Anisotropic Surface Remeshing

Scaling up with Voronoi Parallel Linear Enumeration

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ALICE Géométrie & Lumière
CENTRE INRIA Nancy Grand-Est

OVERVIEW

Part. 1. Goals and Motivations

Part. 2. Centroidal Voronoi Tesselations

Part. 3. Tweaking the Definition of Distance

Part. 4. Constraints & Protecting Balls

1

Goals and Motivations

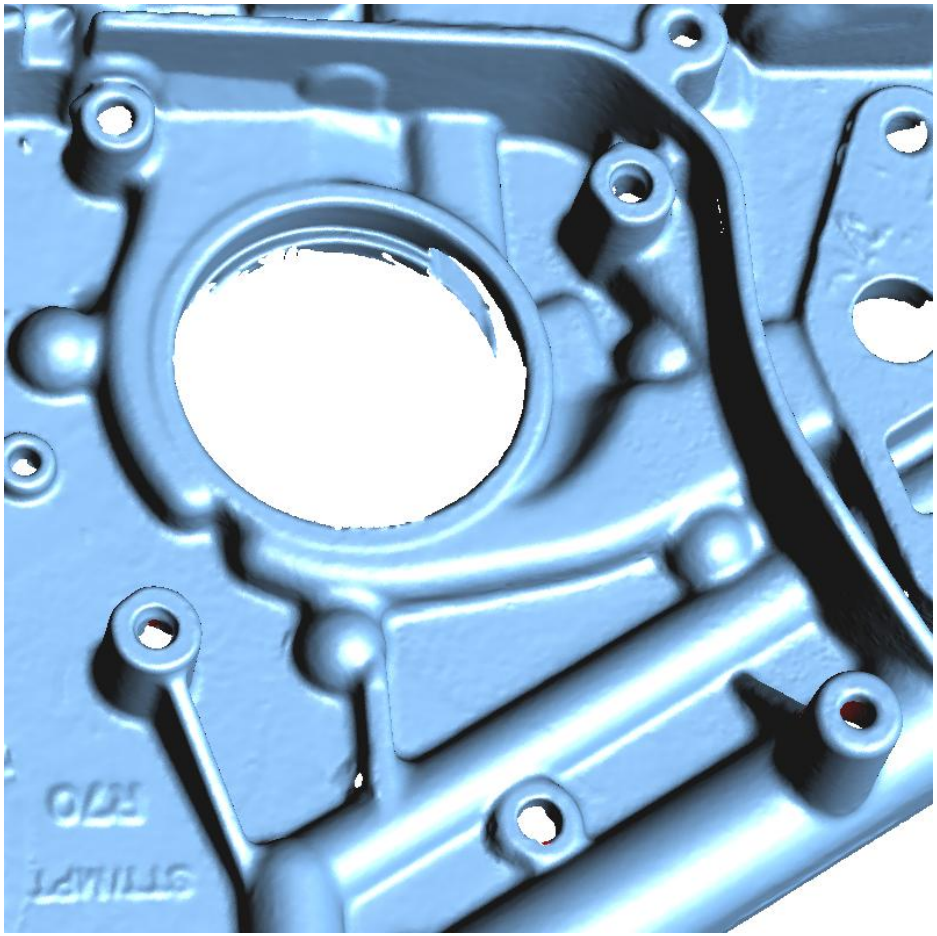
Part. 1. Goals : a “Flexible” mesh generator

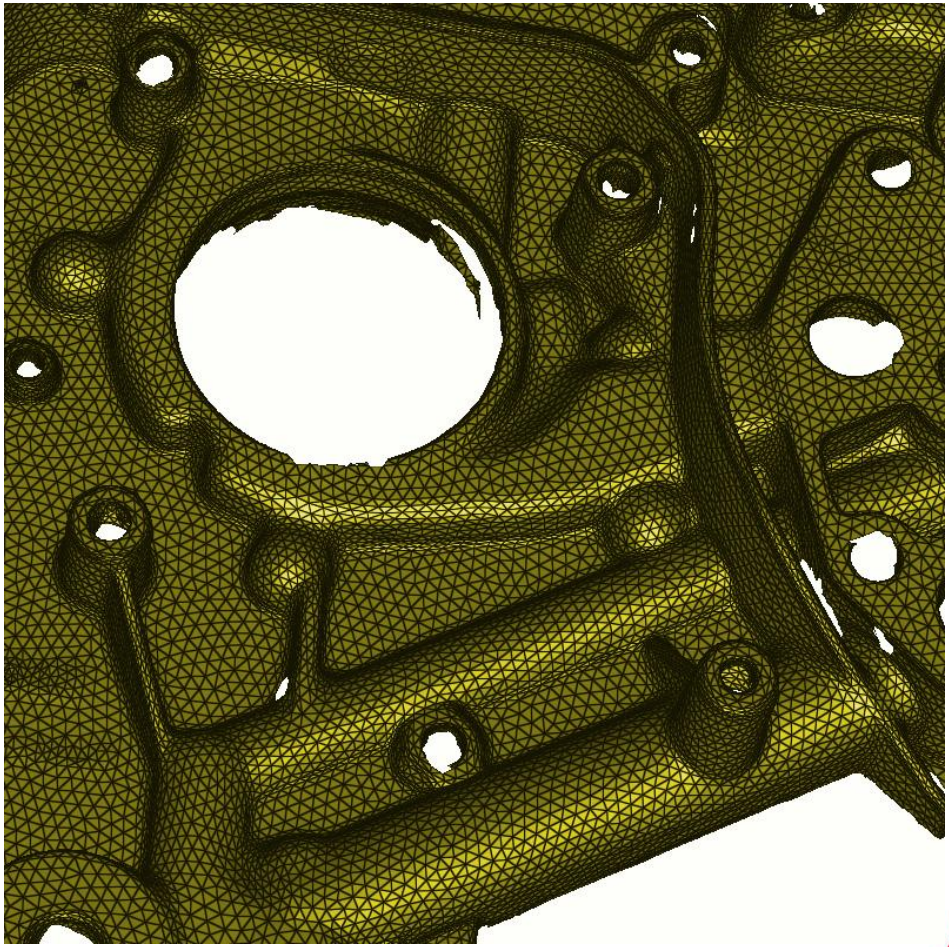
“wish list”

- Tolerant meshing (Scan2FEA, STL2FEA)
- Steerable (orientation, anisotropy, quads/hex)
- Beauty (… of the mesh, … of the approach)

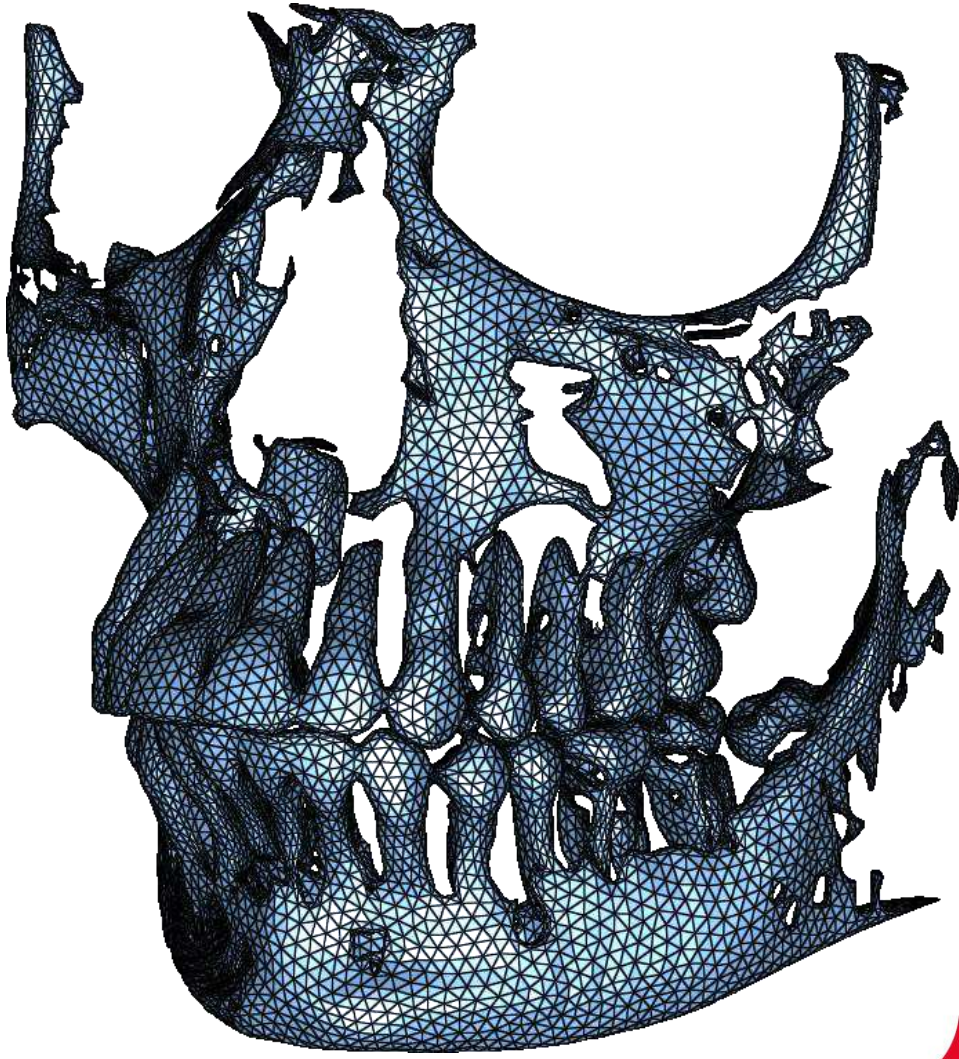
Part. 1. Goals : a “Flexible” mesh generator

Tolerant meshing (Scan2FEA, STL2FEA)





Part. 1. Goals : a “Flexible” mesh generator



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Steerable meshing (orientation, anisotropy, quads/hex)

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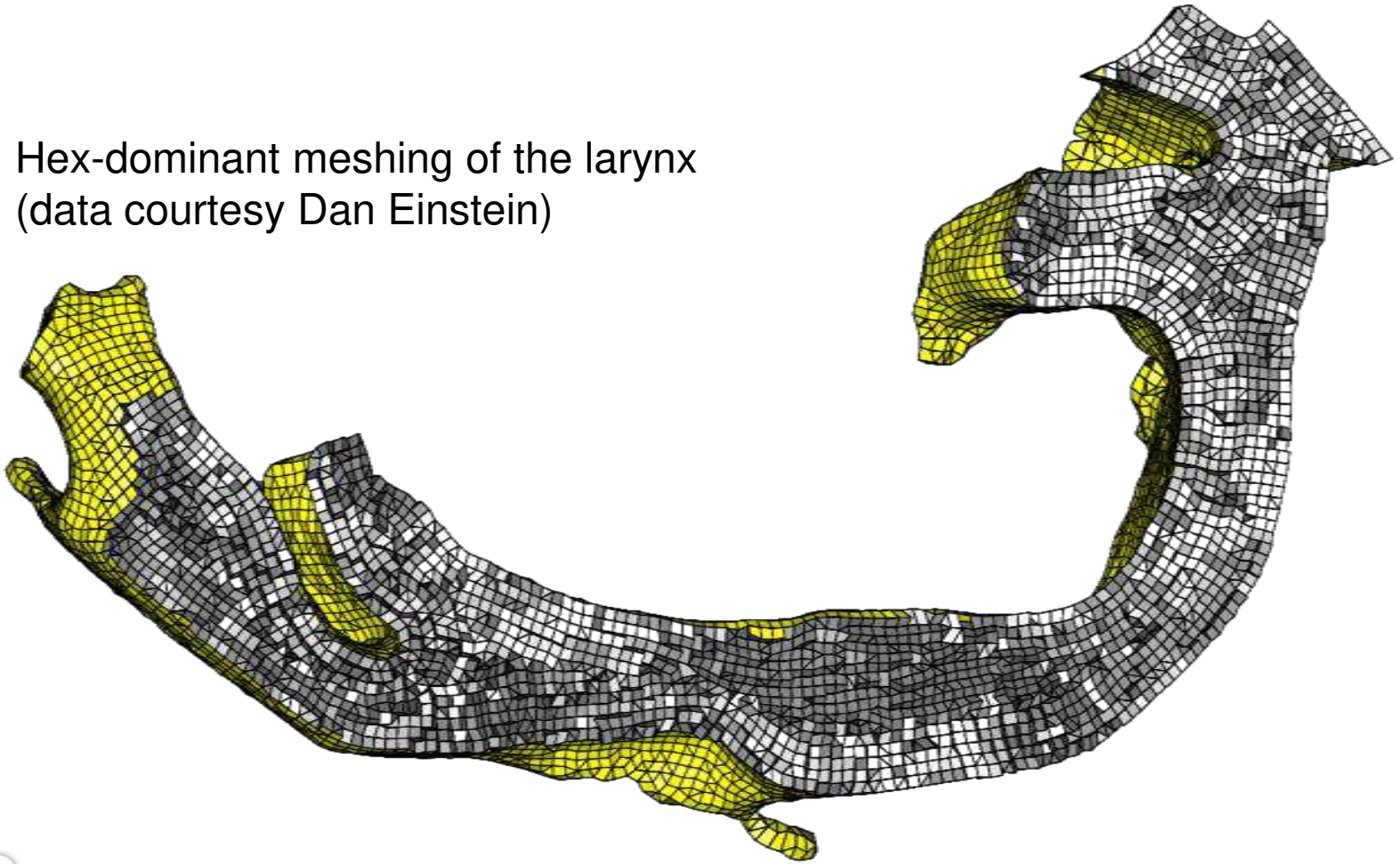
Steerable meshing (orientation, anisotropy, quads/hex)

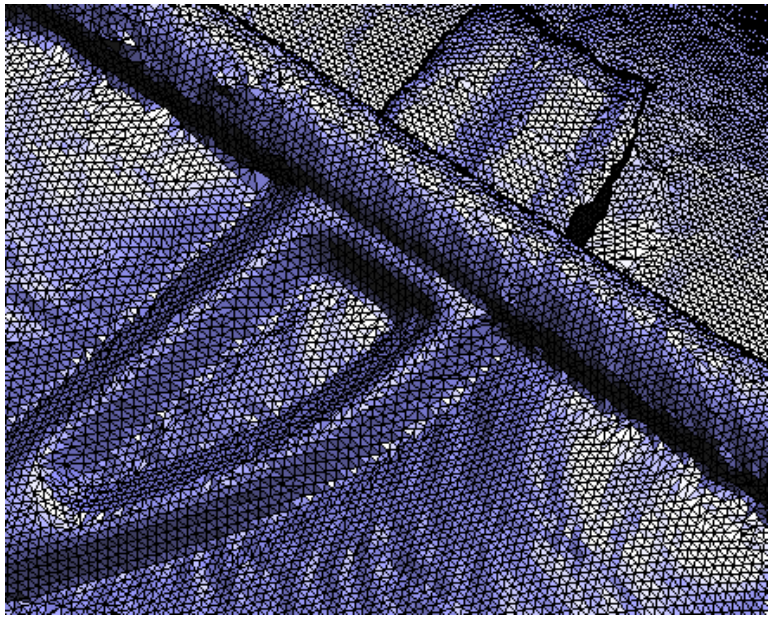
Hex-dominant meshing of the larynx
(data courtesy Dan Einstein)



Part. 1. Goals : a “Flexible” mesh generator

Goal: beauty (… of the mesh, … of the approach)

Input: raw scanned mesh
(courtesy XYZRGB)



