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## Thm (Bratt)

Let  $G$  be a finite group,  $m$  the exponent of  $G$ .  
Then any complex representation is defined over  $\mathbb{Q}(\beta_m)$ .

If let  $\pi_1, \dots, \pi_k$  be representatives of equivalence classes of irreducible rep's of  $G$  over  $\mathbb{Q}(\beta_m)$ .

Note that any one-dim rep of a subgroup of  $G$  is defined over  $\mathbb{Q}(\beta_m)$ .

Hence the induction of any such representation is defined over  $\mathbb{Q}(\beta_m)$ .

It follows by Bratteli's theorem, that if  $\pi$  is any f.d. (comp) rep of  $G$ , then

$$\chi_\pi = \sum_{i=1}^k m_i \chi_{(\pi_i)}_{\mathbb{C}} = \sum_{i=1}^k m_i \chi_{\pi_i} \quad \text{for some } m_i \in \mathbb{Z}.$$

Assume in addition that  $\pi$  is irreducible. Then

$$\begin{aligned} \langle \chi_\pi, \chi_\pi \rangle &= \sum_{i,j} m_i m_j (\chi_{\pi_i}, \chi_{\pi_j}) \\ &= \sum_i m_i^2 (\chi_{\pi_i}, \chi_{\pi_i}) \end{aligned}$$

$\Rightarrow$  only possible if  $\chi_\pi = \chi_{\pi_i}$  for some  $i$ , so  $\pi \sim (\pi_i)_{\mathbb{C}}$ .

$\therefore \pi$  is defined over  $\mathbb{Q}(\beta_m)$

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## Mackey's irreducibility criterion

When is an induced representation irreducible?

We need an (unpleasant) lemma.

Let  $G$  be a finite group and consider  $K, H \subset G$  subgroups.

$\pi$  a representation of  $H$ .

Define a subgroup  $H_g \subset H$  by

$$H_g = gHg^{-1} \cap K \quad g \in G$$

Then for any  $g \in G$ , define a representation  $\pi^g$  by

$$\pi^g(h) = \pi(g^{-1}hg).$$

Lemma  $\text{Res}_K^G \text{Ind}_H^G \pi \sim \bigoplus_{\bar{g} \in K \backslash G / H} \text{Ind}_{H_{\bar{g}}}^K \pi^{\bar{g}}$

where  $K \backslash G / H = G / n$  where  $g_1 \sim g_2$  if  $g_1 = kg_2h$

Pf Let  $V = V_{\pi}$  and  $W$  the underlying space of  $\text{Ind}_H^G \pi$ .

We have  $W = \bigoplus_{\bar{g} \in G / H} W_{\bar{g}}$  and  $W_{\bar{e}} = V$ .

For  $x \in G / H$  and  $y = kx \in G / H$ , define  $W(y)$

$$W(y) = \bigoplus_{x' \in kx} W_{x'}$$

This is a  $\mathbb{K}$ -invariant subspace.

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$$\text{Then } W = \bigoplus_{g \in K^G/H} W(g)$$

So need to show that  ~~$\text{Res}_K^G \text{Ind}_{H^g}^G \pi$~~  is  $\mathbb{K}$ -invariant.

$$(\text{Res}_K^G \text{Ind}_{H^g}^G \pi) |_{W(K^G/H)} \cong \text{Ind}_{H^g}^K \pi^g.$$

The stabilizer  $B$  of  $gH \in G/H$  is exactly  $K^N g H g^{-1}$ .

By our characterization of induced rep's, it follows  $Hg$ .

$$\text{that } (\text{Res}_K^G \text{Ind}_{H^g}^G \pi) |_{W(K^G/H)} \sim [(\text{Res}_{H^g}^K \text{Ind}_H^G \pi) |_{W_{H^g}}]$$

$$\text{Ind}_{H^g}^K$$

$$\text{Thus it remains to show that } (\text{Res}_{H^g}^K \text{Ind}_H^G \pi) |_{W_{H^g}} \cong \pi^g.$$

This is clearly true for  $g = e$ , by definition of  $\text{Ind}_H^G \pi$ .

Moreover, for any  $g \in G$  we have

$$(\text{Ind}_H^G \pi)(g) W_H = W_{gH}$$

It follows that  $(\text{Ind}_H^G \pi)(g)$  defines an equivalence between

$$(\text{Res}_{H^g}^G \text{Ind}_{H^g}^G \pi) |_{W_{H^g}}$$

and

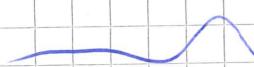
$$(\text{Ind}_H^G \pi)(g) (\text{Res}_{H^g}^G \text{Ind}_{H^g}^G \pi)(e) (\text{Ind}_H^G \pi)(g) |_{W_H}$$

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The value of the last expression at left  $h$  is

$$(Ind_{H \cap \pi}^G(g^{-1}) (Ind_{H \cap \pi}^G(h)) (Ind_{H \cap \pi}^G(g^{-1})) / W_H)$$

$$= (Ind_{H \cap \pi}^G(g^{-1}hg)) / W_H = \pi(g^{-1}hg) \\ = \pi^3(h). \quad \text{!!!!}$$



Thm (Mackey)

