Frictionless contact problem for hyperelastic materials with interior point optimizer

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Nonlinear mechanics

- The deformation of a body $\Omega \subset \mathbb{R}^3$ is described by the application $\phi : \Omega \to \mathbb{R}^3$.

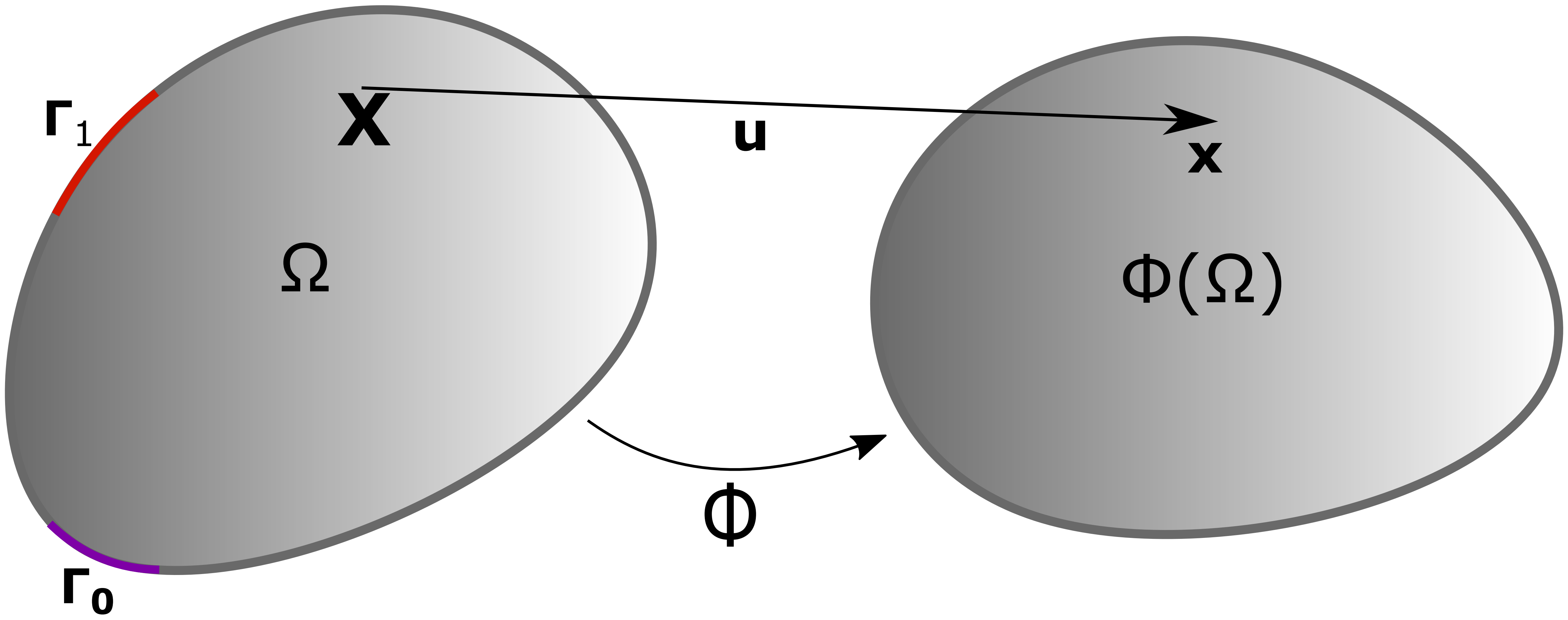


Figure 1: Initial and actual configurations

- The displacement field: $\boldsymbol{u} = \phi(\boldsymbol{X}) - \boldsymbol{X} = \boldsymbol{x} - \boldsymbol{X}$
- The first Piola-Kirchhoff stress tensor **P** describes the forces $d\boldsymbol{f}$ in the current configuration:

$$d\boldsymbol{f} = \boldsymbol{P}.\boldsymbol{N}dS \tag{1}$$

- Boundary conditions:

$$\begin{cases} \mathbf{u} & = \mathbf{u}_0 \quad \text{on} \quad \Gamma_0 \\ \mathbf{P}.\mathbf{N} & = \mathbf{t} \quad \text{on} \quad \Gamma_1 \end{cases} \tag{2}$$

Nonlinear mechanics [4]

- With $\boldsymbol{f}$ the body force per unit mass applied on the body, the displacement field $\boldsymbol{u}$ is the solution of the following equation:

$$\sum_{j=1}^{3} \frac{\partial \mathbf{P}_{ij}}{\partial X_j} + \rho_0 f_i = 0 \quad i = 1,2,3 \text{ (Local balance of angular momentum)} \tag{3}$$

- Hyperelastic materials (Neo-Hookean, Mooney): The strain energy is given by:

$$\mathcal{E}_s(\mathbf{v}) = \int_\Omega \psi \, dV \tag{4}$$

  Where ψ is the strain energy function of the material
- The total potential energy is defined by:

$$\mathcal{E}(\mathbf{v}) = \int_\Omega \psi \, dV - \int_\Omega \rho_0 \mathbf{f}.\mathbf{v} \, dV - \int_{\Gamma_1} \mathbf{t}.\mathbf{v} \, dA \tag{5}$$

- The displacement field is solution of the following minimization problem:

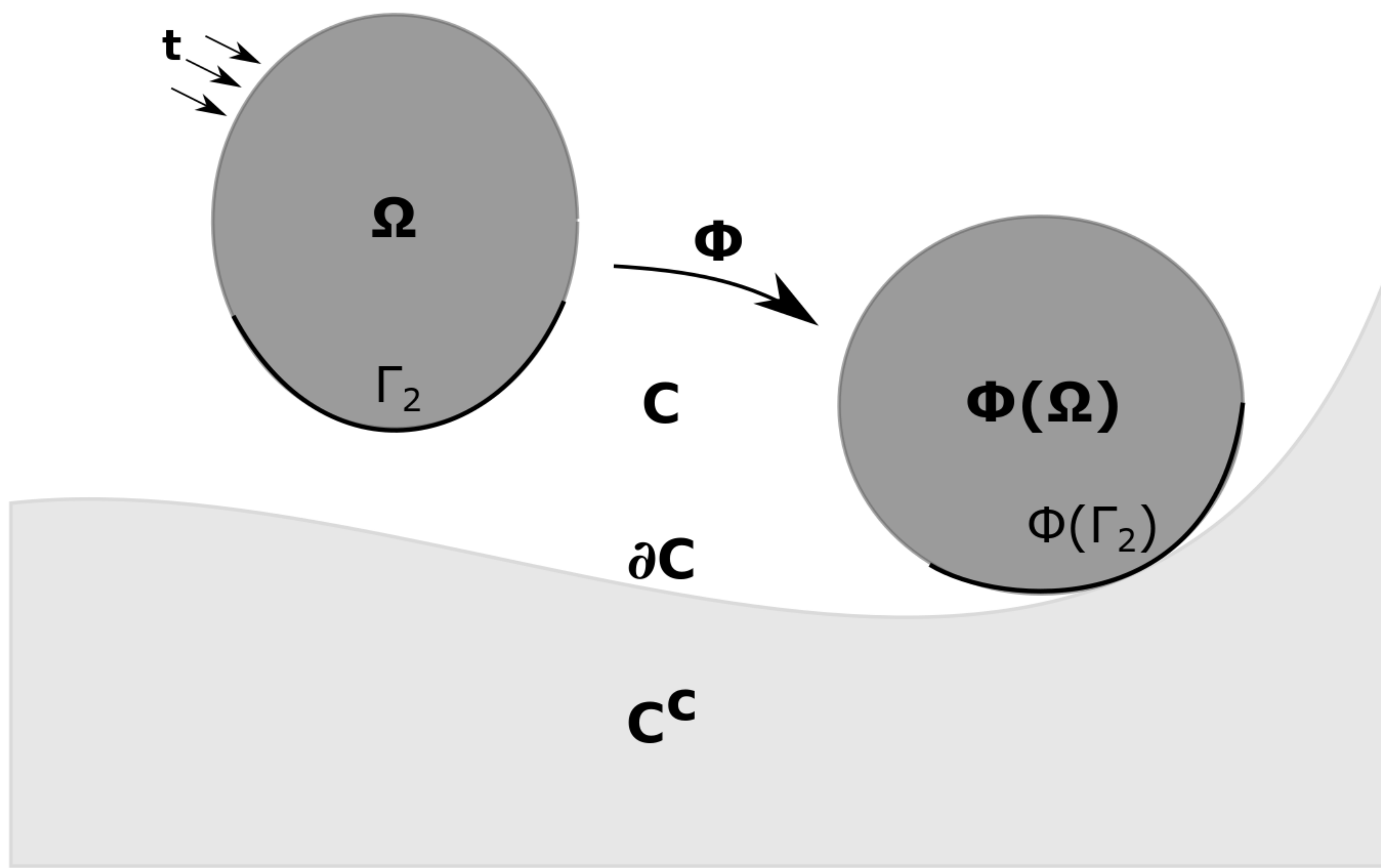
$$\mathbf{u} = \underset{v \in \mathcal{A}}{\operatorname{argmin}}(\mathcal{E}(\mathbf{v})) \tag{6}$$

