①

$d \geq 2$ always

- $f : \mathbb{P}^1 \to \mathbb{P}^1$ d-1 covering maps, branched
- f is of the form $f(z) = \frac{P(z)}{Q(z)}$ z polys w/ no common factors

$d = \max \deg(P, Q)$.

- We have $2(d-1)$ branch points, counting w/ multiplicity

start w/
- $z_0 \in \mathbb{P}^1$. If $f^n(z_0) = z_0$ some n, z_0 a periodic point.

$\Rightarrow$ we classify it as attracting, repelling, irrationally neutral, or parabolic (rationally neutral)
depending on the multiplier $(f^n)'(z_0)$

- If $f^m(z_0)$ is a periodic point, then z_0 is preperiodic.

Key concepts
- z_0 is in the Fatou set $\mathcal{F}$ if $\exists$ open neighbourh. $U \ni z_0$ s.t. $\{f^n\}$ is normal on U.
- The Julia set is the complement $\mathbb{P}^1 \setminus \mathcal{F}$

Facts
① Julia set always non-empty $\mathcal{J} \neq \emptyset$

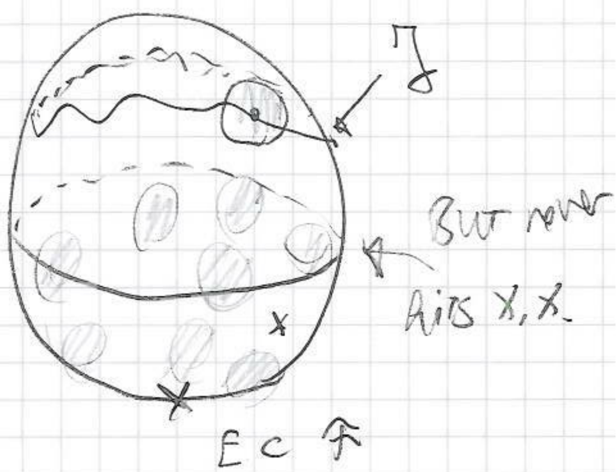
② $\mathcal{J}$ is completely invariant i.e. $f(\mathcal{J}) = \mathcal{J}$ and $f(\mathcal{F}) = \mathcal{F}$.

③ $\mathcal{J}$ always infinite

④ For any $N \in \mathbb{N}$, $\mathcal{F}_f = \mathcal{F}_{f^N}$.

(5) Fix $f: \mathbb{P}^1 \to \mathbb{P}^1$.

Let $z_0 \in \mathcal{J}_f$. Then $\exists$ set $E \subset \mathbb{P}^1$ consisting of at most two pts such that for any open set $U \ni z_0$, $f^n(U) \cap E = \emptyset$ $\forall n$.



- If $\# E = 1$, then f is conjugate to a polynomial map.
- If $\# E = 2$, then $g(z) = z^{\pm d}$.

Thm Let $z_0 \in \mathcal{J}$. Then $\{f^{-n}(z_0)\}_{n=0}^{\infty}$ is dense in $\mathcal{J}$.

Pf Note that $z_0 \notin E$. Fix $w_0 \in \mathcal{J}$. By def, for any open neighb. V of w_0 $\exists\, n \in \mathbb{N}$ s.t. $f^n(V) \ni z_0$. So w is in $\overline{\{f^{-n}(z_0)\}_{n \in \mathbb{N}}}$. ∎

Thm Any completely invariant subset of $\mathcal{J}$ is dense in $\mathcal{J}$.

Pf If $z_0 \in K$, then $\{f^{-n}(z_0)\} \subset K$ $\forall n$. ∎

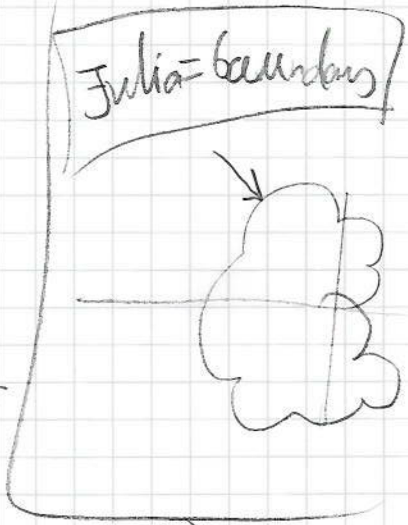
Thm The Julia set is a closed and perfect set. (i.e. no isolated points)

Pf Assume to get a contradiction that $z_0 \in \mathcal{J}$ is isolated in $\mathcal{J}$. Then $\exists$ open $U \ni z_0$ w/ $U \cap \mathcal{J} = \{z_0\}$.
Since the exceptional set $\subseteq \mathcal{F}$, we have that for any $w_0 \in \mathcal{J}$,

∃ n s.t. $w_0 \in f^n(U)$. Then since $\mathcal{F}$ is invariant, we must have $w_0 = f^n(z_0)$.

