A third workshop on generic solvers for PDEs:
A Freefem++ and its Applications

Shape and Topology Optimization of Composite Materials with the level-set method

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Outline

Outline

Motivation: Composite Materials in Aeronautics

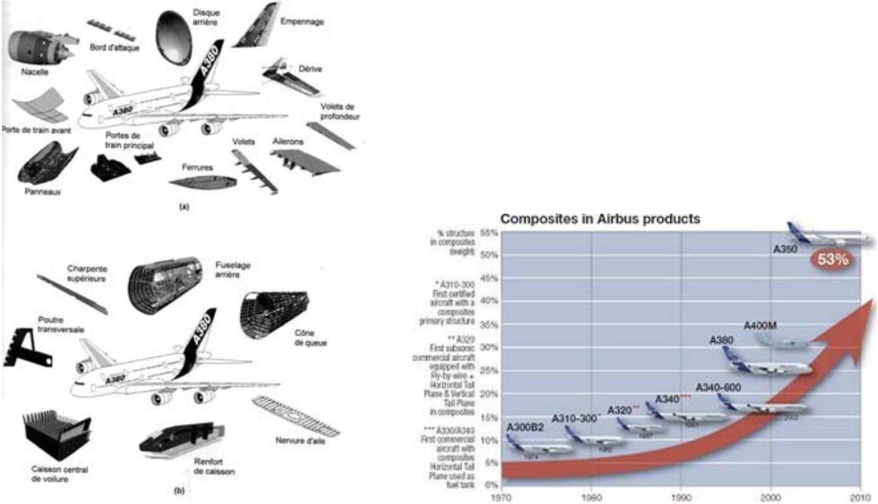
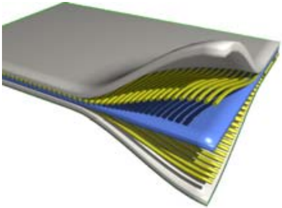
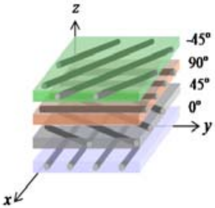


Figure: Composite structures of an A380 and evolution of the use of composites in Airbus.

Motivation: Laminated Composite Materials





Characteristics

- Lamination of a sequence of unidirectionally reinforced plies.
- Each ply is typically a thin sheet of collimated fibers impregnated with a polymer matrix material.
- Strong resistance against severe environmental conditions.
- Less expensive and lighter than metallic alloys.
- Greatest benefit and drawback: Espace of design possibilities.

Motivation: Shape and Topology Optimization examples

Figure: Optimized wing structure of the A380.

Topology optimized knife?

Motivation: Shape and Topology Optimization examples

Figure: Optimized wing structure of the A380.

Topology optimized knife?

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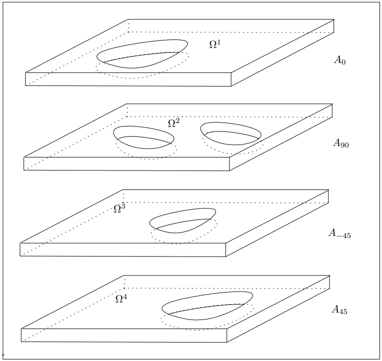
2D Multi-layered composite description

Let be $\mathcal{D} \subset \mathbb{R}^2$ fixed and $\Omega = \left\{\Omega^i\right\}$ a laminated composite structure made from the superposition of N plies under plane stress state, of geometry $\Omega^i \subset \mathcal{D}$, each one composed by two anisotropic elastic phases A^i and A^0 ($\left\|A^0\right\| << 1$). We consider the boundary of $\mathcal{D}$ made of two disjoint parts, $\partial\mathcal{D} = \Gamma_D \cup \Gamma_N$, $|\Gamma_D| \neq 0$.



