BEM in FreeFEM
Introduction to BEM

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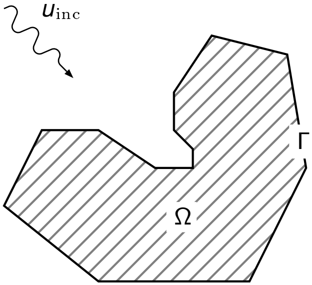
Model problem in 2D : 1st kind BIE



Volume form of the pb :
$\Delta u + \kappa^2 u = 0$ in $\mathbb{R}^3 \setminus \Omega$
$u = -u_{\mathrm{inc}}$ on Γ
+ radiation condition

Green kernel : $\mathscr{G}(\boldsymbol{x}) = \exp(\imath\kappa|\boldsymbol{x}|)/(4\imath\pi|\boldsymbol{x}|)$
We look for the solution under the ansatz,

$$u(\boldsymbol{x}) = \int_\Gamma \mathscr{G}(\boldsymbol{x}-\boldsymbol{y})p(\boldsymbol{y})d\sigma(\boldsymbol{y})$$

A variationnal formulation of the integral equation can be obtained by imposing the Dirichlet in a weak manner : find $p : \Gamma \to \mathbb{C}$ such that

$$\int_{\Gamma\times\Gamma} \frac{\exp(\imath\kappa|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} p(\boldsymbol{y})q(\boldsymbol{x})d\sigma(\boldsymbol{x},\boldsymbol{y}) = -\int_\Gamma u_{\mathrm{inc}}(\boldsymbol{x})q(\boldsymbol{x})d\sigma(\boldsymbol{x}) \quad \forall q : \Gamma \to \mathbb{C}$$

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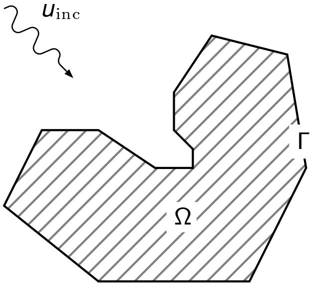

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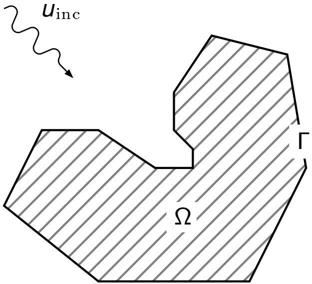

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- this kernel is singular
- it does not vanish : every $\boldsymbol{x}$ are coupled with every $\boldsymbol{y}$

1st kind vs. 2nd kind BIE

Several BIE possible for the same problem. Here, an alternative is an equation of the **2nd kind** : find $v : \Gamma \to \mathbb{C}$ such that

$$\int_{\Gamma\times\Gamma} \frac{\partial}{\partial \boldsymbol{n}(\boldsymbol{y})}\Big(\frac{\exp(\imath\kappa|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\Big) v(\boldsymbol{y})q(\boldsymbol{x})d\sigma(\boldsymbol{x},\boldsymbol{y}) = \mathrm{RHS}(q) \quad \forall q : \Gamma \to \mathbb{C}$$

1st kind BIE

- act as derivation of order ± 1
- more widely applicable
- lead to more accurate methods
- lead to ill conditionned systems
- efficient precondtionners available (both DDM and analytical)

2nd kind BIE

- act as derivation of order 0
- slightly easier to implement
- sometimes not possible (scattering by "screens")

Possible extensions

All integral equations require that an explicit analytical expression of the Green kernel be available. Consequence : only PDEs with constant coefficients are tractable. This includes :

- Laplace, Helmholtz and Yukawa : $\Delta + \kappa^2$ with $\kappa \in \mathbb{C}$
- Maxwell, eddy-current
- Lamé system
- Stokes
- Bi-harmonic

Possible variants :

- transmission problems and piecewise-constant coefficients
- coupling of (standard) volume based formulation with BIE
- time-domain

The four entries of the Calderón projector

The building blocks of all existing integral formulations consist in four operators : in the case of Helmholtz equation, $\mathscr{G}(\boldsymbol{x}) = \exp(\imath\kappa|\kappa|)/(4\pi|\boldsymbol{x}|)$

Single layer operator

$$p, q \mapsto \int_{\Gamma\times\Gamma} p(\boldsymbol{x})q(\boldsymbol{y})\mathscr{G}(\boldsymbol{x}-\boldsymbol{y})\, d\sigma(\boldsymbol{x},\boldsymbol{y})$$

