Two-level Domain decomposition preconditioning for the high-frequency time-harmonic Maxwell equations

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Goal

The convergence analysis for wave propagation problems is challenging

Sign-indefinite time-harmonic Maxwell problem

$$\begin{cases} \nabla \times (\nabla \times \mathbf{E}) - \tilde{\omega}^2 \mathbf{E} = \mathbf{F} & \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

large $\tilde{\omega} \Rightarrow$ large indefinite, non-Hermitian matrix A

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Find a "good" preconditioner for A:

- Seek *independence* of the number of iterations of GMRES from the *wavenumber* $\tilde{\omega} = \omega\sqrt{\mu\varepsilon}$
- parallelisable

$\Rightarrow$ two-level domain decomposition preconditioners:
coarse correction combined with the one-level preconditioner

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We developed a theory for the dissipative case ($\sigma > 0$):

Problem with absorption

$$\begin{cases} \nabla \times (\nabla \times \mathbf{E}) - (\tilde{\omega}^2 + \mathrm{i}\xi) \mathbf{E} = \mathbf{F} & \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

Work for the Helmholtz equation $-\Delta u - (\tilde{\omega}^2 + \mathrm{i}\xi) u = f$:

[Graham, Spence, Vainikko, Domain decomposition preconditioning for high-frequency Helmholtz problems with absorption. Math.Comp. 2017]

Strategy

Solve with GMRES

$$B_\xi^{-1} A\mathbf{x} = B_\xi^{-1}\mathbf{b},$$

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and

(ii) B_ξ^{-1} is a good preconditioner for A_ξ in the sense that both the norm and the distance of the field of values from the origin of $B_\xi^{-1} A_\xi$ are bounded independently of $\tilde{\omega}$.

Remarks

(i) A_ξ^{-1} *is a good preconditioner for* A *in the sense that* $\|I - A_\xi^{-1} A\|$ *is small, independently of* $\tilde{\omega}$

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- "gap" between $|\xi| \sim \tilde{\omega}^2$ for (ii) and $|\xi| \lesssim \tilde{\omega}$ for (i)
  $\Rightarrow$ expected to decrease when replacing PEC interface conditions by impedance conditions, as for Helmholtz
  $\Rightarrow$ future work

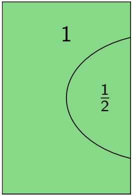
Consider the linear system: $Au = f \in \mathbb{C}^n$.

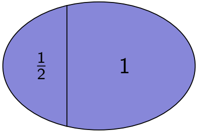
Given a decomposition of $[\![1;n]\!]$, $(\mathcal{N}_1, \mathcal{N}_2)$, define:

- the restriction operator R_i from $[\![1;n]\!]$ into $\mathcal{N}_i$,
- R_i^T as the extension by 0 from $\mathcal{N}_i$ into $[\![1;n]\!]$.

Duplicated unknowns coupled via a *partition of unity*:

$$I = \sum_{i=1}^{N} R_i^T D_i R_i$$





- $M_{\text{AS}}^{-1} := \sum_{i=1}^{N} R_i^T A_i^{-1} R_i$
- $M_{\text{RAS}}^{-1} := \sum_{i=1}^{N} R_i^T D_i A_i^{-1} R_i$ $\qquad A_i = R_i A R_i^T$
- $M_{\text{ORAS}}^{-1} := \sum_{i=1}^{N} R_i^T D_i B_i^{-1} R_i$ Optimized transmission conditions

[B. Després 1991] for Helmholtz

General form of two-level preconditioners

$$M_2^{-1} = QM_1^{-1}P + H, \quad H = ZE^{-1}Z^\dagger$$

($\dagger$ the conjugate transpose)

- M_1^{-1} the one-level preconditioner,
- Z a rectangular matrix with full column rank (coarse space),
- $E = Z^\dagger AZ$ "coarse grid" matrix,
- $H = ZE^{-1}Z^\dagger$ "coarse grid correction" matrix,
- if $P = Q = I$: *additive* two-level preconditioner,
- if $P = I - AH$, $Q = I - HA$: *hybrid* two-level preconditioner, or *balancing* Neumann Neumann (BNN).

Here definition of Z based on a *coarser mesh* of element size H_{cs}:
the *interpolation matrix* from $V_{H_{\text{cs}}}$ to V_h

Convergence rates for GMRES with 2-level Additive Schwarz preconditioner

$$M_{\xi,\mathrm{AS}}^{-1} = \sum_{i=0}^{N_{\mathrm{sub}}} R_i^T (A_\xi^i)^{-1} R_i, \qquad A_\xi^i = R_i A_\xi R_i^T, \quad R_0 = Z^T$$

explicit in wavenumber $\tilde{\omega}$, absorption ξ, coarse mesh size H_{cs}, subdomain diameter H_{sub} and overlap size δ.

