Adjoint representation

Let G be a Lie group. For $g \in G$ consider the automorphism

$h \mapsto ghg^{-1}$ of G.

Its differential at e is denoted by $Ad\,g$ or Ad_g. This way we get a homomorphism

$$Ad: G \longrightarrow GL(\mathfrak{g})$$

called the adjoint representation.

The differential of Ad at e gives a Lie algebra homomorphism

$$ad: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),$$

also called the adjoint representation. Let's compute ad more explicitly.

Take $X, Y \in \mathfrak{g}$. Then by def $(ad\,X)(Y) = \frac{d}{dt}\left((Ad\,\exp(tX))(Y)\right)\Big|_{t=0}$

~~Take~~ $f \in C^\infty(G)$. Then

$$(d_e f)\left(ad(X)(Y)\right) = \frac{d}{dt}(d_e f)\left((Ad\,\exp tX)(Y)\right)\Big|_{t=0}$$

$$= \frac{\partial^2}{\partial t \partial s}\left(f(\exp(tX)\exp(sY)\exp(-tX))\right)\Big|_{s=t=0}$$

chain rule

$$= \frac{\partial^2}{\partial s \partial t}\left(f(\exp(tX)\exp(sY)) - f(\exp(sY)\exp(tX))\right)\Big|_{s=t=0}$$

$$= (d_e f)([X,Y]).$$

Therefore $(ad\,X)(Y) = [X,Y]$.

Of course the adjoint rep. $\mathrm{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $\mathrm{ad}(X)(Y) = [X,Y]$ is well-defined for any Lie algebra $\mathfrak{g}$. Note that the identity

$$\mathrm{ad}[X,Y] = [\mathrm{ad}(X), \mathrm{ad}(Y)]$$

$$= \mathrm{ad}(X)\,\mathrm{ad}\,Y - \mathrm{ad}\,Y\,\mathrm{ad}\,X$$

is exactly the Jacobi identity.

The kernel of ad is the <u>center</u> of $\mathfrak{g}$:

$$\mathfrak{z}(\mathfrak{g}) = \{X \mid [X,Y]=0 \ \forall Y \in \mathfrak{g}\}$$

We say that X,Y <u>commute</u> if $[X,Y]=0$

<u>Prop</u> Let G be a Lie group, $X,Y \in \mathfrak{g}$. TFAE

① $[X,Y]=0$

② $\exp(tX)\exp(sY) = \exp(sY)\exp(tX) \quad \forall s,t$

Furthermore, if these conditions are satisfied then

$$\exp(X+Y) = \exp(X)\exp(Y)$$

<u>Pf</u>

$$\exp(tX)\exp(sY)\exp(-tX) \overset{\text{functoriality}}{=} \exp\big(\mathrm{Ad}\,\exp(tX)(sY)\big)$$

$$= \exp\big(s\, e^{t\,\mathrm{ad}X}\, Y\big) \longleftarrow \text{used}$$

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ad}"] & GL(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{ad}"] & \mathfrak{g} \arrow[u, "\exp"'] \end{tikzcd} \qquad \text{t.c. of} \qquad \begin{tikzcd} G \arrow[r, "c_g"] & G \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{Ad}_g"'] & \mathfrak{g} \arrow[u, "\exp"'] \end{tikzcd}$$

Assume ① holds, so $(\mathrm{ad}X)(Y)=0$, and hence

$e^{t\,\mathrm{ad}X}\, Y = Y$ and $\exp(tX)\exp(sY)\exp(-tX) = \exp(sY)$ ✓

Now assume ②. Then

$$\exp(s e^{t\,\mathrm{ad} X} Y) = \exp(sY) \quad \forall t, s.$$

Applying $\frac{d}{ds}\Big|_{s=0}$ we get $e^{t\,\mathrm{ad}X} Y = Y$. Applying $\frac{d}{dt}\Big|_{t=0}$ we get $\mathrm{ad}(X)(Y) = 0$.

Finally, if ① and ② hold, then

$$t \mapsto \exp(tX)\exp(tY)$$

is a 1-parameter subgroup of G. Hence $\exp(tX)\exp(tY) = \exp(tZ)$ for some $Z \in \mathfrak{g}$. Applying $\frac{d}{dt}\Big|_{t=0}$ we get $X + Y = Z$ ∎

Remark This also follows from the general fact that v.fields X, Y, on a manifold M commute iff the corresponding global flows (if defined) commute.

Prop Any connected abelian Lie group G is isomorphic to $\mathbb{R}^k \times \mathbb{T}^l$ for some $k, l \geq 0$.

Pf Consider $\mathfrak{g}$ as a Lie group under addition. By the previous result, $\exp : \mathfrak{g} \to G$ is a Lie group homomorphism, since the commutator is zero. As $d_0 \exp = \mathrm{id}$, $\exp : \mathfrak{g} \to G$ is a covering map. Hence $\ker \exp$ is a discrete subgroup $\Gamma < \mathfrak{g}$ and $G \cong \mathfrak{g}/\Gamma$ as Lie groups, where on $\mathfrak{g}/\Gamma$ we consider the ! smooth structure making $\mathfrak{g} \to \mathfrak{g}/\Gamma$ a local diffeomorphism.

