], Braun et al. study this hypersurface and a group action on it. They also describe, in detail, a resolution of singularities of $X_{Y'}$.

Remark 4.4.3. In [CD10], the authors study Calabi–Yau complete intersections admitting free actions by finite groups, and certain transitions between them (these are similar to Morrison's extremal transitions). They find that there is a Calabi–Yau with Hodge numbers $(3, 39)$ and a Calabi–Yau with Hodge numbers $(8, 44)$ beloning to the same family of transitions, both admitting $\mathbb{Z}/3$-actions. It is not clear to us if their $(3, 39)$-manifold is the same as our X_1.

4.5 The mirror construction Ansatz

ion_ansatz

In many cases of interest, given a construction of a Calabi–Yau manifold, the following Ansatz produces a mirror.

Let $\mathcal{K}$ be a simplicial complex, with associated Stanley–Reisner scheme X_0. Let G be the automorphism group of $\mathcal{K}$. Then G induces an action on $T^1_{X_0}$ in the following way: each element of $T^1_{X_0}$ can be represented by a $\phi \in \mathrm{Hom}(I/I^2, A)$, and then $g \cdot \phi$ is given by $(g \cdot \phi)(f) = g \cdot \phi(g^{-1} \cdot f)$, the contragredient action.

There is an action of $T_n = (\mathbb{C}^*)^{n+1}/\mathbb{C}^*$ on $\mathbb{P}^n$, and since I_{X_0} is generated by monomials, the action restricts to an action on X_0 as well.

Given a smoothing family with general fiber X and special fiber X_0, we can consider a subfamily with only isolated singularities on which G act. Let $H \subset T_n$ be the subgroup of the torus acting on this family. Then the mirror candidate to X is given by a crepant resolution of $Y_t = X_t/H$.

Though it is often overlooked (or stated in another language) in the literature, even the mirror construction of the famous quintic arises this way. Briefly, the quintic Calabi–Yau is given by the zero locus of a general quintic in $H^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(5))$. The special quintic given by $f = x_0x_1x_2x_3x_4$ is the Stanley–Reisner scheme corresponding to the 3-simplex. The automorphism group is S_5, and an invariant 1-parameter family is given by $f_t = \sum_{i=0}^4 x_i^5 + t x_0x_1x_2x_3x_4$. The fiber at $t = \infty$ is the Stanley–Reisner scheme.

There is a $H \stackrel{\Delta}{=} (\mathbb{Z}/5)^5/\mathbb{Z}^5$-action on $X_t = Z(f_t)$ given by coordinate-wise multiplication by fifth roots of unity. The general element of the family X_t

is smooth, so the only singularities of the quotient $Y_t = X_t/H$ comes from points with non-trivial stabilizer. These can be resolved by methods from toric geometry. For details, see for example the first chapter of Ingrid Fausk's thesis [Fau12].

4.6 Invariant Calabi–Yau's and a mirror construction

In this section, I will explain natural group actions on the X_i's constructed above. Using the mirror construction Ansatz from above, we propose mirror candidates for X_1 and X_2.

Invariant subfamily of X_1

Let us first consider $M = (\mathbb{P}^2 \times \mathbb{P}^2)^{*2}$. Recall that M can be thought of as pairs of rank one 3×3 matrices up to scalar. We will describe several natural finite group actions on M.

There is a natural $\mathbb{Z}/3$-action on M, defined as follows. If E is a 3-dimensional vector space with basis $\{e_0, e_1, e_2\}$, then we can define $e_i \mapsto \omega^i e_i$, where ω^i is a third root of unity. This action extends to an action on $E \otimes E$ by the rule $e_{ij} \mapsto \omega^{i+j} e_{ij}$[5]. Furthermore, it extends to an action on $E \otimes E \oplus E \otimes E$ by $(v, w) \mapsto (gv, gw)$. Call the generator for this group for g.

There is also a non-toric permutation action defined as follows. Let $\mathbb{Z}/3 \subset S_3$ be the cyclic permutation action on $\{e_0, e_1, e_2\}$ defined by $e_i \mapsto e_{i+1}$. Again, we get an action on $E \otimes E$ by $e_{ij} \mapsto e_{i+1,j+1}$, and by extension an action on $E \otimes E \oplus E \otimes E$. Call the generator of this group for σ.

Furthermore, there is a $\mathbb{Z}/2$-action switching the $E \otimes E$-factors. Call the generator for this group for τ.

