Non-Intrusive Reduced Basis Methods Applied to Geotechnics

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Summary

- Reduced Basis Methods: Presentation
- A Non-Intrusive Method (NIRB)
- An Application in Geotechnics: A Non-Linear Model
- Numerical Results
- Resolution Times

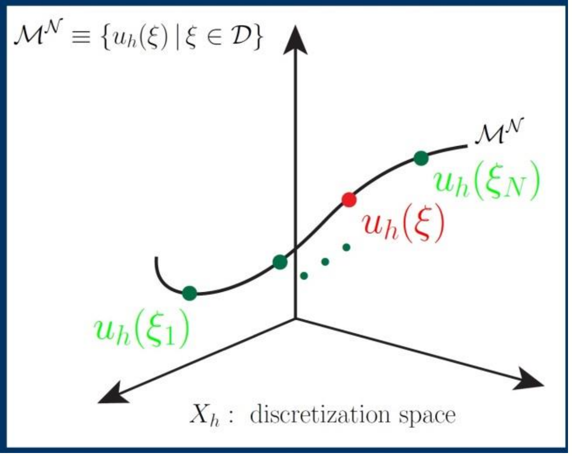
REDUCED BASIS METHODS: PRESENTATION

Motivation

- Consider a parametric PDE in variational formulation:

$$\text{Find } u(\xi) \in X \text{ such that for any } v \in W,\quad a(u(\xi), v; \xi) = l(v; \xi).$$

  - ξ = set of parameters.



- Reasons for model reduction:
  - Costly finite element resolution
  - Many-query and real-time
- Effective method if the Kolmogorov dimension of $\mathcal{M}^{\mathcal{N}}$ is small:

$$\forall \epsilon > 0,\ \exists\ \xi_1, \ldots, \xi_N \text{ in } \mathcal{D} \text{ such that}$$

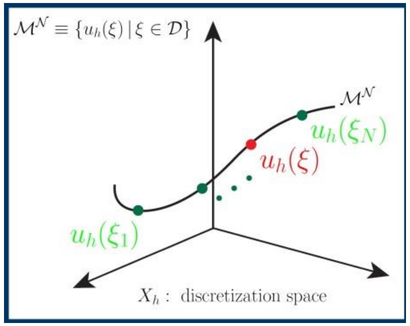
$$\forall \xi \in \mathcal{D},\ \exists\ (\alpha_i(\xi))_{1 \le i \le N} \in \mathbb{R}^N \text{ such that } \|u(\xi) - \sum_{i=1}^{N} \alpha_i(\xi) u(\xi_i)\|_X \le \epsilon.$$

Standard Method: Galerkin

- Discretization space: $X_h = [Y_h]^d$, where

$$Y_h = \{u \in Y \; ; \; \forall\, T \in \mathcal{T}_h,\; u_{|T} \in \mathbb{P}_k(T)\}.$$

- Set $\mathcal{N} = dim(X_h)$



- "Truth approximation" (reference FE solution).

- Galerkin Method over: $X_h^N = \mathrm{span}\{u_h(\xi_1), \ldots, u_h(\xi_N)\}.$

- Given $\xi \in \mathcal{D}$, find $u_h^N(\xi) \in X_h^N$ such that

$$a(u_h^N(\xi), v_h^N; \xi) = l(v_h^N; \xi)\; \forall\, v_h^N \in X_h^N.$$

Standard Method: Implementation

- Linear system of size $N \times N$: $\text{find } \mathbf{U}^N(\xi) \in \mathbb{R}^N \text{ such that}$

$$\boxed{\mathbf{A}^N(\xi)\mathbf{U}^N(\xi) = \mathbf{L}^N(\xi)}$$

- Resolution in $\mathcal{O}(N)$ and not $\mathcal{O}(\mathcal{N})$.

- Efficient method if $N << \mathcal{N}$.

- Offline Step: performed only once
  - Choice of RB basis functions: $u_h(\xi_1), \ldots, u_h(\xi_N)$
  - Orthonormalization of RB functions: $(\phi_1, \ldots, \phi_N)$.
  - Decomposition of the forms is often possible:

$$a(\phi_i, \phi_j; \xi) = \sum_{q=1}^{Q_a} \Lambda_q^a(\xi) a_q(\phi_i, \phi_j),$$

$$l(\phi_i; \xi) = \sum_{q=1}^{Q_l} \Lambda_q^l(\xi) l_q(\phi_i).$$

- Online Step: Calculations depending on a given $\xi \in \mathcal{D}$.

Standard Method: Implementation

Difficulties of the method:

- Updating $\mathbf{A}^N(\xi)$ and $\mathbf{L}^N(\xi)$ without $\mathcal{O}(\mathcal{N})$.

- Modification of the calculation code:
  - Risky
  - Forbidden

A NON-INTRUSIVE METHOD

- For a coarse mesh ($H >> h$), the calculation of $u_H(\xi)$ isn't expensive.

- Projection into the reduced basis space:

$$\Pi_N^H : X_H \to X_h^N$$

$$u_H^N(\xi) = \sum_{i=1}^{N} \gamma_i^H(\xi)\phi_i,$$

$$\gamma_i^H(\xi) = < u_H(\xi), \phi_i >_{L^2(\Omega)}.$$

NIRB : Projection

- Define a transformation

$$\mathcal{R}_N : \mathbb{R}^N \to \mathbb{R}^N$$
$$\left(\gamma_1^H(\xi_i), \dots, \gamma_N^H(\xi_i)\right)^T \mapsto \left(\gamma_1^h(\xi_i), \dots, \gamma_N^h(\xi_i)\right)^T.$$

such that

$$\left(\mathrm{R}^N\right)\begin{pmatrix} \gamma_1^H(\xi_1) & \cdots & \gamma_1^H(\xi_N) \\ \vdots & \vdots & \vdots \\ \gamma_N^H(\xi_1) & \cdots & \gamma_N^H(\xi_N) \end{pmatrix} = \begin{pmatrix} \gamma_1^h(\xi_1) & \cdots & \gamma_1^h(\xi_N) \\ \vdots & \vdots & \vdots \\ \gamma_N^h(\xi_1) & \cdots & \gamma_N^h(\xi_N) \end{pmatrix}$$

- Rectified Solution:

$$\tilde{u}_H^N(\xi) = \sum_{i=1}^{N} \tilde{\gamma}_i^H(\xi)\phi_i, \qquad \tilde{\gamma}_i^H(\xi) = \sum_{k=1}^{N} R_{ik}^N \gamma_k^H(\xi).$$

NIRB : Rectification

What do we need?

F.E. code used as *black box*

Compute snapshots $u_h(\xi_i)$, coarse solution $u_H(\xi)$

Return fine mesh τ_h, coarse mesh τ_H

F.E. library

(FreeFem++)

To compute L^2 and H^1 scalar product

Interpolate from τ_H to τ_h

- Offline Step:
  - Computation of a sample of solutions (Black box software)
  - Selection of set of $(\xi_1, \ldots, \xi_N)$ (F.E. library)
  - Construction of the reduced basis space (F.E. library)
  - Construction of the rectification matrix (F.E. library)

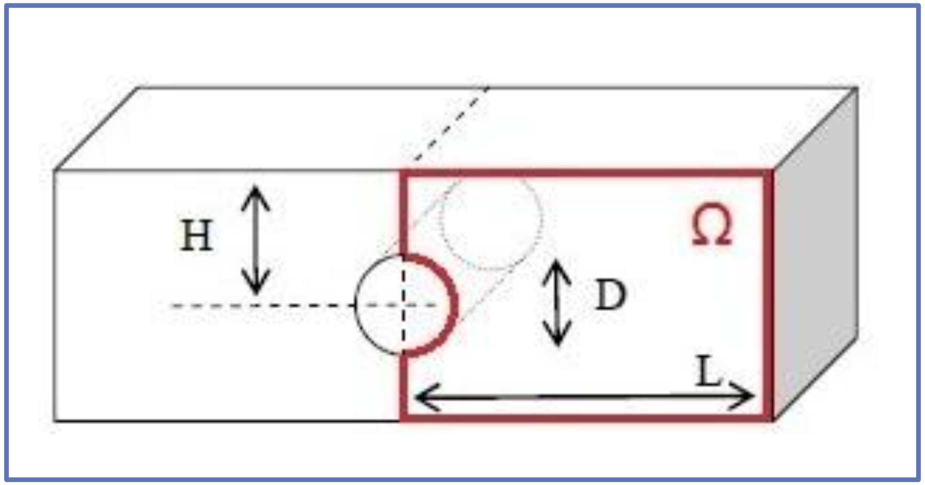
NIRB : Implementation

- Online Step:
  - Solve for $u_H(\xi)$ for a given $\xi \in \mathcal{D}$. (Black box software)
  - Find rectification coefficients. (F.E. library)
  - Obtain $\tilde{u}_H^N(\xi)$. (F.E. library)
  - Build output (function of $\tilde{u}_H^N(\xi)$) (F.E. library)

NIRB : Implementation

AN APPLICATION IN GEOTECHNICS

- Domain Ω :



- The parameters:
  - Material (Young's Module E, Poisson's coefficient υ, volumetric weight of the soil γ, cohesion c, angle of friction φ, angle of dilitancy ψ)
  - Massif's initial stresses K_0.
  - Confinement loss λ.

