FREEFEM'14, December, 11, UPMC

A Finite Element Method for an Ill-posed Oxygen-Balance Model

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Eutrophication

Excess of Organic matter $\Longrightarrow$ Excess of nutrients $\Longrightarrow$ Eutrophication

Le Thouet (F.), Cunnning River (Australia) (Source : WEB!).

Travail en collaboration avec

- LJLL, Paris 6 (C. Bernardi, F. Hecht)
- I2M, Bordeaux I (M. Azaïez, C. Lebot)
- LMA, Reims (S. Salmon)
- ICJ, Lyon I (N. Débit)
- LAMSIN, Tunis (H. EL Fekih)
- LMAC Compiègne (M. Andrle, S. Khiari)

BOD Model

Organic Pollution in a body of water ($\Longrightarrow$) b : Biochemical Oxygen Demand (BOD).

Reaction (Streeter & Phelps, 1925). (Notation : Ω domaine, dim ≥ 2)

$$\partial_t b + Rb = F \qquad \text{in } \Omega \times (0,T).$$

Transport in the River

$$\partial_t b + \operatorname{div}(Vb) + Rb = F \qquad \text{in } \Omega \times (0,T),$$

Dispersion of Oxygen in water

$$\partial_t b - \operatorname{div}(D\nabla b) + \operatorname{div}(Vb) + Rb = F \qquad \text{in } \Omega \times (0,T).$$

DO Model

c : Dissolved Oxygen concentration

c_S : Dissolved Oxygen concentration at saturation (constant)

Pump out Oxygen ($\Longrightarrow$) Oxygen Deficit

($\Longrightarrow$) Oxygen Absorption from Atmosphere

(Dispersion, Transport, Reaction)

$$\partial_t c - \operatorname{div}(D\nabla c) + \operatorname{div}(Vc) + R_* c = G \qquad \text{in } \Omega \times (0,T).$$

BOD-DO Model

Organic Pollution Source (F), Oxygen Source (G)

$$
\begin{aligned}
\partial_t b - \operatorname{div}(D\nabla b) + \operatorname{div}(Vb) + Rb &= F && \text{in } \Omega\times(0,T),\\
\partial_t c - \operatorname{div}(D\nabla c) + \operatorname{div}(Vc) + R_* c + Rb &= G && \text{in } \Omega\times(0,T),\\
D\partial_{\boldsymbol{n}} b &= \gamma && \text{in } \partial\Omega\times(0,T),\\
D\partial_{\boldsymbol{n}} c &= 0 && \text{in } \partial\Omega\times(0,T),\\
b(0) &= 0 && \text{in } \Omega,\\
c(0) &= c_S && \text{in } \Omega.
\end{aligned}
$$

Manuscripts: Brown (EPA, 1987), Okubo (Springer, 1980).
Wider (Huge) Bibliography in this model.

Streeter-Phelps Model (1925)

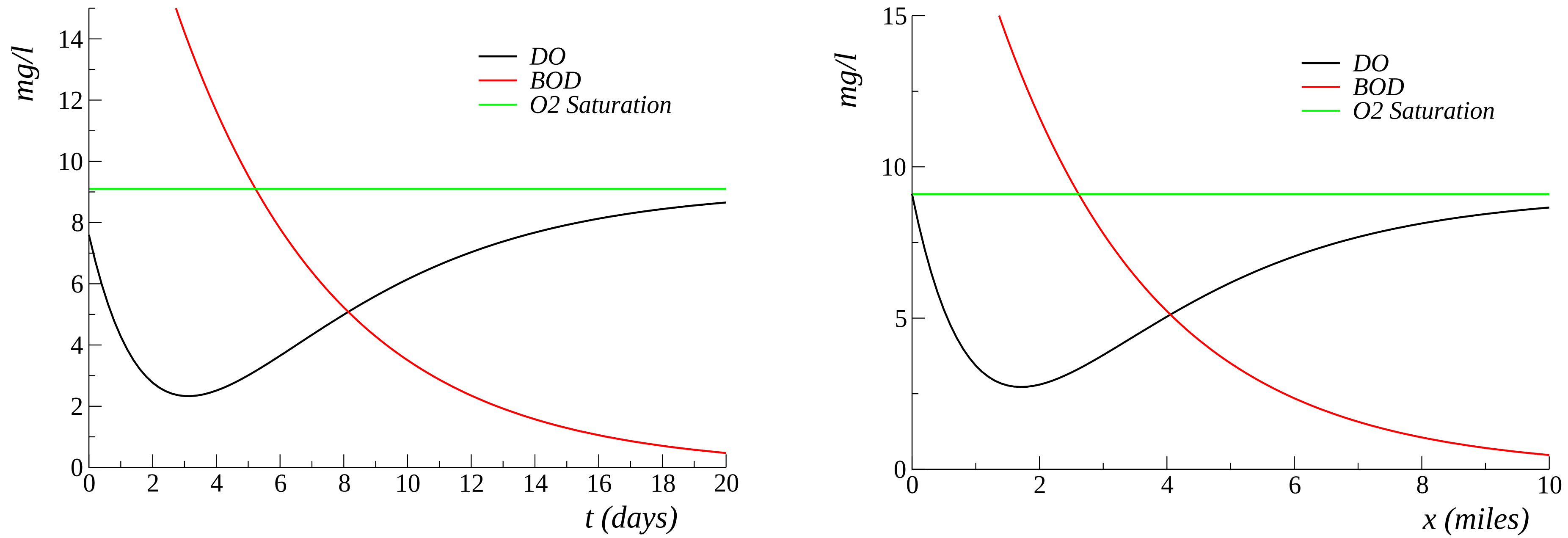


Diagram to the left : Initial pollution, (No Dispersion nor Advection)
Diagram to the right : Steady pollution, (No Dispersion).

Observations on BOD or on DO?

Obtain Experimentally γ $(= \partial_{\boldsymbol{n}} b)$

Long Chemical Protocol $(\Longrightarrow)$ Measurments of the BOD flux γ

$(\Longrightarrow)$ Lasts five Days! Too Long!

Observations c $(= \alpha)$

Use the observations $D\partial_{\boldsymbol{n}} c$ and c on the bondary, available in real-time and directly cope with the inverse Coupled BOD-DO system.

INVERSE BOD-DO MODEL

No boundary condition for b on Γ_C.

Two boundary conditions for c on Γ_C.

$$
\begin{aligned}
\partial_t b - \ \mathrm{div}\ (D\nabla b) + [\ \mathrm{div}\ (Vb)] + Rb &= F && \text{in } \Omega \times (0,T),\\
\partial_t c - \ \mathrm{div}\ (D\nabla c) + [\ \mathrm{div}\ (Vc)] + R_* c + Rb &= G && \text{in } \Omega \times (0,T),\\
D\partial_{\boldsymbol{n}} c &= 0 && \text{in } \partial\Omega \times (0,T),\\
c &= \alpha, && \text{in } \partial\Omega \times (0,T),\\
b(0) &= 0, && \text{in } \Omega,\\
c(0) &= c_S && \text{in } \Omega.
\end{aligned}
$$

The flux (the load) to reconstruct :

$$(D\partial_{\boldsymbol{n}} b)(t)_{|\partial\Omega} = \gamma? \ \ (\text{or } b(t)_{|\partial\Omega} = \eta?) \qquad \text{in } (0,T).$$

Ill-posedness (1D Model)

Consider $\gamma(t) = Db'(0,t)$ as unknown. Solve sequentially

$$\begin{aligned}
\partial_t b_\gamma - (Db'_\gamma)' + Rb_\gamma &= 0 && \text{in } (0,\pi)\times(0,T),\\
Db'_\gamma(0,t) &= 0, && \text{in } (0,T),\\
Db'_\gamma(\pi,t) &= \gamma(t), && \text{in } (0,T),\\
b_\gamma(0,\cdot) &= 0, && \text{in } (0,\pi).
\end{aligned}$$

$$\begin{aligned}
\partial_t c_\gamma - (Dc'_\gamma)' + R_* c_\gamma &= Rb_\gamma && \text{in } (0,\pi)\times(0,T),\\
Dc'_\gamma(0,t) &= 0, && \text{in } (0,T),\\
Dc'_\gamma(\pi,t) &= 0, && \text{in } (0,T),\\
c_\gamma(0,\cdot) &= 0, && \text{in } (0,\pi).
\end{aligned}$$

We have to solve the ill-posed equation on $\gamma \in L^2(0,T)$, that is

$$S\gamma(t) = c_\gamma(\pi,t) = \alpha(t), \qquad \text{in } (0,T).$$

Ill-posedness (II)

Fourier Computations ($D = R = R_* = 1$)

$$b_\gamma(x,t) = \frac{2}{\pi} \sum_{k \in \mathbb{N}} \left(\int_0^t \gamma(s) e^{-k^2(t-s)} \, ds \right) \cos(kx) \,,$$

$$c_\gamma(x,t) = \frac{2}{\pi} \sum_{k \in \mathbb{N}} \left(\int_0^t \gamma(s)(t-s) e^{-k^2(t-s)} \, ds \right) \cos(kx) \,,$$

$$(S\gamma)(t) = \int_0^t K(t-s)\gamma(s) \, ds, \qquad \forall t \in (0,T),$$

$$K(s) = \frac{1}{\pi} s + \frac{2}{\pi} \sum_{k \in \mathbb{N}} s e^{-k^2 s}, \qquad \forall s \in (0,T).$$

$K(s) = \mathcal{O}(\sqrt{s})$ ($\Longrightarrow$) Singular values of S decrease like $k^{-3/2}$.

($S\gamma = \alpha$) is a mildly ill-posed Volterra equation.

Why the Mild Ill-posedness?

