Contacts in an Eulerian Framework

Solved with FreeFem++

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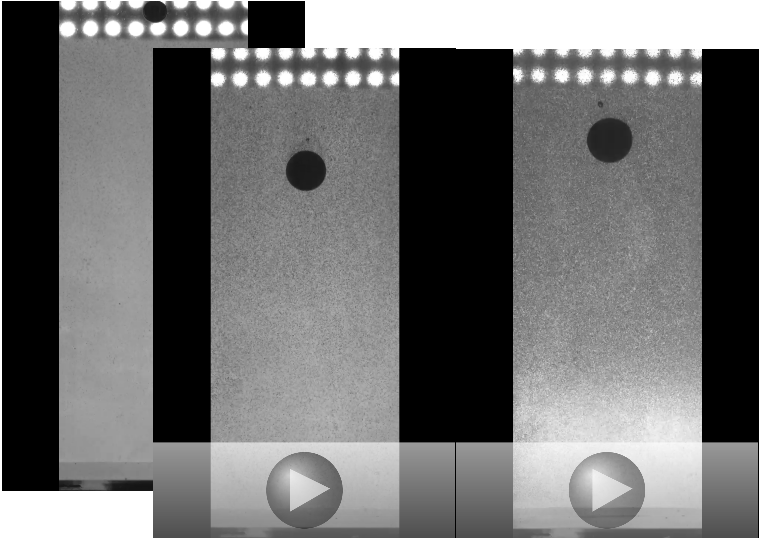
Plan

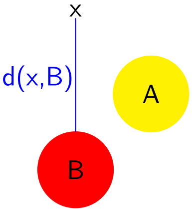
1. Preliminaries:
   - How to detect a Contact,
   - the Characteristic Galerkin method
2. Eulerian Formulations for Solid Mechanics
3. Variational Inequalities for Contacts
4. Numerical Tests

Motivation

By JunZhang, (University of New York at Shanghai)



Preliminary 1: Detection of Contacts



$$d(A,B) = \max[\min_{x\in A}(d(x,B), \min_{x\in B} d(x,A)]$$

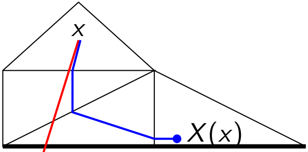
The FreeFem keyword *distance* returns in dist[a] a level set function of the distance between $\mathbf{x}$ and $f=0$:

`distance(th,f,dist[],distmax=100)` returns the $d(\cdot,\{f<0\})$.

Let $B=\{\mathbf{x} \ : \ f(\mathbf{x})<0\}$. Let q^i a vertex of ∂A.

If dist[i]$<\epsilon$ then B ϵ-touches A.

Preliminary 2: The Galerkin - Characteristic Method



The operator `convect( )`

$$D_t u := (\partial_t u + a \cdot \nabla u)|_{x,(m+1)\delta t} \approx \frac{u^{m+1}(x) - u^m(x - a^m(x)\delta t)}{\delta t}$$

A better formulation is to replace $x - a^m(x)\delta t$ by $X^m(x)$ the solution at t_m of

$$\dot{X}(\tau) = a(X(\tau), \tau) \text{ with } X(t^{m+1}) = x. \Rightarrow \quad D_t\mathbf{u} \approx \frac{1}{\delta t}(u^{m+1} - u^m \circ X^m)$$

Consider the problem $\partial_t\mathbf{u} + \mathbf{a} \cdot \nabla\mathbf{u} = 0, \quad \mathbf{u}(t = 0) = \mathbf{u}^0$

Then 3 solutions:

- 1: strong approximation $u^{m+1} = u^m \circ X^m$
- 2: Weak approximation: $\int_\Omega u^{m+1} \cdot \hat{u} = \int_\Omega u^m \circ X^m \cdot \hat{u}, \forall\hat{u}$
- 3: Move the mesh at each time step $q^i \leftarrow q^i + u(q^i)\delta t$ and keep $u[i]$ constant.

