Existence, uniqueness, and numerical analysis of a time-dependent problem for linearly elastic flexural shells

Joint work with Luisa Piersanti and Xiaoqin Shen

Paolo Piersanti

Institute of Mathematics and Scientific Computing
University of Graz, Austria

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Let ω be a domain in $\mathbb{R}^2$ with boundary γ. Let $\gamma_0 \subset \gamma$.

- The map $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{E}^3$ models a surface $\boldsymbol{\theta}(\overline{\omega})$ in the Euclidean space $\mathbb{E}^3$
- $\boldsymbol{a}_\alpha(y) := \partial_\alpha \boldsymbol{\theta}(y) \qquad \boldsymbol{a}^3(y) = \boldsymbol{a}_3(y) := \dfrac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}$
- $\boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta^\alpha_\beta, \qquad a := \det(a_{\alpha\beta})$
- First Fundamental Form: $a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta \quad a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$
- Second Fundamental Form: $b_{\alpha\beta} := \boldsymbol{a}^3 \cdot \partial_\beta \boldsymbol{a}_\alpha, \quad b^\beta_\alpha := a^{\beta\sigma} \cdot b_{\sigma\alpha}$
- $a^{\alpha\beta\sigma\tau,\varepsilon} := \dfrac{2\lambda^\varepsilon \mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon} a^{\alpha\beta} a^{\sigma\tau} + 2\mu^\varepsilon (a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma})$
- $\Gamma^\sigma_{\alpha\beta} := \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}^\sigma$
- $\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \dfrac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma^\sigma_{\alpha\beta} \eta_\sigma - b_{\alpha\beta} \eta_3$
- $$\begin{aligned} \rho_{\alpha\beta}(\boldsymbol{\eta}) := {} & \partial_{\alpha\beta} \eta_3 - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \eta_3 - b^\sigma_\alpha b_{\sigma\beta} \eta_3 + b^\sigma_\alpha (\partial_\beta \eta_\sigma - \Gamma^\tau_{\beta\sigma} \eta_\tau) \\ & + b^\tau_\beta (\partial_\alpha \eta_\tau - \Gamma^\sigma_{\alpha\tau} \eta_\sigma) + (\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma} b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta} b^\tau_\sigma) \eta_\tau \end{aligned}$$

Time-dependent version of linearised change of metric tensor

Recall that

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3.$$

Define functions

$$\tilde{\gamma}_{\alpha\beta} : L^2(0,T;H^1(\omega) \times H^1(\omega) \times L^2(\omega)) \to L^2(0,T;L^2(\omega)),$$

by

$$\tilde{\gamma}_{\alpha\beta}(\boldsymbol{\eta})(t) := \gamma_{\alpha\beta}(\boldsymbol{\eta}(t)) \text{ for all } \boldsymbol{\eta} \in L^2(0,T;H^1(\omega) \times H^1(\omega) \times L^2(\omega)),$$

for a.a. $t \in (0,T)$.
These functions are linear and continuous.

Time-dependent version of linearised change of curvature tensor

Recall that

$$\rho_{\alpha\beta}(\boldsymbol{\eta}) := \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 - b^\sigma_\alpha b_{\sigma\beta}\eta_3 + b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau)$$
$$+ b^\tau_\beta(\partial_\alpha\eta_\tau - \Gamma^\sigma_{\alpha\tau}\eta_\sigma) + (\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma)\eta_\tau$$

Define functions

$$\tilde{\rho}_{\alpha\beta} : L^2(0,T;H^1(\omega) \times H^1(\omega) \times H^2(\omega)) \to L^2(0,T;L^2(\omega)),$$

by

$$\tilde{\rho}_{\alpha\beta}(\boldsymbol{\eta})(t) := \rho_{\alpha\beta}(\boldsymbol{\eta}(t)) \text{ for all } \boldsymbol{\eta} \in L^2(0,T;H^1(\omega) \times H^1(\omega) \times H^2(\omega)),$$

for a.a. $t \in (0,T)$.
These functions are linear and continuous.

