Differential geometry, lecture 2 22/1 - 2015

Tangent vectors "=" tangents of curves

$\parallel$

eq. classes of $\alpha : (-\varepsilon, \varepsilon) \to M$

$\alpha(0) = p$ ← Nothing special about 0 here.

$\alpha'(0) = [\alpha] \in T_pM$

notation

$$\underbrace{(x^{-1} \circ \alpha)'(0) \in \mathbb{R}^n}_{\text{geometric definition}}$$

Another definition

Suppose given differentiable $f : M \to \mathbb{R}$

$(-\varepsilon, \varepsilon) \xrightarrow{\alpha} f$, U

$[\alpha]$

α defines a tangent $v \in T_pM$

Then we can define $df_p(v) \triangleq (f \circ \alpha)'(0)$

Formally, let $\mathcal{D}$ = all smooth functions f on M. Have map

$$\mathcal{D} \times T_pM \to \mathbb{R}$$
$$(f, v) \longmapsto d_pf(v)$$

Fix v, define

$$l_v : \mathcal{O}_p \to \mathbb{R}$$
$$l_v(f) = df_p(v).$$

Easily seen to have the following properties:

- l_v $\mathbb{R}$-linear
- Leibniz rule $\quad l_v(fg) = f(p)\, l_v(g) + g(p)\, l_v(f)$

so l_v is a derivation

Let $\mathcal{L}_p = \{\text{all derivations on } \mathcal{O}_p\}$ (looking at germs of functions at p)

$\mathbb{R}$- vector space

So we have in fact defined a map $T_pM \xrightarrow{l} \mathcal{L}_p$

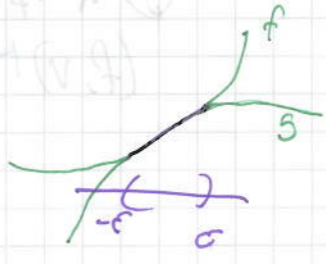
$$v \longmapsto l_v$$

Prop The map $l : T_pM \xrightarrow{l} \mathcal{L}_p$ is an isomorphism

(defined via the pairing $\mathcal{O} \times T_pM \to \mathbb{R}$)

Pf Define $\tilde{\mathcal{F}}_p = \{ f : U \to \mathbb{R} \mid \underset{\text{open}}{U \ni p} \} / \sim$ $\quad f \sim g$ if $f|_W = g|_W$ for some smaller W

clearly there is a map $\mathcal{O}_p \xrightarrow{\mathcal{L}} \tilde{\mathcal{F}}_p$



Can find bump functions.

Let $\sigma : M \to \mathbb{R}^{[0,1]}$ be such that $\sigma \equiv 1$ near p and zero outside the set of equality for f, g.

Then $\sigma \cdot f \equiv \sigma \cdot g$.

<u>Claim</u> If $f \sim g \Rightarrow l_v f = l_v g$.

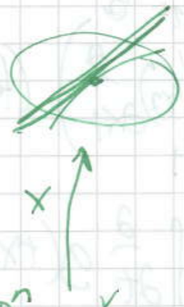
$$l_v(f) = l_v(\sigma f) = \sigma(p)\, l_v(f) + f(p)\, l_v(\sigma)$$
$$= 1 \cdot l_v(f) + f(p) \cdot 0$$
$$= l_v(f)$$

Hence the function $l_v : \mathcal{D}_p \to \mathbb{R}$ factors through $\mathcal{F}_p \to \mathbb{R}$.

$$\begin{tikzcd} \mathcal{D}_p \arrow[r, two heads] \arrow[dr, "l_v"'] & \mathcal{F}_p \arrow[d, "l"] \\ & \mathbb{R} \end{tikzcd}$$

So why is l injective?

$$T_pM \longrightarrow \mathcal{L}_p$$



Any curve can be represented by a curve of the form $x(tv)$.

$\mathbb{R}^n \xrightarrow{v}$ So assume $v \neq 0$. Let $f : \mathcal{F}_p \to \mathbb{R}$ be the function $f(q) = \pi_L \circ x^{-1}$

projection to L

But then $l_v(f) = \left(\pi_L \circ x^{-1}(x(tv))\right)'(0)$
$= 1$.

(surjective after the break)

Need a lemma to prove surjectivity.

