Making things smoother: algebraic geometry and deformation theory

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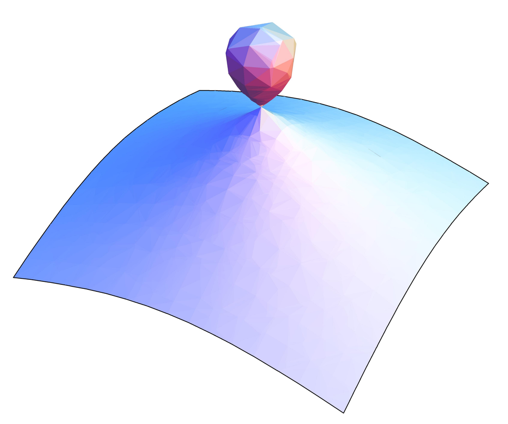
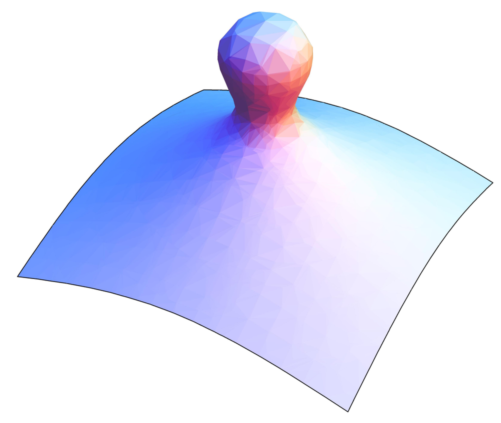
What I study in a sentence

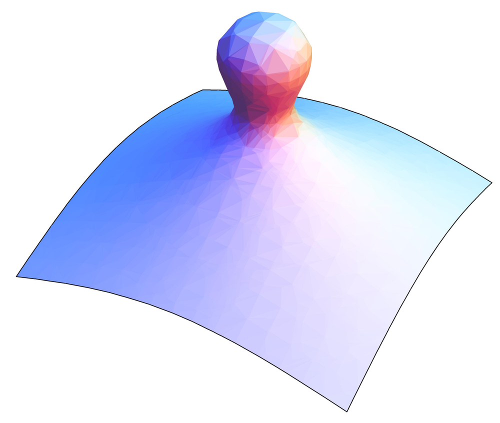
I study geometric objects defined by polynomial equations, and I study how they change under *deformations*.



Plot of $x^2 + y^2 + z^3 - z^2 = 0$.



Plot of $x^2 + y^2 + z^3 - z^2 + 0.1 = 0$.

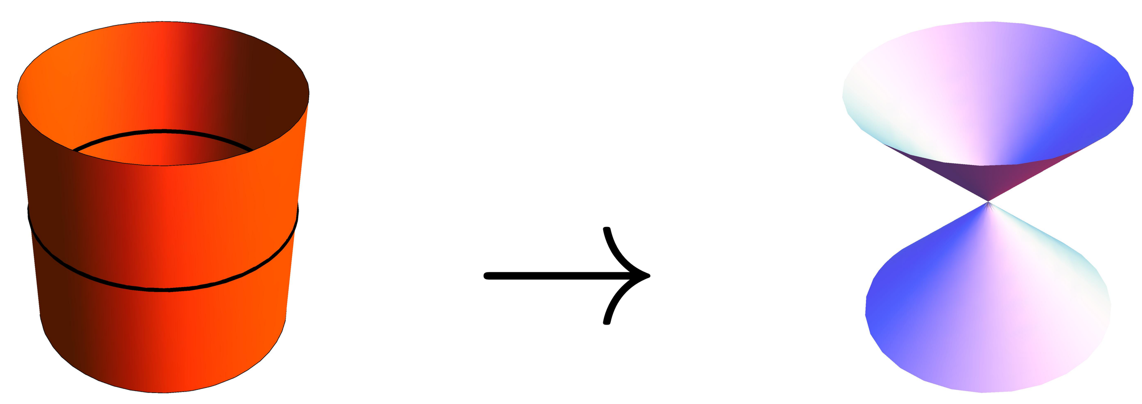


Plot of $x^2 + y^2 + z^3 - z^2 + 1 = 0$.

Deforming these geometric objects correspond to perturbing their equations by a small factor. This often results in less singular objects, as in the illustrations above. One goal of deformation theory is to find smooth deformations of a singular variety. With more variables and more equations, this quickly becomes quite complicated.

Combining algebra with geometry

Algebraic geometry is the study of systems of polynomial equations and the geometric objects they represent. Geometric properties can be translated into algebraic language. For example, singular points of surfaces are points on the surface where all the partial derivatives are zero. Since the partial derivatives are polynomials as well, finding the singular points is the same as solving a (slightly larger) system of polynomial equations.



The resolution of the cone together with its exceptional divisor.

A rational double point, given by the equation $x^2 + y^2 = z^2$.

One way get rid of singularities, is to *resolve* them. This is done by removing the singular point and replacing it with something smooth of higher dimension. In the figure above, the singular point is replaced by the circle in black. This is called *blowing up a point*.