convectmesh.edp

```
// Characteristics Galerkin
border C(t=0, 2*pi) { x=cos(t);  y=sin(t); };
mesh Th = buildmesh(C(100));
fespace Vh(Th,P1);
Vh c, cc, ch, cold = exp(-10*((x-0.3)^2 +(y-0.3)^2)), ccold=cold;

real dt = 0.17;
func u1 = y;
func u2 = -x;

problem aa(c,ch)=int2d(Th)(c*ch) - int2d(Th)(convect([u1,u2],-dt,cold)*ch);

for (real t=0; t<2*pi ; t+=dt) {
    cc=convect([u1,u2],-dt,ccold);
    aa;
    cold=c; ccold=cc;
    plot(c, cc, dim=3, fill=0,cmm=" t="+t+", min="+c[].min+", max="+c[].max);
}
plot(c, cc, dim=3, fill=0,cmm="min="+c[].min+", max="+c[].max, wait=1);

mesh Thm=Th;
fespace Vhm(Thm,P1);
Vhm cc0, c0= exp(-10*((x-0.3)^2 +(y-0.3)^2));

for (real t=0; t<2*pi ; t+=dt) {
    Thm = movemesh(Thm,[x=x+dt*u1,y=y+dt*u2]);
    cc0=cc0;  // cc0 is on new Thm now
    cc0[]=c0[]; // copies the values at the vertices only
    plot(Thm, cc0,dim=2, cmm=" t="+t + ", min=" + cc0[].min + ", max=" +  cc0[].max ,wait=1);
}
```

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Variational Fluid-Structure Eulerian Formulation

Navier-Stokes (ν) + Hyperelastic <u>incompressible</u> 2D Mooney-Rivlin material (c_1):
find velocity, pressure, domain $\mathbf{u}, p, \Omega_t$, such that for all $\hat{\mathbf{u}}, \hat{p}$

$$\boxed{\begin{aligned}&\int_{\Omega_t}[\rho\mathbb{D}_t\mathbf{u}\cdot\hat{\mathbf{u}} - p\nabla\cdot\hat{\mathbf{u}} - \hat{p}\nabla\cdot\mathbf{u}] + \int_{\Omega_t^f}\frac{\nu}{2}\mathrm{D}\mathbf{u}:\mathrm{D}\hat{\mathbf{u}}\\ &\quad + \int_{\Omega_t^s} c_1(\mathrm{D}\mathbf{d} - \nabla\mathbf{d}\nabla^T\mathbf{d}):\mathrm{D}\hat{\mathbf{u}} = \int_{\Omega_t} f\hat{\mathbf{u}}\end{aligned}}$$

- $\mathbf{d}$ = solid displacement

$$\mathbb{D}_t\mathbf{d} := \partial_t\mathbf{d} + \mathbf{u}\cdot\nabla\mathbf{d} = \mathbf{u} \text{ and } \mathrm{D}u = \nabla\mathbf{u} + \nabla^T\mathbf{u}$$

- $\mathbf{u}$ is the velocity field and p is the pressure field.
- The solid domain Ω_t^s is defined by $\mathbf{X}(0)\in\Omega_0^s,\quad \dot{\mathbf{X}} = \mathbf{u}(\mathbf{X},t) \Rightarrow \mathbf{X}(t)\in\Omega_t^s$.

Implicit Euler Discretization with Finite Elements

Discretization in space by the Finite Element Method leads to find $\mathbf{u}_h^{n+1}, p_h^{n+1} \in V_{0h} \times Q_h$ such that for all $\hat{\mathbf{u}}_h, \hat{p}_h \in V_{0h} \times Q_h$,

$$\int_{\Omega_{n+1}} \Big[\rho_{n+1} \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n \circ \mathbb{Y}^{n+1}}{\delta t} \cdot \hat{\mathbf{u}}_h - p_h^{n+1} \nabla \cdot \hat{\mathbf{u}}_h - \hat{p}_h \nabla \cdot \mathbf{u}_h^{n+1} + \mathbf{1}_{\Omega^f_{n+1}} \frac{\mu^f}{2} \mathrm{D}\mathbf{u}^{n+1} : \mathrm{D}\hat{\mathbf{u}}$$
$$+c_1 \mathbf{1}_{\Omega^s_{n+1}} [\mathrm{D}(\tilde{\mathbf{d}}^n + \delta t \mathbf{u}^{n+1}) - \nabla^T(\tilde{\mathbf{d}}^n + \delta t \mathbf{u}^{n+1}) \nabla(\tilde{\mathbf{d}}^n + \delta t \mathbf{u}^{n+1})] : \mathrm{D}\hat{\mathbf{u}} \Big] = \int_{\Omega_{n+1}} \mathbf{f} \cdot \hat{\mathbf{u}}_h, ,$$
$$\Omega_{n+1} = (\mathbb{Y}^{n+1})^{-1}(\Omega_n) = \{x : \ \mathbb{Y}^{n+1}(x) \in \Omega_n\}$$

with $\mathbf{d}_h$ updated by $\mathbf{d}_h^{n+1} = \tilde{\mathbf{d}}_h^n + \delta t \mathbf{u}_h^{n+1}$where $\tilde{\mathbf{d}}_h^n = \mathbf{d}_h^n \circ \mathbb{Y}^{n+1}$ and where