Observation loc. $=$ Reconstruction l.

db'(0) = ?

dc'(0) = 0, c(0) = α

*db'(*π*) = 0*

*dc'(*π*) = 0*

Observa. loc. $\neq$ Reconstruc. loc.

db'(0) = ?

dc'(0) = 0

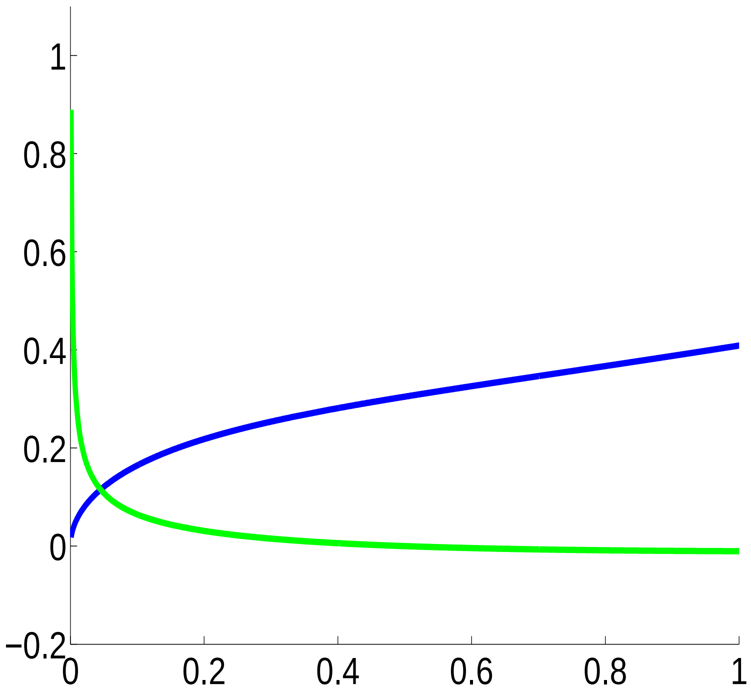
*db'(*π*) = 0*

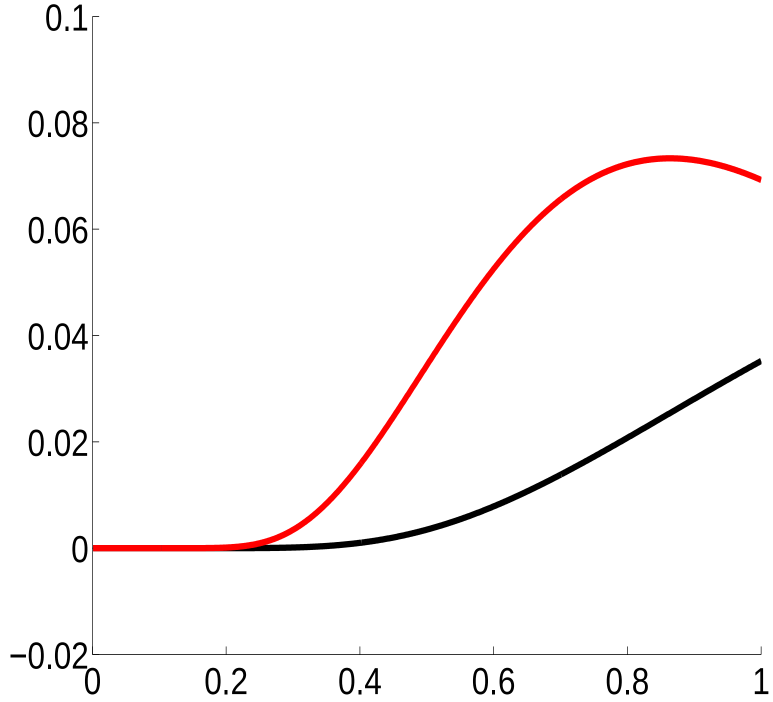
*dc'(*π*) = 0, c(*π*) =* α

Kernels of Volterra Equations (replace 2 by 1 for $k = 0$)

$$K(s) = \frac{s}{\pi} + \frac{2s}{\pi} \sum_{k \geq 1} e^{-k^2 s}, \qquad H(s) = \frac{s}{\pi} + \frac{2s}{\pi} \sum_{k \geq 1} (-1)^k e^{-k^2 s}.$$

Convolution Kernels





Kernels $K(\cdot)$ and $H(\cdot)$ and their first derivatives

!! Different scales in the vertical axis.

Uniqueness : Illustration ($T = \infty$)

Solve equation

$$(S\gamma)(t) = K \star \gamma(t) = 0 \qquad \forall t \in (0, \infty).$$

Use Laplace transform yields that

$$\widehat{K}(p)\hat{\gamma}(p) = 0, \qquad \forall p \in (0, \infty),$$

The Laplace Transform of $K(\cdot)$ is

$$\widehat{K}(p) = \frac{2}{\xi} \sum_{k \in \mathbb{N}} \frac{1}{(p + \lambda_k)^2} > 0, \qquad \forall p \in (0, \infty).$$

We obtain then that

$$(\hat{\gamma}(p) = 0, \ \forall p \in (0, \infty)) \implies (\gamma(t) = 0, \ \forall t \in (0, \infty)).$$

Then

$$b(t, x) = c(t, x) = 0, \qquad \forall t \in (0, \infty), \ \forall x \in (0, L).$$

A Uniqueness Result

Théo. **1** *The Inverse Problem has at most one solution.*

PROOF

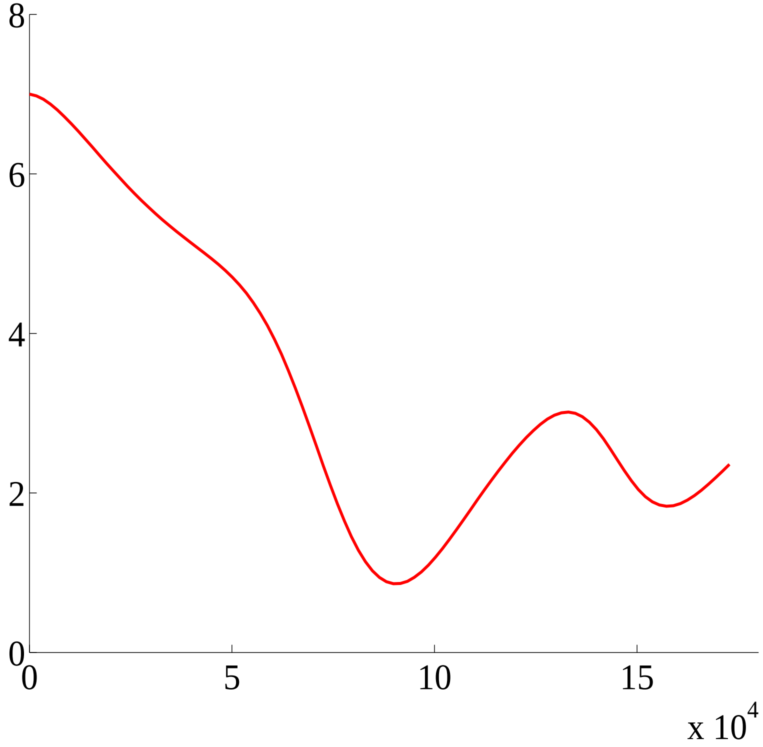
i — Study of the resolvant of the steady problem : Saddle point theory for non symmetric problems (Nicolaides & Bernardi, Canuto, Maday).

ii — Then, use the uniqueness theorem for parabolic problems (A. Pazy).

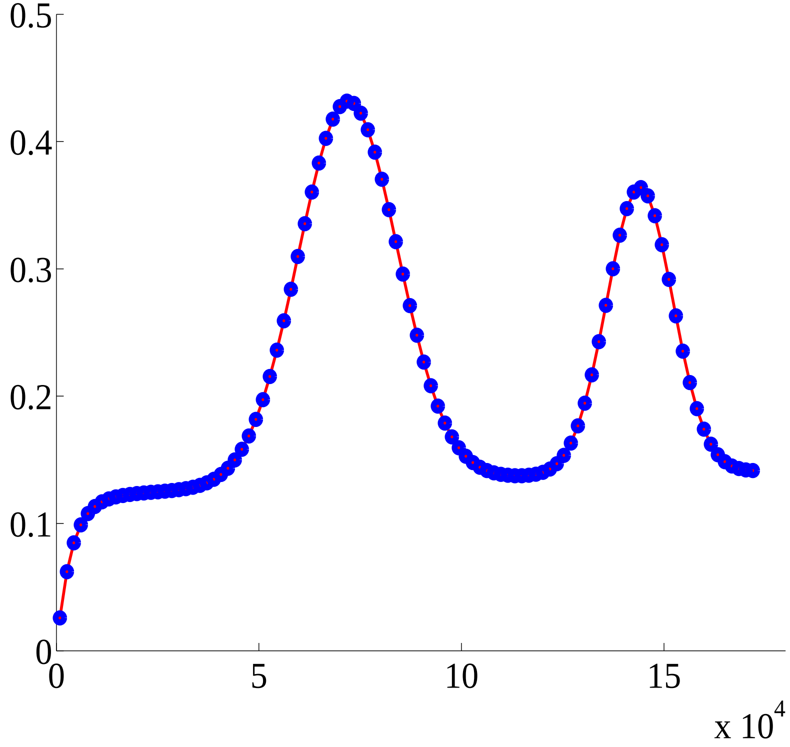
Details in M3AS, Volume 22, Issue 10, October 2012

Numerics (I) : Volterra's equation

Computations based on the MATLAB Library **regu**
developed by P. C. Hansen, http://www.imm.dtu.dk/~pcha

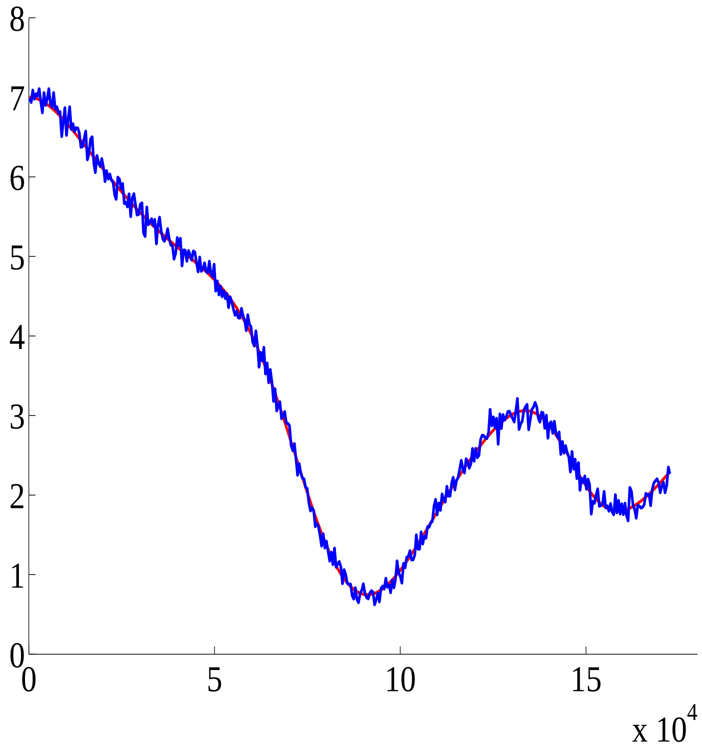


Unnoisy Observation $\alpha(\cdot)$.

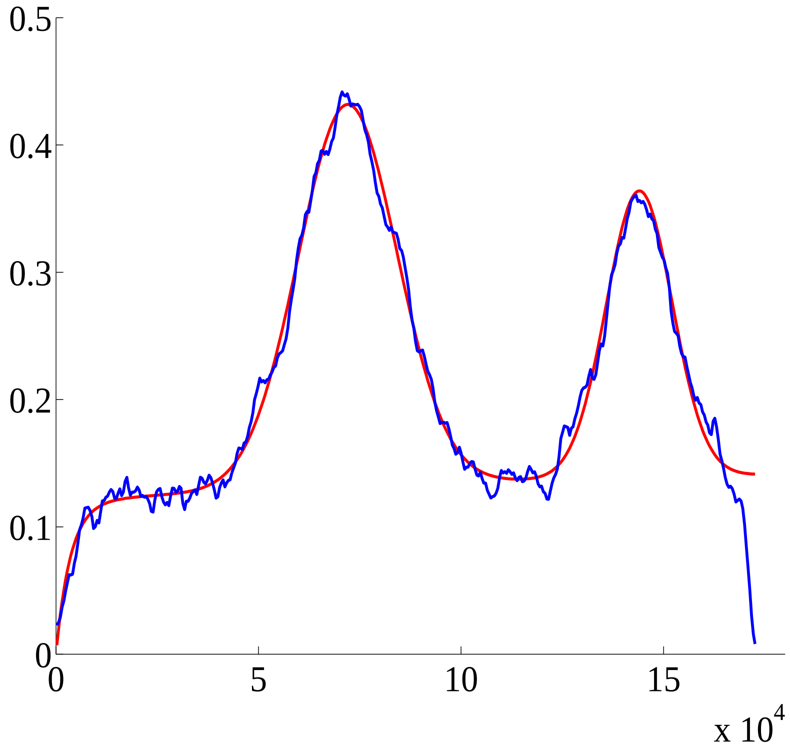


Exact and computed fluxes $\gamma(\cdot)$.

