Manifolds 2014

JAC / FM

1 Introduction; topological manifolds

Manifolds are geometric objects that locally look like some $\mathbb{R}^n$ for some natural number n. In that respect, they are very easy to understand. The interesting things happen when we glue several pieces of $\mathbb{R}^n$ together.

In Calculus, manifolds (in those rare occasions they were encountered) usually appeared embedded in some larger $\mathbb{R}^n$ or $\mathbb{C}^n$. This is not the path we're going to take - for us, manifolds will be objects in their own right, existing independently of some "ambient" space. The way to do this is, exactly, glueing.

We start with the definition of a *topological manifold*.

Definition 1.1. A *topological manifold* M is a topological space such that each $x \in M$ has a neighbourhood U such that U is homeomorphic to $\mathbb{R}^n$. ■

Here are some examples of manifolds.

Example 1.2. Setting $M = \mathbb{R}^n$, set $U = \mathbb{R}^n$ for all $x \in M$. ★

Example 1.3. Any open ball B in $\mathbb{R}^n$: If $x \in B$, let $U = B$. Then I claim that $U \approx \mathbb{R}^n$: just use the map $\vec{x} \mapsto \frac{1}{1-|\vec{x}|}\vec{x}$. ★

Example 1.4. By the previous example, it follows that any open $U \subseteq \mathbb{R}^n$ is a manifold, by definition of an open set as a union of balls. ★

Example 1.5. Similarly, any open subset of a manifold is a manifold. ★

Example 1.6. Up to homeomorphism, there are only two 1-dimensional manifolds. Namely, the circle S^1 and the real line $\mathbb{R}$. The first is compact, the other is not. ★

Example 1.7. By stereographic projection, every n-sphere can be covered by two sheets homeomorphic to $\mathbb{R}^n$. See Figure 1. ★

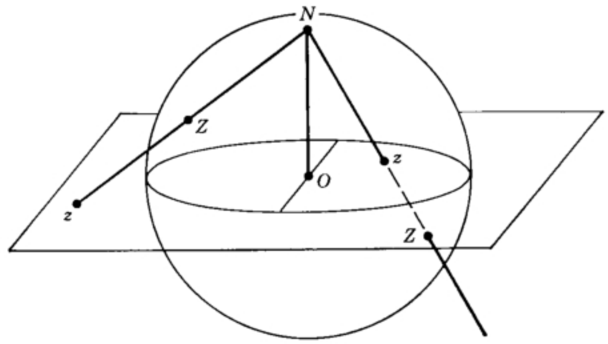

Figure 1: Stereographic projection.

Example 1.8. The $\aleph_0$ number of compact surfaces. These are 2-spheres attached with n handles, for some natural number n. ★

Example 1.9. The *non-orientable* manifolds. An example of a non-orientable manifold is the Möbius band in Figure 2 with the boundary removed. If we keep the boundary, the figure is an example of a *manifold with boundary*.

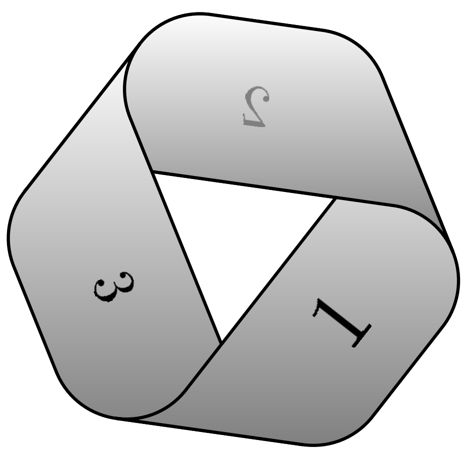

Figure 2: The Möbius band.

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Example 1.10. The last example (for now) is the projective plane over the reals, called $\mathbb{P}^2_{\mathbb{R}}$. By definition, as a set, it is the set of lines through the origin in $\mathbb{R}^3$. However, it has a nice manifold structure. First, note that a line through the origin in $\mathbb{R}^3$ is determined by giving a point on the 2-sphere S^2. This point is however not unique: the antipodal point gives the same line. This means that we can identify $\mathbb{P}^2_{\mathbb{R}}$ with thw quotient space $S^2/\sim$ where $x \sim -x$. We now give the quotient the quotient topology.

This is not completely enlightening though. There is no obvious local homeomorphisms with $\mathbb{R}^2$, nor do we really have a picture of how $\mathbb{P}^2_{\mathbb{R}}$ looks like. The second problem can be solved like this: Note that any line not in the plane $z = 0$, has a *unique* representative in the open north hemisphere. Similarly, every line in the plane $z = 0$, but with $y \neq 0$, has a unique representative on the equator minus a point. Thus $\mathbb{P}^2_{\mathbb{R}}$ can be written as the union of a three cells, namely a B_2 (a 2-ball), a B_1 and a B_0. So $\mathbb{P}^2_{\mathbb{R}} = B_2 \cup B_1 \cup B_0$, but glued in a special way. We can think of this glueing as "adding points/lines at infinity".

There is a natural choice of coordinates on $\mathbb{P}^2_{\mathbb{R}}$. We define *homogeneous coordinates*: a point P can be represented by a 3-tuple $[x_0 : x_1 : x_2]$, and this tuple is unique up to multiplication by $\mathbb{R}\backslash\{0\}$. In other words, every point has a representative of the form $[x_0 : x_1 : x_2]$, where $[x_0 : x_1 : x_2] = [\lambda x_0 : \lambda x_1 : \lambda x_2]$. Define the "basic open sets U_i" as $U_i := \{P \in \mathbb{P}^2_{\mathbb{R}} \,|\, x_i \neq 0\}$. There is a natural bijection between U_0, say, and $\mathbb{R}^2$. In U_0, every point has a *unique* representative of the form $[1 : x_1 : x_2]$, and this can be mapped to $(x_1, x_2) \in \mathbb{R}^2$. Now it is an **exercise** to show that the set U_i is open in the quotient topology.

It is easy to see that the U_i cover $\mathbb{P}^2_{\mathbb{R}}$, and so this defined the structure of a topological manifold on $\mathbb{P}^2_{\mathbb{R}}$. ★

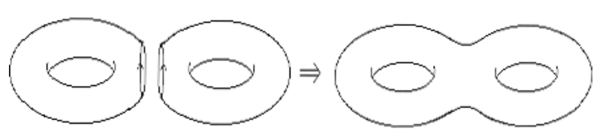

Figure 3: The connected sum of two tori.

Example 1.11. One can make new manifolds from old ones by means of the *connected sum*.

One starts with two manifolds M and N, removes an open disk from both of them. The boundary of a disk is a S^1, and we can identify the two boundaries by an arbitrary homeomorphism $\varphi : S^1 \to S^1$. We glue to get $M\#N$. See Figure 3. ★

For the Mï¿$\frac{1}{2}$bius band example to be a manifold, we have to define a manifold with boundary:

Definition 1.12. Let M be a topological space. We say that M is a *manifold with boundary* if every point of x has a neighbourhood homeomorphic to either $\mathbb{R}^n$ or $H_n := \{(x^1, \ldots, x^n) : x^n \geq 0\}$, the upper half-plane. ■

We call the n in the definition the *dimension* of the manifold. However, to prove that the dimension is well-defined, reduces to proving that if $\mathbb{R}^n \approx \mathbb{R}^m$, then $m = n$. This is a very non-trivial problem, best proved with tools from algebraic topology, using such tools as homology groups and long exact sequences.

2 Differentiable manifolds

To get a course worth something, one has to have something more than just a topological manifold. We do this by adding structure to M, thereby defining the notion of a *differentiable* manifold.

