Geometric constraints for shape and topology optimization in architectural design

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A wee bit of history (I)

- Hooke's theorem (1675):

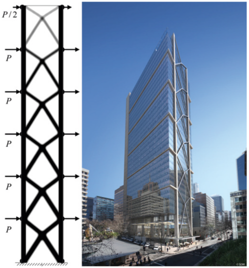
  "As hangs the flexible chain, so but inverted will stand the rigid arch."

- A. Gaudi designed the church of Colònia Güell (1889-1914) by using a $3d$ funicular model to determine a stable arrangement of columns and vaults.

(Left) Experimental device of Gaudi, (right) tentative outline of Colònia Güell ***(Photo courtesy of http://www.gaudidesigner.com).***

A wee bit of history (II)

- Such techniques have been employed and improved by world-renowned architects: Heinz Isler, Gustave Eiffel, Frei Otto, etc.
- Recent shape and topology optimization (S&T) techniques have been used in the device of large-scale structures.

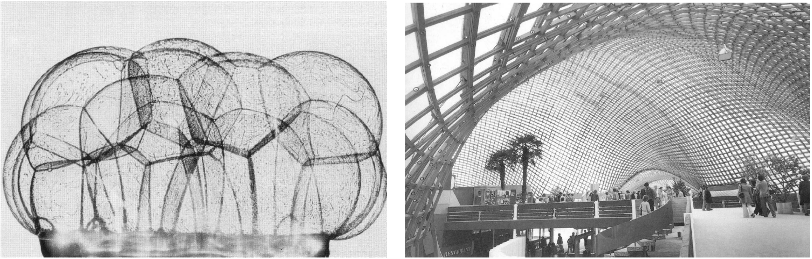


(Left) Front facade of the Qatar National Convention Center in Doha [Sasaki et al]. (right) Design of a 288m tall high-rise in Australia by Skidmore, Owings & Merrill.

Shape and topology optimization for architecture

Applying shape and topology optimization techniques in the field of architectural design is tentative:

- They allow to model and optimize complex criteria of the design, related to aesthetics, manufacturability, or mechanical performance.
- Optimal designs from the mechanical viewpoint often show 'attractive' outlines, and a strongly organic nature which is praised by architects.



(Left) Soap bubble foam structure devised by Frei Otto, (right) interior view of the Manheim Garden festival ***(Excerpted from*** [La]***)***.

Potential limitations

- Architects generally apply construction rules based on intuition to deal with the stringent requirements of stability, robsutness, etc. Except when aesthetics is the priority, or in the construction of exceptional structures (skyscrapers), challenging these rules is lengthy and costly, thus inefficient.

- Results of continuum-based S&T optimization techniques are difficult to use directly, since real-life structures are often assembled from bars or beams.

  $\Rightarrow$ A costly interpretation step of the optimized designs is necessary.

- Architectural design has a lot to do with the personal taste of the architect. On the contrary, most S&T optimization methods only consider mechanical aspects: their results can be reproduced by anyone, and do not leave room for original creation!

Objectives of the present work

- Propose a simplified theoretical and numerical S&T framework which is mechanically relevant for conceptual architectural design.

- Introduce constraint functionals for S&T optimization problems which allow the user to:
  - Encode information about its personal taste;
  - Ease the interpretation of the optimized designs as truss-like structures;
  - Deal with other geometric problems plaguing the optimized structures (visibility, elongated bars, etc.).

Shape optimization of linear elastic shapes (I)

A shape is a bounded domain $\Omega \subset \mathbb{R}^d$, which is

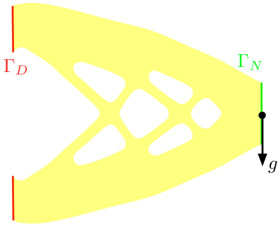
- fixed on a part Γ_D of its boundary,
- submitted to surface loads g, applied on $\Gamma_N \subset \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$.
- submitted to body forces f.

