Manifolds notes

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1 Manifolds

A manifold is "something" that locally looks like $\mathbb{R}^n$ for some natural number n.

Definition 1.1. A manifold is a topological space M such that around each point $p \in M$ there is a neighbourhood U that is homeomorphic to $\mathbb{R}^n$ for some $n \geq 1$. ■

2 Differentiable maps

A differentiable map between two manifolds is what it ought to be. We first define differentiable maps $f : M \to \mathbb{R}$: A map $f : M \to \mathbb{R}$ is *differentiable* if for any charxt $x : M \to \mathbb{R}^n$, the map $f \circ x^{-1} : \mathbb{R}^n \to \mathbb{R}$ is differentiable[1].

Since a function $g : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable if and only if each component $g_i : \mathbb{R}^n \to \mathbb{R}$ is differentiable, our definition of differentiable maps includes these. So the natural definition of a smooth map $f : M \to N$ between to manifolds is that it is smooth on each chart:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "x"'] & N \arrow[d, "y"] \\ \mathbb{R}^n \arrow[r, "yfx^{-1}"'] & \mathbb{R}^m \end{tikzcd}$$

We let the category Diff be the category with objects smooth manifolds and morphisms smooth maps.

[1] Replace the word *differentiable* by the word *smooth* to define *smooth maps*.

3 The tangent bundle

We first define the tangent space of $\mathbb{R}^n$: It is written $T\mathbb{R}^n$ and is $\mathbb{R}^n \times \mathbb{R}^n$. Elements $v_p := (p, v)$ are thought of as v *a tangent vector at* p. The projection map onto the first factor will be denoted by $\pi : T\mathbb{R}^n \to \mathbb{R}^n$. If $p \in \mathbb{R}^n$ is a point, then the fibre $\pi^{-1}(p)$ is denoted $\mathbb{R}^n_p$.

For any differentiable map $f : \mathbb{R}^n \to \mathbb{R}^m$ and $p \in \mathbb{R}^n$, the linear transformation $Df(p) : \mathbb{R}^n \to \mathbb{R}^m$ defines a linear map $f_{*p} : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$:

$$\begin{aligned} f_{*p} : \mathbb{R}^n_p &\to \mathbb{R}^m_{f(p)} \\ v_p &\mapsto \Big[Df(p)(v)\Big]_{f(p)} \end{aligned}$$

This map is defined for all p, so we have a commutative diagram:

$$\begin{tikzcd} T\mathbb{R}^n \arrow[r, "f_*"] \arrow[d, "\pi"'] & T\mathbb{R}^m \arrow[d, "\pi"] \\ \mathbb{R}^n \arrow[r, "f"] & \mathbb{R}^m \end{tikzcd}$$

From the chain rule it follows that $g_* \circ f_* = (g \circ f)_*$. Thus:

Proposition 3.1. *T is "functorial". (in a sense made precise in a moment)*

If we have a curve $c : \mathbb{R} \to \mathbb{R}^n$, then its *tangent vector* at a point t may be defined as
$$c'(t)_{c(t)} \in \mathbb{R}^n_{c(t)}.$$

We want a good way to define tangent spaces to a general manifold M. The naïve way is to embed them into some $\mathbb{R}^N$ and define TM to be the disjoint union

$$T(M, i) = \bigcup_{p \in M} (M, i)_p \subset i(M) \times \mathbb{R}^n \subset T\mathbb{R}^n,$$

where $(M, i)_p = i(M) \times \{p\} \subset T\mathbb{R}^n$. This works, but leads to an awkvard dependence on the choice of embedding.