The Problem: Domain and Parameters

- Variational Problem : Find

$$u \in X = \left\{ v \in (H^1(\Omega))^2 \;\middle|\; \begin{array}{l} v_1 = 0 \text{ on } \Gamma_1, \Gamma_2, \Gamma_3 \\ v_2 = 0 \text{ on } \Gamma_2 \end{array} \right\}$$

such that

$$a(u(\mu), v; \mu) = \mathcal{L}(v; \mu) \qquad v \in X,$$

$$a(u(\mu), v; \mu) = \int_\Omega (\epsilon(u(\mu)) - \epsilon^p(u(\mu))) : C(\mu) : \epsilon(v)\, d\Omega$$

$$\begin{aligned} \mathcal{L}(v; \mu) \;&=\; \int_\Omega \rho F v \, d\Omega - \int_\Omega \sigma^0 : \epsilon(v)\, d\Omega \\ &\quad - \int_{\Gamma_6} \sigma^0 \overrightarrow{n} \cdot v \, d\Gamma \end{aligned}$$

- Yield surface: separates the elastic and plastic domains, defined by the function $f(\sigma_{ij})$

$f(\sigma_{ij}) < 0$	*Interior of the elastic domain (elastic comportement)*
$f(\sigma_{ij}) = 0$	*Boundary of the elastic domain (elastoplastic comportement)*

- Decompose the strain into elastic and plastic parts:

$$\epsilon_{ij} = \epsilon_{ij}^{e} + \epsilon_{ij}^{p}$$

- The yield surface is defined by:

$$\boxed{f(\sigma_{ij}) = (\sigma_\ell - \sigma_s) - (\sigma_s + \sigma_\ell)\sin\varphi - 2c\ \cos\varphi\ = 0}$$

where $\sigma_\ell \geq \sigma_s$ are the principal stresses (eigen values).

- Variational Problem : Find $u \in X = \left\{ v \in (H^1(\Omega))^2 \;\middle|\; \begin{array}{l} v_1 = 0 \text{ on } \Gamma_1, \Gamma_2, \Gamma_3 \\ v_2 = 0 \text{ on } \Gamma_2 \end{array} \right\}$

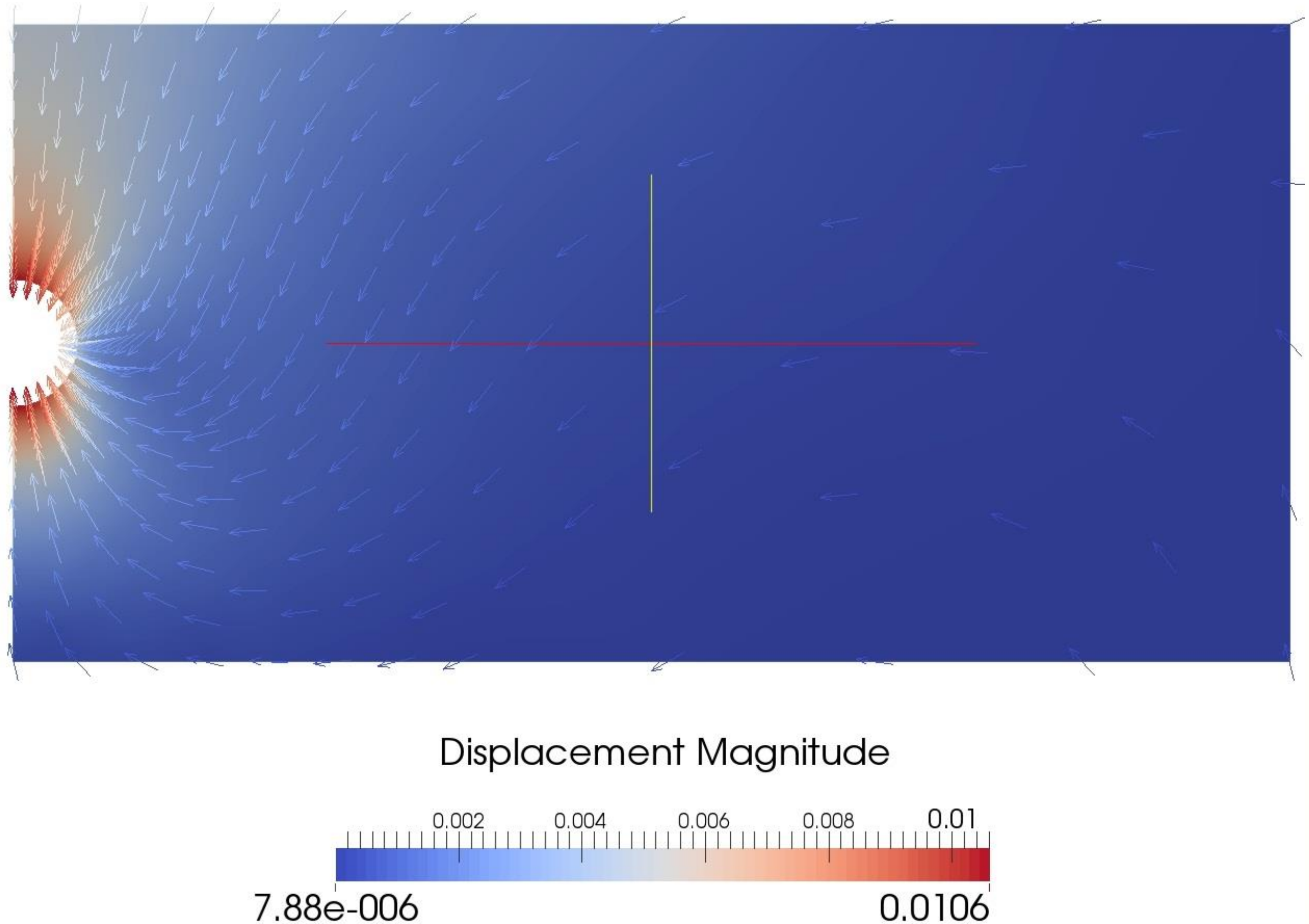
such that

$$a(u(\mu), v; \mu) = \mathcal{L}(v; \mu) \qquad v \in X,$$

$$a(u(\mu), v; \mu) = \int_\Omega (\epsilon(u(\mu)) - \epsilon^p(u(\mu))) : C(\mu) : \epsilon(v)\, d\Omega$$

$$\begin{aligned} \mathcal{L}(v; \mu) \quad &= \quad \int_\Omega \rho F v \, d\Omega - \int_\Omega \sigma^0 : \epsilon(v)\, d\Omega \\ &\quad - \int_{\Gamma_6} \sigma^0 \overrightarrow{n} \cdot v \, d\Gamma \end{aligned}$$

- Approximation by finite elements with CESAR-LCPC: iterative method which calculates $f(\sigma_{ij})$ over each mesh element, term added to right-hand side.
- Resolution will always be linked to the number of DoF.

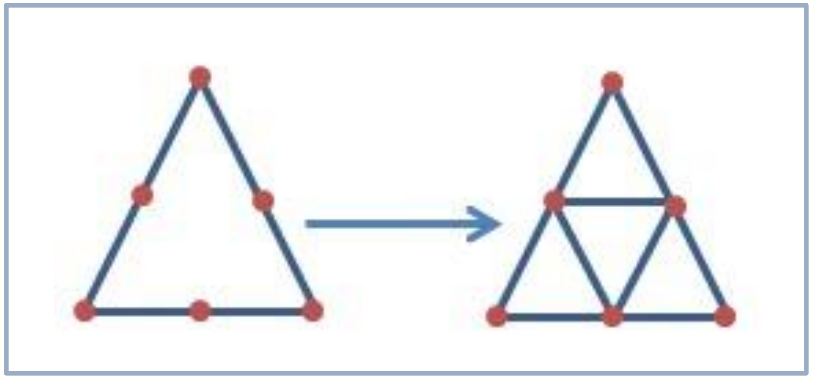


$$\xi = (125, 0.35, 23, 0.04)$$

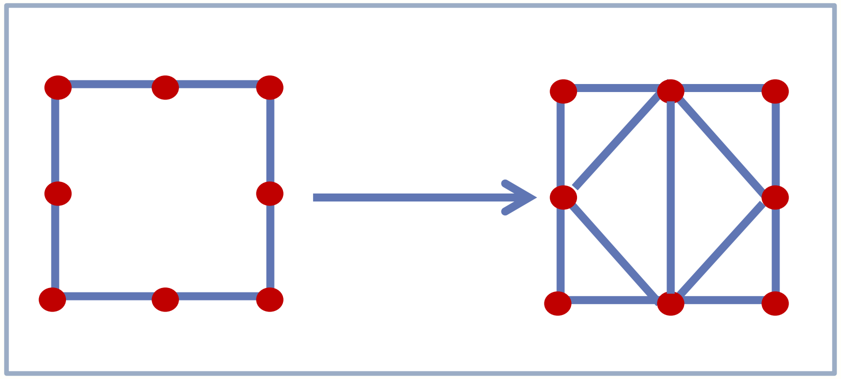
An Example Solution

NUMERICAL RESULTS

- The meshes created for CESAR-LCPC use $\mathbb{P}_2$ or $\mathbb{Q}_2^*$ finite elements in a different format from meshes for FreeFem++.
- We converted the meshes to FreeFem++ $\mathbb{P}_1$ finite elements as follows.



