Solving Maxwell's equations with domain decomposition methods for brain imaging

Pierre-Henri Tournier

Laboratoire Jacques-Louis Lions

INRIA équipe ALPINES

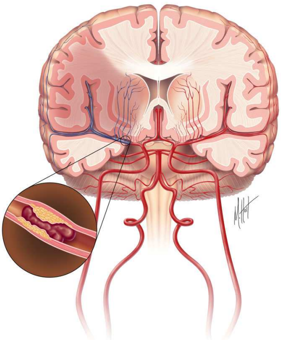
7th Tutorial and Workshop on FreeFem++

December 16, 2015

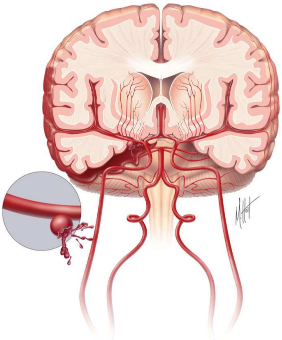
Motivation

2 types of stroke :

ischemic



hemorrhagic



The correct treatment depends on the type of stroke :

$\Longrightarrow$ restore blood flow

$\Longrightarrow$ lower blood pressure

Motivation

In order to differentiate between ischemic and hemorrhagic stroke, CT scan or MRI is typically used.
Microwave tomography is a novel and promising imaging technique, especially for medical and brain imaging.

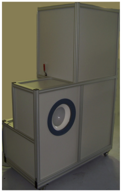
	CT scan	MRI	microwave tomography
resolution	excellent	excellent	good
fast	-	-	✓
safe	-	✓	✓
mobile	-	-	✓
cost	∼ 1 000 000 €	∼1 000 000 €	∼300 000 €

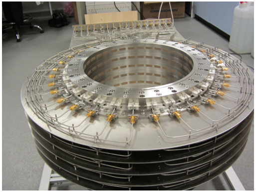
Diagnosing a stroke at the earliest possible stage is crucial for all following therapeutic decisions.

Motivation

EMTensor GmbH, Vienna, Austria.

First-generation prototype: cylindrical chamber composed of 5 rings of 32 antennas (ceramic-loaded waveguides).





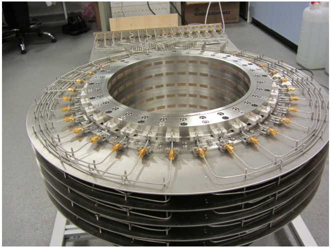
Motivation

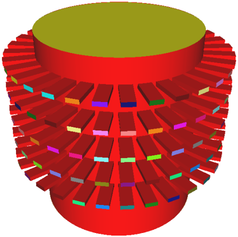
MEDIMAX ANR project: development of a new robust inversion tool associated with the electromagnetic forward problem in highly heterogeneous media based on the high-level integrated development environment FreeFem++.

Partners:

- LEAT : Christian Pichot (project coordinator), Iannis Aliferis, Claire Migliaccio and Ibtissam El Kanfoud;
- JAD : Victorita Dolean, Francesca Rapetti, Richard Pasquetti and Marcella Bonazzoli;
- MAP5 : Maya de Buhan and Marion Darbas;
- LJLL : Frédéric Nataf, Frédéric Hecht, Antoine Le Hyaric and Pierre-Henri Tournier.

The direct problem

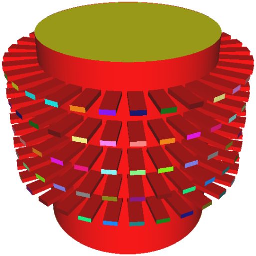




We consider in Ω a linear, isotropic, non-magnetic, dispersive, dissipative dielectric material.
The direct problem consists in finding the electromagnetic field distribution in the whole chamber, given a known material and transmitted signal.