$$\mathbb{Y}^{n+1}(x) = x - \mathbf{u}_h^{n+1}(x)\delta t$$

Conservation of energy in the continuous case works for the discrete case because the map $\mathbf{X} : \Omega_0 \mapsto \Omega_t$ satisfies

$$\mathbf{X}^n = \mathbf{X}^{n+1} \circ \mathbb{Y}^{n+1}, \ \ i.e. \ \mathbf{d}[i] = \mathbf{d}|_{q_i} \ \Rightarrow \ \mathbf{d}^{n+1}[i] = \mathbf{d}^n[i] + \mathbf{u}_h^{n+1}[i]\delta t.$$

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Dry Structures with Possible contact

With $\mathrm{D}u = \nabla \mathbf{u} + \nabla^T \mathbf{u}$. Find $\Omega_t^s, \mathbf{u}, p, \mathbf{d}$ such that, for all $\hat{\mathbf{u}}, \hat{p}$,

$$\int_{\Omega_t^s} [\rho_0^s \mathbb{D}_t \mathbf{u} \cdot \hat{\mathbf{u}} - p\nabla \cdot \hat{\mathbf{u}} - \hat{p}\nabla \cdot \mathbf{u} + c_1(\mathrm{D}\mathbf{d} - \nabla \mathbf{d} \nabla^T \mathbf{d}) : \mathrm{D}\hat{\mathbf{u}}] = \int_{\Omega_t^s} f \hat{\mathbf{u}},$$
$$\mathbb{D}_t \mathbf{d} := \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} = \mathbf{u}.$$

Approximated by $\mathbf{a}([\mathbf{u}^{n+1}, p^{n+1}], [\hat{\mathbf{u}}, \hat{p}]) = \mathcal{L}([\hat{\mathbf{u}}, , \hat{p}]),$, i.e.

$$\int_{\Omega^s} \left[\frac{\mathbf{u}_h^{n+1}}{\delta t} \cdot \hat{\mathbf{u}}_h - p_h^{n+1} \nabla \cdot \hat{\mathbf{u}}_h - \hat{p}_h \nabla \cdot \mathbf{u}_h^{n+1} + c_1 \delta t [\mathrm{D}\mathbf{u}^{n+1} - 2\nabla^T \tilde{\mathbf{d}}^n \nabla \mathbf{u}^{n+1}] : \mathrm{D}\hat{\mathbf{u}} \right] = \dots$$

Rebound on plane $x_2 = 0 \Rightarrow$ at $x_2 = 0$, $\mathbf{d}_2^n \circ \mathbb{Y} + \mathbf{u}_2^{n+1} \delta t \geq 0$. So find

$$[\mathbf{u}^{n+1}, p^{n+1}] + \frac{1}{\delta t}[\tilde{\mathbf{d}}_2^n, 0] \in W := L^2(\Omega^s) \times \{\mathbf{u} \in H^1(\Omega^s)^2 \ : \ \mathbf{u}_2(x_1, 0, t) \geq 0\}$$
$$\mathbf{a}([\mathbf{u}^{n+1}, p^{n+1}], [\hat{\mathbf{u}}, \hat{p}]) \geq \mathcal{L}([\hat{\mathbf{u}}, \hat{p}]), \quad \forall [\hat{\mathbf{u}}, \hat{p}] \in W$$

Which is a variational inequality at each time step.
RUN beam deflected by its own weight, clamped on the right.

Formulation with a Lagrange Multiplier

As the constraint will be active first on the boundary of the solid, and only near to the plane $x_2 = 0$,

$$\lambda \le 0, \ \ \mathbf{u}_2^{n+1} + \frac{1}{\delta t}\tilde{\mathbf{d}}_2^n)|_{\partial\Omega^s} \ge 0, \ \ \lambda(\mathbf{u}_2^{n+1} + \frac{1}{\delta t}\tilde{\mathbf{d}}_2^n)|_{\partial\Omega^s} = 0$$

$$\mathbf{a}([\mathbf{u}^{n+1}, p^{n+1}], [\hat{\mathbf{u}}, \hat{p}]) + \int_{\partial\Omega^s \cap (\mathrm{dist}_{<\epsilon})} \lambda\hat{\mathbf{u}} = \mathcal{L}([\hat{\mathbf{u}}, \hat{p}]), \quad \forall [\hat{\mathbf{u}}, \hat{p}],$$

where $\mathrm{dist}(\mathbf{x})$ is the distance of $\mathbf{x}$ to the plane $x_2 = 0$, i.e. $\mathbf{x}_2$.

