Recall

$S(U_2) \supset T = \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \right\}$

Lie algebra

$su(2) \subset sl_2(\mathbb{C})$

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad T = \{ \exp(i\tau H) \}_{\tau \in \mathbb{R}}$

$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$[H, E] = 2E$
$[H, F] = -2F$
$[E, F] = H.$

Roots

Let G be a compact connected Lie group. Fix a maximal torus. Consider $\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$. This is a complex Lie algebra $[c \otimes X, d \otimes Y] = cd \otimes [X, Y]$. Identify $\mathfrak{g}$ w/ elements of the form $1 \otimes \mathfrak{g} \subset \mathbb{C} \otimes \mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$. So we write elements of $\mathfrak{g}_{\mathbb{C}}$ as $X + iY$.

We extend $Ad : G \to GL(\mathfrak{g})$ to $G \to GL(\mathfrak{g}_{\mathbb{C}})$ and then consider $Ad|_T$.

As T is a compact abelian group we have a decomposition of Ad_T into isotypic components corresponding to characters $\chi \in \hat{T}$.

So $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\chi \in \hat{T}} \mathfrak{g}_{\mathbb{C}}(\chi)$ where $\mathfrak{g}_{\mathbb{C}}(\chi) = \{ X \in \mathfrak{g}_{\mathbb{C}} \mid (Ad\, g) X = \chi(g) X \ \forall g \in T \}$

We view $\mathbb{T} = S^1$ as a subgroup of $\mathbb{C}^* = GL_1(\mathbb{C})$. ②

So the Lie algebra of $\mathbb{T}$ is $i\mathbb{R} \subset \mathbb{C}$.

Any character $\chi \in \hat{T}$ is determined by its differential $\chi_*: \mathfrak{t} \to i\mathbb{R}$.

So we think of $\chi_* \in i\mathfrak{t}^*$ where $\mathfrak{t}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}, \mathbb{R})$.

Then the set of differentials χ_* $(\chi \in \hat{T})$ forms a subset

$\cancel{X}^*(T) \subset i\mathfrak{t}^*$. The set $X^*(T)$ is a lattice in $i\mathfrak{t}^*$, (a discrete subgroup of maximal rank), called the <u>character lattice</u>.

Indeed, if $T = \mathbb{R}^n / \mathbb{Z}^n$ then $\mathfrak{t} = \mathbb{R}^n$, so $\mathfrak{t}^* = \mathbb{R}^n$ and an element $(ia_1, \dots, ia_n) \in i\mathfrak{t}^*$ defines a character from $\mathbb{R}^n \to \mathbb{T} = S^1$

$$(a_1, \dots, a_n) \mapsto e^{ia_1r_1} \cdots e^{ia_nr_n}$$

which factors through $\mathbb{R}^n / \mathbb{Z}^n$ iff $a_1, \dots, a_n \in 2\pi\mathbb{Z}$.

Then $X^*(T) = 2\pi i\,\mathbb{Z}^n \in i\mathbb{R}^n$.

Consider also the complexification $\mathfrak{t}_{\mathbb{C}}$ of the Lie algebra of T, which we denote by $\mathfrak{h}$. ($\mathfrak{h}$ is called a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$).

We extend functionals $\mathfrak{t}^* \to \mathbb{R}$ by complex linearity to $\mathfrak{h}$, so $i\mathfrak{t}^* \subset \mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$. We then have that

$$\mathfrak{g}_{\mathbb{C}}(\chi) = \{X \in \mathfrak{g}_{\mathbb{C}} \mid (\mathrm{Ad}\, g) X = \chi(g) X \quad \forall g \in T\}$$

$$= \{X \in \mathfrak{g}_{\mathbb{C}} \mid [Y,X] = \chi_*(Y)X \quad \forall\, Y \in \mathfrak{t}\}$$

$$= \{X \in \mathfrak{g}_{\mathbb{C}} \mid [Y,X] = \chi_*(Y)X \;\forall\, Y \in \mathfrak{h}\}$$

For $\alpha \in \mathfrak{h}^*$, denote

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_{\mathbb{C}} \mid [H,X] = \alpha(H)X \;\forall\, H \in \mathfrak{h}\}$$

i.e. $\mathfrak{g}_{\chi_*} = \mathfrak{g}_{\mathbb{C}}(\chi)$.

