12th FreeFEM days

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Outline

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FreeFem++ is a software to solve numerically partial differential equations (PDE) in $\mathbb{R}^2$) , $\mathbb{R}^3$) , on Curve or on surface with finite elements methods. We used a user language (DSL) to set and control the problem. The FreeFem++ language allows for a quick specification of linear PDE's, with the variational formulation of a linear steady state problem and the user can write they own script to solve no linear problem and time depend problem. You can solve coupled problem, problem with moving domain, eigenvalue problem, do mesh adaptation , compute error indicator, etc ...

By the way, FreeFem++ is build to play with abstract linear, bilinear form on Finite Element Space and interpolation operator.

FreeFem++ is a freeware and this run on Mac, Unix and Window architecture, in parallel with MPI.
To try of cell phone https://www.ljll.math.upmc.fr/lehyaric/ffjs/

Info: FreeFem++ solve a problem with $22\,10^9$ unknowns in 200 s on 12,000 proc (Thank to P. Jolivet).

Outline

The team

Alpines/LJLL:

- FH F. Hecht: responsable et dévelopeur principal (PR, SU,Alpines, Math, Info,) 50%
- FN F. Nataf: Méthode de decomposition de Domain: HPDDM, FFDDM (DR CNRS, Alpines , Math) 20%
- XC X. Claeys: responsable des méthode BEM (MdC SU, Alpines, Math) 20%
- PHT P.H. Tournier: : dévelopeur FFDDM, BEM, (IR CNRS, Alpines, Math, Info) 60%
- AF A. Fourmont: : dévelopeur Element Surface, Ligne, interface BEM et Language (Ing INRIA,SED, Info) 20%
- PJ P. Jolivet : développeur HPDDM et interface PETSc/SELPc (CR. CNRS, Info) 20%

Airthuim;

- SG S. Garnotel: (Ing. Arthium, restructuration de source, git, ...)
- FL F. Lahaye : (Ing, WEB , Arthium)

and O. Pironneau, G. Sadaka, A. Suzuki, ...

Outline

History

1987 MacFem/PCFem the old ones (O. Pironneau in Pascal) no free.

1992 FreeFem rewrite in C++ (P1,P0 one mesh) O. Pironneau, D. Bernardi, F. Hecht (mesh adaptation , bamg) , C. Prudhomme .

1996 FreeFem+ rewrite in C++ (P1,P0 more mesh) O. Pironneau, D. Bernardi, F. Hecht (algebra of function).

1998 FreeFem++ rewrite with an other finite element kernel and an new language ; F. Hecht, O. Pironneau, K.Ohtsuka.

1999 FreeFem 3d (S. Del Pino) , a first 3d version base on fictitious domaine method.

2008 FreeFem++ v3 use a new finite element kernel multidimensionnels: 1d,2d,3d...

2017 FreeFem++ v3.57 parallel version

2018 FreeFEM v4.1 New matrix type, Surface element (v4.2) , New Parallel tools ...

2019 FreeFEM v4.8 Surface element, Line Element, BEM (v4.5), PETSc interface

For who, for what!

For what

1. R&D
2. Academic Research ,
3. Teaching of FEM, PDE, Weak form and variational form
4. Algorithmes prototyping
5. Numerical experimentation
6. Scientific computing and Parallel computing

For who: the researcher, engineer, professor, student...

A Community https://community.freefem.org

The mailing list mailto:Freefempp@ljll.math.upmc.fr with 551 members with a flux of 1-10 messages per day.

More than 3000 true Users (more than 1000 download / month)

- Wide range of finite elements: continuous P1,P2 elements, discontinuous P0, P1, RT0,RT1,BDM1, elements ,Edge element, vectorial element, mini-element, ...
- Automatic interpolation of data from a mesh to an other one (with matrix construction if need), so a finite element function is view as a function of (x, y, z) or as an array.
- Definition of the problem (complex or real value) with the variational form with access to the vectors and the matrix.
- Discontinuous Galerkin formulation (only in 2d to day).
- LU, Cholesky, Crout, CG, GMRES, UMFPACK, SuperLU, MUMPS , Dissection, PETSc. ... sparse linear solver; eigenvalue and eigenvector computation with ARPACK or SLEPc.
- Online graphics with OpenGL/GLUT/VTK, C++ like syntax.
- Javascript version works straight out of an HTML page, both online or offline (here).

- Analytic description of boundaries, with specification by the user of the intersection of boundaries in 2d.
- Automatic mesh generator, based on the Delaunay-Voronoï algorithm. (2d,3d (tetgen))
- load and save Mesh, solution
- Mesh adaptation based on metric, possibly anisotropic (only in 2d), with optional automatic computation of the metric from the Hessian of a solution. (2d,Surface,3d).
- Link with other soft: parview, gmsh , vtk, medit, gnuplot
- Dynamic linking to add plugin.
- Full MPI interface
- Nonlinear Optimisation tools: CG, Ipopt, NLOpt, stochastic
- Wide range of examples: Navier-Stokes 3d, elasticity 3d, fluid structure, eigenvalue problem, Schwarz' domain decomposition algorithm, residual error indicator ...

on Unix build a "yours.edp" file with your favorite editor : emacs, vi, nedit, etc. Enter `FreeFem++ yours.edp` or `FreeFem++` must be in one directory of your `PATH` shell variable.

on Window, MacOs X build a "yours.edp" file with your favorite text editor (raw text, not word text) : emacs, winedit, wordpad, textmate, bbedit, fraise ... and click on the icon of the application `FreeFem++` and load you file via de open file dialog box or drag and drop the icon of your built file on the application `FreeFem++` icon.

The Changes from dec. . 2018 to dec. 2019

- add matrix and array tools (FH)
- adding of a global variable 'lockOrientation' to allows the building of mesh without checking the orientation elements (AF)
- add plugin tool to build matrix edge/P1 with sign 'mat_edgeP1' (FH)
- new examples 'diffusion-2d-mg.edp' and 'helmholtz-2d-mg.edp' showing how to use user-defined coarse corrections
- support for nonzero scalars in PETSc block matrices
- simpler constructor for sequential HPDDM matrices (no need for the restriction array and the partition of unity)
- array of 'Mat' and 'schwarz' types
- add mpi meshS (serialize object)
- nested fieldsplit example 'examples/hpddm/natural-convection-2d-PETSc-fieldsplit.edp'
- 'int[int][int] array;' is now supported (a size was previously needed, i.e., 'array(0);')
- check selectivity during 'make check', depending on available 3rd party librairies
- new CI/CD tools for 'develop' branch
- new gestion of mesh3 - meshS coupling
- square3, buildSurface... operators for meshS
- Preliminary support for symmetric distributed PETSc matrices (MATMPISBAIJ instead of MATMPIAIJ)
- Interface to AMS, Hiptmair–Xu preconditioner for problems in H(curl), see maxwell-3d-PETSc.edp
- FEM on curve 3D (in test)
- P0, P1, P2 curve 3D FE (scalar for the moment)
- i/o medit and vtk format for curve FE
- checkMesh() function, allow to remove multiple vertices, elements and border elements (argument: precisvertice(double),removeduplicate(bool))
- possible to build a curve mesh from a surface, ThS = buildBdMesh(ThS) and define this new mesh by meshL ThL= ThS.Gamma
- can extract a border part of a meshL (meshL ThL = extract(ThL,label=llabs))
- Support for optimized boundary conditions with PETSc, see helmholtz-2d-PETSc-complex.edp
- buildmeshL() function: build meshL from borders
- 'mpiCommSelf' keyword
- interface to 'TSSolve', DAE/ODE solvers from PETSc
- interface to 'TaoSolve', Toolkit for Advance Optimization from PETSc
- simpler constructor for sequential PETSc matrices (no need for the restriction array and the partition of unity)
- some unit tests

The Changes from dec. 2019 to sep. 2020

- version 4.5
- for windows version: rename under mpi ‘MUMPS‘ in ‘MUMPS_mpi‘
- correct link edition with fortran mpi under windows juste use the msmpi (just use ‘libmsmpi.dll‘)
- new ‘mmg‘ and ‘parmmg‘
- a true 3d anisotropic mesh adaptation
- an example to extract surface mesh from isovalue
- function ‘f.eatspace‘ to reach eof
- function ‘f.length‘ file length
- Interface to ‘PetscLogStagePush()‘/‘PetscLogStagePop()‘
- Ability to directly assemble a ‘Mat‘ using a ‘varf‘
- New ‘bem‘ plugin using htool and BemTool
- New DSL for BEM varfbem
- add int0d to apply Neumann BC (curve FE), dx,dy,...
- add P1dc FE for Border FEM
- PETSc as a subdomain solver for HPDDM
- version 4.6
- new algorithm for searching element containing a point .
- new function ‘hasType‘ to know if a PETSc component
- eigenvalue problems on linear elements,
- ‘–download-cmake‘ in PETSc configure
- interface to ‘KSPSetResidualHistory‘ and ‘KSPGetIterationNumber‘
- interface to ‘mpiWaitAll‘
- new function extract, allows to build a curve mesh from a 2d mesh
- ffglut can plot a vectorial FE function in surface 3d
- distributed ParMmg interface,
- new parallel interpolator on non-matching meshes
- tool to read data form vtk file in 3d
- tool to read/wrile ply file of meshL, mesh3, MeshS ‘load "ioply"‘
- new ‘tgv‘ values: -10 => zero row, -20 => zero row/column
- Windows binary now shipped with PETSc/SLEPc
- BEM examples are now in ‘examples/mpi‘
- plot border type is now in 3d (border 2d and 3d)
- PETSc version 3.13.0

The Changes from sep. 2020 to dec. 2020

- version 4.7
- new way to build matrix beetween 2d Finite element 2d and Curve finite element to do mortar
- add 'Ns' normal vector in R^3 on meshS (normal of the surface) of current point (to day Ns of [x,y,0] plan is [0,0,-1]) .
- add 'Tl' tangent vector in R^3 on meshL (tangent vector of the line/curve) of current point
- compile ffmaster / ffslave example under windows (thanks to johann@ifado.de)
- Boolean parameter 'spiltpbedge' in 'buildmesh' to split in to edge with two boundary vertices
- interface to PETSc DMPlex,
- function 'MatDestroy'
- function 'MatPtAP' and 'transferMat' for parallel interpolation between non-matching grids,PETSc.edp'
- preliminary interface to 'SVDSolve' from SLEPc to compute singular value decompositions,'
- preliminary interface to 'NEPSolve' from SLEPc to solve nonlinear eigenvalue problems, see 'examples/hpddm/nonlinear-2d-SLEPc-complex.edp'
- 'transpose' parameter when constructing a 'Mat' for defining a matrix-free transposed operation
- interface to 'PetscMemoryGetCurrentUsage'
- add P2b, RT0, RT1 surface FE (P2bS, RT0S, RT1S))
- add operator interpolate (2d->3d surface)
- add operator x = A'b; where x, b are array and A 2 dim array (full matrix) and generate an error in case of b'A or b'A expression
- function 'MatLoad' to load a PETSc 'Mat' from disk,
- possibility to assemble a symmetric 'HMatrix<complex>' and to densify a 'HMatrix<complex>' into a 'Mat<complex>'
- version 4.7.1
- Bilaplacian example using Morley FE with PETSc,
- Oseen problem preconditioned by PCD,
- SLEPc polynomial eigenvalue solver

In progress (Near futur or not)

- to get the last develop Version 4.8, do:

```
git clone -b develop https://github.com/FreeFem/FreeFem-sources ff++
```

- Mesh intersection of get conservative formulation in 2d (too hard=> no)
- fast interpolation on Surface and Line Mesh, (done)
- New graphics interface , (in progress)
- Element on curve and Surface (Ok)
- BEM method a first version (see X. Claeys and P-H Tournier, in good progress)
- DG in 3d
- cmake (????)
- rewrite of sparse matrix kernel (Done) ($\Longrightarrow$ new GMRES, new CG, ...), done.
- coupling 3d, surface Finite element and BEM (in progress)
- rewrite of the Finite element kernel for isoparametric FE and to be able mixte surface FE and 3d FE (in future) in a problem (no template).
- debugger (client-server graphics services) (Good idea but technical)
- more general problem (possible with new matrix)
- OpenMP interface ???
- Quadruple precision floating type (see A. Suzuki)

Element of syntax : the script is like in C/C++

First FreeFem++ is a compiler and after it launch the create code (a kind of byte code). The language is polymorphe but it is not an objet oriented language.
The key words are reserved and the operator are like in C exempt for: `^ & |`

`+ - * / ^ // a^b=` a^b

`== != < > <= >= & |//` `a|b` $\equiv$ a or b, `a&b`$\equiv$ a and b

`= += -= /= *=`

`BOOLEAN:` `0 <=> false` , $\neq$ `0 <=> true` $=$ `1`

Automatic cast for numerical value : bool, int, reel, complex

```
func heavyside = real(x>0.);
string two=2; int i2 = atoi(two);
```

```
for (int i=0;i<n;i++) { ... ;}
if ( <bool exp> ) { ... ;} else { ...;};
while ( <bool exp> ) { ... ;}
break continue key words
```

weakless: all local variables are almost static, so no recurtion in function (????))
bug if break before variable declaration in same block.
bug in function argument of fespace value.

Element of syntax: special word for finite element

```
x,y,z                                                    //    current coord.
label,  region                                  //    label of BC (border) , (interior)
N.x, N.y, N.z,                                           //    normal to the border
Ns.x, Ns.y, Ns.z,  //    normal to the surface in case of MeshS (surface mesh 2d in
3d)
Tt.x, Tl.y, Tl.z,   //    tangent to the line in case of MeshL (line mesh 1d in 3d)

int i = 0;                                                          //    an integer
real a=2.5;                                                            //    a reel
bool b=(a<3.);
real[int]  array(10) ;                                       //    a real array of 10 value
mesh Th; mesh3 Th3;  meshS ThS; meshL ThL;      //   a 2d, 3d, surface, line meshes.
fespace Vh(Th,P2);                                        //   Def.  of 2d finite element space;
fespace Vh3(Th3,P1);                                      //   Def.  of 3d finite element space;
fespace VhS(ThS,P1);                                //   Def.  of surface finite element space;
fespace VhL(ThL,P1);                               //   Def.  of line finite element space;
Vh u=x;                                            //    a finite element function or array
Vh3<complex> uc = x+  1i *y;                                    //    complex valued FE
u(.5,.6,.7);                                  //    value of FE function u at point (.5, .6, .7)
u[];                                    //   the array of DoF value assoc.  to FE function u
u[][5];                                       //   6th value of the array (numbering begin
                                                                 //   at 0 like in C)
```

Element of syntax: weak form, matrix and vector 2/4

```
fespace V3h(Th,[P2,P2,P1]);
V3h [u1,u2,p]=[x,y,z];                  //   a vectorial finite element
                                                //    function or array
        //   remark u1[] <==> u2[] <==> p[] same array of unknown.
macro div(u,v) (dx(u)+dy(v))//   EOM a macro
                                                //    (like #define in C )
macro Grad(u) [dx(u),dy(u)]                     //    the macro end with //
varf a([u1,u2,p],[v1,v2,q])=
            int2d(Th)(  Grad(u1)'*Grad(v1) +Grad(u2)'*Grad(v2)
                 -div(u1,u2)*q -div(v1,v2)*p)
            +on(1,2,u1=g1,u2=g2);

matrix A=a(V3h,V3h,solver=UMFPACK);
real[int] b=a(0,V3h);
u2[] =A^-1*b;                   //    or you can put also u1[]= or p[].
```

Element of syntax: matrix and vector tools

```
 func Heaveside=(x>0);                    //    a formal line function
 func real g(int i, real a) { .....; return i+a;}
 A = A + A'; A = A'*A                     //    matrix operation (only 1/1)
 A = [ [ A,0],[0,A'] ];                                //    Block matrix.
int[int] I(15),J(15);                     //    two array for renumbering
      //   the aim is to transform a matrix into a sparse matrix
matrix B;
B = A;                                                //    copie matrix A
B=A(I,J);                                 //    B(i,j) = A(I(i),J(j))
B=A(I^-1,J^-1);                           //    B=0; B(I(i),J(j))+= A(i,j)
B.resize(10,20);                          //    resize the sparse matrix
                                  //    and remove out of bound terms
int[int] I(1),J(1); real[int] C(1);
[I,J,C]=A;                //    get of the sparse term of the matrix A
                                          //    (the array are resized)
A=[I,J,C];                                            //    set a new matrix
matrix D=[diagofA] ;                      //    set a diagonal matrix D
                                          //    from the array diagofA.
real[int] a=2:12;                         //   set a[i]=i+2; i=0 to 10.
```

Element of syntax formal computation on array (done a compilation time)

a formal array is `[ exp1, exp1, ..., expn]`
the Hermitian transposition is `[ exp1, exp1, ..., expn]'`

```
complex a=1,b=2,c=3i;
func va=[ a,b,c];                          //    is a formal array in [ ]
a =[ 1,2,3i]'*va ; cout « a « endl;         //    Hermitian product
matrix<complex>  A=va*[ 1,2,3i]'; cout « A « endl;
a =[ 1,2,3i]'*va*2.;
a =(va+[ 1,2,3i])'*va*2.;
va./va;                                              //    term to term /
va.*va;                                              //    term to term *
trace(va*[ 1,2,3i]') ;                                                 //
(va*[ 1,2,3i]')[1][2] ;                                              //    get coef
det([[1,2],[-2,1]]);                  //    just for matrix 1x1 et 2x2
```

usefull macro to def your edp.

```
macro grad(u) [dx(u),dy(u)] //
macro div(u1,u2) (dx(u1)+dy(u2)) //
```

List of Plugin

```
cd   /usr/local/ff++/mpich-3.3.2/4.7-1/lib/ff++/4.7-1/lib/
```

BEC.so	FreeFemQA.so	ff-AiryBiry.so	mat_psi.so
BernardiRaugel.so	IncompleteCholesky.so	ff-Ipopt.so	medit.so
BinaryIO.so	MUMPS.so	ff-NLopt.so	metis.so
CircumCenter.so	MUMPS_seq.so	ff-cmaes.so	mmg.so
ClosePoints.so	MetricKuate.so	ff-mmap-semaphore.so	mmg3d-v4.0.so
Curvature.so	MetricPk.so	ffnewuoa.so	msh3.so
DxWriter.so	Morley.so	ffrandom.so	mshmet.so
Element_HCT.so	NewSolver.so	freeyams.so	myfunction.so
Element_Mixte.so	SaveHB.so	funcTemplate.so	myfunction2.so
Element_Mixte3d.so	Schur-Complement.so	geophysics.so	pcm2rnm.so
Element_P1bl.so	SuperLu.so	gmsh.so	pipe.so
Element_P1dc1.so	UMFPACK64.so	gsl.so	ppm2rnm.so
Element_P1ncdc.so	VTK_writer.so	ilut.so	qf11to25.so
Element_P2bulle3.so	VTK_writer_3d.so	iohdf5.so	scotch.so
Element_P2pnc.so	addNewType.so	ioply.so	shell.so
Element_P3.so	aniso.so	iovtk.so	splitedges.so
Element_P3dc.so	bfstream.so	isoline.so	splitmesh12.so
Element_P4.so	biofunc.so	lapack.so	splitmesh3.so
Element_P4dc.so	dfft.so	lgbmo.so	splitmesh4.so
Element_PkEdge.so	distance.so	mat_dervieux.so	splitmesh6.so
Element_QF.so	exactpartition.so	mat_edgeP1.so	tetgen.so

Important Plugin

- `qf11to25` add more quadrature formulae in 1d , 2d, 3d and tools to build own quadrature
- `Element_*,Morlay,BernadiRaugel` add new kind finite element
- `UMFPACK64,SuperLu,MUMPS_seq` add sequential sparse solver
- `metis,scotch` mesh Partitioning
- `ffrandom` true random number generator: srandomdev,srandom, random
- `gsl` the gsl lib interface (lot of special function, and random generator)
- `shell,pipe` directory and file interface, pipe interface
- `dfft` interface with fftw3 library for FFT.
- **`msh3,tetgen`** 3d mesh, suface mesh, line mesh tools and tetgen interface
- `lapack` a small interface with lapack library of full linear solver, and full eigen value problem.
- `ff-Ipopt` interface with Ipopt optimisation software
- `ppm2rnm` interface with ppm library to read ppm bitmap.
- `isoline` build a border from isoline.
- `distance` build a the signed distance approximation to an isoline in 2d and 3d (not on surface or curve).
- `mmg`, `mshmet`, `medit` interface of library of P. Frey to adapt mesh, build metric, plot in 3d, .