The direct problem

For each of the 5×32 antennas, the associated electric field $\mathbf{E}_i$ is the solution of Maxwell's equations:



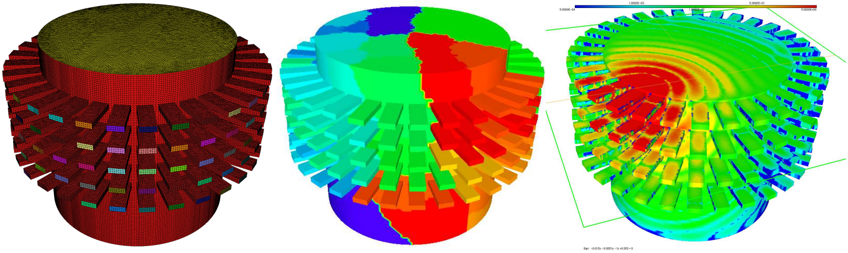
$$(1) \quad \begin{cases} \nabla \times (\nabla \times \mathbf{E}_i) - \mu_0 \left(\omega^2 \varepsilon + \mathrm{i}\omega\sigma\right) \mathbf{E}_i = \mathbf{0} & \text{in } \Omega, \\ \mathbf{E}_i \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_{\text{metal}}, \\ (\nabla \times \mathbf{E}_i) \times \mathbf{n} + \mathrm{i}\beta \mathbf{E}_i \times \mathbf{n} = \mathbf{g} & \text{on } \Gamma_i, \\ (\nabla \times \mathbf{E}_i) \times \mathbf{n} + \mathrm{i}\beta \mathbf{E}_i \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_j \text{ , } j \neq i, \end{cases}$$

where μ_0 is the permittivity of free space, ω is the incident angular frequency , β is the wavenumber of the waveguide, $\varepsilon > 0$ is the dielectric permittivity and $\sigma > 0$ is the conductivity.

The direct problem

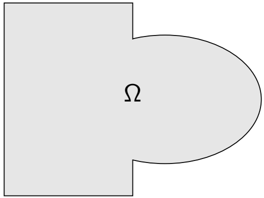
Spatial discretization using Nedelec finite elements yields a large sparse linear system $Au = f_i$ for each transmitting antenna i. We need a robust and efficient solver for second order time-harmonic Maxwell's equations with heterogeneous coefficients.

$\Longrightarrow$ Use domain decomposition methods to produce parallel preconditioners for the GMRES algorithm.



Overlapping domain decomposition methods

Consider the linear system: $Au = f \in \mathbb{R}^n$.



Overlapping domain decomposition methods

Consider the linear system: $Au = f \in \mathbb{R}^n$.
Given a decomposition of $[\![1;n]\!]$, $(\mathcal{N}_1, \mathcal{N}_2)$, define:

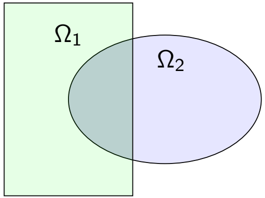
- the restriction operator R_i from $[\![1;n]\!]$ into $\mathcal{N}_i$,
- R_i^T as the extension by 0 from $\mathcal{N}_i$ into $[\![1;n]\!]$.

Then solve concurrently:

$$u_1^{m+1} = u_1^m + A_{11}^{-1} R_1 (f - Au^m) \qquad u_2^{m+1} = u_2^m + A_{22}^{-1} R_2 (f - Au^m)$$

where $u_i = R_i u$ and $A_{ij} := R_i A R_j^T$.

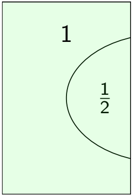
[Schwarz 1870]

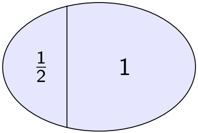


Overlapping domain decomposition methods

Duplicated unknowns coupled via a *partition of unity*:

$$I = \sum_{i=1}^{N} R_i^T D_i R_i.$$





To solve $Au = f$ Schwarz methods can be viewed as preconditioners for a fixed point algorithm:

$$u^{n+1} = u^n + M^{-1}(f - Au^n).$$

- $M_{\text{RAS}}^{-1} := \sum_{i=1}^{N} R_i^T D_i A_i^{-1} R_i$ with $A_i = R_i A R_i^T$
- $M_{\text{ORAS}}^{-1} := \sum_{i=1}^{N} R_i^T D_i B_i^{-1} R_i$ Optimized transmission conditions

