Third Workshop on Generic Solvers for PDEs
FreeFem++ and Applications
Laboratoire Jacques-Louis Lions, Paris, December 5-7, 2011

Buildings Flows with Freefem++

Presented by Eliseo Chacón Vera[a]

Contribution of many people (abc order) :

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Our warmest thanks to Frédéric Hecht to make with FreeFem++ our teaching and research more a pleasure than a duty.

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Outline of the talk:

- High-order stabilized solvers for fluid flows: Primitive Equations of the Ocean
- Domain Decomposition Method with Lagrange Multipliers
- Sustainable architecture: Energy efficiency in buildings

High-order stabilized solvers for fluid flows

For instance, we might consider the **Oseen equations**,

$$\begin{cases} \text{ Find } y : \overline{\Omega} \mapsto \mathbb{R}^d,\, p : \Omega \mapsto \mathbb{R} & \text{such that} \\ u \cdot \nabla y - \nu \Delta y + \nabla p = f,\, \nabla \cdot y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma \end{cases}$$

formulated as

$$\begin{cases} \text{ Obtain } (y, p) \in (H_0^1(\Omega))^d \times L_0^2(\Omega) \text{ such that} \\ L(y, p; v, q) =< F, (v, q) >, \quad \forall (v, q) \in (H_0^1(\Omega))^d \times L_0^2(\Omega); \end{cases}$$

where

$$L(y, p; v, q) = a(y, v) - (p, \nabla \cdot v) + (\nabla \cdot y, q) \;\text{ and }\; < F, (v, q) >=< f, v >;$$

with

$$a(z, v) = (u \cdot \nabla z, v) + (\nabla z, \nabla v) \quad \forall z,\; v \in (H_0^1(\Omega))^d.$$

Triangular finite elements spaces of degree ≥ 2:

$$\begin{cases} Y_h = (V_h^l)^d \cap (H_0^1(\Omega))^d, & \text{degree for discrete velocity } l \geq 2 \\ M_h = V_h^m \cap L_0^2(\Omega), & \text{degree for discrete pressure } m \geq 2 \\ V_h^l = \{r \in C^0(\overline{\Omega}) \text{ such that } r_K \in \mathbb{P}_l(K), \forall K \in \mathcal{T}_h\} \end{cases}$$

Discretization: Find $(y_h, p_h) \in Y_h \times M_h$, such that

$$L_h(y_h, p_h; v_h, q_h) = < F, (v_h, q_h) >, \quad \forall (v_h, q_h) \in Y_h \times M_h.$$

Stabilization formulation has the form

$$\begin{aligned} L_h(y, p; v, q) &= L(y, p; v, q) + \sum_{K \in \mathcal{T}_h} \tau_{pK} (\Pi_h^*(\nabla p), \Pi_h^*(\nabla q))_K \\ &+ \sum_{K \in \mathcal{T}_h} \tau_{BK} (\Pi_h^*(By), \Pi_h^*(Bv))_K. \end{aligned}$$

- B: continuous linear operator on $H_0^1(\Omega)^d$ (e.g., convection, Coriolis, reaction,...).
- $\Pi_h^* = Id - \Pi_h$, with $\Pi_h : L^2(\Omega)^d \to \left[V_h^{l-1}\right]^d$
  - here Π_h is a locally stable interpolation operator (e.g., Clément or Scott-Zhang)
  - and $\left[V_h^{l-1}\right]^d$ is called a buffer space
- The term $\|\Pi_h^*(\nabla q_h)\|_{\tau_p}$ controls the high frequency components of ∇q_h.

Some references:

1. Introduced by Blasco and Codina (2000) with
   - buffer space $[V_h^l]^d$ and
   - L^2 orthogonal projection operation

   this work improves in the sense that

   - buffer space $\left[V_h^{l-1}\right]^d$ and
   - general local projection-stabilization methods and
   - No local orthogonality properties needed.

Freefem++ implements Scott-Zhang operator and $\mathbb{P}_k$ with $k \geq 2$ then... just do math!

Numerical tests: Non-uniformly regular meshes.

Oseen problem on $(0,\pi)^2$ with $\nu = 0.01$ and

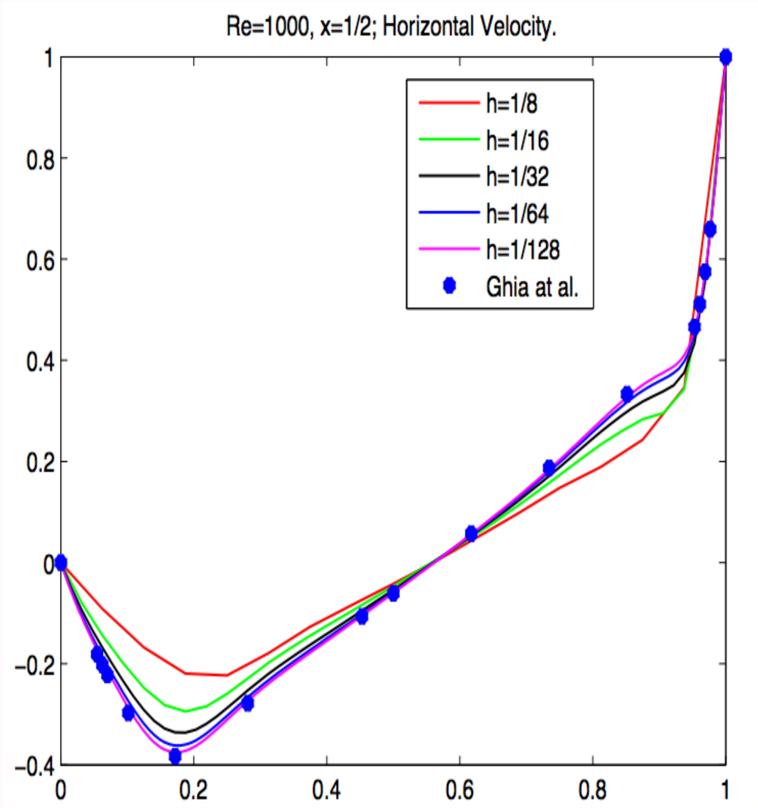
$$u(x,y) = 2\sin(x)^2\cos(x)\sin(y), \quad v(x,y) = -u(y,x).$$

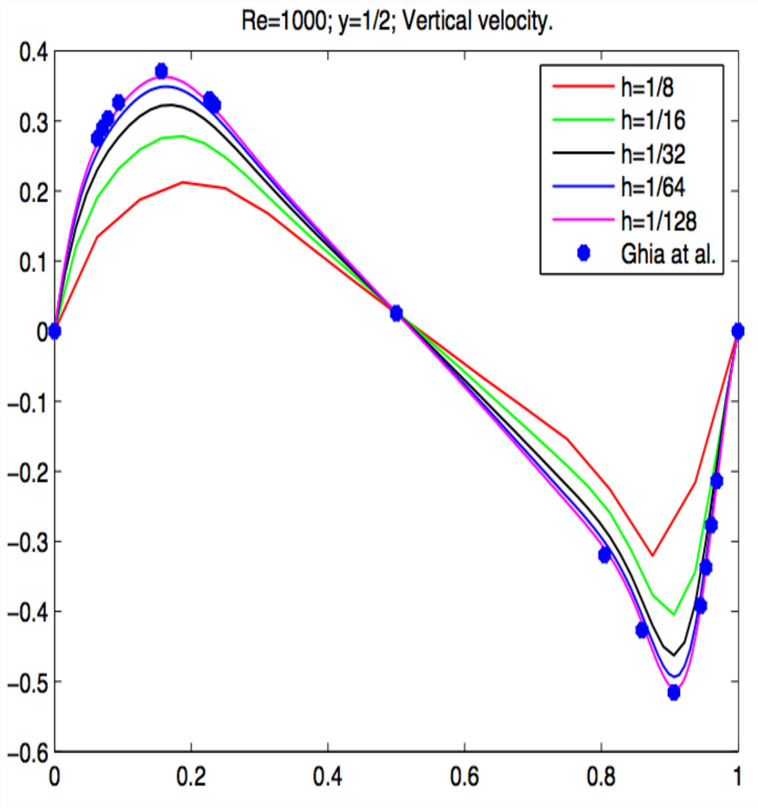
Using a characteristic h given by $h = N^{-1}\sum_{k\in\mathcal{T}_h} h_k$ where N is the number of elements:

$r(1/h_1, 1/h_2)$	Velocity Degree 2	Pressure Degree 2	Velocity Degree 3	Pressure Degree 3	Velocity Degree 4	Pressure Degree 4
r(0.888577,0.403898)	1.9701855	2.5655639	3.0157996	3.1755816	4.0510126	4.7312248
r(0.403898,0.261346)	2.1034308	2.4213809	3.1205766	3.0397084	4.1324295	4.6341359
r(0.261346,0.193169)	2.0853611	2.3892076	3.0892466	3.0269722	4.1032615	4.6401100
r(0.193169,0.153203)	2.0976119	2.2264732	3.0885004	3.0546166	4.0908042	4.4741609
r(0.153203,0.126940)	2.2186965	2.4295187	3.2033248	3.1612073	4.2143989	4.6965576
r(0.126940,0.108363)	2.0595400	2.1205955	3.0481197	3.0509403	4.0458218	4.3530187
r(0.108363,0.094529)	2.2173160	2.3032472	3.1911315	3.1720994	4.1955161	4.5568381

Computed convergence order for smooth solution.

Numerical tests: Navier-Stokes driven lid test $Re = 1000$. $\mathbb{P}_2 - \mathbb{P}_2$+stabilization.

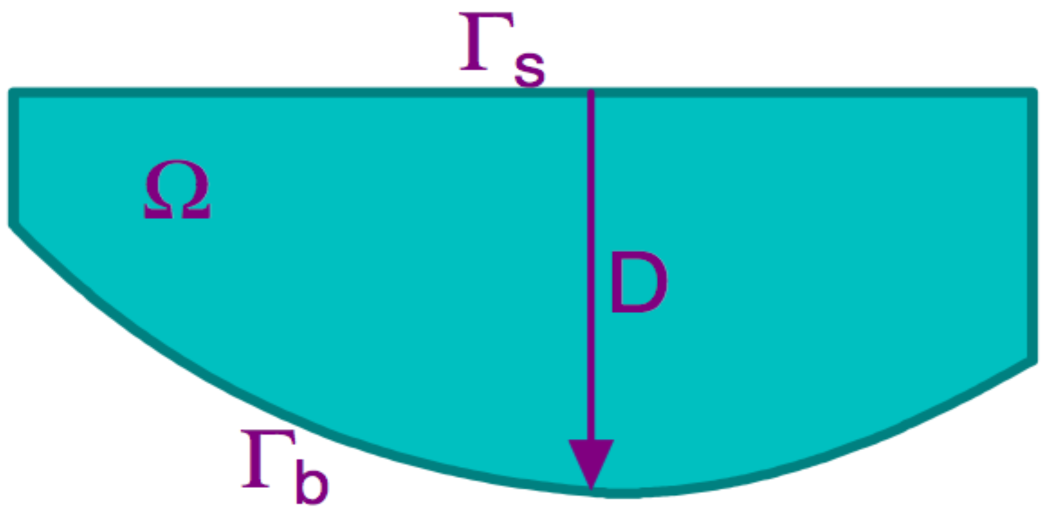




Primitive Equations for the ocean

$$\Omega = \{(\mathbf{x}, z) \in \mathbb{R}^d, \quad x \in \omega, \quad -D(\mathbf{x}) < z < 0\}, \quad \omega \subset \mathbb{R}^{d-1}$$

$\partial\Omega = \Gamma_s \cup \Gamma_b$: $\quad \Gamma_s \equiv \omega \times \{0\}$ Sea surface , $\quad \Gamma_b$ Bottom and sidewalls



Discretization by OSS method.

