# IMPLICITLY CONSTITUTED FLUID FLOW MODELS: ANALYSIS AND APPROXIMATION

Endre Süli

Mathematical Institute
University of Oxford

Acknowledgements:

Miroslav Bulíček (Prague), Josef Málek (Prague),
Lars Diening (Bielefeld), Christian Kreuzer (Dortmund),
Tabea Tscherpel (Bielefeld), Alexei Gazca Orozco (Oxford), Patrick Farrell (Oxford)

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Continuum mechanics of viscous fluids (at uniform temperature):

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(\varrho \boldsymbol{u})_t + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{T} &= \varrho \boldsymbol{f} \qquad \text{with } \boldsymbol{T} = \boldsymbol{T}^{\mathrm{T}}, \\
\varrho_t + \operatorname{div}(\varrho \boldsymbol{u}) &= 0, \\
\boldsymbol{T} : \boldsymbol{D}(\boldsymbol{u}) - \varrho \frac{\mathrm{d}}{\mathrm{d}t} \psi(\varrho) &=: \xi \qquad \text{with } \xi \geq 0.
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| | |
|---|---|
| $\varrho$ | density |
| $\boldsymbol{u}$ | velocity |
| $\boldsymbol{T}$ | Cauchy stress tensor |
| $\boldsymbol{f}$ | density of the external force |
| $\boldsymbol{D}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}})$ | rate of strain tensor |
| $\psi = \psi(\varrho)$ | Helmholtz free energy |
| $\xi$ | entropy production |

Special case: isothermal, homogeneous fluid at steady state:

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$$\begin{aligned} \xi := \boldsymbol{T} : \boldsymbol{D}(\boldsymbol{u}) = \boldsymbol{S} : \boldsymbol{D}(\boldsymbol{u}) = \mu |\boldsymbol{D}(\boldsymbol{u})|^2 + \tfrac{1}{4\mu} |\boldsymbol{S}|^2 &\geq \min\left(\mu, \tfrac{1}{4\mu}\right)(|\boldsymbol{D}(\boldsymbol{u})|^2 + |\boldsymbol{S}|^2) \\ &\geq 0. \qquad \checkmark \end{aligned}$$

# Examples of constitutive relations

- Power-law (Ostwald–de Waele) fluids:

$$\boldsymbol{S} = 2\mu|\boldsymbol{D}(\boldsymbol{u})|^{r-2}\,\boldsymbol{D}(\boldsymbol{u}), \qquad \text{with } 1 < r < \infty;$$

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- Activated fluids, such as Bingham and Herschel–Bulkley fluids ($\boldsymbol{D} := \boldsymbol{D}(\boldsymbol{u})$):

$$|\boldsymbol{S}| \le \tau_* \Leftrightarrow \boldsymbol{D} = \boldsymbol{0} \quad \text{and} \quad |\boldsymbol{S}| > \tau_* \;\Leftrightarrow\; \boldsymbol{S} = \frac{\tau_* \boldsymbol{D}}{|\boldsymbol{D}|} + 2\nu(|\boldsymbol{D}|^2)\,\boldsymbol{D}.$$

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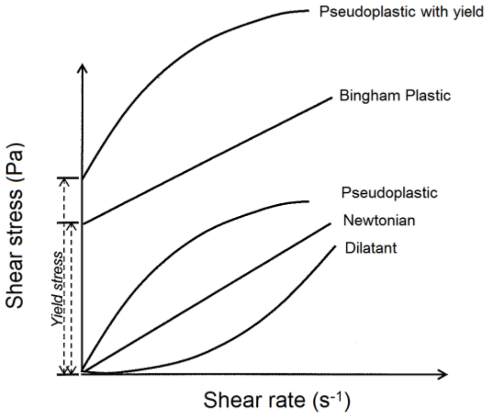
Equivalently:

$$2\nu(|\boldsymbol{D}|^2)\,(\tau_* + (|\boldsymbol{S}| - \tau_*)_+)\,\boldsymbol{D} = (|\boldsymbol{S}| - \tau_*)_+\,\boldsymbol{S}, \quad \tau_* > 0.$$

# Examples of constitutive relations between $S$ and $D$

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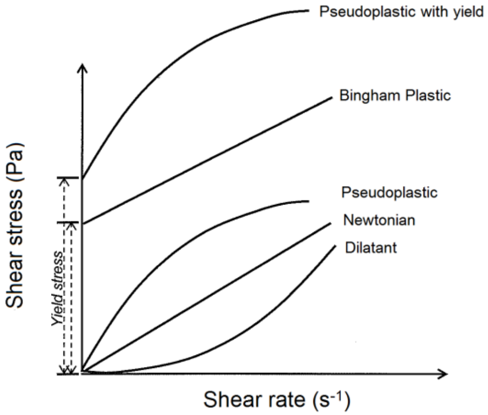


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K. R. Rajagopal. *On implicit constitutive theories for fluids*, J. Fluid Mech. **550** (2006), pp. 243–249.

K. R. Rajagopal & A. R. Srinivasa. *On the thermodynamics of fluids defined by implicit constitutive relations*, Z. Angew. Math. Phys. **59**(4) (2008), pp. 715–729.

