15th FreeFEM days, 8th Dec. 2023

Fictitious domain method with implicit boundary definition by the level set method and a stabilization

Atsushi Suzuki[1]

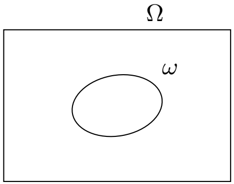
[1] RIKEN R-CCS / Cybermedia Center, Osaka University
atsushi.suzuki.aj@a.riken.jp
atsushi.suzuki@cas.cmc.osaka-u.ac.jp

- fictitious domain method by surface integration of Lagrange multiplier to impose Dirichlet condition on the internal boundary proposed by R. Glowinski and analysed with V. Girault
- `FreeFEM` has domain integration with indicator function on the background mesh and surface integration on the zero level set
- stabilization technique proposed by Burma-Hansbo 2010

fictitious domain method by Lagrange multiplier : 1/5

Error Analysis of a Fictitious Domain Method Applied to a Dirichlet Problem
V. Girault, R. Glowinski, JJIAM 12, 1995, 487-514

Poisson problem with Dirichlet data on the internal boundary



$$\begin{cases} -\nabla \cdot \nabla u = f & \text{in } \Omega \\ u = g & \text{on } \partial\omega \\ u = 0 & \text{on } \partial\Omega \setminus \partial\omega \end{cases}$$

mesh decomposition of Ω, body fitted mesh $\Rightarrow$ uniform mesh of $\Omega \cup \bar{\omega}$
$\widetilde{f}$: extension of f from Ω to $\Omega \cup \bar{\omega}$

$$\begin{cases} -\nabla \cdot \nabla \widetilde{u} = \widetilde{f} & \text{in } \Omega \cup \bar{\omega} \\ \widetilde{u} = g & \text{on } \partial\omega \\ \widetilde{u} = 0 & \text{on } \partial(\Omega \cup \bar{\omega}) \end{cases}$$

find $\widetilde{u} \in H_0^1(\Omega \cup \bar{\omega})$, $\lambda \in H^{-1/2}(\partial\omega)$: Lagrange multiplier

$$\int_{\Omega \cup \bar{\omega}} \nabla \widetilde{u} \cdot \nabla \widetilde{v} + \int_{\partial\omega} \widetilde{v}\, \lambda = \int_{\Omega \cup \bar{\omega}} \widetilde{f}\, \widetilde{v} \quad \forall \widetilde{v} \in H_0^1(\Omega \cup \bar{\omega})$$

$$\int_{\partial\omega} (\widetilde{u} - g)\mu = 0 \qquad \forall \mu \in H^{-1/2}(\partial\omega)$$

fictitious domain method by Lagrange multiplier : 2/5

$$\int_{\Omega\cup\bar{\omega}} \nabla\widetilde{u}\cdot\nabla\widetilde{v} + \int_{\partial\omega} \widetilde{v}\,\lambda = \int_{\Omega\cup\bar{\omega}} \widetilde{f}\,\widetilde{v} \quad \forall\widetilde{v}\in H^1_0(\Omega\cup\bar{\omega})$$

Solution of the weak formulation satisfies the Euler equation with boundary conditions

- taking $\widetilde{v}=0$ in $\bar{\omega}$, and applying integration by parts $\Rightarrow \widetilde{v}=0$ on $\partial\omega$

$$\begin{aligned}\int_{\Omega\cup\bar{\omega}} \nabla\widetilde{u}\cdot\nabla\widetilde{v} &= \int_{\Omega} \nabla\widetilde{u}\cdot\nabla\widetilde{v} + \int_{\omega} \nabla\widetilde{u}\cdot\nabla\widetilde{v} \\ &= -\int_{\Omega} \nabla\cdot\nabla\widetilde{u}\,\widetilde{v} + \int_{\partial\Omega} \nabla\widetilde{u}\cdot n\widetilde{v} - \int_{\omega} \nabla\cdot\nabla\widetilde{u}\,\widetilde{v} - \int_{\partial\omega} \nabla\widetilde{u}\cdot n\widetilde{v} \\ &= -\int_{\Omega} \nabla\cdot\nabla\widetilde{u}\,\widetilde{v} + \int_{\Omega} \widetilde{f}\,\widetilde{v}\end{aligned}$$

  then we have $-\nabla\cdot\nabla\widetilde{u} = \widetilde{f}$ in Ω.

