Reduced basis with FreeFem++
Application to Stokes equations
ANR HANUMAN 18-CE45-0014-01

Pierre Mollo

LMR, Université de Reims-Champagne-Ardenne

19 décembre 2019

Introduction : Hanuman

- Numerical models for cerebro-spinal system
- Human and animal
- Interactions between cerebro-spinal fluid and blood
- ... Stokes, Navier-Stokes eq.

Introduction

- Compute solution for parametrized problem, as fast as possible.
- Parametrized PDE
- Accelerate parametrized-CFD simulation

Introduction : *Offline/Online* strategy

Offline $\Rightarrow$ Create basis

- Pre-processing
- Long time computation
- multiple EF computation

Online $\Rightarrow$ Use basis

- Processing
- Quick solve
- RB solve, very fast

Basis efficenty

Reduced basis efficienty depends on the number and the "quality" of choosen values.

Main idea

We introduce the problem $(\mathcal{P}_\mu)$ parametrized by μ, with $\mu \in \mathcal{D}$ and we call $u(\mu)$ the solution of $(\mathcal{P}_\mu)$ for a given μ.

We take $S = \{\mu_i\}_{i=1}^{P}$ P-sized sample of μ and $\{u(\mu_i)\}_{i=1}^{P}$ the collection of associated solutions. We seek $\alpha \in \mathbb{R}^P$ such that :

$$u(\mu) = \sum_{i=1}^{P} \alpha_i u(\mu_i)$$

How to choose μ value ?

- Random values (naive, no error control)
- Greedy approach (thoughtful, some error control) [4]
- Proper Orthogonal Decomposition
- ...

How to choose μ value ?

- Random values (naive, no error control)
- Greedy approach (thoughtful, some error control) [4]
- Proper Orthogonal Decomposition
- ...

Algorithm 2 Greedy algorithm

Require: : $\varepsilon, \ \mu_1, \ n = 1$

Find $u_\delta(\mu_n)$ solution of $(\mathcal{P}_\mu)$

$V_p \longleftarrow \mathrm{Vect}(\{u(\mu_i)\}_{i=1}^{n})$

...

Greedy algorithm

Algorithm 3 Greedy algorithm

...

for $\mu \in \mathcal{D}_h$ **do**

Find $u_p(\mu) \in V_p$ solution of $(\mathcal{P}_\mu)$

Compute $\eta(\mu) = ||u_\delta(\mu) - u_p(\mu)||$

end for

$\mu_{n+1} = \arg \max_{\mu \in \mathcal{D}_h} \eta(\mu)$

End when $\eta(\mu_{n+1}) < \varepsilon$

Body of the greedy algorithm in FF++

```
int i=0, cv=1;
real threshold = 1e-6, avp;
munp = 0;

while( i<PmaxRB && cv==1){

        // 1. ==========  Add in the RB ==========
        u1RB[i][] = u1Mem[munp][];
        u2RB[i][] = u2Mem[munp][];
         pRB[i][] =  pMem[munp][];
[...]
        // 2. ========== Update RB mat.  ==========
[...]
```

```
// 3. ========== Finding mu(n+1) ==========
for(int k=0; k<Kmu; k++){
        mu = tmu(k);
        [...]
        // ------ RB assembler
        set(MRB, solver=sparsesolver);
        alp = MRB^-1 * FRB;

        // ------ Assembling RB solution
        u1[] = alp[0] * u1RB[0][];
        u2[] = alp[0] * u2RB[0][];
        p[]  = alp[0] * pRB[0][];
        for(int j=1; j<=i; j++){
                u1[]=u1[]+alp[j]*u1RB[j][];
                u2[]=u2[]+alp[j]*u2RB[j][];
                p[] =p[] +alp[j]*pRB[j][];
        }
        [...]
```

```
                // ------ Error
                u1err = abs(u1Mem[k] - u1);
                u2err = abs(u2Mem[k] - u2);
                perr  = abs( pMem[k] -  p);
                [...]
                // global error
                err[k] = sqrt(errH1sp + errL2pr);
        } // End loop on mu values
        // 4. =========  Finding the max error
        munp = 0; errmax=err[0];
        for(int k=0; k<Kmu; k++){
                if(errmax < err[k]){
                        errmax = err[k];
                        munp   = k;
                }
        }
        // 5. ========= Check Threshold
} // End Greedy
```

