Presentation of FreeFem++

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Outline

1. Introduction
2. Tools
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6. Exercices
7. No Linear Problem
8. Technical Remark on freefem++

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1. Introduction
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8. Technical Remark on freefem++

FreeFem++ is a software to solve numerically partial differential equations (PDE) in $\mathbb{R}^2$) and in $\mathbb{R}^3$) with finite elements methods. We used a user language to set and control the problem. The FreeFem++ language allows for a quick specification of linear PDE's, with the variational formulation of a linear steady state problem and the user can write they own script to solve no linear problem and time depend problem. You can solve coupled problem or problem with moving domain or eigenvalue problem, do mesh adaptation , compute error indicator, etc ...

By the way, FreeFem++ is build to play with abstract linear, bilinear form on Finite Element Space and interpolation operator.

FreeFem++ is a freeware and this run on Mac, Unix and Window architecture, in parallel with MPI.
To try of cell phone
https://www.ljll.math.upmc.fr/lehyaric/ffjs/15.8/ffjs.htm

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1. Introduction
   - History
   - The main characteristics
   - The Changes form 12/09 to 06/11
   - Last Changes
   - Basement
   - Weak form

History

1987 MacFem/PCFem the old ones (O. Pironneau in Pascal) no free.

1992 FreeFem rewrite in C++ (P1,P0 one mesh) O. Pironneau, D. Bernardi, F. Hecht , C. Prudhomme (mesh adaptation , bamg).

1996 FreeFem+ rewrite in C++ (P1,P0 more mesh) O. Pironneau, D. Bernardi, F. Hecht (algebra of fucntion).

1998 FreeFem++ rewrite with an other finite element kernel and an new language ; F. Hecht, O. Pironneau, K.Ohtsuka.

1999 FreeFem 3d (S. Del Pino) , a fist 3d version base on fictitious domaine method.

2008 FreeFem++ v3 use a new finite element kernel multidimensionnels: 1d,2d,3d...

2014 FreefEM++ v3.34 parallel version

For who, for what!

For what

1. R&D
2. Academic Research ,
3. Teaching of FEM, PDE, Weak form and variational form
4. Algorithmes prototyping
5. Numerical experimentation
6. Scientific computing and Parallel computing

For who: the researcher, engineer, professor, student...

The mailing list mailto:Freefempp@ljll.math.upmc.fr with 410 members with a flux of 5-20 messages per day.

More than 2000 true Users (more than 1000 download / month)

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- Wide range of finite elements: continuous P1,P2 elements, discontinuous P0, P1, RT0,RT1,BDM1, elements ,Edge element, vectorial element, mini-element, ...
- Automatic interpolation of data from a mesh to an other one (with matrix construction if need), so a finite element function is view as a function of (x, y, z) or as an array.
- Definition of the problem (complex or real value) with the variational form with access to the vectors and the matrix.
- Discontinuous Galerkin formulation (only in 2d to day).
- LU, Cholesky, Crout, CG, GMRES, UMFPack, SuperLU, MUMPS, HIPS , SUPERLU_DIST, PASTIX, PETSc. ... sparse linear solver; eigenvalue and eigenvector computation with ARPACK.
- Online graphics with OpenGL/GLUT/VTK, C++ like syntax.
- An integrated development environment FreeFem++-cs

- Analytic description of boundaries, with specification by the user of the intersection of boundaries in 2d.
- Automatic mesh generator, based on the Delaunay-Voronoï algorithm. (2d,3d (tetgen))
- load and save Mesh, solution
- Mesh adaptation based on metric, possibly anisotropic (only in 2d), with optional automatic computation of the metric from the Hessian of a solution. (2d,3d).
- Link with other soft: parview, gmsh , vtk, medit, gnuplot
- Dynamic linking to add plugin.
- Full MPI interface
- Nonlinear Optimisation tools: CG, Ipopt, NLOpt, stochastic
- Wide range of examples: Navier-Stokes 3d, elasticity 3d, fluid structure, eigenvalue problem, Schwarz' domain decomposition algorithm, residual error indicator ...

on Unix build a "yours.edp" file with your favorite editor : emacs, vi, nedit, etc. Enter`FreeFem++ yours.edp` or `FreeFem++` must be in a directory of your `PATH` shell variable.

on Window, MacOs X build a "yours.edp" file with your favorite text editor (raw text, not word text) : emacs, winedit, wordpad, bbedit, fraise ... and click on the icon of the application `FreeFem++` and load you file via de open file dialog box or drag and drop the icon of your built file on the application `FreeFem++` icon.

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The Changes form 12/09 to 12/10

- add some interface with lapack (inverse of full matrix, eigenvalue of full matrix)
- change the version of tetgen to 1.4.3 download (4 time faster)
- add 3d beam examples `e*3d/beam.edp`
- add dynamic load interface with newuoa fortran optimizer without derivative see ffnewuoa.edp example ins examples++-load
- correct problem of free of mesh in case of gluing meshes $\mathrm{Th} = \mathrm{Tha} + \mathrm{Thb}$;
- add 3d example and interpolation matrix ($\mathrm{e} * 3\mathrm{d}/\mathrm{mat\_interpole.edp}$)
- writing schwarz-nm-3d.edp examples
- add array of 3d mesh
- add seekp, tellp method on ostream type seekg, teeg method on istream type
- correct ' operator do alway in complex case the conj and trans (Hermitian stuff)
- plot of complex field and 3d vector .. (???)
- uniformize named parameter in change, movemesh, in load "msh" , glumesh, ...
- change mesh2 tool to make a renumbering of vertex for periodic 3d mesh.
  $\mathrm{Th} = \mathrm{change}(\mathrm{Th}, \mathrm{renumv} = \mathrm{old2new});$
- correct SubString
- add automatic compilation of download software, (lots of parallel solver)
- Simplify the compilation of dynamics plugin with `ff-c++ -auto`
- correct mistake in mpi
- do asynchronous Isend/Irecv of matrix , meshes.
- 3d mesh adaptation exemple
- correct interface of `mshmet` (compute 3d metric), `freeyams` surface adaptation, `mmg3d` (3d mesh movement)
- add quoting {} in macro argument
- add script to simple the launch of mpi case
- add MPI version on Windows (not tested)
- Compile a version for Windows 64 arch

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The Changes form 09/11 to 06/13

- v 3.16
- cmaes interface in scalar and MPI case (thank to S. Auliac)
- add NLopt interface (thank to S. Auliac)
- build a pkg under macos for distribution .
- rewrite the isoline-P1 plugins
- v 3.18 (11/01/2012)
- add tools for adaptation of P2 and P3 finite elements with metrics
- add plugins with sequential mumps without mpi
- add conversion of re and im part of complex sparse matrix
- add Ipopt interface (thanks to Sylvain Auliac)
- scotch partionner interface see scotch.edp
- v 3.19 (20 april 2012)
- add tool to create Quadrature formular 1d,2d,3d with
- add integration on levelset line (in test)
- add formal tools on array [] or matrix [[],[],] for elasitic problem.
- add new MUMPS parallel version plugin
- add paradiso interface (MKL)
- version 3.22
- add multi windows graphics ; WindowIndex=0 in plot function and add new event in graphic windows * to set/unset default graphics stat to previous plot
- add getenv, setenv , unsetenv function in shell plugins for the management of environnemnt variable for openmp.
- correct pb un trunc for 3d mesh with too flat element (sliver) , and cleanning code .
- correct bug of the domain outsite flag in 3d in case when we use the brute force (searchMethod¿0)
- version 3.23
- do cleanning in version remove x11, glx, std : freefem++
- add flags to remove internal boundary in 2d,3d in function change rmInternalEdges=1
- glumesh in case of no mesh in 2d
- correct extract function of mesh Lo Sala ¡salalo80@gmail.com¿
- correct int2d on levelset see example intlevelset.edp
- correct automake TESTING part (in progress)
- correct typo on the doc with .*= ./= operator
- correct bug in RT0 3d , code in the construction of the DOF.
- add new parameter to ffglut for demo of freefem++
- to version 3.26-2. Correct compile problem, configuration clean up , ...

The Changes form 06/13 to 06/14

- correct misstake in examples++-3d/MeshSurface.idp
- correct a bug the DG with periodic boundary condition with only one layer of element.
- add pluging "bfstream" to write and read in binary file (long, double, complex¡double¿ and array) see bfstream.edp for an example. version 3.30-1 may/2014 (hg rev: 3017)
- add levelset integral on 3d case
- correct problem with Ipopt / lapack configure ...
- add BEC plugin of Bose-Enstein Optimisation
- standardisation movemesh3 -¿ movemesh (same parameter of 2d version)
- correct jump in basic integral to be compatible with varf definition
- version 3.30.
- add binary ios:mode constant, to open file in binary mode under window to solve pb of seekg under windows
- add multy border april 23 2014 , (hg rev : 3004) syntaxe example:
- add ltime() (rev 2982) function returns the value of time in seconds since 0 hours, 0 minutes, 0 seconds, January 1, 1970, (int)
- add new macro tool like in C (rev 2980) FILE,LINE,Stringification() to get line number and edp filename,
- add new int2d on levelset in 3d (int test)
- correct bug in periodic condition in case common dof with periodic.
- correct big bug in memory gestion of sparse matrix
- version 3.32
- correct of problem of plugin and mpi,
- correct of plugin MUMPS.cpp for complex value.
- add vectorial operator a/v and v/a where a est scalar and v vector like real[int], ...
- correct the problem of size of arrow in 2d plot
- version 3.31-2 (rev 3052, 11 july 2014)
- correct stop test function in LinearGC (for zuqi.tang@inria.fr)
- correct bug put in DG formulation (rev 3044) jump, mean , was wrong from Sun Jun 29 22:39:20 2014 +0200 rev 3028 version 3.31-1 (rev 3042, 10 july 2014)
- function to put your stop test in LinearGC and NLGC the prototype is
- add functional interface to arpack (Eingen Value) see examples++-eigen/LapEigenValueFunc.edp for a true example
- version 3.31 (rev 3037, 1 july 2014)
- re-add tan function for complex number
- correct a big mistake in LinearGMRES , the result are completely wrong, correct also the algo.edp
- add sqr functon of O. Pironneau
- correct update of mercurial depot (rev 3034, 1 july 2014)

The Changes form 06/14 to 01/15

- add find of libgsl in configure script
- correct pb of memory leak in case matrix A = ...; in loop
- correct bug in periodic condition in case common dof with periodic.
- correct big bug in memory management of sparse matrix (version 3.32)
- correct of problem of plugin and mpi, build all dynamics lib with and without mpi, the mpi version is install dir lib/mpi
- correct of plugin MUMPS.cpp for complex value.
- add vectorial operator a/v and v/a where a is scalar and v vector like real[int], ...
- correct stop test function in LinearGC (for zuqi.tang@inria.fr) build tar.gz distribution (rev 3050) build version MacOs 3.31-1
- correct bug put in DG formulation ((rev 3044) jump, mean , was wrong from Sun Jun 29 22:39:20 2014 +0200 rev 3028)
- function to put your own stop test in LinearGC and NLGC
- add functional interface to arpack (Eigen Value) see examples++-eigen/LapEigenValueFunc.edp for a true example
- add pluging "bfstream" to write and read in binary file (long, double, complex¡double¿ and array) see bfstream.edp for an example.
- after version 3.33 the some compilation flags a lose, correct configure.ac Please do not use version from [3.33 .. 3.35] included.
- add disable-gmm configure flags version 3.35 dist (12/3/15) (rev 3246:664a6473d705) (warning slow version)
- optional lib search [toto—tyty] in WHERE-LIBRARY seach lib
- change the metis to scotch-metis interface in hips and parms ..
- correct lot of mistake for simple compilation of hpddm interface ..
- add no mandatory lib for petsc write the WHERE-LIBRARY search lib in awk (more simple)
- correct for compilation with g++-4.9.1 -std=c++11 (without downlaod)
- add hd5 interface (13/01/2015) Thank to Mathieu Cloirec CINES - http://www.cines.fr voir example iohd5-beam-2d.edp iohd5-beam-3d.edp
- add find of libgsl in configure script
- correct pb of memory leak in case matrix A = ...; in loop
- correct bug in periodic condition in case common dof with periodic.
- correct big bug in memory management of sparse matrix
- optional lib search [toto—tyty] in WHERE-LIBRARY seach lib

The Changes form 01/15 to 12/15

- correct optimisation of convect (version 3.42)
- add tool do build quadrature formule when we split triangle in 3 triangle for HCT element.
- add new quadrature formula 3d exact form degre 6 to 14 in qf11to25.cpp
- map&key to know if a key exist in a map int[string], ...
- add tools to get mesh of fespace Vh in 2d and 3d
- try to solve problem of compatibility of petsc lib and ff++ lib
- correct hpddm examples
- move of all hpddm code and examples in examples++-hpddm (Own by P. Jolivet and F. Hecht)
- add new C1 Finite element HCT see examples++-load/bilabHCT.edp
- build a windows 64 version under msys64 system
- Add use ArrowSize= of size
- add hppdm explain in the doc.
- function cube to simplify the construction of mesh of cube (like square in 2d) see cube.edp example.
- function ClosePoint in ClosePoint plugin
- += example of varf
- add splitComm for hpddm
- add finite element Edge13d (Finite Element of degrees 1) Thanks to marcella@bonazzoli.it and exemples of wave guide in waveguide.edp
- correct renumbering function in case of rhs
- add new type of optimisation in integral, (version 3.38 25june 2015) optimized=2 =¿ do optimisation without check can be usefull in case of random problem.
- correct freefem++ launch under windows (remove wait when launch freefem++) and add wait in case of launch with launchff++.exe
- in DG linear form with jump or mean in test fonction was wrong like varf a(u,v)=intalledges(Th)(jump(v)*u) ;
- correct problem of try/catch in freefem++ func
- add tools to remove/ renumbering dof (now build in) for hpddm
- add a parameter in renumbering function.
- clean examples-mpi, remove of all usage to RemoveDOF, bb2d bb3d, findDiff (see version 3.36)
- add read of real[int,int], int[int,int], complex[int,int] ..
- after version 3.33 the some compilation flags a lose, correct configure.ac Please do not use version from [3.33 .. 3.35] included.
- add –disable-gmm configure flags

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Element of syntax : the script is like in C/C++

First FreeFem++ is a complier and after it launch the create code (a kind of byte code).
The language is polymorphe but it is not a objet oriented language.

