Recall. $G \supset T$

$$\mathfrak{g} \supset \mathfrak{t}$$

$$\mathfrak{g}_{\mathbb{C}} \supset \mathfrak{t}_{\mathbb{C}} =: \mathfrak{h}$$

For $\alpha \in \mathfrak{h}^*$, we define

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g}_{\mathbb{C}} \mid [H, X] = \alpha(H) X \ \forall H \in \mathfrak{h} \}$$

The roots are $\Delta = \{ \alpha \in \mathfrak{h}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0 \} \subseteq X^*(T) \subset i\mathfrak{t}^* \subset \mathfrak{h}^*$

character lattice

Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$.

recall $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

also: if $X \in \mathfrak{g}_\alpha$, then $X^* \in \mathfrak{g}_{-\alpha}$ $\quad \left((Y + iZ)^* = -Y + iZ \right)$

We also introduced $(\cdot,\cdot)$ a symmetric, inv. under the adjoint representation, negative definite on $\mathfrak{g}$, form on $\mathfrak{g}_{\mathbb{C}}$

(invariance means $([X,Y],Z) + (Y,[X,Z]) = 0$)

Then we set $\mathfrak{h}^* \simeq \mathfrak{h}$, $\alpha \mapsto h_\alpha$ and a symmetric form on $\mathfrak{h}^*$ s.t.

$$(\alpha, \beta) = \alpha(h_\beta) = \beta(h_\alpha) = (h_\alpha, h_\beta)$$

Consider $\quad H_\alpha = \dfrac{2h_\alpha}{(\alpha,\alpha)}$. $\quad$ Choose $E_\alpha \in \mathfrak{g}_\alpha$ s.t. $\quad$ ②

$(E_\alpha, E_\alpha^*) = \dfrac{2}{(\alpha,\alpha)}$

$\boxed{\alpha \in \Delta}$

and $\quad F_\alpha = E_\alpha^*$.

Then these three elements give us an embedding of

$$sl_2(\mathbb{C}) \hookrightarrow \mathfrak{g}.$$

We also showed that $\mathfrak{g}_\alpha = \mathbb{C}E_\alpha$ (one-dim), and

that if $\alpha, c\alpha \in \Delta \Rightarrow c = \pm 1$.
$\quad c \in \mathbb{R}^\times$

Remark The isomorphism $\mathfrak{h}^* \cong \mathfrak{h}$ depends upon the choice of invariant form, but the elements H_α ($\alpha \in \Delta$) does not depend on this choice: H_α is the unique element of the one-dim space $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$ s.t. $\alpha(H_\alpha) = 2$. Then $E_\alpha \in \mathfrak{g}_\alpha$ is unique, up to a phase factor (scalar of modulus 1), element s.t.

$$[E_\alpha, E_\alpha^*] = H_\alpha.$$

3

Def Let V be a fin dim Euclidean v.space (i.e. a real v.space w/ a fixed scalar prod). A finite ~~system~~ subset $\Delta \subset V \setminus \{0\}$ is a *root system* if

① For every $\alpha \in \Delta$, the reflection s_α w.r.t. to the hyperplane $\alpha^\perp$ leaves Δ invariant.

$$\left(\text{note} \quad s_\alpha(v) = v - \frac{2(\alpha, v)}{(\alpha,\alpha)}\alpha \right)$$

② The number $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z} \quad \forall \alpha,\beta \in \Delta$.

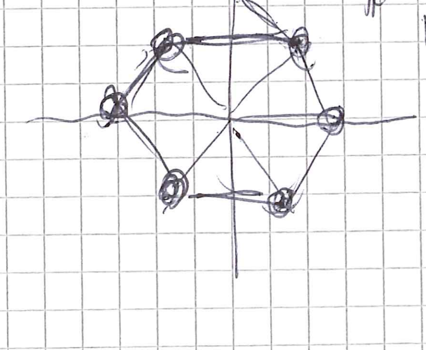
A root system is called *reduced* if

$$③ \quad \alpha, c\alpha \in \Delta \quad (c \in \mathbb{R}^\times) \Rightarrow c = \pm 1.$$

Remark For ③ it is often assumed that it is part of the def of root system. It is also often assumed that the span of Δ is the whole space V, but for us it is important not do this.

Example (root system A_2) (associated w/ $SU(3)$)

$V = \mathbb{R}^2$ Δ regular hexagon of norm 1.



Check conditions!

Def Two root systems (V_1, Δ_1) and (V_2, Δ_2) are isomorphic if $\exists$ a linear isomorphism $T: V_1 \to V_2$ s.t. $T(\Delta_1) = \Delta_2$ and $T s_\alpha = s_{T(\alpha)} T$ $\forall \alpha \in \Delta_1$.

Thm Let G be a compact connected Lie group, $T \subset G$ a maximal torus. Then $(i\mathfrak{t}^*, \Delta)$ is a reduced root system. Up to isomorphism, this root system does not depend on the choice of T and the choice of an invariant form on $\mathfrak{g}$.

