A symmetric algorithm for solving frictional contact problems using FreeFEM

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Signorini's contact problem [Signorini, 1933]

- The deformation of a body $\Omega \subset \mathbb{R}^3$ is described by the application $\phi : \Omega \to \mathbb{R}^3$.

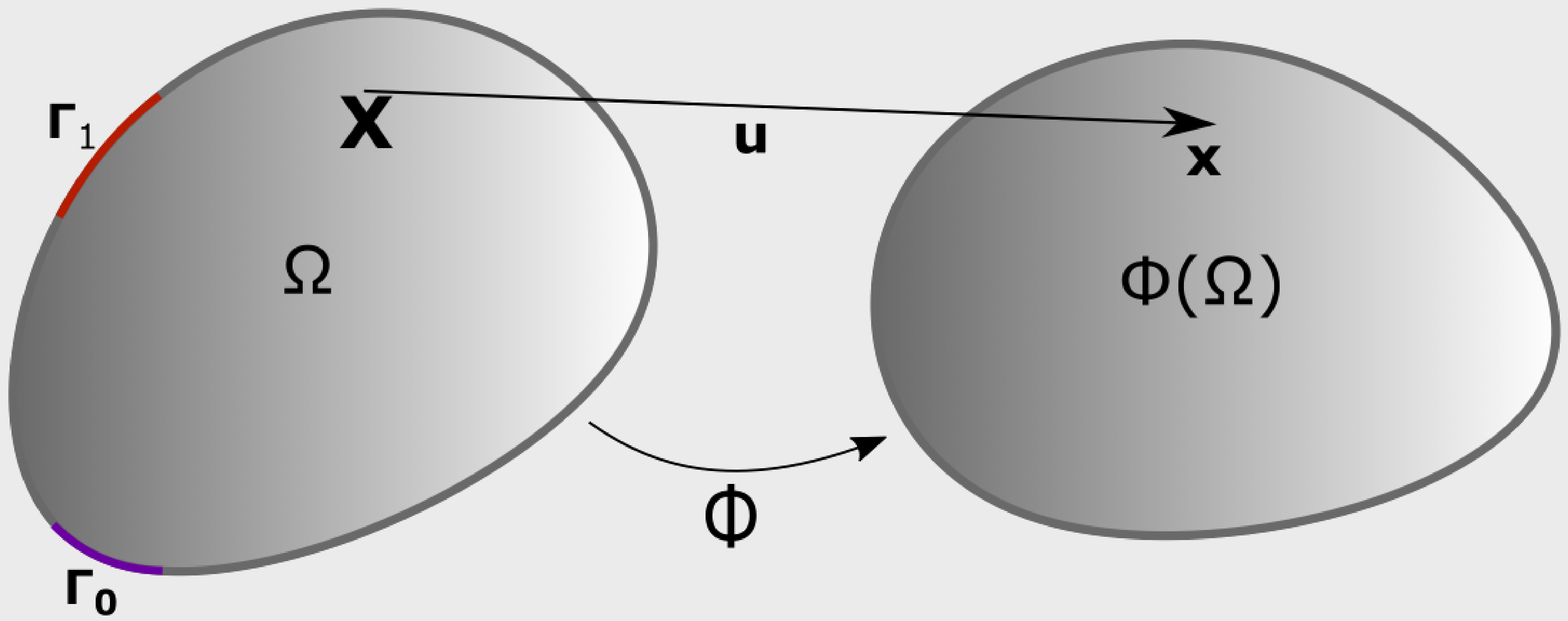

Figure: Initial and actual configurations

- The displacement field: $\boldsymbol{u} = \phi(\boldsymbol{X}) - \boldsymbol{X} = \boldsymbol{x} - \boldsymbol{X}$

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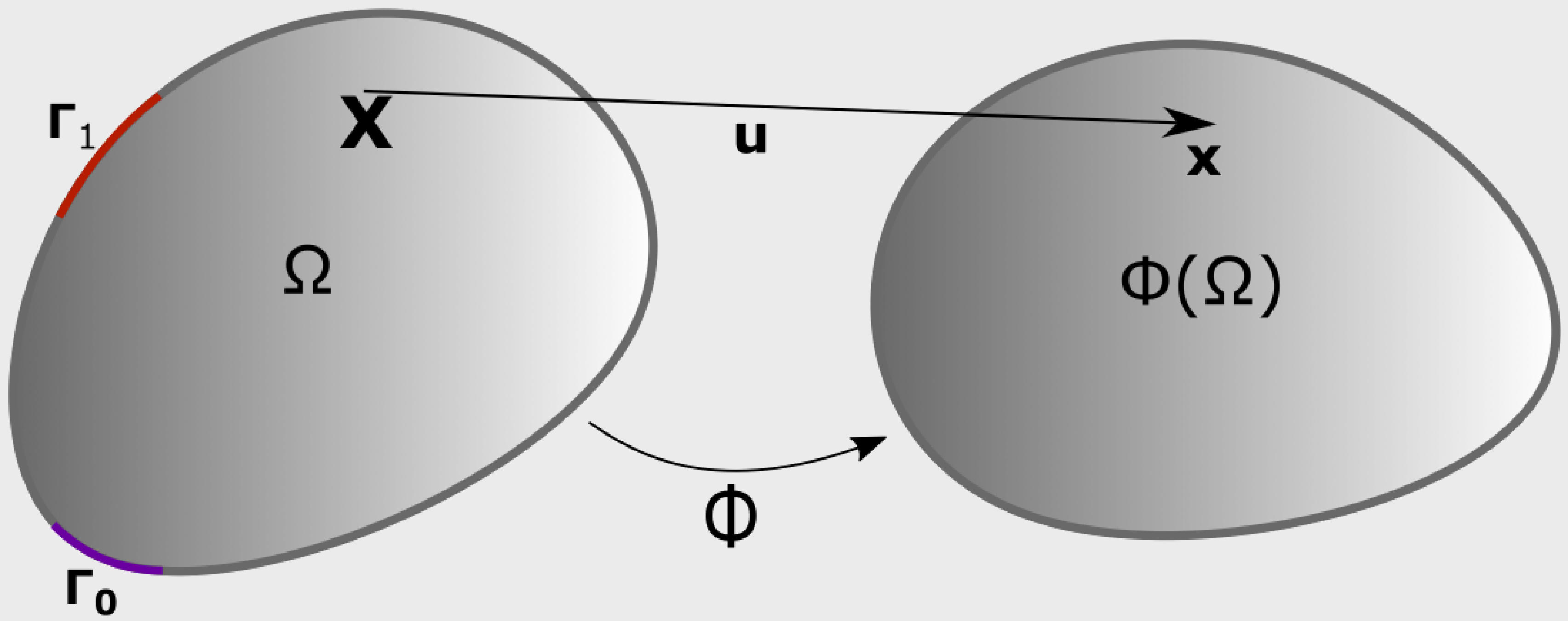

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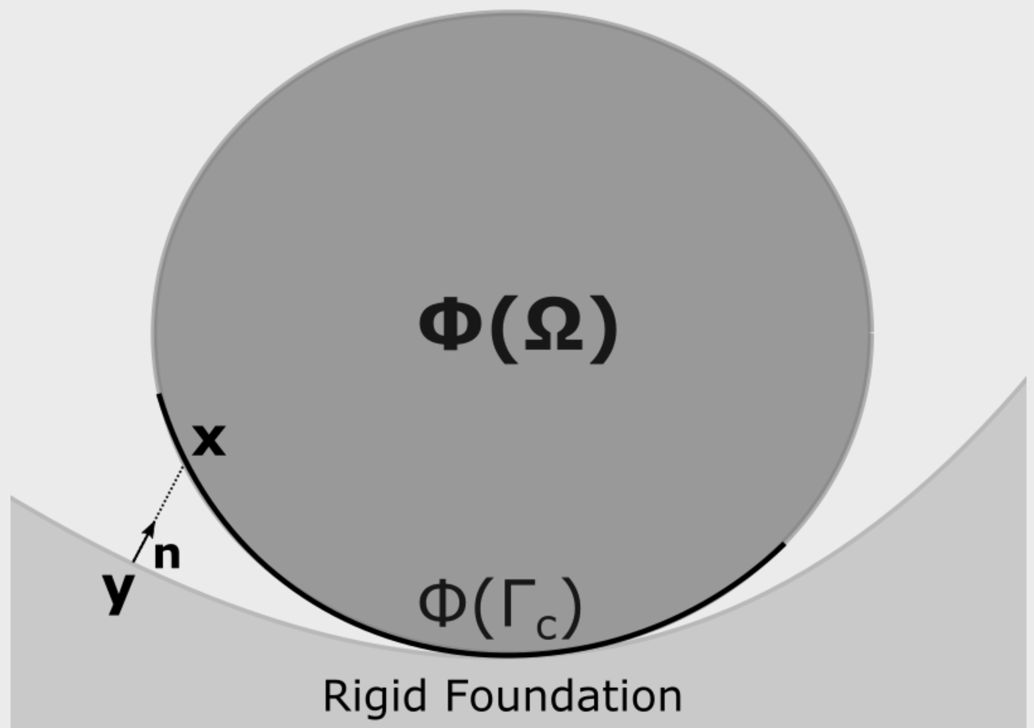