The displacement vector field $u_\Omega : \Omega \to \mathbb{R}^d$ is governed by the linear elasticity system:

$$\left\{ \begin{array}{rcll} -\mathrm{div}(Ae(u_\Omega)) & = & f & \text{in } \Omega \\ u_\Omega & = & 0 & \text{on } \Gamma_D \\ Ae(u_\Omega)n & = & g & \text{on } \Gamma_N \\ Ae(u_\Omega)n & = & 0 & \text{on } \Gamma \end{array} \right. ,$$

where $e(u) = \frac{1}{2}(\nabla u^T + \nabla u)$ is the strain tensor, and A is the Hooke's law of the material:

$$\forall e \in \mathcal{S}_d(\mathbb{R}), \;\; Ae = 2\mu e + \lambda \mathrm{tr}(e) I.$$



A 'Cantilever'

The deformed cantilever

Shape optimization of linear elastic shapes (II)

Goal: Optimize the compliance $C(\Omega)$ of shapes

$$C(\Omega) = \int_\Omega Ae(u_\Omega) : e(u_\Omega)\, dx = \int_\Omega f \cdot u_\Omega \, dx + \int_{\Gamma_N} g \cdot u_\Omega \, ds,$$

under constraints, modelled by a shape functional $P(\Omega)$ (e.g. $P(\Omega) = \mathrm{Vol}(\Omega)$).

Depending on the particular situation, this setting gives rise to two different kinds of optimization problems:

- Unconstrained optimization problems

$$\min_\Omega \mathcal{L}(\Omega), \text{ where } \mathcal{L}(\Omega) = C(\Omega) + \ell P(\Omega),$$

  where the objective criterion $C(\Omega)$ is penalized by the constraint $P(\Omega)$.

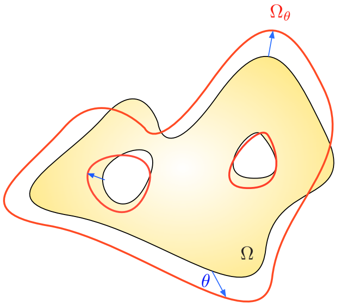
- Constrained optimization problems of the form:

$$\min_{\Omega \text{ s.t. } P(\Omega) \leq \alpha} C(\Omega).$$

Hadamard's boundary variation method describes variations of a reference, Lipschitz domain Ω of the form:

$$\Omega \to \Omega_\theta := (I + \theta)(\Omega),$$

for 'small' $\theta \in W^{1,\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right)$.



Definition 1.

Given a smooth domain Ω, a function $F(\Omega)$ of the domain is shape differentiable at Ω if the function

$$W^{1,\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right) \ni \theta \mapsto F(\Omega_\theta)$$

is Fréchet-differentiable at 0, i.e. the following expansion holds around 0:

$$F(\Omega_\theta) = F(\Omega) + F'(\Omega)(\theta) + o\left(||\theta||_{W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)}\right).$$

Techniques from optimal control theory make it possible to calculate shape gradients; in the case of 'many' functionals of the domain $J(\Omega)$, the shape derivative has the particular structure:

$$\boxed{J'(\Omega)(\theta) = \int_\Gamma v_\Omega \, \theta \cdot n \, ds,}$$

where v_Ω is a scalar field depending on u_Ω, and possibly on an adjoint state p_Ω.

Example: If $J(\Omega) = C(\Omega) = \int_\Omega f \cdot u_\Omega \, dx + \int_{\Gamma_N} g \cdot u_\Omega \, ds$ is the compliance, $v_\Omega = -Ae(u_\Omega) : e(u_\Omega)$ is the (negative) elastic energy density.

The generic algorithm

This shape gradient provides a natural descent direction for $J(\Omega)$: *for instance*, defining θ as

$$\theta = -v_\Omega n$$

yields, for $t > 0$ sufficiently small (*to be found numerically*):

$$J((I + t\theta)(\Omega)) = J(\Omega) - t\int_\Gamma v_\Omega^2 ds + o(t) < J(\Omega)$$

Gradient algorithm: For $n = 0, \dots$ until convergence,

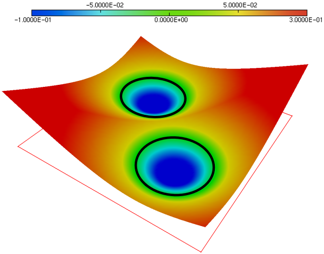
1. Compute the solution u_{Ω^n} (and p_{Ω^n}) of the elasticity system on Ω^n.
2. Compute the shape gradient $J'(\Omega^n)$ thanks to the previous formula, and infer a descent direction θ^n for the cost functional.
3. Advect the shape Ω^n according to θ^n, so as to get $\Omega^{n+1} := (I + \theta^n)(\Omega^n)$.

Definition 2.

The ***signed distance function*** d_Ω *to a bounded domain* $\Omega \subset \mathbb{R}^d$ *is defined by:*

$$\forall x \in \mathbb{R}^d, \begin{cases} -d(x, \partial\Omega) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \partial\Omega \\ d(x, \partial\Omega) & \text{if } x \in {}^c\overline{\Omega} \end{cases},$$

where $d(x, \partial\Omega) = \min_{y \in \partial\Omega} |x - y|$ *is the usual Euclidean distance function to* $\partial\Omega$.



