Representation theory

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Abstract

These are notes from the course MAT4270 on representation theory, autumn 2015. The lectures were held by Sergey Neshyevey. The notes are mine. The first half is about representations of finite groups, the second half about representations of compact Lie groups.

1 Lecture 1

From now on G is always a *finite group* and V a finite-dimensional vector space over the complex number. Most results will hold over any algebraically closed field of characteristic zero.

1.1 Motivating example

In 1896 Dedekind made the following observation. Let $\mathbb{C}[x_g \mid g \in G]$ be the free $\mathbb{C}$-algebra with basis the elements of G. Then one can form the following polynomial:

$$P_G(x_{\rho_1}, \ldots, x_{\rho_n}) = \det\left((x_{\rho_i g_j})_{i,j=1}^n\right).$$

Note that the matrix is just the multiplication table for the group G. The problem is to decompose P_G into irreducible polynomials.

Example 1.1. Let $G = S_3$. Then G is generated by two elements r and s with the relation $srs = r^2$. If we write the elements of G as $\{e, r, r^2, s, sr, sr^2\}$, then the multiplication table looks like

$$\begin{pmatrix} e & r & r^2 & s & sr & sr^2 \\ r & r^2 & e & sr^2 & s & sr \\ r^2 & e & r & sr & sr^2 & s \\ s & sr & sr^2 & e & r & r^2 \\ sr & sr^2 & s & r^2 & e & r \\ sr^2 & s & sr & r & r^2 & e \end{pmatrix}.$$

The determinant is quite long, so I won't write it out. Suffice it to say it is a degree 6 polynomial with 146 terms. Using a computer algebra system such as `Macaulay2`, one can decompose it. It decomposes into three factors: one quadratic and two linear. They are the following (we change notation to avoid confusion about exponents):

$$x_e + x_r + x_{r^2} + x_s + x_{sr} + x_{sr^2}$$

and

$$x_e + x_r + x_{r^2} - x_s - x_{sr} - x_{sr^2}$$

and

$$x_e^2 - x_e x_r + x_{r^2} - x_e x_{r^2} - x_r x_{r^2} + x_{r^2}^2 - x_s^2 + x_s x_{sr} - x_{sr^2}^2 + x_s x_{sr^2} - x_{sr^2}^2 - x_{sr^2}^2.$$

We will see later that the first factor corresponds to the trivial representation, the second to the alternating representation, and the third is the so-called standard represenation of G. ★

Frobenius proved the following theorem:

Theorem 1.2. *Let $P_G = \prod_{i=1}^r P_i^{m_i}$ be the decomposition of P_G into irreducibles. Then*

1. *$m_i = \deg P_i$ for every i.*
2. *r is the number of conjugacy classes in G.*

In particular, since $r = |G|$ if and only if G is abelian, P_G decomposes into linear factors if and only if G is abelian.

In trying to prove the above theorem, Frobenius had to develop representation theory.

1.2 Representations

A **representation of** G **on** V is a homomorphism $\pi : G \to \mathrm{GL}(V)$. We will denote a representation interchangably by $\pi, (\pi, V), V_\pi$ or V, depending upon the situation.

This means that a representation is a *linear* action of G on V.

Example 1.3. The trivial representation ϵ of any group is given by letting $V = \mathbb{C}$ and $\epsilon(g) = e$ for all $g \in G$. ★

Example 1.4 (Left regular representation)**.** Let $\mathbb{C}[G]$ be the space of functions on G. It is a finite-dimensional vector space under pointwise addition. Define $\lambda : G \to \mathrm{GL}(\mathbb{C}[G])$ by

$$g \mapsto \big(f \mapsto (h \mapsto f(g^{-1}h))\big)$$

This vector space have a basis consisting of the characteristic functions e_g for $g \in G$. On these elements on sees that the action is given by $g \cdot e_h = e_{gh}$.

★

Example 1.5. Similarly, one has the **permutation representation**. Let X be a G-set, i.e. a set on which G acts. Then one forms the space $\mathbb{C}[X]$ of X-valued functions. It has a basis e_x of characteristic functions, and the action of G on $\mathbb{C}[X]$ is given by $ge_x = e_{gx}$. ★

The notion of isomorphism is not surprising. An **equivalence** of representations (π, V) and (θ, W) is given by a vector space isomorphism $T : V \to W$ such that $T(g \cdot v) = g \cdot T(v)$ for all $g \in G$.

In case $T : V \to W$ is not an isomorphism, we say that T is an **intertwiner** of V and W. The set of intertwiners is denoted by $\mathsf{Mor}(\pi, \theta)$ or $\mathrm{Hom}_G(V, W)$.

A subspace $W \subset V$ is **invariant** if $g \cdot W \subset W$ for all $g \in G$. Letting $\theta(g) = \pi(g)\big|_W$, we get another representation of G, called a **subrepresentation** of G. We write $\pi\big|_W$ for θ.

If we have two representation (π, V) and (π', V'), we can form the **direct sum representation** by letting $g \in G$ act on $V \oplus V'$ componentwise. Note that π is a subrepresentation of $V \oplus V'$.

Proposition 1.6 (Mascke's theorem)**.** *Let (π, V) be a representation of G and $W \subset V$ an invariant subspace. Then there exists a complementary invariant subspace $W^\perp$. That is, W is also invariant and such that $V = W \oplus W^\perp$.*

Proof. We prove this by "averaging". Let $P : V \to W$ be *any* projection from V to W. Then define

$$\Pi(v) = \frac{1}{|G|} \sum_{g \in G} gP(g^{-1}v).$$

This is a G-linear morphism, because

$$\begin{aligned} \Pi(h \cdot v) &= \tfrac{1}{|G|} \textstyle\sum_{g \in G} gP(g^{-1}h \cdot v) \\ &= \tfrac{1}{|G|} \textstyle\sum_{g \in G} gP((h^{-1}g) \cdot v) \\ &= \tfrac{h}{|G|} \textstyle\sum_{g \in G} h^{-1}gP((h^{-1}g) \cdot v). \end{aligned}$$

It is also surjective, since if $v \in W$, then clearly $\Pi(v) = v$. Hence Π is a G-linear surjection onto W. Then it is clear that $\ker \Pi$ is a G-invariant subspace of V complementary to W. □

We say that a representation is **irreducible** if it has no proper invariant subspaces. A representation is **completely reducible** if it decomposes into a direct sum of irreducible representations.

