Goal: Describe $\mathfrak{g}_{\mathbb{C}}$ from Cartan matrix

1

we introduced huge $\mathfrak{a} = \langle \hat{H}_i, \hat{E}_i, \hat{F}_i \rangle$

w/ relations $[\hat{H}_i, \hat{E}_j] = a_{ij} \hat{E}_j$ . . . . . . $[\hat{E}_i, \hat{F}_j] = 0$ $i \neq j$

Had homomorphism $\pi : \mathfrak{a} \to \mathfrak{g}_{\mathbb{C}}$

$$\hat{\mathfrak{h}} = \text{span} \{\hat{H}_i\}_i$$

$$\mathfrak{a}_\lambda = \{ X \in \mathfrak{a} \mid [\hat{H}, \hat{X}] = \lambda(\hat{H}) \hat{X} \quad \forall H \in \hat{\mathfrak{h}} \}$$

$$\mathfrak{a}_0 = \hat{\mathfrak{h}}$$

$$\mathfrak{a} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{a}_\lambda .$$

$[\ldots, [\hat{E}_{i_1}, \hat{E}_{i_2}], \ldots \hat{E}_{i_k}]$ ← spanned by something like this

If $V \subset \mathfrak{a}$ is an $(\mathrm{ad}\, \hat{\mathfrak{h}})$-invariant subspace, then

$$V = \bigoplus_{\lambda \in \hat{\mathfrak{h}}^*} (V \cap \mathfrak{a}_\lambda) \qquad \mathfrak{a}_0 = \hat{\mathfrak{h}}$$

Hence the sum of all ideals $\mathfrak{p} \subset \mathfrak{a}$ s.t. $\mathfrak{p} \cap \hat{\mathfrak{h}} = 0$ is an ideal

$\underline{m}$ s.t. $\underline{m} \cap \hat{\mathfrak{h}} = 0$.

As $\pi : \mathfrak{a} \to \mathfrak{g}_{\mathbb{C}}$ gives iso $\hat{\mathfrak{h}} \simeq \mathfrak{h}$ we have that

$(\ker \pi) \cap \hat{\mathfrak{h}} = 0$. Hence $\ker \pi \subset \underline{m}$.

Consider the ideal $\mathfrak{p} = \frac{\mathfrak{m}}{\ker \pi}$ in $\mathfrak{g}_{\mathbb{C}}$. (2)

Then $\mathfrak{p} \cap \mathfrak{h} = 0$. On the other hand,

$$\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{h}) \oplus \bigoplus_{\alpha \in \Delta} (\mathfrak{p} \cap \mathfrak{g}_\alpha)$$

If $\mathfrak{p} \neq 0$, then $\mathfrak{p} \cap \mathfrak{g}_\alpha \neq 0$ for some $\alpha \in \Delta$, so $E_\alpha \in \mathfrak{p}$. But then $H_\alpha = [E_\alpha, F_\alpha] \in \mathfrak{p} \cap \mathfrak{h}$, which is a contradiction.

Hence $\ker \pi = \mathfrak{m}$.

Therefore $\mathfrak{g}_{\mathbb{C}} = \frac{\mathfrak{n}}{\mathfrak{m}}$ so $\mathfrak{g}_{\mathbb{C}}$ is described entirely in terms of the Cartan matrix.

Recall that $\mathfrak{g} = \{ X \in \mathfrak{g}_{\mathbb{C}} \mid X^* = -X \}$

The involution $*$ on $\mathfrak{g}$ is completely determined by the properties

$$[X,Y]^* = -[X^*, Y^*].$$

and on generators $H_i^* = H_i$, $E_i^* = F_i$, $F_i^* = E_i$.

Hence we get a complete description of $\mathfrak{g}$ in terms of the Cartan matrix.

Now, if G_1, G_2 are two simply-connected compact Lie groups w/ isomorphic root systems, then their Cartan matrices coincide (up to perm. of indices). Hence $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ and therefore $G_1 \simeq G_2$. ∎

For $i \neq j$, since $\alpha_i - \alpha_j \notin \Delta$,
the set $\{\alpha_i + k\alpha_j \mid k \in \mathbb{Z}\} \cap \Delta$
is $\{\alpha_i, \alpha_i + \alpha_j, \ldots, \alpha_i - \alpha_i(H_j)\alpha_j\}$

In particular $\alpha_i + (1 - \underbrace{\alpha_i(H_j)}_{a_{ji}})\alpha_j \notin \Delta$

Hence $(\mathrm{ad}\, E_j)^{1-a_{ji}}(E_i) = 0$ since it lies in $\mathfrak{g}_{\alpha_i + (1-a_{ji})\alpha_j} = 0$.

