Lect 12

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Any locally compact group $G$ has a ! upto scalar (left-invariant) Haar measure (that is, a nonzero Radon measure inv. under translations).

- Instead of writing $\int_G f \, d\mu(g)$ one usually writes $\int_G f \, dg$. fix measure
- If $G$ is kompact, we have $\mu(G) < \infty$, so we normalize to $\mu(G) = 1$.

- Any compact group is unimodular.

Indeed, for any $g \in G$, the measure $m(-\cdot g)$ is still left-invariant.

Hence $m(\cdot g) = \alpha_g m$ for some $\alpha_g > 0$.

But $1 = m(Gg) = \alpha_g m(G) = \alpha_g$. KOMPAKT USED HERE

so $m(-g) = m$.

Now replace expressions like $\frac{1}{|G|} \sum_{g \in G} f(g)$ by $\int_G f \, dg$ we get the following results for compact groups similar to finite groups:

**Def** rep of a compact grp on a fin-dim space $V$ is a cont's homomorphism $\pi : G \to GL(V)$ (the matrix entries are cont's functions $G \to \mathbb{R}^{n \times n}$)

- Maschke's thm: any fin-dim rep of $G$ is completely reducible
- Any fin-dim rep is unitarizable
- Orthogonality relations. In particular, two fin-dim reps of $G$ are equiv iff their characters coincide

What is more difficult to see is that there are many f.d. reps.

(forced to go to inf. dim spaces)

**Def** A unitary rep of a topological group $G$ on a Hilbert space $H$ is a homomorphism $\pi : G \to U(H)$

$U(H)$ = unitary op of $H$

which is cont's in the strong operator topology.

( that is, if $g_i \to g$ in $G$ then $\pi(g_i)\xi \to \pi(g)\xi \quad \forall \xi \in H$

or the map $G \to H$, $g \mapsto \pi(g)\xi$ is cont's $\forall \xi \in U(H)$ )

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**Example** Let $G$ a locally compact group w/ fixed Haar measure. Consider $L^2(G)$ (v-space of sq. integrable functions) w.r.t. to the measure.

Define $\lambda : G \to U(L^2(G))$ by

$g \mapsto (\lambda(g)f)(h) = f(g^{-1}h)$

Then $\lambda$ is cont's in the above sense (exercise!)

$\lambda$ is called the regular rep.

(**HINT** look first at functions $\xi \in C_c(G)$ compactly supported )

Peter-Weyl-thm

**Thm (Peter-Weyl)**

Let $G$ be a compact group $\pi: G \to U(H)$ a unitary representation. Then $\pi$ decomposes into (in the sense of Hilbert spaces) of finite dim irreps.

**Pf (sketch)**

The key part is to show that $\exists$ a nonzero invariant fin dim subspace $K \subset H$.

Then may assume that $\pi|_K$ is irreducible and by unitarity, the space $K^{\perp}$ is also invariant (exercise)

Cont w/ transfinite induction and Zorn's lemma.

To find $K$, take any nonzero kompakt operator $A \geq 0$.

(recall that being compact means that $\overline{\{A\xi \mid \|\xi\| \leq 1\}}$ is compact)

Example take $A$ to be the orthogonal projection onto a fin-dim nonzero v. space.

Then define
$$B = \int_G \pi(g) A \pi(g^{-1})\, dg$$

more precisely, the operator $B$ s.t.
$$(B\xi, \zeta) = \int_G (\pi(g) A \pi(g^{-1})\xi, \zeta)\, dg \quad \forall \xi, \zeta.$$

Then it's not difficult to show that $B \neq 0$, $B \geq 0$, $B$ is compact and $\pi(g)B = B\pi(g)$

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By the spectral theorem for compact self-adj. operators, $B$ has an eigenvalue $\alpha > 0$ and the corresponding eigenspace $K$ is fin. dim.

As $\pi(g)B = B\pi(g)$ then $\pi(g)K \subset K$. ∎

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**Corollary** For any compact grp $G$, the intersection of all kers of irreps fin. dims reps on $G$ is trivial.

pf [ Take $G$-c. Ker $\lambda = \{e\}$ and $\lambda$ decomposes into a sum of fin. dim irreps. ($\lambda$ is inj.)

Denote by $\mathbb{C}[G]$ the linear span of matrix coeff's of f. dim reps of $G$. This is an algebra under pointwise multiplication. The product of two matrix coeff's is the coeff of the $\otimes$.

