# Multi-Trace approach to Optimized Schwarz Methods

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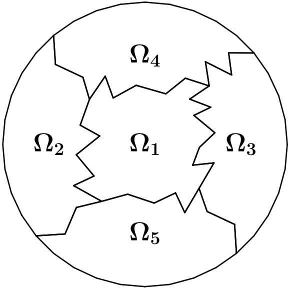
# Scattering in heterogeneous medium

wave number : $\kappa : \mathbb{R}^d \to \mathbb{C}$ bounded
$\Re e\{\kappa(\boldsymbol{x})\} \geq 0, \Im m\{\kappa(\boldsymbol{x})\} \geq 0, \kappa(\boldsymbol{x}) \neq 0$
source : $f \in \mathrm{L}^2(\Omega)$




**Non-overlapping partition**
$\Omega = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_{\mathrm{J}}$,
$\Gamma_j := \partial\Omega_j, \quad \Gamma_j' := \Gamma_j \setminus \partial\Omega$
$\Omega_j$ : Lipschitz, bounded

**Helmholtz bvp**
$-\Delta u - \kappa(\boldsymbol{x})^2 u = f$ in $\Omega$,
$\partial_{\boldsymbol{n}} u - \imath\kappa\, u = 0$ on $\partial\Omega$.

$\Longleftrightarrow$

**local sub-problems** $j = 1 \ldots \mathrm{J}$
$-\Delta u - \kappa^2 u = f$ in $\Omega_j$
$\partial_{\boldsymbol{n}} u - \imath\kappa\, u = 0$ on $\partial\Omega_j \cap \partial\Omega$.

**+**

**transmission conditions**
$\partial_{n_j} u|_{\Gamma_j}^{\mathrm{int}} = -\partial_{n_k} u|_{\Gamma_k}^{\mathrm{int}}$
$u|_{\Gamma_j}^{\mathrm{int}} = u|_{\Gamma_k}^{\mathrm{int}} \quad \forall j, k$

⚠ Cross points allowed

# Optimized Schwarz Method (OSM) [Després, 1991]

Optimized Schwarz Method (OSM) is one of the most established DDM approaches for wave propagation. This is a substructuring method where transmission conditions are imposed through each interface by means of Robin traces involving impedance coefficients.

- operator valued impedance : [Collino, Ghanemi & Joly, 2000]
- second order TC : [Gander, Magoules & Nataf, 2002]
- DtN-like impedance : [Nataf, Rogier & de Sturler, 1995], [Antoine, Boudendir & Geuzaine, 2012], [Antoine, Bouajaj & Geuzaine, 2014]
- large literature : overview article [Gander & Zhang, 2019]

**Cross point issue**
Unappropriate treatment of cross-points may spoil convergence so care must be paid to this issue : [Gander & Kwok, 2013], [Gander & Santugini, 2016], [Després, Nicolopoulos & Thierry, 2020], [Modave, Antoine, Geuzaine & al, 2019 & 2020]. There is also a variant of FETI-DP "à la Després" [Farhat & al, 2005], [Bendali & Boubendir, 2006].

# Outline

**I Review of the Optimized Schwarz Method**

**II New manner to enforce transmission condition**

**III Numerical results**

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# Original OSM (Després)

**Transmission conditions :** with scalar $\Lambda > 0$,

$$\begin{array}{c} \partial_{n_j} u|_{\Gamma_j} = -\partial_{n_k} u|_{\Gamma_k} \\ u|_{\Gamma_j} = u|_{\Gamma_k} \\ \text{on } \Gamma_j \cap \Gamma_k \, \forall j,k \end{array} \iff \begin{array}{c} +\partial_{n_j} u|_{\Gamma_j} + \imath\Lambda u|_{\Gamma_j} = \\ -\partial_{n_k} u|_{\Gamma_k} + \imath\Lambda u|_{\Gamma_k} \\ \text{on } \Gamma_j \cap \Gamma_k \, \forall j,k \end{array} \iff \begin{array}{c} (\partial_{n_j} u|_{\Gamma_j'} - \imath\Lambda u|_{\Gamma_j'})_{j=1}^{\mathrm{J}} = \\ -\Pi_0((\partial_{n_k} u|_{\Gamma_k'} + \imath\Lambda u|_{\Gamma_k'})_{k=1}^{\mathrm{J}}) \end{array}$$

