Thm If $\pi: G \to H$ is cont's homomorphism of Lie groups, then π is smooth.

In particular, for Lie groups there is no difference between fin-dim reps of locally compact groups and Lie groups.

Pf Consider the graph of π:

$$\Gamma = \{(g, \pi(g)) \mid g \in G\} \subset G \times H$$

As π is cont's, Γ is a closed subgroup of the product, hence a submanifold.

Consider the projection $\pi: G \times H \to G$ and let $q = P_1|_\Gamma$.

Then q is a smooth isomorphism between Γ and G.

The homomorphism q (as any Lie group homomorphism) has constant rank, since $\forall x \in \Gamma$, we have $q = l_{q(x)^{-1}} \circ q \circ l_x$ so $\mathrm{rnd}\, d_e q = \mathrm{rnd}\, d_x q$.

By the constant rank theorem a constant rank bijection is a diffeomorphism. Thus q^{-1}, $q^{-1}(g) = (g, \pi(g))$ is smooth.

As π is the composition of q^{-1} w/ $P_2: G \times H \to H$, π is smooth.

$\blacksquare$

Prop 1 If $\pi: G \to H$ is a Lie group homomorphism, then $\ker \pi$ is a closed Lie subgroup of G w/ Lie algebra $\ker(\pi_*: \mathfrak{g} \to \mathfrak{h})$

Pf Clearly $\ker \pi$ is closed, hence it is a closed subgrp.

$X \in \mathfrak{g}$ is in the Lie algebra of $\ker \pi$

$\iff \exp(tX) \in \ker \pi \quad \forall t$

$\iff \pi(\exp(tX)) = e \quad \forall t$

$\iff \exp(t\, \pi_*(X)) = e$

$\iff \pi_*(X) = 0$ ∎

Prop 2 Let $\pi: G \to GL(V)$ be a rpr of a Lie group G on a f.d. v.space (over $\mathbb{C}$/$\mathbb{R}$). Take any $v \in V$. Then $H = \{ g \in G \mid \pi(g)v = v \}$ is a closed Lie subgroup of G and its Lie algebra

$$\mathfrak{h} = \{ X \in \mathfrak{g} \mid \pi_*(X)v = 0 \}$$

Pf Clearly, H is a closed subgrp.

We have, for $X \in \mathfrak{g}$, $X \in \mathfrak{h} \iff \exp(tX) \in H \quad \forall t$

$\iff \pi(\exp(tX))v = v \quad \forall t$

$\qquad e^{t\pi_*(X)}v \qquad \iff \pi_*(X)v = 0$ ∎

Instead of one representation π we can consider

$\pi_1, \dots, \pi_n$ and vectors $v_1, \dots, v_n \in V_{\pi_i}$ and

$$H = \{ g \in G \mid \pi_i(g) v_i = v_i \ \forall i \}$$

is a closed Lie subgroup w/ Lie algebra $\mathfrak{h} = \{ X \in \mathfrak{g} \mid (\pi_i)_*(X) v_i = 0 \ \forall i \}$

Examples (i) Let V be f. dim real/cplx v. space

the <u>special lin group</u> is

$$SL(V) = \{ \varphi \in GL(V) \mid \det \varphi = 1 \}$$

w/ Lie algebra $\mathfrak{sl}(V) = \{ X \in \mathfrak{gl}(V) \mid \operatorname{tr} X = 0 \}$

If $V = \mathbb{R}^n$, then denoted by $SL_n(\mathbb{R})$ or $SL(n; \mathbb{R})$.

(ii) Let V be as in i) and B a bilinear form on V. Consider

$$G = \{ \varphi \in GL(V) \mid B(gv, gw) = B(v,w) \ \forall v, w \in V \}$$

Try to find the Lie algebra. Consider the space W of bilinear forms on V ($W \cong V^* \otimes V^*$)

Define a repr π of $GL(V)$ on W by

$$(\pi(g) B')(v,w) = B'(g^{-1}v, g^{-1}w)$$

Then G is exactly

$$\{ g \in GL(V) \mid \pi(g) B = B \}$$

By Proposition 2, G is a Lie group w/ Lie algebra

(4)

$$\mathfrak{g} = \{ X \in \mathfrak{gl}(V) \,/\, \pi_*(X) B = 0 \}$$

We have $$(\pi_*(X) B)(v,w) = \frac{d}{dt}\Big(\pi(\exp(tX)) B\Big)(v,w)\Big|_{t=0}$$

$$= \frac{d}{dt} B(e^{-tX} v, e^{-tX} w)\Big|_{t=0}$$

$$= - B(Xv, w) - B(v, Xw)$$

Thus $\mathfrak{g} = \{ X \in \mathfrak{gl}(V) \,/\, B(Xv, w) + B(v, Xw) = 0 \ \forall v, w \}$

(iii) $V = \mathbb{R}^n$ and let B be the usual scalar product.

$$(v,w) = \sum_{i=1}^{n} v_i w_i .$$

Then the group G from (ii) is the orthogonal group, denoted by $O(n)$, $O_n(\mathbb{R})$ or $O(n;\mathbb{R})$.

We have $$(Xv, w) + (v, Xw) = (Xv, w) + (X^T v, w)$$

$$= ((X + X^T) v, w)$$

$$\Rightarrow \mathfrak{o}(n) = \{ X \in \mathfrak{gl}_n(\mathbb{R}) \,|\, X + X^T = 0 \}$$

The special orthogonal group is denoted by $SO(n)$, is

$$O(n) \cap SL_n(\mathbb{R})$$

w/ Lie algebra $\mathfrak{so}_n(\mathbb{R}) = \{ X \in \mathfrak{gl}_n(\mathbb{R}) \,|\, X^T = -X, \ \mathrm{Tr}\, X = 0 \}$

We can do the same for $V = \mathbb{C}^n$ and (5)

$(v,w) = \sum_{i=1}^{n} v_i w_i$ and get the complex

orthogonal Lie groups $\wedge$ $O_n(\mathbb{C})$ and $SO_n(\mathbb{C})$.