It can be generalized to contact between 2 moving objects by introducing 2 Lagrange multipliers

$$\mathbf{a}([\mathbf{u}^{n+1}, p^{n+1}], [\hat{\mathbf{u}}, \hat{p}]) + \int_{\partial\Omega_1^s \cap (\mathrm{dist}_1 < \epsilon)} \lambda\hat{\mathbf{u}} + \int_{\partial\Omega_2^s \cap (\mathrm{dist}_2 < \epsilon)} \lambda\hat{\mathbf{u}} = \mathcal{L}([\hat{\mathbf{u}}, \hat{p}])$$

where $\mathrm{dist}_i(\mathbf{x})$ is the distance of $\mathbf{x}$ to the boundary $\partial\Omega_j$ of the j-th object, i,j=1,2. Notice that the formulation is symmetric

The semi-smooth Newton method

Semi-smooth Newton (C. Kunisch et al):
Choose $c \in \mathcal{R}^+$, consider

$$a(\mathbf{u}^{m+1}, \hat{\mathbf{u}}) + (\lambda, \hat{\mathbf{u}}_2) = \mathcal{L}(\hat{\mathbf{u}}) \quad \forall \hat{\mathbf{u}}$$
$$\lambda - \min(0, \lambda + c\ (x_2 + \mathbf{u}_2^{m+1}(x)\delta t)) = 0 \quad \forall x \in \Sigma := \partial\Omega^s \cap (dist(x) \leq \epsilon)$$

Note that this gives $\lambda \leq 0$ and $x_2 + \mathbf{u}_2^{m+1}(x)\delta t \geq 0$.
Iterative solution $k = 0, 1...$,

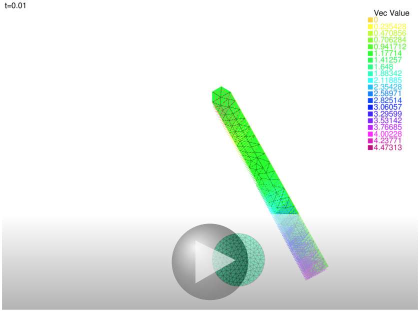
Set $I_k := \{x \ : \ \lambda^k(x) + c\ (x_2 + \mathbf{u}_2^{m+1,k}(x)\delta t) < 0\}$

Get $\mathbf{u}^{m+1,k+1}$ by $a(\mathbf{u}^{m+1,k+1}, \hat{\mathbf{u}}) + (\lambda^k, \hat{\mathbf{u}}_2) = \mathcal{L}(\hat{\mathbf{u}})$,

$$\mathbf{u}_2^{m+1,k+1}(x) = \frac{x_2}{\delta t}, \forall x \in I_k$$

Get λ^{k+1} by $(\lambda^{k+1}, \hat{\mathbf{u}}_2) := \mathcal{L}(\hat{\mathbf{u}}) - a(\mathbf{u}^{m+1,k+1}, \hat{\mathbf{u}}) \quad \forall \hat{\mathbf{u}}$.

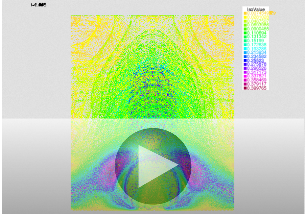
Golf

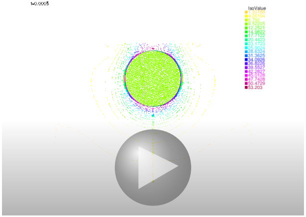


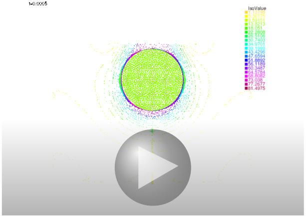
$$Force = 5,\ c_1^{club} = 10^5\mu,\ c_1^{ball} = 10^4\mu, \rho_0^{club} = 50,\ \rho_0^{ball} = 200,$$

RUN Ball hit by a club.

Bouncing Ball in a Liquid







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Soft ball in a fluid under its own weight.

$R = 0.75,\ g = -10,\ \rho^s{=}2,\ c_1 = 167$ $\qquad$ $\rho^f = 1,\ \nu = 0.05,\ c_1 = 83.3$

Conclusion and Perspectives

1. Can be extended to compressible hyperelastic solid
2. Extension to 3D by a C.Y. Chiang at LJLL.

3. Monolithic formulation on the velocity adapted to contact
4. Energy stable algorithm
5. Does not need many grid points
6. `freefem++` is useful to prototype new ideas.
7. Unconditionally stable algorithm? Not there yet (but improving)

Thanks for the invitation!