$$\boxed{\Omega^\varepsilon := \omega \times]-\varepsilon, \varepsilon[}$$

- $\Gamma_0^\varepsilon := \gamma_0 \times [-\varepsilon, \varepsilon]$
- $\Gamma_\pm^\varepsilon := \omega \times \{\pm\varepsilon\}$
- $\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon) := \boldsymbol{\theta}(y) + x_3^\varepsilon \boldsymbol{a}^3(y),$ for all $x^\varepsilon = (y_1, y_2, x_3^\varepsilon) \in \overline{\Omega^\varepsilon}$
- $\boldsymbol{g}_i^\varepsilon(x^\varepsilon) := \partial_i^\varepsilon \boldsymbol{\Theta}(x^\varepsilon), \qquad \boldsymbol{g}^{j,\varepsilon}(x^\varepsilon) \cdot \boldsymbol{g}_i^\varepsilon(x^\varepsilon) = \delta_i^j$

The shell is subjected to *applied body forces* whose density per unit volume is defined by means of its *contravariant* components

$$f^{i,\varepsilon} \in L^\infty(0,T;L^2(\Omega^\varepsilon)),$$

and to *applied surface forces* whose density per unit area is defined by means of its *contravariant* components

$$h^{i,\varepsilon} \in L^\infty(0,T;L^2(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon)).$$

For a.a. $t \in (0,T)$, we define the *dynamic load* acting on the shell via its contravariant components

$$p^{i,\varepsilon}(t) := \left\{ \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}(t)\, \mathrm{d}x_3^\varepsilon + h_+^{i,\varepsilon}(t) + h_-^{i,\varepsilon}(t) \right\} \in L^2(\omega) \text{ for a.a. } t \in (0,T),$$

where $h_\pm^{i,\varepsilon}(t) := h^{i,\varepsilon}(t)(\,\cdot\,, \pm\varepsilon) \in L^2(\omega)$, for a.a. $t \in (0,T)$.

Definition of linearly elastic flexural shell

A linearly elastic shell is said to be a *linearly elastic flexural shell* (from now on *flexural shell*) if the following *two additional assumptions* are satisfied

(1) length $\gamma_0 > 0$, i.e., the homogeneous boundary condition of place is imposed over *a portion of the lateral face* $\gamma \times [-\varepsilon, \varepsilon]$ of the shell,

(2) the space of *admissible linearized inextensional displacements*

$$\boxed{\begin{aligned} \boldsymbol{V}_F(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0 \text{ and } \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\} \end{aligned}}$$

contains nonzero functions; equivalently,

$$\boldsymbol{V}_F(\omega) \neq \{\boldsymbol{0}\}.$$

Two examples of flexural shells

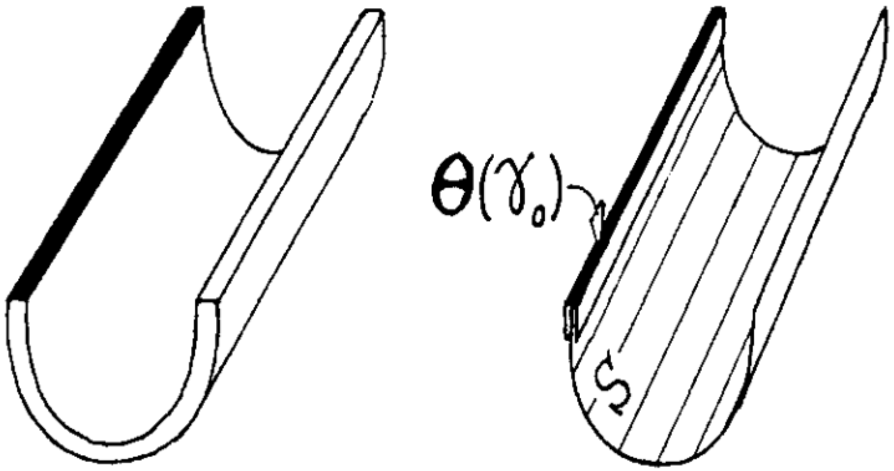

Figure: Figure 6.1-1 of P. G. Ciarlet, Mathematical Elasticity. Volume III: Theory of Shells. *Two linearly elastic flexural shells.* A shell whose middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ is a portion of a cylinder and which is subjected to a boundary condition of place (i.e., of vanishing displacement field) along a portion (darkened on the figure) of its lateral face whose middle curve $\boldsymbol{\theta}(\gamma_0)$ is contained in one or two generatrices of S, provides an instance of a linearly elastic flexural shell, i.e., one for which the space $\boldsymbol{V}_F(\omega)$ contains nonzero functions $\boldsymbol{\eta}$.

