Sparse Autoregressive Reservoir Computers for Dynamic Financial Processes Identification

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Abstract

In this document, some general results in structured matrix approximation theory with applications to autoregressive representation of dynamics financial processes are presented. Firstly, a generic nonlinear time delay embedding is considered for the time series data sampled from a financial or economic system under study. Secondly, sparse least-squares and structured matrix approximation methods are applied to identify approximate representations of the output coupling matrices, that determine the autoregressive representations of the recursive models corresponding to some given financial system under consideration. Prototypical algorithms based on the aforementioned techniques, together with some applications to approximate identification and predictive simulation of dynamic nonlinear financial and economic systems, that may or may not exhibit chaotic behavior are presented.

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1. Introduction

Recursive models and reservoir computers provide interesting computational tools for financial and economic system identification and simulation. Recently, some novel architectures named next generation reservoir computers have been devised. In this document, some elements of the theory and algorithmics corresponding to the computation of some particular types of autoregressive reservoir computers are presented. The study reported in this document is focused on reservoir computers whose architecture can be approximately represented by either linear or nonlinear autoregressive vector models.

The main contribution of the work reported in this document is the application of *colaborative schemes* involving structured matrix approximation methods, together with linear and nonlinear autoregressive models, to the simulation of dynamic financial processes. Some theoretical aspects of the aforementioned methods are described in §3. As a byproduct of the work reported in this document, a toolset of Python programs for semilinear sparse signal model computation based on the ideas presented in §3 and §4 has been developed, and is available in [6].

Even though, the applications of the structure preserving function approximation technology developed as part of the work reported in this document, range from numerical simulation cyber-physical

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systems [7], to climate simulation [5]. We will focus on applications to financial processes simulations in this paper.

A prototypical algorithm for the computation of sparse structured recursive models based on the ideas presented in §3, is presented in §4. Some numerical simulations of financial processes based on the prototypical algorithm presented in §4 are documented in §5.

2. Preliminaries and Notation

The symbols $\mathbb{R}^+$ and $\mathbb{Z}^+$ will be used to denote the positive real numbers and positive integers, respectively. For any pair $p, n \in \mathbb{Z}^+$ the expression $d_p(n)$ will denote the posirive integer $d_p(n) = n(n^p - 1)/(n-1) + 1$. For any finite group G, the expression G will be used to denote the order of G, that is, the number of elements of G. Given $\delta > 0$, let us consider the function defined by the expression

$$H_\delta(x) = \left\{ \begin{array}{ll} 1, & x > \delta \\ 0, & x \leq \delta \end{array} \right. .$$

Given a matrix $A \in \mathbb{C}^{m\times n}$ with singular values [4, §2.5.3] denoted by the expressions $s_j(A)$ for $j = 1, \ldots, \min\{m, n\}$. We will write $\mathrm{rk}_\delta(A)$ to denote the number

$$\mathrm{rk}_\delta(A) = \sum_{j=1}^{\min\{m,n\}} H_\delta(s_j(A)).$$

For a nonzero matrix $A \in \mathbb{R}^{m\times n}$, the symbol A^+ will be used to denote the pseudoinverse[4, §5.5.4] of A.

Given a scalar time series $\Sigma = \{x_t\}_{t\geq 1} \subset \mathbb{R}^\kappa$, a positive integer L and any $t \geq L$, we will write $\mathbf{x}_L(t)$ to denote the vector

$$\mathbf{x}_L(t) = \begin{bmatrix} \mathbf{x}_L^{(1)}(t)^\top & \mathbf{x}_L^{(2)}(t)^\top & \cdots & \mathbf{x}_L^{(n)}(t)^\top \end{bmatrix}^\top \in \mathbb{R}^{nL},$$

with

$$\mathbf{x}_L^{(j)}(t) = \begin{bmatrix} x_{t-L+1}^{(j)} & x_{t-L+2}^{(j)} & \cdots & x_{t-1}^{(j)} & x_t^{(j)} \end{bmatrix}^\top \in \mathbb{R}^L.$$

for $1 \leq j \leq n$, where $x_{j,s}$ denotes the scalar j-component of each element x_s in the vector time series Σ, for $s \geq 1$.

The identity matrix in $\mathbb{R}^{n\times n}$ will be denoted by I_n, and we will write $\hat{e}_{j,n}$ to denote the matrices in $\mathbb{R}^{n\times 1}$ representing the canonical basis of $\mathbb{R}^n$ (each $\hat{e}_{j,n}$ corresponds to the j-column of I_n). For any vector $x \in \mathbb{R}^n$, we will write $\|x\|$ to denote the Euclidean norm of x. Given a matrix $X \in \mathbb{R}^{m\times n}$, the expression $\|X\|_F$ will denote the Frobenius norm of X.

For any integer $n > 0$, in this article we will identify the vectors in $\mathbb{R}^n$ with column matrices in $\mathbb{R}^{n\times 1}$.

Given two matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, the tensor Kronecker tensor product $A \otimes B \in \mathbb{R}^{mp\times nq}$ is determined by the following operation.

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

For any integer $p > 0$ and any matrix $X \in \mathbb{R}^{m\times n}$, we will write $X^{\otimes p}$ to denote the operation determined by the following expression.

$$X^{\otimes p} = \begin{cases} X & ,p = 1 \\ X \otimes X^{\otimes(p-1)} & ,p \geq 2 \end{cases}$$

We will also use the symbol Π_p to denote the operator $\Pi_p : \mathbb{R}^n \to \mathbb{R}^{n^p}$ that is determined by the expression $\Pi_p(x) := x^{\otimes p}$, for each $x \in \mathbb{R}^n$.

For any matrix $A \in \mathbb{R}^{m\times n}$, we will denote by $\mathrm{colsp}(A)$ the columns space of the matrix A. Given a list $A_1, A_2, \ldots, A_m$ such that for $1 \leq j \leq m$, $A_j \in \mathbb{R}^{n_j\times n_j}$ for some integer $n_j > 0$. The expression $A_1 \oplus A_2 \oplus \cdots \oplus A_m$ will denote the block diagonal matrix

$$A_1 \oplus A_2 \oplus \cdots \oplus A_m = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{bmatrix},$$

where the zero matrix blocks have been omitted.