The key words are reserved and the operator is like in C exempt: ^ & |
+ - * / ^ // a^b= a^b
== != < > <= >= & |// a|b $\equiv$ a or b, a&b$\equiv$ a and b
= += -= /= *=

BOOLEAN: 0 <=> false , $\neq$ 0 <=> true = 1

```
//   Automatic cast for numerical value :  bool, int, reel, complex , so
func heavyside = real(x>0.);

for (int i=0;i<n;i++) { ... ;}
if ( <bool exp> )  { ... ;} else {  ...;};
while ( <bool exp> ) { ... ;}
break  continue  key words
```

weakless: all local variables are almost static (????)
bug if break before variable declaration in same block.
bug for fespace argument or fespace function argument

Element of syntax: special word for finite element

```
x,y,z                                              //    current coord.
label,  region                 //    label of BC (border) , (interior)
N.x, N.y, N.z,                                     //    normal
int i = 0;                                         //    an integer
real a=2.5;                                        //    a reel
bool b=(a<3.);
real[int]  array(10) ;                  //    a real array of 10 value
mesh Th; mesh3 Th3;                     //    a 2d mesh and a 3d mesh
fespace Vh(Th,P2);              //   Def.  of 2d finite element space;
fespace Vh3(Th3,P1);            //   Def.  of 3d finite element space;
Vh u=x;                        //    a finite element function or array
Vh3<complex> uc = x+  1i *y;                //    complex valued FE
u(.5,.6,.7);                  //    value of FE function u at point (.5,.6,.7)
u[];              //   the array of DoF value assoc.  to FE function u
u[][5];                  //   6th value of the array (numbering begin
                                                   //   at 0 like in C)
```

Element of syntax: weak form, matrix and vector 2/4

```
fespace V3h(Th,[P2,P2,P1]);
V3h [u1,u2,p]=[x,y,z];              //   a vectorial finite element
                                          //    function or array
        //   remark u1[] <==> u2[] <==> p[] same array of unknown.
macro div(u,v) (dx(u)+dy(v))//   EOM a macro
                                          //    (like #define in C )
macro Grad(u) [dx(u),dy(u)]              //    the macro end with //
varf a([u1,u2,p],[v1,v2,q])=
           int2d(Th)(  Grad(u1)'*Grad(v1) +Grad(u2)'*Grad(v2)
                -div(u1,u2)*q -div(v1,v2)*p)
           +on(1,2,u1=g1,u2=g2);

matrix A=a(V3h,V3h,solver=UMFPACK);
real[int] b=a(0,V3h);
u2[] =A^-1*b;                //    or you can put also u1[]= or p[].
```

Element of syntax: matrix and vector tools

```
 func Heaveside=(x>0);                    //    a formal line function
 func real g(int i, real a) { .....; return i+a;}
 A = A + A'; A = A'*A                     //    matrix operation (only 1/1)
 A = [ [ A,0],[0,A'] ];                                //    Block matrix.
int[int] I(15),J(15);                    //    two array for renumbering
      //   the aim is to transform a matrix into a sparse matrix
matrix B;
B = A;                                                //    copie matrix A
B=A(I,J);                                  //    B(i,j) = A(I(i),J(j))
B=A(I^-1,J^-1);                             //    B(I(i),J(j))= A(i,j)
B.resize(10,20);                          //    resize the sparse matrix
                                  //     and remove out of bound terms
int[int] I(1),J(1); real[int] C(1);
[I,J,C]=A;              //    get of the sparse term of the matrix A
                                          //    (the array are resized)
A=[I,J,C];                                             //    set a new matrix
matrix D=[diagofA] ;                      //    set a diagonal matrix D
                                          //    from the array diagofA.
real[int] a=2:12;                         //   set a[i]=i+2; i=0 to 10.
```

Element of syntax formal computation on array

a formal array is `[ exp1, exp1, ..., expn]`
the Hermitian transposition is `[ exp1, exp1, ..., expn]'`

```
complex a=1,b=2,c=3i;
func va=[ a,b,c];                          //    is a formal array in [ ]
a =[ 1,2,3i]'*va ; cout << a << endl;      //    Hermitian product
matrix<complex>  A=va*[ 1,2,3i]'; cout << A << endl;
a =[ 1,2,3i]'*va*2.;
a =(va+[ 1,2,3i])'*va*2.;
va./va;                                    //    term to term /
va*/va;                                    //    term to term *
trace(va*[ 1,2,3i]') ;                                    //
(va*[ 1,2,3i]')[1][2] ;                    //    get coef
det([[1,2],[-2,1]]);                 //    just for matrix 1x1 et 2x2
```

usefull macro to def your edp.

```
macro grad(u) [dx(u),dy(u)] //
macro div(u1,u2) (dx(u1)+dy(u2)) //
```

Idea of new syntax to set map, array, matrix

```
real[int] tab1;
forall ( i,v;tab1)
  { v=f(i); }                                  //    set tab[i]=f(i); ∀ i.

matrix mat;                                                          //   or
real[int,int] mat;
forall  (auto i,j,v;mat)
 { v=f(i,j);}                                  //   set mat(i,j)=f(i,j); ∀ i,j.

string[string] map;
forall (key,v;map)
   { v=f(key); }                               //    set map[key]=f(key); ∀ key.
```

or

```
for (auto i,v;tab1)  v=f(i);                   //    set tab[i]=f(i); ∀ i.
for (auto i,j,v;mat)  v=f(i,j);   //    set mat(i,j)=f(i,j); ∀ i,j.
for (auto key,v;map)  v=f(key);   //    set map[key]=f(key); ∀ key.
```

List of Plugin

```
ls  /usr/local/lib/ff++/3.20-3/lib/
```

```
BernadiRaugel.dylib
BinaryIO.dylib
DxWriter.dylib
Element_Mixte.dylib
Element_P1dc1.dylib
Element_P3.dylib
Element_P3dc.dylib
Element_P4.dylib
Element_P4dc.dylib
Element_PkEdge.dylib
FreeFemQA.dylib
MPICG.dylib
MUMPS.dylib
MUMPS_FreeFem.dylib
MUMPS_seq.dylib
MetricKuate.dylib
MetricPk.dylib
Morley.dylib
NewSolver.dylib
SuperLu.dylib
UMFPACK64.dylib
VTK_writer.dylib
VTK_writer_3d.dylib
addNewType.dylib
complex_SuperLU_DIST_FreeFem.dylib
complex_pastix_FreeFem.dylib
dSuperLU_DIST.dylib
dfft.dylib
ff-Ipopt.dylib
ff-NLopt.dylib
ff-cmaes.dylib
fflapack.dylib
ffnewuoa.dylib
ffrandom.dylib
freeyams.dylib
funcTemplate.dylib
gmsh.dylib
gsl.dylib
hips_FreeFem.dylib
ilut.dylib
interfacepastix.dylib
iovtk.dylib
isoline.dylib
isolineP1.dylib
lapack.dylib
lgbmo.dylib
mat_dervieux.dylib
mat_psi.dylib
medit.dylib
metis.dylib
mmg3d-v4.0.dylib
mpi-cmaes.dylib
msh3.dylib
mshmet.dylib
myfunction.dylib
myfunction2.dylib
parms_FreeFem.dylib
pcm2rnm.dylib
pipe.dylib
ppm2rnm.dylib
qf11to25.dylib
real_SuperLU_DIST_FreeFem.dylib
real_pastix_FreeFem.dylib
scotch.dylib
shell.dylib
splitedges.dylib
splitmesh3.dylib
splitmesh6.dylib
symmetrizeCSR.dylib
tetgen.dylib
thresholdings.dylib
```

Important Plugin

- `qf11to25` add more quadrature formulae in 1d , 2d, and tools to build own quadrature
- `Element_*`,`Morlay`,`BernadiRaugel` add new kind of 2d finite element
- `SuperLu`,`UMFPACK64`,`SuperLu`,`MUMPS_seq` add sequentiel sparse solver
- `metis`,`scotch` mesh Partitioning
- `ffrandom` true random number generator: srandomdev,srandom, random
- `gsl` the gsl lib interface (lot of special function)
- `shell`,`pipe` directory and file interface, pipe interface
- `dfft` interface with fftw3 library for FFT.
- **`msh3`,`tetgen`** 3d mesh tools and tetgen interface
- `lapack` a small lacpack,interface of full linear solver, full eigen value problem.
- `ff-Ipopt` interface with Ipopt optimisation software
- `ppm2rnm` interface with ppm library to read ppm bitmap.
- `isoline` build a border from isoline.
- `freeyams`, `mesh met`, `mmg3d-v4`, `medit` interface of library of P. Frey to adapt mesh in 3d.

Important Plugin with MPI

- `hpddm` a new parallel linear solver see schwarz.edp example,
- `MUMPS` a new version of MUMPS interface
- `MPICG` parallel version of CG, and GMRES
- `mpi-cmaes` parallel version of stochastic optimization algorithm.
- `hips_FreeFem`,`parms_FreeFem`,`MUMPS_FreeFem` old parallel linear solver iterface.

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Laplace equation, weak form

Let a domain Ω with a partition of $\partial\Omega$ in Γ_2, Γ_e.
Find u a solution in such that:

$$-\Delta u = 1 \text{ in } \Omega, \quad u = 2 \text{ on } \Gamma_2, \quad \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \Gamma_e \tag{1}$$

Denote $V_g = \{v \in H^1(\Omega)/v_{|\Gamma_2} = g\}$.
The Basic variational formulation with is: find $u \in V_2(\Omega)$, such that

$$\int_\Omega \nabla u.\nabla v = \int_\Omega 1v + \int_\Gamma \frac{\partial u}{\partial n} v, \quad \forall v \in V_0(\Omega) \tag{2}$$

The finite element method is just: replace V_g with a finite element space, and the `FreeFem++` code:

Poisson equation in a fish with FreeFem++

The finite element method is just: replace V_g with a finite element space, and the FreeFem++ code:

```
mesh3 Th("fish-3d.msh");                          //    read a mesh 3d
fespace Vh(Th,P1);                          //    define the P1 EF space

 Vh u,v;                    //   set test and unknown function in Vh.
macro Grad(u) [dx(u),dy(u),dz(u)]                    //   EOM Grad def
solve laplace(u,v,solver=CG) =
   int3d(Th)(  Grad(u)'*Grad(v)    )
   - int3d(Th) ( 1*v)
   + on(2,u=2);                                      //    int on γ2
plot(u,fill=1,wait=1,value=0,wait=1);
```

Run:fish.edp Run:fish3d.edp

Important remark:on geometrical item label and region

- All boundary (internal or nor) was define through a label number and this number is define in the mesh data structure. The support of this label number is a edge in 2d and a face in 3d and **so FreeFem++ never use label on vertices**.
- To defined integration you can use the region (resp. label) number if you compute integrate in domain (resp. boundary). They are no way to compute 1d integral in 3d domain.
- Today they are no Finite Element defined on surface.
- You can put list of label or region in integer array (`int[int]`).

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Remark on varf

The functions appearing in the variational form are formal and local to the `varf` definition, the only important think is the order in the parameter list, like in

```
varf vb1([u1,u2],[q]) = int2d(Th)( (dy(u1)+dy(u2)) *q)
                        +int2d(Th)(1*q) + on(1,u1=2);
varf vb2([v1,v2],[p]) = int2d(Th)( (dy(v1)+dy(v2)) *p)
                        +int2d(Th)(1*p) ;
```

To build matrix A from the bilinear part the the variational form a of type `varf` do simply

```
  matrix B1 = vb1(Vh,Wh [, ...] );
  matrix<complex> C1 = vb1(Vh,Wh [, ...] );
//     where the fespace have the correct number of comp.
//     Vh is "fespace" for the unknown fields with 2 comp.
//     ex fespace Vh(Th,[P2,P2]); or fespace Vh(Th,RT);
//     Wh is "fespace" for the test fields with 1 comp.
```

To build a vector, put $u1 = u2 = 0$ by setting 0 of on unknown part.

```
real[int]   b = vb2(0,Wh);
complex[int]   c = vb2(0,Wh);
```

Remark: In this case the mesh use to defined , $\int, u, v$ can be different, and be carefull in case of discontinuity, the result can be wrong.

The boundary condition terms

First `FreeFem++` use only the label number of edge (2d) or faces (3d).

- An "on" scalar form (for Dirichlet) : `on(1, u = g )`

  The meaning is for all degree of freedom i (DoF) of this associated boundary, the diagonal term of the matrix $a_{ii} = tgv$ with the *terrible giant value* `tgv` ($=10^{30}$ by default) and the right hand side $b[i] =$ "$(\Pi_h g)[i]$" $\times\, tgv$, where the "$(\Pi_h g)[i]$" is the boundary DoF value given by the interpolation of g.

- An "on" vectorial form (for Dirichlet) : `on(1,u1=g1,u2=g2)` If you have vectorial finite element like `RT0`, the 2 components are coupled, and so you have :
  $b[i] =$ "$(\Pi_h(g1, g2))[i]$" $\times\, tgv$, where Π_h is the vectorial finite element interpolant.

- a linear form on Γ (for Neumann in 2d)

  `-int1d(Th)( f*w)` or `-int1d(Th,3))( f*w)`

- a bilinear form on Γ or Γ_2 (for Robin in 2d)

  `int1d(Th)( K*v*w)` or `int1d(Th,2)( K*v*w)`.

- a linear form on Γ (for Neumann in 3d)

  `-int2d(Th)( f*w)` or `-int2d(Th,3)( f*w)`

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Build Mesh 2d

First a 10×10 grid mesh of unit square $]0,1[^2$

```
int[int] labs=[10,20,30,40];          //    bot., right, top, left
mesh Th1 =  square(10,10,label=labs,region=0,[x,y]);          //
plot(Th1,wait=1);
int[int] old2newlabs=[10,11,  30,31];       //   10 -> 11, 30 -> 31
Th1=change(Th1,label=old2newlabs) ;                           //
//   do Change in 2d or in 3d.  region=a, fregion=f ,
//    flabel=f
```

a L shape domain $]0,1[^2 \backslash [\frac{1}{2},1[^2$

```
 mesh Th =  trunc(Th1,(x<0.5) | (y < 0.5),label=1);           //
 plot(Th,cmm="Th");
 mesh Thh = movemesh(Th,[-x,y]);
 mesh  Th3  = Th+movemesh(Th,[-x,y]);            //    glumesh ...
 plot(Th3,cmm="Th3");
```

Run:mesh1.edp

Build Mesh 2d, border

a Circle with or without an hole ;
Remark; by default the domain is a left of the border (if the number of is positive of segment positive).

```
border Co(t=0,2*pi) { x=cos(t); y=sin(t); label=1;}
border Ci(t=0,2*pi) { x=cos(t)/2; y=sin(t)/2; label=2;}
plot(Co(30)+Ci(15),wait=1);
mesh Thf=buildmesh(Co(30)+Ci(15));        //    without hole
                                           //    two region:
cout <<" The two Region of Thf : " <<  Thf(0,0).region
     << " " << Thf(0,0.9).region << endl;
plot(Thf,wait=1);
mesh Thh=buildmesh(Co(30)+Ci(-15));       //    without hole
plot(Thh);
```

Get a extern mesh

```
mesh Th2("april-fish.msh");
```

build with `emc2`, `bamg`, modulef, etc...

Run:mesh-circles.edp

Build Mesh 2d, more complicate with implicit loop

a L shape domain $]0,1[^2 \backslash [\frac{1}{2},1[^2$ with 6 multi-borders.

```
 int nn=30; real dd=0.5;
 real[int,int] XX=[[0,0],[1,0],[1,dd],[dd,dd],[dd,1],[0,1]];
 int[int] NN=[nn,nn*dd,nn*(1-dd),nn*(1-dd),nn*dd,nn];
  border bb(t=0,1;i)
  {          //    i is the the index of the multi border loop
    int ii = (i+1)%XX.n; real t1 = 1-t;
    x =  XX(i,0)*t1 + XX(ii,0)*t;
    y =  XX(i,1)*t1 + XX(ii,1)*t;
    label = 1; ;    }
plot(bb(NN),wait=1);
mesh Th=buildmesh(bb(NN));
plot(Th,wait=1);
```

Run:mesh-multi.edp

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Build mesh from image 1/2

```
load "ppm2rnm" load "isoline" load "shell"
string lac="lac-oxford", lacjpg =lac+".jpg", lacpgm =lac+".pgm";
if(stat(lacpgm)<0)  exec("convert "+lacjpg+" "+lacpgm);
real[int,int] Curves(3,1);  int[int] be(1); int nc;
{
  real[int,int] ff1(lacpgm);                          //    read image
  int nx = ff1.n, ny=ff1.m;       //    grey value in 0 to 1 (dark)
  mesh Th=square(nx-1,ny-1,[(nx-1)*(x),(ny-1)*(1-y)]);
  fespace Vh(Th,P1);  Vh f1; f1[]=ff1; //    array to fe function.
  real iso =0.3;            //    try some value to get correct iso
  real[int] viso=[iso];
  nc=isoline(Th,f1,iso=iso,close=0,Curves,beginend=be,
      smoothing=.1,ratio=0.5);
  for(int i=0; i<min(3,nc);++i)
  { int i1=be(2*i),i2=be(2*i+1)-1;
    plot(f1,viso=viso,[Curves(0,i1:i2),Curves(1,i1:i2)],
     wait=1,cmm=i); }}
```

Build mesh from image 2/2

```
int[int] iii=[1,2];                    //    chose to componante ...
int[int] NC=[-300,-300];                          //    2 componante
border G(t=0,1;i) {
P=Curve(Curves,be(2*iii[i]),be(2*iii[i]+1)-1,t);
                        label= iii[i];}
plot(G(NC),wait=1);
mesh Th=buildmesh(G(NC));
plot(Th,wait=1);
real scale = sqrt(AreaLac/Th.area);
Th=movemesh(Th,[x*scale,y*scale]);
```

Run:lac.edp (the Powai lac close to IIT Bombay)

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A cube with buildlayer (simple ???)

```
load "msh3"                                                    //   buildlayer
int[int] ll=[1,1,1,1,  1,2 ];
mesh3 Th=cube(10,10,10,label=ll);
```

or more complex script to get the same (previous version)

```
load "msh3"                                                    //   buildlayer
int nn=10;
int[int]
      rup=[0,2],                    //    label:  upper face 0-> 2 (region -> label)
      rdown=[0,1],                  //    label:  lower face 0-> 1 (region -> label)
      rmid=[1,1 ,2,1 ,3,1 ,4,1 ],                //   4 Vert.  2d label -> 3d label
      rtet= [0,0];                                                              //
real zmin=0,zmax=1;
mesh3 Th=buildlayers(square(nn,nn,),nn,
                     zbound=[zmin,zmax],
                     region=rtet,
                     labelmid=rmid,
                     labelup = rup,
                     labeldown = rdown);
Th= trunc(Th,((x<0.5) |(y< 0.5)| (z<0.5)),label=3);
                                                               //    remove 1/2 cube
plot("cube",Th);
```

