Theoretical Aspects of ANM/FEM Numerical Model, Applied to Nonlinear Elastic, and Thermo-Elastic Analysis of Wrinkles in Film/substrate Systems

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Introduction

- **Film/substrate** (bi-layer) **systems** composed of a **thin film** deposited on a **thicker substrate** are present in numerous applications.

- Under **compression loading**, these systems can exhibit **instabilities** such as **wrinkling**, **folding**, and **creasing**.

- These instabilities are often avoided, but they can be useful in **determining mechanical properties**.

- **Wrinkling phenomena** of **thin membranes** or **film-substrate systems** are very common in the recent literature.

- Here, we will focus on **wrinkling instabilities**: wrinkling is a mechanical instability due to stresses appearing when a **stiff thin layer deposited on a compliant substrate is compressed**. When subjected to such a loading, the **initially smooth surface** bifurcates towards a **periodic pattern with a short wavelength**.

- This phenomenum occurs both in **nature** and **technology**: **wrinkles** of the **skin**, **materials** used in **flexible electronics**.

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Introduction

- **In most research**, the resolution of the problem in 2D and 3D is achieved either using **spectral methods** or with the **Fast Fourier Transform algorithm**. Because of the need of periodic boundary conditions, these methods face some limitations. Later Chen et al (2004), showed that film/substrate systems can be analyzed using the Finite Element Method, allowing for complex geometries.

- Most of the methods used to **follow the equilibrium path** of a nonlinear system of equations are **iterative** from **an initial solution guess**. The best-know are the **Newton-Raphson**, or **pseudo arc-length** methods that use the Jacobian matrix interpolation.

- When **studying wrinkling phenomena**, the **numerical modeling of the instabilities** may be complicated due to the existence of **many equilibrium solutions**.

- Because of the use of **high-order Taylor expansion**, the **ANM continuation algorithm** may be more efficient for following the equilibrium paths when other techniques fail to converge.

- **A key property of ANM**, is the **a posteriori determination of the domain of validity of of the step length** for each ANM branch. It is often associated with discretization techniques like FEM.

- In 2024, **Potier-Ferry writes a review article** that summarizes the key points of ANM and its applications to hyperelasticity, and plasticity.

Introduction

- The coupling between **perturbation technique and FEM** was first proposed by Thompson and Walker (1968). A continuation method together with an efficient procedure to build the Taylor series and convergence acceleration by Padé approximants were introduced By Cochelin, Damil and Potier-Ferry (1994).

- ANM has been already implemented using **various computer languages**: FORTRAN, MATLAB, C++. Applications in fluid mechanics within the open source platform ELMER was also made. Here, we present an new implementation of ANM/FEM numerical model in the FreeFEM++ language. FreeFEM++ uses a high-level language that makes it easy to derive a variational formulation and implement the ANM/FEM algorithm.

- FreeFEM++ also offers MPI parallel solvers and multigrid or domain decomposition tools that reduces the computational time and allow a large number of degrees of freedom to be taken into account.

- We present **the main mathematical aspects of ANM**, the application of ANM in the case of **nonlinear solid elastic and thermo-elastic problems**. We explain what makes FreeFEM++ an efficient tool to implement ANM for various nonlinear mechanical problems, and the implementation of ANM in FreeFEM++ is described.

- **Numerical experiments** are shown for **a planar film/substrate systems under uniaxial loading** and **a spherical film/substrate system subjected to a thermo-mechanical shrinkage of its core**.

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Theoretical aspects of ANM

- Let us consider the general problem to be solved in a Banach space B :

$$\mathrm{R}\left(\mathrm{U},\lambda\right)=0 \quad \text{with, } \mathrm{R} : B\times\mathbb{R}\rightarrow B, \; U\in B, \text{ and } \lambda\in\mathbb{R} \qquad (1)$$

- For example, Sobolev spaces (used for variational formulations of FEM) are Banach spaces.

- Only finite dimensional space $B=\mathbb{R}^n$, usefull for numerical methods, will be considered here.

- The main tool is the **Implicit Function Theorem**:

Assume that the function $\mathrm{R}\left(\mathrm{U},\lambda\right)$ is a class C^p function ($p\geq 1$) from $H\times I\rightarrow\mathbb{R}^n$ where H is an open state of $\mathbb{R}^n$, and I is an open segment of $\mathbb{R}$ and assume that for $\left(\mathrm{U}_0,\lambda\right)\in H\times I$ a solution of (1) the partial differential $\frac{\partial\mathrm{R}}{\partial\mathrm{U}}(\mathrm{U}_0,\lambda_0)$ is a bi-continuous isomorphism. The Implicit Function Theorem states that (1) has a unique solution branch, $\phi:\lambda\rightarrow\mathrm{U}=\phi\left(\lambda\right)$, a class C^p function from a neighborhood of $\lambda_0\in I$ to H. Furthermore, if $\mathrm{R}\left(\mathrm{U},\lambda\right)$ is an analytic function $\phi\left(\lambda\right)$ is also analytic.

- Note that, because of the nonlinearity, the existence and uniqueness is only established locally. In the presence of **bifurcations** the Jacobian matrix $\frac{\partial\mathrm{R}}{\partial\mathrm{U}}(\mathrm{U}_0,\lambda_0)$ is not invertible at the critical point and several forked branches may exist.

