Deformation theory

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1 Motivation

In geometry, and in mathematics in general, one is interested in *classifications*. Two triangles are similar if and only if their angles are equal, two varieties X and Y are birational if and only if their functions fields $K(X)$ and $K(Y)$ are isomorphic, and so on.

After having put objects in the same class, one wants to classify *parameters* or *moduli* of the quotient space. For example, the set of all circles, up to translation, have one parameter, namely the radius, so in this case the moduli space of circles is just $\mathbb{R}^+$, the set of positive real numbers.

In algebraic geometry, the study of moduli spaces is central (there are even connections to mirror symmetry in physics).

As an example, the set of all lines in $\mathbb{P}^2$ is just given by $(\mathbb{P}^2)^* \approx \mathbb{P}^2$, the dual space. But if we increase the dimension, things get more complicated. The set of lines in $\mathbb{P}^3$ is called the Grassmannian $\mathbb{G}(2,4)$, because lines in $\mathbb{P}^3$ correspond bijectively to planes in k^4. To give a plane in k^4 is equivalent to giving a 2×4-matrix M:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix},$$

such that not all of its 2×2-minors vanish. But this matrix M is only unique up to left-multiplication by $\mathrm{GL}_2(k)$. Thus lines in $\mathbb{P}^3$ correspond to the quotient of an open set in $\mathbb{A}^8$ by the group $\mathrm{GL}_2(k)$. In particular $\mathbb{G}(2,4)$ has dimension $8 - 4 = 4$[1]. It turns out (try figuring out how!), that points of $\mathbb{G}(2,4)$ correspond naturally to points on the hypersurface

$$x_{14}x_{23} - x_{13}x_{24} + x_{12}x_{34} = 0$$

[1] In general, $\mathbb{G}(d,n)$ have dimension $d(n-d)$.

in $\mathbb{P}^5$ $(= \mathbb{P}(\wedge^2 k^4))$.

We say that $\mathbb{G}(d,n)$ is the *moduli space of* $d-1$*-planes in* $\mathbb{P}^{n-1}$.

The general definition of a moduli space is somewhat technical, but the meaning is this: a moduli space $\mathcal{M}$ is a scheme (or variety) such that its points correspond to the solution of some classification question. For example, the classification question could be to ask for lines in $\mathbb{P}^n$, or curves of fixed genus g, of which we shall say something about now:

Let $\mathscr{M}_g$ be the set of non-singular curves of genus g. First of all, one wants to put a structure of an algebraic variety on $\mathscr{M}_g$, and then one could ask questions like "what is its dimension?", "is it irreducible?", "is it connected?", "is it singular?", "what are its cohomology groups?", and so on.

It turns out that it is not possible to put a structure of an algebraic variety on $\mathscr{M}_g$. The technical reason for this is that the underlying moduli functor is not representable, and the reason for this is the occurence of so-called "non-trivial automorphisms". To put a geometric structure on $\mathscr{M}_g$, one must turn to the abstract language of "algebraic stacks", which are generalizations of varieties in which the points "remembers automorphisms". However, *locally* $\mathscr{M}_g$ look like a scheme, so it makes sense to ask for properties of $\mathscr{M}_g$ that are local, for example its dimension. The dimension of $\mathscr{M}_g$ is the number of parameters needed to specify isomorphism class of a genus g curve (here is a trivial example: to specify a point in $\mathbb{A}^n$, you need n parameters that can vary freely, so the moduli space of points in $\mathbb{A}^n$ is just $\mathbb{A}^n$ itself).

This is where deformation theory enters the picture. By studying the space of all possible "infinetesimal deformations" of a curve $[C] \in \mathscr{M}_g$, one gets (being careful) the (local) dimension of $\mathscr{M}_g$. In fact, it turns out that $\mathscr{M}_g$ have dimension $3g-3$.

So in this sense, *deformation theory is the local study of how objects vary in a family.*

2 Families and deformations

First of all, what is a family? Naïvely, a family is just a set all of whose members have something in common. A little less naïvely, a family is a variety parametrizing all varieties satisfying some fixed property.

For example, consider the case of Grassmannians: Consider the product $\mathbb{A}^n \times \mathbb{G}(d,n)$ and let $\mathfrak{X}$ be the subset $\{(P,D) \mid P \in D\} \subset \mathbb{A}^n \times \mathbb{G}(d,n)$. Then we have a natural map $\pi : \mathfrak{X} \to \mathbb{G}(d,n)$ onto the second coordinate. This is called the "tautological bundle", by the way. The fibers $\pi^{-1}(D)$ are exactly the d-planes in $\mathbb{A}^n$. This is the prime example of a good family.