Run:Cube.edp

3D layer mesh of a Lac with buildlayer

```
load "msh3"//    buildlayer
load "medit"//    medit
int nn=5;
border cc(t=0,2*pi){x=cos(t);y=sin(t);label=1;}
mesh Th2= buildmesh(cc(100));
fespace Vh2(Th2,P2);
Vh2 ux,uz,p2;
int[int] rup=[0,2],  rdown=[0,1], rmid=[1,1];
func zmin= 2-sqrt(4-(x*x+y*y));   func zmax= 2-sqrt(3.);
//    we get nn*coef layers
mesh3 Th=buildlayers(Th2,nn,
                     coef= max((zmax-zmin)/zmax,1./nn),
                     zbound=[zmin,zmax],
                     labelmid=rmid,  labelup = rup,
                     labeldown = rdown);            //    label def
medit("lac",Th);
```

Run:Lac.edp Run:3d-leman.edp

a 3d axi Mesh with buildlayer

```
func f=2*((.1+(((x/3))*(x-1)*(x-1)/1+x/100))^(1/3.)-(.1)^(1/3.));
real yf=f(1.2,0);
border up(t=1.2,0.){ x=t;y=f;label=0;}
border axe2(t=0.2,1.15) { x=t;y=0;label=0;}
border hole(t=pi,0) { x= 0.15 + 0.05*cos(t);y= 0.05*sin(t);
       label=1;}
border axe1(t=0,0.1) { x=t;y=0;label=0;}
border queue(t=0,1) { x= 1.15 + 0.05*t; y = yf*t; label =0;}
int np= 100;
func bord= up(np)+axe1(np/10)+hole(np/10)+axe2(8*np/10)
         + queue(np/10);
plot( bord);                                  //    plot the border ...
mesh Th2=buildmesh(bord);                     //    the 2d mesh axi mesh
plot(Th2,wait=1);
int[int] l23=[0,0,1,1];
Th=buildlayers(Th2,coef= max(.15,y/max(f,0.05)), 50
  ,zbound=[0,2*pi],transfo=[x,y*cos(z),y*sin(z)]
  ,facemerge=1,labelmid=l23);
```

Run:3daximesh.edp

boundary mesh of a Sphere

```
load "tetgen"
mesh Th=square(10,20,[x*pi-pi/2,2*y*pi]);        //     ]-pi/2, -pi/2[×]0,2π[
func f1 =cos(x)*cos(y); func f2 =cos(x)*sin(y); func f3 = sin(x);
            //    the partiel derivative of the parametrization DF
func f1x=sin(x)*cos(y);     func f1y=-cos(x)*sin(y);
func f2x=-sin(x)*sin(y);    func f2y=cos(x)*cos(y);
func f3x=cos(x);            func f3y=0;
                                                    //     M = DF^t DF
func m11=f1x^2+f2x^2+f3x^2;  func m21=f1x*f1y+f2x*f2y+f3x*f3y;
func m22=f1y^2+f2y^2+f3y^2;
func perio=[[4,y],[2,y],[1,x],[3,x]];
real hh=0.1/R; real vv= 1/square(hh);
Th=adaptmesh(Th,m11*vv,m21*vv,m22*vv,IsMetric=1,periodic=perio);
int[int] ref=[0,L];    //    the label of the Sphere to L ( 0 -> L)
mesh3  ThS= movemesh23(Th,transfo=[f1*R,f2*R,f3*R],orientation=1,
   label=ref);
```

Run:Sphere.edp Run:sphere6.edp

Build 3d Mesh from boundary mesh

```
include "MeshSurface.idp"          //    tool for 3d surfaces meshes
mesh3 Th;
try {  Th=readmesh3("Th-hex-sph.mesh"); }          //   try to read
catch(...) {       //    catch a reading error so build the mesh...
    real hs = 0.2;                                 //    mesh size on sphere
    int[int]  NN=[11,9,10];
    real [int,int]  BB=[[-1.1,1.1],[-.9,.9],[-1,1]]; //    Mesh Box
    int [int,int]  LL=[[1,2],[3,4],[5,6]];          //    Label Box
    mesh3 ThHS = SurfaceHex(NN,BB,LL,1)+Sphere(0.5,hs,7,1);
                                                   //    surface meshes
    real voltet=(hs^3)/6.;                 //    volume mesh control.
    real[int] domaine = [0,0,0,1,voltet,0,0,0.7,2,voltet];
    Th = tetg(ThHS,switch="pqaAAYYQ",
              nbofregions=2,regionlist=domaine);
    savemesh(Th,"Th-hex-sph.mesh"); }        //   save for next run
```

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Mesh tools

- `change` to change label and region numbering in 2d and 3d.
- `movemesh checkmovemesh movemesh23 movemesh3`
- `triangulate` (2d) , `tetgconvexhull` (3d) build mesh mesh for a set of point
- `emptymesh` (2d) built a empty mesh for Lagrange multiplier
- `freeyams` to optimize surface mesh
- `mmg3d` to optimize volume mesh with constant surface mesh
- `mshmet` to compute metric
- `isoline` to extract isoline (2d)
- `trunc` to remove peace of mesh and split all element (2d,3d)
- `splitmesh` to split 2d mesh in no regular way.

2. Tools
   - Remarks on weak form and boundary conditions
   - Mesh generation
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   - Mesh tools
   - Anisotropic Mesh adaptation

In Euclidean geometry the length $|\gamma|$ of a curve γ of $\mathbb{R}^d$ parametrized by $\gamma(t)_{t=0..1}$ is

$$|\gamma| = \int_0^1 \sqrt{< \gamma'(t), \gamma'(t) >} dt$$

We introduce the metric $\mathcal{M}(x)$ as a field of $d \times d$ symmetric positive definite matrices, and the length ℓ of Γ w.r.t $\mathcal{M}$ is:

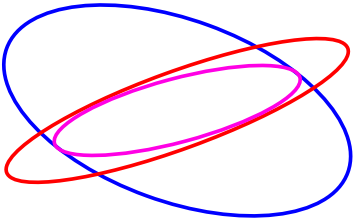
$$\ell = \int_0^1 \sqrt{< \gamma'(t), \mathcal{M}(\gamma(t))\gamma'(t) >} dt$$

The key-idea is to construct a mesh where the lengths of the edges are close to 1 accordingly to $\mathcal{M}$.

The unit ball $\mathcal{BM}$ in a metric $\mathcal{M}$ plot the maximum mesh size on all the direction, is a ellipse.

If you we have two unknowns u and v, we just compute the metric $\mathcal{M}_u$ and $\mathcal{M}_v$, find a metric $\mathcal{M}_{uv}$ call intersection with the biggest ellipse such that:

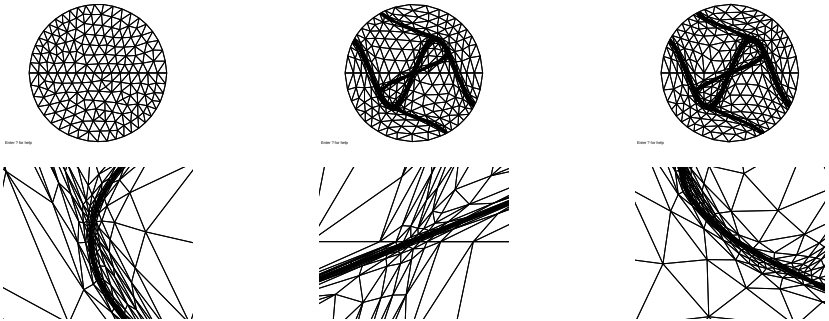
$$\mathcal{B}(\mathcal{M}_v) \subset \mathcal{B}(\mathcal{M}_u) \cap \mathcal{B}(\mathcal{M}_v)$$



Example of mesh

$$u = (10x^3 + y^3) + tanh(500(sin(5y) - 2x)));$$

$$v = (10y^3 + x^3) + tanh(5000(sin(5y) - 2*)));$$

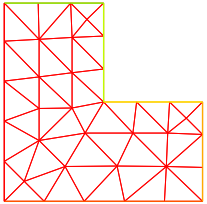


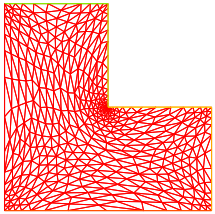
Run:Adapt-uv.edp

The domain is an L-shaped polygon $\Omega =]0,1[^2 \setminus [\frac{1}{2},1]^2$ and the PDE is

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } \quad -\Delta u = 1 \text{ in } \Omega,$$

The solution has a singularity at the reentrant angle and we wish to capture it numerically.





example of Mesh adaptation

FreeFem++ corner singularity program

```
int[int] lab=[1,1,1,1];
mesh Th = square(6,6,label=lab);
Th=trunc(Th,x<0.5 | y<0.5, label=1);

fespace Vh(Th,P1);          Vh u,v;          real error=0.1;
problem Probem1(u,v,solver=CG,eps=1.0e-6) =
       int2d(Th)(  dx(u)*dx(v) + dy(u)*dy(v))
     - int2d(Th)( v) + on(1,u=0);
for (int i=0;i< 7;i++)
{  Probem1;                                    //    solving the pde
   Th=adaptmesh(Th,u,err=error,nbvx=100000);
                              //    the adaptation with Hessian of u
   plot(Th,u,wait=1,fill=1);            u=u;
   error = error/ (1000.^(1./7.)) ;    } ;
```

Run:CornerLap.edp

Build of the metric form the solution u

Optimal metric norm for interpolation error (function `adaptmesh` in freefem++) for P_1 continuous Lagrange finite element

- L^∞ : $\mathcal{M} = \frac{1}{\varepsilon}|\nabla\nabla u| = \frac{1}{\varepsilon}|\mathcal{H}|$ where $\mathcal{H} = \nabla\nabla u$
- L^p : $\mathcal{M} = \frac{1}{\varepsilon}|det(\mathcal{H})|^{\frac{1}{2p+2}}\,|\mathcal{H}|$ (result of F. Alauzet, A. Dervieux)

In Norm $W^{1,p}$, the optimal metric $\mathcal{M}_\ell$ for the P_ℓ Lagrange finite element, Optimal is given by (with only acute triangle) (thank J-M. Mirebeau)

$$\mathcal{M}_{\ell,p} = \frac{1}{\varepsilon}(det\mathcal{M}_\ell)^{\frac{1}{\ell p+2}}\,\mathcal{M}_\ell$$

and (see `MetricPk` plugin and function)

- for P_1: $\mathcal{M}_1 = \mathcal{H}^2$ (sub optimal with acute triangle take $\mathcal{H}$)
- for P_2: $\mathcal{M}_2 = 3\sqrt{\begin{pmatrix} a & b \\ b & c \end{pmatrix}^2 + \begin{pmatrix} b & c \\ c & a \end{pmatrix}^2}$ with $D^{(3)}u(x,y) = (ax^3 + 3bx^2y + 3cxy^2 + dy^3)/3!$,

Run:adapt.edp

Run:AdaptP3.edp

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3 Academic Examples
- Laplace/Poisson
- 3d Poisson equation with mesh adaptation
- Linear elasticty equation
- Stokes equation
- Optimize Time depend schema

Laplace equation with full Neuman B.C. 0/III

We find p in H^1, such that:

$$-\Delta p = f \text{ in } \Omega, \quad \partial_n p = g_n \text{ on } \Gamma.$$

This problem is defined through a constant and mathematically the problem is well pose iff $\int_\Omega f + \int_\Gamma g_n = 0$ and p is in $H^1/\mathbb{R}$. So we can make a small regularization to remove the problem of constant by find $p_\varepsilon \in H^1$ such that

$$\varepsilon p_\varepsilon - \Delta p_\varepsilon = f \text{ in } \Omega, \quad \partial_n p_\varepsilon = g_n \text{ on } \Gamma$$

and the last problem is trivial to be approximate in FreeFem++:

```
Vh p,q; real eps=1e-8;                          //    warning eps
                          //    must be small but no too small.
solve Laplace(p,q)
  =  int2d(Th)(  p*q*eps  + grad(p)'*grad(q))
    - int2d(Th) ( f*q) - int1d(Th)(gn*q) ;
```

Remark: it is hard to put Dirichlet boundary condition on only one point so set the constant due to label definition.

Laplace equation (mixte formulation) I/III

Now we solve $-\Delta p = f$ in Ω, $p = g_d$ on Γ_d, $\partial_n p = g_n$ on Γ_n.
Γ_d, Γ_n is a partition of $\partial\Omega$.
with $\vec{u} = \nabla p$ the problem becomes:
Find $\vec{u}, p$ such that:

$$-\nabla.\vec{u} = f, \ \vec{u} - \nabla p = 0 \ \text{ in } \Omega, \quad p = g_d \ \text{ on } \Gamma_d, \quad \partial_n p = g_n \text{ on } \Gamma_n \tag{3}$$

Mixte variational formulation is: find $\vec{u} \in H_{div}(\Omega)$, $p \in L^2(\Omega)$, $\vec{u}.n = g_n$ on Γ_n such that

$$\int_\Omega q\nabla.\vec{u} + \int_\Omega p\nabla.\vec{v} + \vec{u}.\vec{v} = \int_\Omega -fq + \int_{\Gamma_d} g_d\vec{v}.\vec{n}, \quad \forall(\vec{v}, q) \in H_{div}\times L^2, \text{and } \ \vec{v}.n = 0 \text{ on } \Gamma_n$$

Laplace equation (mixted formulation) II/III

frfr

```
mesh Th=square(10,10); fespace Vh(Th,RT0), Ph(Th,P0);
func gd = 1.; func gn = 1.;  func f = 1.;
Vh [u1,u2],[v1,v2];
Ph p,q;
solve laplaceMixte([u1,u2,p],[v1,v2,q],solver=UMFPACK)
  =  int2d(Th)(  p*q*0e-10  + u1*v1 + u2*v2
                + p*(dx(v1)+dy(v2)) + (dx(u1)+dy(u2))*q )
   + int2d(Th) ( f*q)
   - int1d(Th,1,2,3)( gd*(v1*N.x +v2*N.y))     //     int on Γ_d
   + on(4,u1=gn*N.x,u2=gn*N.y);                //     mean u.n = (g_n n).n
```

Run:LaplaceRT.edp

Laplace equation (Galerking discontinuous formulation) III/III

solve $-\Delta u = f$ on Ω and $u = g$ on Γ

```
macro dn(u) (N.x*dx(u)+N.y*dy(u) ) //    def the normal derivative
mesh Th = square(10,10);                          //    unite square
fespace Vh(Th,P2dc);          //    discontinuous P2 finite element
         //    if pena = 0 => Vh must be P2 otherwise penalization
real pena=0;                                      //    to add penalization
func f=1;     func g=0;
Vh u,v;

problem A(u,v,solver=UMFPACK) =                                    //
     int2d(Th)(dx(u)*dx(v)+dy(u)*dy(v)   )
   + intalledges(Th)(        //    loop on all edge of all triangle
        ( jump(v)*average(dn(u)) - jump(u)*average(dn(v))
          + pena*jump(u)*jump(v) ) / nTonEdge   )
   - int2d(Th)(f*v)
   - int1d(Th)(g*dn(v)   + pena*g*v) ;
 A; //    solve DG
```

Run:LapDG2.edp

A mathematical Poisson Problem with full Neumann BC. with 1D lagrange multiplier

The variationnal form is find $(u, \lambda) \in V_h \times \mathbb{R}$ such that

$$\forall (v, \mu) \in V_h \times \mathbb{R} \qquad a(u, v) + b(u, \mu) + b(v, \lambda) = l(v), \qquad \text{where } b(u, \mu) = \mu \int_\Omega u$$

```
mesh Th=square(10,10);      fespace Vh(Th,P1);    //     P1 FE space
int n = Vh.ndof,  n1 = n+1; func f=1+x-y;
macro Grad(u) [dx(u),dy(u)]                                  //   EOM
varf va(uh,vh) = int2d(Th)( Grad(uh)'*Grad(vh)  ) ;
varf vL(uh,vh) = int2d(Th)( f*vh )  ;
varf vb(uh,vh)=  int2d(Th)(1.*vh);
matrix A=va(Vh,Vh);
real[int] b=vL(0,Vh), B = vb(0,Vh);
real[int]  bb(n1),x(n1),b1(1),l(1); b1=0;
matrix AA = [ [ A ,  B ] , [ B', 0 ] ] ;  bb = [ b, b1];
set(AA,solver=UMFPACK);       //    set the type of linear solver.
x = AA^-1*bb;     [uh[],l] = x;      //    solve the linear systeme
plot(uh,wait=1);                                  //    set the value
```

