1

Denote by Π_λ the irreducible rep of the symmetric group corresp. to the Young diagram λ

—

To describe the character of Π_λ it is convenient to index the conjugacy classes in S_n by the no. of cycles

consider seq. $I = (i_1, i_2, \dots)$, $i_k \geq 0$

$$\sum_{k=1}^{\infty} k\, i_k = n.$$

Denote by C_I the conjugacy class in S_n consisting of elements σ which decompose into i_1 cycles of length 1,

i_2 — " — 2

!

In other words, the corresponding partition of n is

$$(\underbrace{k, \dots, k}_{i_k}, \underbrace{k-1, \dots, k-1}_{k-1}, \dots, \underbrace{1, \dots, 1}_{i_1})$$

where k is the largest # w/ $i_k > 0$.

write $\chi_{\Pi_\lambda}(C_I)$ for the value of some/any element of C_I.

2

Thm (Frobenius character formula)

Assume $\lambda = (\lambda_1, \dots, \lambda_r)$. Take any number $N \geq r$.

Then $\chi_{\pi_\lambda}(C_I)$ is the coefficient of $\prod_{i=1}^{N} x_i^{\lambda_i + N - i}$

in the polynomial

$$\Delta(x) \prod_{k \geq 1} \left(\sum_{i=1}^{N} x_i^{k} \right)^{i_k}$$

where $\Delta(x) = \prod_{1 \leq i < j \leq N} (x_i - x_j)$ $\boxed{\text{Vandermonde}}$

(WON'T prove it) (see Etingof's lectures)

Pf: (= some not particularly pleasant computations)

Idea of pf

Consider also the representation θ_λ defined by the action of S_n on tabloids of shape λ. That is $M_\lambda \simeq \mathbb{C}[S_n]\, a_T \simeq \mathbb{C}[S_n / R_T]$ for any shape λ.

Using the claims in the pf. of thm. above, it is not diff. to check that $\mathrm{Hom}_G(\pi_\mu, \theta_\lambda)$ if $\mu < \lambda$.

and $\dim \mathrm{Hom}(\pi_\lambda, \theta_\lambda) = 1$. Thus $\theta_\lambda \sim \pi_\lambda \oplus \bigoplus_{\mu > \lambda} (\pi_\mu \oplus \dots \oplus \pi_\mu)$

$\boxed{\exists!\ \text{copy of } \pi_\lambda \text{ in } \theta_\lambda}$

This allows one to ~~use~~ compute χ_{π_λ} dominant induction from

χ_{π_μ} for $\mu > \lambda$ and χ_{θ_λ}. (ind. idea of pf) 3

Using this, let us compute $\dim \pi_\lambda$.

This is $\chi_{\pi_\lambda}(e) = \chi_{\pi_\lambda}(C_I)$.

where $I = (n, 0, 0, \ldots)$

Thus $\dim \pi_\lambda$ is the coeff. of $\prod_{i=1}^{n} x_i^{\lambda_i + N - i}$ in

$\Delta(x)(x_1 + \ldots + x_N)^n$.

$$(x_1 + \ldots + x_N)^n \Delta(x) = (x_1 + \ldots + x_N)^n \det\left(\left(x_i^{N-j}\right)_{i,j=1}^{N}\right)$$

$$= (x_1 + \ldots + x_N)^n \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{N} x_i^{N - \sigma(i)}$$

Put $l_i = \lambda_i + N - i$. ($\lambda_i = 0$ for $i > r$)

In other words, we are interested only in σ s.t. $N - \sigma(i) \leq l_i$.

