$G \supset H \qquad (V,\pi)$ repr of H.

$$(W,\theta) = \mathrm{Ind}_H^G (V,\pi)$$

Two def's

① $W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$

$\Updownarrow$

② $W = \left\{ f : G \to V \;\middle|\; f(gh) = \pi(h)^{-1} f(g) \quad \begin{matrix} g \in G \\ h \in H \end{matrix} \right\}$

$$(\theta(g)f)(g') = f(g^{-1}g')$$

The correspondence between ① and ② is

$$f \longmapsto \sum_{g \in G} g \otimes f(g)$$

Example $\mathrm{Ind}_H^G \lambda_H \sim \lambda_G$

Since $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}[H] \approx \mathbb{C}[G]$.

Exercise Assume H acts on a set X and consider the corresp. permutation repr π.

Then $\mathrm{Ind}_H^G \pi$ is the perm. rep. defined by the action of G on $G \times_H X \overset{\text{def}}{=} \dfrac{G \times X}{(g,x) \sim (gh^{-1}, hx)}$

$$g[g',x] = [gg',x]$$

In particular if $X = \{pt\}$. Then we get

$\mathrm{Ind}_H^G \varepsilon_H$ is the perm. repr. defined by $G \curvearrowright G/H$.

Consider (W, θ) defined as (ii). Put $X = G/H$. For every $x \in X$ define $W_x = \{ f \in W \mid f$ supported only on $x \}$.

that is, f nonzero only on $x = gH \subseteq G$

Then ① $W = \bigoplus_{x \in X} W_x$,

② $\theta(g) W_x = W_{gx}$

③ $(\mathrm{Res}_H^G \theta)|_{W_e} \simeq \pi$

(denote by $\bar{g}$ the coset gH)

$\Rightarrow$ consider the map $T : W_e \longrightarrow V$

$f \longmapsto Tf = f(e)$.

Then $T(\mathrm{Res}_H^G \theta)(h) f = \big((\mathrm{Res}_H^G \theta)(h) f\big)(e)$

$= f(h^{-1}) = \pi(h) f(e)$

$= \pi(h) Tf$

Thus T indeed intertwines $(\mathrm{Res}_H^G \theta)|_{W_e}$ and π

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This T is an isomorphism since for every $v \in V$ there exists! $f \in W_e$ s.t. $f(e) = v$; namely $f(h) = \pi(h)v$.

Defin. Given a representation (W, θ) of G and subspaces $W_x \subset W$ $(x \in G/H)$ satisfying (i) and (ii), one says that $\{W_x\}_{x \in G/H}$ is a system of imprimitivity for θ.

Prop Assume (W, θ) is a representation of G and $W_x \subset W$ are subspaces satisfying (0) + (i) + (ii). Then $\theta \sim \mathrm{Ind}_H^G \pi$.

(Therefore, a repr. of G is induced by H iff it admits a system of imprimitivity)

Pf Let T be an equivalence between $(\mathrm{Res}_H^G \theta)|_{W_e}$ and π. We want to extend T to an equivalence betw. θ and $\mathrm{Ind}_G^H \pi$.

1st way Consider $\cancel{1 \otimes T}$ $S : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \longrightarrow W$

$S(a \otimes v) = aT^{-1}(v)$

This is a $\mathbb{C}[G]$-module map and it is surjective by (i) and (ii) $\Rightarrow$ iso for dimension reasons.

2nd way

We extend T to W as follows.

Suppose $v \in W_x$ for some $x \in X = G/H$.

We have $v = \theta(g) v_0$ for some $g \in G$ and $v_0 \in W_{\bar{e}}$.

Then we define $Tv = (\mathrm{Ind}_H^G \pi)(g)\, T v_0$. (∗)

But have to check that this is well-defined. If we have another $g' \in G$ s.t. $\theta(g')^{-1} v \in W_{\bar{e}}$, then $(g')^{-1} g$ maps a nonzero vector of $W_{\bar{e}}$ into itself hence $(g')^{-1} g$ fixes $\bar{e}$ so $(g')^{-1} g \in H$.

That is $g = g' h$ for some $h \in H$.

Then $(\mathrm{Ind}_H^G \pi)(g')\, T\, \theta(h) v_0 = (\mathrm{Ind}_H^G \pi)(g')(\mathrm{Ind}_H^G \pi)(h)(T v_0)$

as $v = \theta(g')\theta(h) v_0$

$= (\mathrm{Ind}_H^G \pi)(g' h)\, T v_0$

$\Downarrow$

same as (∗). ∎

~~So T~~ intertwines θ and $\mathrm{Ind}_H^G \pi$ since

$\cancel{T(\theta(g)}\; T\theta(g)\theta(g') v_0 = T\,\theta(gg')\, v_0$

$= (\mathrm{Ind}_H^G \pi)(gg')\, T v_0$

$= (\mathrm{Ind}_H^G \pi)(g)(\mathrm{Ind}_H^G \pi)(g')\, T v_0$

$= (\mathrm{Ind}_H^G \pi)(g)\, T\theta(g') v_0$ for any $g' \in G$, $v_0 \in W_{\bar{e}}$.