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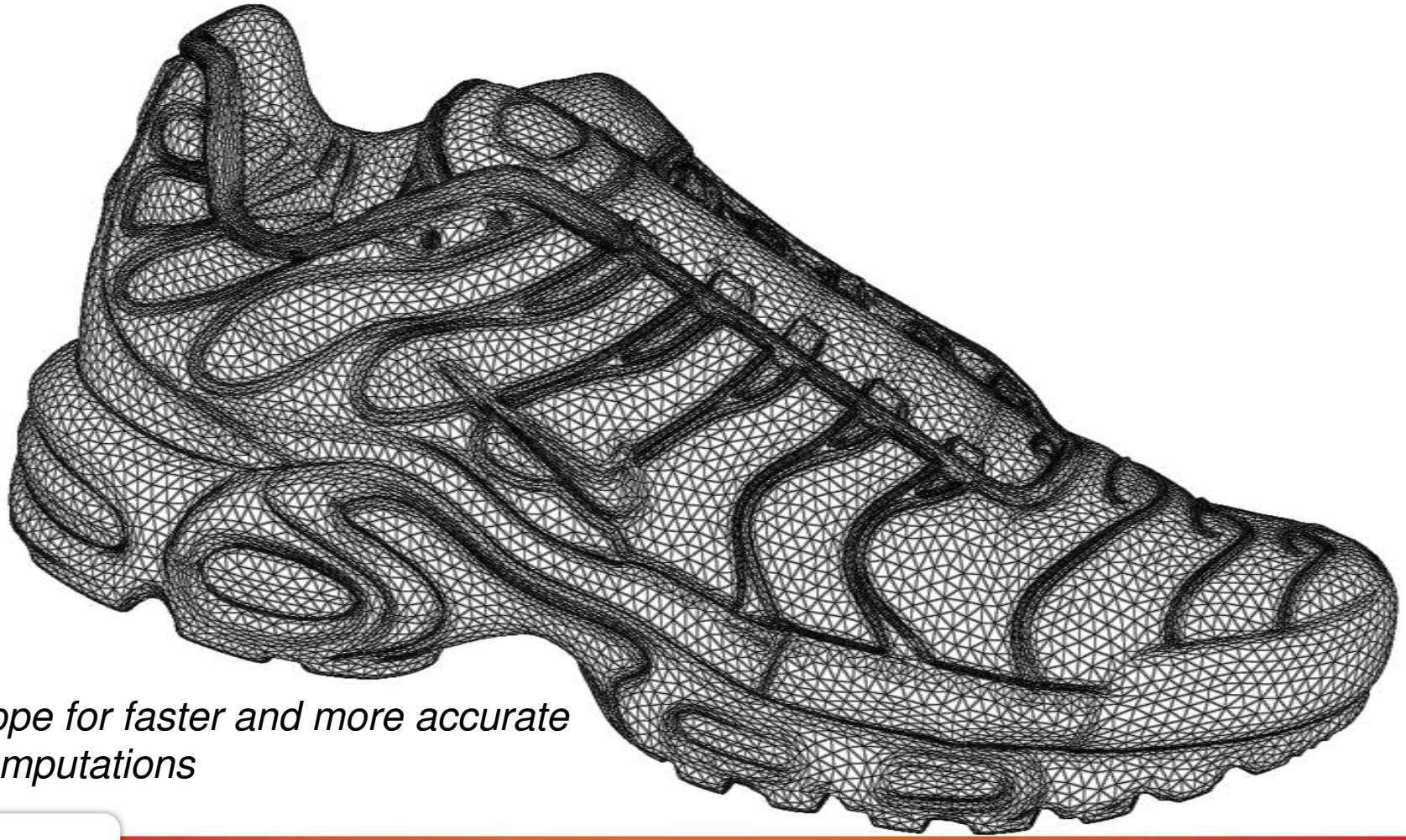
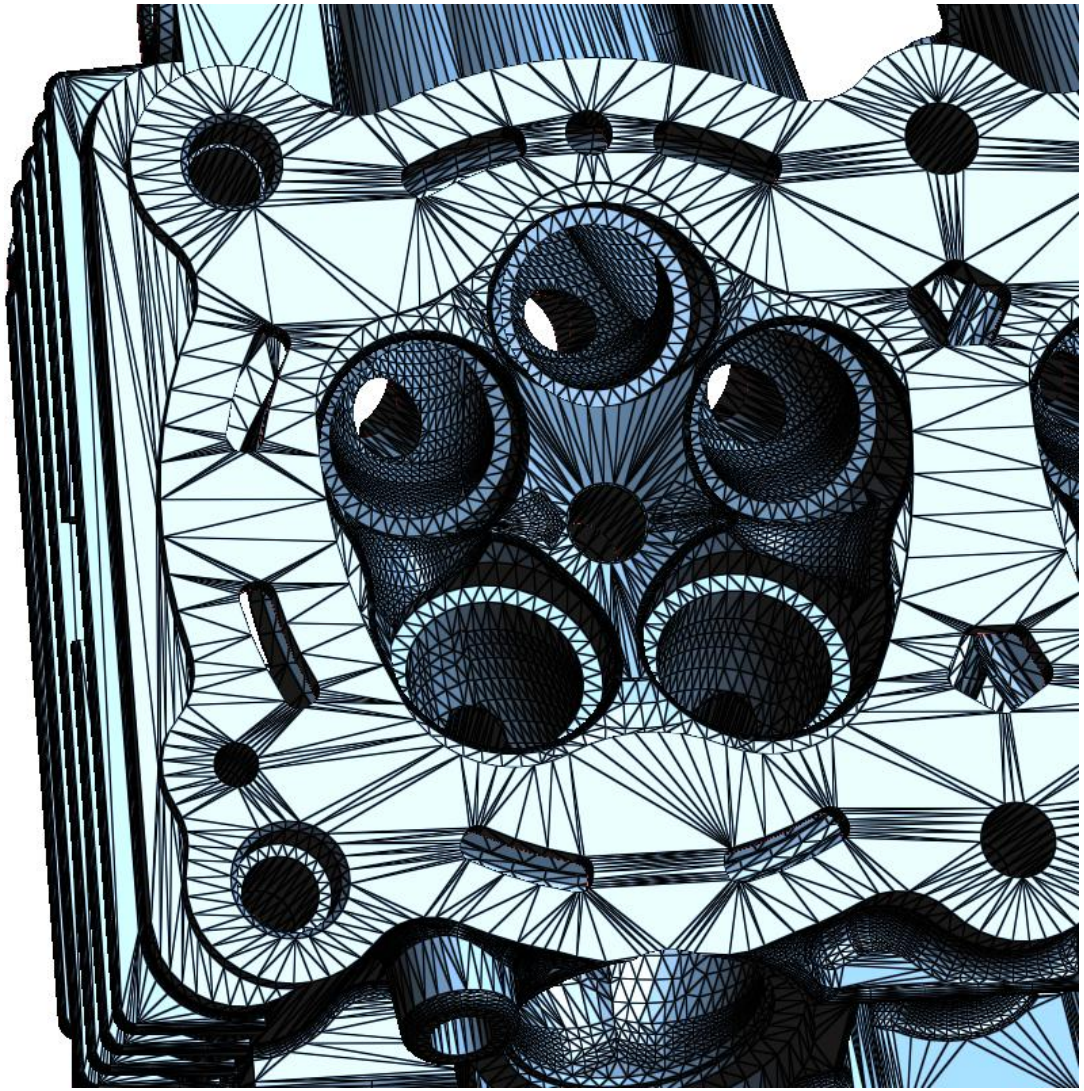


Hope for faster and more accurate computations

Part. 1. Goals : a “Flexible” mesh generator



Goal: beauty

… of the mesh,

… of the approach

Part. 1. Goals : a “Flexible” mesh generator

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Part. 1. Goals : a “Flexible” mesh generator

Part. 1. Goals and Motivations

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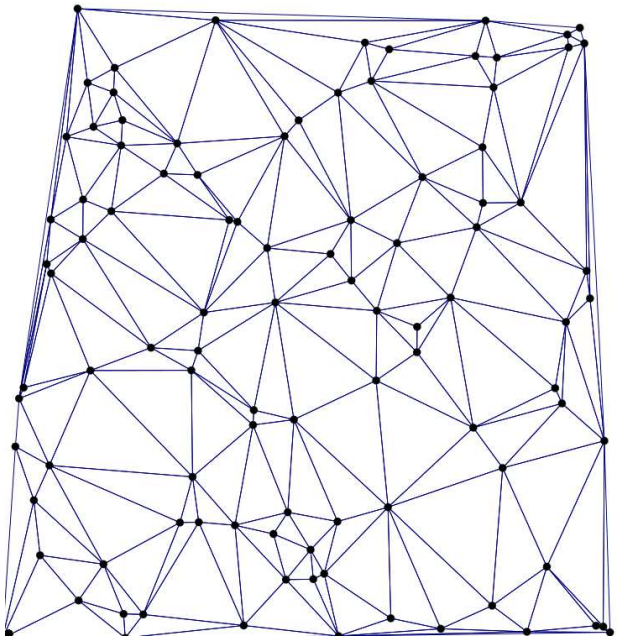
2

Centroidal Voronoi Tesselations

Part. 2. Centroidal Voronoi Tesselation

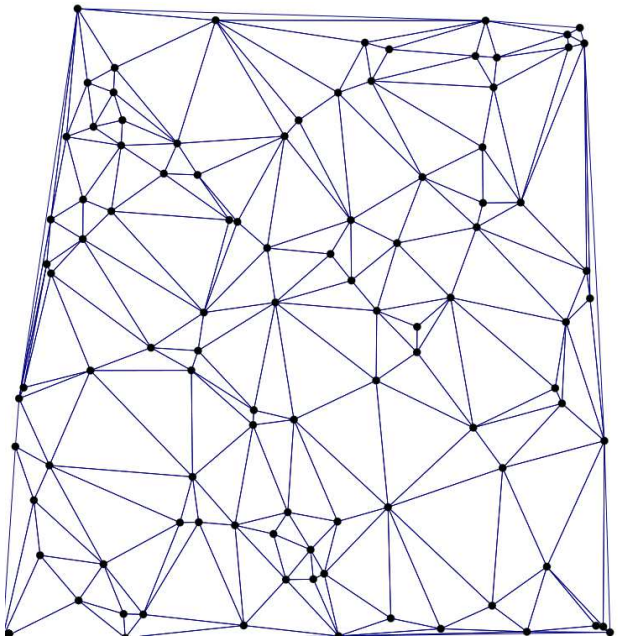
Optimize a Voronoi diagram from the point of view of sampling regularity (quantization noise power)



[Lloyd] least squares quantization in PCM ; [Du], [Iri], [Okabe]

Part. 2. Centroidal Voronoi Tesselation

Optimize a Voronoi diagram from the point of view of sampling regularity (quantization noise power)



$$F = \sum_i \int_{Vor(i)} \left\| x_i - x \right\|^2 dx$$

[Lloyd] least squares quantization in PCM

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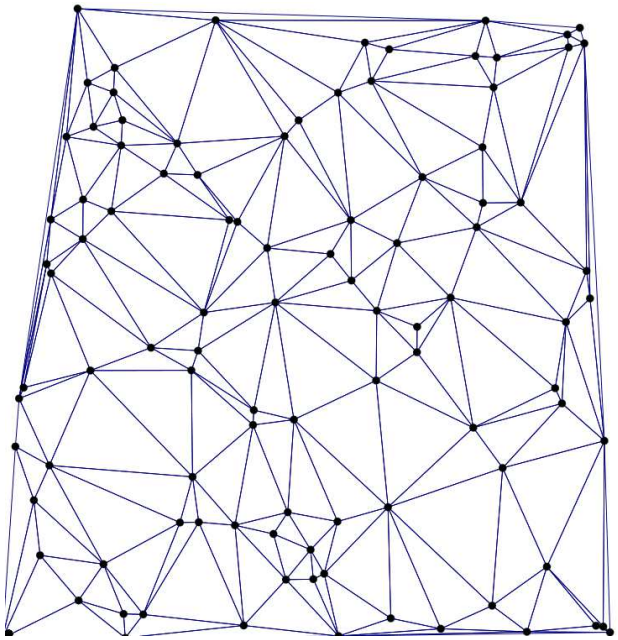

Minimize

$$F = \sum_i \int_{Vor(i)} \| x_i - x \|^2 dx$$

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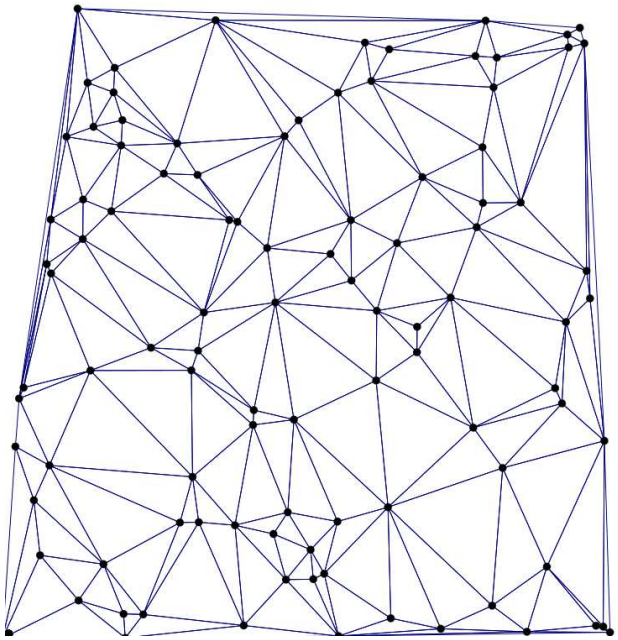
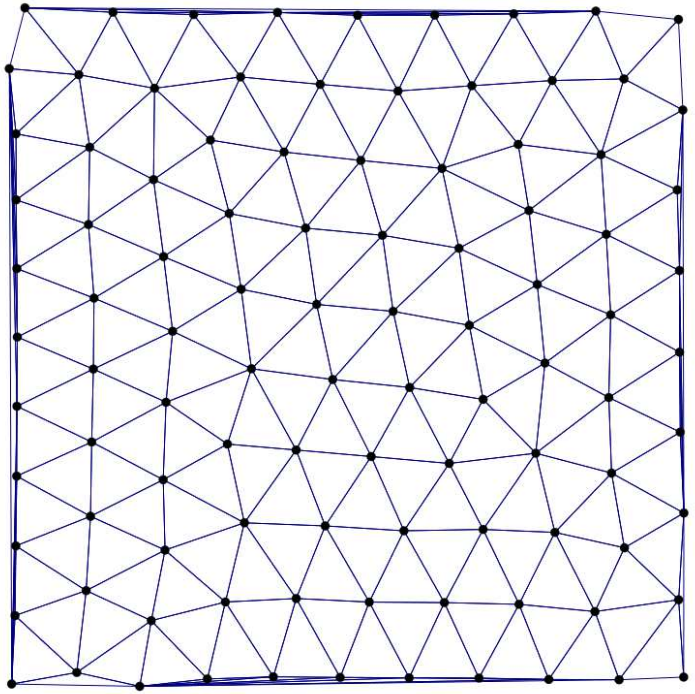


Minimize

$$F = \sum_i \int_{Vor(i)} \left\| \mathbf{x}_i - \mathbf{x} \right\|^2 d\mathbf{x}$$



[Lloyd] least squares quantization in PCM

Part. 2.

Part. 2. Centroidal Voronoi Tesselation

$$F = \sum_i \int_{Vor(i)} \| \mathbf{x}_i - \mathbf{x} \|^2 d\mathbf{x}$$

F: Quantization noise power (measures the quality of the sampling)

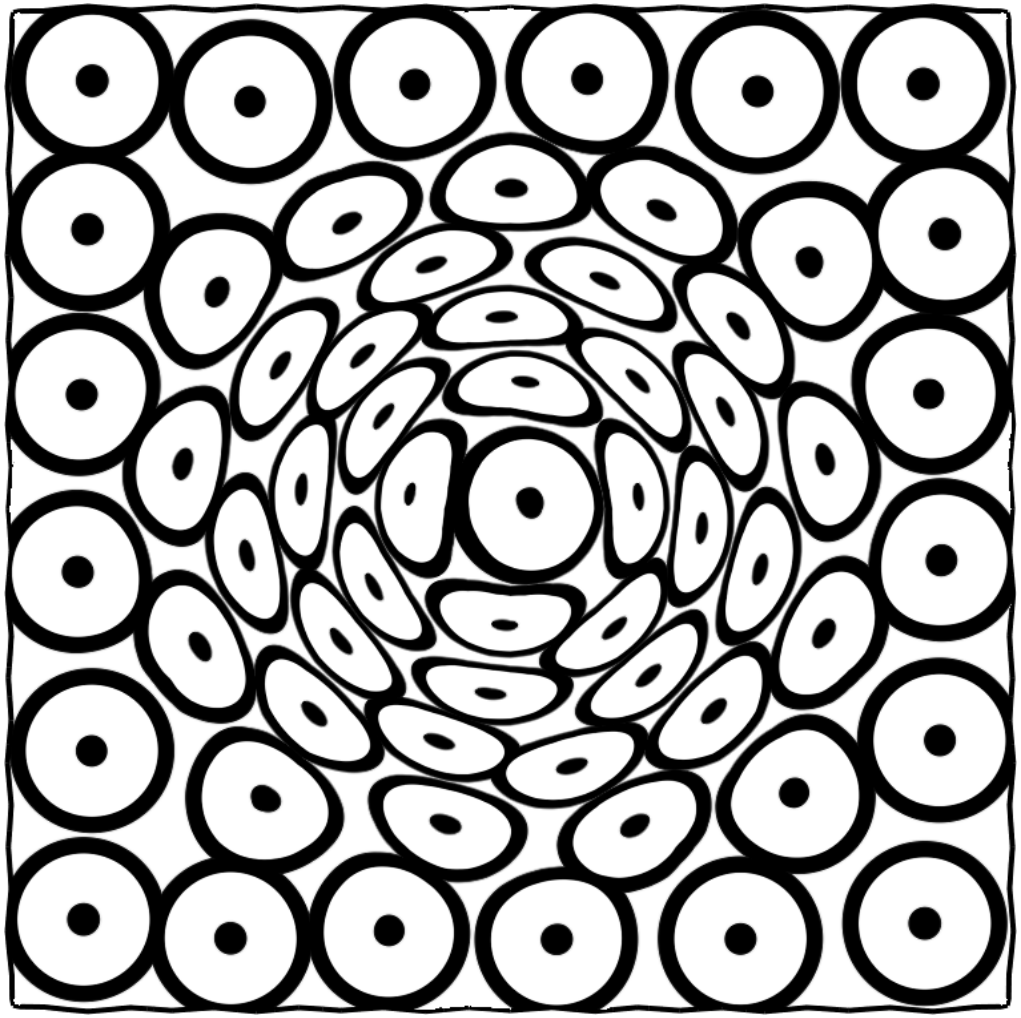
Theorem: F is C^2 almost everywhere

[Liu, Wang, **L**, Sun, Yan, Lu and Yang 09]

3

Tweaking the definition of distances

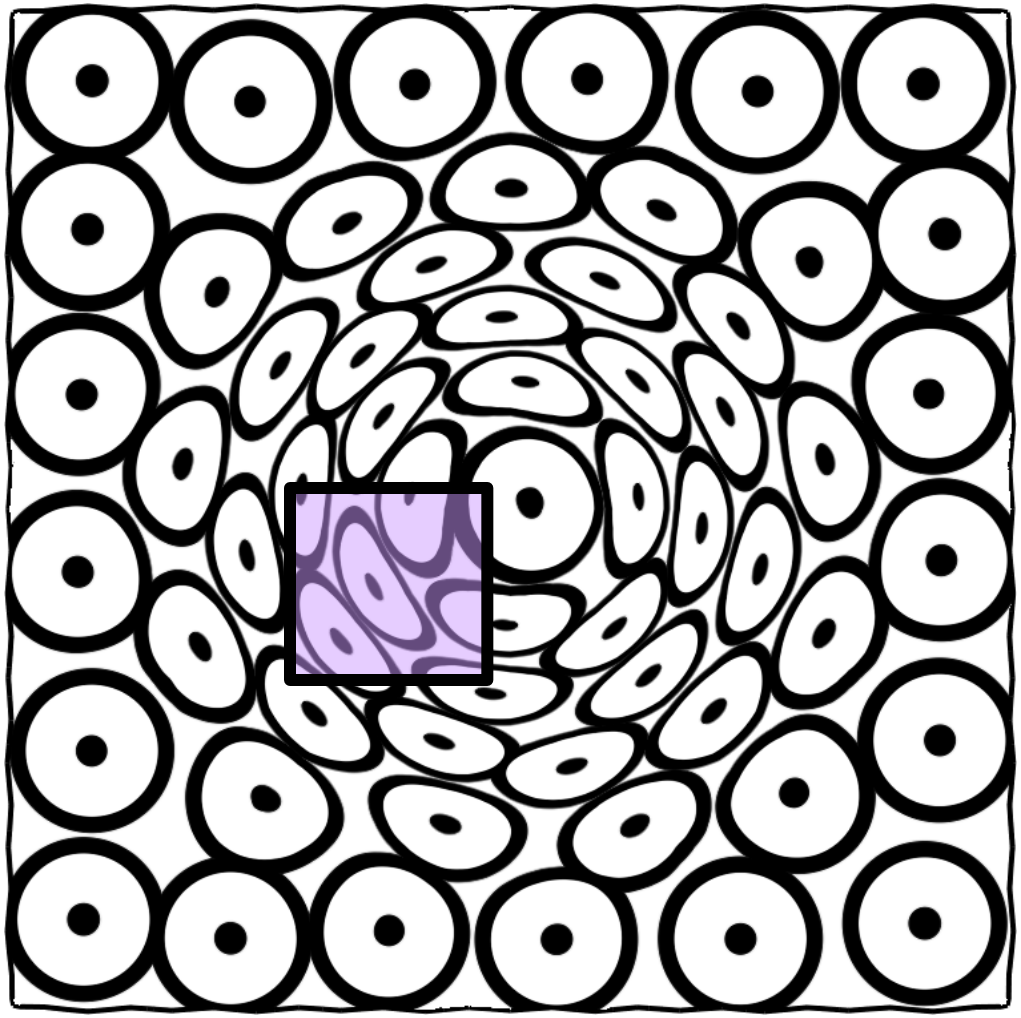
Part. 3. Tweaking distances - Anisotropy



The input: anisotropy field

$$G(x,y) = \begin{bmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{bmatrix}$$