Run:Laplace-lagrange-mult.edp

3. Academic Examples
   - Laplace/Poisson
   - 3d Poisson equation with mesh adaptation
   - Linear elasticty equation
   - Stokes equation
   - Optimize Time depend schema

Poisson equation with 3d mesh adaptation

```
load "msh3" load "tetgen" load "mshmet" load "medit"
int nn  = 6;
int[int] l1111=[1,1,1,1],l01=[0,1],l11=[1,1];                          //   label numbering
mesh3 Th3=buildlayers(square(nn,nn,region=0,label=l1111),
      nn,  zbound=[0,1], labelmid=l11,labelup = l01,labeldown = l01);
Th3=trunc(Th3,(x<0.5)|(y < 0.5)|(z < 0.5) ,label=1);                    //   remove ]0.5,1[^3

fespace Vh(Th3,P1);       Vh u,v;                          //   FE. space definition
macro Grad(u) [dx(u),dy(u),dz(u)]                                      //   EOM
problem Poisson(u,v,solver=CG) =
  int3d(Th3)( Grad(u)'*Grad(v) )  -int3d(Th3)( 1*v ) + on(1,u=0);

real errm=1e-2;                                                  //   level of error
for(int ii=0; ii<5; ii++)
{
  Poisson;         Vh h;
  h[]=mshmet(Th3,u,normalization=1,aniso=0,nbregul=1,hmin=1e-3,
             hmax=0.3,err=errm);
  errm*= 0.8;                                        //   change the level of error
  Th3=tetgreconstruction(Th3,switch="raAQ"
        ,sizeofvolume=h*h*h/6.);
  medit("U-adap-iso-"+ii,Th3,u,wait=1);
}
```

Run:Laplace-Adapt-3d.edp

3 Academic Examples
- Laplace/Poisson
- 3d Poisson equation with mesh adaptation
- Linear elasticty equation
- Stokes equation
- Optimize Time depend schema

Linear Lame equation, weak form

Let a domain $\Omega \subset \mathbb{R}^d$ with a partition of $\partial\Omega$ in Γ_d, Γ_n.
Find the displacement $\boldsymbol{u}$ field such that:

$$-\nabla.\sigma(\boldsymbol{u}) = \boldsymbol{f} \text{ in } \Omega, \quad \mathbf{u = 0} \text{ on } \mathbf{\Gamma_d}, \quad \sigma(\mathbf{u}).\mathbf{n = 0} \text{ on } \mathbf{\Gamma_n} \tag{4}$$

Where $\varepsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + {}^t\nabla \boldsymbol{u})$ and $\sigma(\boldsymbol{u}) = \boldsymbol{A}\varepsilon(\boldsymbol{u})$ with $\boldsymbol{A}$ the linear positif operator on symmetric $d \times d$ matrix corresponding to the material propriety. Denote
$V_{\boldsymbol{g}} = \{\boldsymbol{v} \in H^1(\Omega)^d / \boldsymbol{v}_{|\Gamma_d} = \boldsymbol{g}\}$.
The Basic displacement variational formulation is: find $\boldsymbol{u} \in V_0(\Omega)$, such that:

$$\int_\Omega \varepsilon(\boldsymbol{v}) : \boldsymbol{A}\varepsilon(\boldsymbol{u}) = \int_\Omega \boldsymbol{v}.\boldsymbol{f} + \int_\Gamma ((\boldsymbol{A}\varepsilon(\boldsymbol{u})).n).v, \quad \forall \boldsymbol{v} \in V_0(\Omega) \tag{5}$$

Linear elasticty equation, in FreeFem++

The finite element method is just: replace V_g with a finite element space, and the FreeFem++ code:

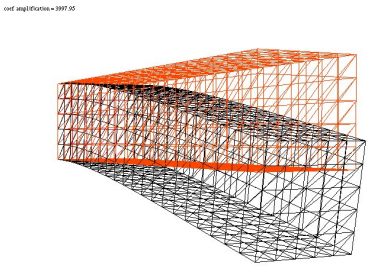
```
load "medit"   include "cube.idp"
int[int]  Nxyz=[20,5,5];
real [int,int]  Bxyz=[[0.,5.],[0.,1.],[0.,1.]];
int [int,int]  Lxyz=[[1,2],[2,2],[2,2]];
mesh3 Th=Cube(Nxyz,Bxyz,Lxyz);
                                                    //    Alu ...
real rhoAlu = 2600, alu11=  1.11e11 , alu12 = 0.61e11 ;
real alu44= (alu11-alu12)*0.5;
func Aalu = [  [alu11, alu12,alu12,   0.   ,0.   ,0.  ],
               [alu12, alu11,alu12,   0.   ,0.   ,0.   ],
               [alu12, alu12,alu11,   0.   ,0.   ,0.   ],
               [0. ,   0. ,   0. ,    alu44,0.   ,0.   ],
               [0. ,   0. ,   0. ,    0.   ,alu44,0.   ],
               [0. ,   0. ,   0. ,    0.   ,0.   ,alu44]   ];
real gravity = -9.81;
```

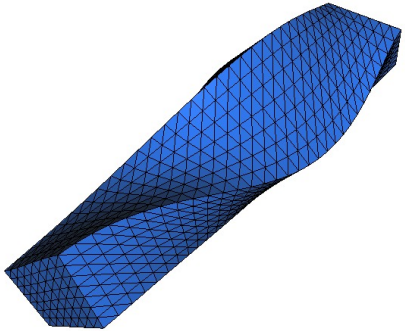
Linear elasticity equation, in FreeFem++

```
fespace Vh(Th,[P1,P1,P1]);
Vh [u1,u2,u3], [v1,v2,v3];
macro Strain(u1,u2,u3)
[ dx(u1), dy(u2), dz(u3), (dz(u2) +dy(u3)), (dz(u1)+dx(u3)),    (dy(u1)+dx(u2)) ]
//    EOM
solve Lame([u1,u2,u3],[v1,v2,v3])=
  int3d(Th)(  Strain(v1,v2,v3)'*(Aalu*Strain(u1,u2,u3))  )
- int3d(Th) ( rhoAlu*gravity*v3)
+ on(1,u1=0,u2=0,u3=0) ;

real coef= 0.1/u1[].linfty;  int[int] ref2=[1,0,2,0];
mesh3 Thm=movemesh3(Th,
     transfo=[x+u1*coef,y+u2*coef,z+u3*coef],
     label=ref2);
plot(Th,Thm, wait=1,cmm="coef  amplification = "+coef );
medit("Th-Thm",Th,Thm);
```

Lame equation / figure





Run:beam-3d.edp Run:beam-EV-3d.edp Run:free-cyl-3d.edp Run:beam-3d-Adapt.edp

3 Academic Examples
- Laplace/Poisson
- 3d Poisson equation with mesh adaptation
- Linear elasticty equation
- Stokes equation
- Optimize Time depend schema

Stokes equation

The Stokes equation is find a velocity field $\boldsymbol{u} = (u_1, .., u_d)$ and the pressure p on domain Ω of $\mathbb{R}^d$, such that

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= 0 \quad \text{in} \quad \Omega \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in} \quad \Omega \\ \boldsymbol{u} &= \boldsymbol{u}_\Gamma \quad \text{on} \quad \Gamma \end{aligned}$$

where $\boldsymbol{u}_\Gamma$ is a given velocity on boundary Γ.
The classical variational formulation is: Find $\boldsymbol{u} \in H^1(\Omega)^d$ with $\boldsymbol{u}_{|\Gamma} = u_\Gamma$, and $p \in L^2(\Omega)/\mathbb{R}$ such that

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^d,\ \forall q \in L^2(\Omega)/\mathbb{R}, \qquad \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} - p \nabla . v - q \nabla . u = 0$$

or now find $p \in L^2(\Omega)$ such than (with $\varepsilon = 10^{-10}$)

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^d,\ \forall q \in L^2(\Omega), \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} - p \nabla . v - q \nabla . u + \varepsilon p q = 0$$

Stokes equation in FreeFem++

```
 ...  build mesh .... Th (3d) T2d ( 2d)
fespace VVh(Th,[P2,P2,P2,P1]);          //    Taylor Hood FE.
macro Grad(u) [dx(u),dy(u),dz(u)]          //    EOM
macro div(u1,u2,u3) (dx(u1)+dy(u2)+dz(u3))      //   EOM
VVh [u1,u2,u3,p],[v1,v2,v3,q] ;
solve vStokes([u1,u2,u3,p],[v1,v2,v3,q]) =
  int3d(Th)(
          Grad(u1)'*Grad(v1)
        +  Grad(u2)'*Grad(v2)
        +  Grad(u3)'*Grad(v3)
      - div(u1,u2,u3)*q - div(v1,v2,v3)*p
      - 1e-10*q*p )
 + on(1,u1=0,u2=0,u3=0)    +  on(2,u1=1,u2=0,u3=0);
```

Run:Stokes-2d.edp Run:Stokes-bug.edp Run:Stokes-UzawaCahouetChabart-bug.edp
Run:Stokes-Pipe.edp Run:Stokes3d.edp

3. Academic Examples
   - Laplace/Poisson
   - 3d Poisson equation with mesh adaptation
   - Linear elasticty equation
   - Stokes equation
   - Optimize Time depend schema

First, it is possible to define variational forms, and use this forms to build matrix and vector to make very fast script (4 times faster here).
For example solve the Thermal Conduction problem of section 3.4. We must solve the temperature equation in Ω in a time interval (0,T).

$$\begin{aligned} &\partial_t u - \nabla \cdot (\kappa \nabla u) = 0 \text{ in } \Omega \times (0,T), \\ &u(x,y,0) = u_0 + x u_1 \\ &u = 30 \text{ on } \Gamma_{24} \times (0,T), \quad \kappa \frac{\partial u}{\partial n} + \alpha(u - u_e) = 0 \text{ on } \Gamma \times (0,T). \end{aligned} \tag{6}$$

The variational formulation is in $L^2(0,T;H^1(\Omega))$; we shall seek u^n satisfying

$$\forall w \in V_0; \qquad \int_\Omega \frac{u^n - u^{n-1}}{\delta t} w + \kappa \nabla u^n \nabla w) + \int_\Gamma \alpha(u^n - u_{ue}) w = 0$$

where $V_0 = \{w \in H^1(\Omega) / w_{|\Gamma_{24}} = 0\}$.

So the to code the method with the matrices $A = (A_{ij})$, $M = (M_{ij})$, and the vectors $u^n, b^n, b', b", b_{cl}$ (notation if w is a vector then w_i is a component of the vector).

$$u^n = A^{-1}b^n, \qquad b' = b_0 + Mu^{n-1}, \quad b" = \frac{1}{\varepsilon}\, b_{cl}, \quad b^n_i = \begin{cases} b"_i & \text{if } \ i \in \Gamma_{24} \\ b'_i & \text{else} \end{cases}$$

Where with $\frac{1}{\varepsilon} = \texttt{tgv} = 10^{30}$:

$$A_{ij} = \begin{cases} \frac{1}{\varepsilon} & \text{if } i \in \Gamma_{24}, \text{and} \quad j = i \\ \int_\Omega w_j w_i/dt + k(\nabla w_j.\nabla w_i) + \int_{\Gamma_{13}} \alpha w_j w_i & \text{else} \end{cases}$$

$$M_{ij} = \begin{cases} \frac{1}{\varepsilon} & \text{if } i \in \Gamma_{24}, \text{and} \quad j = i \\ \int_\Omega w_j w_i/dt & \text{else} \end{cases}$$

$$b_{0,i} = \int_{\Gamma_{13}} \alpha u_{ue} w_i$$

$$b_{cl} = u^0 \quad \text{the initial data}$$

```
 ...
Vh u0=fu0,u=u0;
```

Create three variational formulation, and build the matrices A,M.

```
varf vthermic (u,v)= int2d(Th)(u*v/dt
           + k*(dx(u) * dx(v) + dy(u) * dy(v)))
  +  int1d(Th,1,3)(alpha*u*v)   + on(2,4,u=1);
varf vthermic0(u,v) =   int1d(Th,1,3)(alpha*ue*v);
varf vMass (u,v)= int2d(Th)( u*v/dt)   + on(2,4,u=1);

real tgv = 1e30;
matrix A= vthermic(Vh,Vh,tgv=tgv,solver=CG);
matrix M= vMass(Vh,Vh);
```

Now, to build the right hand size we need 4 vectors.

```
real[int]  b0 = vthermic0(0,Vh);     //   constant part of RHS
real[int]  bcn = vthermic(0,Vh);    //   tgv on Dirichlet part
           //    we have for the node i :  i ∈ Γ24  ⇔  bcn[i] ≠ 0
real[int]  bcl=tgv*u0[];          //    the Dirichlet B.C. part
```

The Fast algorithm:

```
for(real t=0;t<T;t+=dt){
  real[int] b = b0 ;                              //    for the RHS
  b += M*u[];                 //   add the the time dependent part
  b = bcn ? bcl : b;    //   do ∀i:  b[i] = bcn[i] ?  bcl[i] :
b[i] ;
  u[] = A^-1*b;                          //    Solve linear problem
  plot(u);
}
```

Run:Heat.edp

Outline

1. Introduction
2. Tools
3. Academic Examples
4. Numerics Tools
5. Schwarz method with overlap
6. Exercices
7. No Linear Problem
8. Technical Remark on freefem++

4 Numerics Tools
- Connectivity
- Input/Output
- Tricks
- Eigenvalue
- Optimization Tools
- MPI/Parallel

```
mesh Th=square(5,5);
fespace Wh(Th,P2);
cout << " nb of DoF    : " << Wh.ndof << endl;
cout << " nb of DoF / K : " << Wh.ndofK << endl;
 int k= 2, kdf= Wh.ndofK ;;                    //    element 2
 cout << " df of element " << k << ":" ;
 for (int i=0;i<kdf;i++)  cout << Wh(k,i) << " ";
 cout << endl;
```

Remark on local numbering of Dof by element is
for each sub finite element Pk in `[P2,P2,P1]` get fist DoF on vertex, second Dof on edge (opposite to vertex), second on K.

Run:Mesh-info.edp

4 Numerics Tools
- Connectivity
- Input/Output
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- Eigenvalue
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- MPI/Parallel

uses `cout`, `cin`, `endl`, `<<`, `>>`.
To write to (resp. read from) a file,
declare a new variable `ofstream ofile("filename");`
or
`ofstream ofile("filename",append);` (resp. `ifstream ifile("filename");`)
or
`ofstream ofile("filename",append|binary);` (resp. `ifstream ifile("filename",binary);`)
and use `ofile` (resp. `ifile`) as `cout` (resp. `cin`).

You can use pipe to transfer data to a other code here (gnuplot), see pipe.edp example:

Run:pipe.edp

Run:io.edp

4 Numerics Tools
- Connectivity
- Input/Output
- Tricks
- Eigenvalue
- Optimization Tools
- MPI/Parallel

What is simple to do with freefem++ :

- Evaluate variational form with Boundary condition or not.
- Do interpolation
- Do linear algebra
- Solve sparse problem.

Freefem++ Trick: extract Dof list of border

? Question how the list Degree of Freedom (DoF) of border.

Idea Take a function negative function, and increasing on the border, and sort do a simultaneous sort this value and DoF numbering.