GAUSSIAN NOISE. VARIANCE $\sigma = 0.1$

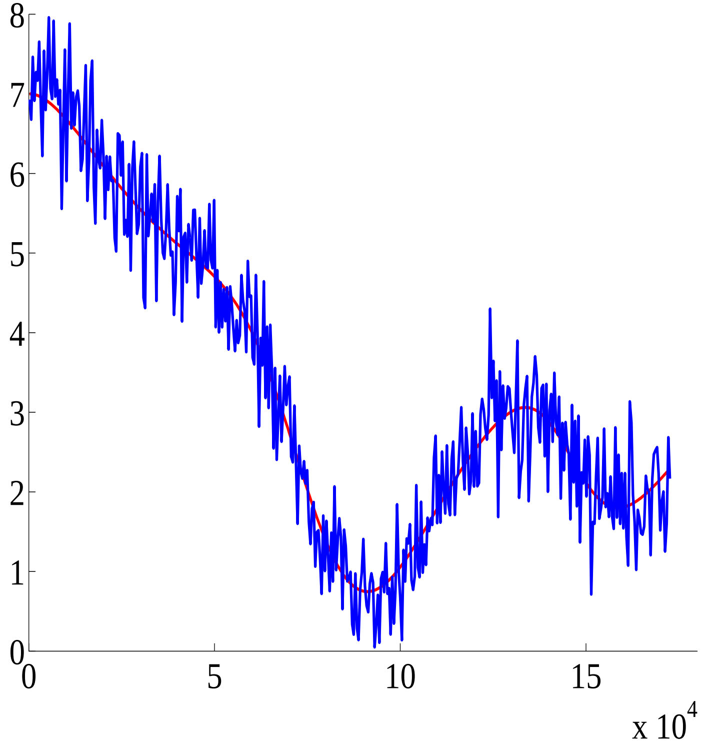


Noisy Observation $\alpha(\cdot)$.

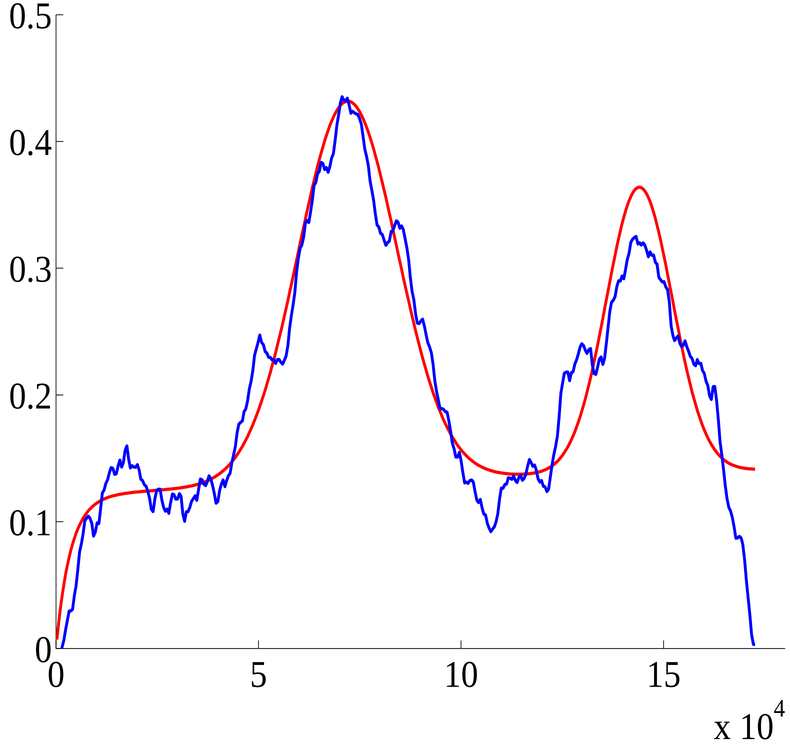


Exact and computed fluxes $\gamma(\cdot)$.

Gaussian Noise. Variance $\sigma = 0.5$



Exact data Observation $\alpha(\cdot)$.



Exact and computed fluxes $\gamma(\cdot)$

Numerics: Steady Model

Ben B., Bernardi, Hecht, Salmon, (FreeFem)

SIAM J. Numerical Analysis, 52-5 (2014), pp. 2207-2226

Steady System

Cancel every thing that is time dependence ($\Longrightarrow$)

$$\begin{aligned} -\operatorname{div}(D\nabla b) + Rb &= F \qquad && \text{in } \Omega \\ -\operatorname{div}(D\nabla c) + R_* c + Rb &= G \qquad && \text{in } \Omega, \\ D\partial_{\boldsymbol{n}} c &= 0 \qquad && \text{in } \partial\Omega, \\ c &= \alpha, \qquad && \text{in } \partial\Omega. \end{aligned}$$

Looks like the Vorticity-Stream functions system in Incompressible Fluid Mechanic.

Difference : A Non-symmetric saddle point problem.

Mixed Finite Elements $\mathcal{P}_1 \times \mathcal{P}_1$

b_h and φ_h, no boundary conditions along $\partial\Omega$.

c_h and ψ_h, Dirichlet conditions along $\partial\Omega$ ($c_h = \alpha_h$ and $\psi_h = 0$ on $\partial\Omega$.).

Find b_h and c_h such that

$$\int_\Omega (D\nabla b_h \nabla \psi_h + R\, b_h \psi_h)\ dx = \int_\Omega f\psi_h\, dx, \qquad \forall \psi_h,$$

$$\int_\Omega (D\nabla \varphi_h \nabla c_h + R_* \varphi_h c_h)\ dx + \int_\Omega R\, b_h \varphi_h\, dx = \int_\Omega g\varphi_h\, dx, \qquad \forall \varphi_h$$

'Abstract form'

$$\kappa\,(\psi_h, b_h) = (f, \psi_h), \qquad \forall \psi_h,$$

$$\kappa_*(c_h, \varphi_h) + a(b_h, \varphi_h) = (g, \varphi_h), \qquad \forall \varphi_h$$

Theoretical Results

inf-sup condition on $\kappa(\cdot,\cdot)$ $(\Longrightarrow)$ OK!

inf-sup condition on $\kappa_*(\cdot,\cdot)$ $(\Longrightarrow)$ OK!

inf-sup condition on $a(\cdot,\cdot)$ on the kernels of $\kappa(\cdot,\cdot)$ and $\kappa_*(\cdot,\cdot)$ $(\Longrightarrow)$??

$R = R_*$ $(\Longrightarrow)$ Things happens exactly as for ψ–ω (Symmetry).

$R \neq R_*$ $(\Longrightarrow)$ High technical difficulties (only partially solved!)

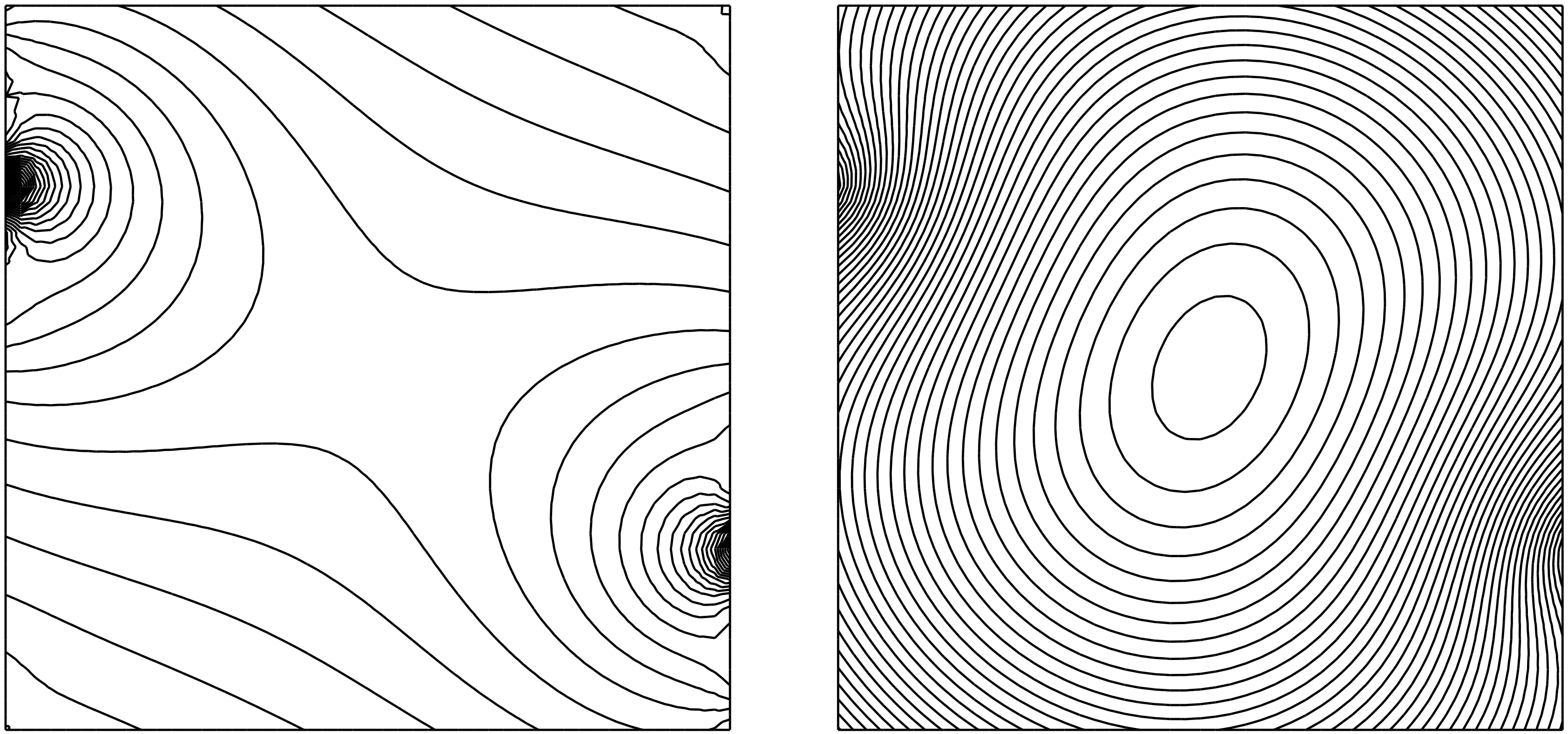
$$(\Longrightarrow)\ \text{osc}\ \left(\sqrt{\frac{R}{R_*}}\right) = \max\sqrt{\frac{R}{R_*}} - \min\sqrt{\frac{R}{R_*}} < 2\ ???!!!!,$$

$(\Longrightarrow)$ The constant blows up for small h.