Given two open subsets U, V of M and homeomorphisms $x : U \to x(U) \subseteq \mathbb{R}^n$ and $y : V \to y(V) \subseteq \mathbb{R}^n$, one has transition maps on the intersection $U \cap V$:

$$\begin{tikzcd} & U \cap V \arrow[r, "y"] \arrow[d, "x"'] & y(U \cap V) \subseteq \mathbb{R}^n \\ \mathbb{R}^n \supseteq x(U \cap V) \arrow[rru, "x \circ y^{-1}"] & & \end{tikzcd}$$

We say that the two charts (x, U) and (y, V) are C^∞*-related* if the map $x \circ y^{-1}$ and the map $y \circ x^{-1}$ are C^∞-maps as maps between subsets of $\mathbb{R}^n$.

Furthermore, we say that a set of charts $\{(x_i, U_i)\}_{i \in I}$ is an C^∞*-atlas* for M if all the x_i, x_j are C^∞-related and the union $\bigcup_i U_i$ cover M.

Here a uniqueness result:

Lemma 2.1. *Given an atlas $\mathscr{A}$ on M, there exists a unique maximal atlas $\mathscr{A}'$ containing $\mathscr{A}$.*

Proof. Define $\mathscr{A}$ to be the set of all charts y which are C^∞-related to all charts $x \in A$. Then $\mathscr{A}$ contains two types of charts: those who belonged to $\mathscr{A}$, and possibly new ones. Every two charts in $\mathscr{A}$ are C^∞-related, by definition of atlas, and every pair with one in each are C^∞-related by definition of $\mathscr{A}'$. Further, evert pair of charts with both in $\mathscr{A}'$ are C^∞-related: one can intersect their domains with a chart in $\mathscr{A}$, compose, and conclude.

Clearly $\mathscr{A}'$ is the unique maximal atlas containing $\mathscr{A}$. □

Thus:

Definition 2.2. A *differentiable manifold* is a pair $(M, \mathscr{A})$, where M is a topological manifold, and $\mathscr{A}$ is a maximal atlas (of C^∞-related transition functions). ■

From now on, when we say "manifold", we will always mean "differentiable manifold". Note that it is defined as a *pair*, and we cannot forget the atlas, because there are, famously, homeomorphic topological manifolds with different differentiable structures (for example, Milnor's "exotic spheres", the first one being the 28 different differentiable structures on S^7).

Here's an example (and probably the only explicit calculation we will do in the course):

Example 2.3. Recall that S^2 (the zero set of the equation $x_1^2+x_2^2+x_3^2=1$ in $\mathbb{R}^3$) can be covered by two charts, by stereographic projection (see Figure 1), by first projecting from the north pole, and then from the south pole. Call these maps φ and ψ, respectively. We are going to compute the transition maps, to verify that this atlas is actually a C^∞-atlas.

We need to find explicit formulae for φ and ψ. The first map has domain $\mathcal{S}^2\backslash\{N\}$ and image $\mathbb{R}^2$. But we want to compute $\psi\circ\varphi^{-1}$, so we start with a point (a,b) in the plane $\mathbb{R}^2$ (embedded in $\mathbb{R}^3$ by $z=0$). We want to connect it with the north pole $(0,0,1)$ by a line, and compute its intersection with the 2-sphere. The line is given parametrically as

$$t(a,b,0)+(1-t)(0,0,1)=(at,bt,1-t).$$

To find the intersection with the sphere, we must compute when the right hand side has norm 1:

$$a^2t^2+b^2t^2+1-2t+t^2=1$$

Getting rid of the 1 and cancelling the t, we end up with the condition

$$t=\frac{2}{1+\|a\|},$$

where $\|a\|=\sqrt{a^2+b^2}$. In other words, explicitly, we have

$$\varphi^{-1}(a,b)=\left(\frac{2a}{1+\|a\|},\frac{2b}{1+\|a\|},\frac{\|a\|-1}{1+\|a\|}\right).$$

To compute ψ, one start with a point on the sphere, and find a formula for the the intersection with the x,y-plane. So let (x,y,z) be on the sphere. Then $\psi(x,y,z)$ is on the line from S (the south pole) to the plane $z=0$. This line is parametrically given by

$$t(x,y,z)+(1-t)(0,0,-1)=(tx,ty,tz-1+t).$$

The z-coordinate is zero when $t=\frac{1}{z+1}$. Thus

$$\psi(x,y,z)=\left(\frac{x}{z+1},\frac{y}{z+1}\right)$$

Finally, we can compute the composition $\psi\circ\varphi^{-1}$:

$$\begin{aligned}
\psi\circ\varphi^{-1}(a,b) &= \psi\left(\frac{2a}{1+\|a\|},\frac{2b}{1+\|a\|},\frac{\|a\|-1}{1+\|a\|}\right)\\
&= \left(\frac{2a}{1+\|a\|}\Big/\frac{2\|a\|}{1+\|a\|},\frac{2b}{1+\|a\|}\Big/\frac{2\|a\|}{1+\|a\|}\right)\\
&= \left(\frac{a}{\|a\|},\frac{b}{\|a\|}\right).
\end{aligned}$$

So the transition maps are just inversion about the origin in the (a,b)-plane. These are clearly C^∞ functions. ★

The moral of the story is that we could just as well defined S^2 as two copies of $\mathbb{R}^2\{0\}$ glued together with the inversion map, because that's exactly how the differentiable structure is defined, and that is all we care about.

Here's another example:

Example 2.4. Recall that we could write the projective plane $\mathbb{P}^2_{\mathbb{R}}$ as the union of three open sets U_i, defined by the non-vanishing of one of the homogeneous coordinates. Here one computes (easily) that the transition maps $\varphi_1\circ\varphi_0^{-1}:\mathbb{R}^2_{(a,b)}\backslash\{a=0\}\to\mathbb{R}^2_{(a,b)}\backslash\{a=0\}$ are given by

$$(a,b)\mapsto\left(\frac{1}{a},\frac{b}{a}\right).$$

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Now we have defined (and become somewhat familiar with) the objects we're going to work it. It is time to define *maps* between them.

Definition 2.5. Let M^n, N^m be manifolds (of dimension n, m, respectively), and let $f: M\to N$ be a continous map between them. We say that f is *differentiable at p* if for all charts (x,U) with $p\in U$ and for all charts (y,V) with $f(p)\in V$, the function $y\circ f\big|_U\circ x^{-1}$ is differentiable (as a function between open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$).

$$\begin{tikzcd}
& U \arrow[r, "f_{|U}"] & V \arrow[dr, "y"] & \\
x(U) \arrow[ur, "x^{-1}"] \arrow[rrr, "y\circ f\circ x^{-1}"'] & & & y(V)
\end{tikzcd}$$

If f is differentiable at all points $p\in M$, then we say that f is *differentiable*. ■

Thus we have a category Diff of differentiable manifolds. Its objects are the differentiable manifolds and the maps are differentiable maps. This gives us at once the notion of an *isomorphism* of manifolds: it is just a pair of differentiable maps that compose to the identity in each direction.

We want to define some properties of maps in Diff. To start off, we introduce some notation. Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable map, then we define

$$D_i f(a) = \lim_{h \to 0} \frac{f(a^1, \ldots, a^i + h, \ldots, a^n)}{h}.$$

This is nothing but the i'th partial derivative. The reason we're not using the usual Leibniz notation is because we want to reserve it for something else. Having established the D_i notation, one recalls the chain rule: Given two maps $g : \mathbb{R}^m \to R^n$ and $f : \mathbb{R}^n \to \mathbb{R}$, the D_j of the composite is given as

$$D_j(f \circ g)(a) = \sum_{i=1}^{n} D_i\left(f\left(g\left(a\right)\right)\right) \cdot D_j\left(g^i\left(a\right)\right)$$

The classical Leibniz notation will be reserved for tangent fields on manifolds. So let $f : M \to \mathbb{R}$ be a map from a manifold to the real line, and let (x, U) be a chart, and let $p \in U$. Then we define

$$\frac{\partial f}{\partial x^i}(p) = \left.\frac{\partial f}{\partial x^i}\right|_p := D_i(f \circ x^{-1})(x(p))$$

Thus $\left.\frac{\partial f}{\partial x^i}\right|_p$ measures the rate of change of the function $f : M \to \mathbb{R}$ *with respect to* the coordinate system (x, U).