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Main theorems:

- upper bound on the norm of $M_{\xi,\mathrm{AS}}^{-1} A_\xi$
- lower bound on the field of values of $M_{\xi,\mathrm{AS}}^{-1} A_\xi$ $\quad \frac{|\langle \mathbf{V}, M_{\xi,AS}^{-1} A_\xi \mathbf{V}\rangle|}{\|\mathbf{V}\|^2}$

$\Rightarrow$ apply Elman-type estimates for the convergence of GMRES

$\tilde{\omega}$-weighted inner product and norm:

$$(\mathbf{v}, \mathbf{w})_{\mathrm{curl},\tilde{\omega}} = (\nabla \times \mathbf{v}, \nabla \times \mathbf{w})_{\mathbf{L}^2(\Omega)} + \tilde{\omega}^2 (\mathbf{v}, \mathbf{w})_{\mathbf{L}^2(\Omega)}$$

[Bonazzoli, Dolean, Graham, Spence, Tournier. Domain decomposition preconditioning for the high-frequency time-harmonic Maxwell equations with absorption. *Submitted* <arXiv:1711.03789>]

Weighted GMRES method: the residual $\mathbf{r}_m$ is minimized in the norm induced by

$$\langle \mathbf{V}, \mathbf{W} \rangle_{D_{\tilde{\omega}}} = (\mathbf{v}_h, \mathbf{w}_h)_{\mathrm{curl},\tilde{\omega}} \quad (\mathbf{v}_h, \mathbf{w}_h \in V_h \text{ with coefficient vectors } \mathbf{V}, \mathbf{W})$$

Theorem (GMRES convergence for left preconditioning)

Ω convex polyhedron. Given $\tilde{\omega}_0 > 0$, there exists $\mathcal{C} > 0$, independent of all parameters, such that, given $0 < a < 1$, if

(i) $\tilde{\omega} \geq \tilde{\omega}_0$,

(ii) $\max\left\{(\tilde{\omega} H_{\mathrm{sub}}),\ (\tilde{\omega} H_{\mathrm{cs}})\left(\left(\frac{\tilde{\omega}^2}{|\xi|}\right)\right)\right\} \leq \mathcal{C}_1 \left(1 + \left(\frac{H_{\mathrm{cs}}}{\delta}\right)^2\right)^{-1} \left(\frac{|\xi|}{\tilde{\omega}^2}\right)$,

(iii) $m \geq \mathcal{C} \left(\left(\frac{\tilde{\omega}^2}{|\xi|}\right)\right)^3 \left(1 + \left(\frac{H_{\mathrm{cs}}}{\delta}\right)^2\right) \log\left(\frac{12}{a}\right)$,

then

$$\frac{\|\mathbf{r}_m\|_{D_{\tilde{\omega}}}}{\|\mathbf{r}_0\|_{D_{\tilde{\omega}}}} \leq a$$

Particular case: when $|\xi| \sim \tilde{\omega}^2$ (max absorption) and $\delta \sim H_{\mathrm{cs}}$ (generous overlap), Condition (ii) is satisfied with $H_{\mathrm{sub}} \sim H_{\mathrm{cs}} \sim \tilde{\omega}^{-1}$, and then bound (iii) implies convergence *independent of* $\tilde{\omega}$.

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$$\frac{\|\mathbf{r}_m\|_{D_{\tilde{\omega}}}}{\|\mathbf{r}_0\|_{D_{\tilde{\omega}}}} \leq a$$

Numerical experiments

- Ω unit cube, regular or METIS decomposition into subdomains,
- mesh diameter $h \sim \tilde{\omega}^{-3/2}$, or $h \sim 2\pi/(g\tilde{\omega})$ if order 2 FEs are used,
- $\mathbf{F} = [f, f, f]$, $f = -\exp(-400((x-0.5)^2 + (y-0.5)^2 + (z-0.5)^2))$,
- GMRES with right preconditioning (tolerance 10^{-6}),
- precondition $A_{\xi_{\text{prob}}}$ by $M^{-1}_{\xi_{\text{prec}}}$, $0 \leq \xi_{\text{prob}} \leq \xi_{\text{prec}}$
- random initial guess $\Rightarrow$ all frequencies are present in the error,
- the local problems in each subdomain and the coarse space problem solved with a direct solver (MUMPS),
- 1 subdomain $\Longleftrightarrow$ 1 processor