Two examples of flexural shells (Continued)

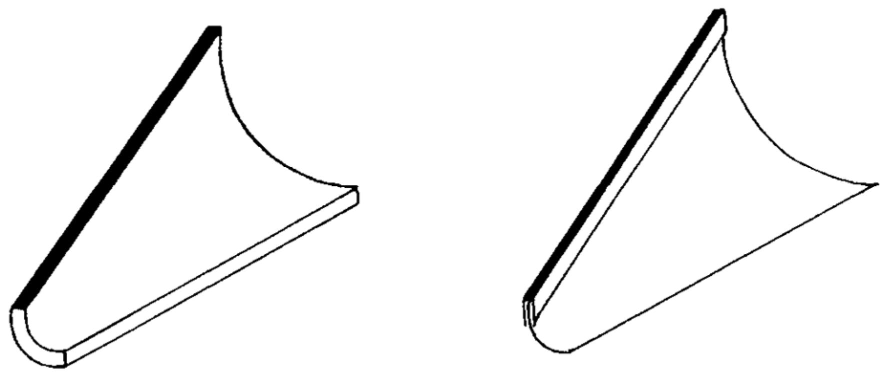

Figure: Figure 6.1-2 of P. G. Ciarlet, Mathematical Elasticity. Volume III: Theory of Shells. *Two linearly elastic flexural shells.* A shell whose middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ is a portion of a cone excluding its vertex and which is subjected to a boundary condition of place along a portion (darkened on the figure) of its lateral face whose middle curve $\boldsymbol{\theta}(\gamma_0)$ is contained in one generatrix of S, provides another instance of a linearly elastic flexural shell.

Korn's inequality for general surfaces

Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ be an immersion. Let γ_0 be a $\mathrm{d}\gamma$-measurable relatively open subset of $\gamma = \partial\omega$ that satisfies

$$\operatorname{length} \gamma_0 > 0$$

and let the space $\boldsymbol{V}_K(\omega)$ be defined as (∂_ν denotes the outer normal derivative operator along γ)

$$\boxed{\boldsymbol{V}_K(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\}.}$$

Then there exists a constant $c_F = c_F(\omega, \gamma_0, \boldsymbol{\theta}) > 0$ such that

$$\left\{\sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + \|\eta_3\|_{2,\omega}^2\right\}^{1/2} \leq c_F \left\{\sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2 + \sum_{\alpha,\beta} |\rho_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2\right\}^{1/2}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_K(\omega)$.

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Problem $\mathcal{P}_F^{\varepsilon}(\omega)$ (Xiao, 2001)

Find a vector field $\boldsymbol{\zeta}^{\varepsilon} = (\zeta_i^{\varepsilon}) : (0,T) \to \boldsymbol{V}_F(\omega)$ such that

$$
\begin{aligned}
\boldsymbol{\zeta}^{\varepsilon} &\in L^{\infty}(0,T;\boldsymbol{V}_F(\omega)),\\
\dot{\boldsymbol{\zeta}}^{\varepsilon} &\in L^{\infty}(0,T;\boldsymbol{L}^2(\omega)),\\
\ddot{\boldsymbol{\zeta}}^{\varepsilon} &\in L^{\infty}(0,T;\boldsymbol{V}_F^*(\omega)),
\end{aligned}
$$

that satisfies the following variational equations

$$
2\varepsilon^3\rho\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\omega}\zeta_i^{\varepsilon}(t)\eta_i\sqrt{a}\,\mathrm{d}y+\frac{\varepsilon^3}{3}\int_{\omega}a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta}^{\varepsilon}(t))\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y=\int_{\omega}p^{i,\varepsilon}(t)\eta_i\sqrt{a}\,\mathrm{d}y,
$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_F(\omega)$, in the sense of distributions in $(0,T)$, and that satisfies the following initial conditions

$$
\begin{cases}
\boldsymbol{\zeta}^{\varepsilon}(0) = \boldsymbol{\zeta}_0,\\
\dot{\boldsymbol{\zeta}}^{\varepsilon}(0) = \boldsymbol{\zeta}_1,
\end{cases}
\tag{IC}
$$

where $\boldsymbol{\zeta}_0 \in \boldsymbol{V}_F(\omega)$ and $\boldsymbol{\zeta}_1 \in \boldsymbol{L}^2(\omega)$ are prescribed.