(and directly)

For $V = \mathbb{R}^{k+l}$ $(k,l \geq 1)$ consider the form

$$B(v,w) = \sum_{i=1}^{k} v_i w_i - \sum_{i=k+1}^{k+l} v_i w_i$$

$$= (Av, w) \quad \text{where} \quad A = \left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} \end{array}\right)$$

Then we get the group $O(k,l)$.

$$B(Xv,w) + B(v,Xw) = (AXv,w) + (Av,Xw)$$

$$= ((AX + X^T A)v, w)$$

so $\mathfrak{o}(k,l) = \{ X \in \mathfrak{gl}_{k+l}(\mathbb{R}) \mid AX + X^T A = 0 \}$

We can also define $SO(k,l) = O(k,l) \cap SL_{k+l}(\mathbb{R})$

(IV) $V = \mathbb{C}^n$, consider the Hermitian form $\underbrace{\cancel{B(v,w)}}_{\langle v,w\rangle} = \sum v_i \overline{w_i}$.

The unitary group is $U(n) = \{ g \in GL_n(\mathbb{C}) \mid \langle gv, gw\rangle = \langle v,w\rangle \ \forall v,w \}$

Then $U(n)$ is a Lie group w/ Lie algebra $\mathfrak{u}(n) = \{ X \in \mathfrak{gl}_n(\mathbb{C}) \mid X = -X^* \}$

We also have the special unitary group (6)

$$SU(n) = U(n) \cap SL_n(\mathbb{C}) \quad w/$$

Lie algebra $su(n) = \{X \in gl_n(\mathbb{C}) \mid X = -X^*, \ \mathrm{Tr}\, X = 0\}$

(v) Let $V = \mathbb{R}^{2n}$ or $V = \mathbb{C}^{2n}$ and consider

$$(v,w) = \sum_{i=1}^{n} (v_i w_{i+n} - v_{i+n} w_i)$$

$$= (Jv, w) \text{ where } J = \begin{pmatrix} 0 & \begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} \\ \begin{matrix} +1 & & \\ & \ddots & \\ & & +1 \end{matrix} & 0 \end{pmatrix}$$

Then we get symplectic Lie groups $Sp(2n, \mathbb{R})$ and $sp(2n, \mathbb{R})$.

The Lie algebra of $Sp(2n, \mathbb{R})$ is

$$sp(2n; \mathbb{R}) = \{X \in gl_{2n}(\mathbb{R}) \mid JX + X^T J = 0\}$$

(and similarly for $\mathbb{C}$)

Last example (vi) Compact symplectic group is $Sp(n) = Sp(2n; \mathbb{C}) \cap U(2n)$

w/ Lie algebra $sp(n) = \left\{X \in gl_{2n}(\mathbb{C}) \;\middle|\; \begin{matrix} X + X^* = 0 \\ JX + X^T J = 0 \end{matrix}\right\}$

It can be shown that $Sp(n)$ is the subgroup of $GL_n(\mathbb{H})$ preserving

$$\sum_{i=1}^{n} v_i \overline{w_i},$$

quaternions

The above examples are <u>classical Lie groups</u> — i.e. the groups of linear transformations preserving non-degenerate symmetric / skew-symmetric / Hermitian forms on real / cplx / quaternionic spaces, as well as their special versions.

$U(k,l) < GL_{k+l}(\mathbb{C})$ preserving

$$\sum_{i=1}^{k} v_i \overline{w_i} - \sum_{i=k+1}^{l} v_i \overline{w_i}$$

It is not difficult to check that the compact groups in the above examples are

$$O(n;\mathbb{R}),\ SO(n;\mathbb{R}),\ U(n),\ SU(n),\ Sp(n)$$

<u>GOAL</u> understand the properties of these groups in a unified way.

Returning to subgroups, we have

<u>Def</u> A <u>Lie subgroup</u> of a Lie group G is a subgroup $H < G$ s.t. it admits the structure of a Lie grp making the embedding $H \to G$ smooth.

<u>Remarks</u> (i) We assume manifolds and Lie groups to be second countable. (otherwise G w/ discrete topology becomes a Lie subgroup)

(ii) Different texts mean different things by Lie subgroups.

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As the embedding $H \hookrightarrow G$ has constant rank, it must be an immersion by the constant rank thm.

Hence H can be identified w/ a Lie subalgebra of $\mathfrak{g}$.

$\boxed{\text{Thm}}$ Assume G is a Lie group and $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie subalgebra. Then there exists a unique connected Lie subgroup $H \subset G$ w/ Lie subalgebra $\mathfrak{h}$.

Recall if M is a manifold, then a rank k distribution is a k-dim subbundle D of TM.

i.e. for every $p \in M$, we are given a k-dim. subspace $D_p \subset T_pM$ depending smoothly on p, in the sense that in a neighb. U of p there ex. smooth v.fields $X_1, \ldots, X_k$ s.t.

$$(X_1)_q, \ldots, (X_k)_q \text{ form a basis of } D_q \quad \forall q \in U.$$

An immersed submanifold $N \subset M$ is integral ~~of~~ D if $T_pN = D_p \quad \forall p \in N$.

D is called integrable if it satisfies the Frobenius integrability condition: if X, Y are v.fields on $U \subset M$ s.t. $X_p, Y_p \in D_p \quad \forall p \in U$, then $[X, Y]_p \in D_p \quad \forall p \in U$.