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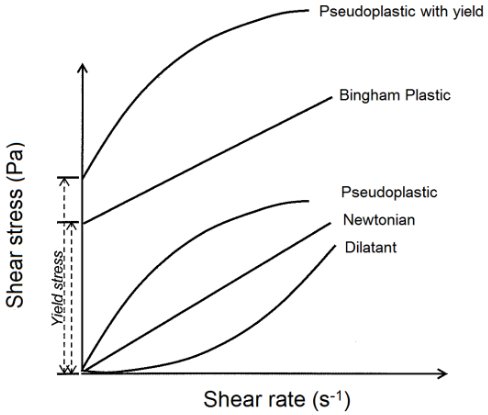


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M. Bulíček, P. Gwiazda, J. Málek, and A. Świerczewska-Gwiazda. *On steady flows of incompressible fluids with implicit power-law-like rheology.* Advances in Calculus of Variations, 2009.

M. Bulíček, P. Gwiazda, J. Málek, and A. Świerczewska-Gwiazda. *On unsteady flows of implicitly constituted incompressible fluids.* SIAM Journal on Mathematical Analysis, 2012.

# Problem

In a bounded open Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, consider:

$$\begin{aligned} \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{S} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega. \end{aligned} \tag{Eq}$$

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**Here:**

- $\boldsymbol{u} : \Omega \to \mathbb{R}^d$: velocity of the fluid;
- $p : \Omega \to \mathbb{R}$: pressure;
- $\boldsymbol{S} = \boldsymbol{S}^{\mathrm{T}} : \Omega \to \mathbb{R}^{d \times d}_{\mathrm{sym},0}$: shear stress and $\boldsymbol{D}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}})$ are assumed to be related through the implicit constitutive relation

$$\mathbf{G}(\boldsymbol{S}, \boldsymbol{D}(\boldsymbol{u})) = \mathbf{0},$$

where $\mathbf{G} : \mathbb{R}^{d \times d}_{\mathrm{sym},0} \times \mathbb{R}^{d \times d}_{\mathrm{sym},0} \to \mathbb{R}^{d \times d}_{\mathrm{sym},0}$.

We identify the implicit relation with a graph $\mathcal{A} \subset \mathbb{R}^{d\times d}_{\mathrm{sym},0} \times \mathbb{R}^{d\times d}_{\mathrm{sym},0}$:

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**(A3)** *$\mathcal{A}$ is a maximal monotone graph;* i.e., for any $(\boldsymbol{D}, \boldsymbol{S}) \in \mathbb{R}^{d\times d}_{\mathrm{sym},0} \times \mathbb{R}^{d\times d}_{\mathrm{sym},0}$,

$$\text{if } (\boldsymbol{S} - \tilde{\boldsymbol{S}}) : (\boldsymbol{D} - \tilde{\boldsymbol{D}}) \geq 0 \text{ for all } (\tilde{\boldsymbol{D}}, \tilde{\boldsymbol{S}}) \in \mathcal{A}, \text{ then } (\boldsymbol{D}, \boldsymbol{S}) \in \mathcal{A};$$

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**(A4)** $\mathcal{A}$ *is an $r$-graph;* i.e., there exists a constant $C_1 > 0$ s.t.:

$$\boldsymbol{S} : \boldsymbol{D} \geq C_1(|\boldsymbol{D}|^r + |\boldsymbol{S}|^{r'}) \text{ for all } (\boldsymbol{D},\boldsymbol{S}) \in \mathcal{A},$$

where $\frac{1}{r} + \frac{1}{r'} = 1$ and $1 < r < \infty$.

# 2. Numerical approximation

- Approximate the implicit constitutive law $\boldsymbol{S}$ by a sequence of explicit laws $\boldsymbol{S}^n$ by convolving a selection $\boldsymbol{S}^*$ with compactly supported $\theta_n \in L^1(\mathbb{R}; \mathbb{R}_{\geq 0})$.

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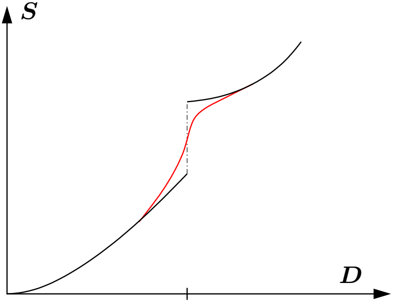

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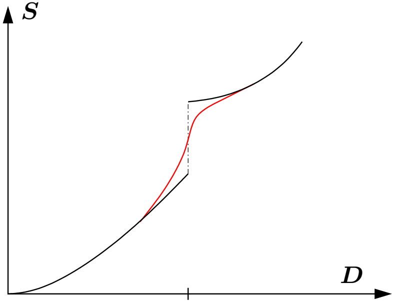
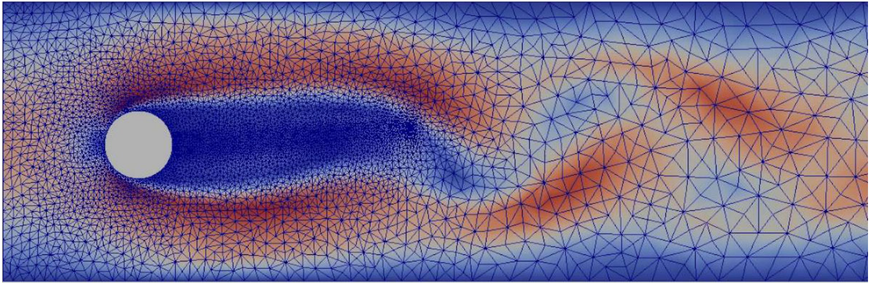