We will write $\mathbf{S}^1$ to denote the set $\{z \in \mathbb{C} : |z| = 1\}$. Given any matrix $X \in \mathbb{R}^{m\times n}$, we will write $X^\top$ to denote the transpose $X^\top \in \mathbb{R}^{n\times m}$ of X. A matrix $P \in \mathbb{C}^{n\times n}$ will be called an orthogonal projector whenever $P^2 = P = P^\top$. The group of orthognal matrices in $\mathrm{R}^{n\times n}$ will be denoted by $\mathbb{O}(n)$ in this study. Given any matrix $A \in \mathbb{R}^{n\times n}$, we will write $\Lambda(A)$ to denote the spectrum of A, that is, the set of eigenvalues of A.

3. Structured recursive model identification

3.1. Low-rank approximation and sparse linear least squares solvers

In this section some low-rank approximation methods with applications to the solution of sparse linear least squares problems are presented.

Definition 3.1. *Given $\delta > 0$ and a matrix $A \in \mathbb{C}^{m\times n}$, we will write $\mathrm{rk}_\delta(A)$ to denote the nonnegative integer determined by the expression*

$$\mathrm{rk}_\delta(A) = \sum_{j=1}^{\min\{m,n\}} H_\delta(s_j(A)),$$

where the numbers $s_j(A)$ represent the singular values corresponding to an economy-sized singular value decomposition of the matrix A.

Lemma 3.2. *We will have that $\mathrm{rk}_\delta\left(A^\top\right) = \mathrm{rk}_\delta(A)$ for each $\delta > 0$ and each $A \in \mathbb{C}^{m\times n}$.*

Proof. Given an economy-sized singular value decomposition

$$U \begin{bmatrix} s_1(A) & & & \\ & s_2(A) & & \\ & & \ddots & \\ & & & s_{\min\{m,n\}}(A) \end{bmatrix} V = A$$

we will have that

$$V^\top \begin{bmatrix} s_1(A) & & & \\ & s_2(A) & & \\ & & \ddots & \\ & & & s_{\min\{m,n\}}(A) \end{bmatrix} U^\top = A^\top$$

is an economy-sized singular value decomposition of $A^\top$. This implies that

$$\operatorname{rk}_\delta\left(A^\top\right) = \sum_{j=1}^{\min\{m,n\}} H_\delta(s_j(A)) = \operatorname{rk}_\delta(A)$$

and this completes the proof. $\square$

Lemma 3.3. *Given $\delta > 0$ and $A \in \mathbb{C}^{m\times n}$ we will have that $\operatorname{rk}_\delta(A) \leq \operatorname{rk}(A)$.*

Proof. We will have that $\operatorname{rk}(A) = \sum_{j=1}^{\min\{m,n\}} H_0(s_j(A)) \geq \sum_{j=1}^{\min\{m,n\}} H_\delta(s_j(A)) = \operatorname{rk}_\delta(A)$. This completes the proof. $\square$

Theorem 3.4. *Given $\delta > 0$ and $y, x_1, \ldots, x_m \in \mathbb{C}^n$, let*

$$X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_m \\ | & | & & | \end{bmatrix}.$$

If $\operatorname{rk}_\delta(X) > 0$ and if we set $r = \operatorname{rk}_\delta(X)$ and $s_{n,m}(r) = \sqrt{r(\min\{m,n\} - r)}$ then, there are a rank r orthogonal projector Q, r vectors $x_{j_1}, \ldots, x_{j_r} \in \{x_1, \ldots, x_m\}$ and r scalars $c_1 \ldots, c_r \in \mathbb{C}$ such that $\|X - QX\|_F \leq (s_{n,m}(r)/\sqrt{r})\delta$, and $\|y - \sum_{k=1}^r c_k x_{j_k}\| \leq \left(\sum_{k=1}^r |c_k|^2\right)^{\frac{1}{2}} s_{n,m}(r)\delta + \|(I_n - Q)y\|$.

Proof. Let us consider an economy-sized singular value decomposition $USV = A$. If u_j denotes the j-column of U, let Q be the rank $r = \operatorname{rk}_\delta(A)$ orthogonal projector determined by the expression $Q = \sum_{j=1}^r u_j u_j^*$. It can be seen that

$$\begin{aligned}\|X - QX\|_F^2 &= \sum_{j=r+1}^{\min\{m,n\}} s_j(X)^2 \\ &\leq (\min\{m,n\} - r)\delta^2 = \frac{s_{n,m}(r)^2}{r}\delta^2.\end{aligned}$$

Consequently, $\|X - QX\|_F \leq \frac{s_{n,m}(r)}{\sqrt{r}}\delta$.

Let us set.

$$\hat{X} = \begin{bmatrix} | & | & & | \\ \hat{x}_1 & \hat{x}_2 & \cdots & \hat{x}_m \\ | & | & & | \end{bmatrix} = QX$$

$$\hat{X}_y = \begin{bmatrix} | & | & & | & | \\ \hat{x}_1 & \hat{x}_2 & \cdots & \hat{x}_m & \hat{y} \\ | & | & & | & | \end{bmatrix} = Q\begin{bmatrix} X & y \end{bmatrix}$$

Since by lemma 3.3 $\operatorname{rk}(X) \geq \operatorname{rk}_\delta(X)$, we will have that $\operatorname{rk}(\hat{X}) = r = \operatorname{rk}_\delta(X) > 0$, and since we also have that $\hat{x}_1, \ldots, \hat{x}_m, \hat{y} \in \operatorname{span}(\{u_1, \ldots, u_r\})$, there are r linearly independent $\hat{x}_{j_1}, \ldots, \hat{x}_{j_r} \in \{\hat{x}_1, \ldots, \hat{x}_m\}$ such that $\operatorname{span}(\{u_1, \ldots, u_r\}) = \operatorname{span}(\{\hat{x}_{j_1}, \ldots, \hat{x}_{j_r}\})$, this in turn implies that $\hat{y} \in \operatorname{span}(\{\hat{x}_{j_1}, \ldots, \hat{x}_{j_r}\})$ and there are $c_1, \ldots, c_r \in \mathbb{C}$ such that $\hat{y} = \sum_{k=1}^r c_k \hat{x}_{j_k}$. It can be seen that for each $z \in \{x_1, \ldots, x_m\}$