Thus we may ask: what happens if we change charts?

Proposition 2.6. *Let $f : M \to \mathbb{R}$ be a map and let $(x, U), (y, V)$ be two overlapping coordinate charts. Then*

$$\frac{\partial f}{\partial y^i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} \frac{\partial x^j}{\partial y^i}.$$

Or in matrix notation:

$$\left[\frac{\partial f}{\partial y^i}\right]_{i=1\ldots n} = \left[\frac{\partial x^j}{\partial y^i}\right]_{i=1\ldots n, j=1\ldots n} \left[\frac{\partial f}{\partial x^j}\right]_{i=1\ldots n}.$$

Proof. For now, see Spivak. $\square$

Remark. *From now on, all maps will be differentiable, unless otherwise stated. Thus, when referring to a "map" above, it should be read as "differentiable map" (or better, "a map in the category* Diff*").*

Let $f : M \to N$ be a map of manifolds, and let $(x, U), (y, V)$ be charts on M, N, respectively. One can define the *Jacobian matrix of f at p* (if $p \in U$):

$$J_f(p) := \left[\left. \frac{\partial(y^i \circ f)}{\partial x^j} \right|_p \right].$$

Then we define the *rank* of $f : M \to N$ to be the rank of the Jacobian matrix.

Proposition 2.7. *The rank is well-defined.*

Proof. **Exercise:** Suppose (x', U') is another chart containing p, and (y', V') is another chart containing $f(p)$. Then by restricting both x and x' to $U \cap U'$, we can assume that $U = U'$ and $V = V'$. Then the rank of f at p is both the rank of the Jacobian

$$\left[\left. \frac{\partial(y^i \circ f)}{\partial x^j} \right|_p \right]$$

and the Jacobian

$$\left[\left. \frac{\partial((y')^i \circ f)}{\partial (x')^j} \right|_p \right].$$

Consider the diagram:

$$\begin{tikzcd}
& M \arrow[r, "f"] & N & \\
& U \arrow[u, hook] \arrow[r, "f|_U"] \arrow[dl, "x'"'] \arrow[d, "x"] & V \arrow[u, hook] \arrow[d, "y"'] \arrow[dr, "y'"] & \\
\mathbb{R}^n \arrow[r, "x \circ x'^{-1}"'] & \mathbb{R}^n \arrow[r, "y \circ f \circ x^{-1}"] & \mathbb{R}^n & \mathbb{R}^n \arrow[l, "y \circ y'^{-1}"]
\end{tikzcd}$$

Then xx'^{-1}, yfx^{-1}, and yy'^{-1} are maps from (open subsets of) $\mathbb{R}^n$ to $\mathbb{R}^n$. Consider the composition $(y'y^{-1}) \circ (yfx^{-1}) \circ (xx'^{-1}) = y'fx'^{-1}$. Applying the chain rule, we get that the second Jacobian matrix is just a conjugate of the first Jacobian by invertible matrices. These have the same rank. $\square$

We say that a point $p \in M$ is *critical for f* if $\operatorname{rank} f < m$ (the dimension of N). Otherwise it is *regular*. We say that a point $q \in N$ is regular if all the points in the preimage $f^{-1}(q)$ are regular points.

Example 2.8. Let $M = N = \mathbb{R}$ and let $f : M \to N$. Then a point $p \in \mathbb{R}$ is regular if and only if $f'(p) \neq 0$. In other words, critical points correspond to either maxima, minima or plateaus of the graph of f inside $\mathbb{R}^2$.

One can also think of critical points as points where the inverse function theorem fails (which is the topic of the next lecture). See Figure 4. ★

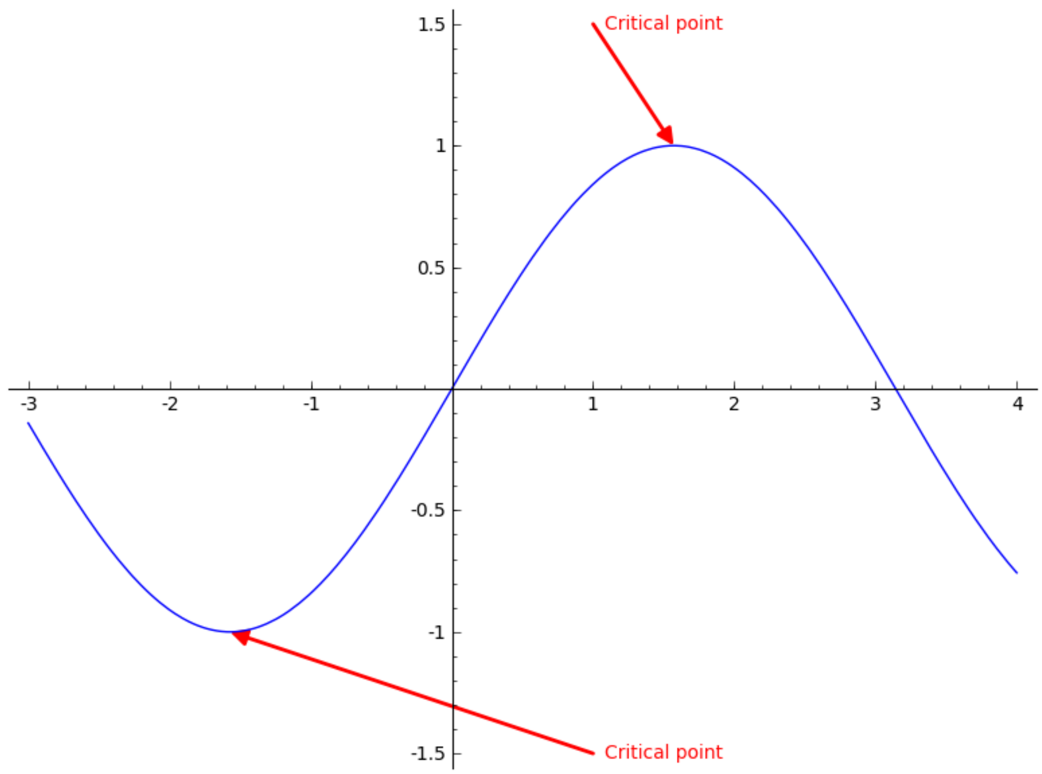


Figure 4: Critical points of the sine function.

3 Sard's theorem and the Inverse Function Theorem

Last time we spoke about critical points, i.e. those points where the Jacobian of a map had non-maximal rank (the maximal rank of a map $\mathbb{R}^n \to \mathbb{R}^m$ is n). Sard's theorem says that most points $p \in M$ are non-critical (=regular).

Think of the case $f : \mathbb{R}^2 \to \mathbb{R}$. The graph of such a function is a surface in $\mathbb{R}^3$. In this case, the theorem says that the maxima of this function constitute no "volume".

