NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

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CONTENTS

MODULE 1 GENERALIZED COORDINATES AND CONSTRAINTS

Unit 1 Constraints
Unit 2 Generalized Coordinates
Unit 3 Virtual work, Virtual Displacement and Generalized Forces

UNIT 1 CONSTRAINTS
CONTENTS

1.0 Introduction

A rigid body is defined as a system of *n* particles for which all the interparticle distances are constrained to fixed constants, $\left|\vec{r}_i - \vec{r}_j\right| = c_{ij}$ and the interparticle potentials are functions only of these interparticle distances. As these distances do not vary, neither does the internal potential energy. These interparticle forces cannot do work, and the internal potential energy may be ignored.
The rigid body is an example of a constrained system, in which the general 3*n* degrees of freedom are restricted by some forces of constraint which place conditions on the coordinates $\vec{r}_i$ perhaps in conjunction with their momenta. In such descriptions we do not wish to consider or specify the forces themselves, but only their (approximate) effect. It is generally assumed, as in the case with the rigid body, that the constraint forces do no work under displacements allowed by the constraints

2.0 Objectives

After studying this unit, you will be able to:

- Define degree of freedom and constraint.
- Distinguish between holonomic and non-holonomic constraints.
- Give examples of holonomic and non-holonomic constraints.
- Identify and Explain the three kinds of non-holonomic constraints.
- Identify and Explain the two kinds of holonomic constraints

3.0 Main Contents

3.1 Degrees of Freedom

The number of degrees of freedom is defined as the number of independent coordinates that is needed to identify uniquely the configuration of the system.

Suppose we have a system of N particles each moving in 3-space and interacting through arbitrary (finite) forces, The number n=3N is called the number of degree of freedom.
There is freedom, of course, in how we specify the degrees of freedom; e.g.:

- choice of origin
- coordinate system: cartesian, cylindrical, spherical, etc.
- center-of-mass vs. individual particles: $\{\vec{r}_k\}$ or $\{\vec{R}, \vec{s}_k = \vec{r}_k = \vec{r}_k - R\}$

But, the number of degrees of freedom is always the same; e.g., in the center-of-mass system, the constraint $\sum_k \vec{s}_k = 0$ applies, ensuring that $\{\vec{r}_k\}$ and $\vec{R}, \vec{s}_k$ have same number of degrees of freedom.
The motion of such a system is completely specified by knowing the dependence of the available degrees of freedom on time.
Simple illustration are: "The beads of a abacus are constrained to one-dimensional motion by the supporting wires" so also "Gas molecules within a container are constrained by the walls of the vessel to move only inside the container".
Throughout this section, we will work two examples alongside the theory. The first consists of an Atwood's machine, and we may allow the rope length to be a function of time, $l = l(t)$
The second consists of point particle sliding on an elliptical wire in the presence of gravity. The Cartesian coordinates of the particle satisfy

$$\left(\frac{x}{a(t)}\right)^2 + \left(\frac{z}{b(t)}\right)^2 = 1 \qquad (1.0)$$

We will at various points consider a and b to be time dependent or constant. The origin of the coordinate system is the stationary centre of the ellipse.

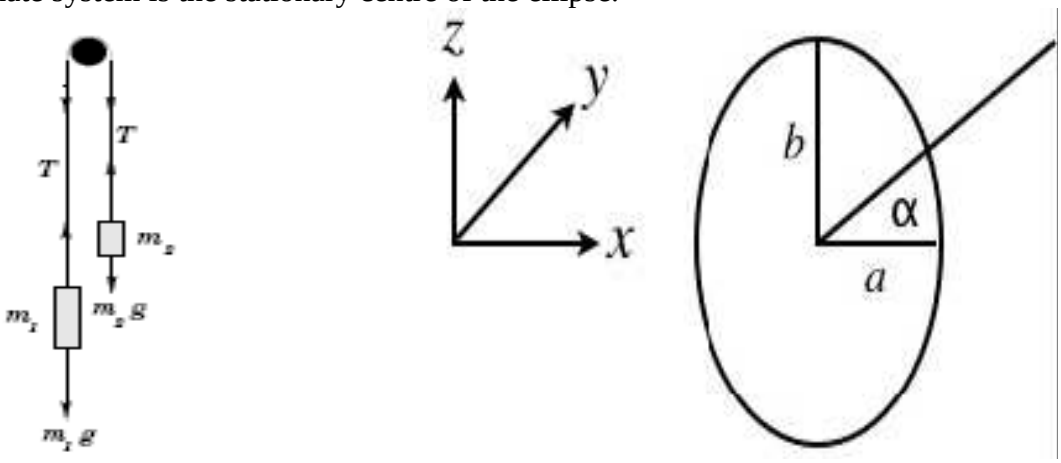


Figure 1.1 (a) Atwood machine (b) Point particle sliding on elliptical wire

Example 1.1:
In the Atwood's machine of figure 1.1(a), there are a priori 2 degrees of freedom; the z coordinates of the two blocks. (We ignore the x and y degrees of freedom because the problem is inherently 1-dimensional.) The inextensible rope connecting the two masses reduces this to one degree of freedom because, when one mass moves by a certain amount, the other one must move by the opposite amount.

Example 1.2:
In the elliptical wire of figure 1.1(b), there are a priori 3 degrees of freedom, the 3 spatial coordinates of the point particle. The constraints reduce this to one degree of freedom, as no motion in y is allowed and the motions in z and x are related. The loss of the y degree of freedom is easily accounted for in our Cartesian coordinate system; effectively, a 2D Cartesian system in x and z will suffice. But the relation between x and z is a constraint that cannot be trivially accommodated by dropping another Cartesian coordinate.

Self Assessment Exercise A

1. Now define Degree of freedom
2. State the number of degrees of freedom in an Atwood's machine and elliptical wire

3.2 Constraints

Constraints mean a restriction on the degree of freedom of motion of a system of particles in the form of a condition.
Constraints may reduce the number of degrees of freedom; e.g., particle moving on a table, rigid body, etc. It is divided into Holonomic and Non-Holonomic constraint.

3.2.1 Holonomic Constraints

Holonomic constraints are those that can be expressed in the form $f(\vec{r}_1, \vec{r}_2, \ldots t) = 0$. For example, restricting a point particle to move on the surface of a table is the holonomic constraint $z - z_0 = 0$, where z_0 is a constant. A rigid body satisfies the holonomic set of constraints $\left|\vec{r}_i - \vec{r}_j\right| - c_{ij} = 0$, where c_{ij} is a set of constants satisfying $c_{ij} = c_{ji} > 0$ for all particle pairs i, j.
A system is called "holonomic" if, in a certain sense, one can recover global information from local information, so the meaning "entire-law" is quite appropriate.
Holonomic constraints may be divided into rheonomic ("running law") and scleronomic ("rigid law") depending on whether time appears explicitly in the constraints:

$$\text{rheonomic:} \quad f\left(\{\vec{r}_k\}, t\right) = 0, \tag{1.1a}$$

$$\text{Scleronomic:} \quad f\left(\{\vec{r}_k\}\right) = 0. \tag{1.1b}$$

At a technical level, the difference is whether $\frac{\partial f}{\partial t} = 0$ or not: the presence of this partial derivative affects some of the relations we will derive later.