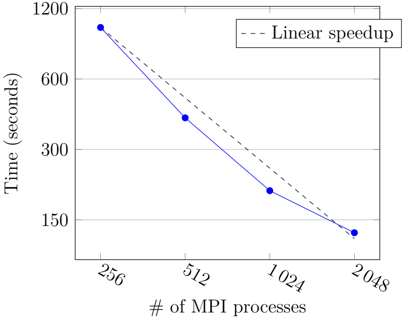
[B. Després 1991] for Helmholtz

HPDDM

HPDDM is an efficient parallel implementation of domain decomposition methods

by Pierre Jolivet and Frédéric Nataf

- header-only library written in C++11 with MPI and OpenMP
- interfaced with FreeFem++



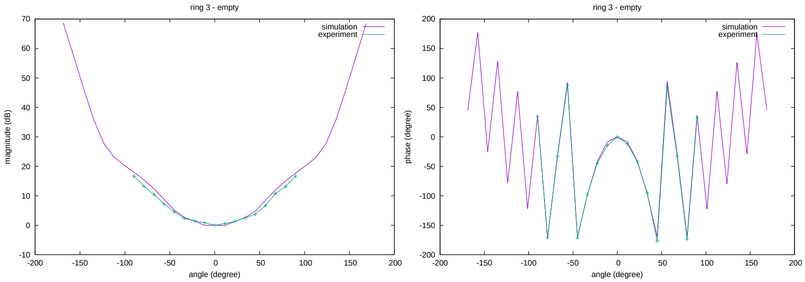
Strong scalability test for Maxwell 3D - 100M d.o.f.

Comparison with experiments

The experimental measurements obtained from the antennas are the reflexion and transmission coefficients

$$S_{ij}^{obs} = \frac{\int_{\Gamma_j} \overline{\mathbf{E}_i} \cdot \mathbf{E}_0 d\gamma}{\int_{\Gamma_j} |\mathbf{E}_0|^2 d\gamma}, \text{ for } i, j = 1, ..., 160,$$

where $\mathbf{E}_0$ is the fundamental mode of the waveguide.



The inverse problem

For given ω, β and $\mathbf{g}$, the inverse problem consists in recovering ε and σ such that for each transmitting antenna i, the solution $\mathbf{E}_i$ to the associated Maxwell's problem matches the observations:

$$\frac{\int_{\Gamma_j} \overline{\mathbf{E}_i} \cdot \mathbf{E}_0 d\gamma}{\int_{\Gamma_j} |\mathbf{E}_0|^2 d\gamma} = S_{ij}^{obs} \quad \text{for each receiving antenna } j.$$

Difficulties :

- inverse problems are ill-posed
- noise in the experimental data
- solving the inverse problem means solving the direct problem multiple times $\Longrightarrow$ time-consuming

The inverse problem

Solving the inverse problem corresponds to minimizing the following cost functional:

$$
\begin{aligned}
J(\kappa) &= \frac{1}{2}\sum_{i=1}^{160}\sum_{j=1}^{160}\left|S_{ij}(\kappa) - S_{ij}^{obs}\right|^2 \\
&= \frac{1}{2}\sum_{i=1}^{160}\sum_{j=1}^{160}\left|\frac{\int_{\Gamma_j}\overline{\mathbf{E}_i(\kappa)}\cdot\mathbf{E}_0\,d\gamma}{\int_{\Gamma_j}|\mathbf{E}_0|^2 d\gamma} - S_{ij}^{obs}\right|^2,
\end{aligned}
$$

where $\kappa(x) \in \mathbb{C}$ and $S_{ij}(\kappa)$ depends on the solution $\mathbf{E}_i(\kappa)$ to

$$
\begin{cases}
\nabla \times (\nabla \times \mathbf{E}_i) - \kappa \mathbf{E}_i = \mathbf{0} & \text{in } \Omega, \\
\mathbf{E}_i \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_{\text{metal}}, \\
(\nabla \times \mathbf{E}_i) \times \mathbf{n} + \mathrm{i}\beta \mathbf{E}_i \times \mathbf{n} = \mathbf{g} & \text{on } \Gamma_i, \\
(\nabla \times \mathbf{E}_i) \times \mathbf{n} + \mathrm{i}\beta \mathbf{E}_i \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_j\ ,\ j \neq i.
\end{cases}
$$