Example 1. If we embed S^1 into $\mathbb{R}^2$ in the usual way: by the set of points p with $|p| = 1$, then the tangent plane to a point p is the set of points

$$\{q_p \,|\, q \cdot p = 0\} \subset i(M) \times \mathbb{R}^2.$$

Having chosen one non-zero q_p, all other vectors in the tangent plane are scalar multiples of it, so we can define a homeomorphism $f_1 : T(S^1, i) \to S^1 \times \mathbb{R}$ by $f_1(\lambda u_p) = (p, \lambda)$ such that the following diagram commutes:

$$\begin{tikzcd} T(S^1,i) \arrow[rr, "f_1"] \arrow[dr, "\pi"'] & & S^1 \times \mathbb{R}^1 \arrow[dl, "\pi'"] \\ & S^1 & \end{tikzcd}$$

This says that the tangent bundle to S^1 is (isomorphic to) the trivial bundle over S^1. One can show that the tangent bundle of S^2 is not isomorphic to the trivial bundle: this is a consequence of the Hairy Ball Theorem. Ξ

But to talk about bundles, we need to define them.

Definition 3.2. An *n-dimensional vector bundle* is a five-tuple

$$\xi = (E, \pi, B, \oplus, \odot),$$

where

1. E and B are spaces, the *total space* and the *base space* of ξ, respectively.
2. $\pi : E \to B$ is a surjective continuous map.
3. $\oplus$ and $\odot$ are maps

$$\oplus : \bigcup_{p\in B} \pi^{-1}(p) \times \pi^{-1}(p) \to E \text{ and } \odot : \mathbb{R} \times E \to E$$

with $\oplus(\pi^{-1}(p) \times \pi^{-1}(p)) \subset \pi^{-1}(p)$ and $\odot(\mathbb{R} \times \pi^{-1}(p)) \subset \pi^{-1}(p)$, making each fibre $\pi^{-1}(p)$ into an n-dimensional vector space over $\mathbb{R}$.

In addition, we have the *local triviality condition*: for each $p \in B$, there is a neighbourhood $U \ni p$ and a homeomorphism $t : \pi^{-1}(U) \to U \times \mathbb{R}^n$ which is a vector space isomorphism from each $\pi^{-1}(q)$ onto $q \times \mathbb{R}^n$ for all $q \in U$. ■

Notice the similarity between vector bundles and sheaves. Any vector bundle on a manifold M can be made into a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is the sheafification of $U \mapsto \{u : U \to \mathbb{R} |\, \text{u smooth}\}$. We do this by letting $\mathcal{F}$ be the sheaf of sections $u : B \to E$.

The simples example of an n-bundle is the *trivial* bundle: it is just $X \times \mathbb{R}^n$ with the projection $\pi : X \times \mathbb{R}^n \to X$. We denote the trivial bundle by $\epsilon^n(X)$.

We say that two vector bundles ξ_1, ξ_2 over the same base space are *equivalent* if there is a commutative diagram and a homeomorphism h:

$$\begin{tikzcd} E_1 \arrow[rr, "h", "\simeq"'] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & B & \end{tikzcd}$$

We call h an *equivalence.* A bundle equivalent to $\epsilon^n(B)$ is *trivial.* The example above showed that $T(S^1, i)$ was trivial.

A *bundle map* from ξ_1 to ξ_2 is a commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd} ,$$

such that $\tilde{f} : \pi_1^{-1}(p) \to \pi_2^{-1}(f(p))$ is a linear map.

We can define the category of vector bundles $\mathsf{VBundles}$: objects are pairs (E_1, B_1) as above and morphisms are bundle maps.

Now, our previous definition of the tangent bundle depended upon an embedding $M \to \mathbb{R}^N$. We don't want this - we want the tangent bundle to be only dependent upon M. This is possible:

Theorem 3.3. *There is a functor $T : \mathsf{Diff} \to \mathsf{VBundles}$ such that*

1. *There are functorial equivalences $t^n : T(\mathbb{R}^N) \to \epsilon^n(\mathbb{R}^n)$, meaning that the following diagram commutes:*

$$\begin{tikzcd} T\mathbb{R}^n \arrow[r, "f_*"] \arrow[d, "t^n"'] & T\mathbb{R}^m \arrow[d, "t^m"] \\ \mathbb{R}^n \arrow[r, "f_*"] & \mathbb{R}^m \end{tikzcd} .$$

   Here the upper f_* is $T(f)$ and the lower f_* is the differential of $f : \mathbb{R}^n \to \mathbb{R}^m$.

