1

Prop Let G be a compact connected Lie group, $T \subset G$ a maximal torus. Then

1. $N(T) = \{ g \in G \mid gTg^{-1} = T \}$ is closed
2. $N(T)^0 = T$;
3. $W = W(G,T) = N(T)/T$ is a fin. ~~subgr~~ group

Pf 1 is obvious.

2 Consider the homomorphism

$$\alpha : N(T) \longrightarrow \mathrm{Aut}(T)$$
$$g \longmapsto \alpha(g)(h) = ghg^{-1}$$

is a homomorphism

$$N(T) \longrightarrow GL_n(\mathbb{Z}) \subset GL_n(\mathbb{R})$$

(indeed, an automorphism of $\mathbb{R}^n/\mathbb{Z}^n$ is completely determined by $d_e\alpha$, that is a linear map $\mathbb{R}^n \to \mathbb{R}^n$.

Note that a linear map $\mathbb{R}^n \to \mathbb{R}^n$, that is, a map $A \in \mathrm{Mat}_n(\mathbb{R})$, defines a homomorphism $\mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ iff $A \in \mathrm{Mat}_n(\mathbb{Z})$. This is an ~~homomorphism~~ automorphism iff $A \in GL_n(\mathbb{Z})$.)

As $GL_n(\mathbb{Z})$ is discrete, ~~$N(T) \to GL_n(\mathbb{Z})$~~ α is locally constant. So that $N(T)^0 \subset \ker\alpha$. Thus $gh = hg$ $\forall g \in N(T)^0$ $\forall h \in T$. Let X be an element of the Lie algebra of $N(T)$. Then $\exp(tX)$ commutes w/ T $\forall t \in \mathbb{R}$. It follows that the closure of the group generated by $\exp(tX)$ $(t \in \mathbb{R})$ is a compact connected abelian Lie group $\Rightarrow$ torus.

But as T is maximal, this torus coincides w/ T, (2)

so $\exp(\tau X) \in T \quad \forall \tau \in \mathbb{R}$.

Thus the Lie algebra of $N(T)$ coincides w/ the Lie algebra of T.

Hence $N(T)^\circ = T$.

(3) By (ii), we have $\dfrac{N(T)}{T} = \dfrac{N(T)}{N(T)^\circ} \leftarrow$ finite b.c. $N(T)$ is compact

Example $G = U(n)$. $T = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subset G$.

Then $N(T) = \{ g \in U(n)$ s.t. every row/column of g contains exactly one nonzero coefficient $\}$.

Thus $\dfrac{N(T)}{T} \cong S_n$.

Digression on Riemannian geometry (he thinks this proof (to come) is originally by Cartan)

Recall that a Riemannian manifold is a manifold M s.t. T_pM is equipped w/ a scalar product depending smoothly on p.

So if X, Y are (smooth) v. fields, then $p \mapsto (X_p, Y_p)$ is smooth.

Equivalently, the scalar product defines a smooth section of $S^2(T^*M)$.

Given a piecewise C^1-curve, we define its length by

$$L(\gamma) \stackrel{\Delta}{=} \int_a^b \|\gamma'(t)\| \, dt.$$

Then we can define a metric on M (assuming M is connected) by

$$d(a,b) = \inf \left\{ L(\gamma) \;\middle|\; \begin{array}{l} \gamma : [0,1] \to M, \text{ piecewise } C^1 \\ \gamma(0) = a, \quad \gamma(1) = b \end{array} \right\} \quad (5)$$

It is not difficult to see that the topology on M coincides w/ the topology defined by this metric d.

A map $\gamma : [a,b] \to M$ is called a geodesic if for any $t_0 \in (a,b)$ there $\exists\, \delta > 0$ s.t. $d(\gamma(s), \gamma(t)) = |s-t|$ for $s,t \in (t_0 - \delta, t_0 + \delta)$

We also allow linear change of variables, so $t \mapsto \gamma(\alpha t)$ is also called a geodesic for $\alpha > 0$.

Fact Let M be a Riemannian manifold. Then

(i) Any geodesic is smooth.

(ii) for any $p \in M$ and $X \in T_pM$, $X \neq 0$, there exists a unique maximal geodesic γ s.t. $\gamma(0) = p$, $\gamma'(0) = X$.

(iii) (Hopf–Rinow thm) If M is compact connected, then any maximal geodesic is defined on the whole line $\mathbb{R}$; and for any $p, q \in M$, there exist a geodesic passing through p and q.

Returning to Lie groups, from now we assume that G is a compact connected Lie group. Consider the adjoint representation $Ad : G \to GL(\mathfrak{g})$. Since G is compact, there exist an $(\mathrm{Ad}\, G)$-invariant scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. (4)

Then $\mathrm{Ad}: G \to O(\mathfrak{g}, \langle,\rangle)$

Recall that the Lie algebra of $O(\mathfrak{g})$ consists of $T \in \mathfrak{gl}(\mathfrak{g})$ s.t.

$$\langle TX, Y\rangle + \langle X, TY\rangle = 0.$$

Therefore $\mathrm{ad}\,\mathfrak{g}$ is contained in the Lie algebra, so

$$\langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle = 0.$$

Now we use left- or right-translations to define a scalar product on $T_g G$ for any $g \in G$.