Therefore in $\mathfrak{g}$ we have the relations

$$(\mathrm{ad}\, E_j)^{1-a_{ji}}(E_i) = 0 \qquad i \neq j$$

and
$$(\mathrm{ad}\, F_j)^{1-a_{ji}}(F_i) = 0 \qquad i \neq j$$

These are called the Serre relations.

Fact: The Serre relations together w/ the relations used in the proof are all relations in $\mathfrak{g}_{\mathbb{C}}$.

As mentioned earlier, conversely, any reduced root system (V, Δ) w/ $\mathrm{Span}\, \Delta = V$, arises from a (simply-connected) compact semi-simple Lie group.

This can be done in different ways.

1. We define $\mathfrak{g}_{\mathbb{C}}$ and then $\mathfrak{g}$ is in the previous proof.

So $\mathfrak{g}_{\mathbb{C}} = \frac{\mathcal{F}}{\mathcal{M}}$. The main difficulty in this approach is to show that $\mathfrak{g}_{\mathbb{C}}$ is fin. dim.

We'll discuss a related problem later.

2) Use classification of reduced root systems

Then we just have to construct 5 exceptional Lie groups.

Reps theory of compact connected Lie groups

Let G be a compact connected Lie group. Fix a maximal torus $T \subset G$. Fix a system of simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset \Delta$. Then for any f. dim rep $\pi: G \to GL(V)$ we have a weight decomposition

$$V = \bigoplus_{\lambda \in P(V)} V(\lambda),$$

where $V(\lambda) = \{v \in V \mid Hv = \lambda(H)v \ \forall H \in \mathfrak{h}\} \neq 0$

and $P(V) \subset X^*(T) \subset i\mathfrak{t}^* \subset \mathfrak{h}^*$.

Yesterday

$E_\alpha V(\lambda) \subset V(\lambda+\alpha)$ $\quad F_\alpha V(\lambda) \subset V(\lambda - \alpha)$

A nonzero vector $v \in V(\lambda)$ is called a highest weight vector (of weight λ) if $E_\alpha v = 0$ $\forall \alpha \in \Delta_+$, equivalently, $E_i v = 0$ for $i = 1, \dots, r$

$E_i \stackrel{\Delta}{=} E_{\alpha_i}$

If in addition, v generates V as a $\mathfrak{g}_{\mathbb{C}}$-module, then V is called a <u>highest weight module</u>

(i.e. the smallest $\mathfrak{g}_{\mathbb{C}}$-invariant subspace of V containing v coincides w/ V)

From the relations $\mathfrak{g}_{\mathbb{C}}$ we see that V is spanned by the elements ~~$F_{i_1} \cdots F_{i_k} v$~~ $F_{i_1} \cdots F_{i_k} v$ $\quad i_1, \dots, i_k \in \{1, \dots, r\}$, $k \geq 0$

(In other words, the claim is that the span of all such vectors is an $\mathfrak{g}_{\mathbb{C}}$-invariant subspace of V. For example,

$$E_k F_i F_j v = (E_k F_i - F_i E_k) F_j + F_i E_k F_j v$$

$$= \underbrace{[E_k, F_i]}_{\delta_{ki} H_k} F_j + F_i \underbrace{[E_k, F_j]}_{\delta_{kj} H_k} v + F_i F_j \underbrace{E_k v}_{0}$$

From this we see that ~~the~~

① the highest weight λ is uniquely determined and $\dim V(\lambda) = 1$, so the highest weight vector is ! up to scalar.

② The weights in $P(V)$ have the form

$\lambda - \alpha_{i_1} - \dots - \alpha_{i_k}$ for $i_1, \dots, i_k \in \{1, \dots, r\}$ $k \geq 0$

Clearly, if V is an irreducible rep, then it is a highest weight module: the highest ~~wt~~ weight is the largest element in $P(V)$ w.r.t. the partial order

$$\mu \geq \eta \iff \eta = \mu - \alpha_{i_1} - \dots - \alpha_{i_k}$$

Thus any irreducible f.d. rep of G has a uniquely assigned weight $\lambda \in X^*(T)$.