Poiseuille's flow example

For $\mu = (\mu_1, \mu_2) \in [10^{-1}, 10^{-5}] \times [0, 10]$, we seek $(\tilde{u}(\mu), p(\mu)) \in (H^1(\Omega))^2 \times L^2(\Omega)$ solution of $(\mathcal{P}_\mu)$ such that :

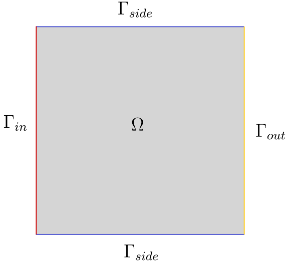


FIGURE – Ω domain

$$\begin{cases} -\mu_1 \Delta \tilde{u}(\mu) + \nabla p(\mu) = 0 & \text{in } \Omega \\ \nabla \cdot \tilde{u}(\mu) = 0 & \text{in } \Omega \\ \tilde{u}(\mu) = 0 & \text{on } \Gamma_{Side} \\ \tilde{u}(\mu) = (\mu_2, 0)^t & \text{on } \Gamma_{In} \\ \mu_1 \dfrac{\partial \tilde{u}(\mu)}{\partial \vec{n}} - p(\mu)\vec{n} = 0 & \text{on } \Gamma_{Out} \end{cases}$$

Lifted problem

We pick $\mu_0 = (0.001, 1) \in \mathcal{D}$, and we set φ_0 solution of $(\mathcal{P}_{\mu_0})$. For $\mu = (\mu_1, \mu_2) \in \mathcal{D}$, we set $u(\mu) = \tilde{u}(\mu) - \mu_2\varphi_0$ to have :

$$\Delta\tilde{u} = \Delta u + \mu_2\Delta\varphi_0$$

$$u|_{\Gamma_{in}} = \tilde{u}|_{\Gamma_{in}} - \mu_2\varphi_0|_{\Gamma_{in}} = 0$$

The new problem is : find $(u(\mu), p(\mu)) \in (H^1(\Omega))^2 \times L^2(\Omega)$ solution of $(\mathcal{P})$ such that :

$$(\mathcal{P}) : \begin{cases} -\mu_1\Delta u(\mu) + \nabla p(\mu) = \mu_2\Delta\varphi_0 & \text{in } \Omega \\ \nabla \cdot u(\mu) = 0 & \text{in } \Omega \\ u(\mu) = 0 & \text{on } \Gamma_{Side} \cup \Gamma_{in} \\ \mu_1 \dfrac{\partial u(\mu)}{\partial \overrightarrow{n}} - p(\mu)\overrightarrow{n} = 0 & \text{on } \Gamma_{out} \end{cases}$$

Weak formulation

We consider these following spaces $V := (H_0^1(\Omega, \Gamma_{Side} \cup \Gamma_{in}))^2$ and $Q := L^2(\Omega)$. $\forall (v, q) \in V \times Q$:

$$\begin{cases} \int_\Omega \mu_1 \nabla u : \nabla v - p(\nabla \cdot v) \ dX = -\mu_2 \int_\Omega \nabla \varphi_0 : \nabla v \ dX \\ \int_\Omega q(\nabla \cdot u) \ dX = 0 \end{cases}$$

$$a(u, v) = \int_\Omega \nabla u : \nabla v \ dX$$

$$b(p, v) = -\int_\Omega p(\nabla \cdot v) \ dX$$

$$f(v) = -\int_\Omega \nabla \varphi_0 : \nabla v \ dX$$

Weak form.

$$\mu_1 \ a(u, v) + b(p, v) = \mu_2 \ f(v)$$
$$b(q, u) = 0$$

Finite Element settings

We take $\mathcal{T}_\delta$ with space step δ.

We take $\{\varphi_k\}_{k=1}^{N_2}$ the mesh-associated hat-functions $P2$ and $\{\psi_k\}_{k=1}^{N_1}$ the $P1$ one. Then we consider these following spaces :

$$V_\delta := (\mathrm{Vect}(\{\varphi_k\}_{k=1}^{N_2})) \qquad Q_\delta := \mathrm{Vect}(\{\psi_k\}_{k=1}^{N_1})$$

$$\begin{cases} \text{Find } (u_\delta, p_\delta) \in V_\delta \times Q_\delta \text{ such that } \forall (v,q) \in V_\delta \times Q_\delta : \\ \mu_1 a(u_\delta, v) + b(p_\delta, v) = \mu_2 f(v) \\ b(q, u_\delta) \qquad\qquad\qquad\quad = 0 \end{cases}$$

We have $N = 2N_2 + N_1$ DOF.