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Theoretical aspects of ANM

- The analytical nature of $\phi(\lambda)$ allows us to expand U and λ with respect to a path parameter a:

$$\begin{cases} \mathrm{U}(a) = \mathrm{U}_0 + \sum_{i=1}^{N} a^i \mathrm{U}_i \\ \lambda(a) = \lambda_0 + \sum_{i=1}^{N} a^i \lambda_i \end{cases} \qquad (2)$$

One of the key point of ANM

- Let us explain some algebraic properties of ANM :

consider a given function : $\hat{\mathrm{Y}} : \mathrm{X} \in \mathbb{R}^n \rightarrow \mathrm{Y} = \hat{\mathrm{Y}}(\mathrm{X}) \in \mathbb{R}^n$

If X is a function of a path parameter a, because of the chained rule, the i-th derivative $\mathrm{Y}^{(i)}$ of $\mathrm{Y}(a)$ with respect to a has the expression:

$$\mathrm{Y}^{(i)} = \mathrm{J}(\mathrm{X}_0) \cdot \mathrm{X}^{(i)} + \mathrm{Y}_{\mathrm{i}}^{\mathrm{nl}}\left(\mathrm{X}^{(0)}, \mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(i-1)}\right) \qquad (3)$$

There is a linear relationship between $\mathrm{Y}^{(i)}$ and $\mathrm{X}^{(i)}$ with the same Jacobian matrix $\mathrm{J}(\mathrm{X}_0)$ for each order i and $\mathrm{Y}_{\mathrm{i}}^{\mathrm{nl}}$ is a multi linear function of previous orders $k \in \{0,1,2,\ldots,i-1\}$.

As a consequence, the matrix $\mathrm{J}(\mathrm{X}_0) = D_{\mathrm{X}}(\mathrm{Y})$ can be obtained from (3) with $i = 1$ since $\mathrm{Y}_1^{\mathrm{nl}} = 0$ and at order $i \in \{2,3,\ldots,N\}$ with the initial assumption $\mathrm{X}^{(i)} = 0$ the expression of $\mathrm{Y}_{\mathrm{i}}^{\mathrm{nl}}\left(\mathrm{X}^{(0)}, \mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(i-1)}\right)$ is given by $\mathrm{Y}^{(i)}$.

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ANM for elastic and thermo-elastic nonlinear problems

- Let us presents first the basics of **ANM** in the case of **geometrically nonlinear elasticity**.
- We consider an elastic domain that obeys **Saint-Venant Kirchhoff law**: linear relationship between the **second Piola-Kirchhoff stress** tensor S and the **Green-Lagrange strain** tensor defined by $\gamma(\mathrm{u})$:

$$\gamma(\mathrm{u}) = \underbrace{\frac{1}{2}\left(\nabla \mathrm{u} + \nabla^{T} \mathrm{u}\right)}_{\gamma_{1}(\mathrm{u})} + \underbrace{\frac{1}{2}\left(\nabla^{T} \mathrm{u} \cdot \nabla \mathrm{u}\right)}_{\gamma_{\mathrm{nl}}(\mathrm{u})} \qquad (4)$$

And, its variation: $$\delta\gamma(\mathrm{u}) = \gamma_{1}(\delta \mathrm{u}) + 2\gamma_{\mathrm{nl}}(\mathrm{u}, \delta \mathrm{u}) \qquad (5)$$

- The formulation of the **full boundary value problem**, is classical and it associates the **elastic constitutive law and the virtual work equations**.

Elastic stiffness tensor

Given force per unit surface on Γ_N

$$\begin{cases} \mathrm{S} = \mathrm{D} : \gamma(\mathrm{u}) \\ \int_{\Omega} \mathrm{S} : \delta\gamma(\mathrm{u})\, d\Omega - \lambda \int_{\Gamma_N} \mathrm{t} \cdot \delta \mathrm{u}\, d\Gamma = 0 \end{cases} \qquad (6)$$

Scalar loading parameter

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ANM for elastic and thermo-elastic nonlinear problems

- Each step of **ANM** relies on **Taylor series**, at the order N according to a well-known **path parameter** a:

$$\begin{cases} \mathrm{u}(a) = \mathrm{u}_0 + a\mathrm{u}_1 + a^2\mathrm{u}_2 + \cdots + a^N\mathrm{u}_N \\ \mathrm{S}(a) = \mathrm{S}_0 + a\mathrm{S}_1 + a^2\mathrm{S}_2 + \cdots + a^N\mathrm{S}_N \\ \lambda(a) = \lambda_0 + a\lambda_1 + a^2\lambda_2 + \cdots + a^N\lambda_N \end{cases} \quad (7)$$

- The most popular choice for the path parameter is **a linearized arclength parameter** that permits to bypass all the extrema of the response curves:

$$a = \frac{1}{\bar{\mathrm{u}}^2}\langle \mathrm{u} - \mathrm{u}_0, \mathrm{u}_1 \rangle + \alpha(\lambda - \lambda_0)\lambda_1 \quad (8)$$

Normalized displacement vector

Normalized loading parameter

- When **Taylor series** (7) **are substituted into equations** (6) and (8), we obtain a **linear system** at each order p in a. The equations involving only linear or quadratic terms with respect to the unknown fields (u, S) the system can be deduced from the **usual Leibnitz rule** to compute high order derivatives or equivalently the Taylor coefficients of a product: $(ab)_p = \sum_{r=0}^{p} a_r b_{p-r}$

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ANM for elastic and thermo-elastic nonlinear problems