Important Plugin with MPI

- `HPDDM` a new parallel linear solver see diffusion-2d.edp example in examples++-hpddm
- `PETSc` a new version of PETSc real interface
- `SLEPc` a new version of SLEPc real interface (include PETSc)
- `PETSc-complex` a new version of complex PETSc interface
- `SLEPc-complex` a new version of complex SLEPc interface (include PETSc-complex)
- `MUMPS` a new version of MUMPS interface
- `MPICG` basic parallel version of CG, and GMRES
- `mpi-cmaes` parallel version of stochastic optimization algorithm.

Laplace (Poisson) equation, weak form

Let a domain Ω with a partition of $\partial\Omega$ in Γ_2, Γ_e.
Find u a solution in such that:

$$-\Delta u = 1 \text{ in } \Omega, \quad u = 2 \text{ on } \Gamma_2, \quad \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \Gamma_e \tag{1}$$

Denote $V_g = \{v \in H^1(\Omega)/v_{|\Gamma_2} = g\}$.
The Basic variationnal formulation with is: find $u \in V_2(\Omega)$, such that

$$\int_\Omega \nabla u.\nabla v = \int_\Omega 1v + \int_\Gamma \frac{\partial u}{\partial n} v, \quad \forall v \in V_0(\Omega) \tag{2}$$

The finite element method is just: replace V_g with a finite element space, and the `FreeFem++` code:

Poisson equation in a fish with FreeFem++

The finite element method is just: replace V_g with a finite element space, and the FreeFem++ script:

```
mesh3 Th("fish-3d.msh");                          //    read a mesh 3d
fespace Vh(Th,P1);                          //    define the P1 EF space

 Vh u,v;                    //   set test and unknown function in Vh.
macro Grad(u) [dx(u),dy(u),dz(u)]                    //   EOM Grad def
solve laplace(u,v,solver=CG) =
   int3d(Th)(  Grad(u)'*Grad(v)    )
   - int3d(Th) ( 1*v)
   + on(2,u=2);                                      //    int on γ2
plot(u,fill=1,wait=1,value=0,wait=1);
```

Run:fish.edp Run:fish3d.edp

New example: Eigen vector of Laplace Beltrami

Let S a surface (here a torus), we want to compute the eigen value of Laplace Beltrami operator, plot the eigen value as the deformation of the surface on the direction of the normal.

The variational form is: find (u_λ, λ) such than

$$\int_S \nabla_S u . \nabla_S v = \lambda \int_S uv \tag{3}$$

where ∇_S is the tangential gradient and at discret level the basic function are constant in normal direction, so $\nabla_S \equiv \nabla$.
remark: Now in FreeFEM the word `Ns` is a global variable to get the normal at surface mesh.

To plot the solution, I propose to deform the surface with vector $u_\lambda N$ where N the unit normal of S.

Run:LapEigenBeltrami.edp

Outline

Important remark: on geometrical item label and region

- All boundary (internal or not) was define through a label number and this number is define in the mesh data structure. The support of this label number is a edge in 2d or Surface meshes and a face in 3d meshes, **so FreeFem++ never use label on vertices**, then it is not easy to set a boundary condition on one point.
- To defined integration you can use the region (resp. label) number if you compute integrate in domain (resp. boundary). They are no way to compute 1d integral on 3d mesh.
- But you have Finite Element defined on surface or curve (new in v4.5) .
- You can put list of label or region in integer array (`int[int]`).

What is a Finite element in FreeFem++

TO DO ...

Remark on varf

The functions appearing in the variational form parameter are formal and local to the `varf` definition, the only important think is the order in the parameter list, like in

```
varf vb1([u1,u2],[q]) = int2d(Th)( (dy(u1)+dy(u2)) *q)
                        +int2d(Th)(1*q) + on(1,u1=2);
varf vb2([v1,v2],[p]) = int2d(Th)( (dy(v1)+dy(v2)) *p)
                        +int2d(Th)(1*p) ;
```

To build matrix A from the bilinear part the the variational form a of type `varf` do simply

```
  matrix B1 = vb1(Vh,Wh [, ...] );
  matrix<complex> C1 = vb1(Vh,Wh [, ...] );
//     where the fespace have the correct number of comp.
//     Vh is "fespace" for the unknown fields with 2 comp.
//     ex fespace Vh(Th,[P2,P2]); or fespace Vh(Th,RT);
//     Wh is "fespace" for the test fields with 1 comp.
```

To build a vector, put $u1 = u2 = 0$ by setting 0 of on unknown part.

```
real[int]   b = vb2(0,Wh);
complex[int]   c = vb2(0,Wh);
```

Remark: In this case the mesh use to defined , $\int, u, v$ can be different.

The boundary condition terms

First `FreeFem++` use only the label number of edge (2d) or faces (3d).

- An "on" scalar form (for Dirichlet) : `on(1, u = g )`
  The meaning is for all degree of freedom i (DoF) of this associated boundary, the diagonal term of the matrix $a_{ii} = tgv$ with the *terrible giant value* `tgv` ($=10^{30}$ by default) and the right hand side $b[i] =$ "$(\Pi_h g)[i]$" $\times\, tgv$, where the "$(\Pi_h g)[i]$" is the boundary DoF value given by the interpolation of g.
- An "on" vectorial form (for Dirichlet) : `on(1,u1=g1,u2=g2)` If you have vectorial finite element like `RT0`, the 2 components are coupled, and so you have :
  $b[i] =$ "$(\Pi_h(g1, g2))[i]$" $\times\, tgv$, where Π_h is the vectorial finite element interpolant.
- a linear form on Γ (for Neumann in 2d)
  `-int1d(Th)( f*w)` or `-int1d(Th,3))( f*w)`
- a bilinear form on Γ or Γ_2 (for Robin in 2d)
  `int1d(Th)( K*v*w)` or `int1d(Th,2)( K*v*w)`.
- a linear form on Γ (for Neumann in 3d)
  `-int2d(Th)( f*w)` or `-int2d(Th,3)( f*w)`

First a 10×10 grid mesh of unit square $]0,1[^2$

```
int[int] labs=[10,20,30,40];          //    bot., right, top, left
mesh Th1 =  square(10,10,label=labs,region=0,[x,y]);          //
plot(Th1,wait=1);
int[int] old2newlabs=[10,11,  30,31];       //   10 -> 11, 30 -> 31
Th1=change(Th1,label=old2newlabs) ;                           //
//   do Change in 2d or in 3d.  region=a, fregion=f ,
//    flabel=f
```

a L shape domain $]0,1[^2 \backslash [\frac{1}{2},1[^2$

```
 mesh Th =  trunc(Th1,(x<0.5) | (y < 0.5),label=1);           //
 plot(Th,cmm="Th");
 mesh Thh = movemesh(Th,[-x,y]);
 mesh  Th3  = Th+movemesh(Th,[-x,y]);            //    glumesh ...
 plot(Th3,cmm="Th3");
```

Run:mesh1.edp

Build Mesh 2d, border

a Circle with or without an hole ;
Remark; by default the domain is a left of the border (if the number segment is positive otherwise at right).

```
border Co(t=0,2*pi) { x=cos(t); y=sin(t); label=1;}
border Ci(t=0,2*pi) { x=cos(t)/2; y=sin(t)/2; label=2;}
plot(Co(30)+Ci(15),wait=1);
mesh Thf=buildmesh(Co(30)+Ci(15));          //    without hole
                                             //    two region:
cout «" The two Region of Thf : " «  Thf(0,0).region« " "
       « Thf(0,0.9).region « endl;
plot(Thf,wait=1,cmm="Thf");
mesh Thh=buildmesh(Co(30)+Ci(-15));         //    without hole
plot(Thh,wait=1,cmm="Thh"));
```

Get a extern mesh

```
mesh Th2("april-fish.msh");
```

build with `emc2`, `bamg`, modulef, etc...

Run:mesh-circles.edp

Build Mesh 2d, more complicate with implicit loop

a L shape domain $]0,1[^2 \backslash [\frac{1}{2},1[^2$ with 6 multi-borders.

```
 int nn=30; real dd=0.5;
 real[int,int] XX=[[0,0],[1,0],[1,dd],[dd,dd],[dd,1],[0,1]];
 int[int] NN=[nn,nn*dd,nn*(1-dd),nn*(1-dd),nn*dd,nn];
  border bb(t=0,1;i)
  {          //    i is the the index of the multi border loop
    int ii = (i+1)%XX.n; real t1 = 1-t;
    x =  XX(i,0)*t1 + XX(ii,0)*t;
    y =  XX(i,1)*t1 + XX(ii,1)*t;
    label = 1; ;    }
plot(bb(NN),wait=1);
mesh Th=buildmesh(bb(NN));
plot(Th,wait=1);
```

Run:mesh-multi.edp

Build mesh from image 1/2

```
load "ppm2rnm" load "isoline" load "shell"
string lac="lock-ness", lacjpg =lac+".jpg", lacpgm =lac+".pgm";
if(stat(lacpgm)<0)  exec("convert "+lacjpg+" "+lacpgm);
real[int,int] Curves(3,1);  int[int] be(1); int nc;
{
  real[int,int] ff1(lacpgm);                          //    read image
  int nx = ff1.n, ny=ff1.m;       //    grey value in 0 to 1 (dark)
  mesh Th=square(nx-1,ny-1,[(nx-1)*(x),(ny-1)*(1-y)]);
  fespace Vh(Th,P1);  Vh f1; f1[]=ff1; //    array to fe function.
  real iso =0.08;           //    try some value to get correct iso
  real[int] viso=[iso];
  nc=isoline(Th,f1,iso=iso,close=0,Curves,beginend=be,
      smoothing=.1,ratio=0.5);
  for(int i=0; i<min(3,nc);++i)
  { int i1=be(2*i),i2=be(2*i+1)-1;
    plot(f1,viso=viso,[Curves(0,i1:i2),Curves(1,i1:i2)],
     wait=1,cmm=i); }}
```

Build mesh from image 2/2

```
int[int] iii=[0];                    //    chose to componant ...
int[int] NC=[-500];                                //    1 componant
border G(t=0,1;i) {
P=Curve(Curves,be(2*iii[i]),be(2*iii[i]+1)-1,t);
                        label= iii[i];}
plot(G(NC),wait=1);
mesh Th=buildmesh(G(NC));
plot(Th,wait=1);
real scale = sqrt(AreaLac/Th.area);
Th=movemesh(Th,[x*scale,y*scale]);
```

Run:lac.edp (the Ali-Bouchta lac close to Kenitra)

A cube with cube or buildlayer (simple)

```
load "msh3"                                                    //   buildlayer
mesh3 Th;
int nn=10;
if(1) { int[int] ll=[1,1,1,1,1,2];
     //   y=0:front, x=1:right, y=1:back, x=0:left, z=0:down,z=1:up
          Th=cube(nn,nn,nn,label=ll); }
   //     Warning bug if no parameter label=ll before version 3.56-1

  Th= trunc(Th,((x<0.5) |(y< 0.5)| (z<0.5)),label=3);
                                                          //    remove 1/2 cube
plot("cube",Th);
```

Run:Cube.edp

3D layer mesh of a Lac with buildlayer

```
load "msh3"//    buildlayer
load "medit"//    medit
int nn=5;
border cc(t=0,2*pi){x=cos(t);y=sin(t);label=1;}
mesh Th2= buildmesh(cc(100));
fespace Vh2(Th2,P2);
Vh2 ux,uz,p2;
int[int] rup=[0,2],  rdown=[0,1], rmid=[1,1];
func zmin= 2-sqrt(4-(x*x+y*y));   func zmax= 2-sqrt(3.);
//    we get nn*coef layers
mesh3 Th=buildlayers(Th2,nn,
                     coef= max((zmax-zmin)/zmax,1./nn),
                     zbound=[zmin,zmax],
                     labelmid=rmid,  labelup = rup,
                     labeldown = rdown);            //    label def
medit("lac",Th);
```

Run:Lac3d.edp Run:3d-leman.edp

a 3d axi Mesh with buildlayer

```
func f=2*((.1+(((x/3))*(x-1)*(x-1)/1+x/100))^(1/3.)-(.1)^(1/3.));
real yf=f(1.2,0);
border up(t=1.2,0.){ x=t;y=f;label=0;}
border axe2(t=0.2,1.15) { x=t;y=0;label=0;}
border hole(t=pi,0) { x= 0.15 + 0.05*cos(t);y= 0.05*sin(t);
        label=1;}
border axe1(t=0,0.1) { x=t;y=0;label=0;}
border queue(t=0,1) { x= 1.15 + 0.05*t; y = yf*t; label =0;}
int np= 100;
func bord= up(np)+axe1(np/10)+hole(np/10)+axe2(8*np/10)
         + queue(np/10);
plot( bord);                                  //    plot the border ...
mesh Th2=buildmesh(bord);                     //    the 2d mesh axi mesh
plot(Th2,wait=1);
int[int] l23=[0,0,1,1];
Th=buildlayers(Th2,coef= max(.15,y/max(f,0.05)), 50
  ,zbound=[0,2*pi],transfo=[x,y*cos(z),y*sin(z)]
  ,facemerge=1,labelmid=l23);
```

Run:3daximesh.edp

boundary mesh of a Sphere

```
load "tetgen"
mesh Th=square(10,20,[x*pi-pi/2,2*y*pi]);        //     ]-pi/2, -pi/2[×]0,2π[
func f1 =cos(x)*cos(y); func f2 =cos(x)*sin(y); func f3 = sin(x);
            //    the partiel derivative of the parametrization DF
func f1x=sin(x)*cos(y);     func f1y=-cos(x)*sin(y);
func f2x=-sin(x)*sin(y);    func f2y=cos(x)*cos(y);
func f3x=cos(x);            func f3y=0;
                                                    //     M = DF^t DF
func m11=f1x^2+f2x^2+f3x^2;  func m21=f1x*f1y+f2x*f2y+f3x*f3y;
func m22=f1y^2+f2y^2+f3y^2;
func perio=[[4,y],[2,y],[1,x],[3,x]];
real hh=0.1/R; real vv= 1/square(hh);
Th=adaptmesh(Th,m11*vv,m21*vv,m22*vv,IsMetric=1,periodic=perio);
int[int] ref=[0,L];    //    the label of the Sphere to L ( 0 -> L)
mesh3  ThS= movemesh23(Th,transfo=[f1*R,f2*R,f3*R],orientation=1,
   label=ref);
```

Run:Sphere.edp Run:sphere6.edp

Build 3d Mesh from boundary mesh

```
include "MeshSurface.idp"          //    tool for 3d surfaces meshes
mesh3 Th;
try {  Th=readmeshS("Th-hex-sph.mesh"); }          //   try to read
catch(...) {       //    catch a reading error so build the mesh...
    real hs = 0.2;                              //    mesh size on sphere
    int[int]  NN=[11,9,10];
    real [int,int]  BB=[[-1.1,1.1],[-.9,.9],[-1,1]]; //    Mesh Box
    int [int,int]  LL=[[1,2],[3,4],[5,6]];          //    Label Box
    meshS ThHS = SurfaceHex(NN,BB,LL,1)+Sphere(0.5,hs,7,1);
                                                    //    surface meshes
    real voltet=(hs^3)/6.;                      //    volume mesh control.
    real[int] domaine = [0,0,0,1,voltet,0,0,0.7,2,voltet];
    Th = tetg(ThHS,switch="pqaAAYYQ",
              nbofregions=2,regionlist=domaine);
    savemesh(Th,"Th-hex-sph.mesh"); }        //   save for next run
```

Build a bottle boundary mesh I/II

```
load "msh3" load "medit" load "gsl" load "tetgen"
....
meshS Ths;                                                    //    the surface
{  meshS Th3c,Th3bottom, Th3top;
 { real[int,int] srneck=
   [ [Htube-0.001,Htube,Htube+Hneck*0.1, Htube+Hneck*0.3  , Htube+Hneck*0.7   ,
Htube+Hneck*0.9, Htube+Hneck+0.1],
     [Rext       ,Rext ,Rext           , Rext*.7+Rneck*.3 , Rext*.1 + Rneck*.9,
Rneck          , Rneck  ]];
  gslspline rneck(gslinterpcspline,srneck);       //    Curve of neck of the bottle

                                                  //    adap mesh of get pretty mesh
  mesh Th2c= square(Hbot/hh,2*pi*Rext/hh,[x*Hbot,y*2*pi]);
  fespace V2x(Th2c,P1);
  func E1 = rneck(x)*cos(y); func E2 = rneck(x)*sin(y); func E3 = x;
  ...
  Th2c=adaptmesh(Th2c,em11,em21,em22,IsMetric=1,periodic=perio,nbvx=100000); }
  Th2c=change(Th2c,fregion=labcyl);
  Th3c = movemesh23(Th2c,transfo=[E1 , E2 , E3]);          //    maillage exterieur
 }
```

Build a bottle boundary mesh II/II

```
                                                //    extraction of the border of the bottle
  int[int] databoder(1);   int ncb= getborder(Th3c,databoder);
  int ktop= Th3c(databoder[databoder[1]]).z < Th3c(databoder[databoder[0]]).z;
  int  kbot=1-ktop;                                                    //    other borber
...
  macro DefBorder(bname,kk,Th3,bb,ll)
    int n#bname= bb[kk+1]-bb[kk];  border bname(t=bb[kk], bb[kk+1])
    {   real iv = int(t); if( iv == bb[kk+1]) iv = bb[kk];
       iv = bb[iv];x= Th3(iv).x ; y= Th3(iv).y ;label = ll;  }              //   EOM

  DefBorder(btop,ktop,Th3c,databoder,1) DefBorder(bbot,kbot,Th3c,databoder,1)

Th3bottom=movemesh23(change(buildmesh(bbot(nbbot),fixeborder=1),fregion=labbottom)
          ,transfo=[x,y,Zbot],orientation=-1);
  Th3top=movemesh23(change(buildmesh(btop(-nbtop),fixeborder=1),fregion=labtop)
           ,transfo=[x,y,Ztop],orientation=1);
  Ths = Th3c + Th3bottom + Th3top; }
...
real[int] domaine = [0,0,Htube,1,hh^3/6.];
mesh3 Th=tetg(Ths,switch="pqaAYY",regionlist=domaine); medit("Th",Th);
```

Run:bottle.edp

Warning now the surface mesh type is `meshS` and not `mesh3` as before version 4.01

- `change` to change label and region numbering in 2d and 3d, surface or line.
- `movemesh checkmovemesh movemesh23 movemesh3 movemeshS`
- `triangulate` (2d) , `tetgconvexhull` (3d) build mesh mesh for a set of point
- `emptymesh` (2d) built a empty mesh for Lagrange multiplier, use now meshL.
- `mmgs` to optimize surface mesh
- `mmg3d` to optimize volume mesh with constant surface mesh in version 4 (you can use le last version but to day you must used file to pass information).
- `mshmet` to compute metric
- `isoline` to extract isoline (2d)
- `trunc` to remove peace of mesh and split all element (2d,3d)
- `splitmesh` to split 2d mesh in no regular way.