3.2.2 Non-holonomic Constraints

The rolling of a ball on a table is non-holonomic, because one rolling along different paths to the same point can put it into different orientations.
Non-holonomic constraints are, obviously, constraints that are not holonomic. There are three kinds of non-holonomic constraints:

i Non-integrable or history-dependent constraints. These are constraints that are not fully defined until the full solution of the equations of motion is known. Equivalently, they are certain types of constraints involving velocities. The classic case of this type is a vertical disk rolling on a horizontal plane. If x and y define the position of the point of contact between the disk and the plane, ϕ defines the angle of rotation of the disk about its axis, and θ defines the angle between the rotation axis of the disk and the x-axis, then one can find the constraints

$$\begin{aligned} \dot{x} &= -r\dot{\phi}\cos\theta, \\ \dot{y} &= -r\dot{\phi}\sin\theta. \end{aligned} \tag{1.2a}$$

The differential version of these constraints is

$$\begin{aligned} dx &= -rd\phi\cos\theta, \\ dy &= -rd\dot{\phi}\sin\theta. \end{aligned} \tag{1.2b}$$

These differential equations are not integrable; one cannot generate from the relations two equations $f_1(x, \theta, \phi) = 0$ and $f_2(y, \theta, \phi) = 0$. The reason is that, if one assumes the functions f_1 and f_2 exist, the above differential equations imply that their second derivatives would have to satisfy

$$\frac{\partial^2 f}{\partial\theta\partial\phi} \neq \frac{\partial^2 f}{\partial\phi\partial\theta} \text{ i.e. for } f = f_1 or\ f_2 \tag{1.3}$$

which is very unpleasant mathematical condition. Explicitly, suppose f_1 existed. Then we would be able to write $f_1(x,\theta,\phi) = 0$

Let us obtain the differential version of the constraint by differentiation:

$$\frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial \theta}d\theta + \frac{\partial f_1}{\partial \phi}d\phi = 0\,. \tag{1.4}$$

This differential constraint should match the original differential constraint $dx = -rd\phi\cos\theta$. Identifying the coefficients of the differential yields

$$\frac{\partial f_1}{\partial x} = 1 \quad \frac{\partial f_1}{\partial \theta} = 0 \quad \frac{\partial f_1}{\partial \phi} = r\cos\phi\,. \tag{1.5}$$

Taking the mixed second partial derivatives gives

$$\frac{\partial^2 f_1}{\partial\phi\partial\theta} = 0 \quad \frac{\partial^2 f_1}{\partial\theta\partial\phi} = -r\sin\phi \tag{1.6}$$

which, clearly, do not match. Such constraints are also called non-integrable because one cannot integrate the differential equation to find a constraint on the coordinates. A differential relation such as the one above is a local one; if the differential relation is integrable, you can obtain the constraint at all points in space, i.e., you can find the "entire law". Clearly, non-integrability is also related to the fact that the constraint is velocity-dependent: a velocity-dependent constraint is a local constraint, and it may not always be possible to determine a global constraint from it.

ii. inequality constraints; e.g., particles required to stay inside a box, particle sitting on a sphere but allowed to roll off,
iii. problems involving frictional forces.

Example 1.3:
For the elliptical wire example in figure 1.1(b), the constraint equation is the one we specified initially:

$$\left(\frac{x}{a(t)}\right)^2 + \left(\frac{z}{b(t)}\right)^2 = 1$$

If a and/or b do indeed have time dependence, then the constraint is rheonomic. Otherwise, it is scleronomic.

Example 1.4:
For the Atwood's machine in figure 1.1(a), the constraint equation is

$$z_1 + z_2 + l(t) = 0$$

Where *l(t)* is the length of the rope (we assume the pulley has zero radius). The signs on z_1 and z_2 are due to the choice of direction for positive z in the example. The constraint is again rheonomic if *l* is indeed time dependent, scleronomic if not

Self Assessment Exercise B
1. Define constraints.
2. Mention two categories of Constraints and Give three examples each.
3. Mention three kinds of non-holonomic constraints.
4. Show that velocity dependent constraints are non-integrable constraints.

4.0 Conclusion

Holonomic constraints are constraints that can be written in the form

$$f\left(\vec{r}_1,\vec{r}_2\ldots\vec{r}_M,t\right) = 0\,,$$

i.e., there is some sort of condition on the coordinates and possibly time. The condition may not

involve the coordinate velocities. Holonomic constraints are also termed integrable because a differential version of the constraint can be integrated to yield the full constraint. The constraint is rheonomic if time appears explicitly, scleronomic if not.

5.0 Summary

- The number of degrees of freedom is defined as the number of independent coordinates that is needed to identify uniquely the configuration of the system.
- Constraints means a restriction on the degree of freedom of motion of a system of particles in the form of a condition
- Constraints may reduce the number of degrees of freedom; e.g., particle moving on a table, rigid body, etc.
- Constraints are divided into holonomic and non-holonomic constraint
- Holonomic constraints are those that can be expressed in the form $f(\vec{r}_1, \vec{r}_2, \ldots t) = 0$
- Holonomic constraints may be divided into rheonomic ("running law") and scleronomic
- Non-holonomic constraints are, obviously, constraints that are not holonomic.
- There are three kinds of non-holonomic constraints: Non-integrable or history-dependent constraints, inequality constraints and problems involving frictional forces

6.0 Tutor Marked Assignment

1. Explain briefly what is meant by degrees of freedom.
2. How many degrees of freedom does a rigid body in the following three dimensional motion has and explain what is meant by the motion: i) heaving, ii) surging, iii)swaying, iv) rolling, v) pitching, vi) yawing.
3. How many degrees of freedom are in atwood machine of figure 1.1(a) and elliptical wire of figure 1.1(b).
4. State the constraint equation of the elliptical wire example in figure 1.1(b), and state if it is scleronomic or rheonomic.
5. In the Atwood's machine in figure 1.1(a), State the constraint equation and what condition could make the constraint equation rheonomic.
6. Show that velocity dependent constraints are non-integrable constraints.

7.0 Further Reading and Other Resources

Classical Mechanics by H.Goldstein, Narosa Publishing Home, New Delhi.

Classical Dynamics of Particles and Systems by Marion and Thomtron, Third Edition, Horoloma Book Jovanovich College Publisher.

Introduction to Classical Mechanics by R.G.Takawale and P.S.Puranik, Tata Mc-Graw Hill Publishing Company Limited, New Delhi.

UNIT 2 GENERALIZED COORDINATES

CONTENTS

1.0 Introduction

In general, if one has j independent constraint equations for a system of M particles with $3M$ degrees of freedom, then the true number of degrees of freedom is $3M - j$. There is dynamical behaviour of the system in only these remaining degrees of freedom. One immediately asks the question that since there are fewer degrees of freedom than position coordinates, is there any way to eliminate those unnecessary degrees of freedom and thereby simplify analysis of the mechanical system? In our example, why keeping both x and z if one of them would suffice? This question leads us to the idea of generalized coordinates, which are a set of $3M - j$ coordinates that have already taken into account the constraints and are independent, thereby reducing the complexity of the system.

