19/1 -2015

Kompleks dynamikk, E. Wold

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[Book is "Complex Dynamics" L. Carleson, T. Gamelin, Chap. 2-4

Basic example $P(z) = a_d z^d + \dots + a_0$

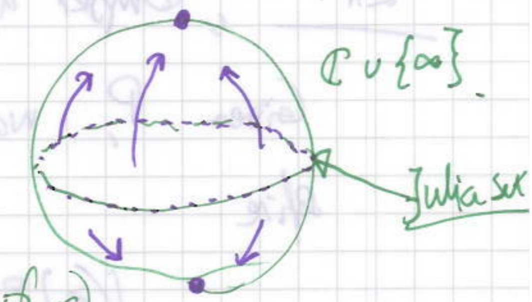
Interested in iterating this map n times,

$$P^n(z) = P\circ \dots \circ P(z) \quad \text{as } n \to \infty.$$

"Long term behaviour of $P^n(z)$".

Example 2 $P(z) = z^2$

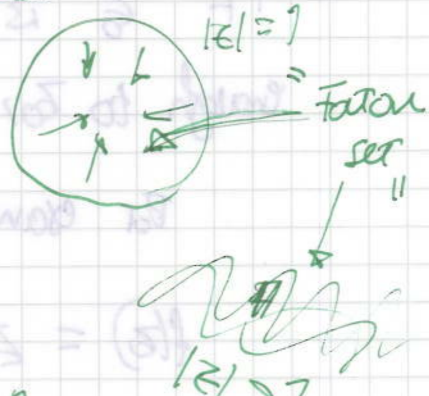
$$P^n(z) = z^{2^n}$$



· If $|z| < 1$, then $P^n(z) \to 0$ (really fast)

· Similarly, if $|z| > 1$, then $P^n(z) \to \infty$ really fast.

But what happens on S^1?



· Let $\xi = e^{2\pi i \frac{p}{q}}$

Then $P^n(\xi) = e^{\frac{2\pi i 2^n p}{q}}$

If $q = 2^n - 1$ for some n, then $\frac{2^n p}{q} = \frac{2^n p}{2^n - 1}$

$$= \frac{(2^n - 1 + 1)p}{2^n - 1} = p + \frac{p}{2^n - 1}$$

Hence this a periodic point! ③

Also, note that the periodic points are dense.

If $z_0 = e^{2\pi i \frac{p}{q}}$ w/ $q = 2^n$, then

$$\varphi^n(z_0) = e^{2\pi i \frac{p}{2^n} 2^n} = 1,$$ hence z_0 is a „backward iterate of 1" (stationary, becomes).

But these are also dense!

We call this S^1 the Julia set of f.

Ex 3 „Complex dynamics originated from Newton's method."

Given P, want to find the roots of P

define

$$f(z) = z - \frac{P(z)}{P'(z)}$$

If z_0 is a root of P, and if you choose w_0 close enough to z_0, then $f^{(n)}(w_0) \to z_0$ (really fast).

For example: if $P(z) = z^2 - 1$, then

$$f(z) = z - \frac{z^2-1}{2z} = \frac{2z^2 - z^2 + 1}{2z} = \frac{z^2+1}{2z}.$$

Note The roots are fixed points of f.

Check that ~~Assume~~ $f'(1)=0$.

Do Taylor expansion of f around $z=1$.

$$f(z) = 1 + a_2(z-1)^2 + a_3(z-1)^3 + \cdots$$

Here 1 is (super)-attracting fixed point of f.

$\boxed{\text{What happens globally?}}$ $\qquad f(z) = \dfrac{z^2+1}{2z}$ ← The Newton map

- Idea Change coordinates, so it is easy to see what happens.

- Conjugacy ~~Instead of~~ If φ is an injective holomorphic map, look instead at $g = \varphi^{-1}\circ f\circ\varphi$. Then $g^n = \varphi^{-1}\circ f^n \circ \varphi$.
  Hence g have the same dynamics as f.

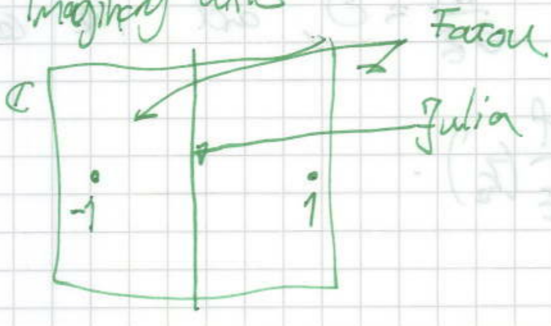
Want φ to move the fixed points $\{-1,1\}$ to $\{0,\infty\}$.

Define $\varphi(z) = \dfrac{z+1}{z-1}$. Then $f(\varphi(z)) = \dfrac{z^2+1}{z^2-1} = \varphi(z^2)$.

Hence $g = z^2$.

Check that $\varphi(S^1)$ = imaginary axis

Here picture is



[B R E A K]

" Some basics and not so basics about complex analysis"

Holomorphic functions / mappings

Let $\Omega \subseteq \mathbb{C}$ be a domain.

