Let G be a compact simply connected semisimple Lie group. (1)

f.dim reps of G = fin. dim $\mathfrak{g}$-modules

= fin. dim $\mathfrak{g}_{\mathbb{C}}$-modules

= fin. $U(\mathfrak{g}_{\mathbb{C}})$-modules

If V is a $U(\mathfrak{g}_{\mathbb{C}})$-module (f.d. or not), then $V\setminus\{0\}$ is a highest w. vector if

- $v \in V(\lambda) = \{w \in V \mid Hw = \lambda(H) w \ \forall H \in \mathfrak{h}\}$
- $E_\alpha v = 0 \quad \forall \alpha \in \Delta_+$.
  ($\Leftrightarrow$ $E_{\alpha_i} = 0$ for $i = 1, \dots, r$ if $\Pi = \{\alpha_1, \dots, \alpha_r\}$)
- If in addition $v \in V$ generates V, so $V = U(\mathfrak{g}_{\mathbb{C}})v$,

then V is a highest weight module.

If V is f.dim, then highest w. vectors exist, and its weight is in P_+.

$$\left\{ \lambda \in \mathfrak{h}^* \;\middle|\; \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \in \mathbb{Z}_+ \ \forall \alpha \in \Delta_+ \right\}$$

For $\lambda \in \mathfrak{h}^*$, we have a 1-dim $\mathfrak{b}$-module $\mathbb{C}_\lambda = \mathbb{C}$,

where $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \subset \mathfrak{g}$. (Borel subalgebra)

$\mathbb{C}_\lambda$ defined by $H\cdot 1 = \lambda(H)1$

$E_\alpha 1 = 0$ for $\alpha \in \Delta_+$.

Define

$$L_\lambda = U(\mathfrak{g}_\mathbb{C}) \underset{U\mathfrak{b}}{\otimes} \mathbb{C}_\lambda$$

Put $v_\lambda = 1 \otimes 1 \in L_\lambda$.

Then v_λ generates L_λ and $Hv_\lambda = \lambda(H)v_\lambda$, $E_\alpha v_\lambda = 0$.

Thus L_λ is a highest weight module with highest weight vector v_λ of w/ λ.

It is called a <u>Verma module</u>.

It has the following universal property: if V is a highest weight module w/ highest weight vector $w \in V(\lambda)$, then there exists a ! morphism $L_\lambda \to V$ s.t. $v_\lambda \mapsto w$.

Indeed $\mathbb{C}_\lambda \to V$, $1 \mapsto w$ is a morphism of $\mathfrak{b}$-modules. Then we define $L_\lambda \to V$ to be the composition

$$L_\lambda = U(\mathfrak{g}_\mathbb{C}) \underset{U\mathfrak{b}}{\otimes} \mathbb{C}_\lambda \longrightarrow U(\mathfrak{g}_\mathbb{C}) \underset{U\mathfrak{b}}{\otimes} V \longrightarrow V$$

By the PBW-theorem, we have $U(\mathfrak{g}_{\mathbb{C}}) \cong$

$$U\mathfrak{n}_- \otimes U\mathfrak{b}$$

where $\mathfrak{n}_- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha$ as $U\mathfrak{n}_- - U\mathfrak{b}$ modules.

Hence

$$L_\lambda = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda = (U\mathfrak{n}_- \otimes U\mathfrak{b}) \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda$$

$$\cong U\mathfrak{n}_- \otimes_{\mathbb{C}} (U\mathfrak{b} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda)$$

$$\cong U\mathfrak{n}_- \otimes_{\mathbb{C}} \mathbb{C}_\lambda$$

$$\cong U\mathfrak{n}_-$$

as $U\mathfrak{n}_-$-modules. In other words, we have an isomorphism of $U\mathfrak{n}_-$-modules $U\mathfrak{n}_- \cong L_\lambda$ $\quad U\mathfrak{n}_- \ni x \mapsto x v_\lambda$.