Exc Assume Γ is a discrete subgroup of $V \cong \mathbb{R}^d$. Then $\exists$ a basis $e_1, \dots, e_d$ and $l \le d$ s.t.

$$\Gamma = \mathbb{Z} e_1 + \dots + \mathbb{Z} e_l.$$

It follows that $V/\Gamma \cong \mathbb{R}^l/\mathbb{Z}^l \times \mathbb{R}^k = \mathbb{T}^l \times \mathbb{R}^k$.

∎

Structure and repr. theory of $SU(2) = \{A \in \mathbb{C}^{2\times 2} \mid AA^* = I\}$

Recall that the Lie algebra of $SU(2)$ is

$$su(2) = \{X \in gl_2(\mathbb{C}) \mid X^* + X = 0, \ \mathrm{Tr}\, X = 0\}$$

$\begin{pmatrix} a & x+iy \\ x+iy & -a \end{pmatrix}$

The cplx space spanned by $su(2)$ in $gl_2(\mathbb{C})$ is

$$sl_2(\mathbb{C}) = \{X \in gl_2(\mathbb{C}) \mid \mathrm{Tr}\, X = 0\}$$

We define a basis in $sl_2(\mathbb{C})$ by

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then as a basis in $su(2)$ (over $\mathbb{R}$) we can take

$$E - F \qquad i(E+F) \qquad iH$$

as a basis of $su(2)$.

Note that $\mathbb{R}iH \subset su(2)$ is the Lie algebra of the torus $T = \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \middle| z \in S^1 \right\} \subset SU(2)$.

The Lie algebra structure on $sl_2(\mathbb{C})$ is described by the rules $[H,E] = 2E \quad [H,F] = -2F \quad [E,F] = H$

We know that $SU(2)$ is simply connected. In fact, $SU(2)$

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \begin{array}{l} \alpha, \beta \in \mathbb{C} \\ |\alpha|^2 + |\beta|^2 = 1 \end{array} \right\}$$

So as a manifold, $SU(2) \simeq S^3$, which has $\pi_1 = 0$.

Therefore, we have a 1-1 correspondence between Lie group homomorphisms $SU(2) \to GL(V)$ and Lie algebra homomorphisms

$$su(2) \to gl(V).$$

For a Lie algebra $\mathfrak{g}$, by a representation of $\mathfrak{g}$ on V we mean a Lie algebra homomorphism $\mathfrak{g} \to gl(V)$. In this case we also say that V is a <u>$\mathfrak{g}$-module</u>.

A subspace $W \subset V$ is called <u>$\mathfrak{g}$-invariant</u> if $xW \subset W \quad \forall x \in \mathfrak{g}$.

A representation ~~of~~ $\mathfrak{g} \to gl(V)$ is called <u>irreducible</u>, or the $\mathfrak{g}$-module V is <u>simple</u> if there are no proper invariant subspaces.

Exc Assume $\pi: G \to GL(V)$ is a repr. of a connected Lie grp G on a f.dim v.space V. Then a subspace $W \subset V$ is G-invariant iff it is $\mathfrak{g}$-invariant. In particular, π is irreducible iff π_* is irreducible.

We want to classify irreducible reps of $SU(2)$ on ~~complex~~ f.dim v.spaces. Equivalently, we want to classify irreps of $su(2)$ on complex f.d. vector spaces.

As $\mathbb{C} \otimes_{\mathbb{R}} su(2) \simeq sl_2(\mathbb{C})$, any Lie algebra

$$\lambda \otimes X \mapsto \lambda X.$$

homomorphism $su(2) \to gl(V)$ (V complex!) extends uniquely to a homomorphism $sl_2(\mathbb{C}) \to gl(V)$ of cplx Lie algebras.

Thus we want to understand irreps of $sl_2(\mathbb{C})$ as a cplx Lie algebra.

Assume V is a f. dim $sl_2(\mathbb{C})$-module. Since it integrates to a representation of $SU(2)$ on V, the space V decomposes into a sum of one-dim spaces invariant under $T = \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \middle| |z|=1 \right\}$

= maximal torus $\subset SU(2)$

This means that the operator H on V is diagonalizable, and as $e^{2\pi i t H} = 1$ in $SU(2)$, all its eigenvalues are integers.

It is common to index the eigenvalues of H on V by half-integers. Thus for $s \in \frac{1}{2}\mathbb{Z}$ we put

$$V(s) = \{\xi \in V \mid H\xi = 2s\xi\}.$$

Elements of $V(s)$ are called vectors of weight s. Thus

$$V = \bigoplus_{s \in \frac{1}{2}\mathbb{Z}} V(s)$$

Recall that $[H,E] = 2E$, $[H,F] = -2F$. This implies that

$$EV(s) \subset V(s+1) \qquad FV(s) \subset V(s-1).$$

Indeed, if $\xi \in V(s)$, then

$$\begin{aligned} HE\xi &= (HE - EH)\xi + EH\xi \\ &= [H,E]\xi + EH\xi \\ &= 2E\xi + 2sE\xi \\ &= 2(s+1)E\xi. \end{aligned}$$

The second inclusion is similar.

In particular, if ξ is a nonzero vector of highest weight, then $E\xi = 0$.

Lemma Assume V is a fin dim $\mathfrak{sl}_2(\mathbb{C})$-module, $\xi \in V(s) \setminus \{0\}$, w/ $E\xi = 0$. Then

① $s \geq 0$

② $F^{2s+1}\xi = 0$ and $F^k\xi \neq 0$ for $k = 0, \dots, 2s$

③ $EF^k\xi = (2s-k+1)k F^{k-1}\xi$.