Recall

$G \supset T$

$$V \cong it^*$$

$$r : N(T)/T \longrightarrow O(it^*)$$

Also $N(T)/T \simeq W$

$N(T)/T$ acts freely ~~transitively~~ on the set of Weyl chambers in $V_0 = \operatorname{Span}\Delta \subset V$.

Assume C is a Weyl chamber fixed by $r(gT)$ for some $g \in N(T)$. We must show that $g \in T$.

Take $\beta \in C$. Replacing, if necessary β by $\sum_{i=0}^{n-1} r(gT)^i \beta$ where n is order of $gT \in \frac{N(T)}{T}$, we may assume that $r(gT)\beta = \beta$. Then $(\mathrm{Ad}\, g)(h_\beta) = h_\beta$.

(where h_β was the image of β under the iso $\mathfrak{h}^* \to \mathfrak{h}$)

Consider the torus $T_1 = \overline{\exp(i\mathbb{R} h_\beta)}$ closure $\subseteq G$.

Then g is in the centralizer of T_1.

Recall that the centralizer of a maximal torus is the maximal torus. Similarly, one can prove that there exists a torus T_2 containing both g and T_1.

(Why it works: take any maximal torus containing g and then conjugate it using Cartan's theorem, inside $C(g)^\circ$ to contain T_1)

Take any $X \in \mathfrak{t}_2$. We can write

$$X = H + \sum_{\alpha \in \Delta} X_\alpha \quad \text{for } X_\alpha \in \mathfrak{g}_\alpha,\ H \in \mathfrak{h}.$$

As $h_B \in \mathfrak{t}_2$, we have

$$0 = [h_B, X] = \sum_{\alpha \in \Delta} \alpha(h_B) X_\alpha$$

As $\alpha(h_B) = (\alpha, B) \neq 0$ $\forall \alpha \in \Delta$, all the $X_\alpha = 0$ $\forall \alpha \in \Delta$.
Therefore $\mathfrak{t}_2 \subset \mathfrak{t}$. So $\Sigma \subset T$. Hence $g \in T$.

$\blacksquare$

Cartan matrices and Dynkin diagrams

Let (V, Δ) be a reduced root system, w/ Span $\Delta = V$.
Fix a system Π of simple roots, $\Pi = \{\alpha_1, \dots, \alpha_r\}$.

The matrix

$$A = (a_{ij})_{i,j=1\dots r} \quad \text{where } a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

called the Cartan matrix of (V, Δ).

Since any two systems of simple roots are related by an orthogonal transformation, the matrix A does not depend on any choices up to permutation of indices.

Also clear that isomorphic root systems have the same Cartan matrix (up to perm. of indices).

Proposition Assume (V_1, Δ_1) and (V_2, Δ_2) are reduced root systems w/ the same Cartan matrix and $\operatorname{Span} \Delta_i = V_i$. Then they are isomorphic. 3

Pf: The assumption means that there exist systems of simple roots $\Pi_1 \subset \Delta_1$ and $\Pi_2 \subset \Delta_2$ and a linear isomorphism

$$T: V_1 \xrightarrow{\sim} V_2 \quad \text{s.t.} \quad T\Pi_1 = \Pi_2$$

and $Ts_\alpha = s_{T\alpha} T \quad \forall \alpha \in \Pi_1$. (*)

As Weyl groups W_1 & W_2 of Δ_1, Δ_2 are generated by simple reflections, we see that

$$\pi(w) = TwT^{-1} \text{ is an isomorphism } \pi: W_1 \xrightarrow{\sim} W_2.$$

Recall from $\Delta_1 = \bigcup_{w \in W_1} w\Pi_1$ (and similarly for Δ_2)

Hence $T\Delta_1 = \Delta_2$, as $Tw\alpha = \pi(w) T\alpha \quad \forall \alpha \in \Pi_1$.

Finally, for any ~~root~~ $\beta = w\alpha$ $(w \in W_1, \alpha \in \Pi_1)$ we have

$$Ts_\beta = Tws_\alpha w^{-1} = \pi(w) s_{T\alpha} \pi(w)^{-1} T$$
$$= s_{\pi(w)T\alpha} T = s_{Tw\alpha} T = s_{T\beta} T. \qquad \blacksquare$$

Instead of the Cartan matrix, we can also consider the Dynkin diagram:

vertices: simple roots $\alpha_1, \dots, \alpha_r$

edges: connect α_i w/ α_j by $a_{ij} a_{ji}$ edges $i \neq j$

orientation: if $|a_{ij}| < |a_{ji}|$ (i.e. $\|\alpha_i\| > \|\alpha_j\|$), then the direction of edges between α_i and α_j is from $\alpha_i \to \alpha_j$.