Graph of the signed distance function to a union of two disks (in black)

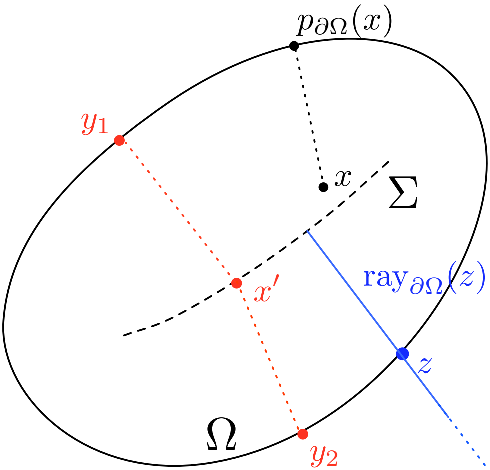
Signed distance function and geometry (I)

Definition 3.

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz, bounded open set;

- *Let $x \in \mathbb{R}^d$; the set of projections $\Pi_{\partial\Omega}(x)$ of x onto $\partial\Omega$ is:*
$$\Pi_{\partial\Omega}(x) = \{y \in \partial\Omega,\ \ d(x, \partial\Omega) = |x - y|\}.$$
- *When this set is a singleton, $p_{\partial\Omega}(x)$ is the projection of x onto $\partial\Omega$.*
- *The skeleton Σ of $\partial\Omega$ is:*
$$\Sigma := \left\{x \in \mathbb{R}^d,\ d_\Omega^2 \text{ is not differentiable at } x\right\}.$$
- *For $x \in \partial\Omega$, the ray emerging from x is:*
$$ray_{\partial\Omega}(x) := p_{\partial\Omega}^{-1}(x).$$

Signed distance function and geometry (II)



x has a unique projection over $\partial\Omega$, whereas x' has two such points y_1, y_2.

Signed distance function and geometry (III)

Proposition 1.

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz, bounded open set;

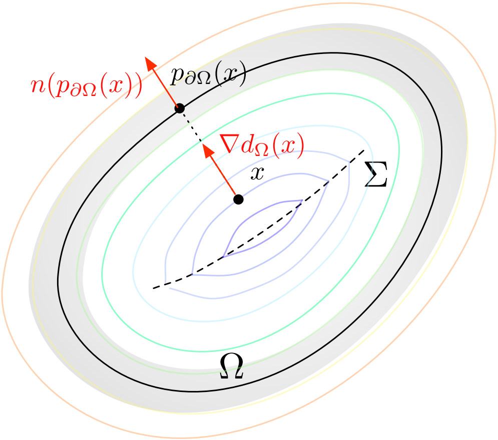
- *A point $x \in \mathbb{R}^d$ has a unique projection point $p_{\partial\Omega}(x)$ iff $x \notin \Sigma$. In such a case, d_Ω is differentiable at x, and its gradient reads:*

$$\nabla d_\Omega(x) = \frac{x - p_{\partial\Omega}(x)}{d_\Omega(x)}.$$

  In particular, $|\nabla d_\Omega(x)| = 1$ wherever it makes sense.

- *If Ω is of class $\mathcal{C}^1$, this last quantity equals $\nabla d_\Omega(x) = n(p_{\partial\Omega}(x))$.*
- *If Ω is of class $\mathcal{C}^k$, $k \geq 2$, then d_Ω is also of class $\mathcal{C}^k$ on a neighborhood of $\partial\Omega$.*

Signed distance function and geometry (IV)

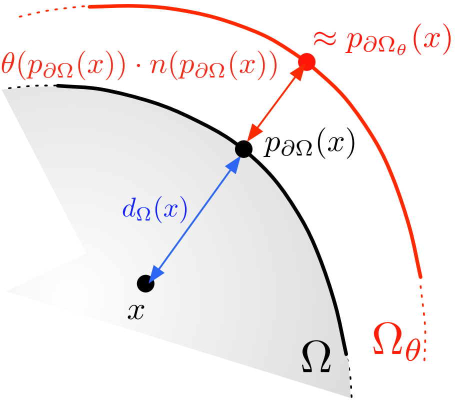


Some level sets of d_Ω *are depicted in color;* d_Ω *is as smooth as the boundary* $\partial\Omega$ *on the shaded area (at least).*

Lemma 2.