All these groups commute up to a scalar, so we get a $\mathbb{Z}/3 \times \mathbb{Z}/2 \times \mathbb{Z}/2$-action on $\mathbb{P}(E \otimes E \oplus E \otimes E)$. Let G be the abelian group generated by g and σ. Let G' be the group generated by g, σ and τ.

For the G-action to restrict to $X_1 = M \cap H$, we must choose H to be invariant under the group action. Here we describe a family of G-invariant $\mathbb{P}^1 1$'s: denote a unit matrix in the first factor of $(E \otimes E) \oplus (E \otimes E)$ by e^0_{ij}, and denote a unit matrix in the second factor by e^1_{ij}, where $0, 1$ are taken modulo 2.

Now consider the $H_t = \mathbb{P}^{11}$ spanned by the following matrices:

$$f^\alpha_{ij} = e^\alpha_{ij} + t^\alpha_{i-j} e^{\alpha+1}_{-i-j,-i-j} \in E \otimes E \oplus E \otimes E,$$

where $i \neq j \in \mathbb{Z}_3$ and $\alpha \in \mathbb{Z}_2$, and t^α_{i+j} is a parameter. Note that $g \cdot f^\alpha_{ij} = \omega^{i+j} f^\alpha_{ij}$, so that H is spanned by eigenvectors of the $\mathbb{Z}/3$-action. This gives us a 4-parameter family of G-invariant planes. However, multiplying all the t^α_{i-j}

[5] We write e_{ij} for $e_i \otimes e_j$.

by the same number yield isomorphic families, so we really have a 3-parameter family.

Denote the intersection between M and H by X_{H_t}. Denote by P_i the coordinate points $(0 : \ldots : 1 : \ldots : 0)$. Then $\mathbb{Z}/3$ act without fixed points outside these points (this can be computed in `Macaulay2`). A `Macaulay2` computation also shows that for $t_i \neq 1, 0$, the family has 48 isolated singularities: the P_i, and 36 other points, which come in two orbits under the group action. These are all double points, which can be verified by local computations.

which group?

Lemma 4.6.1. *There exists a minimal resolution of X_{H_t} ($t \neq 0, 1$), respecting the group action by G, leaving the dualizing sheaf trivial.*

Proof. Analytically, a small resolution is a local operation. The singularities come in 3 orbits under the action, so it is enough to do the resolution on one singularity in each orbit.

check!!!

Since the singularity is small, the change happen in codimension 2. The holomorphic 3-form on X_{H_t} extends holomorphically to all of the resolution by Hartog's theorem. ∎

Lemma 4.6.2. *After resolving the double points as above, the action of g has 24 fixed points on $\widetilde{X_{H_t}}$, two on each of the $\mathbb{P}^1$'s of the initial fixed points.*

Furhermore, the resolution has Euler characteristic 24.

Proof.

explain existence as in Rødland

The Euler characteristic of a small resolution is given by $\chi(\widetilde{X_{H_t}}) = \chi(X_t) + 2s$, where s is the number of double points, and X_t is a smooth member of a smooth smoothing family of X_{H_t} (which we know exists by construction, and is X_1 from above). There are 48 double points, so it follows that the Euler characteristic is $-72 + 2 \cdot 48 = 24$. ∎

These resolutions are still Calabi–Yau manifolds. One reference for this fact is [Cle83].

Let $\mathbb{Z}/3$ denote the torus subgroup group acting on $\widetilde{X_{H_t}}$.

Theorem 4.6.3. *Let X_1° be a minimal resolution of $\widetilde{X_{H_t}}/(\mathbb{Z}/3)$. Such a resolution exists, and it has Euler-characteristic $+72$, making it a potential mirror for X_1.*

Proof. The existence of this kind of quotient singularity is proved in Roan's article [Roa96]. Furthermore, in his article [Roa89], he proves a formula for the Euler characteristic of such resolutions (let $V = \widetilde{X_{H_t}}$):

$$\chi\left(\widetilde{V/(\mathbb{Z}/3)}\right) = \frac{1}{3} \sum_{g,h \in \mathbb{Z}/3} \chi(V^g \cap V^h).$$

For $(g,h) \neq (e,e)$, $\chi(V^g \cap V^h)$ is just the finite set of of fixed points. There are 24 of these. For $(g,h) = (e,e)$, $\chi(V^e \cap V^e) = \chi(V)$ is the Euler characteristic of the resolution of X_{H_t}, which is 24.