- taking $\widetilde{v}=0$ in $\bar{\Omega}$ then we have $-\nabla\cdot\nabla\widetilde{u} = \widetilde{f}$ in ω.
- property of the duality pair between $H^{1/2}(\partial\omega)$ and $H^{-1/2}(\partial\omega)$

$$\int_{\partial\omega} (\widetilde{u}-g)\mu = 0 \quad \forall\mu\in H^{-1/2}(\partial\omega) \;\Rightarrow \widetilde{u} = g \in H^{1/2}(\partial\omega)$$

fictitious domain method by Lagrange multiplier : 3/5

$$\int_{\Omega\cup\bar{\omega}} \nabla\widetilde{u}\cdot\nabla\widetilde{v} + \int_{\partial\omega} \widetilde{v}\,\lambda = \int_{\Omega\cup\bar{\omega}} \widetilde{f}\,\widetilde{v} \quad \forall\widetilde{v}\in H_0^1(\Omega\cup\bar{\omega})$$

Lagrange multiplier λ gives jump of Neumann data on $\partial\omega$

- taking $\widetilde{v}\in H^1(\Omega\cup\bar{\omega})$, and applying integration by parts

$$\begin{aligned}\int_{\Omega\cup\bar{\omega}} \nabla\widetilde{u}\cdot\nabla\widetilde{v} + \int_{\partial\omega} \widetilde{v}\,\lambda &= \int_{\Omega} \nabla\widetilde{u}\cdot\nabla\widetilde{v} + \int_{\omega} \nabla\widetilde{u}\cdot\nabla\widetilde{v} \\ &= -\int_{\Omega} \nabla\cdot\nabla\widetilde{u}\,\widetilde{v} + \int_{\partial\Omega} \nabla\widetilde{u}\cdot n\widetilde{v} - \int_{\omega} \nabla\cdot\nabla\widetilde{u}\,\widetilde{v} - \int_{\partial\omega} \nabla\widetilde{u}\cdot n\widetilde{v} \\ &= \int_{\Omega} \widetilde{f}\,\widetilde{v} + \int_{\omega} \widetilde{f}\,\widetilde{v} + \int_{\partial\omega} (\nabla\widetilde{u}\cdot n|_{\partial\Omega} - \nabla\widetilde{u}\cdot n|_{\partial\omega} + \lambda)\,\widetilde{v}\end{aligned}$$

then we have $\lambda = [\nabla\widetilde{u}\cdot n]_{\partial\omega}$,
here n is taken as outer normal to $\partial\Omega$.

fictitious domain method by Lagrange multiplier : 4/5

$$\int_{\Omega\cup\bar{\omega}} \nabla\widetilde{u}\cdot\nabla\widetilde{v} + \int_{\partial\omega} \widetilde{v}\,\lambda = \int_{\Omega\cup\bar{\omega}} \widetilde{f}\,\widetilde{v} \quad \forall\widetilde{v}\in H_0^1(\Omega\cup\bar{\omega})$$
$$\int_{\partial\omega} (\widetilde{u}-g)\mu = 0 \qquad \forall\mu\in H^{-1/2}(\partial\omega)$$

solvability and uniqueness of the solution by the mixed formulation framework

- coercivity of $a(\widetilde{u},\widetilde{v}) = \int_{\Omega\cup\bar{\omega}} \nabla\widetilde{u}\cdot\nabla\widetilde{v}$
- inf-sup condition on $b(\widetilde{v},\mu) = \int_{\partial\omega} \nabla\widetilde{v}\mu$