- Balance equations:

$$\begin{cases} \nabla.\boldsymbol{\sigma} + \mathbf{f} = 0 & \text{in } \Omega \\ \boldsymbol{\sigma} = C\epsilon & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_0 \\ \boldsymbol{\sigma}.\mathbf{n} = \mathbf{t} & \text{on } \Gamma_1 \end{cases} \quad (1)$$

- Contact conditions:

$$\begin{cases} g := (\mathbf{x} - \mathbf{y}).\mathbf{n} \geq 0 & \text{on } \Gamma_C \\ \sigma_n = (\boldsymbol{\sigma}.\mathbf{n}).\mathbf{n} \leq 0 & \text{on } \Gamma_C \\ \sigma_n.g = 0 & \text{on } \Gamma_C \end{cases} \quad (2)$$

Signorini's contact problem

Finite deformation and linear elasticity

- The admissible set:

$$\mathbf{V} = \left\{ \mathbf{v} \in \left(H^1(\Omega) \right)^3 \; ; \; \mathbf{v} = 0 \text{ on } \Gamma_0 \right\}$$

- Constitutive law of linear elasticity:

$$\boldsymbol{\sigma} = C\boldsymbol{\epsilon} \quad \text{where} \quad \boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla^T \boldsymbol{u} + \nabla \boldsymbol{u} \right) \tag{3}$$

- Linear elasticity: The total potential energy is defined by

$$\mathcal{E}(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - f(\mathbf{v}) \tag{4}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega C\boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v}) \, dV \quad \text{and} \quad f(\mathbf{v}) = \int_\Omega \mathbf{f}.\mathbf{v} \, dV + \int_{\Gamma_1} \mathbf{t}.\mathbf{v} \, dA \tag{5}$$

- Hyperelastic materials (Neo-Hookean, Mooney): The total potential energy is defined by

$$\mathcal{E}(\mathbf{v}) = \int_\Omega \psi \, dV - f(\mathbf{v}) \tag{6}$$

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- The non-penetration set:

$$\mathbf{K} = \{\mathbf{v} \in \mathbf{V} \ ; \ (\mathbf{x} - \mathbf{y}).\mathbf{n} \geq 0 \quad \forall\, \mathbf{x} \in \Phi(\Gamma_c)\} \tag{7}$$

- The displacement field $\boldsymbol{u}$ is a solution of the constrained minimization problem:

$$\mathbf{u} = \underset{v \in \mathbf{K}}{\operatorname{argmin}}\, \mathcal{E}(\mathbf{v}) \tag{8}$$

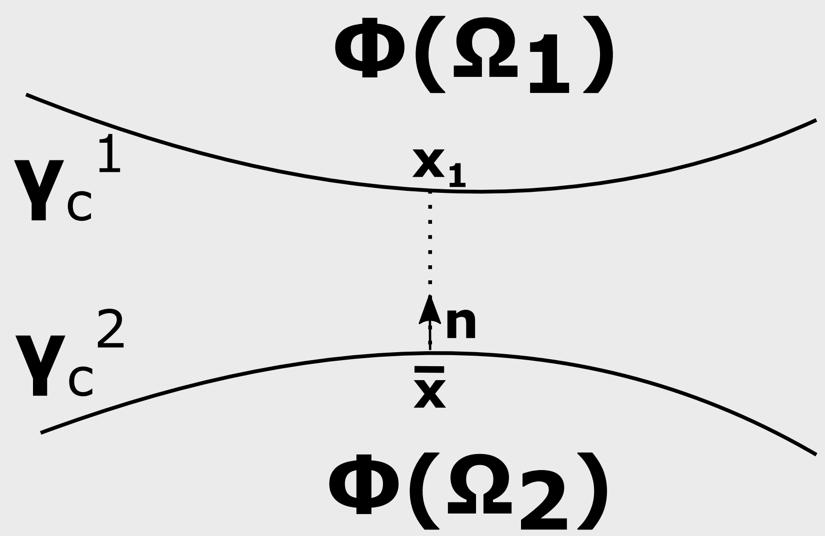
- Variational inequality (linear elasticity):

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq f(\mathbf{v} - \mathbf{u}) \quad \forall\, \mathbf{v} \in \mathbf{K} \tag{9}$$

Contact between two bodies

- Ω_1 and Ω_2 two domains in $\mathbb{R}^3$, representing respectively the first and the second body.
- γ_C^1, γ_C^2 are the actual potential contact areas where the two bodies will probably make contact.
- The non-penetration condition between the two bodies is the following

$$(\mathbf{x}_1 - \bar{\mathbf{x}}).\mathbf{n} \geq 0 \tag{10}$$



where $\mathbf{x}_1$ is the actual position of a material point belonging to γ_C^1, $\bar{\mathbf{x}}$ is the projection of $\mathbf{x}_1$ on γ_C^2, and $\mathbf{n}$ the normal vector at $\bar{\mathbf{x}}$.

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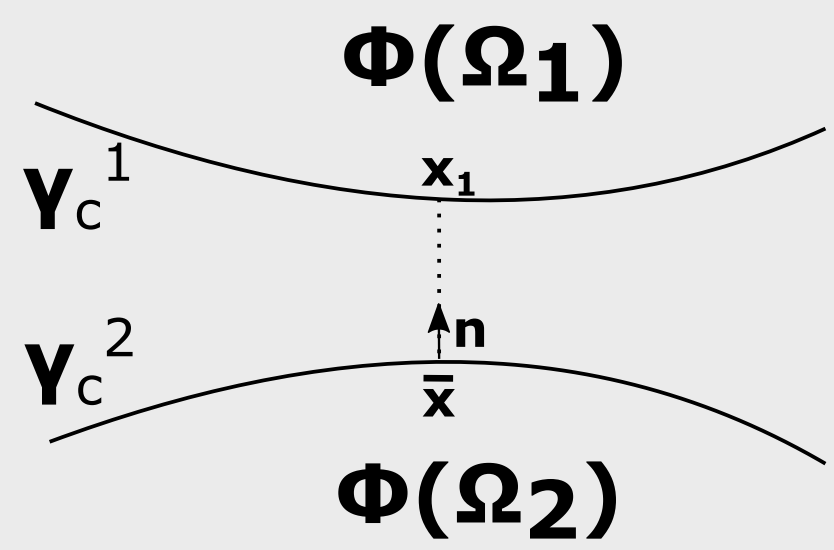

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- Non symmetric formulation $\Rightarrow$ One has to choose the **Slave** and the **Master** bodies.

There are different ways to help the user in defining the slave and the master bodies, for example a body can be chosen as slave if

- The body has the finest mesh
- The body is the less stiff
- The body has a curvature
- ...

In consequence, choosing the best master or slave between the different bodies can be difficult. The same problem occurs in the case of self contact or in the case of a contact between more than two bodies.