2. *T is local, meaning that if we know T on open subsets of M, then we can glue.*

Proof. Idea: One constructs an equivalence class of vector bundles by noting how the tangent space should transform under coordinate change: If

$x, y : U \to \mathbb{R}^n$ are two coordinate systems on M, then $yx^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism between Euclidean spaces. We know how the tangent map should look on Euclidean spaces: $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \ni (x(p), v) \mapsto (yx^{-1}(p), D(yx^{-1})(yx^{-1}(x(p)))(v)) = (yx^{-1}(p), D(yx^{-1})(y(p))(v))$, for p some point in the manifold.

Now, if x and y are coordinate systems, both containing the point p and $v, w \in \mathbb{R}^n$, then we define an equivalence relation by $(x, v) \sim (y, w)$ if $w = D(yx^{-1})(x(p))(v)$. This gives an equivalence class for each $p \in M$ and we define TM to be the union of all these equivalence classes. We have an obvious projection map $\pi : TM \to M$. This provides a bijective map

$$t_x : \pi^{-1}(U) \to U \times \mathbb{R}^n, \text{ by } [x, v]_q \mapsto (q, v).$$

This can be made into a homeomorphism. There is an obvious vector space structure that can be defined on each fibre.

There are lots of things to be checked. □

We are now truly in position to talk about the *tangent bundle* $\pi : TM \to M$.

There are other more intrinsic definitions of tangent vectors. One approach is to consider curves: Let $\epsilon > 0$: one looks at maps $c : (-\epsilon, \epsilon) \to M$ such that $c(0) = p$. We want to think of the derivative of c as tangent vectors.

We say that two curves c_1 and c_2 are equivalent if they have the same tangent vector at 0: $c_1 \sim c_2$ if $(xc_1)'(0) = (xc_2)'(0)$ for some chart U. One needs of course to show that this is independent of choice of chart.

Definition 3.4 (Alternative definition)**.** T'_pM is curves $c : (\epsilon, \epsilon) \to M/\sim$. We have a map $T'_pM \to T_pM$ by $[c]_p \mapsto [x, (xc)'(0)]_p$. ■

One checks that the above map is bijective.

If $f : M \to N$ is a map then we have a pushforward map f_{*p} by composition.

Definition 3.5 (Yet another alternative definition)**.**

$$T''_pM = \{\ell : C^\infty(M) \to \mathbb{R} | \ \ell \text{ is a derivation at } p\},$$

that is, functions satisfying the Leibniz rule: $\ell(fg)(p) = f(p)\ell(g) + g(p)\ell(f)$. ■

We have a map $T_pM \to T_p''$ given by $[x,a]_p \mapsto \ell = \sum a^i \frac{\partial}{\partial x^i}\Big|_p$.

Recall that if $x : U \to \mathbb{R}^n$ is a chart and $g : M \to \mathbb{R}$, then

$$(\frac{\partial}{\partial x_i}\Big|_p)(g) = D_i(gx^{-1})(x(p)).$$

Notice that if we define the tangent space to be the set of point-derivations, then its definitions does not mention any charts at all - so in some sense this is the most intrinsic definition of tangent vectors.

What happens with derivations with maps $f : M \to N$? That is, what should the pushforward map be? Answer: $f_*(\ell)(g) = \ell(gf)$.