$\mathbb{P}_2$ to $\mathbb{P}_1$

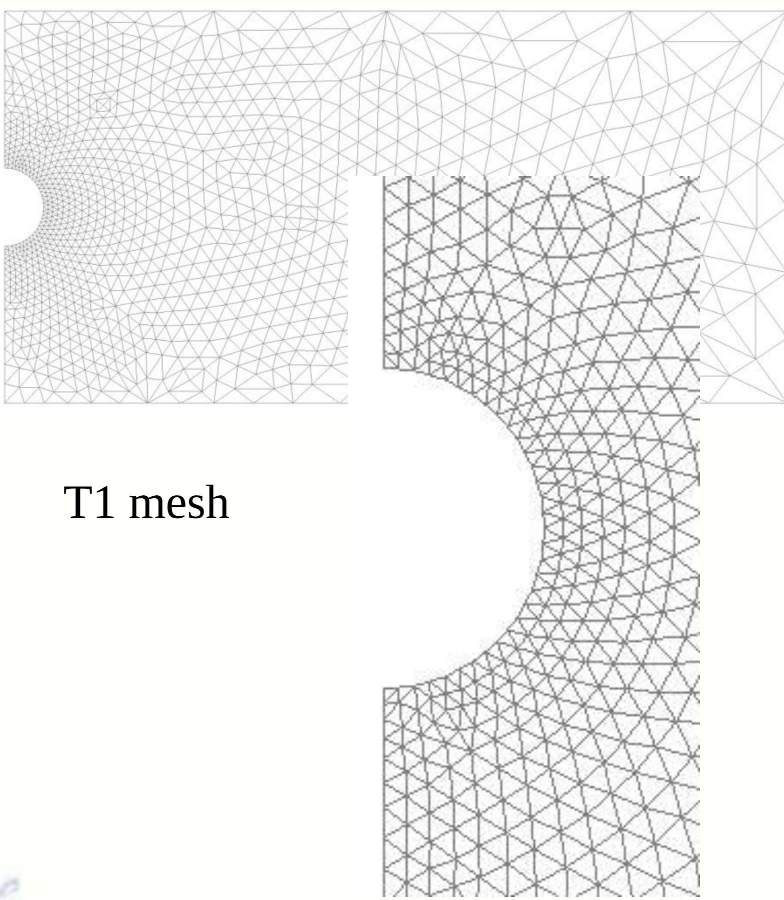


$\mathbb{Q}_2^*$ to $\mathbb{P}_1$

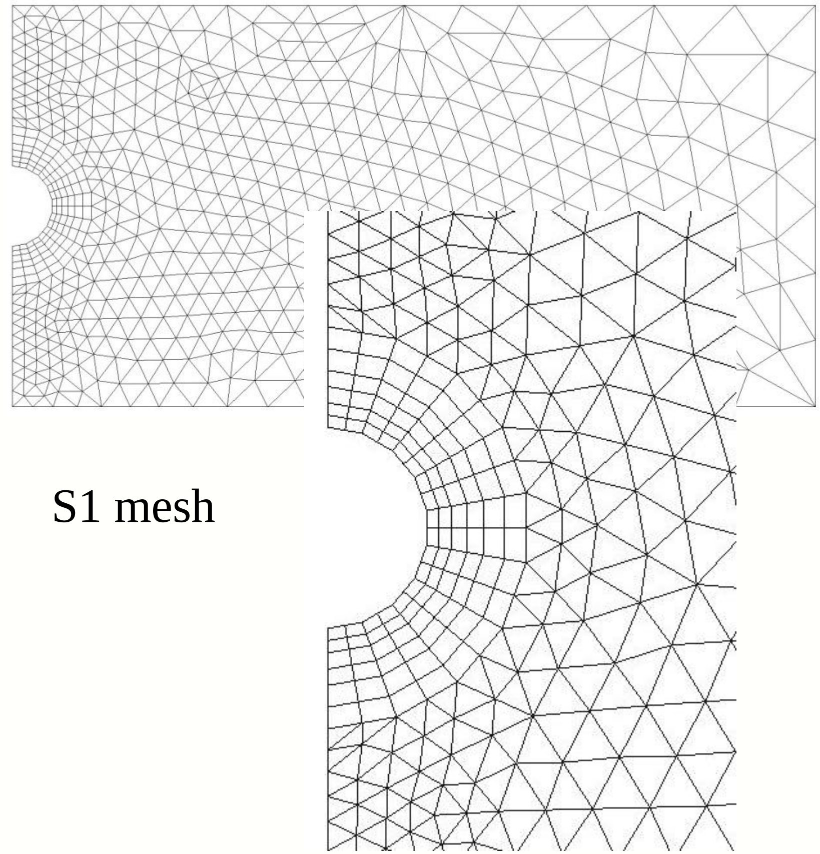
Remark on the Meshes

Meshes

- Example of embedded meshes:



T1 mesh



S1 mesh

- “Test” Solutions:

Y	λ	φ	c
100	0.20	22	0.02
150	0.25	24	0.04
200	0.30	26	0.06
250	0.35	28	
300	0.40	30	
		32	
		34	

Table 1: Set Ξ_{test} of 525 $\xi \in \mathcal{D}$ used to generate the reduced basis.

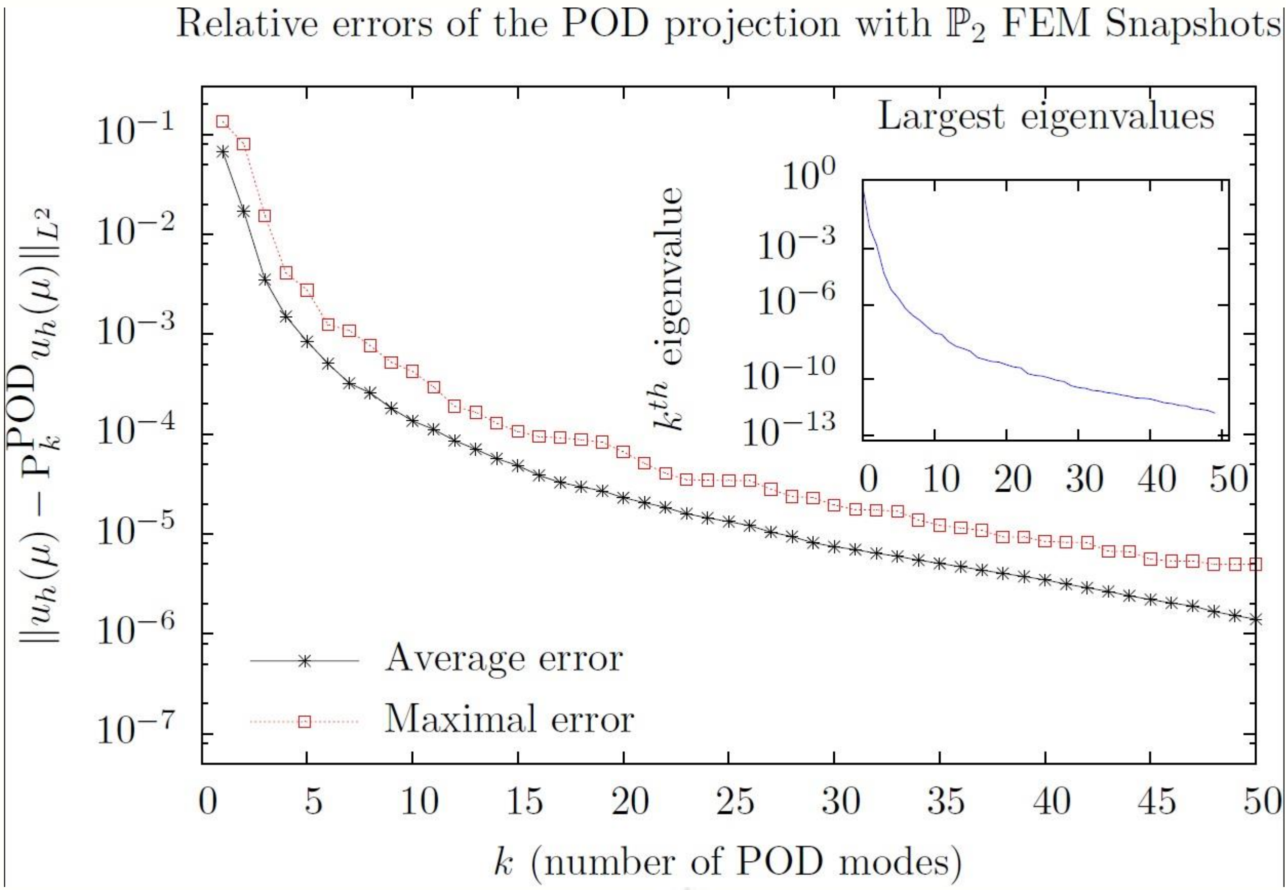
- “Trial” Solutions:

Y	λ	φ	c
125	0.25	23	0.03
275	0.35	33	0.05

Table 2: Set Ξ_{trial} of 16 parameters $\xi \in \mathcal{D}$ used to test our results.

RB Construction

- Evolution of the POD projection error of all "test" solutions (in L^2 norm):



POD Analysis

Construction of the reduced basis:

1. Greedy method with L^2 projection:
   - L^2-projection onto RB space. $P_N : X_h \mapsto X_h^N$

$$< v_h - P_N v_h, v_h^N >_{L^2} = 0 \quad \forall v_h^N \in X_h^N, \; v_h \in X_h.$$

Algorithm 1 : Greedy's algorithm to build the RB space

Given $\Xi_{test} = (\xi_1, \ldots, \xi_{n_{test}}) \in \mathcal{D}^{n_{test}}$, $n_{test} >> 1$:

Choose randomly $\xi_1 \rightarrow \; S_1 = \{\xi_1\}$ and $X_h^1 = \mathrm{span}(u_h(\xi_1))$.

for $N = 2$ to N_{max} do

$$\xi_N = \underset{\xi \in \Xi_{test}}{argmax} \frac{\|u_h(\xi) - P_{N-1} u_h(\xi)\|_{L^2}}{\|u_h(\xi)\|_{L^2}}$$

$$S_N = S_{N-1} \cup \xi_N \text{ and } X_h^N = X_h^{N-1} + \mathrm{span}(u_h(\xi_N))$$

end for

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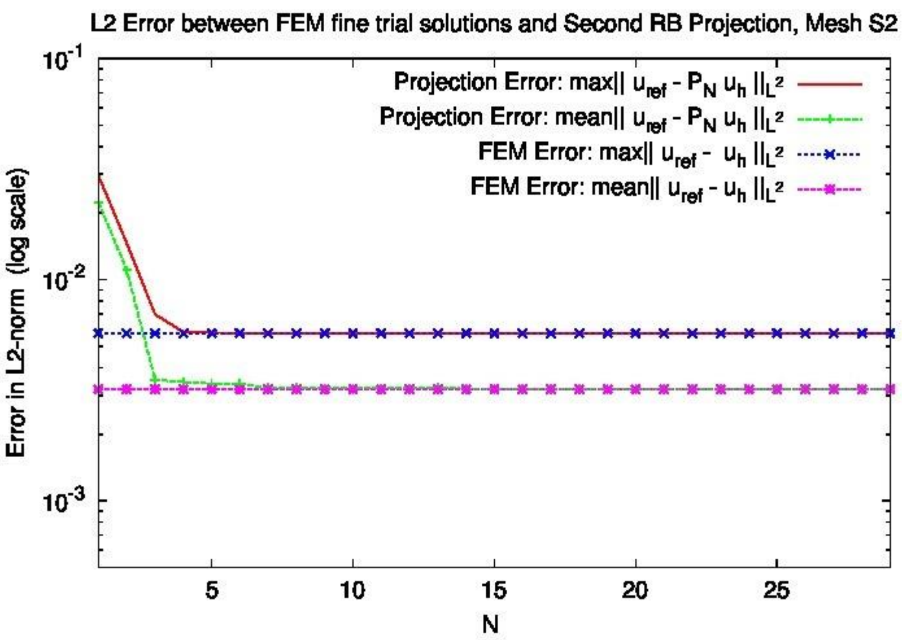
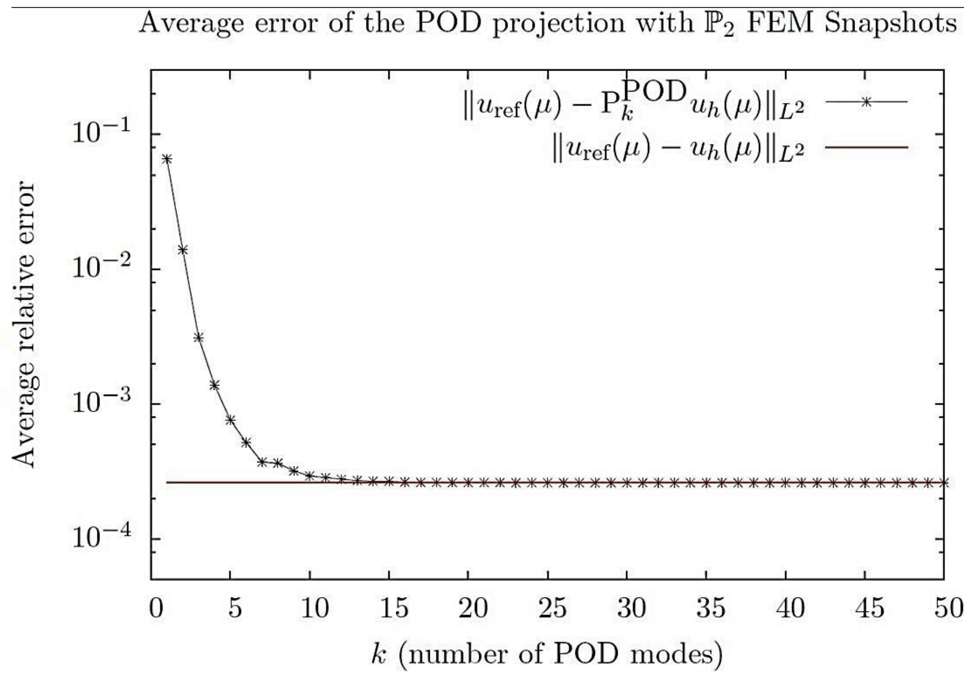
2. Orthogonalization of the RB functions.

- Evolution of the projection error of all “trial” solutions (in L^2 norm):



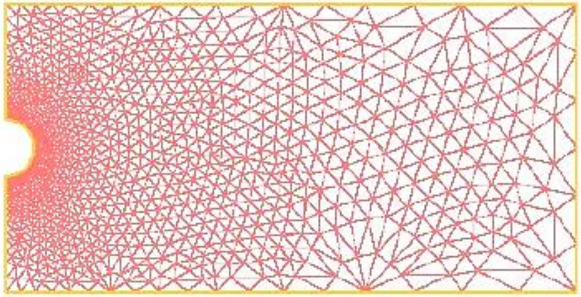
RB Projection



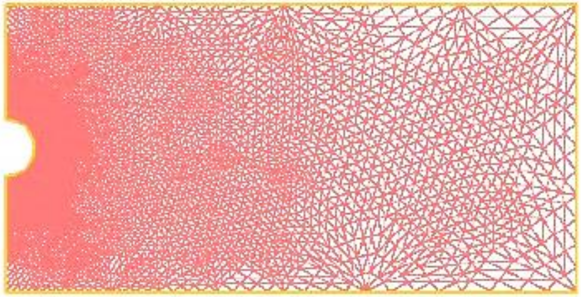
POD Projection

RB Testing

- First group of embedded meshes:



