FREEFEM DAYS
12TH EDITION - PARIS

Sensitivity analysis of limit cycles of Navier–Stokes equations by the Harmonic–Balance method

An application case with PETSc/SLEPc

Javier Sierra[1,2], P. Jolivet[3], F. Giannetti[2], V. Citro[2], D. Fabre[1]

[1] UPS-IMFT, Institut de Mécanique des fluides de Toulouse, Toulouse 31400, France
[2] UNISA, Universitá degli Studi di Salerno,Fisciano 84084, Italy
[3] CNRS-IRIT, 2 rue Charles Camichel, 31071 Toulouse Cedex 7, France

Motivation

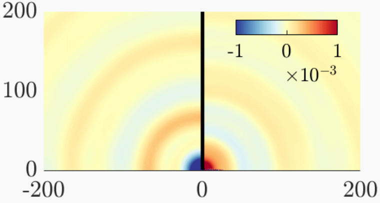
Motivation

Time-periodic evolutionary PDE

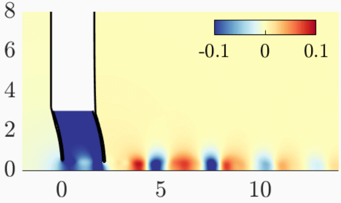
$$\begin{cases} \mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\ \mathbf{q}(t+T) = \mathbf{q}(t) \end{cases} \qquad (1)$$

Two cases of interest:

- Incompressible Navier–Stokes equations: $\mathbf{q} = [\mathbf{u}, p]$
- Compressible Navier–Stokes equations: $\mathbf{q} = [\rho, \mathbf{u}, T, p]$



p' at the far-field



p' at the near-field

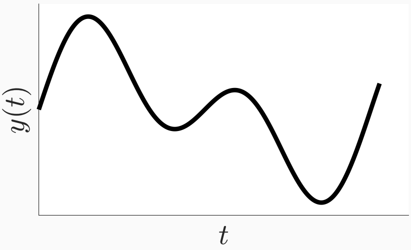
Motivation (II) – A dynamical system detour

Time-periodic evolutionary PDE

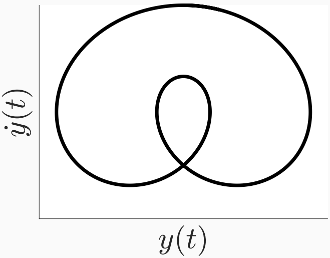
$$\begin{cases} \mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\ \mathbf{q}(t+T) = \mathbf{q}(t) \end{cases}$$

The periodic solution $\mathbf{q}$ can be *stable* or *unstable*

Time evolution of the solution



Phase portrait



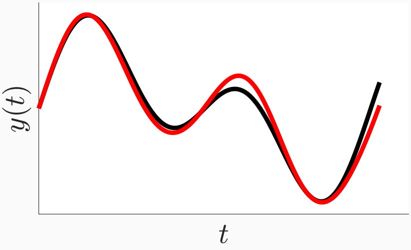
Motivation (II) – A dynamical system detour

Time-periodic evolutionary PDE

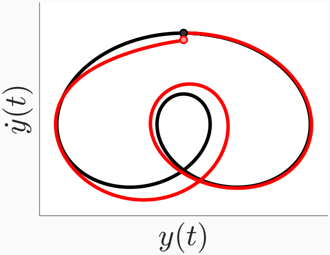
$$\begin{cases} \mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\ \mathbf{q}(t+T) = \mathbf{q}(t) \end{cases}$$

The periodic solution $\mathbf{q}$ can be *stable* or *unstable*

Time evolution of the solution

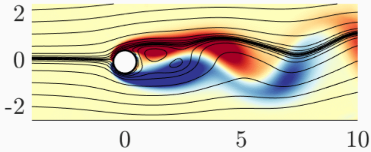


Phase portrait



Motivation (III) – Interests in (unstable) periodic solutions

- Continuation past *self-sustained* instabilities.
- Detection and continuation past instabilities of limit cycles.
- Determination of the *sensitivity quantities*, e.g. the gradient of a property of the system with respect to a control parameter [1].
- Identification of properties of chaotic systems, e.g. *cycle expansions* [2], *Lyapunov exponents*[3].



[1](Giannetti et al. 2019).
[2](Cvitanovic, 1988, Artuso, 1990a,b).
[3](Ott 1993).

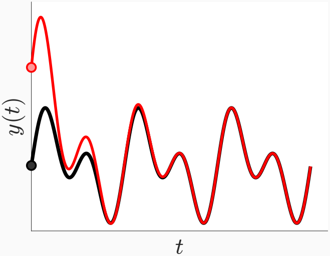
Motivation (IV) – Computing (unstable) periodic solutions

Methods for the computation of the periodic solution

- **Explicit**
  - Time marching
  - Shooting
  - Multi-shooting
- **Implicit**
  - Collocation, e.g. finite differences, orthogonal collocation
  - Spectral. Unknowns in the *frequency* or *time* domain.

Time-periodic evolutionary PDE

$$\begin{cases} \mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\ \mathbf{q}(t+T) = \mathbf{q}(t) \end{cases}$$



$\mathbf{q}(t) = \mathcal{S}^t \mathbf{q}(t_0)$ *stable* if $||S^t|| < 1$

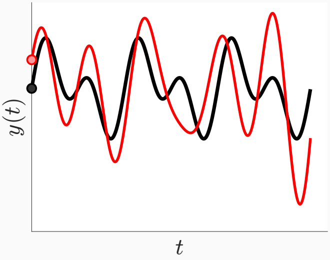
Motivation (IV) – Computing (unstable) periodic solutions

Methods for the computation of the periodic solution

- **Explicit**
  - Time marching
    unstable $\rightarrow$ does not converge
  - Shooting
  - Multi-shooting
- **Implicit**
  - Collocation, e.g. finite differences, orthogonal collocation
  - Spectral. Unknowns in the *frequency* or *time* domain.

Time-periodic evolutionary PDE

$$
\begin{cases}
\mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\
\mathbf{q}(t+T) = \mathbf{q}(t)
\end{cases}
$$



$\mathbf{q}(t) = \mathcal{S}^t \mathbf{q}(t_0)$ *unstable* if $||S^t|| > 1$

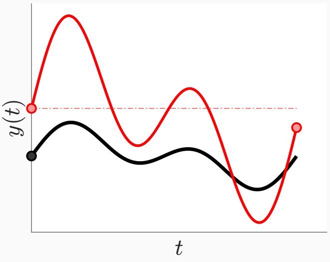
Motivation (IV) – Computing (unstable) periodic solutions

Methods for the computation of the periodic solution

- **Explicit**
  - Time marching
  - Shooting
  - Multi-shooting
- **Implicit**
  - Collocation, e.g. finite differences, orthogonal collocation
  - Spectral. Unknowns in the *frequency* or *time* domain.

Time-periodic evolutionary PDE

$$\begin{cases} \mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\ \mathbf{q}(t+T) = \mathbf{q}(t) \end{cases}$$



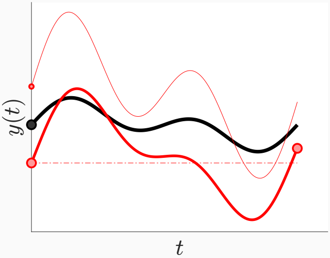
Motivation (IV) – Computing (unstable) periodic solutions

Methods for the computation of the periodic solution

- **Explicit**
  - Time marching
  - Shooting
  - Multi-shooting
- **Implicit**
  - Collocation, e.g. finite differences, orthogonal collocation
  - Spectral: Unknowns in the *frequency* or *time* domain.

Time-periodic evolutionary PDE

$$\begin{cases} \mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\ \mathbf{q}(t+T) = \mathbf{q}(t) \end{cases}$$



A dichotomy process until convergence ...

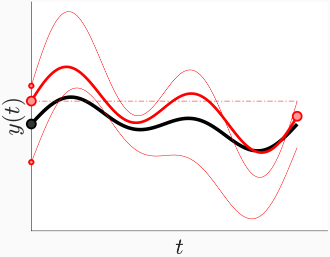
Motivation (IV) – Computing (unstable) periodic solutions

Methods for the computation of the periodic solution

- **Explicit**
  - Time marching
  - Shooting
  - Multi-shooting
- **Implicit**
  - Collocation, e.g. finite differences, orthogonal collocation
  - Spectral. Unknowns in the *frequency* or *time* domain.