  Where $\mathcal{A} = \left\{ \mathbf{v} \in \left(H^1(\Omega)\right)^3 \; ; \; \mathbf{v} = 0 \text{ on } \Gamma_0 \right\}$ is the admissible set.

- Constrained minimization for the incompressible materials.

Signorini's contact problem

- Contact with a rigid foundation with $\Gamma_2 \subset \partial\Omega$, the potential contact part.



- The Signorini's contact problem:

$$\begin{cases} \sum_{j=1}^{3} \frac{\partial \mathbf{P}_{ij}}{\partial X_j} + \rho_0 f_i = 0 \quad \text{in}\,\Omega \quad (i = 1, 2, 3) \\ \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_0 \\ \mathbf{P}.\mathbf{N} = \mathbf{t} \quad \text{on} \quad \Gamma_1 \end{cases} \tag{7}$$

- With the following contact conditions:

$$\begin{cases} \phi(\Gamma_2) \subset C \quad \text{(Non-penetration in the foundation)} \\ P.\mathbf{N} = 0 \quad \text{if}\, X \in \Gamma_2 \,\text{and}\, x = \phi(X) \in \text{int}(C) \\ P.\mathbf{N} = \lambda \mathbf{n} \quad \text{if}\, X \in \Gamma_2 \,\text{and}\, x = \phi(X) \in \partial C \,,\text{where}\, \lambda \le 0 \end{cases} \tag{8}$$

Signorini's contact problem [5]

- The displacement field $\boldsymbol{u}$ is also a solution of the constrained minimization problem [2]:

$$\mathbf{u} = \underset{v \in \mathcal{H}}{\operatorname{argmin}}(\mathcal{E}(v)) \tag{9}$$

Where $\mathcal{H} = \left\{ \mathbf{v} \in \left(H^1(\Omega)\right)^3 \;;\; \mathbf{v} = 0 \text{ on } \Gamma_0 \;;\; \phi(\Gamma_2) \subset C \right\}$.

- An example of a non-penetration condition: The normal gap function.

$$g_n(\mathbf{x}) = (\mathbf{x} - \mathbf{y}).\mathbf{n}(\mathbf{y}) \geq 0 \quad \forall \mathbf{x} \in \Phi(\Gamma_2) \tag{10}$$

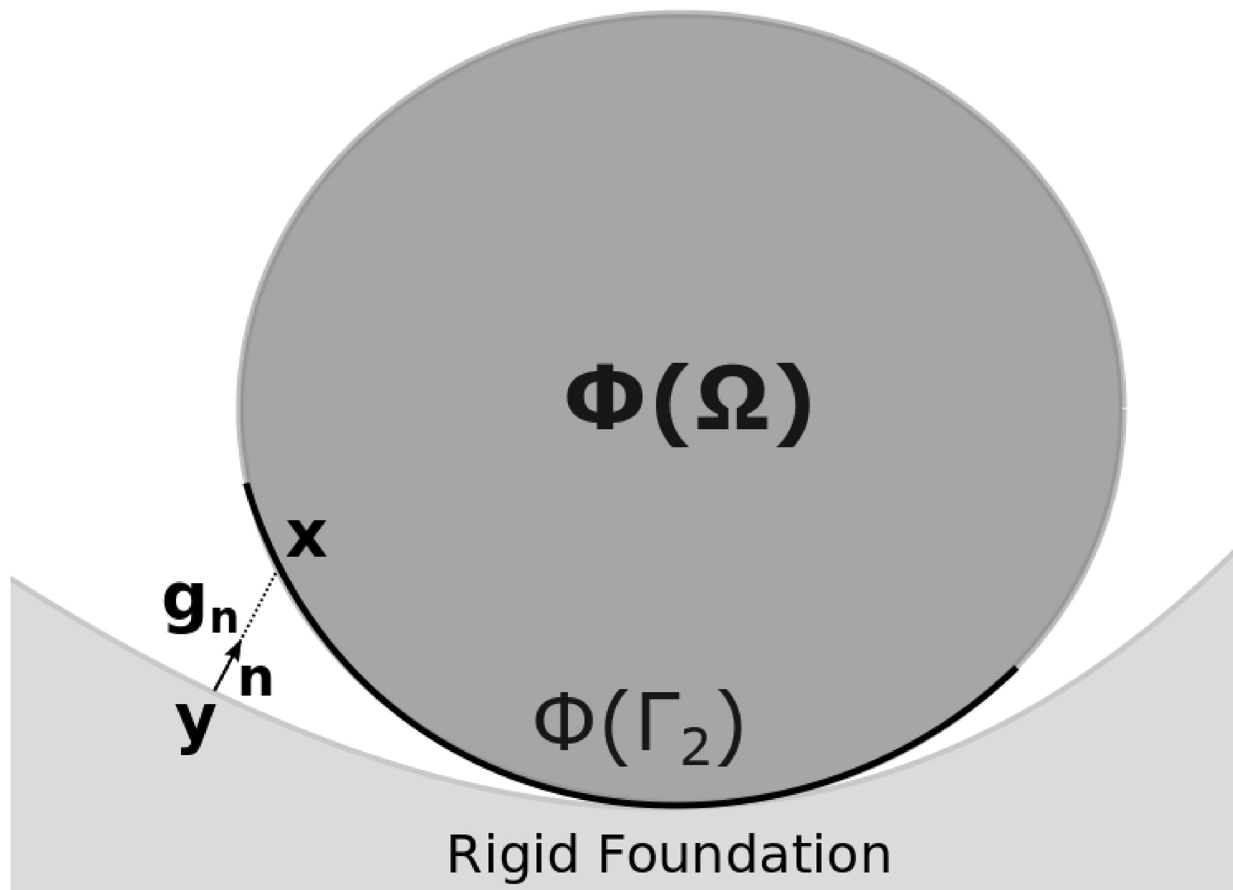


Figure 2: Normal gap function

Interior Point Optimizer

- IPOPT [6] can solve large scale constrained and unconstrained optimization problems:

$$\begin{cases} \underset{u\in\mathbb{R}^n}{Min}\, f(u) \quad \text{such that} \\ g_{Lo} \leq g(u) \leq g_{Up} \\ u_{Lo} \leq u \leq u_{Up} \end{cases} \tag{11}$$

- Line search with the filter method.

