Numerical study of a regularized $3d$ Boussinesq system

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Introduction

This presentation is a brief description of some numerical simulations of a regularized 3d Boussinesq system using Freefeem++.
In fact, we try to refound the analytical results in a previous work published as

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Introduction

Analytical study of attractors to a regularized 3D Boussinesq system

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Abstract

The Leray-alpha model to regularize the forced three-dimensional periodic Boussinesq system is shown to have a global in time unique solution and a universal attractor which is connected. Energy method, compactness argument and absorbing balls are used.

1 Introduction

In this paper, we consider the following Leray-alpha regularized three-dimensional periodic Boussinesq system that we denote by (Bq_α):

$$\partial_t v - \nu\Delta v + (v\cdot\nabla)u = -\nabla p + \theta e_3, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3 \tag{1.1}$$

$$\partial_t\theta - \kappa\Delta\theta + (u\cdot\nabla)\theta = f, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3 \tag{1.2}$$

$$v = u - \alpha^2\Delta u, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3 \tag{1.3}$$

$$\operatorname{div} u = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3 \tag{1.4}$$

$$(u,\theta)|_{t=0} = (u^0,\theta^0), \quad x\in\mathbb{T}^3, \tag{1.5}$$

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Key words and phrases : Three-dimensional regularized periodic Boussinesq system, global compact connected attractor, asymptotic behavior, global existence, global uniqueness, dependence on initial data.

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Problem description

In this work, the following **Leray-α regularized three-dimensional periodic Boussinesq system** is considered and denoted by (Bq_α):

$$\partial_t v - \nu\Delta v + (v\cdot\nabla)u = -\nabla p + \theta e_3, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3, \tag{2.1}$$

$$\partial_t\theta - \kappa\Delta\theta + (u\cdot\nabla)\theta = f, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3, \tag{2.2}$$

$$v = u - \alpha^2\Delta u, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3, \tag{2.3}$$

$$\operatorname{div} u = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3, \tag{2.4}$$

$$(u,\theta)|_{t=0} = (u_0,\theta_0), \quad x\in\mathbb{T}^3. \tag{2.5}$$

where the equation 2.1 represents the momentum equation. The temperature equation is given by equation 2.2. The regularization process is defined by the equation 2.3 to get smoother velocity. The equation 2.4 is the incompressibility condition.

Variables and Functional setting:

Unknown velocity ,the vector u

Pressure is the scalar p

Temperature is the scalar θ

The viscosity is $\nu > 0$

Thermal conductivity is $\kappa > 0$

The regularizing parameter is $\alpha > 0$

The domain is three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/2\pi)^3$

- $u_0 \in \dot{H}^1(\mathbb{T}^3)$ is supposed a divergence free initial velocity. We recall that the homogeneous Sobolev spaces are defined by

$$\dot{H}^s(\mathbb{T}^3) = \{\widehat{u} \in \mathcal{S}'(\mathbb{T}^3) : \sum_{k\in^3} |k|^{2s}|\widehat{u}(k)|^2 < \infty\},$$

and are endowed with the natural norm

$$\|u\|\dot{H}^s(\mathbb{T}^3) = \Big(\sum k \in^3 |k|^{2s}|\widehat{u}(k)|^2\Big)^{1/2}.$$

- $\theta_0 \in L^2(\mathbb{T}^3)$ a mean free initial temperature.
- f a mean free scalar heating source that belongs to $L^2(\mathbb{R}_+; L^2(\mathbb{T}^3))$

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Physical description of the studied model:

- The original problem was derived by Navier-Stokes to describe the fluid motion by application of Newton second law of dynamics.
- A coupling of Navier-Stokes equations with temperature equation is developed rigorously by Boussinesq in the the 19th century.
- The analytical study of Boussinesq systems is based on the investigation of well posedness of Navier-Stokes equations. The theory of existence and unicity of solutions is still incomplete in the $3d$ case.
- Various regularization methods where established to overcome the ill posedness of the problem and construct a rigorous mathematical theory.
- Our regularization method is based on considering the inverse of Helmotz operator in the framework of **Leray-alpha model**. It consists on defining a smoothing velocity. Such regularization is frequently used to study the turbulence flows.

Applications of the attractors

The mathematical study of attractors can be a robust strategy to guarantee a convenient prediction for any evolving system.

Physical applications
- Geophysical fluids like oceanographic turbulence and atmospheric fronts
- Fluid flows
- Water wave dynamics
- Turbulence flow dynamic.

Technological applications The analytical study of attractors consists on studying the states of various dynamic models.

- Analyze network state space
- Studying the behavior of various agent systems in AI.
- Controlling the state stability of players.