We have:

Theorem 3.6. *The set of all linear derivations at $p \in M$ is an n-dimensional real vector space. In fact, if (x, U) is any chart at p, then*

$$\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p$$

span this vector space, and any derivation ℓ can be written

$$\ell = \sum_{i=1}^{n} \ell(x^i) \cdot \frac{\partial}{\partial x^i}\Big|_p$$

3.1 Vector fields

A *vector field* is a section of TM. They are often denoted by capital letters such as X, Y and Z. The vector $X(p) \in M_p$ is often denoted X_p. Thinking of TM as the set of derivations, we have

$$X(p) = \sum_{i=1}^{n} a^i(p) \frac{\partial}{\partial x^i}\Big|_p$$

for some continuous functions $a^i(p)$.

If X and Y are vector fields, then they can be added:

$$(X + Y)(p) = X(p) + Y(p).$$

Similarly, if $f : M \to \mathbb{R}$, we can define the vector field fX by

$$fX(p) = f(p)X(p).$$

If $f : M \to \mathbb{R}$ is a function and X is a vector field, then we can define a new function $\bar{X}(f) : M \to \mathbb{R}$ by letting X operate on f at each point:

$$\bar{X}(f)(p) = X_p(f).$$

4 Tensors

Theorem 4.1. *If (x, U) is a coordinate system and f is a C^∞ function, then on U we have*

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i.$$

Example 2 (Riemannian metric)**.** The Riemannian metric is an example of a section of $\mathcal{T}^2(TM)$. If $i : M \to \mathbb{R}^N$ is an embedding, then $i_{*p} : T_pM \to T_{i(p)}\mathbb{R}^N \cong \mathbb{R}^N$ for each $p \in M$.

This embedding lets us define the *norm* of a tangent vector $v \in T_pM$ as the Euclidean length $\|i_*(v)\|$. More generally, we can define an *inner product*

$$\begin{array}{ccc} T_pM \times T_pM & \xrightarrow{\langle\,,\,\rangle_p^i} & \mathbb{R} \\ (v, w) & \longmapsto & \langle v, w\rangle \stackrel{\Delta}{=} i_*(v) \cdot i_*(w) \end{array}$$

as a bilinear form on T_pM, i.e. an element

$$\langle\,,\,\rangle_p^i \in \mathcal{T}^2(T_pM) = \mathcal{T}^2(TM)_p.$$

Letting p vary we get a section

$$\langle\,,\,\rangle^i : M \to \mathcal{T}^2(TM),$$

that is, a tensor field of order 2.

This section is an example of a *Riemannian metric* on M.

Given a Riemannian metric $\langle\,,\,\rangle$ on M, we can talk about the *length of a tangent vector* $\|x_p\| = \sqrt{\langle x_p, x_p\rangle_p}$ and thus the length of a curve $\lambda : [a, b] \to M$:

$$\|\lambda\| = \int_a^b \|\lambda'(t)\|_{\lambda(t)}\ \mathrm{dt}.$$

Further, this lets us define the distance between to points $p, q \in M$ as the infimum of the lengths of curves starting at p and ending at q. Ξ

4.1 Coordinates

Let $M \supset U : x \to \mathbb{R}^n$ be a chart. Then

$$\left\{dx^1(p), \cdots, dx^n(p)\right\}$$

is a basis of $T^*M_p = (T_pM)^*$. This implies that the n^k elements

$$\left\{dx^{i_1} \otimes \cdots \otimes dx^{i_k}\right\}_{i_1 \cdots i_k = 1}^{n}$$

is a basis for $\mathcal{T}^k(T_pM) = \mathcal{T}^k(TM)_p$. This means that any covariant tensor field A (read: any section of $\mathcal{T}^k(T_pM)$) of order k can be written as

$$A(p) = \sum_{i_1 \cdots i_k}^{n} A_{i_1 \cdots i_k}(p)\, dx^{i_1}(p) \otimes \cdots \otimes dx^{i_k}(p)$$

for $p \in U$, where $A_{i_1 \cdots i_k}(p) \in \mathbb{R}$. Removing the p, we write

$$A = \sum_{i_1 \cdots i_k}^{n} A_{i_1 \cdots i_k}\, dx^{i_1} \otimes \cdots \otimes dx^{i_k}.$$

We see that A is C^∞ if and only if each $A_{i_1 \cdots i_k}$ is C^∞.

