Hyper-Kähler manifolds

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April 19, 2016

Abstract

These are notes from the "summer school" at IMPA, Warzaw, held by Kieran O'Grady.

1 Lecture 1 - Introduction

We will first motivate the definition of hyper-Kähler by looking at K3 surfaces.

By definition, a K3 surface is compact Kähler 2-dimensional complex manifold that is simply connected and has trivial canonical bundle ($K_X \simeq \mathscr{O}_X$).

Example 1.1. Let $X \subset \mathbb{P}^3$ be a smooth quartic surface. Then by Lefschetz:

$$\pi_(X,*) \xrightarrow{\sim} \pi_1(\mathbb{P}^3,*) = \{1\},$$

so X is simply-connected. By adjunction, we have:

$$K_x \simeq (K_{\mathbb{P}^3}\big|_{\times}) \otimes \mathcal{N}_{X/\mathbb{P}^3} = \mathscr{O}_X(-4) \otimes \mathscr{O}_X(4) = \mathscr{O}_X\,.$$

So the last condition is fulfilled also. ★

We list some of the know results about K3's:

1. Any two K3's are deformation equivalent (Kodaira).
2. There is a Hodge-theoretic description of the Kähler cone of a K3 (after having chosen one Kähler class).
3. There is a *Global Torelli Theorem* (Shafarevich Piateski-Shapiro). Namely, a Hodge structure on $H^2(K3;\mathbb{C})$ and a lattice structure on $H^2(K3,\mathbb{Z})$ determines X up to isomorphism.

We now state the definition of a hyper-Kähler manifold:

A hyper-Kähler manifold X is a compact Kähler manifold X, simply connected, such that $H^0(X, \Omega^2, X) = \mathbb{C}\sigma$, where σ is *symplectic*, meanig that $T_xX \times T_xX \xrightarrow{\sigma(x)} \mathbb{C}$ is non-degenerate for all $x \in X$.

Remark. *A HK must have even dimension: the skew-symmetric form σ can be represented by a $n \times n$-matrix A that is skew-symmetric with non-zero determinant. Skew-symmety of A means that $A^T = -A$. Hence $\det A = \det A^T = (-1)^n \det A$. This forces n to be even if we want $\det A \neq 0$.*

In dimension 2, K3's are hyper-Kähler.

To motivate hyper-Käher manifolds, we state the Beaville-Bogomolov decomposition theorem:

Theorem 1.2. *Let Z be a compact Kähler manifold with $c_1(Z) = 0$ (what does this mean?) in $H^2(X, \mathbb{Q})$. Then there exists a finite étale cover $\widetilde{Z} \to Z$ such that*

$$\widetilde{Z} = \mathbb{C}^d/\Lambda \times \prod_i X_i \times \prod_j Y_i$$

where the first factor is a compact torus. The second factor is a product of hyper-Kähler manifolds, and the second is a product of Calabi-Yau manifolds (that is, manifolds with $K_{Y_i} \simeq \mathscr{O}_{Y_i}$ and $h^0(\Omega^p_{Y_i}) = 0$ for all $0 < p < \dim Y_i$).

1.1 First example of a HK

Now we define some higher-dimensional examples of HK's. Terminology: when we say "HK variety", we shall mean a *projective* HK manifold.

Let S be a K3 surface. Then let $S^{(2)}$ be the symmetric square of S (that is, $S \times S/(p,q) \sim (q,p)$). This space comes equipped with two projections, π_1, π_2, and the form $\pi_1^*\sigma + \pi_2^*\sigma \in H^0(\Omega^2_{S\times S})$ is τ-invariant, hence it descends to a holomorphic 2-form on $S^{(2)}$.

But the symmetric square is singular along the (image of the) diagonal, where two points come together. In fact, one can see that it locally looks like $\mathbb{C} \times \mathbb{C}/x \sim -x$. The last factor is a quadric cone, hence a single blowup along the diagonal will resolve the singularities in this case.

Let D be the diagonal. We have a diagram

$$\begin{tikzcd}
Bl_D(S^2) \arrow[r, "\widetilde{\rho}"] \arrow[d] & S^{[2]} \arrow[d, "\gamma"] \\
S^2 \arrow[r, "\rho"] & S^{(2)}
\end{tikzcd}$$

The top right space $S^{[2]}$ can be thought of in two ways: first, notice that the involution on S^2 act on the blowup as well. Hence we can take the quotient of the blowup. This is one description of $S^{[2]}$. Secondly, one can think of $S^{[2]}$ as the blowup of $S^{(2)}$ along the points $2P$ (the image of the diagonal).

Now $S^{[2]}$ is smooth projective and "$\pi_1^*\sigma + \pi_2^*\sigma$" is symplectic.

Then $H^2(S^{[2]})$ is spanned by this symplectic 2-form. We can show that $S^{[2]}$ is simply-connected as well:

$$\pi_1(S^{[2]}\backslash D, *) \twoheadrightarrow \pi_1(S^{[2]}, *).$$

The first group is $\mathbb{Z}/2$, generated by a loop around D (we abuse notation: D denotes the image of the diagonal in $S^{[2]}$), and it is $\mathbb{Z}/2$ because $S^2\backslash D$ is a double cover of $S^{[2]}\backslash D$.