Alternative: Yosida regularization of the graph $\mathcal{A} \subset \mathbb{R}^{d\times d}_{\mathrm{sym},0} \times \mathbb{R}^{d\times d}_{\mathrm{sym},0}$:

$$\mathcal{A}^n := \left\{ \left(\boldsymbol{D} + \frac{1}{n}|\boldsymbol{S}|^{\frac{2-q}{q-1}}\boldsymbol{S}, \boldsymbol{S}\right) \in \mathbb{R}^{d\times d}_{\mathrm{sym},0} \times \mathbb{R}^{d\times d}_{\mathrm{sym},0} : (\boldsymbol{D}, \boldsymbol{S}) \in \mathcal{A} \right\}, \quad q \in (1, \infty).$$

# Computational grid: grid-size $h \ll 1$



Discretize the resulting PDE by a family of finite element spaces $\mathbb{V}_h \times \mathbb{Q}_h$ contained in $W_0^{1,r}(\Omega)^d \times L_0^{\tilde{r}}(\Omega)$, on conforming and shape-regular grids $\mathcal{T}_h$, where the union of $\{\mathbb{V}_h\}_{h>0}$ and $\{\mathbb{Q}_h\}_{h>0}$ is dense in the respective space.

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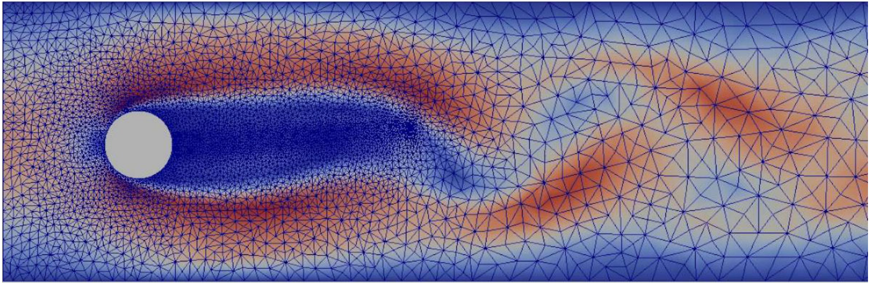

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Notation: $$\tilde{r} = \begin{cases} \min\left\{r', \frac{dr}{2(d-r)}\right\}, & \text{if } r < d, \\ r', & \text{if } r \geq d \end{cases} \qquad \leq r'.$$

# Formal energy (in)equality

Take scalar product of momentum eqn. with $\boldsymbol{v} = \boldsymbol{u}$, $\operatorname{div} \boldsymbol{u} = 0$:

$$\int_\Omega \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \boldsymbol{u} \, \mathrm{d}x + \int_\Omega \boldsymbol{S} : \boldsymbol{D}(\boldsymbol{u}) \, \mathrm{d}x - \int_\Omega p \, (\operatorname{div} \boldsymbol{u}) \, \mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}x$$

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# Formal energy (in)equality

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$$\int_\Omega \boldsymbol{S} : \boldsymbol{D}(\boldsymbol{u}) \, \mathrm{d}x \qquad = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}x$$

$$C_1 \left( \|\boldsymbol{D}(\boldsymbol{u})\|^r_{L^r(\Omega)} + \|\boldsymbol{S}\|^{r'}_{L^{r'}(\Omega)} \right) \leq \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}x$$

$$C_1 \left( \|\boldsymbol{D}(\boldsymbol{u})\|^r_{L^r(\Omega)} + \|\boldsymbol{S}\|^{r'}_{L^{r'}(\Omega)} \right) \leq \|\boldsymbol{f}\|_{W^{-1,r'}(\Omega)} \, \|\boldsymbol{D}(\boldsymbol{u})\|_{L^r(\Omega)}$$

$$\|\boldsymbol{D}(\boldsymbol{u})\|^r_{L^r(\Omega)} + \|\boldsymbol{S}\|^{r'}_{L^{r'}(\Omega)} \leq C_2 \|\boldsymbol{f}\|^{r'}_{W^{-1,r'}(\Omega)}.$$

$\longrightarrow$ This argument must now be replicated at the discrete level.

# Choice of the finite element spaces $\mathbb{V}_h$ and $\mathbb{Q}_h$

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**Problem (in the case of Option (A)):**

In general $\operatorname{div}\boldsymbol{u}_h \not\equiv 0$ pointwise on $\Omega$, and thus

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**Fix (in the case of option (A)): replace $\int_\Omega \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{v})\cdot\boldsymbol{w}\,\mathrm{d}x$ by**

$$\mathcal{B}[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] := \frac{1}{2}\int_\Omega \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{v})\cdot\boldsymbol{w} - \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{w})\cdot\boldsymbol{v}\,\mathrm{d}x$$

$$\swarrow \qquad\qquad\qquad \searrow$$

$$\mathcal{B}[\boldsymbol{v},\boldsymbol{v},\boldsymbol{v}] = 0 \quad \forall\boldsymbol{v}, \qquad \mathcal{B}[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] = \int_\Omega \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{v})\cdot\boldsymbol{w}\,\mathrm{d}x \ \text{ if } \operatorname{div}\boldsymbol{u} \equiv 0.$$