norms of $H^{1/2}(\partial\omega)$ and $H^{-1/2}(\partial\omega)$

$$||v||^2_{H^{1/2}(\partial\omega)} = \inf_{\widetilde{v}\in H_0^1(\omega)} \int_\omega \nabla\widetilde{v}\cdot\nabla\widetilde{v} \leq \inf_{\widetilde{v}\in H_0^1(\omega)} \int_\omega ||\widetilde{v}||^2_{H^1(\omega)}$$

$$||\mu||_{H^{-1/2}(\partial\omega)} = \sup_v \frac{\int_{\partial\omega} v\mu}{||v||_{H^{1/2}(\partial\omega)}}$$

then inf-sup condition is naturally satisfied

$$\forall\mu\in H^{1/2}(\partial\omega) \qquad \sup_{\widetilde{v}\in H_0^1(\omega)} \frac{b(\widetilde{v},\mu)}{||\widetilde{v}||_{H^1(\omega)}} \geq ||\mu||_{H^{-1/2}(\partial\omega)}$$

fictitious domain method by Lagrange multiplier : 5/5

$$\int_{\Omega\cup\bar{\omega}} \nabla\widetilde{u}_h \cdot \nabla\widetilde{v}_h + \int_{\partial\omega} \widetilde{v}_h\,\lambda_h = \int_{\Omega\cup\bar{\omega}} \widetilde{f}\,\widetilde{v}_h \quad \forall\widetilde{v}_h \in H_0^1(\Omega\cup\bar{\omega})$$

$$\int_{\partial\omega} (\widetilde{u}_h - g)\mu_h = 0 \qquad \forall\mu_h \in H^{-1/2}(\partial\omega)$$

finite element approximation to satisfy discrete inf-sup condition

- P1 uniform mesh decomposition to $\Omega\cup\bar{\omega}$
- P0 approximation to $\gamma = \partial\omega$ with large segment size $3h$

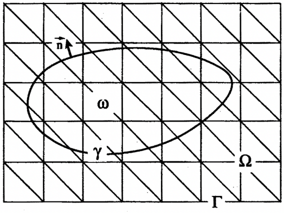


Figure 1.

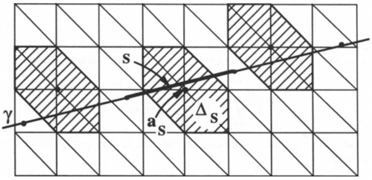


Figure 2.

body fitted mesh and Lagrange multiplier

Body fitted mesh to $\partial\omega$ with polygonal segments of $\partial\omega$

$$\int_{\partial\omega} (\widetilde{u}_h - g)\mu_h = 0$$

will be replaced by point-wise constraint by solving following linear equation with mass matrix on $\partial\omega$

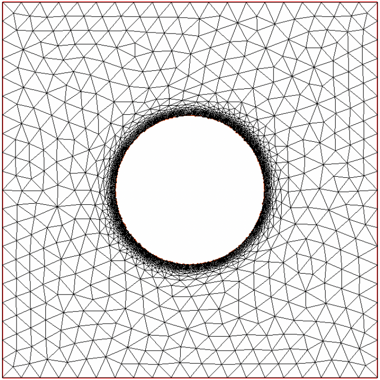
$$M_h(\widetilde{u}_h - g) = 0 \;\Leftrightarrow \widetilde{u}_h = g$$

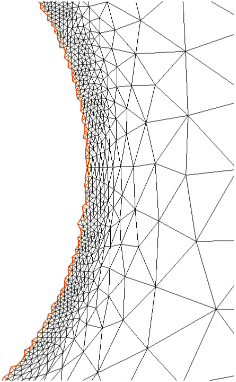
Let B to be a Boolean matrix to pickup DOF on $\partial\omega$

