Recall

$$\exp: T_eG \longrightarrow G$$

$$(t+s)X \longmapsto \exp(sX)\exp(tX) \qquad \frac{d}{dt}\exp(tX)\Big|_{t=0} = X$$

Left-invariant v.fields on G are closed under the commutator. Identifying left-inv. v.fields w/ tangent vectors at e, we get the structure of a real Lie algebra on T_eG. This Lie algebra is ~~described~~ denoted by $\mathfrak{g}$.

In other words, if $L_X : C^\infty(G) \longrightarrow C^\infty(G)$ is the derivation corresponding to the left-invariant v.field defined by $X \in \mathfrak{g}$ then the Lie bracket is def by $L_{[X,Y]} = L_X L_Y - L_Y L_X$.

A Lie algebra has the properties

① $[X,X]=0$

② Jacobi identity $\sum_{\mathbb{Z}_3} [X_1,[X_2,X_3]] = 0$

"replacement of associativity"

How to compute $[X,Y]$ more explicitly?

Take a curve γ_X s.t. $\gamma_X(0)=e$ and $\gamma_X'(0)=X$.

Then for any $g \in G$, the curve $g\gamma_X$ satisfies

$$(g\gamma_X)(0) = g \quad \text{and} \quad (g\gamma_X)'(0) = (d_e l_g)(X)$$

Hence $L_X(f)(g) = \frac{d}{dt} f(g\gamma_X(t))\Big|_{t=0}$. ②

Then $(L_X L_Y)(f)(g) = \frac{d}{dt} L_Y(f)(g\gamma_X(t))\Big|_{t=0}$

$$= \frac{\partial^2}{\partial s \partial t} f(g\gamma_X(t)\gamma_Y(s))\Big|_{s=t=0}$$

Thus $(d_e f)([X,Y]) = (L_X L_Y - L_Y L_X)(f)(e)$

(*)
$$= \frac{\partial^2}{\partial s \partial t}\Big(f(\gamma_X(t)\,\gamma_Y(s)) - f(\gamma_Y(s)\gamma_X(t))\Big)\Big|_{s=t=0}.$$

Quite explicit.

Choose local coord's x around e ~~but~~ takes values in $\mathfrak{g} = T_e G$

s.t. $x(e)=0$ and $d_e x = \mathrm{id}$.

(e.g. $x = \exp^{-1}$)

Consider the map $m : V \times V \longrightarrow \mathfrak{g}$

$(X,Y) \longmapsto x(x^{-1}(X)x^{-1}(Y))$

is well-def for small neighbhds V of $0 \in \mathfrak{g}$.

Take $\gamma_X(t) = x^{-1}(tX)$ $\quad \gamma_Y(s) = x^{-1}(sY)$

Apply (*) to $f = \ell \circ x$ where ℓ is a linear functional on $\mathfrak{g}$

We set

$$\ell([X,Y]) = \frac{\partial^2}{\partial s \partial t} \ell\left(x\left(x^{-1}(tX)\, x^{-1}(sY)\right) - x\left(x^{-1}(sY)\, x^{-1}(tX)\right)\right)\Big|_{\substack{t=\\ s=0}}$$

$$= \ell\left(\frac{\partial^2}{\partial s \partial t}\left(m(tX, sY) - m(sY, tX)\right)\Big|_{t=s=0}\right)$$

therefore

$$[X,Y] = \frac{\partial^2}{\partial s \partial t}\left(m(tX, sY) - m(sY, tX)\right)\Big|_{s=t=0}$$

This can also be written as follows:

As $m(0,0) = 0$, $m(X,0) = X$, $m(0,Y) = Y$ the Taylor expansion of m at $(0,0)$ has the form

$$m(X,Y) = X + Y + B(X,Y) + \text{h.o.t.}$$

where $B: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear map.

Therefore $[X,Y] = B(X,Y) - B(Y,X)$.

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Example Let $G = GL(V)$ for a fin dim real/cplx v-space V.

choose coord's by setting $x(A) = A - 1$.

Then $m(X,Y) = (1+X)(1+Y) - 1$
$= XY + X + Y$

Therefore $[X,Y] = XY - YX$.

The space $\mathrm{End}\, V = T_e GL(V)$ w/ this bracket is denoted as $\mathfrak{gl}(V)$.

(More precisely, if V is complex, then $\mathfrak{gl}(V)$ is viewed as a cplx Lie alg.)

Thm If $\pi : G \to H$ is a Lie grp homomorphism, then $\pi_* = d_e\pi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism. (i.e. preserves the bracket)

$$\pi_*([X,Y]) = [\pi_*(X), \pi_*(Y)]$$

Pf "ok, so why this is the case"

Recall that $\forall X \in \mathfrak{g}$ we have
$$\pi(\exp(tX)) = \exp(t\pi_*(X))$$

As $L_X(f)(g) = \frac{d}{dt}\left(f(g\exp(tX))\right)\Big|_{t=0}$ we have

$$L_{\pi_*(X)}(f)(\pi(g)) = L_X(f\circ\pi)(g).$$

Passing to commutators $L_{[\pi(X),\pi(Y)]}(f)(\pi(g)) = L_{[X,Y]}(f\circ\pi)(g)$

Hence $[\pi_*(X), \pi_*(Y)] = (d_e\pi)([X,Y])$
$= \pi_*([X,Y])$. ▨

Lie subgroups

Def A closed Lie subgroup of a Lie group G is a subgroup H which is also a closed submanifold.