What this looks like on the level of homogeneous coordinate rings is this: the map π corresponds to an inclusion of rings $\pi^* : S \to S \otimes_k k[x_1, \cdots, x_n]$. Locally, the Grassmannian is isomorphic to $\mathbb{A}^{d(n-d)}$, so after going to some localization, the map looks like $\pi^*_{(f)} : k[x_{ij}] \to k[x_{ij}] \otimes_k k[x_1, \cdots, x_n]$. Choosing a maximal ideal $\mathfrak{m} \subset S$ gives a map $k(\mathfrak{m}) \to k(\mathfrak{m}) \otimes_k k[x_1, \cdots, x_n]$, which corresponds to the inclusion of a plane in $\mathbb{A}^n$ (think about why!).

There is something we can say about the map π: it is a locally free sheaf of S-algebras, which implies (Hartshorne, Chapter III, section 9), that π is a *flat* map. This is the only technical condition we in general want to impose on our maps. It turns out that this formalizes what we want the notion of "family" to be.

Definition 2.1. A *family* of varieties consists of a flat map $\pi : \mathfrak{X} \to S$. ■

The space S is called the *base space*. In the examples in the introduction, the base space was also called the "moduli space", but in deformation theory, one rarely gets to touch the moduli space. In particular, in deformation theory on often wants S to correspond to an Artin ring A, i.e. a so-called "fat point", and families where $S = \operatorname{Spec} A$ for an Artin ring are called *infinitesimal deformations*. In particular, if $A = k[\epsilon] := k[x]/(x^2)$, families over $\operatorname{Spec} A$ are called *first-order deformations*.

To be more precise, let X_0 be some variety (for example the two lines $xy = 0$). Then a first-order deformation of X_0 is, by definition, a family $\pi : X \to \operatorname{Spec}(k[\epsilon])$ such that the fiber over the closed point of $\operatorname{Spec}(k[\epsilon])$ is X_0 (the ring corresponding to the closed point of $\operatorname{Spec}(k[\epsilon])$ is just $k[\epsilon]/(\epsilon)$).

More generally, a deformation of X_0 is a family $\pi : X \to S$ such that for some $0 \in S$, the fiber $\pi^{-1}(0) = X_0$. This is what is ment by saying "putting X_0 in a family".

Example 2.2. Let $I_0 = (xy)$ and $I_t = (xy - t)$. Then one can check that the map $k[t] \to k[x, y, t]/(xy - t)$ is a flat map, and so we have a deformation of the variety consisting of the two coordinate axes over $\mathbb{A}^1$. The *special fiber* is singular at the point $(0, 0)$ and the generic fiber is non-singular and irreducible. The conclusion is that the members of a family can look quite different. ★

Even though members of a flat family can vary a lot, under small restrictions, they share an important invariant: the Hilbert polynomial. We quote a theorem from Hartshorne [Har77] (Chapter III, section 9, Theorem 9.9):

Theorem 2.3. *Let T be an integral scheme and let $X \subseteq \mathbb{P}^n \times T$ a closed subscheme, and consider the natural map $\pi : X \to T$. If π is flat, then the Hilbert polynomial of the fibers $X_t \subseteq \mathbb{P}^n$ is constant. The converse also holds.*

Here integral means "irreducible and reduced". Thus, if we know that a family is flat, we know that its members (i.e. the fibers) share at least one invariant. But beware, the Hilbert *function* can be different.

Having fixed some numerical polynomial $P(t) \in \mathbb{Q}[z]$, one may ask: "what is the largest flat family all of whose members are closed subschemes of $\mathbb{P}^n$ with Hilbert polynomial $P(t)$?". This is what is called the *Hilbert scheme* $\mathfrak{Hilb}^n_{P(t)}$. It is a theorem of Grothendieck that the Hilbert scheme is itself a closed projective scheme in some enormous $\mathbb{P}^N$.

We can ask for the dimension of the tangent space of $\mathfrak{Hilb}^n_{P(t)}$ at a point $[V]$ corresponding to the variety V.