2.0 Objectives

After studying this unit, you will be able to:

- Write Physical quantities in terms of generalized coordinates.
- Use constraints equation to define different generalized scheme.
- Calculate azimuthal angle.
- Solve Related Problems.

3.0 Main contents

3.1 Generalized Coordinates

For holonomic constraints, the constraint equations ensure that it will always be possible to define a new set of $3M - j$. generalized coordinates $\{q_k\}$ that fully specify the motion of the system subject to the constraints and that are independent of each other. The independence arises because the number of degrees of freedom must be conserved. The constraint equations yield (possibly implicit) functions

$$\vec{r}_1 = \vec{r}_1(q_1, q_2, \ldots q_{3M-j}, t) \qquad (2.1)$$

that transform between the generalized coordinates and the original coordinates. It may not always be possible to write these functions analytically. Some of these coordinates may be the same as the original coordinates, some may not; it simply depends on the structure of the constraints. We will refer to the original coordinates as position coordinates to distinguish them from the reduced set of independent generalized coordinates. Generalized coordinates are more than just a notational convenience. By incorporating the constraints into the definition of the generalized coordinates, we obtain two important simplifications:

1) the constraint forces are eliminated from the problem; and

2) the generalized coordinates are fully independent of each other. We shall see the application of these simplifications when we discuss virtual work and generalized forces.

Just as the velocity corresponding to a coordinate $\vec{r}_j$ is $\dot{\vec{r}}_j = \frac{d}{dt}\vec{r}_j$, it is possible to define a generalized velocity $\dot{q}_k$ as $\dot{q}_k = \frac{d}{dt}q_k$. Note that in all cases, velocities are defined as total time derivatives of the particular coordinate. Remember that if you have a function $g = g(\{q_k\},t)$ then the total time derivative $\frac{d}{dt}g$ is evaluated by the chain rule:

$$\frac{d}{dt}g(\{q_k\},t) = \sum \frac{\partial g}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial g}{\partial t}. \qquad (2.2)$$

It is very important to realize that, until a specific solution to the equation of motion is found, a generalized coordinate and its corresponding generalized velocity are independent variables. This can be seen by simply remembering that two initial conditions which are $q_k(t=0)$ and $\dot{q}_k(t=0)$ are required to specify a solution of Newton's second law, because it is a second-order differential equation. Higher-order derivatives are not independent variables because Newton's second law relates the higher-order derivatives to $\{q_k\}$ and $\{\dot{q}_k\}$. The independence of $\{q_k\}$ and $\{\dot{q}_k\}$ is a reflection of the structure of Newton's second law, not just a mathematical theorem. Unless otherwise indicated, from here on we will assume all constraints are holonomic.

Example 2.1:
For the elliptical wire, the constraint equation

$$\left(\frac{x}{a(t)}\right)^2 + \left(\frac{z}{b(t)}\right)^2 = 1$$

can be used to define different generalized coordinate schemes. Two obvious ones are
x and z; i.e., let x be the generalized coordinate and drop the z degree of freedom, or vice versa. Another obvious one would be the azimuthal angle α

$$\alpha = \tan^{-1}\left[\frac{z}{b(t)}\frac{a(t)}{x}\right]. \qquad (2.3)$$

The formal definitions $\vec{r}_i(\{q_k\},t)$ are then

$$x = a(t)\cos\alpha \qquad z = b(t)\sin\alpha\,. \qquad (2.4)$$

Here, we see the possibility of explicit time dependence in the relationship between the positions x and z and the generalized coordinate α.

Example 2.2:
For the Atwood's machine, either z_1 or z_2 could suffice as the generalized coordinate. Let's pick z_1, calling it Z to distinguish the generalized coordinate, we have

$$z_1 = Z \qquad z_2 = -l(t) - Z$$

This case is pretty trivial.

Self Assessment Exercise A
1. Write Velocity of any system with cartesian coordinates *x* and *y* in generalized coordinates
2. Express Kinetic energy of a particle with Cartesians coordinates *x* and *y* in a generalized coordinates

3.2 "Dot Cancellation"
For holonomic constraints, there is a very important statement that we will make much use of later

$$\frac{\partial \vec{r}}{\partial q_k} = \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_k}. \tag{2.5}$$

Conceptually, what this says is the differential relationship between a given degree of freedom and a generalized coordinate is the same as the differential relationship between the corresponding velocities. This statement relies on having holonomic constraints. Heuristically, one can understand the rule as simply arising from the fact that holonomic constraints use only the positions and time to define the generalized coordinates; as a result, any relationships between positional velocities and generalized velocities must be determined only by relationships between positions and generalized coordinates. The above relationship is then the simplest possible one.

We derive the result rigorously by starting with the total time derivative $\dot{\vec{r}}_i$:

$$\dot{\vec{r}} = \frac{d}{dt}\vec{r}_i(\{q_k\},t) = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial \vec{r}_i}{\partial t}. \tag{2.6}$$

The last term does not exist if the constraints are scleronomic, but that is not really important here. Now, take the partial derivative with respect to $\dot{q}_1$; this selects out the term in the sum with $k = l$, and drops the t term:

$$\frac{\partial \dot{\vec{r}}}{\partial \dot{q}_l} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\delta_{kl} = \frac{\partial \vec{r}_i}{\partial q_l}. \tag{2.7}$$

So the dot cancellation relation is proven.

Self Assessment Exercise B

1. What is Dot cancellation.
2. Prove that $\frac{\partial \vec{r}}{\partial q_k} = \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_k}$.

4.0 Conclusion

From the Physicist point of view the $q's$ are generalized coordinates in the sense that they need not have dimensions of length. By deriving equations of motion in terms of a general set of generalized coordinates, the results found will be valid for any coordinate system that is ultimately specified.

5.0 Summary

- Given a set of holonomic constraints, one can use the constraints to implicitly define a set of independent generalized coordinates. The transformation relations between the generalized coordinates and the original physical coordinates are of the form
$$\vec{r}_i = \vec{r}_j\left(q_1, q_2 \ldots q_{3M-j}, t\right)$$
where we assume there are j constraint equations. It is assumed that all the constraints are used so that the generalized coordinates truly form an independent set.
- Dot Cancellation

  For holonomic constraints, it holds that $\frac{\partial \vec{r}_i}{\partial q_k} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k}$.

6.0 Tutor Marked Assignment

1. Two particles are connected by a rigid rod so they are constrained to move a fixed distance apart. Write down a constraint equation of the form $f(\vec{r}_1, \vec{r}_2, \ldots t) = 0$ and find suitable generalized coordinates for the system incorporating this holonomic constraint.