Reduced basis settings

We consider $S = \{\mu_p\}_{p=1}^{P}$ a sample of P values of $\mu \in \mathcal{D}$ and $\{(w_i, s_i)\}_{i=1}^{P}$ the associated solutions.
For each μ_i, we have $(w_i, s_i) \in V_\delta \times Q_\delta$, consequently we have :

$$V_p \times Q_p := \mathrm{Vect}(\{w_i\}_{i=1}^{P}) \times \mathrm{Vect}(\{s_i\}_{i=1}^{P}) \subset V_\delta \times Q_\delta \subset V \times Q$$

$$\begin{cases} \text{Trouver } (u_p, p_p) \in V_p \times Q_p \text{ tels que } \forall (v, q) \in V_p \times Q_p : \\ \mu_1 a(u_p, v) + b(p_p, v) = \mu_2 f(v) \\ b(q, u_p) \qquad\qquad\qquad\quad = 0 \end{cases}$$

$$(u_p, p_p) = \sum_{i=1}^{P} \alpha_i \ (w_i, s_i)$$

Algebraic forms & complexity

Finite Element

$$\begin{pmatrix} \mu_1 A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} U_\delta \\ P_\delta \end{pmatrix} = \begin{pmatrix} \mu_2 F \\ 0 \end{pmatrix}$$

$$\text{FE} : \mathcal{O}(N^3)$$

Reduced Basis

$$(\mu_1 A_p + B_p)\alpha_p(\mu) = \mu_2 F_p$$

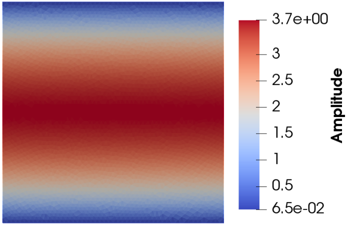
$$\text{RB} : \mathcal{O}(P^3 + PN)$$

Ex : $N \simeq 26 \times 10^3$ and $P = 2$

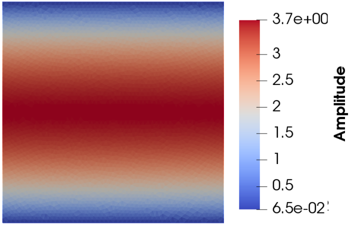
- FE $\Rightarrow 2 \times 10^{13}$
- RB $\Rightarrow 8 + 2 \times 26 \times 10^3$

Results : Velocity field $(\mu_1, \mu_2) = (4 \times 10^{-2}, 0.45)$

Finite elem. sol.



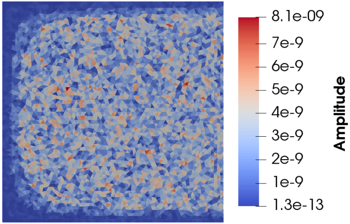
Reduced basis sol.



NDOF : 11969 (P2)
3043 (P1)

N = 26981

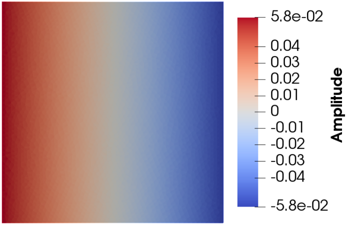
Error



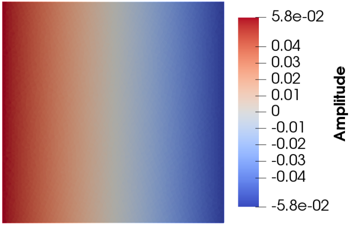
NRB : 2 (<10^-12)

Results : Pressure $(\mu_1, \mu_2) = (4 \times 10^{-2}, 0.45)$

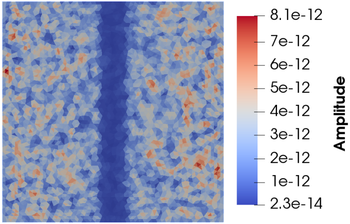
Finite elem. sol.



Reduced basis sol.