Part. 3. Tweaking distances - Anisotropy



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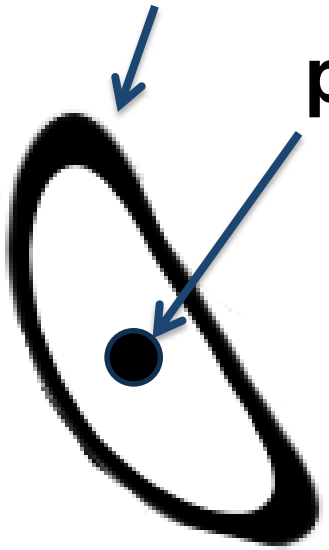
$$\{ \mathbf{q} \mid d_G(\mathbf{p},\mathbf{q}) = 1 \}$$



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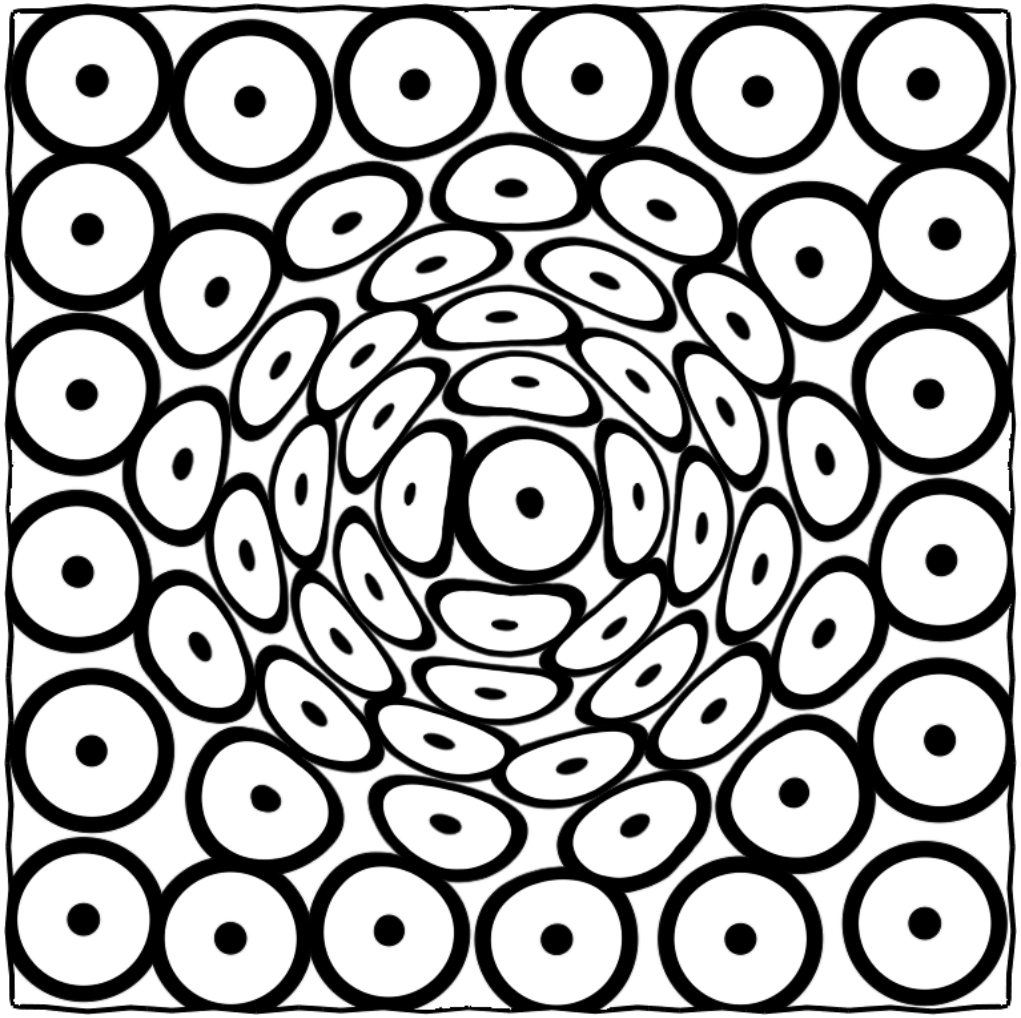


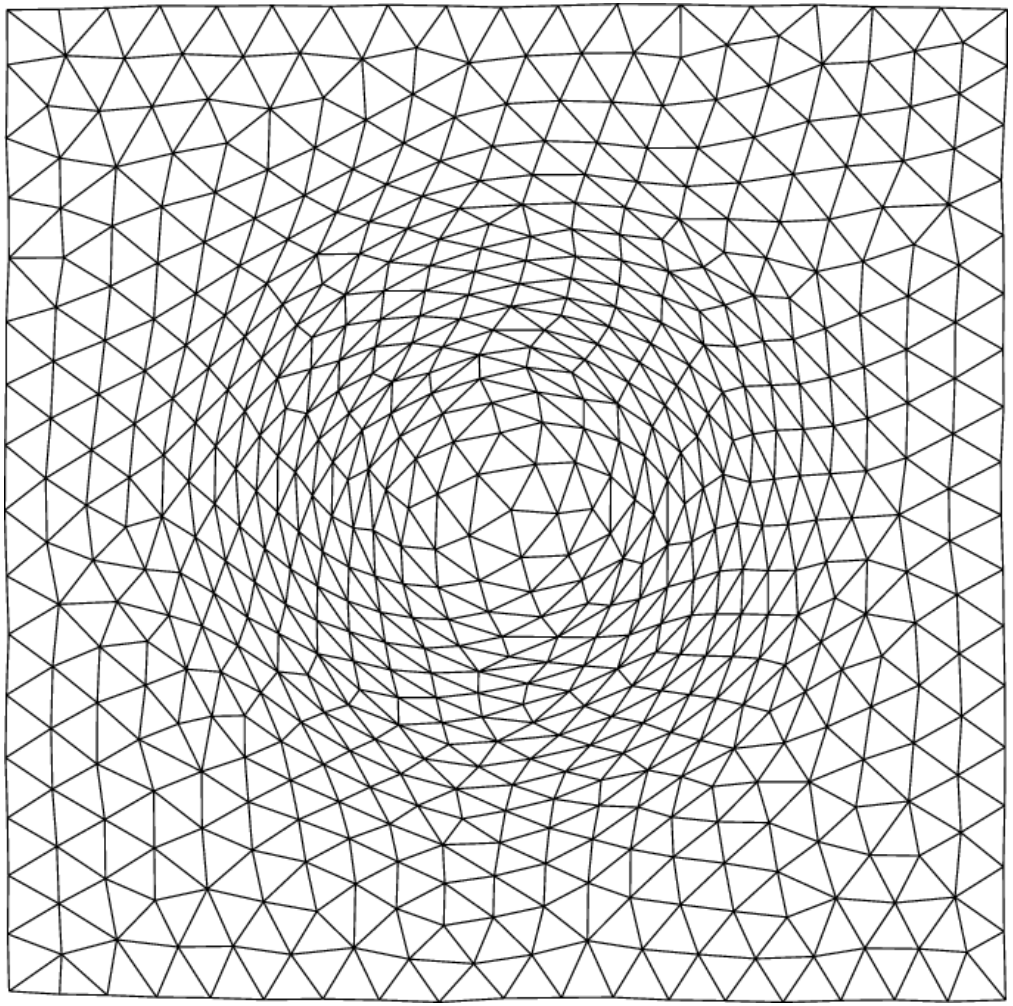
The input: anisotropy field

$$G(x,y) = \begin{bmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{bmatrix}$$

$$\langle v,w \rangle_G = v^t \, G(p) \, w$$

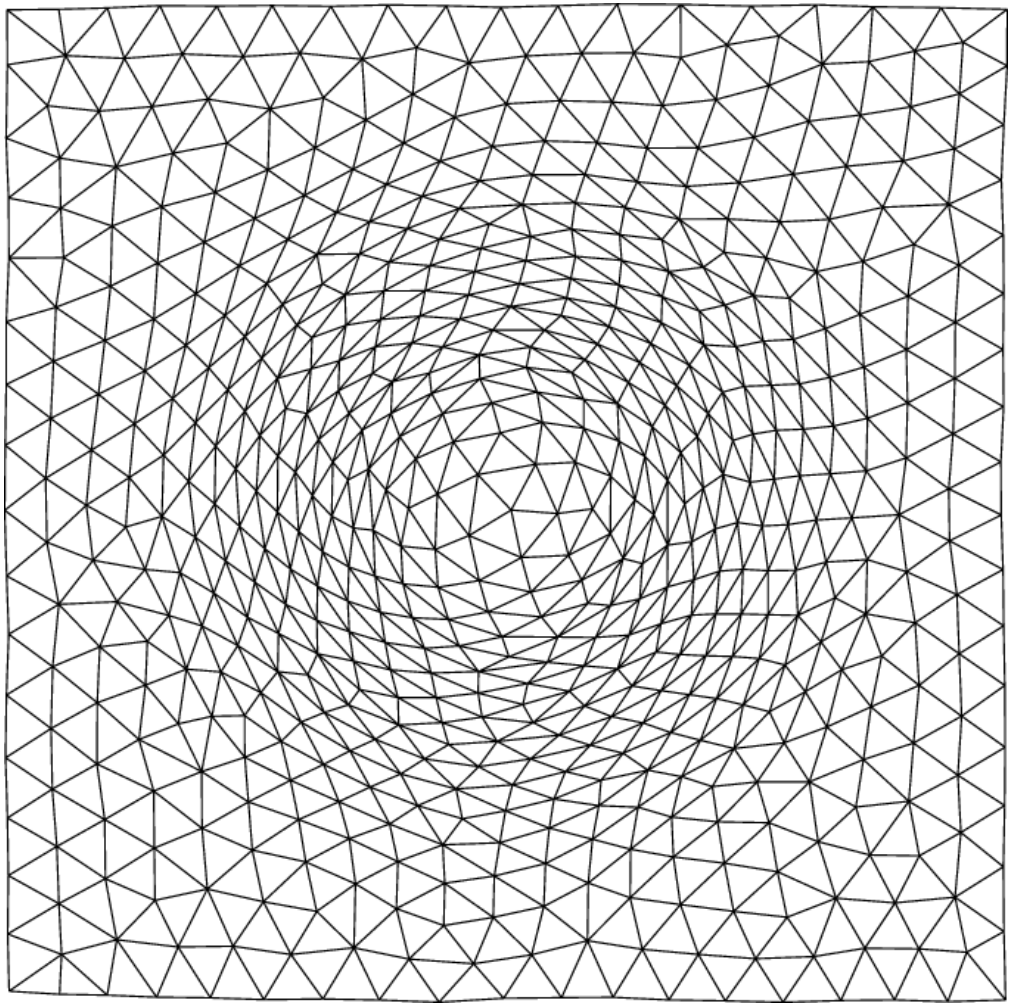
$$l_G(C) = \int_{t=0}^{1} \sqrt{v(t)^t \, G(t) \, v(t)} \, dt$$

Part. 3. Tweaking distances - Anisotropy



The result: triangles are "deformed" by the anisotropy.

Part. 3. Tweaking distances - Anisotropy



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*Q: How to compute an **Anisotropic** Centroidal Voronoi Tessellation ?*

Part. 3 Tweaking distances – Anisotropy

Tweak 1/3

Standard CVT: $F = \sum_i \int_{Vor(i)} \| (x_i - x) \|^2 dx$

Anisotropic CVT: $F = \sum_i \int_{Vor(i)} \| (x_i - x) \|^2_G dx$

[Qiang Du]

[**L** and Bonneel 2012]

Part. 3 Tweaking distances - Anisotropy

The key idea

This example:

Anisotropic mesh in 2d ⟺ Isotropic mesh in 3d

Part. 3 Tweaking distances - Anisotropy

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Replace **anisotropy** with **additional dimensions**

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Note: more dimensions may be needed

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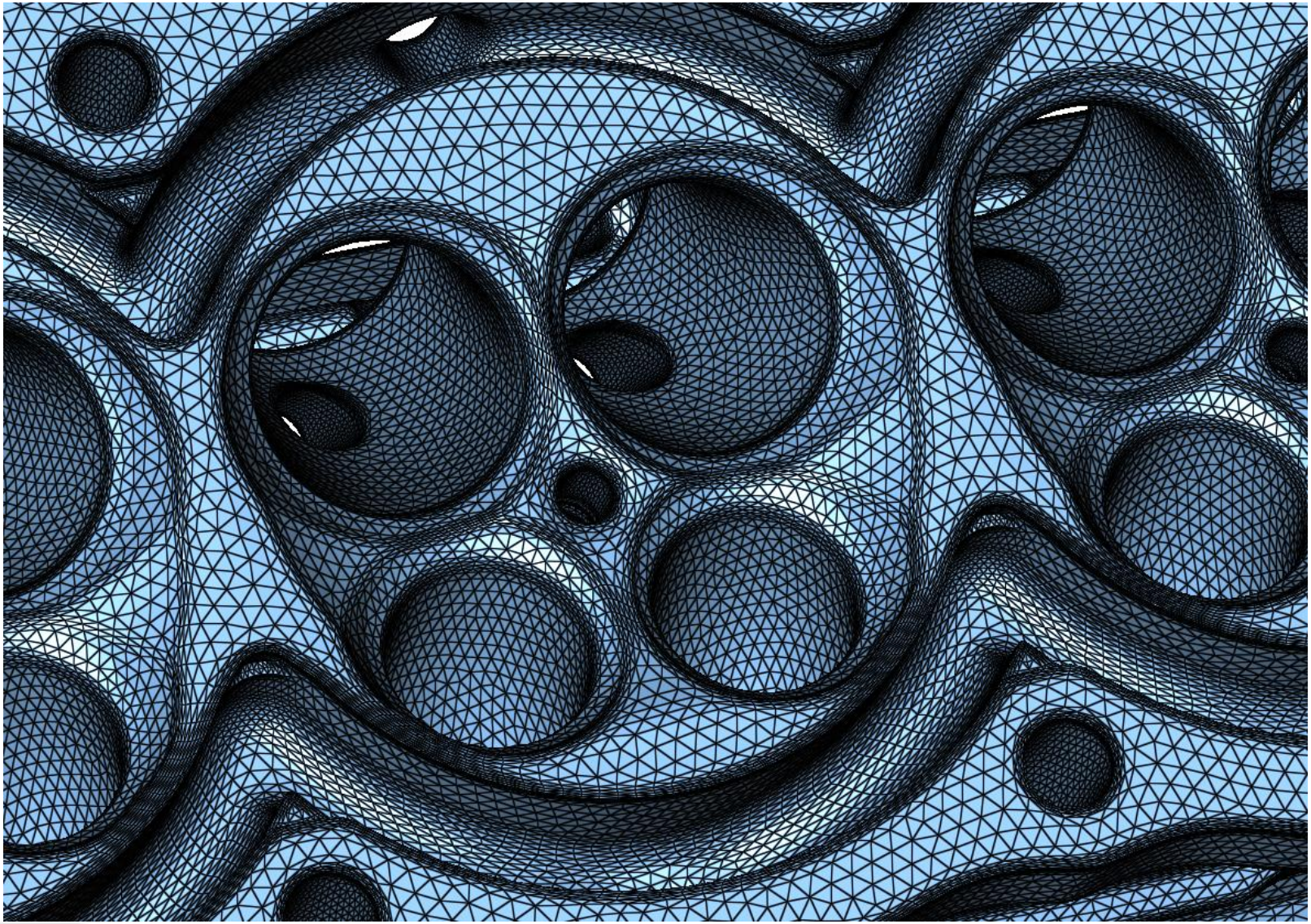
Replace **anisotropy** with **additional dimensions**

Note: more dimensions may be needed
How many ?
John Nash's isometric embedding theorem:

Maximum: depending on desired smoothness
C^1 *: 2n* [Nash-Kuiper]
C^k *: bounded by n(3n+11)/2* [Nash, Nash-Moser]

Part. 3 Tweaking distances - Anisotropy



Part. 3 Anisotropy - the algorithm

Lloyd relaxation in $\mathbb{R}^6$ (Naïve version)

(1) Embed the surface **S** into $\mathbb{R}^6$

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While convergence is not reached

Part. 3 Anisotropy - the algorithm

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    (3) Compute Vor(**X**)

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    (4) Compute Vor(**X**) ∩ S

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    (5) Move each $\mathbf{x}_i$ to the centroid of Vor($\mathbf{x}_i$) ∩ S

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Costs d! for dimension d

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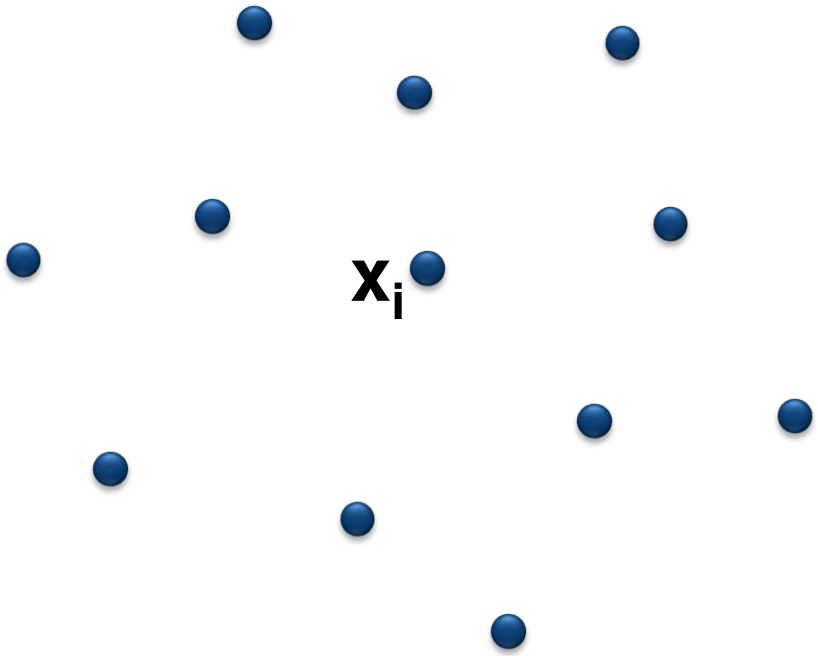
Costs d! for dimension d
d = 6 ; d! = 720

