(glemme ordag-forelæsning)

Recall
$$\lambda = \bigoplus_{[\pi]\in\hat{G}} \underbrace{\pi \oplus \dots \oplus \pi}_{\dim \pi}$$

regular representation

$$\pi : \mathbb{C}[G] \to \mathrm{End}(V_\pi)$$

In fact, stronger result
$$\mathbb{C}[G] \simeq \bigoplus_{[\pi]\in\hat{G}} \mathrm{End}(V_\pi)$$

Consequence
$$\sum_{[\pi]\in\hat{G}} (\dim \pi)^2 = |G|$$

and $|\hat{G}|$ = number of conjugacy classes in G.

+ orthogonality relations

Assume V is a fin. dim v.space w/ a Hermitian scalar product.
We say that a representation of G on V is unitary if the operators $\pi(g)$ are unitary (preserve Hermitian $\langle \cdot, \cdot \rangle$).

Assume $\pi : G \to GL(V)$ is a repr. (G finite, V fin. dim). Then there exist a Hermitian scalar product on V s.t. π is a unitary repr.

Indeed if $\langle , \rangle'$ is any Herm. product, then $\langle v,w \rangle = \frac{1}{|G|}\sum_{g\in G} \langle \pi(g)v, \pi(g)w \rangle'$.

Exc (i) Check that this is indeed an inv. scalar product (2)

(ii) Show that if π is irreducible, then an invariant scalar product is unique up to a factor.

Assume π is a unitary repr. Choose an orthonormal basis $e_1, \dots, e_n \in V$.

Then $\pi(g) = \left(a^{\pi}_{ij}(g) \right)_{ij}$ are unitary matrices.

(i.e. $\pi(g)^* = \pi(g)^{-1} = \pi(g^{-1})$, hence

$$\left\{ \overline{a^{\pi}_{ij}} = a^{\pi}_{ji}(g^{-1}) \right\}$$

Define a scalar product on functions on G by

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

Then the orthogonality relation take the form

Thm For every irreducible representation, choose an inv. scalar product and an orthonormal basis $\{e^{\pi}_1, \dots, e^{\pi}_{\dim \pi}\}$ and corresp. matrix coeff a^{π}_{ij}. Then the functions a^{π}_{ij} are mutually orthogonal.

w/ $$(a^{\pi}_{ij}, a^{\pi}_{ij}) = \frac{1}{\dim \pi}$$

Characters of a representation

Assume $\pi: G \to GL(V)$ is a f.d. rep.

Def The function $\chi_\pi(g) = \operatorname{Tr} \pi(g)$ is called the character of π.

Def A function G is called <u>central</u> if
$$f(hgh^{-1}) = f(g) \quad \forall h, g \in G$$
„class function"

- Characters of a rep. are central.

Note that we also have $\chi_\pi(e) = \dim \pi$.

~~Fa~~ χ_π is <u>constant</u> on equiv. representation.

Thm The characters χ_π, $[\pi] \in \hat{G}$ form an orthonormal basis in the space of central functions.

Pf $\chi_\pi(g) = \sum_{i=1}^{\dim \pi} a_{ii}^\pi(g)$

The orth. relations show that $\{\chi_\pi \mid [\pi] \in \hat{G}\}$ is an orth. system.

e.g.
$$(\chi_\pi, \chi_\pi) = \sum_{i,j} (a_{ii}^\pi, a_{jj}^\pi) = \sum_i \frac{1}{\dim \pi} = 1.$$

So we have to show that χ_π span ④

the whole space of central functions.

Consider the projection P from

$$\mathbb{C}[G] \twoheadrightarrow \{\text{the space of central functions}\}$$

$$f \longmapsto (Pf)(g) = \frac{1}{|G|} \sum_{h \in G} f(hgh^{-1})$$

„obv. a projection"

Take an irred. repr. π. (skip upper index)

$$P(a_{ij})(g) = \frac{1}{|G|} \sum_{h \in G} a_{ij}(hgh^{-1})$$

$$= \frac{1}{|G|} \sum_{h \in G} \sum_{k,l} a_{ik}(h)\, a_{kl}(g)\, a_{lj}(h^{-1})$$

$$= \frac{1}{\dim \pi} \sum_{k,l} a_{kl}(g)\, \delta_{ij}\, \delta_{kl}$$

$$= \frac{1}{\dim \pi} \sum_{k} a_{kk}(g)\, \delta_{ij}$$

$$= \frac{1}{\dim \pi} \delta_{ij}\, \chi_\pi(g)$$

But the elements a_{ij}^π $([\pi] \in \hat{G})$ form a basis in the space of all functions in $\mathbb{C}[G]$, and they consist of $|G|$ elements,

so the image of this projection $= \operatorname{Span}_{[\pi] \in \hat{G}} \chi_\pi$.