$$\|z - Qz\| \leq \|X - QX\|_F \leq \frac{s_{n,m}(r)}{\sqrt{r}}\delta,$$

and this in turn implies that

$$\begin{aligned}\left\|y-\sum_{k=1}^{r}c_kx_{j_k}\right\| &= \left\|y-\sum_{k=1}^{r}c_kx_{j_k}-\left(\hat{y}-\sum_{k=1}^{r}c_k\hat{x}_{j_k}\right)\right\|\\ &= \left\|y-\sum_{k=1}^{r}c_kx_{j_k}-Q\left(y-\sum_{k=1}^{r}c_kx_{j_k}\right)\right\|\\ &\leq \left(\sum_{k=1}^{r}|c_k|^2\right)^{\frac{1}{2}}s_{n,m}(r)\delta+\|(I_n-Q)y\|.\end{aligned}$$

This completes the proof. □

As a direct implication of theorem 3.4 one can obtain the following corollary.

Corollary 3.5. *Given $\delta>0$, $A\in\mathbb{C}^{m\times n}$ and $y\in\mathbb{C}^m$. If $\mathrm{rk}_\delta(A)>0$ and if we set $r=\mathrm{rk}_\delta(A)$ and $s_{n,m}(r)=\sqrt{r(\min\{m,n\}-r)}$ then, there are $x\in\mathbb{C}^n$ and a rank r orthogonal projector Q that does not depend on y, such that $\|Ax-y\|\leq\|x\|s_{n,m}(r)\delta+\|(I_m-Q)y\|$ and x has at most r nonzero entries.*

Proof. Let us set $x=\mathbf{0}_{n,1}$ and $a_j=A\hat{e}_{j,n}$ for $j=1,\ldots,n$. Since $r=\mathrm{rk}_\delta(A)>0$ and $s_{n,m}(r)=\sqrt{r(\min\{m,n\}-r)}$, by theorem 3.4 we will have that there is a rank r orthogonal projector Q such that $\|A-QA\|_F\leq(s_{n,m}(r)/\sqrt{r})\delta$, and without loss of generality r vectors $a_{j_1},\ldots,a_{j_r}\in\{a_1,\ldots,a_n\}$ and r scalars $c_1\ldots,c_r\in\mathbb{C}$ with $j_1\leq j_2\leq\cdots\leq j_r$ (reordering the indices j_k if necessary), such that $\|y-\sum_{k=1}^{r}c_ka_{j_k}\|\leq\left(\sum_{k=1}^{r}|c_k|^2\right)^{\frac{1}{2}}s_{n,m}(r)\delta+\|(I_m-Q)y\|$. If we set $x_{j_k}=c_k$ for $k=1,\ldots,r$, we will have that $\|x\|=\left(\sum_{k=1}^{r}|c_k|^2\right)^{\frac{1}{2}}$ and $Ax=\sum_{k=1}^{r}x_{j_k}a_{j_k}=\sum_{k=1}^{r}c_ka_{j_k}$. Consequently, $\|Ax-y\|\leq\|x\|s_{n,m}(r)\delta+\|(I_m-Q)y\|$. This completes the proof. □

The results and ideas presented in this section can be translated into a sparse linear least squares solver algorithm described by algorithm 4.1.1 in §4.

3.2. Sparse structured nonlinear autoregressive model identification

consider the time series data $\Sigma\subset\mathbb{R}^n$ corresponding to an orbit determined by the difference equation

$$x_{t+1}=\mathcal{T}(x_t), \tag{3.1}$$

for some discrete-time dynamic financial model $(\hat{\mathbf{\Sigma}},\mathcal{T})$ to be identified. One may need to be preprocess the time series data before proceeding with the approximate representation of a suitable evolution operator. For this purpose, given some prescribed integer $L>0$, one can consider the time series $\mathcal{D}_L(\Sigma)$ determined by the expression.

$$\mathcal{D}_L(\Sigma)=\{\mathbf{x}_L(t)\}_{t\geq L}$$

For the dilated time series $\mathcal{D}_L(\Sigma)$, the previously considered recurrence relation $x_{t+1}=\mathcal{T}(x_t)$, $t\geq 1$, induces the following difference equations

$$\mathbf{x}_L(t+1)=\tilde{\mathcal{T}}(\mathbf{x}_L(t)), \tag{3.2}$$

for $t\geq L$. Where $\tilde{\mathcal{T}}$ is some evolution operator to be identified.

For any $p \geq 1$, let us consider the map $\eth_p : \mathbb{R}^n \to \mathbb{R}^{d_p(n)}$ for $d_p(n) = n(n^p - 1)/(n-1) + 1$, that is determined by the expression.

$$\eth_p(x) := \begin{bmatrix} \Pi_1(x) \\ \Pi_2(x) \\ \vdots \\ \Pi_p(x) \\ 1 \end{bmatrix} = \begin{bmatrix} x^{\otimes 1} \\ x^{\otimes 2} \\ \vdots \\ x^{\otimes p} \\ 1 \end{bmatrix}$$

Given integers $p, L > 0$, an orbit $\Sigma = \{x_t\}_{t \geq 1} \subset \mathbb{R}^n$ of a dynamic financial process. For a finite sample $\Sigma_N = \{x_t\}_{t=1}^T \subset \Sigma$, let us consider the matrices:

$$\begin{aligned} \mathbf{H}_L^{(0,p)}(\Sigma_T) &= \begin{bmatrix} \eth_p(\mathbf{x}_L(L)) & \cdots & \eth_p(\mathbf{x}_L(T-1)) \end{bmatrix} \\ \mathbf{H}_L^{(1)}(\Sigma_T) &= \begin{bmatrix} \mathbf{x}_L(L+1) & \cdots & \mathbf{x}_L(T) \end{bmatrix} \end{aligned} \tag{3.3}$$

The operator identification mechanism used in this study for dilated systems of the form (3.2), will be described by the expression:

$$\hat{\mathcal{T}}(\mathbf{x}_L(t)) = W\eth_p(\mathbf{x}_L(t)), \ \ t \geq L, \tag{3.4}$$

for some matrix $W \in \mathbb{R}^{n \times d_p(n)}$ to be determined, with $d_p(n) = n(n^p - 1)/(n-1) + 1$. Applying by the operator theoretic techniques and ideas previously presented in this section, the matrix W in (3.4) can be estimated by approximately solving the matrix equation

$$W\mathbf{H}_L^{(0,p)}(\Sigma_T) = \mathbf{H}_L^{(1)}(\Sigma_T). \tag{3.5}$$

The devices described by (3.4) are called autoregressive reservoir computers (ARC) in this paper.

