Chapter 14

Find the order of the given quotient group

1. $\mathbb{Z}_6/\langle 3\rangle$. The elements here are

$$0+\langle 3\rangle = \{0,3\}$$
$$1+\langle 3\rangle = \{1,4\}$$
$$2+\langle 3\rangle = \{2,5\}$$

Since the union of these is all of $\mathbb{Z}_6$, we are left with three elements.

2. $(\mathbb{Z}_4 \times \mathbb{Z}_{12})/\langle 2\rangle\times\langle 2\rangle$

Since all cosets have the same size, it follows that (for any group G) $\#G = \#\text{number of cosets} \cdot \#\text{size of cosets}$

Thus $\#\text{number of cosets} = \dfrac{\#G}{\#\text{size of cosets}}$.

Hence in this case, the size of the group is

$$\frac{4\cdot 12}{2\cdot 6} = \underline{\underline{4}}$$

3. $(\mathbb{Z}_4 \times \mathbb{Z}_2)/\langle(2,1)\rangle$ have size $\dfrac{4\cdot 2}{2} = 4$ since

$$\langle(2,1)\rangle = \langle(0,0),(2,1)\rangle,$$

since $(2,1)+(2,1) = (4,2) = (0,0)$.

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(4) $\# \dfrac{\mathbb{Z}_3 \times \mathbb{Z}_5}{\{0\} \times \mathbb{Z}_5} = \dfrac{15}{5} = 3$

(5) $\# \dfrac{\mathbb{Z}_2 \times \mathbb{Z}_4}{\langle (1,1) \rangle} = \dfrac{8}{4} = 2.$

$(0,0) + \langle (1,1) \rangle = \{ (0,0), (1,1), (0,2), (1,3) \}$
$(0,1) + \langle (1,1) \rangle = \{ (0,1), (1,2)\ (0,3), (1,0) \}$

(6) $\# \dfrac{\mathbb{Z}_{12} \times \mathbb{Z}_{18}}{\langle (4,3) \rangle} = \dfrac{12 \cdot 18}{6} = \underline{\underline{36}}$

$(4,3) + (4,3) = (8,6)$
$(4,3) + (8,6) = (12,9) = (0,9)$
$(0,9) + (4,3) = (4,12)$
$(4,12) + (4,3) = (8,15)$
$(4,3) + (8,15) = (12,18) = 0$

(7) $\dfrac{\mathbb{Z}_2 \times S_3}{\langle (1,\rho_1) \rangle} = \dfrac{2 \cdot 6}{6} = \underline{\underline{2}}$

$(1,\rho_1) \cdot (1,\rho_1) = (2, \rho_1^2) = (0, \rho_1^2)$
$(0,\rho_1^2) \cdot (1,\rho_1) = (1, \rho_1^3) = (1, e)$
$(1,e) \cdot (1,\rho_1) = (0, \rho_1)$
$(0,\rho_1) \cdot (1,\rho_1) = (1, \rho_1^2)$
$(1,\rho_1^2)(1,\rho_1) = (0, e)$

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(8) $$\frac{\mathbb{Z}_{11} \times \mathbb{Z}_{15}}{\langle(1,1)\rangle} = 0$$ since 11 and 15 are coprime.

~~(1)~~ Section 15 Classify the group

(1) $$\frac{\mathbb{Z}_2 \times \mathbb{Z}_4}{\langle(0,1)\rangle} = \frac{\mathbb{Z}_2 \times \mathbb{Z}_4}{\{0\} \times \mathbb{Z}_4} \simeq \mathbb{Z}_2$$

(2) $$\frac{\mathbb{Z}_2 \times \mathbb{Z}_4}{\langle(0,2)\rangle} = \frac{\mathbb{Z}_2 \times \mathbb{Z}_4}{\{0\} \times 2\mathbb{Z}_4} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$$

(3) $$\frac{\mathbb{Z}_2 \times \mathbb{Z}_4}{\langle(1,2)\rangle} = G$$

$(1,2) + (1,2) = (2,4) = (0,0)$, so the group must have order $\frac{8}{2} = 4$. Hence it is either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The element $\overline{(0,1)}$ have order 4 since $(0,1) \notin \langle(1,2)\rangle$.

Hence $G \cong \mathbb{Z}_4$

(4) $\dfrac{\mathbb{Z}_4 \times \mathbb{Z}_8}{\langle(1,2)\rangle}$ It is of order $\dfrac{4\cdot 8}{4} = 8$, so (4)

the possibilities are

$\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

The element $(0,1)$ have order 8, hence the group is isomorphic to $\mathbb{Z}_8$.

(5) $\dfrac{\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_8}{(1,2,4)} = G$

G have order $\dfrac{4\cdot 4\cdot 8}{4} = 32$.

The possibilities are $\mathbb{Z}_8 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

But the element $(1,0,0)$ have order 4, so $G \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^3$

(6) $\dfrac{\mathbb{Z} \times \mathbb{Z}}{\langle(0,1)\rangle} \simeq \mathbb{Z}$

(8) $\dfrac{\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}}{(1,1,1)} \simeq \mathbb{Z} \times \mathbb{Z}$

(7) $\dfrac{\mathbb{Z} \times \mathbb{Z}}{\langle(1,2)\rangle} \simeq \mathbb{Z}$

(36) $\phi : G \to G' \supseteq N'$

$\phi^{-1}(N')$ normal: Se på $G \xrightarrow{\phi} G' \xrightarrow{\pi} G'/N'$

Da er $\phi^{-1}(N) = (\pi \circ \phi)^{-1}(0)$
$= \ker(\pi \circ \phi)$.