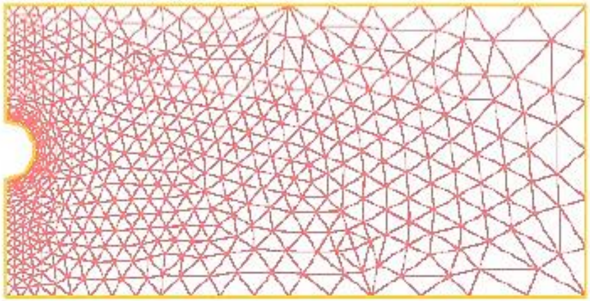
Coarse Mesh T1: 1116 DoF



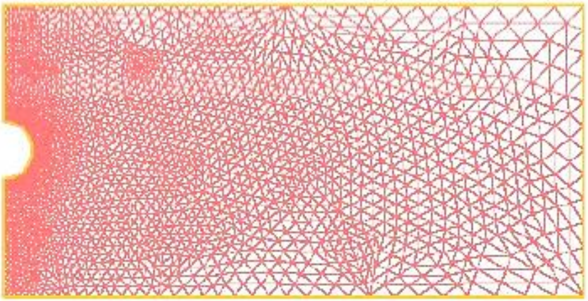
Fine Mesh T2: 4355 DoF

Reference Mesh T3: 17205 DoF

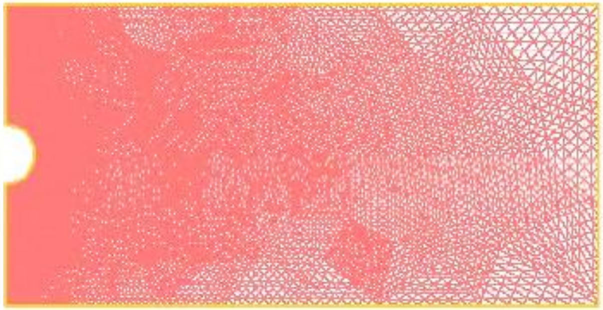
- Second group of embedded meshes:



Coarse Mesh S1: 592 DoF



Fine Mesh S2: 2273 DoF

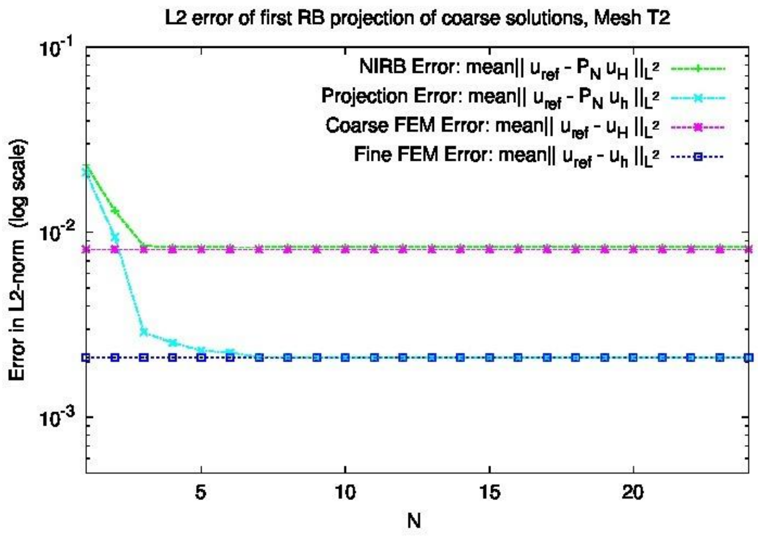


Reference Mesh S3: 8905 DoF

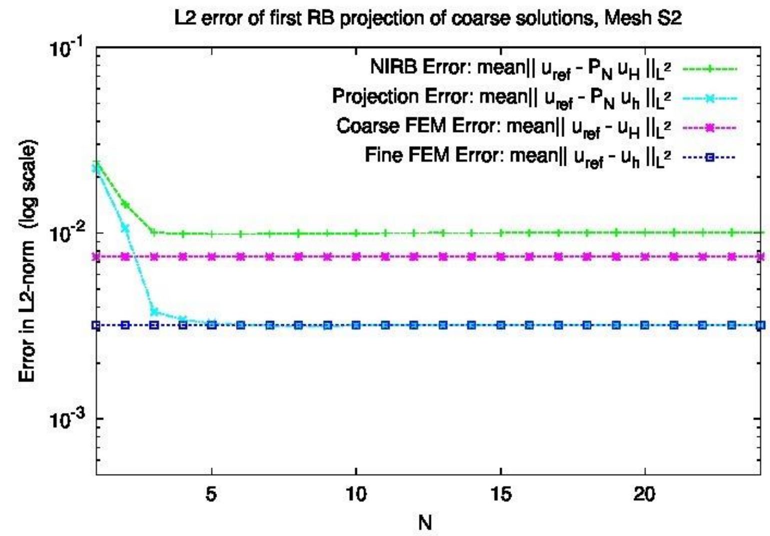
Meshes: FreeFem++ Format

Mesh Testing: Mean Errors

- On the S-meshes (as opposed to differently refined T-meshes used first), the coarse projection error doesn't reach the coarse FEM error.



T2 Mesh



S2 Mesh

- By un-refining, we lost some regularity.

	Coarse FEM Errors	Fine FEM Errors
T-Meshes	0.00807907	0.00209874
S-Meshes	0.00745765	0.00319388

- Offline/Online decomposition of the rectification:

$$\tilde{u}_H^N(\xi) = \sum_{i=1}^{N} (\mathbf{R}^N u_H^N)_i \phi_i$$

$$\begin{aligned}(\mathbf{R}^N u_H^N)_i &= \sum_{k=1}^{N} R_{ik}^N < u_H(\xi), \phi_k >_{L^2} \\ &= \sum_{k=1}^{N} R_{ik}^N \beta_k.\end{aligned}$$

$$\begin{aligned}\beta_k &= < u_H(\xi), \phi_k >_{L^2} \\ &= < \sum_{i=1}^{\mathcal{N}_H} u_H^i \varphi_i^H, \sum_{j=1}^{\mathcal{N}_h} \phi_k^j \varphi_j^h >_{L^2} \\ &= \sum_{i=1}^{\mathcal{N}_H} \sum_{j=1}^{\mathcal{N}_h} u_H^i \phi_k^j \int_\Omega \varphi_i^H \varphi_j^h \\ &= \sum_{i=1}^{\mathcal{N}_H} \sum_{j=1}^{\mathcal{N}_h} u_H^i \mathrm{M}_{ij} \phi_k^j \\ &= \mathrm{u}_H^t \mathrm{M} \vec{\phi}_k,\end{aligned}$$

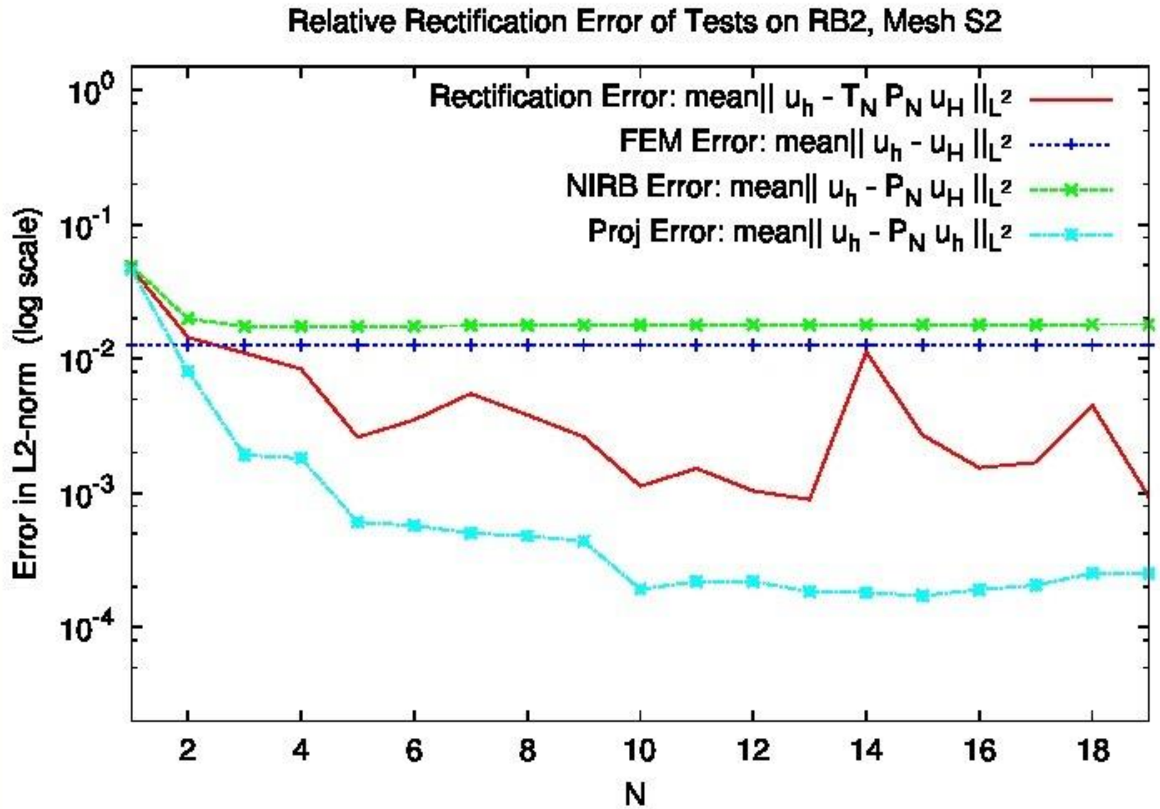
Computed Offline

(φ_j^h Fine FEM basis functions)