# 3. Convergence theorem

For $n \in \mathbb{N}$, compute $(\boldsymbol{u}_{n,h}, p_{n,h}) \in \mathbb{V}_h \times \mathbb{Q}_h \subset W_0^{1,r}(\Omega)^d \times L_0^{\tilde{r}}(\Omega)$, s.t.:

$$\begin{aligned} \int_\Omega \boldsymbol{S}^n(\boldsymbol{D}(\boldsymbol{u}_{n,h})) : \boldsymbol{D}\boldsymbol{v}_h \, \mathrm{d}x + \mathcal{B}[\boldsymbol{u}_{n,h}, \boldsymbol{u}_{n,h}, \boldsymbol{v}_h] &= \int_\Omega p_{n,h} \operatorname{div} \boldsymbol{v}_h + \boldsymbol{f} \cdot \boldsymbol{v}_h \, \mathrm{d}x \quad && \forall \boldsymbol{v}_h \in \mathbb{V}_h, \\ \int_\Omega q_h \operatorname{div} \boldsymbol{u}_{n,h} \, \mathrm{d}x &= 0 && \forall q_h \in \mathbb{Q}_h. \end{aligned}$$

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$$\int_\Omega q_h \operatorname{div} \boldsymbol{u}_{n,h} \,\mathrm{d}x = 0 \quad \forall q_h \in \mathbb{Q}_h.$$

**Theorem**

*Suppose that Case (A): $r > \frac{2d}{d+1}$; or Case (B): $r > \frac{2d}{d+2}$, $\frac{1}{r} + \frac{1}{r'} = 1$ and $f \in W^{-1,\tilde{r}}(\Omega)^d$. Then, as $n \to \infty$, $h \to 0$:*

$$\boldsymbol{u}_{n,h} \rightharpoonup \boldsymbol{u} \quad \text{in } W_0^{1,r}(\Omega)^d,$$

$$\boldsymbol{S}^n(\boldsymbol{D}(\boldsymbol{u}_{n,h})) \rightharpoonup \boldsymbol{S} \quad \text{in } L^{r'}(\Omega)^{d\times d}, \quad \text{and}$$

$$p_{n,h} \rightharpoonup p \quad \text{in } L_0^{\tilde{r}}(\Omega).$$

*Also, $(\boldsymbol{u}, p, \boldsymbol{S}) \in W_0^{1,r}(\Omega)^d \times L_0^{\tilde{r}}(\Omega) \times L^{r'}(\Omega)^{d\times d}$ is such that $(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{S}) \in \mathcal{A}$ and*

$$\int_\Omega \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \boldsymbol{v} \,\mathrm{d}x + \int_\Omega \boldsymbol{S} : \boldsymbol{D}(\boldsymbol{v}) \,\mathrm{d}x - \int_\Omega p\,(\operatorname{div} \boldsymbol{v}) \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall\, \boldsymbol{v} \in W_0^{1,\tilde{r}'}(\Omega)^d,$$

$$\int_\Omega (\operatorname{div} \boldsymbol{u})\, q \,\mathrm{d}x = 0 \quad \forall\, q \in L_0^{r'}(\Omega).$$

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Next, note that, taking $\boldsymbol{v}_h = \boldsymbol{u}_{n,h}$:

$$\|\boldsymbol{D}(\boldsymbol{u}_{n,h})\|_{L^r(\Omega)}^r + \|\boldsymbol{S}^n(\boldsymbol{D}(\boldsymbol{u}_{n,h}))\|_{L^{r'}(\Omega)}^{r'} \leq C \qquad \forall n \in \mathbb{N},\ \forall h > 0.$$

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Hence, by the Bolzano–Weierstrass thm. and Alaoglu's thm. + Minty's method, as $n \to \infty$, $h > 0$ fixed,

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{u}_{n,h}) &\to \boldsymbol{D}(\boldsymbol{u}_h) && \text{in } L^r(\Omega)^{d\times d}, \\ \boldsymbol{S}^n(\boldsymbol{D}(\boldsymbol{u}_{n,h})) &\rightharpoonup \boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}_h)) && \text{in } L^{r'}(\Omega)^{d\times d}, \qquad \tfrac{1}{r} + \tfrac{1}{r'} = 1. \end{aligned}$$

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We also pass to the limit $n \to \infty$ in the above inequality.

STEP 2. $[h \to 0]$

It follows from STEP 1 by (weak) lower semicontinuity that

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Question: $(\boldsymbol{D}(\boldsymbol{u}), \overline{\boldsymbol{S}}) \in \mathcal{A}$ a.e. in $\Omega$?

In other words: is $\overline{\boldsymbol{S}} = \boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}))$?

STEP 3. To show that $(\boldsymbol{D}(\boldsymbol{u}), \overline{\boldsymbol{S}}) \in \mathcal{A}$ a.e. in $\Omega$, we consider

$$a_h := (\boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}_h)) - \boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}))) : \boldsymbol{D}(\boldsymbol{u}_h - \boldsymbol{u}).$$

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We apply Chacon's biting lemma[1] to find a nondecreasing countable sequence of measurable sets $\Omega_1 \subset \cdots \subset \Omega_k \subset \Omega_{k+1} \subset \cdots \subset \Omega$ such that

$$\lim_{k\to\infty} |\Omega \setminus \Omega_k| \to 0$$

and such that for any $k$ there is a subsequence such that

$$a_h := (\boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}_h)) - \boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}))) : \boldsymbol{D}(\boldsymbol{u}_h - \boldsymbol{u}) \quad \text{converges weakly in } L^1(\Omega_k).$$

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E. Acerbi & N. Fusco. An approximation lemma for $W^{1,p}$ functions. In: Material Instabilities in Continuum Mechanics (Edinburgh, 1985-1986), pp. 15, OUP, 1988.