Let $\Omega \subset \mathbb{R}^d$ be a $\mathcal{C}^1$ bounded domain, and $x \notin \Sigma$. The function $\theta \mapsto d_{\Omega_\theta}(x)$, from $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ into $\mathbb{R}$ is Gâteaux-differentiable at $\theta = 0$, with derivative:

$$d'_\Omega(\theta)(x) = -\theta(p_{\partial\Omega}(x)) \cdot n(p_{\partial\Omega}(x)).$$



The anisotropic signed distance function (I)

Let $M \in \mathcal{S}_d(\mathbb{R})$ be a metric tensor, i.e. a symmetric, positive definite matrix. M accounts for another way of evaluating the distance between points:

$$\forall x, y \in \mathbb{R}^d, \;\; |x-y|_M^2 := \langle Mx, y \rangle.$$

Definition 4.

- *The anisotropic (unsigned) distance function $d^M(\cdot, K)$ to a compact subset $K \subset \mathbb{R}^d$ is defined by:*

$$\forall x \in \mathbb{R}^d, \;\; d^M(x, \partial\Omega) = \inf_{y \in K} |x-y|_M.$$

- *The anisotropic signed distance function d_Ω^M to a bounded domain $\Omega \subset \mathbb{R}^d$ is defined by:*

$$\forall x \in \mathbb{R}^d, \;\; d_\Omega^M(x) = \begin{cases} -d^M(x, \partial\Omega) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \partial\Omega, \\ d^M(x, \partial\Omega) & \text{if } x \in {}^c\overline{\Omega}. \end{cases}$$

The anisotropic signed distance function (II)

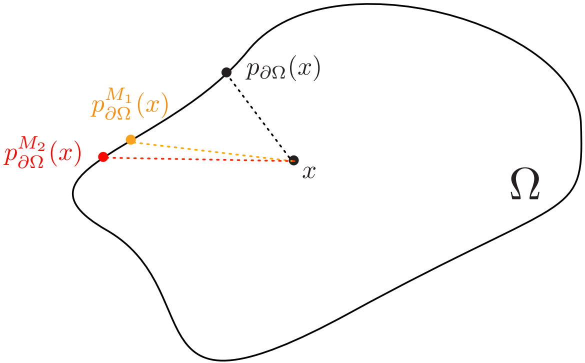
All the previous concepts extend from the isotropic to the anisotropic context, in particular as regards:

- The connections between d_Ω^M and the notions of projection set $\Pi_{\partial\Omega}^M(x)$, or unique projection point $p_{\partial\Omega}^M(x)$, for a point $x \in \mathbb{R}^d$,
- The differentiability of $x \mapsto d_\Omega^M$ outside the (anisotropic) skeleton Σ^M,
- The results about the shape differentiability of $\Omega \mapsto d_\Omega^M(x)$,

... up to giving an 'anisotropic flavour' to the previous formulae for the isotropic case.

Isotropic vs. anisotropic signed distance functions

The anisotropic signed distance function $d_\Omega^M(x)$ primarily evaluates the distance from x to $\partial\Omega$ in the direction of the smallest eigenvalue of M.



Projections $p_{\partial\Omega}(x)$, $p_{\partial\Omega}^{M_1}(x)$ and $p_{\partial\Omega}^{M_2}(x)$ of a point $x \in \Omega$ onto $\partial\Omega$ respectively associated to the metrics I, $M_1 = \begin{pmatrix} 0.1 & 1 \\ 0 & 1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0.01 & 1 \\ 0 & 1 \end{pmatrix}$.

Imposing shapes to resemble a specific design

- Our goal is to impose the resemblance of the optimized design with a user-defined target design Ω_T.

- We rely on a functional used in the previous work [DeBDaFreyNa].

$$P_m(\Omega) = \int_\Omega d_{\Omega_T}\, dx.$$

Intuition: If Ω_T is connected, it is the unique (local and global) minimizer of $P_m(\Omega)$.

Theorem 3.

The function $\Omega \mapsto P_m(\Omega)$ is shape differentiable and its derivative reads:

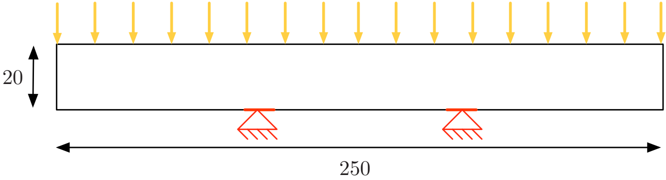
$$P'_m(\Omega) = \int_\Gamma d_{\Omega_T}\, \theta \cdot n\, ds.$$

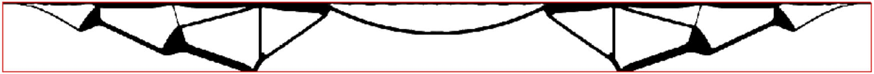
Example: optimization of the design a 2d bridge (I)

We consider the optimization of a 2d bridge under a volume constraint:

$$\begin{array}{cl} \min\limits_{\Omega} & C(\Omega), \\ \text{s.t.} & \text{Vol}(\Omega) \leq V_T, \end{array}$$

where $C(\Omega)$ is the compliance, and the maximum volume is $V_T = 0.163|D|$.





