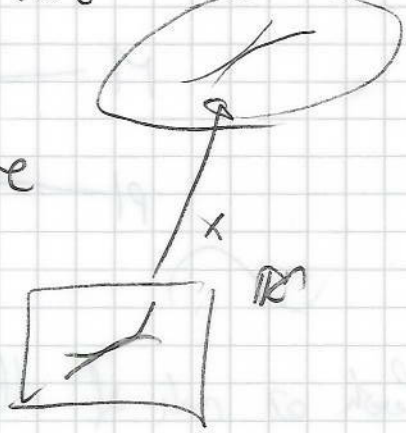
Differentialgeometri

Last time: only „differential topology".

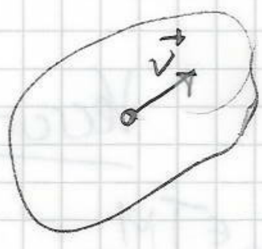
Can not look at „straightness" „up there"



Can say that two curves are tangent „down there".

If we have open $U \subseteq \mathbb{R}^n$. Want to look at T_pU.

$$T_pU \approx \mathbb{R}^n$$
$$[c] \longmapsto c'(0)$$
$$[t \mapsto p+tv] \longmapsto v$$



[Can also look at T_pU as derivatives $D_p \to D_p$]

$$TM = \bigsqcup_{p \in M} T_p$$ ← disjoint union.

← tangent bundle of M

We give it a differentiable structure in the following way:

$$U \times \mathbb{R}^n \longrightarrow TM|x(U) = \pi^{-1}(x(U))$$
$$(q, v) \longmapsto dx_q(v) \in T_{x(q)}M$$

$X: U \to M$ chart

$$
\begin{tikzcd}
T_pM \subseteq TM \arrow[r, "df"] \arrow[d] & TN \supseteq T_pN \arrow[d] \\
M \arrow[r] & N \\
p \arrow[r, mapsto] & f(p)
\end{tikzcd}
$$

Look at rank of df_p. If rank df_p is <u>maximal</u>, then it is maximal on a neighborhood of p as well.

<u>Vector field</u>

$$
\begin{tikzcd}
X_p \in TM \arrow[d] \arrow[d, shift left=4, "\pi"] \\
p \in M \arrow[u, bend right=60, "X"']
\end{tikzcd}
$$

<u>Examples</u> $U \xrightarrow{x} M$

vector field on $x(U)$ given by $\sum a_i \frac{\partial}{\partial x_i}$.

In two charts

$$
\boxed{\frac{\partial}{\partial x_i} = \sum_j \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}}
$$

The Lie bracket

$X_p : \mathcal{D}_p \to \mathbb{R}$ derivation.

Can think of $X(f) \in \mathcal{D}$ as a function $\begin{aligned} M &\to \mathbb{R} \\ p &\mapsto X(f)(p) \end{aligned}$

$\mathcal{D}$ = "functions"

as a $\begin{aligned} X : \mathcal{D} &\to \mathcal{D} \\ f &\mapsto Xf \end{aligned}$

Now if we have two vector fields, can define

$XY : \mathcal{D} \to \mathcal{D}$.

(can show "the set of all vector fields" $\approx$ "set of all derivations")

not a derivation, but

$XY - YX =: [X, Y]$ is.

Lie bracket / "Lie product"

- bilinear / $\mathbb{R}$

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Closing ~~[illegible]~~ w/ a theorem

Let X be a vector field on $\underset{\cup}{\overset{p}{}}M$. Then there is a $\varepsilon > 0$ and a neighbourhood $U \ni p$, and a smooth function

$$\varphi : (-\varepsilon, \varepsilon) \times U \longrightarrow M$$

s.t.

$$\frac{\partial \varphi}{\partial t}(t, q) = X_{\varphi(t,q)}$$

$$\varphi(0, q) = q$$

For each $u \in U \subseteq M$, you get a little curve such that X_q is realized as a tangent vector to that curve.