Physical model description

On Ω^i we denote by Σ^i the interface between both phases. The characteristic function of Ω^i is denoted by $\chi^i(\boldsymbol{x})$, so the composite elastic tensor of Ω at $\boldsymbol{x}$ is described by

$$A_\chi = \sum_{i=1}^{N} A^i \chi^i + \sum_{i=1}^{N} A^0 (1 - \chi^i).$$

The strain and stress tensors are related to the displacement field $\boldsymbol{u}$ as $e(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})$ and $\sigma(\boldsymbol{u}) = A_\chi e(\boldsymbol{u})$. The external charges are denoted by $\boldsymbol{f}$ (volume) and $\boldsymbol{g}$ (surface), and the **displacement field** $\boldsymbol{u}_\chi$ is the solution of the linearized elasticity system in $\mathcal{D}$

$$\begin{cases} -\text{div}(A_\chi e(\boldsymbol{u})) = \boldsymbol{f} & \text{in } \mathcal{D} \\ \boldsymbol{u} = 0 & \text{on } \Gamma_D \\ Ae(\boldsymbol{u})n = \boldsymbol{g} & \text{on } \Gamma_N \end{cases}$$

Optimization problem

Find the best composite structure Ω according to a criteria $J(\Omega, \boldsymbol{u}_\chi)$, where $\boldsymbol{u}_\chi$ is the state variable (displacement) and Ω is the control variable, among a set of admissible shapes of layers $\mathcal{U}_{ad}$ (manufacture and geometric constraints), under a given state of external charges $(\boldsymbol{f}, \boldsymbol{g})$ by changing the geometry of each biphasic layer Ω^i

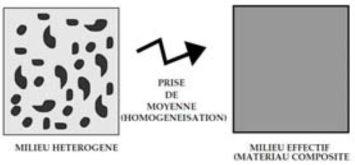
$$\min_{\{\Omega^i\}_{i=1..N} \in \mathcal{U}_{ad},\, \Omega^i \subset \mathcal{D}} J(\Omega, \boldsymbol{u}_\chi)$$

Outline

Main methods in Topology Optimization

Homogenization

The idea is to find the optimal distribution of material inside a structure, by authorizing intermediates densities $(0 \leq \theta^i \leq 1)$ of each material. To do this, we define the set of homogenized tensors G_θ, corresponding to the set of composite materials A^* made from the mixture of the phases A^i in θ^i proportion, where A^* is the law tensor representing the homogenized micro-structure.



Unfortunately the main problem of this method is to find an explicit description of the set G_θ.

Main methods in Topology Optimization

SIMP (Solid Isotropic Material with Penalization)

One of the most popular optimization methods in industry (e.g. Optistruct) because of it's capability to reduce the amount of design variables, simplify the problem, and improve the variety of calculation methods. It corresponds to a simplification of the Homogenization method using as homogenized elastic tensor A^* with density θ

$$A^* = \theta^p A^0 + (1-\theta)^p A^1, \quad p \geq 1.$$

This method suffers of some instabilities as checkerboard patterns and gray intermediate unit values. There is also the ad-hoc choice of the topology penalty factor p and mesh dependency of the solution.

Main methods in Topology Optimization

Level set method

Has as unique advantages to track topology changes and a clear and smooth boundary that can be easily managed.

Problems that arise from a density approach like spurious eigenfrequencies and micro structure stress concentration are avoided.

Coupled to the shape and topological derivative analysis, it makes the level set method a promising research direction in future applications on structures design. Let's take a look of how it works.

Mixed approach based on 3 tools

Structure representation

- **Level set method**
- Precise indirect geometric description in an eulerian framework.
- No re-meshing during the evolution process (low-cost).
- Topology changes easily handled.

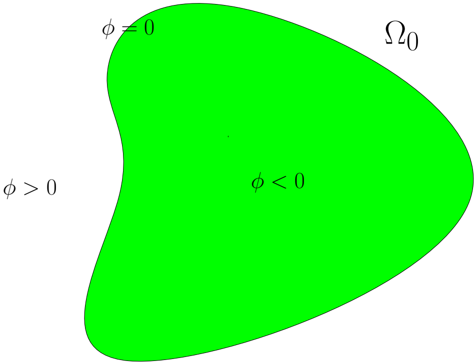
Shape optimization

- **Shape derivative and Hadamard's method.**
- Quasi-Newton method using the shape gradient.