In sum, we find

$$\chi(X_1^\circ) = \frac{1}{3}\,(24 + 8 \cdot 24) = 72.$$

■

Remark 4.6.4. We still have the cyclic permutation action σ. Since σ commutes (up to scalar) with g, it act on the mirror as well. It can be checked that it has no fixed points on X_{H_t}. Thus the induced $\mathbb{Z}/3$-action is free, and we can form the quotiens $X_{H_t}/\langle\sigma\rangle$ and $X_{H_t}^\circ/\langle\sigma\rangle$. These will have Euler characteristics 24 and -24, respectively. However, the fundamental group will be non-trivial.

Invariant subfamily of X_2

Also in this case we are able to produce a mirror candidate. We start by describing natural group actions on N, and then describe a natural invariant subfamily.

Recall that F is a 2-dimensional vector space with basis f_0, f_1. There is, as above, a natural $\mathbb{Z}/2$-action given by $f_i \mapsto (-1)^i f_i$. Concretely, $\mathbb{Z}/2$ act by sending f_0 to itself and multiplying f_1 by -1. This action extend in the natural way to an action on $\mathbb{P}(F^{\otimes 3} \oplus F^{\otimes 3})$.

Furhermore, there is another $\mathbb{Z}/2$-action given by $f_i \mapsto f_{i+1}$ (indices taken modulo 2).

Using the same notation as in the previous section, define K_t to be $\mathbb{P}^{11}$ spanned by the following matrices:

$$g_{ijk}^{\alpha} = e_{ijk}^{\alpha} + t_{i,j,k} e_{i+j+k,i+j+k,i+j+k}^{\alpha+1}$$

for $(i,j,k) \neq (0,0,0), (1,1,1)$ and $\alpha = 0, 1$. Thus these matrices span a $\mathbb{P}^{11}$. Then, as above, the g_{ijk}^{α} are eigenvectors for the $\mathbb{Z}/2$-action.

For $t_{i,j,k} = 1$ for all i,j,k the variety $X_{K_t} \triangleq N \cap K_t$ has 36 double points. Using the same arguments as in the previous section, it follows that a small resolution of X_{K_1} has Euler-characteristic 24 as well. Again, using Roan's formula, we find that a small resolution of the quotient X_{K_1} has Euler-characteristic $+48$. Thus we have a mirror candidate for X_2 as well.

Proposition 4.6.5. *There exists a mirror candidate for X_2 as well. More precisely, there exists a Calabi–Yau desingularization X_2° of the quotient $\widetilde{X_{K_t}}/H$ in such a way that the Hodge numbers satisfy $\chi(X_2) = -\chi(X_2^\circ) = -48$.*

— Denote by D_6 the dihedral group of order 12, the symmetries of a hexagon. It is generated by a rotation ρ of order 6, together with a reflection σ, subject to $\sigma\rho\sigma = \rho^{-1}$. There is an isomorphism $D_6 \simeq S_3 \times \mathbb{Z}_2$: S_3 is identified with $\langle \rho^2, \sigma \rangle$ and $\mathbb{Z}_2$ is identied with ρ^3.

Lemma 4.6.6. *There are D_6-actions on both M and N.*

Proof. Switching the direct summands of $(E \otimes E)^{\oplus 2}$, gives us a $\mathbb{Z}_2$-action. In total we now have a $S_3 \times \mathbb{Z}_2$-action, which by the above remark is a D_6-action. Note that since the action was defined on E, the rank of elements of $E \otimes E$ is preserved, so that we indeed have an action on M.

Similarly, N is the rank $1 + 1$ tensors in $(E \otimes E \otimes E)^{\oplus 2}$, where now E is a 2-dimensional vector space. ∎

4.7 Conclusion

In this final chapter we constructed several smooth Calabi–Yau manifolds. Three of them, X_1, X_2 and X_3 lie in the same flat family (over a necessarily reducible base space). They are all smoothing of X_Y, a complete intersection in a 5-dimensional toric variety Y. This X_Y has a maximally crepant resolution of singularities which is a smooth Calabi–Yau.

We constructed mirror candidates and finite group quotiens of X_1 and X_2.

There are many connections to works by others in the literature. We discuss a few connections below, which we hope to see explained in the future.

1. In [Kap15], the author compiles a list of smooth Calabi–Yau varieties with $\operatorname{Pic} X = \mathbb{Z}$. One of the elements of the list is a Calabi–Yau in $\mathbb{P}^{11}$ with the same Hilbert polynomial as our X_1, and with the Euler characteristic. This Calabi–Yau was however only conjectured to exist, based on the conjecture that to every differential equation of "Calabi–Yau type", there should exist a one parameter family of smooth Calabi–Yau varieties having that equation as its Picard–Fuchs-equation. A list of such equations have been computed by van Straten in [ES06].

