Vortex ring models for studying the fuel injection and computations with FreeFem++

Y. ZHANG   I. DANAILA

Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie-Paris VI

Third Workshop on Freefem++, Paris, December 5-7, 2011.

Outline

Outline

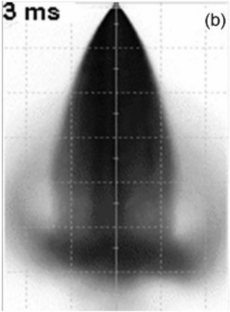
Vortex structures in internal combustion engines

A vortex ring forms ahead of the injected fuel spray.

- New types of gasoline injectors

(courtesy of Continental Automotive)

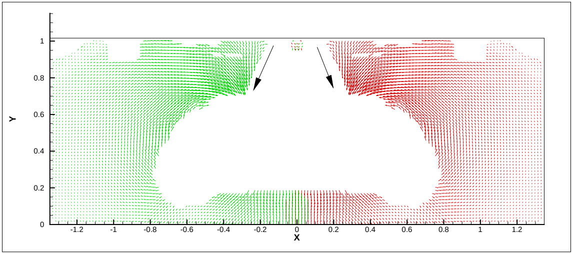
- Diesel injectors



(courtesy of IFP)

Measurements in An Automotive Engine

- Experimental measurements of spray flows by **PIV (Particle Image Velocimetry)** offer a low resolution in the vortex ring region because of high fuel droplet concentration.
- The missing information needs to be reconstructed.



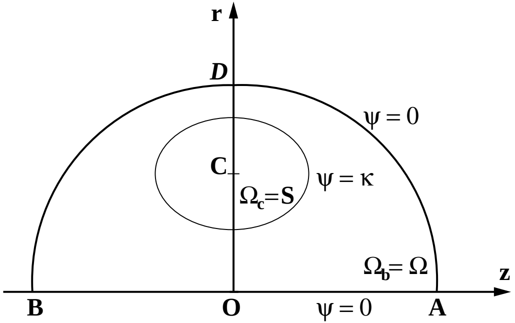
(courtesy of Continental Automotive)

Outline

Mathematical Description of the Problem

- Vortex rings in an elliptic domain $\Omega(\supset \Omega_c = \{x : \psi(x) \geq \kappa\})$

$$\mathcal{L}\psi = -\left(\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \psi}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)\right) = \begin{cases} rf(\psi), & \text{if } \psi \geq \kappa \\ 0, & \text{otherwise} \end{cases}$$



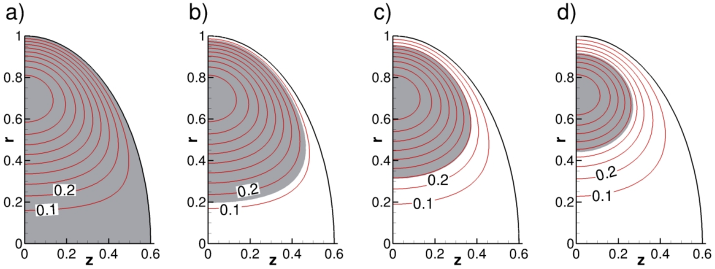
ψ stream function,
κ constant,
$f(\psi)$ arbitrary function
$rf(\psi)$ vorticity distribution.

Vortex ring models

Formulations for $rf(\psi)$ (vorticity distribution) :

- Discontinuous form: $f(\psi) := C\chi_{\Omega_c} = C\,\{\psi \geq k\}$.
- Parameterized 2D Gaussian form: $F(\mathrm{x}, \psi, \Theta, \mathrm{X})$ (Θ = magnitude, X = undetermined parameter set).

