Semisimple compact connected Lie group G ①

$\Leftrightarrow$
- $Z(G)$ is finite
- $\mathfrak{z}(\mathfrak{g}) = 0$
- $\mathrm{span}_{\mathbb{C}} \Delta = \mathfrak{h}^*$ $\quad (\mathrm{span}_{\mathbb{R}} \Delta = i\mathfrak{t}^*)$

Thm $\tilde{G}$ is compact, equivalently, $\pi_1(G)$ is finite.

(Reason $\quad Q \subset X^*(T) \subset P = \left\{ \beta \in i\mathfrak{t}^* \;\middle|\; \underbrace{\frac{2(\alpha,\beta)}{(\alpha,\alpha)}}_{=\beta(H_\alpha)} \in \mathbb{Z} \ \forall \alpha \in \Delta \right\}$

The argument shows that if $X^*(T) = P$, then $\pi_1(G) = 0$.)

(We show that the converse holds tomorrow)

- This can be used to check that $SU(n)$ and $Sp(n)$ are simply connected)

If G is not semisimple, then $G/Z(G)$ is semisimple w/ Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. We showed that $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}]$ (check!) as a Lie algebra.

Prop Any compact connected Lie group G can be written as $\mathbb{T}^n \times H/\Gamma$ where H is a compact ~~semis~~ simply connected semisimple Lie group and Γ is a finite subgroup of $\mathbb{T}^n \times Z(H)$.

Pf $G/Z(G)$ is semisimple w/ Lie algebra

$$\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \cong [\mathfrak{g},\mathfrak{g}].$$

Consider the universal cover H of $G/Z(G)$

Then H is compact semisimple. The identification of the Lie algebra of H w/ $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}$ defines a homomorphism

$$H \xrightarrow{\pi} G.$$

Then we get a homomorphism

$$Z(G)^0 \times H \xrightarrow{\pi'} G$$
$$(g,h) \longmapsto g\,\pi(h)$$

At the Lie algebra level this is an isomorphism. Hence π' is surjective and $\Gamma = \ker \pi'$ is finite (b.c. $\ker \pi' = 0$ on Lie alg. level)

Classification of simply connected semisimple compact Lie groups

Let G be a compact connected Lie group. Fix a maximal torus $T \subset G$.

Consider a f.dim rep $\pi : G \to GL(V)$. (Recall: V is cplx)

We can decompose $\pi|_T$ into isotypic components.

Thus, for $\lambda \in X^*(T)$, define

$$V(\lambda) = \{ v \in V \mid \underset{\pi_*(H)v}{\underset{\shortparallel}{Hv}} = \lambda(H)v \quad \forall H \in \underset{\mathfrak{t}_{\mathbb{C}}}{\underset{\shortparallel}{\mathfrak{h}}} \}$$

The elements of $V(\lambda)$ are called <u>vectors of weight λ.</u> 3

Put $P(V) = \{\lambda \in \mathfrak{h}^* \mid V(\lambda) \neq 0\} \subset X^*(T)$

Then $V = \bigoplus_{\lambda \in P(V)} V(\lambda)$

If $X \in \mathfrak{g}_\alpha$, then $X V(\lambda) \subset V(\lambda + \alpha)$

Indeed, for $H \in \mathfrak{h}$, $v \in V(\lambda)$

$$
\begin{aligned}
HXv &= (HX - XH)v + XHv \\
&= [H,X]v + \lambda(H) Xv \\
&= \alpha(H) Xv + \lambda(H) Xv \\
&= (\alpha(H) + \lambda(H)) Xv
\end{aligned}
$$

<u>Prop</u> Let $\lambda \in P(V)$ and $\alpha \in \Delta$ (root). Then

(i) The set $\{\lambda + k\alpha \in P(V) \mid k \in \mathbb{Z}\}$ has the form

$\lambda - q\alpha, \lambda - (q-1)\alpha, \dots, \lambda + p\alpha$ for some $p, q \geq 0$

s.t. $p - q = \lambda(H_\alpha)$

(ii) $E_\alpha^{p+q} V(\lambda - p\alpha) = V(\lambda + p\alpha)$

"from lowest possible to highest possible"

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Pf

Consider the copy $sl_2(\mathbb{C})_\alpha \subseteq \mathfrak{g}_\mathbb{C}$

spanned by $\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha, H_\alpha$.