   All of these equations have been made searchable in the online database [Str]. Entering the invariants $H^3 = 36$, $H \cdot c_2 = 72$ and $|H| = 12$, yield exactly three matches, corresponding to Calabi–Yau varities with Euler characteristics -72, -60 and -48, respectively. These numbers are exactly the Euler characteristics of our X_i $(i = 1, 2, 3)$.

   Furthermore, their differential operators are Hadamard products, $c * c$, $a * a$ and $a * c$, which according to van Straten (personal communication) is "mirror dual" to join.

   This seems like a perfect match, confirming the existence predicted by the conjecture. The only problem is that our varieties seem to have $h^{11} > 1$.

   Several questions arise: can our X_i still correspond to these equations, without having $h^{11} = 1$? If not, what is their connection to the conjecture?

2. The Calabi–Yau with Hodge numbers $(44, 8)$ in Section 4.4 seem to have some connection with our X_1. Its mirror dual $X_{8,44}$ is a complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, while X_1 is a complete intersection in $(\mathbb{P}^2 \times \mathbb{P}^2) * (\mathbb{P}^2 \times \mathbb{P}^2)$ with the same Euler characteristic. There seem to be some kind of duality going on, which is unfortunaly not described (to my knowledge) in the literature.

   We have a morphism $\pi : X_1 \to \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by $(v \otimes w, r \otimes s) \mapsto v \otimes w \otimes r \otimes s$. The morphism is generically $1-1$. I have not been able to see what the image is (or if the morphism is an isomorphism).

   The same situation occurs with X_2. Here there is a morphism $\pi : X_2 \to (\mathbb{P}^1)^{\times 6}$. I don't know what the image is. Also here there should be a connection with $X_{8,44}$, since $X_{8,44}$ also can be realized as a complete intersection in $(\mathbb{P}^1)^{\times 6}$. See the introduction of [BCD10].

3. It would also be interesting to find proofs of the Euler characteristics being $-72, -60$ and -48 and the Hodge numbers not involving computer calculations.

4. Assuming that my calculation of the Hodge numbers of the X_i are correct, what are representatives of the generators of $\operatorname{Pic} X_i$? (being $\mathbb{Z}^3$, $\mathbb{Z}^5$ and $\mathbb{Z}^4$, respectively)

5. Can my construction via joins be generalized to produce other (potentially new) Calabi–Yau varieties?

Appendices

APPENDIX A

Computer code

mputercode

Extensive use of computer software such as `Macaulay2` [GS] and SAGE [Wil17] has been invaluable during my work. In this Appendix I collect some computer code for reproducing some of my calculations.

A.1 Computing the singular locus

In some cases, equations simplify significantly in affine charts. Therefore, using the naive command `singularLocus` in `Macaulay2` often takes unnecessarily long time (and sometimes the computations never finish), as it computes the minors of a very large Jacobian matrix. Restricting to each affine chart, we can use the command `minimalPresentation` to eliminate variables to produce a new ring isomorphic to the first one, but with fewer equations.

The following code produces a list of the components of the singular locus of the projective scheme with ideal I.

```
fastSingularities = I -> (
    R := ring I;
    n := numgens R;
    gensR := gens R;
    singlist := {};
    for i from 0 to (n-1) do {
        affineChart := I + ideal(gensR_i - 1);
        singloc     := singularLocus minimalPresentation affineChart;
        sing        := radical ideal mingens ideal singloc;
        inv         := affineChart.cache.minimalPresentationMap;
        singlist = singlist | {(homogenize(preimage(inv,sing),gensR_i))};
        };
    saturate intersect(singlist)
    )
```

The method works by computing the singular locus in each affine chart, taking the radical, and then pulling back to the homogeneous coordinate ring.

Finally, we get a list of singular loci in each affine chart. We return the (saturation of) the intersection of the singular loci of the affine charts.

It is especially fast when computing the singular locus of toric varieties with a low-dimensional singular locus.

The following code finds the singular locus of the projectice cone $C(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^9$.

```
R = QQ[x_0..x_8,x_9]
M = genericMatrix(R,3,3)
I = minors(2,M)
time fastSingularities I
time radical ideal singularLocus I
```

Our function performs significantly faster. On a modern Mac, the times are 1.14 seconds versus 4.31 seconds, respectively.