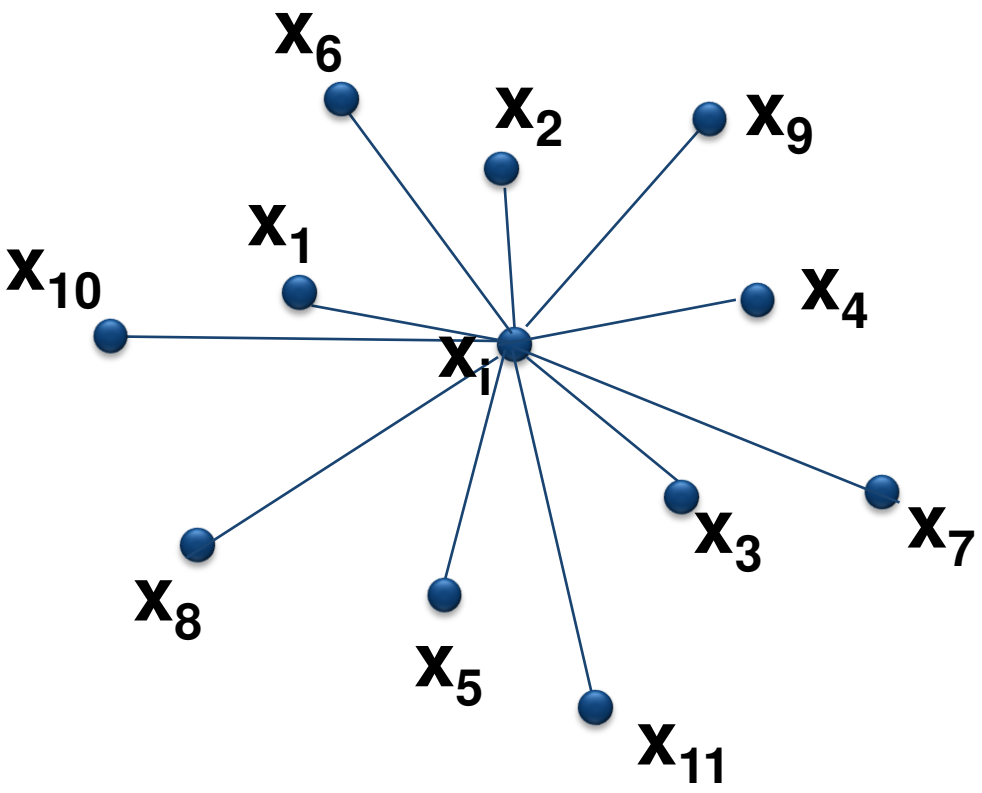
Part. 3 Anisotropy - the algorithm

Voronoi cells as iterative convex clipping



Part. 3 Anisotropy - the algorithm

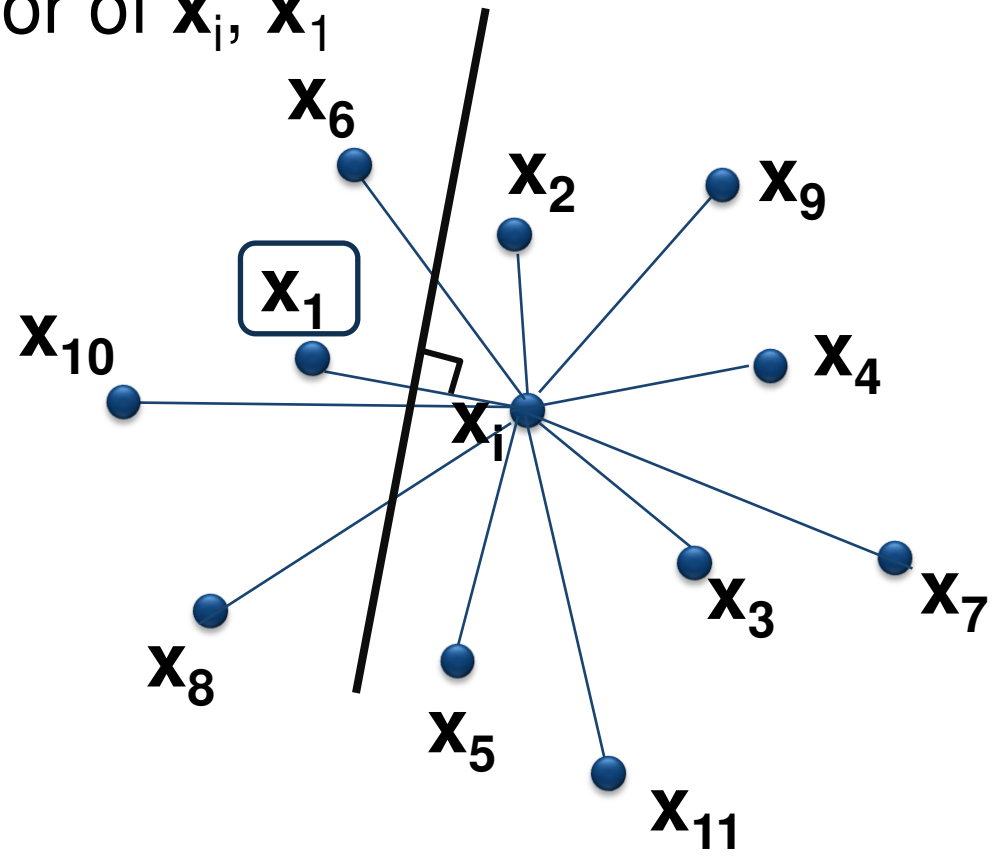
Voronoi cells as iterative convex clipping
Neighbors in increasing (6d) distance from $\mathbf{x}_i$



Part. 3 Anisotropy - the algorithm

Voronoi cells as iterative convex clipping

Bisector of $\mathbf{x}_i$, $\mathbf{x}_1$



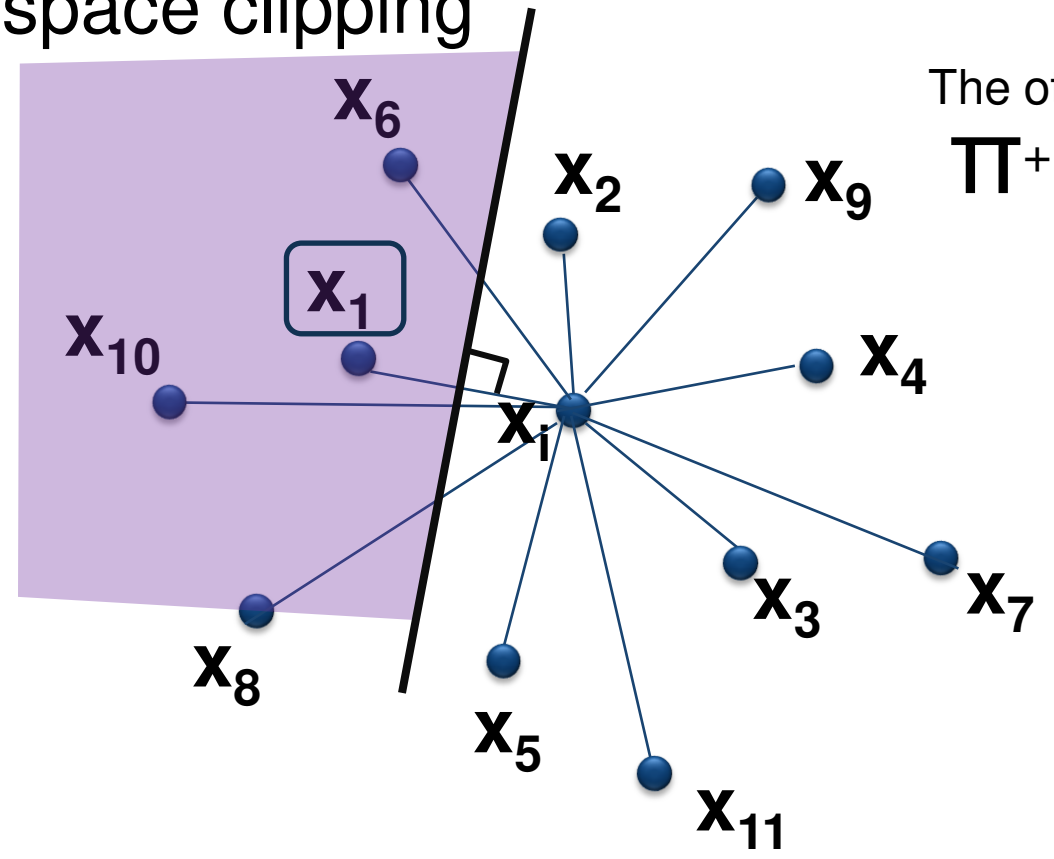
Part. 3 Anisotropy - the algorithm

Voronoi cells as iterative convex clipping

Half-space clipping

This side:

$\Pi^{-}(i,1)$



The other side:

$\Pi^{+}(i,1)$

Part. 3 Anisotropy - the algorithm

Voronoi cells as iterative convex clipping

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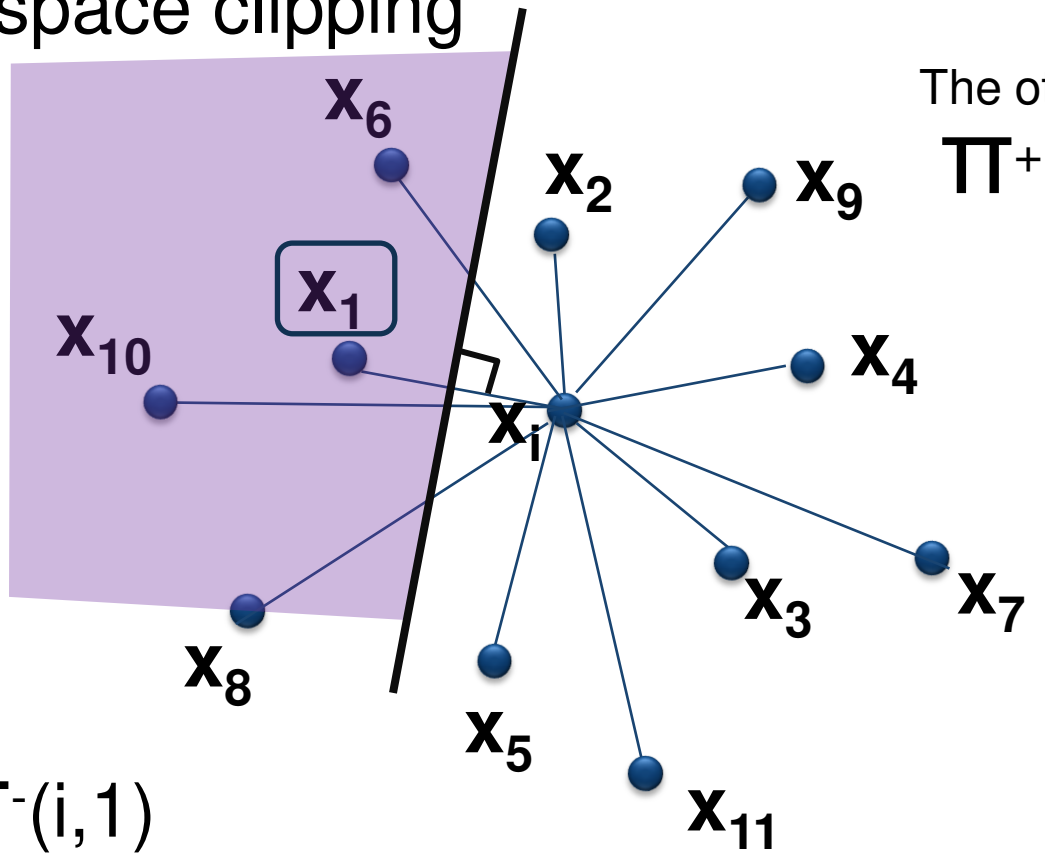
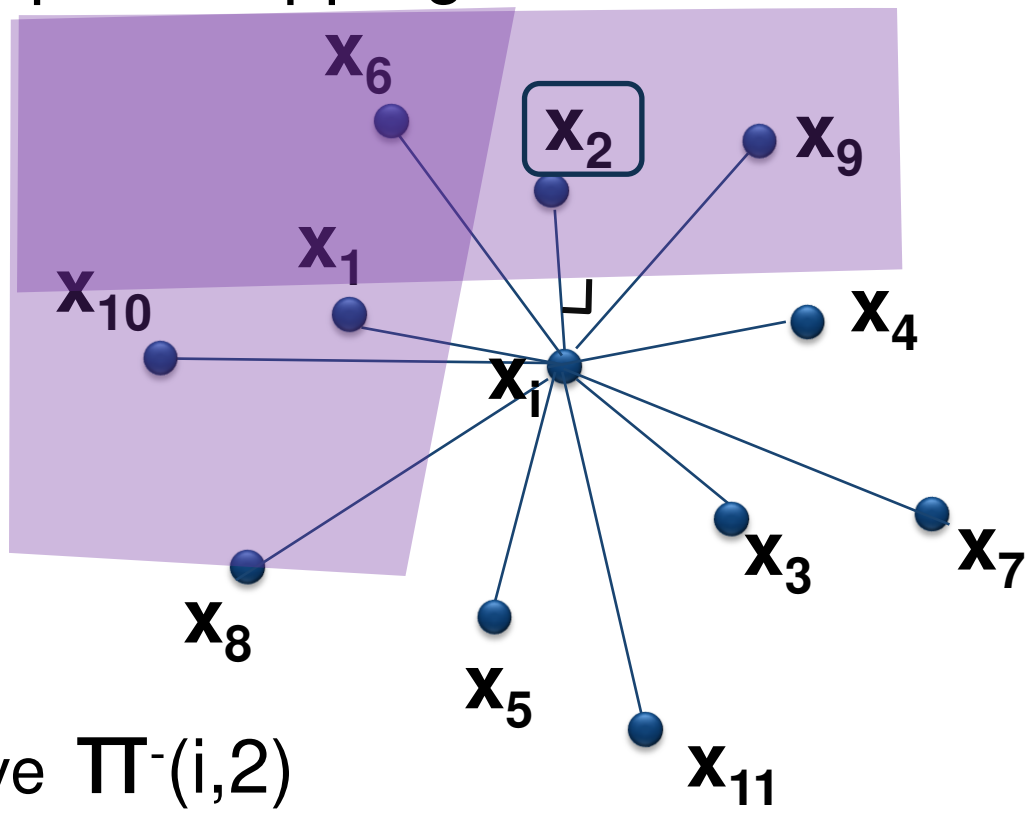