- The objective function f and the constraint function g must be smooth ($\in C^2$)

- FreeFEM [3] command for IPOPT:

$$\textbf{IPOPT}(f, Jacobian(f), Lagrangian(f, g), g, u, Jacobian(g), lb, ub, clb, cub)$$

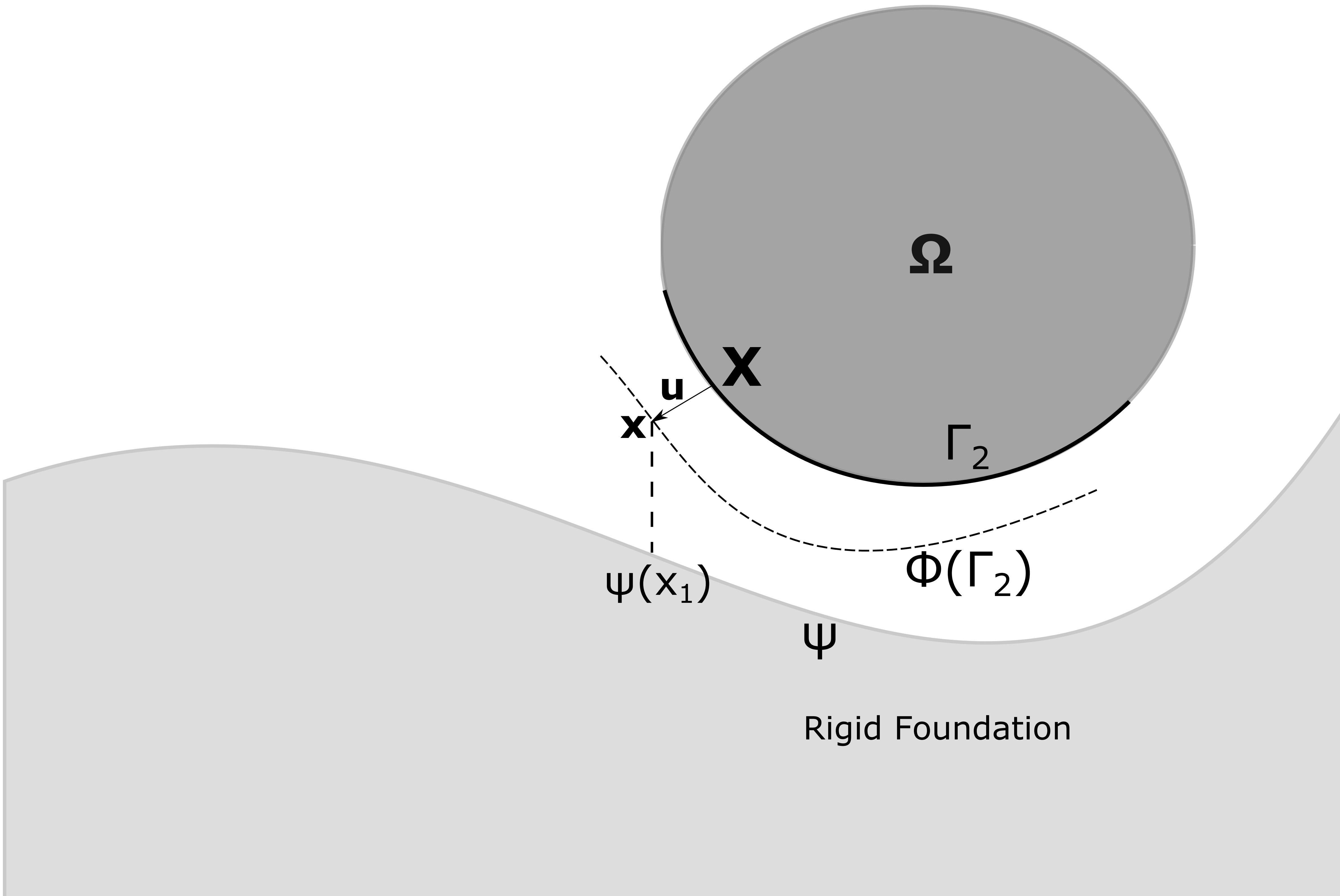
Formulation of the problem

- If the boundary of the foundation can be written in the form of a function ψ of class C^2, then the non-penetration condition is given by:

$$x_2 - \psi(x_1) \geq 0 \tag{12}$$

$$\iff X_2 + u_2 - \psi(X_1 + u_1) \geq 0 \tag{13}$$

Where (x_1, x_2) belongs to the contact border.



Formulation of the problem

- In the finite element method the displacement field **u** is given by:

$$\mathbf{u}(x,y) = \sum_{k=1}^{m} \begin{pmatrix} U_{x,k} \\ U_{y,k} \end{pmatrix} \Phi_k(x,y) \tag{14}$$

  Where Φ_k are the shape functions and $U_{x,k}, U_{y,k}$ the degrees of freedom.

- The unknown of our problem is thus:

$$\mathbf{U} = \left(U_1\ U_2 \ldots U_{2m}\right)^T = \left(U_{x,k}\ U_{y,k} \ldots U_{x,m}\ U_{y,m}\right)^T \tag{15}$$

- The formulation of the contact problem:

$$\begin{cases} \underset{\mathbf{U}\in\mathbb{R}^{2m}}{Min}\ \mathcal{E}(\mathbf{U}) \quad \text{such that} \\ X_{2k} + U_{2k} - \psi(X_{2k-1} + U_{2k-1}) \geq 0\ \forall\ (X_{2k-1}, X_{2k}) \in \Gamma_2 \end{cases} \tag{16}$$

- Jacobian, Hessian of the energy $\mathcal{E}$ and the constraints $\Rightarrow$ IPOPT.

Numerical validations

- Compression of a hyperelastic cube of dimension equal to 1 m, with a pressure of $f = 0.876\ Pa$ applied to its upper face.[1]