So must $J = \{f^n(z_0)\}_{n=0}^{\infty}$

Choose $x_0 \neq z_0$ s.t. $f(x_0) = z_0$. (recall $\deg \geq 2$), then $x_0 \in J$. But then ∃ $m \in \mathbb{N}$ s.t. $f^m(z_0) = x_0$, but then $f^{m+1}(z_0) = z_0$ so J is finite! $\boxed{\text{Contradiction}}$ ∎

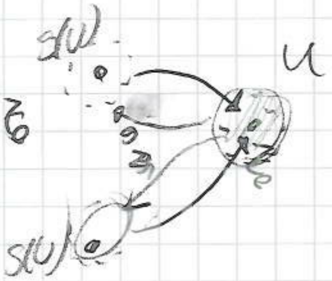


<u>Corollary</u> J is uncountably infinite.
(b.c. any perfect set is)

<u>Prm</u> If J has interior, then $J = \mathbb{P}^1$.

<u>Pf</u> If J has interior $U \subset J$, then any $w_0 \notin E$ ∃ $n \in \mathbb{N}$ s.t. $f^n(U) \ni w_0 \Rightarrow w_0 \in J$. ∎ (P3)

Thm 2 Let $z_0 \in \mathbb{P}^1$, and assume that z_0 is not a critical value.
Assume that there exists an open U of z_0 w/ no periodic points.
Then z_0 is in $\mathcal{F}$.



<u>Pf</u> choose U small enough so that you two different inverses g_1, g_2 of f, f_2 def on U (can choose $U, g_1(U), g_2(U)$ to be disjoint by choosing U small enough)

4

Define
$$F_n(z) = \frac{(f^n(z) - g_2(z))(g_1(z) - z)}{(f^n(z) - g_1(z))(g_2(z) - z)}$$

Claim F_n omits the points 0, 1 and ∞.

If $F_n(z) = 1 \Rightarrow$

$f^n(z) = z$ contrary to ass.

So $\{F_n\}$ is a normal family. (?? why? Standard result?)

It follows that $\{f^n\}$ is a normal family.

Consider a subsequence f^{n_j}. We need to extract a subsequence.

By possibly passing to a subsequence we may assume that $f^{n_j}(z_0) \to w_0$

By conjugation, we may assume $w_0 = 0$.

After passing to yet another subsequence,

$$\frac{(f^{n_j} - g_2)(g_1 - z)}{(f^{n_j} - g_1)(g_2 - z)} = h_j \to h$$

Then
$$\frac{f^{n_j} - g_2}{f^{n_j} - g_1} = \underbrace{h_j \frac{(g_2 - z)}{(g_1 - z)}}_{\phi_j} \to H$$

$f^{n_j} - g_2 = \phi_j (f^{n_j} - g_1)$

$\Rightarrow f^{n_j} = \dfrac{\phi_j g_1 + g_2}{1 - \phi_j}$

f^{n_j} converges

∎

Selfsimilarity of $\mathcal{J}$

- We will show that there are only finitely many attracting or neutral periodic points.

(actually $2(d-1)$ in total)

Assuming this

Thm Repelling periodic points are dense in $\mathcal{J}$, the Julia set.

Pf Let $z_0 \in \mathcal{J}$ and let U be any neighborhood of z_0.

Since z_0 is not isolated in $\mathcal{J}$, we may assume (for our purposes) that z_0 is not a critical value of f.

If there were no periodic points in U, then z_0 would be in $\mathcal{F}$!

Hence there exist a periodic point.

Since there are only finitely many attracting or neutral periodic points the conclusion follows. ∎

This motivates

$\boxed{\text{self-similarity}}$

! **Thm** Let U be any open set s.t. $U \cap J$. Then $\exists N \in \mathbb{N}$ s.t.
$$f^N(U \cap J) = J.$$

Pf Let z_0 be a repelling periodic point in $U \cap J$, i.e. $f^m(z_0) = z_0$ w/ multiplier $|\lambda| > 1$.

Write $g = f^m$. So z_0 is a repelling fixed point for g.

Let V be an open neighb. of z_0 s.t. $g(V) \supset\supset V$.

Then $g^n(V)$ becomes an increasing family of open sets.

Since $E \cap J = \emptyset$, any point in J is in $g^N(V)$ for some n.

Thus
$$J \subset \bigcup_{n=1}^{\infty} g^n(V)$$

By compactness, a finite no. of is enough.

But
$$\bigcup_{n=1}^{M} g^n(V) = g^M(V). = f^{mM}(V)$$

$\blacksquare$