Thus G becomes a Riemannian manifold such that both left and right translations preserve the Riemannian structure.

Thm The geodesics in G are translations (left or right) of one-parameter subgroups of G.

(note that $g\exp(tX) = (g\exp(tX)g^{-1})g = \exp(t(\mathrm{Ad}\,g)(X))g$)

Pf The ~~statement~~ result is true for the tori $\mathbb{R}^n/\mathbb{Z}^n$, as geodesics in $\mathbb{R}^n$ are lines.

In general, it suffices to consider geodesics passing through e.

Let $\gamma: \mathbb{R} \to G$ be a geodesic, $\gamma(0) = e$. Put $X = \dot\gamma(0) \in \mathfrak{g}$.

For any $g \in G$, the map $t \mapsto g\gamma(t)g^{-1}$ the map is also a geodesic, and its derivative at 0 is $\mathrm{Ad}(g)(X)$ (by def of Ad)

Let T be a maximal torus containing $\exp(\tau X)$, $\tau \in \mathbb{R}$. (5)

Then $(\mathrm{Ad}_g)(X) = X \quad \forall g \in T$.

It follows that $\gamma(\tau) = g\gamma(\tau)g^{-1} \quad \forall \tau \in \mathbb{R}, \forall g \in T$.

In particular $\gamma(t) \in N(T)^{\circ} = T$.

Thus $\mathrm{Im}\,\gamma \subset T$. Hence γ is a geodesic in T (where the results is true)

Hence $\gamma(t) = \exp(tX)$, since the thm is true for T. ∎

Thm (Cartan) ← completely fundamental for everything that follows!!

Let G be a compact connected Lie group, $T \subset G$ a maximal torus. Then $\forall g \in G$, $\exists h \in G$ s.t. $hgh^{-1} \in T$.

Pf Let $t_0 \in T$ be a topological generator for T.

Let $H_0 \in \mathfrak{t}$ be such that $\exp(H_0) = t_0$.

Choose a geodesic γ passing thru e and g w/ $\gamma(0) = e$.

Let $X = \gamma'(0)$. Then

$\gamma(\tau) = \exp(\tau X)$ by the prev's thm.

Consider the function $G \ni h \longmapsto \langle (\mathrm{Ad}\,h)(X), H_0 \rangle$

Let h be a pt where this function obtains its maximum.

Claim $[(\mathrm{Ad}\,h)(X), H_0] = 0$.

Assuming the claim, it follows that $\exp(\tau(\mathrm{Ad}\,h)(X)) = h\exp(\tau X)h^{-1}$

commutes w/ $\exp(H_0) = t_0 \quad \forall \tau \in \mathbb{R}$

As g is a topological generator of T, it follows (6)

that $h\exp(tX)h^{-1}$ commutes w/ T.

As T is a maximal torus, it follows that $h\exp(tX)h^{-1} \in T$ $\forall t$.

In particular $hgh^{-1} \in T$.

Remains to prove the claim.

Take $Y \in \mathfrak{g}$. Then the function

$$t \longmapsto \langle \mathrm{Ad}(\exp(tY)h)(X), H_0 \rangle$$

attains its maximum at $t=0$ therefore its derivative at 0 is zero.

so
$$0 = \langle \mathrm{ad}(Y)((\mathrm{Ad}\,h)(X)), H_0 \rangle$$
$$= -\langle [(\mathrm{Ad}\,h)X, Y], H_0 \rangle$$
$$\overset{\text{invariance}}{=} \langle Y, [(\mathrm{Ad}\,h)(X), H_0] \rangle \quad \text{by invariance of the scalar product}$$

(true $\forall Y \Rightarrow \mathcal{F}$ is zero. $\blacksquare$

Corollary: Any two maximal tori in G are conjugate.

Pf: Let $g \in T_1$ be a topological generator. Then $\exists h \in G$ w/ $hgh^{-1} \in T_2$. But then $hT_1h^{-1} \subseteq T_2$.

so $T_1 \subseteq h^{-1}T_2h$ we get $=$ by maximality.

$\blacksquare$

Corr Any element of G is contained in a maximal torus.

In particular $\exp: \mathfrak{g} \to G$ is surjective.

Pf As $g \in G$, T a maximal torus. Then $\exists\, h \in G$ s.t. $hgh^{-1} \in T$ i.e. $g \in \underbrace{h^{-1}Th}_{\text{maximal torus}} \ni g$.

Corr Any maximal torus T in G is a maximal abelian subgroup of G.

Pf Assume g commutes w/ T.

Consider $\underbrace{H = C(g)^{\circ}}_{\text{compact connected Lie group}} = \{h \in G \mid hg = gh\}^{\circ}$

As T is connected, $T \subset H$. As g is contained in a maximal torus in G, we have $g \in H$.

As T is a maximal torus in G, it is a maximal torus in H. Hence $\exists\, h \in H$ s.t. $\underset{\overset{\|}{g}}{hgh^{-1}} \in T$. as $g \in Z(H)$.

~~This implies that the map~~ ☒

$$\alpha: W(G,T) \longrightarrow \mathrm{Aut}(T)$$
$$g \longmapsto \alpha(gT)(h) = ghg^{-1}$$

has trivial kernel. Hence $W(G,T) \hookrightarrow \mathrm{Aut}(T)$.