Adapt the Orthogonal Sub-Scales VMS method (Codina, 2002)

$$\left\{\begin{array}{l} \text{Obtain } (y_h, p_h) \in U_h \text{ such that} \\ B(y_h; y_h, p_h; v_h, q_h) - \sum_{K\in\mathcal{F}_h} \tau_{1K} \left(\mathcal{F}^*(v_h, q_h), (I - \Pi_{\tau_1})(\mathcal{F}(y_h, p_h))\right)_K \\ \qquad\qquad + \sum_{K\in\mathcal{F}_h} \tau_{2K} \left(\nabla_H v_h, (I - \Pi_{\tau_2})(\nabla_H y_h)\right)_K \\ = F(v_h) + \sum_{K\in\mathcal{T}_h} \tau_{1K} \left(\mathcal{F}^*(v_h, q_h), (I - \Pi_{\tau_1})(f)\right)_K, \quad \forall (v_h, q_h) \in U_h \end{array}\right.$$

where $\mathcal{F}(y_h, p_h) = y_h \cdot \nabla y_h - \nu \Delta y_h + \alpha\, y_h^{\perp} + \nabla_H p_h$, and Π_{τ_j}: L^2 orthogonal projections.

- **Interactions large-small scales due to convection are neglected**.
- **Structure similar to adjoint-stabilized method, with projection**.
- **Replacing L^2 orthogonal projections for Π_{τ_j} by local interpolation may be justified.**

Numerical Results: 2D Flows

Solution of discrete non-linear problem through evolution approach

$$\begin{cases} \frac{1}{\Delta t}(y^{n+1} - y^n) + y^n \cdot \nabla y^{n+1} - \mu \Delta y^{n+1} + \partial_x p^{n+1} = f \quad \text{in}\, \Omega \subset \mathbb{R}^2 \\ \partial_x < y^{n+1} >= 0 \quad \text{in}\, \omega \subset \mathbb{R} \\ y^0 = y_0 \\ y^{n+1}_{\Gamma_b} = 0, \quad \mu \partial_{\mathbf{n}} y^{n+1} = \tau_w, \;\; \text{on}\; w = \Gamma_S \end{cases}$$

Implementation with FreeFem++

Primitive Equations of the ocean

Smooth solution on a square

	Horizontal velocity		
h	P1b-P1	OSS	Order in H^1-norm
0,072	0.0586436	0.00369733	
0.036	0.0283502	0.00101718	1.81623
0.018	0.013938	0.00314393	1.67336
0.014	0.011539	0.000211092	1.5564

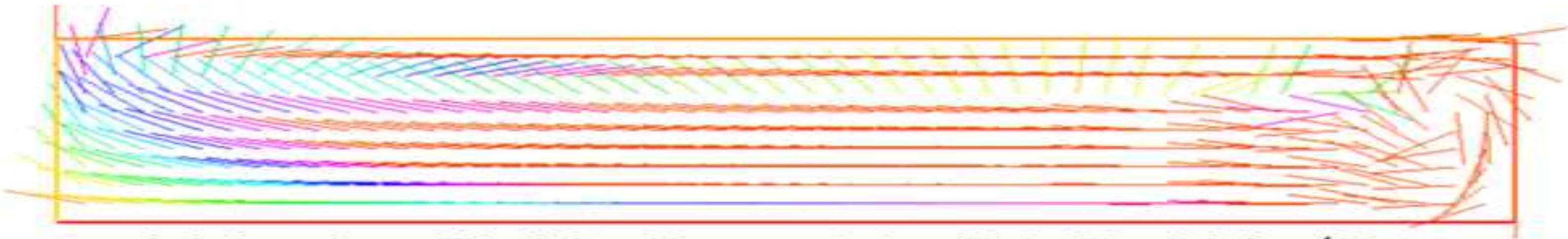
Table 1: Estimated convergence orders for horizontal velocity.

	Pressure		
h	P1b-P1	OSS	Order in L^2-norm
0.072	0.000932524	0.00045671	
0.036	0.000327411	0.00123518	1.84027
0.018	0.000115275	3.7988e-5	1.68045
0.014	7.65232e-5	2.51929e-5	1.60469

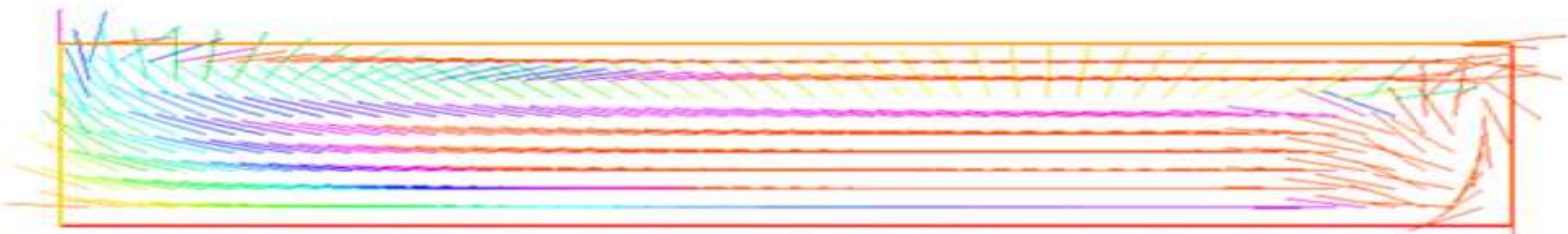
Table 2: Estimated convergence orders for surface pressure.

