T Young tableau of shape λ
(λ is a Young diagram)

$R(T)$ perm. rows, $C(T)$ = perm. col's.

$$a_T = \frac{1}{|R(T)|} \sum_{g \in R(T)} g$$

$$b_T = \frac{1}{|C(T)|} \sum_{g \in C(T)} \operatorname{sgn}(g)\, g$$

$$c_T = a_T b_T \qquad V_T = \mathbb{C}[S_n]c_T.$$

Thm ① Up to a scalar factor, c_T is a minimal idempotent in $\mathbb{C}[S_n]$.
So it defines an irreducible representation π_T of S_n on V_T.

② $\pi_{T_1} \sim \pi_{T_2}$ iff T_1 and T_2 are of the same shape.

Pf ① We have $R(T) \cap C(T) = \{e\}$, hence

$$c_T = \frac{1}{|R(T)||C(T)|} \sum_{g \in R(T)C(T)} \pm g \neq 0 \text{ up to sign.}$$

Hence the map $R(T) \times C(T) \to R(T)C(T)$ is injective.

Furthermore, $c_T^2 \neq 0$, since otherwise $\mathrm{Tr}_{\lambda_{S_n}}(c_T) = 0$. ?

reg rep

but $\mathrm{Tr}_{\lambda_{S_n}}(c_T) = \dfrac{n!}{|C(T)|\,|R(T)|} \neq 0$

Claim 1 $\quad a_T\, \mathbb{C}[S_n]\, b_T = \mathbb{C}\, a_T b_T$.

$\boxed{\begin{array}{l} \text{Recall minimal idempotent:} \\ e \in A,\ e^2 = e \text{ and} \\ eAe = \mathbb{C}e \end{array}}$

Assume $g \in R(T)C(T) \ni g = rc$, $\quad r \in R(T)$, $c \in C(T)$.

$$a_T\, g\, b_T = a_T\, r c\, b_T = a_T b_T \,\mathrm{sgn}(c) \quad ✓$$

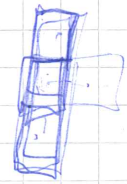
Now assume $g \notin R(T)C(T)$. We will show $\exists$ transposi. $t = (i,j) \in R(T)$

s.t. $g^{-1} t g \in C(T)$.

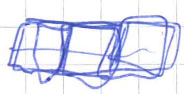
Then $a_T g b_T = a_T t g b_T = a_T g (g^{-1} t g) b_T$

$= -a_T g b_T \quad ✓$

so $a_T g b_T = 0$.



Note that $g^{-1} t g \in C(T) \iff t \in g\, C(T)\, g^{-1} = C(g(T))$.



Assume such a t does not exist.

This means that any two different i, j in the same row of T lie in different columns of $g(T)$.

Let $(i_1, \dots, i_m)$ be the first row of T. ③

Hence $\exists$ $c_1 \in C(g(T))$ s.t. $i_1, \dots, i_m$ belong to the first row of $(c_1 g)(T)$.

Therefore $\exists$ $r_1 \in R(T)$ s.t. $r_1(T)$ and $(c_1 g)(T)$ have the same first row.

Then we apply the same procedure to the second row of $c_1(T)$ and $(c_1 g)(T)$. And so on.

After m steps (# of rows) we get elt's $r_1, \dots, r_m \in R(T)$, $c_1, \dots, c_m \in C(g(T))$ s.t.

$$(r_m \cdots r_1)(T) = (c_m \cdots c_1 g)(T).$$

$$\underset{r(T)}{\parallel} \qquad = \qquad \underset{c' g(T)}{\parallel}$$

so $r = c'g$. Therefore $c = g^{-1} c' g \in C(T)$

and $r = gc$ so $g = rc^{-1} \in R(T)C(T)$,

which is a contradiction. $\boxed{\text{Proves the claim!}}$

Then
$$c_T^2 = a_T b_T a_T b_T = \alpha\, a_T b_T = \alpha\, c_T$$
for some $\alpha \in \mathbb{C}$. $\alpha \neq 0$,

~~so we set~~ here $\frac{c_T}{\alpha}$ is a minimal idemp. in $\mathbb{C}[S_n]$.

$\boxed{\text{Exc}}$ Show that $\alpha = \frac{n!}{|C(T)||R(T)| \dim V_T}$

Second part: $V_T \sim V_{T'}$ iff T' and T have same shape.

Consider the lexicographic order on partitions (= Young diagr)

For a Young tableau T denote by $[T]$ the shape of T.

__Claim 2__ If $[T] > [T']$, then $a_T\, \mathbb{C}[S_n]\, b_{T'} = 0$.

Proved similarly to __Claim 1__. Namely, for any $g \in S_n$, it suffices to show that $\exists$ $t \in R(T)$ s.t. $g^{-1} t g \in C(T')$.

Assume such a t does not exist, that is, any two different elt's in the same row of T belong to different col's of $g(T')$.

If the first row of $[T]$ > than the first row of $[T'] = [g(T')]$ $\Leftarrow$ Clearly impossible!

Hence the lengths of the first rows of T and T' are the same. As in the proof of __Claim 1__ we can find $r_1 \in R(T)$ and ~~$c_1 \in C(T)$~~ $c_1 \in C(g(T'))$ s.t. $r_1(T)$ and $(c_1 g)(T')$ have

have the same first row.

Repeat for second row.