(Top) Setting of the 2d bridge test case, and (bottom) resulting optimized shape.

Example: optimization of the design a $2d$ bridge (II)

We would like to impose that the optimized design resembles the target shape Ω_T, which has unfortunately very poor mechanical performance.

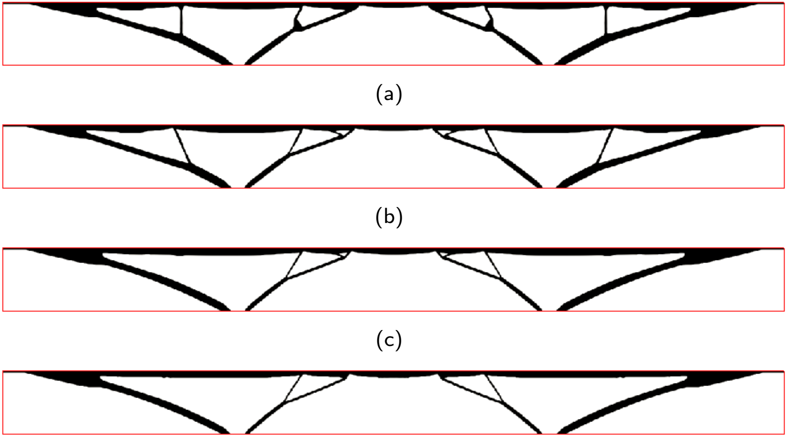
Target shape in the functional $P_m(\Omega)$ for the optimization of the $2d$ bridge.

We now solve the problem:

$$\begin{array}{cl} \min\limits_{\Omega} & \mathcal{L}(\Omega) \\ \text{s.t.} & \text{Vol}(\Omega) \leq V_T, \end{array} \quad \text{where } \mathcal{L}(\Omega) := t\frac{C(\Omega)}{C(\Omega_T)} + (1-t)\frac{P_m(\Omega)}{P_m(\Omega_T)},$$

and $t \in [0,1]$.

Example: optimization of the design a 2d bridge (III)



(a) (b) (c) (d)

Optimized 2d bridges for: (a) $t = 0.35$, *(b)* $t = 0.45$, *(c)* $t = 0.55$, *(d)* $t = 0.70$.

Fitting in a user-defined pattern

- Our aim is to influence the design of a structure Ω with a pattern.
- The pattern is supplied as the 0 level set of a function $g : \mathbb{R}^d \to \mathbb{R}$.
- We rely on the constraint functional

$$P_p(\Omega) = \int_{\partial\Omega} |n_\Omega - n_g|^2 \, ds,$$

where $n_g(x) = \frac{\nabla g(x)}{|\nabla g(x)|}$ is the normal vector field to the isolines of g.

Theorem 4.

The functional $\Omega \mapsto P_p(\Omega)$ is shape differentiable, and:

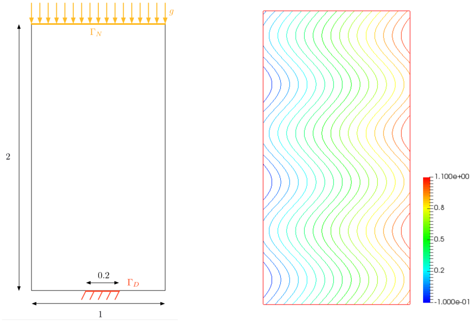
$$P_p'(\Omega)(\theta) = 2 \int_{\partial\Omega} (\kappa - \mathrm{div}_{\partial\Omega}(n_g))\theta \cdot n \, ds,$$

where $\mathrm{div}_{\partial\Omega} V := \mathrm{div}(V) - \langle \nabla V n, n \rangle$ is the tangential divergence of a smooth vector field $V : \partial\Omega \to \mathbb{R}^d$.

