ection 12

1) Describe all symmetries of a point in $\mathbb{R}$. That is, all ~~isometries~~ of $\mathbb{R}$ that leave one point fixed.

We can without loss of generality assume that the point is $0 \in \mathbb{R}$.

Let x be any other point $\in \mathbb{R}$ and let $\alpha: \mathbb{R} \to \mathbb{R}$ be an isometry. Then by definition,

$|\alpha(x) - 0| = |\alpha(x)| = |x|$. But then $\alpha(x) = \pm x$.

Thus α sends x either to itself or to its negative. Suppose $\alpha(x) = x$. Then I claim that $\alpha = id_{\mathbb{R}}$.
Let $y \in \mathbb{R}$. Then we must have $|\alpha(y)| = |y|$ as before, so $\alpha(y) = \pm y$. On the other hand, we also have

$$|\alpha(x) - \alpha(y)| = |x - y|$$

Suppose first $y > 0$, and $x > 0$. If $\alpha(y) = -y$, then $|\alpha(y) - \alpha(x)| > |x - y|$, contrary to α being an isometry.

Suppose now $y > 0$ and $x < 0$. If $\alpha(y) = -y$, then $|\alpha(y) - \alpha(x)| < |x - y|$, contrary to α being an isometry.

The other two cases are similar.
Hence the only automorphisms of a point in $\mathbb{R}$ are $\{\pm$

$\cong \mathbb{Z}/2$.

26 E E E E E E E ...

a) No b) Yes c) No d) No e) $\mathbb{Z}\times\mathbb{Z}_2$

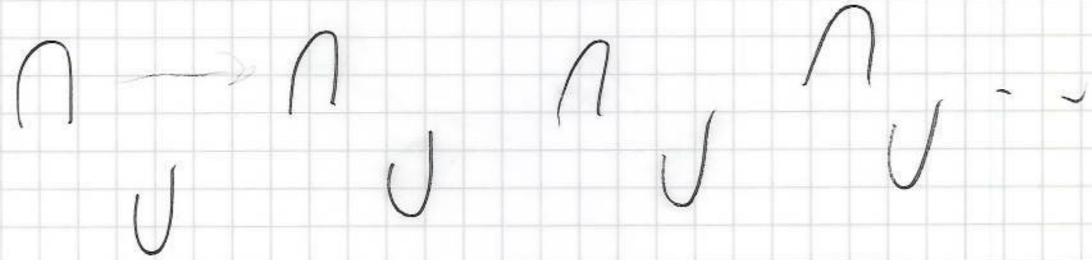
27 Z Z Z Z Z Z Z ...

a) Yes (180°) b) No c) No d) No e) $\mathbb{Z}\times\mathbb{Z}_2$

28 H H H H H H H H 3

a) Yes b) Yes c) Yes d) No e) $D_\infty\times\mathbb{Z}_2$

29 

30 

20) The glide reflections?

No, again you must include translations...

24-30) For each figure, answer the following:

a) Does the group contain a rotation?

b) — // — a reflection across a horizontal line?

c) — // — vertical line?

d) Nontrivial glide reflection?

e) Isomorphic to which group?

$\mathbb{Z}$, D_∞, $\mathbb{Z} \times \mathbb{Z}_2$, $D_\infty \times \mathbb{Z}_2$

24) F F F F F F F F ...

a) No b) No. c) No d) No

e) $\mathbb{Z}$

25) T T T T T T T T ...

a) No. b) No c) Yes d) No e) $\mathbb{Z} \times \mathbb{Z}_2$

We can also reflect through L, i.e the map that sends $(x,y) \longmapsto (x,-y)$. (3)

Thus the full group is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

e) Describe some symmetries of a line segment in $\mathbb{R}^3$.

(16) Do the rotations, together w/ the identity map form a subgroup of the plane isometries?

~~Yes~~ No. (or depends! The rotations w/ fixed center do) Rotations about different centra can be a translation.

(17) Do the translations, —"— ?

Yes.

(18) Rotations w/ fixed center —"— ?

Yes.

(19) Reflections about a fixed line L. —"— ?

No. The composition of two reflections may be a rotation of 180°.

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B) All symmetries of a point in the plane $\mathbb{R}^2$.

We want to find all isometries fixing $(0,0) \in \mathbb{R}^2$.

Suppose first that α preserves orientation. Then since lengths are preserved, α must be ~~[illegible]~~ a rotation about the origin.

Now suppose that α reverses orientation. Then $\alpha \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ preserves orientation, so

$$\alpha \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \rho_\theta = \{\text{rotation } \theta \text{ degrees}\}.$$

But then $\alpha = \rho_\theta \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Hence the group of isometries fixing a point is

$$S^1 \rtimes \mathbb{Z}/2.$$

c) All symmetries of a line segment $\subseteq \mathbb{R}$

<u>Answer</u> Reflection about the midpoint + identity.

d) All symmetries of a line segment $L \subseteq \mathbb{R}^2$.

Suppose wlog that $L = [0,1] \times \{0\} \subseteq \mathbb{R}^2$. We can reflect about the midpoint $(\frac{1}{2}, 0)$ of L, reversing orientation.