Sec 34

(1) $\mathbb{Z}_{12} \xrightarrow{\varphi} \mathbb{Z}_3$

$1 \longmapsto 2$

a) $\ker\varphi = \{ \bar{n} \mid 2n \equiv 0 \bmod 3 \}$ since 3 is prim

$K = \ker\varphi$

$= \{ \bar{n} \in \mathbb{Z}_{12} \mid n \equiv 0 \bmod 3 \}$

$= \{0, 3, 6, 9\}$

$= 3\mathbb{Z}_{12} \simeq \mathbb{Z}_4$

B) The cosets are

$0 + K = \{0, 3, 6, 9\}$

$1 + K = \{1, 4, 7, 10\}$

$2 + K = \{2, 5, 8, 11\}$. These are all.

c) The theorem says there is a ! iso μ from $\mathbb{Z}_{12}/K$ to $\mathbb{Z}_3$ such that

$$\underset{\mathbb{Z}_3}{\underset{\cap}{2}} = \varphi(1) = \mu(\underset{1+K}{\underset{\shortparallel}{\{1, 4, 7, 10\}}})$$

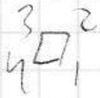
Thus the correspondence is

⑥ Find all Sylow 2-subgroups of S_4 and show that they are conjugate.

- We have $24 = 8 \cdot 3$, so the Sylow 2-subgroups have order 8.

Let G be the subgroup generated by

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

Then $\langle \rho, \sigma \rangle \simeq D_4$, so ~~they~~ G have order 8. 

We can conjugate by any $\alpha \notin G$. For example

$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$. Then we get the group

generated by $\alpha\rho\alpha^{-1}$ and $\alpha\sigma\alpha^{-1}$. Note that $\alpha = \alpha^{-1}$!

So

$$\underset{\rho'}{\underset{\shortparallel}{\alpha\rho\alpha^{-1}}} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \text{ and } \underset{\sigma'.}{\underset{\shortparallel}{\alpha\sigma\alpha^{-1}}} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

Let $G' = \langle \rho', \sigma' \rangle. \simeq D_4$. We know there must be one more Sylow 2 subgroup. This one is found by conjugating w/ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

⑤ Find all Sylow 3-subgroups of S_4, and demonstrate that they are conjugate.

We must look for a subgroup of order 3.

This is easy: Take the subgroups permuting three variables, leaving one fixed, that is, the ~~three~~ four groups generated by (respectively)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}.$$

By the third Sylow theorem, we have found all Sylow 3 subgroups

It is easy to see they are conjugate. We show nr 1 is conjugate to nr 2.:

Let $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$. Then $g^{-1} = g$. So

$$g \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} g^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

Section 36

① Any Sylow 3-subgr is by def a maximal 3-group, ~~and they are~~

so since $12 = 3 \cdot 4$, we must have order 3

② Same. $54 = 9 \cdot 6 = 27 \cdot 2$, so a Sylow-3-subg have order 27.

③ The no. of Sylow 2-subgroups satisfy

$\# S_2 \equiv 1 \mod 2$ and $\# S_2 \mid 24$

so we look at the possibilities:

$\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23\}$

of these, only 1 and 3 divide 24.

④ Same strategy. Answer: 1, 51

1, 85

② ① $\{0\} \overset{2940\mathbb{Z}}{\le} 420\mathbb{Z} \le 60\mathbb{Z} \le 20\mathbb{Z} \le 4\mathbb{Z} \le \mathbb{Z}$

② $\{0\} \le 2940\mathbb{Z} \le 735\mathbb{Z} \le 245\mathbb{Z} \le 49\mathbb{Z} \le 7\mathbb{Z} \le \mathbb{Z}$

The quotients are

① $\mathbb{Z}/7 \quad \mathbb{Z}/7 \quad \mathbb{Z}/3 \quad \mathbb{Z}/5 \quad \mathbb{Z}/4$

② $\mathbb{Z}/4 \quad \mathbb{Z}/3 \quad \mathbb{Z}/5 \quad \mathbb{Z}/7 \quad \mathbb{Z}/7$

③ ① $0 \le \langle 3 \rangle \le \mathbb{Z}_{24}$, $\quad \langle 3 \rangle = 3\mathbb{Z}_{24} \simeq \mathbb{Z}_8$

② $0 \le \langle 8 \rangle \le \mathbb{Z}_{24}$, $\quad \langle 8 \rangle = 8\mathbb{Z}_{24} \simeq \mathbb{Z}_3$

The quotients are

$\mathbb{Z}_8 \qquad \mathbb{Z}_3$

$\mathbb{Z}_3 \qquad \mathbb{Z}_8$

so the series are already isomorphic!

c) The group $\varphi(\mathbb{Z}_{18})$ is

$2\mathbb{Z}_{12}$. $\simeq \mathbb{Z}_6$

d) ... (do this yourself)

Section 35

① Give isomorphic refinements of

$\{0\} < 10\mathbb{Z} < \mathbb{Z}$ and $\{0\} < 25\mathbb{Z} < \mathbb{Z}$

① $\{0\} < 50\mathbb{Z} < 10\mathbb{Z} < 2\mathbb{Z} < \mathbb{Z}$

② $\{0\} < 50\mathbb{Z} < 25\mathbb{Z} < 5\mathbb{Z} < \mathbb{Z}$

The quotients are

①	$50\mathbb{Z}$	$\mathbb{Z}_5$	$\mathbb{Z}/5$	$\mathbb{Z}/2$
②	$50\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}/5$	$\mathbb{Z}/5$

$$0+K=\{0,3,6,9\} \longmapsto 0$$
$$1+K=\{1,4,7,10\} \longmapsto 2$$
$$2+K=\{2,5,8,11\} \longmapsto 1$$

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(2) $$\mathbb{Z}_{18} \xrightarrow{\varphi} \mathbb{Z}_{12}$$
$$1 \longmapsto 10$$

a) $$\ker\varphi = \{n \in \mathbb{Z}_{18} \mid 10n \equiv 0 \mod 12\}$$
$$= \{n \in \mathbb{Z}_{18} \mid 2n \equiv 0 \mod 12\}$$ since 5 is a unit in $\mathbb{Z}_{12}$
$$= \underbrace{\{0,6,12\}}_{= K} \simeq \mathbb{Z}_3$$

b) The cosets are

$$0+\{0,6,12\} = \{0,6,12\}$$
$$1+K = \{1,7,13\}$$
$$2+K = \{2,8,14\}$$
$$3+K = \{3,9,15\}$$
$$4+K = \{4,10,16\}$$
$$5+K = \{5,11,17\}$$