Rigorous definition of the concept of solution

We say that $\boldsymbol{\zeta}^\varepsilon$ is a *weak solution* of Problem $\mathcal{P}_F^\varepsilon(\omega)$ if

$$\begin{aligned}
\boldsymbol{\zeta}^\varepsilon &\in L^\infty(0,T;\boldsymbol{V}_F(\omega)),\\
\dot{\boldsymbol{\zeta}}^\varepsilon &\in L^\infty(0,T;\boldsymbol{L}^2(\omega)),\\
\ddot{\boldsymbol{\zeta}}^\varepsilon &\in L^\infty(0,T;\boldsymbol{V}_F^*(\omega)),
\end{aligned}$$

if $\boldsymbol{\zeta}^\varepsilon$ satisfies the variational equations of Problem $\mathcal{P}_F^\varepsilon(\omega)$ in the sense of distributions in $(0,T)$, and also satisfies the initial conditions (IC).

We say that $\boldsymbol{\zeta}^\varepsilon$ is a *strong solution* of Problem $\mathcal{P}_F^\varepsilon(\omega)$ if

$$\boldsymbol{\zeta}^\varepsilon \in \mathcal{C}^0([0,T];\boldsymbol{V}_F(\omega)) \cap \mathcal{C}^1([0,T];\boldsymbol{L}^2(\omega)),$$

if $\boldsymbol{\zeta}^\varepsilon$ satisfies the variational equations of Problem $\mathcal{P}_F^\varepsilon(\omega)$ in the sense of distributions in $(0,T)$, and also satisfies the initial conditions (IC).

Existence and uniqueness of strong solutions of Problem $\mathcal{P}_F^{\varepsilon}(\omega)$. Classical approach (Lions, 1969)

Define the space

$$\boldsymbol{H}_F(\omega) := \overline{\boldsymbol{V}_F(\omega)}^{\|\cdot\|_{\boldsymbol{L}^2(\omega)}},$$

and observe that it is a closed subspace of $\boldsymbol{L}^2(\omega)$.

Identify the space $\boldsymbol{H}_F(\omega)$ with its dual, and equip it with the same standard inner product $(\cdot\,,\,\cdot)_{\boldsymbol{L}^2(\omega)}$ of $\boldsymbol{L}^2(\omega)$.

Observe that the following chain of compact embeddings holds

$$\boldsymbol{V}_F(\omega) \hookrightarrow\hookrightarrow \boldsymbol{H}_F(\omega) \hookrightarrow\hookrightarrow \boldsymbol{V}_F^*(\omega).$$

Derive the existence and uniqueness.

$\square$

Existence and uniqueness of strong solutions of Problem $\mathcal{P}_F^\varepsilon(\omega)$ by penalty method

From now onward, we identify $L^2(\omega)$ and $\boldsymbol{L}^2(\omega)$ with their respective dual spaces, and we equip them with the following inner products

$$(\eta, \xi) \in L^2(\omega) \times L^2(\omega) \to \int_\omega \eta\xi\sqrt{a}\,\mathrm{d}y,$$
$$(\boldsymbol{\eta}, \boldsymbol{\xi}) \in \boldsymbol{L}^2(\omega) \times \boldsymbol{L}^2(\omega) \to \int_\omega \eta_i\xi_i\sqrt{a}\,\mathrm{d}y.$$

Let $\kappa > 0$ denote the *penalty parameter*.

We consider the following chain of compact embeddings

$$\boldsymbol{V}_K(\omega) \hookrightarrow\hookrightarrow \boldsymbol{L}^2(\omega) \hookrightarrow\hookrightarrow \boldsymbol{V}_K^*(\omega).$$

