IV

1

Dimension of irr reps

Recall G fin grp

- $|\hat{G}|$ = no of conj. classes
- $\sum_{\pi \in \hat{G}} \dim(\pi)^2 = |G|$

Thm The dim. of any irr. divides $\left| G/Z(G) \right|$.

(Later will ~~prove~~ strengthen this by showing that we can take $|G/A|$ where A is any normal abelian subgrp)

Let R be a unital comm. ring and let $S \subset R$ be a unital subring.

Def An el. $a \in R$ is called *integral* $/S$ if it is a root of a monic poly w/ coeff in S.

- A complex nr integral over $\mathbb{Z}$ is called an *algebraic integer*.
- An int. domain S is called integrally closed if any element of $\mathrm{Frac}(S)$ integral $/S$ is in S.

Exc $\mathbb{Z}$ is int. closed.

Lemma Assume $S \subset R$ as above. $a \in R$.

Then a is int. over S iff the subring $S[a]$ is a finitely gen. S-module.

Pf ① Write a^n = sum of lower powers

② Consider the S-module $M \subset R$ gen. by $1, a, \dots, a^{n-1}$.

Then $aM \subset M$.

Hence M is a subring, so $M = S[a]$.

Conv. assume $S[a]$ is fin. gen. as an S-module

Then $\exists$ polys $p_1, \dots, p_m \in S[x]$ s.t. $S[a] =$

$S p_1(a) + \dots + S p_m(a)$

Take any $n > \deg_x p_i(x)$

Then $\exists$ $a_1, \dots, a_m \in S$ s.t.

$$a^n = a_1 p_1(a) + \dots + a_m p_m(a)$$

$\blacksquare$

Exc The set of elements of R int. over S forms a ring.

Prop Assume π is an irr. rep. Let $g \in G$. Let $C(g)$ be the conj. class of g.

Then ① $\chi_\pi(g)$ = algebraic integer

② $\dfrac{|C(g)|}{\dim \pi} \chi_\pi(g)$ is an algebraic integer

Pf (1) We have $g^n = e$ for some $n \geq 1$.

Hence all eigenvalues of $\pi(g)$ are n-th roots of unity.

Hence $\chi_\pi(g) = \mathrm{Tr}\, \pi(g) =$ sum of alg. integers

$=$ alg. integer.

(2) Let p be the char. function of $C(g)$. Then p is in the center of $\mathbb{C}[G]$. Consider the subring $\mathbb{Z}[G] \subset \mathbb{C}[G]$, group ring

Let $R \subset \mathbb{Z}[G]$ be the subring of central functions.

As R is a fin. gen ab. group ($\mathbb{Z}$-module), any element of R is integral over $\mathbb{Z}1 \subset R$.

It follows that $\pi(p)$ is integral over $\mathbb{Z}1 \subset \mathrm{End}(V_\pi)$.

As p is central, $\pi(p) \in \mathrm{End}(\pi) = \mathbb{C}\cdot 1$ (by Schur's lemma.

So $\pi(p) = \alpha \cdot 1$.

This means that α is an algebraic integer.

We have
$$\alpha \cdot \dim \pi = \mathrm{Tr}(\alpha \cdot 1) = \mathrm{Tr}\, \pi(p)$$
$$= \sum_{h \in C(g)} \mathrm{Tr}\, \pi(h)$$
$$= \sum_{h \in C(g)} \chi_\pi(h) = |C(g)|\, \chi_\pi(g)$$

Hence $\alpha = \dfrac{|C(g)|\, \chi_\pi(g)}{\dim \pi}$ is an alg. integer. ∎

Lemma

If π is irr. rep $\Rightarrow$ $\dim \pi \mid |G|$.

Pf Recall that by the orthogonality relations, we have

$$(\chi_\pi, \chi_\pi) = 1.$$

Thus

$$\frac{|G|}{\dim \pi} = \sum_{g \in G} \frac{\chi_\pi(g)\overline{\chi_\pi(g)}}{\dim \pi}$$

For every conj. class C choose a repr. $g_C \in C$.