Time-periodic evolutionary PDE

$$\begin{cases} \mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\ \mathbf{q}(t+T) = \mathbf{q}(t) \end{cases}$$



A dichotomy process until convergence ...

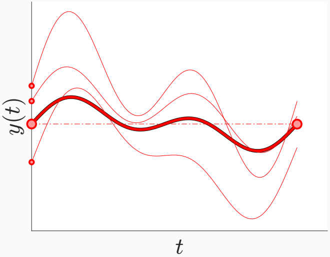
Motivation (IV) – Computing (unstable) periodic solutions

Methods for the computation of the periodic solution

- **Explicit**
  - Time marching
  - Shooting
  - Multi-shooting
- **Implicit**
  - Collocation, e.g. finite differences, orthogonal collocation
  - Spectral: Unknowns in the *frequency* or *time* domain.

Time-periodic evolutionary PDE

$$\begin{cases} \mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\ \mathbf{q}(t+T) = \mathbf{q}(t) \end{cases}$$



A dichotomy process until convergence . . .

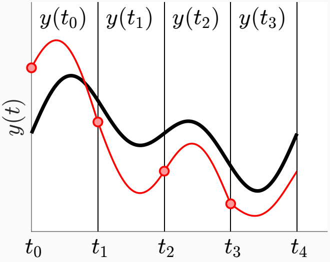
Motivation (IV) – Computing (unstable) periodic solutions

Methods for the computation of the periodic solution

- **Explicit**
  - Time marching
  - Shooting
  - Multi-shooting
- **Implicit**
  - Collocation, e.g. finite differences, orthogonal collocation
  - Spectral: Unknowns in the frequency or time domain

Time-periodic evolutionary PDE

$$\begin{cases} \mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\ \mathbf{q}(t+T) = \mathbf{q}(t) \end{cases}$$



Divide and conquer strategy. The horizon interval is divided in sub-intervals.

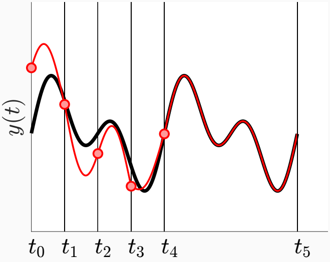
Motivation (IV) – Computing (unstable) periodic solutions

Methods for the computation of the periodic solution

- **Explicit**
    - Time marching
    - Shooting
    - Multi-shooting . It requires to stabilize the system if *unstable*
- **Implicit**
    - Collocation, e.g. finite differences, orthogonal collocation
    - Spectral. Unknowns in the *frequency* or *time* domain.

Time-periodic evolutionary PDE

$$\begin{cases} \mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\ \mathbf{q}(t+T) = \mathbf{q}(t) \end{cases}$$



Divide and conquer strategy. The horizon interval is divided in sub-intervals.

Motivation (IV) – Computing (unstable) periodic solutions

Methods for the computation of the periodic solution

- **Explicit**
  - Time marching
  - Shooting
  - Multi-shooting
- **Implicit**
  - Collocation, e.g. finite differences, orthogonal collocation
  - Spectral. Unknowns in the *frequency* or *time* domain.

Time-periodic evolutionary PDE

$$\begin{cases} \mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\ \mathbf{q}(t+T) = \mathbf{q}(t) \end{cases}$$

Subdivide the interval $[t_0, t_0 + T_0]$ into N intervals:
$t_0 < \tau_1 < \tau_2 < \cdots < \tau_N = t_0 + T_0$
Solve the system:

$$\begin{cases} \mathbf{M}\big(\mathbf{q}^{j+1} - \mathbf{q}^{j-1}\big) - \Delta\tau_j T_0 \mathbf{F}(\tilde{\mathbf{q}}^j) = 0, \\ \mathbf{q}^N - \mathbf{q}^0 = 0, \\ \Psi\big[\mathbf{q}^0, \mathbf{q}^1, \ldots, \mathbf{q}^N\big] = 0 \end{cases}$$

$\Delta\tau_j = (\tau_{j+1} - \tau_{j-1})$ and
$\tilde{\mathbf{q}}^j = \frac{1}{2}\big(\mathbf{q}^{j+1} + \mathbf{q}^j\big)$

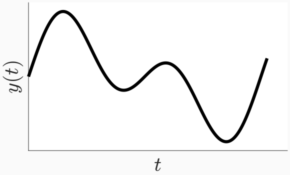
Motivation (IV) – Computing (unstable) periodic solutions

Methods for the computation of the periodic solution

- **Explicit**
  - Time marching
  - Shooting
  - Multi-shooting
- **Implicit**
  - Collocation, e.g. finite differences, orthogonal collocation
  - Spectral. Unknowns in the *frequency* or *time* domain.

Time-periodic evolutionary PDE

$$\begin{cases} \mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\ \mathbf{q}(t+T) = \mathbf{q}(t) \end{cases}$$



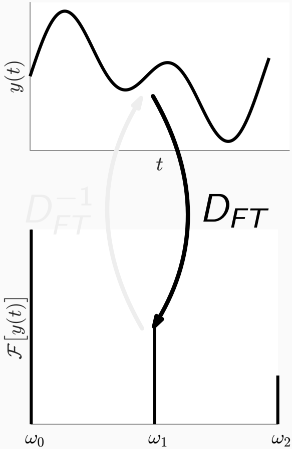
Motivation (IV) – Computing (unstable) periodic solutions

Methods for the computation of the periodic solution

- **Explicit**
  - Time marching
  - Shooting
  - Multi-shooting
- **Implicit**
  - Collocation, e.g. finite differences, orthogonal collocation
  - Spectral. Unknowns in the *frequency* or *time* domain.

Time-periodic evolutionary PDE

$$\begin{cases} \mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\ \mathbf{q}(t+T) = \mathbf{q}(t) \end{cases}$$



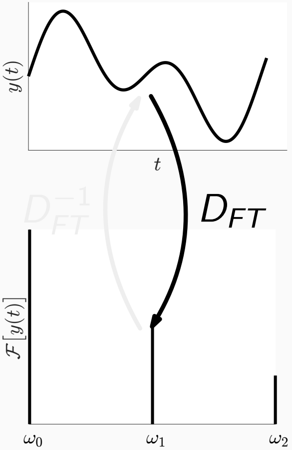
Motivation (IV) – Computing (unstable) periodic solutions

Methods for the computation of the periodic solution

- **Explicit**
  - Time marching
  - Shooting
  - Multi-shooting
- **Implicit**
  - Collocation, e.g. finite differences, orthogonal collocation
  - Spectral. Unknowns in the *frequency* or *time* domain.

Time-periodic evolutionary PDE

$$\begin{cases} \mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\ \mathbf{q}(t+T) = \mathbf{q}(t) \end{cases}$$



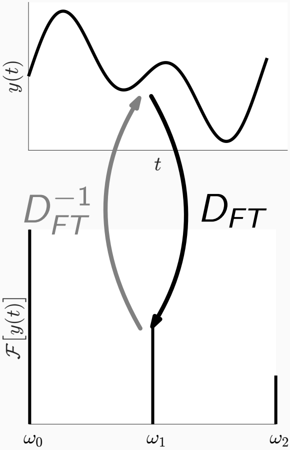
Motivation (IV) – Computing (unstable) periodic solutions

Methods for the computation of the periodic solution

- **Explicit**
  - Time marching
  - Shooting
  - Multi-shooting
- **Implicit**
  - Collocation, e.g. finite differences, orthogonal collocation
  - Spectral. Unknowns in the *frequency* or *time* domain.

Time-periodic evolutionary PDE

$$\begin{cases} \mathbf{M}\dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{F}(\mathbf{q}, \nu) \\ \mathbf{q}(t+T) = \mathbf{q}(t) \end{cases}$$



Motivation (V) – Once computed, can we controlled it?