In Euclidean geometry the length $|\gamma|$ of a curve γ of $\mathbb{R}^d$ parametrized by $\gamma(t)_{t=0..1}$ is

$$|\gamma| = \int_0^1 \sqrt{< \gamma'(t), \gamma'(t) >} dt$$

We introduce the metric $\mathcal{M}(x)$ as a field of $d \times d$ symmetric positive definite matrices, and the length ℓ of Γ w.r.t $\mathcal{M}$ is:

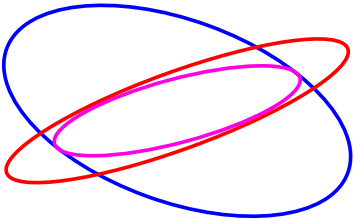
$$\ell = \int_0^1 \sqrt{< \gamma'(t), \mathcal{M}(\gamma(t))\gamma'(t) >} dt$$

The key-idea is to construct a mesh where the lengths of the edges are close to 1 accordingly to $\mathcal{M}$. so $\mathcal{M}$ become a user parameter.

Metric intersection

The unit ball $\mathcal{B}_{\mathcal{M}}$ in a constant metric $\mathcal{M}$ plot the maximum mesh size on all the direction,and it is an ellipse.
If you we have two unknowns u and v, we just compute the metric $\mathcal{M}_u$ and $\mathcal{M}_v$, find a metric $\mathcal{M}_{uv}$ call intersection with the biggest ellipse such that:

$$\mathcal{B}(\mathcal{M}_v) \subset \mathcal{B}(\mathcal{M}_u) \cap \mathcal{B}(\mathcal{M}_v)$$

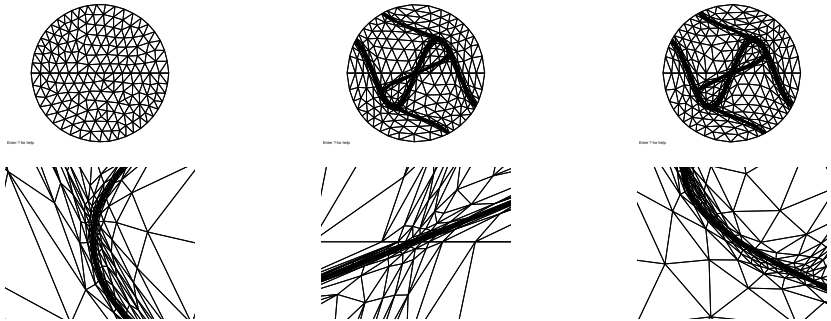


Under the assomption: if the mesh size is decreasing then the error also.

Example of mesh

$$u = (10x^3 + y^3) + tanh(500(sin(5y) - 2x)));$$

$$v = (10y^3 + x^3) + tanh(5000(sin(5y) - 2*)));$$

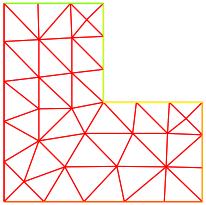


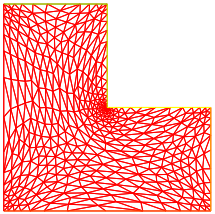
Run:Adapt-uv.edp

The domain is an L-shaped polygon $\Omega =]0,1[^2 \setminus [\frac{1}{2},1]^2$ and the PDE is

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } \quad -\Delta u = 1 \text{ in } \Omega,$$

The solution has a singularity at the reentrant angle and we wish to capture it numerically.





example of Mesh adaptation

FreeFem++ corner singularity program

```
int[int] lab=[1,1,1,1];
mesh Th = square(6,6,label=lab);
Th=trunc(Th,x<0.5 | y<0.5, label=1);

fespace Vh(Th,P1);          Vh u,v;          real error=0.1;
problem Probem1(u,v,solver=CG,eps=1.0e-6) =
       int2d(Th)(  dx(u)*dx(v) + dy(u)*dy(v))
     - int2d(Th)( v) + on(1,u=0);
for (int i=0;i< 7;i++)
{  Probem1;                                    //    solving the pde
   Th=adaptmesh(Th,u,err=error,nbvx=100000);
                              //    the adaptation with Hessian of u
   plot(Th,u,wait=1,fill=1);            u=u;
   error = error/ (1000.^(1./7.)) ;    } ;
```

Run:CornerLap.edp

Example of adaptation process in 3d with mmg3

Let a domain $\Omega =]0,1[^3 \setminus [\frac{1}{2},1[^3$ The border of $\partial\Omega$ is split in 2 part

- Γ_2, if $x = 1$, $y = 1$, or $z = 1$
- Γ_1, else.

Find u a solution in such that:

$$\begin{aligned} -\Delta u &= 1 && \text{in } \Omega, \\ \frac{\partial u}{\partial \vec{n}} &= 0 && \text{on } \Gamma_2, \\ u &= 0 && \text{on } \Gamma_1. \end{aligned}$$

Thank to mmg v5 tools to do 3d mesh adaptation see `http://www.mmgtools.org` .

without mmg (isotrope): Run:Laplace-Adapt-3d.edp

with mmg (anisotrope): Run:Laplace-Adapt-aniso-3d.edp

Build of the metric form the solution u

Optimal metric norm for interpolation error (function `adaptmesh` in freefem++) for P_1 continuous Lagrange finite element

- L^∞ : $\mathcal{M} = \frac{1}{\varepsilon}|\nabla\nabla u| = \frac{1}{\varepsilon}|\mathcal{H}|$ where $\mathcal{H} = \nabla\nabla u$
- L^p : $\mathcal{M} = \frac{1}{\varepsilon}|det(\mathcal{H})|^{\frac{1}{2p+2}}\,|\mathcal{H}|$ (result of F. Alauzet, A. Dervieux)

In Norm $W^{1,p}$, the optimal metric $\mathcal{M}_\ell$ for the P_ℓ Lagrange finite element, Optimal is given by (with only acute triangle) (thanks to J-M. Mirebeau)

$$\mathcal{M}_{\ell,p} = \frac{1}{\varepsilon}(det\mathcal{M}_\ell)^{\frac{1}{\ell p+2}}\,\mathcal{M}_\ell$$

and (see `MetricPk` plugin and function)

- for P_1: $\mathcal{M}_1 = \mathcal{H}^2$ (for sub optimal case with acute triangle take $\mathcal{H}$)
- for P_2: $\mathcal{M}_2 = 3\sqrt{\begin{pmatrix} a & b \\ b & c \end{pmatrix}^2 + \begin{pmatrix} b & c \\ c & a \end{pmatrix}^2}$ with $D^{(3)}u(x,y) = (ax^3 + 3bx^2y + 3cxy^2 + dy^3)/3!$,

Run:adapt.edp

Run:AdaptP3.edp

Outline

3. Academic Examples
   - Laplace/Poisson
   - Mortar Method
   - Linear elasticty equation
   - Stokes equation
   - Optimize Time depend schema

We find p in H^1, such that:

$$-\Delta p = f \text{ in } \Omega, \quad \partial_n p = g_n \text{ on } \Gamma.$$

This problem is defined through a constant and mathematically the problem is well pose iff $\int_\Omega f + \int_\Gamma g_n = 0$ and p is in $H^1/\mathbb{R}$. So we can make a small regularization to remove the problem of constant by find $p_\varepsilon \in H^1$ such that

$$\varepsilon p_\varepsilon - \Delta p_\varepsilon = f \text{ in } \Omega, \quad \partial_n p_\varepsilon = g_n \text{ on } \Gamma$$

and the last problem is trivial to be approximate in FreeFem++:

```
Vh p,q; real eps=1e-8;                          //    warning eps
                          //    must be small but no too small.
solve Laplace(p,q)
  =  int2d(Th)(  p*q*eps  + grad(p)'*grad(q))
   - int2d(Th) ( f*q) - int1d(Th)(gn*q) ;
```

Remark: it is hard to put Dirichlet boundary condition on only one point so set the constant due to label definition.

Laplace equation (mixte formulation) I/III

Now we solve $-\Delta p = f$ in Ω, $p = g_d$ on Γ_d, $\partial_n p = g_n$ on Γ_n.
Γ_d, Γ_n is a partition of $\partial\Omega$.
with $\vec{u} = \nabla p$ the problem becomes:
Find $\vec{u}, p$ such that:

$$-\nabla.\vec{u} = f,\ \vec{u} - \nabla p = 0 \ \text{ in } \Omega, \quad p = g_d \ \text{ on } \Gamma_d, \quad \partial_n p = g_n \text{ on } \Gamma_n \tag{4}$$

Mixte variational formulation is: find $\vec{u} \in H_{div}(\Omega)$, $p \in L^2(\Omega)$, $\vec{u}.n = g_n$ on Γ_n such that

$$\int_\Omega q\nabla.\vec{u} + \int_\Omega p\nabla.\vec{v} + \vec{u}.\vec{v} = \int_\Omega -fq + \int_{\Gamma_d} g_d\vec{v}.\vec{n}, \quad \forall(\vec{v}, q) \in H_{div}\times L^2, \text{and } \vec{v}.n = 0 \text{ on } \Gamma_n$$

Laplace equation (mixte formulation) II/III

```
mesh Th=square(10,10); fespace Vh(Th,RT0), Ph(Th,P0);
func gd = 1.; func gn = 1.;  func f = 1.;
Vh [u1,u2],[v1,v2];
Ph p,q;
solve laplaceMixte([u1,u2,p],[v1,v2,q],solver=UMFPACK)
  =  int2d(Th)(  p*q*0e-10  + u1*v1 + u2*v2
                + p*(dx(v1)+dy(v2)) + (dx(u1)+dy(u2))*q )
   + int2d(Th) ( f*q)
   - int1d(Th,1,2,3)( gd*(v1*N.x +v2*N.y))     //    int on Γ_d
   + on(4,u1=gn*N.x,u2=gn*N.y);                //    mean u.n = (g_n n).n
```

Run:LaplaceRT.edp

Laplace equation (Galerking discontinuous formulation) III/III

solve $-\Delta u = f$ on Ω and $u = g$ on Γ

```
macro dn(u) (N.x*dx(u)+N.y*dy(u) ) //    def the normal derivative
mesh Th = square(10,10);                          //    unite square
fespace Vh(Th,P2dc);          //    discontinuous P2 finite element
         //    if pena = 0 => Vh must be P2 otherwise penalization
real pena=0;                                      //    to add penalization
func f=1;     func g=0;
Vh u,v;

problem A(u,v,solver=UMFPACK) =                                    //
     int2d(Th)(dx(u)*dx(v)+dy(u)*dy(v)   )
   + intalledges(Th)(        //    loop on all edge of all triangle
        ( jump(v)*mean(dn(u)) - jump(u)*mean(dn(v))
          + pena*jump(u)*jump(v) ) / nTonEdge   )
   - int2d(Th)(f*v)
   - int1d(Th)(g*dn(v)   + pena*g*v) ;
 A; //    solve DG
```

Run:LapDG2.edp

A mathematical Poisson Problem with full Neumann BC. with 1D lagrange multiplier

The variationnal form is find $(u, \lambda) \in V_h \times \mathbb{R}$ such that

$$\forall (v, \mu) \in V_h \times \mathbb{R} \qquad a(u, v) + b(u, \mu) + b(v, \lambda) = l(v), \qquad \text{where } b(u, \mu) = \mu \int_\Omega u$$

```
mesh Th=square(10,10);      fespace Vh(Th,P1);    //    P1 FE space
int n = Vh.ndof,  n1 = n+1; func f=1+x-y;
macro Grad(u) [dx(u),dy(u)]                                  //   EOM
varf va(uh,vh) = int2d(Th)( Grad(uh)'*Grad(vh)  ) ;
varf vL(uh,vh) = int2d(Th)( f*vh )  ;
varf vb(uh,vh)=  int2d(Th)(1.*vh);
matrix A=va(Vh,Vh);
real[int] b=vL(0,Vh), B = vb(0,Vh);
real[int]  bb(n1),x(n1),b1(1),l(1); b1=0;
matrix AA = [ [ A ,  B ] , [ B', 0 ] ] ;  bb = [ b, b1];
set(AA,solver=UMFPACK);       //    set the type of linear solver.
x = AA^-1*bb;     [uh[],l] = x;      //    solve the linear systeme
plot(uh,wait=1);                              //    set the value
```

Run:Laplace-lagrange-mult.edp

3 Academic Examples
- Laplace/Poisson
- Mortar Method
- Linear elasticty equation
- Stokes equation
- Optimize Time depend schema

Let be a partition without overlap $\Omega = \cup_{i=0,..,4}\Omega_i$.
Remark Ω is the open set without the skeleton $\mathcal{S}$ and the external boundary is Γ. So the Mortar problem is: Find $u \in H^1(\Omega)$ such that $u_{|\Gamma} = g$ and $\lambda \in H^{-\frac{1}{2}}(\mathcal{S})$ and

$$\forall v \in H^1(\Omega), \quad v_{|\Gamma} = 0, \qquad \int_\Omega \nabla u \nabla v + \int_{\mathcal{S}} [v]\lambda = \int_{\Omega_i} fv$$

$$\forall \mu \in H^{-\frac{1}{2}}(\mathcal{S}), \qquad \int_{\mathcal{S}} [u]\mu = 0$$

For each sub domain Ω_i ,

$$\forall v \in H^1(\Omega_i), \quad v_{|\Gamma} = 0, \quad \int_{\Omega_i} \nabla u \nabla v + \int_{\mathcal{S}\cap\partial\Omega_i} \varepsilon_i \, \lambda v = \int_{\Omega_i} fv$$

$$\forall \mu \in H^{-\frac{1}{2}}(\mathcal{S}), \quad \sum_i \int_{\mathcal{S}\cap\partial\Omega_i} \varepsilon_i \, \mu u = O$$

Where $\varepsilon_i = \mathbf{n}_{\mathcal{S}}.\mathbf{n}_i$, $\varepsilon_i = \pm 1$ and $\sum_i \varepsilon_i = 0$.

$$J'(\lambda)(\mu) = -\int_{\mathcal{S}} [u_\lambda]\mu = 0 \forall \mu$$

$$\forall v \in H^1(\Omega_i), \quad v_{|\Gamma} = 0, \quad \int_{\Omega_i} \nabla u_l \nabla v + \int_{\mathcal{S}\cap\partial\Omega_i} \varepsilon_i \, \lambda v = \int_{\Omega_i} f v$$

For each sub domain Ω_i ,

$$\forall v \in H^1(\Omega_i), \quad v_{|\Gamma} = 0, \quad \int_{\Omega_i} \nabla u \nabla v + \int_{\mathcal{S}\cap\partial\Omega_i} \varepsilon_i \, \lambda v = \int_{\Omega_i} f v$$

$$\forall \mu \in H^{-\frac{1}{2}}(\mathcal{S}), \quad \sum_i \int_{\mathcal{S}\cap\partial\Omega_i} \varepsilon_i \, \mu u = O$$

Where $\varepsilon_i = \mathbf{n}_{\mathcal{S}}.\mathbf{n}_i$, $\varepsilon_i = \pm 1$ and $\sum_i \varepsilon_i = 0$.