For any given integers $L, n, p > 0$. Taking advantage of the maps $\eth_p$, one can find a integer $0 < r_p(n) < d_p(n)$ together with a sparse matrix $R_{p,L}(n) \in \mathbb{R}^{r_p(n) \times d_p(n)}$, such that $R_{p,L}(n)^+ R_{p,L}(n)\eth_p(x) \approx \eth_p(x)$ for $x \in \mathbb{R}^{nL}$. The existence of the pair $r_p(n), R_{p,L}(n)$ is determined by the following theorem.

Theorem 3.6. *Given $\varepsilon \in \mathbb{R}^+$ and $L, n, p \in \mathbb{Z}^+$. There are an integer $0 < \rho_p(n) < d_p(n)$ and a sparse matrix $R_{p,L}(n) \in \mathbb{R}^{\rho_p(n) \times d_p(n)}$ with $d_p(n)$ nonzero entries, such that $\|R_{p,L}(n)^+ R_{p,L}(n)\eth_p(x) - \eth_p(x)\| \leq \sqrt{d_p(nL)}\varepsilon$ for each $x \in \mathbb{R}^{nL}$.*

Proof. Let us consider the symmetric group $\mathfrak{S}_{nL-1}$ on $nL - 1$ letters, and let us consider any finite set of distinct points $\{\hat{x}_1, ..., \hat{x}_{nL}, \hat{x}_{nL+1}\} \subset \mathbb{R}$ such that for each $1 \leq m \leq p$ and every pair on index sets $\{i_1, \ldots, i_m\}, \{j_1, \ldots, j_m\} \subset \{1, \ldots, nL + 1\}$:

$$\prod_{k=1}^m \hat{x}_{j_k} = \prod_{k=1}^m \hat{x}_{i_k}, \Leftrightarrow \exists \sigma \in \mathfrak{S}_{nL-1} : i_k = \sigma(j_k), \forall 1 \leq k \leq m. \tag{3.6}$$

That is, the previously considered products coincide if and only if, the index set $\{i_1, \ldots, i_m\}$ is a permutation of the set $\{j_1, \ldots, j_m\}$, for each $1 \leq m \leq p$.

Let us now set $\mathbf{y} = \nu \begin{bmatrix} \hat{x}_1 & \hat{x}_2 & \cdots & \hat{x}_{nL} \end{bmatrix}^\top$, $d = d_p(n)$ and

$$\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{x}_1 & \cdots & \tilde{x}_d \end{bmatrix}^\top := \eth_p(\mathbf{y}).$$

If in addition we consider the assignments:

$$\tilde{x}_d := \hat{x}_{nL+1},$$

$$R = e_{1,d}^{\top}.$$

Then, for each $j = 2, \ldots, d$ one can find $1 \leq k_1(j), \ldots, k_{n_j}(j) \leq d$ such that $|\tilde{x}_j - \tilde{x}_{k_m(j)}| \leq \epsilon$, for every $1 \leq m \leq n_j$. Consequently, if we set $R_0 := (1/n_j) \sum_{l=1}^{n_j} \hat{e}_{l,d}^T$ and

$$R := \begin{bmatrix} R \\ R_0 \end{bmatrix},$$

whenever $k_1(j) = j$. After iterating on the previously described procedure for $2 \leq j \leq d$, we can define $R_{p,L}(n) := R$ and we can set the value of $\rho_p(n)$ as the number of rows of $R_{p,L}(n)$.

Given $x \in \mathbb{R}^{nL}$. Based on the structure of $R_{p,L}(n)$ determined by the constructive procedure used for its computation, it can be verified that $R_{p,L}(n)^+$ is well defined and in addition:

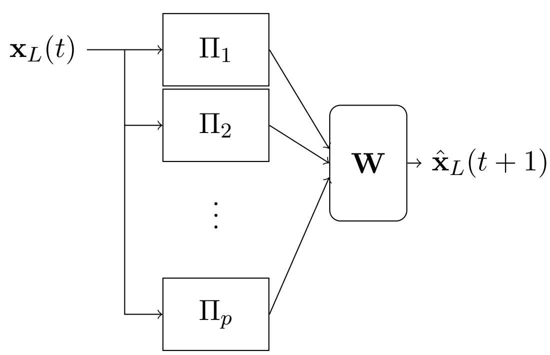
$$\|R_{p,L}(n)^+ R_{p,L}(n) \eth_p(x) - \eth_p(x)\| \leq \sqrt{d_p(n)}\varepsilon.$$

This completes the proof. $\square$

In order to reduce to computational effort corresponding to the solution of (3.5), using the matrix $R_{p,L}(n)$ described by Theorem 3.6, one can obtain an approximate reduced representation of (3.5) determined by the expression.

$$\bar{W} R_{p,L}(n) \mathbf{H}_L^{(0,p)}(\Sigma_T) = \mathbf{H}_L^{(1)}(\Sigma_T) \tag{3.7}$$

The architecture of the autoregressive reservoir computers considered in this study was inspired by next generation reservoir computers [3]. Schematically, the autoregressive models considered in this study can be described by a block diagram of the form,



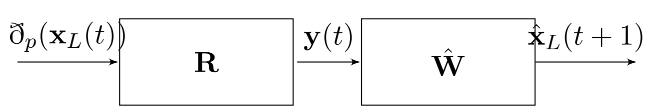
(3.8)

where for each $t \geq L$, the block $\mathbf{W}$ is determined by the expression

$$\begin{aligned} \mathbf{W}(\Pi_1(\mathbf{x}_L(t)), \ldots, \Pi_p(\mathbf{x}_L(t))) &:= \tilde{W} \begin{bmatrix} \Pi_1(\mathbf{x}_L(t)) \\ \vdots \\ \Pi_p(\mathbf{x}_L(t)) \end{bmatrix} + c_W \\ &= \begin{bmatrix} \tilde{W} & c_W \end{bmatrix} \eth_p(\mathbf{x}_L(t)) \end{aligned}$$

and where the matrix $W = \begin{bmatrix} \tilde{W} & c_W \end{bmatrix}$ determined by (3.5).