where the operator $\Pi_0$ swaps traces on both sides of each interfaces :
$(v_0, \ldots, v_{\mathrm{J}}) = \Pi_0(u_0, \ldots, u_{\mathrm{J}}) \iff v_j = u_k$ on $\Gamma_j \cap \Gamma_k$.

**Local scattering operators :**
$\mathrm{S}_0^{\Gamma_j}(\partial_{n_j}\psi|_{\Gamma_j'} - \imath\Lambda\psi|_{\Gamma_j'}) := \partial_{n_j}\psi|_{\Gamma_j'} + \imath\Lambda\psi|_{\Gamma_j'}$
for $\Delta\psi + \kappa^2\psi = 0$ in $\Omega_j$
$\partial_{\boldsymbol{n}}\psi - \imath\kappa\psi = 0$ on $\partial\Omega_j \cap \partial\Omega$

**Wave equations :**
$$(\partial_{n_j} u|_{\Gamma_j'} + \imath\Lambda u|_{\Gamma_j'})_{j=1}^{\mathrm{J}} = \mathrm{S}_0((\partial_{n_k} u|_{\Gamma_k'} - \imath\Lambda u|_{\Gamma_k'})_{k=1}^{\mathrm{J}}) + g$$

with $\mathrm{S}_0 := \mathrm{diag}_{j=1\ldots\mathrm{J}}(\mathrm{S}_0^{\Gamma_j})$.

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stems from the source term of the bvp

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**Optimized Schwarz**

$$(\mathrm{Id} + \Pi_0\mathrm{S}_0)p = -\Pi_0(g)$$

with $p = (\partial_{n_j} u|_{\Gamma_j'} - \imath\Lambda u|_{\Gamma_j'})_{j=1}^{\mathrm{J}}$

# The cross-point issue

$$p^{(n+1)} = (1-r)p^{(n)} - r\Pi_0 \mathrm{S}_0(p^{(n)}) - r\Pi_0(g)$$

Without cross points, geometric convergence can be obtained for appropriate (operator valued) impedance $\Lambda$. In practice, convergence is much slower with cross points i.e. at best algebraic $\|p - p^{(n)}\|_{\mathrm{L}^2} = O(n^{-\gamma})$. The root cause seems related to $\Pi_0$ not being continuous at cross-points in proper trace norms.

This so-called "cross point issue" also arises in the different context of multi-domain boundary integral formulations where Multi-Trace Formalism (MTF) [Claeys & Hiptmair, 2012] now offers a framework that accomodates cross-points, and that is clean as regards function spaces.

**Idea :** use the Multi-Trace Formalism to treat cross-points in OSM. We shall replace $\Pi_0$ by a non-local counterpart $\Pi$ that remains continuous no matter the presence of cross points, following the idea first introduced in :

X.Claeys, "Quasi-local multi-trace boundary integral formulations", Numer. Methods Partial Differential Equations, 31(6) :2043–2062, 2015.

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# Discrete function spaces

Triangulation conforming with $\Omega_j$'s, FE spaces $\mathrm{V}_h(\Omega_j) = \{\, \mathbb{P}_k$-Lagrange on $\Omega_j\}$.

| **Volume functions** | **Tuples of traces** ($\Gamma_j := \partial\Omega_j$) |
|---|---|
| $\mathbb{V}_h(\Omega) := \mathrm{V}_h(\Omega_1) \times \cdots \times \mathrm{V}_h(\Omega_{\mathrm{J}})$ | $\mathbb{V}_h(\Sigma) := \mathrm{V}_h(\Gamma_1) \times \cdots \times \mathrm{V}_h(\Gamma_{\mathrm{J}})$ |
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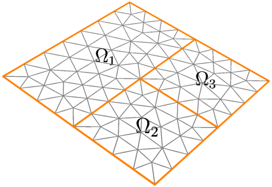