Assume  $\pi$  is an irreducible rep. of  $H \subset G$ . Then

$Ind_{H \cap \pi}^G$  is irreducible iff the representations

$Res_{H \cap \pi}^H \pi$  and  $\pi^g = \pi(g^{-1} - g)$  are disjoint

for all  $g \in G \setminus H$ . (here  $h_g = gHg^{-1} \cap H$ , and disjoint = orthogonal characters)

Let  $\chi = \chi_\pi$ . Then we want to compute

$$(Ind_{H \cap \pi}^G \chi_\pi, Ind_{H \cap \pi}^G \chi_\pi) \stackrel{\text{Frob. rel.}}{=} (Res_{H \cap \pi}^H Ind_{H \cap \pi}^G \chi_\pi, \chi_\pi)$$

Mackey's lemma

$$= \sum_{g \in H \cap \pi^G H} (Ind_{H \cap \pi^g H}^H \chi_{\pi^g}, \chi)$$

Frob. relation

$$= \sum_{g \in H \cap \pi^G H} (\chi_{\pi^g}, Res_{H \cap \pi^g H}^H \chi)$$

We see that  $(Ind_{H \cap \pi}^G \chi, Ind_{H \cap \pi}^G \chi)$

if

we have  $(X_{\pi^g}, X|_{H_g}) = 0$  for  $g \neq 1$ . ⑤

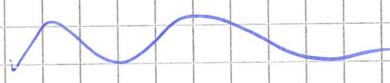
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This criterion is nice when  $H$  is a normal:

[criterion] If  $H \trianglelefteq G$  is normal, and it is a irrep of  $H$ ,  
then  $\text{Ind}_H^G \pi$  is irreducible iff  $\pi \times \pi^g$  for  $g \neq 1$ .

This can also be formulated as follows: we can define a action of  $G$  on  $\mathcal{H}$  by  $g[\pi] = [\pi^g]$ .  $(\pi^g$  is a rppf of  $\pi)$

Then  $\text{Ind}_H^G \pi$  is irreducible iff the stabilizer of  $[\pi]$  in  $G$  is  $H$ .



### Induction from normal subgroups

Assume  $G$  is a finite group,  $H \trianglelefteq G$  normal subgroup.

Q: Can we describe the representation theory in terms of those of  $H$  and  $G/H$ ?

(answers almost)

Assume  $(W, \theta)$  is an irrep of  $G$ . Let  $\pi$  be an irreducible representation contained in  $\text{Res}_H^G W$ .

Then  $\pi^g = \theta(g^{-1})\pi(-)\theta(g)$  is also contained in  $\text{Res}_H^G W$ .

Moreover, if we denote by  $W(\pi) \subset W$  the isotypic component of  $\text{Res}_H^G W$  corresponding to  $\pi$ .

Then  $\theta(g)W(\pi) = W(\pi^g)$ . ⑥

It follows  $\sum_{g \in G} W(\pi^g)$  of  $\theta$

is a  $G$ -invariant subspace of  $W$ . By irreducibility, we

have  $\sum_{g \in G} W(\pi^g) = W$ .

Therefore  $\theta$  determines an orbit  $G[\pi]$  in  $\widehat{H}$  and

$$W = \bigoplus_{x \in G[\pi]} W(x) \quad (\text{it acts on } (ad \theta(g) W(x)) = W(gx))$$

It follows that if  $I(x)$  is the stability of  $x \in \widehat{H}$  in  $G$

(could be inertia subgp) then

$$\theta \sim \text{Ind}_{I(x)}^G (\text{Res}_{I(x)}^G \theta |_{W(x)})$$

What can we say about  $\text{Res}_{I(x)}^G \theta$ ? In other words,  
assume we take  $[\pi] \in I(x)$ , then

$$I = I([\pi]) = \{g \in G / \pi^g \sim \pi\}.$$

Can we describe all irrs  $\eta$  of  $I$  s.t.

$\text{Res}_{I(x)}^I \eta$  is isomorphic to  $\pi$  (decomp of  $\pi$ )?

It can be shown that all such reps of  $I$   
can be described in terms of projective irreps of  $I/H$ .

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(Def) A projective rep. of a group  $K$  on a v.spc  $V$  is  
a homomorphism  $K \rightarrow \text{PGL}(V) = \frac{\text{GL}(V)}{\text{det}^{\star}(\mathbb{C}^\times)^{\text{id}_K}}$

Why do proj. reps appear?

We have that  $\pi^g \sim \pi \forall g \in I$ .

So there exists an inv.  $T_g \in \text{GL}(V_\pi)$  s.t.

$$\pi(g^{-1}hg) = T_g^{-1}\pi(h)T_g + \text{left}$$

by multiplying (and Schur's lemma),  $T_g$  is! up to a scalar factor.

This implies that  $T_{g_1g_2} = (\zeta_{g_1g_2}) T_{g_1} T_{g_2}$  for some  $\zeta_{g_1g_2} \in \mathbb{C}^\times$ .

Therefore the operators  $T$  define a projective rep of  $I$  on  $V$ .

■

In some cases proj. reps do NOT appear. We will make the following assumptions:

-  $H$  abelian (so  $\pi$  is a homomorphism  $H \rightarrow (\mathbb{C}^\times)$ )

-  $\pi$  extends to a homomorphism  $I \rightarrow \mathbb{C}^\times$