The penalised problem: Problem $\mathcal{P}^{\varepsilon}_{F,\kappa}(\omega)$

Find a vector field $\boldsymbol{\zeta}^{\varepsilon}_{\kappa} = (\zeta^{\varepsilon}_{i,\kappa}) : [0,T] \to \boldsymbol{V}_K(\omega)$ such that

$$\boldsymbol{\zeta}^{\varepsilon}_{\kappa} \in \mathcal{C}^0([0,T]; \boldsymbol{V}_K(\omega)) \cap \mathcal{C}^1([0,T]; \boldsymbol{L}^2(\omega)),$$

that satisfies the following variational equations

$$2\varepsilon^3 \rho \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_\omega \zeta^{\varepsilon}_{i,\kappa}(t) \eta_i \sqrt{a} \,\mathrm{d}y + \frac{\varepsilon^3}{3} \int_\omega a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\zeta}^{\varepsilon}_{\kappa}(t)) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} \,\mathrm{d}y$$
$$+ \frac{1}{\kappa} \int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}^{\varepsilon}_{\kappa}(t)) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} \,\mathrm{d}y = \int_\omega p^{i,\varepsilon}(t) \eta_i \sqrt{a} \,\mathrm{d}y,$$

for all $\boldsymbol{\eta} \in \boldsymbol{V}_K(\omega)$, in the sense of distributions in $(0,T)$, and which satisfies the initial conditions

$$\begin{cases} \boldsymbol{\zeta}^{\varepsilon}(0) = \boldsymbol{\zeta}_0, \\ \dot{\boldsymbol{\zeta}}^{\varepsilon}(0) = \boldsymbol{\zeta}_1, \end{cases}$$

where $\boldsymbol{\zeta}_0 \in \boldsymbol{V}_F(\omega)$ and $\boldsymbol{\zeta}_1 \in \boldsymbol{L}^2(\omega)$ are prescribed.

Theorem

Problem $\mathcal{P}^{\varepsilon}_{F,\kappa}(\omega)$ admits a unique strong solution

$$\boldsymbol{\zeta}^{\varepsilon}_{\kappa} \in \mathcal{C}^0([0,T];\boldsymbol{V}_K(\omega)) \cap \mathcal{C}^1([0,T];\boldsymbol{L}^2(\omega)).$$

Besides, up to passing to a suitable subsequence, the sequence $(\boldsymbol{\zeta}^{\varepsilon}_{\kappa})_{\kappa>0}$ satisfies the following convergences

$$\begin{aligned}
\boldsymbol{\zeta}^{\varepsilon}_{\kappa} &\overset{*}{\rightharpoonup} \boldsymbol{\zeta}^{\varepsilon}, && \text{in } L^{\infty}(0,T;\boldsymbol{V}_K(\omega)),\\
\dot{\boldsymbol{\zeta}}^{\varepsilon}_{\kappa} &\overset{*}{\rightharpoonup} \dot{\boldsymbol{\zeta}}^{\varepsilon}, && \text{in } L^{\infty}(0,T;\boldsymbol{L}^2(\omega)),\\
\ddot{\boldsymbol{\zeta}}^{\varepsilon}_{\kappa} &\overset{*}{\rightharpoonup} \ddot{\boldsymbol{\zeta}}^{\varepsilon}, && \text{in } L^{\infty}(0,T;\boldsymbol{V}^*_F(\omega)),\\
\boldsymbol{\zeta}^{\varepsilon}_{\kappa} &\to \boldsymbol{\zeta}^{\varepsilon}, && \text{in } \mathcal{C}^0([0,T];\boldsymbol{L}^2(\omega)),\\
\dot{\boldsymbol{\zeta}}^{\varepsilon}_{\kappa} &\rightharpoonup \dot{\boldsymbol{\zeta}}^{\varepsilon}, && \text{in } \mathcal{C}^0([0,T];\boldsymbol{V}^*_F(\omega)),\\
\tilde{\gamma}_{\alpha\beta}(\boldsymbol{\zeta}^{\varepsilon}_{\kappa}) &\rightharpoonup \tilde{\gamma}_{\alpha\beta}(\boldsymbol{\zeta}^{\varepsilon}), && \text{in } L^2(0,T;L^2(\omega)),
\end{aligned}$$

where $\boldsymbol{\zeta}^{\varepsilon}$ is the unique strong solution of Problem $\mathcal{P}^{\varepsilon}_{F}(\omega)$. □

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Conical shells

Define the domain

$$\omega := \{(y_1, y_2) \in \mathbb{R}^2;\ 0 < y_1 < \pi \text{ and } 0 < y_2 < 1\},$$

and assume that the shell is clamped along

$$\gamma_0 := \{(y_1, y_2) \in \mathbb{R}^2;\ y_1 = \pi, y_2 \in [0, 1]\}.$$

Define the middle surface parametrisation in curvilinear coordinates

$$\boldsymbol{\theta}(y_1, y_2) := (r \cos y_1, r \sin y_1, h y_2), \qquad \text{for all } (y_1, y_2) \in \overline{\omega},$$

where $r > 0$ and $h > 0$.