Symmetric contact formulation

Weak contact formulation

- Let θ be the non-penetration function

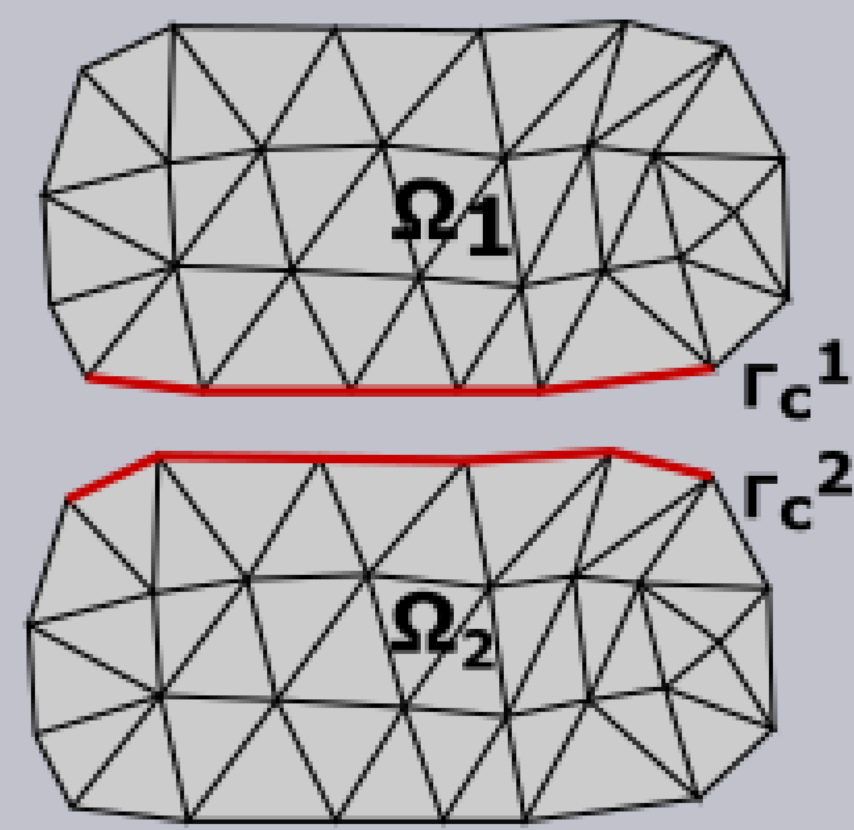
$$\theta_1(\mathbf{x}) = (\mathbf{x}_1 - \bar{\mathbf{x}}_1).\mathbf{n}(\bar{\mathbf{x}}_1) \geq 0 \quad \forall\, \mathbf{x}_1 \in \Phi(\Gamma_C^1)$$

where $\bar{\mathbf{x}}_1$ is the projected point of $\mathbf{x}_1$ on the second body.

- The minimization problem becomes [Belgacem et al., 1998, Hild, 1998, Popp et al., 2009]:

$$\begin{cases} \mathbf{u} = \underset{\mathbf{V}}{argmin}(\mathcal{E}_p) \\ (\theta_1, \phi_i)_{L^2} = \int_{\Gamma_C^1} \theta_1.\phi_i \, d\lambda \geq 0 \quad \forall\, i = 1, \ldots, n_{C_1} \end{cases}$$

where ϕ_i are the shape functions



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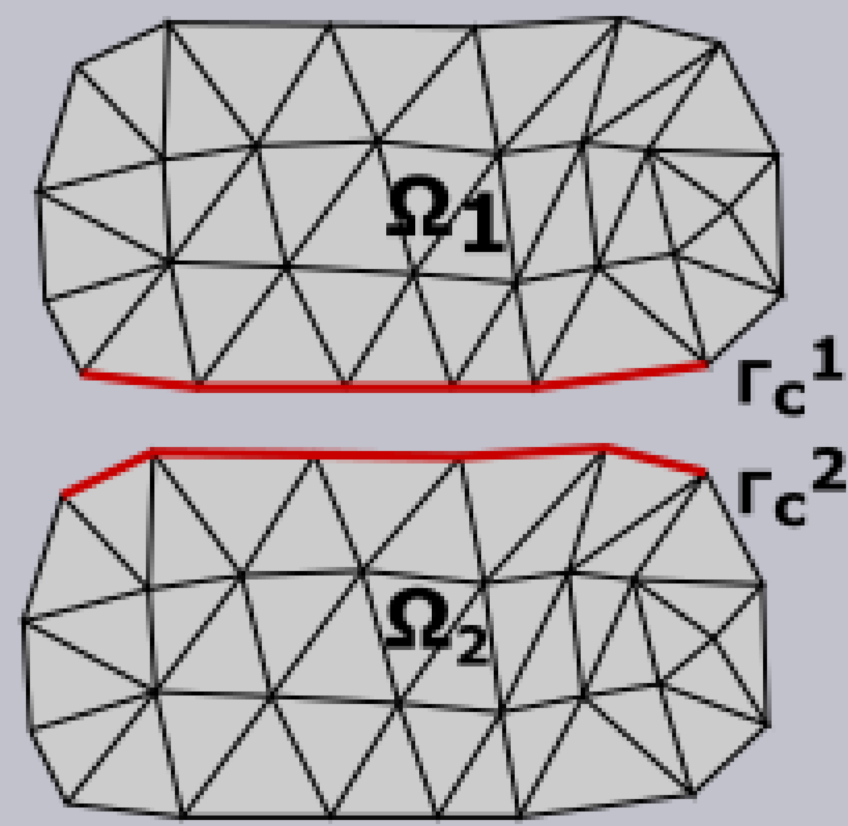

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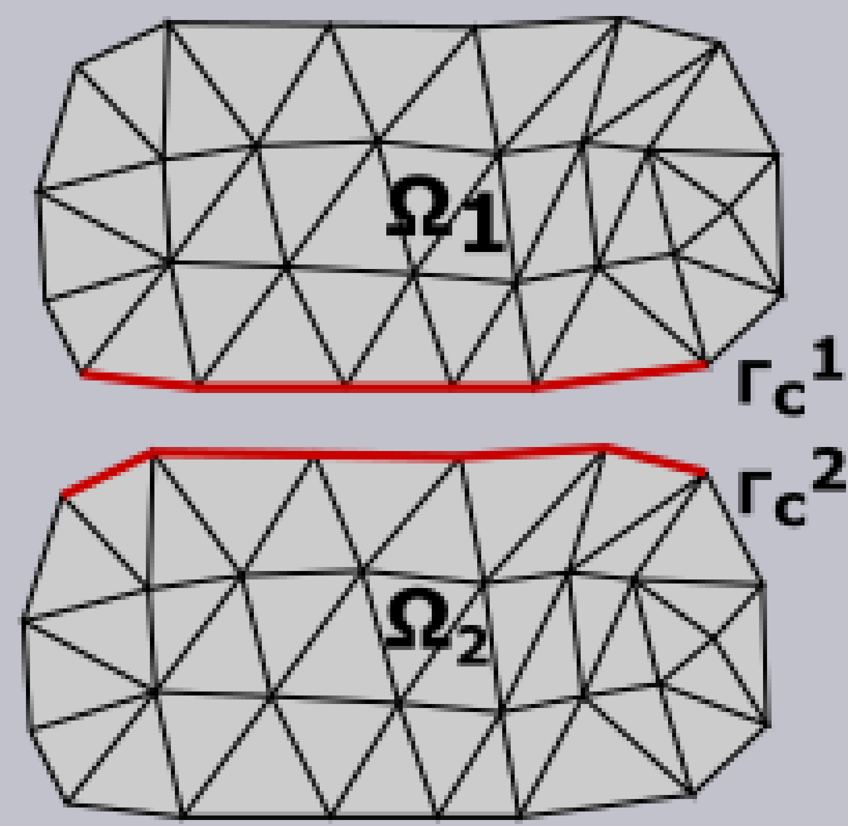

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- After the finite element discretization, the Jacobian matrix of the constraints may be rank-deficient $\Rightarrow$ Problem when using Newton's method $\Rightarrow$ Interior point method [Houssein et al., 2021, Nocedal and Wright, 2006, Wächter and Biegler, 2006]

