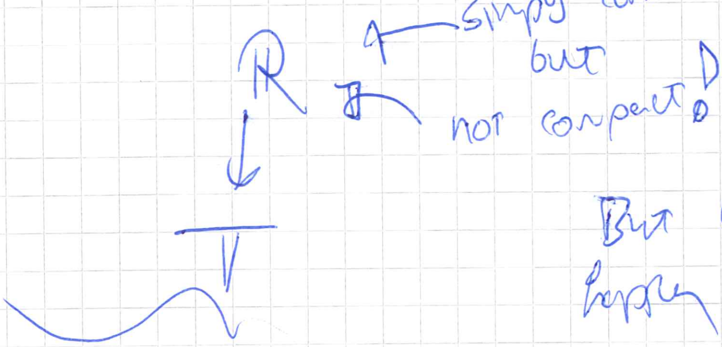
Return to Lie algebras and Lie groups



But basically the only bad thing that happens

~~Semisimple Lie groups~~

An ideal of a Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{a} \subset \mathfrak{g}$ s.t.
$[X, Y] \in \mathfrak{g}$ $\forall\ X \in \mathfrak{g}$, $\forall\ Y \in \mathfrak{a}$

Def • A Lie algebra is called semi-simple if it has no non-zero abelian ideals.

• A Lie group is called semisimple if its Lie algebra is semi-simple.

Exercise Assume G is a connected Lie group and $H \subset G$ a connected Lie subgroup. Then show that $\mathfrak{h} \subset \mathfrak{g}$ is an ideal iff H is normal.

It follows that a connected Lie group G is semisimple iff any normal abelian subgroup of G is discrete.

In particular, if G is a compact connected Lie group, then G is semisimple iff any ~~abeli~~ normal abelian subgroup is ~~discrete~~ finite.

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Recall that if $T \subset G$ is a maximal torus then

$$N(T)^\circ = T.$$

The same argument shows that if a torus T_1 is a normal subgroup, then $T_1 \subset Z(G)$.

(we have a homomorphism $G \to \mathrm{Aut}(T_1) \cong GL_k(\mathbb{Z})$
$$g \mapsto \alpha_g = g \circ g^{-1}$$

but $\mathrm{Aut}(T_1) \cong GL_k(\mathbb{Z})$ so the image consists of
discrete group.

just $e \in \mathrm{Aut}(T_1)$. Hence $ghg^{-1} = h$ $\forall g \in G$, i.e. $h \in Z(G)$))

Hence, if $A \subset G$ is a normal abelian subgroup, then $(\bar{A})^\circ$ is a torus normalized by G, so $(\bar{A})^\circ \subset Z(G)$.

Therefore G is semisimple iff $Z(G)$ is finite.

Recall next that the Lie algebra of $Z(G)$ is

$$\mathfrak{z}(G) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \;\; \forall Y \in \mathfrak{g}\}$$

Thus G is semisimple iff $\mathfrak{z}(\mathfrak{g}) = 0$. $\mathfrak{z}(\mathfrak{g}) = 0$

Fix a maximal torus $T \subset G$ and consider the corresponding root decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$. We have

We have $\mathfrak{z}(\mathfrak{g})_{\mathbb{C}} = \mathfrak{z}(\mathfrak{g}_{\mathbb{C}})$. (check!)

Assume $X = H + \sum_{\alpha\in\Delta} X_\alpha \in \mathfrak{z}(\mathfrak{g}_{\mathbb{C}})$.

Then, for any $H' \in \mathfrak{h}$, we have

$$0 = [X, H'] = \sum_{\alpha\in\Delta} \alpha(H') X_\alpha$$

hence $X_\alpha = 0 \ \forall \alpha \in \Delta_\phi$. In other words $\mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) \subset \mathfrak{h}$.

