Exercises

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I solve and type exercises from different places (read *books*).

1 Algebraic Geometry - Hartshorne

1.1 Chapter I - Varieties

Exercise 1 (Exercise 1.1)**.** a) Let Y be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

c) Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

♠

Solution 1. a) We have $A(Y) = k[x, y]/(y-x^2)$. An isomorphism $A(Y) \to k[t]$ is given by $x \mapsto t$ and $y \mapsto t^2$.

b) We have $A(Z) = k[x, y]/(xy - 1) \simeq k[x, \frac{1}{x}]$. So we must show that $k[x, \frac{1}{x}] \not\approx k[x]$. It can be computed that the first one has automorphisms given by $x \mapsto cx^n$ for c nonzero and $n \neq 0$. The second has as automorphisms $ax + b$ $(a \neq 0)$. So the first one have an abelian automorphism group, the second has not.

c) What is special about $A(Y)$ and $A(Z)$? Staring at pictures, we see that any line in $\mathbb{A}^2$ intersects Y in at least one point, but in the case of Z, there exist two lines which do not intersect Z. We claim that this is the only two things that can happen.

First we claim that if we are in the second situation, that is, if there exist a pair of lines ℓ, ℓ' such that $W \cap \ell = W \cap \ell' = \emptyset$, then $W \simeq Z$.

A general quadric can be written as

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Suppose now $\ell \cap W = \emptyset$. This is equivalent to $I(f, \ell^\vee) = (1)$. Without loss of generality, we can assumme $\ell = \{x = 0\}$. Then

$$I(f, \ell) = (cy^2 + ey + f, x).$$

This generates $k[x, y]$ if and only if $c = e = 0$ and $f \neq 0$. Thus f must be of the form

$$ax^2 + bxy + dx + f = 0$$

with $f \neq 0$. But this can be written as

$$x(ax + by + d) + f = 0.$$

Put $y' = ax + by + d$. Then $I(W)$ takes the form $(xy' + f = 0)$, which is clearly isomorphic to Z after a linear change of coordinates. Note that the other line not meeting W is the line given by $y' = ax + by + d = 0$.

Assume now that we are in the other situation, namely that *every* line in $\mathbb{A}^2$ meets W. Now pick a tangent line ℓ of W. Without loss of generality, we can assume that ℓ is $\{y = 0\}$. This is a tangent line if and only if it meets W doubly, meaning that $I(W) + (\ell^\vee)$ takes the form (l^2, y) for some linear form l. We can also assume that $\ell \cap W = (0, 0)$, so that $I(W) + (\ell^\vee) = (x^2, y)$. But this means that

$$\begin{aligned} I(W) + I(\ell) &= (ax^2 + bxy + cy^2 + dx + ey + f, y) \\ &= (ax^2 + dx + f, y) \end{aligned}$$

We want $ax^2 + dx + f = x^2$. This can happen only if $d = f = 0$ and $a \neq 0$. Thus the quadric takes the form

$$ax^2 + bxy + cy^2 + ey = 0.$$

Now we claim that there exist one line at each point of W that intersect W transversally in exactly one point. This is the case for Y. Consider the pencil of lines through $(0, 0)$ defined by $x = \lambda y$. We want to find λ such that the intersection is transversal and only one point. We have

$$(ax^2 + bxy + cy^2 + ey, x - \lambda y) = \left((a\lambda^2 + b\lambda + c)y^2 + ey, x - \lambda y\right).$$

This have exactly one solution if and only if $a\lambda^2 + b\lambda + c = 0$. This is solvable since $a \neq 0$ and since all lines intersect W. Thus choose λ as above. We can rotate this line such that it becomes $x = 0$. Then the equation takes the form

$$ax^2 + bxy + ey = 0.$$

We have still not arrived at $y = x^2$. Let now $y = \lambda x$ be a general line through the origin. We demand that this intersect W twice for every λ such that the line is not tangent. We get that the intersection is given by

$$ax^2 + b\lambda x + ex = x((a + \lambda b)x + e) = 0.$$

For this to have two solutions for every λ we must have $a + \lambda b \neq 0$ for all λ. But this requires $b = 0$. Thus the equation is

$$ax^2 + ey = 0$$

which is the conic we were looking for.