(φ_j^H Coarse FEM basis functions)

NIRB Testing: Rectification Step

- Evolution of the rectification error of “test” solutions during the offline phase:



Mean Errors in L^2-norm

NIRB Testing

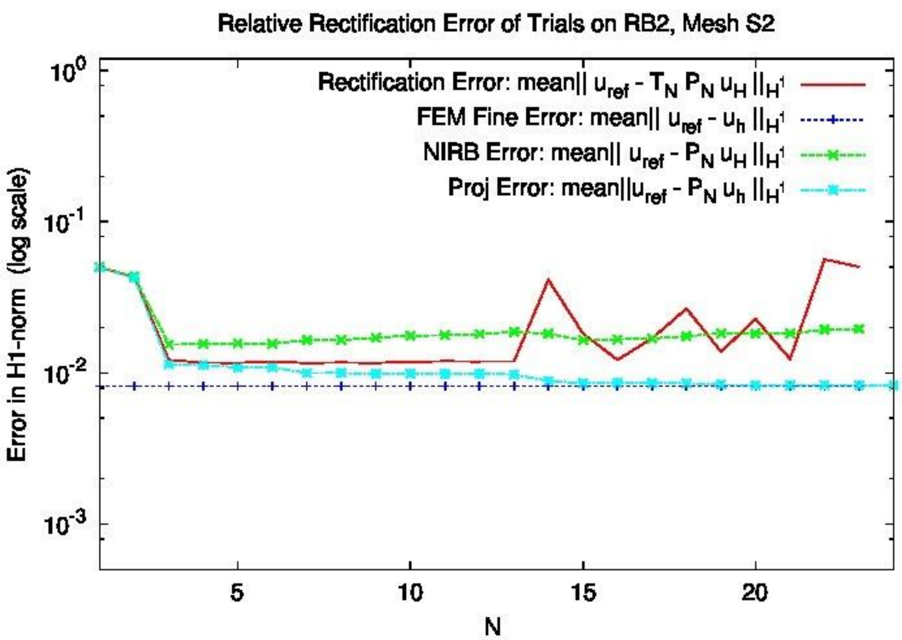
- Very different coarse and fine solutions ⇒ poor rectification: we tried separating linear and nonlinear (with resect to material behavior) test solutions.

- Iterative method of CESAR: 1 iteration ⇒ elastic comportement (linear), multiple iterations ⇒ elastic-plastic comportement (nonlinear).

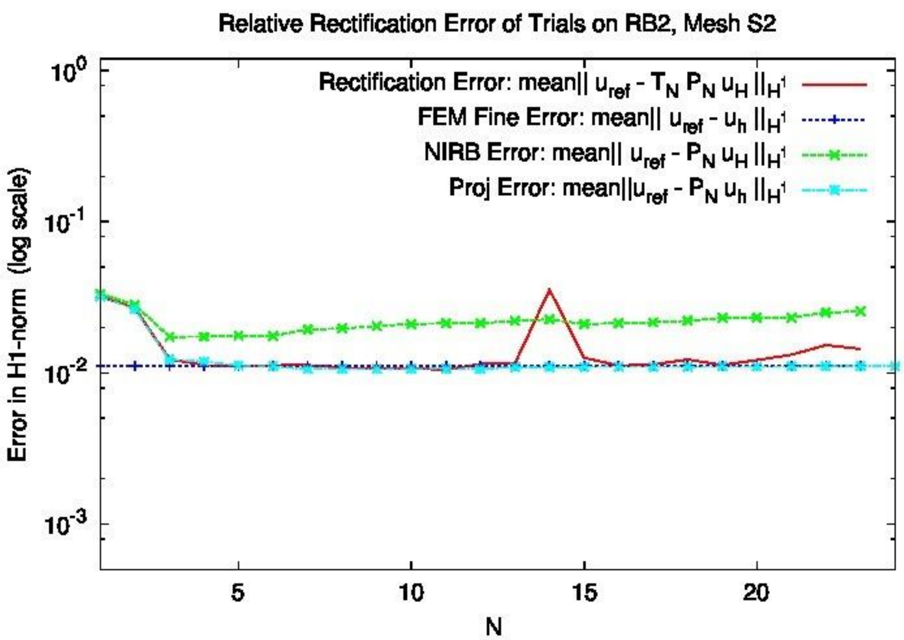
- Set $\tilde{\Xi}_{test}$ of 245 parameter sets which are nonlinear on S1.

Sorting by Material Behavior

- Evolution of the rectification error of “trial” solutions (in H^1 norm) :



Mean Errors of All Trials



Mean Errors of Non-linear Trials

NIRB Testing

- Modify our first rectification matrix:

$$
\left(\mathbf{T}^N\right)\left(\begin{array}{c|c|c}
\gamma_1^H(\xi_1,\psi) & \cdots & \cancel{\gamma_1^H(\xi_N,\psi)} \\
\vdots & \vdots & \vdots \\
\gamma_N^H(\xi_1,\psi) & \cdots & \gamma_N^H(\xi_N,\psi)
\end{array}\right)
=
\left(\begin{array}{c|c|c}
\gamma_1^h(\xi_1,\psi) & \cdots & \cancel{\gamma_1^h(\xi_N,\psi)} \\
\vdots & \vdots & \vdots \\
\gamma_N^h(\xi_1,\psi) & \cdots & \gamma_N^h(\xi_N,\psi)
\end{array}\right),
$$

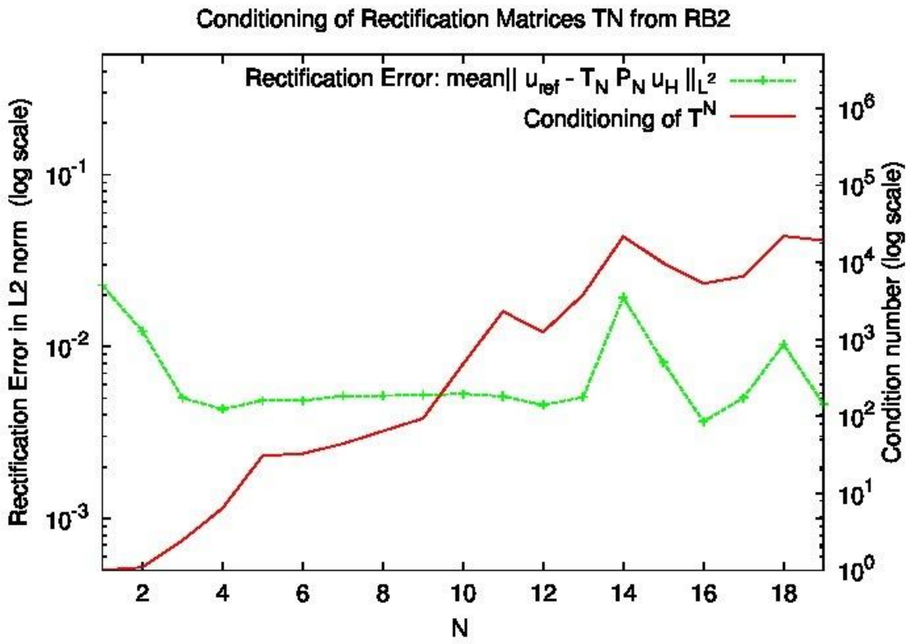
Id Id

- We have a second rectification matrix for $\tilde{N}_{max} = 13$:

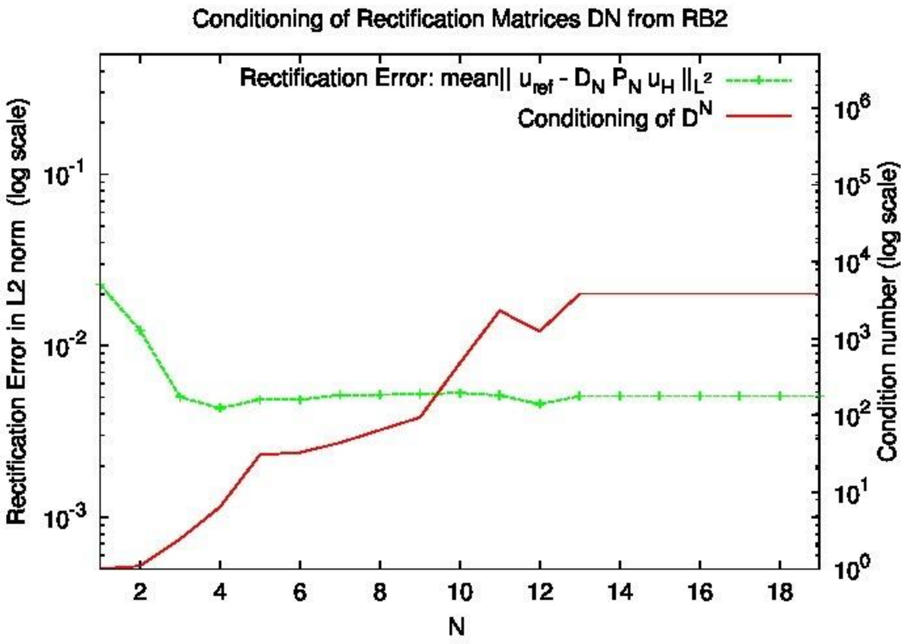
$$
\left(\mathbf{D}^N\right)\left(\begin{array}{c|c|c|ccc}
\gamma_1^H(\xi_1,\psi) & \cdots & \gamma_1^H(\xi_{N_{max}},\psi) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\gamma_{N_{max}}^H(\xi_1,\psi) & \cdots & \gamma_{N_{max}}^H(\xi_{N_{max}},\psi) & 0 & \cdots & 0 \\
\gamma_{N_{max}+1}^H(\xi_1,\psi) & \cdots & \gamma_{N_{max}+1}^H(\xi_{N_{max}},\psi) & 1 & 0\cdots & 0 \\
\vdots & \vdots & \vdots & 0 & \ddots & \vdots \\
\vdots & \vdots & \vdots & 0 & \ddots & 0 \\
\gamma_N^H(\xi_1,\psi) & \cdots & \gamma_N^H(\xi_{N_{max}},\psi) & 0 & \cdots 0 & 1
\end{array}\right)
=
\left(\begin{array}{c|c|c|ccc}
\gamma_1^h(\xi_1,\psi) & \cdots & \gamma_1^h(\xi_{N_{max}},\psi) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\gamma_{N_{max}}^h(\xi_1,\psi) & \cdots & \gamma_{N_{max}}^h(\xi_{N_{max}},\psi) & 0 & \cdots & 0 \\
\gamma_{N_{max}+1}^h(\xi_1,\psi) & \cdots & \gamma_{N_{max}+1}^h(\xi_{N_{max}},\psi) & 1 & 0\cdots & 0 \\
\vdots & \vdots & \vdots & 0 & \ddots & \vdots \\
\vdots & \vdots & \vdots & 0 & \ddots & 0 \\
\gamma_N^h(\xi_1,\psi) & \cdots & \gamma_N^h(\xi_{N_{max}},\psi) & 0 & \cdots 0 & 1
\end{array}\right)
$$

Rectification: Version 2

- Evolution of the rectification error with respect to the conditioning of the rectification matrices:



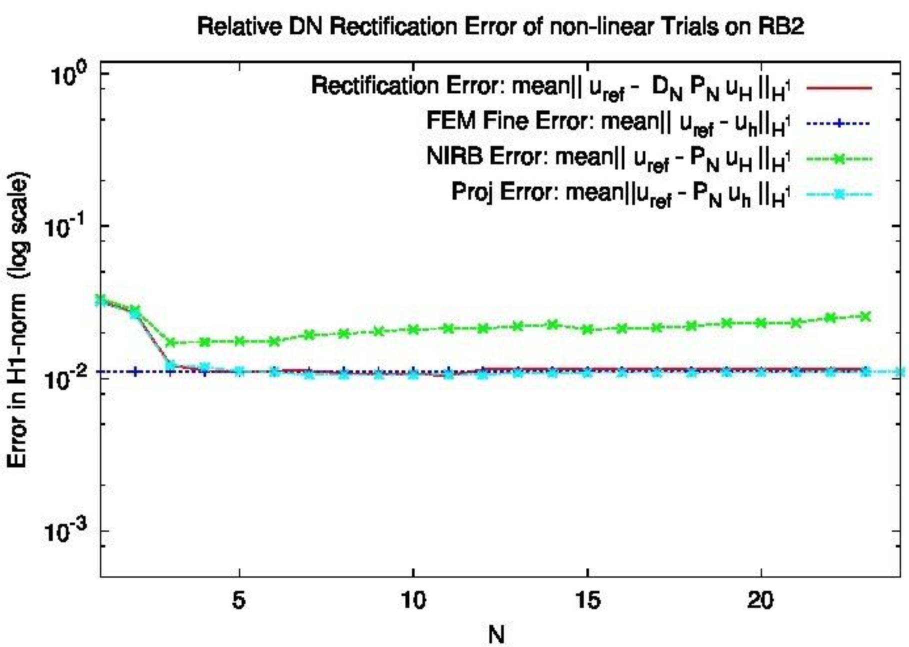
T^N: First Rectification



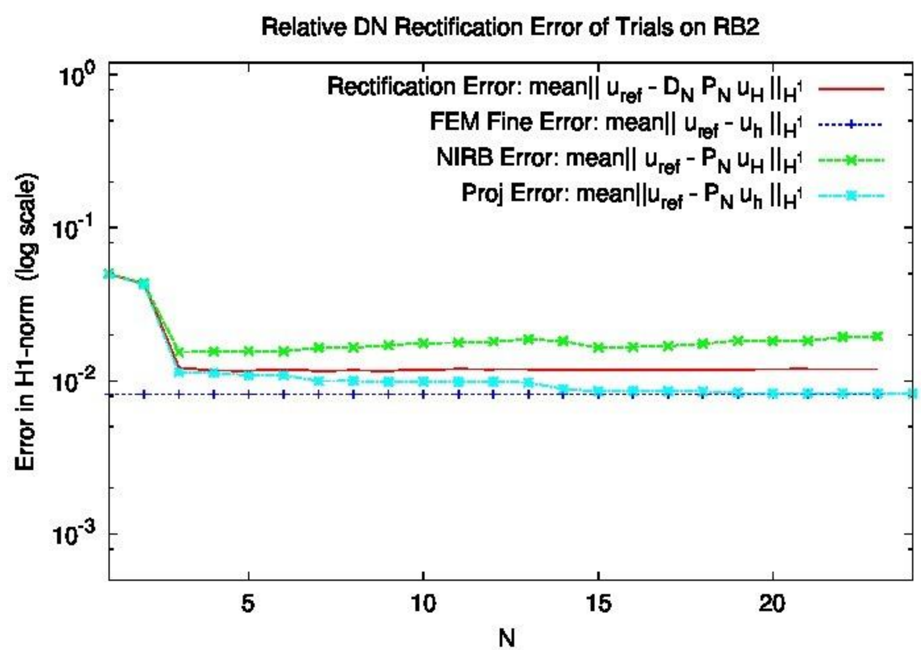
D^N: Second Rectification

Rectification

- Evolution of the rectification error of “trial” solutions (in H^1 norm) :