Answer

Let the position of particle 1 with respect to a stationary Cartesian frame be $\{x_1, y_1, z_1\}$ and that of particle 2 be $\{x_2, y_2, z_2\}$. The rigid rod constraint equation is

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (z_1 - z_2)^2 = l^2 .$$

This reduces the number of degrees of freedom from 6 to 5, which we could take to be position of particle 1, $\{x_1, y_1, z_1\}$ However, the three position coordinates would be the coordinates of the centre of mass whose position vector is

$$r_{cm} = \frac{m_2 r_1 + m_1 r_2}{m_1 + m_2} .$$

7.0 Further Reading and Other Resources

Classical Mechanics by H.Goldstein, Narosa Publishing Home, New Delhi.

Classical Dynamics of Particles and Systems by Marion and Thomtron, Third Edition, Horoloma Book Jovanovich College Publisher.

Introduction to Classical Mechanics by R.G.Takawale and P.S.Puranik, Tata Mc-Graw Hill Publishing Company Limited, New Delhi.

UNIT 3 VIRTUAL DISPLACEMENT, VIRTUAL WORK AND GENERALIZED FORCES
CONTENTS

1.0 Introduction

Our discussion of generalized coordinates essentially was an effort to make use of the constraints to eliminate the degrees of freedom in our system that have no dynamics. Similarly, the constraint forces, once they have been taken account of by transforming to the generalized coordinates, would seem to be irrelevant. We will show how they can be eliminated in favour of generalized forces that contain only the non-constraint forces.
To define generalized forces, we combine the equation relating virtual displacements of position coordinates and generalized coordinates, with the equation relating virtual work and non-constraint forces as it will be shown later in this unit.

2.0 Objectives

After studying this unit, you will be able to:

- Explain virtual displacement, virtual work and generalized forces.
- Express total differential of any set of system position vectors, $\vec{r}_i$ that are functions of other variables, $\{q_1, q_2, \ldots, q_m\}$ and time t in terms of virtual displacement.
- Solve related Problems.

3.0 Main Contents

3.1 Virtual Displacement

We define a virtual displacement $\{\delta\vec{r}_i\}$ as an infinitesimal displacement of the system coordinates $\{\vec{r}_i\}$ that satisfies the following criteria.

1. The displacement satisfies the constraint equations, but may make use of any remaining unconstrained degrees of freedom.
2. The time is held fixed during the displacement.
3. The generalized velocities $\{\dot{q}_k\}$ are held fixed during the displacement.

A virtual displacement can be represented in terms of position coordinates or generalized coordinates. The advantage of generalized coordinates, of course, is that they automatically respect the constraints. An arbitrary set of displacements δq_k can qualify as a virtual displacement if conditions (2) and (3) are additionally applied, but an arbitrary set of displacements $\{\delta\vec{r}_i\}$ may or may not qualify as a virtual displacement depending on whether the displacements obey the constraints. All three conditions will become clearer in the examples.

Explicitly, the relationship between infinitesimal displacements of generalized coordinates and virtual displacements of the position coordinates is

$$\delta\vec{r}_i = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\delta q_k \ . \tag{3.1}$$

This expression has content: there are fewer $\{q_k\}$ than $\{\vec{r}_i\}$, so the fact that $\delta\vec{r}_i$ can be expressed only in terms of the $\{q_k\}$ reflects the fact that the virtual displacement respects the constraints. One can put in any values of the δq_k and obtain a virtual displacement, but not every possible set of $\{\delta\vec{r}_i\}$ can be written in the above way.

Example 3.1:
For the elliptical wire, it is easy to see what kinds of displacements satisfy the first two requirements. Changes in x and z are related by the constraint equation; we obtain the relation by applying the displacements to the constraint equation. We do this by writing the constraint equation with and without the displacements and differentiating the two:

With displacements $$\left(\frac{x+\delta x}{a(t)}\right)^2 + \left(\frac{z+\delta z}{b(t)}\right)^2 = 1, \tag{3.2}$$

without displacement $$\left(\frac{x}{a(t)}\right)^2 + \left(\frac{z}{b(t)}\right)^2 = 1, \tag{3.3}$$

difference $$\left(\frac{x+\delta x}{a(t)}\right)^2 - \left(\frac{x}{a(t)}\right)^2 + \left(\frac{z+\delta z}{b(t)}\right)^2 - \left(\frac{z}{b(t)}\right)^2 = 0, \tag{3.4}$$

$$\frac{x\delta x}{[a(t)]^2} + \frac{z\delta z}{[b(t)]^2} = 0 \Rightarrow \frac{\delta x}{\delta z} = -\frac{a(t)z}{b(t)x}. \tag{3.5}$$

All terms of second order in δx or δz are dropped because the displacements are infinitesimal. The result is that δx and δz cannot be arbitrary with respect to each other and are related by where the particle is in x and z and the current values of a and b; this clearly satisfies the first requirement. We have satisfied the second requirement, keeping time fixed, by treating a and b as constant: there has been no δt applied, which would have added derivatives of a and b to the expressions. If a and b were truly constant, then the second requirement would be irrelevant. The third requirement is not really relevant here because the generalized velocities do not enter the constraints in this holonomic case except if kinetic energy enters the constraints .
The relationship between the virtual displacements in the positions and in the generalized coordinate is easy to calculate:

$$x = a\cos\alpha \Rightarrow \delta x = -a\sin\alpha\delta\alpha, \tag{3.6a}$$
$$z = b\sin\alpha \Rightarrow \delta z = b\cos\alpha\delta\alpha, \tag{3.6b}$$

$$\Rightarrow \frac{\delta x}{\delta z} = -\frac{a}{b}\tan\alpha. \tag{3.7}$$

We see that there is a one-to-one correspondence between all infinitesimal displacements $\delta\alpha$ of the generalized coordinate and virtual displacements of the positional coordinates $(\delta x, \delta z)$, as stated above. The displacements of the positional coordinates that cannot be generated from $\delta\alpha$ by the above expressions are those that do not satisfy the constraints and are disallowed.

Example 3.2:

For our Atwood's machine example, the constraint equation $z_1 + z_2 + l(t) = 0$ is easily converted to differential form, giving $\delta z_1 + \delta z_2 = 0$
Again, remember that we do not let time vary, so *l(t)* contributes nothing to the differential. The equation is what we would have arrived at if we had started with an infinitesimal displacement δz of the generalized coordinate Z

$$\delta z_1 = \delta Z \quad \delta z_2 = -\delta Z \Rightarrow \delta z_1 + \delta z_2 = 0 .$$

Fixing time prevents appearance of derivatives of *l(t)*. Again, also see the one-to-one correspondence between infinitesimal displacements of the generalized coordinates and virtual displacements of the position coordinates.

Self Assessment Exercise A

1. Read and understand the two examples given above properly

3.2 Virtual Work

Using the virtual displacement, we define virtual work as the work that would be done on the system by the forces acting on the system as the system undergoes the virtual displacement $\{\delta\vec{r}_i\}$

$$\delta W \equiv \sum_{ij} \vec{F}_{ij} \cdot \delta\vec{r}_i , \tag{3.8}$$

where $\vec{F}_{ij}$ is the *jth* force acting on the coordinate of the *ith* particle $\vec{r}_i$.