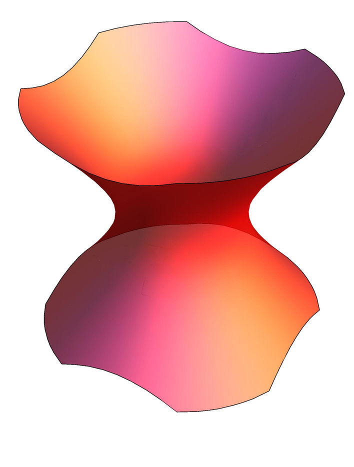
Hilbert schemes

One of the big problems in algebraic geometry is *classification*. One wants to classify geometric objects up to some kind of similarity. One way to do this is to study the *Hilbert scheme*, whose points correspond to different objects with the same Hilbert polynomial.

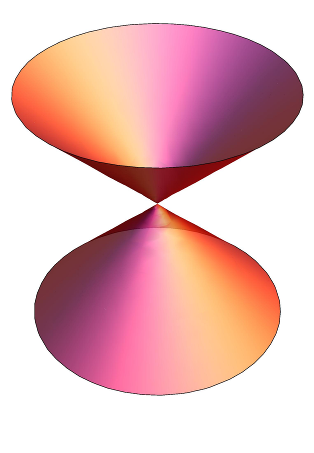
For example, say you want to classify quadric surfaces. These are the surfaces defined by the equation

$$a_0x^2 + a_1xy + a_2xz + a_3xw + a_4y^2 + a_5yz + a_6yw + a_7z^2 + a_8zw + a_9w^2 = 0,$$

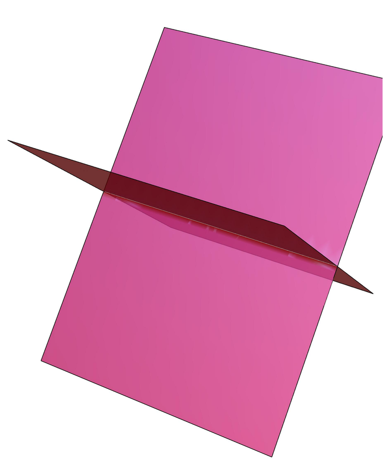
where the a_i's are complex numbers. Counting constants, we see that there are 10 degrees of freedom, except that scaling the equation doesn't change the surface. Hence the *parameters* of the equation constitute a 9-dimensional space. Each point correspond to a quadric surface in $\mathbb{P}^3_{\mathbb{C}}$.



A smooth quadric surface.



A quadric cone.



Two intersecting planes.

A double plane.

The "points" of the Hilbert scheme correspond to the four types of quadric surfaces.

There are exactly four types of quadric surfaces: Most quadric surfaces are smooth. Then there are those with a single singular point (these constitute a 8-dimensional subvariety of the Hilbert scheme). Furthermore, if the polynomial is reducible, the surface splits into two intersecting planes (these make up a 6-dimensional subvariety). Finally, if the polynomial is a square, we get a "double plane", and these constitute a 3-dimensional subvariety.

The Hilbert scheme parametrizes all possible families of deformations of quadrics. This generalizes to higher dimensions, where the Hilbert scheme is much harder to study.

Calabi–Yau varieties

Mathematicians and string theorists are interested in so-called *Calabi–Yau varieties*. These are 3-dimensional algebraic varieties X satisfying some conditions:

- X should be smooth (meaning no singularities).
- The canonical sheaf ω_X is trivial: $\omega_X \simeq \mathcal{O}_X$, meaning that there is an every-where defined volume form on X.
- And also $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.

Any Calabi–Yau variety come with some numbers attached to it. These are called $h^{11} = h^{22}$ and $h^{12} = h^{21}$, and are important invariants. They are often shown as the *Hodge diamond*:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{11} & & 0 & \\ 1 & & h^{21} & & h^{12} & & 1 \\ & 0 & & h^{22} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The general Hodge diamond.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The Hodge diamond of the quintic in $\mathbb{P}^4$.

In the 90's, physicists predicted the existence of "mirror manifolds": to any Calabi–Yau variety X, there should exist a "mirror" $\widetilde{X}$ with switched Hodge numbers. In particular, the quintic has a mirror partner with $h^{11} = 101$ and $h^{12} = 1$. This implied that calculations on one variety could be done by calculating something else on the other variety. This was soon confirmed in special cases by mathematicians.

This spawned a new industry: to find new examples of Calabi–Yau varieties and their mirrors, in order to verify the mirror conjecture.

A new smooth Calabi–Yau variety

I have found a new example X of a smooth Calabi–Yau variety. Here's how:

- Start with a 5-dimensional toric variety Y_0, with 2-dimensional singular locus. By cutting with two hyperplanes, we get a Calabi–Yau variety $X_0 \subset Y_0$ with isolated singularities.
- Deform Y_0 to Y. Then X_0 deforms as well, to X.
- One shows that Y has only 1-dimensional singularities, and it follows that X is smooth.

The hard part is finding the deformation of Y_0. The equations of Y_0 are quite complicated, so one has to use computer algebra software such as `Macaulay2` to do the computations.

I have also computed the Euler characteristic of X. This is the number $2(h^{11} - h^{12})$, and it is -36. Heuristic reasoning suggests that a resolution of singularities $\overline{X}$ of X_0 should have Euler characteristic 36, but I have not been able to prove this yet. If this is the case, X and $\overline{X}$ are *mirror candidates*. I would also like to compute h^{11} and h^{12}, and not just their difference.

To read more

[1] Robin Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York-Heidelberg, 1977.

[2] Ingrid Fausk, *Pfaffian Calabi–Yau threefolds, Stanley-Reisner schemes and mirror symmetry*, PhD thesis, Universty of Oslo, 2012.

... and the references therein.