Run:ListOfDofOnBorder.edp

Computation of error estimate $\eta_K = \sqrt{\int_K \text{blabla}} = \sqrt{\int_\Omega w_k \text{blabla}}$ where w_k is the basic function of `fespace Ph(Th,P0)`.

```
varf vetaK(unused,wK) = int2d(Th)( blabla * wK);
Ph etaK; etaK[] = vetaK(0,Ph);  etaK=sqrt(etaK);
```

to Change Default sparse solver add following line:

```
load += "MUMPS_seq"
```

if `MUMPS-seq` is available in file `\$(HOME)/.freefem++.pref`

Diff How to compute, differential: use of `macro`

$$J(u) = \int_\Omega F(u); \qquad \texttt{macro F(u)} = \sqrt{1 + \nabla u.\nabla u}$$

$$dJ(u)(v) = \int_\Omega dF(u,v); \qquad \texttt{macro dF(u,v)} = \frac{\nabla u.\nabla v}{\sqrt{1 + \nabla u.\nabla u}}$$

$$ddJ(u)(v,w) = \int_\Omega ddF(u,v,w);$$

$$\texttt{macro ddF(u,v,v)} = \frac{\nabla w.\nabla v}{\sqrt{1 + \nabla u.\nabla u}} - \frac{(\nabla u.\nabla v)(\nabla w.\nabla v)}{\sqrt{1 + \nabla u.\nabla u}^3}$$

4 Numerics Tools
- Connectivity
- Input/Output
- Tricks
- Eigenvalue
- Optimization Tools
- MPI/Parallel

Eigenvalue/ Eigenvector example

The problem, Find the first λ, u_λ such that:

$$a(u_\lambda, v) = \int_\Omega \nabla u_\lambda \nabla v = \lambda \int_\Omega u_\lambda v = \lambda b(u_\lambda, v)$$

the boundary condition is make with exact penalization: we put $1e30 = tgv$ on the diagonal term of the lock degree of freedom. So take Dirichlet boundary condition only with a variational form and not on b variational form , because we compute eigenvalue of

$$\frac{1}{\lambda} v = A^{-1} B v$$

Otherwise we get spurious mode.
Arpack interface:

```
int k=EigenValue(A,B,sym=true,value=ev,vector=eV);
```

Eigenvalue/ Eigenvector example code

```
...
fespace Vh(Th,P1);
macro Grad(u) [dx(u),dy(u),dz(u)]                                    //    EOM
varf  a(u1,u2)= int3d(Th)(  Grad(u1)'*Grad(u2) +  on(1,u1=0) ;
varf b([u1],[u2]) = int3d(Th)(  u1*u2 ) ;                            //    no BC
matrix A= a(Vh,Vh,solver=UMFPACK),
        B= b(Vh,Vh,solver=CG,eps=1e-20);

int nev=40;          //    number of computed eigenvalue close to 0
real[int] ev(nev);                    //    to store nev eigenvalue
Vh[int] eV(nev);                      //    to store nev eigenvector
int k=EigenValue(A,B,sym=true,value=ev,vector=eV);
k=min(k,nev);
for (int i=0;i<k;i++)
   plot(eV[i],cmm="ev  "+i+" v =" + ev[i],wait=1,value=1);
```

Run:Lap3dEigenValue.edp Run:LapEigenValue.edp

4 Numerics Tools
- Connectivity
- Input/Output
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- Eigenvalue
- Optimization Tools
- MPI/Parallel

The IPOPT optimizer in a FreeFem++ script is done with the `IPOPT` function included in the `ff-Ipopt` dynamic library. IPOPT is designed to solve constrained minimization problem in the form :

$$\begin{array}{ll} \text{find} & x_0 = \underset{x\in\mathbb{R}^n}{\operatorname{argmin}} f(x) \\ \text{s.t.} & \left\{ \begin{array}{ll} \forall i \le n,\ x_i^{\mathrm{lb}} \le x_i \le x_i^{\mathrm{ub}} & \text{(simple bounds)} \\ \forall i \le m,\ c_i^{\mathrm{lb}} \le c_i(x) \le c_i^{\mathrm{ub}} & \text{(constraints functions)} \end{array} \right. \end{array}$$

Where ub and lb stand for "upper bound" and "lower bound". If for some $i, 1 \le i \le m$ we have $c_i^{\mathrm{lb}} = c_i^{\mathrm{ub}}$, it means that c_i is an equality constraint, and an inequality one if $c_i^{\mathrm{lb}} < c_i^{\mathrm{ub}}$.

```
func real J(real[int] &X) {...}                          //   Fitness Function,
func real[int] gradJ(real[int] &X) {...}                 //   Gradient

func real[int] C(real[int] &X) {...}                     //   Constraints
func matrix jacC(real[int] &X) {...}                     //   Constraints jacobian

matrix jacCBuffer;                          //   just declare, no need to define yet
func matrix jacC(real[int] &X)
{
  ...                                                    //   fill jacCBuffer
  return jacCBuffer;
}
```

The hessian returning function is somewhat different because it has to be the hessian of the lagrangian function : $(x, \sigma_f, \lambda) \mapsto \sigma_f \nabla^2 f(x) + \sum_{i=1}^{m} \lambda_i \nabla^2 c_i(x)$ where $\lambda \in \mathbb{R}^m$ and $\sigma \in \mathbb{R}$. Your hessian function should then have the following prototype :

```
matrix hessianLBuffer;                                   //   just to keep it in mind
func matrix hessianL(real[int] &X,real sigma,real[int] &lambda) {...}
```

Ipopt Call

```
real[int] Xi = ... ;                                  //   starting point
IPOPT(J,gradJ,hessianL,C,jacC,Xi, ... );

IPOPT(J,gradJ,C,jacC,Xi,...);                        //   IPOPT with BFGS
IPOPT(J,gradJ,hessianJ,Xi,...);                         //   Newton IPOPT
                                                  //    without constraints
IPOPT(J,gradJ,Xi, ... );                          //   BFGS, no constraints
IPOPT(J,gradJ,Xi, ... );                          //   BFGS, no constraints
IPOPT([b,A],CC,ui1[],lb=lb1[],clb=cl[]..);            //   affine case
  ...
```

Ipopt interface

```
load  "ff-Ipopt"
varf vP([u1,u2],[v1,v2]) = int2d(Th)(Grad(u1)'*Grad(v1)+ Grad(u2)'*Grad(v2))
- int2d(Th)(f1*v1+f2*v2);

matrix A = vP(Vh,Vh);                                   //   Fitness function matrix...
real[int] b = vP(0,Vh);                                          //   and linear form
int[int] II1=[0],II2=[1];                                     //   Constraints matrix
matrix C1 =  interpolate (Wh,Vh, U2Vc=II1);
matrix C2 =  interpolate (Wh,Vh, U2Vc=II2);
matrix CC = -1*C1 + C2;                                                  //    u2 - u1 >0
Wh cl=0;                              //   constraints lower bounds (no upper bounds)
varf vGamma([u1,u2],[v1,v2]) = on(1,2,3,4,u1=1,u2=1);
real[int] onGamma=vGamma(0,Vh);
Vh [ub1,ub2]=[g1,g2];
Vh [lb1,lb2]=[g1,g2];
ub1[] = onGamma ? ub1[] : 1e19  ;                          //   Unbounded in interior
lb1[] = onGamma ? lb1[] : -1e19  ;
Vh [uzi,uzi2]=[uz,uz2],[lzi,lzi2]=[lz,lz2],[ui1,ui2]=[u1,u2];;
Wh lmi=lm;
IPOPT([b,A],CC,ui1[],lb=lb1[],clb=cl[],ub=ub1[],warmstart=iter>1,uz=uzi[],lz=lzi[],lm=lmi
```

Run:IpoptLap.edp Run:IpoptVI2.edp Run:IpoptMinSurfVol.edp

```
load  "ff-NLopt"
...
if(kas==1)
   mini = nloptAUGLAG(J,start,grad=dJ,lb=lo,ub=up,IConst=IneqC,
            gradIConst=dIneqC,subOpt="LBFGS",stopMaxFEval=10000,
            stopAbsFTol=starttol);
else if(kas==2)
  mini = nloptMMA(J,start,grad=dJ,lb=lo,ub=up,stopMaxFEval=10000,
          stopAbsFTol=starttol);
else if(kas==3)
  mini = nloptAUGLAG(J,start,grad=dJ,IConst=IneqC,gradIConst=dIneqC,
          EConst=BC,gradEConst=dBC,
          subOpt="LBFGS",stopMaxFEval=200,stopRelXTol=1e-2);
else if(kas==4)
  mini = nloptSLSQP(J,start,grad=dJ,IConst=IneqC,gradIConst=dIneqC,
         EConst=BC,gradEConst=dBC,
         stopMaxFEval=10000,stopAbsFTol=starttol);
```

Run:VarIneq2.edp

Stochastic interface

This algorithm works with a normal multivariate distribution in the parameters space and try to adapt its covariance matrix using the information provides by the successive function evaluations. Syntaxe: `cmaes(J,u[],..)` ( )
From `http://www.lri.fr/~hansen/javadoc/fr/inria/optimization/cmaes/package-summary.html`

Stochastic Exemple

```
load  "ff-cmaes"

real mini = cmaes(J,start,stopMaxFunEval=10000*(al+1),
                  stopTolX=1.e-4/(10*(al+1)),
                  initialStdDev=(0.025/(pow(100.,al))));
SSPToFEF(best1[],best2[],start);
```

Run:cmaes-VarIneq.edp

```
load  "mpi-cmaes"

real mini = cmaesMPI(J,start,stopMaxFunEval=10000*(al+1),
                  stopTolX=1.e-4/(10*(al+1)),
                  initialStdDev=(0.025/(pow(100.,al))));
SSPToFEF(best1[],best2[],start);
```

remark, the FreeFem `mpicommworld` is used by default. The user can specify his own MPI communicator with the named parameter "`comm=`", see the MPI section of this manual for more informations about communicators in `FreeFem++`.

4 Numerics Tools
- Connectivity
- Input/Output
- Tricks
- Eigenvalue
- Optimization Tools
- MPI/Parallel

A first way to break complexity

1. Build matrix in parallel by assembling par region remark with the `change` function you change the region numbering to build region.

```
real c = mpisize/real(Th.nt) ;
Th=change(Th,fregion= min(mpisize-1,int(nuTriangle*c)));
```

2. Assemble the full matrix

```
varf vlaplace(uh,vh) =                    //    definition de problem
      int3d(Th,mpirank)( uh*vh+ dt*Grad(uh)'*grad(vh) )
    + int3d(Th,mpirank)( dt*vh*f) + on(1,uh=g) ;
matrix A,Ai = vlaplace(Vh,Vh,tgv=ttgv) ;
mpiAllReduce(Ai,A,mpiCommWorld,mpiSUM);          //   assemble in //
```

3. Solve the linear using a good parallel solver (MUMPS)

```
load "MUMPS_FreeFem"
   uh[] = A^-1*b ;                                  //    resolution
```

Run:Heat3d.edp

Run:NSCaraCyl-100-mpi.edp

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To solve the following Poisson problem on domain Ω with boundary Γ in $L^2(\Omega)$:

$$-\Delta u = f, \text{ in } \Omega, \text{ and } u = g \text{ on } \Gamma,$$

where $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\Gamma)$ are two given functions.
Let introduce $(\pi_i)_{i=1,..,N_p}$ a positive regular partition of the unity of Ω, q-e-d:

$$\pi_i \in \mathcal{C}^0(\Omega): \quad \pi_i \geq 0 \text{ and } \sum_{i=1}^{N_p} \pi_i = 1.$$

Denote Ω_i the sub domain which is the support of π_i function and also denote Γ_i the boundary of Ω_i.
The parallel Schwarz method is Let $\ell = 0$ the iterator and a initial guest u^0 respecting the boundary condition (i.e. $u^0_{|\Gamma} = g$).

$$\forall i = 1.., N_p: \quad -\Delta u_i^\ell = f, \text{ in } \Omega_i, \quad \text{and } u_i^\ell = u^\ell \text{ on } \Gamma_i \tag{7}$$

$$u^{\ell+1} = \textstyle\sum_{i=1}^{N_p} \pi_i u_i^\ell \tag{8}$$

We never use finite element space associated to the full domain Ω because it to expensive. So we use on each domain i we defined $J_i = \{j \in 1, \ldots, N_p \ / \ \Omega_i \cap \Omega_j \neq \emptyset\}$ and we have

$$(u^{\ell+1})_{|\Omega_i} = \sum_{j \in J_i} (\pi_j u_j^\ell)_{|\Omega_i} \tag{9}$$

We denote $u^\ell_{h|i}$ the restriction of u^ℓ_h on V_{hi}, so the discrete problem on Ω_i of problem (7) is find $u^\ell_{hi} \in V_{hi}$ such that:

$$\forall v_{hi} \in V_{0i} : \int_{\Omega_i} \nabla v_{hi} . \nabla u^\ell_{hi} = \int_{\Omega_i} f v_{hi},$$

$$\forall k \in \mathcal{N}_{hi}^{\Gamma_i} \ : \ \sigma_i^k(u^\ell_{hi}) = \sigma_i^k(u^\ell_{h|i})$$

where $\mathcal{N}_{hi}^{\Gamma_i}$ is the set of the degree of freedom (Dof) on $\partial\Omega_i$ and σ_i^k the Dof of V_{hi}.

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Transfer Part equation(5)

To compute $v_i = (\pi_i u_i)_{|\Omega_i} + \sum_{j \in J_i} (\pi_j u_j)_{|\Omega_i}$ and can be write the freefem++ function Update with asynchronous send/recv (Otherwise dead lock).

```
func bool Update(real[int] &ui, real[int] &vi)
{ int n= jpart.n;
  for(int j=0;j<njpart;++j)  Usend[j][]=sMj[j]*ui;
  mpiRequest[int] rq(n*2);
  for (int j=0;j<n;++j)
        Irecv(processor(jpart[j],comm,rq[j  ]), Ri[j][]);
  for (int j=0;j<n;++j)
        Isend(processor(jpart[j],comm,rq[j+n]), Si[j][]);
  for (int j=0;j<n*2;++j)
        int k= mpiWaitAny(rq);
  vi = Pii*ui;                                  //    set to (πi ui)|Ωi
                              //    apply the unity local partition .
   for(int j=0;j<njpart;++j)
     vi += rMj[j]*Vrecv[j][];                   //    add (πj uj)|Ωi
 return true; }
```

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Finally you can easily accelerate the fixe point algorithm by using a parallel GMRES algorithm after the introduction the following affine $\mathcal{S}_i$ operator sub domain Ω_i.

```
func real[int] Si(real[int]& U) {
real[int] V(U.n)    ;  b= onG .* U;
 b  = onG ? b : Bi ;
 V = Ai^-1*b;                                          //      (7)
 Update(V,U);                                          //      (??)
 V -= U;    return V; }
```

Where the parallel MPIGMRES or MPICG algorithm is to solve $A_i x_i = b_i, i = 1, .., N_p$ by just changing the dot product by reduce the local dot product of all process with the following MPI code:

```
template<class R> R ReduceSum1(R s,MPI_Comm * comm)
{   R r=0;
    MPI_Allreduce( &s, &r, 1 ,MPI_TYPE<R>::TYPE(),
                        MPI_SUM,  *comm );
    return r; }
```

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A simple coarse grid is we solve the problem on the coarse grid:

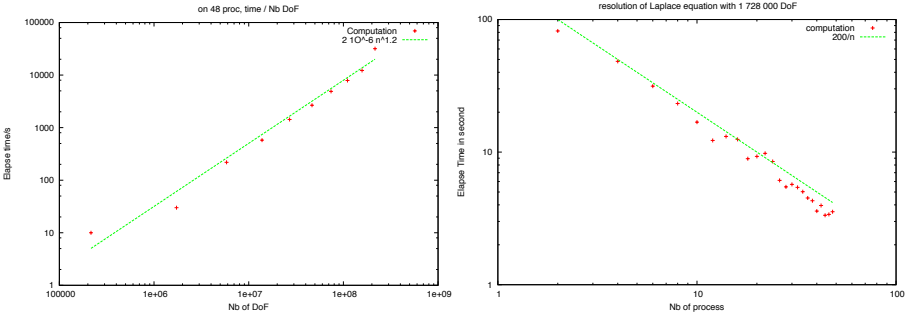
```
func bool  CoarseSolve(real[int]& V,real[int]& U,
             mpiComm& comm)
{
    if(AC.n==0 && mpiRank(comm)==0)          //    first time build
      AC = vPbC(VhC,VhC,solver=sparsesolver);
   real[int] Uc(Rci.n),Bc(Uc.n);
   Uc= Rci*U;                                 //    Fine to Coarse
   mpiReduce(Uc,Bc,processor(0,comm),mpiSUM);
   if(mpiRank(comm)==0)
      Uc = AC^-1*Bc;                          //    solve of proc 0
    broadcast(processor(0,comm),Uc);
   V = Pci*Uc;                                //    Coarse to Fine
}
```

Limitation: if the initial problem, data have oscillation, you must use homogenization technic on coarse problem, or use the F. Nataf and co, preconditionner.

So we finally we get 4 algorithms

1. The basic schwarz algorithm $u^{\ell+1} = \mathcal{S}(u^{\ell})$, where $\mathcal{S}$ is one iteration of schwarz process.
2. Use the GMRES to find u solution of the linear system $\mathcal{S}u - u = 0$.
3. Use the GMRES to solve parallel problem $\mathcal{A}_i u_i = b_i$, $i = 1, \ldots, N_p$, with RAS precondicionneur
4. Use the method with two level precondicionneur RAS and Coarse.