To be able to state the theorem, we have to present some definitions, the first one being, that of "measure 0". We start with a subset $A \subset \mathbb{R}^n$. We say that it has *measure 0* if it can be covered by a sequence of (open) rectangles R_{ab} in such a way that

$$\sum_{i=1}^{n} \mathrm{vol}(R_{ab}) < \epsilon$$

for every $\epsilon > 0$. Here's an example:

Example 3.1. Consider the real line $\{y = 0\} \subseteq \mathbb{R}^2$. Intuitively, this has zero volume, *as a subset of* $\mathbb{R}^2$. Let $B_{an,\frac{1}{n^3}}$ be the rectangle centered at origo with width n and height $\frac{1}{n^3}$. Since the height is finite, clearly the union of all these rectangles (for $n \in \mathbb{N}$) cover the real line. The sum of the volumes is

$$\sum_{i=1}^{\infty} \frac{a}{n^2} = \frac{a\pi^2}{6},$$

so choosing $a < \frac{6\epsilon}{\pi^2}$, gives total volume less than ϵ. ★

Now, a subset A of a manifold M has *measure 0* if there exists a countable sequence of charts (x_i, U_i) with $A \subseteq \bigcup U_i$ such that each $x_i(A \cap U_i)$ has measure 0. To see that this notion is well-defined, one turns to Lemma 6 in Spivak, which says that smooth functions take measure 0 sets to measure 0 sets. [[perhaps a proof will come here later]]

Now, Sard's theorem says the following:

Theorem 3.2. *If $f : M \to N$ is a C^∞-map of n-manifolds, and M has at most countably many components (e.g. one), then the critical values of f form a set of measure zero in N.*

This, in some sense, is similar to a theorem in algebraic geometry, which says that the smooth points of a variety are dense in the Zariski topology.

We will not prove Sard's theorem here, but just note that it is at least intuitively plausable.

3.1 The Inverse Function Theorem

We are used to thinking about the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ as a linear approximation to f near a point p, i.e. a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$. Linear maps live in *vector spaces* and not on manifolds, so we should really think of the derivative as some kind of *functor* taking maps $f : U \to V$ (where U, V are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively) to *vector space maps* $Df(p) : T(\mathbb{R}^n) \to T(\mathbb{R}^m)$, where the T means that we are thinking of just the *vector space* $\mathbb{R}^n$.

For a map $f : \mathbb{R}^n \to \mathbb{R}^m$ and a point $p \in \mathbb{R}^n$, we define the vector space map $Df(p)$ as follows:

Definition 3.3. If it exists, the map $Df(p) : T_p(\mathbb{R}^n) \to T_{f(p)}(\mathbb{R}^m)$ is the unique linear map satisfying

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - D_p f(h)|}{|h|} = 0$$

If we choose the standard coordinate systems on $\mathbb{R}^n$ and $\mathbb{R}^m$, then $D_p f(p)$ has a matrix $\left[D_j f^i(a)\right]$ of partial derivatives (the Jacobian). ■

In particular, the chain rule takes a very nice form with this notation. It just says that given two maps, the derivative of the composition is just the composition of the derivatives:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & \mathbb{R}^m \arrow[r, "g"] & \mathbb{R}^p \end{tikzcd}$$

This gives that

$$\begin{tikzcd} T_p(\mathbb{R}^n) \arrow[r, "D_p f"] \arrow[rr, bend right, "D_p(g\circ f)"'] & \mathbb{R}^m \arrow[r, "D_{f(p)} g"] & \mathbb{R}^p \end{tikzcd}$$

is commutative.

Now we state the inverse function theorem:

Theorem 3.4. *Suppose $p \in U \subseteq \mathbb{R}^n$ and that $f : U \to \mathbb{R}^n$ is a smooth function. Suppose further that the Jacobian is non-singular at p, i.e. that p is a regular value of f (or equivalently that $D_f(p)$ is an isomorphism of vector spaces).*

Then there exists a neighbourhood V of p such that $f\big|_V$ is invertible and such that $f\big|_V^{-1}$ is smooth. Moreover, the following equality holds:

$$J_{f\big|_V^{-1}}(f(p)) = J_f(p)^{-1}.$$

In words (that will have meaning later), this says that if the tangent map is an isomorphism at a point p, then f is locally a diffeomorphism near p. This is a theorem special to differential geometry. Similar situations does not occur in algebraic or complex geometry, because inverse functions are usually not algebraic/complex.

Proof. We skip this for now. It is in the MAT1110 book. $\square$

4 Rank theorems

In this lecture we will elaborate on some "structure theorems" on maps $f : M \to N$, given restrictions on the rank of f, the first one being the following:

Theorem 4.1. *Let $f : M^n \to N^m$ be a map of manifolds.*

1. *If $f : M^n \to N^m$ has rank k at p, then there charts (x, U) and (y, V) containing p and $f(p)$, respectively, such that*

$$y \circ f \circ x^{-1}(a) = \Big(a^1, \ldots, a^k, \psi^{k+1}(a), \ldots, \psi^m(a)\Big)$$

for all $a \in U$.

2. *If furthermore, $f : M^n \to N^m$ has rank k in a neighbourhood of p, then there are charts (x, U) and (y, V) containing p and $f(p)$, respectively, such that*

$$y \circ f \circ x^{-1}(a) = \Big(a^1, \ldots, a^k, 0, \ldots, 0\Big)$$

for all $a \in U$.

Proof. 1. Start by choosing any coordinate system (u, U') around p. Since the rank of the Jacobian $\left[\frac{\partial(y^\alpha \circ f)}{\partial u^\beta}\Big|_p\right]$ is k at p, there is some $k \times k$-minor that is non-zero. So, after permuting coordinates, we can assume that this minor is the upper-left block of the Jacobian.

We define new local coordinates as follows:

$$\begin{aligned} x^\alpha &= y^\alpha \circ f && \text{for } \alpha = 1, \ldots, k \\ x^r &= u^r && \text{for } r = k+1, \ldots, n. \end{aligned}$$

Now consider the change of basis matrix:

$$\left[\frac{\partial x^i}{\partial u^j}\Big|_p\right] = \begin{bmatrix} \frac{\partial(y^\alpha \circ f)}{\partial u^\beta} & & \text{X} & \\ & 1 & & \\ 0 & & 1 & \\ & & & 1 \end{bmatrix}$$

Since the upper corner has determinant non-zero , the whole has determinant non-zero. Hence, by the inverse function theorem, (x, U) is a diffeomorphism in a neighbourhood U of p. Thus $x = (x \circ u^{-1}) \circ u$ is a coordinate system near p, and in fact it is the coordinate system we want:

$$
\begin{aligned}
y \circ f \circ x^{-1}(a^1, \dots, a^n) &= y \circ f \left(f^{\alpha,-1}(y^{\alpha,-1}(a^1), \dots, u^{r-1}(a^n) \right) \\
&= \left(a^1, \dots, a^k, ?, \dots, ? \right).
\end{aligned}
$$

Where the questions marke denote $u^{-1}(a^r)$, which we don't care about. What is important, is that the coordinates have the desired form.

2. Start by choosing coordinate systems (x, U) and (v, V') as in 1). Since the rank of f is k in a neighbourhood, the all of the components of the lower right rectangle of the matrix

$$
\left[\frac{\partial (v^i \circ f)}{\partial x^j} \right] = \begin{bmatrix} 1 & & & & \\ & 1 & & \mathbf{0} & \\ & & 1 & & \\ & & D_{k+1}\psi^{k+1} & \cdots & D_n \psi^{k+1} \\ & \mathrm{X} & \vdots & & \vdots \\ & & D_{k+1}\psi^m & \cdots & D_n \psi^m \end{bmatrix}
$$

must vanish in a neighbourhood of p. In particular, this means that ψ is a function only of the first k coordinates of a, i.e. that it is constant along the last $n-k$ coordinates.

Now define new local coordinates on N by letting

$$
\begin{aligned}
y^\alpha &= v^\alpha & \text{for } r = 1, \dots, k \\
y^r &= v^r - \psi^r \circ (v^1, \dots, v^k) & \text{for } r = k+1, \dots, m.
\end{aligned}
$$

Notice that the last line makes sense because ψ only depends on the first k coordinates. It is easy to see that the change of base matrix has non-zero Jacobian at $v(q)$, so (y, V) is actually a coordinate system, where V is a neighbourhood of $f(p)$.