For $\boldsymbol{v} \in W_0^{1,1}(\Omega)^d$, consider the Hardy–Littlewood maximal fn.

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Indeed,

$$\begin{aligned} |\boldsymbol{v}(x) - \boldsymbol{v}(y)| &\leq c \, (M(\nabla \boldsymbol{v})(x) + M(\nabla \boldsymbol{v})(y)) \, |x - y| \\ &\leq c \, \lambda \, |x - y| \qquad \forall \, x, y \in \mathcal{G}_\lambda. \end{aligned}$$

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Extend $\boldsymbol{v}|_{\mathcal{G}_\lambda}$ to $\boldsymbol{v}^\lambda \in W_0^{1,\infty}(\Omega)^d$ using Kirszbraun's Extension Theorem.

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Consider a locally supported, discretely divergence-preserving, and stable projector $\Pi_h : W_0^{1,1}(\Omega)^d \to \mathbb{V}_h \subset W_0^{1,1}(\Omega)^d$:

$$\int_\Omega (\operatorname{div} \boldsymbol{v})\, q_h \,\mathrm{d}x = \int_\Omega (\operatorname{div} \Pi_h \boldsymbol{v})\, q_h \,\mathrm{d}x \qquad \forall q_h \in \mathbb{Q}_h.$$

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Consider

$$\boldsymbol{e}_h := \Pi_h(\boldsymbol{u}_h - \boldsymbol{u}).$$

# Discrete Lipschitz truncation

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**Theorem (Diening, Kreuzer, Süli)**

*There exists* $\{\lambda_{h,j}\}_j \subset \mathbb{R}$ *with* $2^{2^j} \leq \lambda_{h,j} \leq 2^{2^{j+1}-1}$, *such that:*

$$\boldsymbol{e}_h^j := \Pi_h \boldsymbol{e}_h^{\lambda_{h,j}} \quad \big( + \textit{ terms to ensure that } \operatorname{div} \boldsymbol{e}_h^j = 0 \textit{ discretely} \big)$$

*satisfies:*

1. $\boldsymbol{e}_h^j \in W_0^{1,\infty}(\Omega)^d$ *with* $\|\nabla \boldsymbol{e}_h^j\|_{L^\infty(\Omega)} \leq c\lambda_{h,j}$;
2. $\boldsymbol{e}_h^j \to 0$ *in* $L^\infty(\Omega)^d$ *as* $h \to 0$;
3. $\nabla \boldsymbol{e}_h^j \rightharpoonup^* 0$ *in* $L^\infty(\Omega)^{d\times d}$ *as* $h \to 0$;
4. $\lambda_{h,j} |\boldsymbol{e}_h^j \neq \boldsymbol{e}_h|^{\frac{1}{r}} \lesssim 2^{-\frac{j}{r}} \|\nabla \boldsymbol{e}_h\|_{L^r(\Omega)}$, *with* $r > 1$.

L. Diening, Ch. Kreuzer & E. Süli. *Finite element approximation of steady flows of incompressible fluids with implicit power-law-like rheology.* SIAM J. Numer. Anal., 51(2), 984–1015, 2013.

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Recall:

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For

$$\boldsymbol{e}_h := \Pi_h(\boldsymbol{u}_h - \boldsymbol{u}) \in \mathbb{V}_h$$

let $\boldsymbol{e}_h^j$ be the discrete Lipschitz truncation of $\boldsymbol{e}_h$ at level $2^{2^j}$.

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let $\boldsymbol{e}_h^j$ be the discrete Lipschitz truncation of $\boldsymbol{e}_h$ at level $2^{2^j}$.

For a fixed $j \in \mathbb{N}$, we have by Hölder's inequality that

$$\begin{aligned}
\int_\Omega |a_h|^{1/2}\,\mathrm{d}x &= \int_{\{\boldsymbol{e}_h^j = \boldsymbol{e}_h\}} |a_h|^{1/2}\,\mathrm{d}x + \int_{\{\boldsymbol{e}_h^j \neq \boldsymbol{e}_h\}} |a_h|^{1/2}\,\mathrm{d}x \\
&\leq |\Omega|^{1/2} \underbrace{\Big( \int_{\{\boldsymbol{e}_h^j = \boldsymbol{e}_h\}} |a_h|\,\mathrm{d}x \Big)}_{\lesssim 2^{-j/r}}{}^{1/2} + \underbrace{|\{\boldsymbol{e}_h^j \neq \boldsymbol{e}_h\}|}_{\lesssim 2^{-2^j r}}{}^{1/2} \Big( \int_\Omega |a_h|\,\mathrm{d}x \Big)^{1/2}.
\end{aligned}$$

Hence,

$$\limsup_{h\to 0}\int_\Omega |a_h|^{1/2}\,\mathrm{d}x \le c\,|\Omega|^{1/2}\,2^{-j/(2r)} + c\,2^{-2^j r/2} \qquad \forall\, j\in\mathbb{N}.$$

Therefore, since $j$ was arbitrary, we deduce that for a subsequence:

$$a_h := (\boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}_h)) - \boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}))) : \boldsymbol{D}(\boldsymbol{u}_h - \boldsymbol{u}) \to 0 \qquad \text{a.e. in } \Omega.$$

STEP 5.