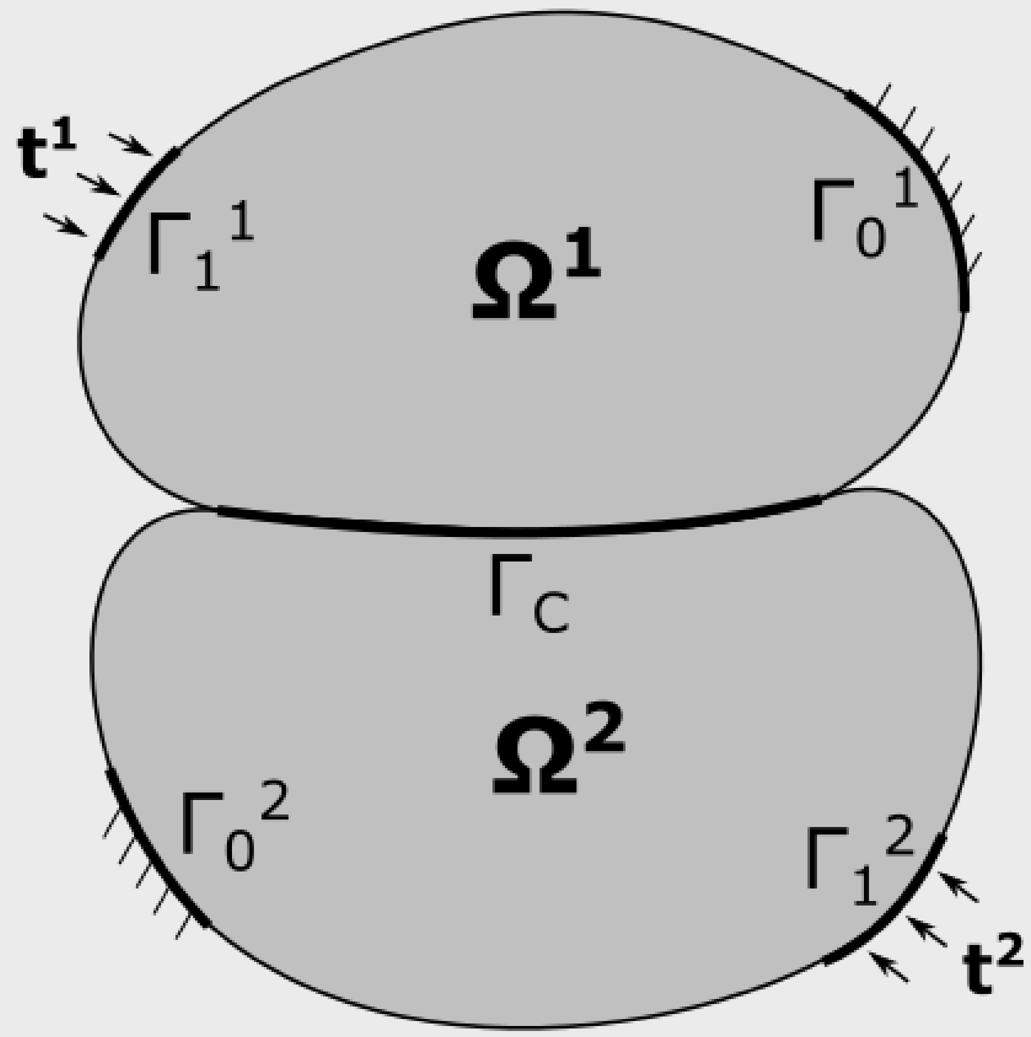
Frictional contact problems description

- Balance equations:

$$\begin{cases} \nabla.\boldsymbol{\sigma}^l + \mathbf{f}^l = 0 & \text{in } \Omega^l \\ \boldsymbol{\sigma}^l = C^l \boldsymbol{\epsilon}^l & \text{in } \Omega^l \\ \mathbf{u}^l = \mathbf{0} & \text{on } \Gamma_0^l \\ \boldsymbol{\sigma}^l.\mathbf{n}^l = \mathbf{t}^l & \text{on } \Gamma_1^l \end{cases} \tag{11}$$

- Contact conditions:

$$\begin{cases} [\mathbf{u}.\mathbf{n}] = \mathbf{u}^1.\mathbf{n}^1 + \mathbf{u}^2.\mathbf{n}^2 = (\mathbf{u}^1 - \mathbf{u}^2).\mathbf{n} \leq 0 & \text{on } \Gamma_C \\ \sigma_n = (\boldsymbol{\sigma}^1.\mathbf{n}^1).\mathbf{n}^1 = (\boldsymbol{\sigma}^2.\mathbf{n}^2).\mathbf{n}^2 \leq 0 & \text{on } \Gamma_C \\ \sigma_n.[\mathbf{u}.\mathbf{n}] = 0 & \text{on } \Gamma_C \end{cases} \tag{12}$$



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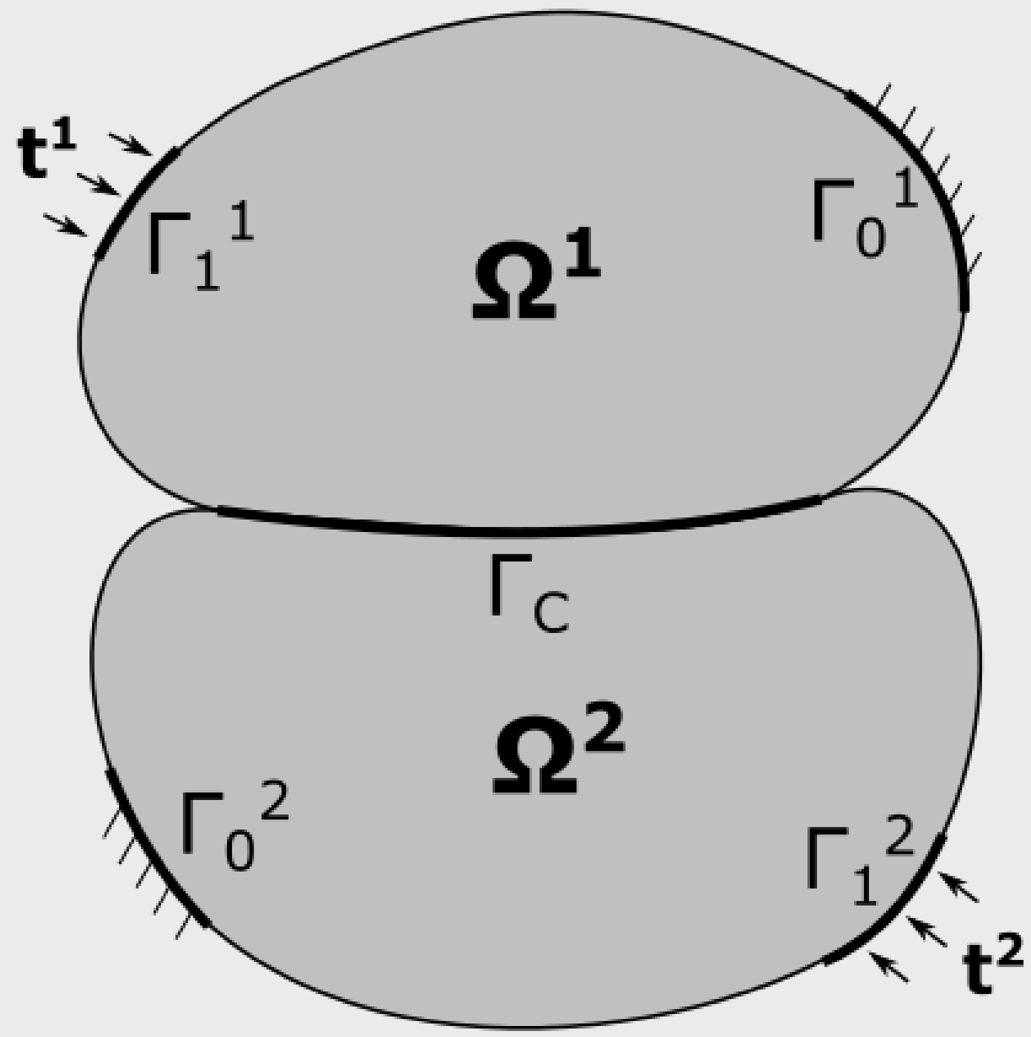

- Coulomb's criterion on Γ_C:

$$\begin{cases} \boldsymbol{\sigma}_T^1 = -\boldsymbol{\sigma}_T^2 \\ |\boldsymbol{\sigma}_T^1| \leq \mu|\sigma_n| \\ \text{if } |\boldsymbol{\sigma}_T^1| < \mu|\sigma_n| \Rightarrow \mathbf{u}_T^1 - \mathbf{u}_T^2 = \mathbf{0} \\ \text{if } |\boldsymbol{\sigma}_T^1| = \mu|\sigma_n| \Rightarrow \exists\lambda \geq 0 \text{ s.t } \mathbf{u}_T^1 - \mathbf{u}_T^2 = -\lambda\boldsymbol{\sigma}_T^1 \end{cases} \quad (13)$$

Frictional contact problems description

Let $\tau \in L^2(\Gamma_C) \geq 0$

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$$\mathbf{V}^l = \{\mathbf{v} \in \mathbf{H}^1(\Omega^l) = H^1(\Omega^l) \times H^1(\Omega^l) \mid \mathbf{v} = 0 \text{ a.e on } \Gamma_0^l\} \tag{15}$$

Let the applications $a : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ and $f : \mathbf{V} \to \mathbb{R}$ be defined by