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

full column rank of B^T i.e., $B^T\mu = 0 \Rightarrow \mu = 0$ ensures solvability of the KKT system with uniqueness of λ

non-matched mesh to the boundary



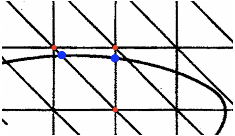


- mesh subdivisions need not to follow the boundary interface completely, but refinement in the vicinity of the boundary is useful

level set expression of the boundary : 1/2

- zero level set function to express the internal boundary $\partial\omega = \{x\,;\,\varphi(x) = 0\}$
- $\varphi_h(x)$: P1 approximation of segment in each triangular element that touches the zero level set.
- meshes in banded domain contains the zero level set $\mathcal{T}_{\partial\omega} = \{K\,;\,K \cap \{x\,;\,\varphi_h(x) = 0\} \neq \emptyset\}$
- replacement of Neumann data in $H^{-1/2}(\partial\omega)$ by interpolated data from neighbouring node closed the zero level set : $\prod_{K\in\mathcal{T}_{\partial\omega}} L^2(K)$

$$\int_{\partial\omega} \widetilde{v}_h \lambda_h \sim \sum_{K\in\mathcal{T}_{\partial\omega}} \int_{K\cap\{x\,;\,\varphi_h(x)=0\}} \widetilde{v}_h \widetilde{\lambda}_h$$



- data on blue node is interpolated from three red data
- Dirichlet data $\widetilde{u}_h$ by P1, Neumann data $\widetilde{\lambda}_h$ by P0

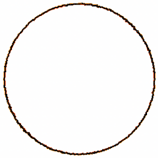
obtained equations system with FEM matrix is a KKT system

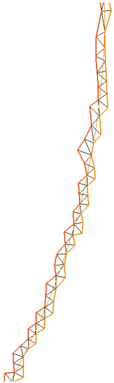
$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

- the system can be factorized with LDL^T decomposition with pivoting, which guarantees the matrix is non-singular
- MUMPS fails, needs to increase working memory

e.g., `INFO(1)=-9 INFOG(2)=1967`

level set expression of the boundary : 2/2





meshes touching the zero-level set $\mathcal{T}_{\partial\omega} = \{K\,;\, K \cap \{x\,;\, \varphi_h(x) = 0\} \neq \emptyset\}$

stabilization technique for fictitious domain method

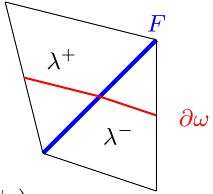
Fictitious domain finite element method using cut elements : I. a stabilized Lagrange multiplier method
E. Burman, P. Hansbo, CMAME 190, 2010, 2680-2686

stabilization term for Neumann data on the elements touching the zero-level set with $\delta > 0$

$$\delta \sum_{F \in \mathcal{F}} \int_F h_F [\lambda][\mu]$$

- $[\lambda] = \lambda^{+1} - \lambda^-$: gap of Neumann data in either side of the edge
- $h_F = \text{diam} F$
- $\mathcal{F} = \{K_1 \cap K_2 \,;\, K_1, K_2 \in \mathcal{T}_{\partial\omega}, K_1 \neq K_2\}$

with approximation of λ by P0 element



$$\begin{aligned}\sum_{F \in \mathcal{F}} \int_F h_F [\lambda][\mu] &= \sum_{F \in \mathcal{F}} h_F^2 (\lambda^+ - \lambda^-)(\mu^+ - \mu^-) \\ &= \sum_{F \in \mathcal{F}} h_F^2 (\lambda_i \psi_i - \lambda_j \psi_j)(\psi_i - \psi_j)\end{aligned}$$

obtained equations system with FEM matrix has additional block diagonal entry

$$\begin{bmatrix} A & B^T \\ B & -\delta D \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

KKT system with constraint : 1/2

$$K\begin{bmatrix}\vec{u}\\ \vec{\lambda}\end{bmatrix}=\begin{bmatrix}A & B^T\\ B & 0\end{bmatrix}\begin{bmatrix}\vec{u}\\ \vec{\lambda}\end{bmatrix}=\begin{bmatrix}\vec{f}\\ \vec{g}\end{bmatrix}$$