Example 3.3:

In our elliptical wire example, $\vec{F}_{11}$ would be the gravitational force acting on the point mass and $\vec{F}_{12}$ would be the force exerted by the wire to keep the point mass on the wire (*i = 1* only because there is only one object involved). The virtual work is

$$\delta W \equiv \sum_{i=1}^{1} \left(\vec{F}_{i1} + \vec{F}_{i2}\right) \cdot \delta\vec{r}_i . \tag{3.9}$$

Example 3.4:

In our Atwood's machine example, the two masses feel gravitational forces $\vec{F}_{11} = -m_1 g\hat{z}$ and $\vec{F}_{21} = -m_2 g\hat{z}$. The tension in the rope is the force that enforces the constraint that the length of rope between the two blocks is fixed, $\vec{F}_{21} = T\hat{z}$ and $\vec{F}_{22} = T\hat{z}$. T may be a function of time if *l* varies with time, but it is certainly the same at the two ends of the rope at any instant.

Because of the possibility that there exist situations of the third type, we use the most generic assumption to proceed with our derivation, which is the third one. We write

$$\sum_{ij} \vec{F}_{ij}^{(c)} \cdot \delta\vec{r}_{ij} = 0 , \tag{3.10}$$

where the $^{(c)}$ superscript restricts the sum to constraint forces but the sum is over all constraint forces and all particles. Mathematically, the assumption lets us drop the part of the virtual work sum containing constraint forces, leaving

$$\delta W = \sum_{ij} \vec{F}_{ij}^{(nc)} \cdot \delta\vec{r}_i , \tag{3.11}$$

where the $^{(nc)}$ superscript indicates that the sum is only over non-constraint forces.

Example 3.5:

In our elliptical wire example, the force exerted by the wire, $\vec{F}_{12}$, acts to keep the point mass *i* on the wire; the force is therefore always normal to the wire. The virtual displacement $\delta\vec{r}_1$ must always be tangential to the wire to satisfy the constraint. So, $\sum_{i=1}^{1} \vec{F}_{i2} \cdot \delta\vec{r}_i = 0$, the only non-constraint force is gravity, $\vec{F}_{11}$, so we are left with

$$\delta W = \sum_{j=1}^{1}\sum_{i=1}^{1} \vec{F}_{ij}^{(nc)} \cdot \delta\vec{r}_i = \sum_{i=1}^{1} \vec{F}_{i1} \cdot \delta\vec{r}_i = \vec{F}_{11} \cdot \delta\vec{r}_1 = -mg\delta z . \tag{3.12}$$

δW will in general not vanish; it gives rise to the dynamics of the problem.

Example 3.6:

In the Atwood's machine example the constraint forces $\vec{F}_{21}$ and $\vec{F}_{22}$ act along the rope. The virtual displacements are also along the rope. Clearly, $\vec{F}_{21}\cdot\delta\vec{r}_1 = f_{21}\delta z_1$ and $\vec{F}_{22}\cdot\delta\vec{r}_2 = f_{22}\delta z_{21}$ do not vanish but the sum does

$$\sum_{i=1}^{2}\vec{F}_{i2}\cdot\delta\vec{r}_i = f_{21}\delta z_1 + f_{22}\delta z_2 = T\left(\delta z_1 + \delta z_2\right) = 0\,. \tag{3.13}$$

Notice that, in this case all the terms pertaining to the particular constraint force have to be summed in order for the result to hold. The virtual work and non constraint force sum is

$$\delta W = \sum_{i=1}^{2}\sum_{j=1}^{1}\vec{F}_{ij}^{(nc)}\cdot\delta\vec{r}_i = \sum_{i=1}^{2}\vec{F}_{i1}\cdot\delta\vec{r}_i = -g(m_1\delta z_1 + m_2\delta z_2) \tag{3.14}$$

Self Assessment Exercise B

1. Express total differential of any set of system position vectors, $\vec{r}_i$ that are functions of other variables, $\{q_1, q_2, \ldots, q_m\}$ and time t in terms of virtual differential displacement

$$dr_i = \frac{\partial r_i}{\partial t}dt + \sum_{j=1}^{m}\frac{\partial r_i}{\partial q_j}dq_j$$

3.3 Generalized Force

To define generalized forces, we combine Equation 3.1, the relationship between virtual displacements of position coordinates and generalized coordinates, with Equation 3.11, the relationship between virtual work and non-constraint forces:

$$\begin{aligned}\delta W &= \sum_{ij}\vec{F}_{ij}^{(nc)}\cdot\delta\vec{r}_i \\ &= \sum_{ij}\vec{F}_{ij}^{(nc)}\cdot\left[\sum_k\frac{\partial\vec{r}_i}{\partial q_k}\delta q_k\right] \\ &= \sum_k\left[\sum_{ij}\vec{F}_{ij}^{(nc)}\cdot\frac{\partial\vec{r}_i}{\partial q_k}\right]\delta q_k \\ &\equiv \sum_k f_k\delta q_k\,,\end{aligned} \tag{3.15}$$

where

$$f_k \equiv \sum_{ij}\vec{F}_{ij}^{(nc)}\cdot\frac{\partial\vec{r}_i}{\partial q_k} = \frac{\delta W}{\delta q_k}\,,$$

is the generalized force along the *kth* generalized coordinate. The last expression, $f_k = \delta W/\delta q_k$ says that the force is simple the ratio of the work done to the displacement when a virtual displacement of only the *kth* generalized coordinate is performed; it is of course possible to displace only the *kth* generalized coordinate because the generalized coordinates are mutually independent. It is important to remember that the generalized force is found by summing only over the non-constraint forces: the constraint forces have already been taken into account in defining generalized coordinates.

Infinitesimally, a force $\vec{F}$ causing a displacement $\delta\vec{r}$ does work $\delta W = \vec{F}\cdot\delta\vec{r}$. The generalized force is the exact analogue: if work δW is done when the ensemble of forces act to produce a generalized coordinate displacement δq_k, then the generalized force f_k doing that work is $f_k = \delta W/\delta q_k$. But the generalized force is a simplification because it is only composed of the non-constraint forces.

Example 3.7:
In the elliptical wire example, the generalized force for the α coordinate (*k=1*) is

$$f_\alpha = \vec{F}_{11} \cdot \frac{\partial r_1}{\partial \alpha} = -mg\hat{z} \cdot \left(\hat{x}\frac{\partial x}{\partial \alpha} + \hat{z}\frac{\partial z}{\partial \alpha} \right) = -mg\hat{z} \cdot \left(-\hat{x}a\sin\alpha + \hat{z}b\cos\alpha \right) \tag{3.16}$$
$$= -mgb\cos\alpha.$$

The constraints force, which acts in both the *x* and *z* direction and is α-dependent, does not appear in the generalized force.