Since $F_{\beta_1}^{a_1} \cdots F_{\beta_m}^{a_m} v_\lambda$ is of weight $\lambda - a_1\beta_1 - \dots - a_m\beta_m$ for $\beta_1, \dots, \beta_m \in \Delta_+$, we have a weight decomposition

$$L_\lambda = \bigoplus_{\mu \leq \lambda} L_\lambda(\mu)$$

where $L_\lambda(\mu) = \{ v \in L_\lambda \mid Hv = \mu(H) v \quad \forall H \in \mathfrak{h} \}$

and $\mu \le \lambda$ means that $\lambda - \mu = \sum_{i=1}^{r} a_i \alpha_i$

for some $a_i \in \mathbb{Z}_+$.

By the PBW-theorem, if $\Delta_+ = \{\beta_1, \dots, \beta_m\}$,

then $\{F_{\beta_1}^{a_1} \dots F_{\beta_m}^{a_m}\}_{a_1, \dots, a_m \in \mathbb{Z}_+}$ is a basis in $U\mathfrak{n}_-$.

It follows that $\dim L_\lambda(\mu)$ = the no. of ways of writing $\lambda - \mu$ as

$\sum_{j=1}^{m} a_j \beta_j \quad a_j \in \mathbb{Z}_+$

(called Kostant's partition function of $\lambda - \mu$)

In particular, $\dim L_\lambda(\mu)$ is ~~always~~ finite $\forall \mu$.

For any submodule $M \subset L_\lambda$ we have

$$M = \bigoplus_{\mu \le \lambda} (M \cap L_\lambda(\mu)).$$

It follows that the sum M_λ of all submodules s.t. $v_\lambda \notin M$ (that is $M \cap L_\lambda(\lambda) = 0$)

still has the same property $v_\lambda \notin M_\lambda$.

Define $V_\lambda = L_\lambda / M_\lambda$.

Denote by ξ_λ the image of v_λ in V_λ.

Thm For any $\lambda \in \mathfrak{h}^*$, V_λ is simple (there is no proper submodule). Furthermore, any simple highest weight module is iso to V_λ for a ! defined $\lambda \in \mathfrak{h}^*$. 5

Pf If there is a proper submodule in V_λ then its preimage $M \subset L_\lambda$ satisfies

$$M_\lambda \subsetneq M \subsetneq L_\lambda .$$

If $v_\lambda \in M$, then $M = L_\lambda$ (which is not the case). Hence $v_\lambda \notin M$, and so $M_\lambda = M$ by maximality of M_λ. Thus V_λ is simple.

If V is a simple highest weight-module w/ highest weight vector $v \in V(\lambda)$. Then we have a morphism

$$\begin{aligned} \pi : L_\lambda &\longrightarrow V \\ v_\lambda &\longmapsto w \end{aligned}$$

Then $v_\lambda \notin \ker \pi$ hence $\ker \pi \subset M_\lambda$. We cannot have $\ker \pi \neq M_\lambda$, as $\pi(M_\lambda)$ is a proper submodule of V. So $\ker \pi = M_\lambda$ and $V \cong V_\lambda$.

The highest weight is ! determined, as all other weights of V satisfy $\leq \lambda$.

As any findim simple module is a highest weight module, to finish the classification of such modules (thus findim reps of $\mathfrak{g}$), it remains to understand when V_λ is findim.

Prop? V_λ is fin.dim $\Leftrightarrow$ $\lambda \in P_+$.
"kinda tricky statement"

Pf

The implication $\Rightarrow$ we already know.

Conversely, take $\lambda \in P_+$.

Claim 1 $F_i^{\lambda(H_i)+1} v_\lambda \in M_\lambda$ for all $i = 1, \dots, r$.