Example: optimization of the design of a $2d$ mast

We aim to optimize the shape of a column, while influencing its outline by the pattern function g defined by:

$$\forall x = (x_1, x_2) \in D, \;\; g(x) = x_1 - \frac{L_1}{10} \sin\left(\frac{2\pi x_2}{3L_2}\right), \;\; L_1 = 1, \;\; L_2 = 2.$$

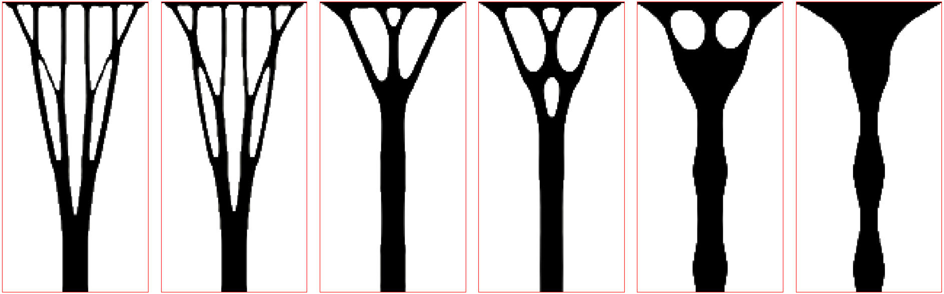


(Left) Boundary conditions for the column example, (right) isovalues of the pattern g.

Example: optimization of the design of a $2d$ mast

We now solve the problem:

$$\begin{array}{ll} \min\limits_{\Omega} & \mathcal{L}(\Omega) \\ \text{s.t.} & \mathrm{Vol}(\Omega) \leq V_T, \end{array} \quad \text{where } \mathcal{L}(\Omega) := t\,C(\Omega) + (1-t)P_p(\Omega)$$



(a) (b) (c) (d) (e) (f)

Optimized columns for: (a) $t = 1.0$*; (b)* $t = 0.90$*; (c)* $t = 0.60$*; (d)* $t = 0.45$*; (e)* $t = 0.35$*; (f)* $t = 0.15$.

Constraining the anisotropic maximum thickness of shapes (I)

We rely on the modelling of thickness of shapes introduced in [AlJouMi].

Definition 5.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, M be metric tensor. Then,

- *Ω has maximum thickness smaller than $\delta > 0$ if:*

$$\forall x \in \Omega, \;\; d_\Omega(x) \geq -\delta/2.$$

- *Ω has anisotropic maximum thickness smaller than $\delta > 0$ if:*

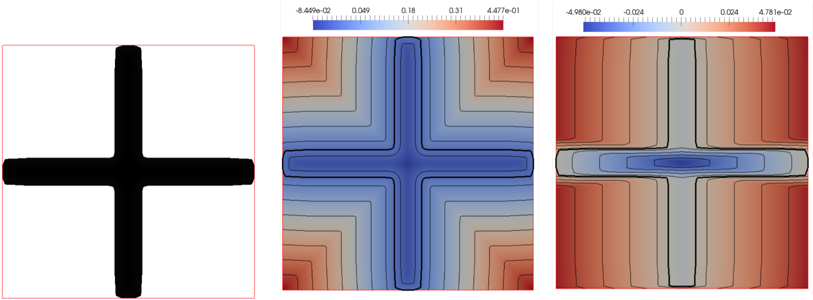
$$\forall x \in \Omega, \;\; d_\Omega^M(x) \geq -\delta/2.$$

Goal: Devise a constraint functional for imposing a bound $\delta > 0$ on the vaues of the anisotropic maximum thickness of Ω.

Constraining the anisotropic maximum thickness of shapes (II)

Recall that d_Ω^M 'sees' the distance from x to $\partial\Omega$ in the direction of the smallest eigenvalue of M.

$\boxed{\textbf{Example:}}$ If $M = \begin{pmatrix} 0.1 & 0 \\ 0 & 1 \end{pmatrix}$, a constraint on the anisotropic maximum thickness of shapes concerns the horizontal length of their features.



(Left) One shape Ω (in black); (middle) isolines of the isotropic distance function d_Ω; (right) isolines of the anisotropic distance function d_Ω^M.

- The maximum thickness constraint reads: $\forall x \in \Omega, \;\; d_\Omega^M(x) \geq -\delta/2$.
- This can be equivalently formulated using an integral penalty function:

$$P(\Omega) = 0, \text{ where } P(\Omega) = \int_\Omega \left(d_\Omega^M - \delta/2\right)_-^2 \, dx,$$

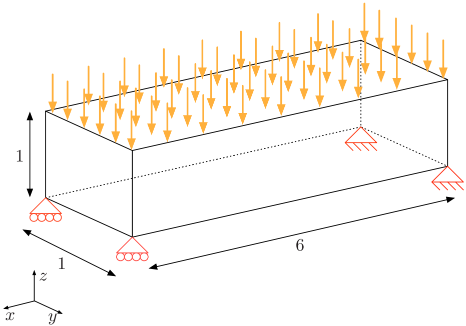
where $t_- = \max(0, -t)$.

