Assume I is a finite group $H \subset I$ an abelian subgroup and $\pi \in \hat{H}$ s.t. $\pi^g = \pi(g^{-1} \cdot g) = \pi$ for all $g \in I$. and s.t. π extends to a homomorphism $\hat{\pi}: I \to \mathbb{C}^\times$

Want to describe all irreps of I s.t. $\mathrm{Res}^I_H \sigma$ is isotypic to π.

Assume σ is such a representation. Then consider

$\tilde{\sigma} = \hat{\pi}^{-1} \otimes \sigma$ so $\tilde{\sigma}$ is the representation on the same space given by $\tilde{\sigma}(g) = \hat{\pi}(g)^{-1} \sigma(g)$.

Then $\tilde{\sigma}$ is irreducible and $H \subset \ker \tilde{\sigma}$. So $\tilde{\sigma}$ is an irrep of I/H.

Conversely, if $\tilde{\sigma}$ is a irrep of I/H, then we can define σ by $\sigma = \hat{\pi} \otimes \tilde{\sigma}$.

We have almost proved:

Thm Let G be a finite group and $H \subset G$ a normal abelian subgroup. Assume that every character $\chi \in \hat{H}$ extends to a homomorphism $\hat{\chi}: I(\chi) \longrightarrow \mathbb{C}^\times$, where $I(\chi)$ is $\{g \in G \,/\, \chi^g = \chi\}$.

Then the irreps of G can be described as follows.

For every G-orbit in $\hat{H}$ fix a representative χ. Then for every $[\sigma] \in \left(I(\chi)/H\right)^\wedge$ define $\mathrm{Ind}^G_{I(\chi)}(\hat{\chi} \otimes \sigma)$

Therefore, as a set $\hat{G}$ can be identified ②

with $\displaystyle\bigsqcup_{[\chi]\in \hat{H}/G} \left(I(\chi)/H\right)^{\wedge}$.

$\ni \theta$

G-orbits

Pf We have already shown that any irrep of G has the form $\mathrm{Ind}_{I(\chi)}^{G}(\tilde{\chi}\otimes\sigma)$ w/ (χ,σ) as above. Namely, $\mathrm{Res}_{H}^{G}\theta$ defines a G-orbit in $\hat{H}$. Once a representative χ of this orbit is fixed, then $\sigma = \left(\mathrm{Res}_{I(\chi)}^{G}\theta\right)\Big|_{V_\theta(\chi)} \otimes \tilde{\chi}^{-1}$.

where $V_\theta(\chi)$ is the isotypic component of $\mathrm{Res}_{H}^{G}\theta$ corresponding to χ.

Remains to show that

① For any (χ,σ) as in formulation, $\mathrm{Ind}_{I(\chi)}^{G}(\hat{\chi}\otimes\sigma)$ is irreducible.

② When we apply the procedure of getting (χ,σ) to $\theta = \mathrm{Ind}_{I(\chi)}^{G}(\tilde{\chi}\otimes\sigma)$ then we recover (χ,σ).

Start w/ ① By Mackey's irreducibility criterion, $\mathrm{Ind}_{I(\chi)}^{G}(\tilde{\chi}\otimes\sigma)$ is irrep iff $\mathrm{Res}_{I(\chi)_g}^{I(\chi)}(\tilde{\chi}\otimes\sigma)$ is disjoint from $(\tilde{\chi}\otimes\sigma)^g$

where $I(\chi)_g = g\, I(\chi)\, g^{-1} \cap I(\chi)$, for all $g \notin I(\chi)$. 3

This is true, since already $\mathrm{Res}^{I(\chi)}_H (\hat{\chi} \otimes \sigma)$ and $\mathrm{Res}^{I(\chi)}_H (\hat{\chi} \otimes \sigma)^g$ are disjoint, since the first rep is isotypic to χ and the second to $\chi^g \neq \chi$.

(2) Consider $\theta = \mathrm{Ind}_{I(\chi)}^G (\hat{\chi} \otimes \sigma)$. Apply the procedure described in the beginning of the proof to θ. As $\mathrm{Res}^G_{H} \theta$ contains χ, we recover the orbit of χ.

Then we get $\theta \sim \mathrm{Ind}_{I(\chi)}^G (\hat{\chi} \otimes \sigma')$ where $\sigma' = (\mathrm{Res}^G_{I(\chi)} \theta) \otimes \hat{\chi}^{-1}$

By def, we see that σ is contained in σ'. As σ, σ' are irreps, they must be equal. $\blacksquare$

Example Consider a finite field $\mathbb{F}_p$, p prime. Consider the corresponding $ax+b$ group. $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle/ \begin{matrix} a \in \mathbb{F}_p^\times \\ b \in \mathbb{F}_p \end{matrix} \right\}$

Take $H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}_{b \in \mathbb{F}_p} \simeq \mathbb{Z}/p\mathbb{Z}$.

We have $\hat{H} \simeq \mathbb{Z}/p$ consists of char's $\chi_0, \dots, \chi_{p-1}$ where $\chi_k(b) = e^{\frac{2\pi i}{p} kb}$. We have $\chi^{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}} = \chi(a^{-1} \cdot)$

The stabilizer of χ_0 is G. The stabilizers of (4)

χ_k $k=1,\dots,p-1$ are trivial in $\widehat{G_0/H}$, and G acts transitively

on $\{\chi_1,\dots,\chi_{p-1}\}$.