♡

Exercise 2 (Exercise 1.2, the twisted cubic curve)**.** Let $Y \subseteq \mathbb{A}^3$ be the set $\{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric equation* $x = t, y = t^2, z = t^3$. ♠

Solution 2. An affine variety is by definition a closed irreducible subset of $\mathbb{A}^3$. So we must find an irreducible ideal I such that $Z(I) = Y$ (forgive the abuse of notation).

I claim that $I(Y) = \langle x^2 - y, x^3 - z\rangle$. Clearly, every $P \in Y$ satisfies these equations. This shows the inclusion $Y \subset Z(I)$. Now suppose $P \in Z(I)$, that is, $f(P) = 0$ for all $f \in I$. In particular $(x^2 - y)(P) = 0$ and $(x^3 - z)(P) = 0$. Thus $y = x^2$ and $z = x^3$. So if $P = (a, b, c) \in k^3$, then $P = (a, a^2, a^3)$, so $P \in Y$. This shows that $Z(I) = Y$. If we can show that I is prime, then it follows that $I(Y) = I$ and that Y is a variety.

In fact, we claim that $k[x, y, z]/I \simeq k[t]$, implying that I is prime. The map φ is given by $x \mapsto t$, $y \mapsto t^2$, $z \mapsto t^3$. Then clearly $I \subseteq \ker \varphi$. We must show equality. So suppose $\varphi(f) = 0$.

First we claim that any $f \in k[x, y, z]$ can be written as $f = R(x) + S(x)y + T(x)z + i(x, y, z)$ where i is a polynomial in I. We prove this by induction on $\deg f$. If $\deg f = 1$, this is trivially true. The rest of the proof proceeds by tedious induction. ♡

1.2 Chapter II - Schemes

Exercise 3 (Excercise 7.1)**.** Let $(X, \mathscr{O}_X)$ be a locally ringed space and let $f : \mathscr{L} \to \mathscr{M}$ be a surjective map of invertible sheaves on X. Show that f is an isomorphism. ♠

Solution 3. Since $\mathscr{L}, \mathscr{M}$ are invertible, we have isomorphisms $\mathscr{L}_x \approx \mathscr{O}_{X,x}$ and $\mathscr{M}_x \approx \mathscr{O}_{X,x}$ for each $x \in X$.

But $\mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{O}_{X,x}, \mathscr{O}_{X,x}) = \mathscr{O}_{X,x}$, that is, all homomorphisms are given by multiplication by some $h \in \mathscr{O}_{X,x}$. But since f was surjective, we conclude that h is outside $\mathfrak{m}_x$, the maximal ideal of $\mathscr{O}_{X,x}$. But then h is a unit, so f is an isomorphism. ♡

1.3 Chapter III - Cohomology

Exercise 4 (Exercise 4.3)**.** Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ and let $U = X \backslash \{(0,0)\}$. Use a suitable open cover of X by open affine subsets to show that $H^1(U, \mathscr{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^i y^j \mid i,j < 0\}$. In particular, it is infinitedimensional, and so U cannot be affine (not projective either). ♠

Solution 4. We can cover U by $U_1 = \mathbb{A}^2 \backslash \{x = 0\}$ and $U_2 = \mathbb{A}^2 \backslash \{y = 0\}$. We have $U_1 \cap U_2 = \mathbb{A}^2 \backslash \{xy = 0\}$. Also, $\mathscr{O}(U_1) = k[x, y, \frac{1}{x}]$ and $\mathscr{O}(U_2) = k[x, y, \frac{1}{y}]$ and $\mathscr{O}(U_1 \cap U_2) = k[x, y, \frac{1}{xy}]$. Then the Čech complex takes the form

$$0 \to k[x, y, \frac{1}{x}] \times k[x, y, \frac{1}{y}] \xrightarrow{d} k[x, y, \frac{1}{xy}] \to 0,$$

