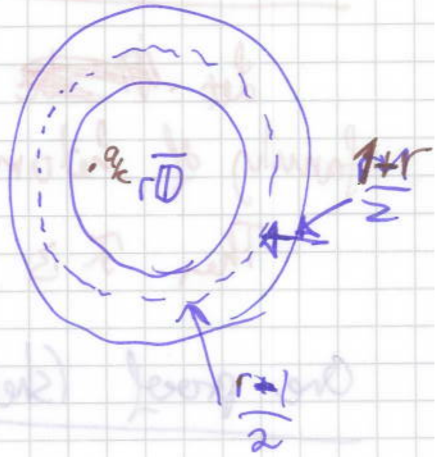
complex dynamics   20/1 - Lecture 2   (1)

Thm 2.7 (Montel's theorem)

Let $\Omega \subseteq \mathbb{C}$ be a domain and let $\mathcal{F}$ be a family of holomorphic functions on Ω s.t. $\forall$ compact set $K \subseteq \Omega$, $\exists\, M_K > 0$ s.t. $\|f\|_K < M_K$ for all $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family (i.e. for any $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{F}$ has a convergent subsequence)

Pf
- Enough to prove it for $\Omega = \mathbb{D}$.
- Enough to prove that $\{f_j\}$ has a convergent subsequence on $r\overline{\mathbb{D}}$ for all $0 < r < 1$.
- Fix $M > 0$ s.t. $\|f\|_{\frac{r+1}{2}\overline{\mathbb{D}}} \le M$ for all $f \in \mathcal{F}$.



- For any $a \in r\overline{\mathbb{D}}$ we have that

$$D_{\frac{1-r}{2}}(a) \subset \frac{r+1}{2}\overline{\mathbb{D}},$$ so by Cauchy-estimate, we get

$$|f'(a)| \le \frac{2M}{1-r} \quad \forall\, a \in r\overline{\mathbb{D}}.$$

$\Rightarrow$ We get a Lipschitz estimate:

$$|f(z)-f(w)| \le \frac{2M}{1-r}|z-w| \quad \forall\, z,w \in r\mathbb{D}$$

and $f \in \mathcal{F}$.

Now choose countable dense subset $A = \{a_j\}_{j\in\mathbb{N}} \subseteq r\mathbb{D}$.

Look at $f_j(a_k)$ for some $k \in \mathbb{N}$.

For each k $\exists$ $\{f_j(a_k)\}_j$ has a convergent subsequence, and by a diagonal argument, there exists a subsequence that converges for all $a \in A$.

So assume $f_j \to f$ on A. From the Lipschitz estimate, it follows that f is cont's. and $f_j \to f$ uniformly on $r\overline{\mathbb{D}}$.

∎

Thm 2.8 (Montel, stronger)

Let $\Omega \subseteq \mathbb{C}$ be a domain and let $\mathcal{F}$ be a family of holomorphic maps $\Omega \to \mathbb{C}\setminus\{p,q\}$.

Then $\mathcal{F}$ is a normal family.

One proof (sketch)

$$\begin{tikzcd} & \mathbb{D} \arrow[d, "\pi"] \\ \mathbb{D} \arrow[ur, dashed, "\tilde f"] \arrow[r, "f"'] & \mathbb{C}\setminus\{0,1\} \end{tikzcd}$$

← universal covering space

And use the previous theorem. ∎

This was more or less refreshing. Maybe something new now?

Goal to prove (2.11)

Thm (Koebe 1/4-Thm)

Let $f: \mathbb{D} \to \mathbb{C}$ be an injective holomorphic map w/ Taylor series expansion $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$

Then $\frac{1}{4}\mathbb{D} \subset f(\mathbb{D})$.

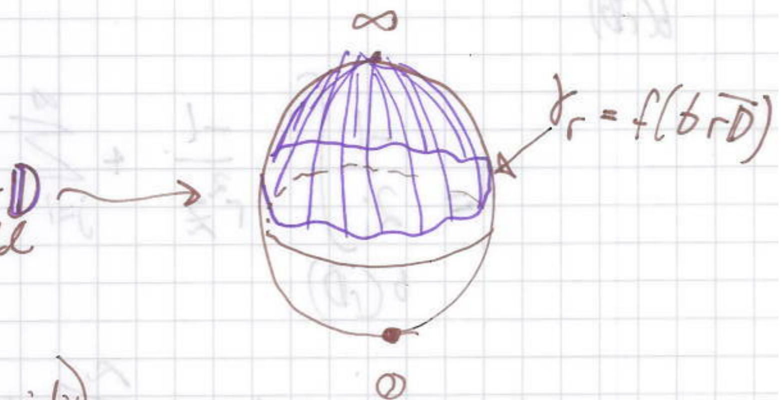
Thm (Area thm)

Let $f: \mathbb{D} \to \mathbb{C}$ be a univalent holomorphic map of the form $f(z) = \frac{1}{z} + \sum_{j=1}^{\infty} a_j z^j$.

Then $\sum_{j=1}^{\infty} j|a_j|^2 \leq 1$.

Fix $0<r<1$.

Pf Let U_r be the bounded domain in $\mathbb{C}$ which is bounded by γ_r. We have



$$\int_{\gamma_r} \bar{z}\, dz = \int_{\gamma_r} (x-iy)(dx+idy)$$

$$= \int_{\gamma_r} (x\,dx + y\,dy) + i(x\,dy - y\,dx)$$

Green's thm $\checkmark$

$$= \iint_{U_r} 2i \, dx\, dy = 2i \, \text{Area}(U_r)$$

④

So $$\text{Area}(U_r) = \frac{1}{2i}\int_{\gamma_r} \bar{z}\, dz = \frac{-1}{2i}\int_{b(r\mathbb{D})} \bar{f}\cdot f'(z)\, dz$$

Recall: $$f(z) = \frac{1}{z} + \sum_{j=1}^{\infty} a_j z^j. \text{ Then}$$

$$f'(z) = -\frac{1}{z^2} + \sum_{j=1}^{\infty} j a_j z^{j-1}.$$

On $b(r\mathbb{D})$, we have $\bar{z} = \frac{\bar{z}z}{z} = \frac{r^2}{z}$.