Mortar method, compute the Normal of martor

```
                                                    //    def of mortar FE..
meshL Thm=buildmeshL(gi(Ng));.
fespace Lh(Thm,P0);                                 //    Thm is a meshL ...
fespace PTh(Thm,[P0,P0]);                            //    FE for normal ...
Phm [Nmx,Nmy];    //   Warning only the tangent is defined on meshL
varf  vNN([ux,uy],[nx,ny]) =      //    axel:  Ns 2D -> Tl 3D curve
       int1d(Thm,1)(( -nx*Tl.y + ny*Tl.x)/lenEdge);
Nmx[]= vNN(0,RTh);
     //    Array each S.D..  ; Thi == mesh of the current subomain
mesh[int] Thsd(nbsd);
for(int sd=0;sd<nbsd;++sd)
  Thsd[sd]=trunc(Tha,region==regi[sd],split=1);      //    the Sub.
Dom.
fespace Vhi(Thi,P1);   fespace Ehi(Thi,P0);
matrix[int] Asd(nbsd),Csd(nbsd),PAsd(nbsd),PIsd(nbsd),PJsd(nbsd);
Vhi[int] usd(nbsd),vsd(nbsd),rhssd(nbsd), pusd(nbsd),bcsd(nbsd);
Ehi[int] epssd(nbsd);
```

Mortar method, next

```
                                                //    for all sub domain ...
for(int sd=0;sd<nbsd;++sd)
{
  Thi=Thsd[sd];  usd[sd]=0;  vsd[sd]=0;
  varf vepsi(u,v)= int1d(Thi,1)( (Nmx*N.x + Nmy*N.y)*v/lenEdge);
  epssd[sd][]= vepsi(0,Ehi);
  epssd[sd] = -real(epssd[sd] <-1e-5) + real(epssd[sd] >1e-5);
  varf cci([l],[u]) = int1d(Thm,1,qforder=3)(l*u*epssd[sd]);
  varf vLapMi([ui],[vi],tgv=tgv) =
         int2d(Thi)(  Grad(ui)'*Grad(vi)   )
      +  int2d(Thi) (f*vi)    +  on(labext,ui=g);
  varf  vrhsMi(ui,vi) =   on(labext,ui=g);
  Csd[sd]  = cci(Lh,Vhi);                       //    new Matrix Stuff
  Asd[sd]  = vLapMi(Vhi,Vhi,solver=sparsesolver);
  rhssd[sd][]=vLapMi(0,Vhi);
}
```

Run:mortar-DN-4-v4.5.edp

3. Academic Examples
   - Laplace/Poisson
   - Mortar Method
   - Linear elasticty equation
   - Stokes equation
   - Optimize Time depend schema

Let a domain $\Omega \subset \mathbb{R}^d$ with a partition of $\partial\Omega$ in Γ_d, Γ_n.
Find the displacement $\boldsymbol{u}$ field such that:

$$-\nabla.\sigma(\boldsymbol{u}) = \boldsymbol{f} \text{ in } \Omega, \quad \mathbf{u = 0} \text{ on } \mathbf{\Gamma_d}, \quad \sigma(\mathbf{u}).\mathbf{n = 0} \text{ on } \mathbf{\Gamma_n} \tag{5}$$

Where $\varepsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + {}^t\nabla \boldsymbol{u})$ and $\sigma(\boldsymbol{u}) = \boldsymbol{A}\varepsilon(\boldsymbol{u})$ with $\boldsymbol{A}$ the linear positif operator on symmetric $d \times d$ matrix corresponding to the material propriety. Denote
$V_{\boldsymbol{g}} = \{\boldsymbol{v} \in H^1(\Omega)^d / \boldsymbol{v}_{|\Gamma_d} = \boldsymbol{g}\}$.
The Basic displacement variational formulation is: find $\boldsymbol{u} \in V_0(\Omega)$, such that:

$$\int_\Omega \varepsilon(\boldsymbol{v}) : \boldsymbol{A}\varepsilon(\boldsymbol{u}) = \int_\Omega \boldsymbol{v}.\boldsymbol{f} + \int_\Gamma ((\boldsymbol{A}\varepsilon(\boldsymbol{u})).n).v, \quad \forall \boldsymbol{v} \in V_0(\Omega) \tag{6}$$

Linear elasticty equation, in FreeFem++

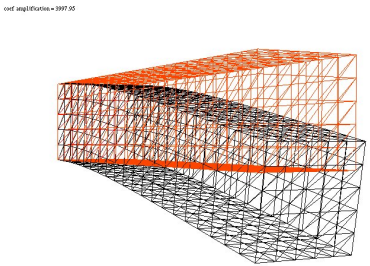
The finite element method is just: replace V_g with a finite element space, and the FreeFem++ code:

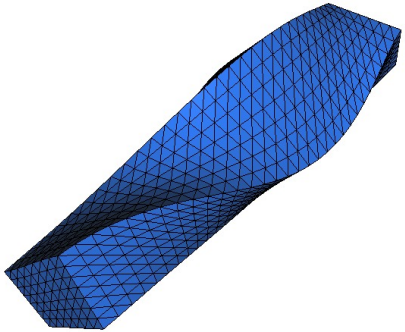
```
load "medit"   include "cube.idp"
int[int]  Nxyz=[20,5,5];
real [int,int]  Bxyz=[[0.,5.],[0.,1.],[0.,1.]];
int [int,int]  Lxyz=[[1,2],[2,2],[2,2]];
mesh3 Th=Cube(Nxyz,Bxyz,Lxyz);
                                                    //    Alu ...
real rhoAlu = 2600, alu11=  1.11e11 , alu12 = 0.61e11 ;
real alu44= (alu11-alu12)*0.5;
func Aalu = [  [alu11, alu12,alu12,   0.   ,0.   ,0.  ],
               [alu12, alu11,alu12,   0.   ,0.   ,0.   ],
               [alu12, alu12,alu11,   0.   ,0.   ,0.   ],
               [0. ,   0. ,   0. ,    alu44,0.   ,0.   ],
               [0. ,   0. ,   0. ,    0.   ,alu44,0.   ],
               [0. ,   0. ,   0. ,    0.   ,0.   ,alu44]   ];
real gravity = -9.81;
```

Linear elasticity equation, in FreeFem++

```
fespace Vh(Th,[P1,P1,P1]);
Vh [u1,u2,u3], [v1,v2,v3];
macro Strain(u1,u2,u3)
[ dx(u1), dy(u2), dz(u3), (dz(u2) +dy(u3)), (dz(u1)+dx(u3)),    (dy(u1)+dx(u2)) ]
//    EOM
solve Lame([u1,u2,u3],[v1,v2,v3])=
  int3d(Th)(  Strain(v1,v2,v3)'*(Aalu*Strain(u1,u2,u3))  )
- int3d(Th) ( rhoAlu*gravity*v3)
+ on(1,u1=0,u2=0,u3=0) ;

real coef= 0.1/u1[].linfty;  int[int] ref2=[1,0,2,0];
mesh3 Thm=movemesh3(Th,
     transfo=[x+u1*coef,y+u2*coef,z+u3*coef],
     label=ref2);
plot(Th,Thm, wait=1,cmm="coef  amplification = "+coef );
medit("Th-Thm",Th,Thm);
```

Lame equation / figure





Run:beam-3d.edp Run:beam-EV-3d.edp Run:free-cyl-3d.edp Run:beam-3d-Adapt.edp

3. Academic Examples
   - Laplace/Poisson
   - Mortar Method
   - Linear elasticty equation
   - Stokes equation
   - Optimize Time depend schema

The Stokes equation is find a velocity field $\boldsymbol{u} = (u_1, .., u_d)$ and the pressure p on domain Ω of $\mathbb{R}^d$, such that

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= 0 \quad \text{in} \quad \Omega \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in} \quad \Omega \\ \boldsymbol{u} &= \boldsymbol{u}_\Gamma \quad \text{on} \quad \Gamma \end{aligned}$$

where $\boldsymbol{u}_\Gamma$ is a given velocity on boundary Γ.
The classical variational formulation is: Find $\boldsymbol{u} \in H^1(\Omega)^d$ with $\boldsymbol{u}_{|\Gamma} = u_\Gamma$, and $p \in L^2(\Omega)/\mathbb{R}$ such that

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^d,\ \forall q \in L^2(\Omega)/\mathbb{R}, \qquad \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} - p \nabla . v - q \nabla . u = 0$$

or now find $p \in L^2(\Omega)$ such than (with $\varepsilon = 10^{-10}$)

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^d,\ \forall q \in L^2(\Omega), \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} - p \nabla . v - q \nabla . u - \varepsilon p q = 0$$

```
 ...  build mesh .... Th (3d) T2d ( 2d)
fespace VVh(Th,[P2,P2,P2,P1]);          //    Taylor Hood FE.
macro Grad(u) [dx(u),dy(u),dz(u)]          //    EOM
macro div(u1,u2,u3) (dx(u1)+dy(u2)+dz(u3))      //   EOM
VVh [u1,u2,u3,p],[v1,v2,v3,q] ;
solve vStokes([u1,u2,u3,p],[v1,v2,v3,q]) =
  int3d(Th)(
          Grad(u1)'*Grad(v1)
        +  Grad(u2)'*Grad(v2)
        +  Grad(u3)'*Grad(v3)
      - div(u1,u2,u3)*q - div(v1,v2,v3)*p
      - 1e-10*q*p )
 + on(1,u1=0,u2=0,u3=0)    +  on(2,u1=1,u2=0,u3=0);
```

Run:Stokes-2d.edp Run:Stokes-bug.edp Run:Stokes-UzawaCahouetChabart-bug.edp
Run:Stokes-Pipe.edp Run:Stokes3d.edp

3. Academic Examples
   - Laplace/Poisson
   - Mortar Method
   - Linear elasticty equation
   - Stokes equation
   - Optimize Time depend schema

First, it is possible to define variational forms, and use this forms to build matrix and vector to make very fast script (4 times faster here).
For example solve the Thermal Conduction problem of section 3.4. We must solve the temperature equation in Ω in a time interval (0,T).

$$\begin{aligned}
&\partial_t u - \nabla \cdot (\kappa \nabla u) = 0 \text{ in } \Omega \times (0,T),\\
&u(x,y,0) = u_0 + x u_1\\
&u = 30 \text{ on } \Gamma_{24} \times (0,T), \quad \kappa \frac{\partial u}{\partial n} + \alpha(u - u_e) = 0 \text{ on } \Gamma \times (0,T).
\end{aligned} \tag{7}$$

The variational formulation is in $L^2(0,T;H^1(\Omega))$; we shall seek u^n satisfying

$$\forall w \in V_0; \qquad \int_\Omega \frac{u^n - u^{n-1}}{\delta t} w + \kappa \nabla u^n \nabla w) + \int_\Gamma \alpha(u^n - u_{ue}) w = 0$$

where $V_0 = \{w \in H^1(\Omega) / w_{|\Gamma_{24}} = 0\}$.

So the to code the method with the matrices $A = (A_{ij})$, $M = (M_{ij})$, and the vectors $u^n, b^n, b', b", b_{cl}$ (notation if w is a vector then w_i is a component of the vector).

$$u^n = A^{-1}b^n, \qquad b' = b_0 + Mu^{n-1}, \quad b" = \frac{1}{\varepsilon}\, b_{cl}, \quad b^n_i = \begin{cases} b"_i & \text{if } \ i \in \Gamma_{24} \\ b'_i & \text{else} \end{cases}$$

Where with $\frac{1}{\varepsilon} = \texttt{tgv} = 10^{30}$:

$$A_{ij} = \begin{cases} \frac{1}{\varepsilon} & \text{if } i \in \Gamma_{24}, \text{and} \quad j = i \\ \displaystyle\int_\Omega w_j w_i/dt + k(\nabla w_j.\nabla w_i) + \int_{\Gamma_{13}} \alpha w_j w_i & \text{else} \end{cases}$$

$$M_{ij} = \begin{cases} \frac{1}{\varepsilon} & \text{if } i \in \Gamma_{24}, \text{and} \quad j = i \\ \displaystyle\int_\Omega w_j w_i/dt & \text{else} \end{cases}$$

$$b_{0,i} = \int_{\Gamma_{13}} \alpha u_{ue} w_i$$

$$b_{cl} = u^0 \quad \text{the initial data}$$

```
 ...
Vh u0=fu0,u=u0;
```

Create three variational formulation, and build the matrices A,M.

```
varf vthermic (u,v)= int2d(Th)(u*v/dt
           + k*(dx(u) * dx(v) + dy(u) * dy(v)))
  +  int1d(Th,1,3)(alpha*u*v)   + on(2,4,u=1);
varf vthermic0(u,v) =   int1d(Th,1,3)(alpha*ue*v);
varf vMass (u,v)= int2d(Th)( u*v/dt)   + on(2,4,u=1);

real tgv = 1e30;
matrix A= vthermic(Vh,Vh,tgv=tgv,solver=CG);
matrix M= vMass(Vh,Vh);
```

Now, to build the right hand size we need 4 vectors.

```
real[int]  b0 = vthermic0(0,Vh);    //   constant part of RHS
real[int]  bcn = vthermic(0,Vh);   //   tgv on Dirichlet part
           //    we have for the node i :  i ∈ Γ24  ⇔  bcn[i] ≠ 0
real[int]  bcl=tgv*u0[];          //    the Dirichlet B.C. part
```

The Fast algorithm:

```
for(real t=0;t<T;t+=dt){
  real[int] b = b0 ;                              //    for the RHS
  b += M*u[];                 //   add the the time dependent part
  b = bcn ? bcl : b;      //   do ∀i:  b[i]=bcn[i]?bcl[i]:b[i];
  u[] = A^-1*b;                          //    Solve linear problem
  plot(u);
}
```

Run:Heat.edp

Outline

With. I. Danaila (Univ. Rouen), G. Vergez (Phd Becasim), P-E Emeriau (Stage ENS)

Just a direct use of `Ipopt` interface (2 day of works to start script see, + n month)
The problem is find a complex field u on domain $\mathcal{D}$ such that:

$$u = \underset{||u||=1}{\operatorname{argmin}} \int_{\mathcal{D}} \frac{1}{2}|\nabla u|^2 + V_{trap}|u|^2 + \frac{g}{2}|u|^4 - \Omega i\overline{u}\left(\begin{pmatrix} -y \\ x \end{pmatrix}.\nabla\right)u$$

to code that in FreeFem++
use

- Ipopt interface (`https://projects.coin-or.org/Ipopt`)
- Adaptation de maillage

The idea to mixte Ipopt and adapt mesh is play this stop criterion, and finally use freefem++ to analyse the result.

Run:BEC.edp

4. Bose Einstein Condensate, result analyse
   - Search all local min
   - Best Fit
   - Delaunay mesh
   - Analyse of a Condensate

Search all local min

The function `findalllocalmin` find all the le local min and use a greedy algorithm to to the local attraction zone, by adding the triangle through the minimal vertices.

```
mesh Th=square(50,50,[x*2-1,y*2-1]);
load "isoline"
fespace Vh(Th,P1), Ph(Th,P0);
int k =2;
Vh u= sin(k*pi*x)*sin(k*pi*y);
plot(u, wait=1);
Ph r;
int[int] lm=findalllocalmin(Th,u[],r[]);
// lm array gives the vertex number of all the local min
// r is function P0 defined the attraction zone of the local min
// (local min number)
plot(r,u,fill=1,wait=1);
// to see where is the minimuns
Ph mx= Th(lm[real(r)]).x -x, my= Th(lm[real(r)]).y -y;
plot([mx,my],u,wait=1,fill=0);
```

Run:findalllocalmin.edp

Run:findalllocalminbec.edp

(With. P-E Emeriau) Just use `ipopt` to find the arg min of $J(\alpha) = \int_\Omega (u - \phi_\alpha)^2)$ where $alpha$ is the set of parameters.

```
real[int]  data0 = [ ux , x0, y0 , s0] ;  // start point

func real J(real[int] & dat){
   alpha=dat;
   return int2d(Th)(square(u-phialpha)) ;}

func real[int]  dJ(real[int] & dat){
   alpha=dat;
    dat[0]=int2d(Th)(-2*(u-phialpha)*d0phialpha) ;
    dat[1]=int2d(Th)(-2*(u-phialpha)*d1phialpha) ;
    dat[2]=int2d(Th)(-2*(u-phialpha)*d2phialpha) ;
    dat[3]=int2d(Th)(-2*(u-phialpha)*d3phialpha) ;
   return dat;  }
 real[int] data=data0;
 verbosity=0;
 int r = IPOPT(J,dJ,data,printlevel=0);
```

Run:fit-ipopt.edp

Best Fit axisymmetric

On the domain with no vortex, all just do the L2 projection on axisymmetric space with a laplace regularisation

```
Ph pok=data7(6,real(r)); // domain with no hole
func r = sqrt(x*x+y*y);
Vh pr = r;
real R = pr[].max;
mesh Th1d=square(200,1,[x*R,y]);
fespace V1d(Th1d,P1,periodic=[[1,x],[3,x]]);// dat axi
mesh The = trunc(Thg,pok==1);  // mesh
   Vh u2=u*u;
varf vM1d(u,v) = int1d(Th1d,1)(dx(u)*dx(v))
     + int2d(The,  mapu=[r,0] ,  mapt=[r,0]  )(u*v);
matrix M=vM1d(V1d,V1d,solver=CG);
varf vb1d(u,v) = int2d(The,mapt=[r,0])(u2*v);
real[int] b1d=vb1d(0,V1d);  V1d u1dt;
u1dt[]=M^-1*b1d;
Vh u20 = u1dt(r,0); // Axi -> 2d
plot(u20,u2,wait=1,dim=3);
```

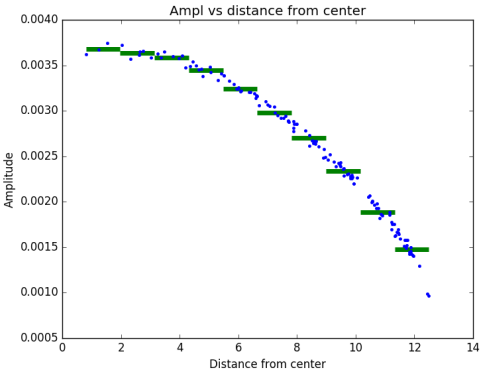
Run:fit-axi.edp

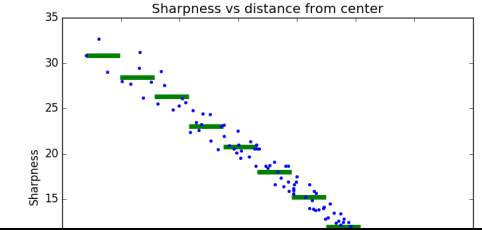
```
mesh Thc=triangulate(data7(0,:),data7(1,:));

fespace Eh(Thc,P0edge);// Element P0 / Edge
varf vedge(u,v) = intalledges(Thc,qforder=1)((nTonEdge==2)*v/
    nTonEdge);
real[int] eih=vedge(0,Eh);
int ei=0;
for(int e=0; e < eih.n; ++e) if(eih[e])  eih[ei++]=eih[e];
eih.resize(ei);
real moy =  eih.sum/ eih.n ;
// Statistic
real[int] dd = eih;dd-= moy;
real variance = dd.l2 /  dd.n;
cout << " moy eih =" << moy<< " standart deviation " << sqrt(
    variance) <<endl;
for(int i=1 ; i<10; ++i)
cout <<" quantile "<<i/10.<< " = " << eih.quantile(i/10.) <<endl;
```

Run:analyssolbec.edp

4. Bose Einstein Condensate, result analyse
   - Search all local min
   - Best Fit
   - Delaunay mesh
   - Analyse of a Condensate

Analyse of a Condensate





Outline

```
mesh Th=square(5,5);
fespace Wh(Th,P2);
cout « " nb of DoF     : " « Wh.ndof « endl;
cout « " nb of DoF / K : " « Wh.ndofK « endl;
 int k= 2, kdf= Wh.ndofK ;;                    //    element 2
 cout « " df of element " « k « ":" ;
 for (int i=0;i<kdf;i++)  cout « Wh(k,i) « " ";
 cout « endl;
```

Remark on local numbering of Dof by element is
for each sub finite element Pk in `[P2,P2,P1]` get fist DoF on vertex, second Dof on edge (opposite to vertex), second on K.

