A distributed Lagrange multiplier based / fictitious domain method to model acoustic wave propagation in granular media in a fluid

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Summary

Section contents

Granular media

- Conglomeration of macroscopic discrete particles called grains

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- Energy loss whenever grains interact due to friction

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Examples

Figure: Dunes

Figure: Rice

Figure: Capsules

Acoustics in granular media

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Acoustics in granular media

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- These events generate acoustic wave sources which can propagate:
  - through the granular medium skeleton when it's a dry one
  - both in skeleton and matrix when the granular medium is submerged

Acoustics in destabilized granular media

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Acoustics in destabilized granular media

- Louder acoustic signals can be recorded: avalanche precursors
- Experimental acoustic measurements have already be done within the framework of the StabInGraM ANR project (*STAbility loss In GRAnular Media*).

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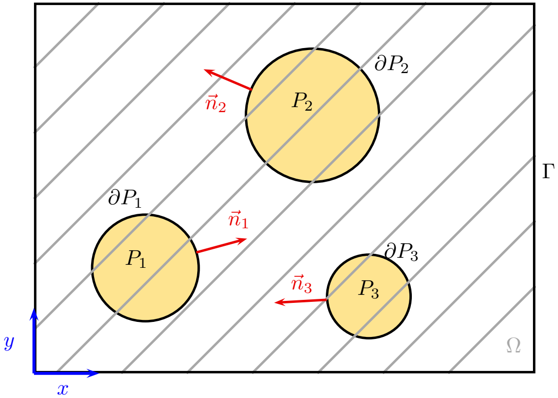
  - *N-body* algorithm which manages a list of particles
  - works fine in vacuum or gas
- Problem: if the matrix is a liquid, sound waves don't only propagate through the packing skeleton, but also through the liquid which is not taken in account by MD

Section contents

Domains

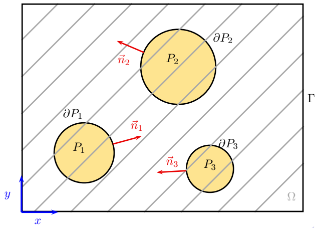


$$P = \bigcup_{i=1}^{N} P_i$$

Acoustic wave equation

$$
\begin{cases}
\rho_f \dfrac{\partial \vec{u}}{\partial t} + \nabla\, p = \vec{0} & \text{in } \Omega \setminus \overline{P(t)} \qquad (1) \\
\dfrac{1}{\rho_f\, c_f^2} \dfrac{\partial p}{\partial t} + \nabla . \vec{u} = 0 & \text{in } \Omega \setminus \overline{P(t)} \qquad (2) \\
\vec{u}.\vec{n}_i = \vec{U}_i.\vec{n}_i & \text{on } \partial P_i(t) \qquad (3) \\
 & \textit{perfectly matched layers} \text{ on } \Gamma
\end{cases}
$$

[BG05, GPHJ99]



Newton's second law

$$\begin{cases} m_{g_i}\,\dfrac{\mathrm{d}\vec{U}_i}{\mathrm{d}t} = \vec{W}_i + \vec{B}_i + \vec{H}_i + \vec{F}_i & (4) \\ I_{g_i}\,\dfrac{\mathrm{d}\omega_i}{\mathrm{d}t} = T_i & (5) \end{cases}$$

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Forces acting on each particle

- weight: $\vec{W}_i = \rho_g \, V_{g_i} \, \vec{g}$
- buoyancy: $\vec{B}_i = -\rho_f \, V_{g_i} \, \vec{g}$
- hydrodynamic force: $\vec{H}_i = -\int_{\partial P_i(t)} p \, \vec{n_i} \, \mathrm{d}\vec{n_i}$
- interaction force: $\vec{F}_i = \sum_{j=i}^{N} (\vec{F_{n_{i,j}}} + \vec{F_{t_{i,j}}})$

  or spring force: $\vec{F}_i = -k(y - y_0)$

Kinematic equations & initial conditions

$$\begin{cases} \dfrac{\mathrm{d}\vec{X}_i}{\mathrm{d}t} = \vec{U}_i & (6) \\ \dfrac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i & (7) \end{cases}$$