Error

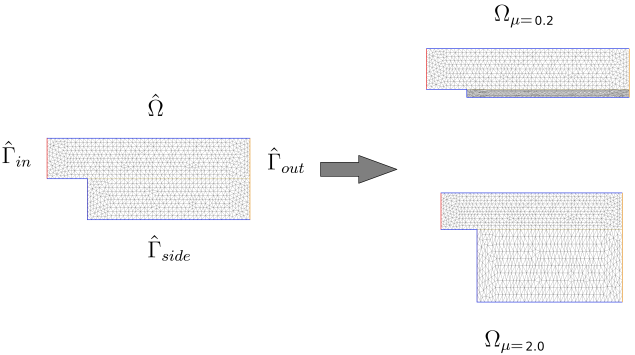


Problem 2 : Step example

For $\mu \in [0.1,\ 1.9]$, we seek $(u(\mu), p(\mu)) \in (H^1(\Omega))^2 \times L^2(\Omega)$ solution of $(\mathcal{P}_\mu)$ such that :

$$(\mathcal{P}_\mu) : \begin{cases} -\nu\Delta u + \nabla p = \nu\Delta\varphi_0 & \text{in } \Omega_\mu \\ \nabla \cdot u = 0 & \text{in } \Omega_\mu \\ u = 0 & \text{on } \Gamma_{Side,\mu} \cup \Gamma_{in,\mu} \\ \nu \dfrac{\partial u}{\partial \overrightarrow{n}} - p\,\overrightarrow{n} = 0 & \text{on } \Gamma_{out,\mu} \end{cases}$$

Step model



$\hat{\Omega}$
$\hat{\Gamma}_{in}$
$\hat{\Gamma}_{out}$
$\hat{\Gamma}_{side}$
$\Omega_{\mu=0.2}$
$\Omega_{\mu=2.0}$

Step model transformation

We consider $\mu \in \mathcal{D} := [0.1, 1.9]$:

$$T_\mu : \qquad \Omega_\mu \longrightarrow \hat{\Omega}$$
$$(x, y) \longrightarrow (x, y(\frac{1}{\mu}\mathbb{1}_{\{y<0\}} + \mathbb{1}_{\{y>0\}}))$$

$$T_\mu^{-1} : \qquad \hat{\Omega} \longrightarrow \Omega_\mu$$
$$(\hat{x}, \hat{y}) \longrightarrow (\hat{x}, \hat{y}(\mu\mathbb{1}_{\{y<0\}} + \mathbb{1}_{\{y>0\}}))$$

Weak formulation

The associated weak formulation :

$$\begin{cases} \text{Trouver } (u(\mu), p(\mu)) \in V \times Q \text{ tels que } \forall (q, v) \in V \times Q : \\ a(u, v; \mu) + b(p, v; \mu) = f(v; \mu) \\ b(q, u; \mu) \qquad\qquad\qquad\quad = 0 \end{cases}$$

with :

$$a(u, v; \mu) = \int_{\Omega_\mu} \nu \nabla u : \nabla v \ dX$$

$$b(p, v; \mu) = \int_{\Omega_\mu} p \nabla \cdot v \ dX$$

$$f(v; \mu) = \int_{\Omega_\mu} \nu \nabla \varphi_0 : \nabla v \ dX$$

J in the step case

Using the expression of T_μ we can compute $J_{T_\mu^{-1}}$ and $J_{T_\mu^{-1}}(J_{T_\mu^{-1}})^t$:

$$J_{T_\mu^{-1}} = \begin{pmatrix} 1 & 0 \\ 0 & \mu \mathbb{1}_{\{y<0\}} + \mathbb{1}_{\{y>0\}} \end{pmatrix}$$

$$J_{T_\mu^{-1}}(J_{T_\mu^{-1}})^t = \begin{pmatrix} 1 & 0 \\ 0 & \mu^2 \mathbb{1}_{\{y<0\}} + \mathbb{1}_{\{y>0\}} \end{pmatrix}$$