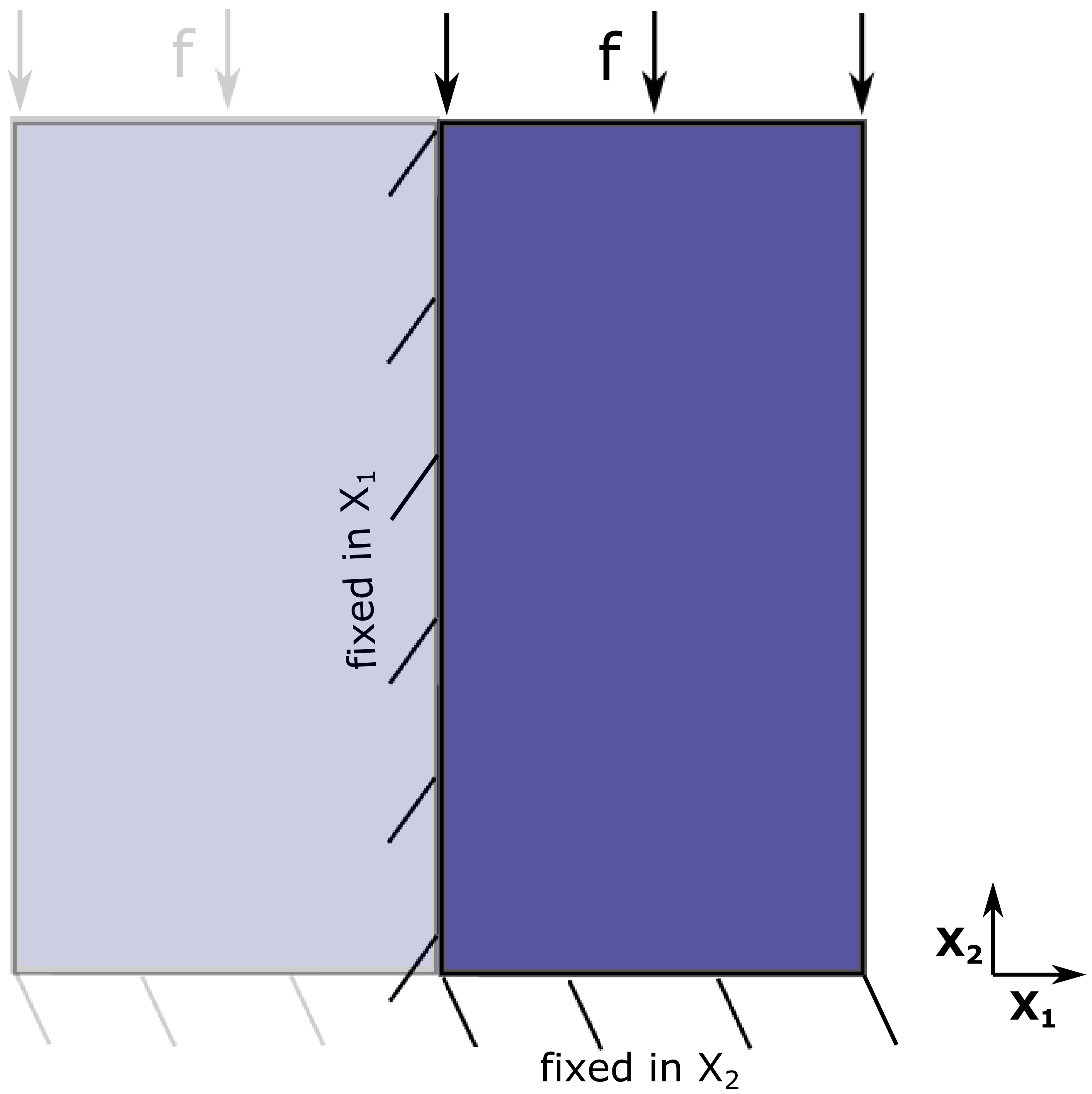


Figure 3: The geometry

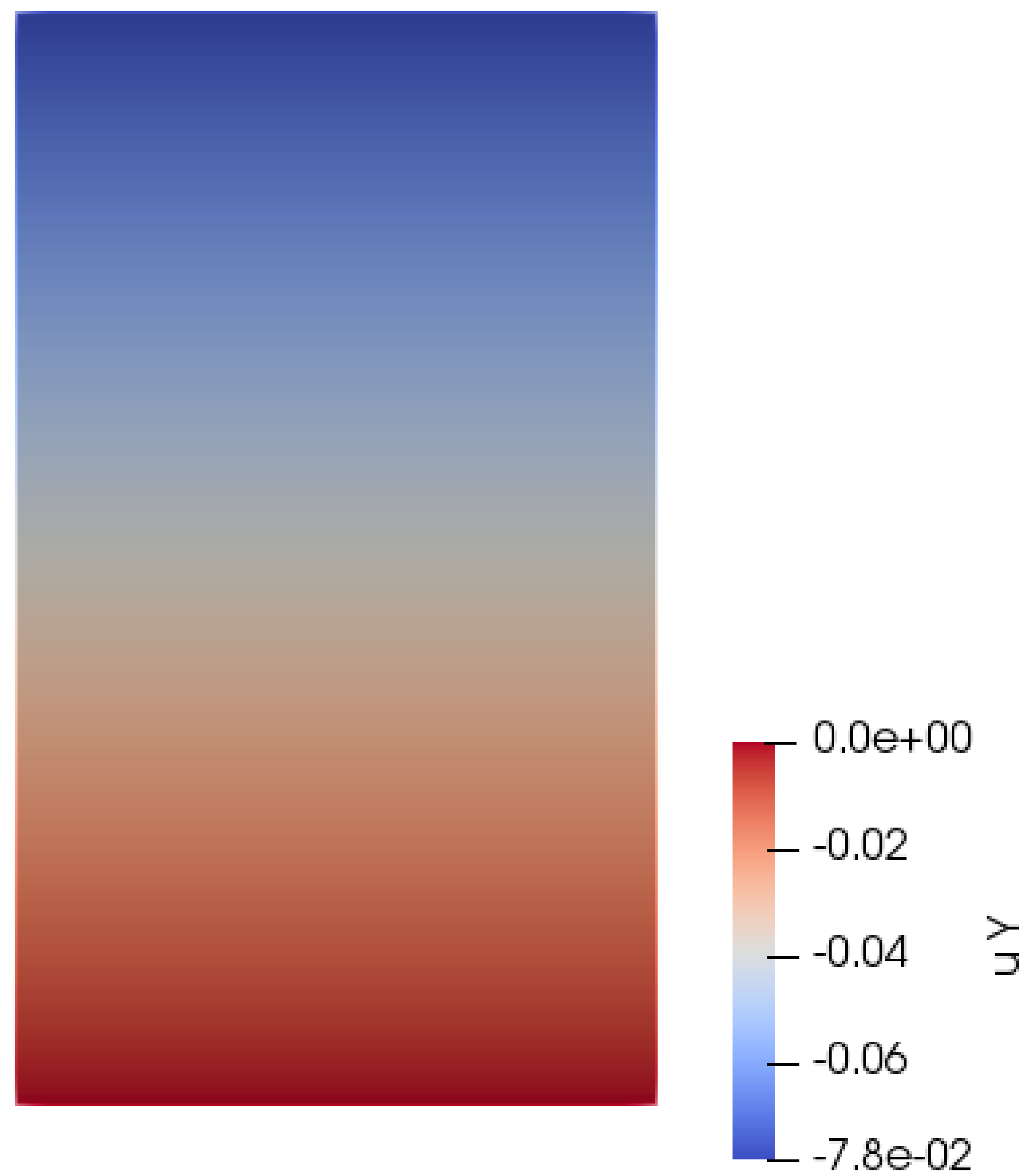


Figure 4: Vertical displacement field (Neo-Hookean)

- Error=0.19% (Same as Code_Aster)
- Similar results if a contact conditions are used.

Numerical validations

- Hertz contact problem: Compression of an elastic cylinder over a rigid plane foundation.