By Vitali's theorem we then deduce that

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The weak convergence of $\boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}_h))$ to $\overline{\boldsymbol{S}}$ and $\boldsymbol{D}(\boldsymbol{u}_h)$ to $\boldsymbol{D}(\boldsymbol{u})$ gives:

$$\lim_{h\to 0} \int_{\Omega_k} \boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}_h)) : \boldsymbol{D}(\boldsymbol{u}_h) \, \mathrm{d}x = \int_{\Omega_k} \overline{\boldsymbol{S}} : \boldsymbol{D}(\boldsymbol{u}) \, \mathrm{d}x.$$

## STEP 5.

By Vitali's theorem we then deduce that

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Finally, by a diagonal procedure and $\lim_{k\to\infty} |\Omega \setminus \Omega_k| \to 0$ we deduce that

$$(\boldsymbol{D}(\boldsymbol{u}), \overline{\boldsymbol{S}}) \in \mathcal{A} \qquad \text{a.e. in } \Omega. \qquad \square$$

# Extensions

- T. Tscherpel & E. Süli. *Finite element approximation of unsteady flows of incompressible implicitly constituted fluids.* IMA J. Numer. Anal., 30(2), 2020.
- P. Farrell, A. Gazca-Orozco & E. Süli. *Numerical analysis of unsteady implicitly constituted incompressible fluids: three-field formulation.* SIAM J. Numer. Anal. (58)1, 757–787, 2020.
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## Note

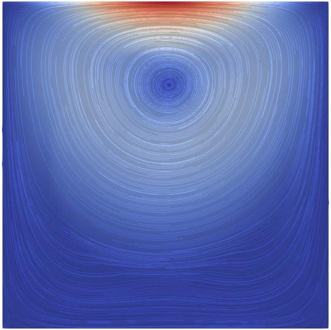
All numerical simulations shown on the next slides were performed by

Alexei Gazca-Orozco (Oxford (now at Erlangen–Nürnberg)).

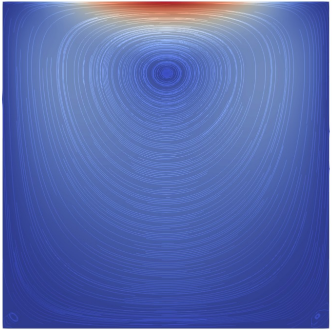
Finite element software: `firedrake`

# Unsteady Navier–Stokes/Euler activated fluid

$$\boldsymbol{D} = \begin{cases} \begin{cases} \delta_s \frac{\boldsymbol{S}}{|\boldsymbol{S}|} + \frac{1}{2\mu}\boldsymbol{S} & \text{if } |\boldsymbol{D}| > \tau_y \\ \boldsymbol{S} = \boldsymbol{0} & \text{if } |\boldsymbol{D}| \leq \tau_y \end{cases} & \text{if } (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \leq \left(\frac{9}{64}\right)^2, \\ \frac{1}{2\mu}\boldsymbol{S} & \text{otherwise} \end{cases}$$



(a) Navier–Stokes fluid



(b) Navier–Stokes/Euler activated fluid

$$\Omega = (0,1)^2, \quad \delta_s = 2.5, \quad \mu = 0.5, \quad \tau_m = 5 \times 10^{-6}, \qquad \mathbb{P}_2 \times \mathbb{P}_1 \times \mathbb{P}_{-1}.$$

# 5. Nonmonotone constitutive laws

The problem under consideration in $Q := (0,T) \times \Omega$ is:

$$\begin{aligned}
\boldsymbol{u}_t - \operatorname{div}(\boldsymbol{S} - \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \boldsymbol{f} \qquad & \text{in } Q, \\
\operatorname{div} \boldsymbol{u} = 0 \qquad & \text{in } Q, \\
\boldsymbol{D}(\boldsymbol{u}) = \mathcal{D}(\boldsymbol{S}) \qquad & \text{in } Q, \\
\boldsymbol{u} = \boldsymbol{0} \qquad & \text{in } (0,T) \times \partial\Omega, \\
\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0(\cdot) \qquad & \text{in } \Omega.
\end{aligned}$$

Here $\mathcal{D} : \mathbb{R}^{d \times d}_{\mathrm{sym},0} \to \mathbb{R}^{d \times d}_{\mathrm{sym},0}$ is a continuous function that satisfies:

- $\mathcal{D}(\boldsymbol{0}) = \boldsymbol{0}$;
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$$\begin{aligned}
c\left(|\boldsymbol{D}|^r + |\boldsymbol{S}|^{r'}\right) \leq \boldsymbol{S} : \boldsymbol{D} \qquad & \text{whenever } \boldsymbol{D} = \mathcal{D}(\boldsymbol{S}), \\
|\mathcal{D}(\boldsymbol{S})| \leq C(1 + |\boldsymbol{S}|^{r'-1}) \qquad & \forall\, \boldsymbol{S} \in \mathbb{R}^{d \times d}_{\mathrm{sym},0}.
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David & Filip (2004), Galindo-Rosales et al. (2011), Fardin et al. (2012), Divoux et al. (2016).

