Algebraiske grupper og moduliteori

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1 Representation theory in general

2 Algebraic groups

Definition 2.1. Let A be a finitely generated k-algebra. An *affine algebraic group* is a quadruple $(A, \mu_A, \epsilon, \iota)$ where $\mu_A : A \to A \otimes_k A$ (the *coproduct*), $\epsilon : A \to k$ (the *coidentity*), $\iota : A \to A$ (the *coinverse*) are k-algebra homomorphisms, satisfying the following conditions:

1. Coassociativity. The following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\mu_A"] \arrow[d, "\mu_A"'] & A \otimes_k A \arrow[d, "\mathrm{id}_A \otimes \mu_A"] \\ A \otimes_k A \arrow[r, "\mu_A \otimes \mathrm{id}_A"'] & A \otimes_k A \otimes_k A \end{tikzcd}$$

2. The following diagram commutes:

$$\begin{tikzcd} & & k \otimes_k A \arrow[dr, "\simeq"] & \\ A \arrow[r, "\mu"] & A \otimes_k A \arrow[ur, "\epsilon \otimes \mathrm{id}_A"] \arrow[dr, "\mathrm{id}_A \otimes \epsilon"'] & & A \\ & & A \otimes_k k \arrow[ur, "\simeq"'] & \end{tikzcd}$$

and is equal to the identity.

3. Inverse. The following diagram commutes:

$$\begin{tikzcd}
A \arrow[rr, "\epsilon"] \arrow[d, "\mu"] & & k \arrow[d] \\
A \otimes_k A \arrow[r, "\mathrm{id}_A \otimes \iota"'] & A \otimes_k A \arrow[r, "."'] & A
\end{tikzcd}$$

Here the right arrow is the morphism making A a k-algebra. The last arrow in the lower sequence is multiplication in A.

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Definition 2.2. An *action* of an affine algebraic group $G = \operatorname{Spec} A$ on an affine variety $X = \operatorname{Spec} R$ is a morphism $G \times X \to X$ defined dually by a k-algebra morphism $\mu_R : R \to R \otimes_k A$ satisfying the following two conditions.

1. The following diagram is commutative:

$$\begin{tikzcd}
R \arrow[r, "\mu_R"] \arrow[dr, "\mathrm{id}_R"] & R \otimes_k A \arrow[d, "\mathrm{id}_R \otimes \epsilon"] \\
& R \simeq R \otimes_k k
\end{tikzcd}$$

2. The diagram

$$\begin{tikzcd}
R \arrow[r, "\mu_R"] \arrow[d, "\mu_R"'] & R \otimes_k A \arrow[d, "\mu_R \otimes \mathrm{id}_A"] \\
R \otimes_k A \arrow[r, "\mathrm{id}_R \otimes \mu_A"'] & R \otimes_k A \otimes_k A
\end{tikzcd}$$

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3 Representations of algebraic groups

Let $G = \operatorname{Spec} A$ be an affine algebraic group over a field k.

Definition 3.1. An *algebraic representation of* G is a pair (V, μ_V) consisting of a k-vector space V and a k-linear map $\mu_V : V \to V \otimes_k A$ satisfying the following two conditions:

1. The diagram

$$\begin{tikzcd} V \arrow[r, "\mu_V"] \arrow[dr, "\mathrm{id}_V"'] & V \otimes_k A \arrow[d, "\mathrm{id}_V \otimes \epsilon"] \\ & V \simeq V \otimes_k k \end{tikzcd} \tag{1}$$

   is commutative.

2. The diagram

$$\begin{tikzcd} V \arrow[r, "\mu_V"] \arrow[d, "\mu_V"'] & V \otimes_k A \arrow[d, "\mu_V \otimes \mathrm{id}_A"] \\ V \otimes_k A \arrow[r, "\mathrm{id}_V \otimes \mu_A"'] & V \otimes_k A \otimes_k A \end{tikzcd}$$

   is commutative. Here μ_A is the coproduct in the coordinate ring of G.

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Remark. *In lieu of Definition 1.2, we see that any action of an algebraic group G on an affine variety $X = \operatorname{Spec} R$ is a representation of G on the infinite-dimensional k-vector space $R = \Gamma(X, \mathscr{O}_X)$.*

We often drop the subcript from μ_V unless confusion may arise. The same comment applies to tensor products. They will always be over the ground field unless otherwise stated. We will sometimes refer to a representation (V, μ_V) sometimes as "a representation $\mu : V \to V \otimes A$" and sometimes as just "a representation V".

Definition 3.2. Let $\mu : V \to V \otimes A$ be a representation of $G = \operatorname{Spec} A$. Then:

1. A vector $x \in V$ is said to be *G-invariant* if $\mu(x) = x \otimes 1$.

2. A subspace $U \subset V$ is called a *subrepresentation* if $\mu(U) \subseteq U \otimes A$.

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Proposition 3.3. *Every representation V of G is locally finite-dimensional. Precisely: every $x \in V$ is contained in a finite-dimensional subrepresentation of G.*

Bevis. Write $\mu(x)$ as a finite sum $\sum_i x_i \otimes f_i$ for $x_i \in V$ and linearly independent $f_i \in A$. This we can always do, by definition of tensor product and bilinearity. Let U be the subspace of V spanned by the vectors x_i.

Now, by the commutativity of the diagram (1) it follows that

$$x = \sum_i \epsilon(f_i) x_i.$$

By the commutativity of the second diagram in the definition, it follows that

$$\sum_i \mu_V(x_i) \otimes f_i = \sum_i x_i \otimes \mu_A(f_i) \in U \otimes A_k \otimes_k A.$$

Because each term of the right-hand-side is contained in $U \otimes A \otimes A$, it follows that $\mu_V(x_i)$ is contained in U since the f_i are linearly independent.

Thus x is contained in the finite-dimensional representation $\mu_V\big|_U : U \to U \otimes A$. $\square$

Repr. of $\mathbb{G}_m$.
Euler operator.
Characters.