Kinematic equations & initial conditions

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Initial conditions:

$$\begin{array}{rr} \vec{u}\big|_{t=0} = \vec{u_0} & p\big|_{t=0} = p_0 \\ \vec{U_i}\big|_{t=0} = \vec{U_{i,0}} & \omega_i\big|_{t=0} = \omega_{i,0} \\ \vec{X_i}\big|_{t=0} = \vec{X_{i,0}} & \theta_i\big|_{t=0} = \theta_{i,0} \end{array}$$

Combined spaces

Combined velocity space

$$\mathbb{U} = \{(\vec{u}, \vec{U}, \omega) | \vec{u} \in \left[L^2(\Omega \setminus \overline{P(t)})\right]^2, \ \vec{U} \in \mathbb{R}^2, \ \omega \in \mathbb{R} \qquad (8)$$
$$\vec{u}.\vec{n} = \vec{U}.\vec{n} \text{ on } \partial P(t)\}$$

Combined variation space

$$\mathbb{V} = \{(\vec{v}, \vec{V}, \xi) | \vec{v} \in \left[L^2(\Omega \setminus \overline{P(t)})\right]^2, \ \vec{V} \in \mathbb{R}^2, \ \xi \in \mathbb{R} \qquad (9)$$
$$\vec{v}.\vec{n} = \vec{V}.\vec{n} \text{ on } \partial P(t)\}$$

Combined equation of motion

Principle [GPHJ99]

Combine:

- wave equation in the fluid
- Newton's second law for grains

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by performing the symbolic operation:

$$\int_{\Omega\setminus\overline{P(t)}} (1).\vec{v}\,\mathrm{d}\vec{x} + (4).\vec{V} + (5)\,\xi = 0$$

$$\rho_f \frac{\partial \vec{u}}{\partial t} + \nabla p = 0 \tag{1}$$

$$m_g \frac{\mathrm{d}\vec{U}}{\mathrm{d}t} - (\rho_g - \rho_f)\, V_g\, \vec{g} + \int_{\partial P(t)} p\,\vec{n}\,\mathrm{d}\ell - \vec{F} = 0 \tag{4}$$

$$I_g \frac{\mathrm{d}\omega}{\mathrm{d}t} - T = 0 \tag{5}$$

Fictitious domain method

Basic idea [GPP94]

- Extend the problem from $\Omega \setminus \overline{P(t)}$ to all of Ω in two steps:
  1. obtain an analogous combined equation of motion for $P(t)$ using a rigid body motion constraint: $\vec{u} = \vec{U}$ in $P(t)$
  2. add it to equation in $\Omega \setminus \overline{P(t)}$ to get the combined equation of motion for all Ω

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- Force the solution to satisfy conditions on $\partial P(t)$ and inside $P(t)$
  1. remove the constraints from the combined velocity space (8)
  2. enforce them as a side constraint using Lagrange multipliers

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In order to keep:

- physical variables in all Ω
- Lagrange multipliers variables in $P(t)$

Section contents

Meshes

- 2 domains of definition : Ω (static) and $P(t)$ (time-dependent)

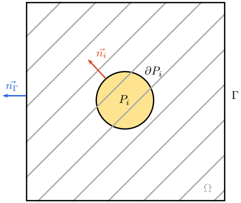


Figure: Domain

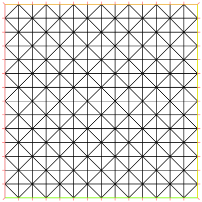


Figure: $\mathscr{T}_{\Omega_h}$

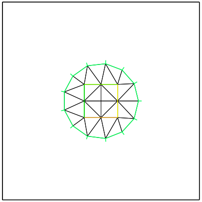
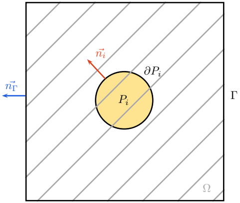


