12/2-14

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- Introduction
- Proj. space $\mathbb{P}^N(x) = \{(x_0 : \cdots : x_N)\}$
- Algebraic variety $X = \{(x) \mid F_1(x) = \ldots = F_a(x) = 0\}$
- Case: X = smooth, irreducible, $\dim_{\mathbb{C}} X = n$
  "algebraic manifold"

• Divisors on X 1) $S \subset X$ $\dim S = n-1$

2) $D = \sum n_i S_i$ $n_i \in \mathbb{Z}$

$\Leftrightarrow$ $X = \bigcup_{i \in I} U_i$ locally, a divisor can be written

$$D|_{U_i} = (f_{|U}) - (g|_U)$$

So to give D, need give $\{D|_{U_i}, U_i\} = D$.

Say $D_1 \sim D_2$ if $D_1 - D_2 = (\varphi) = (\varphi)_0 - (\varphi)_\infty$. With $\varphi \in \Gamma(K, X)$

Canonical divs on X:

Write $X = \cup U_i$ opens. Set $U = U_i$. Let $u_1, \ldots, u_n$ be local holomorphic coord's. Let $\omega_u = du_1 \wedge \ldots \wedge du_n$.

Let $U \xrightarrow{g_{uv}} V$ be a coord change. Then

$$\omega_v = \varphi(v_1 \ldots v_n)\, dv_1 \wedge \ldots \wedge dv_n.$$

Let $(\varphi_u) = \{(\varphi_u)_0 - (\varphi)_\infty\}_{U_i}$ $\to$ get divisor!

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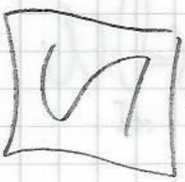
Get divisor K. (independent of data)

Divisor D = ample = ?
By def X alg. manifold of $X = \{(x) \mid F_1 = \dots = F_k = 0\}$

But not unique!

If for some divisor $D = \sum n_i S_i$ on X $\exists$ embedding $X \subseteq \mathbb{P}^N$
w/
$X \cap H \sim D$
then D = very ample!

Example



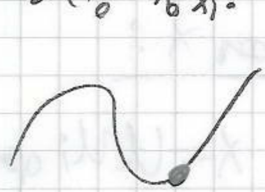
$C_d \subseteq \mathbb{P}^2$
$\deg C \geq 3$

$x \in C$ ample,
but not very ample.

But for some m,

$\mathbb{P}^2 \ni (x_0 : \dots : x_2) \to (x_0^m : x_0^{m-1} x_1 : \dots : x_2^m)$

$C \longrightarrow$



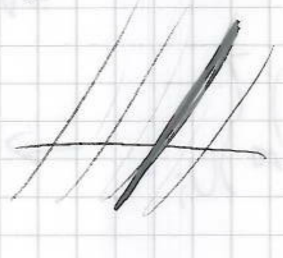
$m \cdot x$

$H \cap C = m \cdot x$

Ex $X = \mathbb{P}^1 \times \mathbb{P}^1$

$D = \mathbb{P}^1 \times \{pt\}$

not ample



Def X is a Fano n fold if $-K_X$ is ample. 3

"Projective spaces are the spaces w/ most negative canonical class"

Ex On $\mathbb{P}^1$ then $K_{\mathbb{P}^1} \sim -2\{pt\}$

b.c. $\mathbb{P}^1 = U_0 \cup U_1$

$u = \frac{x_1}{x_0}$ $v = \frac{x_0}{x_1}$ $u = \frac{1}{v} = g_{uv}$

Then $\omega_u = du \Rightarrow \omega_v = d\left(\frac{1}{v}\right) = -\frac{1}{v^2}\, dv.$

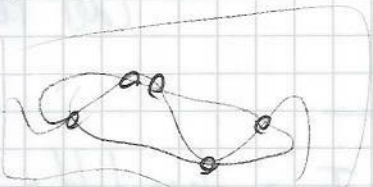
$\Rightarrow$ ~~div~~ $(\omega) = -2 \cdot (x_0 = 0)$

Similarly: On $\mathbb{P}^n$: $K_{\mathbb{P}^n} = -(n+1)H$ H hyperplane

If X = hypersurface $= (F_d = 0) \subseteq \mathbb{P}^n$, then

$$K_{X_d} = (d-n-1)H$$

Ex $X = X_5 \subset \mathbb{P}^2 \Rightarrow K_X = (5-3)H = 2H = X \cap (\text{conic})$



Ex $X = X_2 = (F_2 = 0) \subset \mathbb{P}^2$

$K_X = (2-3)H = -H = -2$ points $\Rightarrow$ conic = Fano 1-fold

But $(x_0 : x_1 : x_2) \mapsto (x_0^2 : x_0 x_1 : x_1^2) = (y_1 : y_2 : y_3)$ (9)

$\mathbb{P}^1 \simeq C$ via $\mathbb{P}^1 \longrightarrow$ Conic $(y_1 y_3 - y_2^2 = 0)$.

$\boxed{\text{If } \dim X = 1, \text{ then } X \text{ is Fano} \iff X \simeq \mathbb{P}^1.}$

Classification of smooth Fano surfaces

1) $\mathbb{P}^2 \quad K_{\mathbb{P}^2} = -3H$

2) $\mathbb{P}^1 \times \mathbb{P}^1 \simeq$ Quadric $Q_2 \subset \mathbb{P}^3$

$$(x_0 : x_1), (y_0 : y_1) \xrightarrow{\sim} (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1)$$

$$z_0 z_3 = z_1 z_2.$$

(blowup) 3) $\mathbb{P}^2_{(x_0 : \dots : x_2)} \longrightarrow (x_0^2 : x_0 x_1 \cdots : x_2^2) \subseteq \mathbb{P}^5$

S + project from pt.