- In practice, we use the relaxed version of the constraint:

$$P_a^M(\Omega) \leq \frac{\delta}{2}, \text{ where } P_a^M(\Omega) = \left(\frac{1}{\int_\Omega h(d_\Omega^M) dx} \int_\Omega h(d_\Omega^M)\,(d_\Omega^M)^2 dx \right)^{\frac{1}{2}}.$$

and h is a regularized characteristic function: $\begin{cases} h(t) \approx 1 & \text{if } |t| \geq \delta/2, \\ h(t) \approx 0 & \text{otherwise.} \end{cases}$

- We consider the optimal design of a three-dimensional bridge.

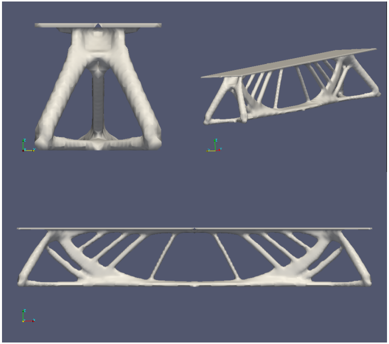


- We solve the simple compliance minimization problem:

$$\begin{array}{cl} \min\limits_{\Omega} & C(\Omega), \\ \text{s.t.} & \text{Vol}(\Omega) \leq V_T, \end{array} \quad \text{where } V_T = 0.1|D|.$$

Application 1: Ease the interpretation of the optimized design (I)

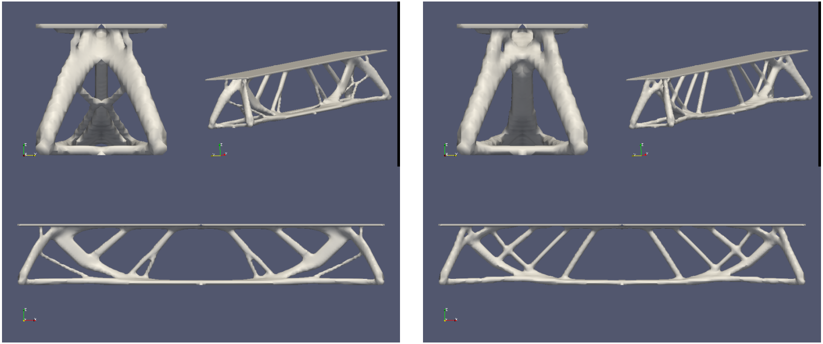
The optimized design contains extended regions in the (yz)-plane, which makes difficult its interpretation as a realistic truss structure.



Application 1: Ease the interpretation of the optimized design (III)

To alleviate this problem, we solve instead

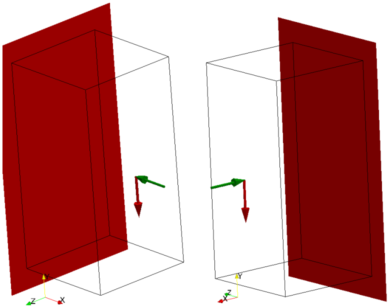
$$\min_{\Omega} \; C(\Omega), \quad \text{s.t.} \; \begin{cases} \mathrm{Vol}(\Omega) \leq V_T, \\ P_a^M(\Omega) \leq \delta/2, \end{cases} \quad \text{, where } M = \begin{pmatrix} m & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$



Optimal bridges obtained with the coefficient (left) $m = 0.1$, and (right) $m = 0.01$.

Application 2: Enhance the visibility of shapes (I)

- We aim to optimize the design of a three-dimensional short cantilever.

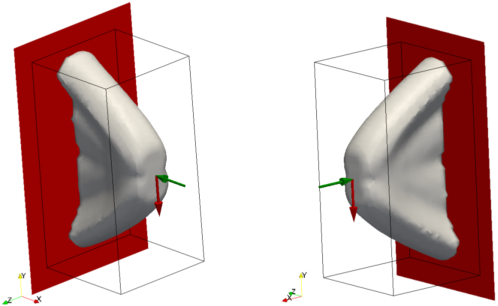


- To this end, we solve the two-load compliance minimization problem:

$$\begin{array}{ll} \min\limits_{\Omega} & C_1(\Omega) + C_2(\Omega), \\ \text{s.t.} & \mathrm{Vol}(\Omega) \leq V_T, \end{array} \quad \text{where } V_T = 0.1|D|.$$

and $C_i(\Omega)$ is the compliance of Ω with respect to the i^{th} load case.

