Recall root system

$\Delta \subset V \setminus \{0\}$

① $s_\alpha(\Delta) = \Delta$ $\quad s_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$

② $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$

Assume that span $\Delta = V$ and Δ is reduced.

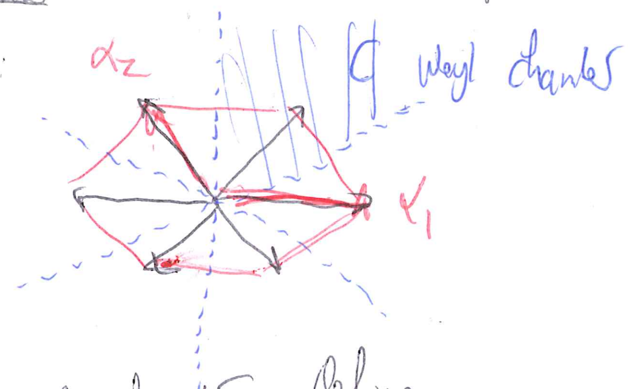
Simple roots $\Pi \subset \Delta$

a) Π is a basis of V

b) $\beta = \sum_\alpha c_\alpha \alpha$ then either $c_\alpha \geq 0$ $\forall \alpha$ or $c_\alpha \leq 0$ $\forall \alpha$.

A Weyl chamber is a connected component of $V \setminus \bigcup_{\alpha \in \Delta} \alpha^\perp$

Keep in mind:



Fix a Weyl chamber. Define

$$\Pi(C) = \left\{ \alpha \;\middle|\; \begin{array}{l} \alpha \text{ is } C\text{-positive, so } (\alpha,\beta) > 0 \ \forall \beta \in C \\ \alpha \neq \beta + \gamma \text{ for any } C\text{-positive } \beta, \gamma \end{array} \right\}$$

Prop $\Pi(C)$ is a system of simple roots. Furthermore, $C \mapsto \Pi(C)$ is a bijection between the Weyl chambers and systems of simple roots.

Pf By def of $\Pi(C)$, any C-positive root β has the form $\sum_{\alpha\in\Pi(C)} c_\alpha \alpha$ for $c_\alpha \in \mathbb{Z}_+$.

Thus the span $\Pi(C) = V$.

Therefore, we have to show that $\Pi(C)$ is a basis. Observe first that if $\alpha, \beta \in \Pi(C)$, then $\alpha - \beta \notin \Delta$. Indeed, otherwise, either $\alpha-\beta$ or $\beta-\alpha$ is C-positive. But then $\alpha = \beta + (\alpha-\beta)$, $\beta = \alpha + (\beta-\alpha)$ shows that we get a contradiction to the def of $\Pi(C)$.

Observe next that if $\alpha, \beta \in \Pi(C)$, $\alpha \neq \beta$, then $(\alpha,\beta) \leq 0$. Indeed, assume $(\alpha,\beta) > 0$. We may assume that $\|\alpha\| \geq \|\beta\|$. Then

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} < \frac{2\,\|\alpha\|\,\|\beta\|}{\|\alpha\|^2} = \frac{2\,\|\beta\|}{\|\alpha\|} \leq 2,$$

Hence $\dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} = 1$. But then $s_\alpha(\beta) = \beta - \alpha \in \Delta$, contradicting the previous observation.

Assume now that $\sum_{\alpha\in\Pi(C)} c_\alpha \alpha = 0$ for some $c_\alpha \in \mathbb{R}$.

Consider $A = \{\alpha \in \Pi(C) \mid c_\alpha > 0\}$, $B = \Pi(C) \setminus A$. and let

$$v = \sum_{\alpha\in A} c_\alpha \alpha = \sum_{\alpha\in B} (-c_\alpha)\alpha.$$

Then

$$(v,v) = \sum_{\substack{\alpha\in A\\ \beta\in B}} \underbrace{c_\alpha(-c_\beta)}_{\text{positive}} \cdot (\alpha,\beta) \leq 0$$

negative

Here $\nu = 0$.

Take any vector $\beta \in C$. Then

$$0 = (\nu, \beta) = \sum_{\alpha \in A} c_\alpha (\alpha, \beta) = \sum_{\alpha \in B} (-c_\alpha)(\alpha, \beta)$$

positive (≥ 0) ; strictly positive ; ~~strictly~~ negative ; str. positive

As $(\alpha, \beta) > 0 \ \forall \alpha \in \Pi(C)$, we conclude that $c_\alpha = 0 \ \forall \alpha \in A$ and $B = \emptyset$. Thus $\Pi(C)$ is a basis$_0$.

For the second part of the proposition, note that for any Weyl chamber C, we have

$$C = \{\beta \in V \,/\, (\alpha, \beta) > 0 \text{ for any } C\text{-positive root } \alpha\}$$
$$= \{\beta \in V \,/\, (\alpha, \beta) > 0 \ \forall \alpha \in \Pi(C)\}$$

Thus we can recover C from $\Pi(C)$, so $C \mapsto \Pi(C)$ is injective.

Assume now that Π is a system of simple roots. As Π is a basis, there exists $\beta_0 \in V$ s.t. $(\alpha, \beta_0) = 1 \quad \forall \alpha \in \Pi$.

Then $(\alpha, \beta_0) \neq 0 \quad \forall \alpha \in \Delta$. Hence β_0 lies in a ! Weyl chamber C. Then $\Pi \subset \Pi(C)$.
! defined

Indeed, if $\alpha \in \Pi$ and $\alpha = \beta + \gamma$ for some C-positive β, γ then β, γ are positive w.r.t. to Π, as $(\beta, \beta_0) > 0$, $(\gamma, \beta_0) > 0$. As β, γ can't be proportional to α (Δ is reduced), decomposing β and γ as $\beta = \sum_{\delta \in \Pi} c_\delta' \delta$, $\gamma = \sum_{\delta \in \Pi} c_\delta'' \delta$ w/ $c_\delta', c_\delta'' \geq 0$ we

Here $\alpha = \sum_{\delta \in \Pi} (c_\delta' + c_\delta'')\delta$ which contradicts

the assumption that Π is a basis.