Multiple solutions for discontinuous vorticity (different C and k)



Outline

Optimal Control Problem (1)

- Minimization problem in the (reconstruction) domain Ω_{R}

$$\operatorname*{Min}_{\mathrm{X}} J(\psi) = \int_{\partial\Omega_{\mathrm{R}}} \left| \frac{1}{r} \left(\frac{\partial \psi}{\partial \vec{\mathrm{n}}} - \frac{\partial \psi_{\mathrm{exp}}}{\partial \vec{\mathrm{n}}} \right) \right|^2 \mathrm{d}S \qquad (1)$$

subject to

$$\begin{cases} \mathcal{L}\psi = & F(\mathrm{x}, \psi, \Theta, \mathrm{X}), \quad \text{in } \Omega_{\mathrm{R}}, \\ \psi = & \psi_{\mathrm{exp}}, \qquad\qquad\quad\ \text{on } \partial\Omega_{\mathrm{R}}. \end{cases} \qquad (2)$$

$\vec{\mathrm{n}}$ is the unit outer normal vector of $\partial\Omega_{\mathrm{R}}$,
ψ_{exp} is provided by experimental results.

Optimal Control Problem (2)

- Explanation of the cost function $J(\psi)$:
  Minimizing $J(\psi)$ indicates that the first order derivatives of ψ and ψ_{exp} should match (with a fixed error bound) on the boundary $\partial\Omega_{\mathrm{R}}$.
- Supplementary constraint: conserve the circulation Γ_{exp}

$$\Gamma_{\mathrm{exp}} = \int_{\partial\Omega_R} \frac{1}{r} \frac{\partial \psi_{\mathrm{exp}}}{\partial \vec{\mathrm{n}}} \mathrm{d}S, \tag{3}$$

This allows to compute the magnitude Θ from

$$\Gamma_{\mathrm{exp}} = \int_{\partial\Omega_R} F(\mathrm{x}, \psi, \Theta, \mathrm{X})\mathrm{dx} = \Theta \int_{\partial\Omega_R} F'(\mathrm{x}, \psi, \mathrm{X})\mathrm{dx}. \tag{4}$$

Outline

Variational Problems

- Functional space

$$H_0(\Omega) = \left\{ u \in L^2(\Omega) : \frac{1}{r}|\nabla u|^2 \in L^1(\Omega) \mid u = 0 \text{ on } \Gamma = \partial\Omega \right\}. \tag{5}$$

- Inner product ($\forall \quad u \in H_0(\Omega), v \in H_0(\Omega)$)

$$< u, v >= \int_\Omega \frac{1}{r} \nabla u \nabla v \, \mathrm{dx} = \int_\Omega v \mathcal{L} u \, \mathrm{dx}. \tag{6}$$

- Variational form

$$\begin{cases} \text{Find: } \psi \in H_0(\Omega) \text{ such that} \\ < \psi, \phi >= (F(\mathrm{x}, \psi, \Theta, \mathrm{X}), \phi), \ \forall \phi \in H_0(\Omega). \end{cases} \tag{7}$$

Discretization

- Variational derivatives of $J(\cdot)$ ($X_i \in \mathrm{X}$)

$$\frac{\delta J}{\delta X_i} = \frac{\delta J}{\delta \psi}\frac{\delta \psi}{\delta X_i} = 2\int_{\partial\Omega_{\mathrm{R}}} \frac{1}{r^2}\left(\frac{\partial\psi}{\partial\vec{\mathrm{n}}} - \frac{\partial\psi_{\mathrm{exp}}}{\partial\vec{\mathrm{n}}}\right)\frac{\partial}{\partial\vec{\mathrm{n}}}\left(\frac{\delta\psi}{\delta X_i}\right)\mathrm{d}S \tag{8}$$

- Calculations of $\delta\psi/\delta X_i$

$$\begin{cases} \mathcal{L}\left(\dfrac{\delta\psi}{\delta X_i}\right) = \dfrac{\delta F(\mathrm{x},\psi,\Theta,\mathrm{X})}{\delta X_i}, & \text{in } \Omega_{\mathrm{R}}, \\ \dfrac{\delta\psi}{\delta X_i} = 0, & \text{on } \partial\Omega_{\mathrm{R}}. \end{cases} \tag{9}$$

The vanishing of $\delta\psi/\delta X_i$ on $\partial\Omega_{\mathrm{R}}$ is natural for $\Omega_{\mathrm{R}} \supset \Omega_{\mathrm{c}}$.