Geometry If one blows up $d = 1, 2, 3, \dots, 8$ points on $\mathbb{P}^2$, then one obtains a alg. surface S_{9-d} $\quad d = 1, \dots, 8$

$\rightarrow$ also Fano surfaces (del Pezzo)

All smooth Fano threefolds were classified in the 80's classical Fano 3-fold

Def^n If $X \subset \mathbb{P}^N$ and $-K_X = H|_X$ is a particular Fano 3-fold

such that $X \cap H_1 \cap H_2$ = canonical curve

$C = C_{2g-2} \subset \mathbb{P}^{g-1}$

$\deg C = 2g-2$

(all div. on $X \sim mH$ $m \in \mathbb{Z}$), so not $\mathbb{P}^1 \times \mathbb{P}^2$)

The complete classification including cases like $\mathbb{P}^2 \times \mathbb{P}^1$, ..., obtained by Mori, Mukai 1983

$$\begin{array}{r} 104 \\ +\ 1 \\ \hline 105 \end{array}$$

Among those 17 types:

classical Fano 3-folds. ($-K_X = rH$, r integer)

- Classification of the 17 Fano 3-folds X w/ $-K_X = rH$, H ample. and any divisor of X is multiple of H. $r > 0$

Geometry If $\dim X = n$, and $-K_X = (n+1)H$ then $X \cong \mathbb{P}^n$!

If $-K_X = nH$, then X is quadric $Q_2 \subset \mathbb{P}^{n+1}$.

Therefore if $\dim X = 3$, $-K_X = rH$

1) $r=4$ $X = \mathbb{P}^3$

2) $r=3$ $X = Q_2$ (quadric) $K_X = -4H$

Relatively easy: $\boxed{r=2}$ $K_X = -2H$. Then $X \cap H = S$ (Fano surface) $K_S = -H$

(these are $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$, S_d $\quad 1\le d\le 8$)

$$\to \exists\ \underbrace{X_1, \dots, X_5}$$

$X_3 \subset \mathbb{P}^4$ cubic threefold

$X_4 = X_{2,2} \subseteq \mathbb{P}^5$ int 2 quadrics

$X_2 \xrightarrow[2:1]{\pi} \mathbb{P}^3$ branched at a quartic

$\Leftrightarrow X_2 = (F_4=0) \subset \mathbb{P}^4(1,1,1,1,2)$

$X_1 = (F_6=0) \subseteq \mathbb{P}^4(1,1,1,2,3)$

X_5 = skew symm. 5×5 matrices

$$\begin{bmatrix} 0 & x_{12} & x_{13} & \cdots & x_{15} \\ -x_{12} & 0 & & & x_{25} \\ & & 0 & & \vdots \\ \vdots & & & 0 & x_{45} \\ & & & & 0 \end{bmatrix} \subseteq \mathbb{P}^9$$

$$G(2,5) = \{X \mid \operatorname{rank} X = 2\}$$

Thm $\quad X_5 = G(2,5) \cap \mathbb{P}^6$

For $K_X = -H$. <u>Result</u> $\quad X\cap H_1 \cap H_2$ is canonical curve

if $d = \deg X = 2g-2 \ge 4$.

Proj geometry $\Rightarrow$ $d=2$ $\quad X_2 = (F_6=0) \subseteq \mathbb{P}^4(1,1,1,1,3)$

$\Leftrightarrow$ $\quad X_2 \xrightarrow{2:1} \mathbb{P}^3 \supset$ branch locus sextic surface

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$d=4$ $(g=3)$, then $X = X_4 \subset \mathbb{P}^4$ ~~quartic~~ quartic.

$\deg X = 6$ $(g=4)$ then $X = X_{2,3} \subset \mathbb{P}^5$

$= (F_2 = 0) \cap (F_3 = 0)$

$\deg X = 8$ $(g=5)$ $X = X_{2,2,2} \subset \mathbb{P}^6$

intersection 3 quadrics.

$\deg X = 10$ $(g=6)$ $X_{10} = G(2,5) \cap \mathbb{P}^7 \cap$ quadric

X_{14} $d=14$ $(\Rightarrow g=8)$

$G(2,6) = \{\text{skew } 6\times 6 \text{ matrices}\} \subseteq \mathbb{P}^{14}$ & dim 8

$\boxed{G(2,6) \cap \mathbb{P}^9 = X_{14}}$ section of $G(2,6)$

$\boxed{?\ g \geq 9}$ $\boxed{\text{Fano}}$ used "double" projection.

Recall if index $r = 2$ $(K_X = -2H)$, then $\exists$ V_d $1 \leq d \leq 5$

For which g, is there a Fano 3-fold w/ $X_d = X_{2g-2}$

If $X = X_{2g-2}$ then on X there is a line $L \subset X \subset \mathbb{P}^{g+1}$

$\Rightarrow g \leq 12$ via geometry

(nice hair....)

something about rationality.

He said it was from Wikipedia.

unirational $\exists f$ $\mathbb{P}^n \dashrightarrow X$ gen. finite

X rational if $\deg f = 1$

dim 1 Lüroth rat = unirational

dim 2 Castelnuovo —//— (very hard 2 prove)

dim 3 counterexamples (Artin, Mumford constructs unirational w/ Tors $H_3(X, \mathbb{Z}) \neq 0$ and this is B invariant.)