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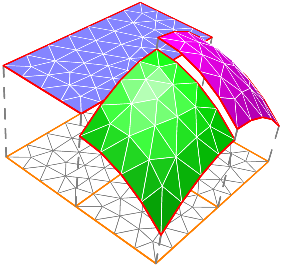
**Piecewise** $\mathrm{H}^1$
Possible jumps through interfaces



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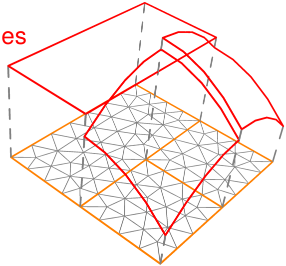
**Multi-traces** = tuples of traces at boundaries



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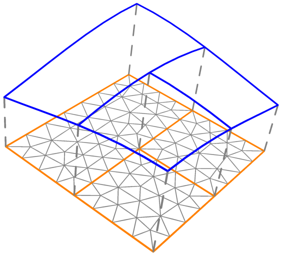
**Single-traces** = tuples of traces that match at interfaces



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**Impedance**

$t_h(\mathfrak{p}, \mathfrak{q}) = t_{\Gamma_1}(p_1, q_1) + \cdots + t_{\Gamma_{\mathrm{J}}}(p_{\mathrm{J}}, q_{\mathrm{J}})$

$t_{\Gamma_j}(\cdot, \cdot) =$ **any** coercive sesquilinear form on $\mathrm{V}_h(\Gamma_j)$

**Choices of impedance :**

- surface mass matrix
- surface order 2 operator
- layer potential, DtN map
- Schur complements
- etc. . .

For the sake of clarity, $t_h(\cdot, \cdot)$ is assumed SPD.

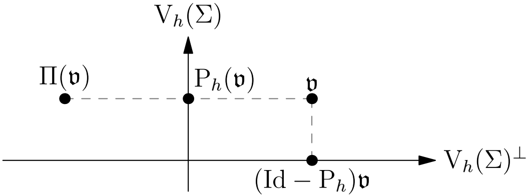
# Matching at interfaces via orthogonal symmetry

The $t_h$-orthogonal projection onto the single-trace space $\mathrm{P}_h : \mathbb{V}_h(\Sigma) \to \mathrm{V}_h(\Sigma)$ can be applied by solving a (**DDM friendly !**) SPD problem

$$\mathfrak{p} = \mathrm{P}_h(\mathfrak{v}) \quad \Longleftrightarrow \quad \begin{aligned} &\mathfrak{p} \in \mathrm{V}_h(\Sigma) \quad \text{and} \\ &t_h(\mathfrak{p}, \mathfrak{w}) = t_h(\mathfrak{v}, \mathfrak{w}) \quad \forall \mathfrak{w} \in \mathrm{V}_h(\Sigma) \end{aligned}$$

**Lemma :** The $t_h$-orthogonal symmetry $\Pi := \mathrm{P}_h - (\mathrm{Id} - \mathrm{P}_h) = 2\mathrm{P}_h - \mathrm{Id}$ satisfies $\|\Pi(\mathfrak{q})\|_{t_h} = \|\mathfrak{q}\|_{t_h} \forall \mathfrak{q} \in \mathbb{V}_h(\Sigma)$ and, for any $\mathfrak{v}, \mathfrak{q} \in \mathbb{V}_h(\Sigma)$,

$$(\mathfrak{v}, \mathfrak{q}) \in \mathrm{V}_h(\Sigma) \times \mathrm{V}_h(\Sigma)^{\perp} \quad \Longleftrightarrow \quad \mathfrak{q} + \imath\mathfrak{v} = \Pi(-\mathfrak{q} + \imath\mathfrak{v}).$$