- **At order 1** in a, we obtain the **classical tangent variational problem**: $k(\mathrm{u}_1, \delta\mathrm{u}) = \lambda_1 P_e(\delta\mathrm{u})$
  where the **bilinear form** $k(.,.)$ is the **stiffness tangent operator** given by:

$$k(\mathrm{u}, \delta\mathrm{u}) = \int_\Omega \Big(\delta\gamma(\mathrm{u}_0) : \mathrm{D} : \big(\gamma_l(\mathrm{u}) + 2\gamma_{\mathrm{nl}}(\mathrm{u}_0, \mathrm{u})\big) + \mathrm{S}_0 : 2\gamma_{\mathrm{nl}}(\mathrm{u}, \delta\mathrm{u})\Big) d\Omega \tag{9}$$

$$\text{with, } P_e(\delta\mathrm{u}) = \int_{\Gamma_D} \mathrm{t} \cdot \delta\mathrm{u} d\Gamma$$

The definition (8) of the path parameter gives an additional equation: $\frac{1}{\bar{\mathrm{u}}^2}\langle \mathrm{u}_1, \mathrm{u}_1 \rangle + \alpha\lambda_1\lambda_1 = 1$

- **At order** $p \geq 2$ in a, we also get a **variational formulation** involving the **same bilinear form**:

$$k(\mathrm{u}_p, \delta\mathrm{u}) = \lambda_p P_e(\delta\mathrm{u}) + \langle \mathrm{F}_p^{\mathrm{nl}}, \delta u \rangle \tag{10}$$

The **last term** is a **linear form** that depends **on the variables at previous orders** $k \leq (p-1)$:

$$\langle \mathrm{F}_p^{\mathrm{nl}}, \delta u \rangle = -2\sum_{r=1}^{p-1} \int_\Omega S_r : \gamma_{\mathrm{nl}}(\mathrm{u}_{p-r}, \delta\mathrm{u}) d\Omega - \sum_{r=1}^{p-1} \int_\Omega \gamma_{\mathrm{nl}}(\mathrm{u}_{p-r}, \delta\mathrm{u}) : D : \delta\gamma(\mathrm{u}_0) d\Omega \tag{11}$$

The stress on order p is: $\mathrm{S}_p = \mathrm{D} : \left[\big(\gamma_l(\mathrm{u}_p) + 2\gamma_{\mathrm{nl}}(\mathrm{u}_0, \mathrm{u}_p)\big) + \sum_{r=1}^{p-1} \gamma_{\mathrm{nl}}(\mathrm{u}_r, \mathrm{u}_{p-r})\right]$

Last the **arclength condition** (8) yields the load parameter λ_p as a function of the displacement u_p:

$$\frac{1}{\bar{\mathrm{u}}^2}\langle \mathrm{u}_p, \mathrm{u}_1 \rangle + \alpha\lambda_p\lambda_1 = 1$$

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ANM for elastic and thermo-elastic nonlinear problems

- Usually the **Taylor expansion order** N is chosen between 15 and 50 to take advantage of the **exponential convergence of power series**. By estimating the **convergence radius of the series**, it is possible to compute the maximum value $a_{\max}$ of the path for each branch.

- The **truncated Taylor series** is tested to **verify a given accuracy**: the quotient of the norm difference between the two last orders in the Taylor series by the norm of the whole Taylor series is tested to be inferior to a given parameter δ, which results to the **following inequality** to be verified : $\left\| a^N \mathrm{u}_N \right\| \leq \delta \left\| a \mathrm{u}_1 \right\|$. This will leads:

$$a_{\max} = \left(\delta \frac{\left\| \mathrm{u}_1 \right\|}{\left\| \mathrm{u}_N \right\|} \right)^{\frac{1}{N-1}}, \text{ with } 10^{-10} \leq \delta \leq 10^{-3} \qquad (12)$$

- In this way, the Taylor expansion (4) gives a part of the solution in the interval $\left[0, a_{\max}\right]$. The continuation procedure may be very simple by chaining several series, the end point of this path being the starting point of the next step.

- It is also possible to improve the accuracy of the end point by using a corrective step via Newton method or by applying a convergence acceleration technique.

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ANM for elastic and thermo-elastic nonlinear problems

- It is easy to take into account an **isotropic thermal expansion** (or shrinkage) of an elastic domain with the **ANM**.

- The **thermo-mechanical effects** results in an **additional contribution** $\lambda \mathrm{I}$ to the **Green-Lagrange strain tensor**.

- **Incorporating the thermal effects**, the virtual work equation results in:

$$\begin{cases} \mathrm{S} = \mathrm{D} : \left(\gamma_{\mathrm{l}}(\mathrm{u}) + \gamma_{\mathrm{nl}}(\mathrm{u},\mathrm{u}) - \lambda \mathrm{I} \right) \\ \int_{\Omega} \mathrm{S} : \delta\gamma(\mathrm{u})\, d\Omega = 0 \end{cases}$$

$\lambda \geq 0$, means a thermo-elastic shrinkage

- at order 1 in a, $k(\mathrm{u}_1, \delta \mathrm{u}) = \lambda_1 P_{\mathrm{th}}(\delta \mathrm{u})$, $P_{\mathrm{th}}(\delta \mathrm{u}) = \int_{\Omega} \mathrm{I} : \mathrm{D} : \left(\gamma_{\mathrm{l}}(\delta \mathrm{u}) + \gamma_{\mathrm{nl}}(\mathrm{u}_0, \delta \mathrm{u}) \right) d\Gamma$
- For $p \geq 1$, $\mathrm{S}_p = \mathrm{D} : \left[\left(\gamma_{\mathrm{l}}(\mathrm{u}_p) + 2\gamma_{\mathrm{nl}}(\mathrm{u}_0, \mathrm{u}_p) \right) + \sum_{r=1}^{p-1} \gamma_{\mathrm{nl}}(\mathrm{u}_r, \mathrm{u}_{p-r}) - \lambda_p \mathrm{I} \right]$
- At order $p \geq 2$: $k(\mathrm{u}_p, \delta \mathrm{u}) = \lambda_p P_{\mathrm{th}}(\delta \mathrm{u}) + \langle \mathrm{F}_p^{\mathrm{nl}}, \delta u \rangle$