Run:Mesh-info.edp

uses `cout`, `cin`, `endl`, «,».
To write to (resp. read from) a file,
declare a new variable `ofstream ofile("filename");`
or
`ofstream ofile("filename",append);` (resp. `ifstream ifile("filename");`)
or
`ofstream ofile("filename",append|binary);` (resp. `ifstream ifile("filename",binary);`)
and use `ofile` (resp. `ifile`) as `cout` (resp. `cin`).

You can use pipe to transfer data to a other code here (gnuplot), see pipe.edp example:

You can use the plugin `bfstream` tp make binary io (see Run: examples++-load/bfstream.edp)

Run:pipe.edp

Run:io.edp

What is simple to do with freefem++ :

- Evaluate variational form with Boundary condition or not.
- Do interpolation
- Do linear algebra
- Solve sparse problem.

? Question, Find the list Degree of Freedom (DoF) of border k for coupling problem.

Idea Take a function increasing negative function $\xi - C_{te}$ on the border, and do a simultaneous sort the to array and DoF numbering, remark we use a PDE on border to build this kind of function

$$\nabla\xi.N^{\perp} = 1 \text{ on } \Gamma_b$$

or use the macro `ExtractDofsonBorder(labs,Wh,doflabs,orient)` defined in "ExtractDofsonBorder.idp",

Run:ListOfDofOnBorder.edp and see:ExtractDofsonBorder.idp.

Computation of error estimate $\eta_K = \sqrt{\int_K \text{blabla}} = \sqrt{\int_\Omega w_k \text{blabla}}$ where w_k is the basic function of `fespace Ph(Th,P0)`.

```
varf vetaK(unused,wK) = int2d(Th)( blabla * wK);
Ph etaK; etaK[] = vetaK(0,Ph);  etaK=sqrt(etaK);
```

to Change Default sparse solver add following line:

```
load += "MUMPS_seq"
```

if `MUMPS-seq` is available in file `\$(HOME)/.freefem++.pref`

Diff How to compute, differential: use of `macro`

$$J(u) = \int_\Omega F(u); \qquad \texttt{macro F(u)} = \sqrt{1 + \nabla u.\nabla u}$$

$$dJ(u)(v) = \int_\Omega dF(u,v); \qquad \texttt{macro dF(u,v)} = \frac{\nabla u.\nabla v}{\sqrt{1 + \nabla u.\nabla u}}$$

$$ddJ(u)(v,w) = \int_\Omega ddF(u,v,w);$$

$$\texttt{macro ddF(u,v,w)} = \frac{\nabla w.\nabla v}{\sqrt{1 + \nabla u.\nabla u}} - \frac{(\nabla u.\nabla v)(\nabla w.\nabla v)}{\sqrt{1 + \nabla u.\nabla u^3}}$$

Eigenvalue/ Eigenvector example

The problem, Find the first λ, u_λ such that:

$$a(u_\lambda, v) = \int_\Omega \nabla u_\lambda \nabla v = \lambda \int_\Omega u_\lambda v = \lambda b(u_\lambda, v)$$

the boundary condition is make with exact penalization: we put $1e30 = tgv$ on the diagonal term of the lock degree of freedom. So take Dirichlet boundary condition only with a variational form and not on b variational form , because we compute eigenvalue of

$$\frac{1}{\lambda} v = A^{-1} B v$$

Otherwise we get spurious mode.
Arpack interface:

```
int k=EigenValue(A,B,sym=true,value=ev,vector=eV);
```

Eigenvalue/ Eigenvector example code

```
...
fespace Vh(Th,P1);
macro Grad(u) [dx(u),dy(u),dz(u)]                              //    EOM
varf  a(u1,u2)= int3d(Th)(  Grad(u1)'*Grad(u2) +  on(1,u1=0) ;
varf b([u1],[u2]) = int3d(Th)(  u1*u2 ) ;                      //    no BC
matrix A= a(Vh,Vh,solver=UMFPACK),
        B= b(Vh,Vh,solver=CG,eps=1e-20);

int nev=40;          //    number of computed eigenvalue close to 0
real[int] ev(nev);                    //    to store nev eigenvalue
Vh[int] eV(nev);                      //    to store nev eigenvector
int k=EigenValue(A,B,sym=true,value=ev,vector=eV);
k=min(k,nev);
for (int i=0;i<k;i++)
   plot(eV[i],cmm="ev  "+i+" v =" + ev[i],wait=1,value=1);
```

Run:Lap3dEigenValue.edp Run:LapEigenValue.edp

The IPOPT optimizer in a FreeFem++ script is done with the `IPOPT` function included in the `ff-Ipopt` dynamic library. IPOPT is designed to solve constrained minimization problem in the form :

$$\begin{array}{ll} \text{find} & x_0 = \underset{x\in\mathbb{R}^n}{\operatorname{argmin}} f(x) \\ \text{s.t.} & \left\{ \begin{array}{ll} \forall i \le n,\ x_i^{\mathrm{lb}} \le x_i \le x_i^{\mathrm{ub}} & \text{(simple bounds)} \\ \forall i \le m,\ c_i^{\mathrm{lb}} \le c_i(x) \le c_i^{\mathrm{ub}} & \text{(constraints functions)} \end{array} \right. \end{array}$$

Where ub and lb stand for "upper bound" and "lower bound". If for some $i, 1 \le i \le m$ we have $c_i^{\mathrm{lb}} = c_i^{\mathrm{ub}}$, it means that c_i is an equality constraint, and an inequality one if $c_i^{\mathrm{lb}} < c_i^{\mathrm{ub}}$.

```
func real J(real[int] &X) {...}                          //   Fitness Function,
func real[int] gradJ(real[int] &X) {...}                 //   Gradient

func real[int] C(real[int] &X) {...}                     //   Constraints
func matrix jacC(real[int] &X) {...}                     //   Constraints jacobian

matrix jacCBuffer;                          //   just declare, no need to define yet
func matrix jacC(real[int] &X)
{
  ...                                                    //   fill jacCBuffer
  return jacCBuffer;
}
```

The hessian returning function is somewhat different because it has to be the hessian of the lagrangian function : $(x, \sigma_f, \lambda) \mapsto \sigma_f \nabla^2 f(x) + \sum_{i=1}^{m} \lambda_i \nabla^2 c_i(x)$ where $\lambda \in \mathbb{R}^m$ and $\sigma \in \mathbb{R}$. Your hessian function should then have the following prototype :

```
matrix hessianLBuffer;                          //   just to keep it in mind
func matrix hessianL(real[int] &X,real sigma,real[int] &lambda) {...}
```

Ipopt Call

```
real[int] Xi = ... ;                                  //   starting point
IPOPT(J,gradJ,hessianL,C,jacC,Xi, ... );

IPOPT(J,gradJ,C,jacC,Xi,...);                        //   IPOPT with BFGS
IPOPT(J,gradJ,hessianJ,Xi,...);                         //   Newton IPOPT
                                                  //    without constraints
IPOPT(J,gradJ,Xi, ... );                             //   BFGS, no constraints
IPOPT(J,gradJ,Xi, ... );                             //   BFGS, no constraints
IPOPT([b,A],CC,ui1[],lb=lb1[],clb=cl[]..);              //   affine case
  ...
```

Ipopt interface

```
load  "ff-Ipopt"
varf vP([u1,u2],[v1,v2]) = int2d(Th)(Grad(u1)'*Grad(v1)+ Grad(u2)'*Grad(v2))
- int2d(Th)(f1*v1+f2*v2);

matrix A = vP(Vh,Vh);                                   //   Fitness function matrix...
real[int] b = vP(0,Vh);                                          //   and linear form
int[int] II1=[0],II2=[1];                                     //   Constraints matrix
matrix C1 =  interpolate (Wh,Vh, U2Vc=II1);
matrix C2 =  interpolate (Wh,Vh, U2Vc=II2);
matrix CC = -1*C1 + C2;                                                  //    u2 - u1 >0
Wh cl=0;                                 //   constraints lower bounds (no upper bounds)
varf vGamma([u1,u2],[v1,v2]) = on(1,2,3,4,u1=1,u2=1);
real[int] onGamma=vGamma(0,Vh);
Vh [ub1,ub2]=[g1,g2];
Vh [lb1,lb2]=[g1,g2];
ub1[] = onGamma ? ub1[] : 1e19  ;                                 //   Unbounded in interior
lb1[] = onGamma ? lb1[] : -1e19  ;
Vh [uzi,uzi2]=[uz,uz2],[lzi,lzi2]=[lz,lz2],[ui1,ui2]=[u1,u2];;
Wh lmi=lm;
IPOPT([b,A],CC,ui1[],lb=lb1[],clb=cl[],ub=ub1[],warmstart=iter>1,uz=uzi[],lz=lzi[],lm=lmi
```

Run:IpoptLap.edp Run:IpoptVI2.edp Run:IpoptMinSurfVol.edp

```
load  "ff-NLopt"
...
if(kas==1)
   mini = nloptAUGLAG(J,start,grad=dJ,lb=lo,ub=up,IConst=IneqC,
            gradIConst=dIneqC,subOpt="LBFGS",stopMaxFEval=10000,
            stopAbsFTol=starttol);
else if(kas==2)
  mini = nloptMMA(J,start,grad=dJ,lb=lo,ub=up,stopMaxFEval=10000,
          stopAbsFTol=starttol);
else if(kas==3)
  mini = nloptAUGLAG(J,start,grad=dJ,IConst=IneqC,gradIConst=dIneqC,
         EConst=BC,gradEConst=dBC,
         subOpt="LBFGS",stopMaxFEval=200,stopRelXTol=1e-2);
else if(kas==4)
  mini = nloptSLSQP(J,start,grad=dJ,IConst=IneqC,gradIConst=dIneqC,
        EConst=BC,gradEConst=dBC,
        stopMaxFEval=10000,stopAbsFTol=starttol);
```

Run:VarIneq2.edp

This algorithm works with a normal multivariate distribution in the parameters space and try to adapt its covariance matrix using the information provides by the successive function evaluations. Syntaxe: `cmaes(J,u[],..)` ( )
From `http://www.lri.fr/~hansen/javadoc/fr/inria/optimization/cmaes/package-summary.html`

Stochastic Exemple

```
load  "ff-cmaes"

real mini = cmaes(J,start,stopMaxFunEval=10000*(al+1),
                  stopTolX=1.e-4/(10*(al+1)),
                  initialStdDev=(0.025/(pow(100.,al))));
SSPToFEF(best1[],best2[],start);
```

Run:cmaes-VarIneq.edp

```
load  "mpi-cmaes"

real mini = cmaesMPI(J,start,stopMaxFunEval=10000*(al+1),
                  stopTolX=1.e-4/(10*(al+1)),
                  initialStdDev=(0.025/(pow(100.,al))));
SSPToFEF(best1[],best2[],start);
```

remark, the FreeFem `mpicommworld` is used by default. The user can specify his own MPI communicator with the named parameter "`comm=`", see the MPI section of this manual for more informations about communicators in `FreeFem++`.

Outline

Full MPI interface I/III

```
int[int] proc1=[1,2,3],proc2=[0,4];
 mpiGroup grp(procs);          //    set MPI_Group to proc 1,2,3 in MPI_COMM_WORLD
 mpiGroup grp1(comm,proc1);          //    set MPI_Group to proc 1,2,3 in comm
 mpiGroup grp2(grp,proc2);                //    set MPI_Group to grp union proc1

 mpiComm  comm=mpiCommWorld;               //    set a MPI_Comm to MPI_COMM_WORLD
 mpiComm  ncomm(mpiCommWorld,grp);                //    set the MPI_Comm form grp
                                                            //    MPI_COMM_WORLD
 mpiComm  ncomm(comm,color,key);             //    MPI_Comm_split(MPI_Comm comm,
                                       //     int color, int key, MPI_Comm *ncomm)
 mpiComm  nicomm(processor(local_comm,local_leader),
                 processor(peer_comm,peer_leader),tag);
          //    build  MPI_INTERCOMM_CREATE(local_comm, local_leader, peer_comm,
                                                //     remote_leader, tag, &nicomm)
 mpiComm  ncomm(intercomm,hight) ;                                  //    build using
                                  //    MPI_Intercomm_merge( intercomm, high, &ncomm)
 mpiRequest rq;                                          //    defined an MPI_Request
 mpiRequest[int] arq(10);                 //    defined an array of 10 MPI_Request

 mpiSize(comm) ;                                  //    return the size of comm (int)
 mpiRank(comm) ;                                  //    return the rank in comm (int)
```

Full MPI interface II/III

```
processor(i)          //    return processor i with no Resquest in MPI_COMM_WORLD
processor(mpiAnySource)                          //    return processor any source
                                              //    with no Resquest in MPI_COMM_WORLD
processor(i,comm)               //    return processor i with no Resquest in comm
processor(comm,i)               //    return processor i with no Resquest in comm
processor(i,rq,comm)            //    return processor i with Resquest rq in comm
processor(i,rq)                          //    return processor i with Resquest rq in
                                                                  //    MPI_COMM_WORLD
processorblock(i)                        //    return processor i in MPI_COMM_WORLD
                                  //    in block mode for synchronously communication
processorblock(mpiAnySource)                        //    return processor any source
           //    in MPI_COMM_WORLD in block mode for synchronously communication
processorblock(i,comm)          //    return processor i in in comm in block mode

int status;                                 //    to get the MPI status of send / recv
processor(10) « a « b;                 //    send a,b asynchronously to the process 1,
processor(10) » a » b;          //   receive a,b synchronously from the process 10,
broadcast(processor(10,comm),a);                    //    broadcast from processor
                                                 //    of com to other comm processor
status=Send( processor(10,comm) , a);                        //    send synchronously
                                                    //   to the process 10 the data a
status=Recv( processor(10,comm) , a);                  //    receive synchronously
                                                  //    from the process 10 the data a;
```

Full MPI interface IIï/III

```
status=Isend( processor(10,comm) , a);              //    send asynchronously to
                                  //    the process 10 , the data a without request
status=Isend( processor(10,rq,comm) , a) ;       //    send asynchronously to to
                                 //    the process 10, the data a with request
status=Irecv( processor(10,rq) , a) ;           //    receive synchronously from
broadcast(processor(comm,a));                    //    Broadcast to all process of comm

mpiBarrier(comm) ;                    //    do a MPI_Barrier on communicator comm,
mpiWait(rq);                                                //    wait on of Request,
mpiWaitAll(arq);                                      //    wait add of Request array,
mpiWtime() ;                                       //    return MPIWtime in second (real),
mpiWtick() ;                                       //    return MPIWTick in second (real),
mpiAlltoall(a,b[,comm]) ;
mpiAllgather(a,b[,comm]) ;
mpiGather(a,b,processor(..) ) ;
mpiScatter(a,b,processor(..)) ;
mpiReduce(a,b,processor(..),mpiMAX) ;
mpiAllReduce(a,b,comm, mpiMAX) ;
mpiReduceScatter(a,b,comm, mpiMAX) ;
....
```

A first way to break complexity

1. Build matrix in parallel by assembling par region remark with the `change` function you change the region numbering to build region.

```
real c = mpisize/real(Th.nt) ;
Th=change(Th,fregion= min(mpisize-1,int(nuTriangle*c)));
```

2. Assemble the full matrix in //

```
varf vlaplace(uh,vh) =                    //    definition de problem
      int3d(Th,mpirank)( uh*vh+ dt*Grad(uh)'*grad(vh) )
    + int3d(Th,mpirank)( dt*vh*f) + on(1,uh=g) ;
matrix A,Ai = vlaplace(Vh,Vh,tgv=ttgv) ;
mpiReduce(Ai,A,processor(0),mpiSUM);            //   assemble in //
```

3. Solve the linear using a good parallel solver (MUMPS)

```
load "MUMPS"
    uh[] = A^-1*b ;                                  //    resolution
```

Run:Heat3d.edp

Run:NSCaraCyl-100-mpi2.edp

To solve the following Poisson problem on domain Ω with boundary Γ in $L^2(\Omega)$:

$$-\Delta u = f, \text{ in } \Omega, \text{ and } u = g \text{ on } \Gamma,$$

where $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\Gamma)$ are two given functions.
Let introduce $(\pi_i)_{i=1,..,N_p}$ a positive regular partition of the unity of Ω, q-e-d:

$$\pi_i \in \mathcal{C}^0(\Omega): \quad \pi_i \geq 0 \text{ and } \sum_{i=1}^{N_p} \pi_i = 1.$$

Denote Ω_i the sub domain which is the support of π_i function and also denote Γ_i the boundary of Ω_i.
The parallel Schwarz method is Let $\ell = 0$ the iterator and a initial guest u^0 respecting the boundary condition (i.e. $u^0_{|\Gamma} = g$).

$$\forall i = 1.., N_p: \quad -\Delta u_i^\ell = f, \text{ in } \Omega_i, \quad \text{and } u_i^\ell = u^\ell \text{ on } \Gamma_i \tag{8}$$

$$u^{\ell+1} = \textstyle\sum_{i=1}^{N_p} \pi_i u_i^\ell \tag{9}$$

We never use finite element space associated to the full domain Ω because it to expensive. So we use on each domain i we defined $J_i = \{j \in 1, \ldots, N_p \ / \ \Omega_i \cap \Omega_j \neq \emptyset\}$ and we have

$$(u^{\ell+1})_{|\Omega_i} = \sum_{j \in J_i} (\pi_j u_j^\ell)_{|\Omega_i} \tag{10}$$

We denote $u_{h|i}^\ell$ the restriction of u_h^ℓ on V_{hi}, so the discrete problem on Ω_i of problem (8) is find $u_{hi}^\ell \in V_{hi}$ such that:

$$\forall v_{hi} \in V_{0i} : \int_{\Omega_i} \nabla v_{hi} . \nabla u_{hi}^\ell = \int_{\Omega_i} f v_{hi},$$

$$\forall k \in \mathcal{N}_{hi}^{\Gamma_i} \ : \ \sigma_i^k(u_{hi}^\ell) = \sigma_i^k(u_{h|i}^\ell)$$

where $\mathcal{N}_{hi}^{\Gamma_i}$ is the set of the degree of freedom (Dof) on $\partial\Omega_i$ and σ_i^k the Dof of V_{hi}.