Applied with a

$$
\begin{aligned}
a(u,v;\mu) &= \int_{\Omega_\mu} \nu \nabla u : \nabla v \ dX \\
&= \int_{\hat{\Omega}} \nu\big((\nabla v \circ T_\mu)^t \nabla u \circ T_\mu\big) : I \ |J_{T_\mu}| \ d\hat{X} \\
&= \int_{\hat{\Omega}} \nu\big((\hat{\nabla}\hat{v})^t \hat{\nabla}\hat{u}\big) : (J_{T_\mu^{-1}} J^t_{T_\mu^{-1}}) |J_{T_\mu}| \ d\hat{X} \\
&= \nu \int_{\hat{\Omega}} \mathbb{1}_{\{y>0\}}(\nabla\hat{u} : \nabla\hat{v}) + \mathbb{1}_{\{y<0\}}\big(\mu \ \partial_x\hat{u}\partial_x\hat{v} + \frac{1}{\mu}\partial_y\hat{u}\partial_y\hat{v}\big) \ d\hat{X} \\
&= \hat{a}^1(\hat{u},\hat{v}) + \mu\hat{a}^2(\hat{u},\hat{v}) + \frac{1}{\mu}\hat{a}^3(\hat{u},\hat{v}) \\
&= \hat{a}(\hat{u},\hat{v};\mu)
\end{aligned}
$$

Reduced basis settings

We consider a sample $S = \{\mu_p\}_{p=1}^{P}$ of P values of $\mu \in \mathcal{D}$ and the family of associated solutions $\{w_{1,p}, w_{2,p}, s_p\}_{p=1}^{P}$. Or for each μ_p, we have : $\{w_{1,p}, w_{2,p}, s_p\} \in \hat{V} \times \hat{Q}$, we can deduct :

$$\hat{V}_P \times \hat{Q}_P := \mathrm{Vect}(\{w_{1,p}, w_{2,p}\}_{p=1}^{P}) \times \mathrm{Vect}(\{s_p\}_{p=1}^{P}) \subset \hat{V} \times \hat{Q}$$

And then :

$$\begin{cases} \text{Trouver } (\hat{u}_p, \hat{p}_p) \in \hat{V}_p \times \hat{Q}_p \text{ tels que } \forall (\hat{q}, \hat{v}) \in \hat{V}_p \times \hat{Q}_p : \\ \hat{a}(\hat{u}_p, \hat{v}; \mu) + \hat{b}(\hat{p}_p, \hat{v}; \mu) = \hat{f}(\hat{v}; \mu) \\ \hat{b}(\hat{q}, \hat{u}_p; \mu) \qquad\qquad\qquad\quad = 0 \end{cases}$$

Affine decomposition $a(.,.;mu)$, $b(.,.;\mu)$ and $f(.;\mu)$

The problem directly admits an affine decomposition

$$\hat{a}(\hat{u},\hat{v};\mu) = \sum_{i=1}^{3} \theta^i(\mu)\ a^i(\hat{u},\hat{v})$$

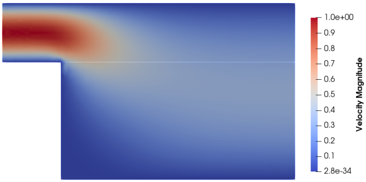
$$\hat{b}(\hat{u},\hat{v}:\mu) = \sum_{i=1}^{2} \theta^i(\mu)\ b^i(\hat{u},\hat{v})$$

$$\hat{f}(\hat{v};\mu) = \sum_{i=1}^{3} \theta^i(\mu)\ f^i(\hat{v})$$

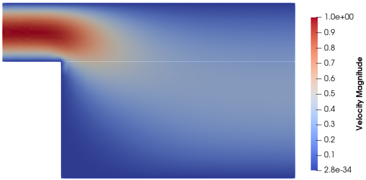
It follows that :

$$\left(\sum_{i=1}^{3} \theta^i(\mu)\ A_i + \sum_{i=1}^{2} \theta^i(\mu) B_i\right)\alpha_p(\mu) = \sum_{i=1}^{3} \theta^i(\mu)\ F_i$$

Finite elem. sol.



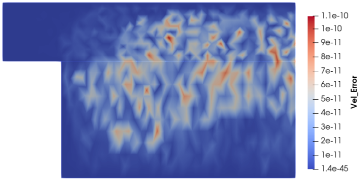
Reduced basis sol.