Recalling the rep. theory of $sl_2(\mathbb{C})$, we also know that $\lambda(H_i) \geq 0$ for all $i = 1, \dots, r$.

A weight $\lambda \in X^*(T)$ s.t. $\lambda(H_i) \geq 0$ for all $i = 1, \dots, r$ is called dominant.

Thm For any compact conn. Lie group G w/ maximal torus T and a fixed system of simple roots, we have:

① For every dominant weight $\lambda \in X^*(T)$ $\exists!$ an irreducible fin.dim rep π_λ of G of highest weight λ

② Any f.dim. irrep. of G is equivalent to π_λ for a uniquely defined λ.

Pf (prob. the longest and most diff. proofs in the whole course) ⓐ

We will concentrate only on the simply connected semisimple case.

And simultaneously we will prove that $X^*(T) = P$.

(st. w/ proof today - finish next week)

Assume that G is such a group. In this case we know that $\mathrm{span}_{\mathbb{C}} \Delta = \mathfrak{h}^*$ and $Q \subset X^*(T) \subset P$.

The set of dominant weights in P, that is, the set

$\{ \lambda \in P \mid (\lambda, \alpha_i) \geq 0 \ \forall i \}$ is denoted by P_+.

The elements of P_+ are called <u>dominant integral weights</u>.

~~We know~~

<u>Note</u> $\boxed{P_+ = P \cap \overline{C}}$ <u>Weyl chamber</u>

Our goal is to show that for any $\lambda \in P_+$, there exist a unique f. dim. irrep highest weight $\mathfrak{g}_{\mathbb{C}}$-module of weight λ.

This will prove the theorem in the semisimple simply-conn. case, as well that $X^*(T) = P$.

Recall that if $\mathfrak{g}$ is a Lie alg over a field k, then the <u>universal enveloping alg</u> $U\mathfrak{g}$ is ~~the~~ a unital associative algebra together w/ a Lie algebra homomorphism

$\mathfrak{g} \to U\mathfrak{g}$ having the universal property

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[dr] & U\mathfrak{g} \arrow[d, dashed, "\exists!"] \\ & A \end{tikzcd}$$

Recall that the PBW-theorem says that if (for f.dim Lie algebras)

$X_1, \dots, X_n$ is a basis of $\mathfrak{g}$, then

$$X_1^{a_1} \cdots X_n^{a_n} \quad (a_1, \dots, a_n \geq 0)$$

form a basis in $U\mathfrak{g}$.

This implies:

- The map $\mathfrak{g} \to U\mathfrak{g}$ is injective, so we can consider $\mathfrak{g}$ as a subspace of $U\mathfrak{g}$.
- If $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are Lie subalgebras and

  $$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b} \text{ as a v. space, then}$$

  the homomorphisms $U\mathfrak{a} \to U\mathfrak{g}$, $U\mathfrak{b} \to U\mathfrak{g}$ induce a linear ~~homom~~ isom. $U\mathfrak{a} \otimes_k U\mathfrak{b} \xrightarrow{\sim} U\mathfrak{g}$.

Returning to semisimple simply-connected Lie groups we have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha .$$

Consider the Lie subalgebras in $\mathfrak{g}_{\mathbb{C}}$:

$$\mathfrak{n}_- = \bigoplus_{\alpha\in} \mathfrak{g}_\alpha \qquad \mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$$

Thus $U(\mathfrak{g}_{\mathbb{C}}) = U\mathfrak{n}_- \otimes_{\mathbb{C}} U\mathfrak{b}$ (as a v.space)

For every $\lambda \in \mathfrak{h}^*$ we have a 1-dim rep. ρ_λ:

$$\rho_\lambda : U\mathfrak{b} \longrightarrow \mathbb{C} \quad \text{determined by} \quad \rho_\lambda(H) = \lambda(H)$$
$$\rho_\lambda(\mathfrak{g}_\alpha) = 0$$

Let's denote this $U\mathfrak{b}$-module by $\mathbb{C}_\lambda$.

Define a $U\mathfrak{g}_{\mathbb{C}}$-module L_λ by <u>induction</u>: (not math induction)

$$L_\lambda = U\mathfrak{g}_{\mathbb{C}} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda$$

Note that $L_\lambda \simeq U\mathfrak{n}_-$ as a v.space.