Example 3.8:
In the Atwood's machine example, the generalized force for the Z coordinate (*k*=1 again) is

$$f_z = \vec{F}_{11} \cdot \frac{\partial \vec{r}_1}{\partial Z} + \vec{F}_{21} \cdot \frac{\partial \vec{r}_2}{\partial Z} = -m_1 g\frac{\partial z_1}{\partial Z} - m_2 g\frac{\partial z_2}{\partial Z} \tag{3.17}$$
$$= -(m_2 - m_1)g$$

Again, the constraint force (the rope tension) is eliminated. Because Z is just z_1 in this case, the generalized force in Z is just the net force on m_1 acting in the z_1 direction.

Self Assessment Exercise C
1. Make sure you understand the above examples very well.
2. Derive generalized force from the expressions of virtual work and virtual displacement.

4.0 Conclusion

Generalized forces are defined via coordinate transformation of applied forces, F_i on a system of n particles, *i*. The concept finds its use in Lagrangian mechanics, where it plays a conjugate role to generalized coordinates.
A convenient equation from which to derive the expression for generalized forces is that of the virtual work, δW caused by applied forces, as seen in D'Alembert's principle in accelerating systems and the principle of virtual work for applied forces in static systems.

$$\delta W = \sum_{i=1}^{n} F_i \cdot \delta r_i \tag{3.18}$$

δr_i is the virtual displacement of the system. Substituting the definition for the virtual displacement (differential), we have

$$\delta r_i = \sum_{j=1}^{m} \frac{\partial r_i}{\partial q_j} \delta q_j \,, \tag{3.19}$$

then we have

$$\delta W = \sum_{i=1}^{n} F_i \cdot \sum_{j=1}^{m} \frac{\partial r_i}{\partial q_j} \delta q_j \,, \tag{3.20}$$

therefore

$$\delta W = \sum_{j=1}^{m} \sum_{i=1}^{n} F_i \cdot \frac{\partial r_i}{\partial q_j} \delta q_j \,. \tag{3.21}$$

From this form, we can see that the generalized applied forces are then defined by

$$f_j = \frac{\delta W}{\delta q_j} = \sum_{i=1}^{n} F_i \cdot \frac{\partial r_i}{\partial q_j} \,,$$

thus, the virtual work due to the applied forces is

$$\delta W = \sum_{j=1}^{n} f_j \delta q_j \,.$$

5.0 Summary

Virtual Displacement and Work

- A virtual displacement is a set of coordinate displacements $\{\delta\vec{r}_i\}$ that obey the following conditions: 1. The displacement satisfies the constraint equations
  2. Time is held fixed during the displacement.
  3. The generalized velocities $\{\dot{q}_k\}$ are held fixed during the displacement.
- Any displacement of a truly independent set of generalized coordinates $\{q_k\}$ satisfies (1) automatically.
- Virtual work is the work done during a virtual displacement. It is defined as

$$\delta W = \sum_{ij} \vec{F}_{ij} \cdot \delta\vec{r}_i \, .$$

  Because the displacement is virtual, we make the assumption that the contribution of constraint forces to the virtual work vanishes, leaving

$$\delta W = \sum_{ij} \vec{F}_{ij}^{(nc)} \cdot \delta\vec{r}_i$$

  where $^{(nc)}$ refers to non constraint forces only.
- The generalized force along the generalized coordinate q_k is defined to be

$$f_k = \sum \vec{F}_i^{(nc)} \cdot \frac{\partial \vec{r}_i}{\partial q_k},$$

  where $\vec{F}_i^{(nc)}$ is the sum of all non-constraint forces acting on particle *i*.

6.0 Tutor Marked Assignment (TMAs)

1. Derive generalized force from the expressions of virtual work and virtual displacement

7.0 Further Reading and Other Resources

Classical Mechanics by H.Goldstein, Narosa Publishing Home, New Delhi.

Classical Dynamics of Particles and Systems by Marion and Thomtron, Third Edition, Horoloma Book Jovanovich College Publisher.

Introduction to Classical Mechanics by R.G.Takawale and P.S.Puranik, Tata Mc-Graw Hill Publishing Company Limited, New Delhi.

Module 2 Lagranges and Hamilton's Formulation of Mechanics

Unit 1 D'Alembert's Principle Of Virtual Work
Unit 2 Lagrangian Mechanics
Unit 3 Hamiltonian Mechanics

Unit 1 D'Alembert's Principle of Virtual Work

CONTENTS

1.0 Introduction

D'Alembert's principle is a statement of the fundamental classical laws of motion. It is named after its discoverer, the French physicist and mathematician Jean le Rond D'Alembert. The principle states that the sum of the differences between the forces acting on a system and the time derivatives of the momenta of the system itself along a virtual displacement consistent with the constraints of the system, is zero and it is the basis of Lagrangian Mechanics.
It is the dynamic analogue to the *principle of virtual work for applied forces* in a static system and in fact is more general than Hamilton's principle, avoiding restriction to holonomic systems.

2.0 Objectives

After studying this unit, you will be able:

- Derive D'Alembert's Principle from Newton's Second Law Of Motion.
- To reformulate Newton's equations system as a system of equations for the generalized coordinates $q(t)$.
- Use D'Alembert's principle to relate generalized forces to the rate of change of the momenta:
- Solve Related Problems.

3.0 Main Contents

3.1 D'Alembert's Principle

D'Alembert's Principle: Our definition of virtual work was $\delta W = \sum \vec{F}_{ij} \cdot \delta \vec{r}_i$ where the sum includes all (constraint and non-constraint) forces. Assuming our position coordinates are in an inertial frame (but not necessarily our generalized coordinates), Newton's second law tells us $\sum_{ij} \vec{F}_{ij} = p_i$ the sum of all the forces acting on a particle give the rate of change of its momentum. We may then rewrite δW :

$$\delta W = \sum_i \sum_j \vec{F}_{ij} \cdot \delta \vec{r}_i = \sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i \ .$$

But, we found earlier that we could write the virtual work as a sum over only non-constraint forces, $\delta W = \sum \vec{F}_{ij}^{(nc)} \cdot \delta \vec{r}_i$.

Thus, we may derive the relation

$$\sum_i \left[\sum_j \vec{F}_{ij}^{(nc)} - \dot{\vec{p}} \right] \cdot \delta \vec{r}_i = 0 \,, \tag{1.1}$$

the above equation 1.1 is referred to as D'Alembert's principle. The expression shows that the rate of change of momentum is determined only by the non-constraint forces. In this form, it is not much use, but the conclusion that the rate of change of momentum is determined only by non-constraint forces is an important physical statement.