- Add a perturbation term into the governing equations, e.g. passive control. [4]

$$\begin{cases} \mathbf{M}\dfrac{\partial(\mathbf{q}+\delta\mathbf{q})}{\partial t} = \mathbf{F}(\mathbf{q}+\delta\mathbf{q},\nu) + \delta\mathbf{H}(\mathbf{q}+\delta\mathbf{q}) \\ \mathbf{q}(t+T) = \mathbf{q}(t). \end{cases}$$

- If $\delta\mathbf{q} = \epsilon\hat{\mathbf{q}}e^{\sigma t}$, with $0 < \mathrm{Re}(\sigma)$. The control is determined by
  - *Structural sensitivity* $\bar{\mathsf{S}}_s$, i.e. sensitivity to perturbations whose Floquet exponent is λ.
  - *Baseflow sensitivity* $\bar{\mathsf{S}}_b$, i.e. sensitivity to variations of the periodic orbit itself.
  - *Frequency sensitivity* $\bar{\mathsf{S}}_\omega$, i.e. sensitivity to variations of the frequency ω (period T).
- Unstable solutions far from equilibrium can be also stabilized, but that is another story ..., e.g. shadowing sensitivity from UPOs [5]

[4] (Giannetti et al. 2019).
[5] (Lasagna et al. 2019).

Motivation (V) – Once computed, can we controlled it?

- Add a perturbation term into the governing equations, e.g. passive control. [4]

$$\begin{cases} \mathbf{M}\dfrac{\partial(\mathbf{q}+\delta\mathbf{q})}{\partial t} = \mathbf{F}(\mathbf{q}+\delta\mathbf{q},\nu) + \delta\mathbf{H}(\mathbf{q}+\delta\mathbf{q}) \\ \mathbf{q}(t+T) = \mathbf{q}(t). \end{cases}$$

- If $\delta\mathbf{q} = \epsilon\hat{\mathbf{q}}e^{\sigma t}$, with $0 < \mathrm{Re}(\sigma)$. The control is determined by
  - *Structural sensitivity* $\bar{\mathsf{S}}_s$, i.e. sensitivity to perturbations whose Floquet exponent is λ.
  - *Baseflow sensitivity* $\bar{\mathsf{S}}_b$, i.e. sensitivity to variations of the periodic orbit itself.
  - *Frequency sensitivity* $\bar{\mathsf{S}}_\omega$, i.e. sensitivity to variations of the frequency ω (period T).
- Unstable solutions far from equilibrium can be also stabilized, but that is another story . . . , e.g. shadowing sensitivity from UPOs [5]

[4](Giannetti et al. 2019).
[5](Lasagna et al. 2019).

Motivation (V) – Once computed, can we controlled it?

- Add a perturbation term into the governing equations, e.g. passive control. [4]

$$\begin{cases} \mathbf{M}\dfrac{\partial(\mathbf{q}+\delta\mathbf{q})}{\partial t} = \mathbf{F}(\mathbf{q}+\delta\mathbf{q},\nu) + \delta\mathbf{H}(\mathbf{q}+\delta\mathbf{q}) \\ \mathbf{q}(t+T) = \mathbf{q}(t). \end{cases}$$

- If $\delta\mathbf{q} = \epsilon\hat{\mathbf{q}}e^{\sigma t}$, with $0 < \mathrm{Re}(\sigma)$. The control is determined by
    - *Structural sensitivity* $\bar{\mathsf{S}}_s$, i.e. sensitivity to perturbations whose Floquet exponent is λ.
    - *Baseflow sensitivity* $\bar{\mathsf{S}}_b$, i.e. sensitivity to variations of the periodic orbit itself.
    - *Frequency sensitivity* $\bar{\mathsf{S}}_\omega$, i.e. sensitivity to variations of the frequency ω (period T).
- Unstable solutions far from equilibrium can be also stabilized, but that is another story . . . , e.g. shadowing sensitivity from UPOs [5]

[4] (Giannetti et al. 2019).
[5] (Lasagna et al. 2019).

Overview

1. Motivation
2. Computation of periodic solutions with spectral methods
3. Stability of periodic solutions
4. Some insights on sensitivity of periodic solutions
5. A FreeFem example
6. Conclusions

Computation of periodic solutions with spectral methods

Spectral methods for limit cycles (I)

The residual is defined as

$$\begin{cases} \mathbf{R} : X \times X \to X, \quad \text{with } X \text{ a suitable Hilbert space,} \\ \mathbf{R}(\mathbf{q}, \frac{\partial \mathbf{q}}{\partial t}) = \mathbf{M}\frac{\partial \mathbf{q}}{\partial t} - \mathbf{F}(\mathbf{q}). \end{cases}$$

And the solution vector $\mathbf{q}$ is projected onto a periodic basis,

$$\begin{aligned} &\pi_N : X \times \mathbb{R} \to X \times (\mathbb{Z}/(2N+1)\mathbb{Z}) \\ &\pi_N(\mathbf{q}) = \mathbf{q}_h(t) = \mathbf{q}_0 + \sum_{n=1}^{N} \left[\mathbf{q}_{c,n}\cos(n\omega t) + \mathbf{q}_{s,n}\sin(n\omega t)\right] \\ &= \underbrace{\left[\mathbf{q}_0, \mathbf{q}_{1,c}, \mathbf{q}_{1,s}, ..., \mathbf{q}_{N,c}, \mathbf{q}_{N,s}\right]}_{(\check{\mathbf{Q}}^{(\tau,N)})^T} \underbrace{\left[1, \cos(\omega t), \sin(\omega t), ..., \cos(N\omega t), \sin(N\omega t)\right]}_{\mathcal{F}_\mathbf{N}}, \end{aligned} \tag{2}$$

A key difference with *collocation methods* is the fact that periodicity is imposed to the basis and not as a constraint.

Spectral methods for limit cycles (II)

Subdivide the interval $[t_0, t_0 + T_0]$ into M intervals with $\tau_m = \frac{2\pi m}{M}$ for $m = 1, \dots M-1$, $t_0 = \tau_1 < \tau_2 < \dots < \tau_M < t_0 + T_0$

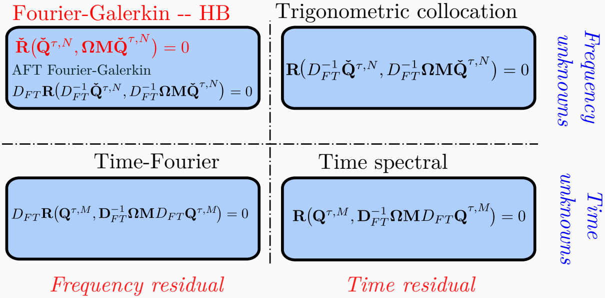
$$\check{\mathbf{R}}^{(\tau,N)} \approx \frac{2}{M} \underbrace{\begin{pmatrix} \frac{1}{2} & \dots & \frac{1}{2} \\ \cos(\frac{2\pi(1)(0)}{M}) & \dots & \cos(\frac{2\pi(1)(M-1)}{M}) \\ \vdots & & \vdots \\ \sin(\frac{2\pi(N)(0)}{M}) & \dots & \sin(\frac{2\pi(N)(M-1)}{M}) \end{pmatrix}}_{D_{FT}} \otimes \mathbf{I} \begin{pmatrix} \mathbf{R}|t = \tau_1 \\ \vdots \\ \mathbf{R}|t = \tau_{M-1} \end{pmatrix}$$

$$\mathbf{R}^{(\tau,M)} \approx \underbrace{\begin{pmatrix} 1 & \cos(\frac{2\pi(1)(0)}{M}) & \dots & \sin(\frac{2\pi(0)(N)}{M}) \\ \vdots & \vdots & & \vdots \\ 1 & \cos(\frac{2\pi(M-1)(1)}{M}) & \dots & \sin(\frac{2\pi(M-1)(N)}{M}) \end{pmatrix}}_{D_{FT}^{-1}} \otimes \mathbf{I} \begin{pmatrix} \check{\mathbf{R}}_0 \\ \check{\mathbf{R}}_{1,c} \\ \vdots \\ \check{\mathbf{R}}_{N,s} \end{pmatrix}$$

Also note that the time derivative is easy to evaluate in the frequency domain:

$$
\begin{aligned}
&\mathbf{M}\frac{\partial \mathbf{q}}{\partial t} \approx \mathbf{M}\frac{\partial \pi_N(\mathbf{q})}{\partial t} \\
&= \sum_{n=1}^{N} n\omega \mathbf{M}\left[-\mathbf{q}_{c,n}\sin(n\omega t) + \mathbf{q}_{s,n}\cos(n\omega t)\right] \\
&= \mathbf{\Omega}\mathbf{M}\check{\mathbf{Q}}^{(\tau,N)}
\end{aligned}
$$