The Dynkin diagram contains the same information as the Cartan matrix:

$a_{ii} = 2$ $\qquad$ $a_{ij} = -1$

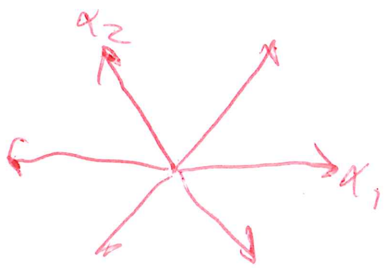
if $0 < |a_{ij}| \leq |a_{ji}|$, then $a_{ji} = -(\#$ of edges between α_i and $\alpha_j)$

Indeed, recall that $a_{ij} \leq 0$ (for $i \neq j$). Hence if $\|\alpha_i\| > \|\alpha_j\|$ and $(\alpha_i, \alpha_j) \neq 0$, we have $\dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = -1$.

$$\left(\text{since } \left|\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\right| \underset{\text{not collinear}}{<} \frac{2\|\alpha_i\|\|\alpha_j\|}{\|\alpha_i\|^2} = \frac{2\|\alpha_j\|}{\|\alpha_i\|} < 2 \right)$$

and $(\alpha_i, \alpha_j) < 0$.

Example

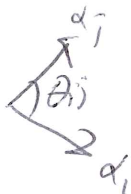


then $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

<u>Remark</u> Note that

$$a_{ij}a_{ji} = \frac{4(\alpha_i,\alpha_j)^2}{(\alpha_i,\alpha_i)(\alpha_j,\alpha_j)} \overset{\text{Schwarz}}{<} 4$$

Therefore, in the Dynkin diagram we can have either 0, 1, 2, 3 edges between vertices.

(if 

then $a_{ij}a_{ji} = 4\cos^2\theta_{ij}$)

<u>Def</u> A root system (V,Δ) is <u>indecomposable</u> if there are no root systems (V_1,Δ_1) (V_2,Δ_2) w/
$$(V,\Delta) \cong (V_1\oplus V_2,\ \Delta_1\times\{0\} \cup \{0\}\times\Delta_2)$$

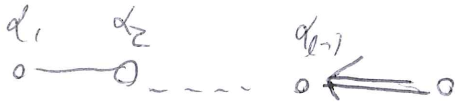
<u>Exc</u> Show that a root system is indecomposable iff its Dynkin diagram is connected.

<u>Fact</u> (Killing, Cartan)
All indecomposable, reduced root systems (apart from $(\mathbb{R},\emptyset)$) are given by the following diagrams:

<u>Lie alg.</u>

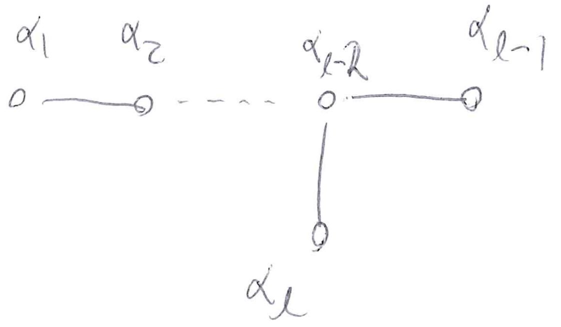
- A_l : 

$(l\geq 1)$ $su(l+1)$
- B_l : 

$(l\geq 2)$ $so(2l+1)$
- C_l : 

$(l\geq 3)$ $sp(2l)$

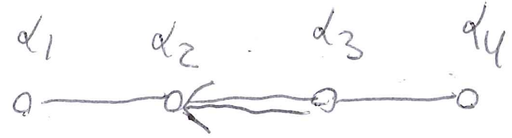
• D_ℓ :



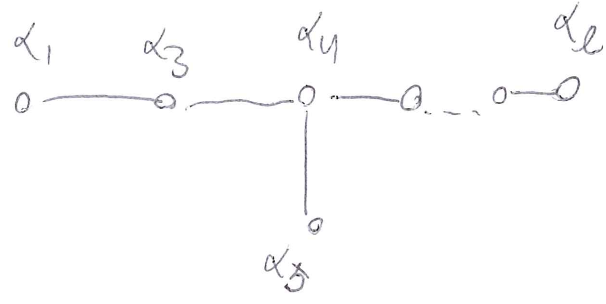
$(\ell \geq 4)$ $so(2\ell)$

• G_2

• F_4



• E_6, E_7, E_8



$(\ell = 6, 7, 8)$

(To recall the details how it is obtained is maybe not the most rewarding experience)