Primitive Equations for the ocean

$\mathbb{P}_1 - \mathbb{P}_1$+bubbles and $\mathbb{P}_1 - \mathbb{P}_1$+OSS

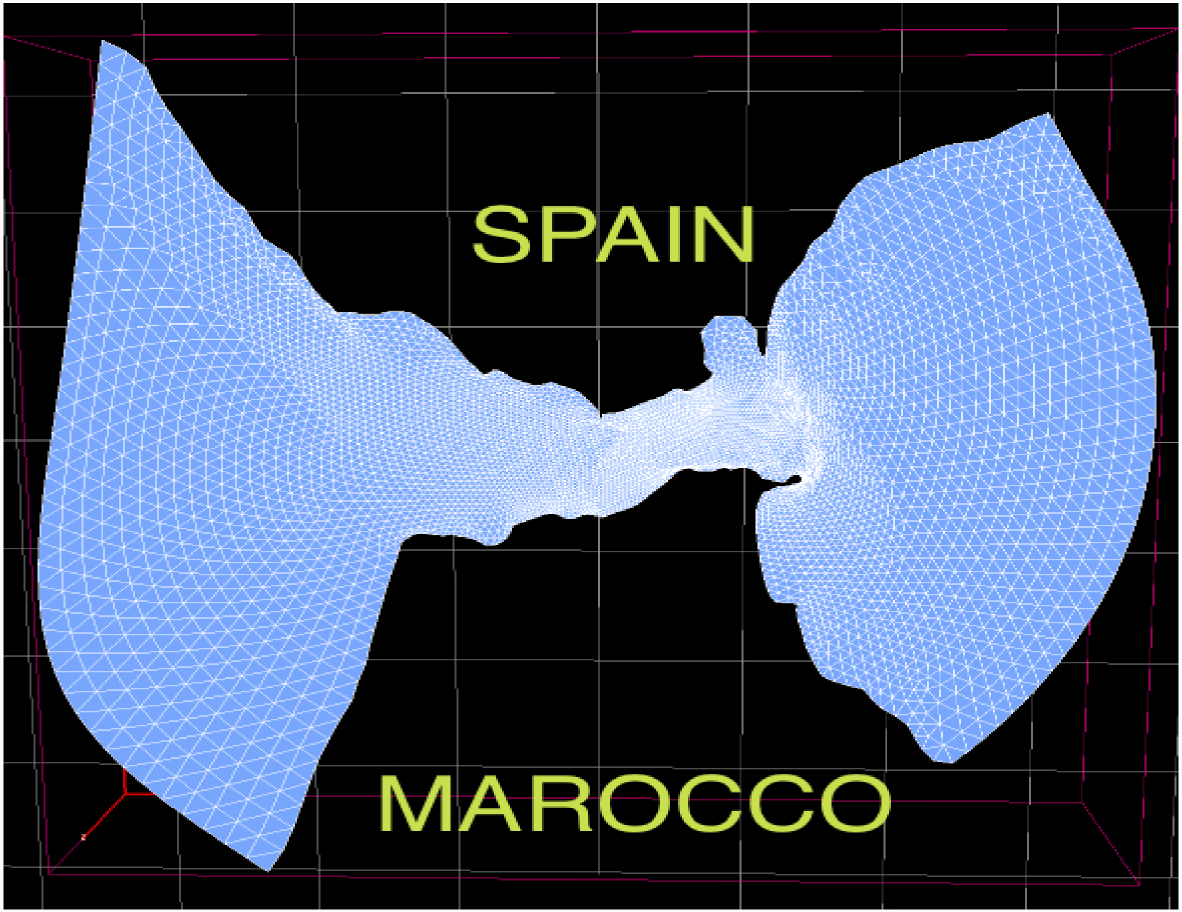


Velocity Field. Re=400. P1-Bubble/P1

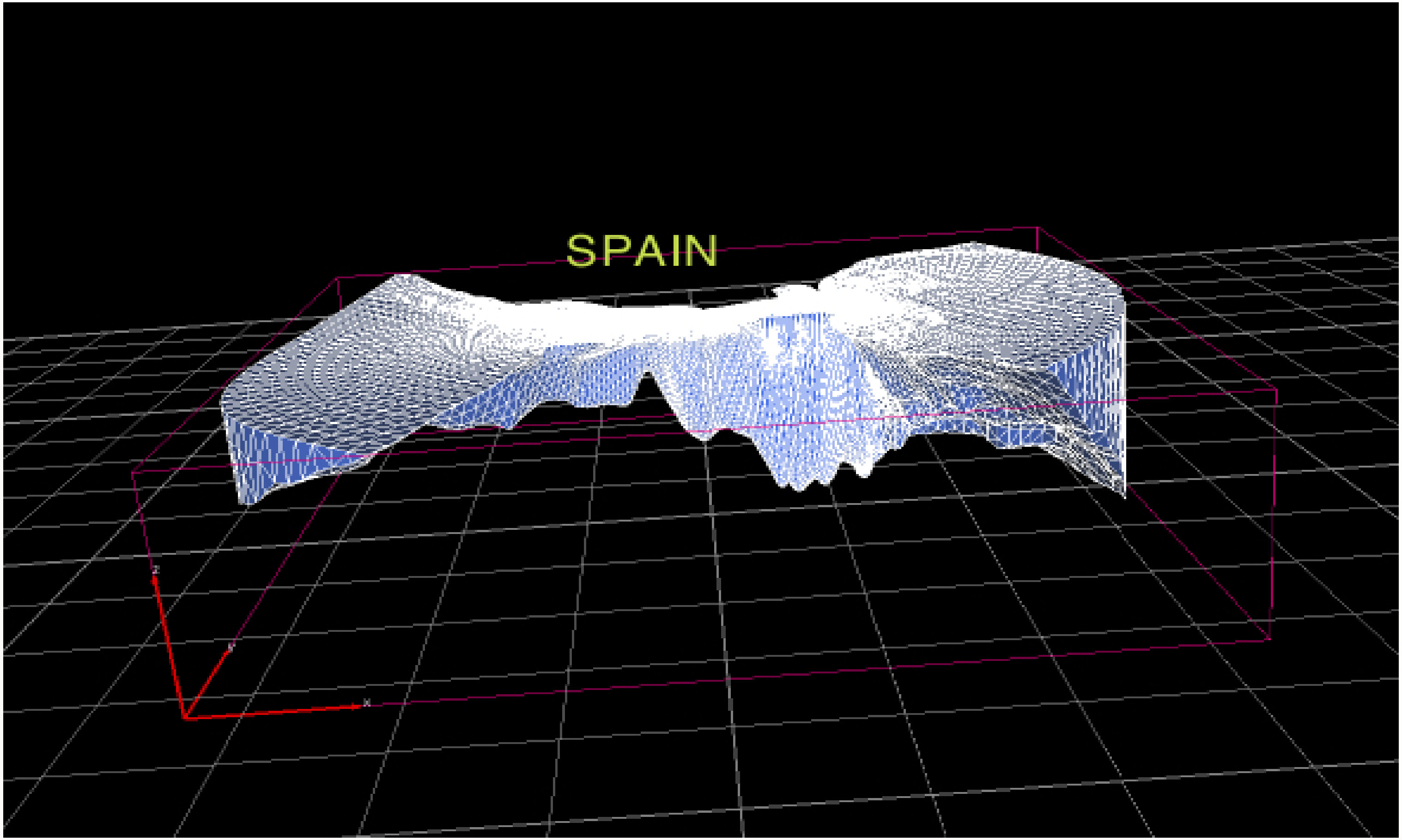


Velocity Field. Re=400. OSS

Study of Strait of Gibraltar

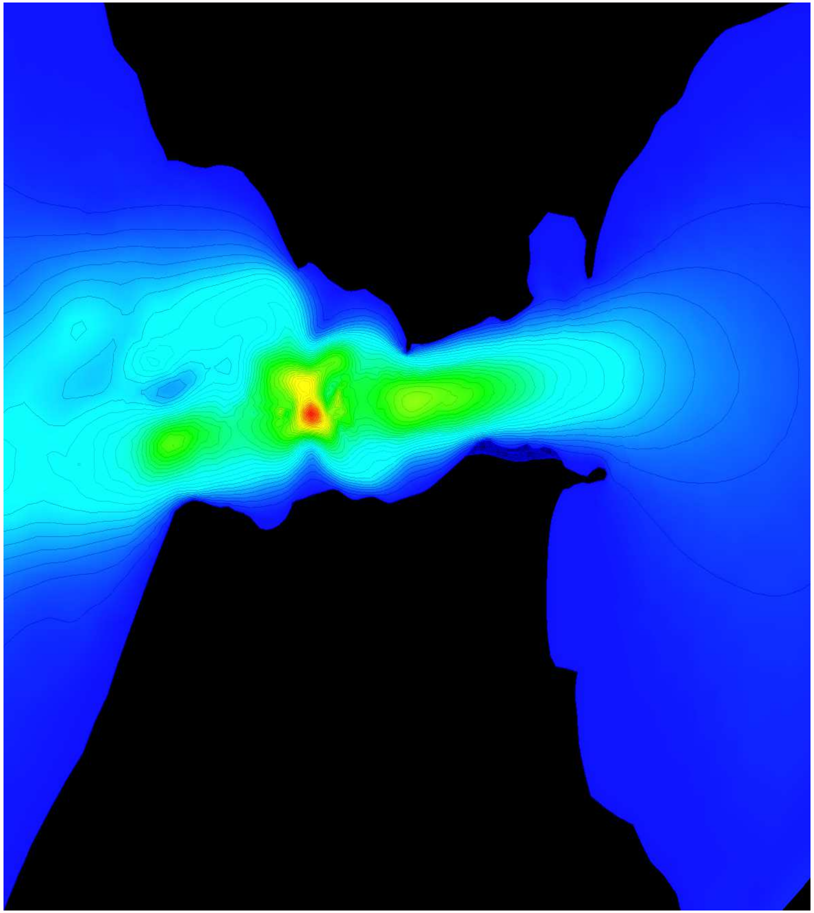


Study of Strait of Gibraltar

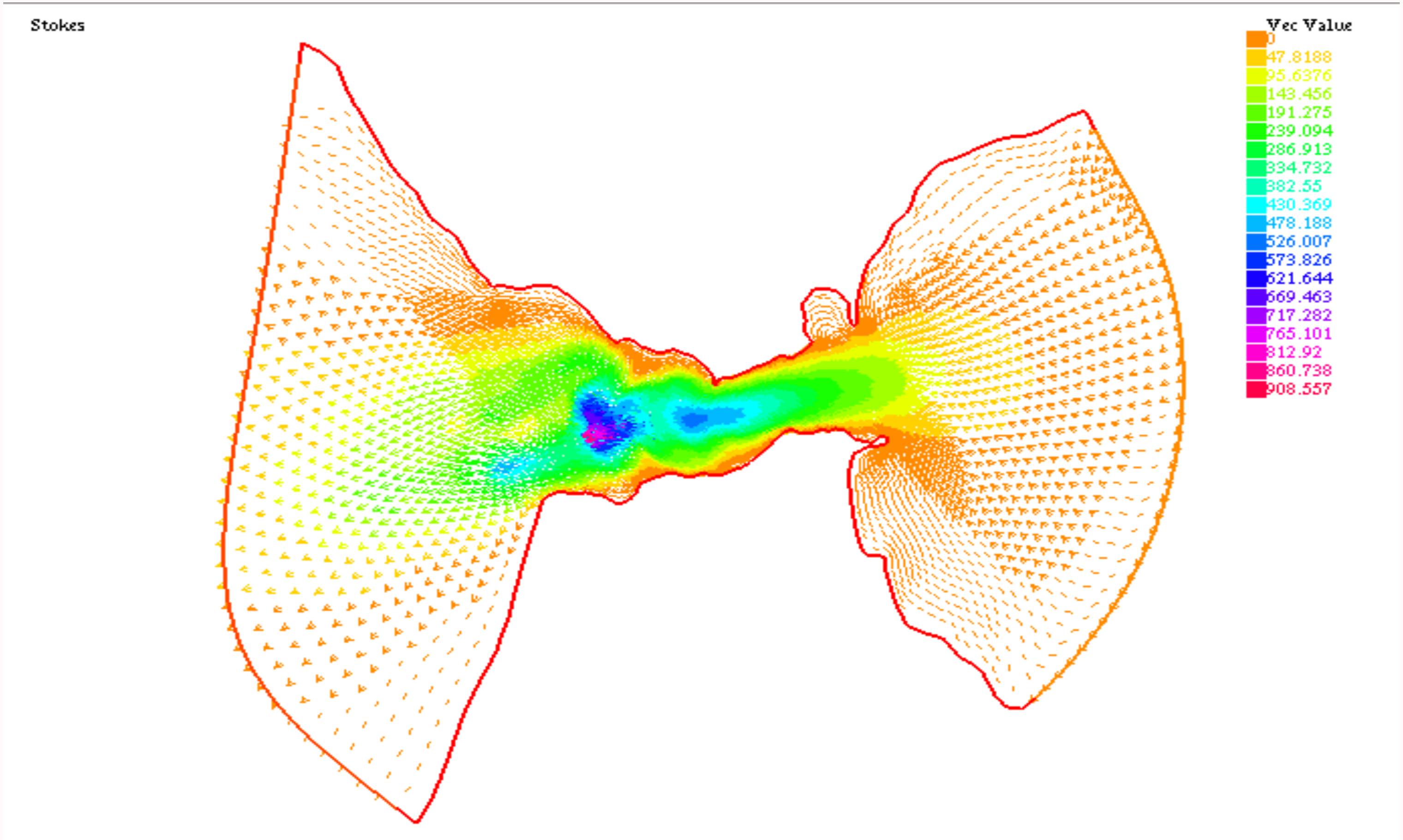


Study of Strait of Gibraltar

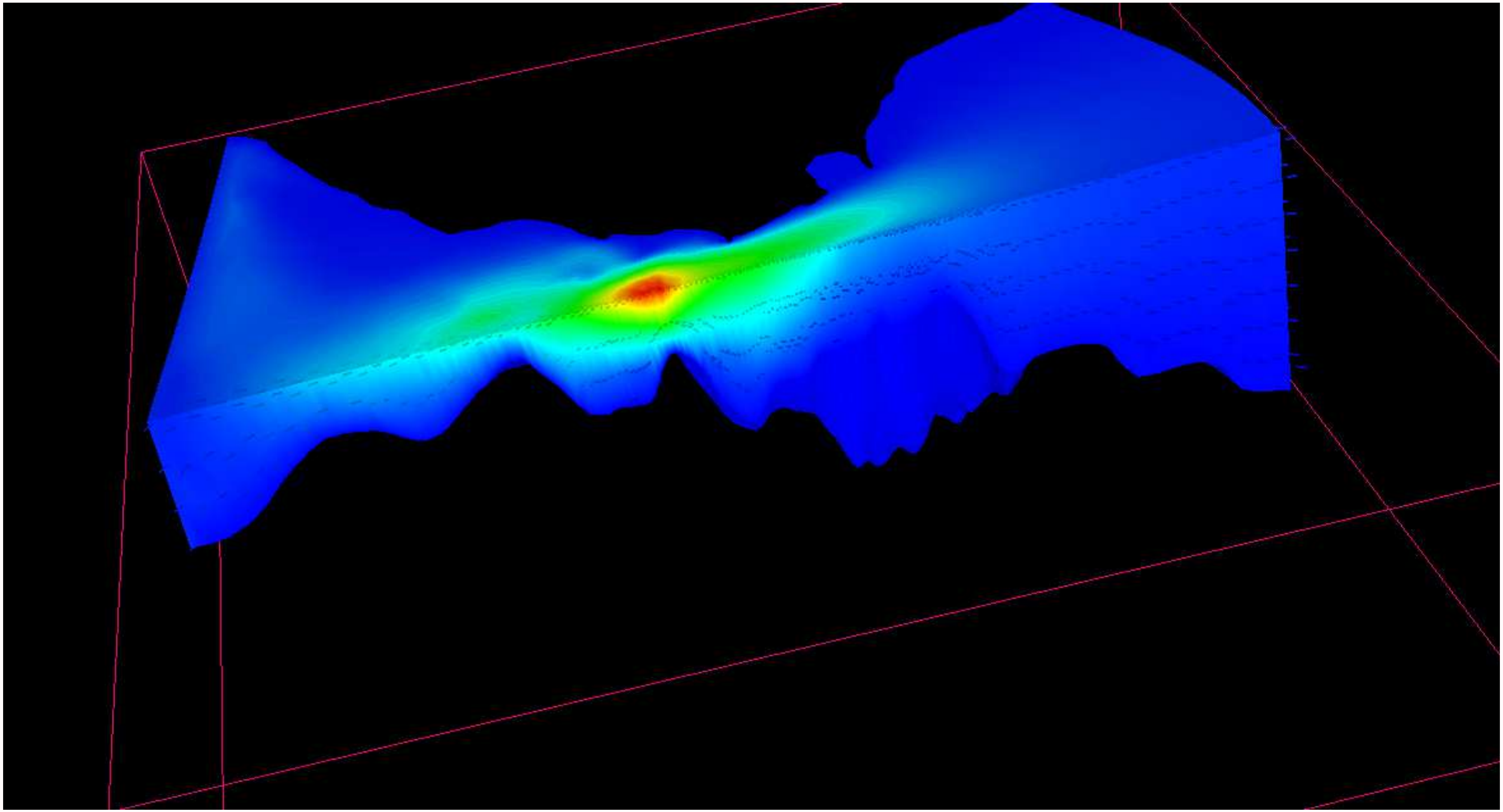
Atlantic water moving out of Mediterranean Sea