- B^T full column rank
$$B^T\vec{\lambda}=\vec{0}\ \Rightarrow \vec{\lambda}=0$$
- A : coercive (symmetric positive definite $\Leftarrow A=A^T$)
$$\exists\alpha>0\quad (A\vec{u},\vec{u})\geq\alpha||\vec{u}||^2\quad\forall\vec{u}$$

$\Rightarrow$ K is invertible on $\mathbb{R}^N$ (in the whole space)
Schur complement system, $B\,A^{-1}B^T$ ensures LDL^T factorization
what kind of procedure is feasible with symmetric pivoting?

- $V_m\subset\mathbb{R}^N$: sequence of subspace by pivoting
- $\dim V_m=m\leq N$, $V_m=\mathrm{span}[\{\vec{e}_{\mu_i}\}_{1\leq i\leq m}]\quad\vec{e}_\mu\in\mathbb{R}^N\ [\vec{e}_\mu]_\lambda=\delta_{\mu\,\lambda}$

find $\vec{u}\in V_m\quad(K\,\vec{u},\vec{v})=(\vec{f},\vec{v})\quad\forall\vec{v}\in V_m$
from m-th step to $m+1$-th with factorized matrix K_{11} in V_m

$$\begin{bmatrix}K_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}=\begin{bmatrix}K_{11} & 0\\ k_{21}K_{11}^{-1} & s\end{bmatrix}\begin{bmatrix}I & K_{11}^{-1}k_{12}\\ 0 & 1\end{bmatrix}\ \text{with } s=k_{22}-k_{21}K_{11}^{-1}k_{12}$$

$s=0\ \Rightarrow K$ is singular on V_{m+1}

KKT system with constraint : 2/2

$$K \begin{bmatrix} \vec{u} \\ \vec{\lambda} \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{\lambda} \end{bmatrix} = \begin{bmatrix} \vec{f} \\ \vec{g} \end{bmatrix}$$

- B^T full column rank

$$B^T \vec{\lambda} = \vec{0} \Rightarrow \vec{\lambda} = 0$$

- A : coercive (symmetric positive definite $\Leftarrow A = A^T$)

$$\exists \alpha > 0 \quad (A\vec{u}, \vec{u}) \geq \alpha ||\vec{u}||^2 \quad \forall \vec{u}$$

- C : symmetric positive semi-definite

$$\Rightarrow \qquad \widetilde{K} = \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} : \quad \text{coercive}$$

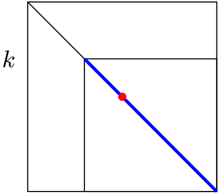
$$\left(\widetilde{K} \begin{bmatrix} \vec{u} \\ \vec{\lambda} \end{bmatrix}, \begin{bmatrix} \vec{u} \\ \vec{\lambda} \end{bmatrix} \right) = (A\vec{u}, \vec{u}) + (B^T \vec{\lambda}, \vec{u}) - (B\vec{u}, \vec{\lambda}) + (C\vec{\lambda}, \vec{\lambda}) \geq 0$$

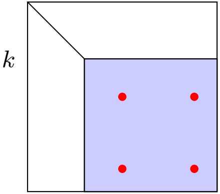
- $\widetilde{K}$ is invertible in any subspace
- LDU without pivoting can factorize $\widetilde{K}$
- $\widetilde{K}$ becomes unsymmetric even for case of $K = K^T$

pivoting strategy

symmetric pivoting : $K = \Pi^T LDU\Pi$
find max$_{k<i\le n}|K(i,i)|$

2×2 pivoting : $K = \Pi^T L\,\tilde{D}\,U\Pi$
find max$_{k<i,j\le n}$det$\begin{vmatrix} K(i,i) & K(i,j) \\ K(j,i) & K(j,j) \end{vmatrix}$





symmetric pivot not always possible for indefinite matrix
how to understand 2×2 pivoting by a framework with weak formulation
D: diagonal, L: lower triangle, $L(i,i)=1$, U: upper tri., $U(i,i)=1$.

