Finish pf of Frobenius' thm)

$G \curvearrowright X$ transitive action

Thm $\exists$ ~~K~~ $\trianglelefteq G$ acting freely transitively on ~~X~~.

$K = \{e\} \cup \{g \mid g \text{ has no fixed pts}\}$

Problem show K is a group. Let $H = G_x$ for $x \in X$.

$$(*) \quad G = K \sqcup \left(\bigsqcup_{y \in X} G_y \setminus \{e\} \right)$$

$$= K \sqcup \left(\bigsqcup_{g \in G/H} gHg^{-1} \setminus \{e\} \right)$$

Start w/ π an irreducible rep. of H.

$$\tilde{\chi}_\pi = \mathrm{Ind}_H^G \chi_\pi - (\dim \pi) \mathrm{Ind}_H^G 1 + (\dim \pi) \cdot 1.$$

Then $\tilde{\chi}_\pi(g) = \begin{cases} \dim \pi & g \in K \\ \chi_\pi(h) & \text{if } g \text{ is conj. to } h \in H \end{cases}$

Using $(*)$ again, we have

$$(\tilde{\chi}_\pi, \tilde{\chi}_\pi) = \frac{1}{|G|} \sum_{g \in G} |\tilde{\chi}_\pi(g)|^2 = \frac{1}{\cancel{|G|}} \frac{\cancel{|G|}}{|H|} \dim \pi^2 + \frac{1}{\cancel{|G|}} \frac{\cancel{|G|}}{|H|} \sum_{h \in H \setminus \{e\}} |\chi_\pi(h)|^2$$

$$= \frac{1}{|H|} \sum_{h \in H} |\chi_\pi(h)|^2 = (\chi_\pi, \chi_\pi) = 1.$$

As $\hat{\chi}_\pi$ is a comb. is a comb. of characters w/ integral coeffs; this is possible only if $\hat{\tilde{\chi}}_\pi = \pm \chi_{\tilde{\pi}}$ for some irrep $\tilde{\pi}$ of G.

As $\tilde{\chi}_\pi(e) = \dim \pi > 0$ we conclude that

$\hat{\chi}_\pi = \chi_{\tilde{\pi}}$.

- Note that as $\chi_{\tilde{\pi}}\big|_H = \chi_\pi$ we have $\mathrm{Res}^G_H \tilde{\pi} \sim \pi$.

<u>Claim</u> We claim that

$$K = \bigcap_{[\pi] \in \hat{H}} \ker \tilde{\pi} \qquad \text{(implies that } K \text{ is a normal subgr!)}$$

$\subseteq$ Indeed, take $g \in K$. Then $\chi_{\tilde{\pi}}(g) = \dim \pi = \dim \tilde{\pi}$

But $\tilde{\pi}(g)^n = 1$ for some $n \geq 1$. So $\tilde{\pi}(g)$ is diag. w/ roots of unity on the diagonal. This is possible only if

$\tilde{\pi}(g) = 1$. So $g \in \ker \tilde{\pi}$.

$\supseteq$ Now take $g \in G \setminus K$. Then g is conj. to an element $h \in H \setminus \{e\}$. Then we can find some irrep π s.t. $\pi(h) \neq 1$. Hence $\tilde{\pi}(h) \neq 1$
$[\pi] \in \hat{H}$

Finally, if $K_0 \subset G$ acts freely on X, then $K_0 \subset K$ by def of K, so if K_0 also acts transitively, then $|K_0| = |X| = |K|$.
Hence $|K| = |K_0|$.

Example Let k be a finite field. Let G be the "$ax+b$"-group, i.e. the group of affine transformations of k.

Thus $G \simeq \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \, a \in k^\times, b \in k \right\}$

Then G satisfies the assumptions of the Thm.

(Frobenius group)

The group K is the group of translations

Then $K = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in k \right\}$.

Main today

Thms of Artin of Brauer

Thm (Artin)

Let G be a finite group, $\mathcal{C}$ a collection of subgroups of G. Then TFAE

① Any character of G is a rational combination of characters of rep's induced from subgrp's in $\mathcal{C}$.

② Any $g \in G$ is conj. to an element in a subgrp in $\mathcal{C}$.

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① can be formulated as follows. For any repr. π of G there $\exists$ $m \in \mathbb{N}$ ~~and~~ and reps π_1, π_2 s.t.

$$\underbrace{\pi \oplus \cdots \oplus \pi}_{m} \oplus \pi_1 \sim \pi_2$$

4

Pf „surprisingly simple"

(i) $\Rightarrow$ (ii)

Recall that the characters span the space of central functions.

At the same time, the characters induced from subgrps in $\mathcal{C}$ are zero on elements that are not conj. to elements of subgrps in $\mathcal{C}$.