The structure of the generic block $\mathbf{W}$ ub (3.8) can be factoted in the form



(3.9)

The layers $\mathbf{R}$ and $\hat{\mathbf{W}}$ of the device (3.9) are determined by the expressions

$$\begin{aligned}\mathbf{R}(\mathbf{x}) &= \hat{R}\mathbf{x},\\ \hat{\mathbf{W}}(\mathbf{y}) &= W\hat{R}^{+}\mathbf{y}\end{aligned}$$

for any pair of suitable vectors $\mathbf{x}, \mathbf{y}$. Where W is a sparse representation of an approximate solution (3.5) and $\hat{R}$ is determined by Theorem 3.6.

Using some suitable training subset Σ_S of a given data sample Σ_T from an arbitrary signal data $\Sigma = \{x_t\}_{t\geq 1} \subset \mathbb{R}^n$ under consideration, the parameters of the block $\mathbf{W}$ of (3.8) are fitted using Σ_S. Simultaneously, a sparse representation of the matrix parameters corresponding to the block $\mathbf{W}$ of the resulting model is computed.

Using the reservoir computer models described by (3.4), (3.8) and (3.9), we can compute approximate representations of evolution operators that satisty (3.2) using the expression

$$\hat{\mathcal{T}}(\mathbf{x}_L(t)) := \hat{K}\left(\hat{\mathbf{W}} \circ \mathbf{R} \circ \eth_p(\mathbf{x}_L(t)) = \hat{K}W\eth_p(\mathbf{x}_L(t))\right), \tag{3.10}$$

for each $t \geq L$. Furthermore, we can use the identified RRC model $\hat{\mathcal{T}}$ to simulate the behavior $x_{t+1} = \mathcal{T}(x_t)$ of the system described by (3.1) for $L \leq t \leq \tau$, by peforming the operation:

$$\mathbf{T}\left(\mathbf{x}_L(t)\right) := \hat{K}\hat{\mathcal{T}}(\mathbf{x}_L(t)) = \hat{K}W\eth_p\left(\mathbf{x}_L(t)\right)), \tag{3.11}$$

for some $\tau > 0$ with

$$\hat{K} = \begin{bmatrix} \hat{e}_{1,nL}^{\top} \\ \hat{e}_{L+1,nL}^{\top} \\ \vdots \\ \hat{e}_{(n-1)L+1,nL}^{\top} \end{bmatrix}.$$

In this article, we will use the following notion of sparse representation. Given $\delta > 0$ and two matrices $A \in \mathbb{R}^{m\times n}$ and $X \in \mathbb{R}^{n\times p}$, a matrix $\hat{X} \in \mathbb{R}^{n\times p}$ is an approximate sparse representation of X, or a sparse representation of X for short, if $\|\hat{X}A - XA\|_F \leq C\delta$ for some $C > 0$ that does not depend on δ, and $\hat{X}$ has fewer nonzero entries than X.

Theorem 3.7. *Given $\delta > 0$, two integers $p, L > 0$, a finite group G, a sample $\Sigma_T = \{x_t\}_{t=1}^{T}$ from a financial modeling's orbit $\Sigma = \{x_t\}_{t\geq 1} \subset \mathbb{R}^n$ with $T > L$, and a matrix solvent $\bar{W} \in \mathbb{R}^{nL\times r_p(nL)}$ of (3.7) with $R_{p,L}(n)$ and $r_p(n)$ determined by Theorem 3.6. If $r = \mathrm{rk}_\delta(R_{p,L}(n)\mathbf{H}_L^{(0,p)}(\Sigma_T)) > 0$, then there is a sparse matrix $\hat{W} \in \mathbb{R}^{nL\times\rho_p(nL)}$ with at most $r\rho_p(nL)$ nonzero entries such that*

$$\|\hat{W}R_{p,L}(n)\mathbf{H}_L^{(0,p)}(\Sigma_T) - \bar{W}R_{p,L}(n)\mathbf{H}_L^{(0,p)}(\Sigma_T)\|_F \leq K\delta, \tag{3.12}$$

for $K = \sqrt{nL(\min\{\rho_p(nL), T-L\} - r)}(\sqrt{r}\|\hat{W}\|_F + \|\bar{W}\|_F)$, where $\rho_p(nL)$ is the integer described by Theorem 3.6.

Proof. Let us set $H = R_{p,L}(n)\mathbf{H}_{L,G}^{(0,p)}(\Sigma_T)^\top$ and $Y = H\bar{W}^\top$. It suffices to prove that there is a sparse matrix $\hat{W} \in \mathbb{R}^{nL\times\rho_p(nL)}$ with at most $r\rho_p(nL)$ nonzero entries such that

$$\|H\hat{W}^\top - Y\|_F \leq K\delta.$$

Since we have that

$$\mathrm{rk}_\delta(H) = \mathrm{rk}_\delta\left(\left(R_{p,L}(n)\mathbf{H}_{L,G}^{(0,p)}(\Sigma_T)\right)^\top\right) = \mathrm{rk}_\delta(R_{p,L}(n)\mathbf{H}_{L,G}^{(0,p)}(\Sigma_T)) > 0$$

by Lemma 3.2. By Corollary 3.5, if we set $r = \mathrm{rk}_\delta(H)$ and $\alpha = \sqrt{r(\min\{\rho_p(nL), T-L\} - r)}$. We will have that there is a rank r orthogonal projector Q such that for each $j = 1, \ldots, nL$, there is $\hat{v}_j \in \mathbb{R}^{nL}$ with at most r nonzero entries, for which $\|H\hat{v}_j - Y\hat{e}_{j,M}\| \leq \alpha\|\hat{v}_j\|\delta + \|(I_{T-L} - Q)Y\hat{e}_{j,nL}\|$. Consequently, if we set

$$\hat{W} = \begin{bmatrix} | & & | \\ \hat{v}_1 & \cdots & \hat{v}_{nL} \\ | & & | \end{bmatrix}^\top$$

we will have that $\hat{W}$ has at most nrL nonzero entries and

$$\begin{aligned} \|H\hat{W}^\top - Y\|_F^2 &= \sum_{j=1}^{nL} \|H\hat{v}_j - Y\hat{e}_{j,nL}\|^2 \\ &\leq M(\alpha\|\hat{W}\|_F\delta + \|(I_{T-L} - Q)Y\|_F)^2, \end{aligned}$$

and this in turn implies that,

$$\|H\hat{W}^\top - Y\|_F \leq \sqrt{nL}(\alpha\|\hat{W}\|_F\delta + \|(I_{T-L} - Q)H\|_F\|\bar{W}\|_F). \tag{3.13}$$