VARIATIONAL FORM IN FREEFEM

```
 macro Grad(b) [dx(b),dy(b)]
 macro dn(b) (N.x*dx(b)+N.y*dy(b))

problem StreeterPhelps ( [b,c],[v,w],solver=LU ) =
 int2d(Th) ( D*(Grad(b)'*Grad(w)) + R*b*w
           + D*(Grad(c)'*Grad(v))+RR*c*v + R*b*v )
 - int2d(Th) (f*v +g*w)-int1d(Th,1) (cN*w)  + on(1, c=cD);
```

FreeFem



Concentrations b (corrupted) and c.

Stabilized MFE $\mathcal{P}_1 \times \mathcal{P}_1$

Find b_h and c_h such that

$$\int_\Omega (D\nabla b_h \nabla \psi_h + R\, b_h \psi_h)\ dx = \int_\Omega f\psi_h\ dx, \qquad \forall \psi_h,$$

$$\int_\Omega (D\nabla \varphi_h \nabla c_h + R_* \varphi_h c_h)\ dx + a_\rho(b_h, \varphi_h) = \int_\Omega g\varphi_h\ dx, \qquad \forall \varphi_h$$

'Abstract form'

$$\kappa\,(\psi_h, b_h) = (f, \psi_h), \qquad \forall \psi_h,$$

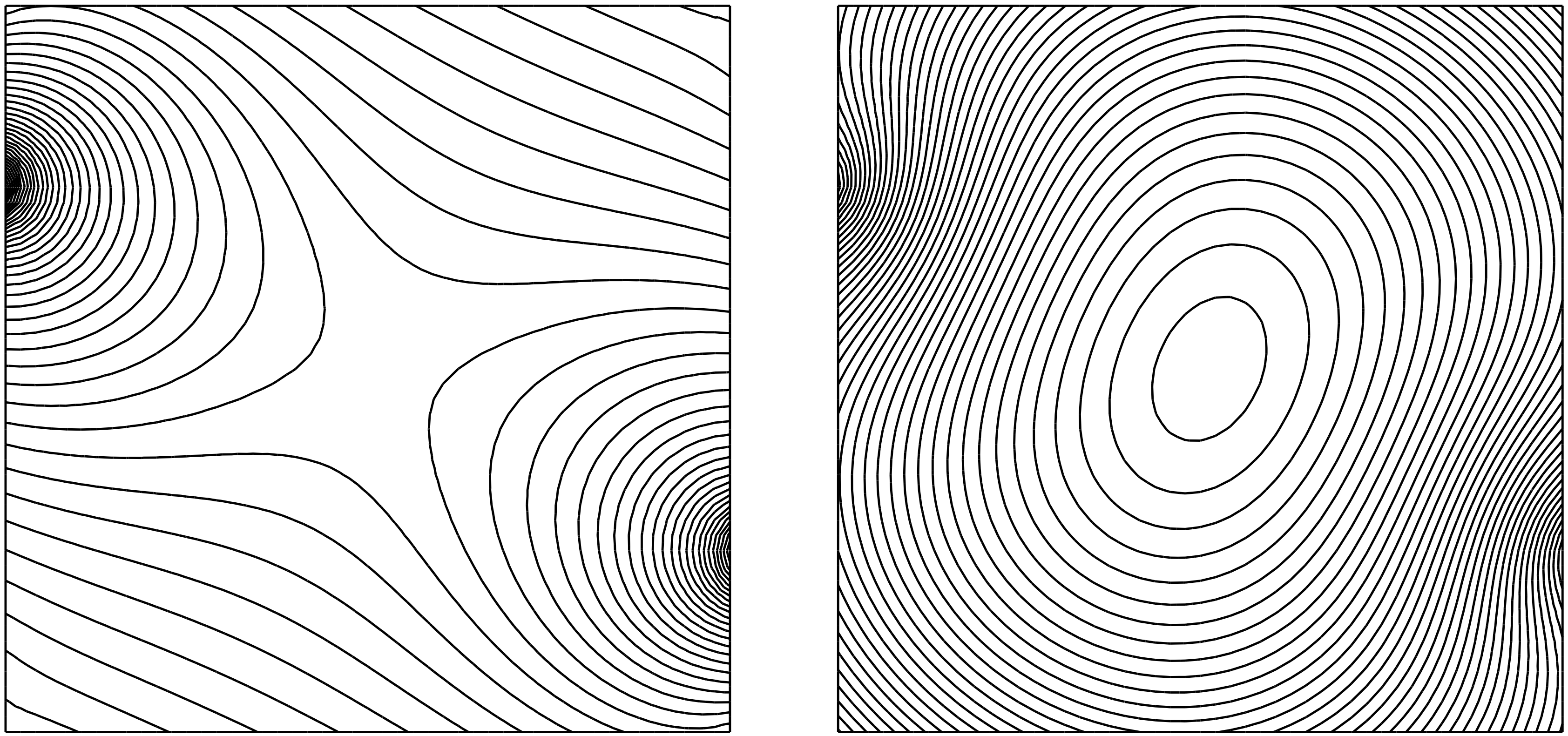
$$\kappa_*(c_h, \varphi_h) + a_\rho(b_h, \varphi_h) = (g, \varphi_h), \qquad \forall \varphi_h$$

Stabilization (Amara-Dabaghi, M2AN 2001) —Constant blows up for small h—

$$a_\rho(b_h, \varphi_h) = a(b_h, \varphi_h) + \rho \sum_{e \not\subset \partial\Omega} h_e \int_e [D\partial_n b_h]_e(\tau)[D\partial_n \varphi_h]_e(\tau)\, d\tau.$$

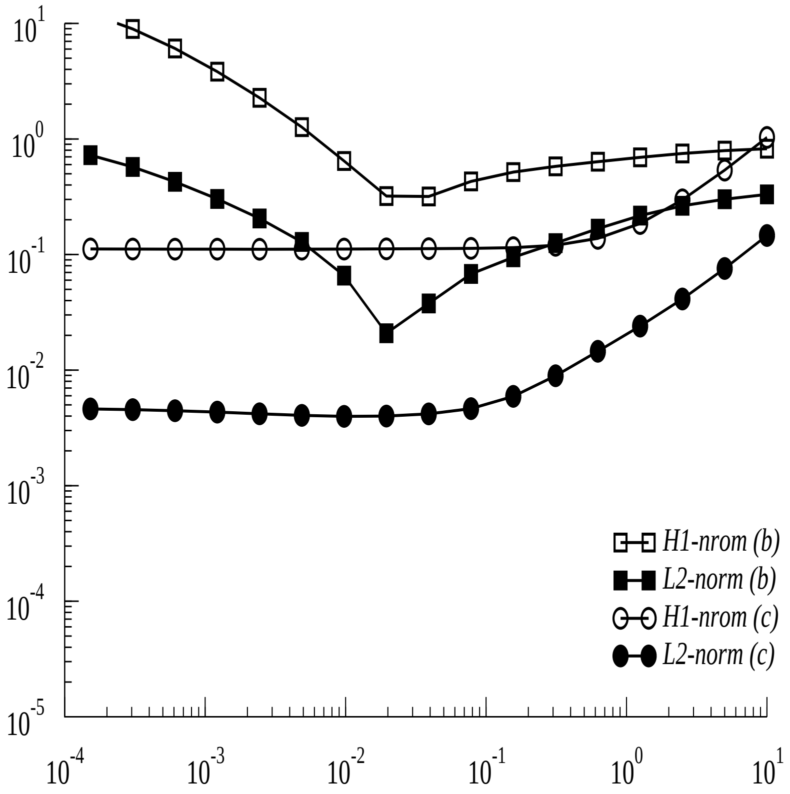
Stabilization in FreeFem

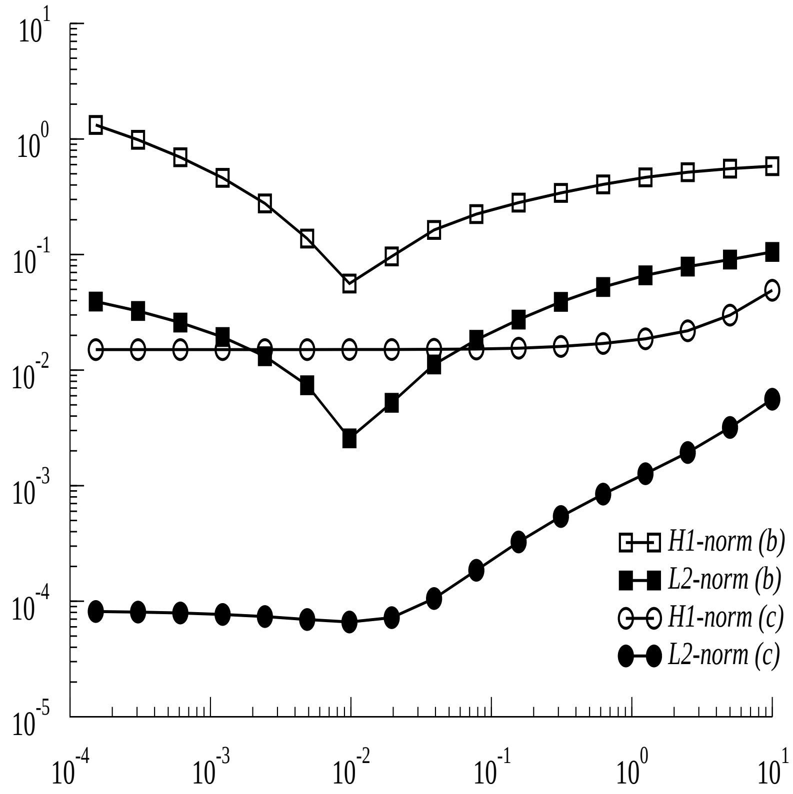
```
problem StreeterPhelps ( [b,c],[v,w],solver=LU ) =
 int2d(Th) ( D*(Grad(b)'*Grad(w)) + R*b*w
           + D*(Grad(c)'*Grad(v))+RR*c*v + R*b*v )

 + intalledges(Th)(rho*D*jump(dn(b))*D*jump(dn(v))*lenEdge*(nTonEdge-1))

 - int2d(Th) (f*v +g*w)-int1d(Th,1) (cN*w)  + on(1, c=cD);
```



Concentrations b (corrected by stabilization) and c.

