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Repth. for. 4

Frobenius determinant problem

G finite group

consider $\mathbb{C}[x_g \mid g\in G]$.

$$P_G = \det\left((x_{gh})_{g,h}\right)$$

Problem decompose P_G into irreducible polynomials.

It is conv. to work w/

$$\tilde{P}_G = \det\left((x_{gh^{-1}})_{g,h}\right)$$

Note $\tilde{P}_G = (\det J) P_G$ where J is the operator on $\mathbb{C}[G]$ given by $\delta_g \longmapsto \delta_{g^{-1}}$. So $\det J = \pm 1$.

The matrix $(x_{gh^{-1}})_{g,h}$ is the matrix of the operator $\sum x_p \lambda(p)$ in the basis δ_p $(p\in G)$. (since $\lambda(gh^{-1})\delta_h = \delta_g$).

We know that the regular rep. decomposes as

$$\lambda \sim \bigoplus_{[\pi]\in \hat{G}} \underbrace{(\pi \oplus \cdots \oplus \pi)}_{\dim \pi}$$

②

Hence

$$\tilde{P}_G = \prod_{[\pi]\in\hat{G}} \det\Big(\sum_g x_g \pi(g)\Big)^{\dim \pi}.$$

Q: Can we decompose further?

Thm (Frobenius)
The polynomials $\det\big(\sum_g x_g \pi(g)\big)$ $[\pi]\in\hat{G}$ are irreducible and neither two of them are associates.

Lemma The polynomial $p = \det(x_{ij})_{i,j=1}^{n}$ is irreducible.

Pf Note that p is linear in every variable x_{ij}.
Assume $p = p_i p_j$.
Then we may assume x_{11} does not appear in p_1.

Then since p has no monomials of the form $x_{11} x_{1i}(\dots)$ and $x_{11} x_{i1}(\dots)$, it follows that p_1 does not depend on x_{1i} and x_{i1}.

For the same reason (using x_{ii} instead of x_{11}) we conclude that p_i does not depend on x_{ji}.

$\Rightarrow$ p_1 is scalar. ∎

Pf of thm

Let us show that if π is irreducible, then $p_\pi \overset{\text{def}}{=} \det\left(\sum_g x_g \pi(g)\right)$ is irreducible.

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Assume that we have a decomposition $p_\pi = p_1 p_2$.

The polys p_1, p_2 are homogeneous.

Fix a basis $e_1, \dots, e_n$ in V_π.

Consider the corresponding matrix units m_{ij}

$$m_{ij} e_k = \delta_{jk} e_i$$

As $\pi(\mathbb{C}[G]) = \operatorname{End}(V_\pi)$ by the density thm, we can find $\alpha_{ij}(g) \in \mathbb{C}$ s.t.

$$m_{ij} = \sum_{g \in G} \alpha_{ij}(g)\, \pi(g)$$

Consider the change of var's $x_g = \sum_{i,j} \alpha_{ij}(g) X_{ij}$

under this change of var's $\sum_g x_g \pi(g)$ becomes

$$\sum_{g,i,j} \alpha_{ij}(g) X_{ij}\, \pi(g) = \sum_{i,j} X_{ij} m_{ij} = (X_{ij})_{i,j}$$

Hence p_π becomes the determinant $\det((X_{ij}))$ and the identity $p_\pi = p_1 p_2$ gives a decomposition $\det(X_{ij}) = \tilde{p}_1 \tilde{p}_2$ ~~$\cdot$~~ /

Still have to check the non-eq. polys give different determinants.

Observe that

$$\text{①} \qquad P_\pi \Big|_{\substack{x_g = 0 \\ g \neq e}} = x_e^{\dim \pi}$$

can recover dim from the poly

and fix $g \in G$

$$\text{②} \qquad P_\pi \Big|_{\substack{x_e = 1 \\ x_h = 0 \\ h \neq g}} = \det(1 + x_g \pi(g))$$

$$= 1 + (\operatorname{Tr} \pi(g)) + \text{h.o.t.}$$

So we can recover the traces, i.e. the character χ_π of π.

Assume π, π' are irreducible and $P_\pi = \alpha P_{\pi'}$ for $\alpha \in \mathbb{C}^*$.

Then by ① $\Rightarrow$ $\alpha = 1$ and $\dim \pi = \dim \pi'$. But then the characters are the same by ②, hence $\pi \sim \pi'$. ∎

Two constructions on reps

Start w/ v. spaces V, W, then $V \otimes W = V \otimes_{\mathbb{C}} W$ is a new vector space together w/ a bilinear map $V \times W \xrightarrow{-\otimes-} V \otimes W$ satisfying the univ. property

$$\begin{array}{ccc} V \times W & \xrightarrow{-\otimes-} & V \otimes W \\ {\scriptstyle \text{bilinear}} \downarrow & \exists! \nearrow & \end{array}$$

If $\{v_1, \dots, v_n\}$, $\{w_1, \dots, w_m\}$ are bases of V, W then (8)

$\{v_i \otimes w_j\}_{i,j}$ is a basis for $V \otimes W$.

Assume (V, π) is a f.dim repr. Consider

$V^* = \mathrm{Hom}(V, \mathbb{C})$.