Figure: $\mathscr{T}_{P_h}$

Meshes

- 2 domains of definition : Ω (static) and $P(t)$ (time-dependent)
- $\Rightarrow$ 2 meshes :
  - a regular grid $\mathscr{T}_{\Omega_h}$ for rectangular domain Ω
  - an unstructured grid $\mathscr{T}_{P_h}$ for the grains



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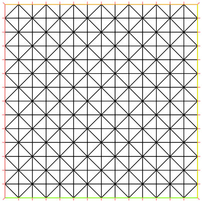

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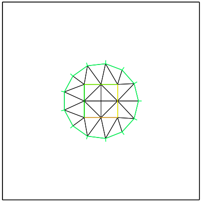

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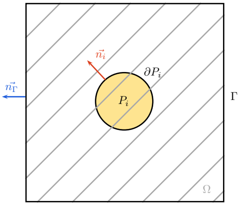
- mesh sizes are related by a condition : $h_P = \kappa\, h_\Omega$ with $1 < \kappa < 2$ which come from results on problems involving Lagrange multipliers [Bab73] (best results are obtained with $\kappa \approx 1.3$)



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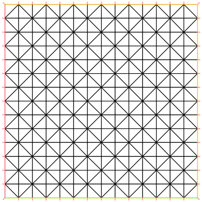

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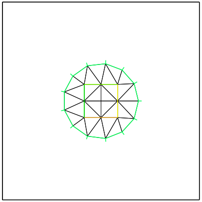

Figure: $\mathscr{T}_{P_h}$

Finite dimensional spaces

- Choice of approximation functions for each space :

$$\mathbb{Q}_h = \{q_h \in H^1(\Omega),\ q_h|_K \in P_1\ \forall K \in \mathcal{T}_{\Omega_h}\} \tag{10}$$

$$\mathbb{W}_h = \{\vec{v}_h \in \left[L^2(\Omega)\right]^2,\ \vec{v}_h|_K \in RT_0\ \forall K \in \mathcal{T}_{\Omega_h}\} \tag{11}$$

$$\Lambda_h = \{\vec{\mu}_h \in \left[L^2(\Omega)\right]^2,\ \vec{\mu}_h|_K \in RT_0\ \forall K \in \mathcal{T}_{P_h}\} \tag{12}$$

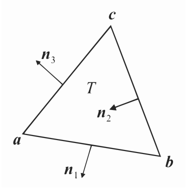

Figure: Raviart-Thomas element RT_0 [Hec]

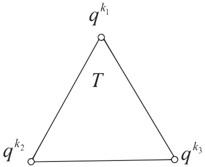

Figure: Lagrangian element P_1 [Hec]

Extra mesh for point-particles variables

- Solution functions $\vec{U}$ and test functions $\vec{V}$ are part of $\mathbb{R}^2$
- A priori FreeFEM++ can't handle those

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Trick

- Use a 3rd grid $\mathscr{T}_{P_{d_h}}$ with only 1 triangle including the grain (triangle's incircle)
- Choice a P_0 constant Lagrangian finite element
- Then correct the area with the factor $\frac{\pi}{\sqrt{27}}$

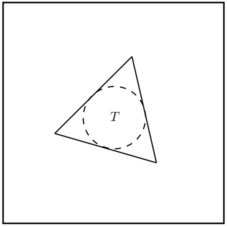

Figure: $\mathscr{T}_{P_{d_h}}$

Operator splitting

- Time-discretization using Finite Differences Method
- Final system is complete but too difficult to solve directly

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Idea

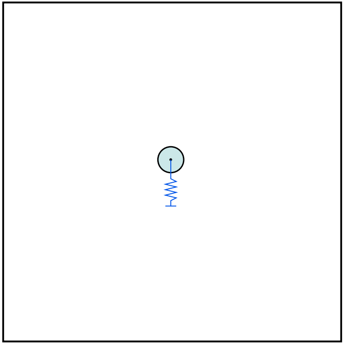
Decouple operators that propagate wave and move the grains from the operators that enforce conditions in $P(t)$.