Velocity on the surface



Vertical cut



Domain Decomposition Method with Lagrange Multipliers

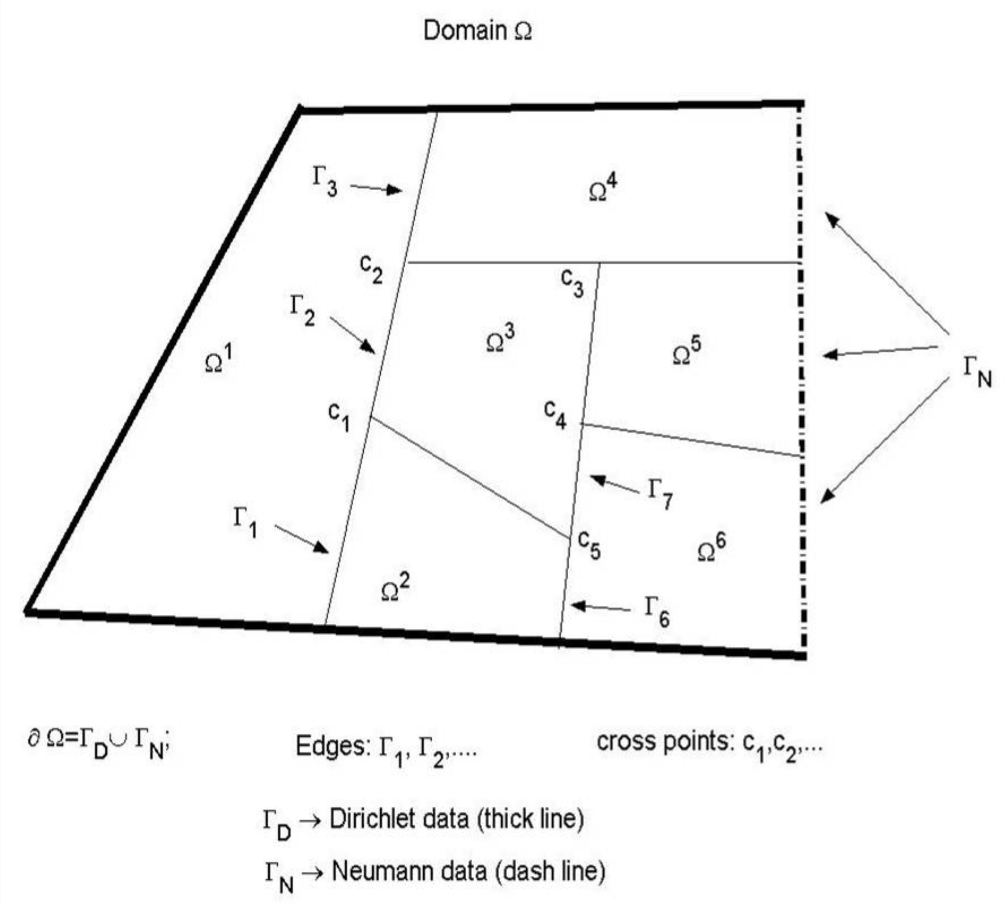
$$\begin{cases} -\Delta u &= f & \text{in } \Omega \\ u &= 0 & \text{on } \Gamma_D \neq \emptyset \\ \partial_n u &= 0 & \text{on } \Gamma_N. \end{cases}$$

Find $u \in H^1_{\Gamma_D}(\Omega)$ s.t. $\forall\, v \in H^1_{\Gamma_D}(\Omega)$

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega$$

$H^1_{\Gamma_D}(\Omega) = \{v \in H^1(\Omega);\ v_{\Gamma_D} = 0\}$.
Set $\Omega = \cup_{s=1}^{N_s} \Omega^s$, each Ω^s polygonal.

- **Cross points:** vertices of Ω^s not on $\overline{\Gamma_D}$.
- **Edges:** $\mathcal{E}_0 = \{\Gamma_e\}_{e=1}^{N_e}$ all edges inside $\overline{\Omega} \backslash \Gamma_D$ and $[v]_{\Gamma_e}$ is the jump across any Γ_e.
- $|\ \Omega^s \ | \sim H^2,\ |\ \Gamma_e\ | \sim H$



Mortar Finite Element Method with Lagrange multipliers

Introduced by Ben Belgacem (1999). Loosely speaking....

Torn apart discrete solution at interfaces...glue them back together using multipliers

Look for discrete functions $(u, \lambda) \in X_h \times M_h$ such that

$$\underbrace{\sum_{s=1}^{N_s} (\nabla u^s, \nabla v^s)_{\Omega^s}}_{=a(u,v)} - \sum_{\Gamma_e \in \mathcal{E}_0} \int_{\Gamma_e} \lambda_e \, (v^r - v^t) \; = \; \sum_{s=1}^{N_s} (f, v^s)_{\Omega^s}, \quad \forall v \in X_h \tag{1}$$

$$\underbrace{-\sum_{\Gamma_e \in \mathcal{E}_0} \int_{\Gamma_e} \mu_e \, (u^r - u^t)}_{=c(u,\mu)} \; = \; 0,, \quad \forall \mu \in M_h. \tag{2}$$

$[u]_{\Gamma_e} = u^r - u^t$ is the jump of u across Γ_e and $\lambda_e \approx \frac{\partial}{\partial n_e} u$ on Γ_e

A saddle point problem $\rightsquigarrow$ $\begin{cases} a(u,v) + c(v,\lambda) \; = \; l(v), & \forall v \in X_h, \\ c(u,\mu) \; = \; 0, & \forall \mu \in M_h. \end{cases}$

Different choices of X_h and M_h are used to satisfy inf-sup.

Now...look at continuous formulation and take into account:
First: **discontinuities of normal vectors on subdomain borders** and
Second: **continuity at cross points**.

Continuous approach: Allow jumps and Integrate by Parts

We look for $u^s \in H^1_b(\Omega^s) := H^1(\Omega_s) \cap H^1_{\Gamma_D}(\Omega)$ and $\lambda_e (= \partial_{\mathbf{n_e}} u) \in H^{-1/2}_{00}(\Gamma_e)$, $\forall \Gamma_e \in \mathcal{E}_0$ s.t.

$$\underbrace{\sum_{s=1}^{N_s} (\nabla u^s, \nabla v^s)_{\Omega^s}}_{=a(u,v)} - \sum_{\Gamma_e \in \mathcal{E}_0} < \lambda_e, v^r - v^t >_{-1/2,00,\Gamma_e} \;=\; \sum_{s=1}^{2} (f, v^s)_{\Omega_s}, \;\forall v_s \in H^1_b(\Omega_s)$$

$$\underbrace{- \sum_{\Gamma_e \in \mathcal{E}_0} < \mu_e, u^r - u^t >_{-1/2,00,\Gamma_e}}_{=c(u,\mu)} \;=\; 0, \quad \forall \mu \in \prod_{\Gamma_e \in \mathcal{E}_0} H^{-1/2}_{00}(\Gamma_e).$$

A continuous saddle point problem. **Trouble with crosspoints and inf-sup:**

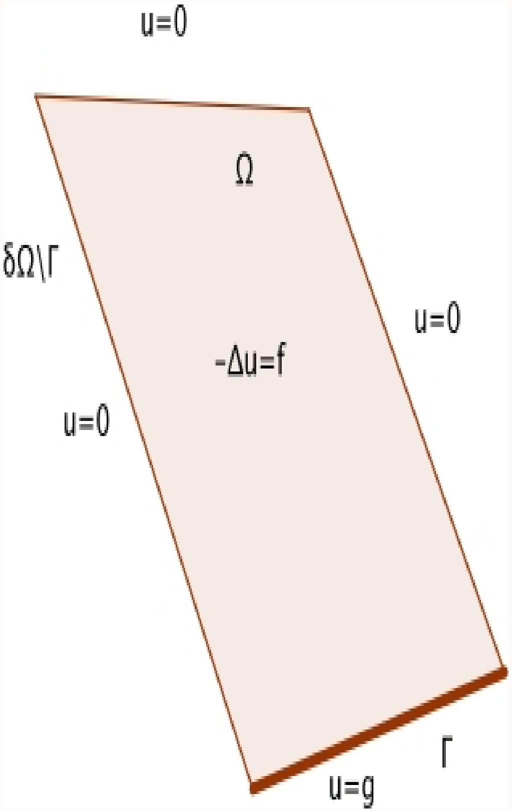
1. $a = \sum_{s=1}^{N_s} (\nabla u^s, \nabla v^s)_{\Omega^s}$ **does not control all jumps**, $[v]_{\Gamma_e} \in H^{1/2}_{00}(\Gamma_e) \subsetneq L^2(\Gamma_e)$

2. $c = \sum_{\Gamma_e \in \mathcal{E}_0} < \mu_e, [u]_{\Gamma_e} >_{-1/2,00,\Gamma_e} = \sum_{\Gamma_e \in \mathcal{E}_0} \int_{\Gamma_e} \mu_e \,(u^t - u^r)$ **L^2 testing...**

L^2 is...much larger than needed: $H^{1/2}_{00}(\Gamma_e)$.

Mesh dependent norms and non standard discrete spaces are used (Braess, Wolmuth, etc...)

About saddle points and Lagrange multipliers...



$$(1)\begin{cases} (\nabla u,\nabla v)_\Omega - \int_\Gamma \lambda\, v \;=\; (f,v)_\Omega, \;\forall v\in H^1_\Gamma(\Omega),\\ \int_\Gamma \mu\, u \;=\; \int_\Gamma \mu\, g, \quad \forall \mu \in L^2(\Gamma).\end{cases}$$

$$(2)\begin{cases} (\nabla u,\nabla v)_\Omega - (\varphi,v)_{1/2,00,\Gamma} \;=\; (f,v)_\Omega, \;\forall v\in H^1_\Gamma(\Omega),\\ (\psi,u)_{1/2,00,\Gamma} \;=\; (\psi,g)_{1/2,00,\Gamma}, \;\forall \psi \in H^{1/2}_{00}(\Gamma).\end{cases}$$

(1) does need stabilization

(2) does NOT need stabilization

Here $(\cdot,\cdot)_{1/2,00,\Gamma}$ is scalar product on $H^{1/2}_{00}(\Gamma)$

Why is that?? Norms balanced in (2) but not in (1)

$$\int_\Gamma \lambda\,\lambda \;=\; \|\lambda\|^2_{0,\Gamma} \qquad \not\geq \qquad C\,\|\nabla E\lambda\|^2_{0,\Omega} \quad \text{BAD!}$$

$$(\psi,\psi)_{1/2,00,\Gamma} \;=\; \|\psi\|^2_{1/2,00,\Gamma} \;\geq\; C\,\|\nabla E\psi\|^2_{0,\Omega} \quad \text{GOOD!}$$

helps the inf-sup condition

How about **identify** $H^{-1/2}_{00}(\Gamma_j) \equiv H^{1/2}_{00}(\Gamma_j)$ **via** R_j (Riesz Theorem) and write

$$\boxed{\int_\Omega \Delta u(x)\, v(x) dx + \int_\Omega \nabla u(x)\, \nabla v(x)\, dx = \sum_j (R_j \partial_{\mathbf{n}_j} u,\, v)_{1/2,00,\Gamma_j}.}$$

Idea $\sharp$ 1: Mortar in $H_{00}^{1/2}(\Gamma)$

Then, **the idea is** the use of this expression in the **mortar condition**:

$$\boxed{< \mu, u_2 - u_1 >_{-1/2,00,\Gamma} = (R\mu, u_2 - u_1)_{1/2,00,\Gamma}.}$$

Meaning of normal derivative is lost but ....**exact testing: in $H_{00}^{1/2}$**.

a **regularization of order 1** for the Lagrange multiplier:

compute an element $R\mu \in H_{00}^{1/2}(\Gamma)$ **instead of** $\mu \in H_{00}^{-1/2}(\Gamma)$

- **Possible if continuity at interfaces endpoints** is maintained.
- **Better behaved with respect to H^1 norm** than scalar product on $L^2(\Gamma)$.

When $\Gamma = [0, 1]$ and $v, w \in H_{00}^{1/2}(\Gamma)$

$$(w, v)_{1/2,00,\Gamma} = \int_0^1 w(x)v(x)dx + \int_0^1 \int_0^1 \frac{(w(x) - w(y))(v(x) - v(y))}{\mid x - y \mid^2} dxdy + \int_0^1 \frac{w(x)v(x)}{x\,(1 - x)}\, dx$$

Gaussian quadrature formulae and **change of variables** allow any $\Gamma \in C^1$.