the differential being difference. Then $H^1(U, \mathscr{O}_U)$ can be computed as the homology at the second term. But nothing on the left side can hit anything of the form $x^i y^j$ with $i, j < 0$. Anything else is hit. Thus we have

$$H^1(U, \mathscr{O}_U) \simeq \{x^i y^j \mid i, j < 0\}$$

as k-vector spaces. ♡

Exercise 5 (Exercise 4.7)**.** Let X be the subscheme of $\mathbb{P}^2_k$ defined by a single homogeneous polynomial $f(x_0, x_1, x_2) = 0$ of degree d. Assume that $(1, 0, 0)$ is not on X. Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Čech complex

$$\Gamma(U, \mathscr{O}_X) \oplus \Gamma(V, \mathscr{O}_X) \to \Gamma(U \cap V, \mathscr{O}_X)$$

explicitly, and thus show that

$$\dim_k H^0(X, \mathscr{O}_X) = 1$$
$$\dim_k H^1(X, \mathscr{O}_X) = \frac{1}{2}(d-1)(d-2).$$

♠

Solution 5. X can be covered by just two open affines since $\mathbb{P}^2 \backslash (U \cup V) = \{(1:0:0)\}$, which was assumed not to lie on the curve.

The open affine subset $\Gamma(U, \mathscr{O}_X)$ can be identified with the polynomial ring $k[u,v]/\langle f(u,1,v)\rangle$, and $\Gamma(V, \mathscr{O}_X) = k[x,y]/f(x,y,1)$. The differential is then given by

$$(g(u,v), h(x,y)) \mapsto g(xy^{-1}, y^{-1}) - h(x,y) \in k[x,y,\frac{1}{y}].$$

We can assume that $f = x_0^d$, since what really matters is the degree, and we are just doing linear algebra.

We first calculate $H^0(X, \mathscr{O}_X)$. So suppose $g(xy^{-1}, y^{-1}) - h(x,y) = 0$ in $k[x,y,y^{-1}]/\langle f(x,y,1)\rangle$. By definition this means that

$$g(xy^{-1}, y^{-1}) - h(x,y) = f(x,y,1) \cdot \tilde{f}(x,y,\frac{1}{y})$$

for some polynomial $\tilde{f}$. Write $\tilde{f}$ as $\tilde{f}_0 + \tilde{f}_1$, where $\tilde{f}_0 = \sum_{j<0} a_{ij} x^i y^j$ and $\tilde{f}_1 \in k[x,y]$. Then we have the equality

$$g(xy^{-1}, y^{-1}) - h(x,y) = \sum_{j<0} a_{ij} x^{i+d} y^j + \sum_{j \geq 0} x^{i+d} y^j.$$

First of all, we see that the constant terms of g and h must be equal, because there are no constant terms on the right hand side. Secondly, $g(xy^{-1}, y^{-1})$ consists solely of terms with $j < 0$. Thus the non constant terms of $g(xy^{-1}, y^{-1})$ must be equal to the left term of the right hand side above. But both terms of the right hand side are zero modulo f, so the constant terms of $g(xy^{-1}, y^{-1})$ are also zero mod f. The same holds for $h(x,y)$. Thus $H^0(X, \mathscr{O}_X) = \{(c,c) \mid c \in k\} \simeq k$.

Now we compute $H^1(X, \mathscr{O}_X)$. Consider a monomial $x^i y^j$ in the target. If both $i, j \geq 0$, then it is hit by $(0, -x^i y^j)$. Likewise, if $j \geq i$, then $(x^i y^{j-i}, 0) \mapsto x^i x^{-j}$. Thus all monomials $x^i y^{-j}$ with $j \geq i$ is zero in the cokernel. Further, if $i \geq d$, then $x^i y^j$ is already zero! Thus, we can draw the non-zero monomials in the cokernel as points in the lattice $\mathbb{Z}^2$. This is a triangle of length $d-2$. Thus the dimension of $H^1(X, \mathscr{O}_X)$ is

$$1+2+\ldots+d-3+d-2 = \frac{1}{2}(d-2)(d-2+1) = \frac{1}{2}(d-2)(d-1).$$

♡

1.4 Chapter IV - Curves

Exercise 6 (Exercise 1.1)**.** Let X be a curve and $P \in X$ a point. Show that there exists a nonconstant rational function $f \in K(X)$ which is regular everywhere except at P. ♠

Solution 6. Let D be the divisor $D = nP$. The linear system

$$\{E = D + f \geq 0\}$$

consists of all divisors linearly equivalent to D. But these are classified by those f with $(f) \geq -nP$, i.e. those f with at most poles of order n at P.