- index set $\{\mu_1, \mu_2, \cdots, \mu_m\}$
- $V_m =$ span$[e_{\mu_1}, e_{\mu_2}, \cdots, e_{\mu_m}] \subset \mathbb{R}^N$
- $P_m : \mathbb{R}^N \to V_m$ orthogonal projection.

find $u \in V_m \quad (Ku - f, v) = 0 \quad \forall v \in V_m$.

$\exists \Pi : K = \Pi^T L\,D\,U\Pi$
$\Rightarrow \exists\{\mu_1, \mu_2, \cdots, \mu_N\}$ s.t. $\quad P_m K\,P_m^T$: invertible on $V_m \quad 1 \le \forall m \le N$.

2×2 pivoting: $\quad V_{m-1}, \quad V_m, \quad V_{m+1}, \quad V_{m+2}, \quad V_{m+3},$ by skipping V_{m+1}.

FreeFEM implementation : 1/2

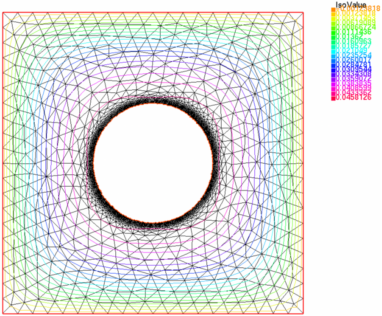
```
mesh Th; fespace Vh(Th, P1);      // background mesh
Vh f = (x * x + y * y) - 0.25;   // level set function
mesh Th0 = adaptmesh(Th, 1.0/(f * f + 1.e-6), hmin = 0.0025);
fesapce Vh0(Th, P1);
Vh0 f0 = f;                       // interpolation to adapted mesh
f0 = f0 > 0.0 ? 1.0 : f0;
f0 = f0 < 0.0 ? -1.0 : f0;
mesh Tha = trunc(Th0, f0 > -0.9; label = 2);
mesh Th1 = trunc(Tha, f0 < 0.9;  label = 3);

fespace Vha(Tha, P1);
fespace Vhd(Th1, P0);
fespace Vhd1(Th1, P1);            // for integration with levelset
Vhd1 f1 = f;
varf aa(u, v)=int2d(Tha)(Grad(u)'*Grad(v)) + on (1,u=0);
varf bb(u, lambda)=int2d(Thd, levelset = f1)(u * lambda);
varf ff(u, v)=int2d(Tha)(force * v);
matrix mata = aa(Vha, Vha);
matrix matb = bb(Vha, Vhd);       // bb(Vha, Vhd1) also works
matrix matbm = - matb;
real[int] zerov(Vhd.ndof); zerov = 0.0; matrix matd = zerov;
matrix matk = [[mata, matb'],     // skewed matrix
               [matb, matd]];
set(matk, solver="MUMPS_seq");
real[int] vecf = ff(0, Vha);
real[int] vecb = [vecf, zerov];
real[int] uu = matk^-1 * vecb;
```

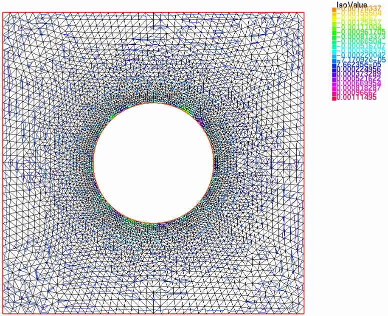
FreeFEM implementation : 2/2

```
int[int] edge0(nedges),edge1(nedges),elm0(nedges),elm1(nedges);
edgemask = 0;
int nnedges = 0;
for (int k = 0; k < Th1.nt; k++) {
  for (int l = 0; l < 3; ++l) {
    int ll = l;       int k1 = Th1[k].adj(ll);
    if (k != k1) {                   // find edge between elements
      int ed0 = Th1[k][l];
      int ed1 = Th1[k][(l + 1)%3];
      // ed0 < ed1 by exchanging if necessary
      // add edge (ed0, ed1) to a list of (edge0, edge1)
    } } }
for (int n = 0; n < nnedges; n++) {
  real edgelen = ((Th1(edge0[n]).x - Th1(edge1[n]).x)^2
                + (Th1(edge0[n]).y - Th1(edge1[n]).y)^2);
  int i = elm0[n];   int j = elm1[n];
  stabdense(i, i) += edgelen;
  stabdense(j, j) += edgelen;
  stabdense(i, j) -= edgelen;
  stabdense(j, i) -= edgelen;
}
matrix stab = stabdense;
real delta = 0.1;
matrix stabd = delta * stab;
matrix matk = [[mata, matb'],    // skewed matrix
               [matb, stabd]];
```