Remove $\Pi^{-}(i,1)$

Part. 3 Anisotropy - the algorithm

Voronoi cells as iterative convex clipping

Half-space clipping

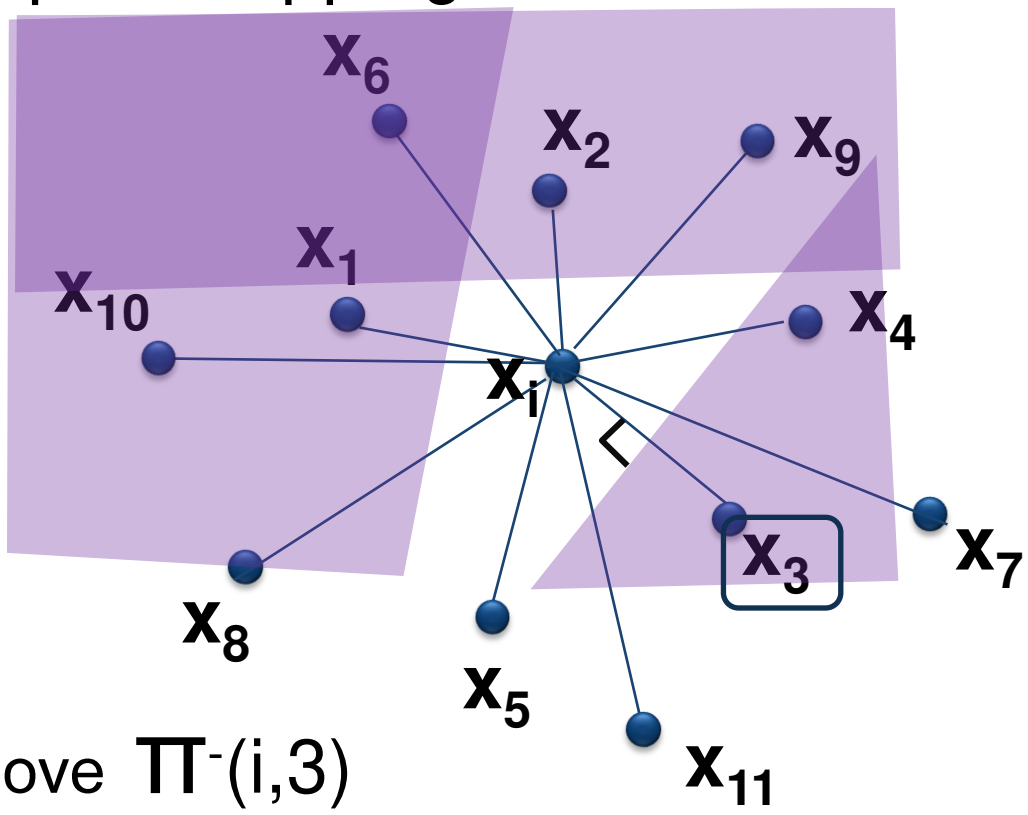


Then remove $\Pi^{-}(i,2)$

Part. 3 Anisotropy - the algorithm

Voronoi cells as iterative convex clipping

Half-space clipping

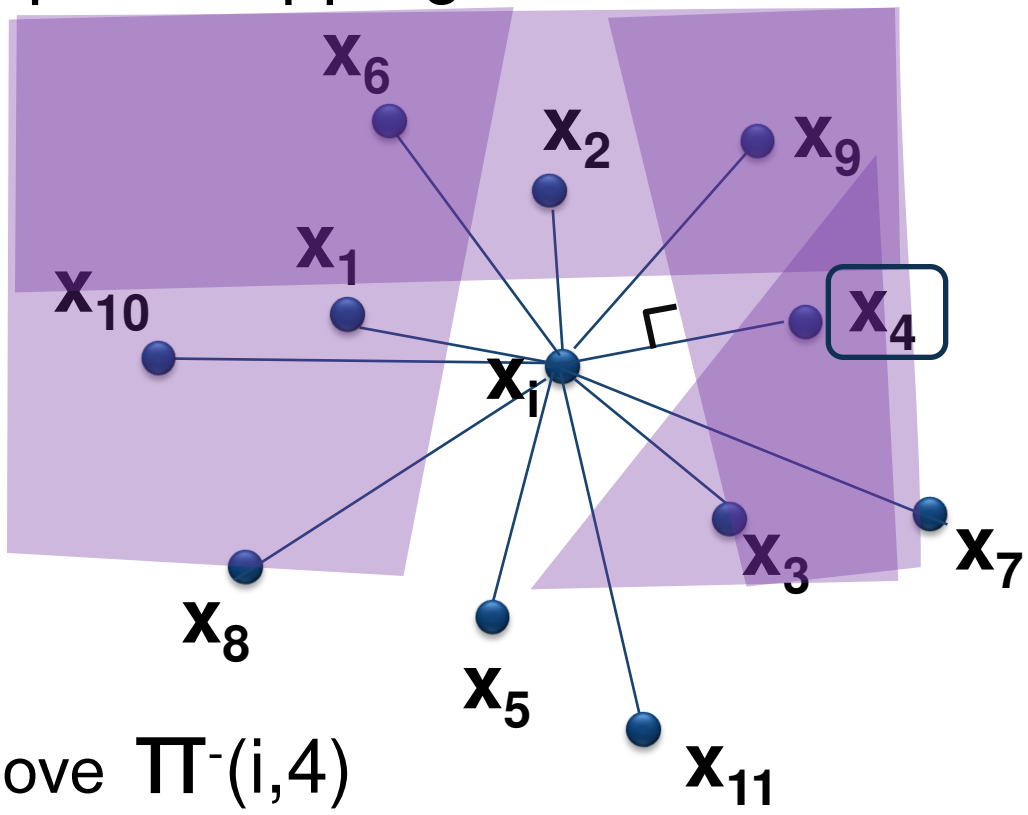


… then remove $\Pi^{-}(i,3)$

Part. 3 Anisotropy - the algorithm

Voronoi cells as iterative convex clipping

Half-space clipping

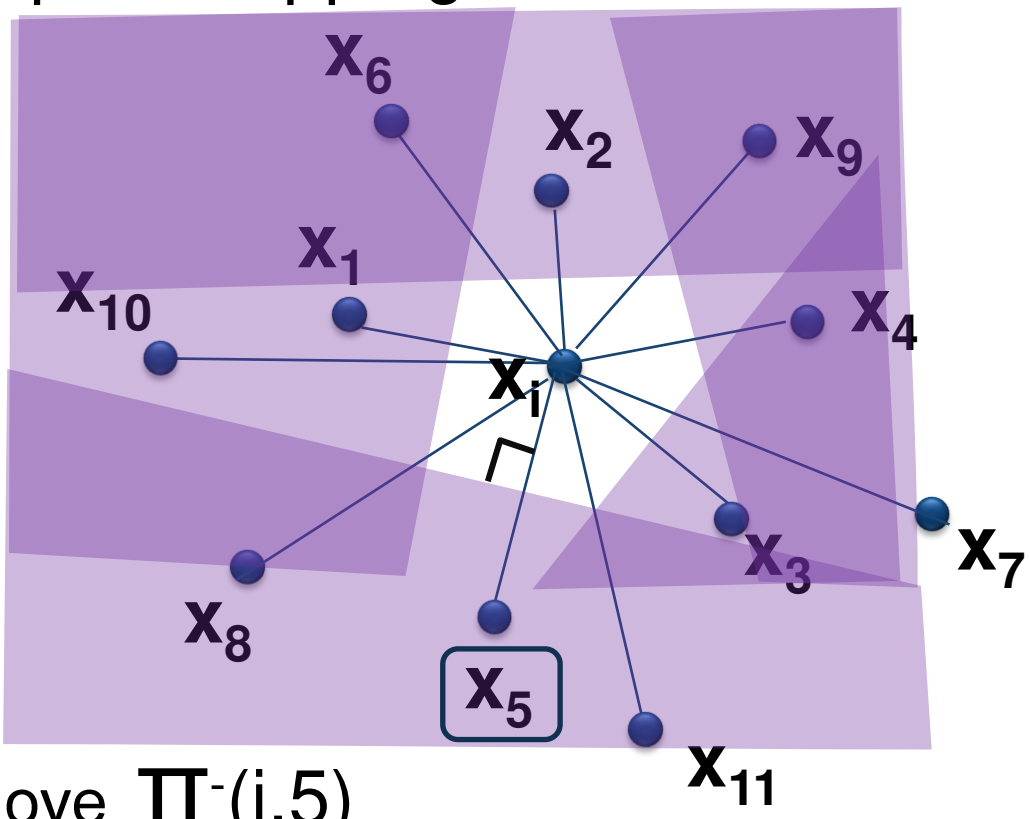


… then remove $\Pi^{-}(i,4)$

Part. 3 Anisotropy - the algorithm

Voronoi cells as iterative convex clipping

Half-space clipping

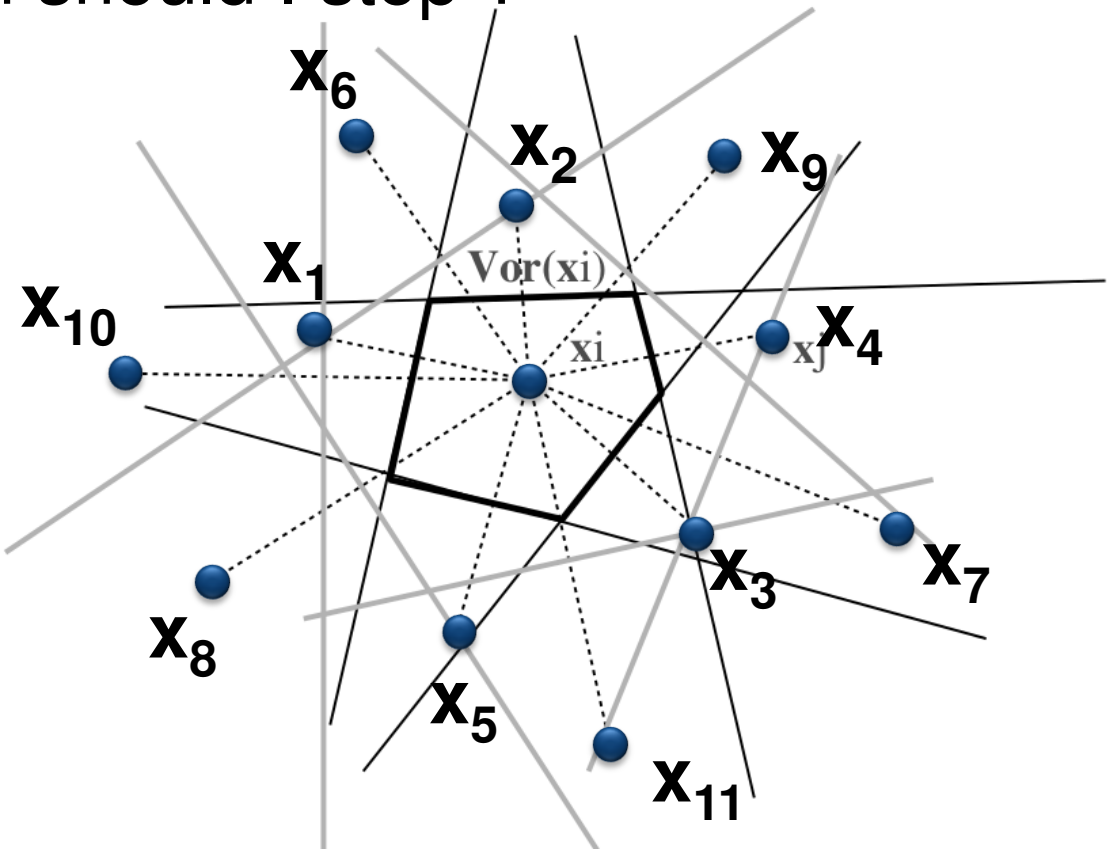


… then remove $\Pi^{-}(i,5)$

Part. 3 Anisotropy - the algorithm

Voronoi cells as iterative convex clipping

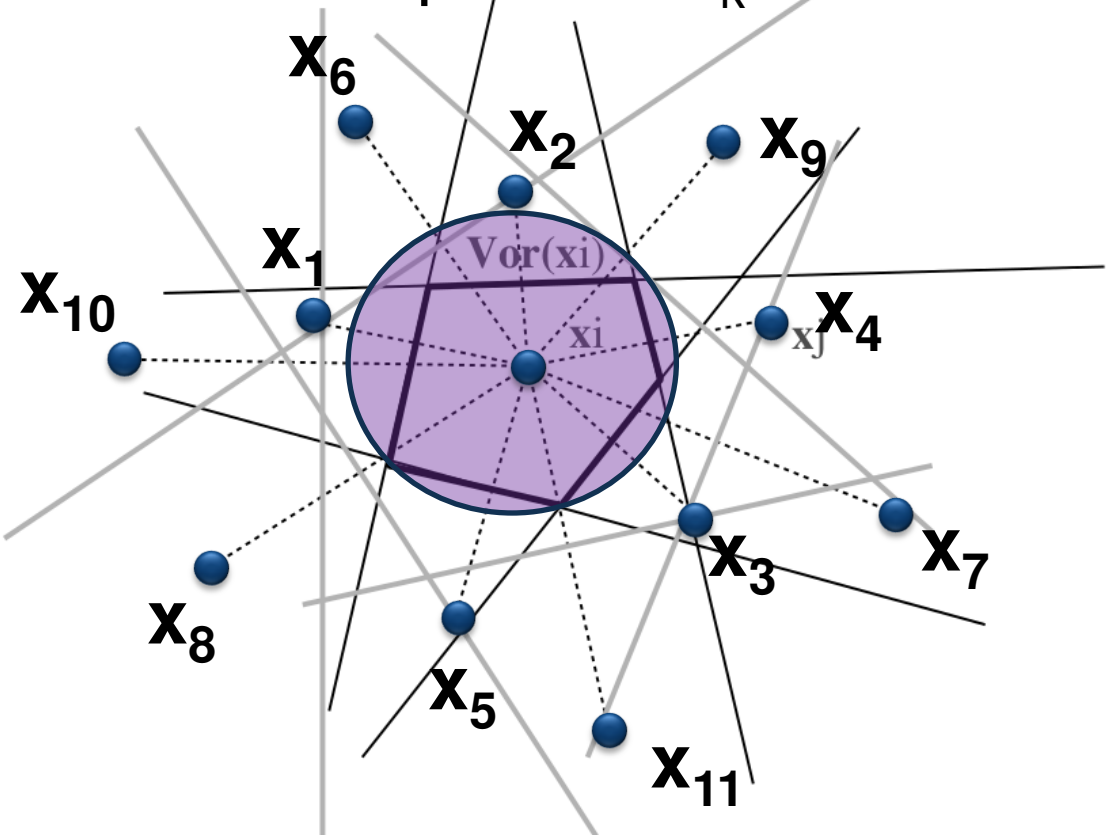
When should I stop ?



Part. 3 Anisotropy - the algorithm

Voronoi cells as iterative convex clipping

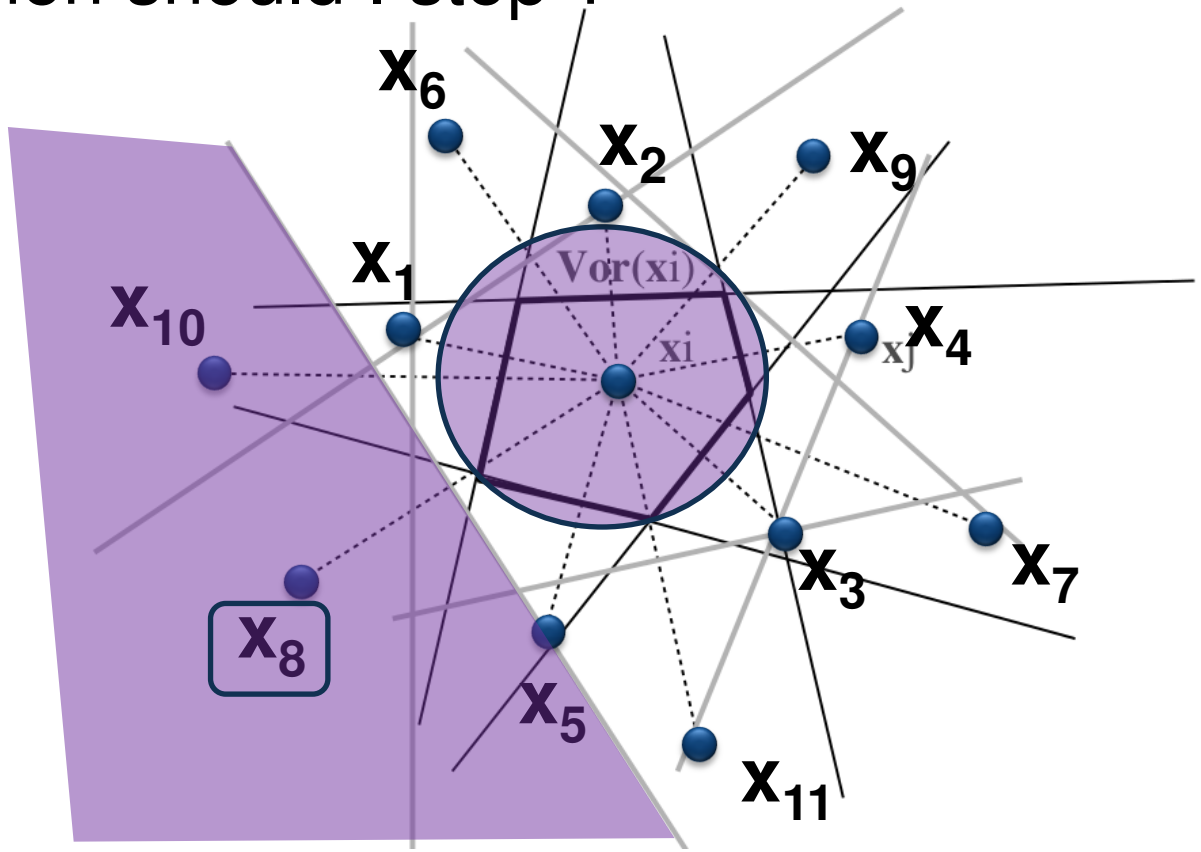
When should I stop ? R_k



Part. 3 Anisotropy - the algorithm

Voronoi cells as iterative convex clipping

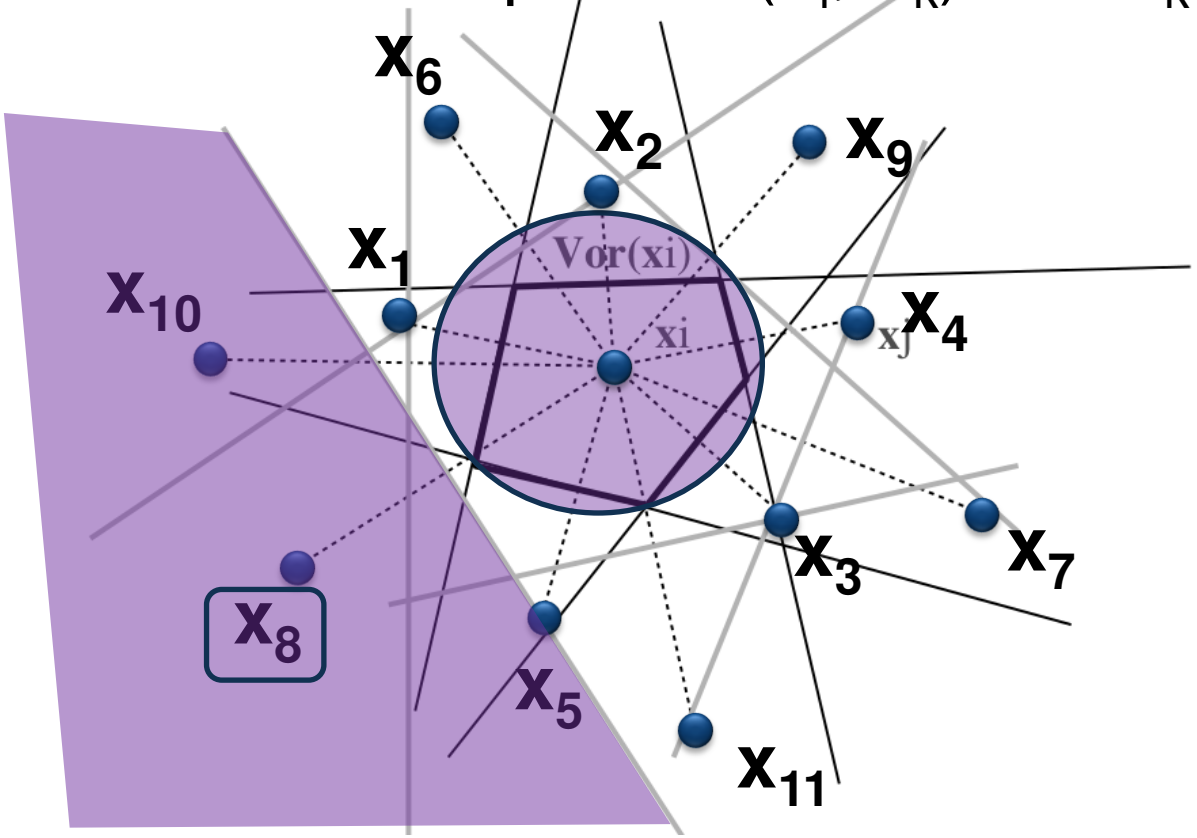
When should I stop ?



Part. 3 Anisotropy - the algorithm

Voronoi cells as iterative convex clipping

When should I stop ? $d(\mathbf{x}_i, \mathbf{x}_k) > 2\, R_k$



Part. 3 Anisotropy - the algorithm

Voronoi cells as iterative convex clipping

Theorem:

$$d(\mathbf{x}_i, \mathbf{x}_{k+1}) > 2R_k \rightarrow \bigcap \boldsymbol{\pi}^+(i,k) = \mathrm{Vor}(\mathbf{x}_i)$$

[**L** and Bonneel 2012]

Part. 3 Anisotropy - the algorithm

Voronoi cells as iterative convex clipping

When should I stop ? $d(\mathbf{x}_i, \mathbf{x}_k) > 2\,Rk$

"Radius of security" is reached

[**L** and Bonneel 2012]

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Advantages:

[**L** and Bonneel 2012]

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(1) Compute $Vor(\mathbf{X}) \cap S$ directly (start with f and clip)

[**L** and Bonneel 2012]

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(2) Replace Delaunay with ANN ! (no d! factor)

[**L** and Bonneel 2012]

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Advantages:

(1) Compute $Vor(\mathbf{X}) \cap S$ directly (start with f and clip)

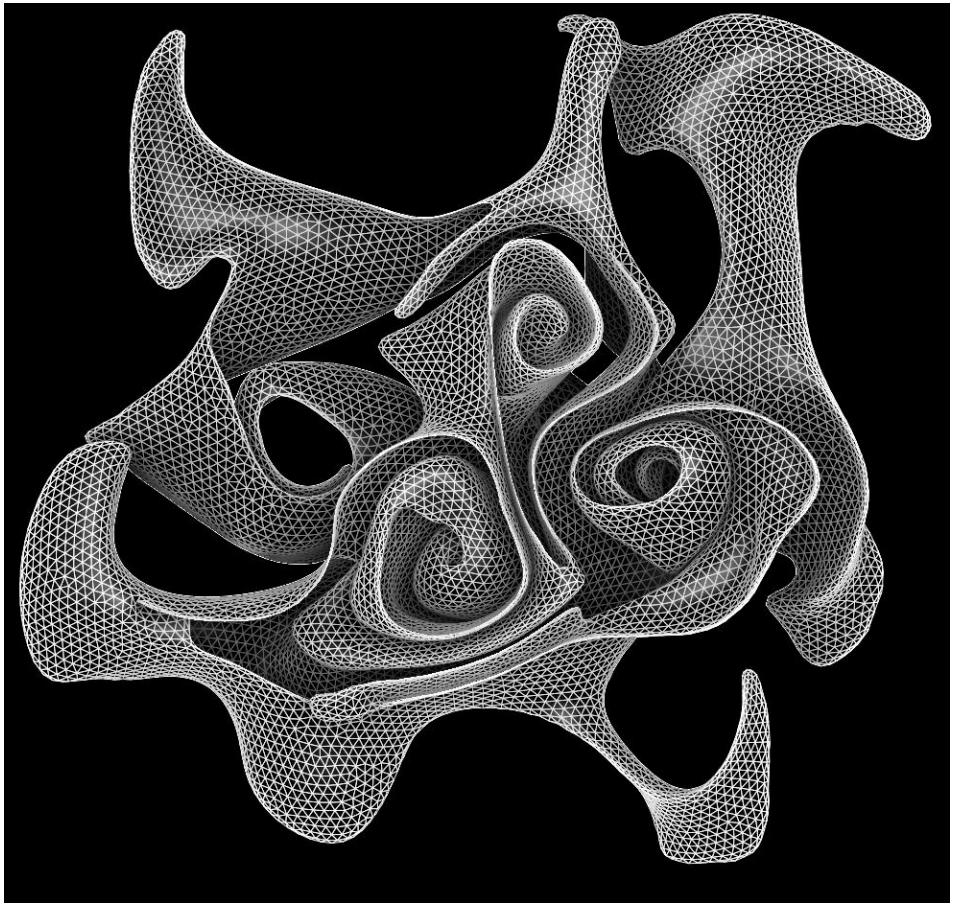
(2) Replace Delaunay with ANN ! (no d! factor)