Outline

Particular formulation of the reconstruction problem Gaussian vorticity distribution

- Minimization problem

$$\operatorname*{Min}_{\alpha,Rc,Zc} J(\psi) = \int_{\partial\Omega_\mathrm{R}} \left| \frac{1}{r}\left(\frac{\partial\psi}{\partial\vec{\mathrm{n}}} - \frac{\partial\psi_\mathrm{exp}}{\partial\vec{\mathrm{n}}} \right) \right|^2 \mathrm{d}S \qquad (10)$$

subject to:

$$\begin{cases} \mathcal{L}\psi = f\left(\Theta, \mathrm{X}\right), & \text{in } \Omega_\mathrm{R}, \\ \psi = \psi_\mathrm{exp}, & \text{on } \partial\Omega_\mathrm{R}. \end{cases} \qquad (11)$$

where $f\left(\Theta, \mathrm{X}\right) = \Theta \cdot \exp\left[-\alpha^2 \left(\left(r - Rc\right)^2 + \left(z - Zc\right)^2\right)\right]$.

Algorithm based on the quasi-Newton BFGS method

1. Set the initial guess $X^{(0)} = \left\{\alpha^{(0)}, R_c^{(0)}, Z_c^{(0)}\right\}$.
   Initialize tolerances $\mathrm{eps_J}$, eps_X and the max iteration steps N.
2. Using the experimental data on $\partial\Omega_R$, calculate Γ_{exp} and get $\Theta^{(n)}$ by the following formula

$$\Gamma = \Gamma_{\mathrm{exp}}$$

$$\Gamma = \int_{\Omega_R} \Theta^{(n)} \cdot \exp\left[-\alpha^2 \left(\left(r - R_c^{(n)}\right)^2 + \left(z - Z_c^{(n)}\right)^2\right)\right] \mathrm{d}z\mathrm{d}r.$$

$$\Gamma_{\mathrm{exp}} = \int_{\partial\Omega_\mathrm{R}} \frac{1}{r} \nabla\psi_{\mathrm{exp}} \cdot \vec{\mathrm{n}}\mathrm{d}S.$$

3. Solve $\mathcal{L}\psi = f(\Theta, \mathrm{X})$ and obtain $\psi^{(n)}$.

Algorithm based on the quasi-Newton BFGS method

4. Get $\nabla J(\psi^{(n)})$ by the following equation

$$\nabla_{\psi} J \cdot \delta\psi^{(n)} = 2 \cdot \int_{\partial\Omega_R} \frac{1}{r^2} \left(\frac{\partial \psi^{(n)}}{\partial \vec{n}} - \frac{\partial \psi_{\mathrm{exp}}}{\partial \vec{n}} \right) \cdot \frac{\partial(\delta\psi^{(n)})}{\partial \vec{n}} \mathrm{d}z\mathrm{d}r. \tag{12}$$

From Eq. (11), we have

$$\mathcal{L}\left(\frac{\partial \psi^{(n)}}{\partial X_i}\right) = \left.\frac{\partial f(\Omega^{(n)}, X)}{\partial X_i}\right|_{X=X^{(n)}} \cdot \left.\frac{\partial \psi^{(n)}}{\partial X_i}\right|_{\partial\Omega_R} = 0. \tag{13}$$

Then, $\nabla_X J(\psi^{(n)})$ is calculated by Eq. (12)

$$\nabla_X J(\psi^{(n)}) = \left\{ \nabla_{\psi} J \cdot \left(\frac{\partial \psi^{(n)}}{\partial X_i} \right) \right\}_{i=0,\ldots,2}. \tag{14}$$