We consider V as a $sl_2(\mathbb{C})_\alpha$ - module.

From the rep. theory of $sl_2(\mathbb{C})$ we know that

the set $\{ \lambda(H_\alpha) + 2k \mid \lambda + k\alpha \in P(V) \}$

has the form

$$\{-n, -n+2, \ldots, n\}$$

for some integer $n = 2s$

This means that

$$\{\lambda + k\alpha \in P(V)\} = \{\lambda - q\alpha, \ldots, \lambda + p\alpha\}$$

w/ $\quad -n = \lambda(H_\alpha) - 2q, \quad n = \lambda(H_\alpha) + 2p$

$$\Rightarrow \quad p - q = \lambda(H_\alpha).$$

(ii) The second claim follows from the property of ~~the~~ spin s $sl_2\mathbb{C}$ - module V_s that $E^{2s}\xi_{-s} = \underset{\text{scalar} \neq 0}{c}\, \xi_s$

∎

Corollary If $\lambda, \lambda+\alpha \in P(V)$ for some $\alpha \in \Delta$, then $E_\alpha V(\lambda) \neq 0$.

In particular applying this to the adjoint rep., we conclude that if $\alpha, \beta, \alpha+\beta \in \Delta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

nonzero

one-dim.

Prop Fix a system of simple roots $\Pi \subset \Delta$. Then every $\alpha \in \Delta_+$ can be written as

$$\alpha_{i_1} + \dots + \alpha_{i_k} \quad \text{s.t.} \quad \alpha_{i_j} \in \Pi \text{ and}$$

$$\alpha_{i_1} + \dots + \alpha_{i_\ell} \in \Delta \quad \text{for all } \ell = 1, \dots, k.$$

Pf This is equivalent to saying that if $\lambda \in \Delta_+ \setminus \Pi$, then $\exists\, \alpha \in \Pi$ s.t. $\lambda - \alpha \in \Delta_+$.

Assume this is not true. So $\lambda - \alpha \notin \Delta_+$ $\forall \alpha \in \Pi$. Then $\lambda - \alpha \notin \Delta$, $\forall \alpha \in \Pi$ (as $\Delta = \Delta_+ \cup \Delta_-$)

By the previous prop, the set $\{\lambda + k\alpha \in \Delta \mid k \in \mathbb{Z}\}$ has the form $\{\lambda - q\alpha, \dots, \lambda + p\alpha\}$ w/ $p - q = -\lambda(H_\alpha)$.

By assumption, we conclude that $q = 0$ and $p = -\lambda(H_\alpha) \geq 0$.

so $\lambda(H_\alpha) \le 0$.

As $\lambda \in \Delta_+$, $\lambda = \sum_{\alpha \in \Pi} n_\alpha \alpha$ for some $n_\alpha \in \mathbb{Z}_+$. 6

Then
$$(\lambda, \lambda) = \sum_{\alpha \in \Pi} n_\alpha (\alpha, \lambda) \le 0 \quad \text{so } \lambda = 0,$$
which is a contradiction since $0 \notin \Delta$.

Combining this w/ the previous corollary we get

Corollary $\mathfrak{g}_{\mathbb{C}}$ is generated by $\mathfrak{h}$ and E_α, F_α for $\alpha \in \Pi$.

Pf If $\lambda = \alpha_{i_1} + \dots + \alpha_{i_k} \in \Delta_+$ as in the proposition, then $\mathfrak{g}_\lambda = [[[E_{\alpha_1}, E_{\alpha_2}], E_{\alpha_3}], \dots, E_{\alpha_k}] \otimes \mathbb{C}$ ∎

(Given main thm in this p. of the course)

Thm Two compact simply connected semisimple Lie groups are isomorphic iff their root systems are isomorphic.

Pf (sketch) Let us describe the Lie algebra of a compact semisimple Lie group in terms of its root system.

Let us fix a maximal torus and a system of simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$

We write H_i, E_i, F_i for $H_{\alpha_i}, E_{\alpha_i}, F_{\alpha_i}$.