Example 1.7. Let $G = (\mathbb{R}, +)$ act on $\mathbb{R}^2$ by the matrices

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

This leaves the x-axis fixed, so the representation is not irreducible. But it does not split into irreducible representations, since there is no complementary subspaces. ★

However, if G is finite (or **compact**, as we shall see later), things are nicer:

Theorem 1.8. *Any finitely-dimensional representation of a finite group G is completely reducible.*

The proof follows directly from Maschke's theorem (1.6).

We next prove Schur's lemma, which although has a very simple proof, has great consequences.

Proposition 1.9 (Schur's lemma)**.** *Let (V, π) and (W, θ) be irreducible representations. Then*

1. *If $\pi \not\sim \theta$, then $\mathrm{Hom}_G(V, W) = 0$.*
2. *If $\pi \sim \theta$, then $\mathrm{Hom}_G(V, W)$ is 1-dimensional.*

Proof. Both the kernel and the image of a morphism of representations is a representation. But since V, W are irreducible, it follows that $\ker \varphi = 0$ or $\ker \varphi = V$ for any $\varphi : V \to W$. This proves i).

Now suppose $\Pi : V \to W$ is an equivalence of representations. Let $\varphi \in \mathrm{Hom}_G(V, W)$. Then φ must be an equivalence as well, since the neither the kernel nor image can be non-zero. Consider $T = \varphi^{-1} \circ \Pi : V \to V$. This is a linear map, and since we work over $\mathbb{C}$, it has an eigenvalue λ. Now consider $T - \lambda \mathrm{id}_V$. This has non-zero invariant kernel, hence $T - \lambda \mathrm{id}_V = 0$. □

Proposition 1.10. *If G is abelian, then any irreducible representation of G is 1-dimensional.*

Proof. Each $g \in G$ gives a map $\pi(g) : V \to V \in \mathrm{End}(V)$. This is even an intertwining map, because $\pi(g)\pi(h) = \pi(gh) = \pi(hg) = \pi(h)\pi(g)$.

If π is not trivial, by Schur's lemma, this map is just multiplication by some constant. But any 1-dimensional subspace of V is invariant under this action, so V must be 1-dimensional in order to be irreducible. □

Thus, up to equivalence, every irreducible representation of a finite abelian group is just a homomorphism $\chi : G \to \mathbb{C}$. But since G is finite, this actually maps into $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$. Such maps are called **characters** of G. Denote by $\hat{G}$ the *set* of irreducible representations of G.

We call the $\hat{G}$ the **Pontryagin dual** of G. It is an abelian group under pointwise multiplication. See Exercise 2 for more on this.

Remark. *If G is non-abelian, there will always exist irreducible representations of dimension ≥ 2. This follows because the regular representation $G \to \mathrm{GL}(\mathbb{C}[G])$ is injective.*

2 Lecture 2 - Density theorems

2.1 Density theorems

Let (V, π) be an irreducible representation. The set of operators $\pi(g)$ for $g \in G$ is quite large. By definition, if $x \neq 0$, then

$$V = \mathrm{Span}\{\pi(g)x \mid g \in G\}.$$

By Schur's lemma, we also know that

$$\mathrm{End}(\pi) = \{T \in \mathrm{End}(V) \mid T\pi(g) = \pi(g)T \,\forall\, g \in G\} = \mathbb{C} \cdot \mathrm{id}_V.$$

Theorem 2.1 (The density theorem)**.** *Assume $\pi_1, \ldots, \pi_n$ are pairwise inequivalent irreducible representations, with $V_i = V_{\pi_i}$. Consider the direct sum representation $V_1 \oplus \ldots \oplus V_n$.*

Then
$$\mathrm{Span}\{\pi(G)\} = \mathrm{End}(V_1) \oplus \ldots \oplus \mathrm{End}(V_2).$$

Proof/exercise: We deduce the first theorem from the second: First break (V, π) into irreducible representations V_i. Then it follows directly from Schur's lemma that $\pi(G)' = \mathbb{C}\mathrm{id}_{V_1} \oplus \ldots \oplus \mathbb{C}\mathrm{id}_{V_n}$.

Now, the elements of $\mathbb{C}\mathrm{id}_{V_1} \oplus \ldots \oplus \mathbb{C}\mathrm{id}_{V_2}$ are block diagonal matrices. It is an easy exercise with matrices to see that the commutator of this set is just
$$\mathrm{End}(V_1) \oplus \ldots \oplus \mathrm{End}(V_2).$$

$\square$

First we need some notation. Let V be some vector space and $X \subset \mathrm{End}(V)$ a subset of the endomorphisms of V. Let

$$X' \triangleq \{T \in \mathrm{End}(V) \mid TS = ST \text{ for all } S \in X\}.$$

We call X' the **commutant** of X. Thus if (V, π) is a representation, then by definition:
$$\mathrm{End}(\pi) = \pi(G)'.$$

Theorem 2.2. *For any finite-dimensional representation (V, π) we have*
$$\pi(G)'' = \mathrm{Span}\{\pi(G)\}.$$

Proof. Let $T \in \pi(G)''$. Let $x_1, \dots, x_n$ be a basis of V. Consider the representation $\theta = \pi \oplus \ldots \oplus \pi$ (n times) on $W = \oplus_{i=1}^n V$. Consider $x = (x_1, \dots, x_n) \in W$ and the operator $S = T \oplus \ldots \oplus T$ on W.

It is an exercise to show that $S \in \Theta(G)''$.

Now $\mathrm{Span}\{\theta(G)x\}$ is an invariant subspace of W. Then there exists an invariant complementary subspace. Let $P : W \to \mathrm{Span}\{\theta(G)x\}$ be the projection with kernel that invariant subspace, so that $P \in \Theta(G)'$ (that is, P is a G-morphism). Then

$$PSx = SPx = Sx$$

since $Px \in \mathrm{Span}\{\theta(G)x\}$. Hence $Sx \in \mathrm{Span}\{\theta(G)x\}$ as well. This means that there exist $\alpha_g \in \mathbb{C}$ such that

$$Sx = \sum_g \alpha_g \theta(g) x$$

This is the same as saying that

$$Tx_i = \sum_g \alpha_g \pi(g) x_i,$$

hence $T = \sum \alpha\pi(g)$. $\square$

2.2 Orthogonality relations

Let (V, π) be a representation and let $\rho \in V^* = \mathrm{Hom}(V, \mathbb{C})$. The functions $\alpha^\pi_{\rho,x}$ on G defined by $\alpha^\pi_{\rho,x}(g) = \rho(\pi(g)x)$ is called a **matrix coefficient** of π. If we fix a basis $e_1, \dots, e_n$ of V and consider the dual basis $e_1^*, \dots, e_n^*$ of V^*, then we write α^π_{ij} instead of $\alpha^\pi_{e_i^*, e_j}$.

