Reps of $SU(2)$ on f.d. cplx v.spcs (1)

are = ~~$\mathfrak{sl}_2(\mathbb{C})$~~ f.dim $\mathfrak{sl}_2(\mathbb{C})$-modules.

(considered as a complex Lie algebra)

Fix basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[H, E] = 2E \quad [H, F] = -2F \quad [E, F] = H.$$

If V is a fin.dim $\mathfrak{sl}_2(\mathbb{C})$-module. For $s \in \frac{1}{2}\mathbb{Z}$, define

$$V(s) = \{\xi \in V \,/\, H\xi = 2s\xi\}$$

$$E\,V(s) \subset V(s+1) \qquad F\,V(s) \subset V(s-1)$$

We have $V = \bigoplus_{s \in \frac{1}{2}\mathbb{Z}} V(s)$ (because $e^{2\pi i H} = 1$ in $SU(2)$)

If $\bar{s}$ is the largest half-integer w/ $V(\bar{s}) \neq 0$, then $E\,V(\bar{s}) = 0$.

Lemma Let V be a fin.dim $\mathfrak{sl}_2(\mathbb{C})$-modul, $\xi \in V(s)$ a nonzero vector s.t. $E\xi = 0$. Then

(i) $s \geq 0$

(ii) $F^{2s+1}\xi = 0$ and $F^k\xi \neq 0$ for $k = 0, \dots, 2s$.

(iii) $E F^k \xi = (2s-k+1)k F^{k-1}\xi \quad \forall k \geq 0$

Pf

We prove (iii) by induction on k. OK for $k=0$. Assume it holds for k.

$$EF^{k+1}\xi = (EF - FE)F^k\xi + FEF^k\xi$$
$$= HF^k\xi + FEF^k\xi$$
$$= 2(s-k)F^k\xi + (2s-k+1)kF^k\xi$$

$\boxed{\text{check!}}$
$$= (2s-k)(k+1)F^k\xi$$

Thus (iii) is proved. Let n be the smallest no. s.t. $F^n\xi = 0$.

Then $0 = EF^n\xi = (2s-n+1)nF^{n-1}\xi$

hence $n = 2s+1$. So (ii) is proved and (i). ∎

In view of this lemma, for any $s \in \frac{1}{2}\mathbb{Z}_+ = \{0, \frac{1}{2}, 1, \dots\}$ define an $sl_2(\mathbb{C})$-module V_s as follows:

V_s is a space w/ basis
$$\{\xi^s_{-s}, \xi^s_{-s+1}, \dots, \xi^s_s\}$$

$$H\xi^s_t = 2t\,\xi^s_t \qquad F\xi^s_t = \xi^s_{t-1} \qquad E\xi^s_{s-k} = (2s-k+1)k\,\xi^s_{s-k+1}$$

[Exc Show that this indeed defines an $sl_2(\mathbb{C})$-module.
Show that $V(s)$ is irreducible.]

(3)

The number s is called the highest weight or the spin of V_s.

The previous lemma tells us that if V is a fin dim $sl_2(\mathbb{C})$-module, $\xi \in V(s)$, $\xi \neq 0$, $E\xi = 0$, then we get an embedding

$V_s \hookrightarrow V$, $\xi^s_{s-k} \mapsto F^k \xi$.

This proves (almost)

Thm The $sl_2(\mathbb{C})$-modules V_s $(s \in \frac{1}{2}\mathbb{Z})$ are irreducible, pairwise non isomorphic, and any f. dim irreducible $sl_2(\mathbb{C})$-module is isomorphic to some V_s.

(pairwise non-iso b.c. $\dim V_s = 2s+1$)

Examples ① $s=0$ Then $\dim V_0 = 1$. And $X\xi = 0$ for any $X \in sl_2(\mathbb{C})$. Corresponds to the trivial rep of $SU(2)$.

② $s = \frac{1}{2}$. $V_{\frac{1}{2}}$ is the canonical rep of $sl_2(\mathbb{C})$ on $\mathbb{C}^2$.
The highest weight vector is $\xi^{\frac{1}{2}}_{\frac{1}{2}} = \binom{1}{0}$.

③ $s=1$. V_1 ~~is~~ defines the adjoint repr. of $sl_2(\mathbb{C})$.
The highest w/ vector is $\xi^1_1 = E$.

Recall that $SU(2)$ is faithfully repr. on $\mathbb{C}^2 = V_{\frac{1}{2}}$ and $\det(g) = 1$ $\forall g$, any irreducible rep of $SU(2)$ ($sl_2(\mathbb{C})$) is a

Exc Let $\rho: G \to GL(V)$, $\eta: G \to GL(W)$ be fin dim reps of the Lie group G.

Show that $(\rho \otimes \eta)_*(X) = \rho_*(X) \otimes 1 + 1 \otimes \eta_*(X)$. (4)

This motivates the notion of tensor product of $\mathfrak{g}$-modules for any Lie algebra $\mathfrak{g}$: if V, W are $\mathfrak{g}$-modules, then

$V \otimes W$ is the $\mathfrak{g}$-module defined by

$$X(v \otimes w) = Xv \otimes w + v \otimes Xw.$$

Take $k \geq 0$ and consider $\xi = \begin{pmatrix} \xi_{\frac{1}{2}} \\ \xi_{\frac{1}{2}} \end{pmatrix}^{\otimes k} \in V_{\frac{1}{2}}^{\otimes k}$.