On the SGI UV 100 of the lab:



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A Parallel Numerical experiment on laptop

We consider first example in an academic situation to solve Poisson Problem on the cube $\Omega =]0,1[^3$

$$-\Delta u = 1, \text{in } \Omega; \qquad u = 0, \text{ on } \partial\Omega. \tag{10}$$

With a cartesian meshes $\mathcal{T}_{hn}$ of Ω with $6n^3$ tetrahedron, the coarse mesh is $\mathcal{T}_{hm}^*$, and m is a divisor of n.
We do the validation of the algorithm on a Laptop Intel Core i7 with 4 core at 1.8 Ghz with $4Go$ of RAM DDR3 at 1067 Mhz,

Run:DDM-Schwarz-Lap-2dd.edp
Run:DDM-Schwarz-Lame-2d.edp
Run:DDM-Schwarz-Lame-3d.edp
Run:DDM-Schwarz-Stokes-2d.edp

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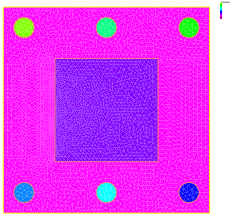
- Benchmark: Navier-Stokes

An exercice: Oven problem

Find the power on the 6 resistors of an oven such that the temperature is close as possible to a given temperature in the region 6.
The equation are the stationary Head equation in 2d with classical Fourier boundary condition. the mesh of the domain :



"oven.msh"

let call the u_p the solution of

$$\begin{aligned} -\nabla.K\nabla u_p &= \sum_{i=0}^{5} p_i * \chi_i \text{ in } \Omega \\ u + K\nabla u_p.n &= 0 \text{ on } \Gamma = \partial\Omega \end{aligned}$$

where χ_i is the characteristics function of the resistance i, $K = 10$ in region 6, $K = 1$ over where.
The problem is find the array p such that

$$p = \operatorname{argmin} \int_{\Omega_6} (u_p - 100)^2 dx$$

build the mesh with multi border trick.

```
  Xh[int]  ur(6);          //   to store the 6 FE. functions Xh
```

FreeFem++ as only linear solver on sparse matrix by default, but in the `lapack` plugin you have access to full matrix solver (see `examples++-load/lapack.edp`) so a way to solve a full matrix problem is for example :

```
  real[int,int] AP(6,6);                    //   a full matrix
  real[int] B(6),PR(6);       //   to array (vector of size 6)

  ... bla bla to compute AP and B

  matrix A=AP;          //    full matrix to sparse of or use of
lapack
  set(A,solver=CG);          //    set linear solver to the C.G.
  PR=A^-1*B;                      //    solve the linear system.
```

The file name of the mesh is `oven.msh`, and the region numbers are 0 to 5 for the resitor, 6 for Ω_6 and 7 for the rest of Ω and the label of Γ is 1.

My solution, build the 6 basics function u_{e_i}

```
int  nbresitor=6;            mesh Th("oven.msh");
real[int] pr(nbresitor+2), K(nbresitor+2);
 K=1;      K[regi]=10;                                  //    def K
int regi=nbresitor, rege=nbresitor+1, lext=1;

macro Grad(u) [dx(u),dy(u)]                             //   EOM
fespace Xh(Th,P2);    Xh u,v; int iter=0;
problem Chaleur(u,v,init=iter)
     =  int2d(Th)( Grad(u)'*Grad(v)* K[region]) +
int1d(Th,lext)(u*v)
     +  int2d(Th)(pr[region]*v) ;

Xh[int]  ur(nbresitor);                    //    to store the 6 u_{e_i}
for(iter=0;iter<nbresitor;++iter)
{  pr=0;pr[iter]=1;
   Chaleur;
   ur[iter][]=u[];
   plot(ur[iter],fill=1,wait=1);  }
```

Computation of the optimal value

```
real[int,int] AP(nbresitor,nbresitor);
real[int] B(nbresitor),PR(nbresitor);

Xh   ui = 100;
for(int i=0;i<nbresitor;++i)
{
    B[i]=int2d(Th,regi)(ur[i]*ui);
    for(int j=0;j<6;++j)
       AP(i,j)= int2d(Th,regi)(ur[i]*ur[j]);
}
  matrix A=AP; set(A,solver=UMFPACK);
  PR=A^-1*B;
  cout << " P R = " << PR << endl;
  u[]=0;
  for (int i=0;i<nbresitor;++i)
    u[] += PR[i]*ur[i][];
```

Run:oven-cimpa.edp

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An exercice: Min surface problem

The geometrical problem: Find a function $u : C^1(\Omega) \mapsto \mathbb{R}$ where u is given on $\Gamma = \partial\Omega$, (e.i. $u_{|\Gamma} = g$) such that the area of the surface S parametrize by $(x, y) \in \Omega \mapsto (x, y, u(x, y))$ is minimal.
So the problem is $\arg\min J(u)$ where

$$\arg\min J(u) = \int_\Omega \left\| \begin{pmatrix} 1 \\ 0 \\ \partial_x u \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \partial_y u \end{pmatrix} \right\| d\Omega = \int_\Omega \sqrt{1 + (\partial_x u)^2 + (\partial_y u)^2} d\Omega$$

So the Euler-Lagrange equation associated to the minimization is:

$$\forall v / v_{|\Gamma} = 0 \quad : \quad DJ(u)v = -\int_\Omega \frac{(\partial_x v \partial_x u + \partial_y v_y \partial_y u)}{\sqrt{1 + (\partial_x u)^2 + (\partial_y u)^2}} d\Omega = 0$$

So find the solution for $\Omega =]0, \pi[^2[$ and $g(x, y) = cos(2 * x) * cos(2 * y)$. by using the Non Linear Conjugate gradient `NLCG` like in the example: `algo.edp` in `examples++-tutorial`, or `IPOPT` interface.

Example of use of NLCG function:

```
func real J(real[int] & xx)        //    the functional to minimized
    {  real s=0;
      ...                                //    add code to copy xx array
                                         //    of finite element function
      return s; }
func real[int] DJ(real[int] &xx)        //    the grad of functional
    { ....                               //    add code to copy xx array
                                         //    of finite element function
      return xx; };          //    return of an existing variable ok
  ...
NLCG(DJ,x,eps=1.e-6,nbiter=20,precon=matId);
```

Two useful operators on array real[int]

```
   real[int]  a(10),b(10);
   ...
   a =  b ? 1. : 0 ;          //    a[i] = 1 if b[i] else a[i]=0.  ∀i
```

To see the 3D plot of the surface

```
 plot(u,dim=3);
```

My solution First the functional

```
func g=cos(2*x)*cos(2*y);                              //    valeur au bord
mesh Th=square(20,20,[x*pi,y*pi]);          //    mesh definition of Ω
fespace Vh(Th,P1);

func real J(real[int] & xx)          //    the functionnal to minimise
  { Vh u;u[]=xx;              //    to set FE.function u from xx array
    return int2d(Th)( sqrt(1 +dx(u)*dx(u) + dy(u)*dy(u) ) ) ; }

func real[int] dJ(real[int] & xx)              //    the grad of the J
  { Vh u;u[]=xx;             //    to set FE. function u from xx array
    varf au(uh,vh) = int2d(Th)( ( dx(u)*dx(vh) + dy(u)*dy(vh) )
              / sqrt(1. +dx(u)*dx(u) + dy(u)*dy(u) )      )
              + on(1,2,3,4,uh=0);
    return xx= au(0,Vh);  } //    warning no return of local array
```

Solution 1:

```
Vh u=G;
verbosity=5;                          //    to see the residual
int conv=NLCG(dJ,u[],nbiter=500,eps=1e-5);
cout << " the  surface =" << J(u[])  << endl;
                                      //    so see the surface un 3D
plot(u,dim=3);
```

Run:minimal-surf.edp

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Heat equation with thermic resistance

let Ω be a domain cut with internal boundary Γ_0 in 2 sub-domain Ω_i,$(i = 1, 2)$
We have Heat equation (Poisson) on Ω , but on Γ_0 we have a jump $[u]$ on the temperature u proportional to the temperature flux which is continue
So the equation to solve is:
Find u such that $u_{|\Omega_i} \in H(\Omega_i)$ for $i = 1, 2$ and

$$-\nabla\kappa\nabla u = f_i, \quad \text{in } \Omega_i$$

$$\alpha[u] - \kappa\nabla u.n = 0, \quad [\kappa\nabla u.n] = 0, \quad \text{on } \Gamma_0$$

$+$ external boundary condition on $\partial\Omega$.
For the test take:
$L = 3$, $\Omega =]-L, L[\times]0, 1[$, $\Gamma_0 = \{sin(\pi y)/5, y \in [0, 1]\}$, take $\kappa = i$ in Ω_i.
The external boundary condition on $\partial\Omega$ are: $\kappa\nabla u.n = 0$ on upper and lower boundary , $u = 0$ at the left part, $u = 1$ at the right part.

Heat equation with thermic resistance

Method 1: Solve 2 coupled problems and use the block matrix tools to defined the linear system of the problem.
Method 2: We suppose the Γ_0 move with time, and Γ_0 is not discretize in the mesh.

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An exercice: Navier-Stokes

Try to make the 2d benchmark of :
http://www.mathematik.tu-dortmund.de/lsiii/cms/papers/SchaeferTurek1996.pdf
The mesh can be set:

```
int n=15;                                          //    parameter ...
real D=0.1, H=0.41;
real cx0 = 0.2, cy0 = 0.2;                        //    center of cyl.
real xa = 0.15, ya=0.2, xe = 0.25,ye =0.2;              //    point for
pressure..

border fr1(t=0,2.2){x=t; y=0; label=1;}
border fr2(t=0,H){x=2.2; y=t; label=2;}
border fr3(t=2.2,0){x=t; y=H; label=1;}
border fr4(t=H,0){x=0; y=t; label=1;}
border fr5(t=2*pi,0){x=cx0+D*sin(t)/2; y=cy0+D*cos(t)/2; label=3;}

mesh Th=buildmesh(fr1(5*n)+fr2(n)+fr3(5*n)+fr4(n)+fr5(-n*3));
plot(Th, wait=1);
```

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- Variational Inequality
- Ground water
- Bose Einstein Condensate
- Hyper elasticity equation
- Periodic Surface Acoustic Waves Transducer Analysis
- Phase change with Natural Convection

Newton methods

To solve $F(u) = 0$ the Newton's algorithm is

1. u^0 a initial guest
2. do
   1. find w^n solution of $DF(u^n)w^n = F(u^n)$
   2. $u^{n+1} = u^n - w^n$
   3. if($||w^n|| < \varepsilon$) break;

The Optimize Newton Method if $F = C + L + N$, where C is the constant part, L is Linear part and N is Non linear part of F. we have $DF = L + DN$ and $DF(u^n)u^{n+1} = DF(u^n)u^n - F(u^n) = DN(u^n)u^n - N(u^n) - C$. So the change in algorithm are:

   2. find u^{n+1} solution of $DF(u^n)u^{n+1} = DN(u^n)u^n - N(u^n) - C$
   3. if($||u^{n+1} - u^n|| < \varepsilon$) break;

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For Navier Stokes problem the Newtow algorithm is: $\forall v, q$,

$$F(u,p) = \int_\Omega (u.\nabla)u.v + u.v + \nu \nabla u : \nabla v - q\nabla.u - p\nabla.v + BC$$

$$\begin{aligned} DF(u,p)(w,w_p) &= \int_\Omega (w.\nabla)u.v + (u.\nabla)w.v \\ &\quad + \int_\Omega \nu \nabla w : \nabla v - q\nabla.w - p_w \nabla.v + BC0 \end{aligned}$$

Run:cavityNewton.edp

Run:NSNewtonCyl-100-mpi.edp

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = 0, \quad \nabla \cdot u = 0$$

with the same boundary conditions and with initial conditions $u = 0$.
This is implemented by using the interpolation operator for the term $\frac{\partial u}{\partial t} + (u \cdot \nabla)u$, giving a discretization in time

$$\begin{array}{rl} \frac{1}{\tau}(u^{n+1} - u^n \circ X^n) - \nu \Delta u^{n+1} + \nabla p^{n+1} & = 0, \\ \nabla \cdot u^{n+1} & = 0 \end{array} \tag{11}$$

The term $X^n(x) \approx x - \tau u^n(x)$ will be computed by the interpolation operator or convect operator.
Or better we use an order 2 schema, BDF1

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u \approx \frac{(3u^{n+1} - 4u^n \circ X_1^n + u^{n-1} \circ X_2^n)}{2\tau}$$

with $u^* = 2u^n - u^{n-1}$, and $X_1^n(x) \approx x - \tau u^*(x), X_2^n(x) \approx x - 2\tau u^*(x)$
Run:NSCaraCyl-100-mpi.edp

```
 real alpha =1./dt;
 varf vNS([uu1,uu2,uu3,p],[v1,v2,v3,q]) =
  int3d(Th)( alpha*(uu1*v1+uu2*v2+uu3*v3)
  + nu*(Grad(uu1)'*Grad(v1)+Grad(uu2)'*Grad(v2)
+Grad(uu3)'*Grad(v3))
  - div(uu1,uu2,uu3)*q - div(v1,v2,v3)*p + 1e-10*q*p )
  + on(1,2,3,4,5,uu1=0,uu2=0,uu3=0)
  + on(6,uu1=4*(1-x)*(x)*(y)*(1-y),uu2=0,uu3=0)
  + int3d(Th)( alpha*(
     u1(X1,X2,X3)*v1  +   u2(X1,X2,X3)*v2 +   u3(X1,X2,X3)*v3 ));
A = vNS(VVh,VVh);   set(A,solver=UMFPACK); //   build and factorize
matrix
real t=0;
for(int i=0;i<50;++i)
  { t += dt;  X1[]=XYZ[]-u1[]*dt;     //    set χ=[X1,X2,X3] vector
    b=vNS(0,VVh);                                //    build NS rhs
    u1[]= A^-1 * b;                    //    solve the linear systeme
    ux= u1(x,0.5,y);  uz= u3(x,0.5,y);  p2= p(x,0.5,y);
    plot([ux,uz],p2,cmm=" cut y = 0.5, time ="+t,wait=0);  }
```

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Variational Inequality

To solve just make a change of variable $u = u^+ - u^-, u > 0$ and $v = u^+ + u^-$, and we get a classical VI problem on u and and the Poisson on v.

So we can use the algorithm of Primal-Dual Active set strategy as a semi smooth Newton Method HINTERMULLER , K. ITO, K. KUNISCH SIAM J. Optim. V 13, I 3, 2002.

In this case , we just do all implementation by hand in FreeFem++ language

Run:VI-2-membrane-adap.edp

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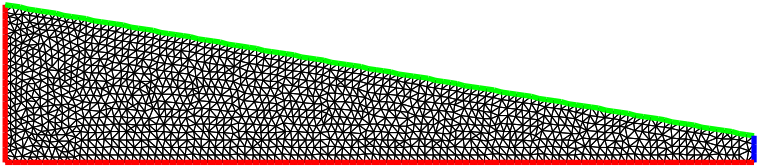
   - Bose Einstein Condensate
   - Hyper elasticity equation
   - Periodic Surface Acoustic Waves Transducer Analysis
   - Phase change with Natural Convection

A Free Boundary problem , (phreatic water)

Let a trapezoidal domain Ω defined in FreeFem++:

```
real L=10;                                                    //    Width
real h=2.1;                                                 //    Left height
real h1=0.35;                                              //    Right height
border a(t=0,L){x=t;y=0;label=1;}; //    impermeable Γa
border b(t=0,h1){x=L;y=t;label=2;};     //    the source Γb
border f(t=L,0){x=t;y=t*(h1-h)/L+h;label=3;}; //    Γf
border d(t=h,0){x=0;y=t;label=4;};     //    Left impermeable Γd
int n=10;
mesh Th=buildmesh (a(L*n)+b(h1*n)+f(sqrt(L^2+(h-h1)^2)*n)+d(h*n));
plot(Th,ps="dTh.eps");
```

The initial mesh



The problem is: find p and Ω such that:

$$\begin{cases} -\Delta p = 0 & \text{in } \Omega \\ p = y & \text{on } \Gamma_b \\ \dfrac{\partial p}{\partial n} = 0 & \text{on } \Gamma_d \cup \Gamma_a \\ \dfrac{\partial p}{\partial n} = \frac{q}{K} n_x & \text{on } \Gamma_f \quad (Neumann) \\ p = y & \text{on } \Gamma_f \quad (Dirichlet) \end{cases}$$

where the input water flux is $q = 0.02$, and $K = 0.5$. The velocity u of the water is given by $u = -\nabla p$.