Thm (Frobenius reciprocity)

Assume $G \supset H$. π a rep. of H and θ a rep. of G. Then $\mathrm{Mor}(\mathrm{Ind}_H^G \pi, \theta) \simeq \mathrm{Mor}(\pi, \mathrm{Res}_H^G \theta)$

$$T \longmapsto T|_{V_\pi}$$

Pf (see books)

Exc Assume $R \supset S$ are rings. A an R-S-bimodule, B a left-S-module. Then (C a left R-mod)

$$\mathrm{Hom}_R(A \otimes_S B, C) \simeq \mathrm{Hom}_S(B, \mathrm{Hom}_R(A, C))$$

Show that Frobenius recipr. follows from this

If χ is the character of π, we denote by $\mathrm{Ind}_H^G \chi$ the character of the induced rep.

Note that for any reps π_1, π_2 of G, note that

$$(\chi_{\pi_1}, \chi_{\pi_2}) = \dim \mathrm{Mor}(\pi_1, \pi_2).$$

Therefore, in terms of characters, Frob. recipr. can be stated as

$$(\mathrm{Ind}_H^G \chi_\pi, \chi_\theta) = (\chi_\pi, \mathrm{Res}_H^G \chi_\theta).$$

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Prop Let π be a rep of $H \subset G$. ; $\chi_\pi = \chi_\pi$

Then
$$\left(\mathrm{Ind}_H^G \chi\right)(g) = \sum_{\bar{s} \in G/H} \chi(s^{-1} g s)$$
where we extend χ to G by 0

Pf Consider the imprimitivity system $\{W_x\}_{x \in G/H}$ for this induced rep. $\theta = \mathrm{Ind}_H^G \pi$.

Then
$$\chi_\theta(g) = \mathrm{Tr}\, \theta(g) = \sum_{\substack{x \in G/H \\ gx = x}} \mathrm{Tr}\left(\theta(g)\big|_{W_x}\right)$$

$$= \sum_{\substack{\bar{s} \in G/H \\ s^{-1} g s \in H}} \mathrm{Tr}\left(\theta(g)\big|_{W_{\bar{s}}}\right)$$

$$= \sum_{\substack{\bar{s} \in G/H \\ s^{-1} g s \in H}} \mathrm{Tr}\left(\theta(s)^{-1} \theta(g) \theta(s)\big|_{W_e}\right) = \sum_{\substack{\bar{s} \in G/H \\ s^{-1} g s \in H}} \mathrm{Tr}\left(\pi(s^{-1} g s)\right) = \sum_{\substack{\bar{s} \in G/H \\ s^{-1} g s \in H}} \chi(s^{-1} g s)$$

Frobenius subgrp thm

Thm Assume a finite group G act trans. on X s.t. every element $g \neq e$ has at most one fixed point.

Then $\exists!$ normal subgr $K \subset G$ acting freely transitively on X.

Pf Put $K \overset{\Delta}{=} \{e\} \cup \{g \in G \mid g \text{ has no fixed points}\}$

We have to show that K is a normal subgrp. And that it acts freely transitively on X.

For the latter, it suffices to show that $|K| = |X|$.

Consider the stabilizer $H = G_x$.

If $g \notin H$, then $gx \neq x$.

and $H \cap gHg^{-1} = G_x \cap G_{gx} = \{e\}$. by assumption.

Hence $G = K \sqcup \left(\bigsqcup_{y \in X} G_y \setminus \{e\} \right) = K \sqcup \left(\bigsqcup_{g \in G/H} gHg^{-1} \setminus \{e\} \right)$

Therefore $\# G = \# K + \frac{|G|}{|H|}(|H| - 1)$ so $|K| = \frac{|G|}{|H|} = |X|$.

We will show that K is the kernel of some repr. of G.

For an irreducible π of H consider the function $\tilde{\hat{\chi}}_\pi$ on G defined by

$$\hat{\chi}_\pi = \mathrm{Ind}_H^G \chi_\pi - (\dim \pi)\, \mathrm{Ind}_H^G \chi_{\varepsilon_H} + (\dim \pi)\, \chi_{\varepsilon_G}$$

$$= \mathrm{Ind}_H^G \chi_\pi - (\dim \pi)\, \mathrm{Ind}_H^G 1 + (\dim \pi)\, 1.$$

since $(\mathrm{Ind}_H^G f)(g) = \sum_{s \in G/H} f(s^{-1} g s)$, so from $(*)$ we

get $$\tilde{\chi}_\pi(g) = \begin{cases} \dim \pi & \text{if } g \in K \\ \chi_\pi(h) & \text{if } g \text{ is conj. to } h \in H \end{cases}$$

(the explains ?)