Moreover:

$$
\begin{aligned}
y \circ f \circ x^{-1}(a^1, \dots, a^n) &= y \circ v^{-1} \circ v \circ f \circ x^{-1}(a^1, \dots, a^n) \\
&= y \circ v^{-1} \left(a^1, \dots, a^k, \psi^{k+1}(a), \dots, \psi^m(a) \right) \\
&= \left(a^1, \dots, a^k, 0, \dots, 0 \right)
\end{aligned}
$$

The first equality is by part 1), and the second follows by definition of y.

□

Thus if the rank is constant in a neighbourhood, the theorem says that the map locally looks like an inclusion of the first k coordinates in $\mathbb{R}^m$.

If the rank is maximal, the theorem says even more:

Theorem 4.2. *1. If $m \leq n$ and $f : M^n \to N^m$ has rank m at p, then for any coordinate system (y, V) around $f(p)$, there is some coordinate system (x, U) around p with*

$$y \circ f \circ x^{-1}(a^1, \ldots, a^n) = \left(a^1, \ldots, a^m\right).$$

2. If $n \leq m$ and $f : M^n \to N^m$ has rank n at p, then for any coordinate system (x, U) around p, there is a coordinate system (y, V) around $f(p)$ with

$$y \circ f \circ x^{-1}(a^1, \ldots, a^n) = \left(a^1, \ldots, a^n, 0, \ldots, 0\right).$$

Proof. Part 1 is just the previous theorem. So we concentrate on part 2. [[comes later]] $\square$

5 Embeddings

What does it mean to put one manifold into another manifold? It turns out that this isn't completely trivial to define.

First off, we want the embedding to respect the smooth structure in some sense. This is achieved by requiring that the map has full rank everywhere:

Definition 5.1. A map $f : M^n \to N^m$ for $m \geq n$ is an *immersion* if $\operatorname{rank} f = n$ everywhere. ■

Example 5.2. Consider the zero set of the equation $y^2 = x^3$, or equivalently, the image of the map $f(t) = (t^2, t^3)$. It is injective everywhere, but it is not an immersion, because the rank is zero at the origin (because the derivative is $(2t, 3t^2)$ which is zero for $t = 0$). See Figure 5. ★

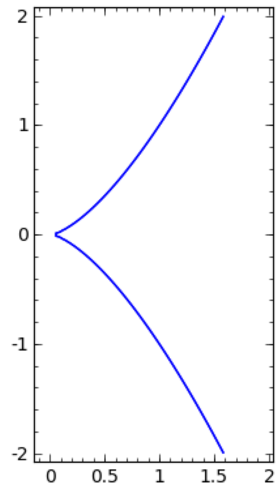


Figure 5: A cusp.

Example 5.3. Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ be parametrized by $\big(t^2 - 1, (t^2 - 1)t\big)$. It is a nodal curve, and its derivative is $\alpha'(t) = \big(2t, 3t^2 - 1\big)$, which is nowhere zero. But it has self-intersections, and we do not want to allow that. See Figure 6. ★

Example 5.4. Another example is given by embedding a curve into the torus with irrational slope: $\alpha(t) = \left(e^{i\sqrt{2}t}, e^{i\sqrt{2}t}\right)$.

This is an immersion, but the image is *dense* in $S^1 \times S^1$! ★

These problems "usually" vanish with the following definition:

Definition 5.5. An *embedding* is a map that is both an immersion and a homeomorphism onto its image. ■

We call the image of an embedding a *submanifold*.

Remark. *This is a sensible definition: it uses both the topological* and *the differentiable structure of the manifolds, whereas the first definition used only the differentiable structure. A similar situation occurs in algebraic geometry, where a map of schemes is defined not just as a map of the underlying topological spaces, but as a* pair $(f, f^\#)$, *where the first is a map of topological spaces, and the second is map of sheaves* $f^\# : f^*\mathcal{O}_Y \to \mathcal{O}_X$.

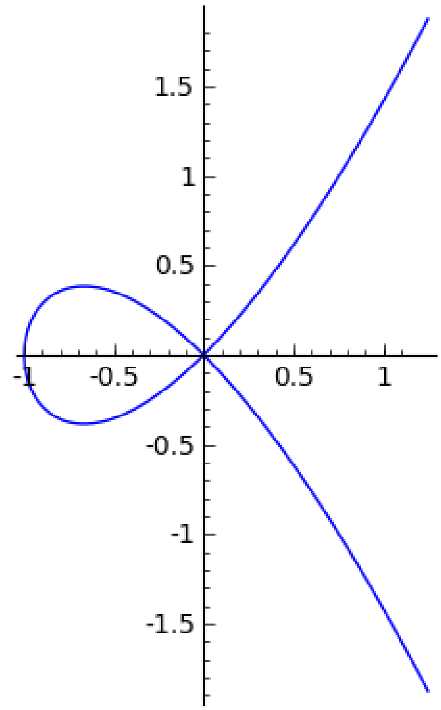


Figure 6: A node

Here's an **exercise**:

Proposition 5.6. *If $f : M \to N$ has constant rank k in a neighbourhood of $f^{-1}(y)$, then $f^{-1}(y)$ is a closed submanifold of M.*

Proof. Since f has rank k in a neighbourhood U of $Z := f^{-1}(y)$, we can cover U by smaller open sets U_α such that $f\big|_{U_\alpha}$ are diffeomorphisms with subsets of N in a such way that $Z \cap U_\alpha$ is just the span of the first k standard basis vectors. [„„] $\square$

Example 5.7. Consider Figure 7. If we define a map $F : \mathbb{R}^2 \to \mathbb{R}$ by $(x, y) \mapsto x^2 - y^2$, we get a function whose graph has a saddle point. As long as we're looking at $F^{-1}(a)$ for a non-zero, the inverse image is a hyperbola, which is a smooth (disconnected) manifold. However, when $a = 0$, the fiber (i.e. the contour line) is a pair of double lines, which is not even a manifold. The reason is that the rank of F is zero at $(0, 0)$. ★

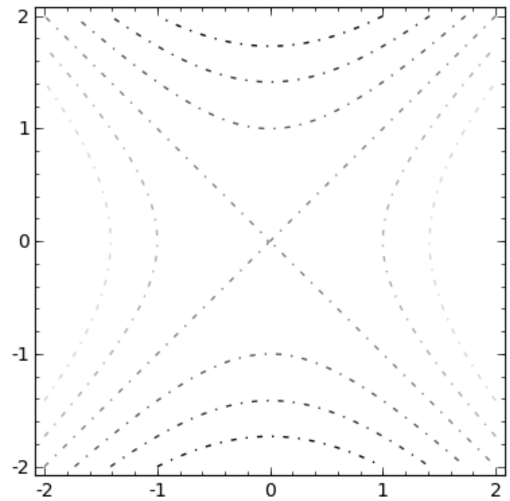


Figure 7: The countour lines of the function $F(x, y) = x^2 - y^2$.

Example 5.8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the map $f(x, y) = x^2 + y^2 - 1$. It has constant rank 1 as long as we're away from the origin. Then the theorem says that $f^{-1}(0) = \{x^2 + y^2 = 1\} \approx S^1$ is a closed submanifold av $\mathbb{R}^2$. ★

6 Useful functions and partitions of unity

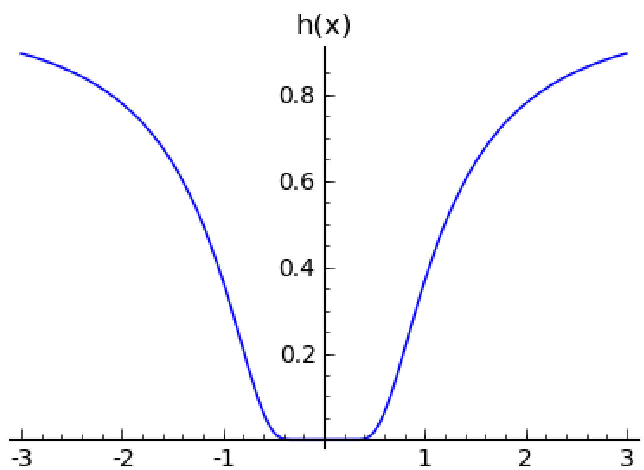
In this section we introduce some useful C^∞-functions that will be useful later on. They are in some sense the secret ingredient of differential geometry, as their existence allows for many constructions that are completely unthinkable in other categories, such as the analytic or algebraic category.