<u>Def</u> The <u>contragradient representation</u> π^c to π is the representation on the dual space define by

$$(\pi^c(g) f)(v) = f(\pi(g^{-1}) v)$$

$f \in V^*, v \in V$

If $v_1, \dots, v_n$ is a basis for V. and $\pi(g) = (a_{ij}(g))_{ij}$ in this basis

$$\pi^c(g) = (a_{ji}(g^{-1}))_{i,j}$$

In particular, the character $\chi_{\pi^c}(g) = \chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$

since π can be unitary / unitarized

Assume now (V, π) and (W, θ) are repr's.

Then the tensor product of π and θ is the representation $\pi \otimes \theta$ defined by $(\pi \otimes \theta)(g)(v \otimes w) = \pi(g)v \otimes \theta(g) w$.

Exercise

① $\pi^{cc} = \pi$ (upon identifying V^{**} w/ V)

② $(\pi \otimes \theta)^c = \pi^c \otimes \theta^c$ (up to identifying)

③ π^c is irreducible $\Leftrightarrow \pi$ is irreducible

Assume U, V, W are f.d. v.spaces. Then we have

$$* \quad \mathrm{Hom}(U, W \otimes V^*) = \mathrm{Hom}(U \otimes V, W)$$

(We prove this)

Exc: Show that $*$ is iso.

In the simplest case $(W = V)$ and $U = \mathbb{C}$ we get

$$V \otimes V^* = \mathrm{Hom}(\mathbb{C}, V \otimes V^*) = \mathrm{Hom}(V, V) = \mathrm{End}(V)$$

$$v \otimes f \longmapsto (w \longmapsto f(w)v)$$

(every operator is a sum of rank 1 operators)

Prop Assume we have f.d. reps $\pi: G \to GL(U)$

$\eta: G \to GL(W)$

$\theta: G \to GL(V)$

Then the iso $(*)$ an iso of rep's.

$$\mathrm{Mor}(\pi, \eta \otimes \theta^c) \simeq \mathrm{Mor}(\pi \otimes \theta, \eta)$$

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this is something called

„Frobenius reciprocity"

but more often by Frob. reciprocity one means a different isomorphism to be disc. later.

Pf Let us check that if $T \in \mathrm{Mor}(\pi, \eta \otimes \theta^c)$ then the image S of T is in $\mathrm{Mor}(\pi \otimes \theta, \eta)$.

Take $u \in U$, $v \in V$, $Tu = \sum_i w_i \otimes f_i$

Then $T\pi(g)u = \sum_i \eta(g) w_i \otimes \theta^c(g) f_i$

Hence $S((\pi \otimes \theta)(g)(u \otimes v)) = S(\pi(g)u \otimes \theta(g)v)$

$$= \sum_i (\theta^c(g) f_i)(\theta(g)v)\, \eta(g) w_i$$

$$= \sum_i f_i(v)\, \eta(g) w_i$$

$$= \eta(g)\, S(u \otimes v)$$

✓

$\square$

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Denote by $R(G)$ the abelian group generated by classes $[\pi] \in \hat{G}$ of fin dim rep's and relations $[\pi] + [\pi'] = [\pi \oplus \theta]$

[**Exc** Show that $R(G)$ is a free abelian group w/ basis $[\pi] \in \hat{G}$.

We can define a product on $R(G)$ by
$$[\pi]\cdot[\theta] = [\pi \otimes \theta].$$
This makes $R(G)$ into a ring, called the repr. ring of G (fusion)

(It is a nontrivial problem for many groups. Compute for S_3 ?)

[**Exc** Assume π, θ are irreducible reps. Show that (using Frob rep)
$$\pi^c \sim \theta \iff Mor(\varepsilon, \pi \otimes \theta) \neq 0.$$
And in this case, the space $Mor(\varepsilon, \pi\otimes\theta)$ is 1-dim.
Thus describe this unique up to scalar morphism $\varepsilon \to \pi \otimes \theta = \pi\otimes\pi^c$.

In other words, if π is irreducible, then $\pi \otimes \pi^c \sim \varepsilon \oplus \bigoplus_{\theta \in \hat{G}} \underbrace{\theta \oplus \dots \oplus \theta}_{n_\theta}$

If π, θ are irreducible, then
$$\pi \otimes \theta \sim \bigoplus \eta \oplus \dots \oplus \eta,$$

The rules describing this decomposition, that is, the coeff's $m^{\eta}_{\pi,\theta}$ are called the **fusion rules** of G. 9

(the multiplication in $R(G)$)

$$\boxed{\begin{array}{l}\text{Exc Show that} \\ \quad Mor(\pi, \theta) \underset{\substack{\uparrow \\ \text{non-canonical !}}}{\simeq} Mor(\theta, \pi)\end{array}}$$

Then $Mor(\pi \otimes \theta^c, \eta) \simeq Mor(\pi, \eta \otimes \theta)$

implies $m^{\pi}_{\eta,\theta} = \dim Mor(\pi, \eta \otimes \theta)$

So $m^{\eta}_{\pi,\theta^c} = m^{\pi}_{\eta,\theta}$

for any irreducible π, η, θ.