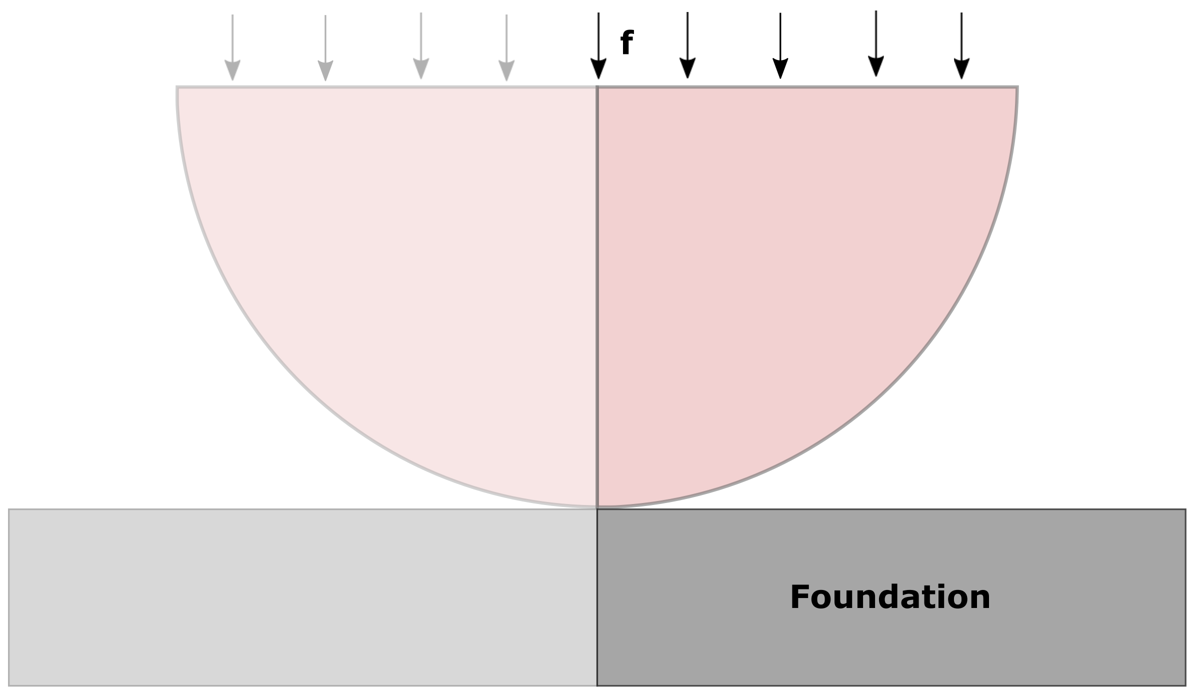


Figure 5: The geometry

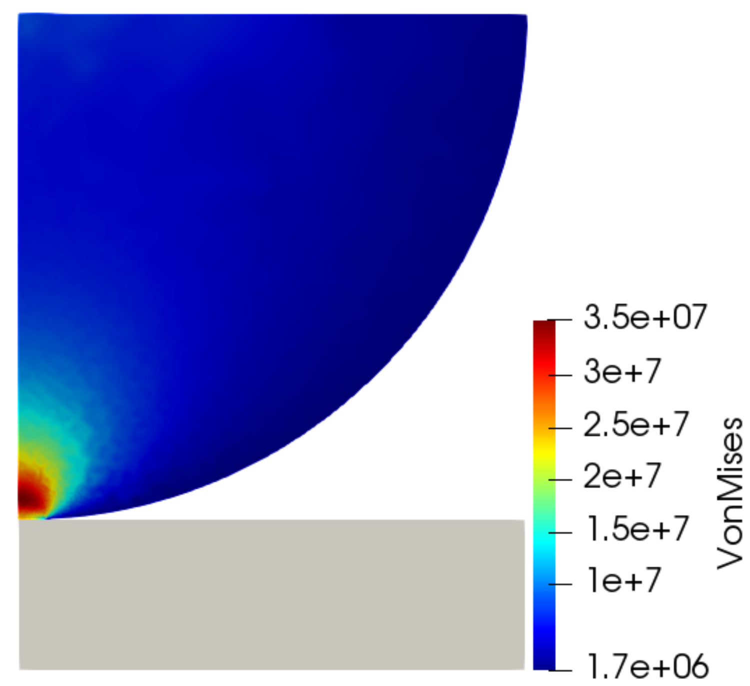
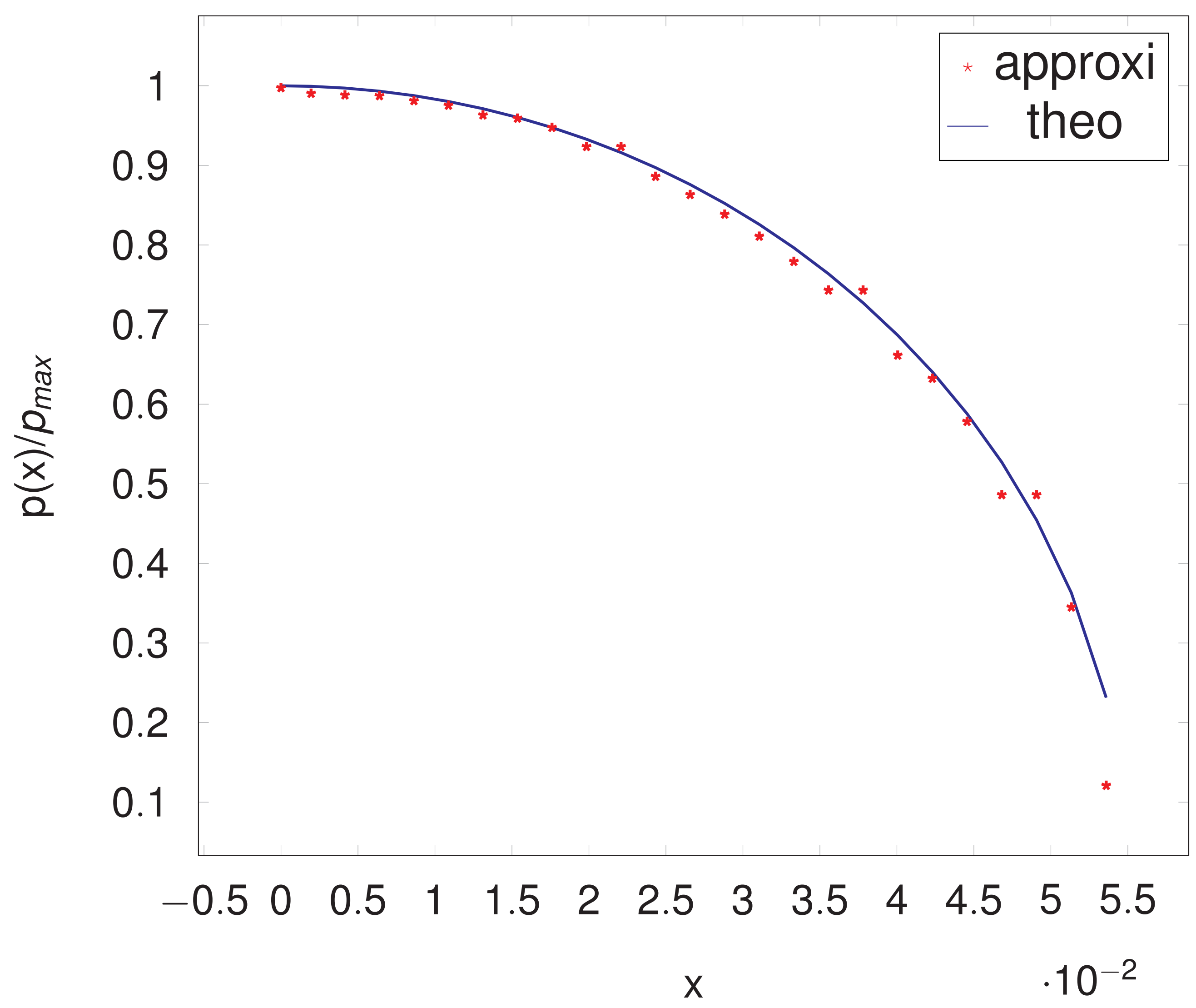


Figure 6: Deformation and Von mises stress

The pressure at the contact area:



New contact formulation

Non-penetration of the contact boundary of the first body with the triangles of the second body.

- Indicator function of a triangle:

Let $F_1, F_2, F_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ be defined as follow:

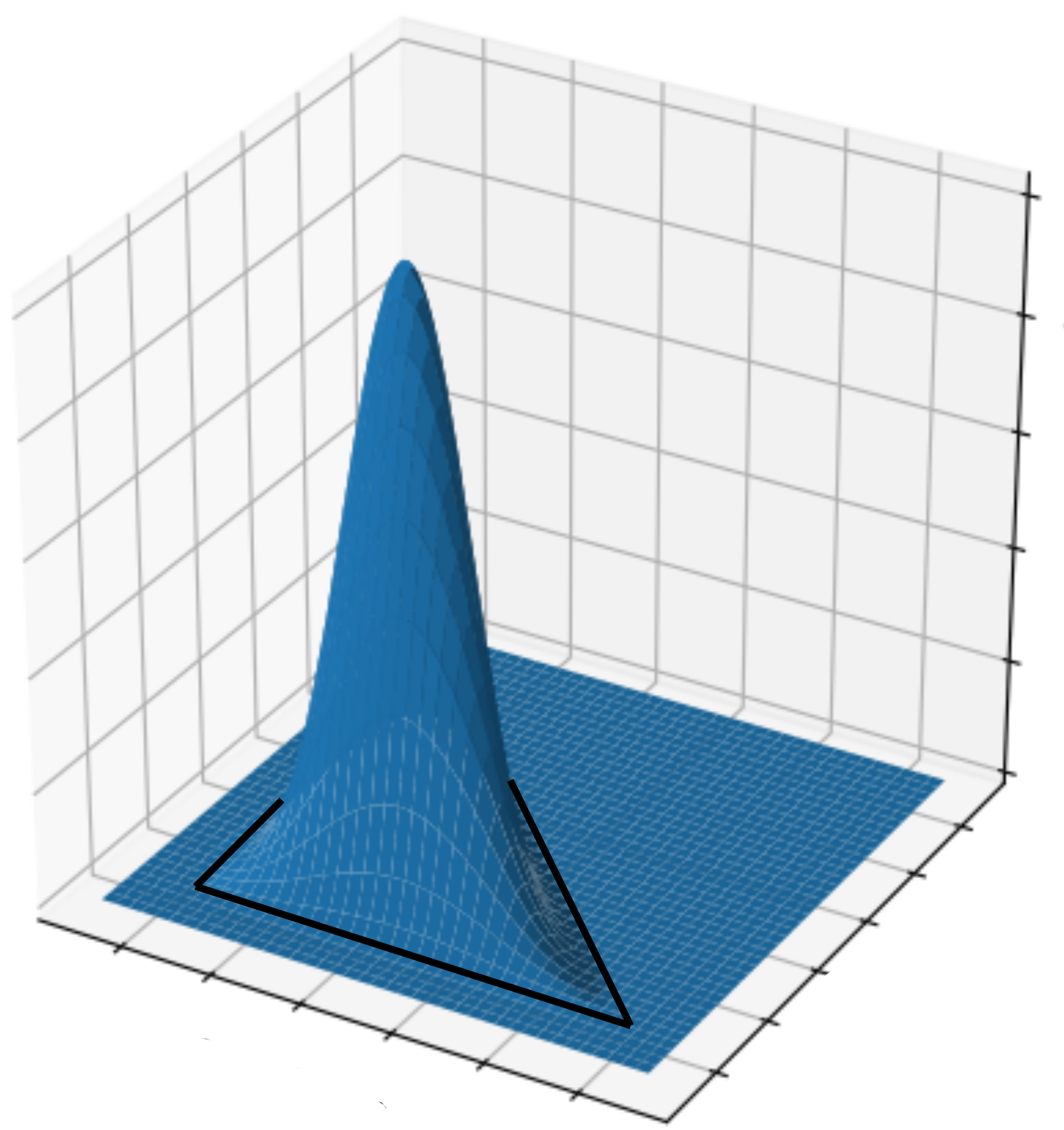
$$F_1(x,y) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^3 & \text{if } x > 0 \end{cases} \quad F_2(x,y) = \begin{cases} 0 & \text{if } y \leq 0 \\ y^3 & \text{if } y > 0 \end{cases} \quad F_3(x,y) = \begin{cases} -(y+x-1)^3 & \text{if } y+x-1 < 0 \\ 0 & \text{if } y+x-1 \geq 0 \end{cases}$$

The function $F_p : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$:

$$F_p(x,y) = F_1(x,y).F_2(x,y).F_3(x,y) \tag{17}$$

is $C^2(\mathbb{R}^2)$ and if $(X,Y) \in \mathbb{R}^2$:

$$\begin{cases} F_p(X,Y) > 0 & \text{if } (X,Y) \in \mathring{\mathcal{T}_0} \text{ (Interior of the reference triangle)} \\ F_p(X,Y) = 0 & \text{otherwise} \end{cases} \tag{18}$$



Impenetrability condition

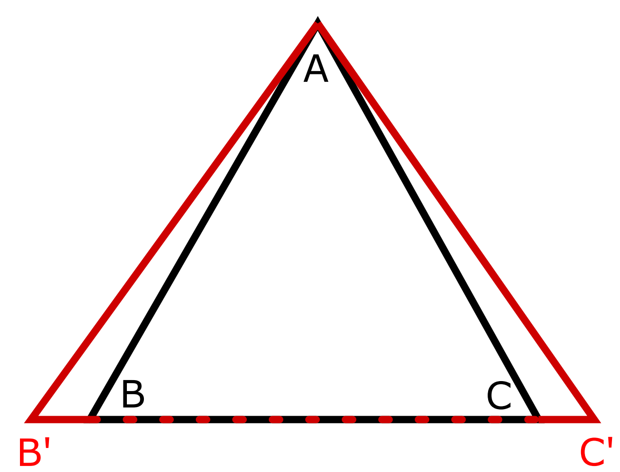
- For an arbitrary triangle $\mathcal{T}$:

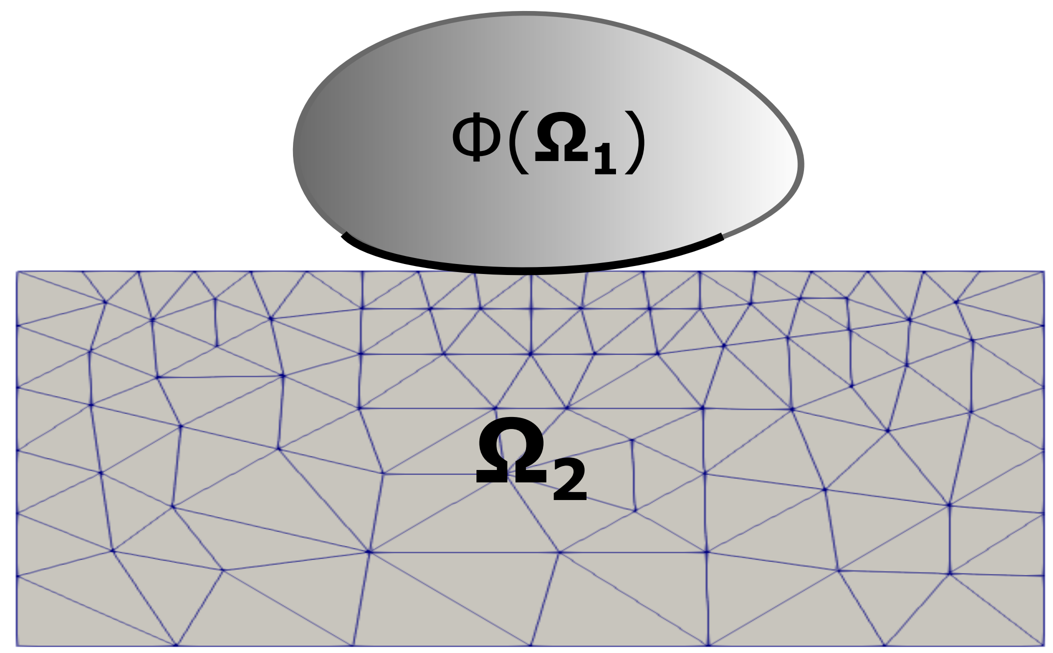
$$\begin{cases} F_{\mathcal{T}}(X,Y) > 0 & \text{if } (X,Y) \in \overset{\circ}{\mathcal{T}} \\ F_{\mathcal{T}}(X,Y) = 0 & \text{otherwise} \end{cases} \tag{19}$$

- Indicator function Φ_{Ω_2} of the domain Ω_2:

$$\begin{cases} \Phi_{\Omega_2}(X,Y) > 0 & \text{if } (X,Y) \in \bigcup_{j=1}^{nT} \overset{\circ}{\mathcal{T}}_j \\ \Phi_{\Omega_2}(X,Y) = 0 & \text{otherwise} \end{cases} \tag{20}$$

- Indicator function Φ_{Ω_2} of the domain Ω_2 after a sligth modification of the triangles:





$$\begin{cases} \Phi_{\Omega_2}(X,Y) > 0 & \text{if } (X,Y) \in \text{contact area neighborhood of } \Omega_2 \\ \Phi_{\Omega_2}(X,Y) = 0 & \text{otherwise} \end{cases} \tag{21}$$

- Impenetrability condition:

$$\Phi_{\Omega_2,i}(\mathbf{U}) = \Phi_{\Omega_2}(X_i + U_i^c, Y_i + U_{i+1}^c) = 0 \quad \forall\, i = 1, \ldots, n_C \tag{22}$$

Impenetrability condition using integral over the contact border

- Alternative impenetrability condition:

$$\int_{\Gamma_C} \Phi_{\Omega_2}(\mathbf{X} + \mathbf{u})\, ds = 0 \tag{23}$$

  Where Γ_C is the contact border of the first body in the reference configuration, and $\mathbf{u}$ is the displacement field.