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Numerical algorithm and implementation in FreeFEM++

- The **ANM/FEM** algorithm :

Initialize u_0 and λ_0
for step=1 until Nstep
Assemble $\left[K_t(u_0,S_0)\right]$ and $\{F\}$
$\{\hat{u}\} \leftarrow \left[K_t(u_0,S_0)\right]^{-1}\{F\}$
$\lambda_1 \leftarrow 1/\sqrt{1+\{\hat{u}\}^T\{\hat{u}\}}$
$\{u_1\} \leftarrow \lambda_1\{\hat{u}\}$
For each Gauss point: $\{S_1\} \leftarrow [D]\left(\{\gamma_l(u_1)\}+\{\gamma_{nl}(u_0,u_1)\}\right)$
For each Gauss point: $\{S_2^{nl}\} \leftarrow [D]\{\gamma_{nl}(u_1,u_1)\}$
For each Gauss point: $\{S_2^*\} \leftarrow \left[A(u_1)\right]^T\{S_1\}$
Assemble $\{F_2^{nl}\}$
for p=2 until N
$\{u_p^{nl}\} \leftarrow \left[K_t(u_0,S_0)\right]^{-1}\{F_p^{nl}\}$
$\lambda_p \leftarrow -\lambda_1\{u_p^{nl}\}^T\{u_1\}$
$\{u_p\} \leftarrow \lambda_p\{\hat{u}\}+\{u_p^{nl}\}$
For each Gauss point) $\{S_p\} \leftarrow [D]\left(\{\gamma_l(u_p)\}+\{\gamma_{nl}(u_0,u_p)\}\right)+\{S_p^{nl}\}$
For each Gauss point) $\{S_{p+1}^{nl}\} \leftarrow [D]\left(\sum_{r=1}^{p}\gamma_{nl}(u_r,u_{p+1-r})\right)$
For each Gauss point) $\{S_{p+1}^*\} \leftarrow \left(\left[A(u_{p+1-r})\right]^T\{S_r\}\right)$
Assemble $\{F_{p+1}^{nl}\}$
end for
Compute a_{max}
Compute $\lambda(a)$ and $u(a)$ with $a \in [0, a_{max}]$
Compute the normalized residual error
Actualize u_0 and λ_0
end for

Numerical algorithm and implementation in FreeFEM++

- **Macro for differential operators**:

```
macro GammaL(u,v,w) [dx(u),dy(v),dz(w),(dy(u)+dx(v)),(dz(u)+dx(w)),(dz(v)+dy(w))] //

macro GammaNL(u1,v1,w1,u2,v2,w2) [(dx(u1)*dx(u2)+dx(v1)*dx(v2)+dx(w1)*dx(w2))*0.5,
(dy(u1)*dy(u2)+dy(v1)*dy(v2)+dy(w1)*dy(w2))*0.5,  (dz(u1)*dz(u2)+dz(v1)*dz(v2)+dz(w1)*dz(w2))*0.5,
(dy(u1)*dx(u2)+dx(u1)*dy(u2)+dy(v1)*dx(v2)   +dx(v1)*dy(v2)+dy(w1)*dx(w2)+dx(w1)*dy(w2))*0.5,
(dz(u1)*dx(u2)+dx(u1)*dz(u2)+dz(v1)*dx(v2)   +dx(v1)*dz(v2)+dz(w1)*dx(w2)+dx(w1)*dz(w2))*0.5,
(dz(u1)*dy(u2)+dy(u1)*dz(u2)+dz(v1)*dy(v2)   +dy(v1)*dz(v2)+dz(w1)*dy(w2)+dy(w1)*dz(w2))*0.5] //

macro Gamma(u,v,w) (GammaL(u,v,w)+GammaNL(u,v,w,u,v,w)) //

macro dGammaNL(u,v,w,uu,vv,ww) (2.0*GammaNL(u,v,w,uu,vv,ww)) //

macro dGamma(u,v,w,uu,vv,ww) (GammaL(uu,vv,ww)+dGammaNL(u,v,w,uu,vv,ww)) //
```

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Numerical algorithm and implementation in FreeFEM++

- **P2 Lagrange FEM interpolation** is assumed and defined on the mesh Th3D (**tetrahedrons**) via the command:

```
fespace Vh(Th3D,[P2,P2,P2]);
```

- The displacement field at each ANM order is declared:

```
Vh[int] [u,v,w](Norder+1);
```

- The command **varf** facilitates the computation of the **tangent matrix**, and the **linear right-hand side**:

```
varf PbTg ([u1,v1,w1],[uu,vv,ww]) =   int3d(Th3D) (
(dGamma(u[0],v[0],w[0],uu,vv,ww))'(D(GammaL(u1,v1,w1)+2*GammaNL(u[0],v[0],w[0],u1,v1,w1)))
+(dGammaNL(u1,v1,w1,uu,vv,ww))'*(D*(GammaL(u[0],v[0],w[0])+GammaNL(u[0],v[0],w[0],u[0],v[0],w[0]))) )  +
on(lencasmid,u1=0.,v1=0.,w1=0.);
```

```
varf PbF ([u1,v1,w1],[uu,vv,ww]) = int2d(Th3D,lrigthmid) (Pa*ww) +on(lencasmid,u1=0.,v1=0.,w1=0.);
```

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Numerical algorithm and implementation in FreeFEM++