We may use D'Alembert's principle to relate generalized forces to the rate of change of the momenta:

$$\sum_k F_k \delta q_k = \delta W = \sum_k \dot{\vec{p}} \cdot \delta \vec{r}_i = \sum_{i,k} \dot{\vec{p}} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \delta q_k \ . \tag{1.2}$$

Now unlike the $\{\delta \vec{r}_i\}$, the $\{\delta q_k\}$ are mutually independent. Therefore, we may conclude that equality holds for each of the sum separately providing a different version of D'Alembert's principle

$$\sum_{ij} \vec{F}_{ij}^{(nc)} \cdot \frac{\partial \vec{r}_i}{\partial q_k} = f_k = \sum_i \dot{\vec{p}} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \,. \tag{1.3}$$

This is now a very important statement: the generalized force for the *kth* generalized coordinate, which can be calculated from the non-constraint forces only, is related to a particular weighted sum of momentum time derivatives (the weights being the partial derivatives of the position coordinates with respect to the generalized coordinates). Effectively, we have an analogue of Newton's second law, but including only the non-constraint forces.
This can be a major simplification in cases where the constraint forces are complicated or simply not known. D'Alembert's principle is not yet useful in the current form because the left side contains generalized forces depending on the generalized coordinates but the right side has derivatives of the momenta associated with the position coordinates. We need time derivatives with respect to the generalized coordinates on the right hand side so that all the dynamics can be calculated in the generalized coordinates.

Self Assessment Exercise A
1. State and Derive the D'Alembert's Principle of virtual work

3.2 Generalized Equations of Motion

Here we perform the manipulation needed to make D'Alembert's principle useful. We know from Newtonian mechanics that work is related to kinetic energy, so it is natural to expect

$$T \equiv \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = T(\{q_k\}, \{\dot{q}_k\}, t). \tag{1.4}$$

T should be obtained by first writing T in terms of position velocities $\{\dot{\vec{r}}_i\}$ and then using the definition of the position coordinates in terms of generalized coordinates to re-write T as a function of the generalized coordinates and velocities. T may depend on all the generalized coordinates and velocities and on time because the $\{\vec{r}_i\}$ depend on the generalized coordinates and time and a time derivative is being taken, which may introduce dependence on the generalized velocities. The partial derivatives of T are:

$$\frac{\partial T}{\partial q_k} = \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k} = \sum_i \vec{p}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k}, \tag{1.5}$$

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} = \sum_i \vec{p}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}, \tag{1.6}$$

where in the last step we have made use of dot cancellation because the constraints are assumed to be holonomic. Now, we have $\vec{p}_i$ floating around but we need $\dot{\vec{p}}_i$. The natural thing is then to take a time derivative. We do this to $\partial T/\partial \dot{q}_k$ (instead of $\partial T/\partial q_k$) because we want to avoid second order time derivatives if we are to obtain any expression algebraically similar to the right hand side of D'Alembert's principle. We find

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} + \sum_i \vec{p}_i \cdot \frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_k}. \tag{1.7}$$

Referring back to the second form of D'Alembert's principle (equation 1.1), we see that the first term in the expression is the generalized force F_k for the kth coordinate. Continuing onward, we need to evaluate the second term. We have

$$\frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_k} = \sum_l \left[\frac{\partial^2 \vec{r}_i}{\partial q_l \partial q_k}\dot{q}_l + \frac{\partial^2 \vec{r}_i}{\partial \dot{q}_l \partial q_k}\ddot{q}_l\right] + \frac{\partial^2 \vec{r}_i}{\partial t \partial q_k}. \tag{1.8}$$

When we exchange the order of the derivatives in the second term, we see that the second term vanishes because our holonomic constraint assumption that the generalized velocities do not enter the constraints, and thus do not enter the relationship between position and generalized coordinates implies $\partial \vec{r}_i/\partial \dot{q}_k = 0$. In the last term, we can trivially exchange the order of the partial derivatives. We can bring $\partial/\partial q_k$ outside the sum in the first term because $\partial \dot{q}_l/\partial q_k = 0$. Thus, we have

$$\frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_k} = \frac{\partial}{\partial q_k}\left\{\sum_l \left[\frac{\partial \vec{r}_i}{\partial q_l}\dot{q}_l\right] + \frac{\partial \vec{r}_i}{\partial t}\right\} = \frac{\partial \dot{\vec{r}}_i}{\partial q_k}, \tag{1.9}$$

where the last step simply used the chain rule for evaluation of $\dot{\vec{r}}_i = d\vec{r}_i/dt$. Essentially, we have demonstrated that the total time derivative d/dt and the partial derivative $\partial/\partial q_k$ commute when acting on $\vec{r}_i$ for holonomic constraints, which is a nontrivial statement because q_k is time-dependent. We emphasize that it was the assumption of holonomic constraints that allow us discard the second term above. Had that term remained, the dependence on $\ddot{q}_1$ would have made it impossible to bring $\partial/\partial q_k$ outside the sum because $\ddot{q}_l$ in general may depend on q_l (via Newton's second law). So we now have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) &= \sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} + \sum_i \vec{p}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k}, \\ &= \sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} + \frac{\partial T}{\partial q_k},\end{aligned} \tag{1.10}$$

or

$$\sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}. \tag{1.11}$$

Recalling the D'Alembert's principle (equation 1.1), we may re-write the above:

$$\sum_i \vec{F}_i^{(nc)} \cdot \frac{\partial \vec{r}_i}{\partial q_k} = f_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}. \tag{1.12}$$

This is the generalized equation of motion. The left hand side is completely determined by the non-constraint forces and the constraint equations. The right hand side is just derivatives of the

kinetic energy with respect to the generalized coordinates and velocities. Thus, we obtain a differential equation for the motion in the generalized coordinates.

Example 1.1:
For the elliptical wire example, the kinetic energy in terms of position coordinate velocities is

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{z}^2\right). \tag{1.13}$$

We have previously obtained formulae for $\dot{x}$ and $\dot{z}$ in terms of $\dot{\alpha}$:

$$\dot{x} = \dot{a}\cos\alpha - a\sin\alpha\dot{\alpha} \qquad \dot{z} = \dot{b}\sin\alpha + b\cos\alpha\dot{\alpha} \tag{1.14}$$

Let us specialize to the case $\dot{a} = 0$ and $\dot{b} = 0$; so that

$$\dot{x} = -a\sin\alpha\dot{\alpha} \qquad \dot{z} = b\cos\alpha\dot{\alpha} \tag{1.15}$$

We use these to rewrite the kinetic energy in terms of $\dot{\alpha}$:

$$T = \frac{m}{2}\left[a^2\dot{\alpha}^2\sin^2\alpha + b^2\dot{\alpha}^2\cos^2\alpha\right] \tag{1.16}$$

This is an important example of how to convert T from a function of the position velocities to a function of the generalized coordinates and velocities. Now take the prescribed derivatives:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\alpha}}\right) - \frac{\partial T}{\partial\alpha} &= \frac{d}{dt}\left[m\dot{\alpha}\left(a^2\sin^2\alpha + b^2\cos^2\alpha\right)\right] - 2m\dot{\alpha}^2\left(a^2 - b^2\right)\sin\alpha\cos\alpha \\ &= m\ddot{\alpha}\left(a^2\sin^2\alpha + b^2\cos^2\alpha\right)\end{aligned} \tag{1.17}$$

In taking the total derivative in the first term, we obtain two terms: the one displayed in the last line above, and one that exactly cancels the last term in the first line. We have the generalized force f_α from equation 3.16, $f_\alpha = -mg\cos\alpha$, so the generalized equation of motion is

$$f_\alpha = \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\alpha}}\right) - \frac{\partial T}{\partial\alpha} \tag{1.18}$$