Then $\dim \pi_\lambda = \sum_{\substack{\sigma \in S_n \\ \text{s.t. } l_i \geq N - \sigma(i) \\ \text{for all } i}} \mathrm{sgn}(\sigma) \dfrac{n!}{\prod_{i=1}^{N} (l_i - N + \sigma(i))!}$

(+ a good exp. of $\dfrac{n!}{l_1! \ldots l_N!}$)

$$= \frac{n!}{l_1! \cdots l_n!} \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \prod_{i=1}^{N} \underbrace{l_i (l_i - 1) \cdots (l_i - N + \sigma(i) + 1)}$$

for $\sigma(i) = N$ this should be interpreted as 1

$$= \frac{n!}{l_1! \cdots l_n!} \det \left(l_i (l_i - 1) \cdots (l_i - N + j + 1) \right)_{i,j=1}^{N}$$

$$= \frac{n!}{l_1! \cdots l_n!} \begin{vmatrix} l_1(l_1-1)\cdots(l_1-N+2) & \cdots & \cdots & l_1 & 1 \\ \vdots & & & & \vdots \\ l_N(l_N-1)\cdots & \cdots & \cdots & \cdots & 1 \end{vmatrix}$$

use right side to get rid of higher powers

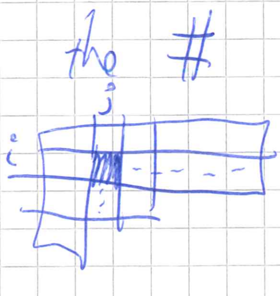
claim

$$= \frac{n!}{l_1! \cdots l_n!} \begin{vmatrix} l_1^{N-1} & l_1^{N-2} & \cdots & l_1 & 1 \\ \vdots & & & & \vdots \\ l_N^{N-1} & l_N^{N-2} & \cdots & l_N & 1 \end{vmatrix}$$

$$= \frac{n!}{l_1! \cdots l_n!} \prod_{i<j} (l_i - l_j) .$$

$\boxed{\text{Phew!}}$

To make this differently, for a box (i,j) in λ, let $h(i,j)$ be the # of boxes in the <u>hook</u> defined



by (i,j)

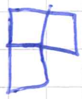
Lemma For any i, we have that

$$\frac{l_i!}{\prod_{j=i+1}^{N}(l_i-l_j)} = \prod_{k=1}^{\lambda_i} h(i,k)$$

From this we get the following

Thm (Hook length formula)

$$\dim \pi_\lambda = \{\text{no. of s.t. Young tableaux of shape } \lambda\} = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}$$

Ex $\lambda =$  $\quad \dim \pi_\lambda = \frac{3!}{1\cdot 1\cdot 3} = 2$

Pf of lemma Can assume $i=1$.

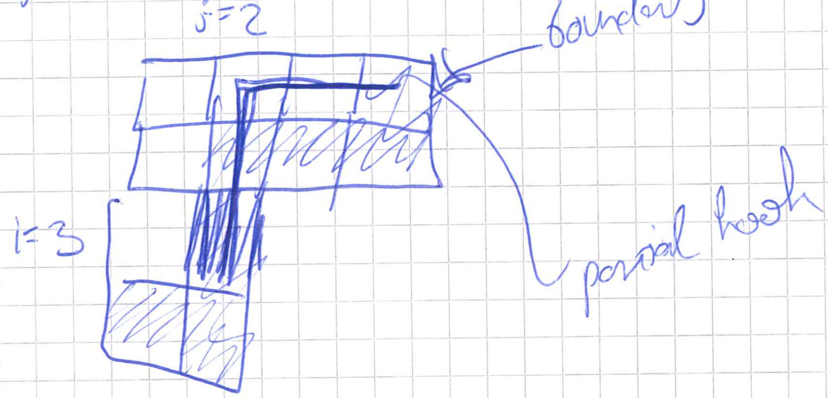
The LHS does not depend on N. If we replace N by $N+1$, then $l_1,\dots,l_N$ get replaced by $l_1+1,\dots,l_N+1$.

Recall: $l_i = \lambda_i + N - i$. So we may assume that $N = r = \#$ of rows.

Consider the lower right boundary B of λ.