- In order to **assemble the nonlinear right-hand side vector** $\left\{F_2^{nl}\right\}$, we define intermediate tensors $\left\{S_1\right\}$ and $\left\{S_2^{nl}\right\}$, written in vectorial form and defined at the Gauss points

```
fespace QFh6(ThL3D,[FEQF53d,FEQF53d,FEQF53d,FEQF53d,FEQF53d,FEQF53d]);
QFh6[int][Sxx,Syy,Szz,deuxSxy,deuxSxz,deuxSyz](Norder+1);
QFh6[Snlxx,Snlyy,Snlzz,deuxSnlxy,deuxSnlxz,deuxSnlyz](Norder+1);

[Sxx[1],Syy[1],Szz[1],deuxSxy[1],deuxSxz[1],deuxSyz[1]] =
DL*(GammaL(u[1],v[1],w[1])+2*GammaNL(u[0],v[0],w[0],u[1],v[1],w[1]));

[Snlxx,Snlyy,Snlzz,deuxSnlxy,deuxSnlxz,deuxSnlyz] = DL*(GammaNL(u[1],v[1],w[1],u[1],v[1],w[1]));

varf PbFnl2 ([u1,v1,w1],[uuu,vvv,www]) =  - int3d(Th3D) (
(dGammaNL(u[1],v[1],w[1],uuu,vvv,www))'*([Sxx[1],Syy[1],Szz[1],deuxSxy[1],deuxSxz[1],deuxSyz[1]])+
(dGamma(u[0],v[0],w[0],uuu,vvv,www))'*([Snlxx,Snlyy,Snlzz,deuxSnlxy,deuxSnlxz,deuxSnlyz]) )+
on(lencasmid,u1=0.,v1=0.,w1=0.);

Fnlu[]=PbFnl2(0,Vh);
```

- Equivalent commands are used at order $p \geq 2$

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Numerical experiments

Film/substrate Systems Under Uniaxial Compression

- Film-substrate system under axial compression: the film-substrate system being studied consists of a stiff film deposited on a soft rectangle parallelepiped substrate. It has already been studied with the help of ANM using a coupled between shell finite element and hexahedron finite elements by Fan Xu et al.

F. Xu, M. Potier-Ferry, S. Belloeuttar, Y. Cong, 3D finite element modeling for instabilities in thin films on soft substrate, International of Solids and Structures 51 (2014) 3619-3632.

- Geometrical nonlinear elasticity is assumed for the stiff film, and linear elasticity for the soft substrate.

- Young's modulus and Poisson's coeffcient of the substrate (1.8 Mpa, 0.48), and of the film (1.3×10^5 Mpa, 0.3).

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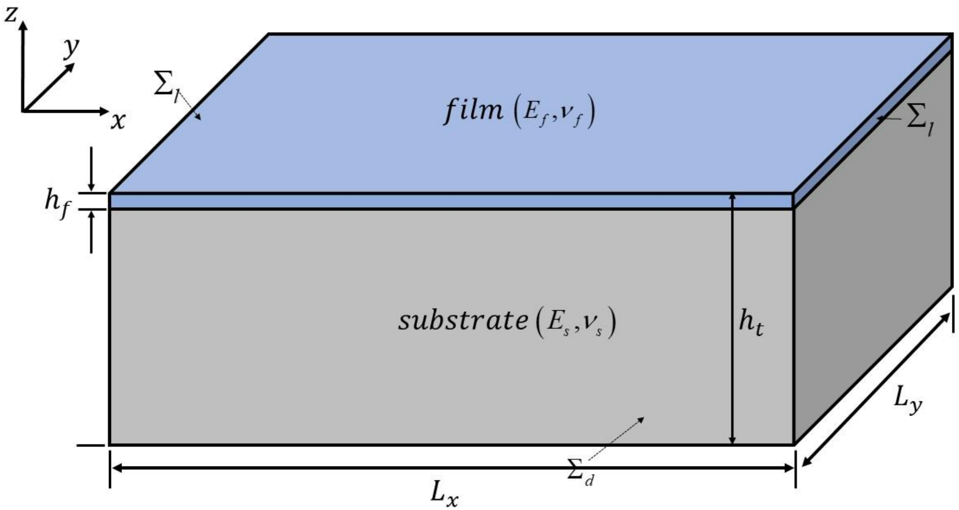
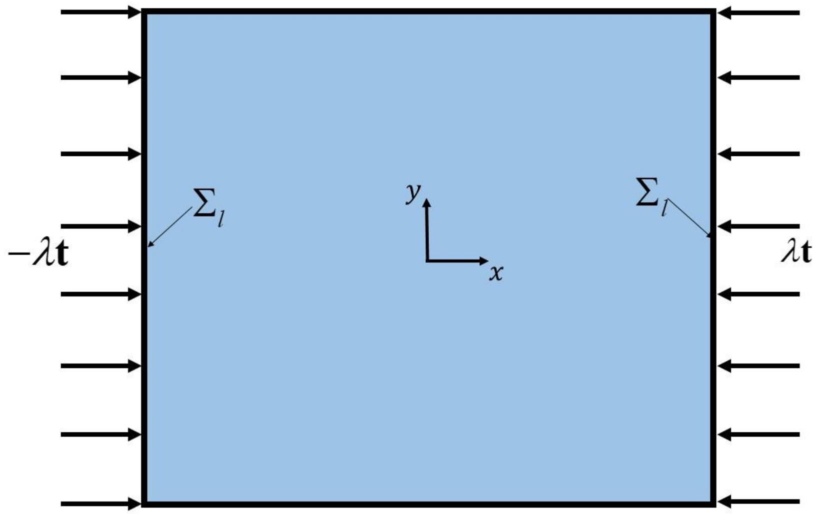
Numerical experiments