By (3.13) and by Theorem 3.4 we will have that

$$\begin{aligned} \|H\hat{W}^\top - Y\|_F &\leq \sqrt{nL}(\alpha\|\hat{W}\|_F\delta + (\alpha/\sqrt{r})\|\bar{W}\|_F\delta) \\ &= \alpha\sqrt{(nL/r)}(\sqrt{r}\|\hat{A}\|_F + \|A\|_F)\delta = K\delta. \end{aligned}$$

This completes the proof. □

4. Algorithms

The sparse model identification methods presented in §3.1 can be translated into prototypical algorithms that will be presented in this section, some programs for data reading and writing, synthetic signals generation, and predictive simulation are also include as part of the **SPORT** tool-set available in [6].

4.1. Sparse linear least squares solver and structured assembling matrix identification algorithms

As an application of the results and ideas presented in §3.1 one can obtain a prototypical sparse linear least squares solver algorithm like algorithm 4.1.1.

The least squares problems $c = \arg\min_{\hat{c}\in\mathbb{C}^K} \|\hat{A}\hat{c} - y\|$ to be solved as part of the process corresponding to algorithm 4.1.1 can be solved with any efficient least squares solver available in the language or program where the sparse linear least squares solver algorithm is implemented. For the Matlab and Julia implementations of algorithm 4.1.1 written as part of this research project the `backslash "\"` operator is used, and for the Python version of algorithm 4.1.1 the function `lstsq` is implemented.

In this section we focus on the applications of the structured matrix approximation methods presented in S3, to dynamical financial systems identification via autoregressive reservoir computers.

4.2. Structured coupling matrix identification algorithm

Given discrete-time dynamic financial model $(\Sigma, \mathscr{T})$ and a structured data sample $\Sigma_T \subset \Sigma$, we can apply Algorithm 4.1.2 and Algorithm 4.1.1, in order to to compute the otuput coupling matrix that can be used to obtain an approximate representation of the evoulution operator $\mathcal{T}$, corresponding to the orbit Σ. For this purpose one can use the following Algorithm.

Algorithm 4.1.1 SLRSolver: Sparse linear least squares solver algorithm

Data: $A \in \mathbb{C}^{m \times n}$, $Y \in \mathbb{C}^{m \times p}$, $\delta > 0$, $N \in \mathbb{Z}^+$, $\varepsilon > 0$
Result: $X = \mathbf{SLRSolver}(A, Y, \delta, N, \varepsilon)$

1. Compute economy-sized SVD $USV = A$
2. Set $s = \min\{m, n\}$
3. Set $r = \mathrm{rk}_\delta(A)$
4. Set $U_\delta = \sum_{j=1}^{r} U\hat{e}_{j,s}\hat{e}_{j,s}^*$
5. Set $T_\delta = \sum_{j=1}^{r} (\hat{e}_{j,s}^* S\hat{e}_{j,s})^{-1}\hat{e}_{j,s}\hat{e}_{j,s}^*$
6. Set $V_\delta = \sum_{j=1}^{r} \hat{e}_{j,s}\hat{e}_{j,s}^* V$
7. Set $\hat{A} = U_\delta^* A$
8. Set $\hat{Y} = U_\delta^* Y$
9. Set $X_0 = V_\delta^* T_\delta \hat{Y}$
10. **for** $j = 1, \ldots, p$ **do**
    - Set $K = 1$
    - Set error $= 1 + \delta$
    - Set $c = X_0 \hat{e}_{j,p}$
    - Set $x_0 = c$
    - Set $\hat{c} = \begin{bmatrix} \hat{c}_1 & \cdots & \hat{c}_n \end{bmatrix}^\top = \begin{bmatrix} |\hat{e}_{1,n}^* c| & \cdots & |\hat{e}_{n,n}^* c| \end{bmatrix}^\top$
    - Compute permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that: $\hat{c}_{\sigma(1)} \geq \hat{c}_{\sigma(2)} \geq \cdots \geq \hat{c}_{\sigma(n)}$
    - Set $N_0 = \max\left\{\sum_{j=1}^{n} H_\varepsilon\left(\hat{c}_{\sigma(j)}\right), 1\right\}$
    - **while** $K \leq N$ **and** error $> \delta$ **do**
        - Set $x = \mathbf{0}_{n,1}$
        - Set $A_0 = \sum_{j=1}^{N_0} \hat{A}\hat{e}_{\sigma(j),n}\hat{e}_{j,N_0}^*$
        - Solve $c = \arg\min_{\tilde{c} \in \mathbb{C}^{N_0}} \|A_0\tilde{c} - \hat{Y}\hat{e}_{j,p}\|$
        - **for** $k = 1, \ldots, N_0$ **do**
            - Set $x_{\sigma(k)} = \hat{e}_{k,N_0}^* c$
        - **end for**
        - Set error $= \|x - x_0\|_\infty$
        - Set $x_0 = x$
        - Set $\hat{c} = \begin{bmatrix} \hat{c}_1 & \cdots & \hat{c}_n \end{bmatrix}^\top = \begin{bmatrix} |\hat{e}_{1,n}^* x| & \cdots & |\hat{e}_{n,n}^* x| \end{bmatrix}^\top$
        - Compute permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that: $\hat{c}_{\sigma(1)} \geq \hat{c}_{\sigma(2)} \geq \cdots \geq \hat{c}_{\sigma(n)}$
        - Set $N_0 = \max\left\{\sum_{j=1}^{n} H_\varepsilon\left(\hat{c}_{\sigma(j)}\right), 1\right\}$
        - Set $K = K + 1$
    - **end while**
    - Set $x_j = x$
11. **end for**
12. Set $X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_p \\ | & | & & | \end{bmatrix}$

return X

Algorithm 4.1.2 Compression matrix computation algorithm

Data: $n, p, L \in \mathbb{Z}^+$, $\nu, \varepsilon \in \mathbb{R}^+$.
Result: Compression matrix factor: $R_{p,L}(n)$