4.2 Invariant description of tensor fields

One way of giving a tensor field is to give it in coordinates and check that it agrees on coordinate overlaps - however, this breaks with the philosophy of today: we want all our descriptions to be independent of any choices made.

Let $A : M \to \mathcal{T}^k(TM)$ be a covariant tensor field of order k. Given k vector fields $X_1, \cdots, X_k$ we can evaluate A on $X_1, \cdots, X_k$ to get a function on M:

$$p \longmapsto A(p)\big(X_1(p), \cdots, X_k(p)\big) \in \mathbb{R}$$

Let $\mathcal{V}$ be the set of vector fields on M (sections of TM) and let $\mathcal{F}$ be the set of C^∞ functions $M \to \mathbb{R}$. Letting p and the vector fields above vary, we get a function $\bar{A}$ assigning to k vector fields an element of $\mathcal{F}$:

$$\bar{A} : \mathcal{V} \times \cdots \times \mathcal{V} \to \mathcal{F}$$

The theorem is the following:

Theorem 4.2. *The rule $A \mapsto \bar{A}$ gives a $1-1$ correspondence between covariant tensor fields of order k and k-multilinear maps*

$$\mathcal{A} : \mathcal{V} \times \cdots \mathcal{V} \to \mathcal{F}$$

that are $\mathcal{F}$-linear. That is, given a map as above, there is a unique tensor field A with $\mathcal{A} = \bar{A}$.

Proof. One direction is easy: it is just the rule $A \mapsto \bar{A}$. We need to construct a tensor field A from $\mathcal{A}$.

If $v \in M_p$ is a tangent vector, then there is a vector field $X \in \mathcal{V}$ with $X(p) = v$: In fact, if (x, U) is a coordinate system and

$$v = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}\Big|_p,$$

then we can just let

$$X = \begin{cases} f \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i} & \text{on } U \\ 0 & \text{outside } U \end{cases}$$

where f is a bump function with $f(p) = 1$ and $\operatorname{supp} f \subset U$.

This way we can extends k tangent vectors $v_1, \cdots, v_k \in M_p$ to vector fields $X_1, \cdots, X_k \in \mathcal{V}$. Since we demand that $\mathcal{A} = \bar{A}$, we are forced to define

$$A(p)(v_1, \cdots, v_k) = \mathcal{A}(X_1, \cdots, X_k)(p).$$

The problem is to check that this is well-defined: If $X_i(p) = Y_i(p)$ for each i, we claim that

$$\mathcal{A}(X_1, \cdots, X_k)(p) = \mathcal{A}(Y_1, \cdots, Y_k)(p)$$

for each p. To avoid cumbersome notation, we only prove the case $k = 1$. So we need to prove that $\mathcal{A}(X)(p) = \mathcal{A}(Y)(p)$ when $X(p) = Y(p)$.

(1) Suppose first that $X = Y$ in a neighbourhood U of p. Let f be a C^∞ bump function with $f(p) = 1$ and $\operatorname{supp} f \subset U$. Then $fX = fY$, so

$$f\mathcal{A}(X) = \mathcal{A}(fX) = \mathcal{A}(fY) = f\mathcal{A}(Y),$$

where in first and third equality we used that $\mathcal{A}$ was $\mathcal{F}$-linear. Evaluating the left-most and right-most expressions at p gives

$$\mathcal{A}(X)(p) = \mathcal{A}(Y)(p).$$

(2) To prove the result, it suffices to show that $\mathcal{A}(X)(p) = 0$ if $\times(p) = 0$. Let (x, U) be a coordinate system around p, so that on U we can write

$$X = \sum_{i=1}^{n} b^i \frac{\partial}{\partial dx^i} \quad \text{where all } b^i(p) = 0.$$