Assume $H \in \mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) \subset \mathfrak{h}$. We have

$$0 = [H, X] = \alpha(H) X \quad \text{for } X \in \mathfrak{g}_\alpha$$

hence $H \in \ker\alpha$. So we see that

$$\mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) = \bigcap_{\alpha\in\Delta} \ker\alpha$$

Thus $\mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) = 0 \iff \bigcap_{\alpha\in\Delta} \ker\alpha = 0$

$$\iff \operatorname{span}_{\mathbb{C}} \Delta = \mathfrak{h}^* .$$

Thus, for any compact connected Lie group TFAE

1. G is semisimple
2. $Z(G)$ is finite
3. $\mathfrak{z}(\mathfrak{g}) = 0$
4. $\operatorname{span}_{\mathbb{C}} \Delta = \mathfrak{h}^* \iff$ (4') $\operatorname{span}_{\mathbb{R}} \Delta = i\mathfrak{t}^* \subset \mathfrak{h}^*$

[Example $Z(SU(n)) = \left\{ \begin{pmatrix} z & & \\ & \ddots & \\ & & z \end{pmatrix} \middle| \ z^n = 1 \right\}$

$\simeq \mathbb{Z}/n$ (finite)]

Let G be a $\underbrace{\text{semisimple compact connected Lie group}}_{\text{scc group}}$

Fix a maximal torus $T \subset G$ and consider the corresponding root system Δ inside $\Delta \subset i\mathfrak{t}^*$.

As a system of simple roots $\Pi \subset \Delta$ form a basis in $i\mathfrak{t}^*$, the additive subgroup $Q < i\mathfrak{t}^*$ generated by Δ is a lattice (discr. subgr. of maximal rank) called the root lattice

Define

$$P = \left\{ \beta \in i\mathfrak{t}^* \,\middle|\, \underset{\substack{\| \\ \beta(H_\alpha)}}{\frac{2(\alpha,\beta)}{(\alpha,\alpha)}} \in \mathbb{Z}, \ \forall \alpha \in \Delta \right\}$$

(so not depending upon choice of scalar product)

- P is clearly a subgroup of $i\mathfrak{t}^*$ w/ $Q < P$.
- P is a lattice: indeed, if $\beta \in P$ is close to zero, then $(\alpha,\beta) = 0$, $\forall \alpha \in \Delta$, so $\beta = 0$ since $\operatorname{span} \Delta = i\mathfrak{t}^*$.

Hence P is a lattice, it is called the weight lattice

In particular, the quotient P/Q is finite abelian.

Example Consider the root system A_1 corresponding to $SU(2)$.

$$it^* = \mathbb{R}$$

$$\Delta = \{-1, 1\}.$$

Then $Q = \mathbb{Z}$. Here $P = \frac{1}{2}\mathbb{Z}$

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ & P \arrow[r] \arrow[d, "2"] & Q \arrow[r] \arrow[d, "2"] & P/Q \arrow[d, "2"] \\ 0 \arrow[r] & 2\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/2 \\ & 0 & 0 & \end{tikzcd}$$

so $P/Q =$

Recall that the character lattice $X^*(T) \subset it^*$ satisfies

$$Q \subset X^*(T) \subset P$$

differentials from the torus

Prop. Assume G is semisimple compact connected Lie group. Then its universal cover $\tilde{G}$ is still compact.

Pf. Consider the covering map $\pi: \tilde{G} \to G$. Then

- $\ker \pi = \pi_1(G)$
- As G is a compact manifold, $\pi_1(G)$ is finitely generated.

As $\ker \pi$ is a normal discrete subgroup of $\tilde{G}$, we have $\ker \pi \subset Z(\tilde{G})$. Hence $\ker \pi \cong \mathbb{Z}^n \oplus \Gamma$ where Γ is fin. abelian.

Now to show that $n = 0$. Assume not.

Then we can find a subgroup $H \subset \ker \pi$ s.t. $\frac{\ker \pi}{H}$ is finite, but arbitrarily large.

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Pf. Consider $G_1 = \tilde{G}/H$. We have a homomorphism $\pi_1 : G_1 \xrightarrow[\text{finite 1-compact}]{} G$ w/ $\ker \pi_1 = \ker \pi/H$.

Thus G_1 is compact, but $|\ker \pi_1|$ can be chosen arbitrarily large. This will lead to a contradiction.

Fix a maximal torus $T \subset G$.

We identify $\mathfrak{g}_1$ w/ $\mathfrak{g}$ using $(\pi_1)_*$. Consider the maximal torus $T_1 = \exp_{G_1} \mathfrak{t} \subset G_1$.

Consider the root and weight lattices $Q, P \subset i\mathfrak{t}^* = i\mathfrak{t}_1^*$.