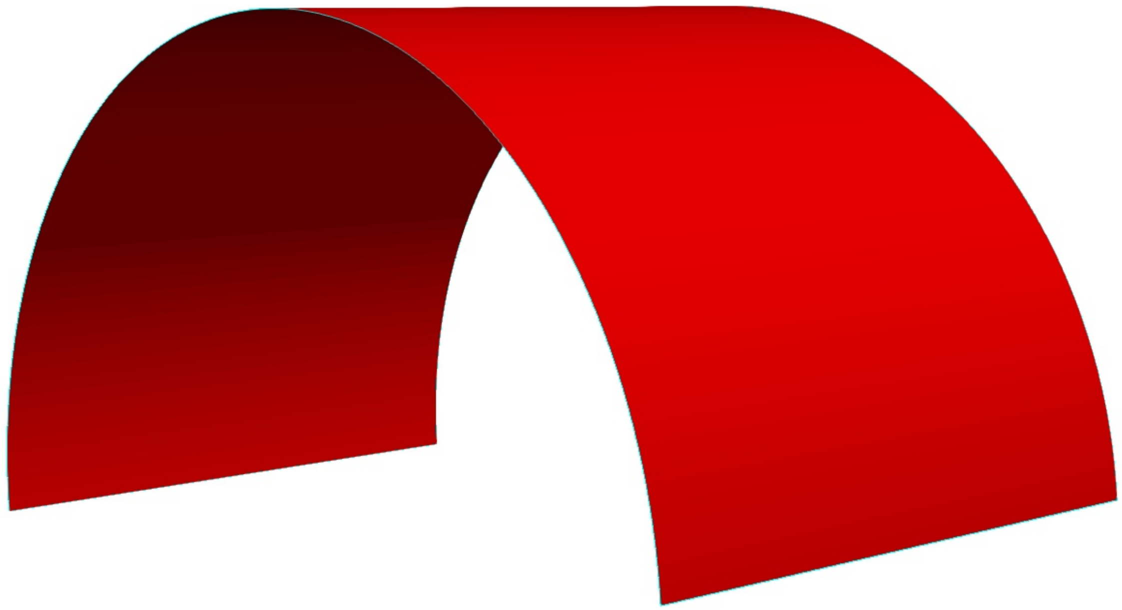


Figure: A cylindrical shell.

Choice of parameters

$$
\begin{aligned}
r &= 0.20\text{m}, \\
h &= 0.40\text{m}, \\
\varepsilon &= 0.002\text{m}, \\
E &= 2.1 \times 10^{11}\text{Pa}, \\
\nu &= 0.3, \\
\rho &= 7.85 \times 10^{3}\text{kg}/\text{m}^{3}, \\
\kappa &= 10^{-6}, \\
\gamma &= 0.6, \\
\beta &= \frac{(1/2+\gamma)^2}{4}.
\end{aligned}
$$

We consider the following dynamic loads

$$p^{1,\varepsilon}(t, y_1, y_2) = p^{2,\varepsilon}(t, y_1, y_2) = 0, \quad p^{3,\varepsilon}(t, y_1, y_2) = 20 t y_1.$$

Newmark's scheme

For each $n = 0, \ldots, N-2$, find $\boldsymbol{\zeta}_h^n \in \boldsymbol{V}_h$ such that

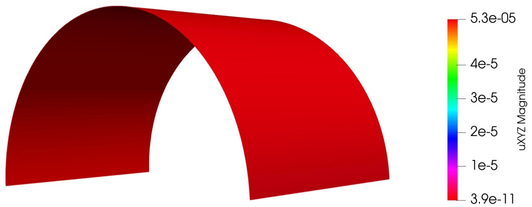
$$\frac{2\varepsilon^3\rho}{(\Delta t)^2}(\boldsymbol{\zeta}_h^{n+2,\kappa} - 2\boldsymbol{\zeta}_h^{n+1,\kappa} + \boldsymbol{\zeta}_h^{n,\kappa}, \boldsymbol{e}_h)_{\boldsymbol{L}^2(\omega)}$$
$$+ a_\kappa\left(\beta\boldsymbol{\zeta}_h^{n+2,\kappa} + \left(\frac{1}{2} - 2\beta + \gamma\right)\boldsymbol{\zeta}_h^{n+1,\kappa} + \left(\frac{1}{2} + \beta - \gamma\right)\boldsymbol{\zeta}_h^{n,\kappa}, \boldsymbol{e}_h\right)$$
$$= \left(\beta\boldsymbol{p}^{n+2} + \left(\frac{1}{2} - 2\beta + \gamma\right)\boldsymbol{p}^{n+1} + \left(\frac{1}{2} + \beta - \gamma\right)\boldsymbol{p}^{n}, \boldsymbol{e}_h\right)_{\boldsymbol{L}^2(\omega)},$$