Here is a more involved example. Let Y' be four-dimensional toric variety from Chapter 4. It is defined by the 2×2-minors two matrices. In `Macaulay2` we can define it as follows:

```
S = QQ[x_1..x_6,z_1..z_6,y]
M1 = matrix{{y,x_1,x_2},{x_4,y,x_3},{x_5,x_6,y}}
M2 = matrix{{y,z_1,z_2},{z_4,y,z_3},{z_5,z_6,y}}
J = minors(2,M1) + minors(2,M2)
```

Here the performance difference is even more impressive. Our function computes the singular locus in 7.29 seconds, but the built-in function `singularLocus` used more than 22 minutes (at which point I interrupted the computation).

A.2 Torus action

The following lines checks if a projective scheme with ideal sheaf `IX` admits an action of a subtorus of $G = (\mathbb{C}^*)^n \subset \mathbb{P}^n$. To check this, we check if the equations are still valid after a torus action. Since G is abelian, it act on functions by $\lambda \cdot f(x_1, \ldots, x_n) = f(\lambda_1 x_1, \ldots, \lambda_n x_n)$.

Lemma A.2.1. *Suppose $\{f_1, \ldots, f_r\}$ is a homogeneous generating set for $I_X =$ `IX`. Then subgroup of G acting on $X \subset \mathbb{P}^n$ is generated by those $\lambda \in G$ such that $\lambda \cdot f_i = cf_i$ for some $c \in \mathbb{C}^*$.*

Proof. Let H be the subgroup of G fixing the ideal I_X. Let H' be the subgroup of $g \in G$ acting on the f_i by scalar multiplication: $g \cdot f_i = cf_i$. Clearly $H' \subseteq H$. Now suppose $g \in H$. Then

$$g \cdot f_1 = \sum_j a_j f_j$$

for some constants a_j. Now $g \cdot f_1 = f_1(\lambda_1 x_1, \ldots, \lambda_n x_n)$. Suppose the leading term of f_1 is $x_1^{a_1} \cdots x_n^{a_n}$. Then comparing leading terms in the left hand side and the right hand side, we see that $a_1 = \lambda_1^{a_1} \cdots \lambda_n^{a_n} := \lambda^m$. Hence the right hand side is $\lambda^m f_1 +$ other terms. But now there are the same number of terms on each side of the equation, so there are no other terms. Hence $H = H'$. ■

It follows that to find the subgroup of G acting on X, we have to find the $\lambda \in G$ such that the f_i are simultaneous eigenvectors for them.

mple:torus **Example A.2.2.** Let X be defined by $f = x_0x_1x_2x_3x_4 + \sum_{i=0}^{5} x_i^5$ in $\mathbb{P}^4$. Then for $\mathbb{C}^4$ to act on it, we must have $\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4 = \lambda_0^5 = \ldots = \lambda_4^5$. By setting $\lambda_0 = 1$, we see that all the λ_i are fifth roots of unity. Hence the subgroup acting on H is the subgroup of $(\mathbb{Z}/5)^5/\mathbb{Z}_5$ given by $\{(a_0, \ldots, a_5) \mid \sum a_i = 0\}$. ♡

The following code find the subtori of G acting on X in this way, by equating terms in the polynomials defining X.

```
loadPackage "Binomials"
torus = ideal apply(flatten apply(apply(apply(flatten entries gens IX,
     monomials), v ->  flatten entries v), j -> subsets(j,2)),    s -> s_0-s_1)
toruskomps = BPD torus
toruskomps = select(toruskomps, I -> dim I == 1)
```

Explanation. In order to have $g \cdot f = \lambda f$, all terms of the polynomial must be eigenvectors of g. Then as in Example A.2.2, this translates into equating all monomials in the generators.

The ideal `torus` is the ideal generated by the differences of terms in the polynomials defining X.

The `Macaulay2` package `Binomials` [Kah12] can decompose binomials over cyclic extensions of $\mathbb{Q}$ with the command `BPD`. In the last line we select the components corresponding to finite subgroups of the torus.

Then we check manually if these actually correspond to non-trivial actions. There will be one component for each generator of the cyclic group acting on X. ■

A.3 Computing fixed points

Computing fixed points of torus action is often just as easy to do by hand, but to save time and potential for error, we mostly did this in `Macaulay2`.

To check if a point $P \in \mathbb{P}^n$ is a fixed point of a group action, we lift it to $\overline{P} \in \mathbb{C}^{n+1}$. Then P is a fix point if and only if $g \cdot \overline{P} = \lambda \overline{P}$ for some $\lambda \in \mathbb{C}^*$.