A function $f = u + iv \in C^1(\Omega)$ is holomorphic if

$$\left[\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right]$$

(Cauchy-Riemann eq's)

- Hence the real tangent map have the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ which is a rotation matrix.

Introduce differential operators $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. Then $f \in \mathcal{O}(\Omega)$ iff $\frac{\partial f}{\partial \bar{z}} = 0$, and the complex derivative at z_0 is $\frac{\partial f}{\partial z}(z_0)$.

Also introduce differentials

$$dz = dx + i\,dy \quad \text{and} \quad d\bar{z} = dx - i\,dy$$

Thm $f \in \mathcal{O}(\Omega) \iff$ locally at all $a \in \Omega$, we have

$$f(w) = \frac{1}{2\pi i} \int_{bD_\varepsilon(a)} \frac{f(z)}{z-w}\, dz.$$

$\iff$ $f(z)$ is locally represented by a power series $f(z) = \sum_{j=0}^{\infty} a_j (z-a)^j$ ∎

Prop Let $\Omega \subseteq \mathbb{C}$ be a domain, and let $f_j \in \mathcal{O}(\Omega)$ s.t. $f_j \to f$ uniformly on compacts as $j \to \infty$. Then $f \in \mathcal{O}(\Omega)$.

Pf

For $w \in D_\varepsilon(a)$, we have

$$f(w) = \lim_{j\to\infty} f_j(w) = \lim_{j\to\infty} \frac{1}{2\pi i} \int_{bD_\varepsilon(a)} \frac{f_j(z)}{z-w}\, dz$$

$$= \frac{1}{2\pi i} \int_{bD_\varepsilon(a)} \frac{f(z)}{z-w}\, dz \quad \text{since } f_j \to f \text{ uniformly on}$$

compacts. ∎

Prop Let $\Omega \subseteq \mathbb{C}$ be a domain. Let
$f_j \in \mathcal{O}^*(\Omega)$ (i.e. $f_j(z) \neq 0$ for $z \in \Omega$)
and assume $f_j \to f$ uniformly on compacts as $j \to \infty$.
Then either $f \equiv 0$ or $f \in \mathcal{O}^*(\Omega)$.

Pf Assume f nonconstant but $f(a) = 0$ for some $a \in \Omega$.
Choose small ε disc around a s.t. a is the only zero of f on $bD_\varepsilon(a)$. Then $\exists\ \delta > 0$ s.t. $|f_j(z)| \geq \delta$ for all j long enough.

Claim Then $g_j = \frac{1}{f_j}$ is convergent on $D_\varepsilon(a)$.

Then
$$|g_n(z) - g_m(z)| = \left| \frac{1}{f_n(z)} - \frac{1}{f_m(z)} \right| = \left| \frac{f_m(z) - f_n(z)}{f_n(z) f_m(z)} \right| \leq \frac{|f_m(z) - f_n(z)|}{\delta^2} \to 0$$

So $\{g_j\}$ is a Cauchy sequence on $bD_\varepsilon(a)$, hence also on $\overline{D_\varepsilon(a)}$ by the maximum principle. (??)

So $g_j \to g \in \mathcal{O}(D_\varepsilon(a)) \cap C(\overline{D_\varepsilon(a)})$.

Hence $g \cdot f = 1$ on the boundary $\Rightarrow g f = 1$ on $D_\varepsilon(a)$ by the identity principle.

Contradiction b.c. $f(a) = 0$! ∎

Prop. Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $f_j \in \mathcal{O}(\Omega)$ be injective holomorphic, and assume that $f_j \to f$ u.o.c. Then f is constant or injective. (7)

Pf. Assume f is nonconstant but not injective. So $f(z_0) = f(w_0)$ for some $z_0 \neq w_0$.

Look at $g_j(z) = f_j(z) - f_j(w_0)$.

Then $g_j(z) = 0 \iff z = w_0$.

Again, choose a small disc. But $g_j(z) \to f(z) - f(w_0)$ which is zero for $z = z_0$. Contra! ∎

Prop 2.6 (Cauchy estimates)

Let $f \in \mathcal{O}(D_r(a)) \cap C(\overline{D_r(a)})$. Then

$$|f^{(k)}(a)| \leq \frac{k!\, \|f\|_{D_r(a)}}{r^k}.$$

$$F(z_0) = \int_{bD_\varepsilon(a)} g(z_0, w)\, dw$$

Pf. Write

$$f(z) = \frac{1}{2\pi i} \int_{bD_r(a)} \frac{f(w)}{w - z}\, dw.$$

$$F'(z_0) = \lim_{\delta \to 0} \frac{F(z_0+\delta) - F(z_0)}{\delta}$$

$$f'(z) = \frac{1}{2\pi i} \int_{bD_r(a)} \frac{+ f(w)}{(w - z)^2}\, dw$$

etc. . .

$$= \lim_{\delta \to 0} \int_{bD_\varepsilon(a)} \frac{g(z_0+\delta, w) - g(z_0, w)}{\delta}$$

⋮

etc

må bytte på disse
(ue konvergensgreier)

(no more time - continue tomorrow)

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