The assumptions of the theorem are satisfied, as

χ_0 extends to G as $\hat{\chi}_0(g)=1$.

Consider the representatives of the two G-orbits on $\hat{H}$:

$\underline{\chi_0}$

$I(\chi_0)=G$ We have $G/H \simeq \mathbb{F}_p^\times$

This leads to $|\mathbb{F}_p^\times| = p-1$ one-dim reps.

namely $G \ni \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \longmapsto \chi(a)$, $\chi \in \widehat{\mathbb{F}_p^\times} = \mathbb{Z}/_{p-1}$

$\underline{\chi_1}$

$I(\chi_1)=H$ Then we get one irreducible $(p-1)$-dimensional

representation $\mathrm{Ind}_H^G \chi_1$.

$\boxed{\text{m}}$

Exercise

1. Check that the same holds for $\mathbb{F}_{p^n}$.

2. Let χ be a non-trivial character of the additive group k.

Then show that $\mathrm{Ind}_H^G \chi$ can be described as follows

Consider the space $V = \{ f : k \to \mathbb{C} \mid \sum f(x) = 0 \}$

and the permutation rep. π of $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \begin{array}{l} a \in k^* \\ b \in k \end{array} \right\}$

on V.

$$(\pi(g) f)(x) = f(g^{-1} x)$$

("I recommend to do this exercise")

Theorem Let G be a finite group, $H \subset G$ a normal abelian subgrp. and π an irrep of G.

Then $\dim \pi \mid \frac{|G|}{|H|}$.

Pf Note that we proved this for $H = Z(G)$. For general H, we know that $\pi \sim \mathrm{Ind}_I^G \sigma$ where $I \supset H$ and $\mathrm{Res}_H^I \sigma$ is isotypic to $\chi \in \hat{H}$.

Consider the groups $\tilde{I} = \sigma(I)$ and $\tilde{H} = \sigma(H)$
$= \{ \chi(h) \cdot \mathrm{id} \mid h \in H \}$.

Then $\tilde{H} \subseteq Z(\tilde{I})$ and the representation of $\tilde{I}$ on V_σ is irreducible (as σ is).

Hence $\sigma \,\Big|\, \left|\tilde{I}/\tilde{H}\right| = \left[I/H\ker\sigma\right]$ 6

so $\dim \sigma \,\Big|\, \left[I/H\right]$.

Hence $\dim \pi = \left[G/I\right] \dim \sigma \,\Big|\, \left[I/H\right]\left[G/I\right]$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \left[G/H\right]$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mid$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad G$ ■

"generally done w/ finite grps"

Part II Rep theory of compact groups

There are several problems in developing rep theory for infinite groups.

- fin. dim reps are not completely reducible

$\underline{\underline{Ex}}$ $\mathbb{Z} \to GL_n(\mathbb{C})$

$\qquad n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

- the class of fin. dim reps is too small

$\underline{\underline{Ex}}$ There are no nontrivial cont's fin. dim reps of $SL_2(\mathbb{Q}_p)$

There are discrete countable groups w.o. nontrivial fin. dim reps

Rep. theory mostly studies unitary rep's on Hilbert spaces.

The theory closest to fin. groups. is that of compact groups.

<u>Def</u> <u>A topological grp G</u> is a group G w/ a topology s.t. $G \times G \xrightarrow{\mu} G$ and $\underset{g \mapsto g^{-1}}{G \longrightarrow G}$ are cont's.

<u>Key fact</u> (Haar, von Neumann, A. Weil)

For any locally compact group G there exists a nonzero left-invariant Radon measure on G (! up to scalar).

The <u>Haar measure</u>

- Recall that a Radon measure μ on a locally compact space is a Borel measure which is outer regular, inner regular on open sets and $\mu(K) < \infty$ for any compact K.
- Left-invariance means that the measure is invariant under left-multiplication.
- Recall also that to be given a Radon measure on a locally compact space X is equivalent to be given a linear functional

$$F : \underbrace{C_c(X)}_{\text{compactly supp. cont's functions}} \longrightarrow \mathbb{C} \quad \text{s.t.} \quad F(f) \geq 0 \quad \forall f \geq 0.$$

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Simplest example

$G = \mathbb{R}$

Then the Haar measure is the usual Lebesgue measure.

There are also right-invariant measures. In general, they are not left-invariant.

If they are G is called <u>unimodular</u>.

<u>Exercise</u> Find left- and right- invariant Haar measures on $GL_2(\mathbb{R})$, $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \begin{array}{l} a > 0 \\ b \in \mathbb{R} \end{array} \right\}$

Check that $GL_2(\mathbb{R})$ is unimodular but G is not.

<u>Hint</u> $A = (a_{ij})$ $\quad f(A)\, da_{11}\, da_{12}\, da_{21}\, d_{22}$

$$\int F(A)\, f(A)\, da_{11} \cdots d_{22}$$

so find F s.t.