(3) Parallelization is trivial (partition S and // in parts)

[**L** and Bonneel 2012]

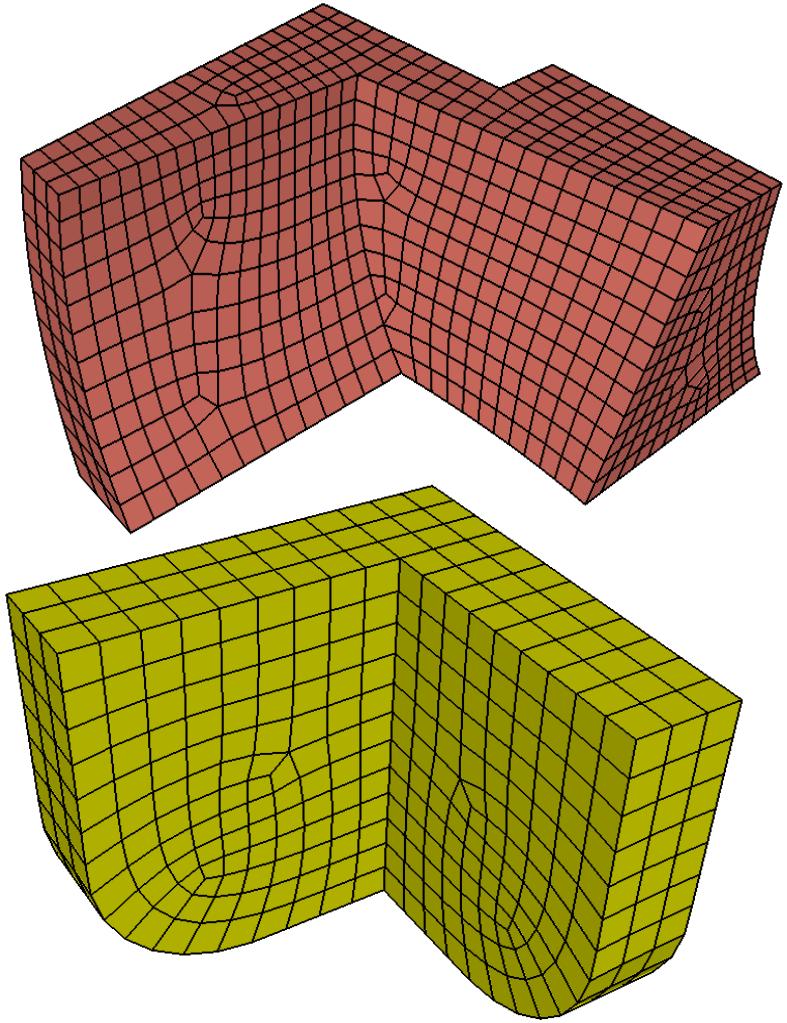
Part. 3 Tweaking distances - Anisotropy

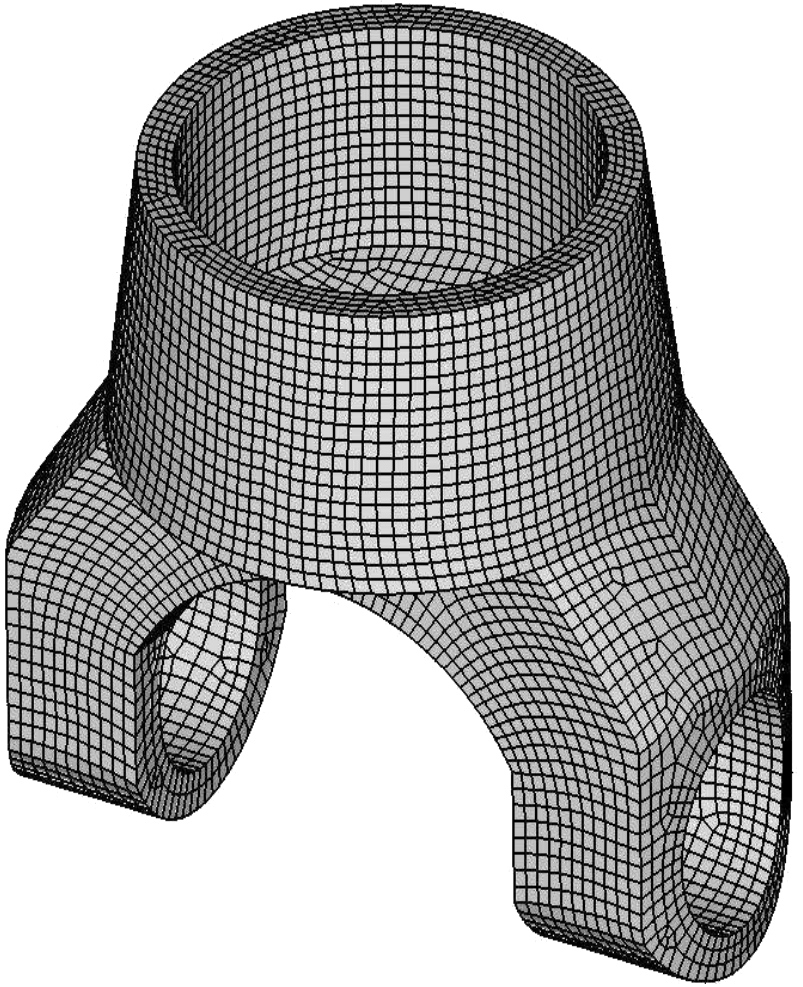
A 6d embedding for curvature-adapted meshing



David Lopez

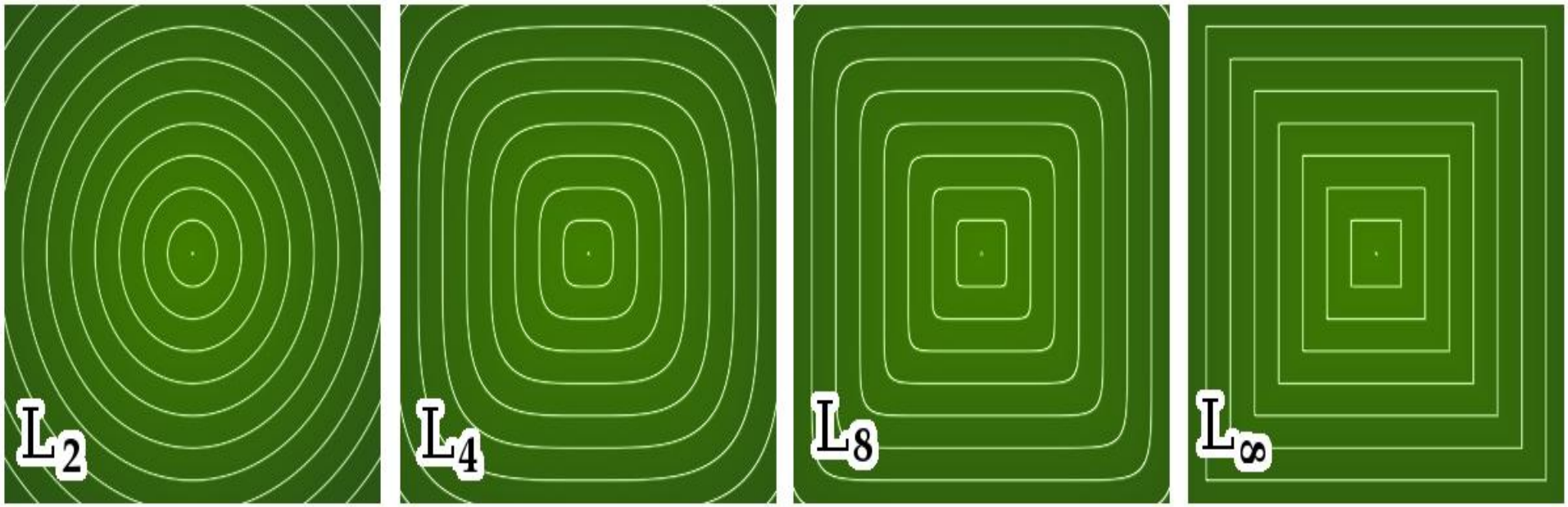
Part. 3 Tweaking distances – Lp norm





Part. 3 Tweaking distances – Lp norm

p=2 p=4 p=8 ….



Lp norm: $|| x ||_p = \sqrt[p]{|x|^p + |y|^p + |z|^p}$

Part. 3

Part. 3 Tweaking distances – Lp norm

Tweak 2/3

Standard CVT: $F = \sum_i \int_{Vor(i)} \left\| (x_i - x) \right\|^2 dx$

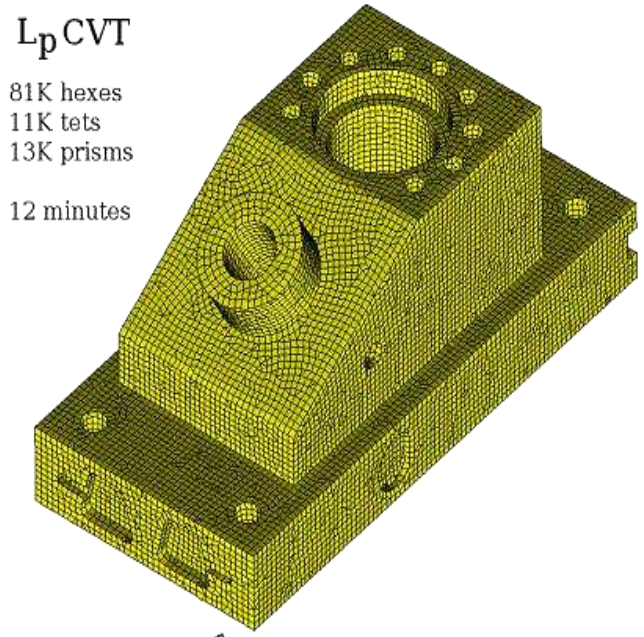
Lp CVT:
[**L** and Liu 2010]

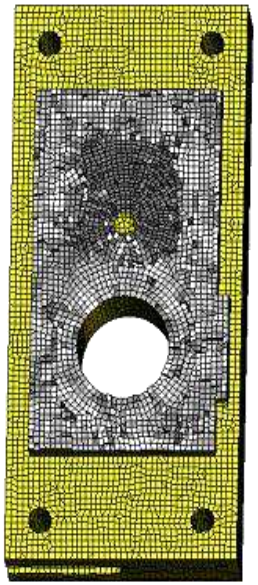
$$F = \sum_i \int_{Vor(i)} \left\| \mathbf{M(x)} (x_i - x) \right\|_p^p dx$$

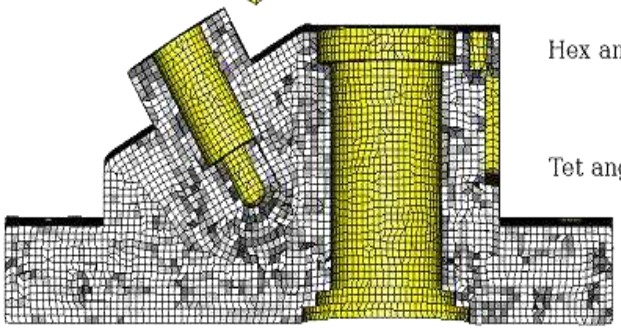
Anisotropy and desired orientation

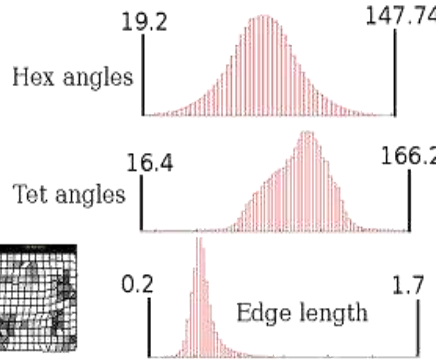
Lp norm: $\| x \|_p = \sqrt[p]{|x|^p + |y|^p + |z|^p}$

Part. 3 Tweaking distances – Lp norm

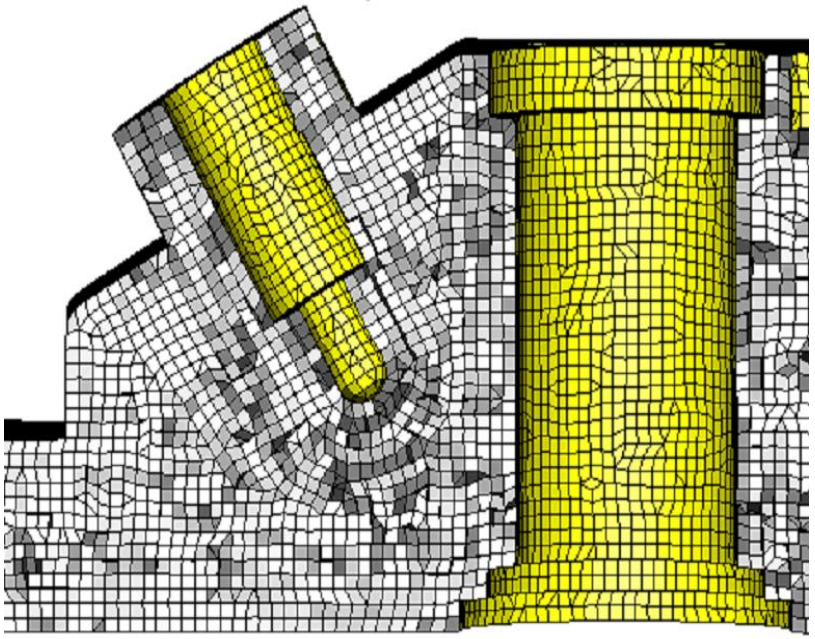








+ many other examples in paper and supplemental material.



4

Preserving sharp features

with protecting balls

Part. 4 Tweaking distances – Protecting balls

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Voronoi diagram: $\mathrm{Vor}(x_i) = \{\, x \mid d^2(x,x_i) < d^2(x,x_j) \,\}$

Part. 4 Tweaking distances – Protecting balls

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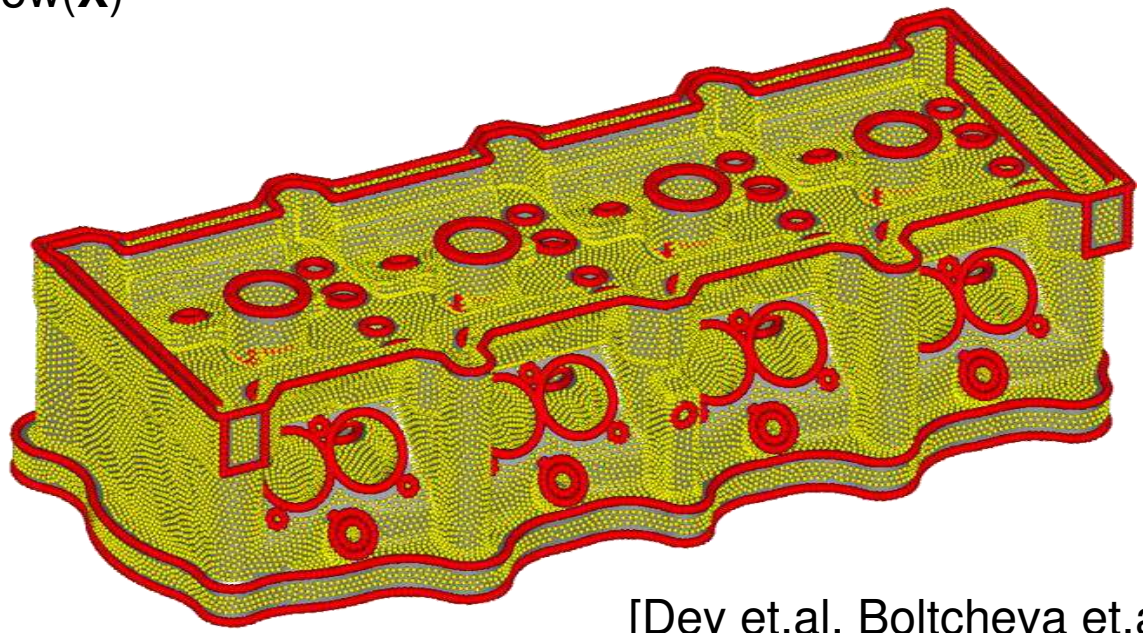
Power diagram: $\mathrm{Pow}(x_i) = \{\, x \mid d^2(x,x_i) - \mathbf{w_i^2} < d^2(x,x_j) - \mathbf{w_j^2} \,\}$

Part. 4 Tweaking distances – Protecting balls

Voronoi diagram: $\mathrm{Vor}(x_i) = \{ x \mid d^2(x,x_i) < d^2(x,x_j) \}$

Power diagram: $\mathrm{Pow}(x_i) = \{ x \mid d^2(x,x_i) - \mathbf{w_i^2} < d^2(x,x_j) - \mathbf{w_j^2} \}$

Thm: [Dey et.al] if $B(x_i, w_i)$ and $B(x_j,w_j)$ intersect and are empty of other x_k's, then $[x_i,x_j]$ appears in $\mathrm{Pow}(\mathbf{X})$



[Dey et.al, Boltcheva et.al]

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Problem for 6d remeshing:

"Security Radius" theorem does not work with power diagrams

Part. 4 Tweaking distances – Protecting balls

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Problem for 6d remeshing:

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Idea: use 7d embedding: $x_i \rightarrow [\, x_i \,;\, \sqrt{(W^2 - w_i^2)}\,]$ where $W = \mathrm{Max}(w_i)$

Part. 4 Tweaking distances – Protecting balls

Tweak 3/3

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Problem for 6d remeshing:

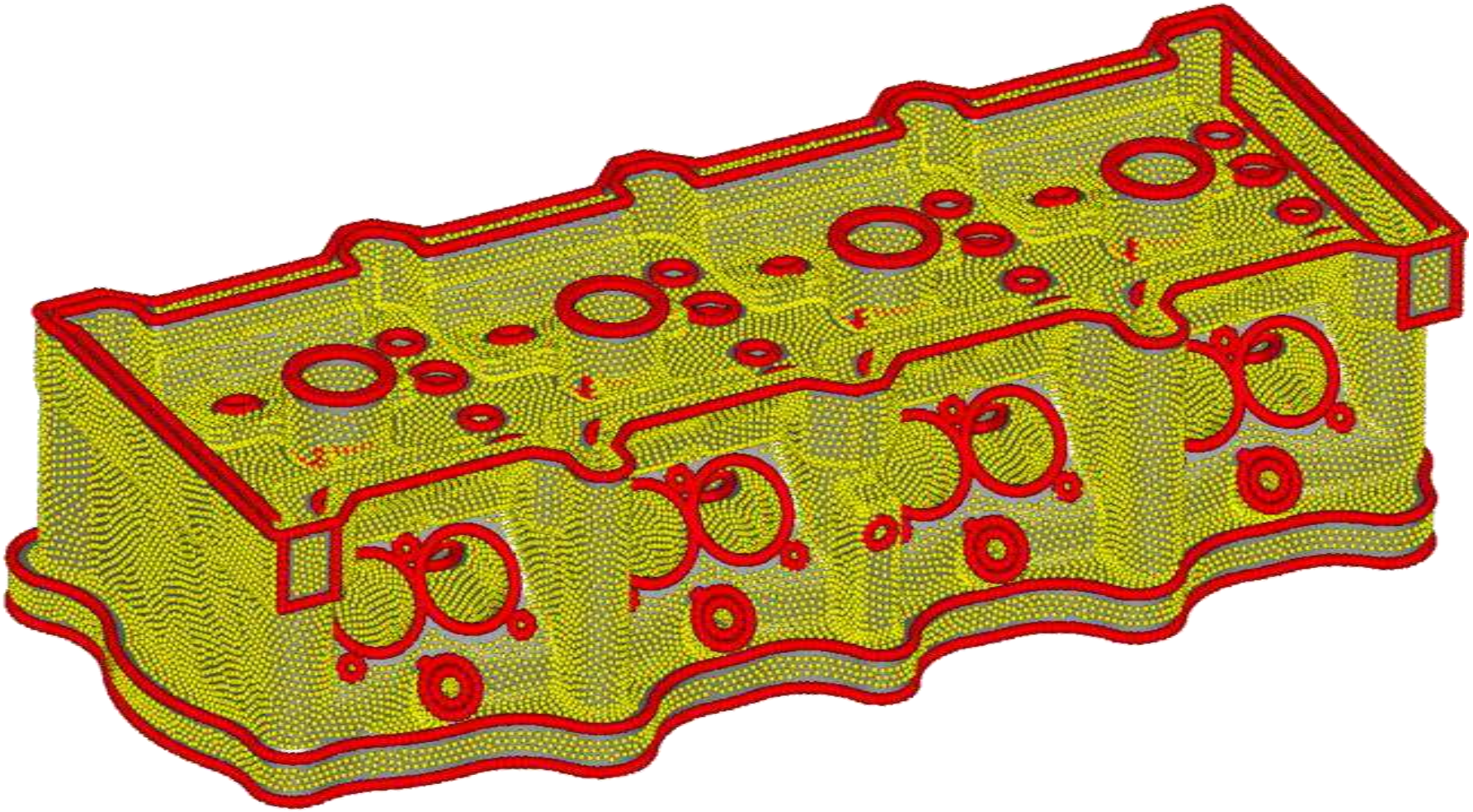
"Security Radius" theorem does not work with power diagrams

Idea: use 7d embedding: $x_i \rightarrow [x_i ; \sqrt{(W^2 - w_i^2)}]$; $x \rightarrow [x ; 0]$

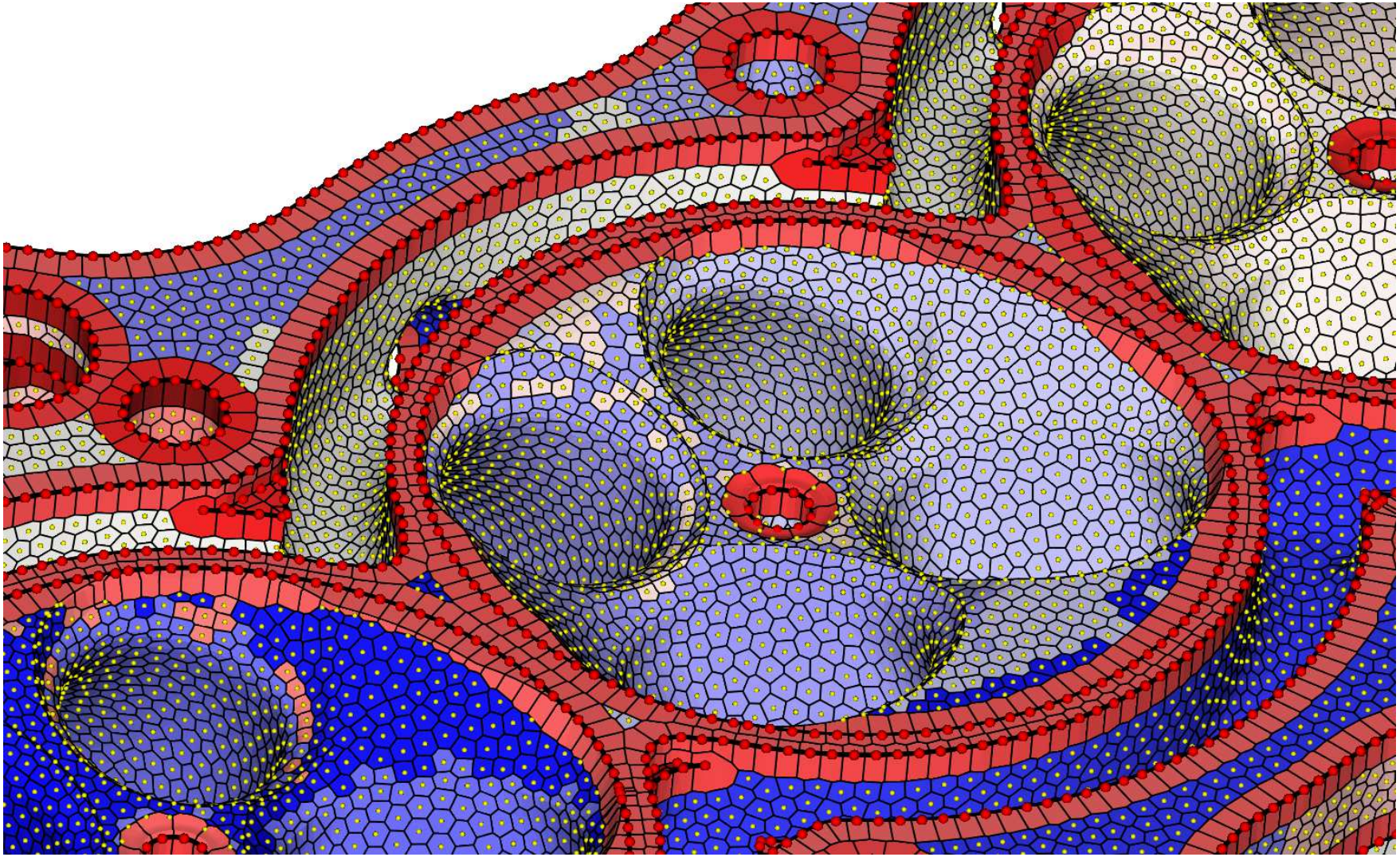
where $W = Max(w_i)$

$$d(x,x_i)_{(Voronoi\ 7d)} = d(x,x_i)_{(Power\ 6d)} + W$$

Part. 4 Constraints – Protecting balls

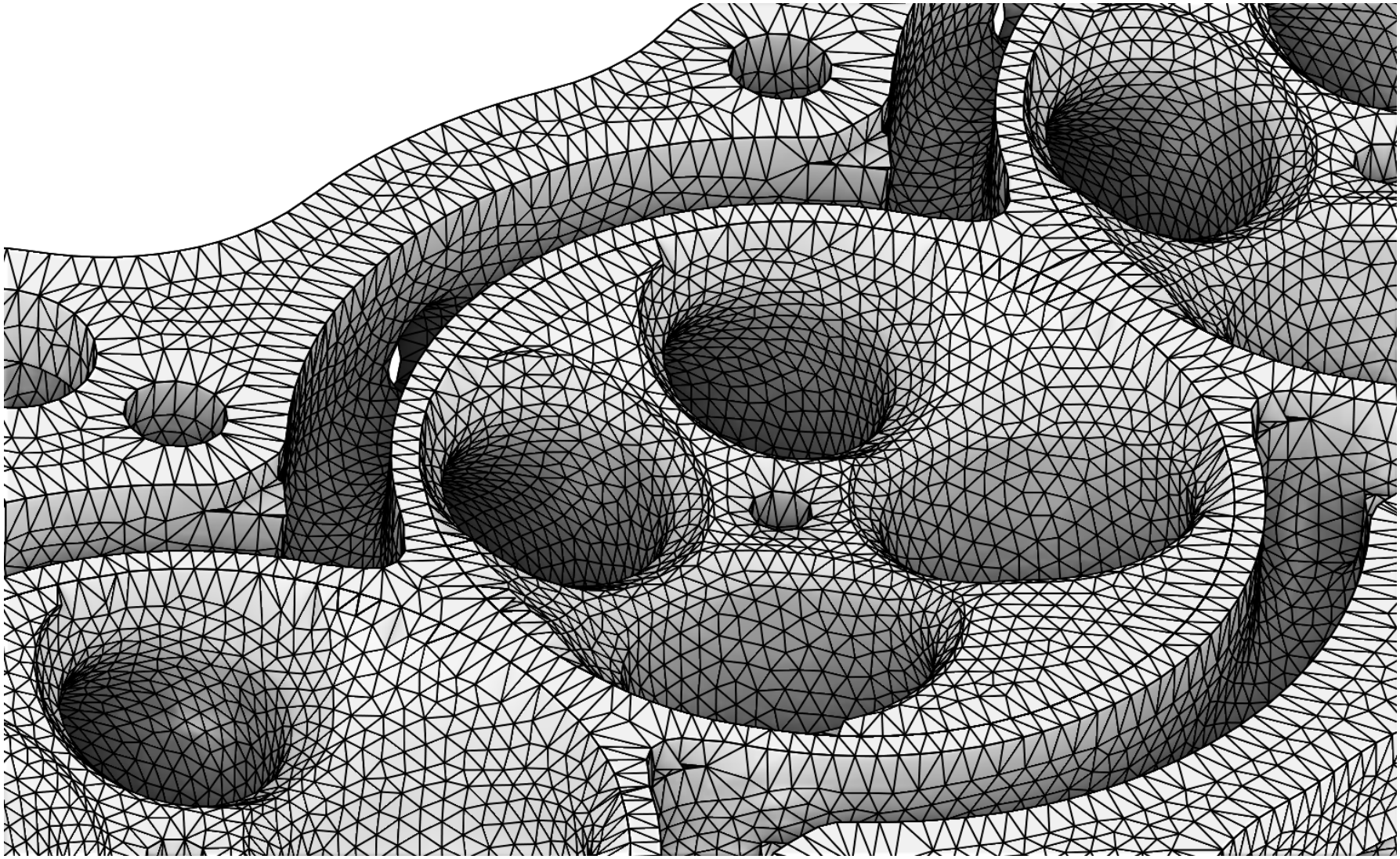


Part. 4 Constraints – Protecting balls

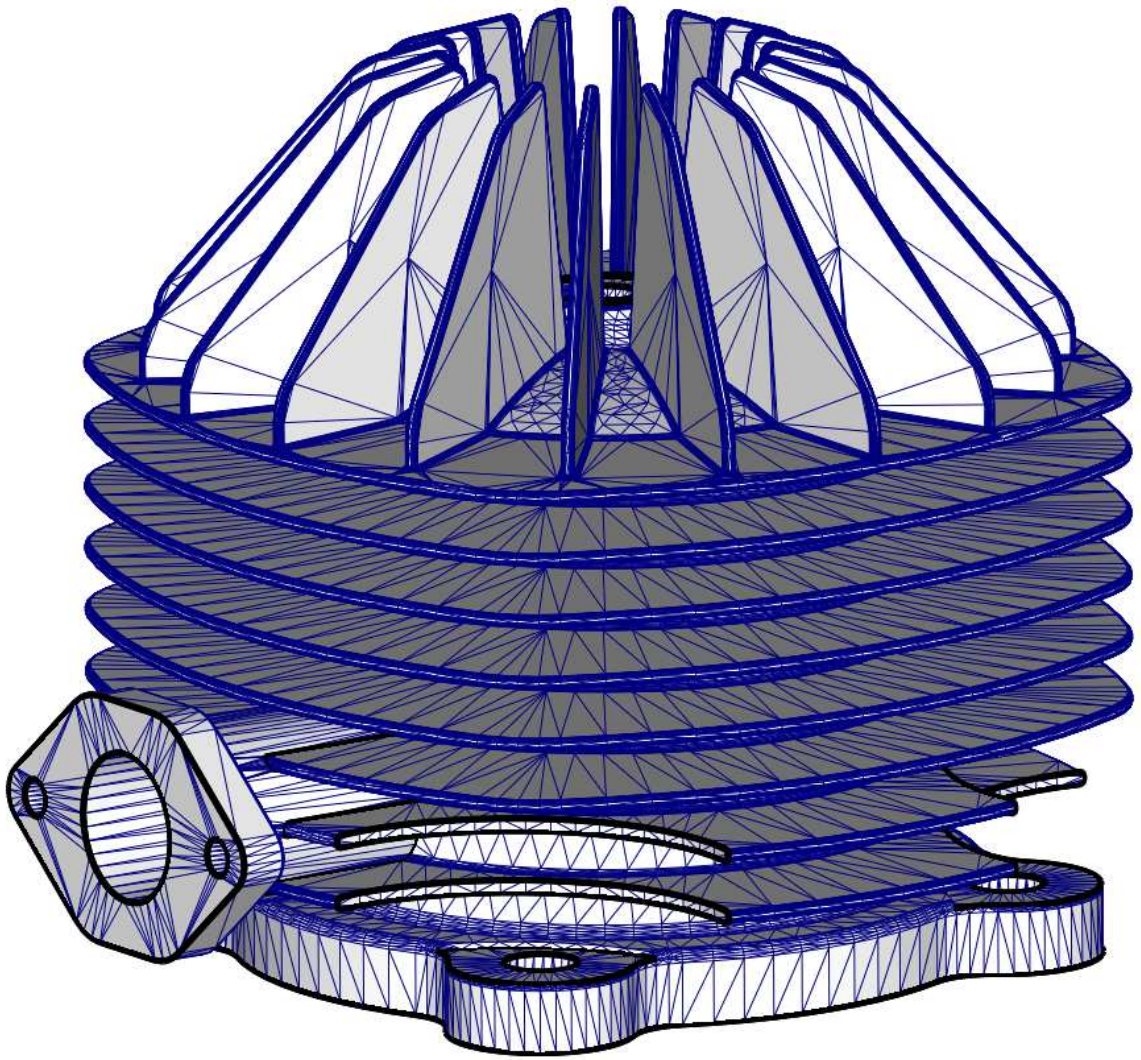


6d power diagram (7d Voronoi diagram)

Part. 4 Constraints – Protecting balls



Part. 4 Constraints – Protecting balls



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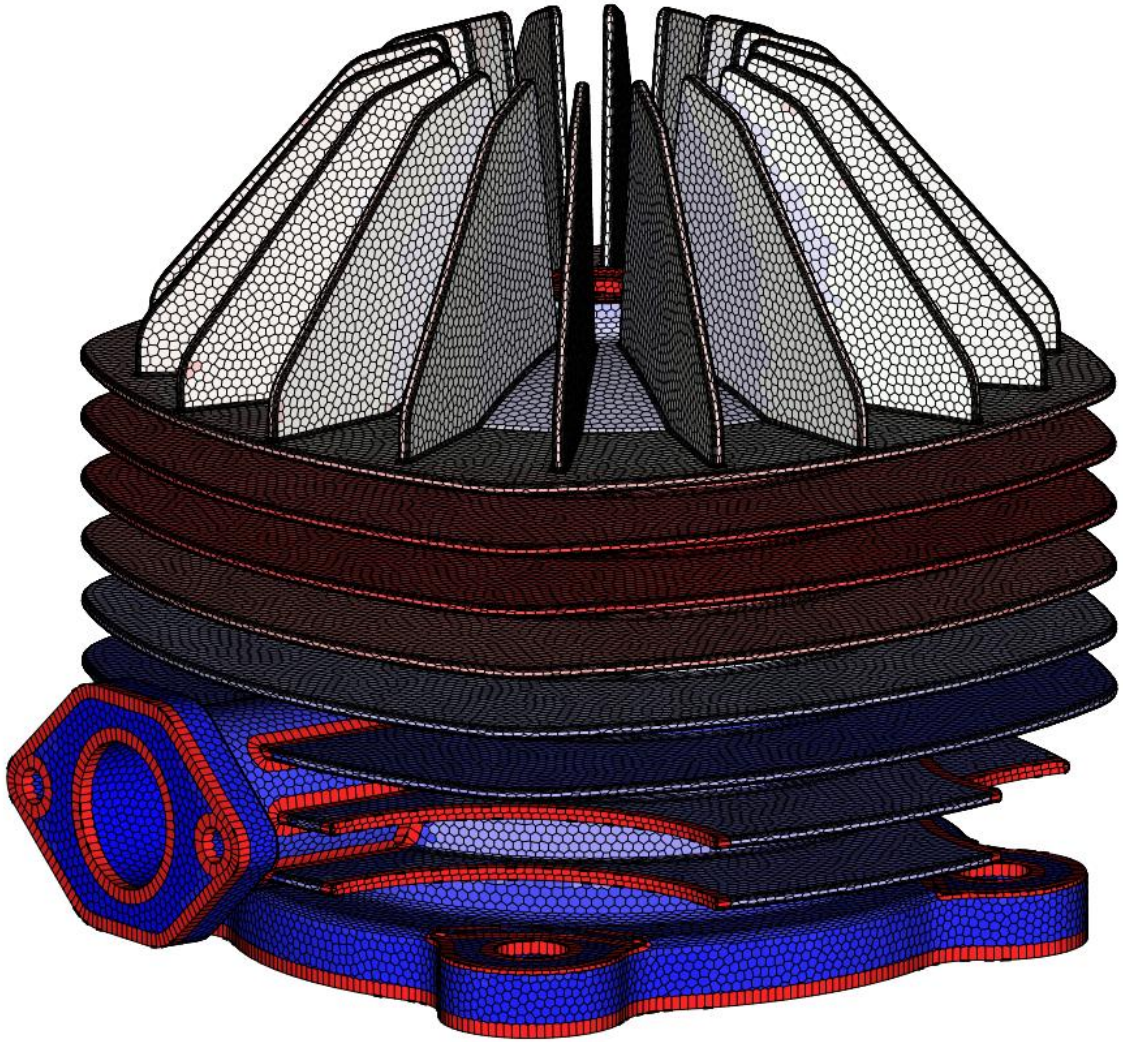

