Recall distributions:

$D \subset TM$ a subbundle

An immersed submanifold $N \subset M$ is called an integral manifold of D if $T_pN = D_p \ \forall p \in N$.

Fact (Frobenius integrability thm)

If D is integrable, then $\forall p_0 \in M$ there exists a coordinate chart (x, U) around p_0 s.t.

$$D_p = \mathrm{Span}\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^k}\Big|_p \right\} \quad \forall p \in U.$$

Pf surprisingly short

This implies that locally there exists a ! integral manifold passing through a given point.

$$N = \{ p \in U \mid x^{k+1}(p) = x^{k+1}(p_0), \dots, x^n(p) = x^n(p_0) \}$$

Uniqueness: if N' is another integral manifold passing through p_0 then $N \cap N'$ is open in N and N'.

This implies that every point lies in a ! maximal connected integral manifold called a leaf of the foliation defined by D.

Pf of the thm on Lie subgroups: Consider the distribution D on G defined by $D_g = (d_e l_g)(\mathfrak{h})$. As $\mathfrak{h}$ is a Lie subalgebra, D is integrable.

Since left-translations map D into itself they map leaves into leaves of the corresponding foliation.

It follows that if H is the leaf passing through e, then $H = hH \ \forall h \in H$. (B.c. both hH and H are leaves passing through h)

Therefore $H \hookrightarrow G$ is a subgroup.

It is not difficult to see that H is a Lie group.

If $K \subset G$ is another connected Lie subgroup $\blacksquare$ w/ Lie algebra $\mathfrak{h}$, then K would be an integral manifold of D passing through e. So K is an open subgroup $\Rightarrow$ & closed small.

$$K = H \setminus \left(\bigcup_{h \notin K} hK \right).$$ As H is connected, $H < K$. $\blacksquare$

Simply connected Lie groups

Prop Assume $\pi : G \to H$ is a Lie group homomorphism and G connected. Then π is completely determined by

$$\pi_* : \mathfrak{g} \to \mathfrak{h}.$$

Pf Consider the subgroup K generated by $\exp \mathfrak{g} \subseteq G$.
As $\exp \mathfrak{g}$ contains an open neighborhood of e, K is an open subgroup of G. $\Rightarrow$ K closed $\Rightarrow$ $K = G$ since G is connected.
This proves the result since $\pi(\exp X) = \exp \pi_*(X)$. $\blacksquare$

But it is not true that any homomorphism of Lie algebras defines or integrates to a homomorphism of Lie groups.

Example $G = \mathbb{R}/\mathbb{Z} \simeq S^1$ $\quad H = \mathbb{R}$

$\mathfrak{g} = \mathfrak{h}$ is abelian $[X,X]=0$

But the $id : \mathfrak{g} \to \mathfrak{h}$ does not integrate to a homomorphism $\mathbb{R}/\mathbb{Z} \to \mathbb{R}$

Recall that a map $p: N \to M$ is a <u>covering map</u> if p is cont's and surjective and a <u>local homeomorphism</u>

s.t. the preimage of $p \in M$, $U \ni p$: $p^{-1}(U) \simeq \bigsqcup_{i \in I} V_i$.

<u>Example</u> Assume $\pi: G \to H$ is a Lie group homomorphism, H connected and $\pi_*: \mathfrak{g} \to \mathfrak{h}$ is a iso.

Then π is a covering map:

– We can find $U \ni e \in G$ s.t. $\pi|_U \to \pi(U)$ is a diffeomorphism. Choose a neighborhood V of $e \in G$ s.t. $VV^{-1} \subset U$

Note that as π is open and H connected, π is surjective.

Then for $\forall g \in G$, we have

$$\pi^{-1}\left(\pi(g)\,\pi(V)\right) = \bigsqcup_{h \in \ker \pi} hgV$$

We have to check that $h_1gV \cap h_2gV = \phi$ for $h_1, h_2 \in \ker\pi$, $h_1 \neq h_2$.

Assume $h_1 g v_1 = h_2 g v_2$ for some $v_1, v_2 \in V$.

$\Rightarrow$ $g^{-1} h_2^{-1} h_1 g = v_2 v_1^{-1} \in VV^{-1} \subset U$

Hence $\underbrace{g^{-1} h_2^{-1} h_1 g}_{\in \ker \pi} = e$ so $h_1 = h_2$.