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) = a^1(\mathbf{u}, \mathbf{v}) + a^2(\mathbf{u}, \mathbf{v}) \\ f(\mathbf{v}) = f^1(\mathbf{v}) + f^2(\mathbf{v}) \end{cases} \qquad \text{where for } l = 1, 2 \qquad \begin{cases} a^l(\mathbf{u}, \mathbf{v}) = \displaystyle\int_{\Omega^l} \boldsymbol{\sigma}(\mathbf{u}^l) : \boldsymbol{\epsilon}(\mathbf{v}^l)\, dv \\ f^l(\mathbf{v}) = \displaystyle\int_{\Omega^l} \mathbf{f}^l.\mathbf{v}^l\, dv + \int_{\Gamma_1^l} \mathbf{t}^l.\mathbf{v}^l\, ds \end{cases}$$

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$$j_\tau(\mathbf{v}) = \int_{\Gamma_C} \tau |\mathbf{v}_T^1 - \mathbf{v}_T^2|\, ds \tag{16}$$

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- The problem can be expressed as

Find $\mathbf{u} \in \mathbf{K}$ such that

$$a(\mathbf{u},\mathbf{v}-\mathbf{u}) + j_\tau(\mathbf{v}) - j_\tau(\mathbf{u}) \geq f(\mathbf{v}-\mathbf{u}) \quad \forall\, \mathbf{v} \in \mathbf{K} \qquad \Longleftrightarrow \qquad J_\tau(\mathbf{u}) \leq J_\tau(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{K}$$

where $J_\tau(\mathbf{v}) := \mathcal{E}_p(\mathbf{v}) + j_\tau(\mathbf{v}) = \frac{1}{2}a(\mathbf{v},\mathbf{v}) - f(\mathbf{v}) + j_\tau(\mathbf{v})$

Coulomb's criterion as a fixed point problem

- Coulomb criterion can be equivalent to the fixed point of the following application [Raous, 1999]

$$T(\tau) = -\mu\sigma_N(\mathbf{u}_\tau) \tag{17}$$

with $\mathbf{u}_\tau$ solution of

Find $\mathbf{u}_\tau \in \mathbf{K}$ such that

$$a(\mathbf{u}_\tau, \mathbf{v} - \mathbf{u}_\tau) + j_\tau(\mathbf{v}) - j_\tau(\mathbf{u}_\tau) \geq f(\mathbf{v} - \mathbf{u}_\tau) \quad \forall\, \mathbf{v} \in \mathbf{K}$$

or

Find $\mathbf{u}_\tau \in \mathbf{K}$ such that

$$J_\tau(\mathbf{u}_\tau) \leq J_\tau(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{K}$$

Regularization of the frictional problem

- The module vector $|\cdot|$ is approximated by an application $\eta_\alpha \in \Xi_\alpha$
- Ξ_α is defined by

$$\eta_\alpha \in \Xi_\alpha \iff \begin{cases} \eta_\alpha \in C^2(\mathbb{R}^d) \\ \eta_\alpha \text{ is convex} \\ \eta_\alpha(\mathbf{v}) = \eta_\alpha(-\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbb{R}^d \\ \eta_\alpha(\mathbf{v}) \geq 0 \quad \forall\, \mathbf{v} \in \mathbb{R}^d \\ |\eta_\alpha(\mathbf{v}) - |\mathbf{v}|| \leq \alpha \quad \forall\, \mathbf{v} \in \mathbb{R}^d \\ |\eta_\alpha(\mathbf{v}_1) - \eta_\alpha(\mathbf{v}_2)| \leq \big|\, |\mathbf{v}_1| - |\mathbf{v}_2| \,\big| \quad \forall\, \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^d \end{cases} \tag{18}$$

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For $\alpha > 0$, $\bar{\eta}_\alpha(\mathbf{v}) = \sqrt{|\mathbf{v}|^2 + \alpha^2} \quad \forall\, \mathbf{v} \in \mathbb{R}^d$ belongs to Ξ_α

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$$j_\tau(\mathbf{v}) = \int_{\Gamma_C} \tau |\mathbf{v}_T^1 - \mathbf{v}_T^2|\, ds \quad \text{is replaced by} \quad j_{\alpha,\tau}(\mathbf{v}) = \int_{\Gamma_C} \tau.\eta_\alpha(\mathbf{v}_T^1 - \mathbf{v}_T^2)\, ds$$

- The regularized Tresca problem becomes

Find $\mathbf{u} \in \mathbf{K}$ such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j_{\alpha,\tau}(\mathbf{v}) - j_{\alpha,\tau}(\mathbf{u}) \geq f(\mathbf{v} - \mathbf{u}) \quad \forall\, \mathbf{v} \in \mathbf{K} \qquad \iff \qquad J_{\alpha,\tau}(\mathbf{u}) \leq J_{\alpha,\tau}(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{K}$$

where $J_{\alpha,\tau}(\mathbf{v}) := \mathcal{E}_p(\mathbf{v}) + j_{\alpha,\tau}(\mathbf{v}) = \frac{1}{2}a(\mathbf{v}, \mathbf{v}) - f(\mathbf{v}) + j_{\alpha,\tau}(\mathbf{v})$

Regularization of the frictional problem

Let $\mathbf{u} \in \mathbf{K}$ be sufficiently regular, satisfying

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j_{\alpha,\tau}(\mathbf{v}) - j_{\alpha,\tau}(\mathbf{u}) \geq f(\mathbf{v} - \mathbf{u}) \quad \forall\, \mathbf{v} \in \mathbf{K} \qquad \Longleftrightarrow \qquad J_{\alpha,\tau}(\mathbf{u}) \leq J_{\alpha,\tau}(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{K}$$

Then $\mathbf{u}$ satisfies the following equations for $l = 1, 2$

- Balance equations:

$$\begin{cases} \nabla.\boldsymbol{\sigma}^l + \mathbf{f}^l = 0 & \text{in } \Omega^l \\ \boldsymbol{\sigma}^l = C^l \boldsymbol{\epsilon}^l & \text{in } \Omega^l \\ \mathbf{u}^l = \mathbf{0} & \text{on } \Gamma_0^l \\ \boldsymbol{\sigma}^l.\mathbf{n}^l = \mathbf{t}^l & \text{on } \Gamma_1^l \end{cases}$$

- Contact conditions:

$$\begin{cases} [\mathbf{u}.\mathbf{n}] = \mathbf{u}^1.\mathbf{n}^1 + \mathbf{u}^2.\mathbf{n}^2 = (\mathbf{u}^1 - \mathbf{u}^2).\mathbf{n} \leq 0 & \text{on } \Gamma_C \\ \sigma_n = (\boldsymbol{\sigma}^1.\mathbf{n}^1).\mathbf{n}^1 = (\boldsymbol{\sigma}^2.\mathbf{n}^2).\mathbf{n}^2 \leq 0 & \text{on } \Gamma_C \\ \sigma_n.[\mathbf{u}.\mathbf{n}] = 0 & \text{on } \Gamma_C \end{cases}$$

- Regularized frictional criterion on Γ_C:

$$\begin{cases} \boldsymbol{\sigma}_T^1 & = -\boldsymbol{\sigma}_T^2 \\ \boldsymbol{\sigma}_T^1 & = -\tau.\nabla\eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2) \\ & = -\tau \frac{(\mathbf{u}_T^1 - \mathbf{u}_T^2)}{\sqrt{|\mathbf{u}_T^1 - \mathbf{u}_T^2|^2 + \alpha^2}} \text{ if } \eta_\alpha(\mathbf{v}) = \sqrt{|\mathbf{v}|^2 + \alpha^2} \end{cases}$$

Regularization of the frictional problem

Let $\mathbf{u} \in \mathbf{K}$ be sufficiently regular, satisfying

$$a(\mathbf{u}, \mathbf{v}-\mathbf{u}) + j_{\alpha,\tau}(\mathbf{v}) - j_{\alpha,\tau}(\mathbf{u}) \geq f(\mathbf{v}-\mathbf{u}) \quad \forall\, \mathbf{v} \in \mathbf{K} \qquad \Longleftrightarrow \qquad J_{\alpha,\tau}(\mathbf{u}) \leq J_{\alpha,\tau}(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{K}$$

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- Error between Tresca's solution and regularized Tresca's solution, $\exists\, C \geq 0$

$$\|\mathbf{u}_\alpha - \mathbf{u}\|_1 \leq C\alpha$$

Regularization of the frictional problem

Let $\mathbf{u} \in \mathbf{K}$ be sufficiently regular, satisfying

$$a(\mathbf{u}, \mathbf{v}-\mathbf{u}) + j_{\alpha,\tau}(\mathbf{v}) - j_{\alpha,\tau}(\mathbf{u}) \geq f(\mathbf{v}-\mathbf{u}) \quad \forall\, \mathbf{v} \in \mathbf{K} \qquad \Longleftrightarrow \qquad J_{\alpha,\tau}(\mathbf{u}) \leq J_{\alpha,\tau}(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{K}$$