Proposition 2.4. *The dimension of the tangent space of $[V] \in \mathfrak{Hilb}^n_{P(t)}$ is the the dimension of the space of first-order embedded deformations of V.*

Proof. For notational simplicity, set $H = \mathfrak{Hilb}^n_{P(t)}$. To give a tangent vector $v \in T(H)_{[V]}$ is the same as giving a morphism $\operatorname{Spec}(k[\epsilon]) \to H$ with image $[V]$ (Hartshorne, Chapter II, Exercise 2.7). Because of the universal property of the Hilbert scheme, this corresponds to a first-order deformation of $[V]$ over $k[\epsilon]$, and the correspondence is one-one. $\square$

To be more precise, if $\mathcal{N}_{V/\mathbb{P}^n}$ is the normal sheaf of V in $\mathbb{P}^n$, first-order embedded deformations of V are classified by elements of $H^0(Y, \mathcal{N}_{V/\mathbb{P}^n})$. This is Theorem 2.4 in [Har10].

So the moral is this: if you're only interested in local properties of a family, deformation theory might be what you want to study.

3 Embedded versus abstract deformations

Generally, there are two types of deformations to consider. Say you have some variety X embedded in $\mathbb{P}^n$. Then you can ask, what are the possible deformations of X *as a subscheme* of $\mathbb{P}^n$? Or you can ask, what are the possible deformations of X *as abstract schemes*? Here you consider two varieties equal if they are abstractly isomorphic.

Example 3.1. Consider again the deformation of the coordinate axes in $\mathbb{A}^2$: i.e. the zero sets $X_t = V(\{xy - t = 0\}) \subseteq \mathbb{A}^2$ as t varies. As embedded schemes, all the deformations are different. However, as abstract schemes, all X_t are isomorphic for $t \neq 0$. Hence there are a lot fewer abstract deformations than embedded deformations. ★

4 First-order deformations

Let Y be a closed subscheme of $\mathbb{A}^n$, that is, Y is defined by an ideal $I \subseteq k[x_1, \cdots, x_n]$. Then we can ask, what are the possible first-order deformations of Y as a subscheme of $\mathbb{A}^n$? Recall that this means that we are asking for the possible flat families $\pi : \mathfrak{X} \to k[\epsilon]$ such that $\pi^{-1}(0) = Y$.

In algebra language, this means that we want an ideal $I_\epsilon \subseteq k[x_1, \cdots, x_n, \epsilon]$ such that $I_\epsilon \otimes_{k[\epsilon]} k[\epsilon]/(\epsilon) = I$ *and* such that the map $k[\epsilon] \to k[x_1, \cdots, x_n, \epsilon]/I_\epsilon$ is flat. By Proposition 2.2 in [Har10], this is equivalent to the exactness of the sequence

$$0 \to B/I \xrightarrow{\cdot t} B[\epsilon]/I_\epsilon \to B/I \to 0.$$

Set $B := k[x_1, \cdots, x_n]$ and consider the commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \arrow[r, "t"] \arrow[d] & I_\epsilon \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r, "t"] \arrow[d] & B[\epsilon] \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B/I \arrow[r, "t"] \arrow[d] & B[\epsilon]/I_\epsilon \arrow[r] \arrow[d] & B/I \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The exactness of the bottom row is equivalent to the exactness of the top row. Then we have the following result:

Proposition 4.1. *Finding I_ϵ such that the extension above is flat is equivalent to picking an element $\varphi \in \mathrm{Hom}_B(I, B/I)$. Choosing $\varphi = 0$ correspond to the trivial deformation* $\operatorname{Spec} k[\epsilon] \times Y \to \operatorname{Spec} k[\epsilon]$.

Proof. First suppose given I_ϵ as above. We will produce a $\varphi \in \mathrm{Hom}_B(I, B/I)$. So let $x \in I$ and lift it to an element of I_ϵ, which can be written uniquely as $x + \epsilon y$ for some $y \in B$. The liftings are unique up to elements of the form tz with $z \in I$. Thus y is not unique, but its image $\bar{y} \in B/I$ is. This defines our map $\varphi : I \to B/I$: it is simply given by $x \mapsto \bar{y}$.

