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Thm (Brauer)

Let G be a finite group, m the exponent of G. Then any complex representation is defined over $\mathbb{Q}(\zeta_m)$

Pf Let $\pi_1, \ldots, \pi_k$ be the representatives of equivalence classes of irreducible rep's of G over $\mathbb{Q}(\zeta_m)$.

Note that any one-dim rep of a subgroup of G is defined over $\mathbb{Q}(\zeta_m)$.

Hence the induction of any such representation is defined over $\mathbb{Q}(\zeta_m)$.

It follows by Brauer's theorem, that if π is any f.d. (complex) rep of G, then

$$\chi_\pi = \sum_{i=1}^{k} m_i \chi_{(\pi_i)_{\mathbb{C}}} = \sum_{i=1}^{k} m_i \chi_{\pi_i} \quad \text{for some } m_i \in \mathbb{Z}.$$

Assume in addition that π is irreducible. Then

$$1 = (\chi_\pi, \chi_\pi) = \sum_{i,j} m_i m_j (\chi_{\pi_i}, \chi_{\pi_j})$$

$$= \sum_i m_i^2 (\chi_{\pi_i}, \chi_{\pi_i})$$

$\Rightarrow$ only possible if $\chi_\pi = \chi_{\pi_i}$ for some i, so $\pi \sim (\pi_i)_{\mathbb{C}}$.

So π is defined over $\mathbb{Q}(\zeta_m)$ ∎

(back to $\mathbb{C}$) ②

Mackey's irreducibility criterion

When is an induced representation irreducible?

> **Example**
>
> $\mathrm{Ind}_H^G \varepsilon_H = \mathbb{C}[G/H]$
>
> never irreducible

We need an (unpleasant) lemma.

Let G be a finite group and consider $K, H \subset G$ subgroups.

π a representation of H.

Define a subgroup $H_g \subset K$ by

$$H_g = gHg^{-1} \cap K \qquad g \in G$$

Then for any $g \in G$, define a representation π^g by

$$\pi^g(h) = \pi(g^{-1}hg).$$

Lemma $\mathrm{Res}_K^G \mathrm{Ind}_H^G \pi \sim \bigoplus_{\bar{g} \in K \backslash G/H} \mathrm{Ind}_{H_g}^K \pi^g$

where $K \backslash G/H = G/\sim$ where $g_1 \sim g_2$ if $g_1 = kg_2h$

Pf Let $V = V_\pi$ and W the underlying space of $\mathrm{Ind}_H^G \pi$.

We know $W = \bigoplus_{\bar{g} \in G/H} W_{\bar{g}}$ and $W_{\bar{e}} = V$.

For $x \in G/H$ and $y = Kx \overset{\text{orbit}}{\subset} G/H$, define

$$W(y) = \bigoplus_{x' \in Kx} W_{x'}$$

This is a K-invariant subspace.

Then $$W = \bigoplus_{g \in K\backslash G/H} W(g)$$

So need to show that ~~$W(g) = \mathrm{Ind}_{H_g}^{K}\pi^g$~~.

$$\left(\mathrm{Res}_K^G \mathrm{Ind}_H^G \pi\right)\Big|_{W(KgH)} \simeq \mathrm{Ind}_{H_g}^{K} \pi^g.$$

The stabiliser is K of $gH \in G/H$ is exactly $K \cap gHg^{-1}$.
$$\| \\ H_g.$$

By our characterization of induced rep's, it follows that
$$\left(\mathrm{Res}_K^G \mathrm{Ind}_H^G \pi\right)\Big|_{W(KgH)} \sim \underbrace{\left[\left(\mathrm{Res}_{H_g}^{K} \mathrm{Ind}_H^G \pi\right)\Big|_{W_{gH}}\right]}_{\mathrm{Ind}_{H_g}^{K}}$$

Thus it remains to show that $\left(\mathrm{Res}_{H_g}^G \mathrm{Ind}_H^G \pi\right)\Big|_{W_{gH}} \sim \pi^g$.

This is clearly true for $g = e$, by definition of $\mathrm{Ind}_H^G \pi$.

More generally, for any $g \in G$ we have
$$\left(\mathrm{Ind}_H^G \pi\right)(g) W_H = W_{gH}$$

It follows that $\left(\mathrm{Ind}_H^G \pi\right)(g^{-1})$ defines an equivalence between
$$\left(\mathrm{Res}_{H_g}^G \mathrm{Ind}_H^G \pi\right)\Big|_{W_{gH}} \quad \text{and}$$
$$\left(\mathrm{Ind}_H^G \pi\right)(g^{-1}) \left(\mathrm{Res}_{H_g}^G \mathrm{Ind}_H^G \pi\right)(\cdot) \left(\mathrm{Ind}_H^G \pi\right)(g)\Big/_{W_H}$$