# Reformulations of the scattering problem

Find $u_h \in \mathrm{V}_h(\Omega)$ and
$a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in \mathrm{V}_h(\Omega)$

$$a(u,v) := \sum_{j=1}^{\mathrm{J}} \int_{\Omega_j} \nabla u \cdot \nabla \overline{v} - \kappa^2 u \, \overline{v} \, d\boldsymbol{x} - \int_{\partial\Omega_j} \imath\kappa \, u \, \overline{v} \, d\sigma$$

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**Non trivial theorem :**

- relaxing constraints with Lagrange multipliers
- Use $\Pi$ to enforce continuity

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$$\Longleftrightarrow$$

**Skeleton formulation**
$\mathfrak{p}_h \in \mathbb{V}_h(\Sigma)$ and
$(\mathrm{Id} + \Pi\mathrm{S})\mathfrak{p}_h = g_h$

Unknowns $u_h$ are eliminated in all subdomains in parallel by local "ingoing -to-outgoing" solves, applying a (block diagonal) scattering operator.

**Proposition :** Define $\mathrm{S}(\mathfrak{p}) := \mathfrak{p} + 2\imath w|_\Sigma$ where $w \in \mathbb{V}_h(\Omega)$ satisfies $a(w, v) - \imath t_h(w|_\Sigma, v|_\Sigma) = t_h(\mathfrak{p}, v|_\Sigma) \forall v \in \mathbb{V}_h(\Omega)$. Then $\|\mathrm{S}(\mathfrak{p})\|_{t_h} \leq \|\mathfrak{p}\|_{t_h}$ for all $\mathfrak{p} \in \mathbb{V}_h(\Sigma)$.

## Convergence estimate

**Theorem :**

**1)** Boundedness : $\|\mathrm{Id} + \Pi\mathrm{S}\|_{t_h} \leq 2$

**2)** Coercivity : $\Re e\{t_h(\mathfrak{v}, (\mathrm{Id} + \Pi\mathrm{S})\mathfrak{v})\} \geq \gamma_h^2 \|\mathfrak{v}\|_{t_h}^2 \quad \forall \mathfrak{v} \in \mathbb{V}_h(\Sigma)$

**Coercivity constant**

$$\gamma_h := \frac{\alpha}{\lambda_h^+ + 2\,\|a\|/\lambda_h^-}$$

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$$\lambda_h^+ := \sup_{\substack{j=1\dots J \\ v_h \in \mathrm{V}_h(\Gamma_j)\setminus\{0\}}} \frac{t_{\Gamma_j}(v_h, v_h)}{\|v_h\|^2_{\mathrm{H}^{1/2}_{\kappa,h}(\Gamma_j)}} \qquad \lambda_h^- := \inf_{\substack{j=1\dots J \\ v_h \in \mathrm{V}_h(\Gamma_j)\setminus\{0\}}} \frac{t_{\Gamma_j}(v_h, v_h)}{\|v_h\|^2_{\mathrm{H}^{1/2}_{\kappa,h}(\Gamma_j)}}$$

# Convergence estimate

**Theorem :**

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The exact solution $\mathfrak{p}^{(\infty)} \in \mathbb{V}_h(\Sigma)$ to the skeleton formulation can be computed with e.g. a Richardson iteration : given $r \in (0, 1)$, compute

$$\mathfrak{p}^{(n+1)} = (1 - r)\mathfrak{p}^{(n)} - r\Pi\mathrm{S}\mathfrak{p}^{(n)} + rg_h.$$

**Proposition :** convergence of Richardson's solver

$$\frac{\|\mathfrak{p}^{(n)} - \mathfrak{p}^{(\infty)}\|_{t_h}}{\|\mathfrak{p}^{(0)} - \mathfrak{p}^{(\infty)}\|_{t_h}} \leq \left( 1 - 2r(1 - r)\gamma_h^2 \right)^{n/2}.$$

**Important consequence :** If the $t_{\Gamma_j}(\cdot, \cdot)$'s yield norms that are $h-$uniformly equivalent to $\|\cdot\|_{\mathrm{H}^{1/2}(\Gamma_j)}$, then we have $h-$uniform geometric convergence.