We use the following fix point method: (*with bad main B.C. Run:freeboundaryPB.edp*) let be, $k = 0$, $\Omega^k = \Omega$. First step, we forgot the Neumann BC and we solve the problem: Find p in $V = H^1(\Omega^k)$, such $p = y$ on Γ_b^k et on Γ_f^k

$$\int_{\Omega^k} \nabla p \nabla p' = 0, \quad \forall p' \in V \text{ with } p' = 0 \text{ on } \Gamma_b^k \cup \Gamma_f^k$$

With the residual of the Neumann boundary condition we build a domain transformation $\mathcal{F}(x, y) = [x, y - v(x)]$ where v is solution of: $v \in V$, such than $v = 0$ on Γ_a^k (bottom)

$$\int_{\Omega^k} \nabla v \nabla v' = \int_{\Gamma_f^k} (\frac{\partial p}{\partial n} - \frac{q}{K} n_x) v', \quad \forall v' \in V \text{ with } v' = 0 \text{ sur } \Gamma_a^k$$

remark: we can use the previous equation to evaluate

$$\int_{\Gamma^k} \frac{\partial p}{\partial n} v' = -\int_{\Omega^k} \nabla p \nabla v'$$

Implementation

The new domain is: $\Omega^{k+1} = \mathcal{F}(\Omega^k)$ Warning if is the movement is too large we can have triangle overlapping.

```
problem Pp(p,pp,solver=CG) =
    int2d(Th)( dx(p)*dx(pp)+dy(p)*dy(pp))
  + on(b,f,p=y) ;
problem Pv(v,vv,solver=CG) =
    int2d(Th)( dx(v)*dx(vv)+dy(v)*dy(vv))
  + on (a, v=0)
  + int1d(Th,f)(vv*
       ((Q/K)*N.y-(dx(p)*N.x+dy(p)*N.y)));
while(errv>1e-6)
{   j++;  Pp;  Pv;    errv=int1d(Th,f)(v*v);
     coef = 1;
//     Here french cooking if overlapping see the example
    Th=movemesh(Th,[x,y-coef*v]);                    //    deformation
}
```

Run:freeboundary.edp

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Bose Einstein Condensate

Just a direct use of `Ipopt` interface (2day of works)
The problem is find a complex field u on domain $\mathcal{D}$ such that:

$$u = \underset{||u||=1}{\operatorname{argmin}} \int_{\mathcal{D}} \frac{1}{2}|\nabla u|^2 + V_{trap}|u|^2 + \frac{g}{2}|u|^4 - \Omega i\overline{u}\left(\begin{pmatrix} -y \\ x \end{pmatrix}.\nabla\right)u$$

to code that in FreeFem++
use

- Ipopt interface (`https://projects.coin-or.org/Ipopt`)
- Adaptation de maillage

Run:BEC.edp

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Hyper elasticity equation

The Hyper elasticity problem is the minimization of the energy $W(I_1, I_2, I_3)$ where I_1, I_2, I_3 are the 3 invariants. For example The Ciarlet Geymonat energy model is

$$W = \int_\Omega \kappa_1(J_1 - 3) + \kappa_2(J_2 - 3) + \kappa(J - 1) - \kappa \ln(J)$$

where $J_1 = I_1 I_3^{-\frac{1}{3}}$, $J_2 = I_2 I_3^{-\frac{2}{3}}$,$J = I_3^{\frac{1}{2}}$,
let u the deplacement, when

- $F = I_d + \nabla u$
- $C = {}^tFF$
- $I_1 = \mathrm{tr}(C)$
- $I_2 = \frac{1}{2}(\mathrm{tr}(C)^2 - \mathrm{tr}(C^2))$
- $I_3 = \det(C)$

The problem is find

$$u = \underset{u}{\mathtt{argmin}}\, W(I_1, I_2, I_3)$$

Hyper elasticity equation

```
fespace Wh(Th,[P2,P2]);
                                                //    methode de Newton ..
Wh [d1,d2]=[0,0];
Wh [w1,w2],[v1,v2];
for(int i=0;i<Nnewton;++i)
 {
    solve dWW([w1,w2],[v1,v2]) =
          int2d(Th)( ddW2d([d1,d2],[w1,w2],[v1,v2]) )
        - int2d(Th)( dW2d([d1,d2],[v1,v2]) -[v1,v2]'*[f1,f2] )
        + on(1,w1=0,w2=0);

     d1[] -= w1[];
     real err = w1[].linfty;
     if(err< epsNewton) break;
 }
```

Run:Hyper-Elasticity-2d.edp Run:ElasticLaw2d.edp Run:CiarletGemona.edp

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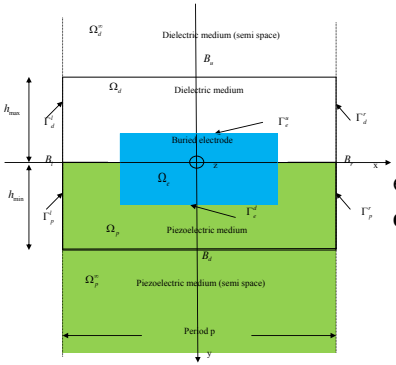
Periodic Surface Acoustic Waves Transducer Analysis

A true industrial numerical problem (2d) :

1. 3 EDP: Dielectric, Elasticity, Piezoelectric , Linear, harmonic approximation
2. α-periodic B.C. (New simple idea)
3. semi-infinite Dielectric domain (classical Fourier/Floquet transforme)
4. semi-infinite Piezoelectric domain (Hard)

In 9 month, we build with P. Ventura (100%, me 10%) a numerical simulator form scratch (8 months for the validation), The only thing to add to freefem++ is a interface with `lapack` to compute eigenvector of full 8×8 matrix.

A good message : Les calculs des paramètres physiques des transducteurs dans la bande d'arrêt et les évaluations de capacité statiques sont très satisfaisants par rapport aux résultats expérimentaux !



In the **dielectric medium** Ω_d,

$$\boldsymbol{D} = \varepsilon_d \boldsymbol{E} \tag{12}$$

In the **elastic medium** Ω_e

$$\boldsymbol{T} = \boldsymbol{C}_e : \boldsymbol{S} \tag{13}$$

With $\boldsymbol{C}_e$ is the elastic tensor for the elastic metallic domain. In the **piezoelectric domain**

$$\begin{cases} \boldsymbol{T} = \boldsymbol{C}_p^{\boldsymbol{E}} : \boldsymbol{S} - \boldsymbol{e}\boldsymbol{E} \\ \boldsymbol{D} = \boldsymbol{e}^T \boldsymbol{S} + \varepsilon^{\boldsymbol{S}} \boldsymbol{E} \end{cases} \tag{14}$$

The material domain Ω_m obeys Newton's second law:

$$\nabla \cdot \boldsymbol{T} = \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \tag{15}$$

The quasi static Maxwell's equation is assumed for the whole domain Ω :

$$\nabla \cdot \boldsymbol{D} = 0 \tag{16}$$

By using the divergence relationship and the Green's integral formula, it results the general weak formulation of the periodic γ-harmonic problem:

Find $(\boldsymbol{u}, \phi)$ in $V_\gamma^3(\Omega_m) \times V_\gamma(\Omega)$(verifying the equipotential boundary condition in the electrode), such that for all $(\boldsymbol{v}, \psi)$ in $V_\gamma^3(\Omega_m) \times V_\gamma^3(\Omega)$, satisfying the zero equi potential boundary condition), we have:

$$\int_{\Omega_m} \overline{\boldsymbol{S}(\boldsymbol{v})} : \boldsymbol{T}(\boldsymbol{u})\ d\Omega - \omega^2 \int_{\Omega_m} \rho\, \overline{\boldsymbol{v}} \cdot \boldsymbol{u}\, d\Omega - \int_{\Omega} \overline{\boldsymbol{E}(\psi)} \cdot (\boldsymbol{e}\boldsymbol{S}(\boldsymbol{u}) + \varepsilon \boldsymbol{E}(\phi))\ d\Omega - \int_{\Gamma_d} \overline{\boldsymbol{v}} \cdot (\boldsymbol{T}(\boldsymbol{u}) \cdot \boldsymbol{n})\, d\Gamma - \int_{\Gamma_u \cup \Gamma_d} \overline{\psi}\, (\boldsymbol{D}(\phi) \cdot \boldsymbol{n})\, d\Gamma = 0 \quad (17)$$

With, $V_\gamma(\Omega)$is the mathematical space of $L^2(\Omega)$ with the derivative in $L^2(\Omega)$ satisfying γ-harmonic periodic boundary conditions.

The γ-harmonic periodic boundary trick

Let us first define $\varphi_\gamma(x) = e^{-j2\pi\gamma\frac{x}{p}}$, $\varphi_\gamma(x)$ is a γ-harmonic periodic function satisfying:

$$\varphi_\gamma(x+p) = e^{-j2\pi\gamma}\varphi_\gamma(x) \tag{18}$$

We just do the change of variable:

$$\begin{cases} \boldsymbol{u}(x,y) = \varphi_\gamma(x)\,\boldsymbol{u}^\diamond(x,y) \\ \phi(x,y) = \varphi_\gamma(x)\,\phi^\diamond(x,y) \end{cases} \tag{19}$$

Where $\boldsymbol{u}^\diamond(x)$ and $\phi^\diamond(x)$ are p-periodic functions.
The main idea is to define a new differential operator ∇_γ by:

$$\nabla_\gamma \boldsymbol{u}^\diamond = \nabla(\varphi_\gamma \boldsymbol{u}^\diamond) = \varphi_\gamma \nabla \boldsymbol{u}^\diamond + \varphi_\gamma{}' \boldsymbol{u}^\diamond \tag{20}$$

Because the physical fields $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{T}$, and $\mathbf{S}$ are expressed using partial derivative of $\boldsymbol{u}$, and, ϕ, it is possible to define the operators $\boldsymbol{E}_\gamma(\phi^\diamond) = \boldsymbol{E}(\varphi_\gamma\phi^\diamond)$, $\boldsymbol{D}_\gamma(\phi^\diamond) = \boldsymbol{D}(\varphi_\gamma\phi^\diamond)$, $\boldsymbol{T}_\gamma(\boldsymbol{u}^\diamond) = \boldsymbol{T}(\varphi_\gamma\boldsymbol{u}^\diamond)$, ...

The new variational form with period BC.

Find $(\boldsymbol{u}^{\diamond}, \phi^{\diamond})$in $V_1^3(\Omega_m) \times V_1(\Omega)$ (verifying the equipotential boundary condition), such that for all $(\boldsymbol{v}^{\diamond}, \psi^{\diamond})$in $V_1^3(\Omega_m) \times V_1^3(\Omega)$, satisfying the zero equipotential boundary condition), we have:

$$\int_{\Omega_m} \overline{\boldsymbol{S}_\gamma(\boldsymbol{v}^{\diamond})} : \boldsymbol{T}_\gamma(\boldsymbol{u}^{\diamond})\, d\Omega - \omega^2 \int_{\Omega_m} \rho\, \overline{\boldsymbol{v}^{\diamond}} \cdot \overline{\boldsymbol{u}^{\diamond}}\, d\Omega - \int_{\Omega} \overline{\boldsymbol{E}_\gamma(\psi^{\diamond})} \cdot (\boldsymbol{e}\boldsymbol{S}_\gamma(\boldsymbol{u}^{\diamond}) + \varepsilon \boldsymbol{E}_\gamma(\phi^{\diamond}))\, d\Omega - \int_{\Gamma_d} \overline{\varphi_\gamma \boldsymbol{v}^{\diamond}} \cdot (\boldsymbol{T}_\gamma(\boldsymbol{u}^{\diamond})\, \boldsymbol{n})\, d\Gamma - \int_{\Gamma_u \cup \Gamma_d} \overline{\varphi_\gamma \psi^{\diamond}} (\boldsymbol{D}_\gamma(\phi^{\diamond}) \cdot \boldsymbol{n})\, d\Gamma = 0 \quad (21)$$

Where, $V_1(\Omega)$ is the mathematical space of $L^2(\Omega)$ with derivative in $L^2(\Omega)$ satisfying p-periodic boundary conditions.

We have to modelized the following term:

$$-\int_{\Gamma_d} \overline{\varphi_\gamma \boldsymbol{v}^\diamond} \cdot \left(\boldsymbol{T}_\gamma\left(\boldsymbol{u}^\diamond\right)\boldsymbol{n}\right) d\Gamma - \int_{\Gamma_u \cup \Gamma_d} \overline{\varphi_\gamma \psi^\diamond} \left(\boldsymbol{D}_\gamma\left(\phi^\diamond\right) \cdot \boldsymbol{n}\right) d\Gamma, \tag{22}$$

also called border terms.
First from (22), let us look at the boundary integral A_{Γ_u}, at the interface Γ_u of the semi-infinite dielectric semi-space.

$$A_{\Gamma_u} = \int_{\Gamma_u} \overline{\varphi_\gamma \psi^\diamond}\, \boldsymbol{D}_\gamma\left(\phi^\diamond\right) \cdot \boldsymbol{n}\, d\Gamma \tag{23}$$

The elementary coefficients to compute are for all finite element basic functions ${w^\diamond}_i$ introduce in (**??**), only for node $i \in \mathcal{N}_u$ the set of node on Γ_u.

$$\forall (i,j) \in \mathcal{N}_u^2, \qquad (A_{\Gamma_u})_{ij} = -\varepsilon_d \int_{\Gamma_u} \overline{\varphi_\gamma {w^\diamond}_i}\, \partial_n \left(\varphi_\gamma {w^\diamond}_j\right) d\Gamma \tag{24}$$

According [2], it is possible to expand, at the interface Γ_u the γ-harmonic periodic $\varphi_\gamma {w^\diamond}_j$ into the Floquet's basis function

$$f_m(x,y) = e^{-2\pi(j(m+\gamma)x - |m+\gamma|((y-y_u))/p} = \varphi_\gamma(x)\, {f^\diamond}_m(x,y). \tag{25}$$

where y_u is the y coordinate of Γ_u.

$$\varphi_\gamma(x)\, {w^\diamond}_j(x,y) = \sum_{m=-\infty}^{+\infty} c_m^j f_m(x,y) \tag{26}$$

With the $L^2(\Gamma_u)$ orthogonality of Fourier's basis ${f^\diamond}_m$, we have:

$$c_m^j = \frac{1}{p} \int_{\Gamma_u} {w^\diamond}_j \overline{{f^\diamond}_m} d\Gamma, \tag{27}$$

and on Γ_u the normal derivative $\partial_n f_m(x,y) = \partial_y f_m(x,y)$ satisfies:

$$\partial_n f_m = -g_m f_m, \qquad \text{with } g_m = \frac{2\pi}{p}\,|\gamma + m| \tag{28}$$

Leading to the relationship:

$$\partial_n \left(\varphi_\gamma {w^\diamond}_j\right) = -\sum_{m=-\infty}^{+\infty} c_m^j g_m f_m \tag{29}$$

Finally the term $(A_{\Gamma_u})_{ij}$ is

$$(A_{\Gamma_u})_{ij} = \frac{\varepsilon_d}{p} \sum_{m=\infty}^{+\infty} g_m \int_{\Gamma_u} \overline{w^\diamond} i {f^\diamond}_m d\Gamma \int_{\Gamma_u} {w^\diamond}_j \overline{f^\diamond}_m d\Gamma \tag{30}$$

Run:BEM.edp

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Phase change with Natural Convection

The starting point of the problem is Brainstorming session (part I) of the third FreeFem++ days in december 2011, this is almost the Orange Problem is describe in web page `http://www.ljll.math.upmc.fr/~hecht/ftp/ff++days/2011/Orange-problem.pdf`.
The coupling of natural convection modeled by the Boussinesq approximation and liquid to solid phase change in $\Omega =]0,1[^2$, No slip condition for the fluid are applied at the boundary and adiabatic condition on upper and lower boundary and given temperature θ_r (resp θ_l) at the right and left boundaries.
The model is: find the field : the velocity $\boldsymbol{u} = (u_1, u_2)$, the pressure p and temperature θ :

$$\begin{cases} \boldsymbol{u} & \text{given} & \text{in } \Omega_s \\ \partial_t \boldsymbol{u} + (\boldsymbol{u}\nabla)\boldsymbol{u} + \nabla.\mu\nabla\boldsymbol{u} + \nabla p & = -c_T \boldsymbol{e}_2 & \text{in } \Omega_f \\ \nabla.\boldsymbol{u} & = 0 & \text{in } \Omega_f \\ \partial_t\theta + (\boldsymbol{u}\nabla)\theta + \nabla.k_T\nabla\theta & = \partial_t S(T) & \text{in } \Omega \end{cases} \tag{31}$$

Where Ω_f is the fluid domain and the solid domain is $\Omega_s = \Omega \setminus \Omega_f$.