<u>Lemma</u> $\mathfrak{g}_0 = \mathfrak{h}$

<u>pf</u> $\mathfrak{t}$ is a maximal abelian Lie subalgebra of $\mathfrak{g}$, so

$$\{X \in \mathfrak{g} \mid [H,X] = 0 \;\forall\, H \in \mathfrak{t}\} = \mathfrak{t}.$$

Now extend by linearity to complexifications. ∎

The elements $\alpha \in X^*(T) \setminus \{0\}$ s.t. $\mathfrak{g}_\alpha \neq 0$ are called <u>roots</u> of G or $\mathfrak{g}_{\mathbb{C}}$.

The set of roots is denoted by Δ, or φ. (Φ)

To remember the position of $\mathfrak{g}$ in $\mathfrak{g}_{\mathbb{C}}$ it is convenient to introduce an involution (antilinear operator of ~~□~~ on $\mathfrak{g}_{\mathbb{C}}$) by

$$(X + iY)^* = -X + iY \quad \text{so}$$

$$\mathfrak{g} = \{X \in \mathfrak{g}_{\mathbb{C}} \mid X = -X^*\}$$

We can introduce an involution $\alpha \mapsto \bar{\alpha}$ on the $\mathfrak{h}^*$ by $\bar{\alpha}(H) = \overline{\alpha(H^*)}$.

So $\bar{\alpha}$ will be linear.

Then we see $it^* = \{\alpha \in \mathfrak{h}^* \mid \alpha(t) \subset i\mathbb{R}\}$

$= \{\alpha \in \mathfrak{h}^* \mid \bar{\alpha} = \alpha\}$.

Indeed, if $H \in \mathfrak{t}$, so $H^* = -H$, then $\bar{\alpha}(H) = \overline{\alpha(H^*)}$

$= -\overline{\alpha(H)}$

which equals $\alpha(H)$ iff $\alpha(H) \in i\mathbb{R}$.

Summary of notation

$$\begin{array}{ccc} \mathfrak{g}_{\mathbb{C}} & \supset & \mathfrak{g} = \{X \in \mathfrak{g}_{\mathbb{C}} \mid X^* + X = 0\} \\ \cup & \text{Lie alg. of torus} & \cup \\ \mathfrak{h} & \supset & \mathfrak{t} \end{array}$$

$\mathfrak{h}^*$

$\cup$

$it^* = \{\alpha \in \mathfrak{h}^* \mid \alpha(\mathfrak{t}) \subset i\mathbb{R}\}$

$= \{\alpha \mid \bar{\alpha} = \alpha\}$

$\cup$

$X^*(T)$ = character lattice of T

$\cup$

roots $\to$ Δ We have $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$

$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_{\mathbb{C}} \mid [H, X] = \alpha(H) X\}$

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Lemma ① $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$

② $\mathfrak{g}_\alpha^* \doteq \{X^* \mid X \in \mathfrak{g}_\alpha\} = \mathfrak{g}_{-\alpha}$

Pf (i) Take $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, $H \in \mathfrak{h}$. Then

$$[H, [X,Y]] = [[H,X],Y] + [X,[H,Y]] \qquad (\text{Jac.})$$

$$= (\alpha(H) + \beta(H)) [X,Y].$$

(ii) If $X \in \mathfrak{g}_\alpha$ and $H \in \mathfrak{h}$, then

$$[H,X]^* = -[H^*, X^*]$$

for $X \in \mathfrak{g}$, $H \in \mathfrak{t}$ and extended when
$X^* = -X$, $H^* = -H$ linearity.

∎

Therefore, replacing H by H^*, we get $[H, X^*] = -\alpha(H^*) X^*$

$$= -\bar{\alpha}(H) X^*$$

for $\alpha \in \Delta \subset X^*(T)$.

∎

To proceed, it is convenient to fix an Ad-invariant scalar product on $\mathfrak{g}$. In fact, it is convenient to change its sign, so consider a negative definite symmetric form $(\cdot,\cdot)$; by linearity we extend to $\mathfrak{g}_\mathbb{C}$. Thus we get a symmetric bilinear form on $\mathfrak{g}_\mathbb{C}$ s.t. $([X,Y],Z) + (Y,[X,Z]) = 0 \;\forall X,Y,Z \in \mathfrak{g}_\mathbb{C}$ and $(\cdot,\cdot)$ is neg. definite on $\mathfrak{g} \subset \mathfrak{g}_\mathbb{C}$.

<u>Lemma</u> $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\alpha+\beta=0$.