For every $(i,j) \in B$ let L_{ij} be the partial hook connecting (i,j) w/ $(1, \lambda_1)$.

Ex



Then $l_1! = (\lambda_1 + r - 1)! = \prod_{(i,j) \in B} |L_{ij}|$ for $(i,j) \in B$

A partial hook L_{ij} is NOT a hook iff $\lambda_{i+1} = j$

And then $|L_{ij}| = \lambda_i - j + 1 = l_1 - l_{i+1}$

Then

$$\frac{l_1!}{\prod_{i=2}^{r} (l_1 - l_i)!} = \frac{\prod_{(i,j) \in B} |L_{ij}|}{\prod_{\substack{(i,j) \in B \\ L_{ij} \text{ not a hook}}} |L_{ij}|} = \prod_{k=1}^{\lambda_1} h(1,k).$$

$\blacksquare$

Return of the General Theory

7

Induced rep's (several lectures)

Assume G finite and $H \subset G$ a subgrp. Assume (π, V) is a rep of H. Want to construct rep of G out of this.

def The induced rep $\mathrm{Ind}_H^G \pi$ is the rep' defined by the $\mathbb{C}[G]$-module $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$.

„somehow surprisingly fundamental"

Essentially by def. $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ is the quotient of $\mathbb{C}[H] \otimes_{\mathbb{C}} V$ by the subspace spanned by the vectors $xh \otimes v - x \otimes \pi(h) v \quad \forall x \in \mathbb{C}[G]$, $v \in V$, $h \in H$.

Equivalently, by the subspace spanned

by

$$x \otimes v - xh^{-1} \otimes \pi(h)v$$

Consider the right reg. rep. ρ_G of G on $\mathbb{C}[G]$.

$$(\rho_G(g) f)(h) = f(hg)$$ ✓

In other words $\rho_G(g) x = xg^{-1}$ for $x \in \mathbb{C}[G]$.

Therefore $x \otimes v - xh^{-1} \otimes \pi(h)v = x \otimes v - (\rho_G|_H \otimes \pi)(h)(x \otimes v)$

Lemma For any rep. (W, θ) of H, the space spanned by the vectors $w - \theta(h)w$, $w \in W$ is a complementary subspace of $W^H \overset{\text{def}}{=} \{w \mid \theta(h)w = w \ \forall h\}$.

Pf Consider the projection $P = \frac{1}{|H|} \sum_{h \in H} \theta(h) : W \to W^H$

suffices to show that $\{w - \theta(h)w\}_{w,h} = \ker P$.

$\subseteq$ $\cancel{\not\subseteq}$!

Conversely assume $w \in \ker P$. Then $w = w - Pw$

$$= w - \frac{1}{|H|} \sum_{h \in H} \theta(h)w$$

$$= \frac{1}{|H|} \sum_{h \in H} w - \theta(h)w$$

9

Therefore $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ can

be identified w/ the subspace of $\mathbb{C}[G] \otimes V$

of vectors v fixed by

$\rho_G(h) \otimes \pi(h)$, $h \in H$.

Moreover, $\mathbb{C}[G]$ still acts on the left on this submodule, so this is an identification of modules.

Next, identify $\mathbb{C}[G] \otimes V$ w/ the space of V-valued functions on G.

$$f \otimes v \longmapsto (g \mapsto f(g)v)$$

In this picture, the repr. $\Theta = \rho_G|_H \otimes \pi$

is given by

$$(\Theta(h) f)(g) \quad \boxed{\text{acting on } f}$$
$$\|$$
$$\pi(h) f(gh)$$

To summarize: the induced rep. can be defined as follows: the underlying space is the space of functions $f: G \to V$ s.t. $f(gh) = \pi(h)^{-1} f(g)$ $\forall g \in G$ $\forall h \in H$.

Then
$$\left[\left(\mathrm{Ind}_H^G \pi\right)(g) f\right](g') = f(g^{-1} g').$$