Then the sum is

$$\cancel{\sum_{g_C \in C(g)} |C(g)|\, \chi_\pi(g)\overline{\chi_\pi(g)}}$$

$$\sum_C \sum_{h \in C} \frac{\chi_\pi(h)\chi_\pi(h^{-1})}{\dim \pi} = \sum_C \underbrace{\frac{|C|\,\chi_\pi(g_C)}{\dim \pi}}_{\text{alg int}} \underbrace{\chi_\pi(g_C^{-1})}_{\text{alg int.}}$$

$\Rightarrow$ hence sum is alg. integer.

so $\frac{|G|}{\dim \pi} \in \mathbb{Z}$. $\blacksquare$

Pf of thm (recall want to show $\left(\dim \pi \,\Big|\, \left|\frac{G}{Z(G)}\right|\right)$)

2

For every $n \in \mathbb{N}$ consider the rep of

$$G^n = G \times \dots \times G \quad \text{on} \quad V_\pi \otimes \dots \otimes V_\pi$$

defined by $\pi_n(g_1, \dots, g_n) = \pi(g_1) \otimes \dots \otimes \pi(g_n)$.

This rep is irreducible since $\pi(\mathbb{C}[G]) = End(V_\pi)$

we have $\pi_n(\mathbb{C}[G^n]) = End(V_\pi) \otimes \dots \otimes End(V_\pi)$

$$= End(V_\pi \otimes \dots \otimes V_\pi).$$

exc!

Consider the subgroup $Z_n = \{ (g_1, \dots, g_n) \in Z(G)^n \mid g_1 \cdots g_n = e \}$

$$\cong Z(G)^{n-1}.$$

If $g \in Z(G)$, then by Schur $\pi(g)$ is a scalar operator.

So if $(g_1, \dots, g_n) \in Z_n$, then $\pi(g_i) = \alpha_i 1$, hence

$$\pi_n(g_1, \dots, g_n) = \alpha_1 \cdots \alpha_n 1_{V_\pi^{\otimes n}}$$

$$= 1_{V_\pi^{\otimes n}} \text{ as } g_1 \cdots g_n = e.$$

Therefore $Z_n \subset \ker \pi_n$, thus is π_n a repr. of G^n / Z_n.

Then by previous lemma, $(\dim \pi)^n \Big| \frac{|G|^n}{|Z(G)|^{n-1}}$ thus is

$$\left(\frac{|G|}{\dim \pi \cdot |Z(G)|} \right)^n \in \mathbb{Z} \frac{1}{|Z(G)|} \text{ for any } n.$$

Therefore
$$\mathbb{Z}\left[\frac{|G|}{\dim \pi \cdot |Z(G)|}\right] \subset \mathbb{Z}\frac{1}{|Z(G)|}$$

Hence $\frac{|G|}{\dim\pi\cdot|Z(G)|}$ is an algebraic integer

So it is an integer. $\blacksquare$

Repr theory of symm. group

It is easy to understand conjugacy classes in S_n:

every $\sigma \in S_n$ defines a partition of $\{1, \dots, n\}$ into orbits of σ.

Let $X_1, \dots, X_m$ be these orbits.

$$|X_1| \geq |X_2| \geq |\dots| \geq |X_m|.$$

Write $n_i = |X_i|$.

Def A partition of n are numbers $n_1 \geq \dots \geq n_m \geq 1$ s.t. $n = n_1 + \dots + n_m$.

Thus every $\sigma \in S_n$ gives a partition of n. Two elements of S_n are conjugate iff they define the same partition.

Our goal is to construct an irrep. for any partition of n.

Recall for any fin.g.p $\mathbb{C}[G] \simeq \bigoplus_{\pi \in \hat{G}} \mathrm{End}(V_\pi)$

Recover space from matrix algebra??

Def An idempotent e in an algebra A is called <u>minimal</u> if $e \neq 0$ and $eAe = \mathbb{C}e$.