Double layer operator

$$p, q \mapsto \int_{\Gamma\times\Gamma} p(\boldsymbol{x})q(\boldsymbol{y})\frac{\partial}{\partial \boldsymbol{n}(\boldsymbol{y})}\mathscr{G}(\boldsymbol{x}-\boldsymbol{y})\, d\sigma(\boldsymbol{x},\boldsymbol{y})$$

Adjoint double layer operator

$$p, q \mapsto \int_{\Gamma\times\Gamma} p(\boldsymbol{x})q(\boldsymbol{y})\frac{\partial}{\partial \boldsymbol{n}(\boldsymbol{x})}\mathscr{G}(\boldsymbol{x}-\boldsymbol{y})\, d\sigma(\boldsymbol{x},\boldsymbol{y})$$

Hypersingular operator

$$p, q \mapsto \int_{\Gamma\times\Gamma} p(\boldsymbol{x})q(\boldsymbol{y})\frac{\partial}{\partial \boldsymbol{n}(\boldsymbol{x})}\frac{\partial}{\partial \boldsymbol{n}(\boldsymbol{y})}\mathscr{G}(\boldsymbol{x}-\boldsymbol{y})\, d\sigma(\boldsymbol{x},\boldsymbol{y})$$

Numerical considerations

Possible discretisation procedures

- Nyström $\simeq$ finite difference. Not applicable for :
  - in 3D
  - when the hypersingular operator is involved
  - on non-smooth geometries (ex : polygonal)
- Collocation
- Galerkin method : our method of choice.
  $\mathrm{V}_h(\Gamma) = \{\mathbb{P}_1\text{-Lagrange on a surface triangulation}\}$

  Find $p_h \in \mathrm{V}_h(\Gamma)$ such that

$$\int_{\Gamma\times\Gamma} p_h(\boldsymbol{x}) q_h(\boldsymbol{y}) \frac{\exp(\imath\kappa|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} d\sigma(\boldsymbol{x},\boldsymbol{y}) = RHS(q_h) \quad \forall q_h \in \mathrm{V}_h(\Gamma).$$

FEM vs. BEM

- matrices are large sparse in FEM, small dense in BEM
- with the same computational effort, classical BEM is usually more accurate than classical FEM
- BEM can deal with unbounded domains
- No pollution effect : BEM remains robust at high frequency

Characteristic numerical features of BEM

- Integral formulation reduce the dimension of the problem by 1 i.e. in 3D we have to deal with surfaces. In particular mesh generation is easier to handle.

- The matrices are dense. Consequences :
  - ⋆ storage is $\mathcal{O}(N^2)$
  - ⋆ matrix-vector product costs $\mathcal{O}(N^2)$

- The density issue can be circumvented by using sophisticated hierarchical storage format for representing such dense operators : Fast Multipole Methods (FMM) or H-matrices. $\Rightarrow$ storage and MV-prod reduced to $\mathcal{O}(N \log^p(N))$.

- Singular kernel : $\lim_{\boldsymbol{x}\to\boldsymbol{y}} |\mathscr{G}(\boldsymbol{x}-\boldsymbol{y})| = +\infty \quad \Rightarrow$ quasi-singular integrals. Special quadrature rules (Sauter-Schwab) required for close interactions.

- The condition number does not deteriorate for 2nd kind formulations ! It does for 1st kind formulations, although more slowly than FEM matrices.

Existing general purpose BEM libraries

In contrast to FEM, there exist nearly no established general purpose library for BEM. Which open access initiatives I am aware of :

- BEM++, developped at UCL (T.Betcke), version 3, full python, densely populated matrices,
- Gipsylab, developped at Ecole Polytechnique (M.Aussal), full matlab, various acceleration techniques,
- NEMOH, developped at Central Nantes, Fortran, only Laplace in 3D, densely populated matrices

Besides :

- many ad-hoc codes available
- many fine tuned industrial codes
- general BEM $\neq$ FMM

What is BEMTool

BEMTool is a general purpose BEM library, written in C++ (98 mainly), under GNU LGPL. Supported through the ANR research project NonlocalDD. It handles :

- Laplace, Yukawa, Helmholtz, Maxwell
- both in 2D and 3D
- 1D, 2D and 3D triangulations (not necessary flat)
- $\mathbb{P}_k$-Lagrange $k = 0, 1, 2$ and surface $\mathbb{RT}_0$
- does not handle acceleration (FMM or H-Matrix)
- available on Github : `https://github.com/xclaeys/BemTool`
- easy to install (header only) + rather few files
- heavily relies on templates
- currently depends on :
  - ⋆ special function : Boost (GSL in the future)
  - ⋆ linear algebra : `eigen3` (easy to remove)
  - ⋆ mesh generation : `Gmsh`

A quick view on BEMTool