- Because $\Phi_2 \geq 0$:

$$\int_{\Gamma_C} \Phi_{\Omega_2}(\mathbf{X} + \mathbf{u})\, ds = 0 \iff \Phi_{\Omega_2}(\mathbf{X} + \mathbf{u}) = 0 \text{ almost everywhere on } \Gamma_C \tag{24}$$

  $\Rightarrow$ More stronger than the first impenetrability condition.

- Numerical integration: The integral over a segment $[q_i, q_{i+1}]$ gives:

$$\int_{[q_i, q_{i+1}]} \Phi_{\Omega_2}(\mathbf{X} + \mathbf{u})\, ds = \sum_{k=1}^{K} \omega_k \Phi_{\Omega_2}(\mathbf{x}_{|\xi_k}) \tag{25}$$

  Where $\mathbf{x}_{|\xi_k}$ are the current positions of the integration points at $[q_i, q_{i+1}] \subset \Gamma_C$.

- Interpretation: impenetrability of each segment of the contact border Γ_C with the second body.

Penalty method

First Formulation:

- The contact problem formulation:

$$\begin{cases} Min(\mathcal{E}_p(\mathbf{U})) \quad \text{subjected to} \\ \Phi_{\Omega_2,i}(\mathbf{U}) = \Phi_{\Omega_2}(X_i + U_i^c, Y_i + U_{i+1}^c) = 0 \quad \forall\, i = 1, \ldots, n_C \end{cases} \tag{26}$$

Where $\mathcal{E}_p$ denotes the total potential energy:

$$\begin{cases} \mathcal{E}_p = \mathcal{E}_{p1} & \text{Contact between one body and a foundation} \\ \mathcal{E}_p = \mathcal{E}_{p1} + \mathcal{E}_{p2} & \text{Contact between two bodies} \end{cases} \tag{27}$$

- The penalty function:

$$G_{\mu_k}(\mathbf{U}) = \mathcal{E}_p(\mathbf{U}) + \mu_k \sum_{i=1}^{nC} \Phi_{\Omega_2,i}(\mathbf{U}) \tag{28}$$

- Solving the unconstrained minimization problem where $\mu_k \to +\infty$:

$$Min(G_{\mu_k}(\mathbf{U})) \tag{29}$$

Second Formulation:

- The contact problem can be reformulated as the following:

$$\begin{cases} Min(\mathcal{E}_p(\mathbf{U})) \quad \text{subjected to} \\ \int_{\Gamma_C} \Phi_{\Omega_2}(\mathbf{X} + \mathbf{u})\, ds = 0 \end{cases} \tag{30}$$

- The penalty function becomes:

$$G_\mu(\mathbf{U}) = \mathcal{E}_p(\mathbf{U}) + \mu \int_{\Gamma_C} \Phi_{\Omega_2}(\mathbf{X} + \mathbf{u})\, ds \tag{31}$$

Numerical validations

- Hertz contact problem: Compression of an elastic cylinder over an elastic block.