If g is 1 in a neighbourhood V of p, and $\operatorname{supp} g \subset U$, then

$$Y = g \sum_{i=1}^{n} b^i \frac{\partial}{\partial x^i} = \sum_{i=1}^{n} b^i g \frac{\partial}{\partial x^i}$$

is a well-defined C^∞ vector field on all of M which equals X on V, so that

$$\mathcal{A}(X)(p) = \mathcal{A}(Y)(p), \quad \text{by (1)}.$$

Now

$$\begin{aligned} \mathcal{A}(Y)(p) &= \sum_{i=1}^{n} b^i(p) \cdot \mathcal{A}\big(g \frac{\partial}{\partial x^i}\big)(p) \\ &= 0, \quad \text{since } b^i(p) = 0. \end{aligned}$$

This proves that A is well-defined. That A is smooth can be seen by seeing that $A_{i_1 \cdots i_k} = \mathcal{A}(\partial/\partial x_{i_1}, \cdots, \partial/\partial x_{i_k})$. □

If $f : M \to N$ is a C^∞ map of manifolds, then we get a map f^* taking covariant tensor fields A of order k on N to covariant tensor fields f^*A of order k on M:

$$\begin{aligned} A \in \mathcal{T}^k(N) &\xrightarrow{\quad f^* \quad} \mathcal{T}^k(M) \\ A(p)(X_{1_p}, \cdots, X_{k_p}) &\longmapsto A(f(p))(f_* X_{1_p}, \cdots, f_* X_{k_p}). \end{aligned}$$

5 Vector fields and differential equations

Let $X : M \to TM$ be a vector field. Then one can ask: is there a curve $\rho : (-\epsilon, \epsilon) \to M$ with $\rho(0) = p \in M$ such that

$$\rho'(t) = X_{\rho(t)} \quad \forall t \in (-\epsilon, \epsilon)?$$

This is a local question, so we can assume that $M = \mathbb{R}^n$. Then a vector field X on $\mathbb{R}^n$ is a just a smooth function $f : V \to \mathbb{R}^n$, where V is some open neighbourhood, which we assume contains zero. Then the above equation just reads:

$$\rho'(t) = f(\rho(t)).$$

This is just an ordinary differential equation of order one. In the beginning of this section we will study the existence and uniqueness of solutions of these in $\mathbb{R}^n$.

Two examples will illuminate the kind of pathologies (or problems, if you like) that can arise:

Example 3. Set $n = 1$ and $f(y) = -y^2$. We seek a curve $c(t)$ with $c(0) = x \in \mathbb{R}$. Here f corresponds to the vector field X that assigns to every number y a vector pointing backward of length y^2. We get the equation

$$-\frac{1}{c^2}\frac{dc}{dt} = 1.$$

Integrating both sides:

$$\begin{aligned} \int -\frac{1}{c^2}\frac{dc}{dt}dt &= \int 1dt \\ \int -\frac{1}{c^2}dc &= \int 1dt \\ \frac{1}{c} &= t + C, \end{aligned}$$

yielding either $c(t) = 1/(c + C)$ as a solution, for some constant C, or $c(t) = 0$ for all t, the latter alternative occuring if $c(0) = 0$. But if $c(0) \neq 0$, the solution is not defined for all t! It does not extend outside $(-\frac{1}{x}, \frac{1}{x})$ in both directions. This is one of the problems that can occur. Ξ

Example 4. Again $n = 1$. Now let $f(y) = y^{2/3}$. Set $c(0) = 0$. Then we get the equation

$$\frac{dc}{dt} = c^{2/3}, \quad c(0) = 0.$$

But there are two solutions! The curve is not unique! Ξ

There is a theorem, however:

Theorem 5.1. *Let $V \subset \mathbb{R}^n$ be open and $f : V \to \mathbb{R}^n$. Let $x_0 \in V$ and $a > 0$ be such that $\bar{B_{2a}}(x_0) \subset V$. If there are K, L such that*