Regularisation parameter

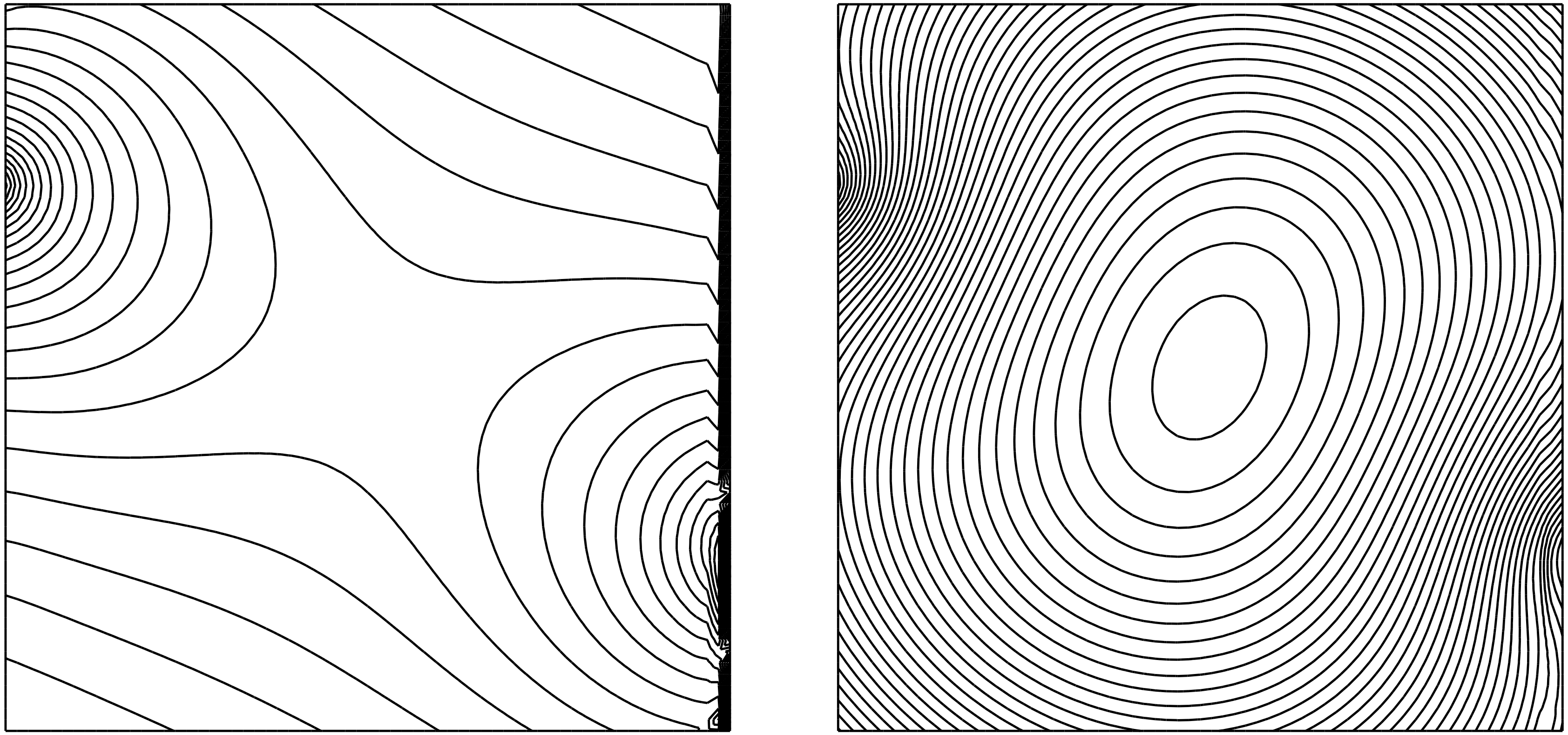




Mesh size $h = 1/16$ and $h = 1/128$.

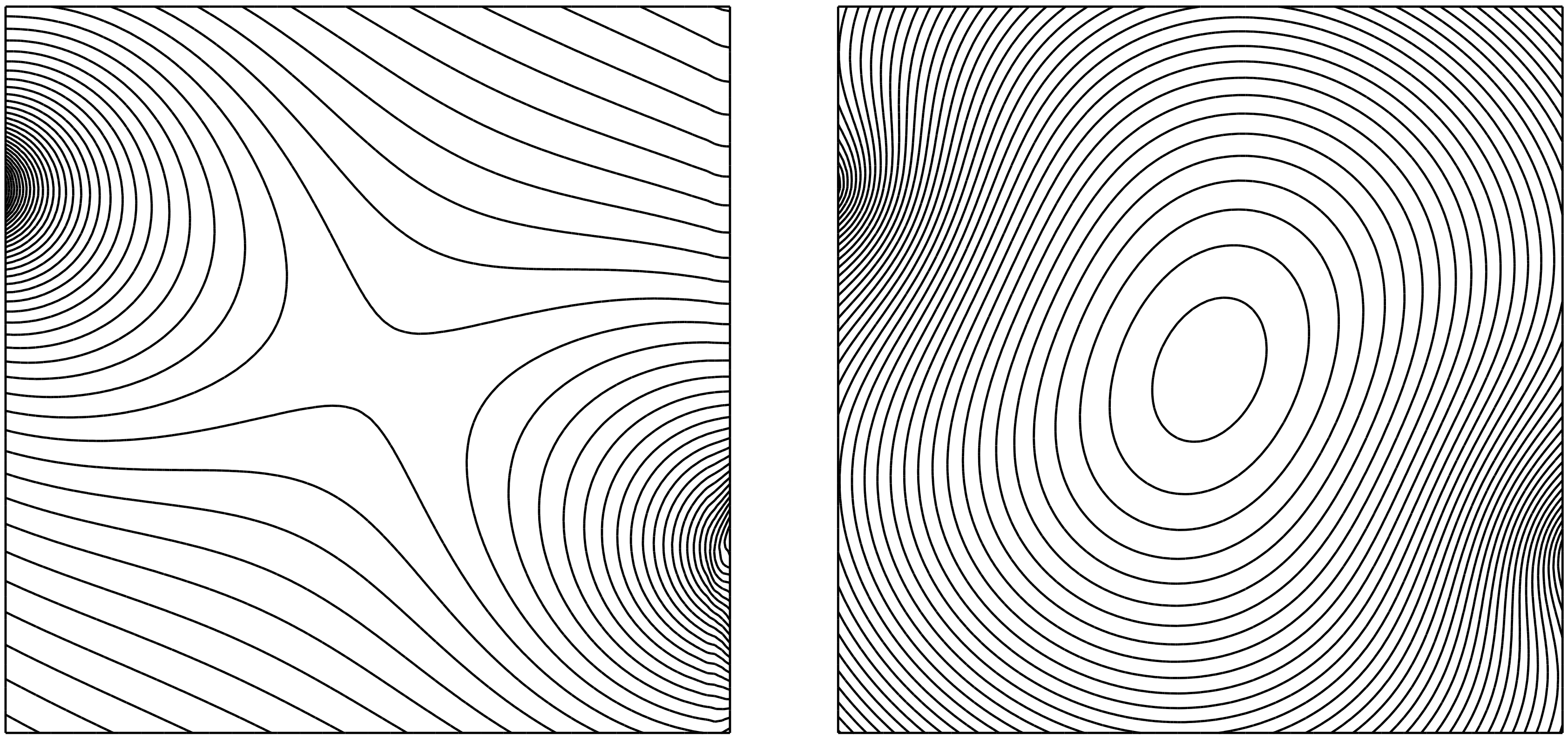
Choice of the parameter ρ ($\Longrightarrow$) Not that Clear?

WITH ADVECTION



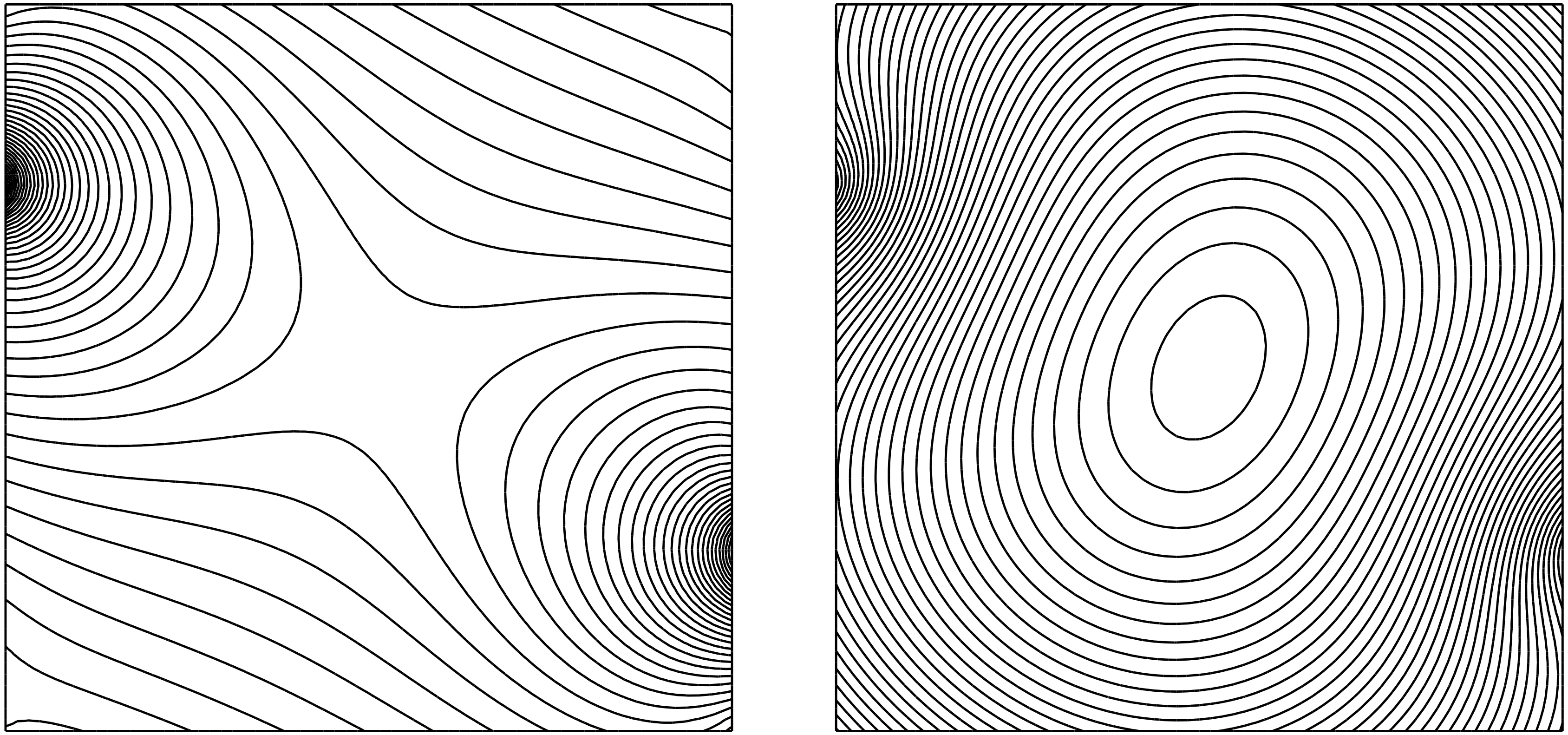
Without stablization.

With advection



With stablization.

Example, osc $(\sqrt{\frac{R}{R_*}}) > 13$



Satabilized Concentrations b and c.

Numerics : (Un)Steady Model

Ben B., Débit, El Fekih, Khiari, (Matlab)
Journal of Inverse and Ill-posed Problems, In revision.