- Marchuk's fractional step [Mar90]

Section contents

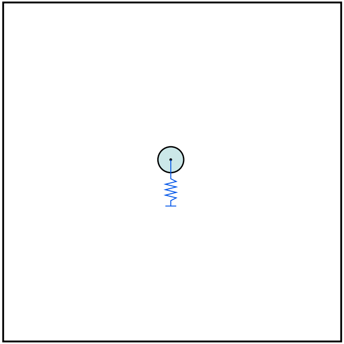
1 grain

- Fixed to a spring with a stiffness k



1 grain

- Fixed to a spring with a stiffness k
- Moved forward from its equilibrium position



Parameters

- Medias:

c_f	$1,500\,\mathrm{m/s}$
c_g	$5,300\,\mathrm{m/s}$
ρ_f	$1.0\,\mathrm{g/cm^3}$
ρ_g	$2.4\,\mathrm{g/cm^3}$

- Domains size:

Ω	$5 \times 5\,\mathrm{cm}$
r	$2\,\mathrm{mm}$

- Forces:

$\vec{g}$	$0\,\mathrm{m/s^2}$
$\vec{F}$	$-k(y - y_0)$

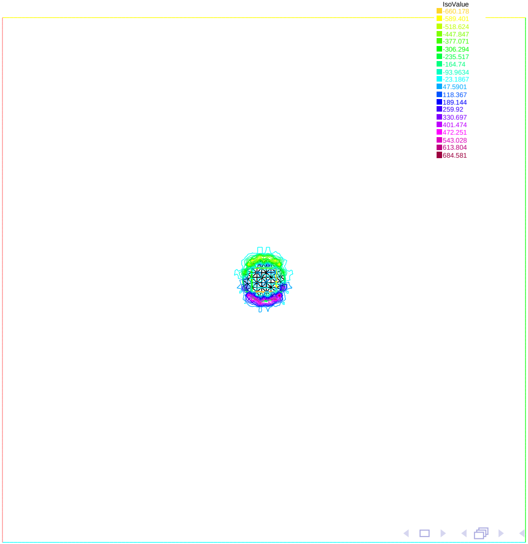
- Spring eigenfrequency:

$f_{0_{res}}$	$0.2\,\mathrm{MHz}$

- Discretization:

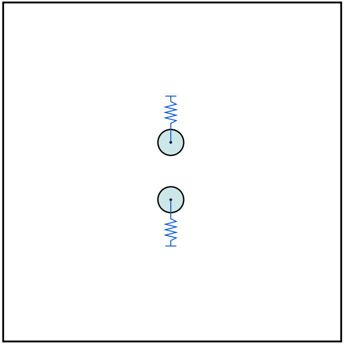
$N_x \times N_y$	136×136
N_t	346
Δx	$0,37\,\mathrm{mm}$
Δt	$43\,\mathrm{ns}$

Acoustic pressure field: 1 grain



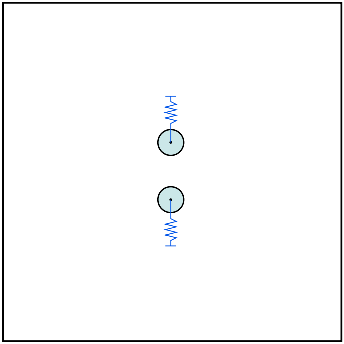
2 grains

- Both fixed to their own spring with the stiffness k



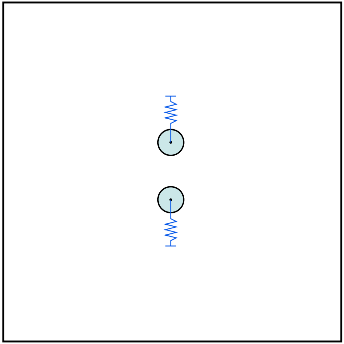
2 grains

- Both fixed to their own spring with the stiffness k
- The bottom one is moved forward from its equilibrium position