Phase change with Natural Convection

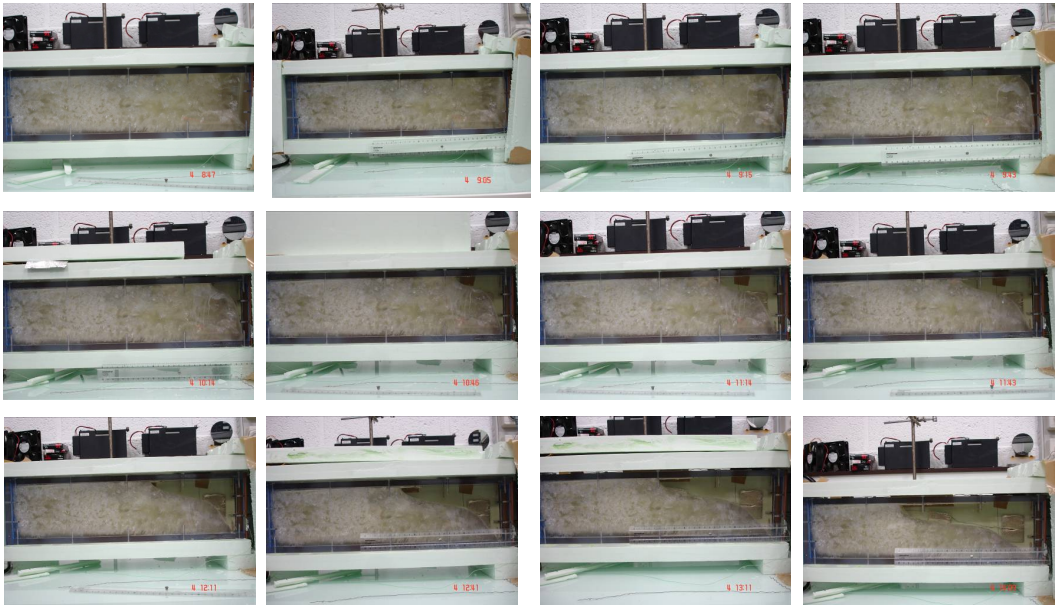
The enthalpy of the change of phase is given by the function S; μ is the relative viscosity, k_T the thermal diffusivity.
In $\Omega_f = \{x \in \Omega; \theta > \theta_f\}$, with θ_m the melting temperature the solid has melt.
We modeled, the solid phase as a fluid with huge viscosity, so :

$$\mu = \begin{cases} \theta < \theta_f & \sim & 10^6 \\ \theta \geq \theta_m & \sim & \frac{1}{\mathrm{Re}} \end{cases},$$

The Stefan enthalpy S_c with defined by $S_c(\theta) = H(\theta)/S_{th}$ where S_{the} is the stefan number, and H is the Heaviside function with use the following smooth the enthalpy:

$$S(\theta) = \frac{\tanh(50(\theta - \theta_m)))}{2S_{te}}.$$

The true device



We apply a fixed point algorithm for the phase change part (the domain Ω_f is fixed at each iteration) and a full no-linear Euler implicit scheme with a fixed domain for the rest. We use a Newton method to solve the non-linearity.

- if we don't make mesh adaptation, the Newton method do not converge
- if we use explicit method diverge too,
- if we implicit the dependance in Ω_s the method also diverge.

This is a really difficult problem.

The finite element space to approximate $u1, u2, p, \theta$ is defined by

```
fespace Wh(Th,[P2,P2,P1,P1]);
```

We do mesh adaptation a each time step, with the following code:

```
Ph ph = S(T), pph=S(Tp);
Th= adaptmesh(Th,T,Tp,ph,pph,[u1,u2],err=errh,
              hmax=hmax,hmin=hmax/100,ratio = 1.2);
```

This mean, we adapt with all variable plus the 2 melting phase a time $n+1$ and n and we smooth the metric with a ratio of 1.2 to account for the movement of the melting front.

the fixed point are implemented as follows

```
real err=1e100,errp ;
 for(int kk=0;kk<2;++kk) // 2 step of fixe point on Ωs
 { nu = nuT; // recompute the viscosity in Ωs,Ωf
   for(int niter=0;niter<20; ++ niter) // newton loop
    { BoussinesqNL;
      err = u1w[].linfty;
      cout << niter << " err NL " << err <<endl;
      u1[] -= u1w[];
      if(err < tolNewton) break;  }// convergence ..
  }
```

The linearized problem

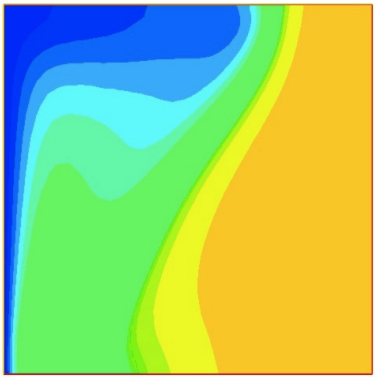
```
problem BoussinesqNL([u1w,u2w,pw,Tw],[v1,v2,q,TT])
= int2d(Th) (
      [u1w,u2w,Tw]'*[v1,v2,TT]*cdt
      + UgradV(u1,u2,u1w,u2w,Tw)' * [v1,v2,TT]
      + UgradV(u1w,u2w,u1,u2,T)' * [v1,v2,TT]
      + ( Grad(u1w,u2w)'*Grad(v1,v2)) * nu
      + ( Grad(u1,u2)'*Grad(v1,v2)) * dnu* Tw
      + cmT*Tw*v2  + grad(Tw)'*grad(TT)*kT
      - div(u1w,u2w)*q -div(v1,v2)*pw - eps*pw*q
      + dS(T)*Tw*TT*cdt )
    - int2d(Th)(
      [u1,u2,T]'*[v1,v2,TT]*cdt
      + UgradV(u1,u2,u1,u2,T)' * [v1,v2,TT]
      + ( Grad(u1,u2)'*Grad(v1,v2)) * nu
      + cmT*T*v2 - eps*p*q  +  grad(T)'*grad(TT)*kT
      - div(u1,u2)*q -div(v1,v2)*p
      +  S(T)*TT*cdt - [u1p,u2p,Tp]'*[v1,v2,TT]*cdt
      -  S(Tp)*cdt*TT)
  + on(1,2,3,4, u1w=0,u2w=0)+on(2,Tw=0)+on(4,Tw=0) ;
```

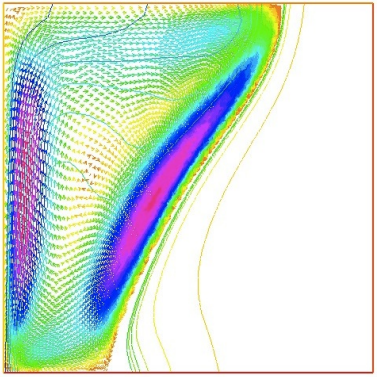
The parameters of the computation

take case 2 from
Shimin Wang, Amir Faghri, and Theodore L. Bergman. A comprehensive numerical model for melting with natural convection. *International Journal of Heat and Mass Transfer*, January 2010.

$\theta_m = 0$, $\mathrm{Re} = 1$, $S_{te} = 0.045$, $P_r = 56.2$, $R_a = 3.27\ 10^5$, $\theta_l = 1, \theta_r = -0.1$ so in this case $\mathrm{cmT} = c_T = -R_a/P_r$, $\mathrm{kT} = k_T = 1/P_r$, $\mathrm{eps} = 10^{-6}$, time step $\delta t = 10^{-1}$, $\mathrm{cdt} = 1/\delta t$, at time $t = 80$ and we get a good agreement with the article.

Phase change with Natural Convection





So now, a real problem, get the physical parameter of the real experiment.
Run:Orange-Newton.edp

Outline

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2. Tools
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8 Technical Remark on freefem++
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- FreeFem++ et C++ type

compilation process on Windows

1. Download and install MINGW32 see
   http://sourceforge.net/projects/mingw/files/Installer/mingw-get-inst/mingw-get-inst-20120426/
2. Under mingw32 install `wget` and `unzip`
   - `mingw-get install msys-wget`
   - `mingw-get.exe install msys-unzip`
3. To install freeglut of win32 for the graphics part

```
wget http://files.transmissionzero.co.uk/software/development/GLUT/freeglut-MinGW.zip
unzip freeglut-MinGW-2.8.0-1.mp.zip
cp freeglut/include/* /c/MinGW/include/GL/.
cp freeglut/lib*.a /c/MinGW/lib/.
cp freeglut/freeglut.dll /bin
```

4. install a good blas (OpenBlas) http://xianyi.github.com/OpenBLAS/
5. install MPI for // version HPC Pack 2008 SDK and HPC Pack 2008 R2 Service Pack 2
6. install inno setup to build installer: http://www.xs4all.nl/~mlaan2/ispack/isetup-5.4.0.exe
7. GSL for gsl interface http://sourceforge.net/projects/mingw-cross/files/%5BLIB%5D%20GSL/mingw32-gsl-1.14-1/mingw32-gsl-1.14-1.zip/download

Finaly, the configure argument are:

```
./configure '--enable-download' 'FC=mingw32-gfortran' 'F77=mingw32-gfortran' 'CC=mingw32-gcc'
'CXX=mingw32-g++' '-with-blas=/home/hecht/blas-x86/libgoto2.dll' 'CXXFLAGS=-I/home/hecht/blas-x86'
'--enable-generic' '--with-wget=wget' 'MPIRUN=/c/Program Files/Microsoft HPC Pack 2008 R2/Bin/mpiexec.exe'
```

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Dynamics Load facility

Or How to add your C++ function in FreeFem++.
First, like in cooking, the first true difficulty is how to use the kitchen.
I suppose you can compile the first example for the examples++-load

```
numermac11:FH-Seville hecht#  ff-c++ myppm2rnm.cpp
...
```

add tools to read pgm image

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The interesting code

```
#include "ff++.hpp"
typedef KNM<double> * pRnm;                                   //    real[int,int] array variable type
typedef KN<double> * pRn;                                     //    real[int] array variable type
typedef string ** string;                                     //    the ff++ string variable type
                                                              //    the function to read image
pRnm  read_image( pRnm  const & a,const pstring & b);
                                                              //    the function to set 2d array from 1d array
pRn  seta( pRn  const & a,const pRnm & b)
{ *a=*b;
  KN_<double> aa=*a;
  return a;}

void Init(){                                                  //    the link with FreeFem++
//    add ff++ operator "<-" constructor of real[int,int] form a string
TheOperators->Add("<-",
       new OneOperator2_<KNM<double> *,KNM<double> *,string*>(&read_image) );
//    add ff++ an affection "=" of real[int] form a real[int,int]
TheOperators->Add("=",
       new OneOperator2_<KN<double> *,KN<double> *,KNM<double>* >(seta));
}
LOADFUNC(Init); //    to call Init Function at load time
```

Remark, `TheOperators` is the ff++ variable to store all world operator, `Global` is to store function.

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How to extend

A true simple example How to make dynamic gnuplot
Idea: use a pipe to speak with gnuplot the C code :

```
FILE * gp = popen("gnuplot");
 for( double f=0; f < 3.14; f += 0.01)
   fprintf(gp,"plot sin(x+%f)\n",f);
```

To do this add a new constructor of ofstream in freefem++

A way to pass info between to code

Make a pipe, under unix (with a use of pstream tools)

```
#include "ff++.hpp"
#include "pstream.h"
typedef redi::pstream pstream;
typedef std::string string;
static pstream **  pstream_init(pstream **const & p,string * const & a)
{ *p =  new pstream(a->c_str());
  return p;};

void initttt()
{
                                                                    //    add new pointer type * pstream
  Dcl_TypeandPtr<pstream*>(0,0,::InitializePtr<pstream*>,::DeletePtr<pstream*>);
                                                  //    add cast operation to make std iostream read and write
  atype<istream* >()->AddCast( new E_F1_funcT<istream*,pstream**>(UnRef<istream* >));
  atype<ostream* >()->AddCast( new E_F1_funcT<ostream*,pstream**>(UnRef<ostream* >));
                                                                  //    the constructor from a string .
  TheOperators->Add("<-",new OneOperator2_<pstream**,pstream**,string*>(pstream_init) );
                                                                    //    add new keyword type pstream
  zzzfff->Add("pstream",atype< pstream ** >());
}
LOADFUNC(initttt);
t


MBP-FH:plugin hecht$ ff-c++ pipe.cpp
/usr/local/bin/g++ -c -g -m64 -fPIC -DNDEBUG -O3 -DBAMG_LONG_LONG -DNCHECKPTR -fPIC -I/usr/local/lib/ff++/3.20/include
/usr/local/bin/g++ -bundle -undefined dynamic_lookup -g -m64 -fPIC -DNDEBUG -O3 -DBAMG_LONG_LONG -DNCHECKPTR -fPIC 'pip
```

a small test : Run:gnuplot.edp

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FreeFem++ et C++ type

The tools to add a operator with 2 arguments:

```
OneOperator2_<returntype ,typearg1 ,typearg2>(& thefunction ));
returntype thefunction(typearg1 const &, typearg2 const &)
```

To get the C++ type of all freefem++ type, method, operator, just do in examples++-tutorialdirectory

```
c++filt -t < lestables
     Cmatrix 293 Matrice_Creuse<std::complex<double> >
     R3 293 Fem2D::R3
     bool 293 bool*
     complex 293 std::complex<double>*
     element 293 (anonymous namespace)::lgElement
     func 294 C_F0
       ifstream 293 std::basic_istream<char, std::char_traits<char> >**
     int 293 long*
     matrix 293 Matrice_Creuse<double>
     mesh 293 Fem2D::Mesh**
     mesh3 293 Fem2D::Mesh3**
     ofstream 293 std::basic_ostream<char, std::char_traits<char> >**
     problem 294 Problem
     real 293 double*
     solve 294 Solve
     string 293 std::basic_string<char, std::char_traits<char>, std::allocator<char> >**
     varf 294 C_args
     vertex 293 (anonymous namespace)::lgVertex
```

FreeFem++ Triangle/Tet capabylity

```
                                        //    soit T un Element de sommets A,B,C ∈ R^2
                                          //    ---------------------------------
Element::nv ;                           //    number of vertices of triangle (here 3)
const Element::Vertex & V =  T[i];                //    the vertex i of T (i ∈ 0,1,2
double a = T.mesure() ;                                        //    mesure of T
Rd AB = T.Edge(2);                                             //    edge vector
Rd hC = T.H(2) ;                                        //    gradient of 2 base fonction
R l = T.lenEdge(i);                                     //    length of i edge oppose of i
(Label) T   ;                                             //    label of T (region number)
R2 G(T(R2(1./3,1./3)));                                    //    The barycentre of T in 3d
```

```
Mesh Th("filename");                                  //    read the mesh in "filename"
Th.nt ;                                         //    number of element (triangle or tet)
Th.nv ;                                                //    number of vertices
Th.neb or  Th.nbe ;                           //    numbe rof border element (2d) or(3d)
Th.area;                                             //    area of the domain (2d)
Th.peri;                                                //    length of the border
typedef Mesh::Rd Rd;                                                  //    R2 or R3
Mesh2::Element & K = Th[i];                             //    triangle i , int i∈ [0,nt[
Rd A=K[0];                                        //    coor of vertex 0 of triangle K
Rd G=K(R2(1./3,1./3)):                                  //    the barycentre de K.
Rd DLambda[3];
K.Gradlambda(DLambda);                            //    compute the 3 ∇λ_i^K for i = 0,1,2
Mesh::Vertex   & V = Th(j);                              //    vertex j , int j∈ [0,nv[
Mesh::BorderElement   &   BE=th.be(l) ;                  //    border element l∈ [0,nbe[
Rd B=BE[1];                                       //    coord of vertex 1 on Seg BE
Rd M=BE(0.5);                                                //    middle of BE.
int j = Th(i,k);            //   global number of vertex k∈ [0,3[ of tria.  i∈ [0,nt[
Mesh::Vertex & W=Th[i][k];                    //    vertex k∈ [0,3[ of triangle i∈ [0,nt[

int ii = Th(K) ;                                             //   number of triangle K
int jj = Th(V) ;                                             //    number of triangle V
int ll = Th(BE) ;                                       //    number of Seg de bord BE
assert( i == ii && j == jj) ;                                                //    check.
```

Freefem++ v3.38 is

- very good tool to solve non standard PDE in 2D/3D
- to try new domain decomposition domain algorithm

The the future we try to do:

- Build more graphic with VTK, paraview , ... (in progress)
- Add Finite volume facility for hyperbolic PDE (just begin C.F. FreeVol Projet)
- 3d anisotrope mesh adaptation
- automate the parallel tool

Thank for you attention.