Some hints for the exercises

FM

1 Ark 2, 1-14

These are hints for the exercises at `uio.no/studier/emner/matnat/math/MAT2200/v14/ark2.pdf`.

Exercise 1. Write a general element of $G = \langle a \rangle$ as a^ℓ for some natural number ℓ. ♠

Exercise 2. Think of relatively prime. ♠

Exercise 3. Hint 1: $7 \cdot 7 = 48 + 1$. Hint 2: $11 \equiv 1 \pmod{12}$. Hint 3: $gcd(7, 12) = 1$. ♠

Exercise 4. Look at divisors. For the subgroup diagram, think least common multiple and greatest common divisor. Why? ♠

Exercise 5. Same. ♠

Exercise 6. Look at the hint in the exercise and count. ♠

Exercise 7. Think relatively prime. ♠

Exercise 8. Same. ♠

Exercise 9. The kernel

$$\ker \phi = \{n \in \mathbb{Z}_p \,|\, \phi(n) \equiv 0 \pmod{p}\}$$

is a subgroup of $\mathbb{Z}_p$. ♠

Exercise 10. The element $\phi(1)$ has order p in $\mathbb{Z}_q$. ♠

Exercise 11. Again, look at the kernel of ϕ. ♠

Exercise 12. $\langle a \rangle \cap \langle b \rangle$ is a subgroup of $\langle a \rangle \approx \mathbb{Z}_p$. ♠

Exercise 13. Define a homomorphism $\phi : \langle x \rangle \to \mathbb{Z}_p \times \mathbb{Z}_q$ by $x \mapsto (1, 1)$. Show that it is injective $\Rightarrow$ bijective. ♠

Exercise 14. List all 8 elements, and compute their powers, recalling that each non-trivial subgroup have order ≤ 4. ♠

2 Permutations, 15-23

Exercise 15. Here's how to compute $\tau\sigma$:

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 3 & 4 & 2 & 6 \end{pmatrix}.$$

The order of each of the element is the length of the longest orbit of a number $n \in \{1, 2, 3, 4, 5, 6\}$.

One computes that $|\sigma| = 6$. Now, write $100 = 16 \cdot 6 + 4$. Then $\sigma^{100} = \sigma^4$. Compute. ♠

Exercise 16. Hint a): Think of a permutation group. Hint b): Note that $\tau^{-1} = \tau$. Also note that the element of $\tau G \tau$ are precisely those of the form $\tau g \tau$ for $g \in G$. For example: $\tau g \tau(4) = \tau g(3) = \tau(3) = 4$. ♠

Exercise 17. Try a counting argument. How much freedom is it for the first element (answer: 4), for the next element (answer: 3), and so on? ♠

Exercise 18. Does the identity of $Sym(X)$ fix Y? Does the inverses of elements fixing Y fix Y? And if two elements fix Y, does their product? ♠

Exercise 19. Hint a). This is easy if you did 16a). ♠

Exercise 20. Read the hint in the exercise. ♠

Exercise 21. Suppose γ commutes with all $\sigma \in S_n$. Then, in particular, γ commutes with all permutations permuting only the first three numbers $\{1, 2, 3\}$. These permutations form a subgroup isomorphic to S_3, and S_3 is generated by the two permutations

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

So we both $\alpha^{-1}\gamma\alpha = \gamma$ and $\beta^{-1}\gamma\beta = \gamma$. Suppose

$$\gamma = \begin{pmatrix} 1 & 2 & 3 \\ x & y & z \end{pmatrix}.$$

Then

$$\alpha^{-1}\gamma\alpha(3) = \alpha^{-1}(x) = \gamma(3) = z.$$

Hence $\alpha(z) = x$. Also

$$\beta^{-1}\gamma\beta(3) = \beta^{-1}(z) = z.$$

Hence $\beta(z) = z$, but the only fix point of β is 3, so $z = 3$. Hence $x = \alpha(z) = 1$, and thus must be equal to 2. So $\gamma = \mathrm{id}$.

This proves that γ fixes the first three elements of $\{1, 2, \cdots, n\}$. We could have chosen any three elements, so it fixes all of them. $\spadesuit$

Exercise 22. Hint: The group S_3 can be realized as the group of symmetrices of an equilateral triangle in the plane, and is generated by rotations and reflections through the vertices. $\spadesuit$

Exercise 23. Hint a): Take determinants or stare at it. Hint b): $e_i \cdot e_j = \delta_{ij}$. Hint c): For orthogonal matrices, we have $aa^T = I$. $\spadesuit$

3 Orbits and cycles, 24 - 34

Exercise 24. The first one goes like this: $1 \mapsto 5 \mapsto 2 \mapsto 1$, $3 \mapsto 3$, $4 \mapsto 6 \mapsto 4$. So the orbits are $\{1, 5, 2\}$, $\{3\}$ and $\{4, 6\}$. Note that their lengths sum to 6. $\spadesuit$

Exercise 25. Start on the right:

$$(1,4,5)(7,8)(2,5,6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 5 & 6 & 2 \end{pmatrix}.$$

$\spadesuit$

Exercise 26. This is just writing up the orbits. For the first one:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} = (1,8)(3,6,4)(5,7).$$

$\spadesuit$

Exercise 27. Write down the orbits. For example, $(1,2)(1,3) = (1,3,2)$. $\spadesuit$

Exercise 28. Similar. $\spadesuit$

Exercise 29. Hint: Write $(1,2,3,4,5)\sigma = (1,5)$ as $\tau\sigma = \gamma$. Then, since S_5 is a group? $\spadesuit$

Exercise 30. Write $\sigma \in S_n$ as a product of k disjoint cycles, say the i'th cycle has length l_i. Then $\sum_{i=1}^k l_i = n$. Each cycle can be broken into $l_i - 1$ transpositions. So in total we have

$$\sum_{i=1}^{k}(l_i - 1) = \sum_{i=1}^{k} l_i - \sum_{i=1}^{k} 1 = n - k$$

transpositions. If σ is a cycle, then $k = 1$. ♠

Exercise 31. Let $B_n = \{$odd permutations $\}$. Define a map $l_\sigma : A_n \to B_n$ by $\tau \mapsto \sigma\tau$. We have $|A_n| = |B_n|$ and? ♠

Exercise 32. Hint: The order is the least order of the (disjoint) cycles. Why?? ♠

Exercise 33. Write an element as a product of disjoint cycles. ♠

Exercise 34. Hint: $\sigma \in S_n$ is a cycle of length k if and only if there are integers $i_1, \cdots i_k$ such that $\sigma(i_j) = i_{j+1}$ where j is counted modulo k. Thus $\sigma^2(i_j) = i_{j+2}$. In cycle notation

$$\sigma = (i_1, i_2, \cdots i_k) \Rightarrow \sigma^2 = (i_1, i_3, \cdots, i_k, i_2).$$

What happens if k is even/odd? For the counterexample, consider $(1234)^2$.
♠