- $H_{\text{sub}} \sim H_{\text{cs}} \sim \tilde{\omega}^{-1}$
- overlap $\delta \sim H_{\text{sub}}$
- $\xi_{\text{prob}} = \xi_{\text{prec}} = \tilde{\omega}^2$
- PEC B.Cs on $\partial\Omega$
- order 1 edge FEs
- $h \sim \tilde{\omega}^{-3/2}$
- regular partitioning

	$\alpha = 1 = \alpha'$					
$\tilde{\omega}$	n	N_{sub}	n_{cs}	#AS	#RAS	#HRAS
10	4.6×10^5	1000	7.9×10^3	53(57)	26(37)	12
15	1.5×10^6	3375	2.6×10^4	59(64)	28(42)	12
20	1.2×10^7	8000	6.0×10^4	76(105)	29(57)	17

n: size of the fine grid problem
N_{sub}: number of subdomains
n_{cs}: size of the coarse grid problem
$\#$: number of iterations 2-level (1-level)

- $H_{\text{sub}} \sim \tilde{\omega}^{-\alpha}$
  $H_{\text{cs}} \sim \tilde{\omega}^{-\alpha'}$
- overlap $\delta \sim H_{\text{sub}}$
- $\xi_{\text{prob}} = \xi_{\text{prec}} = \tilde{\omega}^2$
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- regular partitioning

	$\alpha = 0.8,\ \alpha' = 1$					
$\tilde{\omega}$	n	N_{sub}	n_{cs}	#AS	#RAS	#HRAS
10	$3.4\ \times 10^5$	216	$7.9\ \times 10^3$	37	20	11
20	$7.1\ \times 10^6$	1000	$6.0\ \times 10^4$	57	24	11
30	$4.1\ \times 10^7$	3375	$2.0\ \times 10^5$	53	32	16
40	$2.0\ \times 10^8$	6859	$4.6\ \times 10^5$	53	33	16

: number of iterations 2-level

- $H_{sub} \sim \tilde{\omega}^{-0.6}$
  $H_{cs} \sim \tilde{\omega}^{-0.9}$
- overlap $\delta \sim \cancel{H_{sub}}\ 2h$
- $\xi_{prob} = \cancel{\tilde{\omega}^2}\ 0$, $\xi_{prec} = \cancel{\tilde{\omega}^2}\ \tilde{\omega}$
- ~~PEC~~ impedance B.Cs on $\partial\Omega$

- order 1 edge FEs
- $h \sim \tilde{\omega}^{-3/2}$
- regular partitioning

			$\xi_{prec} = \tilde{\omega}$				
$\tilde{\omega}$	n	N_{sub}	2-level	n_{cs}	Time	1-level	Time
10	2.6×10^5	27	20	2.9×10^3	16.2(1.6)	37	13.7(2.6)
15	1.5×10^6	125	26	1.0×10^4	25.5(4.0)	70	26.1(8.9)
20	5.2×10^6	216	29	2.1×10^4	52.0(9.1)	94	60.6(25.6)
25	1.4×10^7	216	33	4.4×10^4	145.5(29.5)	105	191.2(88.1)
30	3.3×10^7	343	38	6.9×10^4	380.4(128.4)	132	673.5(443.1)

2-level: # OHRAS, 1-level: # ORAS, Total Time (GMRES Time)

Numerical experiments
order 2 FEs + fixed number of points per wavelength

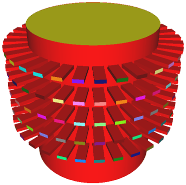
- $H_{\text{sub}} \sim \cancel{\tilde{\omega}^{-0.6}}\ (20\,\tilde{\omega}/(2\pi))^{-0.5}$
  $H_{\text{cs}} \sim \cancel{\tilde{\omega}^{-0.9}}\ 2\pi/(2\,\tilde{\omega})$
- overlap $\delta \sim 2h$
- $\xi_{\text{prob}} = 0$, $\xi_{\text{prec}} = \tilde{\omega}$
- impedance B.Cs on $\partial\Omega$
- order $\cancel{1}$ 2 edge FEs
- $h \sim \cancel{\tilde{\omega}^{-3/2}}\ 2\pi/(20\,\tilde{\omega})$
- ~~regular~~ METIS partitioning