Topology optimization

- **Topological derivative.**
- Allowance of topology changes by the nucleation of inclusions.

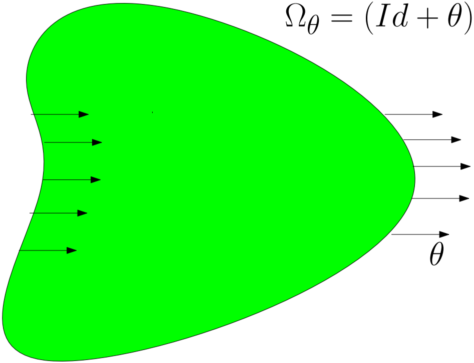
General explanation: Level set method



General explanation: Shape sensitivity

$$J(\Omega_\theta) < J(\Omega_0)$$

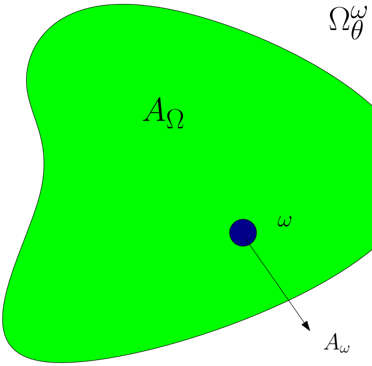
$$\Omega_\theta = (Id + \theta)(\Omega_0)$$



Boundary movement

General explanation: Topological sensitivity

$$J(\Omega_\theta^\omega) < J(\Omega_\theta)$$



Nucleation of an inhomogeneity

Outline

Cost function and the adjoint problem

Let's take a general cost function $J(\Omega, \boldsymbol{u}_\chi(\Omega))$ as

$$J(\Omega, \boldsymbol{u}_\chi(\Omega)) = \int_{\mathcal{D}} j(\boldsymbol{x}, \boldsymbol{u}_\chi(\Omega))dx + \int_{\partial\mathcal{D}} h(\boldsymbol{x}, \boldsymbol{u}_\chi(\Omega))ds.$$

This criteria can represent for example the compliance $(j(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{f} \cdot \boldsymbol{u}, h(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{g} \cdot \boldsymbol{u})$, the volume $(j = 1)$, a least square target displacement $(j(\boldsymbol{x}, \boldsymbol{u}) = |\boldsymbol{u} - \boldsymbol{u}_0|^2)$, a stress dependent function $(j(\boldsymbol{x}, \boldsymbol{u}) = |\sigma(\boldsymbol{u})|^2)$, etc.

Furthermore, in order to avoid the calculation of the explicit variation of $\boldsymbol{u}_\chi$ w.r.t. the domain, we introduce the adjoint state problem

$$\begin{cases} -\mathrm{div}(A_\chi e(\boldsymbol{p}_\chi)) = -j'(\boldsymbol{x}, \boldsymbol{u}_\chi) & \text{in } \mathcal{D} \\ \boldsymbol{p}_\chi = 0 & \text{on } \Gamma_D \\ A_\chi e(\boldsymbol{p}_\chi)n = -h'(\boldsymbol{x}, \boldsymbol{u}_\chi) & \text{on } \Gamma_N \end{cases}$$

The level-set method

We define the level set function $\phi^i(\boldsymbol{x})$ on $\mathcal{D} \subset \mathbb{R}^2$ as

$$\begin{cases} \phi^i(\boldsymbol{x}) = 0 \Leftrightarrow & \boldsymbol{x} \in \partial\Omega^i \cap \mathcal{D}, \\ \phi^i(\boldsymbol{x}) < 0 \Leftrightarrow & \boldsymbol{x} \in \Omega^i, \\ \phi^i(\boldsymbol{x}) > 0 \Leftrightarrow & \boldsymbol{x} \in (\mathcal{D} \setminus \bar{\Omega}^i). \end{cases}$$

From this definition we can easily deduce

- Outward normal vector $n^i = \nabla\phi^i / |\nabla\phi^i|$
- Curvature $\kappa^i = \text{div } n^i$