Algorithm based on the quasi-Newton BFGS method

5. Calculate maximum acceptable Euclidean norm of the gradient $\|\nabla_X J(\psi^n)\|$. If $\|\nabla_X J(\psi^n)\| \leq \mathrm{eps_J}$, the iteration is terminated. Otherwise, go to Step 6.
6. From Hessian matrix $H\left(\psi^{(n)}\right)$ obtained by BFGS

$$\nabla_X J\left(\psi^{(n+1)}\right) \approx \nabla_X J\left(\psi^{(n)}\right) + H\left(\psi^{(n)}\right) \cdot \delta X^{(n+1)}. \tag{15}$$

$$\delta X^{*,(n)} = -H^{-1}\left(\psi^{(n)}\right) \cdot \nabla_X J\left(\psi^{(n)}\right). \tag{16}$$

Update $X^{(n+1)}$ by

$$X^{(n+1)} = X^{(n)} + \beta^{(n)} \delta X^{*,(n)}, \tag{17}$$

where $\beta^{(n)}$ is an acceptable step size in the direction $\delta X^{*,(n)}$.

Algorithm based on the quasi-Newton BFGS method

7. If $\left\| X^{(n+1)} - X^{(n)} \right\| / \left\| X^{(n)} \right\| \leq \mathrm{eps}_\mathrm{X}$ or $n + 1 > N$, the iteration is terminated. Otherwise, go to Step 2.

For current computation tests, $\mathrm{eps}_\mathrm{J} = \mathrm{eps}_X = 10^{-6}$ and $N = 50$.

Remark on Step 2

The parameter Θ appears as an extra constraint to guarantee
the existence of the vortex core and
the consistency of the circulation.

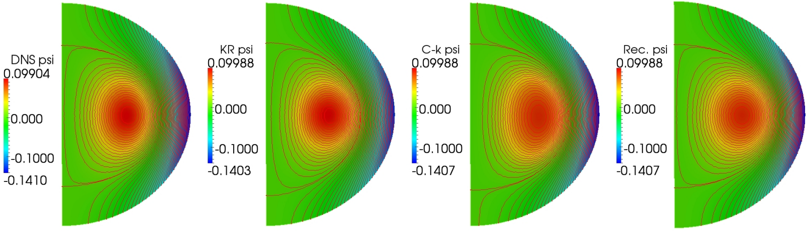
Basic quantities

- The reconstruction domain has an elliptic shape.
  R_c and Z_c are initialized by the center coordinates of the computational domain.
  α is set as the half-length inverse of semi-minor axis.

Table: Comparison between the fitting/reconstructed results and the DNS results. K-R: Kaplanski-Rudi.

Methods	Γ	I	E	R_c	Z_c
NSEs	1.17592	1.33698	0.216944	0.66194	3.51000
K-R	1.12217	1.29092	0.224925	0.71384	3.51000
C-κ	1.17488	1.55884	0.217152	0.71384	3.49190
Recon.	1.17592	1.73646	0.175927	0.65307	3.50953

Streamline figures



(a) DNS (b) K-R (c) $C-\kappa$ (d) Recon.

Figure: Stream function contours: comparison between different vortex ring models, the reconstructed method and DNS data.

Streamline profiles along $z = Z_c$

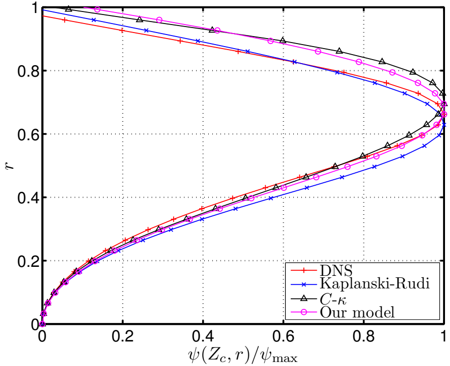


Figure: The profiles of ψ along the line $z = Z_c$ and $r \in (0, 1)$.

Conclusion

- FreeFem++ is a powerful tool to handle this problem.
- The missing information can be reconstructed easily in FreeFem++.
- The reconstruction can provide the necessary vortex ring information for practical/engineering studies of fuel injectors.

Future Work with FreeFem++

- Validate our novel numerical scheme of Navier-Stokes equations and theoretical analysis.
- Implement a time-dependent two-phase flow model .

Thanks for your attention.