Lemma $l \in \mathcal{L}_p$

1) $l(\text{constant}) = 0$

2) $f(p) = g(p) = 0 \Rightarrow l(fg) = 0$

Example $p \in M$

$$\begin{array}{c} M \\ \uparrow x \\ \mathbb{R}^n \end{array}$$

$$x^{-1} \circ \alpha(t) = (x_1(t), \dots, x_n(t))$$

$$l_v(f) = (f \circ \alpha)'(0) = (f x) \circ (x^{-1} \alpha)(t)$$

$$= (f x)'(x_1(t), \dots, x_n(t))$$

$$= \sum \frac{\partial x_i}{\partial t}\Big|_0 \cdot \frac{\partial (f \circ x)}{d x_i}\Big|_0$$

$$= \left(\sum x_i'(0) \frac{\partial}{\partial x_i} \right) (f \circ x)$$

Need $\underbrace{f \circ x}_{g}(x_1, \dots, x_n) - g(0, \dots, 0) = \int_0^1 \frac{\partial}{\partial t} g(t x_1, \dots, t x_n)\, dt$

$$= \int_0^1 \sum \frac{\partial g}{\partial x_i} x_i \, dt = \sum x_i \underbrace{\int_0^1 \frac{\partial g}{\partial x_i}(t x \dots)\, dt}_{h_i(x_1 \dots, x_n)}$$

$$= \sum x_i \left(h_i(0) + \sum x_j h_{ij}(x_1, \dots, x_n) \right)$$

$$= \sum x_i \underbrace{h_i(0)}_{\frac{\partial g}{\partial x_i}} + \sum_{ij} x_i x_j h_{ij} (\dots)$$

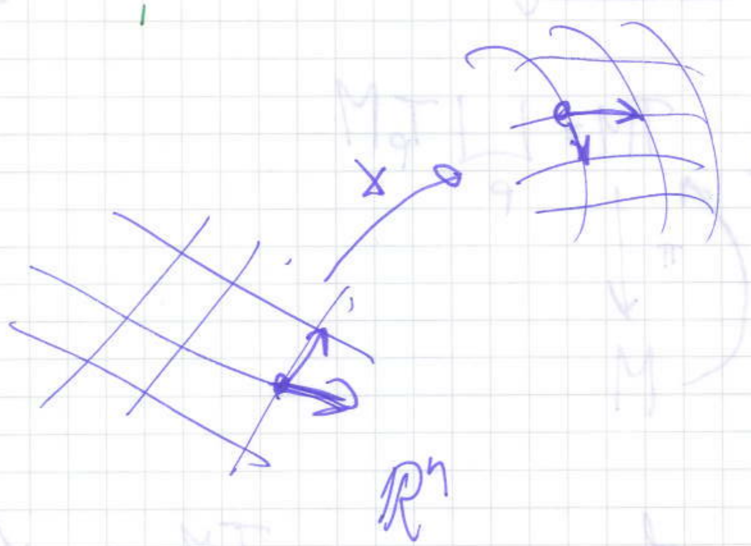
So $g(x_1,\dots,x_n) = g(0) + \sum_i x_i \frac{\partial g}{\partial x_i}(0) + \sum_{ij} x_i x_j k_{ij}$

Can rewrite to get

$$f(q) = f(p) + \sum_i x_i(q) \frac{\partial (f\circ x)}{\partial x_i}(p) + \sum_{ij} x_i(q) x_j(q) k_{ij}(q)$$

$$\text{So } \quad l(f) = 0 + \sum_i l(x_i) \frac{\partial f}{\partial x_i}\Big|_p + \underset{\text{o.c.}}{\overset{\text{constant}}{0}}$$

$$= \sum_i l(x_i) \frac{\partial f}{\partial x_i}\Big|_p = l_v(f) \qquad \blacksquare$$



(a little about the notation $\frac{\partial}{\partial x_i}$)

Vectorfields X

is a rule $p \mapsto X_p \in T_pM$. (or just a section of $TM \xrightarrow{p} M$:-)

of course $X_p = \sum \eta_i(p) \frac{\partial}{\partial x_i}(p)$

we can say that X is smooth if all $\eta_i(p)$ are smooth.

There is a better way: to introduce the tangent bundle

$(p, v) \mapsto v_p$

$U \times \mathbb{R}^n \cong TU \subseteq$ Take $TM = \bigsqcup_p T_pM$

$$\begin{tikzcd} TU \arrow[d] & TM \arrow[d, "\pi"] \\ U \arrow[r, phantom, "\subseteq"] & M \arrow[u, bend left, "X"] \end{tikzcd}$$

Want smooth structure on TM so that we can define smoothness of X.