for all $\boldsymbol{e}_h \in \boldsymbol{V}_h$,

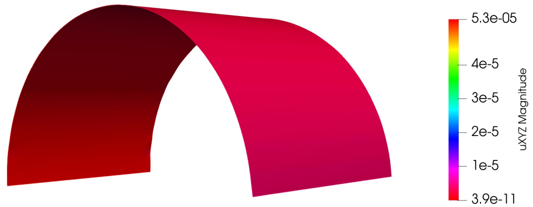
where β and γ are given nonnegative real constants.
The vector field $\boldsymbol{\zeta}_h^{1,\kappa}$ is obtained as the unique solution of the following variational equations

$$\frac{2\varepsilon^3\rho}{(\Delta t)^2}(\boldsymbol{\zeta}_h^{1,\kappa} - \boldsymbol{\zeta}_{0,h} - \Delta t\boldsymbol{\zeta}_{1,h}, \boldsymbol{e}_h)_{\boldsymbol{L}^2(\omega)}$$
$$+ a_\kappa\left(\beta\boldsymbol{\zeta}_h^{1,\kappa} + \left(\frac{1}{2} - \beta\right)\boldsymbol{\zeta}_{0,h}, \boldsymbol{e}_h\right)$$
$$= \left(\beta\boldsymbol{p}^{1} + \left(\frac{1}{2} - \beta\right)\boldsymbol{p}^{0}, \boldsymbol{e}_h\right)_{\boldsymbol{L}^2(\omega)}, \quad \text{for all } \boldsymbol{e}_h \in \boldsymbol{V}_h.$$

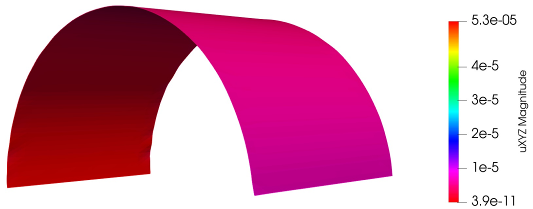
Numerical Experiments for a Cylindrical Shell