6.1 Useful functions

Recall that the *support* of a function $f : X \to \mathbb{R}$ is the closure of the set $\{x \mid f(x) \neq 0\}$.

Example 6.1. The first function is an example of a non-zero function whose Taylor series around any point is zero (thus it is an non-analytical function).

$$h(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0. \end{cases}$$



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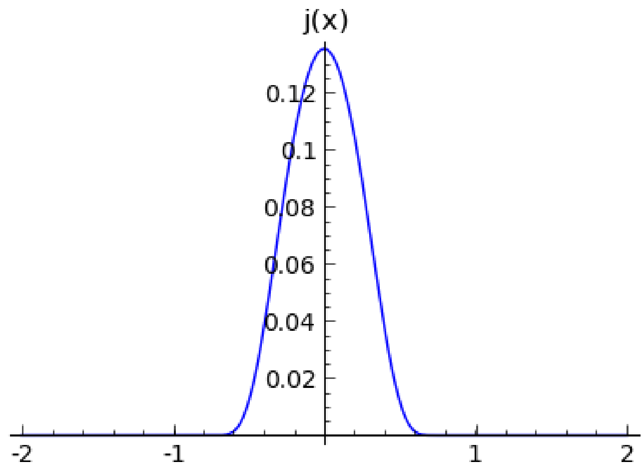
Example 6.2. This is an example of a function whose support is $[-1, 1]$, but is zero everywhere else.

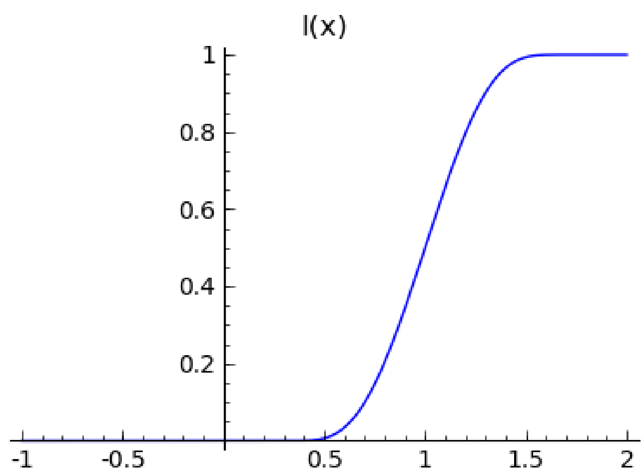
$$j(x) = \begin{cases} e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}} & x \in (-1, 1) \\ 0 & x \notin (-1, 1) \end{cases}$$



Note that this function is, roughly, the same as $h(x-1) \cdot h(x+1)$ inside $(-1, 1)$. By composing with a linear change of coordinates, we get a function which is positive on $(0, \delta)$ and 0 elsewhere. ★

Example 6.3. There is a function $l : \mathbb{R} \to \mathbb{R}$ which is zero for $x \leq 0$, strictly increasing on $(0, \delta)$, and equal to 1 for $x \geq \delta$:

$$l(x) = \frac{\int_0^x k(x)\mathrm{dx}}{\int_0^\delta k(x)\mathrm{dx}}.$$



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Example 6.4. By mirroring the function $l(x)$ from the previous example around $x = \delta + 1$, we get a function $f(x)$ which is constantly equal to 1 on $(\delta, \delta + 1)$, and has support $[0, frm{-}e\delta + 1]$. By affine transformations, the function can be made to be 1 on any bounded interval K and support on any interval containing K. ★

Example 6.5. There is a function $g : \mathbb{R}^n \to \mathbb{R}$ which is positive on the open square $(-\epsilon, \epsilon) \times \cdots \times (-\epsilon, \epsilon)$ and zero elsewhere:

$$g(x) = \prod_{i=1}^{n} j(x^i/\epsilon).$$

★

Example 6.6. Generalizing Example 6.4, by defining $\phi : \mathbb{R}^n \to \mathbb{R}$ as

$$\phi(x) = \prod_{i=1}^{n} f(x^i),$$

we get a function which is constantly equal to one on a closed, bounded square, and has support in a slightly larger square. ★

The last example can be generalized to general manifolds:

Proposition 6.7. *Let M be a smooth manifold and K a compact subset of M and U an open subset containing K. Then there exists a smooth function $\beta : M \to [0, 1]$ that is constant equal to 1 on K and compact support contained in U.*

Proof. We first do the case $M = \mathbb{R}^n$. In this case, K is compact, so it is closed and bounded. For each $p \in K$, let U_p be an open square of radius ϵ_p centered at p and contained in U. The set of all these U_p is an open cover of K, and since K is compact, we can choose finitely many such p. By translation, the function in the last example, can be made such that f_p is positive in the interior of U_p and zero outside (so its support is $\bar{U_p}$), and constant 1 on $K \cap U_p$. Now define the following function:

$$\beta(x) = 1 - \prod_{p} (1 - f_p(x)),$$

where the product ranges over those finitely many p needed to cover K. Then, if $x \in K$, x is contained in one of the $U_p \cap K$, hence $\beta(x) = 1$. The support is clearly bounded, hence compact.

Now to the general case. If K is contained in a single chart, we are done by the above. If not, K is contained in finitely many charts (U_i, x_i), hence we can find compact sets $K_1, \cdots, K_k$ with $K \subset \bigcup_{i=1}^k K_i$, $K_i \subset U_i$ and $\bigcup_i U_i \subset U$. Let ϕ_i be identically 1 on K_i and zero on $M \backslash U_i$. Then define

$$\beta(x) = 1 - \prod_{i=1}^{k}(1 - \phi_i(x)).$$

□

6.2 Partition of unity

Partitions of unity is an extremely important tool, but to define it, we need some technical definitions. To motivate all this, we will note that when we are done, we will be able to prove that any compact manifold can be embedded into real Euclidean space $\mathbb{R}^N$ for some large N.

Definition 6.8. We say that a family $\mathscr{U}$ of open sets is an *open cover* of M if

$$\bigcup_{U \in \mathscr{U}} U = M.$$

■

Definition 6.9. We say that $\mathscr{U}'$ is a *refinement* of $\mathscr{U}$ if for all $U \in \mathscr{U}'$, there exists some $V \in \mathcal{O}$ with $U \subseteq V$. ■

Example 6.10. Cover $\mathbb{R}$ with the single open set $\mathbb{R}$, so that $\mathscr{U} = \{\mathbb{R}\}$. Now consider the cover given by $\mathscr{U}' = \{(-\infty, 1), (-1, \infty)\}$. Then $\mathscr{U}'$ is a refinement of $\mathscr{U}$.

Let $\mathscr{U}'' = \{(-2, 2), (0, \infty), (-\infty, 0)\}$. Then $\mathscr{U}''$ is *not* a refinement of $\mathscr{U}'$. ★

Definition 6.11. We say that an open cover $\mathscr{U}$ is *locally finite* if for every $p \in M$, there are only finitely open sets U in $\mathscr{U}$ with $\{p\} \cap U \neq \emptyset$. ■

Theorem 6.12. *If $\mathscr{U}$ is an open cover of a connected manifold M, then there exists a locally finite refinement $\mathscr{U}'$ of $\mathscr{U}$.*

Moreover, we can choose these $\mathscr{U}'$ in such a way that $\mathscr{U}' \approx \mathbb{R}^n$ as differentiable manifolds.