The inverse problem

$$J(\kappa) = \frac{1}{2}\sum_{i=1}^{160}\sum_{j=1}^{160}\left|S_{ij}(\kappa) - S_{ij}^{obs}\right|^2 = \frac{1}{2}\sum_{i=1}^{160}\sum_{j=1}^{160}\left|\frac{\int_{\Gamma_j}\overline{\mathbf{E}_i(\kappa)}\cdot\mathbf{E}_0 d\gamma}{\int_{\Gamma_j}|\mathbf{E}_0|^2 d\gamma} - S_{ij}^{obs}\right|^2$$

For $i = 1, ..., 160$, we introduce the adjoint problem

$$\begin{cases} \nabla \times (\nabla \times \mathbf{F}_i) - \kappa \mathbf{F}_i = \mathbf{0} & \text{in } \Omega, \\ \mathbf{F}_i \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_{\text{metal}}, \\ (\nabla \times \mathbf{F}_i) \times \mathbf{n} + \mathrm{i}\beta \mathbf{F}_i \times \mathbf{n} = \dfrac{S_{ij}(\kappa) - S_{ij}^{obs}}{\int_{\Gamma_j}|\mathbf{E}_0|^2 d\gamma}\overline{\mathbf{E}}_0 & \text{on } \Gamma_j. \end{cases}$$

We have

$$DJ(\kappa, \delta\kappa) = \sum_{i=1}^{160}\Re\left[\int_\Omega \delta\kappa\, \mathbf{E}_i \cdot \mathbf{F}_i dx\right].$$

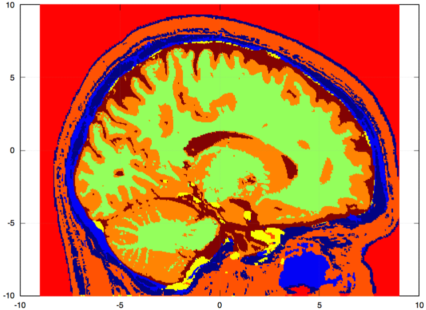
We can then compute the gradient to use in a gradient-based optimization algorithm.

some FreeFem++ code

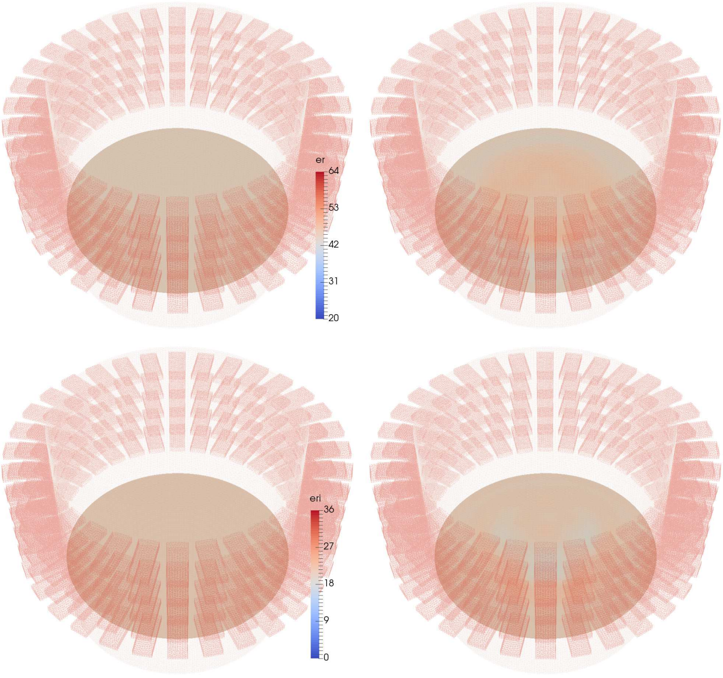
```
fespace Wh(Th,Edge03d); // local FE space
fespace WhP1(Th,P1);
macro generateTh(nm)nm = readmesh3("global.mesh");// EOM
build(generateTh, Th, ThBorder, ThOverlap, s, D,
    numberIntersection, arrayIntersection,
    restrictionIntersection, Wh, Pk, commrhs, 0)
...
macro Varf(varfName, meshName, PhName)
    varf varfName([Ex,Ey,Ez], [vx,vy,vz]) =
    int3d(meshName,qforder=qforder)(Curl(vx,vy,vz)'*Curl(Ex,
        Ey,Ez))
    -int3d(meshName,qforder=qforder)((2*pi*f/c0)^2*(er+
        kappariaug+1.i*kappaiiaug)*[vx,vy,vz]'*[Ex,Ey,Ez])
    -int2d(meshName,qforder=qforder,tlabwps)(1i*betawg*
        CrossN(vx,vy,vz)'*CrossN(Ex,Ey,Ez))
    CLtopbottom(meshName,er+kappariaug+1.i*kappaiiaug)
    + on(labmetal,Ex=0,Ey=0,Ez=0); // EOM
...
assemble(mat, rhs, Wh, Th, ThBorder, Varf, VarfOpt)
zschwarz A(mat, arrayIntersection, restrictionIntersection,
    scaling = D,communicator = commrhs);
...
DDM(A, nEx, Bi, O=Opt, excluded=0, timing=stats);
```

