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# III.3 Symmetric powers of a curve



$V$ variety $/k$

$V^r = V \times \dots \times V$

$S_r$ = Symmetric group on $r$ letters

A morphism $\varphi : V^r \to T$ is „symmetric" if $\varphi\sigma = \varphi \quad \forall \sigma \in S_r$.

**Prop 3.1** There exists a variety $V^{(r)}$ and a symmetric morphism $\pi : V^r \to V^{(r)}$ such that

1. $(V^{(r)}, \pi)$ is $V^r/S_r$ as a topological space.
2. For any open affine $U \subseteq V$, then $U^{(r)} = \pi(U)$ is open affine $\subseteq V^{(r)}$ and $\Gamma(V^{(r)}, \mathcal{O}_{V^{(r)}}) = \Gamma(U^r, \mathcal{O}$

s.t. any symmetric $k$-morphism factors through $\pi$.

In addition, $\pi$ is finite, surjective and separable.

Pf If $V$ is affine, $V = \mathrm{Specm}(A)$, we may let

$$V^{(r)} = \mathrm{Spec}\left( (A \otimes \dots \otimes A)^{S_r} \right).$$

Cover, glue, see Mumford. ∎

**Prop 3.2** If $C$ is a nonsingular curve, then $C^{(r)}$ is also nonsingular. Q

**Pf** Assume $k = \bar{k}$. A point $Q$ in $C^{(r)}$ is an image of some point

$$Q = \pi\Big(\big(\underbrace{P_1, \dots, P_1}_{r_1}, \underbrace{P_2, \dots, P_2}_{r_2}, \dots, \underbrace{P_m, \dots, P_m}_{r_m}\big)\Big)$$

with $P_1, \dots, P_m$ different points on $C$.

Then $\hat{\mathcal{O}}_{P_i} \cong k[[x]]$.

and
$$\hat{\mathcal{O}}_P = k[[x]] \hat{\otimes} \cdots \hat{\otimes} k[[x]]$$
$$= k[[x_1, \dots, x_{r_1}]] \hat{\otimes} \cdots \hat{\otimes} k[[x_1 \dots x_{r_m}]]$$

$$\Rightarrow \hat{\mathcal{O}}_Q = k[[x_1, \dots, x_{r_1}]]^{S_{r_1}} \hat{\otimes} \cdots \hat{\otimes} k[[x_1, \dots, x_{r_m}]]^{S_{r_m}}$$
$$\cong k[[\sigma_1 \dots \sigma_{r_1}]] \hat{\otimes} \cdots \hat{\otimes} k[[\sigma_1 \dots, \sigma_{r_m}]]$$

symmetric poly's.

**Ex** $C = \mathbb{P}^1 \quad C^{(r)} = \mathbb{P}^r$

If $C$ has genus $\geq 1$, then $C^{(r)}$ is an „approximation" of $J_C$.

There is a canonical map

$$C^{(r)} \longrightarrow J$$
$$(P_1 \cdots P_r) \longmapsto f(P_1) + \dots + f(P_r)$$

Let $K \supset k$ be fields.

- If $K = \overline{K}$, then

$$C^{(r)}(K) = S_r \backslash C(K)^r$$

orbit space

„=" set of effective divisors of degree $r$ on $C_K$.

(Kristian: „think of $k = \mathbb{R}$, $K = \mathbb{C}$. $(\mathbb{P}^1)^{(r)} = \mathbb{P}^r$

poly's of deg $r$ in one variable)

$X/k$ Cartier divisor $D$ effective

$$0 \to \mathcal{J}(D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$
$$\mathcal{J}(D) = \mathcal{L}(-D)$$

<u>Def</u> If $\pi : X \to T$ is a morphism of $k$-schemes, then relative effective Cartier divisor of $X/T$ is a Cartier divisor of $X$, flat over $T$.

(no „vertical" components in the fiber)

<u>Lemma</u> If $D_1$ and $D_2$ are such rel. effective Cartier divisors, then $D_1 + D_2$ is also.

(long story)