∎

Note that a_{ij}^{π} span the whole space of functions Λ: can be easily seen as follows:

Span $\mathbb{C}[G]$

Since λ decomposes into copies of π $([\pi] \in \hat{G})$ it suffices to show that matrix coeff. of λ span $\mathbb{C}[G]$.

But this is true

$$a^{\lambda}_{\delta_g^*, \delta_e}(h) = \delta_g^*(\lambda(h)\delta_e)$$

??

$$= \delta_g^*(\delta_h) = \delta_{g,h}.$$

This allows us to prove some results using the density thm

Cor. $|\hat{G}| =$ number of conj. classes in G

Pf $|\hat{G}| = \dim(\operatorname{span}_{\pi \in \hat{G}} \chi_\pi) = \dim\{\text{space of central f.}\}$

$=$ number of conj. classes.

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Corr 2

Two fin. dim rep, π, and θ, are equiv. iff $\chi_\pi = \chi_\theta$.

Pf $\pi \simeq \bigoplus_{[\eta]\in\hat{G}} \eta^{m_\eta}$

Then $\chi_\pi = \sum_{\eta\in\hat{G}} m_\eta \chi_\eta$.

Therefore $m_\eta = (\chi_\pi, \chi_\eta)$. Thus the multiplicities m_η can be recovered from χ_π.

But the eq. class of π can be recovered by ~~the~~ m_η. $\blacksquare$

Corr 3 For any $[\pi]\in\hat{G}$ the multiplicity of π in the reg. repr. λ equals $\dim\pi$.

Pf

$$\chi_\lambda(g) = \sum \mathrm{Tr}(\delta_h \mapsto \delta_{hg}) = \begin{cases} 0 & g\neq e \\ |G| & e=g \end{cases}$$

Thus $(\chi_\lambda, \chi_\pi) = \frac{1}{|G|}\chi_\lambda(e)\overline{\chi_\pi(e)}$

$= \overline{\chi_\pi(e)} = \dim\pi$. $\blacksquare$

Let (V,π) be a fin. dim $\overset{\text{irr.}}{\text{rep.}}$ and (W,θ) another rep.

Denote by $W(\pi)$ the subspace of W spanned by all vectors of the form Tv, $v\in V$ and $T\in \mathrm{Mor}(\pi,\theta)$.

This is an inv. subspace of W.

and $\theta\big|_{W(\pi)}$ is called the <u>isotypic component</u> corresp. to π.

EXC Consider the operator $P_\pi : W \to W$ defined by

$$P_\pi = \frac{\dim \pi}{|G|} \sum_{g\in G} \chi_\pi(g^{-1})\,\theta(g) = \frac{\dim \pi}{|G|} \sum_{g\in G} \overline{\chi_\pi(g)}\,\theta(g).$$

<u>Show</u> (i) P_π is the projection onto $W(\pi)$ along

$$\sum_{\substack{[\pi']\in\hat{G} \\ [\pi']\neq[\pi]}} W(\pi')$$

this is mainly killed

(ii) $W = \bigoplus_{[\pi]\in\hat{G}} W(\pi)$

(iii) $\theta\big|_{W(\pi)} \sim \underbrace{\pi\oplus\cdots\oplus\pi}_{m_\pi}$ for some m_π.

Let us return again to the decomposition of λ. 8

Let π be an irreducible rep. and fix a basis $e_1, \dots, e_n$ in V_π.

For a fixed index i, consider the functions

$$f_j(g) = a_{ij}^{\pi}(g^{-1})$$

we have

$$(\lambda(h) f_j)(g) \overset{\text{def of reg. rep.}}{=} f_j(h^{-1} g)$$

$$= a_{ij}^{\pi}(g^{-1} h)$$

$$= \sum_k a_{ik}^{\pi}(g^{-1})\, a_{kj}^{\pi}(h)$$

$$= \sum_k a_{kj}^{\pi}(h)\, f_k(g)$$

so $$\lambda(h) f_j = \sum_k a_{kj}^{\pi}(h)\, f_j$$

Therefore we get an embedding of π into λ by

$$e_j \longmapsto f_j.$$

By varying i we get $n = \dim \pi$ ~~diff~~ linearly indep. embeddings.

(....... lots over!)

This gives a concrete decomposition 9

of λ as $\bigoplus_{[\pi]\in\hat{G}} [\pi]^{\dim\pi}$

Next time return to

motivating thm.

determinant smooth.