Numerical experiment - hemorrhagic stroke

- Brain model from X-ray data ($362 \times 434 \times 362$)

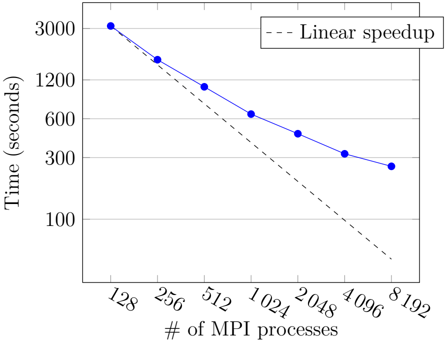


- $f = 1GHz$
- waveguides (ceramic) : $\epsilon_r = 59$
- matching liquid : $\epsilon_r = 44 + 20i$
- 10% white Gaussian noise on synthetic data
- reconstruction obtained after 45 minutes using 4096 processors of the Curie supercomputer (CEA).

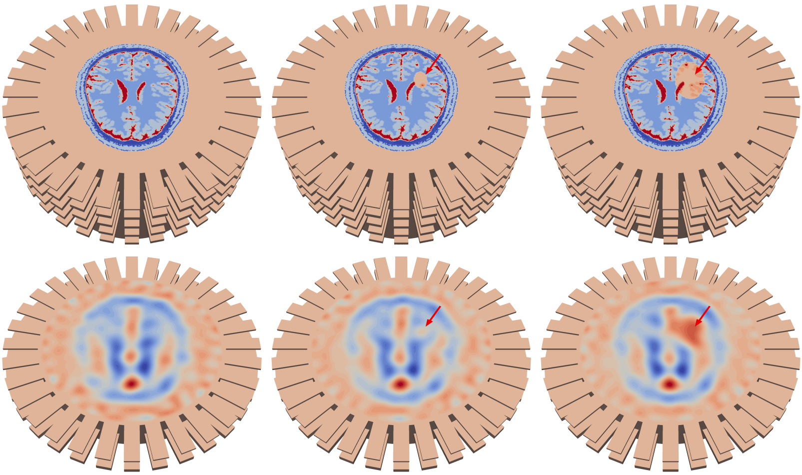

er
64
53
42
31
20
eri
36
27
18
9
0

Numerical experiment - hemorrhagic stroke

Idea: reconstruct the permittivity slice by slice, by taking into account the transmitting antennas corresponding to only one ring and truncating the computational domain.
$\Longrightarrow$ Reconstructed images corresponding to one ring obtained in less than 5 minutes.



Numerical experiment - hemorrhagic stroke



Current work and perspectives

- experimental data from EMTensor
- use recycling techniques for multiple right hand sides and between iterations during the optimisation process
- choose a good coarse space for a two-level scalable preconditioner for Maxwell's equations
- high-order edge elements