In the basis $e_1, \dots, e_n$ we have

$$\pi(g) = \left(\alpha^\pi_{ij}(g)\right)_{ij}$$

Theorem 2.3. *We have:*

1. *If π and θ are inequivalent irreducible representations, then*

$$\frac{1}{|G|} \sum_g \alpha^\pi_{f,x}(g) \alpha^\theta_{\rho,y}(g^{-1}) = 0$$

for all $f \in V_\pi^*$, $\rho \in V_\theta^*$, $x \in V_\pi$ and $y \in V_\theta$.

2. *If π is irreducible, then*

$$\frac{1}{|G|} \sum_g \alpha^\pi_{f,x}(g) \alpha^\pi_{\rho,y}(g^{-1}) = \frac{f(x)\rho(y)}{\dim \pi}.$$

Proof. It is the avering trick again. First observe that if $T : V_\theta \to V_\pi$ is any linear operator, then

$$S = \frac{1}{|G|} \sum_{g \in G} \pi(g) T \theta(g^{-1})$$

is in $\mathrm{Hom}_G(\theta, \pi)$ by averaging. Thus, since π and θ are in-equivalent, it follows from Schur's lemma that this must be zero.

Now let $T(v) = \rho(v)x$. This is a linear operator. Then

$$0 = f(Sy) = \frac{1}{|G|} \sum_g \alpha^\pi_{f,x}(g) \alpha^\theta_{\rho,y}(g^{-1}).$$

This proves part 1.

Now suppose $\pi = \theta$. Then $S \in \mathrm{End}(\pi) = \mathbb{C} \cdot \mathrm{id}_V$. Thus $S = \alpha \cdot \mathrm{id}_V$. Taking traces on both sides, we get $\alpha = TrT / \dim \pi$ (note that T and S have the same trace).

Then

$$\frac{1}{|G|} \sum_g \alpha^\pi_{f,x}(g) \alpha^\pi_{\rho,y}(g^{-1}) = f(S(y)) = f(y) \frac{TrT}{\dim \pi}$$

(make this nicer some time) $\square$

Instead of writing expressions like $\sum_g \alpha_g \pi(g)$ it is convenient to introduce the following notation. Define the **convolution** of two functions $f_1, f_2 \in \mathbb{C}[G]$ by

$$(f_1 * f_2)(g) = \sum_{g = g_1 g_2} f_1(g_1) f_2(g_2) = \sum_{h \in G} f_1(h) f_2(h^{-1} g).$$

We have $\delta_{g_1} * \delta_{g_2} = \delta_{g_1 g_2}$ (easy check!). The convolution product is associative and makes $\mathbb{C}[G]$ into an algebra, called the **group algebra** of G.

Given a representation $\pi : G \to \mathrm{GL}(V)$ we can define an algebra homomorphism $\mathbb{C}[G] \to \mathrm{End}(V)$ by $\pi(f) = \sum_{g \in G} f(g) \pi(g)$. Thus V becomes a left $\mathbb{C}[G]$-module. Conversely, given any united algebra homomorphism $\pi : \mathbb{C}[G] \to \mathrm{End}(V)$, we get a representation by $g \mapsto \pi(\delta_g)$. Thus we have a correspondence between representations of G and left $\mathbb{C}[G]$-modules. Using these notions, the Density Theorem (Theorem 2.1) becomes $\pi(\mathbb{C}[G]) = \oplus_i \mathrm{End}(V_i)$.

Theorem 2.4. *Let $\hat{G}$ be the set of equivalence classes of irreducible representations of G. Then $\hat{G}$ is finite and $\sum_{[\pi] \in \hat{G}} (\dim \pi)^2 = |G|$. Furthermore, we have*

$$\lambda \sim \bigoplus_{[\pi] \in \hat{G}} \pi^{\oplus \dim \pi}.$$

Here λ is the left regular representation of G.

Proof to come later.

3 Lecture 3 - More on orthogonality

Where does the name "orthogonality relations" come from? We'll see now.

First of all, note that if V_π is a representation of a finite group G, and $\langle , \rangle'$ is *any* Hermitian scalar product on V_π, then

$$\langle v, w \rangle := \frac{1}{\langle G \rangle} \sum_{g \in G} \langle \pi(g)v, \pi(g)w \rangle'$$

is another Hermitian scalar product on V_π, making the representation **unitary**.

If π is any unitary representation, choose an orthonormal basis $e_1, \ldots, e_n \in V$. Then the matrices $\pi(g) = \left(a_{ij}^\pi(g)\right)$ are unitary matrices. Hence $\overline{a_{ij}^\pi} = a_{ji}^\pi(g^{-1})$.

There is a standard scalar product on $\mathbb{C}[G]$ defined by

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}.$$

Then the orthogonality relations take the form:

Theorem 3.1. *For every irreducible representation π choose an invariant scalar product and an orthonormal basis $\{e_1^\pi, \ldots, e_n^{\dim \pi}\}$ and corresponding matrix coeffecients a_{ij}^π. Then the functions a_{ij}^π are mutually orthogonal with*

$$\left(a_{ij}^\pi, a_{ij}^\pi\right) = \frac{1}{\dim \pi}.$$

3.1 Characters of a representation

Let (π, V) be a representation.

The function $\chi_\pi : G \to \mathbb{C}$ defined by $\operatorname{tr} \pi(g)$ is called the **character of** π.

We say that a function $f : G \to \mathbb{C}$ is **central** if $f(hgh^{-1}) = f(g)$ for all $h, g \in G$. Note that characters are central. Also note that $\chi_\pi(e) = \dim \pi$.