Transfer Part equation(5)

To compute $v_i = (\pi_i u_i)_{|\Omega_i} + \sum_{j \in J_i} (\pi_j u_j)_{|\Omega_i}$ and can be write the freefem++ function Update with asynchronous send/recv (Otherwise dead lock).

```
func bool Update(real[int] &ui, real[int] &vi)
{ int n= jpart.n;
  for(int j=0;j<njpart;++j)  Usend[j][]=sMj[j]*ui;
  mpiRequest[int] rq(n*2);
  for (int j=0;j<n;++j)
        Irecv(processor(jpart[j],comm,rq[j  ]), Ri[j][]);
  for (int j=0;j<n;++j)
        Isend(processor(jpart[j],comm,rq[j+n]), Si[j][]);
  for (int j=0;j<n*2;++j)
        int k= mpiWaitAny(rq);
  vi = Pii*ui;                                    //    set to (πi ui)|Ωi
                              //    apply the unity local partition .
   for(int j=0;j<njpart;++j)
     vi += rMj[j]*Vrecv[j][];                     //    add (πj uj)|Ωi
 return true; }
```

simple parallel GMRES

Finally you can easily accelerate the fixe point algorithm by using a parallel `GMRES` algorithm after the introduction the following affine $\mathcal{S}_i$ operator sub domain Ω_i.

```
func real[int] Si(real[int]& U) {
real[int] V(U.n)    ;  b= onG .* U;
 b  = onG ? b : Bi ;
 V = Ai^-1*b;                                          //      (8)
 Update(V,U);                                          //      (??)
 V -= U;    return V; }
```

Where the parallel `MPIGMRES` or `MPICG` algorithm is to solve $A_i x_i = b_i, i = 1, .., N_p$ by just changing the dot product by reduce the local dot product of all process with the following MPI code:

```
template<class R> R ReduceSum1(R s,MPI_Comm * comm)
{   R r=0;
    MPI_Allreduce( &s, &r, 1 ,MPI_TYPE<R>::TYPE(),
                        MPI_SUM,  *comm );
    return r; }
```

A simple coarse grid is we solve the problem on the coarse grid:

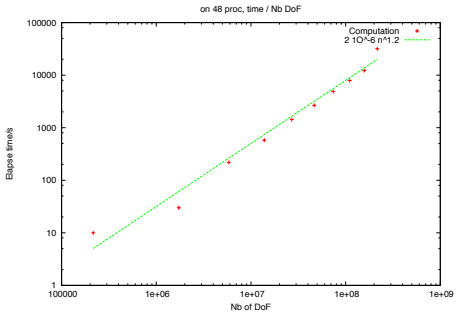
```
func bool  CoarseSolve(real[int]& V,real[int]& U,
             mpiComm& comm)
{
    if(AC.n==0 && mpiRank(comm)==0)          //    first time build
      AC = vPbC(VhC,VhC,solver=sparsesolver);
   real[int] Uc(Rci.n),Bc(Uc.n);
   Uc= Rci*U;                                 //    Fine to Coarse
   mpiReduce(Uc,Bc,processor(0,comm),mpiSUM);
   if(mpiRank(comm)==0)
      Uc = AC^-1*Bc;                          //    solve of proc 0
    broadcast(processor(0,comm),Uc);
   V = Pci*Uc;                                //    Coarse to Fine
}
```

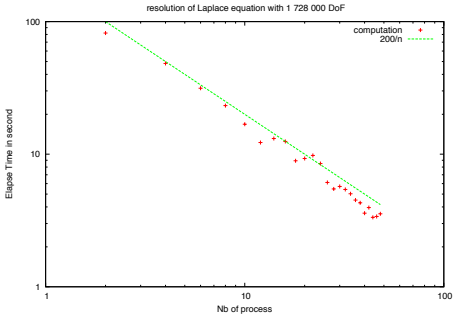
Limitation: if the initial problem, data have oscillation, you must use homogenization technic on coarse problem, or use the F. Nataf and co, preconditionner.

So we finally we get 4 algorithms

1. The basic schwarz algorithm $u^{\ell+1} = \mathcal{S}(u^{\ell})$, where $\mathcal{S}$ is one iteration of schwarz process.
2. Use the GMRES to find u solution of the linear system $\mathcal{S}u - u = 0$.
3. Use the GMRES to solve parallel problem $\mathcal{A}_i u_i = b_i$, $i = 1, \ldots, N_p$, with RAS precondicionneur
4. Use the method with two level precondicionneur RAS and Coarse.

On the SGI UV 100 of the lab:





A Parallel Numerical experiment on laptop

We consider first example in an academic situation to solve Poisson Problem on the cube $\Omega =]0,1[^3$

$$-\Delta u = 1, \text{ in } \Omega; \qquad u = 0, \text{ on } \partial\Omega. \tag{11}$$

With a cartesian meshes $\mathcal{T}_{hn}$ of Ω with $6n^3$ tetrahedron, the coarse mesh is $\mathcal{T}^*_{hm}$, and m is a divisor of n.
We do the validation of the algorithm on a Laptop Intel Core i7 with 4 core at 1.8 Ghz with $4Go$ of RAM DDR3 at 1067 Mhz,

Run:DDM-Schwarz-Lap-2dd.edp
Run:DDM-Schwarz-Lame-2d.edp
Run:DDM-Schwarz-Lame-3d.edp
Run:DDM-Schwarz-Stokes-2d.edp

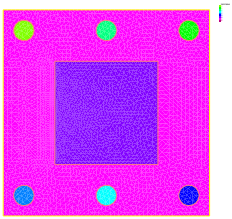
Reader the book of V. Dolean, P. Jolivet and F. Nataf, An Introduction to Domain Decomposition Methods: algorithms, theory and parallel implementation SIAM bookstore , 2015. (see pdf), (see Erratum)

Run:diffusion-3d.edp
Run:diffusion-3d-PETSc.edp
Run:elasticity-3d.edp
Run:elasticity-3d-PETSc.edp
Run:Stokes-3d.edp
Run:Stokes-3d-PETSc.edp
Run:helmholtz-2d.edp
Run:helmholtz-2d-PETSc.edp

Outline

An exercice: Oven problem

Find the power on the 6 resistors of an oven such that the temperature is close as possible to a given temperature in the region 6.
The equation are the stationary Head equation in 2d with classical Fourier boundary condition. the mesh of the domain :



"oven.msh"
Run:mesh-oven.edp

let call the u_p the solution of

$$\begin{aligned} -\nabla.K\nabla u_p &= \sum_{i=0}^{5} p_i * \chi_i \text{ in } \Omega \\ u + K\nabla u_p.n &= 0 \text{ on } \Gamma = \partial\Omega \end{aligned}$$

where χ_i is the characteristics function of the resistance i, $K = 10$ in region 6, $K = 1$ over where.
The problem is find the array p such that

$$p = \operatorname{argmin} \int_{\Omega_6} (u_p - 100)^2 dx$$

build the mesh with multi border trick.

```
  Xh[int]  ur(6);          //   to store the 6 FE. functions Xh
```

FreeFem++ as only linear solver on sparse matrix by default, but in the `lapack` plugin you have access to full matrix solver (see `examples++-load/lapack.edp`) so a way to solve a full matrix problem is for example :

```
  real[int,int] AP(6,6);                    //   a full matrix
  real[int] B(6),PR(6);       //   to array (vector of size 6)

  ... bla bla to compute AP and B

  matrix A=AP;          //    full matrix to sparse of or use of
lapack
  set(A,solver=CG);          //    set linear solver to the C.G.
  PR=A^-1*B;                      //    solve the linear system.
```

The file name of the mesh is `oven.msh`, and the region numbers are 0 to 5 for the resitor, 6 for Ω_6 and 7 for the rest of Ω and the label of Γ is 1.

My solution, build the 6 basics function u_{e_i}

```
int  nbresitor=6;            mesh Th("oven.msh");
real[int] pr(nbresitor+2), K(nbresitor+2);
 K=1;      K[regi]=10;                                  //    def K
int regi=nbresitor, rege=nbresitor+1, lext=1;

macro Grad(u) [dx(u),dy(u)]                             //   EOM
fespace Xh(Th,P2);    Xh u,v; int iter=0;
problem Chaleur(u,v,init=iter)
     =  int2d(Th)( Grad(u)'*Grad(v)* K[region]) +
int1d(Th,lext)(u*v)
     +  int2d(Th)(pr[region]*v) ;

Xh[int]  ur(nbresitor);                    //    to store the 6 u_{e_i}
for(iter=0;iter<nbresitor;++iter)
{  pr=0;pr[iter]=1;
   Chaleur;
   ur[iter][]=u[];
   plot(ur[iter],fill=1,wait=1);  }
```

Computation of the optimal value

```
real[int,int] AP(nbresitor,nbresitor);
real[int] B(nbresitor),PR(nbresitor);

Xh   ui = 100;
for(int i=0;i<nbresitor;++i)
{
    B[i]=int2d(Th,regi)(ur[i]*ui);
    for(int j=0;j<6;++j)
       AP(i,j)= int2d(Th,regi)(ur[i]*ur[j]);
}
  matrix A=AP; set(A,solver=UMFPACK);
  PR=A^-1*B;
  cout « " P R = " « PR « endl;
  u[]=0;
  for (int i=0;i<nbresitor;++i)
    u[] += PR[i]*ur[i][];
```

Run:oven.edp

An exercice: Min surface problem

The geometrical problem: Find a function $u : C^1(\Omega) \mapsto \mathbb{R}$ where u is given on $\Gamma = \partial\Omega$, (e.i. $u_{|\Gamma} = g$) such that the area of the surface S parametrize by $(x, y) \in \Omega \mapsto (x, y, u(x, y))$ is minimal.
So the problem is $\arg\min J(u)$ where

$$\arg\min J(u) = \int_\Omega \left\| \begin{pmatrix} 1 \\ 0 \\ \partial_x u \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \partial_y u \end{pmatrix} \right\| d\Omega = \int_\Omega \sqrt{1 + (\partial_x u)^2 + (\partial_y u)^2} d\Omega$$

So the Euler-Lagrange equation associated to the minimization is:

$$\forall v / v_{|\Gamma} = 0 \quad : \quad DJ(u)v = -\int_\Omega \frac{(\partial_x v \partial_x u + \partial_y v_y \partial_y u)}{\sqrt{1 + (\partial_x u)^2 + (\partial_y u)^2}} d\Omega = 0$$

So find the solution for $\Omega =]0, \pi[^2[$ and $g(x, y) = cos(2 * x) * cos(2 * y)$. by doing fixed point method , by using the Non Linear Conjugate gradient `NLCG` like in the example: `algo.edp` in `examples++-tutorial`, `IPOPT` interface, or Newton method.

Fixed Point algorithm (Trivial Algorithm)

```
int nn=10;
mesh Th=square(nn,nn);
fespace Vh(Th,P1);  Vh u=0,up,v;// up previous mvalue
func g = cos(pi*x)*cos(2*pi*y);
for(int i=0; i< 100; ++i)
{
    up[]=u[];// set the previous value by copiing the arroy of dof
    solve Pb(u,v) = int2d(Th)( (dx(u)*dx(v) + dy(u)*dy(v) )
                      /sqrt( 1+ (dx(up)*dx(up) + dy(up)*dy(up) )))
    +on(1,2,3,4,u=g);
    real area = int2d(Th)( sqrt(1+ (dx(u)*dx(u) + dy(u)*dy(u) )));
    real err= sqrt(int2d(Th)( (u-up)^2)); // Error L2
    cout << i << " surface = "<< area<< " err L2 = " << err <<endl
        ;
    plot(u, dim=3,fill=3, cmm=i+" area="+area+" err= "+err);
    if(err<1e-5) break;
}
```

Run:minimal-surf-fixed-Point.edp

Example of use of NLCG function:

```
func real J(real[int] & xx) // the functional to minimized
    {  real s=0;
      ...// add code to copy xx array of finite element function
      return s; }
func real[int] DJ(real[int] &xx)// the grad of functional
    { .  ...// add code to copy xx array of finite element
       function
      return xx; }; // return of an existing variable ok
  ...
NLCG(DJ,xx,eps=1.e-6,nbiter=20,precon=matId);
```

Useful operator on array real[int]

```
real[int]  a(10),b(10);
...
a =  b ? 1. : 0 ; // a[i] = 1 if b[i]; else a[i]=0.  ∀i
```

To see the 3D plot of the surface

```
plot(u,dim=3);
```

a solution with NLCG: first functional

```
func g=cos(2*x)*cos(2*y); // valeur au bord
mesh Th=square(20,20,[x*pi,y*pi]);  // mesh definition of Ω
fespace Vh(Th,P1);


func real J(real[int] & xx)// the functionnal to minimise
  { Vh u;u[]=xx; // to set FE.function u from xx array
    return int2d(Th)( sqrt(1 +dx(u)*dx(u) + dy(u)*dy(u) ) ) ; }


func real[int] dJ(real[int] & xx)// the grad of the J
  { Vh u;u[]=xx; // to set FE. function u from xx array
    varf au(uh,vh) = int2d(Th)( ( dx(u)*dx(vh) + dy(u)*dy(vh) )
             / sqrt(1. +dx(u)*dx(u) + dy(u)*dy(u) )      )
             + on(1,2,3,4,uh=0);
    return xx= au(0,Vh);  } // warning no return of local array
```

Solution 1:

```
Vh u=G;
verbosity=5;                              //    to see the residual
int conv=NLCG(dJ,u[],nbiter=500,eps=1e-5);
cout « " the  surface =" « J(u[])  « endl;
                                          //    so see the surface un 3D
plot(u,dim=3);
```

Run:minimal-surf.edp

Heat equation with thermic resistance

let Ω be a domain cut with internal boundary Γ_0 in 2 sub-domain Ω_i,$(i = 1, 2)$
We have Heat equation (Poisson) on Ω , but on Γ_0 we have a jump $[u]$ on the temperature u proportional to the temperature flux which is continue
So the equation to solve is:
Find u such that $u_{|\Omega_i} \in H(\Omega_i)$ for $i = 1, 2$ and

$$-\nabla\kappa\nabla u = f_i, \quad \text{in } \Omega_i$$

$$\alpha[u] - \kappa\nabla u.n = 0, \quad [\kappa\nabla u.n] = 0, \quad \text{on } \Gamma_0$$

+ external boundary condition on $\partial\Omega$.
For the test you can use:
$L = 3$, $\Omega =]-L, L[\times]0, 1[$, $\Gamma_0 = \{sin(\pi y)/5, y \in [0, 1]\}$, take $\kappa = i$ in Ω_i.
The external boundary condition on $\partial\Omega$ are: $\kappa\nabla u.n = 0$ on upper and lower boundary , $u = 0$ at the left part, $u = 1$ at the right part.

Method 1: Solve 2 coupled problems and use the block matrix tools to defined the linear system of the problem.

An exercice: Navier-Stokes

Try to make the 2d benchmark of :
http://www.mathematik.tu-dortmund.de/lsiii/cms/papers/SchaeferTurek1996.pdf
The mesh can be set:

```
int n=15;                                          //    parameter ...
real D=0.1, H=0.41;
real cx0 = 0.2, cy0 = 0.2;                        //    center of cyl.
real xa = 0.15, ya=0.2, xe = 0.25,ye =0.2;              //    point for
pressure..

border fr1(t=0,2.2){x=t; y=0; label=1;}
border fr2(t=0,H){x=2.2; y=t; label=2;}
border fr3(t=2.2,0){x=t; y=H; label=1;}
border fr4(t=H,0){x=0; y=t; label=1;}
border fr5(t=2*pi,0){x=cx0+D*sin(t)/2; y=cy0+D*cos(t)/2; label=3;}

mesh Th=buildmesh(fr1(5*n)+fr2(n)+fr3(5*n)+fr4(n)+fr5(-n*3));
plot(Th, wait=1);
```

Outline

To solve $F(u) = 0$ the Newton's algorithm is

1. u^0 a initial guest with correct B.C.
2. do
   1. find w^n solution of $DF(u^n)w^n = F(u^n)$, with 0 B.C.
   2. $u^{n+1} = u^n - w^n$
   3. if($||w^n|| < \varepsilon$) break;

The Optimize Newton Method if $F = C + L + N$, where C is the constant part,L is Linear part and N is Non linear part of F. we have $DF = L + DN$ and $DF(u^n)u^{n+1} = DF(u^n)u^n - F(u^n) = DN(u^n)u^n - N(u^n) - C$. So the change in algorithm are:

1. u^0 a initial guest
2. do
   1. find u^{n+1} solution of $DF(u^n)u^{n+1} = DN(u^n)u^n - N(u^n) - C$
   2. if($||u^{n+1} - u^n|| < \varepsilon$) break;

The main advantage is now you can use classical B.C. on the tangent problem, we compute directly the solution not and increment.