But $\#\Pi = \#\Pi(C)$, so they are equal. $\blacksquare$

Given a Weyl chamber C, we say that a hyperplane $L \subset V$ is a *wall of* C if $L \cap \bar{C}$ has nonempty interior in L.

Exc ① $\forall \alpha \in \Delta$, $\alpha^\perp$ is a *wall* of some Weyl chamber.

② For any Weyl chamber C, the walls of C are the hyperplanes $\alpha^\perp$, $\alpha \in \Pi(C)$.

① and ② imply that any $\alpha \in \Delta$ lies in $\Pi(C)$ for some C.

Def The *Weyl group* W of (V, Δ) is the subgroup of $O(V)$ generated by reflections s_α, $\alpha \in \Delta$.

As $w(\Delta) = \Delta$ $\forall w \in W$ and $\operatorname{span} \Delta = V$, the Weyl group W is finite. As W maps the system of hyperplanes $\alpha^\perp$, $\alpha \in \Delta$ into itself, W maps Weyl chambers into Weyl chambers.

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<u>Thm</u> We <u>have</u>

① W acts freely and transitively on the set of Weyl chambers.

② For any Weyl chamber C, W is generated by the reflections s_α for $\alpha \in \Pi(C)$, called the <u>simple reflections</u>.

Pf Let C be a Weyl chamber. Let C' be an adjacent Weyl chamber. So there exists an hyperplane L s.t. $L \cap \bar{C} \cap \bar{C}'$ has nonempty interior in L.

Then $L = \alpha^\perp$ for some $\alpha \in \Pi(C)$ and $C' = s_\alpha(C)$.

Then $s_\alpha(\Pi(C)) = \Pi(C')$

It follows that for any $\overset{\beta'}{\overset{\shortparallel}{s_\alpha(\beta)}} \in \Pi(C')$ (so $\beta \in \Pi(C)$)

we have $s_{\beta'} = s_{s_\alpha(\beta)} = s_\alpha s_\beta s_\alpha$

(since for any orthogonal ~~reflection~~ transformation we have $s_{T\alpha} = T s_\alpha T^{-1}$.)

Thus reflections $s_{\beta'}$ for $\beta' \in \Pi(C')$ lie in the group generated by s_α, $\alpha \in \Pi(C)$.

Now for any Weyl chamber C'' we can find a sequence of adjacent Weyl chambers $C_0, \dots, C_n = C''$. Using the above argument and induction on n, we conclude that $\exists\, w \in W$ s.t. $w(C) = C''$ and the reflections $s_{\beta''}$ for $\beta'' \in \Pi(C'')$ lie in the group generated by s_α for $\alpha \in \Pi(C)$.

As any root in Δ lies in some $\Pi(C')$ for some C', we see that W is generated by s_α, $\alpha \in \Delta$.

So the action of W on the Weyl chambers is transitive.

We will not prove (nor need) that the action is free in general. For root systems arising from compact groups, this will follow from the proof of the next result.

$\square$

Assume now that G is a compact connected Lie group and $T \subset G$ a maximal torus. Consider the corresponding root system Δ in $V = i\mathfrak{t}^* \subset \mathfrak{h}^*$.

Let $V_0 = \operatorname{span}_{\mathbb{R}} \Delta \subset V$. Note that for any $\alpha \in \Delta$, the reflection s_α acts trivially on $V_0^\perp$. It follows that the group generated by s_α, $\alpha \in \Delta$ can be identified w/ the Weyl group W generated by $s_\alpha|_{V_0}$.

We have a homomorphism

$$r : N(T)/T \longrightarrow O(V)$$

defined by $r(gT)(\alpha) = \alpha \circ (\operatorname{Ad} g^{-1})$

Thm The homomorphism r defines an isomorphism (7)

$$N(T)/T \simeq W,$$

so our two def's of the Weyl group are equivalent.

Pf Recall that for any $g \in N(T)$, we have that

$$(\mathrm{Ad}\, g)(\mathfrak{g}_\alpha) = \mathfrak{g}_{\alpha \circ \mathrm{Ad}\, g^{-1}}$$

This implies that $\mathrm{Im}\, r$ leaves Δ (hence V_0) invariant.

Recall that for every root $\alpha \in \Delta$, we constructed an element $w_\alpha \in N(T)$ (the image of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SU(2)$) s.t.

$S_\alpha(\beta) = \beta \circ \mathrm{Ad}\, w_\alpha \quad \forall \beta.$

Thus $S_\alpha \in \mathrm{Im}\, r$, ~~and~~ hence $W \subset \mathrm{Im}\, r$.

We claim that the action of $N(T)/T$ on the Weyl chambers in V_0 is free.

Assuming this, we are done. Indeed, first of all this implies that r is injective (which in fact we already know: $N(T)/T \hookrightarrow \mathrm{Aut}\, T$)

Next, if C is a Weyl chamber (in V_0) and $g \in N(T)$, then $r(gT)C$ is a Weyl chamber. Hence $r(gT)C = wC$ for some $w \in W$ by the previous theorem. Then $w = r(hT)$ for some $h \in N(T)$, so $r(h^{-1}gT)C = C$, hence $h^{-1}gT = T$ by the claim. Thus $r(gT) = r(hT) = w$, so that $\mathrm{Im}\, r \subset W$.