To compute all fix points, we consider the ideal generated by $x_i - \lambda(g \cdot x_i)$ for each generator x_i. The fixed points correspond to a primary decomposition of this ideal.

Below is the code to compute the fixed points of the $\mathbb{Z}/2$-action on the invariant subfamily of X_2. We create the ideal, then saturate by the maximal ideal $(x_1, \ldots, x_n)$ (since not all coordinates are allowed to be zero). Then we use the `decompose` command in `Macaulay2` to get a primary decomposition.

```
S  = R[lambda]
M1 =  matrix{{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_10,x_11,x_12}}
M2 =  matrix{{x_1*lambda,x_2*lambda,-x_3*lambda,x_4*lambda,-x_5*lambda,-x_6*
    lambda,-lambda*x_7,-x_8*lambda,lambda*x_9,-lambda*x_10,lambda*x_11,lambda*
    x_12}}

Ifiks = saturate(ideal (M1-M2), sub(ideal gens R,S))
decompose(Ifiks + IX)
```

The result is a list of 12 ideals, corresponding to the 12 fixed points.

A.4 Computing the Gaifullin triangulation

mpute_gaifullin

Below is a short SAGE script computing the 15 vertex triangulation of $\mathbb{CP}^2$ as described in [Gai09]. The last line returns a `SimplicialComplex` object in SAGE.

```
#Defines the Klein 4 group.
V4 = PermutationGroup([Permutation("(1,2)(3,4)"),Permutation("(1,3)(2,4)")])

def isValidFace(F):
    '''
    Assumes the first vertex is a permutation.
    Then checks if F satisfies the condition in the
    definition of T.
    '''
    g = F[0]
    for v in (1,2,3,4):
        if (F[g(v)][1] == F[v][1]):
            return False
    return True


# Makes a list of all possible maximal faces of the correct form
candidates = [(g,(1,a1),(2,a2),(3,a3),(4,a4)) for g in V4.list()[1:] for a1 in
      (1,2,3) for a2 in (1,2,3) for a3 in (1,2,3) for a4 in (1,2,3)]

# Filters out the faces not fullfilling the condition
maximalFacets = filter(lambda F: isValidFace(F), candidates)

# Renames the vertices
S = SimplicialComplex(maximalFacets)
vertexSet = S.vertices()
D = dict([(F,i) for i,F in enumerate(vertexSet)])
renamedMaximalFacets = [[D[v] for v in F] for F in maximalFacets]
SS = SimplicialComplex(renamedMaximalFacets)
```

To get the Stanley–Reisner ideal, one can write:

```
list(SS.stanley_reisner_ring().defining_ideal().gens())
```

The returned value is a list of the monomials generating the Stanley–Reisner ideal of $\mathcal{T}$. This can then be copied into `Macaulay2` for further computation.

A.5 Construction of the X_i

In this section we describe an efficient way to present the Calabi–Yau varieties X_i from Chapter 4 in `Macaulay2`.

Construction of X_1

Recall the construction of X_1: it is the intersection of a toric variety $M \subset \mathbb{P}^{17}$ with a generic $\mathbb{P}^{11}$. The variety X_1 parametrized pairs of rank $1+1$ tensors in this $\mathbb{P}^{11}$.

We can think of elements of $E \otimes E \oplus E \otimes E$ as pairs of 3×3 matrices, which we denote by (A, B). To span the $\mathbb{P}^{11}$, we choose block matrices $(A, B)_i$ $(i = 1, \ldots, 12)$. Then we form the sum

$$A \triangleq \sum_{i=1}^{12} (A, B)_i x_i,$$

with variables x_i. This matrix has rank $1+1$ if all the 2×2-minors of A and B vanish.

Below is a short `Macaulay2` script implementing this construction.

```
kk = ZZ/3001
R = kk[x_1..x_12]

generateX2 = () -> (
    K = random(R^18,R^12);
    a = transpose gens gb K; -- same image
    b = entries a;
    b = apply(0..11, i-> apply(b#i, z -> z*x_(i+1)));
    bb = sum toList b;
    bb1 = bb_{0..8};
    bb2 = bb_{9..17};
    M1 = matrix toList apply(0..2,
            i-> toList apply(0..2, j-> bb1#(3*i+j)));
    M2 = matrix toList apply(0..2,
            i-> toList apply(0..2, j-> bb2#(3*i+j)));
    I1 = minors(2, M1);
    I2 = minors(2, M2);
    I1+I2
```

```
    )
```

Listing A.1: Code for X_1

We explain each step. First we create a random 18×12-matrix with coefficients from the field `kk`. Then we replace the random matrix with a its Gaussian reduced form, which have the same image in k^{18}, but is much simpler.