By Riemann-Roch we have

$$l(D) - l(K-D) = \deg D + 1 - g = n + 1 - g.$$

If n is large enough, $K - D$ will have negative degree, so $l(K-D) = 0$. Thus for large n, we can get $l(D)$ as big as we want.

♡

2 Commutative Algebra - Eisenbud

2.1 Chapter 16 - Modules of Differentials

Exercise 7 (Exercise 16.1)**.** Show that if $b \in S$ is an idempotent ($b^2 = b$), and $d : S \to M$ is any derivation, then $db = 0$. ♠

Solution 7. This is trivial. $db = d(b^2) = 2db$. If $2 = 0$, then the statement is automatically true. If not, then $db = 0$ by subtraction. ♡

3 Deformation Theory - Hartshorne

3.1 Chapter I.3 - The T^i functors

Exercise 8 (Exercise 3.1)**.** Let $B = k[x,y](xy)$. Show that $T^1(B/k, M) = M \otimes k$ and $T^2(B/k, M) = 0$ for any B-module M. ♠

Solution 8. Since B is defined by a principal ideal in $P = k[x, y]$, it follows that $L_2 = 0$ in the cotangent complex. Thus $T^2(B/k, M)$ is automatically zero.

We have that $L_1 = B$ and $L_0 = Bdx \oplus Bdy$ with d_1 being $f \mapsto (fy, fx)$. Applying $\mathrm{Hom}(-, M)$, we get $\mathrm{Hom}(L_0, M) = M \oplus M$ and $\mathrm{Hom}(L_1, M) = M$.

We have $\mathrm{Hom}(B \oplus B, M) \simeq M \oplus M$ by $\phi \mapsto (\phi(1, 0), \phi(0, 1)$. We have a diagram

$$\begin{tikzcd}
\mathrm{Hom}(B \oplus B, M) \arrow[r, "\psi^*"] \arrow[d, "\simeq"'] & \mathrm{Hom}(B, M) \arrow[d, "\simeq"] \\
M \oplus M \arrow[r] & M
\end{tikzcd}$$

Under these isomorphisms, it is easy to see that the bottom map is given by

$$(\phi(1, 0), \phi(0, 1)) \mapsto y\phi(1, 0) + x\phi(0, 1).$$

Thus since T^1 is the cokernel of this map, we must have $T^1(B/k, M) = M \otimes k$. ♡

Exercise 9 (Exercise 3.3)**.** Let $B = k[x, y]/(x^2, xy, y^2)$. Show that $T^0(B/k, B) = k^4$, $T^1(B/k, B) = k^4$ and $T^2(B/k, B) = k$. ♠

Solution 9. Let's compute L_2 first. For that we need part of a resolution of I. We have in fact

$$0 \to \mathrm{im} \begin{pmatrix} -y & 0 \\ x & -y \\ 0 & x \end{pmatrix} \to P(-2)^3 \to I \to 0.$$

The Koszul relations are given by

$$\mathrm{im} \begin{pmatrix} -y^2 & -xy & 0 \\ 0 & x^2 & -y^2 \\ x^2 & 0 & xy \end{pmatrix}.$$

Let's compute Q/F_0 (relations modulo Koszul relations). Since Q is generated in degree 3, and F_0 is of degree 4, we have $\dim_k(Q/F_0)_3 = 2$. Let's consider degree 4. As a k-vector space Q_4 is spanned by the four elements

$$\begin{pmatrix} -y^2 \\ xy \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -y^2 \\ xy \end{pmatrix}, \begin{pmatrix} -yx \\ x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -yx \\ x^2 \end{pmatrix}.$$