Remark. *We skip the proof. Now it's time to note that we are hiding details under carpets: for this theorem to be true, we must assume that M is σ-compact, meaning that we can write M as a countable union of compact* *subsets. In particular, this is clearly true if M itself is compact. Note also that all subsets of $\mathbb{R}^n$ have this property.*

Theorem 6.13. *Let $\mathscr{U}$ be an open cover of M. Then it is possible to choose for each $U \in \mathscr{U}$ an open set U' with $\bar{U}'subsetU$ in such a way that the new collection $\{U'\}$ is also an open cover M.*

Remark. *In particular, this new open cover is a refinement of $\mathscr{U}$.*

Theorem 6.14 (Existence of partitions of unity)**.** *Let $\mathscr{U}$ be an open locally finite cover of a manifold M. Then there is a collection of C^∞-functions $\varphi_U : M \to [0,1]$ for each $U \in \mathscr{U}$ such that*

1. $\operatorname{supp} \varphi_U \subset U$ *for each U.*
2. $\sum_{U \in \mathscr{U}} \varphi_U(p) = 1$ *for all $p \in M$. (this makes sense because the cover is locally finite!)*

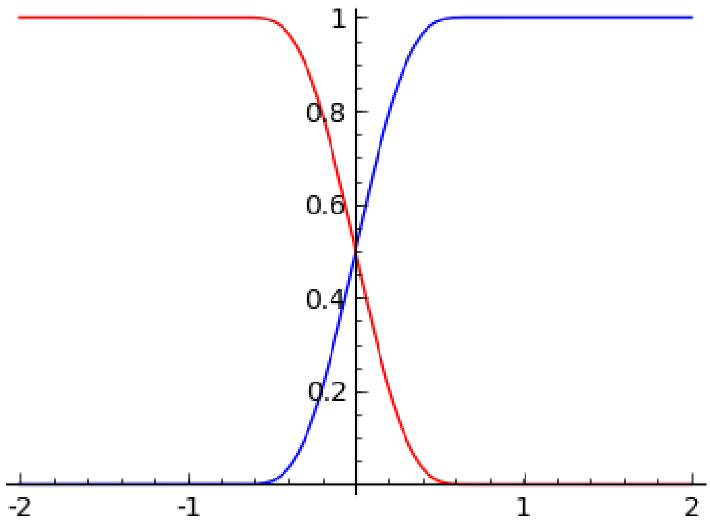
You should think of this theorem as saying that open covers of a manifold can be separated by continous functions. (?)

Definition 6.15. A collection $\{\varphi_i : M \to [0,1]\}$ is called a *partition of unity* if (1) the collection $\{p \,|\, \varphi_i(p) \neq 0\}$ is locally finite and (2) if $\sum_i \varphi_i(p) = 1$ for all $p \in M$.

If for each i there is an $U \in \mathscr{U}$ such that $\operatorname{supp} \varphi_i \subset U$, then we say that the collection is *subordinate to $\mathscr{U}$*. ■

So the theorem says that given a locally finite cover $\mathscr{U}$ of a manifold M, there exists a partition of unity subordinate to $\mathscr{U}$.

Example 6.16. Let $M = \mathbb{R}$ and cover M by $U_1 = (-1, \infty)$ and $U_2 = (-\infty, 1)$. Let $\varphi_1(x) = \frac{l(x+1)}{l(x+1)+l(-x-1)}$. And similarly, $\varphi_2(x) = \frac{l(-x+1)}{l(x+1)+l(-x+1)}$.



Then the partition of unity looks like the graph above. Notice that it sums to 1 everywhere. ★

Theorem 6.17. *Let M^n be a compact manifold. Then there exists an embedding $f : M \to R^N$ for some N.*

Remark. *Note that this embedding is highly non-canonical. It contains several layers of choices. The proof begins by* choosing *a finite open cover, and then* choosing *a refinement, and then choosing a partition of unity. None of these choices are natural in any sense.*

7 The tangent bundle

Think of a surface $S \subseteq \mathbb{R}^3$ and let p be a point on S. Then the tangent plane of S is a linear subspace of $\mathbb{R}^3$. Thus, in the case of embedded manifold, it is easy to assign a "tangent space" to each point $p \in S$. However, for general manifolds, we are not given an embedding anywhere. What we want is some rule that assigns to each point $p \in S$ a vector space of the same dimension as S, called the "tangent space". Also, we want this rule to reflect the global structure of S in a natural way.

More formally, we want a functor from manifolds to vector bundles, such that its restriction to chart domains, the result is just $\mathbb{R}^n$, as a vector space. To carry this out, we first formally introduce the "tangent space" of R^n (or open subsets thereof).

We define $T(\mathbb{R}^n) := \mathbb{R}^n \times \mathbb{R}^n$, and we write its elements as (p, v) or v_p, where we think of the left factor as the *manifold* $\mathbb{R}^n$, and the right factor as the *vector space* $\mathbb{R}^n$. Thus, to each *point* p in $\mathbb{R}^n$, we attach a *vector* v_p.

So far so good. Now we have defined the functor T on objects. Now we define it on morphisms, so let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function. Then we define $Tf : T(\mathbb{R}^n) \to T(\mathbb{R}^m)$ by $Tf(p, v) = (f(p), Df(p)(v))$. By the chain rule, this is really a functor, because if $g : \mathbb{R}^m \to \mathbb{R}^k$ is another smooth function, we have

$$\begin{aligned}
T(g) \circ T(f)(p, v) &= T(g)(f(p), Df(p)(v)) \\
&= (g(f(p)), Dg(f(p))(Df(p)(v))) \\
&= (g \circ f(p), D(g \circ f)(v)) \\
&= T(g \circ f).
\end{aligned}$$

To generalize this construction to general manifolds, we need to notion of a *vector bundle*:

Definition 7.1. A *vector bundle* over M is a map $\pi : E \to M$, where E is a manifold, such that for each $p \in M$, there is an open neighbourhood U, and a homeomorphism $e_q : \pi^{-1}(U) \to U \times \mathbb{R}^n$ in such a way that

$$t\big|_{\pi^{-1}(\{q\})} : \pi^{-1}(\{q\}) \to \{q\} \times \mathbb{R}^n$$

is a vector space isomorphism for all $q \in U$.

A *map of vector bundles* is just a commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd}$$

such that, when restricted to each fiber, the map $\tilde{f} : \pi_1^{-1}(p) \to \pi_2^{-1}(f(p))$ is a linear map. ■

Thus we have a category of vector bundles, which we shall denote by $\mathsf{VBundles}$. We have a natural map $p : \mathsf{VBundles} \to \mathsf{Diff}$ that sends a vector bundle to its base. The fiber $p^{-1}(M)$ is a category, called the category of *vector bundles over* M, which we shall often denote by $\mathsf{VBundles}(M)$.

We call a vector bundle E *trivial* if it is isomorphic to $M \times \mathbb{R}^n$ (with the obvious maps).

This really generalizes the $T(\mathbb{R}^n)$ above: it is trivally a vector bundle over $\mathbb{R}^n$, and it is easy to see that it restricts to a sub-vector bundle for any open subset $U \subset \mathbb{R}^n$.

Requring only that $T(U) = U \times \mathbb{R}^n$ for U an open subset of $\mathbb{R}^n$ and functoriality, there is a unique extension of T from open subsets of $\mathbb{R}^n$ to general manifolds.