(this proves the implication)

(immediate consequence of $(\mathrm{Ind}_H^G \chi)(g) = \sum_{g \in G/H} \chi(s^{-1} g s)$)

(ii) Denote by $R_{\mathbb{C}}(G)$ the space of central functions on G w/ the usual scalar product.

$$(f_1, f_2) = \frac{1}{|G|} \sum_g f_1(g) \overline{f_2(g)}$$

Consider the map $T : \bigoplus_{H \in \mathcal{C}} R_{\mathbb{C}}(H) \longrightarrow R_{\mathbb{C}}(G)$

$$T\left((f)_{H \in \mathcal{C}}\right) = \sum_{H \in \mathcal{C}} \mathrm{Ind}_H^G f_H$$

same formula as for characters

Recall that by Frobenius reciprocity

Res_H^G is adjoint to Ind_H^G.

Therefore the adjoint map $T^* : R_{\mathbb{C}}(G) \to \bigoplus_{H \in \mathcal{C}} R_{\mathbb{C}}(H)$ is given by

$$T^*(f) = \left(\mathrm{Res}_H^G f\right)_{H \in \mathcal{C}}$$

By assumption $\ker T^{\vee} = 0$

(b.c. any element is conj. to something in $\mathcal{H} \in \mathcal{C}$.

$\ker T^{\vee} = 0$ iff ~~im~~ T ~~is~~ is surjective,

"we didn't do any real work"

This means that we can find a basis in $R_{\mathbb{C}}(G)$ consisting of characters $\chi_1, \dots, \chi_n$ of reps induced from subgrps in $\mathcal{H} \in \mathcal{C}$.

The char's $\chi_1', \dots, \chi_n'$ of irreps of G form a basis in $R_{\mathbb{C}}(G)$.

Therefore the transition matrix from $\{\chi_1, \dots, \chi_n\}$ to $\{\chi_1', \dots, \chi_n'\}$ has integer entries $\Rightarrow A^{-1}$ has rational entries.

Therefore any χ_i' is a $\mathbb{Q}$-linear comb. of $\chi_1, \dots, \chi_n$.

$\blacksquare$

Exc If $f: A \to B$ is a homomorphism of abelian grp s.t. $f \otimes \mathbb{C} : A \otimes \mathbb{C} \to B \otimes \mathbb{C}$ is surjective, then $f \otimes 1_{\mathbb{Q}} : A \otimes \mathbb{Q} \to B \otimes \mathbb{Q}$ is surjective.

As $\mathcal{C}$ we can take the collection of all cyclic subgroups.

Corollary Any character can be written as a $\mathbb{Q}$-linear comb. w/ rational coeff's of characters induced from cyclic subgps.

(Q: is this monomial?)

The following result is more difficult to prove (see e.g. Serre) or Lang's Algebra book)

Fact (Brauer) For any finite G, the character of G can be written as a linear comb. w/ integer coeff's of characters $\mathrm{Ind}_H^G \chi_\pi$ for one-dim rep's π of subgps $H \subset G$. (as H can take products of cyclic and p-groups)

Field of ~~de~~ def' of a rep

Let $\pi : G \to GL(V)$ be a rep over cplx nr's.

Let $K \subseteq \mathbb{C}$. We say that π is <u>def over K</u>

if $\exists$ vspace V' over K ~~st~~ and a rep $\pi' : G \to GL(V')$

s.t. π is equivalent to $\pi'_{\mathbb{C}}$ where

$$\pi'_{\mathbb{C}}(g) = 1 \otimes \pi(G) : \mathbb{C} \otimes_K V' \to \mathbb{C} \otimes_K V'.$$

Example ① $G = \mathbb{Z}/n\mathbb{Z}$ But reps are chars, powers of $e^{\frac{2\pi i}{n}}$.

So any rep of G is defined over $\mathbb{Q}(\zeta_n)$

② But if $G = S_n$ then it is def over $\mathbb{Q}$.

(b.c. the Specht modules are def / $\mathbb{Q}$

Thm (Brauer)

Let G be a finite group, n the exponent of G

($=$ lcm of ords of elements in G).

Then any cplx rep of G is defined over $\mathbb{Q}(\zeta_n)$.

The main difference of rep th. over K vs over $\mathbb{C}$ is that if π is an irrep over K, then $End(\pi)$ form a division algebra over K rather than K. But many results remain true for K. E.g.

- Maschke's thm (complete reducibility)
- If π', π'' are irreducible, over $\pi' \not\cong \pi''$ then $Mor(\pi', \pi'') = 0$

8

and $(\chi_{\pi'}, \chi_{\pi''}) = 0$.

Note that $\chi_{\pi'_\sigma} = \chi_{\pi'}$

This, together w/ the last propos, implies that if π', π'' are irreps over K, then π'_σ, π''_σ are disjoint.

w/ $\pi' \not\cong \pi''$

i.e. $\mathrm{Mor}(\pi'_\sigma, \pi''_\sigma) = 0$.

Next week

ind. rep

after that finish w/ finite grps