III. Repr. th. of compact Lie groups

Def A Lie group is a grp G which is also a smooth manifold s.t. $G \times G \xrightarrow{\mu} G$ and $G \xrightarrow{i} G$ are smooth maps. (smooth = C^∞)

Ex (i) Let V ~~be a fin dim~~ be a fin dim. real v-space. The $GL(V)$ is a Lie group.

Smooth structure inherited from $End(V)$.

(ii) $(\mathbb{R}^n, +)$, $(\mathbb{C}^n, +)$

First develop some general results on Lie groups. Idea of these results: „linearize the group".

Exponential map

Let M be a smooth manifold, $p \in M$. The tangent space T_pM can be equivalently defined as

- Equivalence classes of smooth curves passing through p

$$\gamma_i : (a_i, b_i) \to M, \quad a_i < 0 < b_i \quad \gamma_i(0) = p$$

$$\gamma_1 \sim \gamma_2 \iff (x \circ \gamma_1)'(0) = (x \circ \gamma_2)'(0) \text{ for}$$

some/any coordinate system (x, U) around p

We denote by $\gamma'(0)$ the equiv. class of γ. (2)

— More algebraic definition: space of derivations of $C^\infty(M)$ at p.

A derivation at p is a linear map s.t.

$$L : C^\infty(M) \longrightarrow \mathbb{R}$$
$$L(fg) = f(p)L(g) + L(f)g(p)$$

— The relation between the two def's is that the derivation L defined by $\gamma'(0)$ is $L(f) = \frac{d}{dt}(f\circ\gamma)\Big|_{t=0}$

In local coord's x any tangent vector can be written as

$$\sum \alpha^i(p) \frac{\partial}{\partial x^i}\Big|_p$$

— Recall A (smooth) v. field is a smooth section of the tangent bundle.

— ~~An integral curve~~ The space of v. fields can be identified w/ the space of derivations of $C^\infty(M)$, that is maps $L : C^\infty(M) \to C^\infty(M)$ w/ Leibniz.

— Given a v.field X on M, an integral curve of X is a smooth map curve $\gamma : (a,b) \to M$ s.t. $\gamma'(t) = X_{\gamma(t)}$ $\forall\ t \in (a,b)$.

For any $p \in M$, there exists an integral curve γ passing through p. And if γ_1, γ_2 are two integral curves with $\gamma_1(t_0) = \gamma_2(t_0)$ for some t_0, then $\gamma_1 \equiv \gamma_2$ on the intersection of their sources.

Hence for any $p \in M$ $\exists$ a maximal integral curve through p.

If $f: M \to N$ is a smooth map, then the differential of f at $p \in M$ is the linear map $T_pM \xrightarrow{d_pf} T_{f(p)}N$ defined by

$$\gamma'(0) \longmapsto (f \circ \gamma)'(0)$$

Let now G be a Lie group. Define diffeomorphism

$$l_g : G \longrightarrow G$$
$$h \longmapsto gh$$

Then l_g induces a map $(l_g)_*$ on v.fields by

$$(l_g)_*(X)_p = \left(d_{g^{-1}(p)} l_g\right)\left(X_{g^{-1}(p)}\right)$$

Def A v.field X is called <u>left-invariant</u> if $(l_g)_*(X) = X$ $\forall g \in G$.

Such a v.field is determined by its value at $e \in G$: $X_g = (d_e l_g)(X_e)$

This way we get a 1-1 correspondence between left-invariant v.fields and $v \in T_eG$.

Almost tautological exercise: Show that X is left-inv. iff the corresponding derivation L_X commutes w/ the maps

$$l_g^* : C^\infty(G) \to C^\infty(G)$$
$$f \longmapsto f(g^{-1}G)$$

Prop Let G be any group, $X \in T_e G$, and

$$\alpha_X : (a,b) \to G$$

be the maximal integral curve of the corresp. left-inv. v.field s.t. $\alpha_X(0) = e$.

Then $(a,b) = \mathbb{R}$ and $\alpha_X(s+t) = \alpha_X(s)\alpha_X(t)$ $\forall s,t \in \mathbb{R}$

Pf Fix $s \in (a,b)$. Consider two curves: $\gamma_1 : (a,b) \to G$

$$\gamma_1(t) = \alpha_X(s)\,\alpha_X(t)$$

$$\gamma_2 : (a-s, b-s) \to G$$
$$\gamma_2(t) = \alpha_X(s+t)$$

Then $\gamma_1(0) = \alpha_X(s) = \gamma_2(0)$. Both curves are integral curves of our v.field. Clear for γ_2, while for γ_1 we have

$$\gamma_1'(t) = \left(l_{\alpha_X(s)} \circ \alpha_X\right)'(t) = d_{\alpha_X(t)} l_{\alpha_X(s)}\left(\alpha_X'(t)\right)$$

$$= \left(d_{\alpha_X(t)} l_{\alpha_X(s)}\right)\left(\tilde{X}_{\alpha_X(t)}\right) = \tilde{X}_{\alpha_X(s)\alpha_X(t)} = \tilde{X}_{\gamma_1(t)}$$

Hence $\gamma_1 = \gamma_2$ on their domain of def.