This formulas have a meaning over all $\mathcal{D}$ and not only on Σ^i. If the domain $\Omega^i(t)$ evolves in a pseudo-time $t \in \mathbb{R}^+$ with velocity $\boldsymbol{\theta}(\boldsymbol{x}, t)$, then the level set transport equation is

$$\frac{\partial\phi^i}{\partial t}(\boldsymbol{x}, t) + \boldsymbol{\theta}(\boldsymbol{x}, t) \cdot \nabla\phi^i(\boldsymbol{x}, t) = 0$$

Level-set method: Advection

This equation can be solved by a Galerkin-characteristic method for example (**Freefem++** *convect()*). Nevertheless, if we take only the normal component of the velocity $\mathcal{V}(\boldsymbol{x}, t)$, the equation simplifies into

$$\frac{\partial \phi^i}{\partial t} + \mathcal{V}|\nabla \phi^i| = 0.$$

This is a **Hamilton-Jacobi type equation** and it does not suffer of tangential numerical diffusion. We use a *finite differences upwind* numerical scheme to solve it over a fixed structured mesh $\mathcal{D}_h$. This scheme was programmed as a routine in C++ and then linked to **Freefem++**.

Level-set method: Reinitialization

In order to regularize the level-set function (which may become too flat or too steep), we reinitialize it periodically by solving

$$\begin{cases} -\frac{\partial \phi}{\partial t} + \mathrm{sign}(\phi_0)(1 - |\nabla \phi|) = 0 & \text{in } \mathcal{D} \times \mathbb{R}^+ \\ \phi(t = 0, \boldsymbol{x}) = \phi_0(\boldsymbol{x}) & \text{in } \mathcal{D} \end{cases}$$

which admits as a stationary solution the signed distance to the initial interface $\{\phi_0 = 0\}$. Reinitialize the level function is really important to obtain a good approximation of the normal n^i and the curvature κ^i of Σ^i.

Question: Find $\mathcal{V}$ that minimizes $J(\Omega, u_\chi)$.
Hint: Shape derivative.

Shape derivative: Variation of the cost function under boundary changes

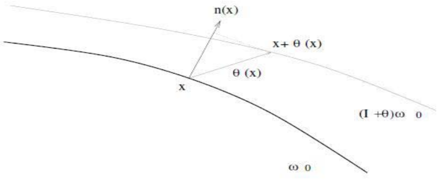
We are interested on the variations of the functional $J(\Omega, \boldsymbol{u}_\chi(\Omega))$ when the position of the interface Σ^i on each ply changes. Let be ω_0 an open smooth set such that $\omega_0 \subset \mathcal{D}$. We denote by χ_0^ω the characteristic function of ω_0, and we consider the variations on the form

$$\chi_\theta^\omega = \chi_0^\omega \circ (Id + \theta), \text{ i.e. } \chi_\theta^\omega = \chi_0^\omega \circ (\boldsymbol{x} + \theta(\boldsymbol{x})),$$

where $\theta \in W^{1,\infty}(\mathcal{D}, \mathbb{R}^2)$ (Lipschitz functions with uniformelly bounded derivatives), $\|\theta\|_{W^{1,\infty}} < 1$ and $(\theta \cdot n)_{|\partial\mathcal{D}} = 0$. We define the set of admissible shapes $\mathcal{U}_{ad}$ by

$$\mathcal{U}_{ad} = \{\omega_\theta \subset \mathcal{D} : \chi_\theta^\omega = \chi_0^\omega \circ (Id + \theta)\}.$$

Shape derivative



Definition

Let be the functional $J(\omega) : \mathcal{U}_{ad} \to \mathbb{R}$. **The shape derivative** of $J(\omega)$ at ω_0 is defined as the Fréchet derivative in $W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ at 0 of the l'application $\theta \to J(\omega_0 \circ (Id + \theta))$, i.e.

$$J(\omega_0 \circ (Id + \theta)) = J(\omega_0) + J'(\omega_0)(\theta) + o(\theta) \text{ with } \lim_{\theta \to 0} \frac{|o(\theta)|}{\|\theta\|_{W^{1,\infty}}} = 0$$

where $J'(\omega)$ is a continuous linear form on $W^{1,\infty}(\mathcal{D}; \mathbb{R}^2)$.