The two in the middle are already Koszul relations, so that $(Q/F_0)_4$ have dimension ≤ 2. But we also have

$$\begin{pmatrix} -y^2 \\ xy \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ yx \\ -x^2 \end{pmatrix} + \begin{pmatrix} -y^2 \\ 0 \\ x^2 \end{pmatrix}.$$

Thus $\dim_k (Q/F_0)_4 = 1$, since the second term above is a Koszul relation. Similarly we find that $\dim_k (Q/F_0)_5 = 0$. Hence, L_2 is the 3-dimensional k-vector space spanned by Q_3 and one more relation. L_1 is $F \otimes B = B^3$, and L_0 is $B \oplus B$, spanned by dx, dy.

Taking duals, we get that $L_2 = \operatorname{Hom}(Q/F_0, B)$. This set can be identified with

$$\begin{aligned} \operatorname{Hom}(Q/F_0, B) &= \{\varphi : Q \to B \mid \varphi\big|_{F_0} = 0\} \\ &= \{\varphi : Q \to P \mid \operatorname{im} f\big|_{F_0} \subseteq I\} \end{aligned}$$

Thus, since $I = \mathfrak{m}^2$, we must have that φ sends the two generators of Q to something of degree 1 (degree 0 is not ok, since then F_0 would be sent outside I). Thus $\operatorname{Hom}(Q/F_0, B)$ is $2 \times 2 = 4$-dimensional, spanned by

$$\operatorname{im} \begin{pmatrix} y & x & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}.$$

But d_2 is the dual of the inclusion $Q \to F$ from the exact sequence above. The dual is given by transposing, and we are left with one column - in conclusion, $T^2(B/k, B)$ is one-dimensional.

The Jacobian of I is given by

$$\begin{pmatrix} 2x & y & 0 \\ 0 & x & 2y \end{pmatrix},$$

and it is easily seen that the kernel of $\mathrm{Jac} \otimes B$ is given by $\mathfrak{m} \oplus \mathfrak{m} \oplus \mathfrak{m} \subset B^3$. The two relations kill off two dimensions, so $\dim_k T^1(B/k, B) = \dim_k \mathfrak{m}^{\oplus 3} - 2 = 6 - 2 = 4$.

Also $T^0(B/k, B)$ is B^2 modulo the image of the Jacobian. The constants are left untouched, so $\dim_k T^0(B/k, B) = 2 + 2 + 2 - 3 = 3$. A basis is given by $(1, 0), (0, 1)$ and (x, y). (thus Hartshorne is wrong?) ♡

4 Introduction to Commutative Algebra - Atiyah-MacDonald

4.1 Chapter 1 - Rings and ideals

Exercise 10. Let x be a nilpotent element of a ring A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit. ♠

Solution 10. Suppose $x^{n+1}=0$ and that $x^n \neq 0$. Consider

$$s = 1 - x + x^2 - x^3 + \ldots + x^n$$

Then

$$sx = x - x^2 + x^3 - x^4 + \ldots - x^n$$

since $x^{n+1}=0$. But then $s + sx = 1$, so that $s(1+x)=1$. Hence $1+x$ is a unit. To prove that the sum of any unit and any nilpotent is a unit, note that if u is any unit, then $u^{-1}x$ is still nilpotent. So since $u+x = u(1+u^{-1}x)$ and product of units are units, the claim follows. ♡

4.2 Chapter 2 - Modules

Exercise 11 (Excercise 1)**.** Show that $\mathbb{Z}/m \otimes_Z \mathbb{Z}/n = 0$ if m, n are coprime. ♠

Solution 11. Write $1 = am + bn$. Then

$$\begin{aligned} 1 \otimes 1 = (am+bn) \otimes 1 &= am \otimes 1 + bn \otimes 1 \\ &= 0 + bn \otimes 1 = 1 \otimes bn = 1 \otimes 0 = 0. \end{aligned}$$

