Nasjonalt algebramøte, 2014 (1)

Roy Skjelnes 12:50

Will talk about $\mathrm{Quot}^n_{\mathcal{F}/\mathbb{P}^r/S} = Q$ ← proj. scheme

- $\mathcal{F} = \bigoplus_{i=1}^{p} \mathcal{O}$ on $\mathbb{P}^r \xrightarrow{f} S$ ← „no big restriction"

constant polynomial „n".

$\mathrm{Quot}^n_{\mathcal{F}/\mathbb{P}^r/S} \ni$ points $\leftrightarrow$ $0 \to \mathcal{R} \to \mathcal{F} \to \mathcal{E} \to 0$

flat over S, relative rank n/S

$\Rightarrow$ $\mathrm{supp}(\mathcal{E})$ is finite over S

and $f_*\mathcal{E}$ locally free rk n.

[Particular cases: When $p=1$, $\mathrm{Hilb}^n_{\mathbb{P}^r/S}$.

$r=0$ $(f=\mathrm{id}_S)$ $\mathrm{Grass}^n(\mathcal{F})$]

Regularity $d \geq n$ (Castelnuovo-Mumford-reg.)

„tensor + push down" $\Rightarrow$

$$0 \to f_*\mathcal{R}(d) \to f_*\mathcal{F}(d) \to f_*\mathcal{E}(d) \to 0$$

$\Rightarrow$ get map $Q \xrightarrow{i} \mathrm{Grass}^n(f_*\mathcal{F}(d)) = G$

Grothendieck: i is a closed immersion. (2)

(abstract proof)

On G we have the universal sequence

$$0 \to R_d \to f_* \mathcal{F}(d) \to \mathcal{E}_d \to 0 \quad \text{(modules on } G\text{)}$$

Can multiply with $f_* \mathcal{O}(1) \otimes -$.

Let R_{d+1} be the image of $R_d \otimes f_* \mathcal{O}(1)$ in $f_* \mathcal{F}(d+1)$.

This gives a graded submodule $R \subseteq \bigoplus_{s \geq d} f_* \mathcal{F}(s)$ and a graded quotient $\mathcal{E} = \bigoplus_{s \geq d} \mathcal{E}_s$.

$\boxed{\text{Thm}}$ Let $d \geq n$. $\mathcal{F} = \bigoplus^{p} \mathcal{O}_{\mathbb{P}^r}$.

Then $\mathrm{Quot}^n_{\mathcal{F}/\mathbb{P}^r/S} = \bigcap_{i \geq 1} \mathrm{Fitt}_{n-1}(\mathcal{E}_{d+i})$

What are the Fitting ideals? A fin. generated A-module M.

$$0 \leftarrow M \leftarrow \bigoplus^{N} A \xleftarrow{\varphi} \bigoplus_{\alpha} A$$

Then $\mathrm{Fitt}_i(M)$ is closed in $\mathrm{Spec}\, A$ given by the ideal of $(N-i)$-minors of φ.

- M loc. free rank i $\iff$ $\mathrm{Fitt}_i(M) = \emptyset$ (3)
and $\mathrm{Fitt}_{i-1}(M) = \emptyset$ Spec A

As top. spaces "reduced part"

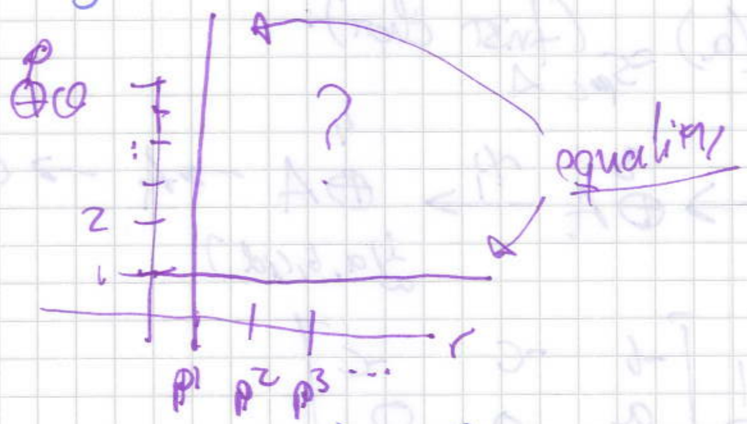
$$|\mathrm{Quot}^n_{\mathcal{F}/\mathbb{P}^r/S}| = |\mathrm{Fitt}_{n-1}(\mathcal{E}_{d+1})|$$

We have

$$\mathrm{Quot}^n_{\mathcal{F}/\mathbb{P}^r/S} = \mathrm{Fitt}_{n-1}(\mathcal{E}_{d+1})$$

$$\mathrm{Hilb}^n_{\mathbb{P}^r/S} = \mathrm{Fitt}_{n-1}(\mathcal{E}_{d+1})$$

(Gotzmann persistence thm.)



G. Ståhl (student of his): have equality.

i.e. $\mathrm{Quot}^n_{\mathcal{F}/\mathbb{P}^r/S} = \mathrm{Fitt}_{n-1}(\mathcal{E}_{d+1})$

(4)

Ex $Q = \mathrm{Quot}^1_{\mathcal{F} = \overset{2}{\oplus}\mathcal{O}/\mathbb{P}^1/k}$

$S = k[x,y]$ $\quad F = S \oplus S$

Choose $d = 1 \geq 1$.

Then $f_* \mathcal{F}(1) = V \oplus V$ $\quad V = \{x, y\}$ v. space of 1-forms

$$\mathrm{Grass}^1(f_* \mathcal{F}(1)) = \mathbb{P}^3 = \mathrm{Proj}\, k[a,b,c,d]$$

$$0 \to R_1 \to f_* \mathcal{F}(1) \longrightarrow \mathcal{E}_1 = \mathcal{O}(1) \longrightarrow 0$$

Over $D_+(a) = \operatorname{Spec} A$ (first chart):

$$0 \longrightarrow \overset{3}{\oplus} A \xrightarrow{\varphi_1} \overset{4}{\oplus} A \to A \to 0$$

$\frac{1}{a}[a, b, c, d]$

$$\varphi_1 = \frac{1}{a}\begin{bmatrix} -b & -c & -c \\ a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

$$\overset{6}{\oplus} A \xrightarrow{\varphi_2} \overset{6}{\oplus} A \to \mathcal{E}_2 \to 0$$

$$\varphi_2 = \frac{1}{a}\begin{bmatrix} -b & 0 & \\ a & -b & \\ 0 & a & \\ 0 & 0 & \cdots \\ 0 & 0 & \\ 0 & 0 & \end{bmatrix}$$

So $\mathrm{Fix}_0(\mathcal{E}_2)$ is given by $\det \frac{1}{2}$.

$$\frac{\overset{N}{bc-ad}}{a^2}$$

$\to$ $\mathrm{Fix}_0(\mathcal{E}_2) = \mathrm{Quot}$ given by $bc-ad \cong \mathbb{P}^1 \times \mathbb{P}^1$ via Segre.