Tips, tricks

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A note about computer code: I will remove blank lines where I find that they take too much space.

1 Number theory

1.1 Compute the discriminant of a polynomial

The discriminant Δ_f of a polynomial $f \in \mathbb{Q}[x]$ is defined as follows: let $\{\alpha_i\}_{i=1}^n$ be the roots of f in some finite algebraic extension E of $\mathbb{Q}$, where n is the degree of f. Then

$$\Delta_f := \prod_{i \neq j} (\alpha_i - \alpha_j)^2.$$

Then $\Delta_f \in \mathbb{Q}$, because it is invariant under the action of $\mathrm{Gal}(E/\mathbb{Q})$. To compute the discriminant in `Macaulay2`, the following code can be used:

```
i1 : R = QQ[x,a,b,c,d]
o1 = R
o1 : PolynomialRing
i2 : f = x^3+a*x^2+b*x+c
      3    2
o2 = x  + x a + x*b + c
o2 : R
i3 : discriminant(f,x)
        2 2     3      3                2
o3 = - a b  + 4a c + 4b  - 18a*b*c + 27c
o3 : R
```

Here `Macaulay2` uses the fact that $\mathrm{Res}(f, f') = \Delta_f$, which is easily proved. The resultant $\mathrm{Res}(f, g)$ is defined as the determiant of the coefficient matrix of the system of polynomial equations $f, zf, \cdots, z^{n-1}f, g, zg, \cdots, z^{m-1}g = 0$ with respect to the auxiliary variable z, where n, m denotes the degree of f and g, respectively.

2 Algebraic Geometry

2.1 Projecting to linear subspaces

Let $X \subseteq \mathbb{P}^n$ be a variety defined by an ideal $I \subseteq k[x_0, \cdots, x_n]$. To compute the equations of the image of the projection $\pi : \mathbb{P}^n \to \mathbb{P}^{n-1}$, one must eliminate one variable from the equations.

In `Macaulay2` one can use the command `eliminate` to achieve this. For the example, we use the twisted cubic curve, which is the 3'rd Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$. We compute the projection to the plane $\{z = 0\} \simeq \mathbb{P}^2$.

```
i1 : R = QQ[x,y,z,w]
o1 = R
o1 : PolynomialRing
i2 : S = QQ[s,t]
o2 = S
o2 : PolynomialRing
i3 : f = map(S,R,basis(3,S))
                 3   2      2   3
o3 = map(S,R,{s , s t, s*t , t })
o3 : RingMap S <--- R
i45 : I = ker f
               2                    2
o4 = ideal (z  - y*w, y*z - x*w, y  - x*z)
o4 : Ideal of R
i5 : J = eliminate(I,z)
              3    2
o5 = ideal(y  - x w)
o5 : Ideal of R
```

`Macaulay2` uses Gröbner basis techniques to do this, see for example [1, Chapter 3] for a very readable introduction. If the ideal is generated by two elements, one can type `ideal resultant(f,g,x)`, where x is the variable one wants to eliminate.

References

[1] D. Cox, J. Little, D. O'Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra.* Springer 2nd Edition, 2006.