The nonmonotone response seems crucial in modelling complex non-Newtonian phenomena such as shear banding.

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|\mathcal{D}(\boldsymbol{S})| &\leq C(1 + |\boldsymbol{S}|^{r'-1}) && \forall\, \boldsymbol{S}\in\mathbb{R}^{d\times d}_{\mathrm{sym},0}.
\end{aligned}$$

Notation: For a probability measure $\nu\in\mathcal{P}(\mathbb{R}^{d\times d}_{\mathrm{sym},0})$ we write

$$\langle \nu, f(\tilde{\boldsymbol{S}})\rangle := \int_{\mathbb{R}^{d\times d}_{\mathrm{sym},0}} f(\boldsymbol{S})\,\mathrm{d}\nu(\boldsymbol{S}) \qquad \forall\, f\in C_0(\mathbb{R}^{d\times d}_{\mathrm{sym},0}).$$

## Definition

A couple $(\boldsymbol{S}, \boldsymbol{u})$ is called a Young-measure solution of the problem if

(a) $\boldsymbol{S} \in L^{r'}_{\mathrm{sym},0}(Q)^{d\times d}, \qquad \boldsymbol{u} \in L^r(0,T; W^{1,r}_{0,\mathrm{div}}(\Omega)^d) \cap C_w([0,T]; L^2_{\mathrm{div}}(\Omega)^d),$

$\boldsymbol{u}_t \in L^{\tilde{r}'}(0,T; (W^{1,\tilde{q}}_{0,\mathrm{div}}(\Omega)^d)')$ satisfy the eqn.

$$-\int_0^T \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}_t + \int_0^T \int_\Omega (\boldsymbol{S} - \boldsymbol{u} \otimes \boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) = \int_0^T \langle \boldsymbol{f}, \boldsymbol{v} \rangle + \int_\Omega \boldsymbol{u}_0 \cdot \boldsymbol{v}(0,\cdot)$$

$$\forall\, \boldsymbol{v} \in C^\infty_{0,\mathrm{div}}((-T,T) \times \Omega)^d;$$

(b) There exists a Young-measure $\{\nu_{\boldsymbol{z}}\}_{\boldsymbol{z}\in Q} \subset \mathcal{P}(\mathbb{R}^{d\times d}_{\mathrm{sym},0})$ such that

$$\boldsymbol{S} = \langle \nu_\bullet, \tilde{\boldsymbol{S}} \rangle, \qquad \boldsymbol{D}(\boldsymbol{u}) = \langle \nu_\bullet, \mathcal{D}(\tilde{\boldsymbol{S}}) \rangle \qquad \text{a.e. on } Q.$$

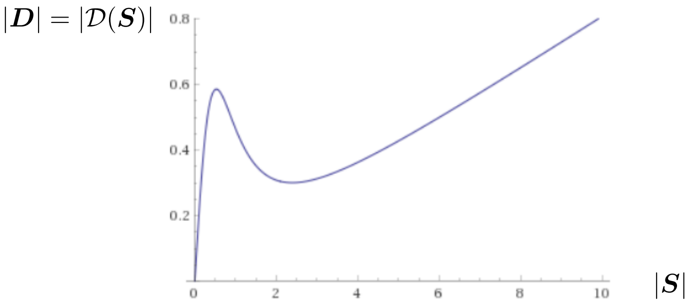
### Theorem

*Suppose* $r > \frac{2d}{d+2}$, $\boldsymbol{u}_0 \in L^2_{\mathrm{div}}(\Omega)^d$, *and* $\boldsymbol{f} \in L^{r'}(0,T;W^{-1,r'}(\Omega)^d)$. *Then, the numerical method admits a solution* $(\boldsymbol{S}_j^{m,n}, \boldsymbol{u}_j^{m,n}) \in \Sigma^n \times V^n_{\mathrm{div}}$, $j \in \{1,\dots,T/\tau_m\}$, $m \in \mathbb{N}$. *Furthermore, there is a subsequence (not indicated) such that the piecewise constant in time interpolants satisfy*

$$\begin{aligned}
\overline{\boldsymbol{S}}^{m,n} &\rightharpoonup \boldsymbol{S} && \text{weakly in } L^{r'}_{\mathrm{sym},0}(Q)^{d\times d},\\
\overline{\boldsymbol{u}}^{m,n} &\rightharpoonup \boldsymbol{u} && \text{weakly* in } L^{\infty}(0,T;L^2_{\mathrm{div}}(\Omega)^d),\\
\overline{\boldsymbol{u}}^{m,n} &\rightharpoonup \boldsymbol{u} && \text{weakly in } L^{r}(0,T;W^{1,r}_{0,\mathrm{div}}(\Omega)^d),\\
\overline{\boldsymbol{u}}^{m,n} &\to \boldsymbol{u} && \text{strongly in } L^{2}(Q)^d,
\end{aligned}$$

*where* $(\boldsymbol{S}, \boldsymbol{u})$ *is a Young-measure solution of the problem.*