Then $\mathbf{u}$ satisfies the following equations for $l = 1, 2$

- Balance equations:

$$\begin{cases} \nabla.\boldsymbol{\sigma}^l + \mathbf{f}^l = 0 & \text{in } \Omega^l \\ \boldsymbol{\sigma}^l = C^l \boldsymbol{\epsilon}^l & \text{in } \Omega^l \\ \mathbf{u}^l = \mathbf{0} & \text{on } \Gamma_0^l \\ \boldsymbol{\sigma}^l.\mathbf{n}^l = \mathbf{t}^l & \text{on } \Gamma_1^l \end{cases}$$

- Contact conditions:

$$\begin{cases} [\mathbf{u}.\mathbf{n}] = \mathbf{u}^1.\mathbf{n}^1 + \mathbf{u}^2.\mathbf{n}^2 = (\mathbf{u}^1 - \mathbf{u}^2).\mathbf{n} \leq 0 & \text{on } \Gamma_C \\ \sigma_n = (\boldsymbol{\sigma}^1.\mathbf{n}^1).\mathbf{n}^1 = (\boldsymbol{\sigma}^2.\mathbf{n}^2).\mathbf{n}^2 \leq 0 & \text{on } \Gamma_C \\ \sigma_n.[\mathbf{u}.\mathbf{n}] = 0 & \text{on } \Gamma_C \end{cases}$$

- Regularized frictional criterion on Γ_C:

$$\begin{cases} \boldsymbol{\sigma}_T^1 & = -\boldsymbol{\sigma}_T^2 \\ \boldsymbol{\sigma}_T^1 & = -\tau.\nabla\eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2) \\ & = -\tau\frac{(\mathbf{u}_T^1 - \mathbf{u}_T^2)}{\sqrt{|\mathbf{u}_T^1 - \mathbf{u}_T^2|^2 + \alpha^2}} \text{ if } \eta_\alpha(\mathbf{v}) = \sqrt{|\mathbf{v}|^2 + \alpha^2} \end{cases}$$

- Error between Tresca's solution and regularized Tresca's solution, $\exists\, C \geq 0$

$$\|\mathbf{u}_\alpha - \mathbf{u}\|_1 \leq C\alpha$$

- Fixed point for the regularized Coulomb problem

The Algorithm

- Consider the following spaces, for $l = 1, 2$

$$\mathbf{V}_h^l = \left\{ \mathbf{v} = (v_1, v_2) \in C^0(\Omega_h^l) \times C^0(\Omega_h^l) \mid \mathbf{v}_{|T_i^l} \in P_r \times P_r, \forall i = 1, \ldots, n_T^l \text{ and } \mathbf{v} = 0 \text{ on } \Gamma_0^l \right\} \tag{19}$$

$$\mathbf{V}_h = \mathbf{V}_h^1 \times \mathbf{V}_h^2 \tag{20}$$

- Let $\mathbf{u}_h = (\mathbf{u}_h^1, \mathbf{u}_h^2) \in \mathbf{V}_h$, the displacement vector field $\mathbf{u}_h^l$ on the mesh Ω_h^l is given by

$$\mathbf{u}_h^l = \sum_i \begin{pmatrix} U_i^x \\ U_i^y \end{pmatrix} \hat{w}_i^l \tag{21}$$

Algorithm 1 Regularized frictional algorithm using the fixed point method

Set the error tolerance $\epsilon_{tol} = 10^{-6}$
Compute $\sigma_{n,0}$ the normal stress pressure at the contact area for the frictionless problem
Compute $\tau_0 = -\mu\sigma_{n,0}$, the first sliding limit
while *error* $\geq \epsilon_{tol}$ **do**
1. For a given sliding limit τ_k, solve Tresca's regularized problem, given in the algorithm 2
2. Retrieve the displacement field $\mathbf{u}_h$
3. Compute the normal pressure $\sigma_{n,k}(\mathbf{u}_h)$ on the contact surface
4. Compute the new sliding limit $\tau_{k+1} = -\mu\sigma_{n,k}$
5. error=$\dfrac{\|[\![\tau_{k+1}]\!] - [\![\tau_k]\!]\|_\infty}{\|[\![\tau_k]\!]\|_\infty}$

end while

The Algorithm

Algorithm 2 Symmetric algorithm using the fixed point method for Tresca's regularized problem

Initialization of the displacement $\mathbf{U}_0$ and setting the tolerance $\epsilon_{tol} = 10^{-6}$

while *error* $\geq \epsilon_{tol}$ **do**

1. Using the displacement vector $\mathbf{U}_n$ of the previous iteration n:
   - Compute the projection points' parameters $\{\eta_i^* \,|\, i = 1, \ldots, nS\}$ of all slave integration points
   - Compute the normal at the projection points $\{n_i \,|\, i = 1, \ldots, nS\}$ (Using smoothing techniques)
   - Compute the contact area γ_C^n
2. For each integration point, its projection point $\bar{\mathbf{x}}_i$ depends linearly on the actual displacement
3. Reverse the role of the master and the slave bodies
4. Form the Energy E_{n+1} and the symmetric linear constraints
5. Use the interior point method in order to solve the minimization problem with linear constraints and to obtain the actual displacement $\mathbf{U}_{n+1}$
6. error=$\dfrac{\|\mathbf{U}_{n+1} - \mathbf{U}_n\|_\infty}{\|\mathbf{U}_n\|_\infty}$

end while

IPOPT [Wächter and Biegler, 2006] in FreeFEM

Minimization of $(x-1)^2$

```
// Minimization of f(x)=(x-1)^2

load "ff-Ipopt"

// Energy
func real f(real[int] &xx) {
  return (xx[0]-1)^2.;}

// Jacobian
func real[int] df(real[int] &xx) {
  real[int] id=[2.*(xx[0]-1)];
  return id;}

// Hessian
real[int,int] idd1(1,1);
idd1=0;
matrix idd=idd1;
func matrix ddf(real[int] &xx) {
  idd(0,0)=2.;
  return idd;}

// Initial solution
real[int] xx=[0.4];

// Solve
IPOPT(f,df,ddf,xx);

// Print
cout << "Solution= " << xx[0] <<endl;
```

Numerical validations (*Using FreeFEM [Hecht, 2012]*)

- **Validation of the regularized friction law**:

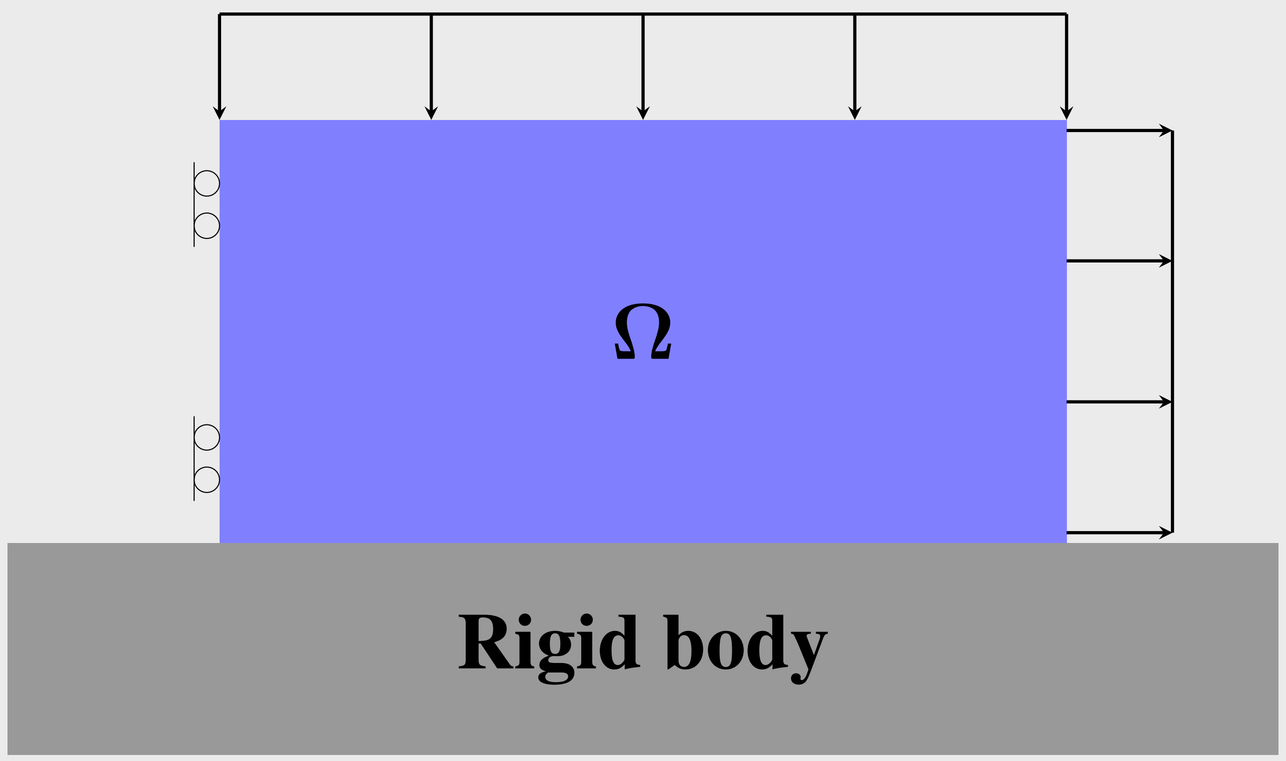

Figure: Problem geometry

Numerical validations (*Using FreeFEM [Hecht, 2012]*)

- **Validation of the regularized friction law**:

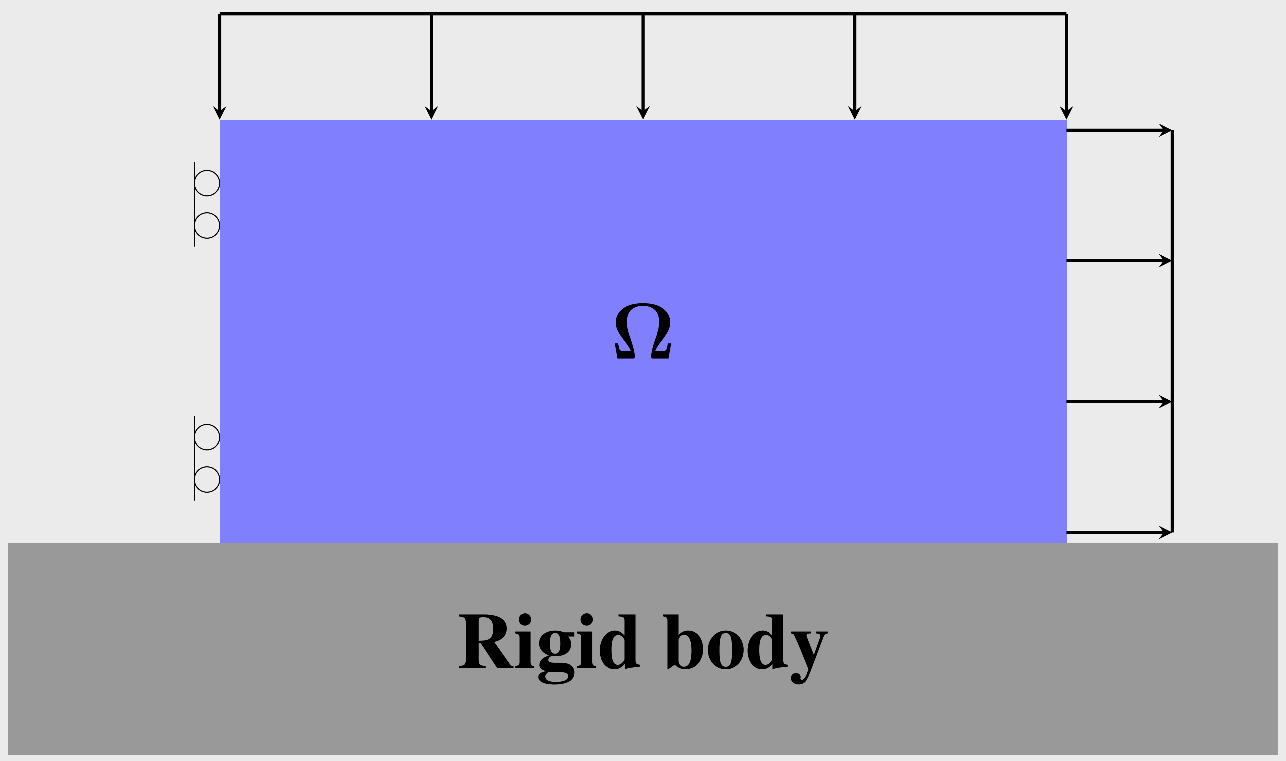

Figure: Problem geometry

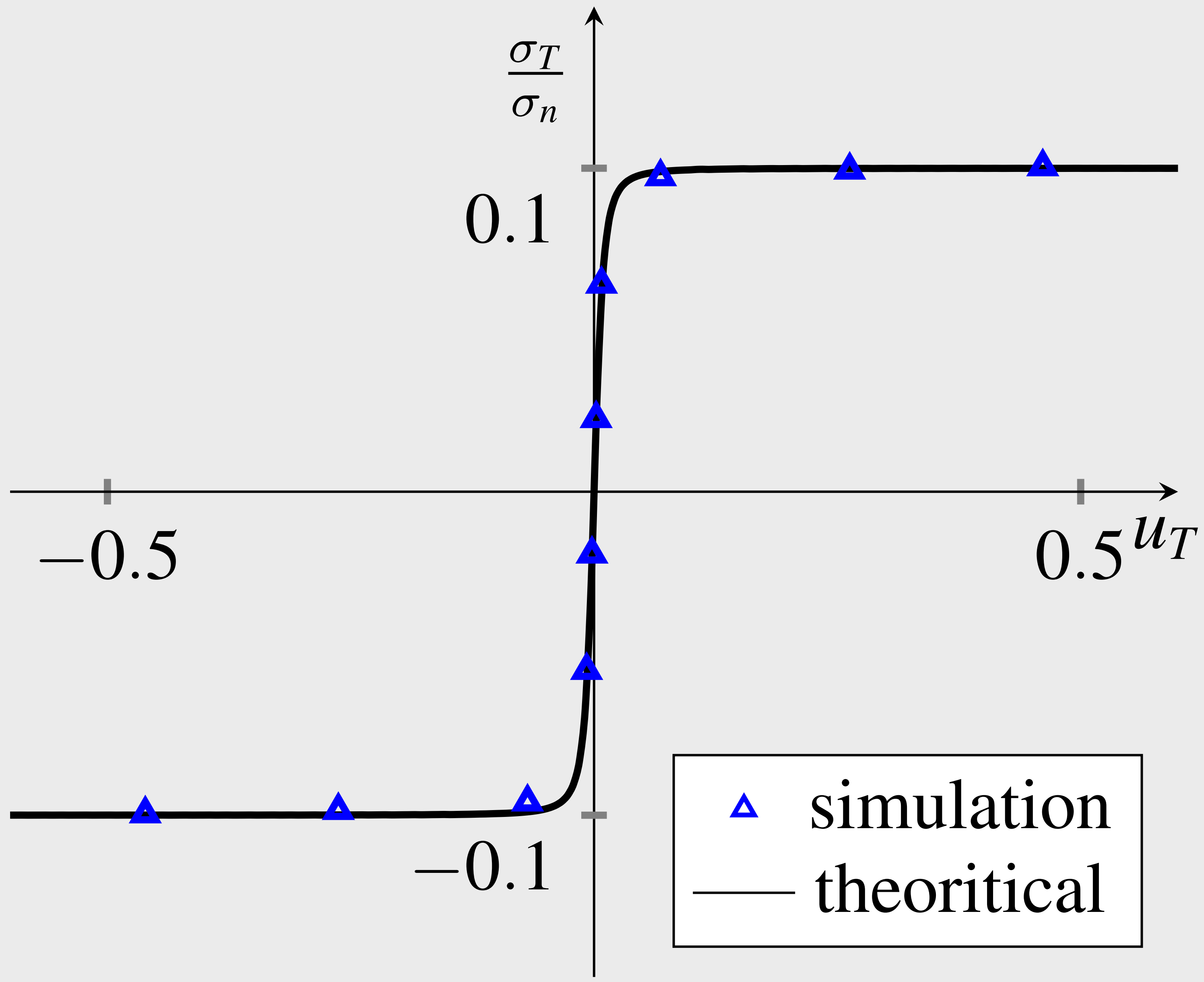

Figure: $\frac{\sigma_T}{\sigma_n}$ vs u_T

Numerical validations (*Using FreeFEM [Hecht, 2012]*)