Next, we use the matrix to create 18 random linear forms in the variables x_i. These are then inserted into two 3×3 matrices M_1 and M_2. Finally, we return the ideal which is the sum of the ideal of the minors of the two matrices M_1 and M_2. This is the ideal of X_1.

Remark A.5.1. Replacing the matrix K with its Gaussian reduced form is the same as letting $\mathrm{GL}(k^{12})$ act on the left. This *significantly* reduces the size of the resulting Gröbner basis. Without this simplification, the resulting Gröbner basis have 49 elements, but with it, it has 19 elements.

As an example, computing the degree zero part of $T^1(S_{X_1}/k, S_{X_1})$ takes about a week on a modern computer before simplification. With the smaller Gröbner basis, the same computation takes just a couple of hours.

Construction of X_2

The construction of X_2 is very similar. Again, we create 12 random elements of $(F \otimes F \otimes)^{\oplus 2}$ spanning a $\mathbb{P}^{11}$. This correspond to the 12 columns of the random matrix K.

As with X_1, we replace K with its Gaussian reduced form. This matrix spans the same $\mathbb{P}^{11}$, but has a lot more zeroes.

Then we form the sum

$$\sum_{i=1}^{12} (T_1, T_2)_i x_i,$$

where T_1 and T_2 are $2 \times 2 \times 2$-tensors. We return the ideal generated by the "minors" of this sum.

```
minors222tensor = (L) -> ( -- L is a list of lists of lists
    eqs = {L#0#0#0*L#1#0#1 - L#0#0#1*L#1#0#0,
      L#1#0#0*L#1#1#1 - L#1#1#0*L#1#0#1,
      L#1#1#0*L#0#1#1 - L#1#1#1*L#0#1#0,
      L#0#1#0*L#0#0#1 - L#0#1#1*L#0#0#0,
      L#1#0#1*L#0#1#1 - L#1#1#1*L#0#0#1,
      L#1#0#0*L#0#1#0 - L#1#1#0*L#0#0#0};
    eqs = eqs | {L#0#0#0 * L#1#1#1 - L#0#0#1*L#1#1#0,
      L#1#0#0*L#0#1#1 - L#1#0#1*L#0#1#0,
      L#0#0#1*L#1#1#0 - L#1#0#1*L#0#1#0};
    ideal eqs
    )

generateX2 = () -> (
```

```
    K = random(R^16,R^12);
    a = transpose gens gb K;
    b = entries transpose K;
    b = entries a;
    b = apply(0..11, i-> apply(b#i, z -> z*x_(i+1)));
    bb = sum toList b;
    bb1 = bb_{0..7};
    bb2 = bb_{8..15};
    I1 = minors222tensor {{{bb1#0,bb1#1},{bb1#2,bb1#3}},
                          {{bb1#4,bb1#5},{bb1#6,bb1#7}}};
    I2 = minors222tensor {{{bb2#0,bb2#1},{bb2#2,bb2#3}}
                          {{bb2#4,bb2#5},{bb2#6,bb2#7}}};
    I1+I2
    )
```

Listing A.2: Code for X_2

Constructing X_3 in `Macaulay2` is entirely similar to the above two constructions, so we omit the code.

Remark A.5.2. Using a finite field is essential. Without a limit on the size of the coefficients, the amount of necesserary computer RAM is way beyond current technology.

APPENDIX B

Triangulations of spheres

c:spheres8 In the article [GS67], Grünbaum enumerates simplicial 4-polytopes with 7 and 8 vertices. There are 5 combinatorial types of triangulations of the 4-sphere with 7 vertices, and there are 37 combinatorial types of triangulations with 8 vertices.

In her thesis [Fau12], Ingrid Fausk considered the polytopes with 7 vertices, and their associated Stanley–Reisner schemes. She showed that four out of the five possible triangulations of 4-spheres with seven vertices admit a smoothing. These smoothings correspond to Calabi–Yau varieties with Hodge numbers $(1, 73)$, $(1, 73)$, $(1, 61)$ and $(1, 50)$, respectively. The last one is Rødland's construction.

In this Appendix, we perform deformation theoretic calculations on the 37 triangulations with 8 vertices. Unfortunaly, most of them appear to be non-smoothable.

Unfortunaly, there seems to be a mistake in Grünbaums list. Two of spheres (name which ones) listed have $H^3(K; k) = 0$, which should not occur if they were spheres.