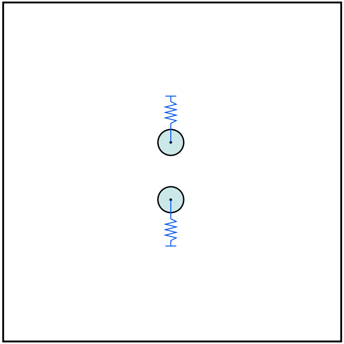
2 grains

- Both fixed to their own spring with the stiffness k
- The bottom one is moved forward from its equilibrium position
- The top one is at its equilibrium position

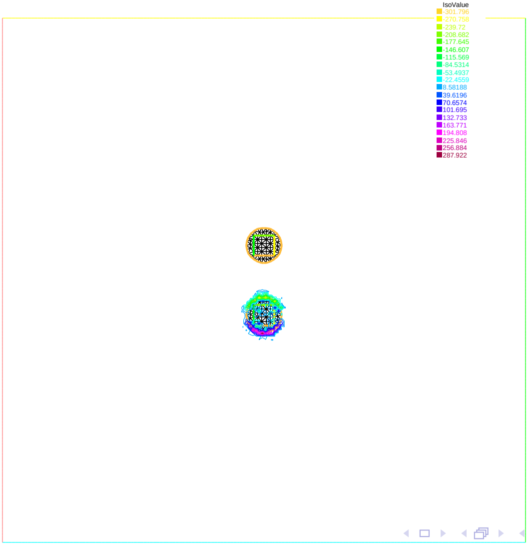


2 grains

- Both fixed to their own spring with the stiffness k
- The bottom one is moved forward from its equilibrium position
- The top one is at its equilibrium position
- Parameters are unchanged



Acoustic pressure field: 2 grains



Mechanical energy of each grain

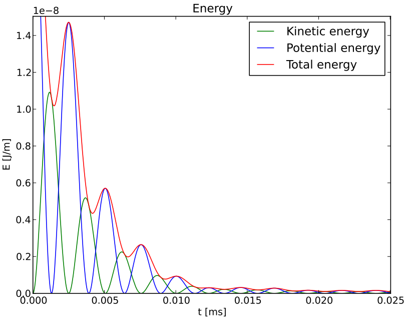


Figure: Bottom grain

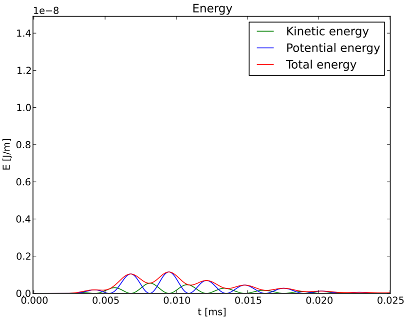


Figure: Top grain

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Conclusions & Outlook

1. Conclusions
   - Molecular Dynamic's philosophy preserved (infinite velocity inside the grain, elasticity in interaction forces)
   - Good geometry representation of the grains
   - *Perfectly Matched Layer* version done

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   - Molecular Dynamic's philosophy preserved (infinite velocity inside the grain, elasticity in interaction forces)
   - Good geometry representation of the grains
   - *Perfectly Matched Layer* version done
2. Outlook
   - Stability analysis of the scheme
   - Contact forces
   - Granular packing modelling
   - Simulate acoustic emission of a destabilized granular medium
   - Parallel version

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