Idea ♯ 2: Change Primal Space X to control jumps

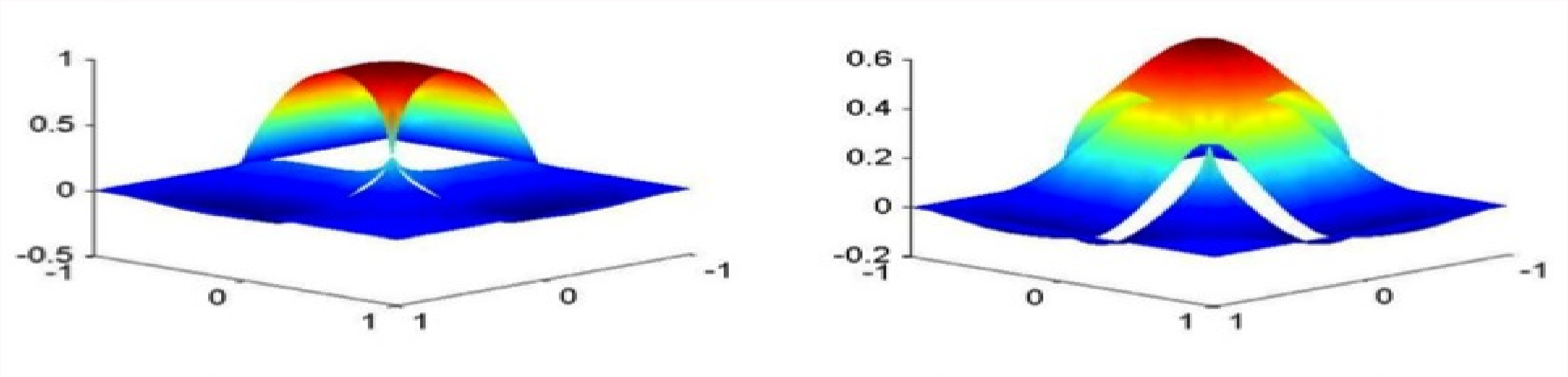
With **weak continuity at all cross-points**

$$\Omega^s \rightsquigarrow H^1_b(\Omega^s) = \{v^s \in H^1(\Omega^s); v^s = 0 \text{ on } \partial\Omega^s \cap \Gamma_D\}, \quad 1 \le s \le N_s,$$
$$\Omega \rightsquigarrow X = \{v \in L^2(\Omega); v^s = v_{|\Omega^s} \in H^1_b(\Omega^s),\ [v]_{\Gamma_e} = v^t - v^r \in H^{1/2}_{00}(\Gamma_e),\ \forall \Gamma_e \in \mathcal{E}_0\}$$

Hilbert space normed by $\|v\|^2_X = (v, v)_X$ where $(\cdot,\cdot)_X$ is the scalar product

$$(u, v)_X = \sum_{s=1}^{N_s} (\nabla u^s, \nabla v^s)_{\Omega^s} + \sum_{\Gamma_e \in \mathcal{E}_0} ([u]_{\Gamma_e}, [v]_{\Gamma_e})_{1/2,00,\Gamma_e}, \quad u, v \in X.$$

X is the **natural space where the discrete solution lives** (Braess, Ben Belgacem, etc...)



Proposed method for Laplace operator

We look for $(u,\lambda) \in X \times N (N = \prod_{\Gamma_e \in \mathcal{E}_0} H_{00}^{1/2}(\Gamma_e))$ such that for all $(v,\mu) \in X \times N$

$$\begin{aligned}
\underbrace{\sum_{s=1}^{N_s} (\nabla u^s, \nabla v^s)_{\Omega^s} + \sum_{\Gamma_e \in \mathcal{E}_0} ([u]_{\Gamma_e}, [v]_{\Gamma_e})_{1/2,00,\Gamma_e}}_{a(u,v)=(u,v)_X} + \sum_{\Gamma_e \in \mathcal{E}_0} (\lambda_e, [v]_{\Gamma_e})_{1/2,00,\Gamma_e} &= \sum_{s=1}^{N_s} (f, v^s)_{\Omega^s}, \\
\underbrace{\sum_{\Gamma_e \in \mathcal{E}_0} (\mu_e, [u]_{\Gamma_e})_{1/2,00,\Gamma_e}}_{c(u,\mu)=([u],\mu)_M} &= 0.
\end{aligned}$$

$u \in H^1_{\Gamma_D}(\Omega)$ solves original problem; $\lambda = \{\lambda_e\}_e$; λ_e the Riesz repr. of $\pm\partial_{\mathbf{n}_e} u \in H_{00}^{-1/2}(\Gamma_e)$.

When Galerkin...FETI-DP mortar method.

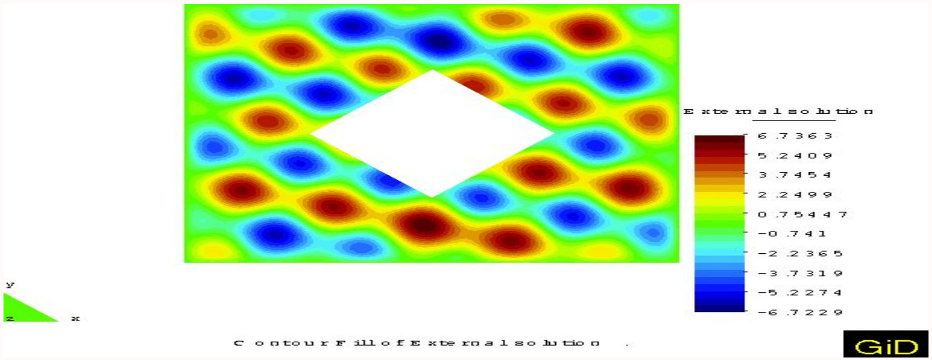
Recall that when $\Gamma = [0,1]$ and $v, w \in H_{00}^{1/2}(\Gamma)$

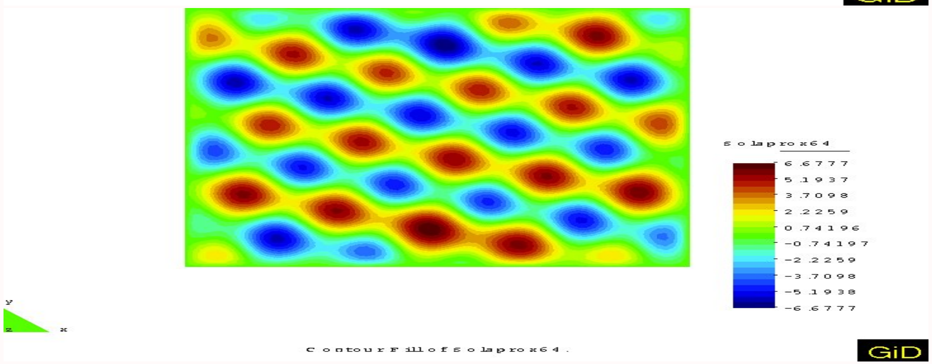
$$(w,v)_{1/2,00,\Gamma} = \int_0^1 w(x)v(x)dx + \int_0^1 \int_0^1 \frac{(w(x)-w(y))(v(x)-v(y))}{\mid x-y \mid^2} dxdy + \int_0^1 \frac{w(x)v(x)}{x\,(1-x)}\,dx$$

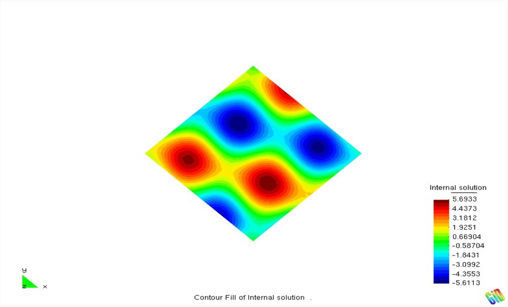
On FreeFem++: nonlinear boundary integrals coupled with stiffness matrix

Test with floating subdomain for Laplace

with good convergence results...