1. Choose nL pseudorandom numbers $\hat{x}_1, \ldots, \hat{x}_{nL} \in \mathbb{R}$ from $N(0,1)$
2. Set $\mathbf{y} = \nu \begin{bmatrix} \hat{x}_1 & \hat{x}_2 & \cdots & \hat{x}_{nL} \end{bmatrix}^\top$
3. Set $d = d_p(n)$
4. Set $\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{x}_1 & \cdots & \tilde{x}_d \end{bmatrix}^\top := \eth_p(\mathbf{y})$
5. Choose a pseudorandom number $\alpha \in N(0,1)$;
6. Set $\tilde{x}_d := \alpha$
7. Set $R = e_{1,d}^\top$
8. **for** $j = 2, \ldots, d$ **do**
   Find $1 \leq k_1, \ldots, k_{n_j} \leq d$ such that $|\tilde{x}_j - \tilde{x}_{k_m}| \leq \varepsilon$, for each $1 \leq m \leq n_j$
   if $k_1 = j$ **then**
   Set $R_0 := (1/n_j)\sum_{l=1}^{n_j} \hat{e}_{l,d}^T$
   Set $R := \begin{bmatrix} R \\ R_0 \end{bmatrix}$
   end if
9. **end for**
10. Set $R_{p,L}(n) := R$

return $R_{p,L}(n)$

Algorithm 4.2.1 ARC Model: ARC model identification

Data: $\Sigma_N = \{x_t\}_{t=1}^T \subset \mathbb{R}^n$, $\boldsymbol{\pi} : G \to \mathbb{O}(n)$
Result: Output coupling and compression matrices: $\hat{W}, \tilde{W}, R_{p,L}(n)$

1. Choose or estimate the lag value L using auto-correlation function based methods
2. Set a tensor order value p
3. Compute compression matrix $R_{p,L}(n)$ applying Algorithm 4.1.2
4. Compute matrices:

$$\mathbf{H}_0 := \mathbf{H}_{L,G}^{(0,p)}(\Sigma_T)$$
$$\mathbf{H}_1 := \mathbf{H}_{L,G}^{(1)}(\Sigma_T)$$

5. Approximately solve:

$$\hat{W}\left(R_{p,L}(n)\mathbf{H}_0\right) = \mathbf{H}_1$$

   for $\hat{W}$ applying Algorithm 4.1.1

return $\hat{W}, R_{p,L}(n)$

5. Numerical Simulations

In this section we will present some numerical simulations computed using the **SPORT** toolset available in [6], that was developed as part of this project, the toolset consists of a collection of Matlab programs for structured sparse identification and numerical simulation of discrete-time dynamical financial systems.

The numerical experiments documented in this section were performed with Matlab R2023a 64-bit. All the programs written for real world data reading, synthetic data generation and sparse model identification as part of this project are available at [6].

The numerical simulations reported in this section were computed on a Linux Ubuntu Server 20.04 PC equiped with an Intel Xeon E3-1225 v5 (8M Cache, 3.30 GHz) processor and with 40GB RAM.

5.1. Sparse autoregressive reservoir computers for dynamical nonlinear financial system behavior identification

In this section we consider a nonlinear dynamical system described by the model

$$\begin{aligned}
\dot{x}_1 &= x_3 + (x_2 - s)x_1, \\
\dot{x}_2 &= 1 - cx_2 - x_1^2, \\
\dot{x}_2 &= -x_1 - ex_3, \\
x_1(0) &= x_0, x_2(0) = y_0, x_3(0) = z_0.
\end{aligned} \tag{5.1}$$

As observed in [1] systems of the form (5.1), can exhibit among others behavior types, chaotic and eventually approximately periodic dynamic behavior depending on the configuration of parameters and initial conditions considered for (5.1).

5.1.1. Chaotic behavior identification

For $s = 3, c = 0.1, e = 1$, let us consider the initial conditions $x_0 = 2$,$y_0 = 3$,$z_0 = 2$. For this configuration one can obtain synthetic time series data $\Sigma_{14633} \subset \mathbb{R}$ obtained by applying a fourth order adaptive numerical integration method to (5.1) for the configuration determined by the previous choice of parameters, followed by a spline interpolation to guarantee that the elements in Σ_{14633} are uniformly distributed with respect to the time interval [0,120].

The training orbit's data set corresponding to the first 61% of the data in Σ_{14633}, together with the remaining data used for model validation are illustrated in Figure 1. The factorization for the output coumpling matrix $W = \hat{W}R$ determined by Theorems 3.6 and 3.7 are illustrated in Figure 2.

5.1.2. Eventually approximately periodic behavior identification

For $s = 0.5, c = 0.1, e = 0.1$, let us consider the initial conditions $x_0 = 1$,$y_0 = 1$,$z_0 = 1$. For this configuration one can obtain synthetic time series data $\Sigma_{22901} \subset \mathbb{R}$ obtained by applying a fourth order adaptive numerical integration method to (5.1) for the configuration determined by the previous choice of parameters, followed by a spline interpolation to guarantee that the elements in Σ_{22901} are uniformly distributed with respect to the time interval [0,120].

The training orbit's data set corresponding to the first 10% of the data in Σ_{22901}, together with the remaining data used for model validation are illustrated in Figure 3. The factorization for the output coumpling matrix $W = \hat{W}R$ determined by Theorems 3.6 and 3.7 are illustrated in Figure 4.

The computational setting used for the experiments performed in this section is documented in the Matlab programs `SpaRC.m` and `NLFinancialSystem.m` in [6] that can be used to replicate these results.