1. *$|f(x)| \leq L$ on $\overline{B_{2a}(x_0)}$, i.e. f is L-bounded.*

2. *$|f(x) - f(y)| \leq K|x - y|$ on $\overline{B_{2a}(x_0)}$, i.e. f is K-Lipschitz.*

then choose $b > 0$ such that

3. *$b \leq 1/L$ and*

4. *$b \leq 1/K$.*

Then for each $x \in \overline{B_a(x_0)}$, there exists a unique curve $\alpha_x : (-b, b) \to U$ such that

$$\alpha_x'(t) = f(\alpha_x(t)) \text{ and } \alpha_x(0) = x.$$

Sketch of proof. Topologize the set of maps $Y = \{\alpha : (-b,b) \to \overline{B_{2a}(x_0)}\}$ by the sup metric. Then M is a complete metric space. One defines an operator $S : Y \to Y$ by

$$(S\alpha)(t) = x + \int_0^t f(\alpha(u))du.$$

Then one sees that if S has a fixed point, then it is a solution to our differential equation. One shows that S is a contraction, and this implies that there exists a unique fixed point solution by the contraction lemma of analysis.

Then there is some checking to do, and the proof is complete. $\square$

Write $\alpha_x(t)$ as $\alpha(t,x)$ to get a map

$$\begin{aligned} \alpha : (-b,b) \times \overline{B_a(x_0)} &\longrightarrow V \\ (t,x) &\longmapsto \alpha(t,x) = \alpha_x(t) \end{aligned}$$

satisfying $\alpha(0,x) = x$ and $\frac{d}{dt}\alpha(t,x) = f(\alpha(t,x))$. This map α is called the **local flow** for f in $(-b,b) \times B_a(x_0)$.

Suppose $y = \alpha_x(t_0)$ for some $t_0 \in (-b,b)$ (think of this as starting at x and following the flow for a time t_0). Then the reparametrized integral curve

$$t \longmapsto \beta(t) := \alpha_x(t + t_0)$$

satisfies $\beta'(t) = f(\alpha(t_0 + t)) = f(\beta(t))$, with $\beta(0) = \alpha_x(t_0) = y$. This means that β satisfies the conditions that uniquely determine α_y, so $\beta(t) = \alpha_y(t)$ for t near 0 on

$$(-b,b) \cap (-b - t_0, b - t_0).$$

This means that $\alpha_x(t + t_0) = \alpha_y(t)$ for t near zero. Thus:

Proposition 5.2. *For each $t \in (-b,b)$ we get a map*

$$\begin{aligned} \phi_t : B_a(x_0) &\longrightarrow V \\ x &\longmapsto \alpha(t,x) \end{aligned}$$

Such that $\phi_0(x) = x$ and $\phi_{s+t}(x) = \phi_s(\phi_t(x))$ for $s, t, s+t \in (-b,b)$. In particular, $\phi_{-t} = \phi_t^{-1}$, so each ϕ_t is a bijection.

Theorem 5.3. *The flow*

$$\alpha : (-b,b) \times B_a(x_0) \longrightarrow V$$

is cont's. Hence each ϕ_t is also cont's.

Sketch of proof. Let S denote the operator used in the previous theorem. Using a geometric series tricks, one proves that

$$\sup_t |\alpha(t,x) - \alpha(t,y)| = \|\alpha_x - \alpha_y\| \leq \frac{1}{1-bK}|x-y|,$$

where α_x is a solution of the differential equation starting at x and α_y a solution starting at y. This inequality implies continuity of α. $\square$

Proposition 5.4 (Spivak cites Lang)**.** *If $f : V \to \mathbb{R}^n$ is C^∞, then the flow is also C^∞. Hence each ϕ_t is smooth.*

Proof. "Introduction to Differentiable Manifolds" by Serge Lang. $\square$