NDOF : 4009 (P2)
1038 (P1)

N : 9056

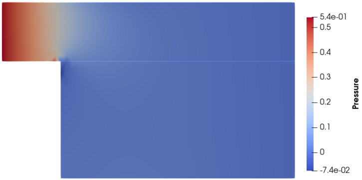
Error



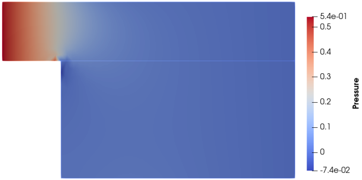
NRB :		
	8	(10^-3)
	10	(10^-4)
	12	(10^-5)
	15	(10^-6)
	18	(10^-8)

Results : Pressure field $\mu = 0.6$

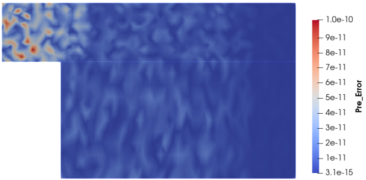
Finite elem. sol.



Reduced basis sol.



Error

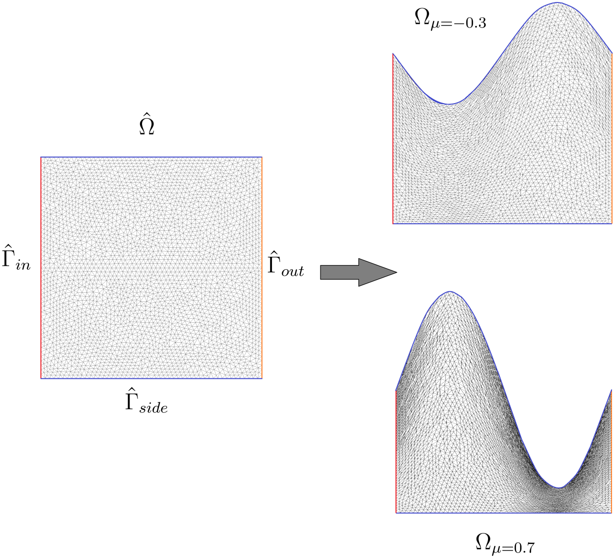


Problem (RHP08)

For $\mu \in [-0.8, 0.8]$, we seek $(u(\mu), p(\mu)) \in (H^1(\Omega))^2 \times L^2(\Omega)$ solution of $(\mathcal{P}_\mu)$ such that :

$$(\mathcal{P}_\mu) : \begin{cases} -\nu \Delta u + \nabla p = 0 & \text{in } \Omega_\mu \\ \nabla \cdot u = 0 & \text{in } \Omega_\mu \\ u = 0 & \text{on } \Gamma_{Side,\mu} \\ \nu \dfrac{\partial u}{\partial \overrightarrow{n}} - p \overrightarrow{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{on } \Gamma_{in,\mu} \\ \nu \dfrac{\partial u}{\partial \overrightarrow{n}} - p \overrightarrow{n} = 0 & \text{on } \Gamma_{out,\mu} \end{cases}$$

Stenosis model (RHP08)



Stenosis model (RHP08)

We consider $\mu \in \mathcal{D} := [-0.8, 0.8]$:

$$\begin{aligned} T_\mu : \qquad \Omega_\mu &\longrightarrow \hat{\Omega} \\ (x, y) &\longrightarrow (x, (1 + \mu \sin(2\pi x))y) \end{aligned}$$

$$\begin{aligned} T_\mu^{-1} : \qquad \hat{\Omega} &\longrightarrow \Omega_\mu \\ (\hat{x}, \hat{y}) &\longrightarrow \left(\hat{x}, \frac{1}{1 + \mu \sin(2\pi \hat{x})} \hat{y}\right) \end{aligned}$$

J in the stenose case

Using the expression of T_μ we can compute $J_{T_\mu^{-1}}$ and $J_{T_\mu^{-1}}(J_{T_\mu^{-1}})^t$:

$$J_{T_\mu^{-1}} = \begin{pmatrix} 1 & 0 \\ -\frac{2\pi\mu\cos(2\pi x)}{(1+\mu\sin(2\pi x))} y & \frac{1}{1+\mu\sin(2\pi x)} \end{pmatrix}$$

$$J_{T_\mu^{-1}}(J_{T_\mu^{-1}})^t = \begin{pmatrix} 1 & -\frac{2\pi\mu\cos(2\pi x)}{(1+\mu\sin(2\pi x))} y \\ -\frac{2\pi\mu\cos(2\pi x)}{(1+\mu\sin(2\pi x))} y & \left(\frac{2\pi\mu\cos(2\pi x)}{(1+\mu\sin(2\pi x))} y\right)^2 + \left(\frac{1}{1+\mu\sin(2\pi x)}\right)^2 \end{pmatrix}$$