Four kinds of spectral methods



Also note that the time derivative is easy to evaluate in the frequency domain:

$$\begin{aligned}
&\mathbf{M}\frac{\partial \mathbf{q}}{\partial t} \approx \mathbf{M}\frac{\partial \pi_N(\mathbf{q})}{\partial t} \\
&= \sum_{n=1}^{N} n\omega \mathbf{M}\left[-\mathbf{q}_{c,n}\sin(n\omega t) + \mathbf{q}_{s,n}\cos(n\omega t)\right] \\
&= \mathbf{\Omega}\mathbf{M}\check{\mathbf{Q}}^{(\tau,N)}
\end{aligned}$$

Four kinds of spectral methods

Fourier-Galerkin -- HB

$\check{\mathbf{R}}\left(\check{\mathbf{Q}}^{\tau,N}, \mathbf{\Omega}\mathbf{M}\check{\mathbf{Q}}^{\tau,N}\right) = 0$

AFT Fourier-Galerkin

$D_{FT}\mathbf{R}\left(D_{FT}^{-1}\check{\mathbf{Q}}^{\tau,N}, D_{FT}^{-1}\mathbf{\Omega}\mathbf{M}\check{\mathbf{Q}}^{\tau,N}\right) = 0$

Trigonometric collocation

$\mathbf{R}\left(D_{FT}^{-1}\check{\mathbf{Q}}^{\tau,N}, D_{FT}^{-1}\mathbf{\Omega}\mathbf{M}\check{\mathbf{Q}}^{\tau,N}\right) = 0$

Frequency unknowns

Time-Fourier

$D_{FT}\mathbf{R}\left(\mathbf{Q}^{\tau,M}, \mathbf{D}_{FT}^{-1}\mathbf{\Omega}\mathbf{M}D_{FT}\mathbf{Q}^{\tau,M}\right) = 0$

Time spectral

$\mathbf{R}\left(\mathbf{Q}^{\tau,M}, \mathbf{D}_{FT}^{-1}\mathbf{\Omega}\mathbf{M}D_{FT}\mathbf{Q}^{\tau,M}\right) = 0$

Time unknowns

Frequency residual | *Time residual*

Also note that the time derivative is easy to evaluate in the frequency domain:

$$
\begin{aligned}
&\mathbf{M}\frac{\partial \mathbf{q}}{\partial t} \approx \mathbf{M}\frac{\partial \pi_N(\mathbf{q})}{\partial t} \\
&= \sum_{n=1}^{N} n\omega \mathbf{M}\left[-\mathbf{q}_{c,n}\sin(n\omega t) + \mathbf{q}_{s,n}\cos(n\omega t)\right] \\
&= \mathbf{\Omega}\mathbf{M}\check{\mathbf{Q}}^{(\tau,N)}
\end{aligned}
$$

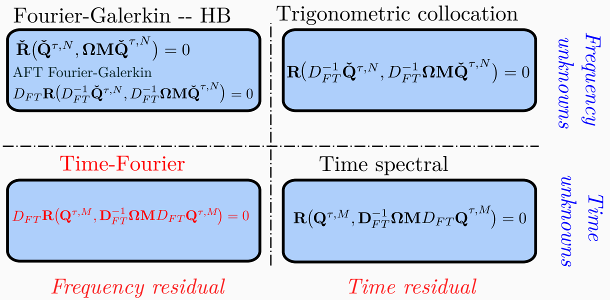
Four kinds of spectral methods

	Frequency residual	*Time residual*
Frequency unknowns	Fourier-Galerkin -- HB $\check{\mathbf{R}}\left(\check{\mathbf{Q}}^{\tau,N}, \mathbf{\Omega}\mathbf{M}\check{\mathbf{Q}}^{\tau,N}\right) = 0$ AFT Fourier-Galerkin $D_{FT}\mathbf{R}\left(D_{FT}^{-1}\check{\mathbf{Q}}^{\tau,N}, D_{FT}^{-1}\mathbf{\Omega}\mathbf{M}\check{\mathbf{Q}}^{\tau,N}\right) = 0$	Trigonometric collocation $\mathbf{R}\left(D_{FT}^{-1}\check{\mathbf{Q}}^{\tau,N}, D_{FT}^{-1}\mathbf{\Omega}\mathbf{M}\check{\mathbf{Q}}^{\tau,N}\right) = 0$
Time unknowns	Time-Fourier $D_{FT}\mathbf{R}\left(\mathbf{Q}^{\tau,M}, \mathbf{D}_{FT}^{-1}\mathbf{\Omega}\mathbf{M}D_{FT}\mathbf{Q}^{\tau,M}\right) = 0$	Time spectral $\mathbf{R}\left(\mathbf{Q}^{\tau,M}, \mathbf{D}_{FT}^{-1}\mathbf{\Omega}\mathbf{M}D_{FT}\mathbf{Q}^{\tau,M}\right) = 0$

Also note that the time derivative is easy to evaluate in the frequency domain:

$$
\begin{aligned}
&\mathbf{M}\frac{\partial \mathbf{q}}{\partial t} \approx \mathbf{M}\frac{\partial \pi_N(\mathbf{q})}{\partial t} \\
&= \sum_{n=1}^{N} n\omega \mathbf{M}\left[-\mathbf{q}_{c,n}\sin(n\omega t) + \mathbf{q}_{s,n}\cos(n\omega t)\right] \\
&= \mathbf{\Omega}\mathbf{M}\check{\mathbf{Q}}^{(\tau,N)}
\end{aligned}
$$

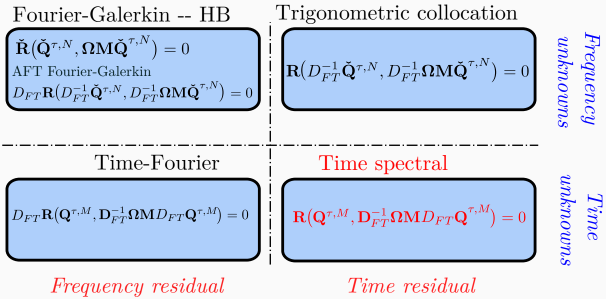
Four kinds of spectral methods



Also note that the time derivative is easy to evaluate in the frequency domain:

$$
\begin{aligned}
&\mathbf{M}\frac{\partial \mathbf{q}}{\partial t} \approx \mathbf{M}\frac{\partial \pi_N(\mathbf{q})}{\partial t} \\
&= \sum_{n=1}^{N} n\omega \mathbf{M}\left[-\mathbf{q}_{c,n}\sin(n\omega t) + \mathbf{q}_{s,n}\cos(n\omega t)\right] \\
&= \boldsymbol{\Omega}\mathbf{M}\check{\mathbf{Q}}^{(\tau,N)}
\end{aligned}
$$

Four kinds of spectral methods



Four kinds of spectral methods

- Free of aliasing ✓
- Low CPU time ✓
- Intensive memory consumption ✗
- Evaluation of analytic residual ⚙

Fourier-Galerkin -- HB	Trigonometric collocation	
$\check{\mathbf{R}}\big(\check{\mathbf{Q}}^{\tau,N}, \boldsymbol{\Omega}\mathbf{M}\check{\mathbf{Q}}^{\tau,N}\big) = 0$ AFT Fourier-Galerkin $D_{FT}\mathbf{R}\big(D_{FT}^{-1}\check{\mathbf{Q}}^{\tau,N}, D_{FT}^{-1}\boldsymbol{\Omega}\mathbf{M}\check{\mathbf{Q}}^{\tau,N}\big) = 0$	$\mathbf{R}\big(D_{FT}^{-1}\check{\mathbf{Q}}^{\tau,N}, D_{FT}^{-1}\boldsymbol{\Omega}\mathbf{M}\check{\mathbf{Q}}^{\tau,N}\big) = 0$	*Frequency unknowns*
Time-Fourier	**Time spectral**	
$D_{FT}\mathbf{R}\big(\mathbf{Q}^{\tau,M}, \mathbf{D}_{FT}^{-1}\boldsymbol{\Omega}\mathbf{M}D_{FT}\mathbf{Q}^{\tau,M}\big) = 0$	$\mathbf{R}\big(\mathbf{Q}^{\tau,M}, \mathbf{D}_{FT}^{-1}\boldsymbol{\Omega}\mathbf{M}D_{FT}\mathbf{Q}^{\tau,M}\big) = 0$	*Time unknowns*
Frequency residual	*Time residual*	

- Inherent aliasing ✗
- Low CPU time ✓
- Intensive memory consumption ✗
- Evaluation of analytical time-derivative ✓

It can be reduced increasing the number of snapshots M, but higher memory consumption ...

