Numerical analysis and simulation of the spreading and dispersion using FreeFem++

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Mathematical modeling and stochastic simulation are important tools in the prediction of pollutant transport in groundwater. We consider here the steady flow in a porous medium without source:

$$\begin{cases} v(\omega,x) = -a(\omega,x)\nabla p(\omega,x), & \text{in } \Omega \times D, \\ \operatorname{div}(v)(\omega,x) = 0 & \text{in } \Omega \times D, \\ \text{Boundary conditions} \end{cases} \tag{1}$$

where $(\Omega, \mathcal{F}, dP)$ is a complete probability space, $D \subset \mathbb{R}^d$. The permeability $a = e^G$ is log-normal field, where G is a Gaussian field defined by an exponential covariance function. The velocity v is used in a transport model of an inert solute governed by advection-dispersion:

$$\begin{cases} \partial_t c(\omega,t,x) + v(\omega,x)\nabla c(\omega,t,x) - D_m \triangle c(\omega,t,x) = 0, & \text{in } \Omega \times [0,T] \times D \\ c(\omega,0,x) = c_0(x), \quad x \in D \\ \text{Boundary conditions,} \end{cases} \tag{2}$$

where c is the concentration of the solute, D_m is the diffusion coefficient and c_0 is the initial condition at $t = 0$. Equations (2), (1) can be completed with Dirichlet, mixed or periodic boundary conditions on ∂D. From this model, we are interested in the approximation of the mean spread and the mean dispersion of the solute[1, 3, 4].

The flow model is solved using a mixed finite element method in the physical space. The transport equation is computed thanks to a random walk method, where the concentration of the solute is the law of a stochastic process. This process is a solution of a stochastic differential equation (SDE), which is discretized using an Euler scheme. Then, the mean of the spread and of the dispersion are expressed as functions of the approximate stochastic process. A priori error analysis is provided in [3]. We use the Freefem++ software [2] to solve the flow equation where the linear flow system is solved with the direct algorithm implemented in the software UMFPACK. We implemented, also with Freefem++ framework, the random walk of the transport equation.

References

[1] A. Beaudoin, J.-R. de Dreuzy, J. Erhel and G. Pichot *"Convergence analysis of macro spreading in 3D heterogeneous porous media"*, ESAIM Proceedings. December 2013, Vol. 41, p. 59-76. 251-265. 65Y15

[2] J. Erhel, Z. Mghazli, M. Oumouni. *" Numerical analysis of stochastic advection-diffusion equation via Karhunen-Loève expansion"*. [INRIA Research Report] 2014, hal-01056785.

[3] J. Erhel, Z. Mghazli, M. Oumouni *" A combined collocation and Monte Carlo method for advection-diffusion equation of a solute in random porous media"* ESAIM-Proceedings, Vol. 45, p. 328-337, 2014.

[4] F. Hecht, O. Pironneau, A. Le Hyaric and K. Ohtsuka. *"FreeFem++ Manual"*, 2011. http://www.freefem.org/ff++/index.htm

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