If a topological space M is nice (e.g. a manifold) connected then there exists a universal covering

$$\pi: \tilde{M} \to M$$

$$\begin{tikzcd}
\tilde{M} \arrow[r, dashed, "\exists!\, \tilde{f}"] \arrow[d, "\pi"'] & P \arrow[d, "p"] \\
M \arrow[r, "f"] & N \\
\tilde{p}_0 \arrow[u, phantom, sloped, "\in"] &
\end{tikzcd}$$

s.t. $\tilde{f}(\tilde{p}_0) = p_0$

Furthermore a covering ~~space~~ map $\pi: Q \to M$ is universal iff Q is simply connected (Q connected + $\pi_1(Q) = 0$).

Explicitly $\tilde{M}$ can be defined as the set of equiv. classes $[\gamma]$ of curves $\gamma: [0,1] \to M$ s.t. $\gamma(0) = p_0$ and $\gamma_1 \sim \gamma_2$ if $\gamma_1(1) = \gamma_2(1)$ and γ_1 and γ_2 are homotopic within the class of curves ending at $\gamma_1(1) = \gamma_2(1)$.

Prop 5 Let G be a connected Lie group. $\pi: \tilde{G} \to G$ a universal covering. Then we can introduce a Lie group structure on $\tilde{G}$ so that π becomes a Lie group homomorphism.

Pf. sketch (1) Let $\tilde{e} \in \pi^{-1}(e)$. Then using that $\pi \times \pi: \hat{G} \times \tilde{G} \to G \times G$ is a universal cover, there exist a unique lift of the product map $m: G \times G \to G$ to a map $\hat{m}: \hat{G} \times \hat{G} \to \hat{G}$ s.t. $\tilde{m}(\hat{e}, \hat{e}) = \hat{e}$. Then $\hat{G}$ is a Lie group, w/ inverse given by the lift of the map st. $\hat{e} \mapsto \tilde{e}$.

More concretely, if we think of $\hat{G}$ as the set of equivalence classes of paths γ w/ $\gamma(0) = e$ then

$$[\gamma_1] \cdot [\gamma_2] = [t \mapsto \gamma_1(t)\gamma_2(t)]$$
$$[\gamma]^{-1} = [t \mapsto \gamma(t)^{-1}].$$

Thm Let G, H be Lie groups w/ G simply connected. Then any Lie algebra homomorphism
$$\rho: \mathfrak{g} \to \mathfrak{h}$$ integrates to a Lie group homomorphism $\Pi: G \to H$.

Pf Consider the Lie subalgebra

$$\mathfrak{h}_\rho = \{(X, \rho(X)) \mid X \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{h}.$$

Let $P \subset G \times H$ be the connected Lie subgp w/ Lie algebra $\mathfrak{h}_\rho$. Consider the projection $p_1: G \times H \to G$ and $q = p_1|_P$. Then $q_*: \mathfrak{h}_\rho \to \mathfrak{g}$ is an iso.

Hence $q: P \to G$ is a covering map. As G is simply connected, it does not have non-trivial coverings. So q is an iso.

Then we can define Π as the composition of
$q^{-1}: G \to P \subset G \times H$ w/ the projection $p_2: G \times H \to H$ $\boxed{\text{N}}$

Corollary Two simply connected Lie groups are isomorphic G, H

$\Leftrightarrow \mathfrak{g} \simeq \mathfrak{h}$.

(due to Cartan)

Fact ("Lie's third fundamental thm")

For any fin. dim. real Lie algebra $\mathfrak{g}$ there exists a simply connected Lie group G w/ Lie algebra $\mathfrak{g}$.

$\boxed{\text{not what Lie proved}}$ that's for sure

One possible pf is based on

Fact (Ado's thm)

Any f.-dim. real Lie algebra is isomorphic to a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$.

Given this $\dagger$, we can take $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$, we can take G to be the Lie subgroup of $GL_n(\mathbb{R})$ corresp. to $\mathfrak{g}$ and then pass to the universal covering.

Remark Not every Lie group can be embedded into $GL_n(\mathbb{R})$.

To summarize the last results, we can say that the category of simply connected Lie groups is equiv. to the categ. of fin. real Lie algebras.

If G is connected, but not simply connected, then

$G \simeq \tilde{G}/\Gamma$ for a discrete normal subgrp $\Gamma \subset \tilde{G}$.

Exc If G is a connected top grp, and $\Gamma \subset G$ is a discrete normal subgrp, then $\Gamma \subset Z(G)$.

Therefore $G \cong \tilde{G}/\Gamma$ w/ Γ a discrete subgrp of $Z(\tilde{G})$.

Example $SU(n)$, $Sp(n)$ are simply connected.

$SO(n,\mathbb{R})$ is connected, but not simply connected.

$\pi_1(SO(2)) = \pi_1(S^1) = \mathbb{Z}$

$n>2$ $\pi_1(SO(n)) = \mathbb{Z}/2$ (universal cover = „Spin group")

We will prove (almost) this later.