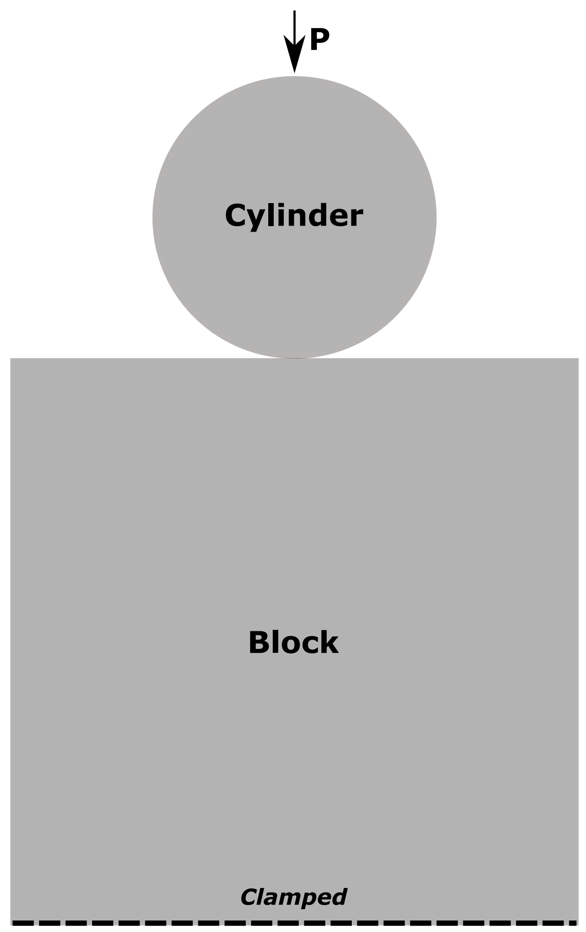


Figure 7: The geometry

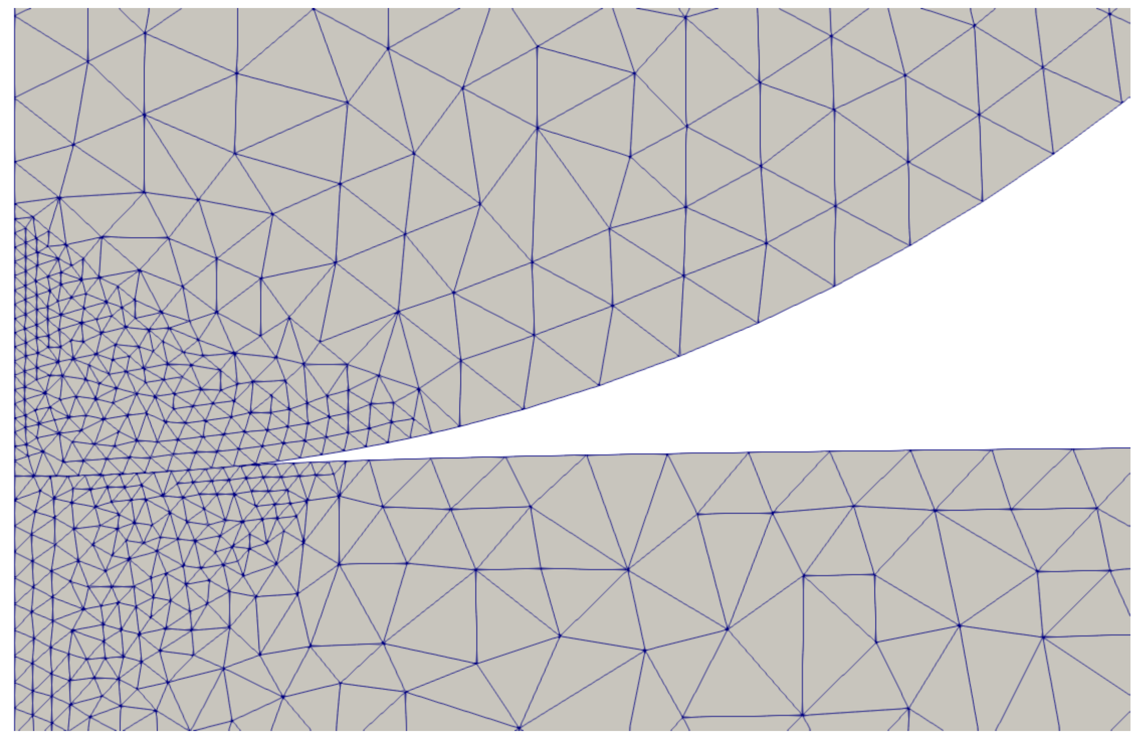
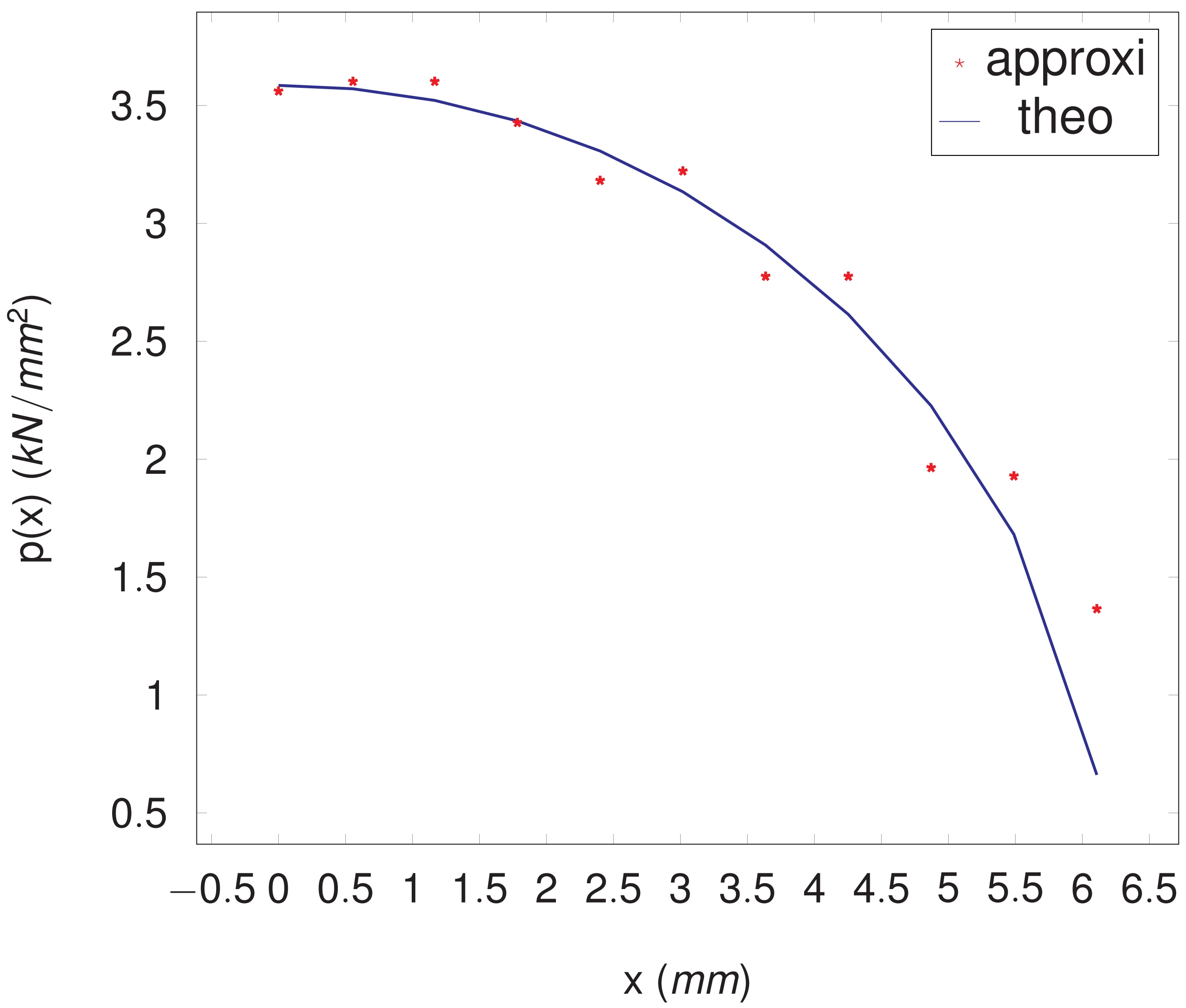


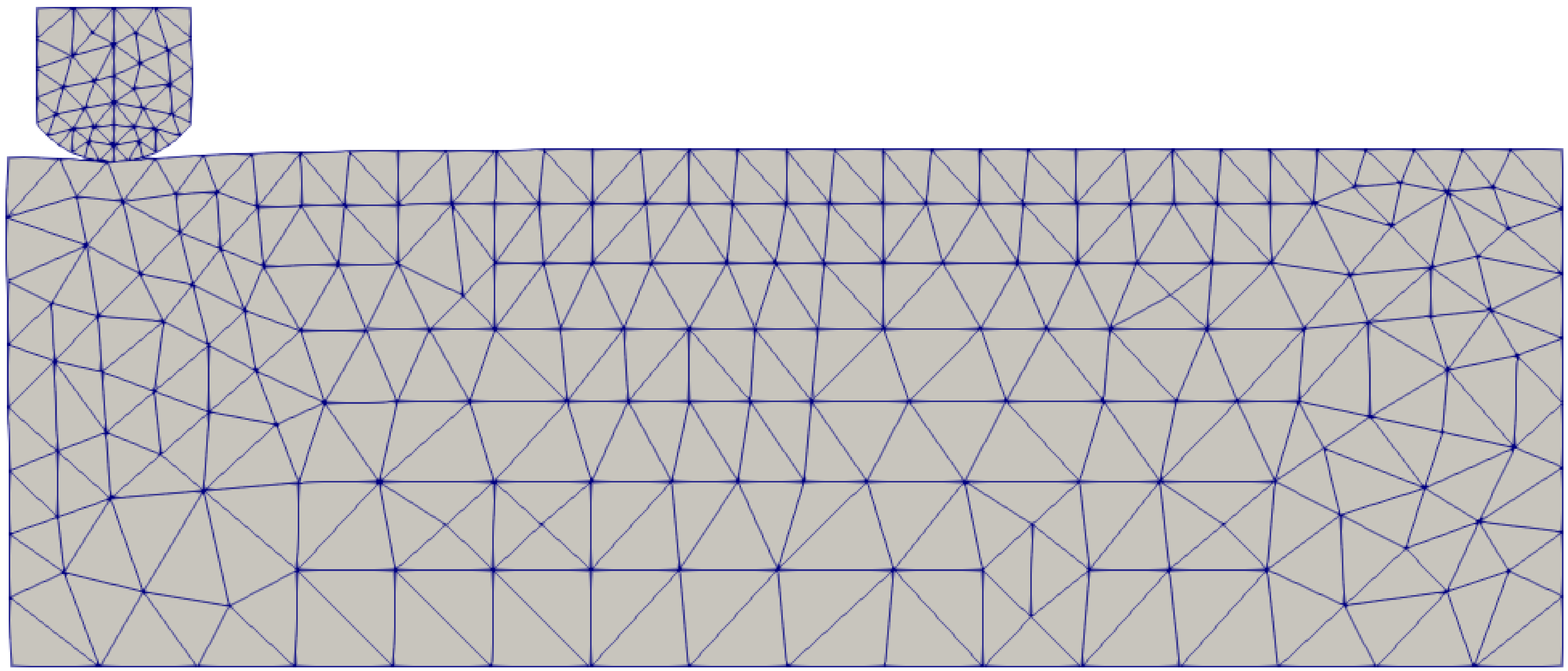
Figure 8: Deformation

The pressure at the contact area:



Numerical validations

- Shallow ironing:



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THANKS FOR YOUR ATTENTION