Sooner or later we get a contradiction.

Thus <u>Claim</u> is proved.

So $[T] > [T'] \Rightarrow a_T\, \mathbb{C}[S_n]\, b_{T'} = 0.$

Note also by applying the anti-automorphism $g \mapsto g^{-1}$ on $\mathbb{C}[S_n]$, we also get $b_{T'}\, \mathbb{C}[S_n]\, a_T = 0.$

<u>Exercise</u> Assume A is an algebra, $e \in A$ an idempotent, M a left A-module. Then

$$\mathrm{Hom}_A(Ae, M) \simeq eM$$
$$f \longmapsto f(e)$$

Hence if $[T] > [T']$, then $\mathrm{Hom}_{\mathbb{C}[S_n]}(V_{T'}, V_T)$

$$= \mathrm{Hom}_{\mathbb{C}[S_n]}\left(\mathbb{C}[S_n]c_{T'}, \mathbb{C}[S_n]c_T\right)$$

$$= c_{T'}\, \mathbb{C}[S_n]\, c_T = a_{T'} b_{T'}\, \mathbb{C}[S_n]\, a_T b_{T'} = 0$$

Hence $\pi_T \not\simeq \pi_{T'}$.

Can be seen otherwise by noting that if $[T] = [T']$ then $c_{T'}$ is conj. to c_T

Exercise: We have an iso of $\mathbb{C}[S_n]$-mod's

$$V_T = \mathbb{C}[S_n]c_T = \mathbb{C}[S_n]a_T b_T$$
$$\simeq \mathbb{C}[S_n]\, b_T a_T.$$

A bit more explicitly the Specht moduls can be described as follows:

Fix a Young diagram λ. Introduce an equivalence relation on Young tableaux of shape λ:

$$T_1 \sim T_2 \text{ if } T_1 = r(T_2) \text{ if } r \in R(T_2).$$

~~The~~ The equivalence class of a Young tabl. T is called a tabloid (of shape λ) and denoted by $\{T\}$.

The group S_n still acts on the set of tabloids.

Then we can consider the corresp. perm. representation

Thus we consider the space M_λ w/ basis consisting of tabloids of shape λ. and the repr. on M_λ: $\theta_\lambda(g)\{T\} = \{g(T)\}$.

The action on tabloids w/ fixed λ is transitive.

Thus the basis can be identified w/ $S_n/R(T)$.

Thus we get isomorphisms of $\mathbb{C}[S_n]$-modules.

ex.
$$\begin{cases} M_\lambda \cong \mathbb{C}\left[S_n/R(T)\right] \cong \mathbb{C}[S_n]a_T \\ \{g(T)\} \longleftarrow\!\shortmid\ \delta_{gR(T)} \longleftarrow\!\shortmid\ g a_T \end{cases}$$

Under this isomorphism the submodule $\mathbb{C}[S_n]\,b_T a_T \subseteq \mathbb{C}[S_n]a_T$ we get the submodule $V_\lambda \subset M_\lambda$ spanned by elt's

$$g b_T \mapsto g b_T\{T\} = b_{g(T)}\, g\{T\}$$
$$= b_{g(T)}\{g(T)\}$$

In other words, V_λ is spanned by all elements

$$e_{T'} \overset{\text{def}}{=} b_{T'}\{T'\} \quad \text{by all tableaux of shape } \lambda.$$

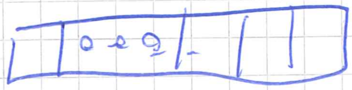
This gives a description of $V_\lambda \cong V_T$ only in terms of λ.

Def A tableau T is called *standard* if the numbers in every row and column are increasing.

Fact For any Young diagram λ the elt's e_T for all standard Y. tabl.x. of shape λ form a basis in V_λ.

Problem If T standard, then $g e_T = e_{g(T)}$ is a combination of standards?

Examples

(i) $\lambda = (n)$ 

(one row of n boxes)

For any T of shape λ, we get

$$C(T) = \{e\}$$
$$R(T) = S_n$$

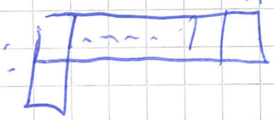
Thus $$c_T = \frac{1}{n!} \sum_{g \in S_n} g$$

Then $g c_T = c_T g = c_T$

Thus we get the trivial representation.

(i.e. $\mathbb{C}[S_n] c_T = \mathbb{C} c_T$ and

$\pi_T \sim \varepsilon$.

Exc λ:  $n-1$

The corresp. rep is the <u>standard</u> rep π_{ST} on $\{x \in \mathbb{C}^n \mid \sum x_i = 0\}$
$\cong \mathbb{C}^n / (1 \dots 1)$

(ii) Let $\lambda = (1, \dots, 1)$ i.e. $T =$ 

Then for any T of shape λ, we get

$$C(T) = S_n \quad R(T) = \{e\}.$$

Therefore $$c_T = \frac{1}{n!} \sum \mathrm{sgn}(g) \cdot g$$

so $g c_T = c_T g = \mathrm{sgn}(g)\, c_T$.

Thus $\mathbb{C}[S_n] c_T = \mathbb{C} c_T$ and π_T is the 1-dim rep $\boxed{\mathrm{sgn}}: \mathbb{C} \to \mathbb{C}$.

Exc ε and the <u>sgn</u> are the only 1-dim reps of S_n.