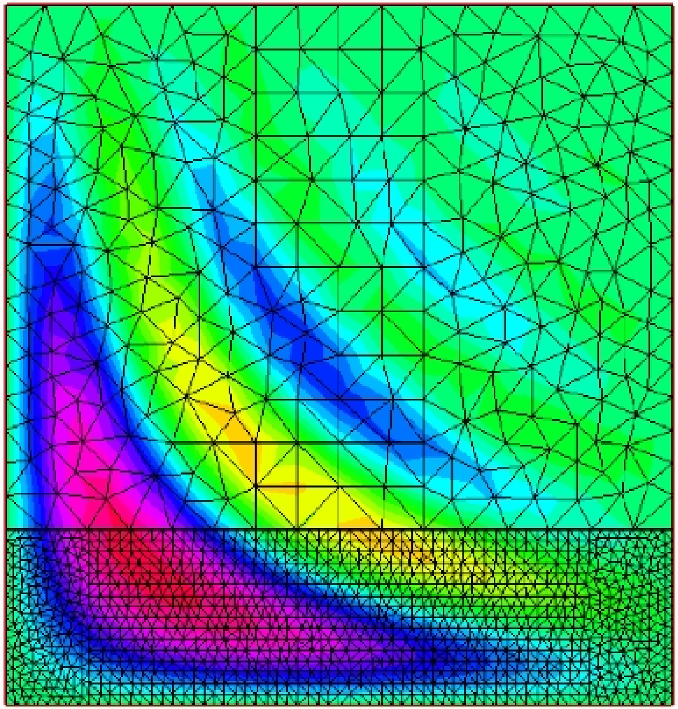


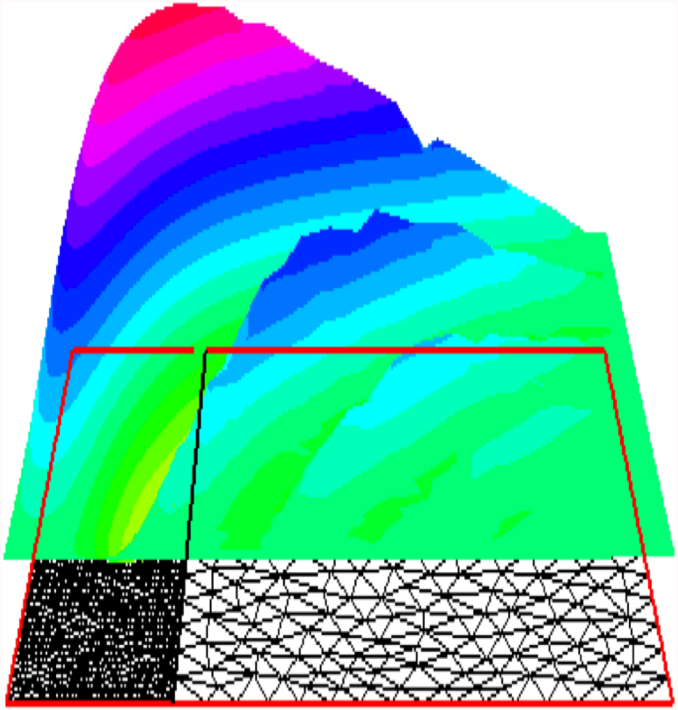




Numerical test with non-conforming mesh for Laplace

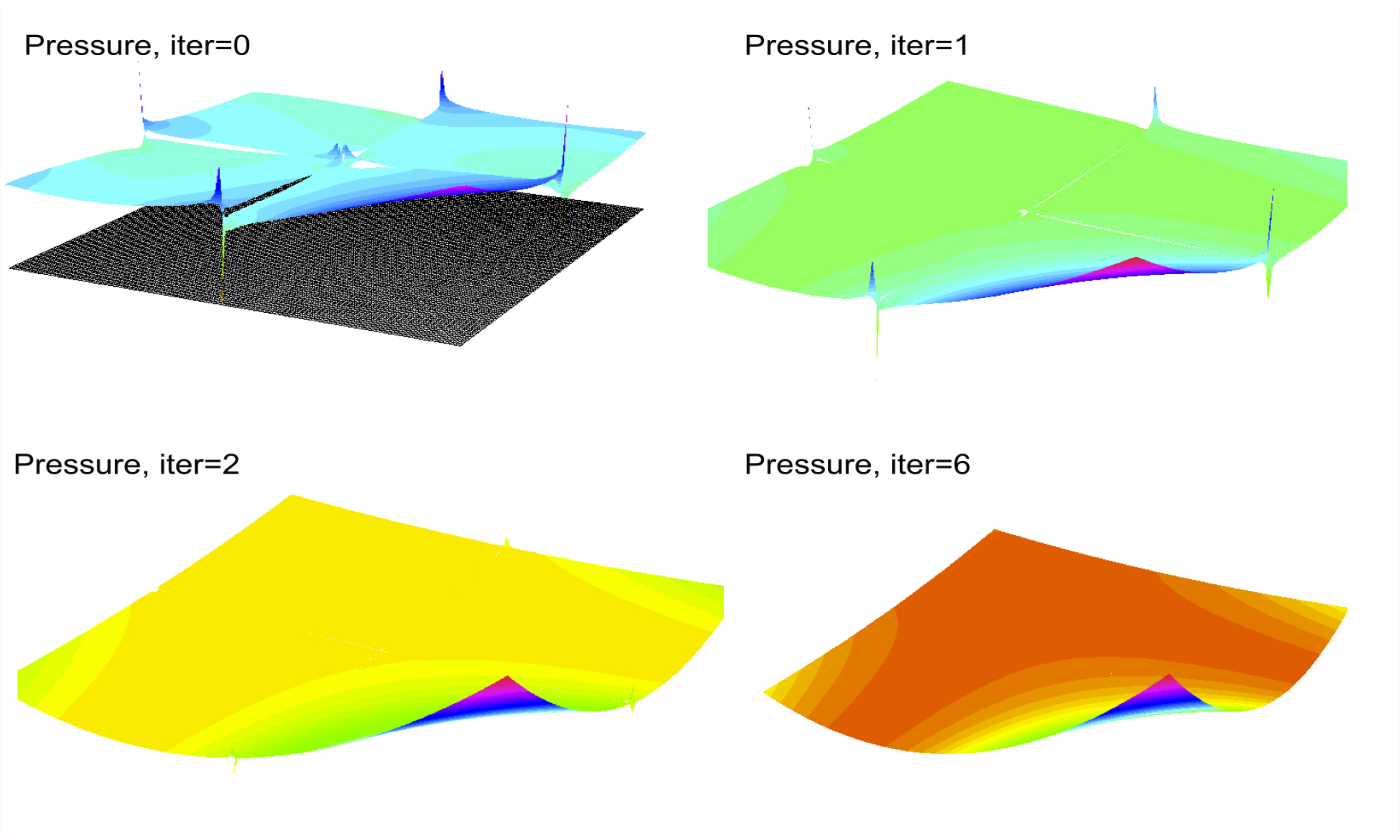
with good convergence results...





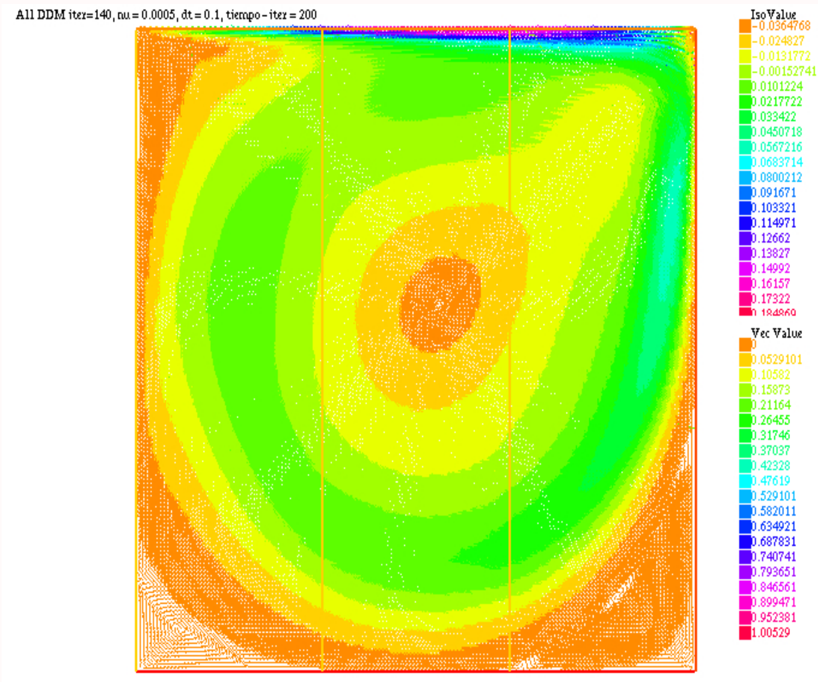
Numerical test with conforming mesh for Stokes

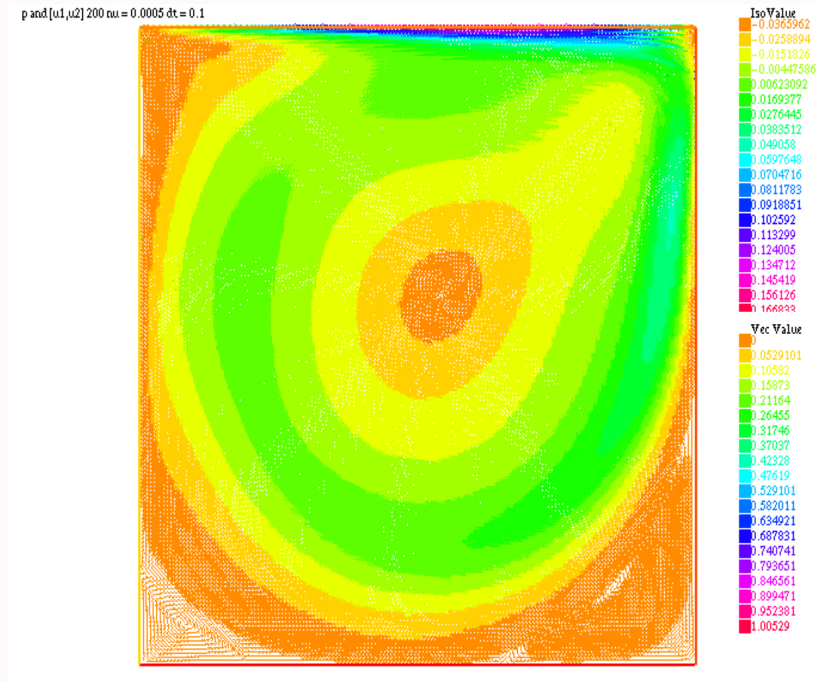
with similar convergence results...



Navier-Stokes using characteristics: driven lid test

On $\Omega = (0,1)^2$ with uniform conforming meshes and $\mu = 0.0005$.

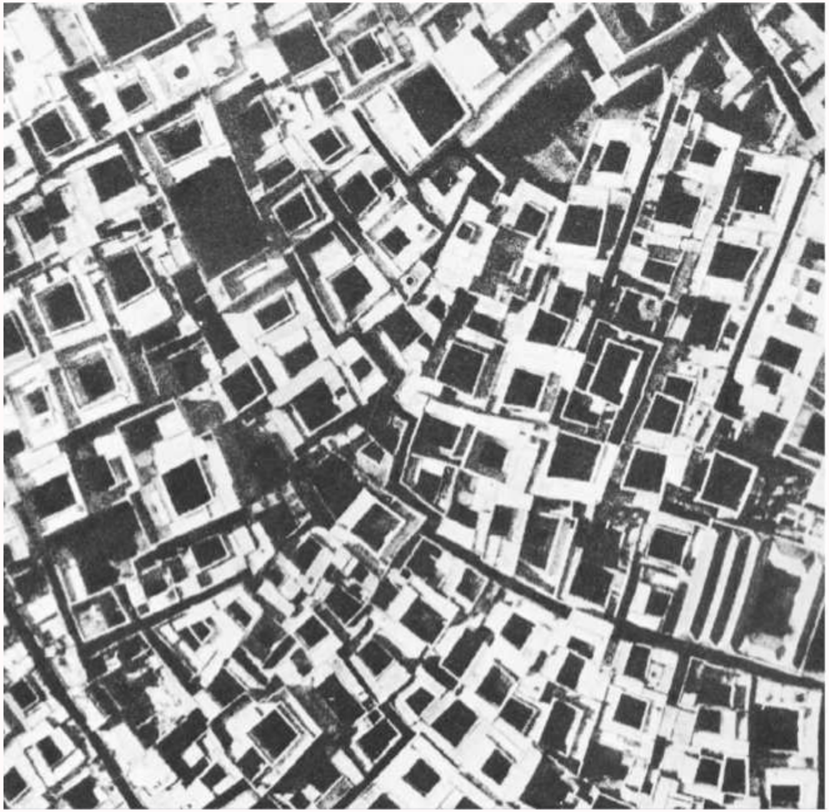




Sustainable architecture: Example I

Mediterranean "patio" (courtyard) for natural cooling of buildings

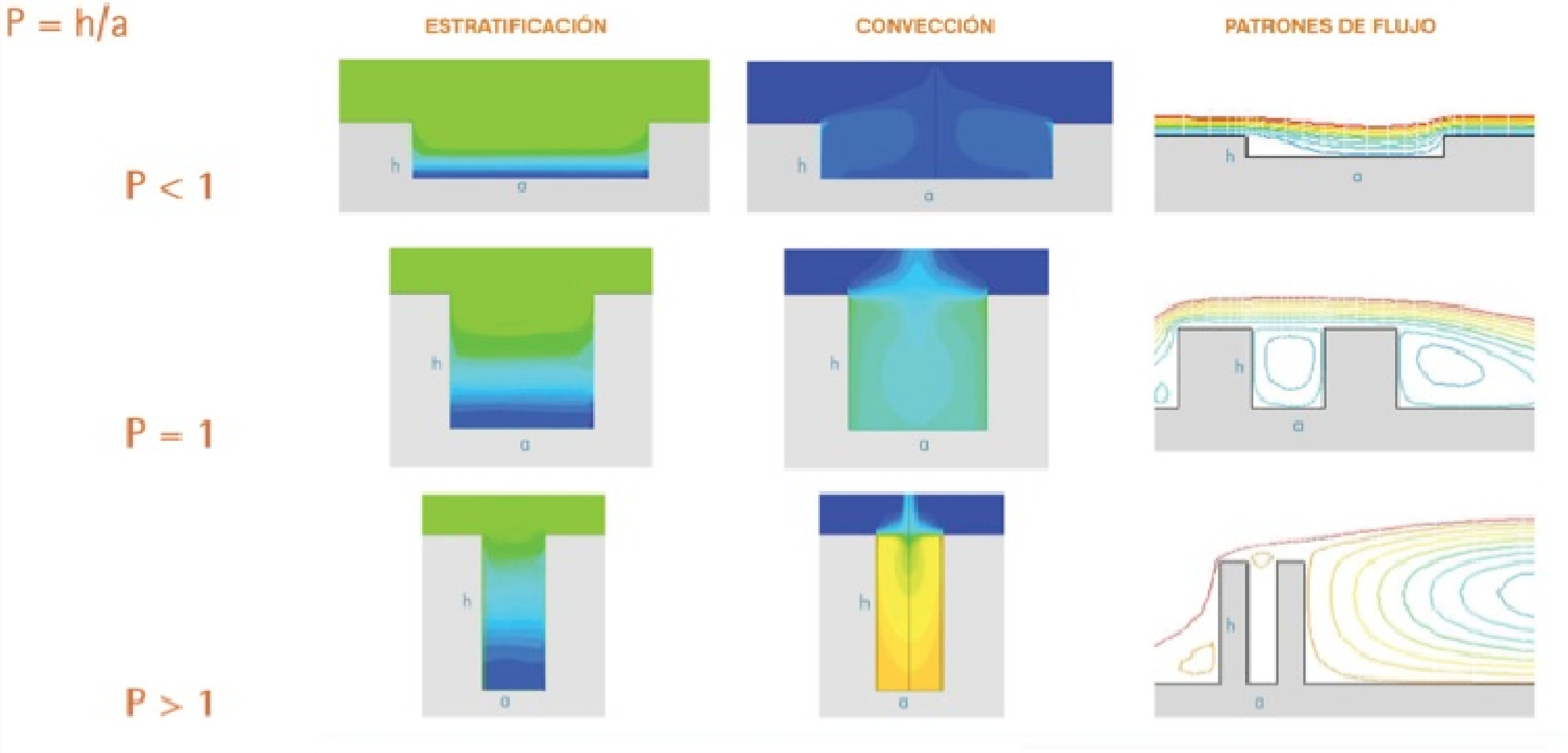
Sevilla (Spain)



Marrakesh(Marocco)

Very old technique...study to design energy-efficient buildings

Thermodynamical behaviour of patios



Ratio P=h/a Stratification Convection Wind flow pattern

Research in Nanosciences & technologies

Research in Materials

Research in Production

Public Private Partnerships in research

Research Fund for Coal and Steel (RFCS)

Energy-efficient Buildings

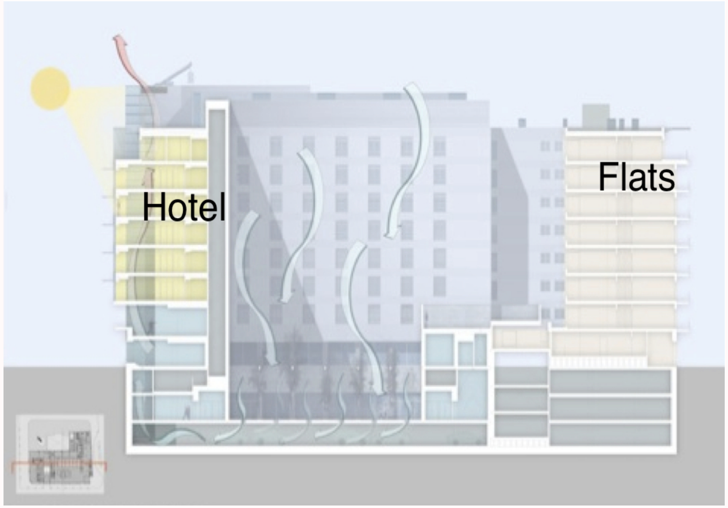
Energy-efficient buildings (EeB) consists of a financial envelope of € 1 billion to boost the construction sector, and aims at promoting green technologies and the development of energy efficient systems and materials in new and renovated buildings - this, with a view to radically reducing their energy consumption and CO2 emissions.