Spectral methods for limit cycles (III)

Fourier–Galerkin or Harmonic–Balance is a Galerkin method:

$$\begin{aligned}
&\check{\mathbf{R}} : X \times (\mathbb{Z}/(2N+1)\mathbb{Z}) \to X \times (\mathbb{Z}/(2N+1)\mathbb{Z}) \\
&\check{\mathbf{R}}(\check{\mathbf{Q}}^{(\tau,N)}, \omega) = \int_0^{\frac{2\pi}{\omega}} \check{\mathbf{R}}(\mathbf{q}_h, \tfrac{\partial \mathbf{q}_h}{\partial t}, t)^T \mathcal{F}_N \, dt = \mathbf{0},
\end{aligned} \tag{3}$$

[6] Fourier coefficients of the non-linear operator detailed in (Sierra et al. 2021).

Spectral methods for limit cycles (III)

Fourier–Galerkin or Harmonic–Balance is a Galerkin method:

$$\begin{aligned} &\check{\mathbf{R}} : X \times (\mathbb{Z}/(2N+1)\mathbb{Z}) \to X \times (\mathbb{Z}/(2N+1)\mathbb{Z}) \\ &\check{\mathbf{R}}(\check{\mathbf{Q}}^{(\tau,N)}, \omega) = \int_0^{\frac{2\pi}{\omega}} \check{\mathbf{R}}(\mathbf{q}_h, \tfrac{\partial \mathbf{q}_h}{\partial t}, t)^T \mathcal{F}_N \, dt = \mathbf{0}, \end{aligned} \tag{3}$$

if $\mathbf{F}(\mathbf{q}) = \mathbf{L}\mathbf{q} + \mathbf{N}(\mathbf{q})$ where $\mathbf{L}$ is a linear operator and $\mathbf{N}$ is a non-linear operator, then *Harmonic–Balance equations* ($\check{\mathbf{R}}(\check{\mathbf{Q}}^{(\tau,N)}, \omega) = 0$) [6]

$$\begin{aligned} 0 &= \mathbf{L}\mathbf{q}_0 + \mathbf{N}_0 \\ n\omega \mathbf{M}\mathbf{q}_{n,s} &= \mathbf{L}\mathbf{q}_{n,c} + \mathbf{N}_{n,c}, \quad n = 1, \ldots, N \\ -n\omega \mathbf{M}\mathbf{q}_{n,c} &= \mathbf{L}\mathbf{q}_{n,s} + \mathbf{N}_{n,s}, \quad n = 1, \ldots, N \\ \Psi(\mathbf{q}_h) &= 0. \end{aligned} \tag{4}$$

[6] Fourier coefficients of the non-linear operator detailed in (Sierra et al. 2021).

Spectral methods for limit cycles (III)

Fourier–Galerkin or Harmonic–Balance is a Galerkin method:

$$\begin{aligned}&\check{\mathbf{R}} : X \times (\mathbb{Z}/(2N+1)\mathbb{Z}) \to X \times (\mathbb{Z}/(2N+1)\mathbb{Z}) \\ &\check{\mathbf{R}}(\check{\mathbf{Q}}^{(\tau,N)}, \omega) = \int_0^{\frac{2\pi}{\omega}} \check{\mathbf{R}}(\mathbf{q}_h, \tfrac{\partial \mathbf{q}_h}{\partial t}, t)^T \mathcal{F}_N \, dt = \mathbf{0},\end{aligned} \tag{3}$$

if $\mathbf{F}(\mathbf{q}) = \mathbf{L}\mathbf{q} + \mathbf{N}(\mathbf{q})$ where $\mathbf{L}$ is a linear operator and $\mathbf{N}$ is a non-linear operator, then *Harmonic–Balance equations* ($\check{\mathbf{R}}(\check{\mathbf{Q}}^{(\tau,N)}, \omega) = 0$) [6]

$$\begin{aligned} 0 &= \mathbf{L}\mathbf{q}_0 + \mathbf{N}_0 \\ n\omega \mathbf{M}\mathbf{q}_{n,s} &= \mathbf{L}\mathbf{q}_{n,c} + \mathbf{N}_{n,c}, \quad n = 1, \dots, N \\ -n\omega \mathbf{M}\mathbf{q}_{n,c} &= \mathbf{L}\mathbf{q}_{n,s} + \mathbf{N}_{n,s}, \quad n = 1, \dots, N \\ \Psi(\mathbf{q}_h) &= 0. \end{aligned} \tag{4}$$

Resolution of an implicit non-linear problem $\longrightarrow$ *Newton–Krylov* method

[6] Fourier coefficients of the non-linear operator detailed in (Sierra et al. 2021).

Spectral methods for limit cycles (IV)

Newton–Krylov iteration

At each step of the procedure, starting from an estimate $\left[\check{\mathbf{Q}}_n^{(\tau,N)}, \omega_n\right]^T$ of the solution, we look for an improved estimate defined as $\check{\mathbf{Q}}_{n+1}^{(\tau,N)} = \check{\mathbf{Q}}_n^{(\tau,N)} + \delta\check{\mathbf{Q}}_n^{(\tau,N)}$, $\omega_{n+1} = \omega_n + \delta\omega_n$, from the linear system

$$D\check{\mathbf{R}}\left[\delta\check{\mathbf{Q}}_n^{(\tau,N)}, \delta\omega\right]^T = -\left[\check{\mathbf{R}}(\check{\mathbf{Q}}_n^{(\tau,N)}), \Psi(\check{\mathbf{Q}}_n^{(\tau,N)})\right]^T$$

Solved with a Krylov-method + preconditioning,

$$\mathbf{P}^{-1}D\check{\mathbf{R}}\left[\delta\check{\mathbf{Q}}_n^{(\tau,N)}, \delta\omega\right]^T = -\mathbf{P}^{-1}\left[\check{\mathbf{R}}(\check{\mathbf{Q}}_n^{(\tau,N)}), \Psi(\check{\mathbf{Q}}_n^{(\tau,N)})\right]^T$$

Choices for $\mathbf{P}$: Block Jacobi, Gauss Seidel or block–circulant (*not explored yet*). Inner blocks solved by a direct method or preconditioning again, e.g. ASM, Augmented Lagrangian[7]

[7] Moulin et al. (2019)

Spectral methods for limit cycles (IV)

Newton–Krylov iteration

At each step of the procedure, starting from an estimate $\left[\check{\mathbf{Q}}_n^{(\tau,N)}, \omega_n\right]^T$ of the solution, we look for an improved estimate defined as $\check{\mathbf{Q}}_{n+1}^{(\tau,N)} = \check{\mathbf{Q}}_n^{(\tau,N)} + \delta\check{\mathbf{Q}}_n^{(\tau,N)}$, $\omega_{n+1} = \omega_n + \delta\omega_n$, from the linear system

$$D\check{\mathbf{R}}\left[\delta\check{\mathbf{Q}}_n^{(\tau,N)}, \delta\omega\right]^T = -\left[\check{\mathbf{R}}(\check{\mathbf{Q}}_n^{(\tau,N)}), \Psi(\check{\mathbf{Q}}_n^{(\tau,N)})\right]^T$$

Solved with a Krylov-method + preconditioning,

$$\mathbf{P}^{-1}D\check{\mathbf{R}}\left[\delta\check{\mathbf{Q}}_n^{(\tau,N)}, \delta\omega\right]^T = -\mathbf{P}^{-1}\left[\check{\mathbf{R}}(\check{\mathbf{Q}}_n^{(\tau,N)}), \Psi(\check{\mathbf{Q}}_n^{(\tau,N)})\right]^T$$

Choices for $\mathbf{P}$: Block Jacobi, Gauss Seidel or block–circulant (*not explored yet*). Inner blocks solved by a direct method or preconditioning again, e.g. ASM, Augmented Lagrangian[7]

[7] Moulin et al. (2019)

Spectral methods for limit cycles (IV)