Azaïez, Ben B., Hecht, Le Bot, (FreeFem)
Inverse Problems, (2014) 30 015002

Space Discretization

Find $b_h(t)$ and $c_h(t)$ such that : $c_h = 0$ on $\partial\Omega$ and

$$\int_\Omega \partial_t b_h(t)\psi_h + \int_\Omega (D\nabla b_h(t)\nabla\psi_h + R\,b_h(t)\psi_h) = \int_\Omega f(t)\psi_h,$$

$$\int_\Omega \varphi_h\partial_t c_h(t) + \int_\Omega (D\nabla\varphi_h\nabla c_h(t) + R_*\varphi_h c_h(t)) + \int_\Omega R\,b_h(t)\varphi_h = \int_\Omega g(t)\varphi_h,$$

$$b_h(0) = 0, \qquad c_h(0) = c_S.$$

'Abstract form'

$$(\partial_t b_h(t), \psi_h) + \kappa\,(\psi_h(t), b_h) = (f(t), \psi_h), \qquad \forall\psi_h \quad (c_h),$$

$$(\partial_t c_h(t), \varphi_h) + \kappa_*(c_h(t), \varphi_h) + a_\rho(b_h(t), \varphi_h) = (g(t), \varphi_h), \qquad \forall\varphi_h \quad (b_h),$$

$$b_h(0) = 0, \qquad c_h(0) = c_S.$$

Euler Time scheme (I)

Vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ of Degrees of Freedom. Set $\boldsymbol{Y}^T = (\boldsymbol{b}, \boldsymbol{c})$.

$$\boldsymbol{M}\,\boldsymbol{Y}'(t) + \boldsymbol{K}\boldsymbol{Y}(\boldsymbol{t}) = \boldsymbol{G}(t), \qquad \text{in } (0,T)$$
$$\boldsymbol{Y}(0) = \boldsymbol{Y}_0.$$

Crude Euler time scheme (τ: time step)

$$\boldsymbol{M}\frac{\boldsymbol{Y}^{p+1} - \boldsymbol{Y}^p}{\tau} + \boldsymbol{K}\boldsymbol{Y}^{p+1} = \boldsymbol{G}^{p+1},$$
$$\boldsymbol{Y}^0 = \boldsymbol{Y}_0.$$

Is the induction well defined?

EULER TIME SCHEME (II)

Setting $\lambda = \tau^{-1}$, we obtain the full discrete problem

$$(\lambda \boldsymbol{M} + \boldsymbol{K})\boldsymbol{Y}^{p+1} = \boldsymbol{G}^{p+1} + \lambda \boldsymbol{M}\boldsymbol{Y}^{p},$$

$$\boldsymbol{Y}^{0} = \boldsymbol{Y}_{0}.$$

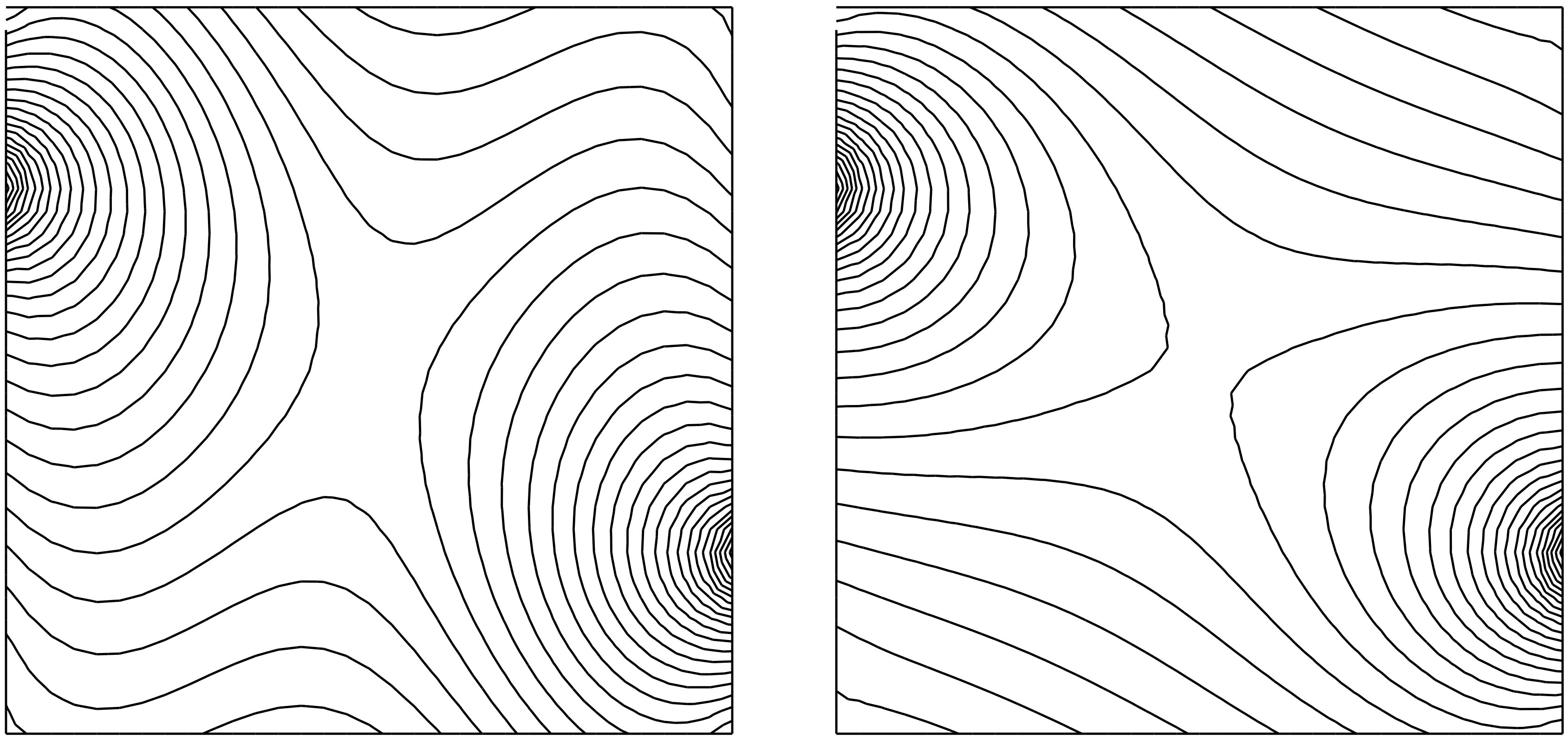
($\Longrightarrow$) Is $(\lambda \boldsymbol{M} + \boldsymbol{K})$ invertible?

($\Longrightarrow$) Look at the mixed problem with $R := R + \lambda$ and $R_* := R_* + \lambda$.

($\Longrightarrow$) $\operatorname{osc}\left(\sqrt{\dfrac{R}{R_*}}\right) = \max\sqrt{\dfrac{R}{R_*}} - \min\sqrt{\dfrac{R}{R_*}} < 2.$, EASY !!! FAAACILE!

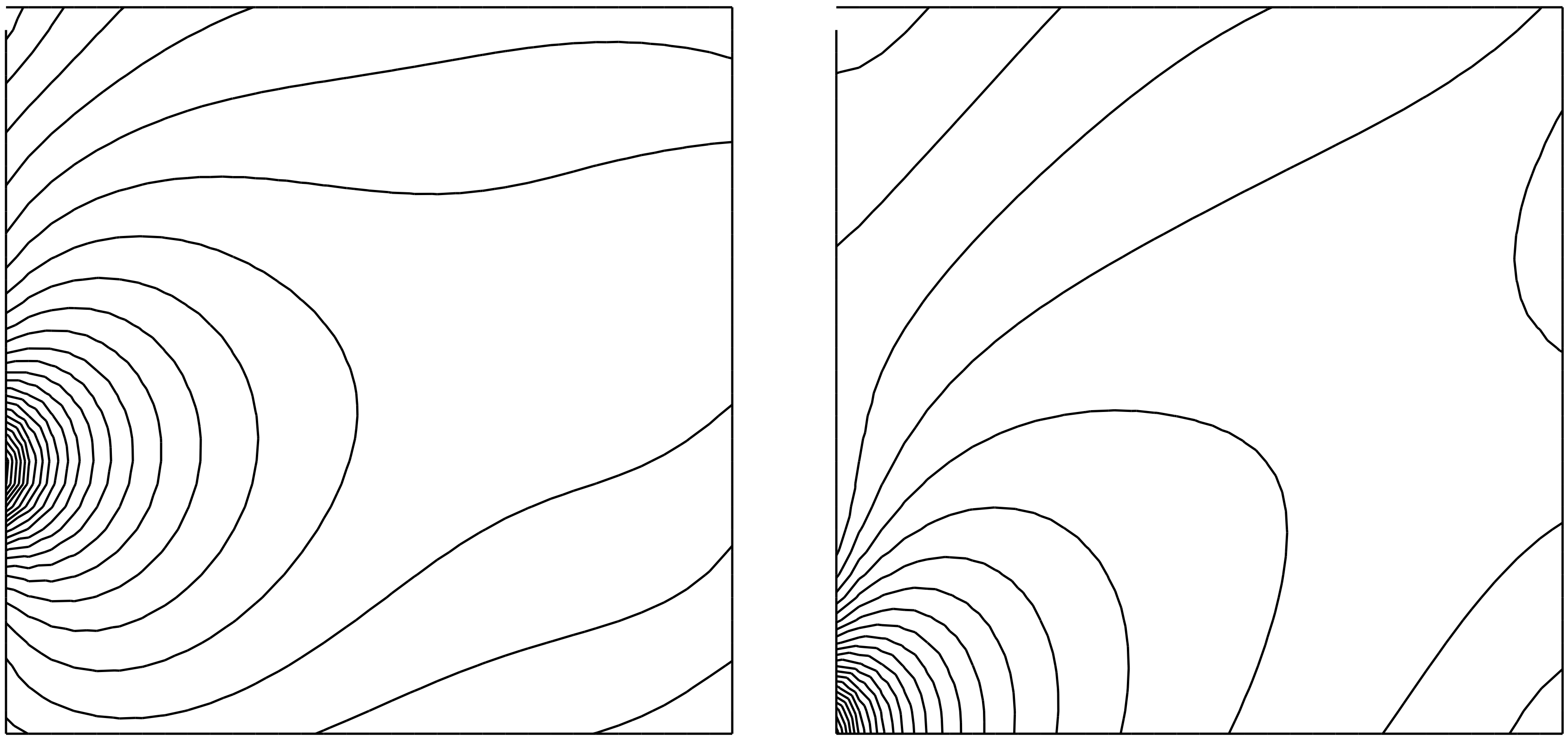
PROP. **2** *The discrete problem has a unique solution.*

FreeFem (Exact data)