And we are done. ♡

Exercise 12 (Exercise 2)**.** Let A be a ring, $\mathfrak{a}$ an ideal, and M an A-module. Then $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$. ♠

Solution 12. Start with

$$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0.$$

Tensoring with M gives

$$\mathfrak{a} \otimes M \to M \to A/\mathfrak{a} \otimes_A M \to 0.$$

But $\mathfrak{a} \otimes_A M \simeq \mathfrak{a}M$, so that the sequence reads $A/\mathfrak{a} \otimes M \simeq M/\mathfrak{a}M$. ♡

Exercise 13 (Exercise 3)**.** Let A be a local ring, M, N finitely generated A-modules. Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$. ♠

Solution 13. First a counterexample if A is not a local ring. Let $A = k[x]$ and $M = k[x]/(x-1)$ and $N = k[x]/(x)$. We can write $1 = -(x-1)+x$. Then $M \otimes_A N = 0$ by the same method as in Exercise 1 ($1 \otimes 1 = (-x+1+x) \otimes 1 = x \otimes 1 = 1 \otimes x = 0$).

Let $M_k := M \otimes k = M/\mathfrak{m}M$. By Nakayama's lemma, $M_k = 0 \Rightarrow M = 0$.

So suppose $M \otimes_A N = 0$. Then $(M \otimes_A N)_k = 0$. But this is isomorphic to $M_k \otimes_A N_k$ since $k \otimes_A k = k$. But $M_k \otimes_A N_k \simeq M_k \otimes_k N_k$, as k-modules, since everything in $\mathfrak{m}$ acts trivially on M_k. But these are vector spaces over a field, now we must have $M_k = 0$ or $N_k = 0$, and by Nakayama we are done. ♡

Exercise 14 (Exercise 24)**.** If M is an A-module, the following are equivalent:

i) M is flat.

ii) $\mathrm{Tor}_n^A(M, N) = 0$ for all $n > 0$ and A-modules N.

iii) $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N.

Solution 14. To compute $\mathrm{Tor}_A^n(M, N)$, one takes an A-resolution of N and tensor it with M and take homology. But M is flat, so the sequence stays exact, so the homology is zero. This shows $i) \Rightarrow ii)$.

The implication $ii) \Rightarrow iii)$ is trivial.

Now let

$$0 \to N' \to N \to N'' \to 0$$

be any exact sequence of A-modules. Then by properties of the Tor functor, we have an exact sequence

$$\mathrm{Tor}_1(M, N'') \to N' \otimes M \to N \otimes M \to N'' \otimes M \to 0.$$

But $\mathrm{Tor}_1(M, N'') = 0$, so the sequence is short exact. Hence M is flat. ♡

Exercise 15 (Exercise 25)**.** Let

$$0 \to N' \to N \to N'' \to 0$$

be an exact sequence with N'' flat. Then N' is flat if and only if N is flat. ♠

Solution 15. We have from the Tor exact sequence

$$0 \to \mathrm{Tor}_1(N', M) \to Tor_1(N, M) \to 0$$

since $\mathrm{Tor}_2(N'', M) = \mathrm{Tor}_1(N'', M) = 0$. The statement follows. ♡

♠

4.3 Chapter III - Rings and modules of fractions

Exercise 16 (Exercise 1)**.** Let S be a multiplicatively closed subset of a ring A, and let M be a finitely-generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$. $\spadesuit$

Solution 16. Suppose there exists such s. Let $m/s' \in S^{-1}M$. This is zero if and only if there exists $s \in M$ such that $s(s'm) = 0$. But $ss'm = s'sm = s'0 = 0$. So $m = 0$ in $S^{-1}M$. (note that we did not use finite generation)

Now let $m_1, \ldots, m_r$ be a set of generators for M and suppose that $S^{-1}M = 0$. Then for each i ($i = 1, \ldots, r$), there exists s_i such that $s_i m_i = 0$. Since every element of M is an A-linear combination of the m_i, it follows that the product $s_1 s_2 \cdots s_r$ makes $sM = 0$. $\heartsuit$