Theorem 3.2. *The characters χ_π (for $[\pi] \in \widehat{G}$) form an orthonormal basis for the space of central functions on G.*

Proof. We have

$$\chi_\pi(g) = \sum_{i=1}^{\dim \pi} a_{ii}^\pi(g),$$

and the orthogonality relations show that $\{\chi_\pi \mid [\pi] \in \widehat{G}\}$ is an orthonormal system:

$$(\chi_\pi, \chi_\pi) = \sum_{ij}^{\dim \pi} (a_{ii}^\pi, a_{jj}^\pi) = \sum_{i=1}^{\dim \pi} \frac{1}{\dim \pi} = 1.$$

Thus we need to show that the characters *span* the space of central functions. Consider the projection P from $\mathbb{C}[G]$ to the space of central functions defined by

$$f \mapsto (Pf)(g) = \frac{1}{|G|} \sum_{h \in G} f(hgh^{-1}).$$

It is obviously a projection. Now consider $P(a_{ij})(g)$ (we skip the upper index π). Then this is equal to

$$\frac{1}{|G|} \sum_h a_{ij}(hgh^{-1}) = \frac{1}{|G|} \sum_{h \in G} \sum_k \sum_l a_{ik}(h) a_{kl}(g) a_{lj}(h^{-1}).$$

But after using that the orthogonality relations, this simplifies to

$$\frac{1}{\dim \pi} \delta_{ij} \chi_\pi(g).$$

But the a_{ij} constitute a basis of $\mathbb{C}[G]$. [[HM!? TRUE??]] □

[[some comment about why the last claim is true...]]

Corollary 3.3. *Let $c(G)$ be the number of conjugacy classes in G. We have*

$$|\widehat{G}| = c(G).$$

Proof. The left hand side is equal to $\dim_{\mathbb{C}} \operatorname{span}_{\pi \in \widehat{G}} \{\chi_\pi\}$. But this is, as just proved, equal to the dimension of the space of central functions. □

Corollary 3.4. *Two finite-dimensional representations π and θ are equivalent if and only if $\chi_\pi = \chi_\theta$.*

Proof. You can read off the multiplicities from the character. □

Corollary 3.5. *For any $[\pi] \in \widehat{G}$, the multiplicity of π in the regular representation λ equals $\dim \pi$.*

Proof. We are interested in (χ_λ, χ_π). But note that

$$\chi_\lambda(g) = \sum_{h \in G} \operatorname{tr}(\delta_h \mapsto \delta_{hg}) = \begin{cases} 0 & \text{if } g \neq e \\ |G| & \text{if } g = e. \end{cases}$$

Then $(\chi_\lambda, \chi_\pi) = \frac{1}{|G|} \chi_\lambda(e) \cdot \overline{\chi_\pi(e)} = \dim \pi$. □

3.2 Isotypic component and can. decomposition of V

[[isotypic component ...]]

4 Lecture 4 - The Frobenius determinant problem

Let G be a finite group and $\mathbb{C}[x_g \mid g \in G]$ the algebra on generators indexed by $g \in G$. Then one can form the following determinant:

$$P_G = \det\left((x_{gh})_{gh}\right).$$

It is the determinant of the multiplication table of G. The problem is to decompose P_G into irreducible polynomials.

It is conventient to instead work with $\widetilde{P_G} = \det((x_{gh^{-1}})_{gh})$. This has the same determinant as P_G up to multiplication by ± 1 since this just corresponds to permuting some of rows of the multiplication table.

Now here comes the *key* observation: the matrix $(x_{gh^{-1}})$ is the matrix of the operator

$$\sum_{g\in G} x_g \lambda(g)$$

in the basis $\delta_g \in \mathbb{C}[G]$. Here you are supposed to view elements of $\mathbb{C}[x_g \mid g \in G]$ as **functions** on G.

That this operator actually has matrix $(x_{gh^{-1}})$ can be seen by applying the operator to basis elements:

$$\left(\sum_{g\in G} x_g \lambda(g)\right)(\delta_h) = \sum g \in G x_g \delta_{gh}$$

by definition of the left regular action. Reparametrizing this last sum by setting $g' = gh$ we get

$$\sum_{g\in G} x_g \delta_{gh} = \sum_{g'=gh} x_{gh^{-1}} \delta_g,$$

as wanted.

Now, the regular representation decomposes as

$$\lambda \sim \bigoplus_{[\pi]\in\widehat{G}} \pi^{\oplus \dim \pi}$$

Hence the determinant decomposes as well (since the operator is G-invariant) and we get

$$\widehat{P_G} = \prod_{[\pi]\in\widehat{G}} \det\left(\sum_{g\in G} x_g \pi(g)\right)^{\dim \pi}$$

The question is now of course if we can decompose further. The answer is no:

Theorem 4.1 (Frobenius)**.** *The polynomials* $\det\left(\sum_{g\in G} x_g\pi(g)\right)$ *(*$[\pi]\in\widehat{G}$*) are irreducible and no two of them are associates.*

We begin with a lemma:

Lemma 4.2. *The polynomial* $p = \det(x_{ij})$ *is irreducible.*

Proof. To come. □

Proof of theorem. Suppose that $P_\pi = p_1p_2$. The polynomials p_1, p_2 are homogeneous. Fix a basis $e_1, \dots, e_n$ in V_π. Consider the corresponding matrix units m_{ij} defined by

$$m_{ij}e_k = \delta_{jk}e_i.$$

Since $\pi(\mathbb{C}[G]) = \mathrm{End}(V_\pi)$ by the density theorem,we can find $a_{ij}(g) \in \mathbb{C}$ such that

$$m_{ij} = \sum_{g\in G} \alpha_{ij}(g)\pi(g).$$

Consider the ring map $\mathbb{C}[x_g \mid g \in G] \to \mathbb{C}[x_{ij}]$ defined by

$$x_g \mapsto \sum_{ij} \alpha_{ij}(g)x_{ij}.$$

Under this map the operator $\widehat{P_G}$ becomes

$$\sum_{g\in G} x_g\pi(g) = \sum_{g\in G}\sum_{i,j} \alpha_{ij}(g)x_{ij}\pi(g) = \sum_{ij} m_{ij}x_{ij} = (x_{ij}).$$

Since the determinant is functorial with respect to ring maps, a decomposition $P_G = p_1p_2$ would induce a decomposition of $\det(x_{ij})$, which was just proved to be impossible.

We still have to check that no two of the determinants are equal (up to associates). But note that $P_\pi\big|_{x_g=0, g\neq e} = x_e^{\dim\pi}$. So we can recover the dimension of π from the determinant. Also note that if we fix $g \in G$, then

$$P_\pi\big|_{x_e=1, x_h=0 \text{ for } h\neq g} = \det(1 + x_g\pi(g)) = 1 + \operatorname{tr}\pi(g)x_g + h.o.t.$$

, so we can recover the trace as well. But a representation is determined by its character, so if the polynomials are equal, then the representations are equivalent. The converse is similar. □

4.1 Two constructions on representations

Recall that the tensor product of V and W have a basis $v_i \otimes v_j$. Hence $\dim_{\mathbb{C}} V \otimes W = \dim_{\mathbb{C}} V \cdot \dim_{\mathbb{C}} W$.