Pf Take $\alpha \in \Delta$. Consider the corresponding ~~[illegible]~~ embedding

$$i_\alpha : \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}_\mathbb{C}$$

As $SU(2)$ is simply connected, it defines a homomorphism $SU(2) \to G$.

Choose that $i_\alpha(\mathfrak{su}(2)) \subset \mathfrak{g}$ $\Leftrightarrow$ $i_\alpha(X)^* = i_\alpha(X^*)$, and $\mathfrak{su}(2) = \{X \in \mathfrak{sl}_2(\mathbb{C}) \mid X = -X^*\}$, $\mathfrak{g} = \{X \in \mathfrak{g}_\mathbb{C} \mid X = -X^*\}$, so $i_\alpha(\mathfrak{su}(2)) \subset \mathfrak{g}$

Let w_α be the image of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SU(2)$ under this homomorphism. Then $(\mathrm{Ad}\, w_\alpha)(H_\alpha) = -H_\alpha$, $(\mathrm{Ad}\, w_\alpha)(E_\alpha) = -E_{-\alpha}$ and $(\mathrm{Ad}\, w_\alpha)(E_{-\alpha}) = -E_\alpha$. (Check!)

On the other hand, if we take $H \in \ker \alpha$ then

H commutes with $i_\alpha(\mathfrak{sl}_2(\mathbb{C}))$ (check!).

Hence $(\mathrm{Ad}\, g)(H) = H$ for any g in the image of $SU(2)$.

In particular, $(\mathrm{Ad}\, w_\alpha)(H) = H$.

We thus see that as $\mathfrak{h} = \mathbb{C}H_\alpha \oplus \ker \alpha$,

we have $(\mathrm{Ad}(w_\alpha))(\mathfrak{h}) = \mathfrak{h}$. Hence $w_\alpha T w_\alpha^{-1} = T$ so $w_\alpha \in N(T)$.

It is not difficult to see that for any $g \in N(T)$, we have

$(\mathrm{Ad}\, g)(\mathfrak{g}_\beta) = \mathfrak{g}_{\beta \circ \mathrm{Ad}\, g^{-1}}$ and $(\mathrm{Ad}\, g)(h_\beta) = h_{\beta \circ \mathrm{Ad}\, g^{-1}}$ $\forall \beta \in \mathfrak{h}^*$

We claim now that $s_\alpha(\beta) = \beta \circ \mathrm{Ad}\, w_\alpha$. (*) (exercise!)

(Note then $\mathrm{Ad}\, w_\alpha = \mathrm{Ad}\, w_\alpha^{-1}$ as $w_\alpha^2 \in T$ (w_α^2 image of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$)

and $\mathrm{Ad}\,T$ is trivial on $\mathfrak{h}$.)

If we prove this, then

$$(\mathrm{Ad}\, w_\alpha)(\mathfrak{g}_\beta) = \mathfrak{g}_{\beta \circ \mathrm{Ad}\, w_\alpha} = \mathfrak{g}_{s_\alpha(\beta)}$$ so $s_\alpha(\Delta) = \Delta$.

To prove (*) we have to check that $h_{s_\alpha(\beta)} = h_{\beta \circ \mathrm{Ad}\, w_\alpha} = (\mathrm{Ad}\, w_\alpha)(h_\beta)$

(exc!)

But this is for $\beta = \alpha$ as $s_\alpha(\alpha) = -\alpha$ and and

$(\mathrm{Ad}\, w_\alpha)(h_\alpha) = -h_\alpha$. On the other hand, if $(\beta, \alpha) = 0$, then

$s_\alpha(\beta) = \beta$ and $(\mathrm{Ad}\, w_\alpha)(h_\beta) = h_\beta$ as $h_\beta \in \ker \alpha$.

As $\mathfrak{h}^* = \mathbb{C}\alpha \oplus \alpha^\perp$, (*) is proved.

Next we have to check that $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \frac{2\beta(h_\alpha)}{(\alpha,\alpha)} = \beta(H_\alpha)$ (6)

are integers $\forall\ \alpha,\beta \in \Delta$.

We will show a bit more:

$$\beta(H_\alpha) \in \mathbb{Z} \quad \forall\ \beta \in X^*(T).$$

This true since $e^{2\pi i \mathring{H}} = 1$ in $SU(2)$, so $\exp(2\pi i H_\alpha) = 1$ in G, hence $e^{2\pi i \beta(H_\alpha)} = 1$ $\forall\ \beta \in X^*(T)$

which is true only if $\beta(H_\alpha) \in \mathbb{Z}$.

Note

$$\begin{tikzcd} \mathfrak{su}(2) \arrow[r, hook, "i_\alpha"] \arrow[d, "\exp"] & \mathfrak{g} \arrow[d, "\exp"] \\ SU(2) \arrow[r, "\pi_\alpha"] & G \end{tikzcd}$$

$iH \mapsto iH_\alpha$, $\quad 1$

Thus $(i\mathfrak{t}^*, \Delta)$ is a root system. We already know that it is reduced.