As γ_2 is maximal it follows that $(a,b) \subset (a-s, b+s)$.

Only possible for $(a,b) = \mathbb{R}$. $\blacksquare$

Def A 1-param subgrp of G is a smooth map

$$\gamma : \mathbb{R} \to G$$

that is a group homomorphism $(\mathbb{R}, +) \to (G, \mu)$.

Corollary The map $X \mapsto \alpha_X$ is a bijection between T_eG and one-parameter subgroups of G.

Pf Assume $\gamma : \mathbb{R} \to G$ is a one-param subgrp. Put $X = \gamma'(0) \in T_eG$. Then $\gamma = \alpha_X$.

$$\gamma'(t) = \left(d_e \gamma(t)\right)\left(\gamma'(0)\right) \quad \text{as} \quad \gamma(t+s) = \gamma(t)\gamma(s)$$

$\blacksquare$

Remark In general, $\gamma(\mathbb{R})$ is not a closed subgrp of G.

Example $\mathbb{R} \to \mathbb{T} \times \mathbb{T}$ $\quad t \mapsto (e^{it}, e^{i\alpha t})$

is dense in $\mathbb{T} \times \mathbb{T}$)

Def The exponential map is the map

$$\exp : T_eG \longrightarrow G$$
$$X \longmapsto \alpha_X(1)$$

Note that for any $s \in \mathbb{R}$ $\quad \frac{d}{dt}\alpha_X(st) = s\,\alpha_X'(st)$

$$= s\,\tilde{X}_{\alpha_X(st)}$$

Thus $t \mapsto \alpha_X(st)$ is the integral curve of v. field corresponding to sX, hence $\alpha_X(st) = \alpha_{sX}(t)$.

Therefore $\exp(tX) = \alpha_{tX}(1) = \alpha_X(t)$.

Therefore all 1-param subgrps of G have the form

$$t \mapsto \exp(tX)$$

We have $\left.\frac{d}{dt}\exp(tX)\right|_{t=0} = X$

In other words, if we identify $T_0(T_eG) \leftrightarrow T_eG$, we conclude that $d_0\exp = \mathrm{id}_{T_eG}$.

Note that the map $\exp: T_eG \to G$ is smooth, as the map

$$T_eG \times \mathbb{R} \to G \qquad (X, t) \mapsto \alpha_X(t)$$

(by smooth dependence of solutions of ODE on initial cond's and param's)

Thus we have proved the following theorem:

Thm The exp. map is the ! smooth map $\exp: T_eG \to G$ s.t. $\exp((s+t)X) = \exp(sX)\exp(tX)$ and

$$\left.\frac{d}{dt}\left(\exp(tX)\right)\right|_{t=0} = X.$$

As $d_e \exp = \mathrm{id}_e$, the map $\exp$ defines a diffeomorphism between a neighborhood of $0 \in T_eG$ and a neighb. hood of $e \in G$.

In genl. $\exp$ is neither inj. nor surjective, even for connected groups.

Example $G = GL(V)$

As $GL(V) \subset \mathrm{End}(V)$ is open, the tangent space at every $M \in GL(V)$ is identified w/ $\mathrm{End}(V)$ $\forall M \in GL(V)$.

Then $\exp : \mathrm{End}(V) \to GL(V)$ is given by

$$X \longmapsto e^X = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

Def A homomorphism of Lie groups is a homomorphism $\pi : G \to H$ which is also smooth.

Thm Assume $\pi : G \to H$ is a Lie grp homomorphism. Then the diagram

$$\begin{tikzcd} G \arrow[r, "\pi"] & H \\ T_eG \arrow[u, "\exp"] \arrow[r, "d_e\pi"'] & T_eH \arrow[u, "\exp"'] \end{tikzcd}$$

commutes.

Pf

Example $G = GL(V)$, consider the map

$$\det : GL(V) \to K^\times = GL(k)$$

Then $d_e \det = \text{Trace}$

Therefore the theorem gives us that

$$\det(e^A) = e^{\operatorname{Tr} A} \quad \forall A \in \operatorname{End}(V)$$

Still some time to start w/ main construction.

The Lie algebra of a Lie group

Recall that for v.fields on a manifold M we have a commutator, or Lie bracket.

If X, Y are v.fields, and L_X, L_Y are the corresp. derivations, then the bracket is defined as

$$L_{[X,Y]} = L_X L_Y - L_Y L_X .$$

In local coords, if $X = \sum f^i \frac{\partial}{\partial x^i}$ and $Y = \sum g^i \frac{\partial}{\partial x^i}$

then
$$[X,Y] = \sum_{i,j} \left(f^i \frac{\partial g^j}{\partial x^i} - g_i \frac{\partial f^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

Def A *Lie algebra* over a field k is a v.space $\mathfrak{g}$ over k equipped w/ a bilinear map $[\cdot,\cdot]$ called the Lie bracket s.t. it is skew symmetric, and satisfies the Jacobi identity.

$$[[A,B],C] + [[C,A],B] + [[B,C],A] = 0$$

v.fields on G form a Lie algebra.