We have $Q \subset X^*(T) \subset X^*(T_1) \subset P$ (b.c. $T_1 \to T$)

Hence $\left| X^*(T_1)/X^*(T) \right| \leq |P/Q|$

$\boxed{\text{Ex}}$ Assume T_1 is a torus $\Gamma \subset T_1$ a finite subgroup and $T = T_1/\Gamma$. Show that any character of Γ extends to a character of T_1. Conclude that $\hat{\Gamma} \simeq \hat{T_1}/\hat{T}$

true since $\ker \pi_1 \subset Z(G_1) \subset T_1$

By the exercise, we have that $\widehat{\ker(\pi_1 : G_1 \to G)} = \ker(\pi_1|_{T_1} : T_1 \to T)\hat{}$

$= X^*(T_1)/X^*(T)$

Thus $|\ker \pi_1| = \left| \left(X^*(T_1)/X^*(T) \right) \right| \leq |P/Q|$ so we have a contradiction, since $|\ker \pi_1|$ can be made arb. large. $\blacksquare$

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Remark The arguments suppose that the character lattice of a maximal torus in $\tilde{G}$ is P while char of $G/Z(G)$ is Q.

True, too will see this later.

Let now G be a compact connected Lie group.

Fix a maximal torus $T \subset G$.

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha .$$

$$\cup\!|$$

$$\mathfrak{z}(\mathfrak{g})_{\mathbb{C}}$$

Consider the Lie group $Ad(G) = G/Z(G) \subset GL(\mathfrak{g})$.

The Lie algebra of $Ad(G)$ is $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$.

We know that by def. $\mathfrak{z}(\mathfrak{g}) = \ker(ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}))$

So all we need to know is that

$ad : \mathfrak{g} \to (\text{Lie alg. of } Ad(G))$ is surjective.

But this follows from the construct. with thm.

We have

$$\left(\mathfrak{g}/\mathfrak{z}(\mathfrak{g})\right)_{\mathbb{C}} = \mathfrak{h}/\mathfrak{z}(\mathfrak{g})_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

$$= \mathfrak{h}/\bigcap_{\alpha\in\Delta} \ker\alpha \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha \quad (\star)$$

We thus see that $\mathfrak{h}/\bigcap_{\alpha\in\Delta}\ker\alpha$ is a maximal abelian subalgebra of $\left(\mathfrak{g}/\mathfrak{z}(\mathfrak{g})\right)_{\mathbb{C}}$ so $T/Z(G)$ is a maximal torus in $Ad(G) = G/Z(G)$.

and $(\star)$ is ~~a root~~ the root decomposition ~~of~~ (more precisely, every $\alpha\in\Delta$ defines a linear functional $\tilde{\alpha}$ on $\mathfrak{h}/\bigcap_{\beta\in\Delta}\ker\beta$ and $\tilde{\alpha}$ are the new roots)

~~Hence~~ we see that $\bigcap_{\tilde{\alpha}\in\Delta}\ker\tilde{\alpha} = 0$ in $\mathfrak{h}/\mathfrak{z}(\mathfrak{g})_{\mathbb{C}}$. Thus $Ad(G)$ is semisimple.
$$\overset{\shortparallel}{G/Z(G)}$$

Exe If $\mathfrak{g}$ is a Lie algebra, the the linear span of elements $[X,Y]$ for $X,Y\in\mathfrak{g}$ is an ideal in $\mathfrak{g}$, called the commutator ~~of~~ ideal. Denoted $[\mathfrak{g},\mathfrak{g}]$.

Remark $T([G,G])_0 = [\mathfrak{g},\mathfrak{g}]$ if G is connected.

Prop Assume G is a compact connected Lie group. Then

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}].$$

Pf Equivalently have to show

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{z}(\mathfrak{g})_{\mathbb{C}} \oplus [\mathfrak{g}_{\mathbb{C}},\mathfrak{g}_{\mathbb{C}}]$$

$$= \bigcap_{\alpha\in\Delta} \ker\alpha \oplus [\mathfrak{g}_{\mathbb{C}},\mathfrak{g}_{\mathbb{C}}]$$

Exc $[\mathfrak{sl}_2\mathbb{C}, \mathfrak{sl}_2\mathbb{C}] = \mathfrak{sl}_2\mathbb{C}$

For every $\alpha\in\Delta$, we have a copy of $\mathfrak{sl}_2\mathbb{C}$ in $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{sl}_2\mathbb{C} \approx \mathfrak{g}_{-\alpha} \oplus \mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha$. Applying the exercise show that

$$[\mathfrak{g}_{\mathbb{C}},\mathfrak{g}_{\mathbb{C}}] = \operatorname{span}\{H_\alpha \mid \alpha\in\Delta\} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha.$$

$\blacksquare$