Mean Errors of Non-linear Trials

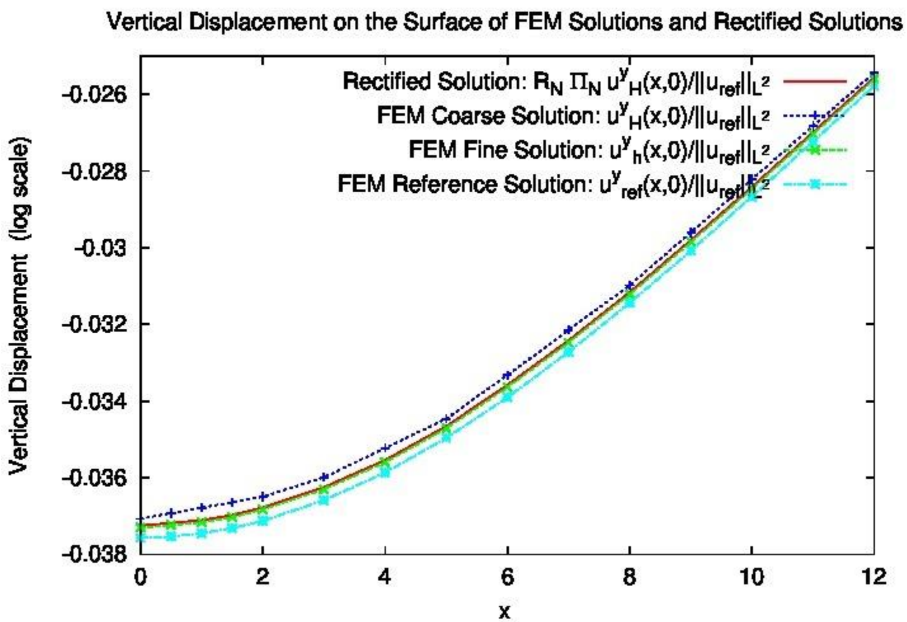


Mean Errors of All Trials

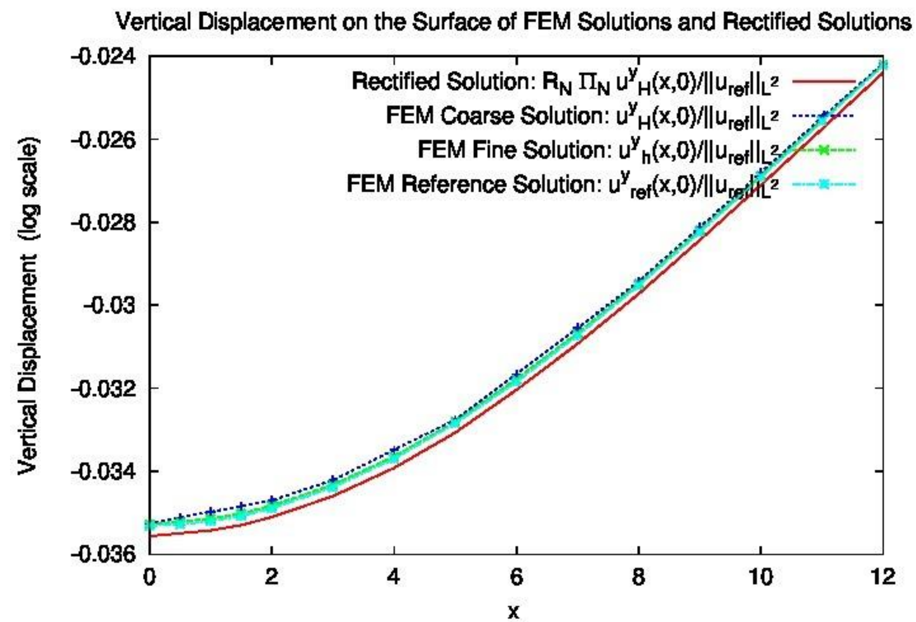
Rectification: Version 2

Evaluation at the Surface

- Quantity of interest: vertical displacement at the surface.

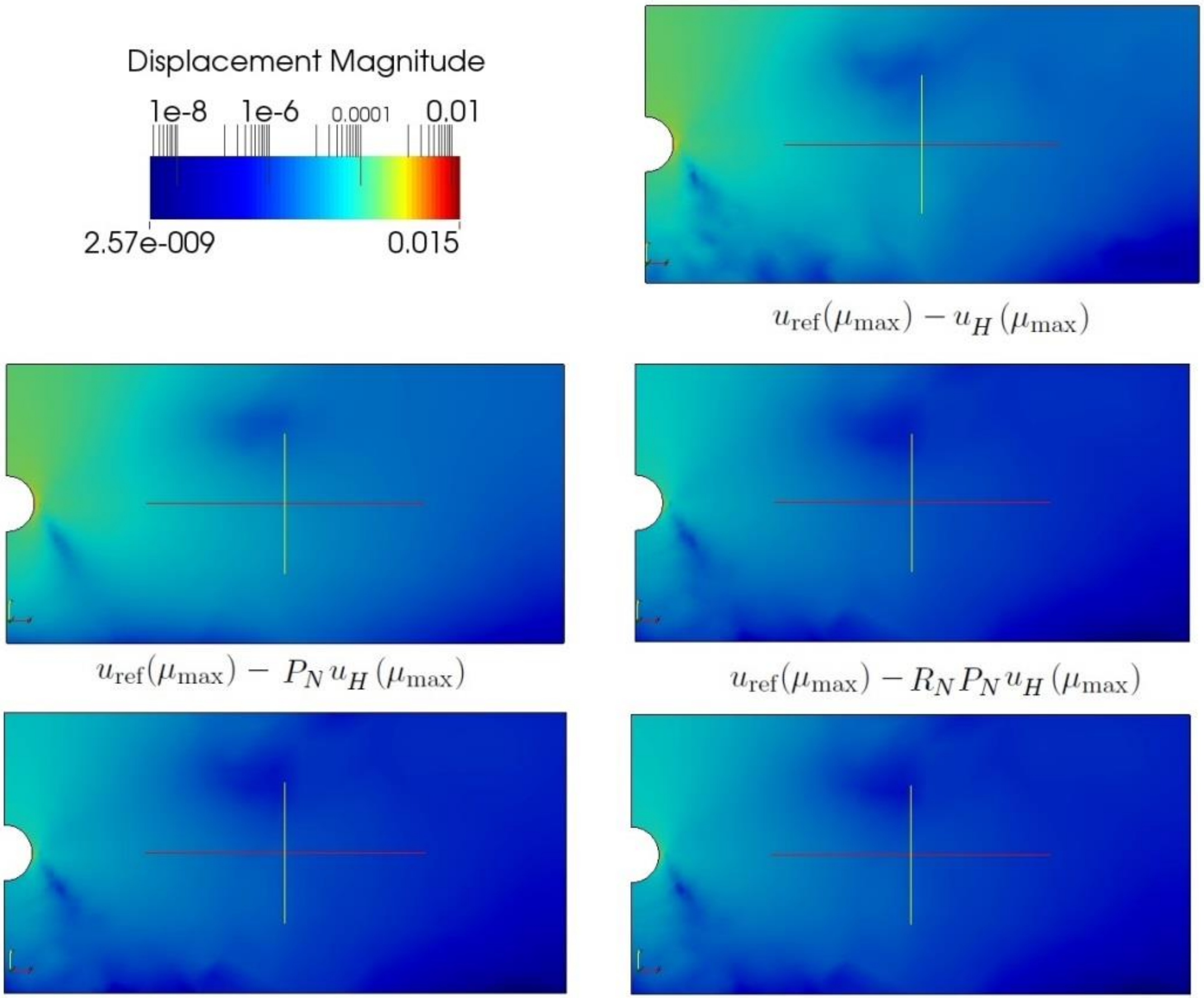


$\xi = (E, \lambda, \varphi, c) = (125, 0.35, 23, 0.03)$. (non-linear)



$\xi = (275, 0.25, 23, 0.05)$ (linear)

Error Plots



$u_{\mathrm{ref}}(\mu_{\max}) - u_H(\mu_{\max})$

$u_{\mathrm{ref}}(\mu_{\max}) - P_N u_H(\mu_{\max})$

$u_{\mathrm{ref}}(\mu_{\max}) - R_N P_N u_H(\mu_{\max})$

$u_{\mathrm{ref}}(\mu_{\max}) - P_N u_h(\mu_{\max})$

$u_{\mathrm{ref}}(\mu_{\max}) - u_h(\mu_{\max})$

RESOLUTION TIMES

NIRB Full Online: Resolution Time

- Resolution times by the NIRB method, optimized algorithm with $N = 13$:

FEM	NIRB
4.42051s	0.978888s

Average calculation times (s) for a single approximate solution : FEM on S2 mesh and the corresponding full NIRB method.

- Calculation times of the rectified solution step-by-step:

Total Time	Coarse FEM Calculation	Determining Coefficients	Reconstruction of Solution
0.978888s	0.697638s	0.0925s	0.203125s

CONCLUSIONS

- An Application of a Non-Intrusive Reduced Basis Method in Geotechnics: Mohr-Coulomb model

- Results
  - Resolution times greatly reduced.
  - Tested with different meshes : $\mathbb{P}_2$ and $\mathbb{P}_2/\mathbb{Q}_2^*$
  - Rectification by sorting by material behavior: works with nonlinear solutions.

- Perspectives
  - In Progress: rectification without sorting by material behavior.

Conclusion

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- Gram-Schmidt Orthonormalization $(\psi_i)_{1\leq i\leq N}$
- $L^2 - H^1$ orthogonalization:
  - Eigen Values/vectors of: $\mathbf{S}\mathbf{v} = \lambda\mathbf{M}\mathbf{v}$

$$(\mathrm{S})_{ij} =< \nabla\psi_i, \nabla\psi_j >_{L^2(\Omega)} \;\; 1 \leq i, j \leq N,$$
$$(\mathrm{M})_{ij} =< \psi_i, \psi_j >_{L^2(\Omega)} \;\; 1 \leq i, j \leq N.$$

  - $L^2 - H^1$ orthogonal and L^2-normal basis functions:
    - $v_i(j) = i^{th}$ vector, j^{th} component.

$$\phi_i = \sum_{j=1}^{N} v_i(j)\psi_j, \; 1 \leq i \leq N.$$

- $u_h^N(\xi) \in X_h^N \quad u_h^N(\xi) = \sum_{j=1}^{N} \alpha_i^h(\xi)\phi_i$

Standard Method: Stabilization