Assume (V, π) is a finite-dimensional representation and consider $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Then we define the **contragradient representation** π^c to be the representation of the dual space defined by

$$(\pi(g)f)(v) = f(\pi(g^{-1})v).$$

If $v_1, \dots, v_n$ is a basis for V and $\pi(g) = (a_{ij}(g))$, then in this basis

$$\pi^c(g) = (a_{ji}(g^{-1}))_{ij}.$$

In particular, the character $\chi_{\pi^c}(g) = \chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$.

Also, we can define the **tensor product of two representations** π and θ by $(\pi \otimes \theta)(g)(v \otimes w) = \pi(g)v \otimes \theta(g)w$.

See the exercises for some properties.

Also: $\mathrm{Hom}(U, W \otimes V^*) = \mathrm{Hom}(U \otimes V, W)$.

Frobenius reciprocity:

$$\mathsf{Mor}(\pi, \eta \otimes \theta^c) \simeq \mathsf{Mor}(\pi \otimes \theta, \eta).$$

Denote by $R(G)$ the abelian group generated by classes $[\pi] \in \widehat{G}$ of finite-dimensional representations and relations $[\pi] + [\pi'] = [\pi \oplus \pi']$. Then it is an exercise to show that $R(G)$ is a free abelian group with basis $[\pi] \in \widehat{G}$.

Via the tensor product we can form a product on $R(G)$, making it into a ring.

5 Lecture 5 - Dimension of irreps and the symmetric group

Recall that

$$\sum_{[\pi] \in \widehat{G}} (\dim \pi)^2 = |G|.$$

In this lecture we will prove the following theorem:

Theorem 5.1. *The dimension of any irreducible representation divides the number $|G/Z(G)|$, where $Z(G)$ is the center of G.*

Later we will strengthen this and show that $Z(G)$ can be replaced by any normal abelian subgroup of G.

In order to prove this, we need to introduce some results from commutative algebra. Let R be a unital commutative ring and $S \subset R$ a united subring. Then we say that an element $a \in R$ is **integral over**

S if it satisfies a monic polynomial $a^n + s_1 a^{n-1} + \ldots + s_{n-1} = 0$ with coefficients in S.

We say that a complex number integral over $\mathbb{Z}$ is an **algebraic integer**. An integral domain S is called **integrally closed** if any element in the fraction field of S integral over S is already in S. It is an easy exercise to see that $\mathbb{Z}$ is integrally closed (and in fact any UFD).

Lemma 5.2. *Let $S \subset R$ be as above. Then $a \in R$ is integral over S if and only if the subring $S[a]$ is a finitely-generated S-module.*

Proof. This is Proposition 5.1 in Atiyah-MacDonald. □

It is an exercise to see that the set of elements of R integral over S actually form a ring. Alternatively, consult Corollary 5.3 in Atiyah-MacDonald.

Proposition 5.3. *Assume that π is an irreducible representation. Let $g \in G$. Let $C(g)$ be the conjugacy class of G. Then*

1. *The number $\chi_\pi(g)$ is an algebraic integer.*
2. *The number $\frac{|C(g)|}{\dim \pi}\chi_\pi(g)$ is also an algebraic integer.*

Proof. i). Since G is finite, we have $g^n = e$ for some n. Hence all eigenvalues of g are nth roots of unity which are algebraic integers. But $\chi_\pi(g)$ is the sum of the eigenvalues, and the set of algebraic integers form a subring of $\mathbb{C}$, hence $\chi_\pi(g)$ is an algebraic integer as well.

The second part is more difficult. Let p be the characteristic function of $C(g)$. Then p lies in the center of $\mathbb{C}[G]$. Consider the subring $\mathbb{Z}[G] \subset \mathbb{C}[G]$ (the **group ring** of G). Let $R \subset \mathbb{Z}[G]$ be the subring of *central functions*. As R is a finitely generated abelian group, any element of R is integral over $\mathbb{Z} \cdot 1 \subset R$.

It follows that $\pi(p)$ is integral over $\mathbb{Z} \cdot \mathrm{id}_{V_\pi} \subset \mathrm{End}(V_\pi)$. Since p is central $\pi(p) \in \mathrm{End}_G(V_\pi) = \mathbb{C} \cdot 1$. Hence $\pi(p) = \alpha \cdot 1$ for some complex number α. This α is an algebraic integer.

But now

$$\begin{aligned}
\alpha \cdot \dim \pi &= \mathrm{tr}(\alpha \cdot 1) \\
&= \mathrm{tr}\,\pi(p) \\
&= \sum_{h \in C(g)} \mathrm{tr}\,\pi(h) \\
&= \sum_{h \in C(g)} \chi_\pi(h) \\
&= |C(g)|\chi_\pi(g).
\end{aligned}$$

The last equality is because characters are constant on conjugacy classes. This proves the claim. □

Lemma 5.4. *If π is irreducible, then* $\dim \pi \mid |G|$.

Proof. Recall that
$$(\chi_\pi, \chi_\pi) = 1.$$
This can be rewritten as
$$\frac{|G|}{\dim \pi} = \sum_{g \in G} \frac{\chi_\pi(g)\chi_\pi(g^{-1})}{\dim \pi}.$$

Now let $C(G)$ denote the set of conjugacy classes in G. C is then a partition of G. Then the above sum can be rewritten as
$$\sum_{C \in C(G)} \sum_{h \in C} \frac{\chi_\pi(h)\chi_\pi(h^{-1})}{\dim \pi} = \sum_{C \in C(G)} \frac{|C|\chi_\pi(g)}{\dim \pi} \chi_\pi(g^{-1}).$$

The right-hand-side is a sum of product that we know are algebraic integers. They form a ring, hence the sum is an algebraic integer as well.

But the left hand side is a fraction, so we must have $|G|/\dim \pi \in \mathbb{Z}$, which is equivalent to what was to be proven. □

Now we can prove Theorem 5.1. Recall that we want to show that $\dim \pi$ divides $|G/Z(G)|$.

Proof. For every $n \in \mathbb{N}$, consider the representation of $G^n = G \times \cdots \times G$ on $V_\pi \otimes \ldots \otimes V_\pi$ defined by $\pi_n(g_1, \ldots, g_n) = \pi_1(g_1) \otimes \ldots \otimes \pi_n(g_n)$. It is an **exercise** to see that this representation is irreducible as well.

Now consider the subgroup $Z_n = \{(g_1, \ldots, g_n) \in Z(G)^n \mid \prod g_i = e\}$. It is isomorphic to $Z(G)^{n-1}$.