Low-contrast adapted shape derivative (fixed mesh)

Let be $\mathcal{D}_h$ and $\Omega_h = \left\{\Omega_h^i\right\}$ polygonal approximations of $\mathcal{D}$ and Ω. Given a triangulation $\mathcal{T} = \{K_l\}_l$ of $\mathcal{D}_h$ and $\theta \in W^{1,\infty}(\mathcal{D}_h;\mathbb{R}^2)$, the total shape derivative for a fixed mesh of $J(\Omega_h, \boldsymbol{u}_\chi)$ in the direction θ is (according to Einstein notation on l)

$$
\begin{aligned}
dJ^H(\Omega)(\theta) = &\int_{K_l} A_\chi e(\boldsymbol{u}_\chi) : e(\boldsymbol{p}_\chi)(\nabla\cdot\theta)dx + \int_{K_l} A_\chi E^H(\theta,\boldsymbol{u}_\chi) : e(\boldsymbol{p}_\chi) \\
&+ \int_{K_l} A_\chi E^H(\theta,\boldsymbol{p}_\chi) : e(\boldsymbol{u}_\chi)dx - \int_{\partial K_l} A_\chi e(\boldsymbol{u}_\chi) : e(\boldsymbol{p}_\chi)\theta\cdot n ds
\end{aligned}
$$

with the tensor

$$(E^H(\theta,\boldsymbol{u}))_{i,j} = \nabla(e(\boldsymbol{u})_{i,j})\cdot\theta$$

How do we chose $\mathcal{V}$ using the shape sensitivity?

We will take θ^* smooth ($\|\theta^*\|_{W^{1.\infty}} = 1$) such that

$$J'(\Omega)(\theta^*) < 0.$$

For that purpose, we will solve with **Freefem++** the variational problem

$$\langle \theta^*, v \rangle_{W(\mathcal{D})} = -J'(\Omega)(v), \ \forall v \in W(\mathcal{D}),$$

for some Hilbert space $W(\mathcal{D})$ (H^1 for instance). Then we will chose as advection velocity for the level-set $\mathcal{V} = \theta^* n$. If we take a small enough advection step $t << 1$, we can assure that the objective function will decrease using the shape derivative definition:

$$J(\Omega_0 \circ (Id + t\theta)) = J(\Omega_0) + tJ'(\Omega_0)(\theta) + o(t).$$

Drawback: Tendency to local minima, no change of topology (nucleation).
Hint: Topological gradient.

The topological derivative

Let ω a smooth open subset of $\mathbb{R}^2$. Let $\rho > 0$ be a small positive parameter which is intended to go to zero. For a point $\boldsymbol{z} \in \mathcal{D}$ with $d(\boldsymbol{z}, \partial\mathcal{D}) < \rho$ and $d(\boldsymbol{z}, \Sigma^i) < \rho$, $\forall i < N$, we define the rescaled inclusion

$$\omega_\rho = \left\{ \boldsymbol{x} \in \mathbb{R}^2 : \frac{\boldsymbol{x} - \boldsymbol{z}}{\rho} \in \omega \right\},$$

which, for a small enough ρ does not intersect any Σ^i. We are interested on the variation of the objective function when we embed an inclusion ω_ρ of Hooke law A^* on the i-layer obtaining a composite Ω_ρ.

Definition

If the objectif function J admits the following so-called topological asymptotic expansion for small $\rho > 0$:

$$J(\Omega_\rho) - J(\Omega) - \rho^2 DJ(\boldsymbol{z}) = o(\rho^2),$$

then the number $DJ(\boldsymbol{z})$ is called the topological derivative of J at $\boldsymbol{z}$ for the inclusion shape ω.

Theorem

The topological derivative $DJ(\boldsymbol{z})$ of the general cost function J, evaluated at $\boldsymbol{z}$ for an inclusion shape ω, has the following expression

$$DJ(\boldsymbol{z}) = -M(A, A^*, \omega) e(\boldsymbol{u}_\chi)(\boldsymbol{z}) : e(\boldsymbol{p}_\chi)(\boldsymbol{z})$$

where $M(A, A^*, \omega)$ is the so-called **elastic momentum tensor***.*

The optimization algorithm

1. **Initialization** of each level set function ϕ_0^i corresponding to an initial guess χ_0^i (most of time trivial guess $\chi_0^i \equiv 1$).