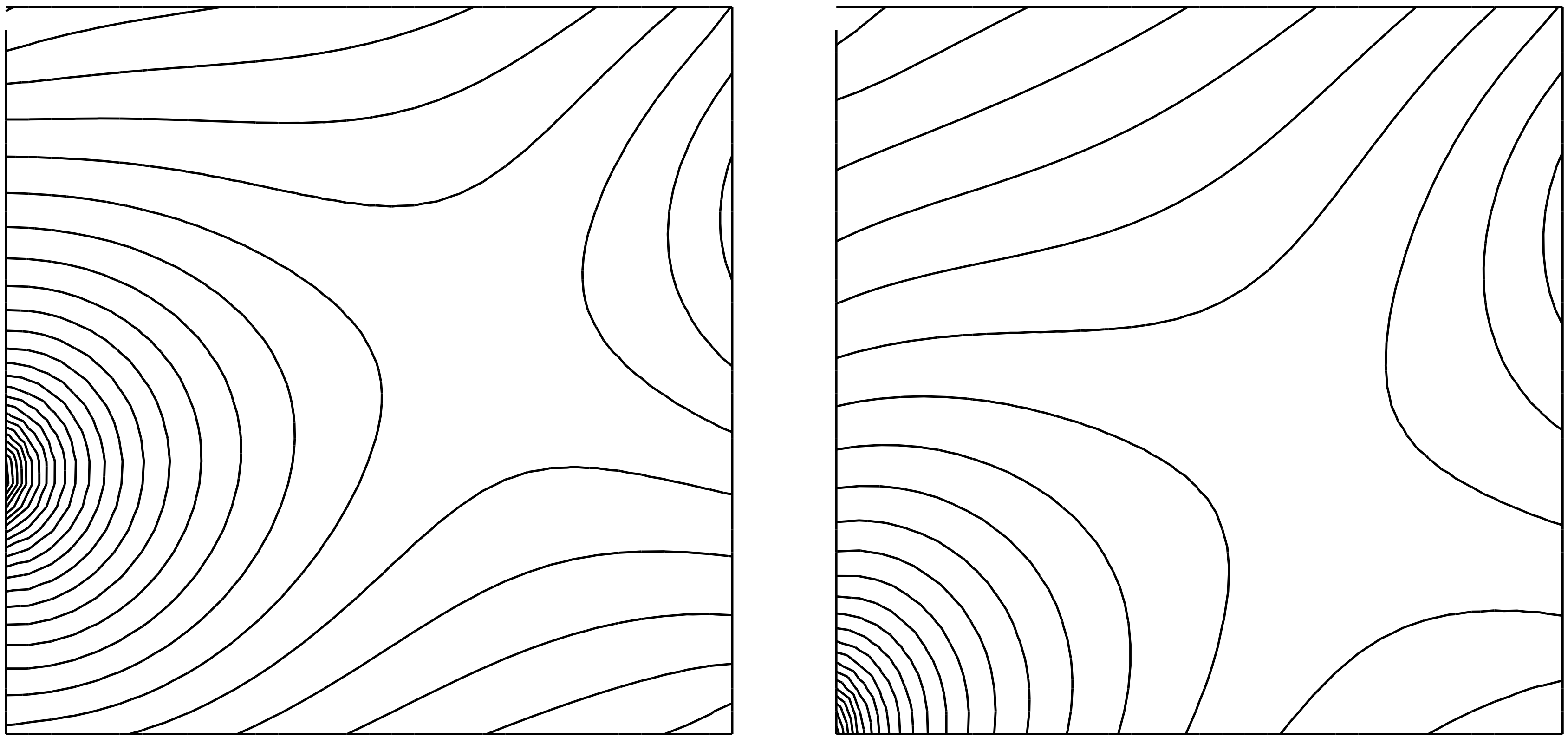
Steady solutions (b, c), $t = 1.25$ and $t = 2.5$).

FreeFem (Exact data)



Unsteady solutions (b, c), $t = 0.25$ and $t = 0.5$.

FreeFem



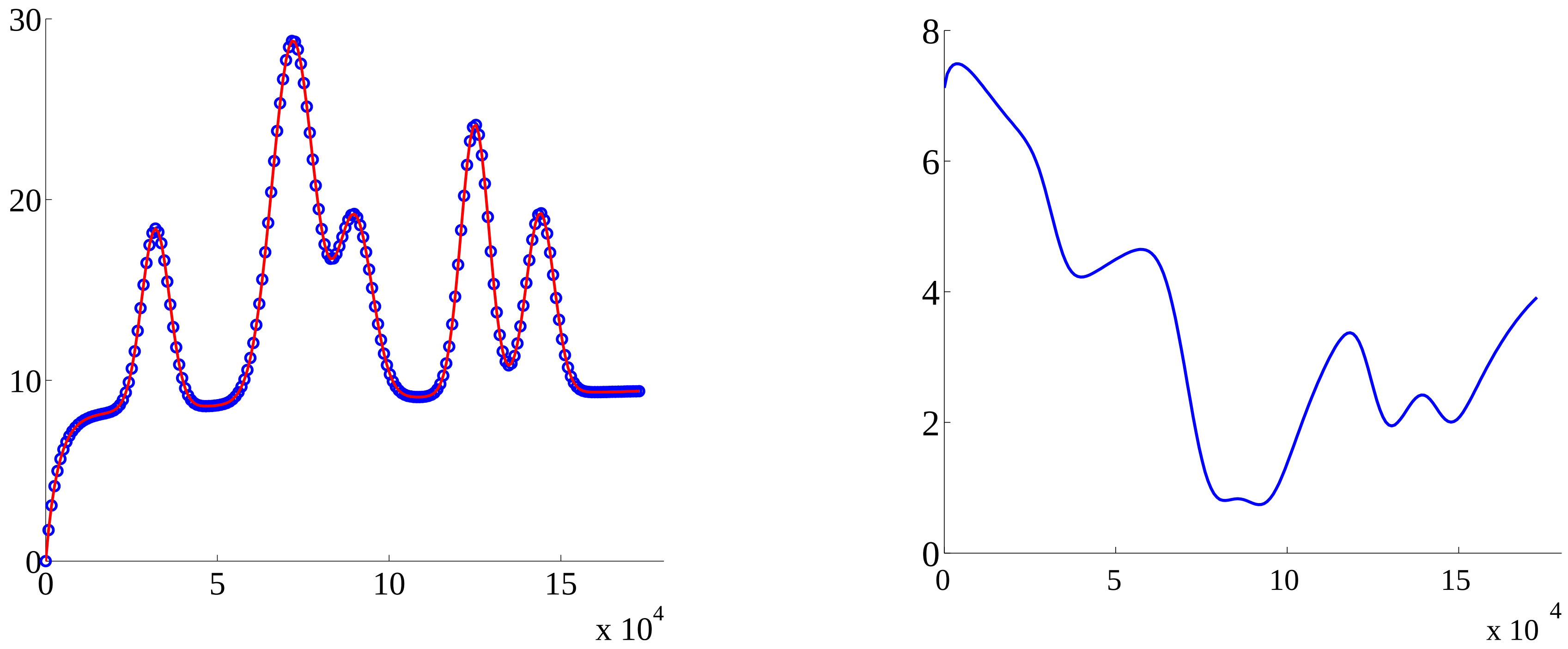
Unsteady solutions (ω, ψ), $t = 0.25$ and $t = 0.5$.

Quantitative Comparison

	H^1	L^2	H^1	L^2	L^∞	L^∞
Oxyg. balance	1.533 (b)	0.040 (b)	0.478 (c)	0.017 (c)	1.022 (b)	0.012 (c)
Fluid Flow	0.197 (ω)	0.037 (ω)	0.028 (ψ)	0.002 (ψ)	0.112 (ω)	0.002 (ψ)

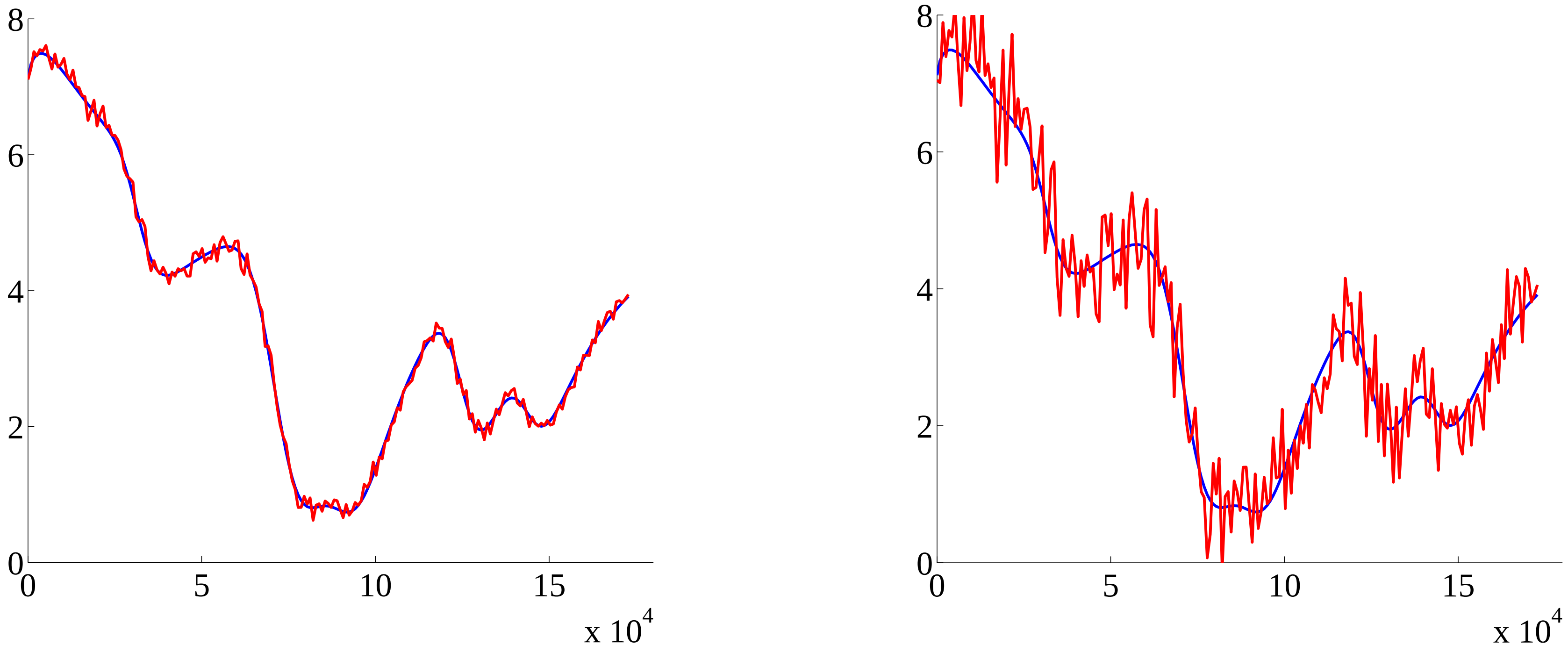
Various relative errors.

Simulations 1D (Matlab)



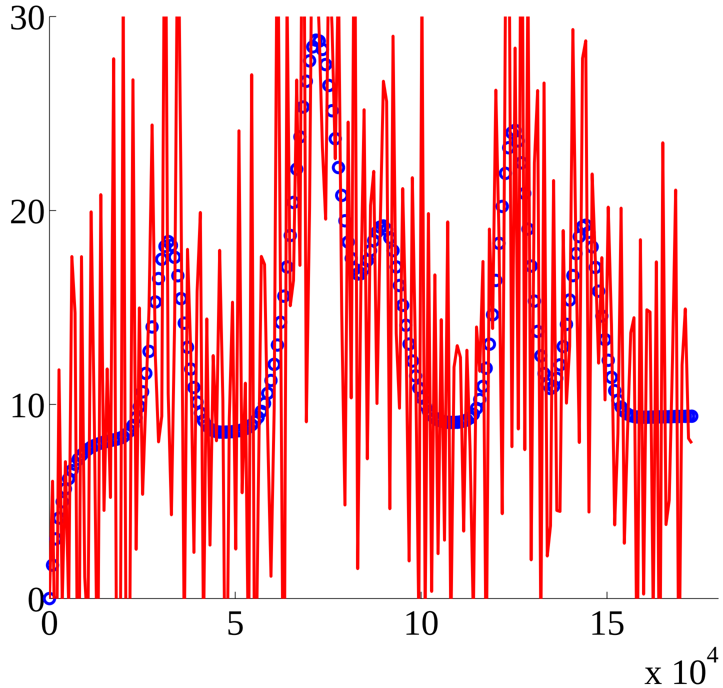
BOD load $b(0,t)$. Synthetized DO profile $\alpha(t) = c(0,t)$.