Then H is itself a Lie group. We can identify $\mathfrak{h} = T_eH$ w/ the subspace of $\mathfrak{g} = T_eG$.

As the embedding $H \hookrightarrow G$ is a Lie group homomorphism, this gives an identification of $\mathfrak{h}$ w/ a Lie subalgebra. (a subspace closed under Lie ~~of the~~ bracket).

Thm 5 If G is a Lie group and H is a closed subgroup, then H is a closed submanifold.

Pf Consider $V \subset \mathfrak{g}$ consisting of v's X s.t. there exist $X_n \in \mathfrak{g}$ and $\alpha_n \in \mathbb{R}$ s.t. $\exp(X_n) \in H$ and $\alpha_n X_n \xrightarrow[n\to\infty]{} X$ and $|\alpha_n| \xrightarrow[n\to\infty]{} \infty$.

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Claim 1 If $X \in V$ then $\exp(tX) \in H$ $\forall t \in \mathbb{R}$.

Take $t \in \mathbb{R}$. As $|\alpha_n| \to \infty$ we can find $m_n \in \mathbb{Z}$ s.t. $\frac{m_n}{\alpha_n} \to t$ as $n \to \infty$ (e.g. let $m_n = [\alpha_n t]$)

Then

$$H \ni \exp(X_n)^{m_n} = \exp(m_n X_n)$$

$$= \exp\left(\frac{m_n}{\alpha_n} \alpha_n X_n\right)$$

$$\xRightarrow{n \to \infty} \exp(tX) \in H$$

since H closed.

Claim 2 $V \subset \mathfrak{g}$ is a v.space

By claim 1, V is closed under mult. by scalars.

Therefore, we have to show that if $X, Y \in V$, then $X+Y \in V$.

Consider $\gamma(t) = \exp(tX)\exp(tY)$.

Then $\gamma'(0) = X+Y$

For small $|t|$ we can write $\gamma(t) = \exp(f(t))$ where f is a smooth $\mathfrak{g}$-valued function.

Then $f'(0) = X+Y$ so $\frac{f(t)}{t} \to X+Y$ as $t \to \infty$.

In particular, $n f(\frac{1}{n}) \to X+Y$ as $n \to \infty$.

and $\exp(f(\frac{1}{n})) \in H$. By definition of X, $X+Y \in V$.

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<u>Claim 3</u> H is a closed manifold w/ $T_e H = V$.

Let V' be a complementary subspace to V in $\mathfrak{g}$,

i.e. $\mathfrak{g} = V \oplus V'$.

Consider the map $\phi : V \times V' \to G$, $\phi(X, Y) = \exp(X)\exp(Y)$

Then $d_{(0,0)}\phi = id$, so we can find neighborhoods Ω of 0 in V

and Ω' of 0 in V' s.t. ϕ is a diffeomorphism of

$\Omega \times \Omega'$ onto $\phi(\Omega \times \Omega')$.

We will show that for n large enough

$$\phi(\Omega \times \tfrac{1}{n}\Omega') \cap H = \exp(\Omega)$$

$$\exp(\Omega) = \phi(\Omega \times \{0\})$$

Suppose this is not true. Then we can find $(X_n, X_n') \in \Omega \times \Omega'$

s.t. $\exp(X_n)\exp\left(\frac{X_n'}{n}\right) \in H$ and $X_n' \neq 0$.

Take any limit point X' of the sequence $\dfrac{X_n'}{\|X_n'\|}$

Then $X' \neq 0$, $X' \in V'$, $\exp\left(\frac{X_n'}{n}\right) \in H$ and $\dfrac{n}{\|X_n'\|} \cdot \dfrac{X_n'}{n} \to X'$

Then by def, $X' \in V$, which is

a contradiction.

Thus in the neighborhood $\underbrace{\phi(\Omega \times \tfrac{1}{n}\Omega')}_{U}$ of $e \in G$,

we can take the coordinate system $x = \phi^{-1}$ s.t. $U \cap H = \{ g \in U \mid$ the coord's $x(g)$ in V' are zero $\}$

For arbitrary $h \in H$, we can take the neighborhood hU of $h \in G$ and coord's

$x_h = x(h^{-1} -)$ s.t. $U_h \cap H = \{g \in G \mid$ the coord's of $x_h(g)$ in V is $0\}$

(8)

(tomorrow: cont'd to discuss this.

Consequence: a cont's homom. of Lie grps is automatically smooth)