   Iteration **k**

2. **Solve** the direct (u_χ^k) and adjoint (p_χ^k) state problems posed on Ω_k.
3. **Shape/Topology sensitivities and level set advection**

   For each layer **i**

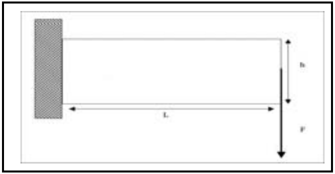
   - *If* k *mod*$(Nopt) = 0$
   - Calculate the topological derivative DJ and remove a small percentage of the area where it reaches the minimum.
   - *If* k *mod*$(Nopt) \neq 0$
   - Computation of a regular velocity $\mathcal{V}_k^i$ such that the shape derivative satisfies $J'(\chi_k^i)(\mathcal{V}_k^i \boldsymbol{n}) \leq 0$.
   - Deformation of the shape by solving the level-set equation with a time step of Δs^k, chosen as $J(\chi^{k+1}) \leq J(\chi^k)$.

Outline

Airbus simplified fuselage test case

Description

Find the stiffest (lightest) four layered (0°,90°,45°,-45°) composite structure, under a weight (compliance) constraint for the following cantilever-type problem



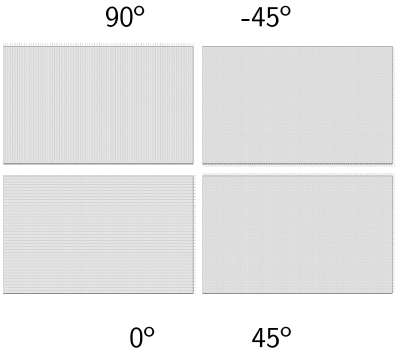
This corresponds to an equivalent model of a flattened section of a fuselage.

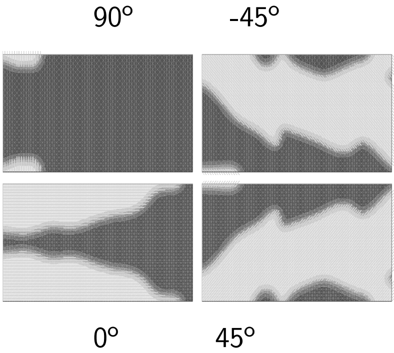
Freefem++ parameters

Description

- Unique structured isotropic rectangular mesh (80x40) for the level-set and FE analysis created with *square(...,flags=1)*.
- Truncation of the original mesh around the level set for the shape derivative calculation (*trunc(),interpolate()*).
- $\boldsymbol{u}_\chi$, $\boldsymbol{p}_\chi = \boldsymbol{u}_\chi$ and θ were taken as $[P_1(\Omega), P_1(\Omega)]$ finite elements.
- $DJ(\boldsymbol{z})$ was taken $\mathbb{P}_{1dc}(\Omega)$ and then regularized.
- Adapted augmented Lagrangian method used for the constrained optimization.
- Number of iterations 500, time 20 min.

Results





Outline

Conclusions

1. A shape and topology optimization algorithm has been developed in **Freefem++** for composite materials, where the design variable is the shape and topology of each ply.
2. Difficulties related to the elastic low contrast and anisotropy of the materials were successfully overcome.
3. In order to check the feasibility of the method for a simplified test-case, **Freefem++** has proven to be a powerful and flexible tool for the optimization algorithm.

Perspectives

1. Add more plies to the composite and enhance the numerical performance of the optimization algorithm (processing time and constraint management).
2. Explore two other design variables: the orientation of each ply and the stacking sequence.
3. Introduce a fiber orientation optimizer based on Homogenization and a new constraint measuring the ratio between the bending stiffness and the in-plane stress of the composite (commonly used in Airbus).
4. New developments will take place in **Freefem++** in a first instance, and then transfered to specialized FE solvers used in the industry (NASTRAN,ABAQUS,etc).