In [Kap15], the author compiles a list of smooth Calabi–Yau varieties with $\operatorname{Pic} X = \mathbb{Z}$. Several of the smoothings we find below occur in that list. There is also the paper [Cou+16], where the authors make a list of arithmetically Gorenstein Calabi–Yau threefolds in $\mathbb{P}^7$, which they conjecture is the complete list of such threefolds.

Technique

We manually entered the maximal facets from each triangulation P_i^8 (in Grünbaum's notation) into `Macaulay2`. Then we used Nathan Ilten's package [Ilt12] to compute their first order deformations and the obstruction spaces, T^1 and T^2, respectively.

Those with $T^2 = 0$ are perhaps the most interesting, as they correspond to smooth points on the Hilbert scheme. Having $T^2 = 0$ means that all first-order deformations lift to a second-order deformation. In many cases this implies that it lifts automatically to an honest family over $\operatorname{Spec}\mathbb{C}[t_1,\dots,t_N]$ (where $N = \dim T^1$).

However, even in obstructed cases, we might have power series solutions, meaning that lifting the equations one step at a time will never terminate.

Then we compute the T^i modules for the other triangulations. We also compute their automorphism groups, using SAGE.

B.1 Table of information

Here is the whole table of T^i-dimensions together with some other information. Compare with the list in [Kap15].

Number	degree	$c_2 \cdot H$	T^1	T^2	$\mathrm{Aut}(T)$	Comment
P^8_1	14	–	–	–	–	Not a sphere.
P^8_2	14	68	98	9	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
P^8_3	14	68	108	24	D_6	
P^8_4	15	66	95	17	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
P^8_5	15	64	88	32	$\mathbb{Z}/2 \times D_4$	
P^8_6	15	66	88	9	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
P^8_7	15	–	–	–	–	Not a sphere.
P^8_8	16	64	78	9	1	
P^8_9	16	64	82	17	$\mathbb{Z}/2$	
P^8_{10}	16	64	92	32	$\mathbb{Z}/4$	
P^8_{11}	17	62	74	18	$\mathbb{Z}/2$	
P^8_{12}	17	62	77	25	$\mathbb{Z}/2$	
$\mathbf{P^8_{13}}$	15	66	83	0	$S_3 \times D_5$	Smooths to $X_{113} \subset \mathbb{G}(2,5)$.
P^8_{14}	16	64	80	18	D_4	
P^8_{15}	16	64	88	32	$\mathbb{Z}/2 \times \mathbb{Z}/4$	
$\mathbf{P^8_{16}}$	16	64	72	0	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
$\mathbf{P^8_{17}}$	16	64	72	0	$\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$	
P^8_{18}	17	62	72	17	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
P^8_{19}	17	62	72	17	$\mathbb{Z}/2$	
P^8_{20}	17	62	67	9	$\mathbb{Z}/2$	
P^8_{21}	17	62	80	32	D_4	
$\mathbf{P^8_{22}}$	17	62	62	0	$\mathbb{Z}/2$	
P^8_{23}	18	60	63	17	$\mathbb{Z}/2$	
P^8_{24}	18	60	18	18	$\mathbb{Z}/2$	
P^8_{25}	18	60	67	25	1	
$\mathbf{P^8_{26}}$	17	62	62	0	D_6	
P^8_{27}	18	60	58	9	$\mathbb{Z}/2$	
P^8_{28}	18	60	58	9	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
P^8_{29}	18	60	58	9	$\mathbb{Z}/2 \times \mathbb{Z}/2$	

Number	degree	$c_2 \cdot H$	T^1	T^2	Aut(T)	Comment
P^8_{30}	19	58	63	33	$\mathbb{Z}/2$	
P^8_{31}	19	58	59	26	1	
P^8_{32}	19	58	55	18	$\mathbb{Z}/2$	
P^8_{33}	19	58	60	27	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
$\mathbf{P^8_{34}}$	16	64	72	0	.	Smooths to $X_{2222} \subset \mathbb{P}^7$.
P^8_{35}	20	56	72	64	D_8	
P^8_{36}	20	56	64	50	$\mathbb{Z}/4$	
P^8_{37}	20	56	61	43	$\mathbb{Z}/2$	
$\mathcal{M}$	20	56	53	27	S_3	

Regne ut $(T^1)^G$ og $(T^2)^G$

B.2 The five unobstructed triangulations

There are five triangulations in Grünbaums list with $T^2 = 0$.

compute - find smooth?? fiber

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