numerical example : 1/2



fictitious domain (3,339 +892 DOF)



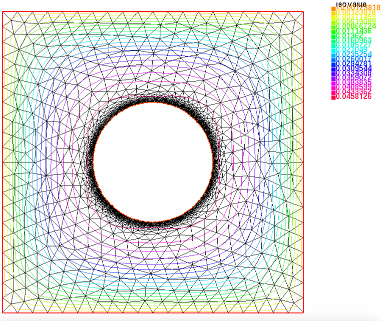
boundary fitted mesh (4,625 DOF)

$\delta = 0.1$, comparison with the solution in boddy fitted mesh
$||u_F - u_B||_2/||u_B||_2 = 0.0123319$

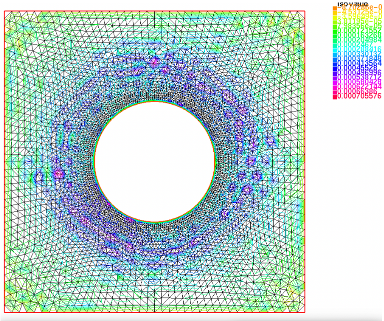
future work

- verification of the Stokes equation where Lagrange multiplier corresponding to the stress of the outer normal direction
- P1 approximation to the Largange multiplier ?

numerical example : 2/2



fictitious domain (3,339 +892 DOF)



boundary fitted mesh (4,625 DOF)

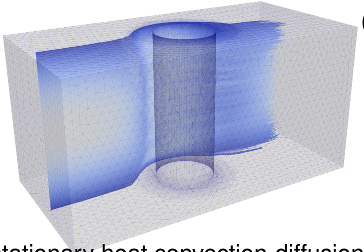
comparison with the solution in boddy fitted mesh
$||u_F - u_B||_2/||u_B||_2 = 0.0255968$

- `plugin/seq/MUMPS_seq.cpp` with modification by adding `ICNTL(14) = 200;` in `void fac_numeric()`
- approximation of Lagrange multiplier by P1 element without stabilization also finds solution
- how to introduce stabilization term for the multiplier in P1

heat exchanger with cylinder shape in Navier-Stokes flow

boundary $\Gamma_{\text{cylinder}} = \partial\omega$, $\omega \subset \Omega$: target sub-domain, $\Omega := \widetilde{\Omega} \setminus \overline{\omega}$

stationary Navier-Stokes equations without buoyancy term:



$$(u \cdot \nabla u) - \frac{1}{Re}\nabla \cdot \nabla u + \nabla p = 0 \text{ in } \Omega$$

$$\nabla \cdot u = 0 \text{ in } \Omega$$

$$u = u_0 \text{ on } \Gamma_0 := \Gamma_{\text{in}}$$

$$u = 0 \text{ on } \Gamma_1 := \Gamma_{\text{side}} \cup \Gamma_{\text{t/b}} \cup \partial\omega$$

$$\frac{2}{Re} D(u)n - n\,p = 0 \text{ on } \Gamma_2 := \Gamma_{\text{out}}$$

stress free boundary conditions

stationary heat convection-diffusion equation

$$(u \cdot \nabla)\theta - \kappa \nabla \cdot \nabla\theta = 0 \text{ in } \Omega \quad \kappa = \frac{1}{Re\,Pr}, \; Pr = \frac{\nu}{\alpha}$$

