G. Ellingsrud,

(1)

- Goes back to Matsusaka (1952).

- Milne

Let A abelian variety$/k$

Thm Assume k infinite.

$\exists$ curve C/k, nonsingular, connected. Then $\exists$

$\varphi: J_C \twoheadrightarrow A$ def. over k.

Jacobian

We have $g(C) = \dim J(C) \ggg \dim A$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad g$

$g!(g-1)$ „very, very big"

(1) Assume $k = \bar{k}$.

We know $A \subseteq \mathbb{P}^n$

Let $C = A \cap H_1 \cap \dots \cap H_{g-1}$ (H_i generic hyperplanes)

By Bertini $\Rightarrow$ C nonsing, connected.

Have canonical map

$$\begin{tikzcd} C \arrow[r, "\text{can}"] \arrow[dr, "\forall f"'] & J_C \arrow[d, dashed, "\exists \varphi"] \\ & A \text{ abelian} \end{tikzcd}$$

Have φ. Q) Is φ surjective?

Let $A_0 = \varphi(J_C) = \langle C \rangle \subseteq A$. If $A_0 \neq A$ then $\exists A_1$

w/ $A_1 \cap A_0 = \mu$ w/ $\# \mu < \infty$. s.t. $A_1 + A_0 = A$.

$(a,b) \longmapsto a+b$

Look at $A_1 \times A_e \xrightarrow{\pi} A \to 0$ finite map

$0 \to \mu \nearrow$

Interested in $\pi^{-1}(C)$.

Let $a \in A_0$. $C_a = C - a = \{ x-a / x \in C \}$

Now if $a \in A_0 \cap A_1$.

Then
$$\begin{array}{ccc} C_a \times \{a\} & \subseteq & A_0 \times A_1 \\ (x,a) & & \\ \downarrow + & & \downarrow \\ x & & C \end{array}$$

Hence $\pi^{-1}(C) = \bigsqcup_{a \in A_0 \cap A_1} C_a \times \{a\} = \pi^{-1}(C)$

so if $|A_0 \cap A_1| \geq 2 \Rightarrow \pi^{-1}(C)$ not connected!

contradicts a connectedness theorem.

(a trick for the case $\# |A_0 \cap A_1| = 1$)

Thm $f : X \to \mathbb{P}^n$, X normal variety w/ $\dim f(X) \geq 2$. $H \subseteq \mathbb{P}^n$ hyperplane $\Rightarrow f^{-1}(H) = Z$ connected.

Pf Ex. seq $0 \to \mathcal{O}_X(-nZ) \to \mathcal{O}_X \to \mathcal{O}_{nZ} \to 0$

+ Serre vanishing. + $\mathcal{O}_{nZ}$ and Z same topology.

Prop If $U \subseteq \mathbb{P}^n_k$ open dense, then $U(k) \neq \emptyset$. 3

Pf Induction on n.

$\#(\mathbb{P}^1 \setminus U) < \infty$.

$U(k) \neq \emptyset$ since k infinite.

$\mathbb{P}^n \setminus U$ = union of finitely many hypersurfaces.

There are only finitely many hyperplanes $\subseteq \mathbb{P}^n \setminus U$.

$\Rightarrow \exists \infty$ many H's def over k.

$\Rightarrow \exists H$ s.t. $U \cap H \neq \emptyset$. $\Rightarrow$ $U(k) \neq \emptyset$ by induction

Let $\tilde{A} \triangleq Bl_0 A$
$$\tilde{A} \xrightarrow{\pi} A$$
ex. divisor.

If $\mathcal{L}$ v.a. on A,

look at $\pi^* \mathcal{L}^N(-E)$ Very ample.

gives $\tilde{A} \subseteq \mathbb{P}^M$ s.t. $E \subseteq \mathbb{P}^M$

is linear since $\mathcal{O}_{\tilde{A}}(-E)|_E \simeq \mathcal{O}_E(1)$

Then $\tilde{A}$ is def $/k$. Same w/ E.

look at $C = \tilde{A} \cap H_1 \cap \dots \cap H_{g-1}$

and $E \cap H_1 \cap \dots \cap H_{g-1} = \{pt\}$

see point on the curve. $\Rightarrow$ $C(k)$

If $\#k < \infty$

Thm (O. G., Poonen)

If $\# k < \infty$, $X \subseteq \mathbb{P}^n_k$ nonsing

Then $\exists S$ hypersurface, def $/k$ s.t. $\bar{X} \cap S$ is non-singular (over $\bar{k}$)

(something about the proof)

(it involved ζ-functions and sieves)