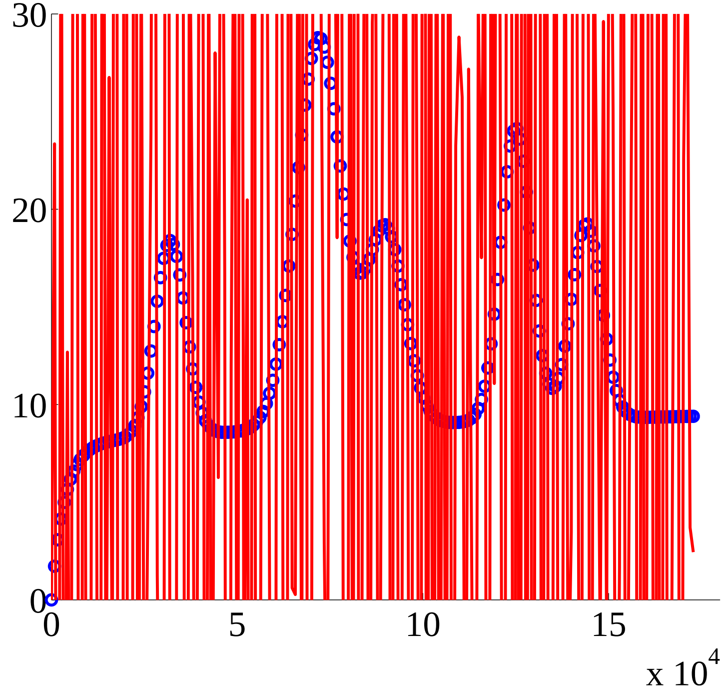
Noisy DO profiles $\alpha(\cdot)$



Standard deviation $\sigma = 0.1$ (left) and $\sigma = 0.5$ (right)

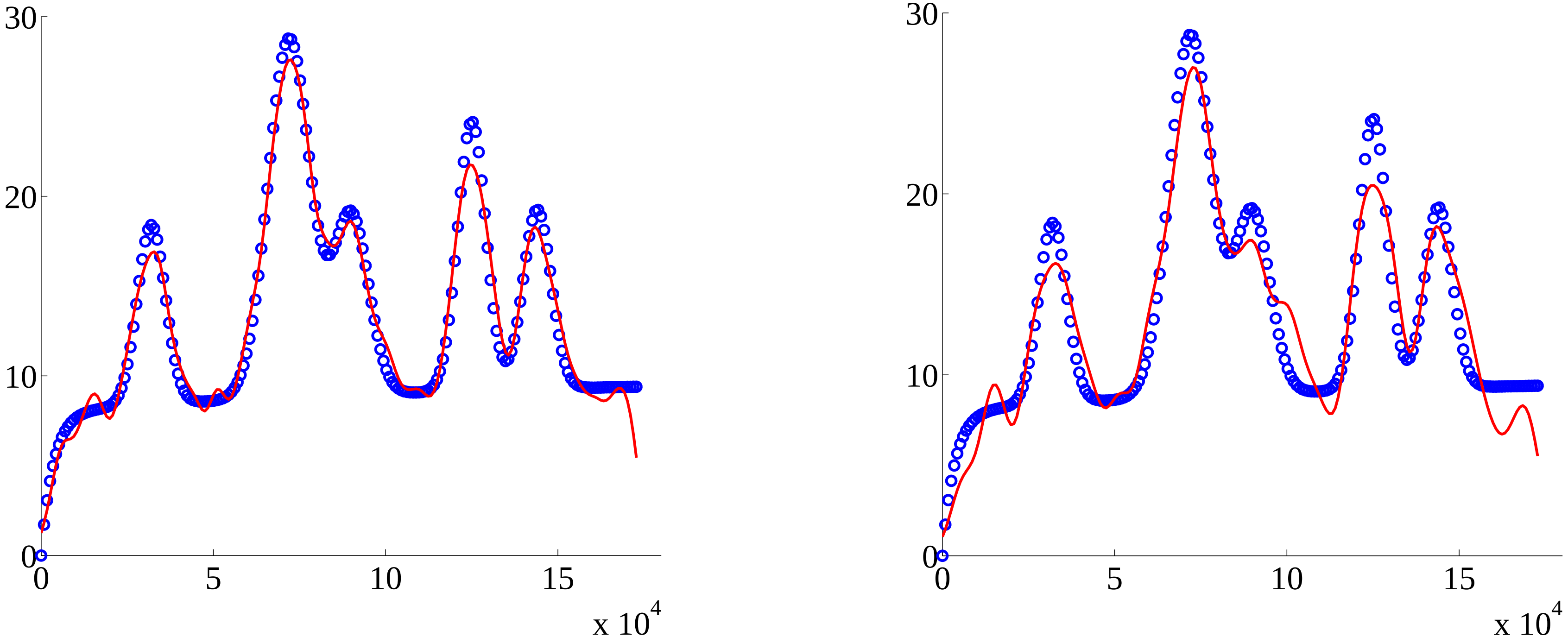
POLLUTED BOD LOADS $b(0,t)$





$\sigma = 0.1$ —left— and $\sigma = 0.5$ —right—

Filtered BOD loads $b(0,t)$



Gaussian Filter, $\sigma = 0.1$ —left— and $\sigma = 0.5$ —right—

Back to the Semi-Discrete Problem

Vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ of Degrees of Freedom ($\Longleftrightarrow$) FE functions b_h and c_h.

$$\dim \boldsymbol{b} = p = DOF(\Omega), \qquad \dim \boldsymbol{c} = q = DOF(\Omega \setminus \partial\Omega)$$

The discrete problem can be expressed as a block linear system

$$\begin{pmatrix} M & 0 \\ 0 & M^T \end{pmatrix} \partial_t \begin{pmatrix} \boldsymbol{b}(t) \\ \boldsymbol{c}(t) \end{pmatrix} + \begin{pmatrix} K & 0 \\ A_\rho & K_*^T \end{pmatrix} \begin{pmatrix} \boldsymbol{b}(t) \\ \boldsymbol{c}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g}(+)\boldsymbol{\alpha}(t) \end{pmatrix}, \qquad \text{in } (0,T).$$

Set $\boldsymbol{Y}^T = (\boldsymbol{b}, \boldsymbol{c})$. Then the following condensed form

$$\boldsymbol{M}\,\boldsymbol{Y}'(t) + \boldsymbol{M}\boldsymbol{Y}(\boldsymbol{t}) = \boldsymbol{G}(t), \qquad \text{in } (0,T)$$

$$\boldsymbol{Y}(0) = \boldsymbol{Y}_0.$$

Looks like a differential equation. But ... It is not!

Why it is not?

Prop. 3 $\quad \dim \mathcal{N}(\boldsymbol{M}) = (p - q)$. Then $\boldsymbol{M}$ is singular.

Why? $\quad$ Take $\boldsymbol{Y} \in \mathcal{N}(\boldsymbol{M})$ $(\Longrightarrow)$ (b_h, c_h) : F.E. functions.

The following orthogonalities hold : $b_h(p), \quad c_h(q), \quad p - q = DOF(\partial\Omega)$

$$\int_\Omega b_h \psi_h \, dx = 0, \qquad \forall \psi_h(q),$$
$$\int_\Omega c_h \varphi_h \, dx = 0, \qquad \forall \varphi_h(p).$$

$(\Longrightarrow)$ $c_h = 0$.
$(\Longrightarrow)$ $b_{h|\partial\Omega}$ fully specifies b_h $(\Longrightarrow)$ $(p - q)$ $(DOF) = \dim \mathcal{N}(\boldsymbol{M})$.

KRONECKER WEIERSTRASS CANONICAL FORM

Problem in $\boldsymbol{Y}$ is an IMPLICIT differential algebraic equation —DAE—.

Important! Uncouple the DE from the AE.

PROP. **4** *There exist* $\boldsymbol{LY} = (\boldsymbol{u}, \boldsymbol{v})^T$ *and* $\boldsymbol{HG} = (\boldsymbol{q}, \boldsymbol{r})^T$ *such that*

$$\begin{pmatrix} I_{(2q)} & 0 \\ 0 & N \end{pmatrix} \partial_t \begin{pmatrix} \boldsymbol{u}(t) \\ \boldsymbol{v}(t) \end{pmatrix} + \begin{pmatrix} W & 0 \\ 0 & I_{(p-q)} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}(t) \\ \boldsymbol{v}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{q}(t) \\ \boldsymbol{r}(t) \end{pmatrix}.$$

$\boldsymbol{u}$: *differential variable.*

$\boldsymbol{v}$: *algebraic variable.*

N: *square of dimension* $(p-q)$. *It is nilpotent of order* μ.

W: *square matrix of dimension* $(2q)$.

Uncoupling

Solve the ordinary differential problem

$$\partial_t \boldsymbol{u}(t) + W\boldsymbol{u}(t) = \boldsymbol{q}(t) \qquad \text{in } (0,T),$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0,$$

Unique solution

$$\boldsymbol{u}(t) = e^{-Wt}\boldsymbol{u}_0 + \int_0^t e^{-W(t-s)}\boldsymbol{q}(s)\, ds.$$

Solve the algebraic equation

$$N\partial_t \boldsymbol{v}(t) + \boldsymbol{v}(t) = \boldsymbol{r}(t) \qquad \text{in } (0,T),$$
$$\boldsymbol{v}(0) = \boldsymbol{v}_0.$$

Hypothetic solution

$$\boldsymbol{v}(t) = \sum_{j\le\mu-1} (-1)^j N^j \boldsymbol{r}^{(j)}(t) \; (\Longrightarrow) \; \sum_{j\le\mu-1} (-1)^j N^j \boldsymbol{r}^{(j)}(0) = \boldsymbol{v}_0.$$

Existence an Uniqueness

Prop. 5 *The semi-discrete problem has at most one solution.*

Rem. 1 *Existence is guaranteed only for consistent initial data* $(\boldsymbol{b}_0, \boldsymbol{c}_0)$ *and the boundary condition* $\boldsymbol{\alpha}$

Kronecker index, Time Scheme

Prop. 6 *The Kronecker index is one,* $\mu = 1$*. As a consequence,* $N = 0$*.*

Rem. 2 *Euler should be sufficient* $(\Longrightarrow)$ *Yes! May be .... and may be NO!*

Conclusion

A lot of computational work is waiting!

Perspective

Achieve that work!

Point source detection.