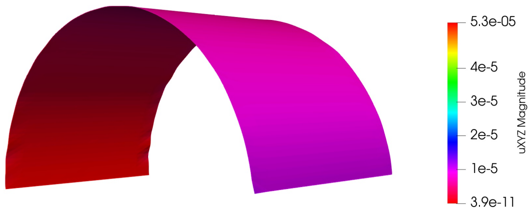
(a) $t = 0.00\mathrm{s}$



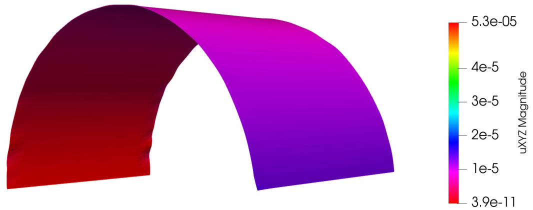
(b) $t = 0.10\mathrm{s}$



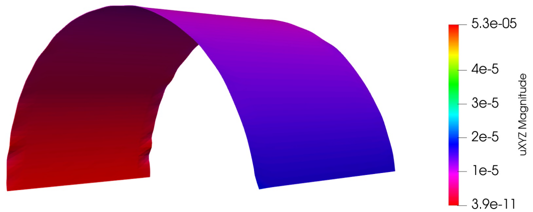
(c) $t = 0.20\mathrm{s}$



(d) $t = 0.30\mathrm{s}$

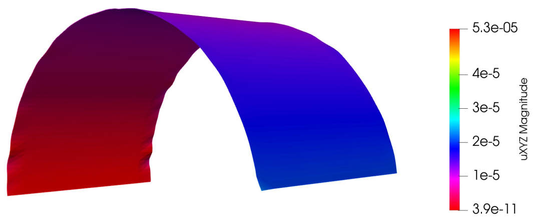


(e) $t = 0.40\mathrm{s}$

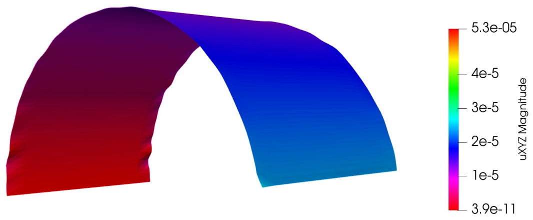


(f) $t = 0.50\mathrm{s}$

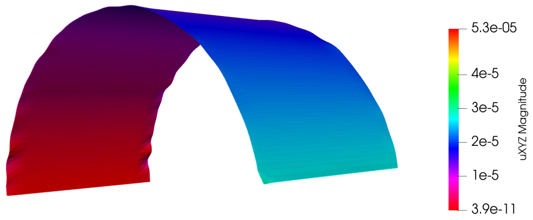
Numerical Experiments for a Cylindrical Shell (Continued)



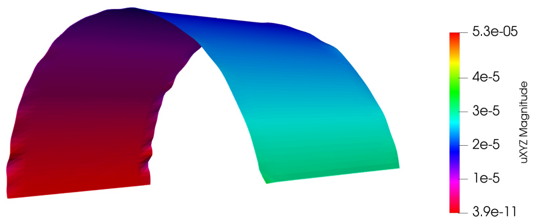
(g) $t = 0.60\text{s}$



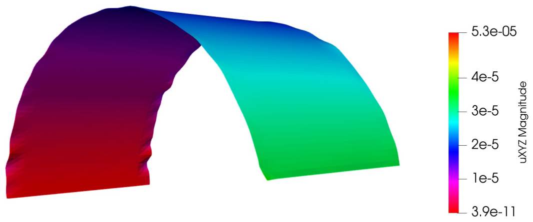
(h) $t = 0.70\text{s}$



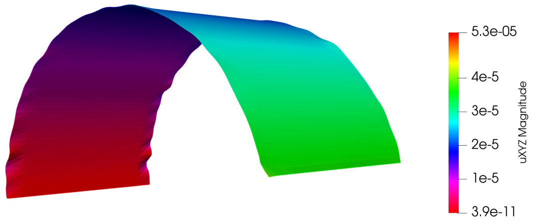
(i) $t = 0.80\text{s}$



(j) $t = 0.90\text{s}$



(k) $t = 1.00\text{s}$



(l) $t = 1.10\text{s}$

Verification of the convergence $\zeta_\kappa^\varepsilon \to \zeta^\varepsilon, \quad$ in $\mathcal{C}^0([0,T];\boldsymbol{L}^2(\omega))$

Time	$\kappa = 10^{-6}$ and $\kappa' = 10^{-8}$
0.10s	8.923964821442561e-09
0.20s	2.9976395853899877e-08
0.30s	6.320529449051073e-08
0.40s	1.0911183801720238e-07
0.50s	1.6750799347387358e-07
0.60s	2.3799061869290796e-07
0.70s	3.2029549991047165e-07
0.80s	4.150932011593057e-07
0.90s	5.233865140892298e-07
1.00s	6.43545490928386e-07
1.10s	7.780641080033482e-07
1.20s	9.20683782879014e-07
1.30s	1.0773057672601421e-06
1.40s	1.248193238692263e-06

Time	$\kappa = 10^{-8}$ and $\kappa' = 10^{-12}$
0.10s	9.092922670431935e-13
0.20s	3.0551785666233502e-12
0.30s	6.469394300280324e-12
0.40s	1.1151143158076022e-11
0.50s	1.7097534957261284e-11
0.60s	2.4308028522236183e-11
0.70s	3.2755013818368175e-11
0.80s	4.2462762579475536e-11
0.90s	5.351133923228302e-11
1.00s	6.576490708244588e-11
0.10s	7.913665593178143e-11
1.20s	9.4053912832205e-11
1.30s	1.101622383552461e-10
1.40s	1.2757241762698354e-10

END OF THE PRESENTATION

paolo.piersanti@uni-graz.at