# Only 3 hypothesis required

**H1)** $\Im m\{a(v,v)\} \leq 0 \ \forall v \in \mathrm{V}_h(\Omega)$ i.e. the medium can only absorb or propagate. The case of "purely" propagative media $\Im m\{a(v,v)\} = 0$ $\forall v \in \mathrm{V}_h(\Omega)$ is covered in particular.

**H2)** Unique solvability of the discrete pb :

$$\inf_{u_h \in \mathrm{V}_h(\Omega)} \sup_{v_h \in \mathrm{V}_h(\Omega)} \frac{|a(u_h, v_h)|}{\|u_h\|_{\mathrm{H}^1_\kappa(\Omega)} \|v_h\|_{\mathrm{H}^1_\kappa(\Omega)}} > 0.$$

**H3)** The impedance $t_h(\cdot,\cdot)$ must be chosen coercive.

**No assumption on :**

- the shape constant of the mesh
- the frequency regime

# Outline

I Review of the Optimized Schwarz Method

II New manner to enforce transmission conditions

**III Numerical results**

# Numerical experiments : Helmholtz in 2D

Constant wave number $\kappa > 0$ in a disc $\Omega = \mathrm{D}(0,1)$ and impedance boundary condition $(\partial_{\boldsymbol{n}} - \imath\kappa)u^{\mathrm{ex}} = g$ with $g(\boldsymbol{x}) = (\partial_{\boldsymbol{n}} - \imath\kappa)\exp(-\imath\kappa\mathbf{d}\cdot\boldsymbol{x})$, discretization with $\mathrm{V}_h(\Omega) = \mathbb{P}_1$-Lagrange.

$$u_h^{\mathrm{ex}} \in \mathrm{V}_h(\Omega) \quad \text{and} \quad a(u_h^{\mathrm{ex}}, v_h) = \ell(v_h) \quad \forall v_h \in \mathrm{V}_h(\Omega)$$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla\overline{v} - \kappa^2 u\overline{v}d\boldsymbol{x} - \imath\kappa\int_{\partial\Omega} u\overline{v}d\sigma$$
$$\ell(v) = \int_{\partial\Omega}\overline{v}gd\sigma.$$

With $u_0^{(0)} \equiv 0$, we denote $u_h^{(n)}$ the iterates of the linear solver. The measured error is given by

$$(\text{relative error})^2 = \frac{\sum_{j=1}^{\mathrm{J}} \|u_h^{(n)} - u_h^{\mathrm{ex}}\|^2_{\mathrm{H}^1(\Omega_j)}}{\sum_{j=1}^{\mathrm{J}} \|u_h^{(0)} - u_h^{\mathrm{ex}}\|^2_{\mathrm{H}^1(\Omega_j)}}.$$

**Remarks :**
- global linear solver is GMRes, relative tolerance $= 10^{-8}$
- sequential computations on a 6 core workstation
- FEM & DDM code `NIDDL` (in `Julia`) + `BemTool` (in C++) for integral operators
- exchange operator $\Pi$ computed with PCG.

# Choices of impedance

Recall that $t_h(\mathfrak{p},\mathfrak{q}) = t_{\partial\Omega_1}(p_1,q_1) + \cdots + t_{\partial\Omega_\mathrm{J}}(p_\mathrm{J},q_\mathrm{J})$. We tested several choices of local impedances.

**Choice 1 :** $\mathrm{M}$ = **surface mass matrix**
$t_{\partial\Omega_j}(p_h,q_h) = \int_{\partial\Omega_j} p_h(\boldsymbol{x})\overline{q}_h(\boldsymbol{x})d\sigma(\boldsymbol{x})$
This is the impedance originally considered by Després.