Pf Indeed, ~~for~~ $H \in \mathfrak{h}$, $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$ we have

$$0 = ([H,X],Y) + (X,[H,Y])$$
$$= (\alpha(H)+\beta(H))(X,Y)$$

$\blacksquare$

This form defines a linear isomorphism $\mathfrak{h}^* \to \mathfrak{h}$, $\alpha \mapsto h_\alpha$.

s.t. $\alpha(H) = (h_\alpha, H)$ $\forall H \in \mathfrak{h}$

here we use non-degeneracy

We then define a bilinear symmetric form on $\mathfrak{h}^*$ by

$$(h_\alpha, h_\beta) = (\alpha, \beta).$$

The map $\alpha \mapsto h_\alpha$ respects the involution, so that $h_\alpha^* = h_{\bar\alpha}$.

Indeed, we have
$$\beta(h_\alpha^*) = \overline{\bar\beta(h_\alpha)} = \overline{(\bar\beta, \alpha)}$$
$$\beta(h_{\bar\alpha}) = (\beta, \bar\alpha).$$

It follows that if $\alpha = \bar\alpha$ then $h_\alpha = h_\alpha^*$ so $h_\alpha \in i\mathfrak{t}_0$

Therefore $(\alpha,\alpha) = (h_\alpha, h_\alpha) = -(ih_\alpha, ih_\alpha) \geq 0$,

thus the form $(\cdot,\cdot)$ is pos. def. on the space $\{\alpha \in \mathfrak{h}^* \mid \alpha = \bar\alpha\}$

$$= i\mathfrak{t}_0^*$$

Lemma For $\alpha \in \Delta$, $X \in \mathfrak{g}_\alpha$, we have
$$[X, X^*] = (X, X^*) h_\alpha$$

Pf For $\beta \in \mathfrak{h}^*$, we have (recall that $[h, X^*] \in \mathfrak{g}_{-\alpha}$ so $[X, X^*] \in \mathfrak{g}_0 = \mathfrak{h}$).
$$\beta([X, X^*]) = (h_\beta, [X, X^*])$$
$$= -([X, h_\beta], X^*)$$
$$= +\alpha(h_\beta)(X, X^*)$$
$$= \beta(h_\alpha)(X, X^*)$$
Since this is true for any $\beta \in \mathfrak{h}^*$, we get $[X, X^*] = (X, X^*) h_\alpha$

$\blacksquare$

Note that if any $X = Y + iZ \neq 0$ in $\mathfrak{g}_\alpha$, we have
$$(X, X^*) = (Y + iZ, -Y + iZ)$$
$$= -(Y, Y) - (Z, Z) < 0$$

Now define the following elements for $\alpha \in \Delta$
$$H_\alpha = \frac{2h_\alpha}{(\alpha, \alpha)} \quad E_\alpha \in \mathfrak{g}_\alpha \text{ s.t. } (E_\alpha, E_\alpha^*) = \frac{2}{(\alpha, \alpha)}$$
$$F_\alpha = E_\alpha^*.$$

Then $[H_\alpha, E_\alpha] = \alpha(H_\alpha) E_\alpha = \dfrac{2\alpha(h_\alpha)}{(\alpha, \alpha)} E_\alpha = 2E_\alpha$

$[H_\alpha, F_\alpha] = \ldots$ and $[E_\alpha, F_\alpha] = \ldots = H_\alpha$

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Thus we get an embedding

$$i_\alpha : \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g} \quad \forall \alpha \in \Delta.$$

(Two minutes! One more lemma!)

Lemma ① The spaces $\mathfrak{g}_\alpha$ $(\alpha \in \Delta)$ are one-dimensional;

② If $\alpha \in \Delta$ $(\alpha \in \Delta$, $c \in \mathbb{R}^\times$ then $c = \pm 1)$

Assume $\alpha \in \Delta$ s.t. $c\alpha \notin \Delta$ if $c \in (0,1)$. Consider the space

$$V = \mathbb{C}E_\alpha \oplus \mathfrak{h} \oplus \Big(\bigoplus_{\substack{c>0 \\ c\alpha \in \Delta}} \mathfrak{g}_{c\alpha} \Big).$$

This space is invariant under the adjoint action of $i_\alpha(\mathfrak{sl}_2(\mathbb{C}))$

But we know that as fin-dim. $\mathfrak{sl}_2(\mathbb{C})$-module the spectrum of H is symmetric around zero, counting w/ multiplicities.

Hence $\mathfrak{g}_\alpha$ is 1-dim and $\mathfrak{g}_{c\alpha} = 0$ for $c > 1$. ∎