Newton–Krylov iteration

At each step of the procedure, starting from an estimate $\left[\check{\mathbf{Q}}_n^{(\tau,N)}, \omega_n\right]^T$ of the solution, we look for an improved estimate defined as $\check{\mathbf{Q}}_{n+1}^{(\tau,N)} = \check{\mathbf{Q}}_n^{(\tau,N)} + \delta\check{\mathbf{Q}}_n^{(\tau,N)}$, $\omega_{n+1} = \omega_n + \delta\omega_n$, from the linear system

$$D\check{\mathbf{R}}\left[\delta\check{\mathbf{Q}}_n^{(\tau,N)}, \delta\omega\right]^T = -\left[\check{\mathbf{R}}(\check{\mathbf{Q}}_n^{(\tau,N)}), \Psi(\check{\mathbf{Q}}_n^{(\tau,N)})\right]^T$$

Solved with a Krylov-method + preconditioning,

$$\mathbf{P}^{-1}D\check{\mathbf{R}}\left[\delta\check{\mathbf{Q}}_n^{(\tau,N)}, \delta\omega\right]^T = -\mathbf{P}^{-1}\left[\check{\mathbf{R}}(\check{\mathbf{Q}}_n^{(\tau,N)}), \Psi(\check{\mathbf{Q}}_n^{(\tau,N)})\right]^T$$

Choices for $\mathbf{P}$: Block Jacobi, Gauss Seidel or block–circulant (*not explored yet*). Inner blocks solved by a direct method or preconditioning again, e.g. ASM, Augmented Lagrangian[7]

[7] Moulin et al. (2019)

Stability of periodic solutions

Some insights in stability of limit cycles

Consider the periodic solution $\mathbf{q}^*(t+T) = \mathbf{q}^*(t)$,
Add a small perturbation $\delta\mathbf{q}(t)$
What is the response of the system? *stable* or *unstable*?

Answer: Floquet theory,

$$\delta\mathbf{q}(t) = \sum_{n=1}^{N} c_n \delta\mathbf{q}_n(t) \text{ where } \delta\mathbf{q}_n(t) = e^{\lambda_n t}\mathbf{v}_n(t),$$

here $\mathbf{v}_n$ is a T-periodic vector and λ_n are called the *Floquet exponents*.

Thus, the stability of the system is determined by the following $T-$periodic linear problem,

$$\lambda_n \mathbf{M}\mathbf{v}_n = \Big[-\mathbf{M}\frac{\partial}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{q}^*, \lambda)}{\partial \mathbf{q}} \Big] \mathbf{v}_n.$$

and the periodic solution $\mathbf{q}$ is *stable* if $\mathrm{Re}(\lambda_n) < 0$ and *unstable* if $\mathrm{Re}(\lambda_n) > 0$.

Some insights in stability of limit cycles

Consider the periodic solution $\mathbf{q}^*(t+T) = \mathbf{q}^*(t)$,
Add a small perturbation $\delta\mathbf{q}(t)$
What is the response of the system? *stable* or *unstable*?

Answer: Floquet theory,

$$\delta\mathbf{q}(t) = \textstyle\sum_{n=1}^{N} c_n \delta\mathbf{q}_n(t) \text{ where } \delta\mathbf{q}_n(t) = e^{\lambda_n t}\mathbf{v}_n(t),$$

here $\mathbf{v}_n$ is a T-periodic vector and λ_n are called the *Floquet exponents*.

Thus, the stability of the system is determined by the following $T-$periodic linear problem,

$$\lambda_n \mathbf{M}\mathbf{v}_n = \Big[-\mathbf{M}\frac{\partial}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{q}^*, \lambda)}{\partial \mathbf{q}} \Big]\mathbf{v}_n.$$

and the periodic solution $\mathbf{q}$ is *stable* if $\mathrm{Re}(\lambda_n) < 0$ and *unstable* if $\mathrm{Re}(\lambda_n) > 0$.

Some insights in stability of limit cycles

Consider the periodic solution $\mathbf{q}^*(t+T) = \mathbf{q}^*(t)$,
Add a small perturbation $\delta\mathbf{q}(t)$
What is the response of the system? *stable* or *unstable*?

Answer: Floquet theory,

$$\delta\mathbf{q}(t) = \sum_{n=1}^{N} c_n \delta\mathbf{q}_n(t) \text{ where } \delta\mathbf{q}_n(t) = e^{\lambda_n t}\mathbf{v}_n(t),$$

here $\mathbf{v}_n$ is a T-periodic vector and λ_n are called the *Floquet exponents*.

Thus, the stability of the system is determined by the following $T-$periodic linear problem,

$$\lambda_n \mathbf{M}\mathbf{v}_n = \Big[-\mathbf{M}\frac{\partial}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{q}^*, \lambda)}{\partial \mathbf{q}}\Big]\mathbf{v}_n.$$

and the periodic solution $\mathbf{q}$ is *stable* if $\mathrm{Re}(\lambda_n) < 0$ and *unstable* if $\mathrm{Re}(\lambda_n) > 0$.

Stability of limit cycles with Harmonic Balance

Floquet exponents λ_n and *Floquet vectors* $\mathbf{v}_n$ are determined by solving a generalized eigenvalue problem:

$$\begin{aligned}
\lambda \mathrm{diag}\big(\mathbf{M}_{2N+1}\big)\mathbf{V}^{(\tau,N)} &= D\check{\mathbf{R}}(\mathbf{Q}^{(\tau,N)})\mathbf{V}^{(\tau,N)} \\
\text{where } D\check{\mathbf{R}}(\mathbf{Q}^{(\tau,N)})\mathbf{V}^{(\tau,N)} &= \Big[-\omega\tilde{\mathbf{M}} + \tilde{\mathbf{L}} + D\tilde{\mathbf{N}}(\mathbf{Q}^{(\tau,N)})\Big]\mathbf{V}^{(\tau,N)},
\end{aligned}$$

here $\mathbf{V}^{(\tau,N)} = \big[\mathbf{v}_0, \mathbf{v}_{1,c}, \mathbf{v}_{1,s}, ..., \mathbf{v}_{N,c}, \mathbf{v}_{N,s}\big]^T$.

The same preconditioning techniques are adopted.

Stability of limit cycles with Harmonic Balance

Floquet exponents λ_n and *Floquet vectors* $\mathbf{v}_n$ are determined by solving a generalized eigenvalue problem:

$$\begin{aligned} \lambda \mathrm{diag}\big(\mathbf{M}_{2N+1}\big)\mathbf{V}^{(\tau,N)} &= D\check{\mathbf{R}}(\mathbf{Q}^{(\tau,N)})\mathbf{V}^{(\tau,N)} \\ \text{where } D\check{\mathbf{R}}(\mathbf{Q}^{(\tau,N)})\mathbf{V}^{(\tau,N)} &= \Big[-\omega\tilde{\mathbf{M}} + \tilde{\mathbf{L}} + D\tilde{\mathbf{N}}(\mathbf{Q}^{(\tau,N)})\Big]\mathbf{V}^{(\tau,N)}, \end{aligned}$$

here $\mathbf{V}^{(\tau,N)} = \big[\mathbf{v}_0, \mathbf{v}_{1,c}, \mathbf{v}_{1,s}, ..., \mathbf{v}_{N,c}, \mathbf{v}_{N,s}\big]^T$.

The same preconditioning techniques are adopted.

Some insights on sensitivity of periodic solutions

Sensitivity of periodic solutions (I)

Consider the case where $|\text{Re}(\lambda)| \ll 1$, it is said that the system is near an *instability*. In these cases we may wonder if we can control the system

$$\mathbf{M}\frac{\partial(\mathbf{q}^* + \delta\mathbf{q})}{\partial t} = \mathbf{F}(\mathbf{q}^* + \delta\mathbf{q}, \nu) + \delta\mathbf{H}(\mathbf{q}^* + \delta\mathbf{q}), \quad \mathbf{q}^*(t+T) = \mathbf{q}^*(t).$$

by the introduction of a force-feedback perturbation $\delta\mathbf{H}(\mathbf{q}^* + \delta\mathbf{q})$ [8].

Consider a localised feedback $\mathbf{H}(\mathbf{q}^* + \delta\mathbf{q}) = \delta(\mathbf{x} - \mathbf{x}_0)\mathbf{C} \cdot \mathbf{B}\big(\mathbf{q}^* + \delta\mathbf{q}\big)$ where $\mathbf{B}$ is a input/output matrix and $\mathbf{C}$ is a gain matrix.

The variation of the *Floquet exponent* is determined by

$$\delta\lambda = \mathbf{C} : \mathrm{S}_{tot}$$

here $\mathrm{S}_{tot} = \mathrm{S}_b + \mathrm{S}_s$, the *baseflow* and *structural sensitivities.*

[8] (Giannetti et al. 2019).