We call $S^{[2]}$ the *Hilbert square* of S. It is a HK variety of dimension 4. It can also be realized as the Hilbert scheme parametrizing length 2 subschemes of S.

In general,

$$Y^{[n]} := \{Z \subset Y \mid l(Z) = n\},$$

is smooth, irreducible of dimension $2n$ (if Y is a surface).

Beauville shows that if S is a K3, then $S^{[n]}$ is always a HK variety of dimension $2n$. Hence we have examples of hyper-Kählers in each even dimension!

If $n \geq 2$, then $b_2(S^{[2]}) = 23$, which is $22 + 1$, the last divisor coming from the blowup.

1.2 Second example of a HK

The analogue of $S^{[2]}$ with S replaced by A, an abelian surface.

The space $A^{[2]}$ has a holomorphic symplectic form, but is far from being simply connected. But consider the maps

$$A^{[2]} \xrightarrow{\gamma_2} A^{(2)} \xrightarrow{s_2} A.$$

The first map sends a points z to the sum $\sum_{p\in A} l(\mathscr{O}_{Z,P}\, P$. The second map sends $P + Q$ to $(p) + (q) \in A$, where the parenthesis means that we actually consider the sum in the *group* A.

Then the composition $s_2 \circ \gamma_2$ is a locally trivial fibration in the étale/analytic topology. It follows that the cohomology group $H^1(A) = \mathbb{C}^4 \hookrightarrow H^1(A^{[2]})$.

Now look at the fiber $s_2 \circ \gamma_2^{-1}(0)$. This is a smooth Kummer surface $\approx A/\langle -1 \rangle$. These are K3 surfaces!

In general we look at the sequence

$$A^{[n+1]} \xrightarrow{\gamma_{n+1}} A^{(n+1)} \xrightarrow{s_{n+1}} A$$

defined analogously, and we define the *generalized Kummer* to be $Kum^{[n]}(A) := (s_{n+1} \circ \gamma_{n+1})^{-1}(0)$.

Beauville proved that $K^{[n]}$ is a HK variety of dimension $2n$. For $n \geq 2$, we have $b_2(K^{[n]}(A)) = 7 = b_2(A) + 1$, so we do actually have two topologically distinct families.

1.3 Third example, lines on a cubic 4-fould

Let $Y \subset \mathbb{P}^n$ be some algebraic variety. Let $X = F(Y)$ be the set of lines contained in Y. It is a closed subset of the Grassmannian $\mathbb{G}(1, \mathbb{P}^n)$, which we can think of as embedded via Plücker in $\mathbb{P}^{\binom{n-1}{2}-1}$.

Theorem 1.3. *Let $Y \subset \mathbb{P}^n$ be a smooth cubic hypersurface. Then $X = F(Y)$ is a smooth connected variety of dimension $2n-6$ and $K_X \simeq \mathscr{O}_X(5-n)$.*

Since we are interested in HK varieties, we put $n = 5$.

Remark. *All HK's have trivial canonical bundle: the $n/2$th power of the symplectic form gives a trivialization of $K_X = \Omega^n_X$.*

We want to look at a incidence correspondence $\mathcal{I} \subset Y \times X$:

$$\begin{tikzcd}
& \mathcal{I} = \{(y, L) \in Y \times X \mid y \in L\} \arrow[dl, "\rho"'] \arrow[dr, "\pi"] & \\
Y & & X
\end{tikzcd}$$

The fibers of ρ are $\mathbb{P}^1$s (how to see this?). In general we get a map

$$H^{n-1}(Y) \xrightarrow{c} H^{n-3}(X)$$

given by $\alpha \mapsto \pi_*(\rho^*\alpha)$. So for $n = 5$, we get a map $H^4(Y) \to H^2(X)$.

Beauville and Donagi showed that if $Y \subset \mathbb{P}^5$ is a smooth cubic hypersurface, then $X = F(Y)$ is a HK variety of type $K3^{[2]}$. Moreover, the restriction of c to the primitive cohomology

$$H^4(Y)_0 := \{\alpha \in H^4(Y) \mid \alpha - c_1(\mathscr{O}_Y(1)) = 0\}$$

is an isomorphism to $H^2(X)_0$ of Hodge structures.

We get an isomorphism $H^{p,q}(Y)_0 \simeq H^{p-1,q-1}(X)_0)$, and there exists a bilinear symmetric form $\langle , \rangle$ on $H^2(X)_0$ such that

$$\langle c(\alpha), c(\beta) \rangle = -\int_Y \alpha \wedge \beta$$

for all $\alpha, \beta \in H^4(Y)_0$.

One consequence: if $Y \subset \mathbb{P}^5$ is very general, then $H^2(X)_0$ has no nonzero integral $(1,1)$-classes. [explanation follows]

Main point: if Y is very general, then $F(Y) \not\simeq S^{[2]}$ (not isomorphic) (S is K3).

Beauville proved however that if Y is a pfaffian cubic

$$\{(t_0 : \cdots : t_5) \in \mathbb{P}^5 \mid Pf(t_0 A_0 + \ldots t_5 A_5)\},$$

where the A_i are skewsymmetric matrices, then $F(Y) \simeq S^{[2]}$.

2 Lecture 2 - some of the main general resutls