The value of the last expression at $h \in H_g$ is

$$\left(\mathrm{Ind}_H^G \pi\right)(g^{-1}) \left(\mathrm{Ind}_H^G \pi\right)(h) \left(\mathrm{Ind}_H^G \pi\right)(g^{-1})\Big|_{W_H}$$

$$= \left(\mathrm{Ind}_H^G \pi\right)(g^{-1} h g)\Big|_{W_H} = \pi(g^{-1} h g)$$

$$= \pi^g(h). \blacksquare$$

Thm (Mackey)

Assume π is an irreducible rep. of $H < G$. Then $\mathrm{Ind}_H^G \pi$ is irreducible iff the representations $\mathrm{Res}^H_{H_g} \pi$ and $\pi^g = \pi(g^{-1} \cdot g)$ are disjoint for all $g \in G \setminus H$. (here $H_g = g H g^{-1} \cap H$, and disjoint = orthogonal characters)

Pf Let $\chi = \chi_\pi$. Then we want to compute

$$\left(\mathrm{Ind}_H^G \chi_\pi, \mathrm{Ind}_H^G \chi_\pi\right) \overset{\text{Frob. rec.}}{=} \left(\mathrm{Res}^G_H \mathrm{Ind}_H^G \chi, \chi\right)$$

Mackey's lemma $K=H$

$$= \sum_{g \in H \backslash G / H} \left(\mathrm{Ind}^H_{H_g} \chi_{\pi^g}, \chi\right)$$

Frob. reciprocity

$$= \sum_{g \in H \backslash G / H} \left(\chi_{\pi^g}, \mathrm{Res}^H_{H_g} \chi\right).$$

We see that $\left(\mathrm{Ind}_H^G \chi, \mathrm{Ind}_H^G \chi\right) = 1$

H

we have $\left(\chi_{\pi^g}, \chi|_{H_g}\right) = 0$ for $g \notin H$.

$\blacksquare$

This criterion is nice when $H \triangleleft G$ is normal:

Corollary If $H \subset G$ is normal, and π is a irrep of H, then $\mathrm{Ind}_H^G \pi$ is irreducible iff $\pi \not\sim \pi^g$ for $g \notin H$.

This can also be formulated as follows: we can define a action of G on $\hat{H}$ by $g[\pi] = [\pi^g]$. (π^g is a irrep of H)

Then $\mathrm{Ind}_H^G \pi$ is irreducible iff the stabilizer of $[\pi]$ in G is H.

Induction from normal subgroups

Assume G is a finite group, $H \subset G$ normal subgroup.

Q: Can we describe the representation theory in terms of those of H and G/H?

(answer: almost)

Assume (W, θ) is an irrep of G. Let π be an irreducible representation contained in $\mathrm{Res}_H^G \theta$.

Then $\pi^g = \theta(g^{-1})\pi(-)\theta(g)$ is also contained in $\mathrm{Res}_H^G \theta$.

Moreover, if we denote by $W(\pi) \subset W$ the isotypic component of $\mathrm{Res}_H^G \theta$ corresponding to π.

Then $\theta(g) W(\pi) = W(\pi^g)$.

It follows $\sum_{g \in G} W(\pi^g)$

is a G-invariant subspace of W. By irreducibility of θ, we have $\sum_{g \in G} W(\pi^g) = W$.

Therefore θ determines an orbit $G[\pi]$ in $\hat{H}$ and

$$W = \bigoplus_{x \in G[\pi]} W(x)$$

(and it acts as $\theta(g) W(x) = W(gx)$)

It follows that if $I(x)$ is the stabilizer of $x \in \hat{H}$ in G (called the inertia subgroup of x) then

$$\theta \sim \mathrm{Ind}_{I(x)}^{G} \left(\mathrm{Res}_{I(x)}^{G} \theta \big|_{W(x)} \right)$$

What can we say about $\mathrm{Res}_{I(x)}^{G} \theta$? In other words, assume we ~~have~~ take $[\pi] \in \hat{H}$, then

$$I = I([\pi]) = \{ g \in G / \pi^g \sim \pi \}.$$

Can we describe all irreps η of I s.t.

$\mathrm{Res}_H^I \eta$ is isotypic to π (decomp. as copies of π)?

It can be shown that all such reps of I can be described in terms of projective irreps of I/H. 7

Def A projective rep. of a group K on a v. space V is a homomorphism $K \to PGL(V) \underset{def}{=} GL(V)/\mathbb{C}^* \cdot id_V$

Why do proj. reps appear?

We have that $\pi^g \sim \pi$ $\forall g \in I$.

So there exists an invertible $T_g \in GL(V_\pi)$ s.t.

$$\pi(g^{-1}hg) = T_g^{-1}\pi(h)T_g \quad \forall h \in H$$

by irreducibility (and Schur's lemma), T_g is ! up to a scalar factor.

This implies that $T_{g_1 g_2} = c(g_1, g_2) T_{g_1} T_{g_2}$ for some $c(g_1,g_2) \in \mathbb{C}^*$.

Therefore the operators T define a projective rep of I on V. ∎

In some cases proj. reps do NOT appear. We will make the following assumptions:

- H abelian (so π is a homomorphism $H \to \mathbb{C}^*$)
- π extends to a homomorphism $I \to \mathbb{C}^*$