Conversely, suppose given $\varphi \in \mathrm{Hom}_B(I, B/I)$. Then define:

$$I_\epsilon := \{x + \epsilon y \,|\, x \in I, y \in B \text{ such that } \varphi(x) = \bar{y} \in B/I\}$$

Then one checks that I_ϵ is an ideal in $B[\epsilon]$ and that $I_\epsilon \otimes_{k[\epsilon]} k[\epsilon]/(\epsilon) = I$ and that there is an exact sequence as the first row in the diagram above. This then implies the exactness of the bottom row, and we are done. $\square$

The conclusion is this: first-order embedded deformations of an affine scheme are in $1-1$ correspondence with elements of $\mathrm{Hom}_B(I, B/I)$. A little calculation shows that $\mathrm{Hom}_B(I, B/I) = \mathrm{Hom}_{B/I}(I/I^2, B/I)$, and this latter module is called the *normal module* of Y in $\mathbb{A}^n$.

Lets compute an example:

Example 4.2. Let Y be the line $\{x = 0\}$ in $\mathbb{A}^2$. Then the ideal $I = (x) \approx k[x, y]$ as $k[x, y]$-modules. Hence $\mathrm{Hom}_B(I, B/I) = \mathrm{Hom}_B(B, B/I) = B/I = k[y]$. Thus to give φ as above is the same as giving a polynomial $h(y)$ only in y. Hence I_ϵ in the proof above is

$$I_\epsilon = \{f(x, y) + \epsilon g(x, y) \mid f(x, y) \in \langle x\rangle \text{ and such that } g(0, y) = h(y)\}$$

More concretely, suppose $h(y) = y^2$. Then

$$\begin{aligned} I_\epsilon &= \{f(x, y) + \epsilon g(x, y) \mid f(x, y) \in \langle x\rangle \text{ and such that } g(0, y) = y^2\} \\ &= \langle x + \epsilon y^2\rangle. \end{aligned}$$

Thus, all first-order-deformations of the x-axis are given by a polynomial only in y. Note, however, that all the varieties obtained this way are isomorphic - just project down to the x axis. ★

If one wants to work with non-affine schemes, for example projective schemes, one have to globalize this construction. For projective schemes this is done as follows: if X is a subscheme of $\mathbb{P}^n$, one can cover X by opens U_i such that each $X \cap U_i$ is affine. This way one gets a sheaf of $\mathcal{O}_X$-modules, and then the result is that first-order deformations are given by global sections of $\mathcal{N}_{X/\mathbb{P}^n}$, i.e. elements of $\Gamma(X, \mathcal{N}_{X/\mathbb{P}^n})$.

A Glossary

Definition A.1. A functor $F : \mathsf{Sch} \to \mathsf{Set}$ is **representable** if there exists a scheme $X \in \mathsf{Sch}$ such that $F(Y) \simeq \mathrm{Hom}_{\mathsf{Sch}}(Y, X)$ for all $Y \in \mathsf{Sch}$. ■

Example A.2. Let F be the functor sending a variety Y to the set of units in its coordinate ring: $Y \mapsto \Gamma(\mathcal{O}_Y, Y)^\times$. Then F is represented by the variety $\mathbb{A}^1 \backslash \{0\}$, because if R is the coordinate ring of Y, then

$$R^\times = \mathrm{Hom}_{\mathsf{C-Alg}}(\mathbb{Z}[x, x^{-1}], R) = \mathrm{Hom}_{\mathsf{Sch}}(R, \mathbb{A}^1 \backslash \{0\}) = F(Y).$$

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Definition A.3. A **sheaf** is a contravariant functor $F : \mathsf{Top} \to \mathsf{Set}$ satisfying a *glueing condition*. If $s \in F(U)$, and $U_i \subset U$, we write $s\big|_{U_i}$ (read "the restriction of s to U_i") for the element $F(i)(s) \in F(U_i)$ induced by the inclusion $U_i \to U$. If $\{U_i\}_{i \in I}$ is an open cover of U and $g_i \in F(U_i)$ are elements such that $g_i\big|_{U_j \cap U_i} = g_j\big|_{U_i \cap U_j}$ for all i, j, then there exists a *unique* element $g \in F(U)$ such that its restriction to each U_i is g_i. ■

Example A.4. Let X be a variety and let F be the sheaf sending each open set to the set of regular functions $U \to k$. This is clearly a sheaf, and it is the *structure sheaf* of X. It is usually written $\mathcal{O}_X$. ★

References

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Har10] Robin Hartshorne. *Deformation theory*, volume 257 of *Graduate Texts in Mathematics*. Springer, New York, 2010.