**Choice 2 :** $\mathrm{K}$ = **surface** $\mathrm{H1}$**-scalar product**
$t_{\partial\Omega_j}(p_h,q_h) = \int_{\partial\Omega_j} \kappa^{-1}\nabla p_h(\boldsymbol{x})\cdot\nabla\overline{q}_h(\boldsymbol{x})/2 + \kappa\, p_h(\boldsymbol{x})\overline{q}_h(\boldsymbol{x})d\sigma(\boldsymbol{x})$

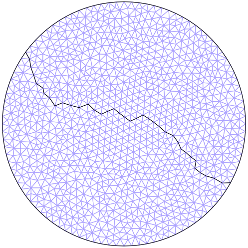
**Choice 3 :** $\mathrm{W}$ = **positive hypersingular integral operator**

$$\begin{aligned} t_{\partial\Omega_j}(p_h,q_h) = \int_{\partial\Omega_j\times\partial\Omega_j} &\exp(-\kappa|\boldsymbol{x}-\boldsymbol{y}|)/(4\pi|\boldsymbol{x}-\boldsymbol{y}|)[\\ &\kappa^{-1}\,\boldsymbol{n}(\boldsymbol{x})\times\nabla_{\partial\Omega_j}p_h(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{y})\times\nabla_{\partial\Omega_j}q_h(\boldsymbol{y})\\ &+\kappa\,\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{y})p_h(\boldsymbol{x})q_h(\boldsymbol{y})\,]d\sigma(\boldsymbol{x},\boldsymbol{y}) \end{aligned}$$

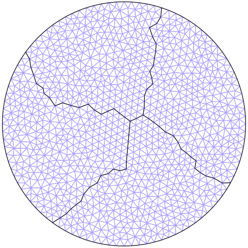
**Choice 4 :** $\Lambda$ = **schur complement**($\simeq$ discrete DtN) associated with the interior numerical solution to the **positive** problem $-\Delta v + \kappa^2 v = 0$ in $\Omega_j$.

# Mesh partitionning

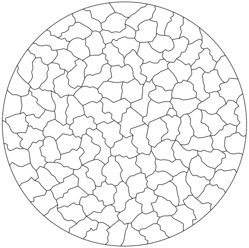
Meshes were generated a priori on the whole computational domain with GMSH. Partitionning is obtained a posteriori with Metis.



2 subdomains



4 subdomains

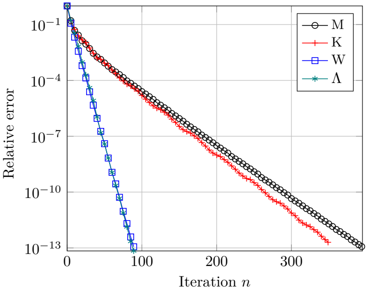


128 subdomains

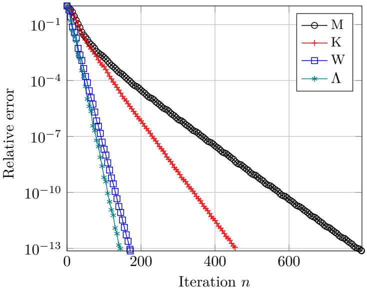
# Convergence history

$\kappa = 5, \quad \lambda = 2\pi/\kappa \simeq 1.25$
$N_\lambda = \lambda/h = 40$ points/wavelength.

4 subdomains

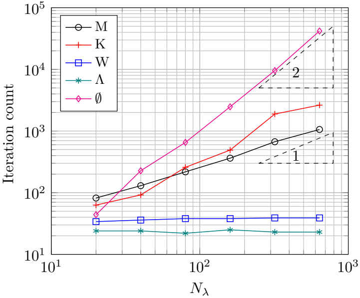



128 subdomains



# Iteration count vs. $N_\lambda$ = points/wavelength

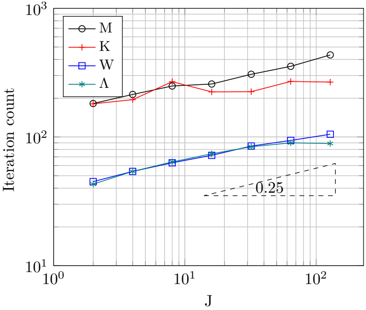
$\kappa = 1$, $\lambda = 2\pi/\kappa \simeq 6.28$, $N_\lambda = \lambda/h$, 4 subdomains.
Relative tolerance of GMRes $= 10^{-8}$, $\emptyset$ = no DDM.