Sensitivity of periodic solutions (I)

Consider the case where $|\mathrm{Re}(\lambda)| \ll 1$, it is said that the system is near an *instability*. In these cases we may wonder if we can control the system

$$\mathbf{M}\frac{\partial(\mathbf{q}^* + \delta\mathbf{q})}{\partial t} = \mathbf{F}(\mathbf{q}^* + \delta\mathbf{q}, \nu) + \delta\mathbf{H}(\mathbf{q}^* + \delta\mathbf{q}), \quad \mathbf{q}^*(t+T) = \mathbf{q}^*(t).$$

by the introduction of a force-feedback perturbation $\delta\mathbf{H}(\mathbf{q}^* + \delta\mathbf{q})$ [8].

Consider a localised feedback $\mathbf{H}(\mathbf{q}^* + \delta\mathbf{q}) = \delta(\mathbf{x} - \mathbf{x}_0)\mathbf{C} \cdot \mathbf{B}\big(\mathbf{q}^* + \delta\mathbf{q}\big)$ where $\mathbf{B}$ is a input/output matrix and $\mathbf{C}$ is a gain matrix.

The variation of the *Floquet exponent* is determined by

$$\delta\lambda = \mathbf{C} : \mathrm{S}_{tot}$$

here $\mathrm{S}_{tot} = \mathrm{S}_b + \mathrm{S}_s$, the *baseflow* and *structural sensitivities*.

[8](Giannetti et al. 2019).

Sensitivity of periodic solutions (I)

Consider the case where $|\text{Re}(\lambda)| \ll 1$, it is said that the system is near an *instability*. In these cases we may wonder if we can control the system

$$\mathbf{M}\frac{\partial(\mathbf{q}^* + \delta\mathbf{q})}{\partial t} = \mathbf{F}(\mathbf{q}^* + \delta\mathbf{q}, \nu) + \delta\mathbf{H}(\mathbf{q}^* + \delta\mathbf{q}), \quad \mathbf{q}^*(t+T) = \mathbf{q}^*(t).$$

by the introduction of a force-feedback perturbation $\delta\mathbf{H}(\mathbf{q}^* + \delta\mathbf{q})$ [8].

Consider a localised feedback $\mathbf{H}(\mathbf{q}^* + \delta\mathbf{q}) = \delta(\mathbf{x} - \mathbf{x}_0)\mathbf{C} \cdot \mathbf{B}\big(\mathbf{q}^* + \delta\mathbf{q}\big)$ where $\mathbf{B}$ is a input/output matrix and $\mathbf{C}$ is a gain matrix.

The variation of the *Floquet exponent* is determined by

$$\delta\lambda = \mathbf{C} : \mathsf{S}_{tot}$$

here $\mathsf{S}_{tot} = \mathsf{S}_b + \mathsf{S}_s$, the *baseflow* and *structural sensitivities*.

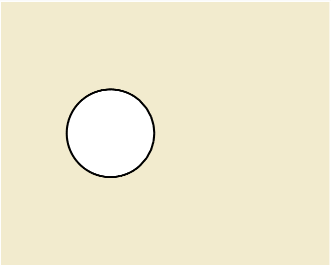
[8](Giannetti et al. 2019).

Sensitivity of periodic solutions (II)

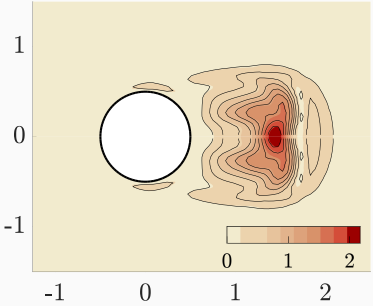
Sensitivity maps are determined from the periodic solution $\mathbf{q}$, direct and adjoint Floquet modes $\mathbf{v}$ and $\mathbf{v}^\dagger$,

$$\mathsf{S}_s(\mathbf{x}, t) = \frac{\mathbf{Bv} \otimes \mathbf{Bv}^\dagger}{\int_t^{T+t} \int_\Omega \mathbf{Bv} \cdot \mathbf{Bv}^\dagger \mathrm{d}\mathbf{x}\mathrm{d}t}, \quad \bar{\mathsf{S}}_s(\mathbf{x}) = \frac{\int_t^{T+t} \mathbf{Bv} \otimes \mathbf{Bv}^\dagger}{\int_t^{T+t} \int_\Omega \mathbf{Bv} \cdot \mathbf{Bv}^\dagger \mathrm{d}\mathbf{x}}.$$

Instantaneous structural sensitivity



Averaged structural sensitivity



A FreeFem example

Computation of periodic solutions with FreeFem(I)

Example: *Harmonic–Balance of Navier–Stokes equations with $N = 1$*

The governing equations are quadratic in the state variables $\mathbf{q} = [\mathbf{u}, p]$,

$$\mathbf{M}\frac{\partial \mathbf{q}}{\partial t} = \mathbf{L}\mathbf{q} + \mathbf{N}(\mathbf{q}, \mathbf{q})$$

with the linear operator

$$\mathbf{L}\mathbf{q} \equiv \begin{array}{l} \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\left[\mathbf{u}_0 \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}_0\right] + \nabla p - \nabla \cdot \tau(\mathbf{u}) \\ \nabla \cdot \mathbf{u}, \end{array}$$

and the non-linear quadratic operator

$$\mathbf{N}(\mathbf{q}_1, \mathbf{q}_2) \equiv \mathbf{u}_1 \cdot \nabla \mathbf{u}_2 + \mathbf{u}_1 \cdot \nabla \mathbf{u}_2$$

Thus, the governing HB equations are as follows,

$$\begin{array}{ll} 0 & = \mathbf{L}\mathbf{q}_0 + \frac{1}{4}\left[\mathbf{N}(\mathbf{q}_{1,c}, \mathbf{q}_{1,c}) + \mathbf{N}(\mathbf{q}_{1,s}, \mathbf{q}_{1,s})\right] \\ \omega \mathbf{M}\mathbf{q}_{1,s} & = \mathbf{L}\mathbf{q}_{1,c}, \\ -\omega \mathbf{M}\mathbf{q}_{1,c} & = \mathbf{L}\mathbf{q}_{1,s}, \\ \Psi(\mathbf{q}_h) & = 0. \end{array}$$

Computation of periodic solutions with FreeFem (II)

```
fespace XXMh(th,[P2,P2,P1]);
XXMh defQ(u0),defQ(u1c),defQ(u1s);
```

```
macro NSL(nu,u,v) (-2*nu*(D(u):D(v)) + u#p*div(v)
+ v#p*div(u) - .5*Conv(U,u,v) ) //EOM. Linear operator
macro F20(u,v,w) Conv(u,v,w) // EOM. Nonlinear operator
```

```
varf vDR0x0(defQ(du),defQ(v)) =
int2d(th)( NSL(nu,u0,du,v) ) + BC0(du,v);
varf vDR0x1c(defQ(du),defQ(v))=
int2d(th)( -0.5*F20(u1c,du,v) );
varf vDR0x1s(defU(du),defU(v))=
int2d(th)( -0.5*F20(u1s,du,v) );
```

```
matrix<real> ffDR0x0 = vDR0x0(XXMh,XXMh);
matrix<real> ffDR0x1c = vDR0x1c(XXMh,XXMh);
matrix<real> ffDR0x1s = vDR0x1s(XXMh,XXMh);
Mat<real> DR0x0(ffDR0x0,intersection,DDD, bs=1);
Mat<real> DR0x1; // MatNest
DR0x1 = [ [ffDR0x1c, ffDR0x1s, matNull] ];
```

```
varf vRHS0(defQ(du),defQ(v))= int2d(th)(NSL(nu,u0,u0,v))
+ BC0(du,v)+ int2d(th)(-0.25*F20(u1c,u1c,v)
-0.25*F20(u1s,u1s,v) );
real[int] RHS0(u0x[].n); RHS0 = vRHS0(0,XXMh);
real[int] RHS = [RHSF0, RHSF1c, RHSF1s, RHSphase];
```

```
KSPSolve(funcMatVec,RHS,dSol,precon=funcGS,
sparams = "-ksp_type gmres -ksp_monitor
-ksp_view_final_residual -ksp_converged_reason
-ksp_gmres_restart 1000 -pc_type shell");
```

```
func real[int] funcMatVec(real[int] & x)
{ int n = x.n;
real[int] u(n);u=0;
u(0:(n-1)) += DR0x0*x(0:(n-1));
u(0:(n-1)) += DR0x1* x(n:(3*n));
u(n:(3*n)) += DR1x0* x(0:(n-1));
u(n:(3*n)) += DR1x1*x(n:(3*n));
return u; }
```

```
func real[int] funcGS(real[int] & x)
{ int n = x.n;
real[int] u(x.n); u=0;
real[int] v(x.n); v=0;
real[int] w(x.n); w=0;
v(n:(3*n)) = DR1x1^-1*VecIN(n:(3*n));
u(n:(3*n)) += v(nDOF:(3*nDOF));
v(0:(n-1)) = DR0x1* u(nDOF:(3*nDOF));
w(0:(n-1)) = -v(0:(n-1));
w(0:(n-1)) += x(0:(n-1));
v(0:(n-1)) = DR0x0^-1*w(0:(n-1));
u(0:(n-1)) += v(0:(n-1));
return u; }
```