$\mathbb{C}[G]$ is closed under cplx conjugation. Indeed, it suffices to consider unitary reps. If $\pi$ is unitary, then $\pi(g) = (\pi(g)\xi, \zeta)$

then $\overline{\pi(g)} = \overline{(\pi(g)\xi, \zeta)} = \cancel{(\xi, \pi(g^{-1})\zeta)} = (\overline{\pi(g^{-1})}, \overline{\zeta})$

is the matrix coeff. of the contragradient representation $\pi^c$ on $\overline{H} = H^*$ defined by $\pi^c(g)\overline{\xi} = \overline{\pi(g^{-1})}$

For a fin dim Hilbert space $H$, the dual $H^*$ can be identified w/ $\overline{H^*} = \{\overline{\xi} \mid \xi \in H\}$. The linear structure is defined by

$\bar{\xi} + \bar{\zeta} = \overline{\zeta + \xi}$.

And $\alpha \bar{\xi} = \overline{\bar{\alpha}\xi}$ for $\alpha \in \mathbb{C}$.

and $(\bar{\xi}, \bar{\zeta}) = \overline{(\xi, \zeta)} = (\zeta, \xi)$

The duality is given by $\langle \xi, \bar{\zeta} \rangle = (\xi, \zeta)$.

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**Corollary 2** The algebra $\mathbb{C}[G]$ is norm-dense in $C(G)$.

**Pf** This follows e.g. from the previous corollary since $\mathbb{C}[G]$ separates points by the corollary ( for $g \neq h$ $\exists$ $a \in \mathbb{C}[G]$ ~~and~~ s.t. $a(g) \neq a(h)$ ). So $\mathbb{C}[G]$ is dense in $C(G)$ by the Stone-Weierstrass thm.

The algebra $\mathbb{C}[G]$ is called the algebra of regular, or polynomial functions on $G$.

**Thm** (Peter-Weyl) For every compact group $G$, we have

$$\lambda \sim \bigoplus_{[\pi] \in \hat{G}} \pi^{\dim \pi}$$

**Pf** For every irrep $\pi$ (unitary) choose an orthonormal basis $\{e_1, \ldots, e_{\dim \pi}\}$ and consider the corresponding matrix coeffs $a_{ij}^{\pi}(g) = (\pi(g)e_i, e_j)$

Then by orthogo. relations, we have isometric ~~matrices~~ embeddings

$$V_\pi \hookrightarrow L^2(G)$$

$$e_j \longmapsto (\dim \pi)^{1/2}\, a^\pi_{ij}$$

for all $j$.

This gives an isometric embedding of $\bigoplus_{[\pi]\in\hat{G}} \pi^{\oplus \dim \pi}$ into $\lambda$.

~~But the img~~ As $\mathbb{C}[G]$ is norm-dense in $C(G)$, hence in $L^2(G)$, this embedding is unitary.

$\blacksquare$

Just as in the fin. grp case it is shown that the characters of f.dim irreps form an orthonormal basis in the space

$$\{ f \in L^2(G) \mid f(g \cdot g^{-1}) = f(\cdot)\ \forall g \} \subseteq L^2(G)$$

With a bit more effort, it can be proved that we have

**FACT** For any compact grp $G$, the characters of f.dim reps of $G$ span a <u>dense</u> subspace in the space of cont's central functions on $G$.

(equiv. $g, h$ are conj. iff $\chi_\pi(g) = \chi_\pi(h)\ \forall [\pi]\in\hat{G}$)

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<u>Thm</u> Assume $G$ is a compact subgrp of the unitary group $U(n) \subset \mathbb{C}^{n\times n}$.

Then $\mathbb{C}[G]$ is the unital algebra generated by matrix coeff's $a_{ij}$ (so $g = (a_{ij}(g))_{ij}$) and $\det^{-1}$.

<u>Pf</u> Since matrix coeffs of $\left[a_{ij}(g)_{ij}\right]^{-1}$ can be expressed as poly's in $a_{ij}$ and $\det^{-1}$ and

$$\left[a_{ij}(g)_{ij}\right]^{-1} = (\overline{a_{ji}(g)})_{ij}$$

by unitarity, we get that the algebra $A \subset \mathbb{C}[G]$ generated by $a_{ij}$ and $\det^{-1}$ is closed under cplx conjugation.

It also obviously separates pt's of $G$.

Hence it is dense in $C(G)$ hence in $L^2(G)$.

By the orthogonality relations, it follows that $A = \mathbb{C}[G]$

Equivalently, we can say that any irrep of $G$ is a subrep of the $(\det^{-1})^{\otimes k} \otimes \pi^{\otimes l}$ for some $k, l \geq 0$.

(Tomorrow: classical grp's)