Then $E\xi = 0$, $H\xi = k\xi$

Hence we get an embedding of $V_{\frac{k}{2}} \hookrightarrow V_{\frac{1}{2}}^{\otimes k}$ uniquely

determined by $\xi_{\frac{k}{2}} \longmapsto (\xi_{\frac{1}{2}})^{\otimes k}$.

The image is the subspace

$S^k(V_{\frac{1}{2}}) \subset V_{\frac{1}{2}}^{\otimes k}$ of symmetric tensors.

Indeed $S^k(V_{\frac{1}{2}})$ is $\underline{\text{invariant under}}$ $SL(2) / sl_2(\mathbb{C})$

and it contains $\begin{pmatrix} \xi_{\frac{1}{2}} \\ \xi_{\frac{1}{2}} \end{pmatrix}^{\otimes k}$ and is of dim $k+1$.

Hence $V_{\frac{k}{2}} \simeq S^k(V_{\frac{1}{2}})$.

Tori

By a torus, or a compact torus, we mean a Lie group isom. to $\mathbb{T}^n = (S^1)^n$.

Recall that any compact connected abelian Lie group is a torus

Prop Let G be a compact connected Lie group. Then there exists a maximal torus in G. Furthermore, we have a 1-1 correspondence between the maximal tori in G and the maximal abelian Lie subalgebras of $\mathfrak{g}$

Pf: As $\mathfrak{g}$ is f.dim, maximal abelian Lie subalgebras exist. Let $\mathfrak{a} \leq \mathfrak{g}$ be maximal abelian. Then $\exp \mathfrak{a}$ is a connected abelian Lie subgroup of G.

Then $H = \overline{\exp \mathfrak{a}}$ is a closed abelian connected subgroup of G. Hence H is a torus. Then $\mathfrak{h}$ is abelian and $\mathfrak{h} \supset \mathfrak{a}$ so $\mathfrak{h} = \mathfrak{a}$ and $H = \exp \mathfrak{a}$.

If $T < G$ is a maximal torus, then $\mathfrak{t} \subset \mathfrak{g}$ is an abelian Lie algebra, hence it is contained in a maximal abelian Lie algebra $\mathfrak{a}$. Then $\exp \mathfrak{a}$ is a torus containing T, hence $T = \exp \mathfrak{a}$ and $\mathfrak{t} = \mathfrak{a}$.

∎

Our goal is to show that any two maximal tori are conjugate.

This is often not difficult to show in concrete examples.

Ex $G = U(n)$. Then $\left\{ \begin{pmatrix} z & & 0 \\ & \ddots & \\ 0 & & z \end{pmatrix} \right\} \cap SU(n) \simeq \mathbb{T}^n$

Assume $H \subset G$ is a torus. As H is compact abelian, $\mathbb{C}^n$ decomposes into a direct sum of 1-dim H-invariant subspaces, hence there exists $g \in U(n)$ s.t. $gHg^{-1} \subset T$.

This shows that ~~any other~~ T is maximal and any maximal torus is conj. to T.

We need some preparation for the general case.

If T is a torus, we say that $g \in T$ is a topological generator if $\overline{\{g^n \mid n \in \mathbb{Z}\}}$ is dense in T.

Thm (Kronecker) Consider $T = \mathbb{T}^n$. $g = (e^{2\pi i t_1}, \dots, e^{2\pi i t_n})$

Then g is a topological generator iff the numbers $1, t_1, \dots, t_n$ are linearly independent over $\mathbb{Q}$.

Pf Consider $H = \overline{\{g^n \mid n \in \mathbb{Z}\}}$. If $H \neq T$. Then T/H is a nontrivial compact abelian group, hence T/H has a nontrivial character χ, so by composition, T have a non-trivial character χ vanishing on H: $\chi(H) = 1$. Any character of T has the form

$$\chi\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right) = e^{2\pi i m_1 t_1} \cdots e^{2\pi i m_n t_n}$$ (7)

for some $m_1, \ldots, m_n \in \mathbb{Z}$. (follows from the 1-dim case)

$\therefore$ Then $e^{2\pi i (m_1 t_1 + \cdots + m_n t_n)} = 1$

$\Updownarrow$

$\{m_1, \ldots, m_n\}$ lin. dep $/ \mathbb{Q}$.

$\boxed{\text{Corollary: The set of top. generators is dense in } T.}$

Prop Let G be a compact connected Lie group, $T \subset G$ a maximal torus. Then

(1) $N(T) \stackrel{\text{def}}{=} \{ g \in G \mid gTg^{-1} = T \}$ is a closed subgroup of G

(2) $N(T)^0 = T$ — the connected component of the identity in $N(T)$

(3) The group $W(G,T) \stackrel{\text{def}}{=} N(T)/T$ is finite.

It is called the Weyl group of G.