Film/substrate Systems Under Uniaxial Compression

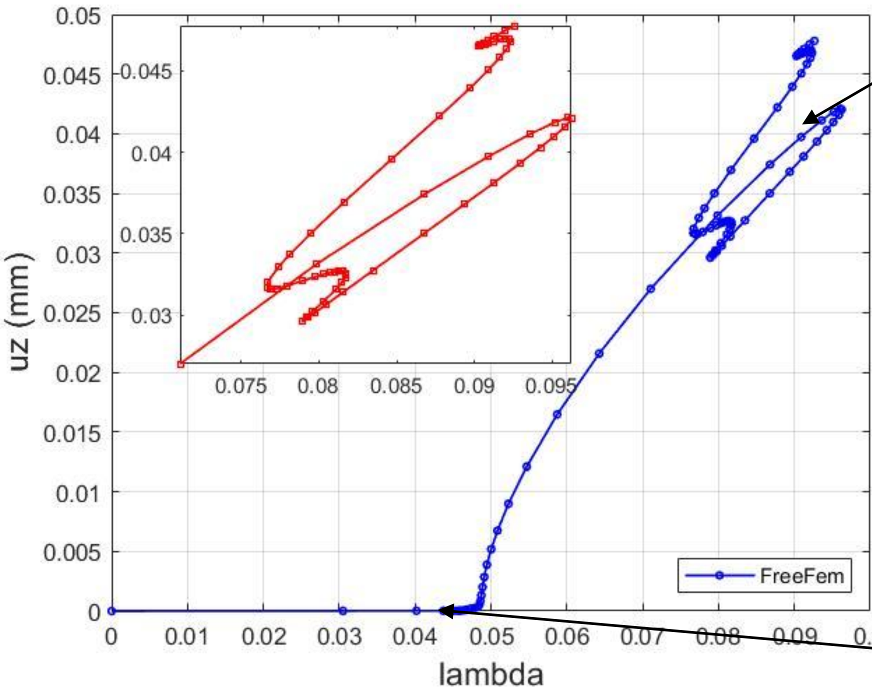
- Film-substrate system under axial compression