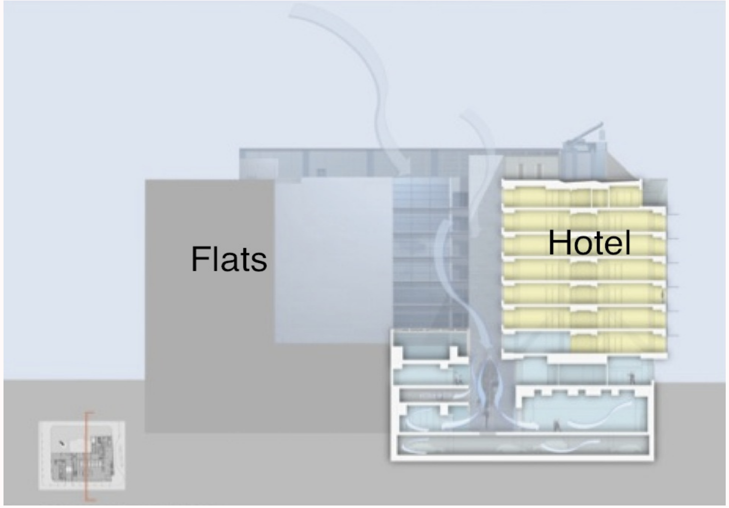
The programme is financed jointly by industry and the European Commission under the Seventh Framework Programme for Research (FP7). The research programme has started in July 2009 with coordinated calls for research proposals, jointly implemented by DG Research and Innovation, DG Energy and DG Information Society and Media.
Further calls for proposals have been launched in July 2010 and will be launched every year in July until 2012.

Hotel Monte Malaga, Malaga - Spain

Building block top view

Nominated to CONSTRUMAT 2007 Awards (Cataluña-Spain) for the use of patios at different depth levels and use of solar energy

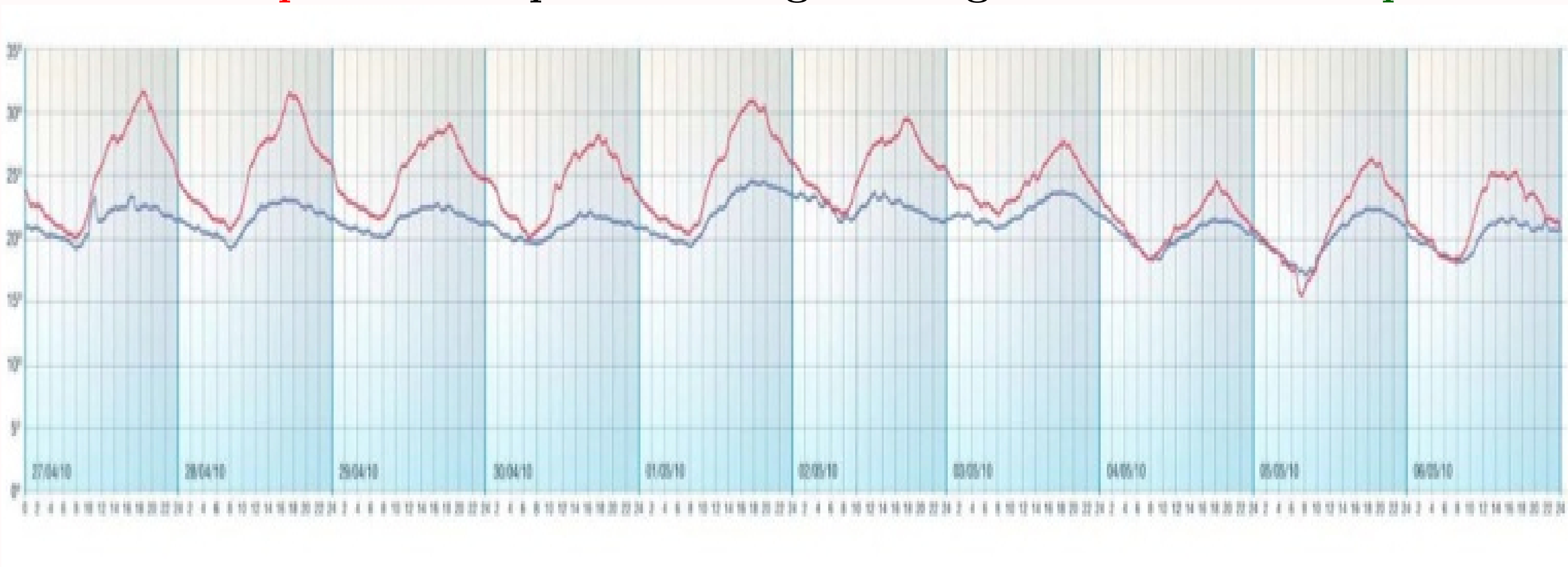




Cross Sections of the Building+Patio

View of the lowest Patio

Outdoor temperatures Up to nine degrees larger than Patio temperatures



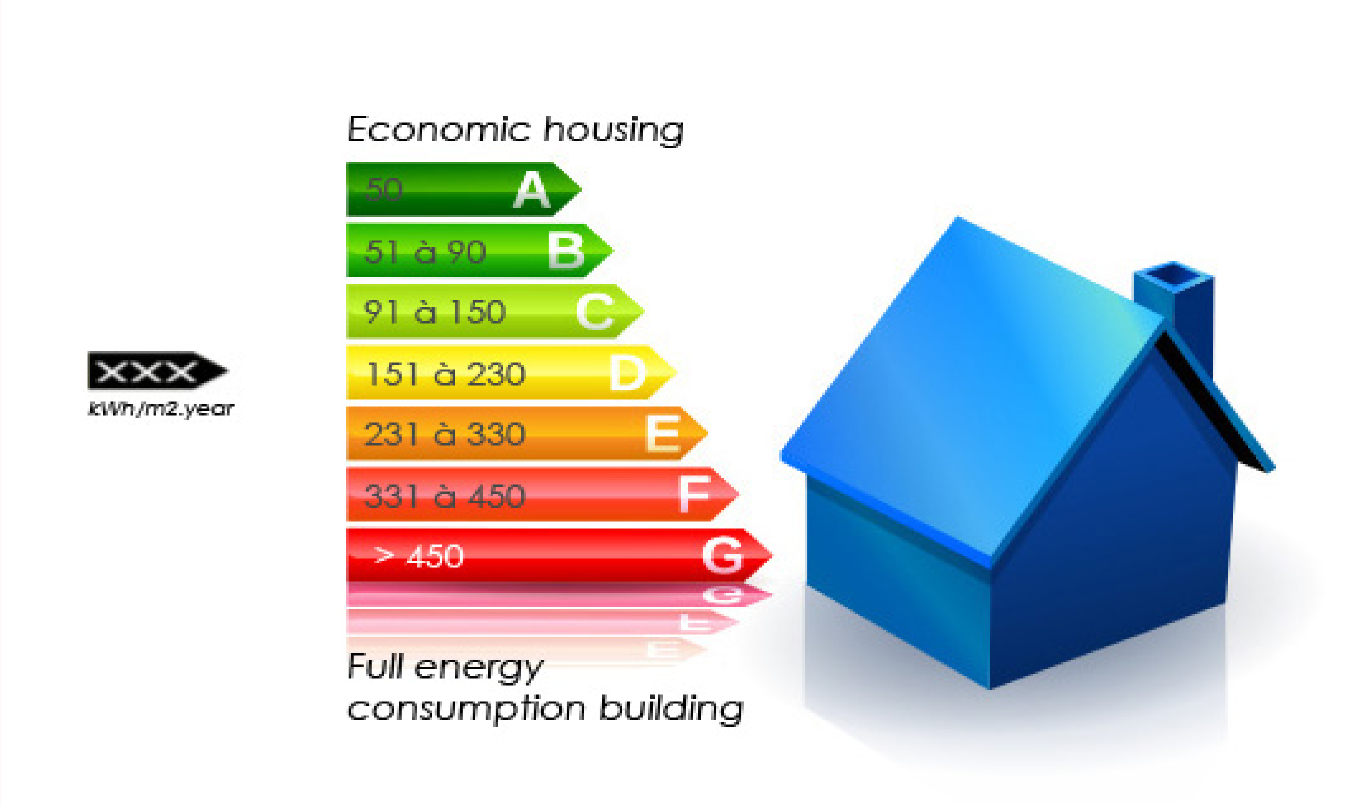
Air intakes for the cooling machines from the patio are more energy efficient

Building used as a benchmark with FreeFem++ to test the efficiency of the patio: run demo

Estratificacion.mpg

Sustainable architecture: Example II

Energetic Classification of Buildings



Application to design energy-efficient buildings

Idea: Integrate FreeFem++ with current softwares

Energetic Classification of Buildings

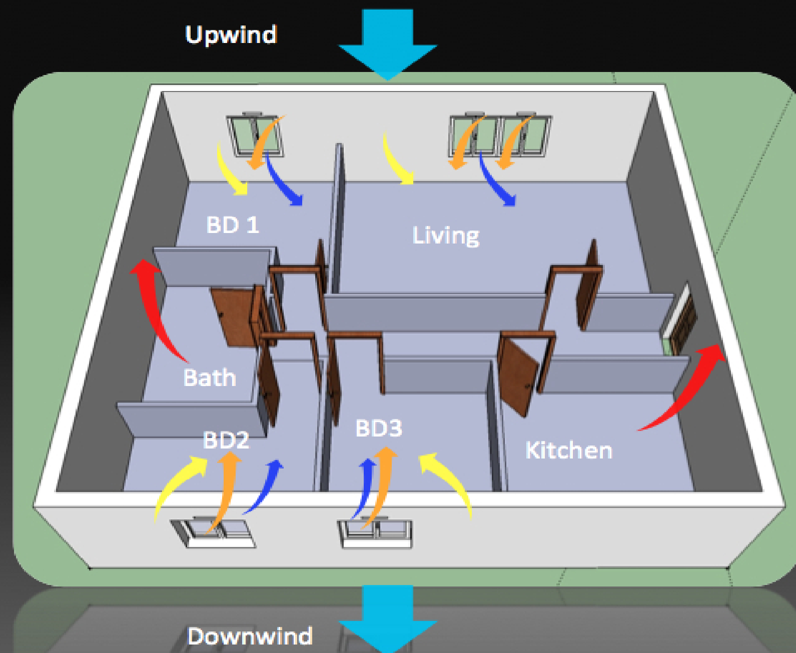
1.- Ventilation in Buildings

Driven Forces:

- Mechanical
- Natural
  - Wind
  - Temperature differences

Inlets/Outlets:

- Facade leakage.
- Windows leakage.
- Vents.
- Exhaust fans.