$$\begin{cases} \text{Find } (\hat{u}, \hat{p}) \in \hat{V} \times \hat{Q} \text{ such that } \forall(\hat{q}, \hat{v}) \in \hat{V} \times \hat{Q} : \\ \hat{a}(\hat{u}, \hat{v}; \mu) + \hat{b}(\hat{p}, \hat{v}; \mu) = \hat{f}(\hat{v}; \mu) \\ \hat{b}(\hat{q}, \hat{u}; \mu) \qquad\qquad\qquad = 0 \end{cases}$$

Reduced basis settings

We consider a sample $S = \{\mu_p\}_{p=1}^{P}$ of P values of $\mu \in \mathcal{D}$ and the family of associated solutions $\{w_{1,p}, w_{2,p}, s_p\}_{p=1}^{P}$. Or for each μ_p, we have : $\{w_{1,p}, w_{2,p}, s_p\} \in \hat{V} \times \hat{Q}$, we can deduct :

$$\hat{V}_P \times \hat{Q}_P := \mathrm{Vect}(\{w_{1,p}, w_{2,p}\}_{p=1}^{P}) \times \mathrm{Vect}(\{s_p\}_{p=1}^{P}) \subset \hat{V} \times \hat{Q}$$

And then :

$$\begin{cases} \text{Trouver } (\hat{u}_p, \hat{p}_p) \in \hat{V}_p \times \hat{Q}_p \text{ tels que } \forall (\hat{q}, \hat{v}) \in \hat{V}_p \times \hat{Q}_p : \\ \hat{a}(\hat{u}_p, \hat{v}; \mu) + \hat{b}(\hat{p}_p, \hat{v}; \mu) = \hat{f}(\hat{v}; \mu) \\ \hat{b}(\hat{q}, \hat{u}_p; \mu) \qquad\qquad\qquad\quad = 0 \end{cases}$$

Affine decomposition $a(.,.;mu)$, $b(.,.;\mu)$

Using the Empirical Interpolation Method (EIM,[4]), we can find :

$$\hat{a}(\hat{u},\hat{v};\mu) = \sum_{i=1}^{19} \theta^i(\mu)\ a^i(\hat{u},\hat{v})$$

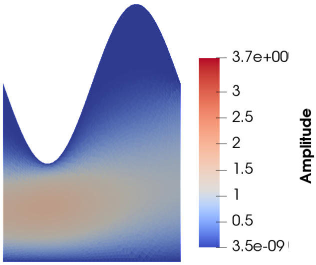
$$\hat{b}(\hat{u},\hat{v}:\mu) = \sum_{i=1}^{4} \theta^i(\mu)\ b^i(\hat{u},\hat{v})$$

It follow that :

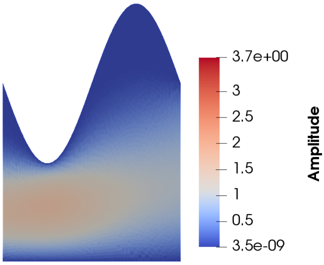
$$\left(\sum_{i=1}^{19} \theta^i(\mu)\ A_i + \sum_{i=1}^{4} \theta^i(\mu) B_i\right)\alpha_p(\mu) = F$$

Results : Velocity field $\mu = 0.6$

Finite elem. sol.



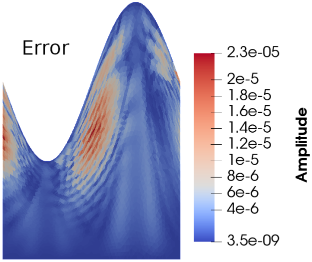
Reduced basis sol.