Stability of periodic solutions with FreeFem

```
string DRParams = "-ksp_type gmres -ksp_view_final_residual
-ksp_converged_reason -ksp_gmres_restart 1000 -pc_type shell ";
set(DR, sparams = DRParams, precon = funcGS);
```

```
ssparams =
" -eps_nev " + nEig /* Number of eigenvalues*/
+" -eps_type krylovschur " /* Type of Eigen Problem Solver */
+ " -eps_target " + shiftValue /* Shift value */
+" -eps_gen_non_hermitian " /* Not hermitian (A*!=A) by default */
+" -st_type sinvert " /* Spectral Transformation */
+" -st_ksp_type gmres " /* Iterative lin. solver */
+" -eps_monitor "
+" -eps_view ";
```

```
nComputed = EPSSolve
(DR, /* matrix OP = DR - shift M */
 M, /* M matrix */
array   = EigenVEC, /* Array to store the FEM-EigenFunctions */
values  = EigenVAL, /* Array to store the EigenValues */
sparams = ssparams  /* Parameters for the distributed EigenValue solver */
);
```

Conclusions

Conclusions and perspectives

- **Conclusions**
  - Set of *spectral* tools for the computation of periodic solutions and the determination of its *stability* and the *sensitivity* to force-feedback terms. Some tools available in *StabFem* [9].
  - This strategy was implemented in FreeFem using the linear algebra brackends PETSc/SLEPc [10].
  - Implemented for *incompressible* and *compressible* flows. Large speed-up in CPU time for *low Mach* flows (acoustics).
  - Large speed-up for the computation of *sensitivity* maps in comparison with classical time marching methods.
- **Perspectives**
  - Block Toeplitz preconditioning to boost GMRES convergence with high number of Fourier modes.
  - Extension to other flow configurations: three-dimensional flows in cylindrical coordinates, free-surface flows, fluid-structure . . .

[9] Sierra et al., Journal of Computational Physics, 2021 (under review)
[10] `https://stabfem.gitlab.io/StabFem/`

Questions?

Adjoint-based sensitivity analysis of periodic orbits by the Fourier–Galerkin method

J. Sierra[a,b], P. Jolivet[c], F. Giannetti[b], V. Citro[b]

[a]*Institut de Mécanique des fluides de Toulouse (IMFT), Toulouse 31400, France*
[b]*Dipartimento di Ingegneria (DIIN),Universitá degli Studi di Salerno,Fisciano 84084, Italy*
[c]*CNRS-IRIT, 2 rue Charles Camichel, 31071 Toulouse Cedex 7, France*

Efficient computation of periodic compressible flows with spectral techniques

Javier Sierra[a,b], Vincenzo Citro[a], Flavio Giannetti[a], David Fabre[b]

[a]*DIIN, Via Giovanni Paolo II, 132, 84084 Fisciano SA, Italy*
[b]*IMFT, 2 Allé du Professeur Camille Soula, 31400 Toulouse, France*

Quadratic non-linear Fourier coefficients

$$
\begin{aligned}
\mathbf{N}_0 = \quad & \mathbf{N}(\mathbf{q}_0, \mathbf{q}_0) + \tfrac{1}{2}\textstyle\sum_{i=1}^{N} \mathbf{N}(\mathbf{q}_{i,s}, \mathbf{q}_{i,s}) + \mathbf{N}(\mathbf{q}_{i,c}, \mathbf{q}_{i,c}) \\
\mathbf{N}_{i,c} = \quad & \left[\mathbf{N}(\mathbf{q}_{i,c}, \mathbf{q}_0) + \mathbf{N}(\mathbf{q}_0, \mathbf{q}_{i,c})\right] \\
& + \tfrac{1}{2}\textstyle\sum_{j=1}^{i-1} \left[\mathbf{N}(\mathbf{q}_{j,c}, \mathbf{q}_{i-j,c}) - \mathbf{N}(\mathbf{q}_{j,s}, \mathbf{q}_{i-j,s})\right] \\
& + \tfrac{1}{2}\textstyle\sum_{j=i+1}^{N} \left[\mathbf{N}(\mathbf{q}_{j,c}, \mathbf{q}_{j-i,c}) + \mathbf{N}(\mathbf{q}_{j-i,s}, \mathbf{q}_{j,s})\right] \\
& + \tfrac{1}{2}\textstyle\sum_{j=i+1}^{N} \left[\mathbf{N}(\mathbf{q}_{j-i,c}, \mathbf{q}_{j,c}) + \mathbf{N}(\mathbf{q}_{j,s}, \mathbf{q}_{j-i,s})\right] \\
\mathbf{N}_{i,s} = \quad & \left[\mathbf{N}(\mathbf{q}_{i,s}, \mathbf{q}_0) + \mathbf{N}(\mathbf{q}_0, \mathbf{q}_{i,s})\right] \\
& + \tfrac{1}{2}\textstyle\sum_{j=1}^{i-1} \left[\mathbf{N}(\mathbf{q}_{j,c}, \mathbf{q}_{i-j,s}) + \mathbf{N}(\mathbf{q}_{j,s}, \mathbf{q}_{i-j,c})\right] \\
& - \tfrac{1}{2}\textstyle\sum_{j=i+1}^{N} \left[\mathbf{N}(\mathbf{q}_{j,c}, \mathbf{q}_{j-i,s}) + \mathbf{N}(\mathbf{q}_{j-i,s}, \mathbf{q}_{j,c})\right] \\
& + \tfrac{1}{2}\textstyle\sum_{j=i+1}^{N} \left[\mathbf{N}(\mathbf{q}_{j-i,c}, \mathbf{q}_{j,s}) + \mathbf{N}(\mathbf{q}_{j,s}, \mathbf{q}_{j-i,c})\right].
\end{aligned}
$$

Matrices of Fourier–Galerkin

$$\tilde{\mathbf{B}}\mathbf{Q}_N = \begin{bmatrix} \mathbf{0} & & & \\ & \mathbf{B}_1 & & \\ & & \ddots & \\ & & & \mathbf{B}_N \end{bmatrix} \begin{bmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \vdots \\ \mathbf{q}_N \end{bmatrix} \text{ with } \mathbf{B}_n = \begin{bmatrix} \mathbf{0} & n\mathbf{B} \\ -n\mathbf{B} & \mathbf{0} \end{bmatrix} \text{ for } n = 1, \dots, N \tag{5}$$

$$\tilde{\mathbf{L}}\mathbf{Q}_N = \begin{bmatrix} \mathbf{L} & & & \\ & \mathbf{L} & & \\ & & \ddots & \\ & & & \mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \vdots \\ \mathbf{q}_N \end{bmatrix} \text{ with } \mathbf{q}_n = \begin{bmatrix} \mathbf{q}_{n,c} \\ \mathbf{q}_{n,s} \end{bmatrix} \text{ for } n = 1, \dots, N. \tag{6}$$

$$\tilde{\mathbf{N}}(\mathbf{Q}_N, \mathbf{Q}_N) = \begin{bmatrix} \mathbf{N}_0 \\ \mathbf{N}_1 \\ \vdots \\ \mathbf{N}_N \end{bmatrix} \text{ with } \mathbf{N}_n = \begin{bmatrix} \mathbf{N}_{n,c} \\ \mathbf{N}_{n,s} \end{bmatrix} \text{ for } n = 1, \dots, N. \tag{7}$$