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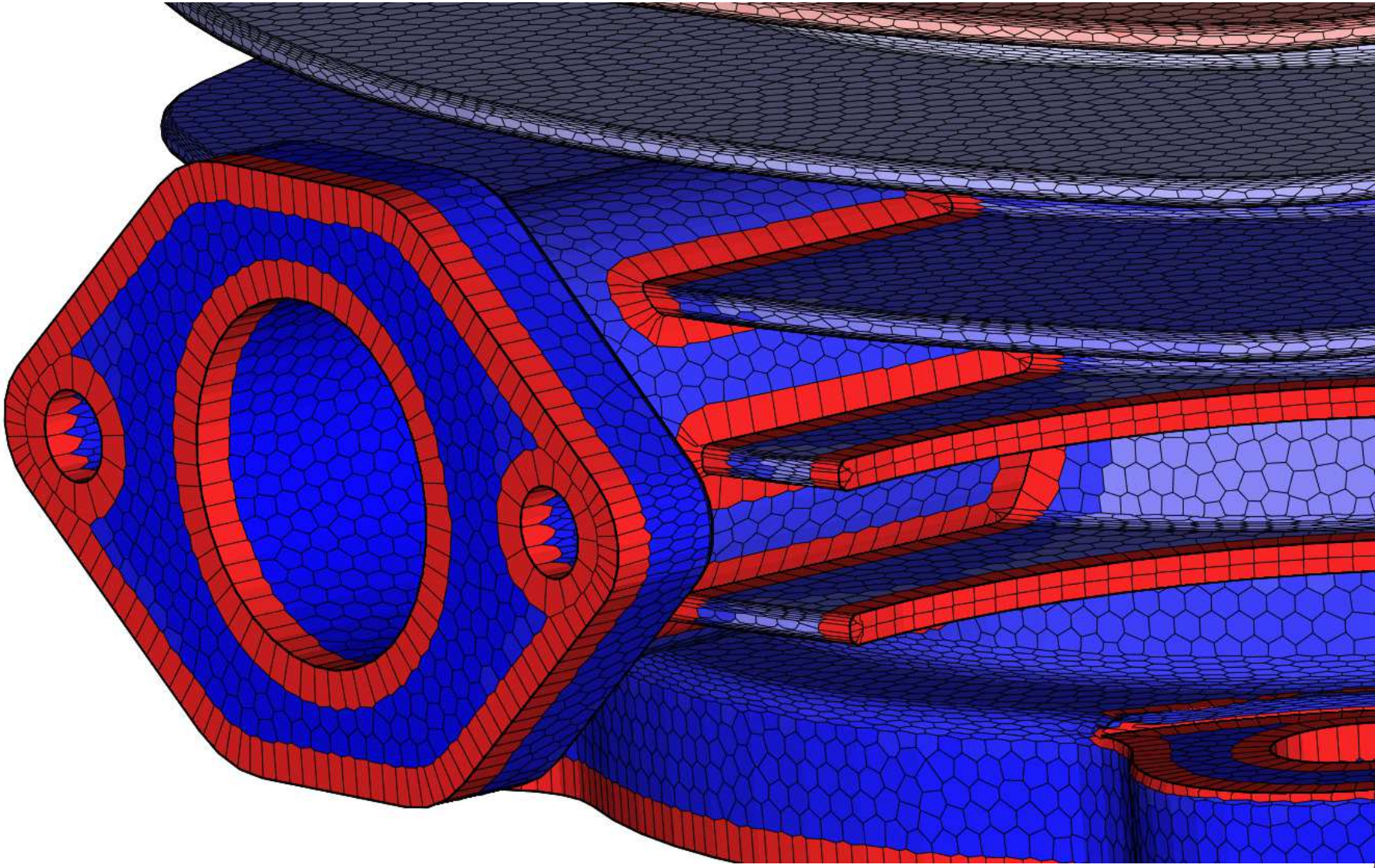

6d power diagram (7d Voronoi diagram)

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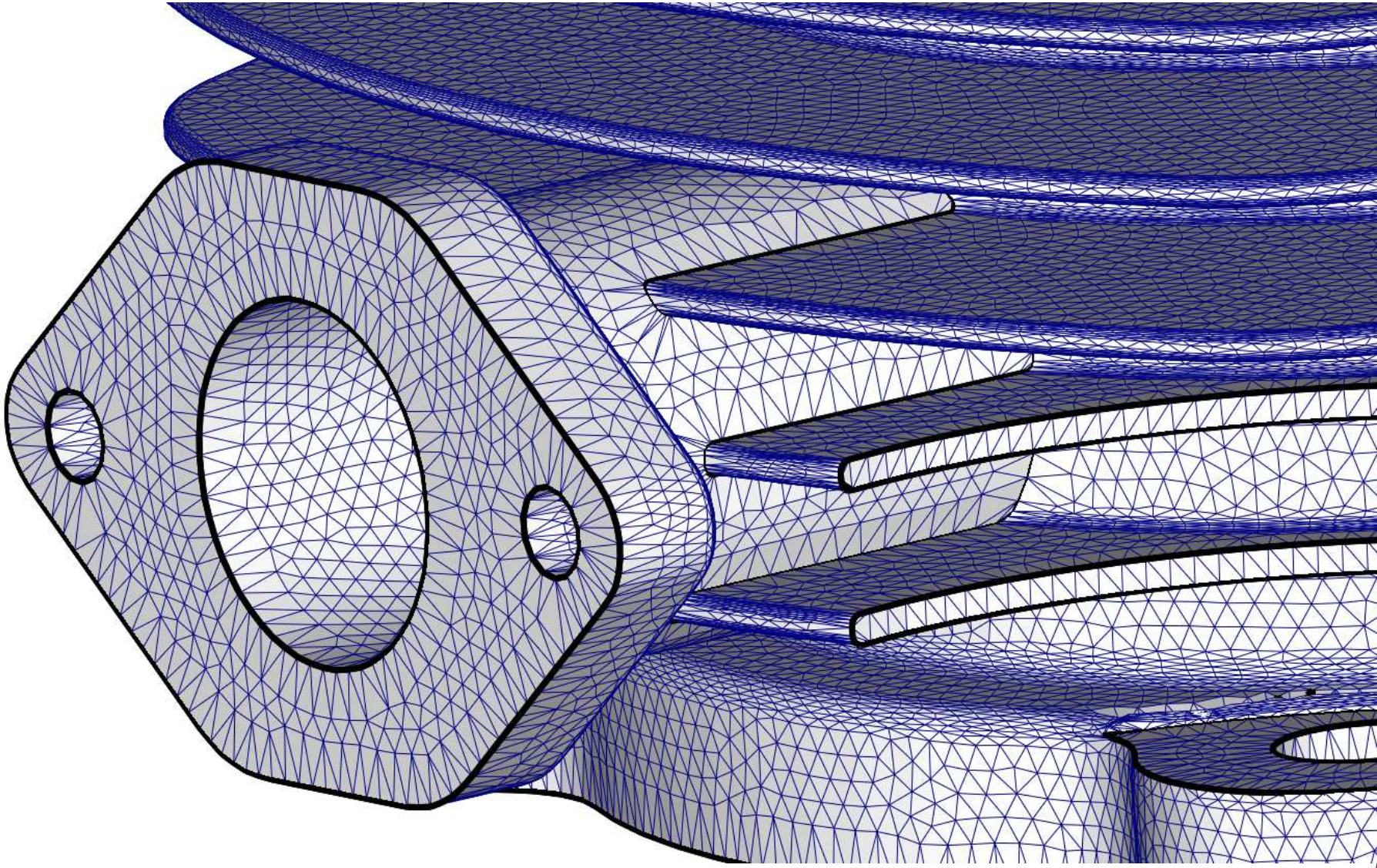
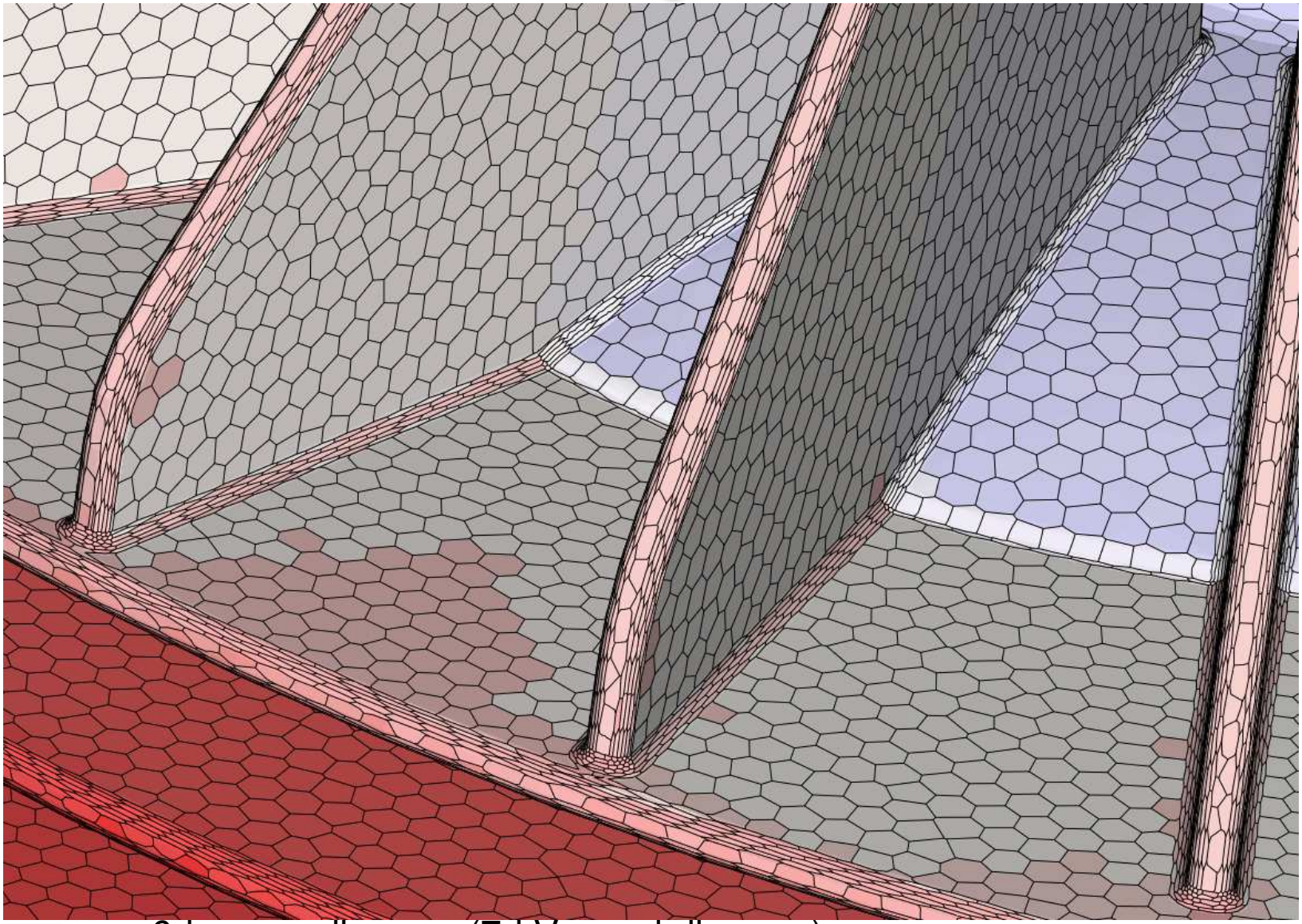
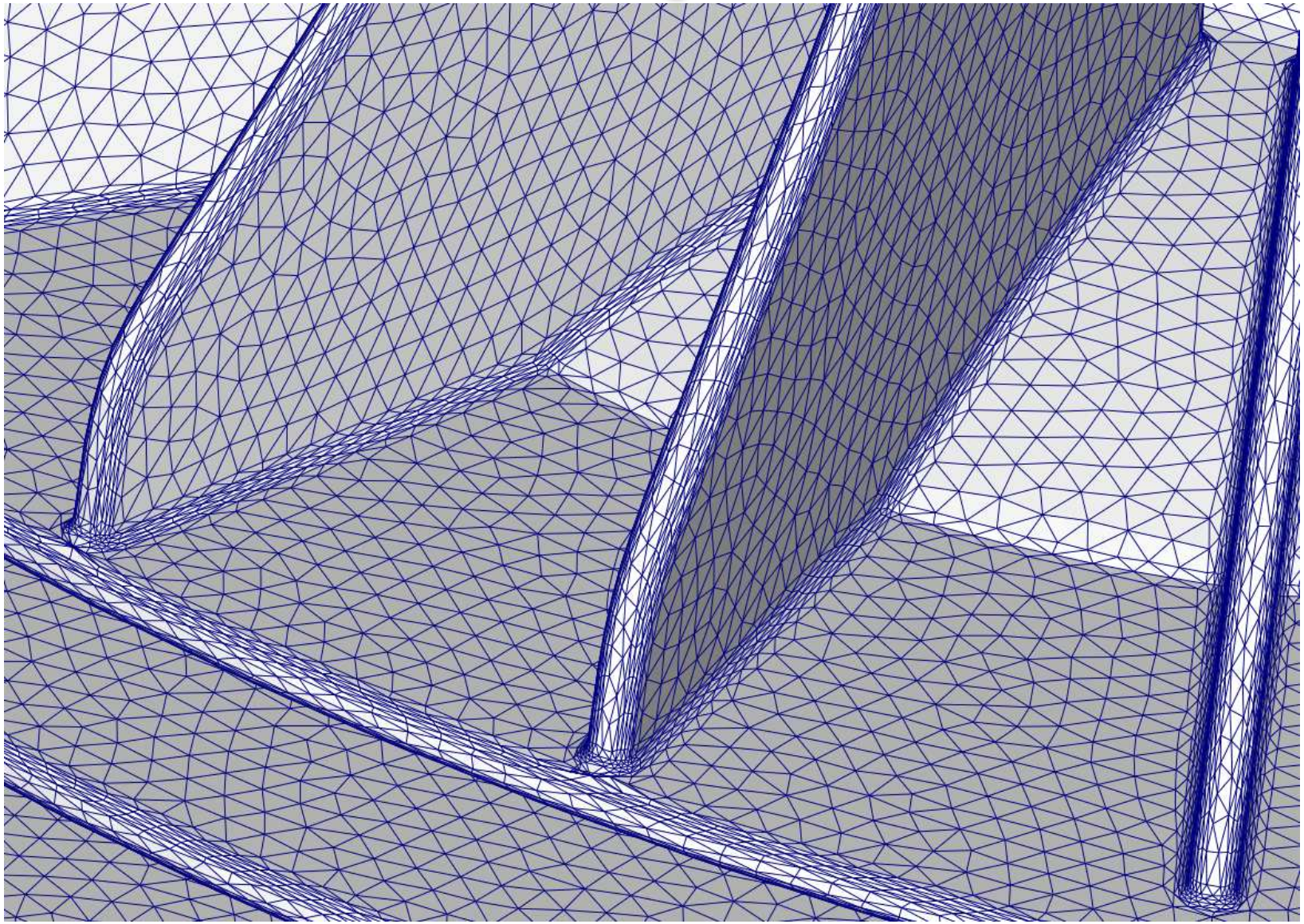


Part. 4 Constraints – Protecting balls



6d power diagram (7d Voronoi diagram)

Part. 4 Constraints – Protecting balls



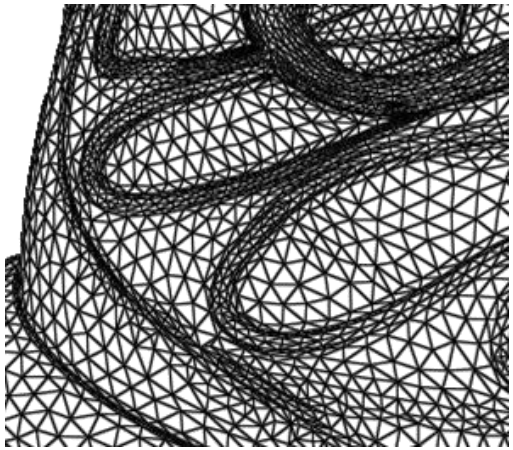
Summary

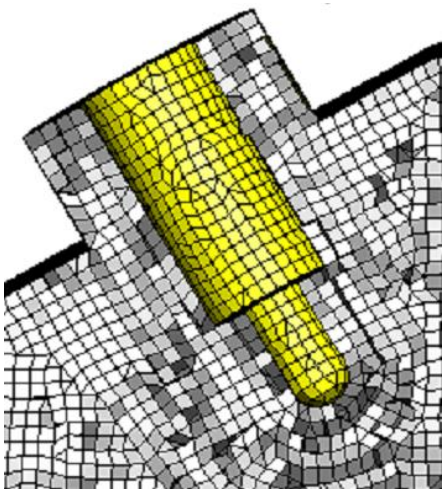
$$F = \sum_i \int_{Vor(i)} \left\| (x_i - x) \right\|^2 dx \quad \text{is of class } C^2 \ ^*$$

Tweaking distances:

Tweak #1: anisotropic distance
(through higher-d embedding)

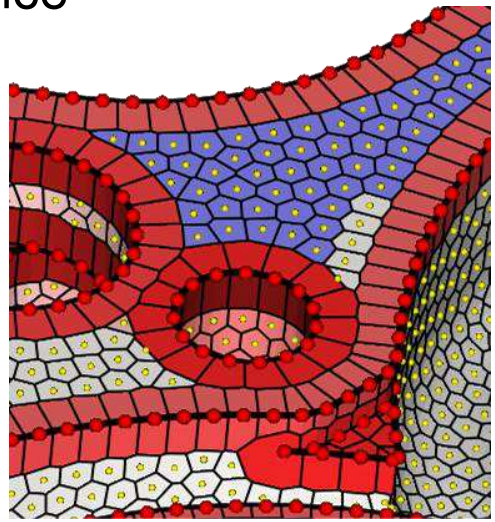
Readily available
"Vorpaline" software





Tweak #2: hex/quads, L_p distance

Prototype



Tweak #3: constraints, power diagrams
(through d+1 dim. embedding)

Prototype (note: we can do simpler)

* almost everywhere

Software roadmap

Vorpaline 1.0: (current version)

Isotropic surface meshing
Anisotropic surface meshing
Support for mesh gradation

Vorpaline 2.0: (planned Q1 2014)

Constrained surface meshing
Structural model meshing
Quad-dominant surface meshing

Vorpaline 3.0: (planned Q4 2014)

Anisotropic volumetric meshing
Hex-dominant volumetric meshing

Acknowledgements

- Bytes : ANN [Mount et.al], libmesh5 [Marechal], tetgen [Si], CGAL
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  VORPALINE ERC-PoC-334829
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Nicolas Saugnier, DongMing Yan, Loic Maréchal,
Tamal Dey, Pierre Alliez, David Bommes, Leif Kobbelt

merci

http://alice.loria.fr