Exc Let V be a fin.dim v.space. Show that $e \in \mathrm{End}(V)$ is a minimal idempotent iff it is a projection onto a 1-dim subspace $\mathbb{C}v \subset V$.

Then we have an iso

$$\mathrm{End}(V)e \simeq V \qquad Te \longmapsto Tv.$$

of $\mathrm{End}(V)$-modules.

Therefore finding irreps of G is same as finding minimal idempotents in the group algebra $\mathbb{C}[G]$: if $e \in \mathbb{C}[G]$ is a ~~...~~ minimal idempotent, then the $\mathbb{C}[G]$-module

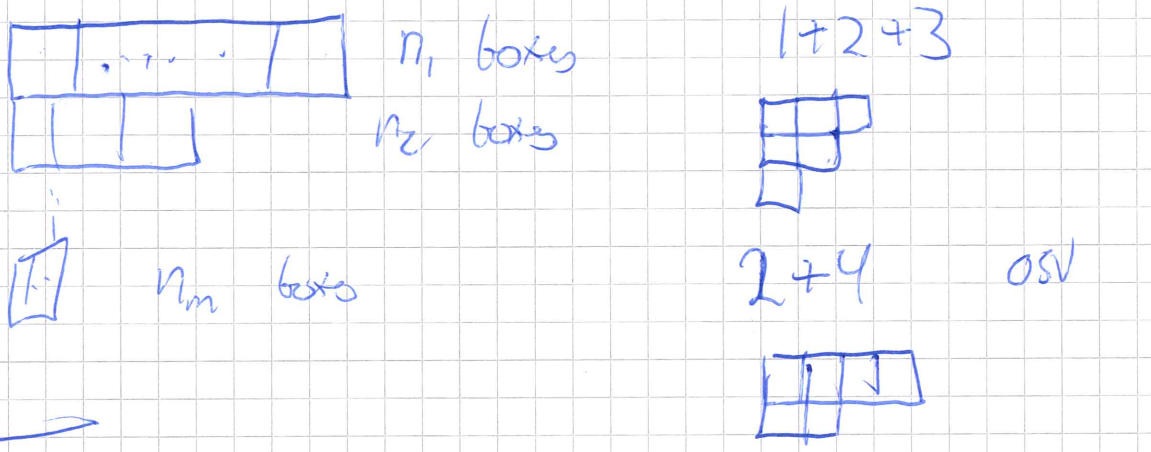
$\underset{\uparrow}{\mathbb{C}[G]}e$ defines and irreducible rep.

convolution

Thus, returning to S_n, we want to construct minimal idempotents in $\mathbb{C}[S_n]$ for every partition.

It is conventional to present partitions as Young diagrams.

Given a partition $(n_1, \dots, n_m)$ of n, we draw the diagram



Def: A Young tableau is a Young diagram consisting of n boxes filled w/ numbers $1, \dots, n$ (w.o. reps)

Example | $n=3$ $\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$

Two Young tableaux are said to be of the same shape if they arise from the same Young diagram.

The symm. group acts on Young tableaux of same shape. (σ repl. i w/ $\sigma(i)$)

9

Fix a Young tableau T (of n boxes). Let $R(T) \subset S_n$ be the subgroup of elements permuting numbers in the rows of T.

Same $C(T) \subset S_n$ for columns.

Define the following elt's in ~~the group algebra~~ of $\mathbb{C}[S_n]$

$$a_T = \frac{1}{|R(T)|} \sum_{g \in R(T)} g$$

$$b_T = \frac{1}{|C(T)|} \sum_{g \in C(T)} \operatorname{sgn}(g)\, g$$

$$c_T = a_T b_T.$$

Thm ① For any T, the element c_T is (up to scalar) a minimal idempotent in $\mathbb{C}[S_n]$, so it defines an irrep. of S_n.

② The reps corresponding to two T_1 and T_2 are equivalent $\iff$ T_1 and T_2 have same shape.

The modules $\mathbb{C}[S_n]\, c_T$ are called Specht modules.