NDOF : 11969 (P2)
3043 (P1)

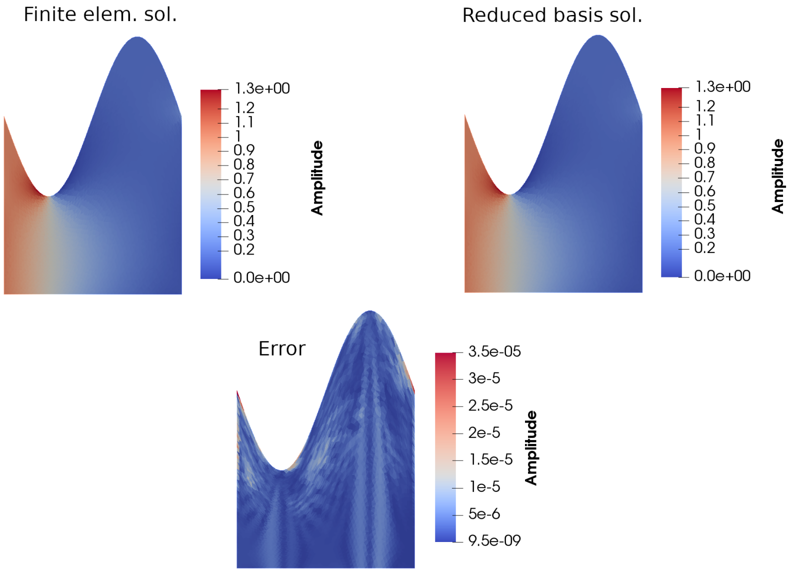
N = 26981

Error



NRB : 22 (10^-2)
30 (10^-3)
35 (10^-4)
43 (10^-5)

Results : Pressure field $\mu = 0.6$



Results : Convergence

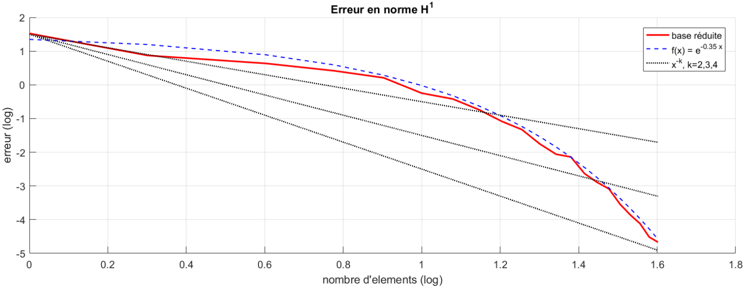


FIGURE – Error evolution

Conclusion & Perspectives

- Unsteady problem (with Greedy-POD)
- Better use of estimations
- Navier-Stokes equations

Conclusion

Thank you

Biblio

- J.S. Hesthaven, G. Rozza, B. Stamm, «Certified reduced basis methods for Parametrized partial differential equations», Springer 2016.
- [RHP08] G. Rozza, D.B.P. Huynh, A.T. Patera, «Reduced basis approximation and a posteriori error estimation for affinely parametrized elliptic coercive partial differential equations », Springer 2008.
- A. Quarteroni, A. Manzoni, F. Negri, «Reduced basis methods for partial differential equations», Springer, 2016.
- [EIM] M. Barrault, Y. Maday, N.C. Nguyen, A. Patera, « An 'empirical interpolation' method : application to efficient reduced-basis discretization of partial differential equations», C. R. Acad. Sci. Paris, Ser. I 339 (2004).

Transformation Lemma 1

Lemme

Let Ω_0 be an open subset of $\mathbb{R}^n$, T a smooth diffeomorphism and $1 \leq p \leq +\infty$. Then $f \in L^p(T(\Omega_0))$ if and only if $f \circ T \in L^p(\Omega_0)$, we also have :

$$\int_{T(\Omega_0)} f \; dX = \int_{\Omega_0} f \circ T \; |J_T| \; dX$$

Furthermore, $f \in W^{1,p}(T(\Omega_0))$ if and only if $f \circ T \in W^{1,p}(\Omega_0)$, we also :

$$(\nabla f) \circ T = \left((J_T)^{-1}\right)^t \nabla(f \circ T)$$

Transformation Lemma 2

Lemme

Let Ω_0 be a bounded open subset $\mathbb{R}^n$, T a smooth diffeomorphism. If $f \in L^1(\partial T(\Omega_0))$ then $f \circ T \in L^1(\partial\Omega_0)$, and we have :

$$\int_{\partial T(\Omega_0)} f \; dS = \int_{\partial\Omega_0} f \circ T \; |J_T| \; \left\| \left((J_T)^{-1}\right)^t \cdot \overrightarrow{n} \right\|_{\mathbb{R}^n} dS$$