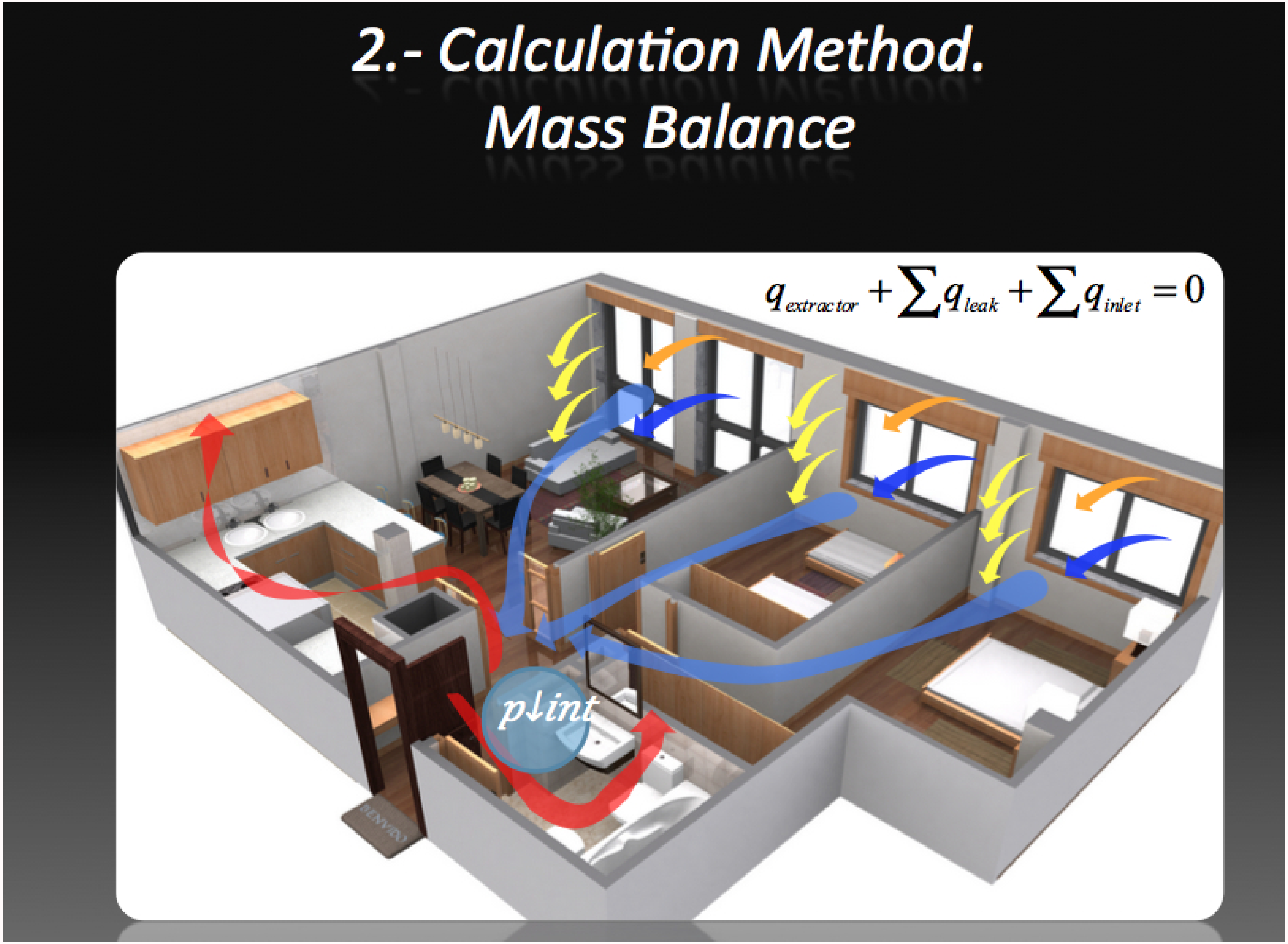
Facade leakage.

Windows leakage.

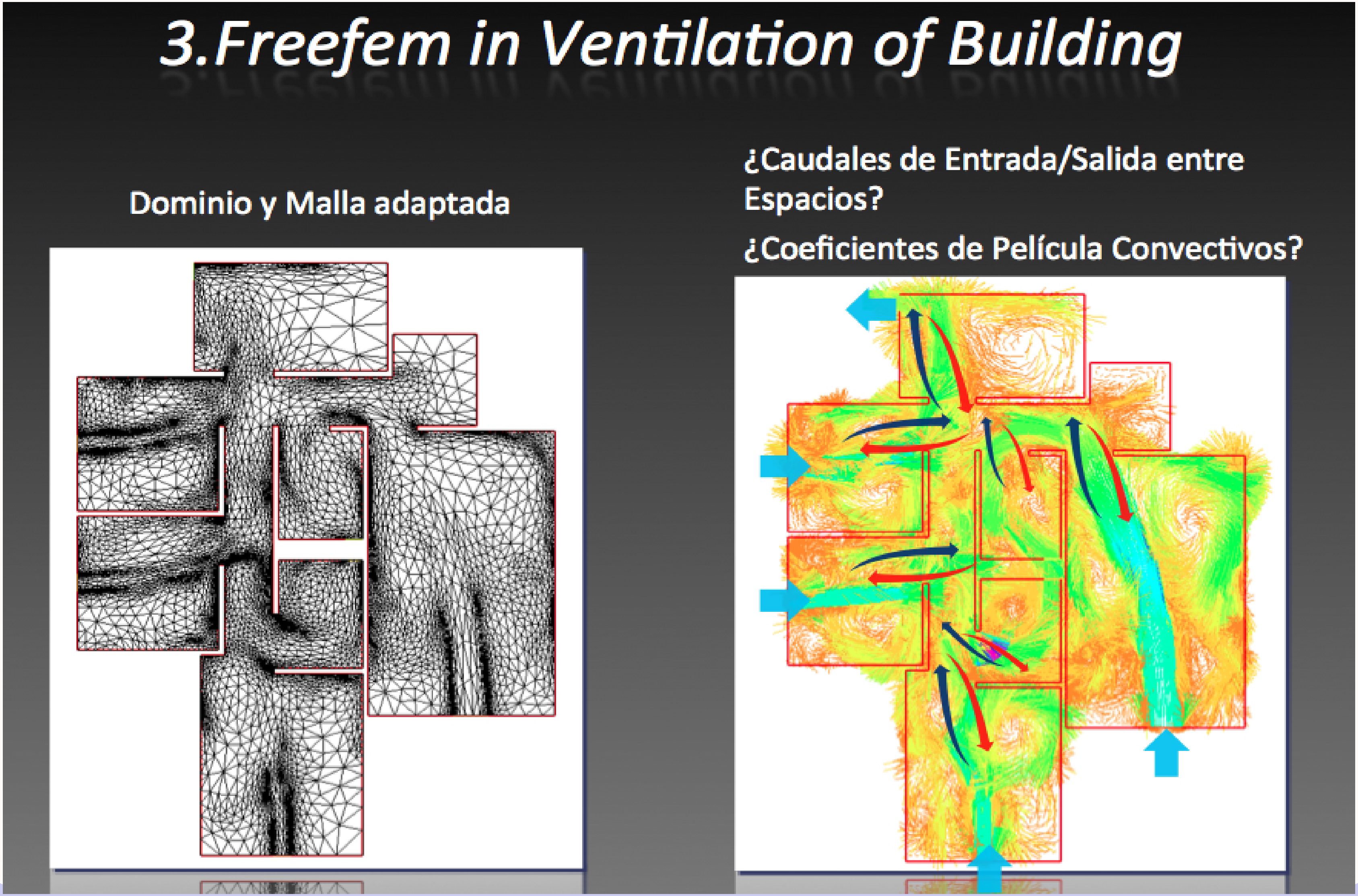
Vents.

Exhaust fans.

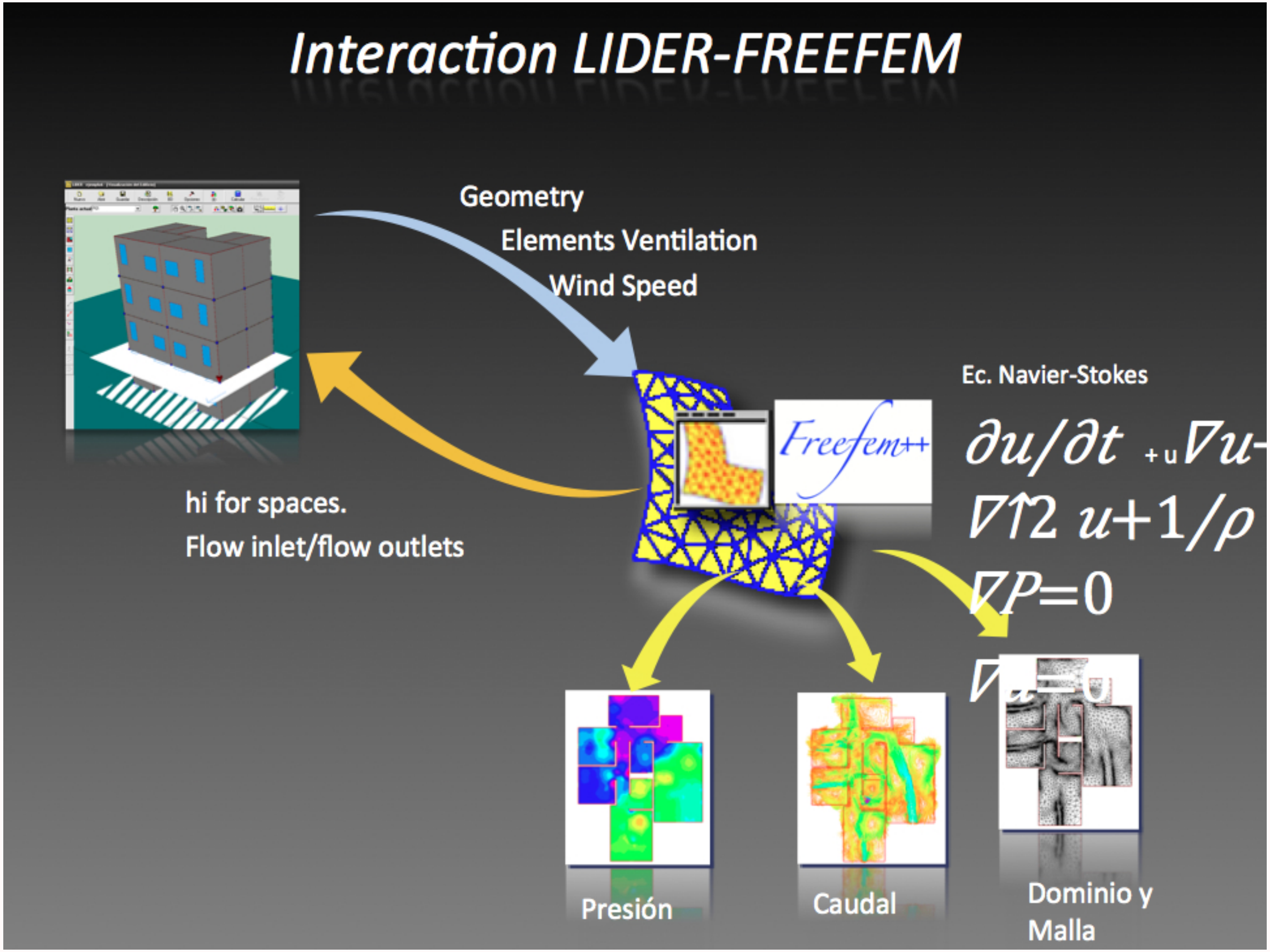
Energetic Classification of Buildings



Energetic Classification of Buildings



Energetic Classification of Buildings



Conclusions:

Computations with FreeFem++...

Our deepest thanks to Prof. Hecht and collaborators.

Thank you for your attention (?)