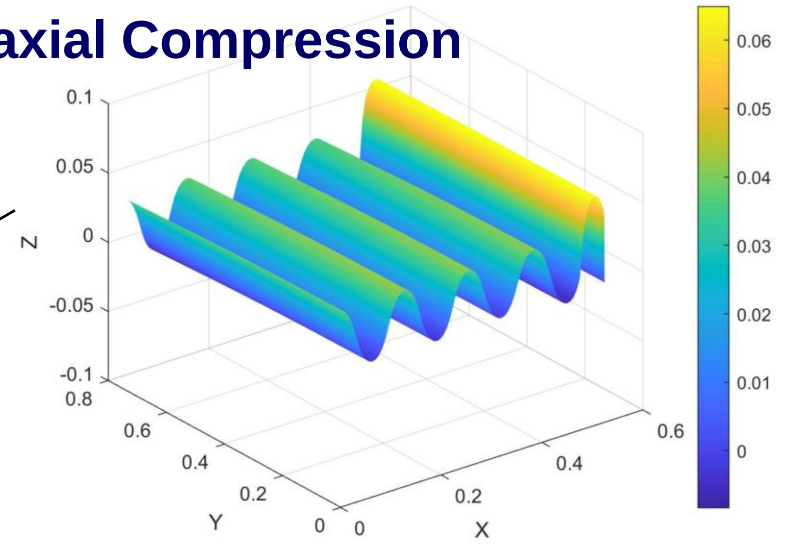


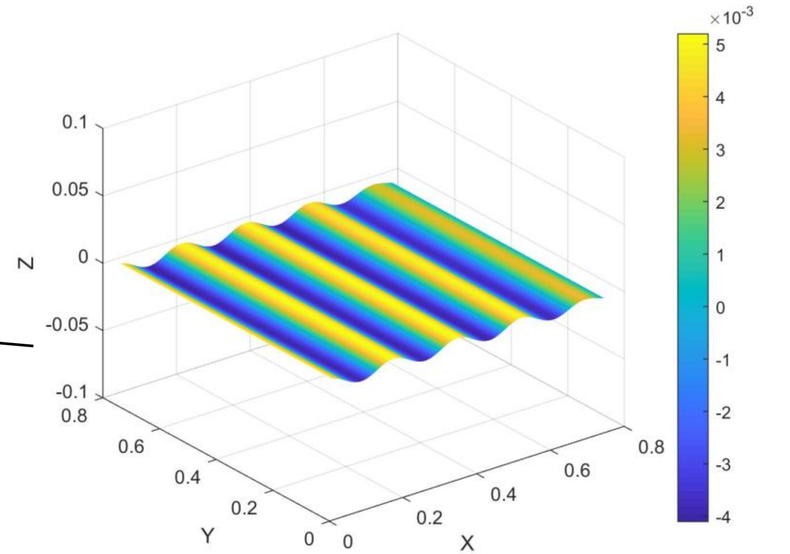
Numerical experiments

Film/substrate Systems Under Uniaxial Compression



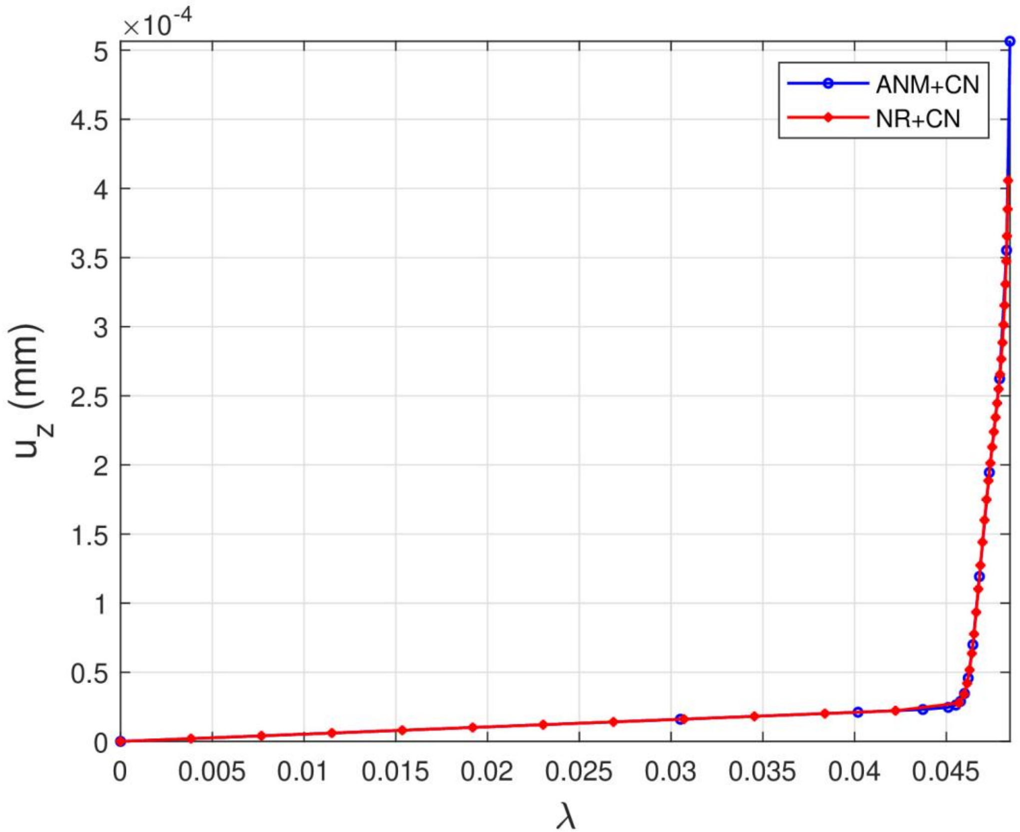
Bifurcation diagram: vertical displacement of the center of the film-substrate system (top surface) with $E_f / E_s = 0.72.10^5$





Numerical experiments

Film/substrate Systems Under Uniaxial Compression



Bifurcation diagram: vertical displacement of the center of the film-substrate system (top surface) with $E_f / E_s = 0.72.10^5$
Comparison ANM full algorithm and Newton-Raphson continuation algorithm with Newton-Riks like corrections.

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Numerical experiments

Spherical Film/substrate Under Thermal Loading

- Spherical film-substrate system under thermal loading: thermo-mechanical shrinkage of its compliant core.
- Xu et al. have presented an analytical model, experimental, and numerical results. The numerical results have been obtained with the commercial software ABAQUS using both the Riks method and the pseudo dynamix method.

F. Xu, S. Zhao, L. Conghua, M. Potier-Ferry, Pattern selection in core-shell spheres, Journal of the Mechanics and Physics of Solids 137 (2020) 103892.

- The relevant parameter to characterize surface morphological pattern formation of core-shell spheres upon shrinkage of core is