So $$\overline{f(z)} = \frac{z}{r^2} + \sum \bar{a}_j \frac{r^{2j}}{z^j}$$

Now $$\frac{-1}{2i}\int_{b(r\mathbb{D})} \left(\frac{z}{r^2} + \sum_{j=1}^{\infty} \bar{a}_j \frac{r^{2j}}{z^j}\right)\left(\frac{-1}{z^2} + \sum j a_j z^{j-1}\right) dz$$

$$= \frac{-1}{2i}\int_{b(r\mathbb{D})} \frac{-1}{r^2 z} + \sum_{j=1}^{\infty} |a_j|^2 \frac{j r^{2j}}{z}\, dz$$

$$= \underbrace{\cancel{-\frac{2\pi i}{2i}}\left(\frac{-1}{r^2} + \sum_{j=1}^{\infty} j|a_j|^2 r^{2j}\right)}_{\text{area}}$$

Since we are computing an area, the expression must be positive. Hence $\sum_{j=1}^{\infty} j\,|a_j|^2 r^{2j} < \frac{1}{r^2}$

Now let $r \to 1$. $\blacksquare$

Thm 2.10 Let $f : \mathbb{D} \to \mathbb{C}$ be a univalent map of the form
$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j .$$
Then $|a_2| \le 2$. (also true that $|a_j| \le j \ \forall j$)

Pf Define the function $g(z) = \frac{1}{\sqrt{f(z^2)}}$.

Well-defined:
$$f(z^2) = z^2 + \sum_{j=2}^{\infty} a_j z^{2j} = z^2 \underbrace{\left(1 + \sum_{j=2}^{\infty} a_j z^{2j-2}\right)}_{\text{observe: this is non zero!}}$$

So the expression have a square root! In fact,
$$h(z) = \sqrt{f(z^2)} = z\sqrt{1 + \sum_{j=2}^{\infty} a_j z^{2j-2}} \quad \text{is a root for } f(z^2).$$

So
$$g(z) = \frac{1}{z\sqrt{1 + \sum_{j=2}^{\infty} a_j z^{2j-2}}} = \frac{1}{z}\left(1 - \frac{a_2}{2} z^2 + \text{h.o.t.}\right) = \frac{1}{z} - \frac{a_2}{2} z + \text{h.o.t.}$$

This a function of the form in the previous theorem. 6

Claim · $h(z)$ is univalent.

· It is injective near the origin

Let r be the sup over all no's $0 < r < 1$ such that $h: r\overline{\mathbb{D}} \to \mathbb{C}$ is injective.

Assume $r < 1$.

Then there is z_0 on $\partial(r\overline{\mathbb{D}})$ s.t. $h(z_0) = h(-z_0)$.

...

Pf of Koebe $\frac{1}{4}$-thm

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

Assume $f(z) \neq c \quad \forall\, z \in \mathbb{D}$. Goal prove $|c| \geq \frac{1}{4}$.

$$g(z) := \frac{c\, f(z)}{c - f(z)} \quad \text{Then}$$

$$= f(z) \frac{1}{1 - \frac{1}{c} f(z)} = z + \left(a_2 + \frac{1}{c}\right) z^2 + \text{h.o.t.}$$

Now we have that $|a_2| \leq 2$ but also $\left|a_2 + \frac{1}{c}\right| \leq 2$.

$$\Rightarrow \frac{1}{|c|} = \left| \frac{1}{c} + a_2 - a_2 \right| \leq \left| \frac{1}{c} + a_2 \right| + |a_2| \leq 4.$$

$$\Rightarrow |c| \geq \frac{1}{4}. \checkmark$$

Not much more time, but lets make a definition.

Fixed points

Let $f : \Omega \to \mathbb{C}$ be a holomorphic mapping, and let $z_0 \in \Omega$ be a point.

If $$f(z) = z_0 + \lambda(z - z_0) + \sum_{j=2}^{\infty} a_j (z - z_0)^j$$

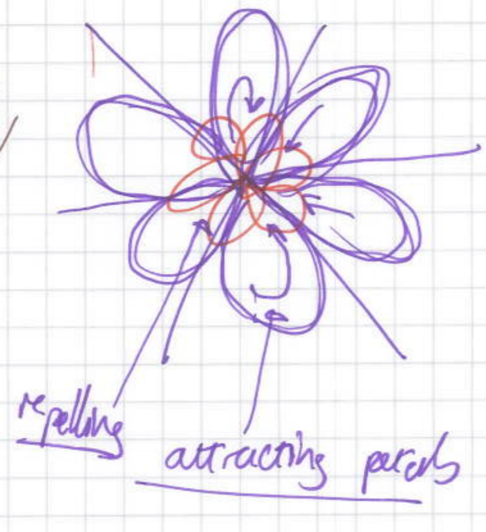
we call z_0 a fixed point w/ multiplier λ.

(i) If $|\lambda| < 1$ we say it is a attracting fixed point. (if $\lambda = 0$, we say superattracting)

(ii) If $|\lambda| > 1$, we say it is a repelling fixed point.

(iii) If $\lambda^n = 1$ for some n, it is a rationally neutral fixed point.

(iv) If $|\lambda| = 1$ (and λ is not a root of unity), it is an irrational neutral point.



(no lectures next week)