If $g \in Z(G)$, then by Schur's lemma, $\pi(g)$ is a scalar operator. So if $(g_1, \ldots, g_n) \in Z_n$, then $\pi(g_i) = \alpha_i \cdot 1$, hence
$$\pi_n(g_1, \ldots, g_n) = \alpha_1, \ldots, \alpha_n \mathrm{id}_{V^{\otimes n}} = \mathrm{id}_{V^{\otimes n}}$$
since $g_1 \cdots g_n = e$. Therefore $Z_n \subset \ker \pi_n$. It follows that π_n induces a representation of G^n/Z_n. Then by the previous lemma
$$(\dim \pi)^n \mid \frac{|G|^n}{|Z(G)|^{n-1}}$$
That is:
$$\left(\frac{|G|}{\dim \pi |Z(G)|}\right)^n \in \mathbb{Z}[\frac{1}{|Z(G)|}]$$
for any n. But this implies that the left-hand-side is a finitely-generated $\mathbb{Z}$-module, hence what is inside the brackets is an algebraic integer. But is also a fraction, so it must be an integer. □

6 Lecture 6 - Representation theory of S_n

It is easy to understand conjugacy classes in S_n. Every $\sigma \in S_n$ defined a partition of $\{1, \dots n\}$ into orbints of σ. Let $O_1, \dots, O_m$ be these orbits. Assume that they are decreasing in size. Let $n_i = |O_i|$. These numbers n_i sum to n.

We say that a **partition of** n is a decreasing sequence of integers $n_1 geq \dots \geq n_m \geq 1$ such that $n = \sum n_i$.

Thus every $\sigma \in S_n$ gives a partition of n. Two elements of S_n are conjugate if and only if they define the same partition. This is straightforward to see. First suppose that σ and τ are conjugate, that is, that $\sigma = g^{-1}\tau g$ for some $g \in S_n$. The orbit of 1 under σ is defined by $O(\sigma, 1) = \{\sigma^n(1) \mid n \geq 0\}$. But this is $g^{-1}O(\tau, g(1))$. This defines a bijection the orbit of 1 under σ and the orbit of 1 under τ. Hence their sizes are equal. Conversely, assume that the orbits of τ and σ have the same sizes. Let g be an arbitrary bijection $[n] \to [n]$ between sending orbits to orbits. [[HOW TO PROCEED?????]]

Our goal is to produce an irreducible representation for every partition of n.

Recall that for any finite group G, we have

$$\mathbb{C}[G] \simeq \bigoplus_{[\pi] \in \widehat{G}} \mathrm{End}(V_\pi),$$

the isomorphism sending g to the corresponding endomorphism of $V = \oplus V_\pi$. Now one can ask if it is possible to recover the space V_π from its matrix algebra? (essentially, yes)

We need a few definitions. An idempotent e in an algebra A is called **minimal** if $e \neq 0$ and $eAe = \mathbb{C}e$.

Here's an exercise (to be proven in the appendix!):

Exercise 1. Let V be a finite-dimensional vector space. Show that $e \in \mathrm{End}(V)$ is a minimal idempotent if and only if it is a projecting onto a 1-dimensional subspace $\mathbb{C}v \subset V$. Then we have an isomorphism

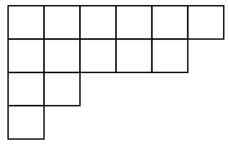
$$\mathrm{End}(V)e \simeq V$$

of $\mathrm{End}(V)$-modules given by $Te \mapsto Tv$. ♠

Therefore finding irreducible representations of G is the same as finding minimal idempotents in the group algebra $\mathbb{C}[G]$: if $e \in \mathbb{C}[G]$ is a minimal idempotent, then $\mathbb{C}[G]e$ defines an irreducible representation ([[WHY IS IT IRREDUCIBLE??]].

Thus, returning to S_n, we want to to construct minimal idempotents in $\mathbb{C}[S_n]$ for every partition.

It is convenient to present partitions as **Young diagrams**. Given a partition $(n_1, \dots, n_m)$ of n, we draw a diagram having n_1 boxes in the first row, n_2 boxes in the second, and so on. For example, for the partition $(6, 5, 2, 1)$ of 14, we draw the following diagram:



We can also fill these with numbers: a **Young tableau** (plural *tableaux*) is a Young diagram filled with numbers $i \in [n]$ without repetitions. For example:

$$\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

Two Young tableaux are said to be of the same **shape** if they arise from the same underlying Young diagram.

Note that the symmetric group acts on the set of Young tableaux of the same shape λ.

Fix a Young tableaux T. Let $R(T) \subset S_n$ be the subgroup of elements of S_n permuting numbers in the *rows* of T. Let $C(T)$ be the same group, but permuting columns instead. Define the following two elements of $\mathbb{C}[S_n]$:

$$a_T = \frac{1}{|R(T)|} \sum_{g \in R(T)} g$$

and

$$b_T = \frac{1}{|C(T)|} \sum_{g \in C(T)} sgn(g) \cdot g.$$

Then we define $c_T = a_T b_T$. The element c_T is called a **Young symmetrizer**. Our main theorem is this:

Theorem 6.1. *1. For any T, the element c_T is (up to a scalar) a minimal idempotent in $\mathbb{C}[S_n]$, so it defines an irreducible representation of S_n.*

2. If T_1, T_2 are two Young tableaux, then the representations corresponding to them are equivalent if and only if T_1 and T_2 have the same shape.

The modules $\mathbb{C}[S_n]c_T$ are called **Specht modules**.

Proof. TOO LONG FOR NOW. WILL PROB. COME LATER $\square$

6.1 Tabloids

A bit more explicitly the Specht modules can be described as follows.

Fix a Young diagram λ. Introduce an equivalence relation on Young tableaux of shape λ:

$$T_1 \sim T_2$$

if $T_1 = r(T_2)$ for some $r \in R(T_2)$. The equivalence class of a Young tableau T is called a **tabloid** (of shape λ) and denoted by $\{T\}$.

Note that S_n act on the set of tabloids. Let (M_λ, S_n) be the corresponding permutation representation. Note also that action on the set of tabloids is transitive. Thus the basis of M_λ can be identified with $S_n/R(T)$[1].

Thus we get isomorphisms of $\mathbb{C}[S_n]$-modules:

$$\mathbb{C}[S_n]a_T \simeq \mathbb{C}[S_n/R(T)] \simeq M_\lambda.$$

The first isomorphism is given by sending ga_T to $\partial_{gR(T)}$ (the delta function), and the second is given by sending a coset $\partial_{gR(T)}$ to $\{g(T)\}$.