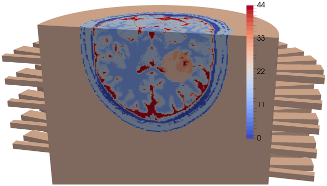
			$g = 20,\ \alpha = 0.5,\ g_{cs} = 2$				
$\tilde{\omega}$	n	N_{sub}	2-level	n_{cs}	Time	1-level	Time
10	1.1×10^6	125	38	1.3×10^3	37.7(7.5)	80	36.6(14.8)
20	8.3×10^6	343	36	9.3×10^3	85.8(18.9)	123	161.7(72.1)
30	2.8×10^7	729	41	3.0×10^4	155.7(39.8)	162	267.3(174.5)
40	6.6×10^7	1331	51	7.0×10^4	272.3(77.6)	> 200	453.8(305.2)

2-level: # OHRAS, 1-level: # ORAS, Total Time (GMRES Time)

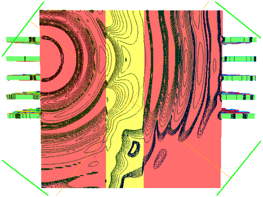
Numerical experiments
Medimax

- order 1 edge FEs, 40 points per wavelength
- $n \approx 1.6 \times 10^7$
- $n_{cs} \approx 3.8 \times 10^4$
- 729 subdomains


homogeneous gel

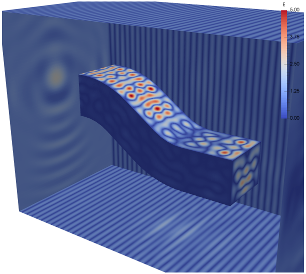


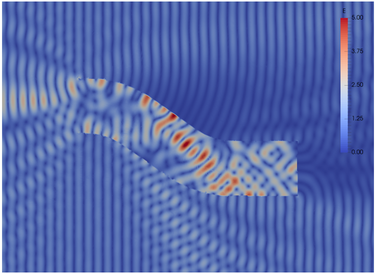
brain model


plastic cylinder

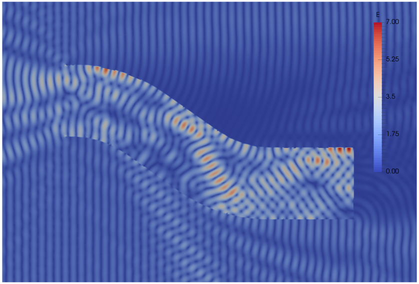
	#2-level	Time	#1-level	Time
homogeneous gel	28	63.4(8.6)	30	53.1(6.4)
brain model	28	64.1(9.2)	32	53.4(6.9)
non-conducting cylinder	29	62.3(9.4)	125	83.5(38.2)

10 GHz





16 GHz



- order 2 edge FEs, 10 points per wavelength
- coarse mesh: 3.33 points per wavelength
- $\xi_{\text{prob}} = 0, \xi_{\text{prec}} = \tilde{\omega}$
- $f = 10\ GHz$: $n \approx 1.07 \times 10^8$, $n_{\text{cs}} \approx 4 \times 10^6$
- $f = 16\ GHz$: $n \approx 1.98 \times 10^8$, $n_{\text{cs}} \approx 7.4 \times 10^6$

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Advantages:

- one-level method already efficient for dissipative pbs (depending on ξ_{prec})
- naturally load-balanced parallel implementation
- same communication pattern between neighbors for coarse and fine levels
- R^0 and $(R^0)^T$ involve no communication
- GMRES tolerance of 10^{-2} gives good results

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		$g = 10$, $g_{cs} = 3.33$					
			Total #	Total times (seconds)			
f	N_{sub}	# it	inner it	Total	Setup	GMRES	inner
10GHz	1536	134	4116	1128.9	415.7	713.1	309.3
10GHz	3072	139	6162	659.6	239.4	420.2	209.9
16GHz	3072	183	10645	1504.2	373.9	1130.3	568.5
16GHz	6144	198	14438	1037.0	267.7	769.4	433.8

- speedup of 1.7 for $f = 10\ GHz$, 1.45 for $f = 16\ GHz$

Perspectives

Implementation

For now:

- outer fine problem solved in FreeFem++
- inner coarse problem: solved exactly with MUMPS, or inexactly with HPDDM (one-level preconditioner)

$\Rightarrow$ add support for 2-level preconditioners based on coarse mesh in HPDDM ?

Recycling

Use Krylov-subspace recycling techniques between successive (inexact) solutions of the coarse problem ?
(recycling techniques are already implemented in HPDDM)

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Thank you for your attention !