Enough to show that $F_i^{\lambda(H_i)+1} v_\lambda$ is a highest weight vector (of weight $\lambda - (\lambda(H_i)+1)\alpha_i$). Therefore it generates a submodule with weights $\leq \lambda - (\lambda(H_i)+1)\alpha_i$, hence this submodule is contained in M_λ.

We have to check that

$$E_j F_i^{\lambda(H_i)+1} v_\lambda = 0 \quad \text{for } j = 1, \dots, r.$$

If $j \neq i$, then $[F_i, E_j] = 0$ so

$$E_j F_i^{\lambda(H_i)+1} v_\lambda = F_i^{\lambda(H_i)+1} E_j v_\lambda = 0.$$

For $j = i$, recall that we have

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$$E_i F_i^n v_\lambda = (\lambda(H_i) - (n-1))n F_i^{n-1} v_\lambda$$

(same proof as for $SU(2)$)

Thus for $n = \lambda(H_i)+1$, we get

$$E_i F_i^{\lambda(H_i)+1} v_\lambda = 0$$

✓

Claim 2 For every $i = 1, \dots, r$, V_λ decomposes as a sum of fin. dim $sl_2(\mathbb{C})_i$ - modules, where $sl_2(\mathbb{C})_i = \text{Span}\{F_i, H_i, E_i\} \simeq sl_2(\mathbb{C})$.

By claim 1, we have $F_i^{\lambda(H_i)+1} \xi_\lambda = 0$.

This implies that $\xi_\lambda, F_i\xi_\lambda, \dots, F_i^{\lambda(H_i)}\xi_\lambda$ span a fin. dim $sl_2(\mathbb{C})_i$ -module (of spin $\frac{\lambda(H_i)}{2}$)

Let $V \subset V_\lambda$ be the sum of f.dim $sl_2(\mathbb{C})_i$ -modules (i fixed). Then $\xi_\lambda \in V$, so if we prove that V is a $\mathfrak{g}_\mathbb{C}$-submodule, then $V = V_\lambda$.

For this, it suffices to show that if $M \subset V$ is a f.dim $sl_2(\mathbb{C})_i$-module, then $E_j M + E_i E_j M + \dots + E_i^{-a_{ij}} E_j M$ $j \neq i$ is still a $sl_2(\mathbb{C})_i$-module (and similar statements hold for H_j, F_j)

Then E_jM, F_jM, $H_jM \subset V_i$ so V is a $\mathfrak{g}_p$-module.

Recall that we have the Serre relation

$$(\mathrm{ad}\, E_i)^{1-a_{ij}}(E_j) = 0.$$

Let us observe that for all $k \geq 0$, we have

$$E_i^k E_j M \subset (\mathrm{ad}\, E_i)^k (E_j) M + E^{k-1} E_j M + \dots + E_j M,$$

i.e.

$$(\mathrm{ad}\, E_i)^k E_j M \subset E_i^k E_j M + E_i^{k-1} E_j M + \dots + E_j M$$

The proof is by induction on k. Nothing to prove for $k=0$.

Assume the claim is true for k.

$$E_i^{k+1} E_j M \subset E_i\left((\mathrm{ad}\, E_i)^k (E_j) M + E_i^{k-1} M + \dots + E_j M \right)$$

$$\subset (\mathrm{ad}\, E_i)^{k+1} (E_j) M + (\mathrm{ad}\, E_i)^k M + E_i^{k+1} E_j M + \dots + E_j M$$

$$\overset{\text{induction}}{\subset} (\mathrm{ad}\, E_i)^{k+1} (E_j) M + E_i^k E_j M + \dots + E_j M.$$

Similarly

$$(\mathrm{ad}\, E_i)^{k+1} (E_j) M \subset E_i (\mathrm{ad}\, E_i)^k (E_j) M + (\mathrm{ad}\, E_i)^k (E_j) M$$

$$\overset{\text{ind.}}{\subset} E_i^{k+1} E_j M + \dots + E_j M.$$

This proves the induction step

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Thus, for $k = 1 - a_{ij}$, we get (Uses the Serre relation)

$$E_i^{1-a_{ij}} E_j M \subset E_i^{-a_{ij}} E_j M + \dots + E_j M$$

This shows that $E_i^{-a_{ij}} E_j M + \dots + E_j M$ is invariant under E_i

(BTW not big sum. Recall $|a_{ij}| \leq 3$)

It is even easier to see that this space is invariant under F_i and H_i.