Consider the Cartan ~~Shtsls~~ matrix (7)

„huunge"
$$A = (a_{ij})_{i,j=1}^r \qquad a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

Let $\mathfrak{a}$ be the universal Lie algebra w/ generators

$$\hat{H}_i,\ \hat{E}_i,\ \hat{F}_i \qquad (i = 1, \dots, r)$$

and the relations

$$[\hat{H}_i, \hat{H}_j] = 0 \quad \forall i,j, \qquad [\hat{E}_i, \hat{F}_i] = \hat{H}_i$$

$$[\hat{H}_i, \hat{E}_j] = a_{ij}\hat{E}_j \qquad \text{and} \quad [\hat{E}_i, \hat{F}_j] = 0 \text{ for } i \neq j$$

$$[\hat{H}_i, \hat{F}_j] = -a_{ij}\hat{F}_j \qquad \forall i,j.$$

(b.c. $[E_i, F_j] \in \mathfrak{g}_{\alpha_i - \alpha_j} = 0$, neither pos nor negative)

(To construct $\mathfrak{a}$, we take the free non associative algebra w/ generators $\hat{H}_i, \hat{E}_i, \hat{F}_i$ and divide by the two-sided ideal generated by the elements XX and $(XY)Z + (ZX)Y + (YX)Z$)

Then we have a Lie algebra homomorphism $\pi: \mathfrak{a} \to \mathfrak{g}_{\mathbb{C}}$
$\hat{H}_i \mapsto H_i$ etc

How to describe $\ker \pi$?

Let $\hat{\mathfrak{h}} = \mathrm{span}\{\hat{H}_i\}$, $\mathfrak{a}_+ \subset \mathfrak{a}$ the Lie subalgebra generated by $\hat{E}_i$ $(i = 1, \dots, r)$, $\mathfrak{a}_- \subset \mathfrak{a}$ gen. by $\hat{F}_i$ $(i = 1, \dots, r)$

(someth. to explain ...)

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Then it is not difficult to see that

$$\mathfrak{a} = \mathfrak{a}_- \oplus \hat{\mathfrak{h}} \oplus \mathfrak{a}_+ \quad \text{as a v. space}$$

Define $\hat{\alpha}_i \in \hat{\mathfrak{h}}^*$ by $\hat{\alpha}_i(\hat{H}_j) = a_{ji}$

Then we can define root spaces

$$\mathfrak{a}_\lambda = \{\hat{X} \in \mathfrak{a} \mid [\hat{H}, \hat{X}] = \lambda(\hat{H})\hat{X} \quad \forall \hat{H} \in \hat{\mathfrak{h}}\}$$

Then $\mathfrak{a} = \hat{\mathfrak{h}} \oplus \mathfrak{a}_+ \oplus \mathfrak{a}_-$ and $\mathfrak{a}_+ = \bigoplus_{\substack{\lambda = n_1\hat{\alpha}_1 + \dots + n_r\hat{\alpha}_r \\ n_i \in \mathbb{Z}_+}} \mathfrak{a}_\lambda$

and $\mathfrak{a}_- = \bigoplus_{\substack{\lambda = -n_1\hat{\alpha}_1 - \dots - n_r\hat{\alpha}_r \\ n_i \in \mathbb{Z}_+}} \mathfrak{a}_\lambda$

(exe!)

Now, if $\mathfrak{p} \subset \mathfrak{a}$ is an ideal, then $\mathfrak{p} = \bigoplus_\lambda (\mathfrak{a}_\lambda \cap \mathfrak{p})$

It follows that the sum of all ideals $\mathfrak{p}$ in $\mathfrak{a}$ such that $\mathfrak{p} \cap \hat{\mathfrak{h}} = 0$ we still get an ideal s.t. $\mathfrak{m} \cap \hat{\mathfrak{h}} = 0$

<u>Claim</u>: the kernel$(\pi : \mathfrak{a} \to \mathfrak{g}_{\mathbb{C}}) = \mathfrak{m}$. $\Rightarrow \mathfrak{g}_{\mathbb{C}} = \mathfrak{a}/\mathfrak{m}$.