- **Sliding on an inclined interface**:

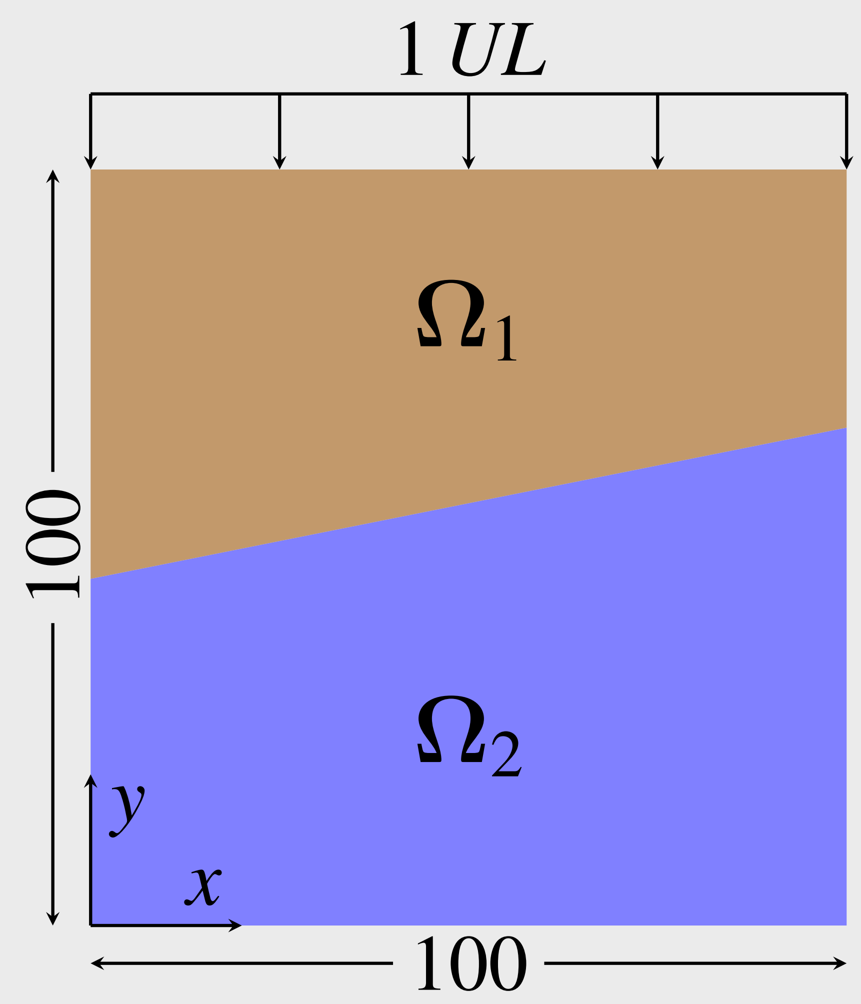

Figure: The geometry of the two bodies and the imposed displacement

Numerical validations (*Using FreeFEM [Hecht, 2012]*)

- **Sliding on an inclined interface**:

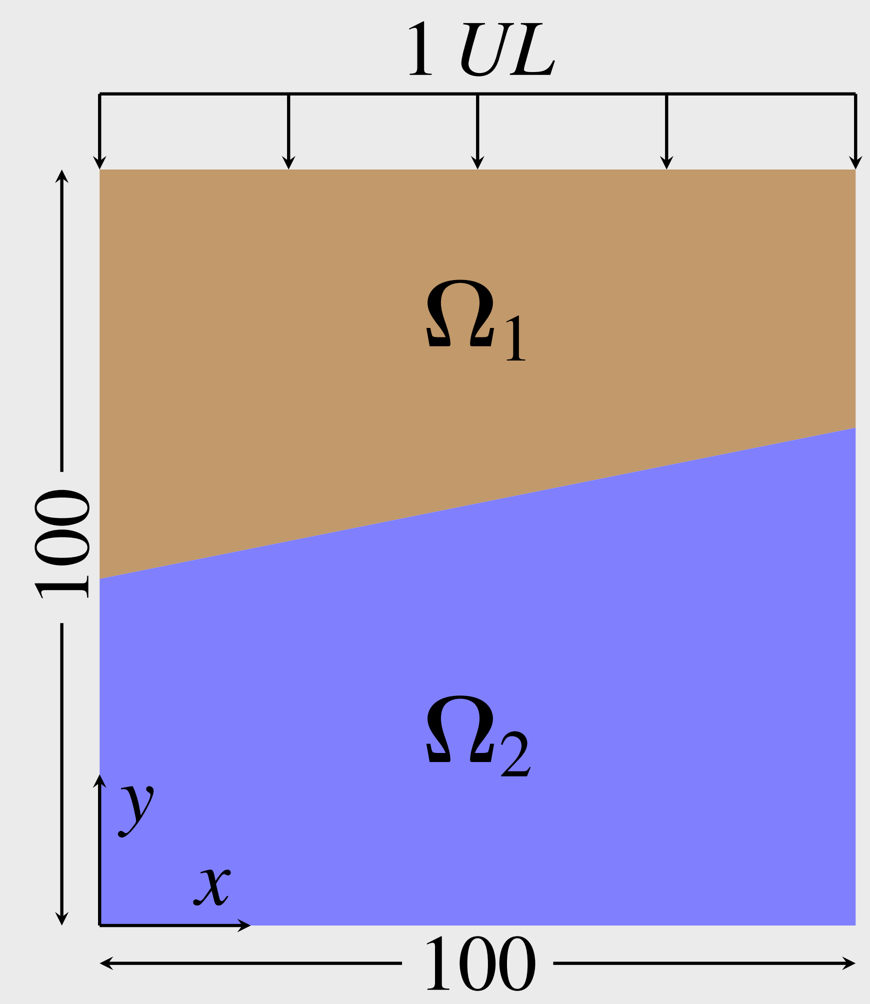

Figure: The geometry of the two bodies and the imposed displacement

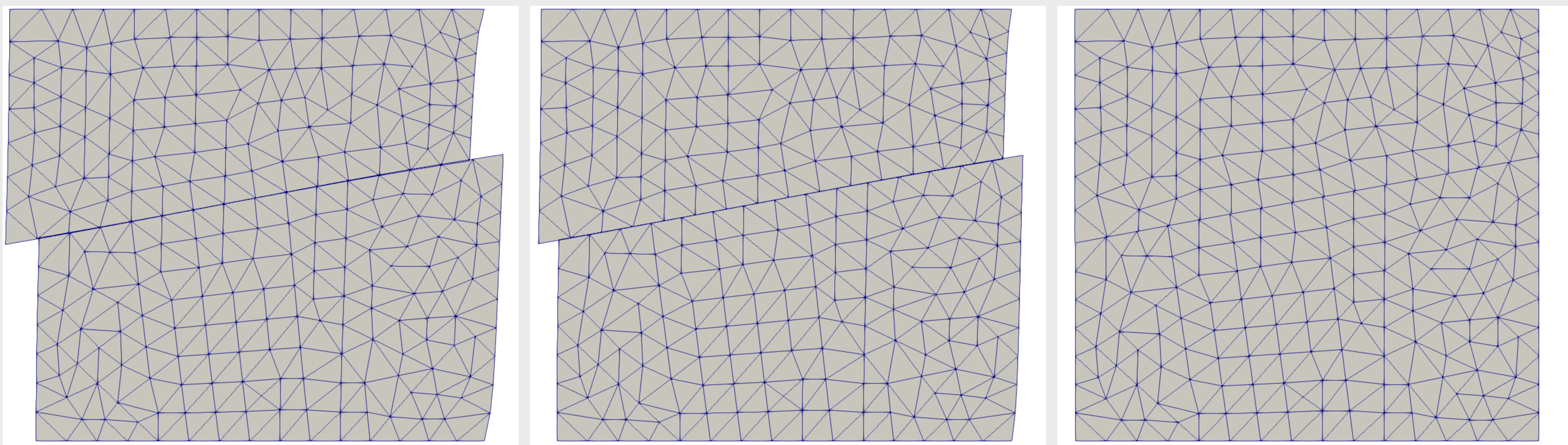

Figure: The deformation states (amplification factor =5) for $\mu = 0, 0.1, 0.2$

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THANK YOU FOR YOUR ATTENTION