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Our results:

We are trying to refound the first theorem the existence of a unique solution global in time and proving the convergence of the regularized problem to the original Boussinesq system.

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First theorem (Existence of a global in time solution)

Theorem

Let $u_0 \in \dot{H}^1(\mathbb{T}^3)$ be a divergence free initial velocity, $\theta_0 \in L^2(\mathbb{T}^3)$ a mean free initial temperature and f a mean free scalar heating source that belongs to $L^2(\mathbb{R}_+; L^2(\mathbb{T}^3))$. Then a global in time weak solution $(u_\alpha, \theta_\alpha)$ exists to system (2.1)–(2.5), such that

$$u_\alpha \in C(\mathbb{R}_+; \dot{H}^1(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; \dot{H}^2(\mathbb{T}^3)),$$

$\theta_\alpha \in C(\mathbb{R}_+; L^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; \dot{H}^1(\mathbb{T}^3))$. Moreover, this solution satisfies the energy estimate

$$\begin{aligned}\|\theta_\alpha(t)\|_{L^2}^2 + \|u\alpha(t)\|_{L^2}^2 + \alpha^2\|\nabla u\alpha(t)\|_{L^2}^2 &+ \kappa \int_0^t \|\nabla\theta_\alpha(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \\ &+ \nu \int_0^t \|\nabla u_\alpha(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \\ &+ 2\nu\alpha^2 \int_0^t \|\Delta u_\alpha(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leq C,\end{aligned} \tag{5.1}$$

First theorem (Existence of a global in time solution

where

$$C = C(\alpha, \nu, \kappa, u_0, \theta_0, f) = \|u_0\|_{L^2}^2 + \alpha^2 \|\nabla u_0\|_{L^2}^2 \\ + \Big(1 + \frac{1}{2\nu\kappa}\Big)\Big(\|\theta_0\|_{L^2}^2 + \frac{1}{\kappa}\|f\|_{L^2(L^2)}^2\Big). \tag{5.2}$$

Furthermore, if the initial velocity is mean free this weak solution is continuously dependent on the initial data (u_0, θ_0) for all $t \in \mathbb{R}_+$. In particular, it is unique.

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Theorem 2 : Existence of attractor

Theorem

Under the hypothesis of Theorem 1, there is a compact global attractor $\mathcal{A} \subset L^2(\mathbb{T}^3) \times \dot{H}^1(\mathbb{T}^3)$*, in terms of the solution* $(u_\alpha, \theta_\alpha)$*, for the system* (2.1)–(2.5)*. Moreover, this attractor is connected.*

The proof uses the absorbing balls method and is based on energy estimates in appropriate functional spaces.

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We consider as a first mesh example a cubic mesh, we will later consider a 3d Torus.
we used P1 discretisation spaces for the velocity, temperature, and we are testing also the P2 elements to study the mesh refinement iterations.
As a first result, we compute the L2 norm error associated to the velocity and the L2 norm associated to the temperature and we considered many values of alpha.
we found that as the regularization parameter goes to zero, the error norms decrease which is confirm with analytical results of convegence.

Challlenges

1. I found many challenges in this study, especially with visualization plots of the velocity, temperature, i have error when i try to load MUMPS , or do visualization internally in freefem script.
2. The second challenge is the large time iterations , because i would like to retrieve the global aspect of long time behaviour of solutions as the regularization parameter goes to zero.
3. Im working with the Helmotz operator on velocity and i am testing until now the most performant associated numerical discretization

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Conclusion

This job aims to create a robust numerical investigation of the existence and convergence of solutions of the 3d regularized Boussinesq system conformed with theoretical results.

- The study is based on some technical computations and assumptions to avoid the blow up of the studied solutions. A big challenge is the right choice of the initial boundary conditions, some physical parameters as Rayley numbers, convection, to ensure a mesh refeninement and convergence.

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Perspectives

1. The second part of this study will be the numerical proof of the second theorem where we established the existence of a global attractor which is connected.
2. Our motivation of investigation on attractors is justified by the perfect description of the stability of flow states and the study of the evolution flow pattern.
3. we aim to implement the two main assumptions(The mean free forcing term and the mean free initial temperature) to guarantee the construction of a global in time solution. we then overcome the blow up of energy caused by the buoyancy force θe_3 as $T \to \infty$.

- Another perspective is a rich characterization of attractors of a complex fluid system by focusing its dimension, geometrical properties, visualization,...
- The study of possible bifurcations related to the Rayleigh number which control the Buoyancy term.
- A numerical study of attractors can be convenient to ameliorate the Flows dynamics approximations in a long term behavior.
- Investigation of the technological systems based on a dynamic state spaces.

In conclusion; our objective is to obtain robust numerical method based on Freefem++ computation compatible with real life applications.

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