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- Hyper elasticity equation
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For Navier Stokes problem the Newtow algorithm is: $\forall v, q$,

$$F(u,p) = \int_\Omega (u.\nabla)u.v + u.v + \nu \nabla u : \nabla v - q\nabla.u - p\nabla.v + BC$$

$$\begin{aligned} DF(u,p)(w,w_p) &= \int_\Omega (w.\nabla)u.v + (u.\nabla)w.v \\ &\quad + \int_\Omega \nu \nabla w : \nabla v - q\nabla.w - p_w\nabla.v + BC0 \end{aligned}$$

Run:cavityNewton.edp

Run:NSNewtonCyl-100-mpi.edp

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = 0, \quad \nabla \cdot u = 0$$

with the same boundary conditions and with initial conditions $u = 0$.
This is implemented by using the interpolation operator for the term $\frac{\partial u}{\partial t} + (u \cdot \nabla)u$, giving a discretization in time

$$\begin{array}{rl} \frac{1}{\tau}(u^{n+1} - u^n \circ X^n) - \nu \Delta u^{n+1} + \nabla p^{n+1} & = 0, \\ \nabla \cdot u^{n+1} & = 0 \end{array} \tag{12}$$

The term $X^n(x) \approx x - \tau u^n(x)$ will be computed by the interpolation operator or convect operator.
Or better we use an order 2 schema, BDF1

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u \approx \frac{(3u^{n+1} - 4u^n \circ X_1^n + u^{n-1} \circ X_2^n)}{2\tau}$$

with $u^* = 2u^n - u^{n-1}$, and $X_1^n(x) \approx x - \tau u^*(x), X_2^n(x) \approx x - 2\tau u^*(x)$
Run:NSCaraCyl-100-mpi.edp Run:NSUzawaCahouetChabart-3d-aorte.edp

The generalise Stokes problem is find u, p solution of

$$Au + Bp = f, \quad {}^tBu = 0$$

with $A \equiv (\alpha Id + \nu\Delta)$ and $B \equiv \nabla$. remark,if A est symmetric positive the you can use a conjugate gradient to solve the the following problem

$${}^tBA^{-1}B\,p = {}^tBA^{-1}f$$

Now in a periodic domain, all differential operators commute and the Uzawa algorithm comes to solving the linear operator $-\nabla.((\alpha Id - \nu\Delta)^{-1}\nabla$, where Id is the identity operator. So the preconditioner suggested is $-\alpha\Delta^{-1} + \nu Id$.
the term $\frac{\partial u}{\partial t} + (u \cdot \nabla)u$ is the total derivative and discretization in time

$$\begin{array}{rl} \frac{1}{\tau}(u^{n+1} - u^n \circ X^n) - \nu\Delta u^{n+1} + \nabla p^{n+1} & = 0, \\ \nabla \cdot u^{n+1} & = 0 \end{array} \tag{13}$$

The term $X^n(x) \approx x - \tau u^n(x)$ will be computed with convect operator.
Run:NSUzawaCahouetChabart.edp Run:NSUzawaCahouetChabart-3d-aorte.edp

```
 real alpha =1./dt;
 varf vNS([uu1,uu2,uu3,p],[v1,v2,v3,q]) =
  int3d(Th)( alpha*(uu1*v1+uu2*v2+uu3*v3)
  + nu*(Grad(uu1)'*Grad(v1)+Grad(uu2)'*Grad(v2) +Grad(uu3)'*Grad(v3))
  - div(uu1,uu2,uu3)*q - div(v1,v2,v3)*p + 1e-10*q*p )
  + on(1,2,3,4,5,uu1=0,uu2=0,uu3=0)
  + on(6,uu1=4*(1-x)*(x)*(y)*(1-y),uu2=0,uu3=0)
  + int3d(Th)( alpha*(
     u1(X1,X2,X3)*v1  +   u2(X1,X2,X3)*v2 +   u3(X1,X2,X3)*v3 ));
A = vNS(VVh,VVh);   set(A,solver=UMFPACK);          //   build and factorize matrix
real t=0;
for(int i=0;i<50;++i)
  { t += dt;  X1[]=XYZ[]-u1[]*dt;                     //    set χ=[X1,X2,X3] vector
    b=vNS(0,VVh);                                            //    build NS rhs
    u1[]= A^-1 * b;                                  //    solve the linear systeme
    ux= u1(x,0.5,y);  uz= u3(x,0.5,y);  p2= p(x,0.5,y);
    plot([ux,uz],p2,cmm=" cut y = 0.5, time ="+t,wait=0);  }
```

Run:NSI3d.edp

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Variational Inequality

To solve just make a change of variable $u = u^+ - u^-, u > 0$ and $v = u^+ + u^-$, and we get a classical VI problem on u and and the Poisson on v.

So we can use the algorithm of Primal-Dual Active set strategy as a semi smooth Newton Method HinterMuller , K. Ito, K. Kunisch SIAM J. Optim. V 13, I 3, 2002.

In this case , we just do all implementation by hand in FreeFem++ language

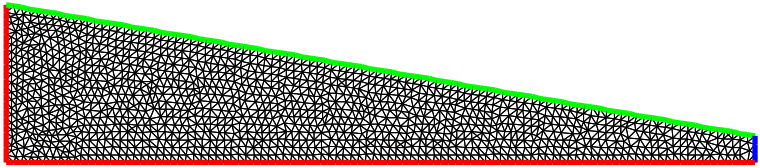
Run:VI-2-membrane-adap.edp

A Free Boundary problem , (phreatic water)

Let a trapezoidal domain Ω defined in FreeFem++:

```
real L=10;                                                    //    Width
real h=2.1;                                                 //    Left height
real h1=0.35;                                              //    Right height
border a(t=0,L){x=t;y=0;label=1;}; //    impermeable Γ_a
border b(t=0,h1){x=L;y=t;label=2;};     //    the source Γ_b
border f(t=L,0){x=t;y=t*(h1-h)/L+h;label=3;}; //    Γ_f
border d(t=h,0){x=0;y=t;label=4;};     //    Left impermeable Γ_d
int n=10;
mesh Th=buildmesh (a(L*n)+b(h1*n)+f(sqrt(L^2+(h-h1)^2)*n)+d(h*n));
plot(Th,ps="dTh.eps");
```

The initial mesh



The problem is: find p and Ω such that:

$$\begin{cases} -\Delta p = 0 & \text{in } \Omega \\ p = y & \text{on } \Gamma_b \\ \dfrac{\partial p}{\partial n} = 0 & \text{on } \Gamma_d \cup \Gamma_a \\ \dfrac{\partial p}{\partial n} = \frac{q}{K} n_x & \text{on } \Gamma_f \quad (Neumann) \\ p = y & \text{on } \Gamma_f \quad (Dirichlet) \end{cases}$$

where the input water flux is $q = 0.02$, and $K = 0.5$. The velocity u of the water is given by $u = -\nabla p$.

We use the following fix point method: (*with bad main B.C. Run:freeboundaryPB.edp*) let be, $k = 0$, $\Omega^k = \Omega$. First step, we forgot the Neumann BC and we solve the problem: Find p in $V = H^1(\Omega^k)$, such $p = y$ on Γ_b^k et on Γ_f^k

$$\int_{\Omega^k} \nabla p \nabla p' = 0, \quad \forall p' \in V \text{ with } p' = 0 \text{ on } \Gamma_b^k \cup \Gamma_f^k$$

With the residual of the Neumann boundary condition we build a domain transformation $\mathcal{F}(x, y) = [x, y - v(x)]$ where v is solution of: $v \in V$, such than $v = 0$ on Γ_a^k (bottom)

$$\int_{\Omega^k} \nabla v \nabla v' = \int_{\Gamma_f^k} (\frac{\partial p}{\partial n} - \frac{q}{K} n_x) v', \quad \forall v' \in V \text{ with } v' = 0 \text{ sur } \Gamma_a^k$$

remark: we can use the previous equation to evaluate

$$\int_{\Gamma^k} \frac{\partial p}{\partial n} v' = -\int_{\Omega^k} \nabla p \nabla v'$$

Implementation

The new domain is: $\Omega^{k+1} = \mathcal{F}(\Omega^k)$ Warning if is the movement is too large we can have triangle overlapping.

```
problem Pp(p,pp,solver=CG) =
    int2d(Th)( dx(p)*dx(pp)+dy(p)*dy(pp))
  + on(b,f,p=y) ;
problem Pv(v,vv,solver=CG) =
    int2d(Th)( dx(v)*dx(vv)+dy(v)*dy(vv))
  + on (a, v=0)
  + int1d(Th,f)(vv*
       ((Q/K)*N.y-(dx(p)*N.x+dy(p)*N.y)));
while(errv>1e-6)
{   j++;  Pp;  Pv;    errv=int1d(Th,f)(v*v);
     coef = 1;
//     Here french cooking if overlapping see the example
    Th=movemesh(Th,[x,y-coef*v]);                    //    deformation
}
```

Run:freeboundary.edp

Bose Einstein Condensate

Just a direct use of `Ipopt` interface (2day of works)
The problem is find a complex field u on domain $\mathcal{D}$ such that:

$$u = \underset{||u||=1}{\operatorname{argmin}} \int_{\mathcal{D}} \frac{1}{2}|\nabla u|^2 + V_{trap}|u|^2 + \frac{g}{2}|u|^4 - \Omega i\overline{u}\left(\begin{pmatrix} -y \\ x \end{pmatrix}.\nabla\right)u$$

to code that in FreeFem++
use

- Ipopt interface (`https://projects.coin-or.org/Ipopt`)
- Adaptation de maillage

Run:BEC.edp

Hyper elasticity equation

The Hyper elasticity problem is the minimization of the energy $W(I_1, I_2, I_3)$ where I_1, I_2, I_3 are the 3 invariants. For example The Ciarlet Geymonat energy model is

$$W = \int_\Omega \kappa_1(J_1 - 3) + \kappa_2(J_2 - 3) + \kappa(J - 1) - \kappa \ln(J)$$

where $J_1 = I_1 I_3^{-\frac{1}{3}}$, $J_2 = I_2 I_3^{-\frac{2}{3}}$,$J = I_3^{\frac{1}{2}}$,
let u the deplacement, when

- $F = I_d + \nabla u$
- $C = {}^tFF$
- $I_1 = \mathrm{tr}(C)$
- $I_2 = \frac{1}{2}(\mathrm{tr}(C)^2 - \mathrm{tr}(C^2))$
- $I_3 = \det(C)$

The problem is find

$$u = \underset{u}{\mathtt{argmin}}\, W(I_1, I_2, I_3)$$

Hyper elasticity equation

```
fespace Wh(Th,[P2,P2]);
                                                //    methode de Newton ..
Wh [d1,d2]=[0,0];
Wh [w1,w2],[v1,v2];
for(int i=0;i<Nnewton;++i)
 {
    solve dWW([w1,w2],[v1,v2]) =
          int2d(Th)( ddW2d([d1,d2],[w1,w2],[v1,v2]) )
        - int2d(Th)( dW2d([d1,d2],[v1,v2]) -[v1,v2]'*[f1,f2] )
        + on(1,w1=0,w2=0);

     d1[] -= w1[];
     real err = w1[].linfty;
     if(err< epsNewton) break;
 }
```

Run:Hyper-Elasticity-2d.edp see:ElasticLaw2d.idp see:CiarletGemona.idp

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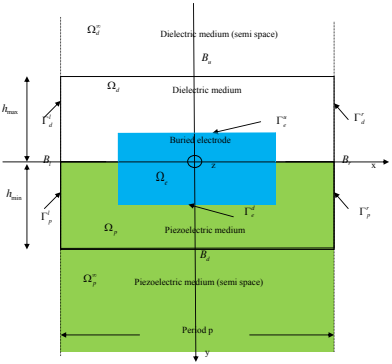
Periodic Surface Acoustic Waves Transducer Analysis

A true industrial numerical problem (2d) :

1. 3 EDP: Dielectric, Elasticity, Piezoelectric , Linear, harmonic approximation
2. α-periodic B.C. (New simple idea)
3. semi-infinite Dielectric domain (classical Fourier/Floquet transforme)
4. semi-infinite Piezoelectric domain (Hard)

In 9 month, we build with P. Ventura (100%, me 10%) a numerical simulator form scratch (8 months for the validation), The only thing to add to freefem++ is a interface with `lapack` to compute eigenvector of full 8×8 matrix.

A good message : Les calculs des paramètres physiques des transducteurs dans la bande d'arrêt et les évaluations de capacité statiques sont très satisfaisants par rapport aux résultats expérimentaux !



In the **dielectric medium** Ω_d,

$$\boldsymbol{D} = \varepsilon_d \boldsymbol{E} \tag{14}$$

In the **elastic medium** Ω_e

$$\boldsymbol{T} = \boldsymbol{C}_e : \boldsymbol{S} \tag{15}$$

With $\boldsymbol{C}_e$ is the elastic tensor for the elastic metallic domain. In the **piezoelectric domain**

$$\begin{cases} \boldsymbol{T} = \boldsymbol{C}_p^{\boldsymbol{E}} : \boldsymbol{S} - \boldsymbol{e}\boldsymbol{E} \\ \boldsymbol{D} = \boldsymbol{e}^T \boldsymbol{S} + \varepsilon^{\boldsymbol{S}} \boldsymbol{E} \end{cases} \tag{16}$$

The material domain Ω_m obeys Newton's second law:

$$\nabla \cdot \boldsymbol{T} = \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \tag{17}$$

The quasi static Maxwell's equation is assumed for the whole domain Ω :

$$\nabla \cdot \boldsymbol{D} = 0 \tag{18}$$

By using the divergence relationship and the Green's integral formula, it results the general weak formulation of the periodic γ-harmonic problem:

Find $(\boldsymbol{u}, \phi)$ in $V_\gamma^3(\Omega_m) \times V_\gamma(\Omega)$(verifying the equipotential boundary condition in the electrode), such that for all $(\boldsymbol{v}, \psi)$ in $V_\gamma^3(\Omega_m) \times V_\gamma^3(\Omega)$, satisfying the zero equi potential boundary condition), we have:

$$\int_{\Omega_m} \overline{\boldsymbol{S}(\boldsymbol{v})} : \boldsymbol{T}(\boldsymbol{u})\ d\Omega - \omega^2 \int_{\Omega_m} \rho\, \overline{\boldsymbol{v}} \cdot \boldsymbol{u}\, d\Omega - \int_{\Omega} \overline{\boldsymbol{E}(\psi)} \cdot (\boldsymbol{e}\boldsymbol{S}(\boldsymbol{u}) + \varepsilon \boldsymbol{E}(\phi))\ d\Omega - \int_{\Gamma_d} \overline{\boldsymbol{v}} \cdot (\boldsymbol{T}(\boldsymbol{u}) \cdot \boldsymbol{n})\, d\Gamma - \int_{\Gamma_u \cup \Gamma_d} \overline{\psi}\, (\boldsymbol{D}(\phi) \cdot \boldsymbol{n})\, d\Gamma = 0 \quad (19)$$

With, $V_\gamma(\Omega)$is the mathematical space of $L^2(\Omega)$ with the derivative in $L^2(\Omega)$ satisfying γ-harmonic periodic boundary conditions.

Let us first define $\varphi_\gamma(x) = e^{-j2\pi\gamma\frac{x}{p}}$, $\varphi_\gamma(x)$ is a γ-harmonic periodic function satisfying:

$$\varphi_\gamma(x+p) = e^{-j2\pi\gamma}\varphi_\gamma(x) \tag{20}$$

We just do the change of variable:

$$\begin{cases} \boldsymbol{u}(x,y) = \varphi_\gamma(x)\,\boldsymbol{u}^\diamond(x,y) \\ \phi(x,y) = \varphi_\gamma(x)\,\phi^\diamond(x,y) \end{cases} \tag{21}$$

Where $\boldsymbol{u}^\diamond(x)$ and $\phi^\diamond(x)$ are p-periodic functions.
The main idea is to define a new differential operator ∇_γ by:

$$\nabla_\gamma \boldsymbol{u}^\diamond = \nabla(\varphi_\gamma \boldsymbol{u}^\diamond) = \varphi_\gamma \nabla \boldsymbol{u}^\diamond + \varphi_\gamma{}' \boldsymbol{u}^\diamond \tag{22}$$

Because the physical fields $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{T}$, and $\mathbf{S}$ are expressed using partial derivative of $\boldsymbol{u}$, and, ϕ, it is possible to define the operators $\boldsymbol{E}_\gamma(\phi^\diamond) = \boldsymbol{E}(\varphi_\gamma\phi^\diamond)$, $\boldsymbol{D}_\gamma(\phi^\diamond) = \boldsymbol{D}(\varphi_\gamma\phi^\diamond)$, $\boldsymbol{T}_\gamma(\boldsymbol{u}^\diamond) = \boldsymbol{T}(\varphi_\gamma\boldsymbol{u}^\diamond)$, ...

Find $(\boldsymbol{u}^{\diamond}, \phi^{\diamond})$in $V_1^3(\Omega_m) \times V_1(\Omega)$ (verifying the equipotential boundary condition), such that for all $(\boldsymbol{v}^{\diamond}, \psi^{\diamond})$in $V_1^3(\Omega_m) \times V_1^3(\Omega)$, satisfying the zero equipotential boundary condition), we have:

$$\int_{\Omega_m} \overline{\boldsymbol{S}_\gamma(\boldsymbol{v}^{\diamond})} : \boldsymbol{T}_\gamma(\boldsymbol{u}^{\diamond})\ d\Omega - \omega^2 \int_{\Omega_m} \rho\, \overline{\boldsymbol{v}^{\diamond}} \cdot \overline{\boldsymbol{u}^{\diamond}}\, d\Omega - \int_{\Omega} \overline{\boldsymbol{E}_\gamma(\psi^{\diamond})} \cdot (\boldsymbol{e}\boldsymbol{S}_\gamma(\boldsymbol{u}^{\diamond}) + \varepsilon \boldsymbol{E}_\gamma(\phi^{\diamond}))\, d\Omega - \int_{\Gamma_d} \overline{\varphi_\gamma \boldsymbol{v}^{\diamond}} \cdot (\boldsymbol{T}_\gamma(\boldsymbol{u}^{\diamond})\,\boldsymbol{n})\, d\Gamma - \int_{\Gamma_u \cup \Gamma_d} \overline{\varphi_\gamma \psi^{\diamond}} (\boldsymbol{D}_\gamma(\phi^{\diamond}) \cdot \boldsymbol{n})\, d\Gamma = 0 \quad (23)$$

Where, $V_1(\Omega)$ is the mathematical space of $L^2(\Omega)$ with derivative in $L^2(\Omega)$ satisfying p-periodic boundary conditions.

We have to modelized the following term:

$$-\int_{\Gamma_d} \overline{\varphi_\gamma \boldsymbol{v}^\diamond} \cdot (\boldsymbol{T}_\gamma (\boldsymbol{u}^\diamond)\, \boldsymbol{n})\, d\Gamma - \int_{\Gamma_u \cup \Gamma_d} \overline{\varphi_\gamma \psi^\diamond} (\boldsymbol{D}_\gamma (\phi^\diamond) \cdot \boldsymbol{n})\, d\Gamma, \tag{24}$$

also called border terms.