Theorem 7.2. *Let M be a smooth manifold. Then there is a functor T : $\mathsf{Diff} \to \mathsf{VBundles}$ that to each manifold M associates a vector bundle over M in such a way that if (U, x) is a chart domain of M, we have $T(U) = U \times \mathbb{R}^n$.*

Proof. Here is a proof sketch. Cover M by open chart domains (x_i, U_i). Each of these are isomorphic to $\mathbb{R}^n$, so we define $TM\big|_{U_i} := x_i(U_i) \times \mathbb{R}^n$, with the obvious projection map. On the intersections $U_i \cap U_j$ we have a map $x_j \circ x_i^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, and we define

$$TM\big|_{U_i \cap U_j} \to TM\big|_{U_i \cap U_j}$$

by

$$(u, v) \mapsto \left(x_j \circ x_i^{-1}(u), D(x_j \circ x_i^{-1})(u)(v)\right).$$

That this is well-defined on triple overlaps should follow from the chain rule, and all these glueing maps make TM into a manifold. Local triviality is clear by construction. □

We call the vector bundle $\pi : TM \to M$ the *tangent bundle* of M.

Example 7.3. Consider the circle S^1. It is covered by two open sets $S^1\backslash N$ and $S^2\backslash S$. The map $\phi_1 : S^1\backslash N$ is given by $(x, y) \mapsto \frac{x}{1-y}$, and the inverse map is given by $\phi_1^{-1}(x) = \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\right)$. Similarly, $\phi_2(x, y) = \frac{x}{1+y}$, and the inverse is $\phi_2^{-1}(x) = \left(\frac{2x}{x^2+1}, -\frac{x^2-1}{x^2+1}\right)$. Thus $T(S^1)$ is covered by two open sets $U = T(R^1)$ and $V = T(R^1)$. The transition map ϕ_{UV} is given by $(x, v) \mapsto (\phi_2 \circ \phi_1^{-1}(x), D(\phi_2 \circ \phi_1^{-1})(x)(v))$, which we compute to be

$$(x, v) \mapsto \left(\frac{1}{x}, -\frac{v}{x^2}\right).$$

This is well-defined, since $x \in \mathbb{R}\backslash\{0\}$.

Then I claim that $TS^1 \approx S^1 \times \mathbb{R}^1$, that is, the tangent bundle of the circle is trivial. To do this, we cover the circle with the same open sets as above, so that S^1 is covered by the two open sets $U' \times \mathbb{R}$ and $V' \times \mathbb{R}$ with transition maps $\phi_{U'V'} : (x, v) \mapsto (\frac{1}{x}, v)$. This is truly a product, because the second factor does not interact with the first.

An isomorphism $TS^1 \approx S^1 \times \mathbb{R}^1$ is the same thing as maps from the charts from one manifold to the other that agrees on the overlaps. So we define, from $U \times \mathbb{R}$ to $U' \times \mathbb{R}$ a map given by $(x, v) \mapsto (x, v)$, and from $V \times \mathbb{R}$ to $V' \times \mathbb{R}$, a map given by $(x, v) \mapsto \left(x, -\frac{b}{a^2}\right)$. Then the reader can check that the following diagram commutes, and so we have defined a map on manifolds, that is an isomorphism on charts and agrees on overlaps, hence an isomorphism of manifolds:

$$\begin{tikzcd}
U \times \mathbb{R} \arrow[r, "\phi_{UV}"] \arrow[d, "(x,v)\mapsto(x,v)"'] & V \times R \arrow[d, "(x,v)\mapsto(x,-\frac{b}{a^2})"] \\
U' \times \mathbb{R} \arrow[r, "\phi_{U'V'}"'] & V' \times \mathbb{R}
\end{tikzcd}$$

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The construction of the tangent space here is the most intuitive one: locally it looks like $U \times \mathbb{R}^n$. However, there are other views one can take. One is to consider points of TM as equivalence classes of curves, as follows: let $[x, v]_p \in T_pM$. Then we say that this corresponds to a curve $x^{-1} \circ \gamma$ where γ is a curve in $\mathbb{R}^n$ with $\gamma'(0) = v$ (here x is a coordinate chart). It is not so difficult to see for every vector v in T_pM, there is a curve with derivative v. Taking equivalence classes resolves uniqueness.

The other view is to consider elements of T_pM to be *derivations*. To define these, let first $\mathcal{O}_p$ consist of all functions $f : U \to \mathbb{R}$ where U is a neighbourhood of p, where we consider $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ to be equal if there is a neighbourhood $W \subset U \cap W$ such that $f\big|_W = g\big|_V$. More formally, $\mathcal{O}_p$ is really the direct limit $\varinjlim \mathcal{O}(U)$, where $\mathcal{O}(U)$ is the sheaf of C^∞ functions $f : U \to \mathbb{R}$.

Then we claim that T_pM can be considered as the set of *derivations* $\ell : \mathcal{O}_p \to \mathcal{O}_p$. Formally, a derivation at p is a function from $\mathcal{O}_p$ to $\mathcal{O}_p$ that is linear and satisfies the Leibniz rule

$$\ell(fg)(p) = f(p)\ell(g)(p) + g(p)\ell(f)(p).$$

To see that derivations actually land in $\mathcal{O}_p$, we must show the following lemma:

Lemma 7.4. *If $f : W \to \mathbb{R}$ and $g : W \to R$ represent the same function in $\mathcal{O}_p$, and ℓ is a derivation, then $\ell(f) = \ell(g)$.*

Proof. Cleary we can assume $g = 0$. So we need to prove that if f is zero in a neighbourhood of p, then $\ell(f)$ is also zero in a neighbourhood of p.

Choose a C^∞ function $h : M \to \mathbb{R}$ with $h(p) = 1$ and support contained in $f^{-1}(\{0\})$. Such a function exists by Proposition 6.7. Let $q \in W$, where W is the smallest neighbourhood of p such that $f \equiv 0$ on W.

$$0 = \ell(0)(q) = \ell(fh)(0) = f(p)\ell(h)(q) + h(p)\ell(f)(q) = 0 + \ell(f)(q).$$

Thus $\ell(f)(q) = 0$ for all $q \in W$, hence $\ell(f) = 0$. $\square$

Here's a lemma:

Lemma 7.5. *Let $f \in \mathcal{O}(U)$ [1], where $U \subset \mathbb{R}^n$ and U is convex. Then there are C^∞ functions $g_i : U \to \mathbb{R}$ with*

1. *$f(x^1, \cdots, x^n) = \sum_{i=1}^n x^i g_i(x^1, \cdots, x^n)$ for all $x \in U$.*
2. *$g_i(0) = D_i(f)(0)$.*

Note that if (x, U) is a chart domain of M around p, then $\frac{\partial}{\partial x^i}\big|_p$ is a derivation at p. In fact, the derivations $\frac{\partial}{\partial x^i}\big|_p$ span the space of derivations at p.

[1]From now on, unless otherwise stated, the notation $\mathcal{O}(U)$ will denote the set of C^∞-functions from U to $\mathbb{R}$.

Theorem 7.6. *The set of derivations at $p \in M^n$ is an n-dimensional vector space. In fact, if (x, U) is a coordinate system around p, then*

$$\left.\frac{\partial}{\partial x^1}\right|_p, \cdots, \left.\frac{\partial}{\partial x^n}\right|_p$$

span this vector space, and any derivation ℓ can be written

$$\ell = \sum_{i=1}^{n} \ell(x^i) \cdot \left.\frac{\partial}{\partial x^i}\right|_p .$$

If (y, V) is another coordinate system, then we know that $\left.\frac{\partial}{\partial y^i}\right|_p = \sum_{j=1}^n \left.\frac{\partial x^j}{\partial y^i}\right|_p \frac{\partial}{\partial x^j}\Big|_p$, so the derivations transform exactly the same way as the charts of TM. So we identify them.