Up to isomorphism, this root system does not depend on the choice of invariant form: $s_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha$.

and H_α does not depend on the invariant form.

If T' is another maximal torus, then $T' = gTg^{-1}$ for some $g \in G$ by Cartan's thm. Then the adjoint action Ad_g defines an isomorphism of the corresponding root system. $\blacksquare$

Our goal is to prove (more or less) the following fundamental result:

Thm

(1) Any compact connected Lie group G is completely determined (up to iso) by the triple $(i\mathfrak{t}^*, \Delta, X^*(T))$ consisting of the root system $(i\mathfrak{t}^*, \Delta)$ defined by a maximal torus $T \subset G$ and the character lattice $X^*(T) \subset i\mathfrak{t}^*$.

(2) Any triple (V, Δ, X^*) where (V, Δ) is a reduced root system, $X^* \subset V$ is a lattice such that $\Delta \subset X^*$ and $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ $\forall \alpha \in \Delta$, $\beta \in X^*$, arises from a compact connected Lie group, up to iso.

Simple roots, Weyl chambers and the Weyl group

Let (V, Δ) be a root system and assume $\operatorname{Span} \Delta = V$.

Def A system of simple roots, or a base, in Δ is a ~~system~~ subset $\Pi \subset \Delta$ s.t.

(1) Π is a basis in V

(2) If $\beta = \sum_{\alpha \in \Pi} c_\alpha \alpha \in \Delta$, then either all $c_\alpha \geq 0$ or all $c_\alpha \leq 0$

Reflections s_α defined by simple roots are called simple reflections.

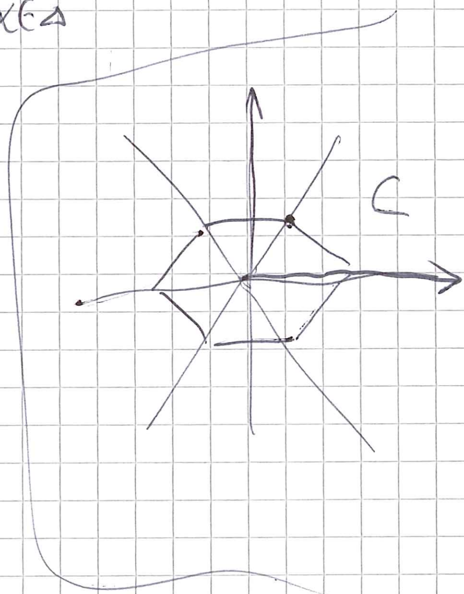
When Π is fixed, then the roots

$$\beta = \sum_\alpha c_\alpha \alpha \in \Delta \quad \text{s.t.} \quad c_\alpha \geq 0 \quad \forall \alpha \in \Pi$$

are called positive, and the set of positive roots is denoted by Δ_+. Thus $\Delta = \Delta_+ \cup -(\Delta_+)$.

In order to construct Π, consider the set $V \setminus \bigcup_{\alpha \in \Delta} \alpha^\perp$

The connected components of are called (open) Weyl chambers.



Let C be such a Weyl chamber.

Since $(\alpha, \beta) \neq 0$ $\forall \alpha \in \Delta$ and $\forall \beta \in C$, we have for any $\alpha \in \Delta$ that either $(\alpha, \beta) > 0$ $\forall \beta \in C$ or $(\alpha, \beta) < 0$ $\forall \beta \in C$.

In the first case we say that α is C-positive. Let $\Pi(C) \subset \Delta$ be the set of C-positive roots α s.t. it is impossible to write α as $\beta + \gamma$ for C-positive $\beta, \gamma \in \Delta$.

Prop For any Weyl chamber C, the set $\Pi(C)$ is a system of simple roots, and furthermore the map $C \to \Pi(C)$ between {Weyl chambers} $\longrightarrow$ {systems of simple roots}.

Exc

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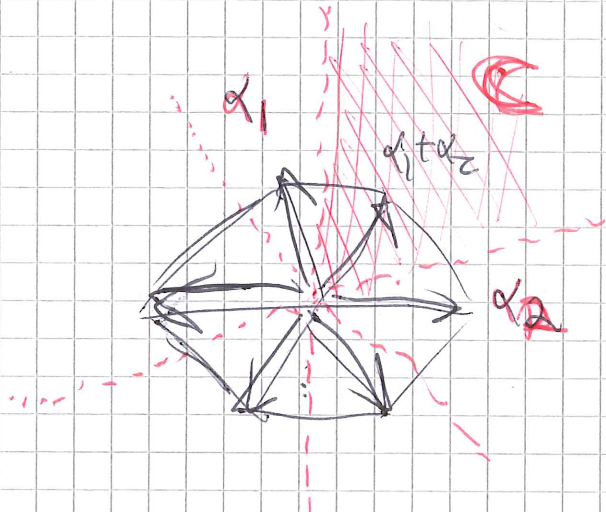

$\Pi(C) = \{\alpha_1, \alpha_2\}$

So $\Delta_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$