First from (24), let us look at the boundary integral A_{Γ_u}, at the interface Γ_u of the semi-infinite dielectric semi-space.

$$A_{\Gamma_u} = \int_{\Gamma_u} \overline{\varphi_\gamma \psi^\diamond}\, \boldsymbol{D}_\gamma (\phi^\diamond) \cdot \boldsymbol{n}\, d\Gamma \tag{25}$$

The elementary coefficients to compute are for all finite element basic functions $w^{\diamond}{}_i$ introduce in (**??**), only for node $i \in \mathcal{N}_u$ the set of node on Γ_u.

$$\forall (i,j) \in \mathcal{N}_u^2, \qquad (A_{\Gamma_u})_{ij} = -\varepsilon_d \int_{\Gamma_u} \overline{\varphi_\gamma w^{\diamond}{}_i}\, \partial_n \left(\varphi_\gamma w^{\diamond}{}_j\right) d\Gamma \tag{26}$$

According [2], it is possible to expand, at the interface Γ_u the γ-harmonic periodic $\varphi_\gamma w^{\diamond}{}_j$ into the Floquet's basis function

$$f_m(x,y) = e^{-2\pi(j(m+\gamma)x - |m+\gamma|((y-y_u))/p} = \varphi_\gamma(x)\, f^{\diamond}{}_m(x,y)\,. \tag{27}$$

where y_u is the y coordinate of Γ_u.

$$\varphi_\gamma(x)\, w^{\diamond}{}_j(x,y) = \sum_{m=-\infty}^{+\infty} c_m^j f_m(x,y) \tag{28}$$

With the $L^2(\Gamma_u)$ orthogonality of Fourier's basis $f^{\diamond}{}_m$, we have:

$$c_m^j = \frac{1}{p} \int_{\Gamma_u} w^{\diamond}{}_j \overline{f^{\diamond}}_m d\Gamma, \tag{29}$$

and on Γ_u the normal derivative $\partial_n f_m(x,y) = \partial_y f_m(x,y)$ satisfies:

$$\partial_n f_m = -g_m f_m, \qquad \text{with } g_m = \frac{2\pi}{p}\,|\gamma + m| \tag{30}$$

Leading to the relationship:

$$\partial_n \left(\varphi_\gamma {w^\diamond}_j\right) = -\sum_{m=-\infty}^{+\infty} c_m^j g_m f_m \tag{31}$$

Finally the term $(A_{\Gamma_u})_{ij}$ is

$$(A_{\Gamma_u})_{ij} = \frac{\varepsilon_d}{p} \sum_{m=\infty}^{+\infty} g_m \int_{\Gamma_u} \overline{w^\diamond}_i {f^\diamond}_m d\Gamma \int_{\Gamma_u} {w^\diamond}_j \overline{f^\diamond}_m d\Gamma \tag{32}$$

Run:BEM.edp

8 No Linear Problem
- Newton Method
- Navier-Stokes
- Variational Inequality
- Ground water
- Bose Einstein Condensate
- Hyper elasticity equation
- Periodic Surface Acoustic Waves Transducer Analysis
- Phase change with Natural Convection

The starting point of the problem is Brainstorming session (part I) of the third FreeFem++ days in december 2011, this is almost the Orange Problem is describe in web page `http://www.ljll.math.upmc.fr/~hecht/ftp/ff++days/2011/Orange-problem.pdf`. The coupling of natural convection modeled by the Boussinesq approximation and liquid to solid phase change in $\Omega =]0,1[^2$, No slip condition for the fluid are applied at the boundary and adiabatic condition on upper and lower boundary and given temperature θ_r (resp θ_l) at the right and left boundaries.
The model is: find the field : the velocity $\boldsymbol{u} = (u_1, u_2)$, the pressure p and temperature θ :

$$\left\{ \begin{array}{rll} \boldsymbol{u} & \text{given} & \text{in } \Omega_s \\ \partial_t \boldsymbol{u} + (\boldsymbol{u}\nabla)\boldsymbol{u} + \nabla.\mu\nabla\boldsymbol{u} + \nabla p & = -c_T \boldsymbol{e}_2 & \text{in } \Omega_f \\ \nabla.\boldsymbol{u} & = 0 & \text{in } \Omega_f \\ \partial_t \theta + (\boldsymbol{u}\nabla)\theta + \nabla.k_T\nabla\theta & = \partial_t S(T) & \text{in } \Omega \end{array} \right. \tag{33}$$

Where Ω_f is the fluid domain and the solid domain is $\Omega_s = \Omega \setminus \Omega_f$.

Phase change with Natural Convection

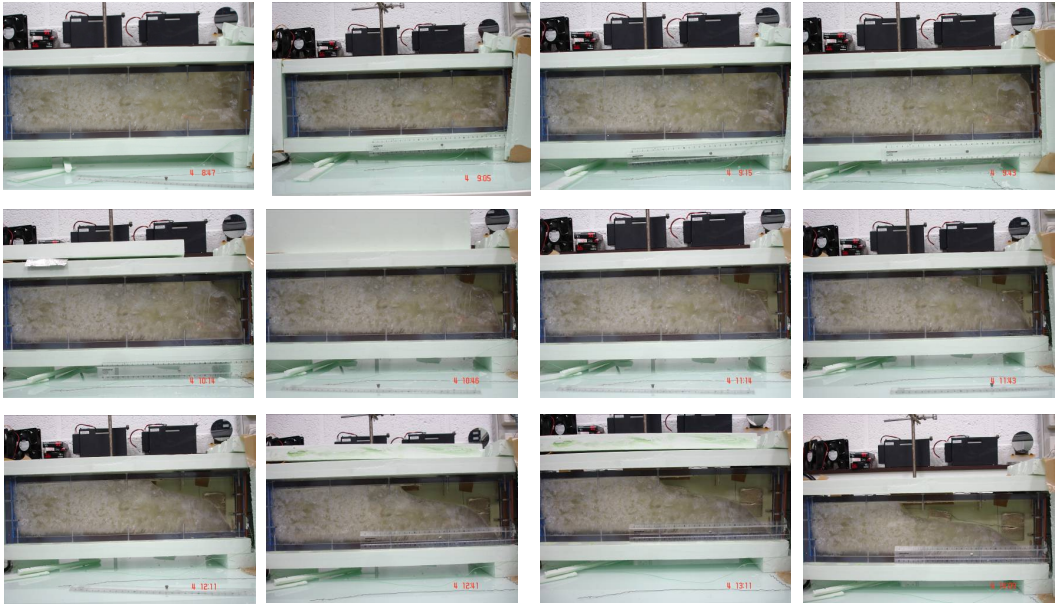
The enthalpy of the change of phase is given by the function S; μ is the relative viscosity, k_T the thermal diffusivity.
In $\Omega_f = \{x \in \Omega; \theta > \theta_f\}$, with θ_m the melting temperature the solid has melt.
We modeled, the solid phase as a fluid with huge viscosity, so :

$$\mu = \begin{cases} \theta < \theta_f & \sim & 10^6 \\ \theta \geq \theta_m & \sim & \frac{1}{\mathrm{Re}} \end{cases},$$

The Stefan enthalpy S_c with defined by $S_c(\theta) = H(\theta)/S_{th}$ where S_{the} is the stefan number, and H is the Heaviside function with use the following smooth the enthalpy:

$$S(\theta) = \frac{\tanh(50(\theta - \theta_m)))}{2S_{te}}.$$

The true device



We apply a fixed point algorithm for the phase change part (the domain Ω_f is fixed at each iteration) and a full no-linear Euler implicit scheme with a fixed domain for the rest. We use a Newton method to solve the non-linearity.

- if we don't make mesh adaptation, the Newton method do not converge
- if we use explicit method diverge too,
- if we implicit the dependance in Ω_s the method also diverge.

This is a really difficult problem.

The finite element space to approximate $u1, u2, p, \theta$ is defined by

```
fespace Wh(Th,[P2,P2,P1,P1]);
```

We do mesh adaptation a each time step, with the following code:

```
Ph ph = S(T), pph=S(Tp);
Th= adaptmesh(Th,T,Tp,ph,pph,[u1,u2],err=errh,
              hmax=hmax,hmin=hmax/100,ratio = 1.2);
```

This mean, we adapt with all variable plus the 2 melting phase a time $n+1$ and n and we smooth the metric with a ratio of 1.2 to account for the movement of the melting front.

the fixed point are implemented as follows

```
real err=1e100,errp ;
 for(int kk=0;kk<2;++kk) // 2 step of fixe point on Ωs
 { nu = nuT; // recompute the viscosity in Ωs,Ωf
   for(int niter=0;niter<20; ++ niter) // newton loop
    { BoussinesqNL;
      err = u1w[].linfty;
      cout << niter << " err NL " << err <<endl;
      u1[] -= u1w[];
      if(err < tolNewton) break;  }// convergence ..
  }
```

The linearized problem

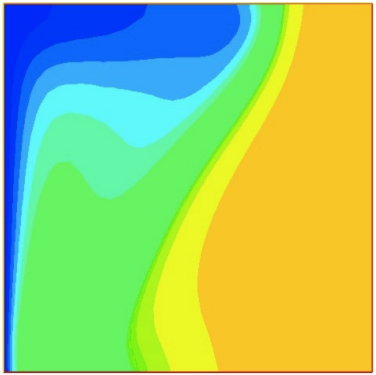
```
problem BoussinesqNL([u1w,u2w,pw,Tw],[v1,v2,q,TT])
= int2d(Th) (
      [u1w,u2w,Tw]'*[v1,v2,TT]*cdt
      + UgradV(u1,u2,u1w,u2w,Tw)' * [v1,v2,TT]
      + UgradV(u1w,u2w,u1,u2,T)' * [v1,v2,TT]
      + ( Grad(u1w,u2w)'*Grad(v1,v2)) * nu
      + ( Grad(u1,u2)'*Grad(v1,v2)) * dnu* Tw
      + cmT*Tw*v2  + grad(Tw)'*grad(TT)*kT
      - div(u1w,u2w)*q -div(v1,v2)*pw - eps*pw*q
      + dS(T)*Tw*TT*cdt )
   - int2d(Th)(
      [u1,u2,T]'*[v1,v2,TT]*cdt
      + UgradV(u1,u2,u1,u2,T)' * [v1,v2,TT]
      + ( Grad(u1,u2)'*Grad(v1,v2)) * nu
      + cmT*T*v2 - eps*p*q  +  grad(T)'*grad(TT)*kT
      - div(u1,u2)*q -div(v1,v2)*p
      +  S(T)*TT*cdt - [u1p,u2p,Tp]'*[v1,v2,TT]*cdt
      -  S(Tp)*cdt*TT)
 + on(1,2,3,4, u1w=0,u2w=0)+on(2,Tw=0)+on(4,Tw=0) ;
```

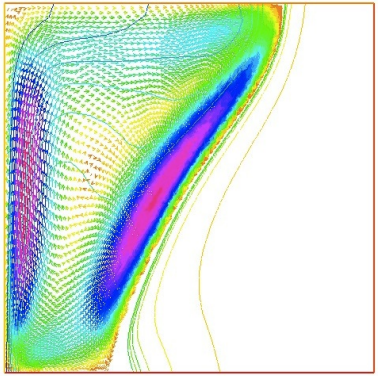
The parameters of the computation

take case 2 from
Shimin Wang, Amir Faghri, and Theodore L. Bergman. A comprehensive numerical model for melting with natural convection. *International Journal of Heat and Mass Transfer*, January 2010.

$\theta_m = 0$, $\mathrm{Re} = 1$, $S_{te} = 0.045$, $P_r = 56.2$, $R_a = 3.27\ 10^5$, $\theta_l = 1, \theta_r = -0.1$ so in this case $\mathrm{cmT} = c_T = -R_a/P_r$, $\mathrm{kT} = k_T = 1/P_r$, $\mathrm{eps} = 10^{-6}$, time step $\delta t = 10^{-1}$, $\mathrm{cdt} = 1/\delta t$, at time $t = 80$ and we get a good agreement with the article.

Phase change with Natural Convection





So now, a real problem, get the physical parameter of the real experiment.
Run:Orange-Newton.edp

Outline

Read the page `http://www3.freefem.org/ff++/windows.php`

Dynamics Load facility

Or How to add your C++ function in FreeFem++.
First, like in cooking, the first true difficulty is how to use the kitchen.
I suppose you can compile the first example for the examples++-load

```
numermac11:FH-Seville hecht#  ff-c++ myppm2rnm.cpp
...
```

add tools to read pgm image

The interesting code

```
#include "ff++.hpp"
typedef KNM<double> * pRnm;                                   //    real[int,int] array variable type
typedef KN<double> * pRn;                                     //    real[int] array variable type
typedef string ** string;                                     //    the ff++ string variable type
                                                              //    the function to read image
pRnm  read_image( pRnm  const & a,const pstring & b);
                                                      //    the function to set 2d array from 1d array
pRn  seta( pRn  const & a,const pRnm & b)
{ *a=*b;
  KN_<double> aa=*a;
  return a;}

void Init(){                                                  //    the link with FreeFem++
//    add ff++ operator "<-" constructor of real[int,int] form a string
TheOperators->Add("<-",
       new OneOperator2_<KNM<double> *,KNM<double> *,string*>(&read_image) );
//    add ff++ an affection "=" of real[int] form a real[int,int]
TheOperators->Add("=",
       new OneOperator2_<KN<double> *,KN<double> *,KNM<double>* >(seta));
}
LOADFUNC(Init); //    to call Init Function at load time
```

Remark, `TheOperators` is the ff++ variable to store all world operator, `Global` is to store function.

How to extend

A true simple example How to make dynamic gnuplot
Idea: use a pipe to speak with gnuplot the C code :

```
FILE * gp = popen("gnuplot");
 for( double f=0; f < 3.14; f += 0.01)
   fprintf(gp,"plot sin(x+%f)\n",f);
```

To do this add a new constructor of ofstream in freefem++

A way to pass info between to code

Make a pipe, under unix (with a use of pstream tools)

```
#include "ff++.hpp"
#include "pstream.h"
typedef redi::pstream pstream;
typedef std::string string;
static pstream **  pstream_init(pstream **const & p,string * const & a)
{ *p =  new pstream(a->c_str());
  return p;};

void inittt()
{
                                                                    //    add new pointer type * pstream
  Dcl_TypeandPtr<pstream*>(0,0,::InitializePtr<pstream*>,::DeletePtr<pstream*>);
                                                  //    add cast operation to make std iostream read and write
  atype<istream* >()->AddCast( new E_F1_funcT<istream*,pstream**>(UnRef<istream* >));
  atype<ostream* >()->AddCast( new E_F1_funcT<ostream*,pstream**>(UnRef<ostream* >));
                                                                  //    the constructor from a string .
  TheOperators->Add("<-",new OneOperator2_<pstream**,pstream**,string*>(pstream_init) );
                                                                    //    add new keyword type pstream
  zzzfff->Add("pstream",atype< pstream ** >());
}
LOADFUNC(inittt);
t

MBP-FH:plugin hecht$ ff-c++ pipe.cpp
/usr/local/bin/g++ -c -g -m64 -fPIC -DNDEBUG -O3 -DBAMG_LONG_LONG -DNCHECKPTR -fPIC -I/usr/local/lib/ff++/3.20/include
/usr/local/bin/g++ -bundle -undefined dynamic_lookup -g -m64 -fPIC -DNDEBUG -O3 -DBAMG_LONG_LONG -DNCHECKPTR -fPIC 'pip
```

a small test : Run:gnuplot.edp

FreeFem++ et C++ type

The tools to add a operator with 2 arguments:

```
OneOperator2_<returntype ,typearg1 ,typearg2>(& thefunction ));
returntype thefunction(typearg1 const &, typearg2 const &)
```

To get the C++ type of all freefem++ type, method, operator, just do in examples++-tutorialdirectory

```
c++filt -t < lestables
     Cmatrix 293 Matrice_Creuse<std::complex<double> >
     R3 293 Fem2D::R3
     bool 293 bool*
     complex 293 std::complex<double>*
     element 293 (anonymous namespace)::lgElement
     func 294 C_F0
       ifstream 293 std::basic_istream<char, std::char_traits<char> >**
     int 293 long*
     matrix 293 Matrice_Creuse<double>
     mesh 293 Fem2D::Mesh**
     mesh3 293 Fem2D::Mesh3**
     ofstream 293 std::basic_ostream<char, std::char_traits<char> >**
     problem 294 Problem
     real 293 double*
     solve 294 Solve
     string 293 std::basic_string<char, std::char_traits<char>, std::allocator<char> >**
     varf 294 C_args
     vertex 293 (anonymous namespace)::lgVertex
```

```
                                                //    soit T un Element de sommets A,B,C ∈ R^2
                                                                //    ------------------------
Element::nv ;                                   //    number of vertices of triangle (here 3)
const Element::Vertex & V =  T[i];                          //    the vertex i of T (i ∈ 0,1,2
double a = T.mesure() ;                                                 //    mesure of T
Rd AB = T.Edge(2);                                                      //    edge vector
Rd hC = T.H(2) ;                                                 //    gradient of 2 base fonction
R l = T.lenEdge(i);                                             //    length of i edge oppose of i
(Label) T   ;                                                    //    label of T (region number)
R2 G(T(R2(1./3,1./3)));                                           //    The barycentre of T in 3d
```

FreeFem++ Mesh/Mesh3 capabylity

```
Mesh Th("filename");                                  //    read the mesh in "filename"
Th.nt ;                                          //    number of element (triangle or tet)
Th.nv ;                                                    //    number of vertices
Th.neb or  Th.nbe ;                              //    numbe rof border element (2d) or(3d)
Th.area;                                                  //    area of the domain (2d)
Th.peri;                                                  //    length of the border
typedef Mesh::Rd Rd;                                                    //    R2 or R3
Mesh2::Element & K = Th[i];                               //    triangle i , int i∈ [0, nt[
Rd A=K[0];                                          //    coor of vertex 0 of triangle K
Rd G=K(R2(1./3,1./3)):                                    //    the barycentre de K.
Rd DLambda[3];
K.Gradlambda(DLambda);                              //    compute the 3 ∇λ_i^K  for i = 0, 1, 2
Mesh::Vertex    & V = Th(j);                              //    vertex j , int j∈ [0, nv[
Mesh::BorderElement  &   BE=th.be(l) ;                    //    border element l∈ [0, nbe[
Rd B=BE[1];                                         //    coord of vertex 1 on Seg BE
Rd M=BE(0.5);                                                    //    middle of BE.
int j = Th(i,k);            //   global number of vertex k∈ [0, 3[ of tria.  i∈ [0, nt[
Mesh::Vertex & W=Th[i][k];                    //    vertex k∈ [0, 3[ of triangle i∈ [0, nt[

int ii = Th(K) ;                                          //   number of triangle K
int jj = Th(V) ;                                          //    number of triangle V
int ll = Th(BE) ;                                    //    number of Seg de bord BE
assert( i == ii && j == jj) ;                                           //    check.
```

Freefem++ v4 is

- very good tool to solve non standard PDE in 2D/3D and of surface
- to try new domain decomposition domain algorithm

The the future we try to do:

- Build more graphic with VTK, paraview , ... (in progress)
- 3d anisotrope mesh adaptation (see new version mmg3d software at page 64)
- automate the parallel tool (in progress)
- Add integral method (A. Fourmont) , and finite volume (See G. Sadaka).

Thank for you attention.