Under these isomorphisms, the image of the submodule $\mathbb{C}[S_n]b_Ta_T \subset \mathbb{C}[S_n]a_T$ is spanned by the elements $b_{g(T)}\{g(T)\}$. In other words, V_λ is spanned by $e_{T'} := b_{T'}\{T'\}$ by all elements of shape λ. This gives us a description of $V_\lambda \simeq V_T$ only in terms of λ.

This is a spanning set. We would like a basis. We say that that tableau T is **standard** if the numbers in every row and column are increasing.

Here's a theorem that we wont prove:

Theorem 6.2. *For any Young diagram λ the elements e_T for all standard tableaux T of shape λ form a basis in V_λ.*

The complication is of course that $ge_T = c_{g(T)}$ is a combination of standard tableaux!

[[two examples]]

6.2 Characters and the Hook length formula

Here we prove two theorems.

Let $I = (i_1, i_2, \ldots,)$ consist of non-negative integers such that

$$\sum_{k=1} \infty k i_k = n.$$

Denote by c_I the conjugacy class in S_n consisting of elements σ which decompose into i_1 orbits of length 1, i_2 orbits of length two and so on.

This corresponds to an obvious partition of n.

Let $\chi_{\pi_\lambda}(C_I)$ denote the value of the character of π_λ on any representating of the conjugacy class of C_I. Then

[1] Every $\{T\}$ can be written as $\sigma\{T_0\}$ for some fixed T_0. But the ambiguity lies in $R(T)$.

Theorem 6.3 (Frobenius character formula)**.** *Assume* $\lambda = (\lambda_1, \dots, \lambda_r)$*. Take any number* $N \geq r$*. Then* $\chi_{\pi_\lambda}(C_I)$ *is the coefficient of* $\prod_{i=1}^N x_i^{\lambda_i + N - i}$ *in the polynomial*

$$\Delta(x) \prod_{i \geq 1} \sum_{i=1}^{N} (x_i^k)^{i_k}.$$

where $\Delta(x) = \prod(x_i - x_j)$ *is the Vandermonde determinant.*

This was not proved, but the idea of the proof was given.

Now we quote the **hook length** formula. Let λ be Young diagram and let (i, j) be the element in position row i and column j. Then the **hook length** $h(i, j)$ is the number of boxes below and to the right of (i, j) including (i, j) itself. Then:

Theorem 6.4. *Let* λ *be partition and let* π_λ *be the corresponding irreducible representation of* S_n*. Then*

$$\dim \pi_\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i, j)}.$$

Prooof later.

7 Induced representations

Let G be a finite group and $H \subset G$ a subgroup. Assume (π, V) is a representation of H. We want to construct a representation of G out of this.

We define the **induced representation** $\mathrm{Ind}_H^G \pi$ to be the representation defined by the $\mathbb{C}[G]$-module

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V.$$

It is "somehow completely fundamental".

[[DEFINITION AND SOMETHING ABOUT EQUIVALENCE]]

A Exercises

A.1 Lecture 1

Exercise 2. Suppose $V_1 \oplus \ldots \oplus V_r \sim W_1 \oplus \cdots \oplus W_s$ as representations and that the V_i, W_i are irreducible. Show that $n = m$ and there exists $\rho \in S_r$ such that $V_i \sim W_{\rho(i)}$ for all i. ♠

Solution 1. First note a lemma: If $V \oplus W \sim V' \oplus W'$ and $V \sim V'$, then $W \sim W'$. This follows from the five-lemma.

Now consider the composition $V_1 \hookrightarrow V_1 \oplus \cdots \oplus V_r \to W_1 \oplus \cdots \oplus W_s \to W_i$. This has to be non-zero for at least one i. Hence $V_1 \sim W_i$ for some i. By rearring, we are in the situation of the lemma above. Hence, inductively, if $r \leq s$, we find $0 \sim W_{r+1} \oplus \cdots \oplus W_s$, which is impossible. Similarly for $s \leq r$. Hence $r = s$ and we conclude. ♡

Exercise 3. Let $\hat{G}$ denote the Pontryagin dual of G. Then, for any finite abelian group G, we have $G \simeq \hat{\hat{G}}$. ♠

Solution 2. We first do this when $G = \mathbb{Z}/n$ for some $n \geq 0$. Let $\varphi : \mathbb{Z}/n \to \mathbb{C}^*$ be a character. Then $\phi(1) = z$ for some z, but $\phi(n) = \phi(0) = z^n = 1$, so z must be a n'th root of unity.

Realize $\mathbb{Z}/n$ as the nth roots of unity. Then we can define a homomorphism $\widehat{\mathbb{Z}/n} \to \mathbb{Z}/n$ by $\varphi \mapsto \varphi(1)$. Similarly, we can define an inverse map by sending $e^{2\pi i/n}$ to the character $m \mapsto e^{2\pi i m/n}$. These are inverses.

Now, every finite abelian group is a product of these groups. So it remains to show (by induction) that if G, H are two finite abelian groups, then $\widehat{G \times H} \simeq \widehat{G} \times \widehat{H}$. The inclusion maps $G \to G \times H$ and $H \to G \times H$ induces a map $\widehat{G} \times \widehat{H} \to \widehat{G \times H}$ by $(\varphi_1, \varphi_2) \mapsto ((g,h) \mapsto \varphi_1(g)\varphi_2(h))$.

It is easy to see that this map is injective. To see that it is surjective, let $\varphi : G \times H \to \mathbb{C}$ be a character. Write φ as $gh \mapsto \varphi(gh)$. Then we can define characters on G, H by $g \mapsto \varphi(g \cdot 1)$ which maps to φ. Ok. ♡

Exercise 4. Show that the S defined in the proof of Theorem (2.1) lies in $\Theta(G)''$. ♠

Solution 3. This is, I think, definition-hunting. DETAILS COME LATER ♡

Exercise 5.

1. Check that indeed
$$\langle v, w \rangle := \frac{1}{\langle G \rangle} \sum_{g \in G} \langle \pi(g)v, \pi(g)w \rangle'$$
is an invariant Hermitian scalar product.
2. Show that if π is irreducible, then an invariant scalar product is unique up to a factor.

♠

Solution 4. i). Invariant means that $\langle \pi(h)v, \pi(h)w \rangle = \langle v, w \rangle$. This is clear from the definitions, since if $g \in G$ ranges over all of G, then so does gh.

