Numerical Modeling of Kirchhoff-Love Plates with Rotational Inertia

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Journées `FreeFEM++`

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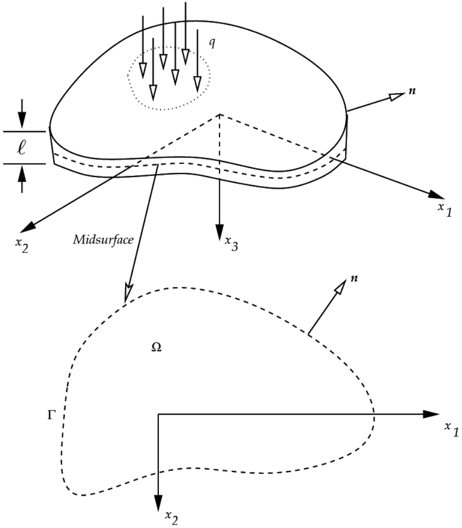
Outline

1. **Introduction**
   - Problem Statement
   - Mathematical Setting

2. **Numerical Treatment**
   - General Method
   - Dynamics
   - Statics
   - Remarks
   - Conclusions

Problem Statement



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- This problem derives from a simplified model of a magneto-electro-thermo-elastic 3D plate-like body (sensor/actuator) *[F. B., G. Geymonat, F. Krasucki, M. Serpilli, Math. Mech. Solids, 2015].*

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- D is the *flexural rigidity* of the plate; it takes into account all these couplings and does not depend only on the mechanical behavior.

Mathematical Setting

- Let $q \in L^2(0,T;L^2(\Omega))$ and

$$V = H_0^2(\Omega) = \{u \in H^2(\Omega) : u = \partial_n u = 0 \text{ on } \partial\Omega\}, \quad H = H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}.$$

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- The following problem is well-posed (J. L. Lions):

$$\boxed{\begin{gathered} \text{Find } u \in C^0([0,T];V) \cap C^1([0,T];H) \text{ such that} \\ \forall v \in V, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}(u(t),v) + a(u(t),v) = L_t(v), \\ u(0) = u_0 \in V, \ \frac{\mathrm{d}u}{\mathrm{d}t}(0) = u_1 \in H. \end{gathered}}$$

General Method

- Let $V_h \subset V$ be a N-dimensional subspace of V, $\{\varphi_j\}_{j=1,\dots,N}$ a base of V_h, and $u_h(t) := \sum_{j=1}^{N} \xi_j(t)\varphi_j$.

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- Recall: $(\varphi_i, \varphi_j) = \rho\ell \int_\Omega \left(\varphi_i\varphi_j + \frac{\ell^2}{12}\boldsymbol{\nabla}\varphi_i \cdot \boldsymbol{\nabla}\varphi_j\right) \mathrm{d}\Omega.$

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- Space discretization: **HCT finite elements** (C^1-class finite elements).
- Time discretization: **Newmark's (constant) average acceleration method**.

Dynamics I

- $q \equiv 0$.

Clamped circular plate

- Initial conditions:

$$u_0(x,y) = 0,$$
$$u_1(x,y) = \alpha(1-(x^2+y^2)).$$

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Simply supported circular plate

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$$u = 0 \quad \text{and} \quad \mathbf{Mn}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega,$$

$$\mathbf{M} := -D\Big((1-\nu)\boldsymbol{\nabla\nabla} u + \nu\Delta u\,\mathbf{I}\Big).$$

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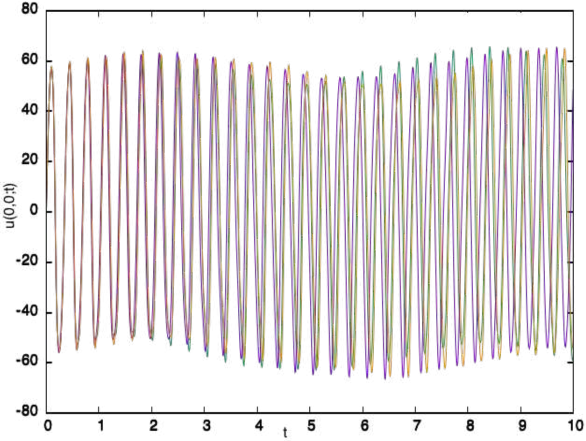
Number N of boundary mesh nodes: $\boldsymbol{N = 60}$, $\boldsymbol{N = 90}$, $\boldsymbol{N = 120}$.



Dynamics III

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- Let us consider a test case in the static regime.

Statics: clamped circular plate I

- Data set: $D = 1831.5\, kN \cdot m$, $\ell = 0.1\, m$, $R = 10\, m$, $q = 1\, kN/m^2$.
- $N = 60$ boundary mesh nodes.