The proof classifies Cartan matrices and implies the following result:

<u>Fact</u> An integer $r \times r$ matrix $A = (a_{ij})_{i,j=1}^{r}$ is a Cartan matrix of a root system iff

1. $a_{ii} = 2$
2. $a_{ij} < 0$ $i \neq j$
3. There exists a matrix $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{pmatrix}$ w/ $d_i > 0$ s.t. DA is symmetric and positive def.

Remark If $A = \left(\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}\right)_{i,j}$ is the Cartan matrix then we can take $d_i = \frac{(\alpha_i,\alpha_i)}{2}$.

The classification is based on a series of elementary observations about configurations of vectors in Euclidean spaces.

For example let us prove the following:

Lemma Any Dynkin diagram has no loops apart from multiple edges, that is, there are no different vertices $\alpha_1, \dots, \alpha_n$, $n \geq 3$ w/ α_i and α_{i+1} are connected for $i=1,\dots,n-1$, as well as α_n and α_1 are connected.

Pf Consider $v_i = \frac{\alpha_i}{\|\alpha_i\|}$. Note that

Note that $4(v_i,v_j)^2 = a_{ij}a_{ji} \in \{0,1,2,3\}$.

Recall also that $(v_i,v_j) \leq 0$ $\forall i \neq j$.

Hence either $(v_i,v_j)=0$ or $(v_i,v_j) \leq -\frac{1}{2}$ ($i \neq j$)

Now consider $v = v_1 + \dots + v_n$.

$$(v,v) = n + \sum_{i\neq j}(v_i,v_j) \leq n + 2\left(\sum_{i=1}^{n-1}(v_i,v_{i+1}) + (v_n,v_1)\right)$$

$$\leq 0$$

Hence $v=0$. But this contradicts linear independence of $\alpha_1,\dots,\alpha_n$. ∎

So the only type of allowed diagrams are trees.

Example

$G = SU(n)$ $\quad T = \{\text{diagonal matrices in } SU(n)\}$
$\cong \mathbb{T}^{n-1}$

$\mathfrak{g} = su(n) = \{X \in gl_n(\mathbb{C}) \mid X + X^*, \ Tr X = 0\}$

$\mathfrak{g}_{\mathbb{C}} = sl_n(\mathbb{C}) = \{X \in gl_n(\mathbb{C}) \mid Tr X = 0\}$

$\mathfrak{h} = \{\text{diag. matrices of trace zero}\}$

Define $\varepsilon_i \in \mathfrak{h}^*$ by $\varepsilon_i(H) = H_{ii}$

We have $\varepsilon_1 + \dots + \varepsilon_n = 0$.

Then $\Delta = \{\alpha_{ij} = \varepsilon_i - \varepsilon_j \mid i \neq j\}$.

w/ root vectors $\mathfrak{g}_{\alpha_{ij}} = \mathbb{C} e_{ij}$.

Because $[H, e_{ij}] = (H_{ii} - H_{jj}) e_{ij}$

Need an Ad-inv. sc. product: Define the Ad-invariant scalar product by $(X, Y) = Tr(XY)$.

It is Ad-invariant since $(Ad\, g)(X) = gXg^{-1}$. For $X \in su(n)$ we have $(X, X) = Tr(X^2) = Tr(-X^* X) < 0$ for $X \neq 0$.

Thus $(\cdot, \cdot)$ is neg. definite on $su(n)$.

Then $h_{\alpha_{ij}} = H_{\alpha_{ij}} = e_{ii} - e_{jj}$.

As a system of simple roots, we can take

$$\alpha_1 = \varepsilon_1 - \varepsilon_2 \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n.$$

"obviously a system of simple roots"

$$(\alpha_i, \alpha_k) = (h_{\alpha_i}, h_{\alpha_k}) = \mathrm{Tr}\left((e_{ii} - e_{i+1,i+1})(e_{kk} - e_{k+1,k+1})\right)$$

$$= \begin{cases} 2 & i = k \\ -1 & i = k \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the Dynkin diagram is

(as the classification claimed)