The only (slightly) nontrivial thing to check is that $\langle , \rangle$ is positive definite. But this is so.

ii) If $\langle , \rangle$ is invariant, then it is an element of $\mathrm{Hom}_G(V \otimes V, \mathbb{C})$. This is canonically isomorphism to $\mathrm{Hom}_G(V \otimes, V^*)$. If we can show that V and V^* are isomorphic as representations, then we are done by Schur's lemma.

But we are given an inner product $\langle , \rangle$. We can define a map $V \to V^*$ by sending $v \in V$ to the function $w \mapsto \langle v, w \rangle$. This is clearly a linear isomorphism, and it is also a map of representations. Recall that the action of G on V^* is defined by $\pi^*(g)\varphi(v) = \varphi(g^{-1}v)$. Then in the diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\cdot g"] & V^* \arrow[d, "\cdot g"] \\ V \arrow[r] & V^* \end{tikzcd}$$

we want $\langle gv, w \rangle$ to be equal to $\langle v, g^{-1}w \rangle$. But this is true since $\langle , \rangle$ is invariant:

$$\langle gv, w \rangle = \langle gvgg^{-1}w \rangle = \langle v, g^{-1}w \rangle.$$

Hence V and V^* are isomorphic as representations and then $\mathrm{Hom}_G(V, V^*)$ (space of bilinear forms) is one-dimensional. ♡

A.2 Lecture 4

Exercise 6. 1. $\pi^{cc} = \pi$.

2. $(pi \otimes \theta)^c = \pi^c \otimes \pi^c$.
3. π is irreducible if and only if π^c is.

♠

B Some worked examples

B.1 The representation ring of S_3

We want to explicitly compute the representation ring of S_3.

First we find all irreducible representations. The first one is the trivial representation $\epsilon : S_3 \to \mathbb{C}$. We also have the *sign* representation given by $g \mapsto sgn(g) \in \mathbb{C}$. Both of these are one-dimensional representations.

S_3 have a natural action on $\mathbb{R}^3$ given by permuting the basis vectors. But it acts trivially on the subspace spanned by $e_1 + e_2 + e_3$, hence $\mathbb{R}^3$ decomposes as $\epsilon \oplus V$, where V is some 2-dimensional representation. It is irreducible: if not, any $v \in V$ would be sent to a scalar multiple of itself, but this is not the case. This representation is called the **standard representation** of S_3. We have also now found all representations of S_3, since $1^2 + 1^2 + 2^2 = 6$.

Thus the representation ring $R(S_3)$ is $\mathbb{Z}[A, S]$ modulo some relations to be found. Here A is the alternating representation and S is the standard representation. The trivial representation is 1. It is easy to see that $A \otimes A \sim \epsilon$, so $A^2 = 1 \in R(S_3)$. The representation $S \otimes S$ is 4-dimensional. To compute how it decomposes, we use character theory.

Recall that characters are class functions on G, that is, they only depend on the conjugacy classes of G. So we write a character table:

	e	τ	σ
ϵ	1	1	1
A	1	1	-1
S	2	-1	0

To compute the character of the standard representation, we first note that it is, as a vector space given by $\mathbb{R}^3/(1,1,1)$. A basis is then given by the images of e_1, e_2. Let τ be the transposition (123), sending e_i to e_{i+1}. Let σ be reflection fixing e_1 and exchanging e_2 and e_3. In this basis that means e_2 is sent to $-e_1 - e_2$. Writing up the corresponding matrices lets us find the value of the character.

Now one can compute by hand (or use a result on characters), that the character of $S \otimes S$ is given by $\chi_S \cdot \chi_S$, so that its entry in the character table is $(4, 1, 0)$. If $S \otimes S = V_1 \oplus V_2$, then $\chi_S = \chi_{V_1} + \chi_{V_2}$. Using also that the characters are linearly independent, we see that the only option is $S \otimes S \sim \epsilon \oplus A \oplus S$.

Hence $S^2 = 1 + A + S$ in the representation ring. Similarly, we find that $AS = S$ in the representation ring. All in all

$$R(S_3) = \mathbb{Z}[A, S]/(A^2 - 1, S^2 - 1 - A - S, AS - S).$$

B.2 Explicit Specht modules

Again, we work with $G = S_3$. Consider the following Young-diagram:

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

We want to use the theorem from Lecture 5 (?) to find the standard representation of S_3. The elements of S_3 are generated by $\rho = (123)$ and $s = (23)$ with the relations $\rho^3 = e$ and $s\rho s = \rho^2$.

First off, the elements permuting the rows is the subgroup consisting of e and $(12) = s\rho^2$. The elements permuting the columns is the subgroup generated by $(13) = s\rho$. Then

$$a_T = \frac{1}{2}\left(e + (12)\right)$$

and

$$b_T = \frac{1}{2}\left(e + (13)\right).$$

Thus

$$c_T = \frac{1}{4}\left(e - (13) + (12) - (132)\right).$$

This gives us as in the lecture a map

$$\mathbb{C}[S_3] \to V \subseteq \mathbb{C}[S_3]$$

given by $g \mapsto gc_T$, whose image is supposed to be an irreducible representation of S_3. Let's find this. One computes the action of c_T on the basis elements of $\mathbb{C}[S_3]$ by direct computation:

$$\begin{aligned}
c_T &= \frac{1}{4}\left(e - s\rho + s\rho^2 - \rho^2\right)\\
\rho c_T &= \frac{1}{4}\left(\rho - s + s\rho - e\right)\\
\rho^2 c_T &= \frac{1}{4}\left(\rho^2 - s\rho^2 + s - \rho\right)\\
s c_T &= \frac{1}{4}\left(s - \rho + \rho^2 - s\rho^2\right)\\
s\rho c_T &= \frac{1}{4}\left(s\rho - e + \rho - s\right)\\
s\rho^2 c_T &= \frac{1}{4}\left(s\rho^2 - \rho^2 + e - s\rho\right)
\end{aligned}$$

We claim that c_T and ρc_T span the image. For we have $\rho^2 c_T = -c_T - \rho c_T$. And $s c_T = \rho^2 c_T$. And $s\rho c_T = \rho c_T$. And $s\rho^2 c_T = c_T$.

Thus we have some 2-dimensional representation of S_3 with basis $c_T, \rho c_T$. We compute the matrices of S_3 with respect to this basis:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad \rho^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$s = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \qquad s\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad s\rho^2 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}.$$

As expected, the character of this representation is exactly the character of the standard representation from the previous example (as is seen by computing traces). This is no coincidence.