This proves the second claim

Claim 3 For any $w \in W$ (Weyl group) and $\mu \in P$, we have

$$\dim V_\lambda(\mu) = \dim_\lambda (w\mu)$$

For any i, we have that V_λ is a sum of f. dim $sl_2(\mathbb{C})_i$ - submodules. Hence we get a representation of $SU(2)$ on V_λ defined by the ~~~~ isomorphism

$$sl_2\mathbb{C} \simeq sl_2(\mathbb{C})_i$$

Consider the action by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SU(2)$ for this representation.

It maps $V_\lambda(\mu)$ onto $V_\lambda(s_i\mu)$ (see the proof of the fact that Δ is a root system)

Thus ~~~~ $\dim V_\lambda(\mu) = \dim V_\lambda(s_i\mu)$ for any simple reflection s_i.

As $\{s_i\}$ generate W, this holds for all $w \in W$.

$\boxed{\text{Claim 4}}$ There are finitely many weights $\mu \in P$ s.t. $V_\lambda(\mu) \neq 0$.

By Claim 3, if $V_\lambda(\mu) \neq 0$, then $V_\lambda(w\mu) \neq 0$ for any $w \in W$.

We can choose w s.t. $w\mu \in P_+ = P \cap \overline{C}$

weight lattice

(as W acts transitively on the w. chambers)

Therefore, it suffices to show that there are only fin. many weights $\mu \in P_+^{\#}$ s.t. $\mu \leq \lambda$.

Indeed we have $\lambda - \mu = \sum_{i=1}^{r} c_i \alpha_i \qquad c_i \in \mathbb{Z}_+$.

Then $(\lambda, \lambda) - (\mu, \mu) = (\lambda + \mu, \lambda - \mu)$

$$= \sum_{i=1}^{r} c_i \underbrace{(\lambda + \mu, \alpha_i)}_{\geq 0 \text{ since in } P_+} \geq 0$$

Thus $\|\mu\| \leq \|\lambda\|$. As P is a lattice, there are only fin. many elements $\mu \in P$ s.t. $\|\mu\| \leq \|\lambda\|$.

As $\dim V_\lambda(\mu) \leq \dim L_\lambda(\mu) < \infty$

we conclude that $\dim V_\lambda < \infty$

$$\mathfrak{g}_{\alpha} = \frac{\mathbb{C}}{\mathfrak{n}}$$

A more careful look at the proof gives the Fact:

Fact For $\lambda \in P_+$, M_λ is generated by the vectors $F_i^{\lambda(H_i)+1} v_\lambda$, $i = 1, \dots, r$.

Deep fact The following is very useful:

Thm Any fin. dim. highest w. module is ~~irreducible~~ simple, hence iso to V_λ for some $\lambda \in P_+$.

Pf Let V be a high. weight fin. dim module w/ highest weight $\xi \in V(\lambda)$ $(\lambda \in P_+)$.

As any fin dim rep. of G decomposes into irreducibles, we have $V \cong V_{\lambda_1} \oplus \dots \oplus V_{\lambda_n}$, $\lambda_i \in P_+$.

But $V(\lambda) = \mathbb{C}\xi$

Hence, under the identification of V w/ $V_{\lambda_1} \oplus \dots \oplus V_{\lambda_n}$ we have $\xi \in V_{\lambda_i}(\lambda)$ for some i. As ξ generates V, we get $V \cong V_{\lambda_i}$

$\blacksquare$