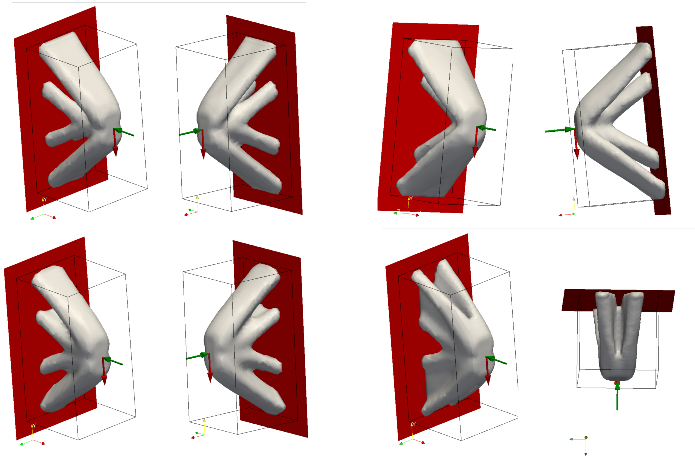
The optimized shape presents an extended surface in the (xy)-plane, which hinders the path of light.



To alleviate this problem, we add one or several constraints of the form

$$P_a^M(\Omega) \leq \delta/2, \text{ where } M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}.$$

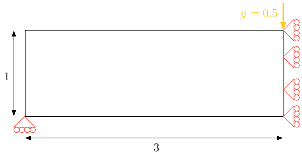
Application 2: Enhance the visibility of shapes (III)

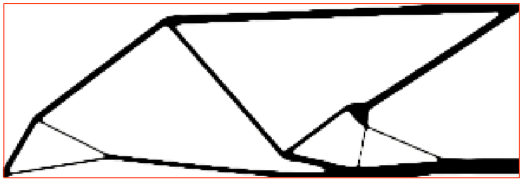


Optimized short cantilevers using different metric tensors M: (top-left) $m_1 = 0.01$; (top-right) $m_1 = 0.01, m_2 = 0.01$; (bottom-left) one constraint with $m_1 = 0.01$, $m_2 = 1.0$ and a second one with $m_1 = 1.0$, $m_2 = 0.01$; (bottom-right) $m_3 = 0.01$.

In order to optimize a $2d$ MBB beam, we minimize the volume $\mathrm{Vol}(\Omega)$ of shapes, under a compliance constraint:

$$\begin{array}{ll} \min\limits_{\Omega} & \mathrm{Vol}(\Omega), \\ \text{s.t.} & C(\Omega) \leq C_T, \end{array} \quad \text{where } C_T = 150.$$

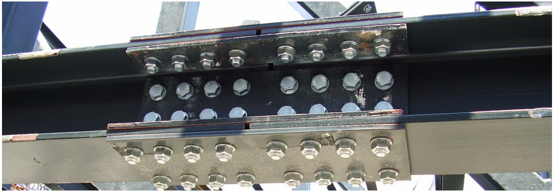




(Left): Setting of the MBB beam example, and (right) resulting optimized shape.

Application 3: penalization of the appearance of long bars (II)

- The optimized shape shows long, horizontal bars.
- In practical realizations, these are assembled by joining two (or more) smaller bars together, constructing strong connections between them.



Example of a junction between two HEA beams.

- Besides the increased cost of such connections, the final beam may not behave mechanically in the predicted way.

  $\Rightarrow$ Structural engineers generally wish to avoid such features.

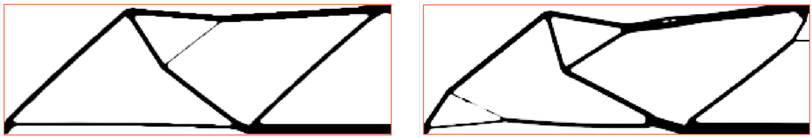
Application 3: penalization of the appearance of long bars (III)

To prevent the appearance of such bars, we solve:

$$\min_{\Omega} \mathrm{Vol}(\Omega) \text{ s.t. } \begin{cases} C(\Omega) \leq C_T \\ P_a^M(\Omega) \leq \delta/2. \end{cases},$$

with a threshold value $\delta = 1.0$, and where the anisotropy is dictated by the metric tensor:

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & 1 \end{pmatrix}.$$



Optimized shapes in the MBB beam example with a constraint on the anisotropic maximum thickness, and parameters (left) $m_1 = 0.1$; (right) $m_1 = 0.01$.

Thank you for your attention!

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