| $N_\lambda$ | $\emptyset$ | M | K | W | Λ |
|---|---|---|---|---|---|
| 20 | 44 | 82 | 63 | 34 | 24 |
| 40 | 227 | 130 | 92 | 36 | 24 |
| 80 | 654 | 218 | 258 | 38 | 22 |
| 160 | 2474 | 363 | 491 | 38 | 25 |
| 320 | 9559 | 671 | 1888 | 39 | 23 |
| 640 | 41888 | 1060 | 2633 | 39 | 23 |

## Iteration count vs. $\mathrm{J}$ = number of subdomains

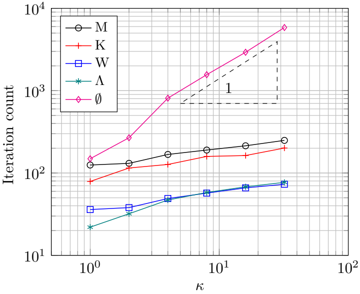
$\kappa = 5$, $\lambda = 2\pi/\kappa \simeq 1.26$, $N_\lambda = \lambda/h = 40$.
Relative tolerance of GMRes $= 10^{-8}$.




| J | M | K | W | Λ |
|---|---|---|---|---|
| 2 | 182 | 181 | 45 | 43 |
| 4 | 214 | 195 | 54 | 54 |
| 8 | 249 | 269 | 63 | 64 |
| 16 | 258 | 224 | 72 | 74 |
| 32 | 307 | 225 | 85 | 84 |
| 64 | 354 | 270 | 94 | 90 |
| 128 | 434 | 267 | 105 | 89 |

# Iteration count vs. $\kappa =$ wavenumber

$\lambda = 2\pi/\kappa$, $N_\lambda = \lambda/h = 30$, 4 subdomains.
Relative tolerance of GMRes $= 10^{-8}$, $\emptyset =$no DDM.




| $\kappa$ | $\emptyset$ | M | K | W | $\Lambda$ |
|---|---|---|---|---|---|
| 1 | 149 | 125 | 79 | 36 | 22 |
| 2 | 268 | 131 | 115 | 38 | 32 |
| 4 | 811 | 168 | 127 | 49 | 47 |
| 8 | 1563 | 190 | 159 | 57 | 58 |
| 16 | 2926 | 214 | 163 | 66 | 68 |
| 32 | 5846 | 249 | 201 | 73 | 77 |

# Conclusion

We proposed a new way of imposing transmission conditions involving another choice of exchange operator. This yields $h$-uniform convergence of iterative solvers, and accomodates cross-points.

In addition this approach appears as a natural generalization of the classical OSM à la Després, and allows to propose a full theoretical framework, which was not available so far.

**Also available :**

- other boundary conditions (Dirichlet, Neumann),
- other equations (3D Helmholtz, Maxwell),
- analysis of non-infsup-stable impedances.

**Future investigations**

- fine properties of the exchange operator
- large scale optimized parallel implementation
- multi-level strategy
- non-conforming DDM

## Thank you for your attention

## Questions ?


X.Claeys, F.Collino, P.Joly and E.Parolin, "A discrete domain decomposition method for acoustics with uniform exponential rate of convergence using non-local impedance operators", proceedings of the DD25 conference.

X.Claeys, "A new variant of the Optimised Schwarz Method for arbitrary non-overlapping subdomain partitions", ESAIM Math. Model. Numer. Anal. 55 (2021), no. 2, 429–448.

X.Claeys and E.Parolin, "Robust treatment of cross points in Optimized Schwarz Methods", submitted, preprint Arxiv 2003.06657

X.Claeys, "Non-self adjoint impedance in Generalized Optimized Schwarz Methods", submitted, preprint Arxiv 2108.03652

X.Claeys, F.Collino and E.Parolin, Matrix form of nonlocal OSM for electro-magnetics, preprint Arxiv 2108.11352