# An example

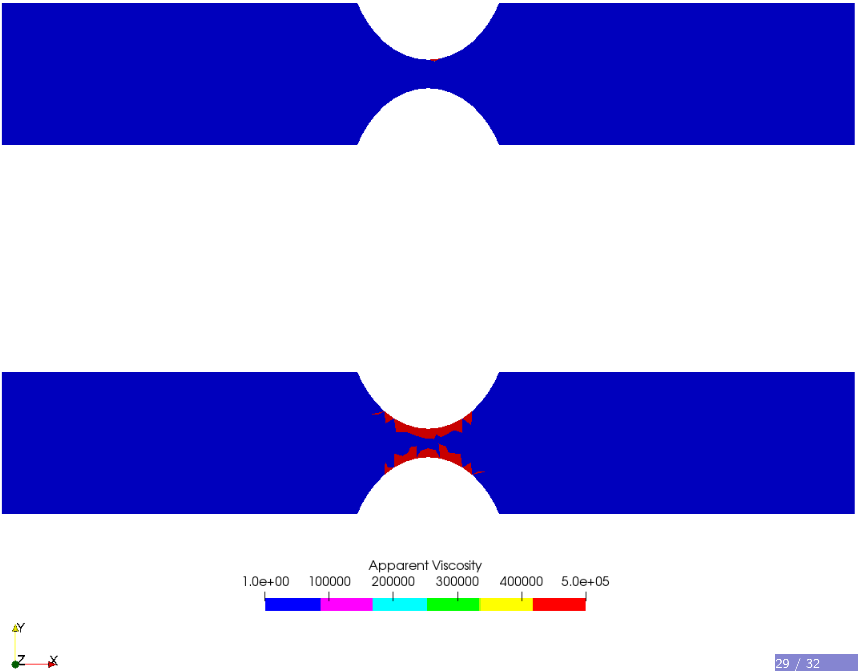


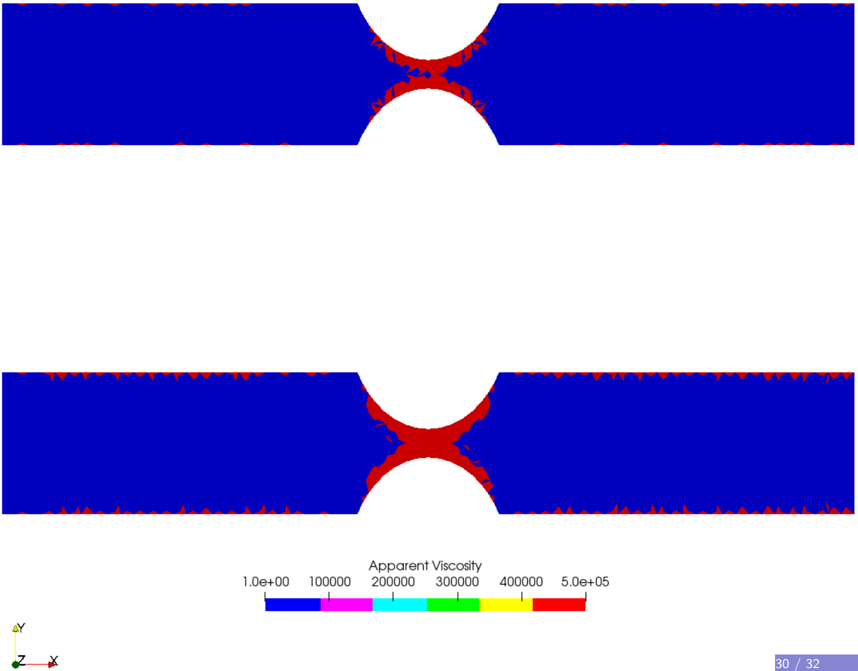

$$\mathcal{D}(\boldsymbol{S}) = \alpha_1 (1 + \beta_1 |\boldsymbol{S}|^2)^{\frac{2-q}{2(q-1)}} \boldsymbol{S} + \alpha_2 (1 + \beta_2 |\boldsymbol{S}|^2)^{\frac{2-r}{2(r-1)}} \boldsymbol{S}$$

where $q \in (-\infty, -1)$, $r \in (1, \infty)$, and $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$.

We shall plot the *apparent viscosity* $|\boldsymbol{S}|/|\boldsymbol{D}|$.

A. Janečka, J. Málek, V. Průša & G. Tierra, Acta Mech., 230, 729–747 (2019).


Apparent Viscosity
1.0e+00
100000
200000
300000
400000
5.0e+05
Y
Z
X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1.0e+00
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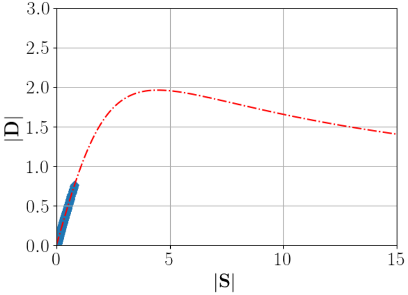
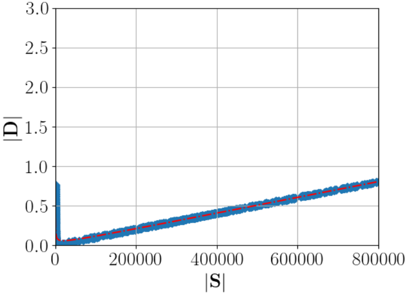



(e) Monotone ↑ part 1 of the graph




(f) Monotone ↑ part 2 of the graph

Numerical simulations by Alexei Gazca (Oxford/Erlangen–Nürnberg)

finite element software: `firedrake`

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