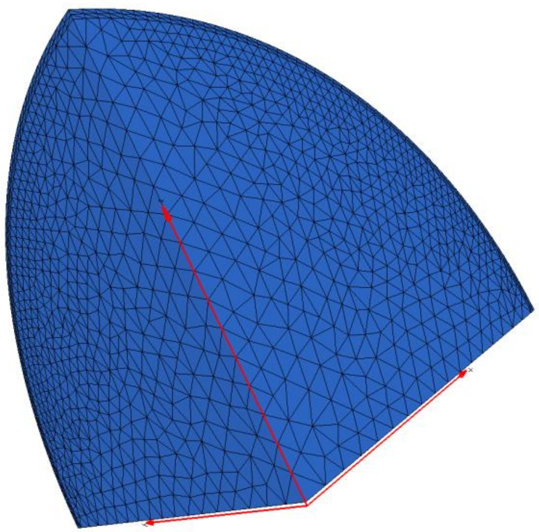
Substrate Young modulus

Sphere radius

$$C_s = \left(\frac{E_s}{E_f}\right)\left(\frac{R}{h_f}\right)^{\frac{3}{2}}$$

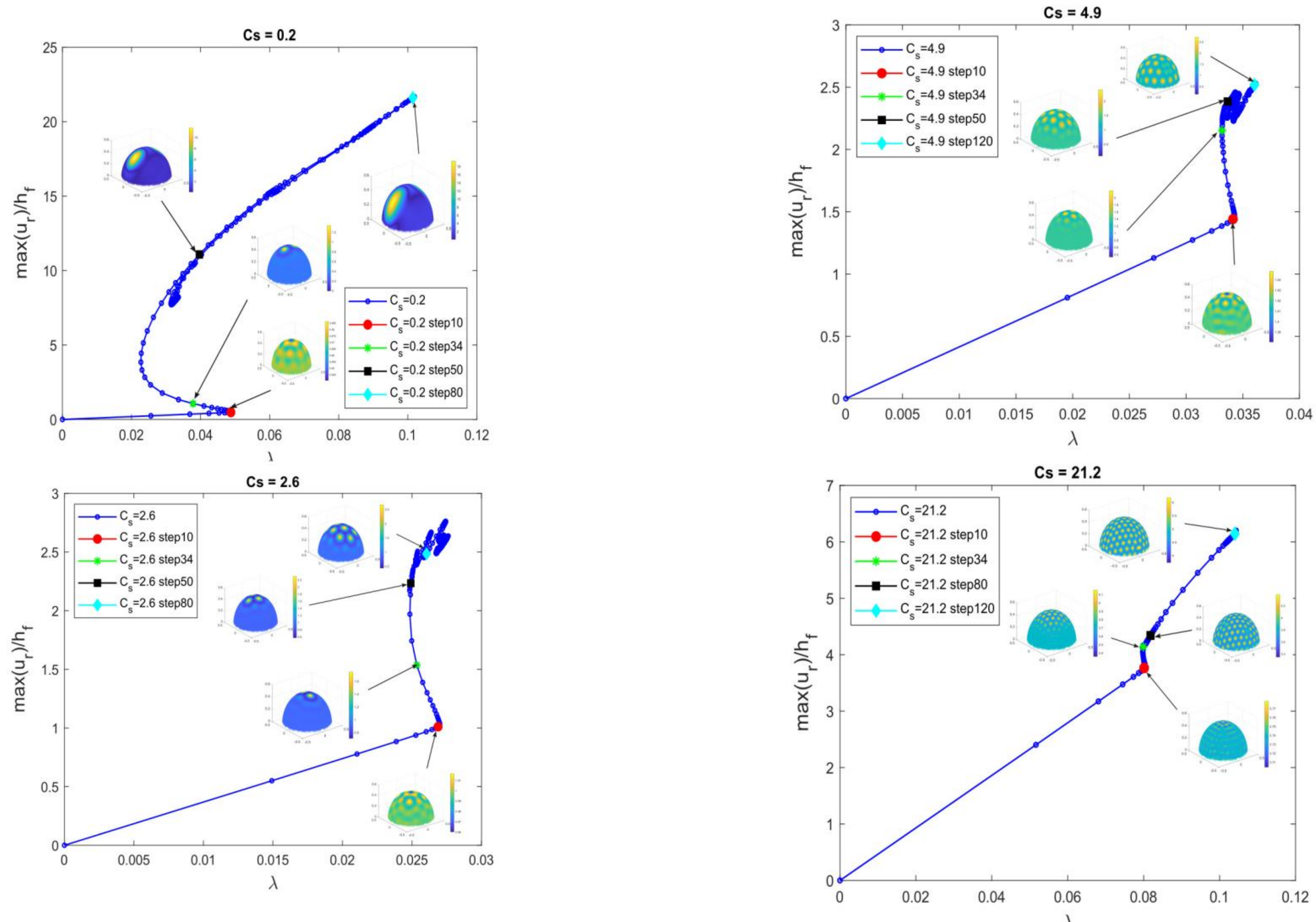
Film Young modulus

Film thickness



Numerical experiments

Spherical Film/substrate Under Thermal Loading



Bifurcation diagrams (maximum normalized deflection at the upper face of the film versus the thermal loading parameter λ) for a spherical film/substrate system with a parameter $C_s = 0.2,\ 2.6,\ 4.9,\ 21.2$ plots of the deformation of the film profile at different steps are inserted into the figures.

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Conclusions

- This work has shed a **new light on the mathematical and algorithmic aspects of ANM** based on Potier-Ferry's review article (2024).

- It has been shown that ANM traces its roots back to **the Implicit Function Theorem on a Banach space**.

- ANM uses **Taylor-series expansion** of both the unkown vectorial field U , and the scalar load parameter λ. The **chaine rule for derivation of function composition** is a key point for deriving an efficient numerical algorithm for the ANM prediction part.

- The **ANM continuation method**, based on a **perturbation technique**, has proven historically to be **a very efficient technique for the numerical simulation of nonlinear instability problems**.

- **A new implementation of the ANM has been presented within the user-friendly finite element environment FreeFEM++**. Its high-level Designed Specific Language dedicated to variational formulations and finite elements makes it easier to write scripts for **solving nonlinear solid or fluids mechanics problems using ANM**.

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Conclusions

- ANM **was applied to two film/substrate systems**. There are a few papers in the literature to analyze the post-bifurcation response of such systems: these authors have underlined **the difficulty of these computations via a continuation procedure**, and often they switched to dynamical approaches.

- The first case is the study of **a planar film/substrate system subjected to axial compression**, and the second case is **a spherical film substrate system subjected to the thermo-mechanical shrinkage of its core**.

- **Both film/substrate numerical simulations have demonstrated the efficiency of the implementation of ANM in FreeFEM++ for the following of equilibrium paths when solving solid mechanical problems**. In the case of the spherical film/substrate system, it has been possible to **improve the following of the equilibrium path for large thermal loading with the ANM compared to previous works**.

- The **main strengths of ANM are the high-order Taylor expansion**, which offers **better interpolation than a first differential** order like the classical Newton-Raphson method, and also a **a posteriori estimation of the domain of validity for the path parameter**, which results in **a consistent chaining of the ANM branches**.

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Perspective

- **Create a documented module** for **ANM/FEM applied nonlinear solid mechanics problems** in the FreeFEM++ website.

- Use PETSc or HPDDM library available in FreeFEM++ to implement **Multigrid** or **Domain Decomposition Technique** to **solve large problems**.

- **Investigate perturbed bifurcation problem**: the purpose is to compute the first branch using the **ANM/FEM continuation algorithm** and to determine **a point solution in the second branch** (not connected to the first, often close to the first one) **using local bifurcation theory** and finally **to make a new continuation to obtain the whole second bifurcated branch**.

A paper will be presented, on this topic with **a simple nonlinear scalar Ordinary Differential Equation**, at CSMA 2026.

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Thank you very much for your attention !