$$\theta = 0 \text{ on } \widehat{\Gamma}_0 := \Gamma_{\text{in}}$$

$$\theta = 1 \text{ on } \widehat{\Gamma}_1 := \Gamma_{\text{t/b}} \cup \partial\omega$$

$$\nabla\theta \cdot n = 0 \text{ on } \widehat{\Gamma}_2 := \Gamma_{\text{side}} \cup \Gamma_{\text{out}}$$

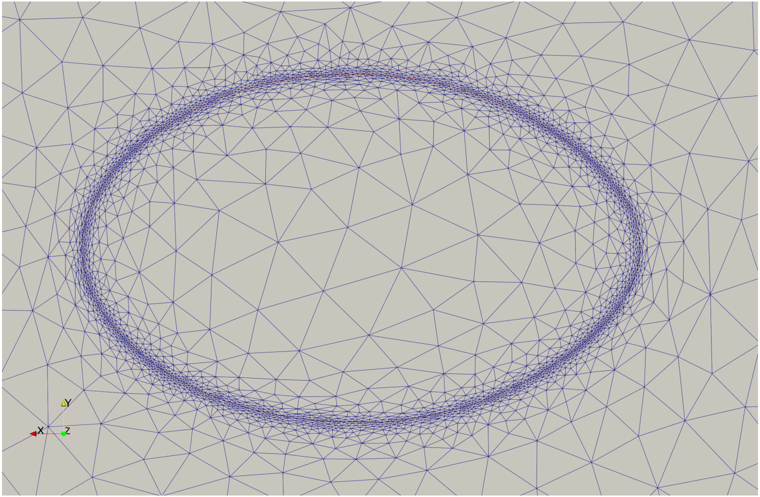
ϕ : shape deformation $\omega(\phi) = \phi(\widehat{\omega})$ from the reference sub-domain $\widehat{\omega}$

cost function:

$$J(\phi, \{u, p, \theta\}) = -\kappa \int_{\partial\omega(\phi)} \nabla\theta \cdot n \;\to \min$$

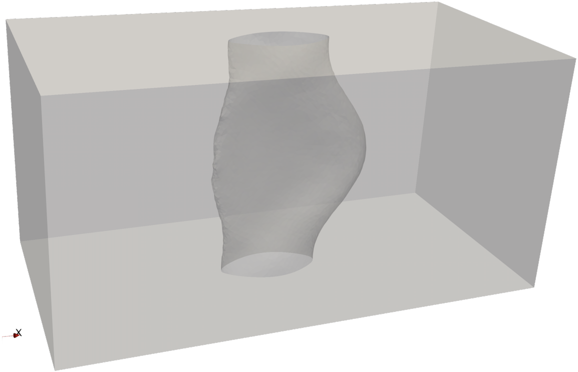
volume constraint + pressure drop for optimization

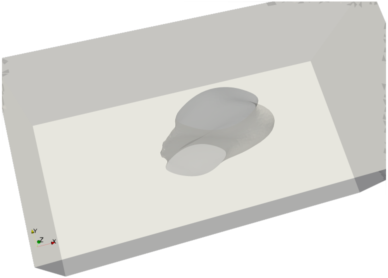
an-isotropic mesh near the interface

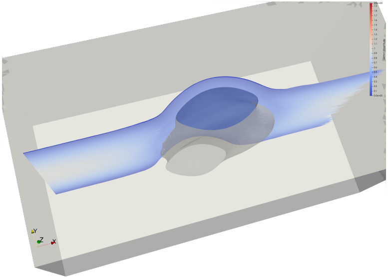


- mesh subdivisions need not to follow the boundary interface completely, but an-isotropic mesh refinement in the vicinity of the boundary is effective

$Re = 1$ without constraints on volume and pressure drop







$Re = 50$ without constraints on volume and pressure drop