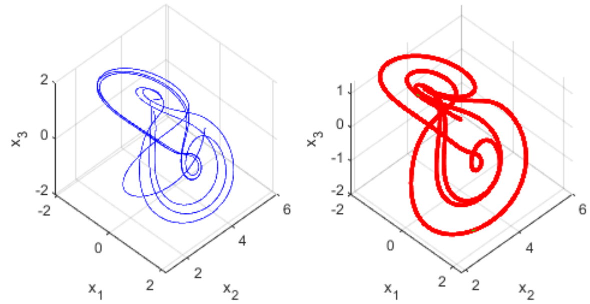


Figure 1: Training orbits data (left), validation orbits data (right). The green line corresponds to validation data, and the red dotted line corresponds to the model's predictions.

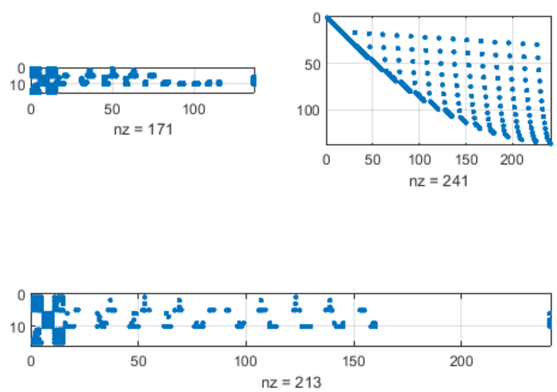


Figure 2: Matrix factors $\hat{W}$ (top-left) and R (top-right), output coupling matrix $W = \hat{W}R$ (bottom).

5.1.3. Sparse identification of interest rate forecasting models

For this example we will use Honduran lending interest rates time series dataset Σ_{40} recorded by the International Monetary Fund and the World Bank, that was collected between 1982 and 2021 and is available in [2] and also as part of the SPORT in [6]. For this experiment we will consider two sparse autoregressive reservoir computer models. One of the models consists of a fully linear sparse autoregressive reservoir computer model determined by the expression

$$\left(1 - \sum_{j=1}^{12} L^j\right) \lambda_t = e_\lambda(t),$$

where λ_t denotes the lending interest rates corresponding to year t, and where L denotes the lag operator determined for this case by the local condition $L^j \pi_t = \pi_{t-j}$, and where e_λ denotes the residual term determined by the covariance matrix and innovations corresponding to the model. The other model under consideration is a sparse nonlinear autoregressive reservoir computer model. Both models will be trained with the the first 75% of the data in the time series sample Σ_{40}.

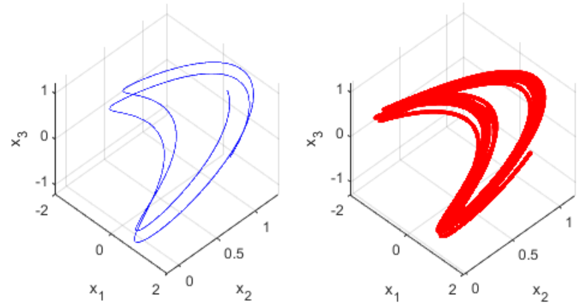


Figure 3: Training orbits data (left), validation orbits data (right). The green line corresponds to validation data, and the red dotted line corresponds to the model's predictions.

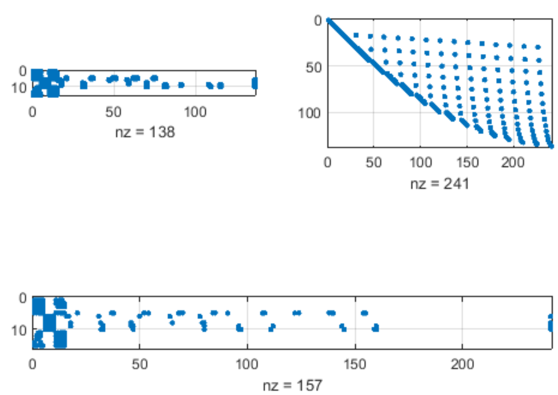
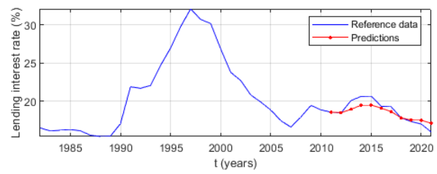


Figure 4: Matrix factors $\hat{W}$ (top-left) and R (top-right), output coupling matrix $W = \hat{W}R$ (bottom).

The reference data along with the models predictions are shown in figure 5.

The combination of fully linear with nonlinear sparse ARC models considered in this experiment is aimed to provide an example of the potential that this model combination has to capture both short term and long term behavior of a given financial system under consideration.

The computational setting used for the experiments performed in this section is documented in the Matlab program `IntRatePredictor.m` in [6], and this program can be used to replicate these experiments.



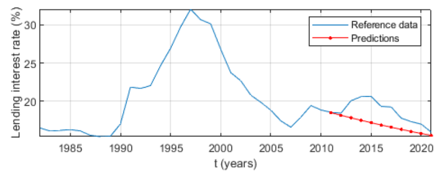


Figure 5: Reference signal (blue line) and model prediction (red dotted line) computed with sparse fully linear autoregressive reservoir computer model (top). Reference signal (blue line) and model prediction (red dotted line) computed with sparse nonlinear autoregressive reservoir computer (bottom).

6. Conclusion

The results in §3.1 and §3.2 in the form of algorithms like the ones described in §4, can be effectively used for the sparse structured identification of financial dynamical models that can be used to compute data-driven predictive numerical simulations.

7. Future Directions

Further implementations of the the structured sparse model identification algorithms presented in this document to the computation of dynamic general equilibrium models, will be the subject of future communications. The applications of sparse ARC models to inflation dynamics forecasting will be studied in future communications.

Data Availability

The programs and data that support the findings of this study are openly available in the SPORT repository, reference number [6].

Conflicts of Interest

The author declares that he has no conflicts of interest.

Acknowledgment

The structure preserving matrix computations needed to implement the algorithms in §4, were performed with Matlab 9.14.0.2206163 (R2023a) 64-bit, with the support and computational resources of the National Commission of Banks and Insurance Companies (**CNBS**) of Honduras.

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