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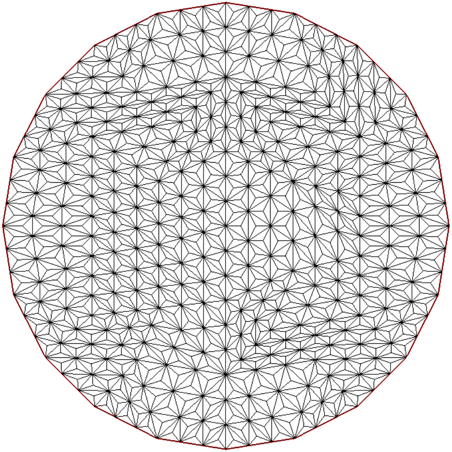

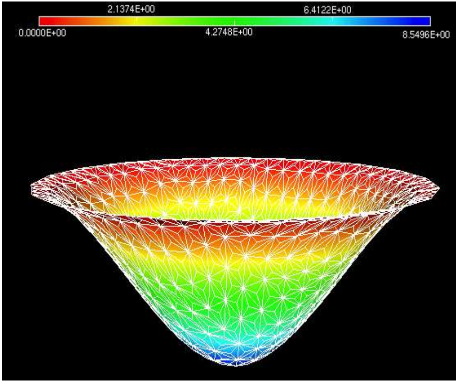

Statics: clamped circular plate II

- Exact solution:

$$u(x,y) = \frac{q}{64D}(R^2 - x^2 - y^2)^2, \quad u(0,0) = \frac{qR^4}{64D} \simeq 8.531\,cm.$$

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Table: Percentage error between $u_h(0,0)$ and $u(0,0)$.

Boundary mesh nodes	$N = 75$	$N = 90$	$N = 105$	$N = 120$	$N = 135$
$\boldsymbol{u_h(0,0)}$	$8.691\,cm$	$8.850\,cm$	$8.811\,cm$	$8.849\,cm$	$8.876\,cm$
Percentage error	1.89065%	3.74634%	3.28311%	3.73569%	4.0511%

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- **There is something wrong with the implementation of HCT in `FreeFEM++ 3.41`.**

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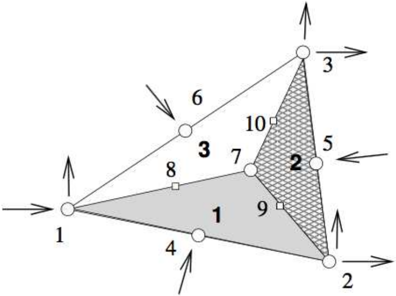

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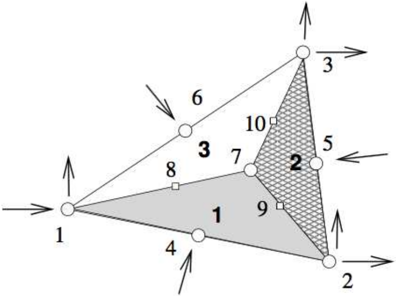

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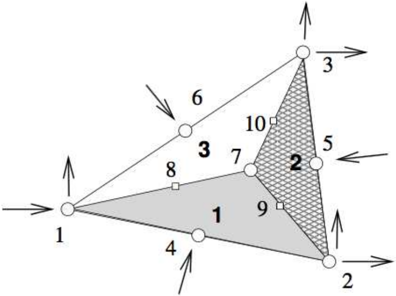

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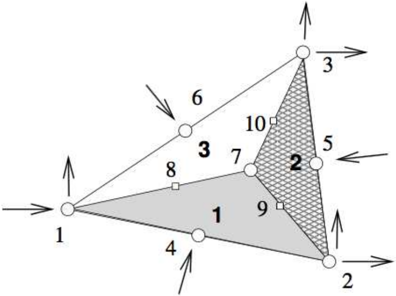

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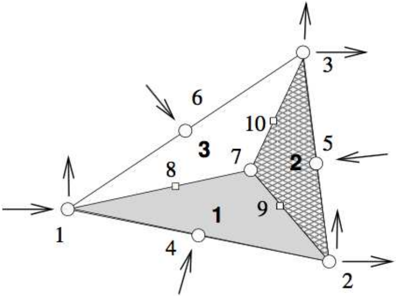

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- **Corrections have been carried out in `FreeFEM++ 3.42`, available online since December 10th.**

Statics: clamped circular plate with corrections

- Exact solution:

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Percentage error	2.5295%	1.70338%	1.2146%	0.924838%	0.719373%

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$\boldsymbol{u_h(0,0)}$	$8.691\, cm$	$8.850\, cm$	$8.811\, cm$	$8.849\, cm$	$8.876\, cm$
Percentage error	1.89065%	3.74634%	3.28311%	3.73569%	4.0511%

Table: Percentage error between $u_h(0,0)$ and $u(0,0)$ with corrections.

Boundary mesh nodes	$N = 75$	$N = 90$	$N = 105$	$N = 120$	$N = 135$
$\boldsymbol{u_h(0,0)}$	$8.315\, cm$	$8.385\, cm$	$8.427\, cm$	$8.452\, cm$	$8.469\, cm$
Percentage error	2.5295%	1.70338%	1.2146%	0.924838%	0.719373%

- Percentage error **decreases** as mesh is refined.

Conclusions

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Future work

As HCT finite elements are computationally expensive, we would like to use a different space discretization (e.g., HHO methods [Di Pietro, Ern & Lemaire, Comput. Meth. Appl. Math., 2014]).

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Motivation

- The problem results from a dimension-reduction procedure carried out on a problem posed in the 3D domain

$$\mathcal{P}^\varepsilon := \Omega \times \left(-\frac{\ell^\varepsilon}{2}, \frac{\ell^\varepsilon}{2}\right), \quad \ell^\varepsilon = \varepsilon\ell, \quad \ell := \mathrm{diam}(\mathcal{P}^\varepsilon) \simeq \mathrm{diam}(\Omega).$$

- Thus, ℓ^ε is the real thickness of the plate-like domain under study, whereas ℓ is a scaled thickness, i.e. the thickness of the domain

$$\mathcal{P} := \Omega \times \left(-\frac{\ell}{2}, \frac{\ell}{2}\right).$$

- In other words, ℓ **is not *small***.

$$\Downarrow$$

$$\boxed{\textbf{Rotational inertia is not negligible}}$$