Substituting equation 1.17 into equation 1.18, we have

$$-mgb\cos\alpha = m\ddot{\alpha}\left(a^2\sin^2\alpha + b^2\cos^2\alpha\right). \tag{1.19}$$

Solving for $\ddot{\alpha}$, we get

$$\ddot{\alpha} = -g\frac{b\cos\alpha}{a^2\sin^2\alpha + b^2\cos^2\alpha}. \tag{1.20}$$

Specializing to $a = b = r$ (circular wire), this simplifies to

$$\ddot{\alpha} = -g\frac{\cos\alpha}{r}. \tag{1.21}$$

Example 1.1:
For the Atwood's machine example, things are significantly simpler. The kinetic energy is

$$T = \frac{1}{2}\left(m_1\dot{z}_1^2 + m_2 z_2^2\right). \tag{1.22}$$

Re-writing using the generalized coordinate Z gives

$$T = \frac{1}{2}\left(m_1 + m_2\right)\dot{Z}^2. \tag{1.23}$$

The kinetic energy derivatives term is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{Z}}\right) - \frac{\partial T}{\partial Z} = \left(m_1 + m_2\right)\ddot{Z}. \tag{1.24}$$

Using $f_z = (m_2 - m_1)g$ from equation 3.17 above, the generalized equation of motion is

$$\begin{aligned} f_z &= \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{Z}}\right) - \frac{\partial T}{\partial Z} \\ \left(m_2 - m_1\right)g &= \left(m_1 + m_2\right)\ddot{Z} \\ \ddot{Z} &= -\frac{m_1 - m_2}{m_1 + m_2}g. \end{aligned} \tag{1.25}$$

Self Assessment Exercise B

1. Make sure you study and understand the examples given above properly.
2. Given that kinetic energy $T \equiv \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = T(\{q_k\},\{\dot{q}_k\},t)$ derive the generalized equation of motion.

3.3 The Lagrangian and the Lagrange's Equations

For conservative non-constraint forces, we can obtain a slightly more compact form of the generalized equation of motion, known as the Euler-Lagrange equations.

3.3.1 Generalized Conservative Forces

Now let us specialize to non-constraint forces that are conservative; i.e. $\vec{F}_i^{(nc)} = -\vec{\nabla}_i U(\{\vec{r}_j\})$ where $\vec{\nabla}_i$ indicates the gradient with respect to $\vec{r}_i$. Whether the constraint forces are conservative is irrelevant; we will only explicitly need the potential for the non-constraint forces. U is assumed to be a function of the coordinate positions only; there is no explicit dependence on time or on velocities, $\partial U/\partial t = 0$ and $\partial U/\partial \dot{\vec{r}}_i = 0$. Let us use this expression in writing out the generalized force:

$$f_k = \sum_i \vec{F}_i^{(nc)} \cdot \frac{\partial \vec{r}_i}{\partial q_k} = -\sum_i \vec{\nabla}_i U(\{\vec{r}_j\}) \cdot \frac{\partial \vec{r}_i}{\partial q_k} = -\frac{\partial}{\partial q_k} U(\{q_l\},t). \qquad (1.26)$$

We made use of the holonomic constraints to re-write U as a function of the $\{q_l\}$ and possibly *t*, and realize that the previous line is just the partial derivative of U with respect to q_k. Thus, rather than determining the equation of motion by calculating the generalized force from the non-constraint forces and the coordinate transformation relations, we can re-write the potential energy as a function of the generalized coordinates and calculate the generalized force by gradients thereof.

Example 1.2:
For the elliptical wire the potential energy is due to gravity,

$$U(z) = mgz\,. \qquad (1.27)$$

Re-writing in terms of α, gives

$$U(\alpha,t) = mgb(t)\sin\alpha\,. \qquad (1.28)$$

The generalized force is then

$$f_\alpha = \frac{\partial U(\alpha;t)}{\partial \alpha} = -mgb(t)\cos\alpha\,. \qquad (1.29)$$

as obtained in equation 3.16. Note that we may allow b to be a function of time without ruining the conservative nature of the potential energy U becomes a function of t through the definition of the generalized coordinate but, obviously, if it is initially a conservative potential, a transformation of coordinates cannot change that.

Example 1.3:
For the atwood's machine, the potential energy function is

$$U(z_1,z_2) = g(m_1 z_1 + m_2 z_2) \qquad (1.30)$$

Re-writing in terms of Z gives

$$U(Z,t) = g[(m_1 - m_2)Z - m_2 l(t)] \qquad (1.31)$$

The generalized force is

$$f_z = -\frac{\partial U(Z;t)}{\partial Z} = g(m_2 - m_1) \qquad (1.32)$$

as found earlier. Again, l is allowed to be function of time without ruining the conservative nature of the potential energy.

3.3.2 The Euler-Lagrange Equations

An even simpler method exists, we may re-write the generalized equation of motion using the above relationship between generalized force and gradient of the potential energy as

$$\frac{\partial U}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}. \tag{1.33}$$

Define the Lagrangian as

$$L \equiv T - U$$

sincewe have assumed holonomic constraints, we have $\partial U/\partial \dot{q}_k = 0$. This allows the replacement of $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)$ with $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$ giving

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \tag{1.34}$$

This is the Euler-Lagrange equation, there is one for each generalized coordinate q_k.

Self Assessment Exercise C

1. Given $U(z) = mgz$ where $z = b\sin\alpha$ calculate the generalized force.
2. From the generalized equation of motion prove the Euler-Lagrangian equation.

4.0 Conclusion

At the end we have been able to use D'Alembert's principle to obtain a closed set of equations for only the generalized coordinates; they are known as the *Euler-Lagrange equations*. Writing these equations explicitly only requires the knowledge of a scalar valued function: the Lagrangian in terms of the generalized coordinates and the generalized velocities.

5.0 Summary

- D'Alembert's Principle
  D'Alembert's Principle is
  $$\sum\left[\vec{F}_i^{(nc)} - \dot{\vec{p}}_i\right]\cdot \delta\vec{r}_i = 0$$
  where $\delta\vec{r}_i$ is a virtual displacement that is differential and satisfies the constraints. D'Alembert's principle may be re-written in terms of generalized coordinates and forces
  $$f_k = \sum \dot{\vec{p}} \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$
- Generalized equation of motion
  D'Alembert.s principle can be used to prove the generalized equation of motion
  $$f_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k},$$
  where $T = T(\{q_k\},\{\dot{q}_k\},t)$ is the kinetic energy written as a function of the generalized coordinates.
- Euler-Lagrange's Equation

When the non-constraint forces are conservative, they can be written as gradients of a time independent potential, $\vec{F} = -\vec{\nabla}_i U(\{\vec{r}_j\})$. From this, we can prove that the generalized forces can also be written as gradients

$$f_k = -\frac{\partial}{\partial q_k} U((q_l).t).$$

If we then define the Lagrangian as L = T – U,
then, we can prove the Euler-Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0.$$

In some cases, it is possible to write non conservative forces using a potential function. If the non conservative force can be written in terms of function $U(\{q_k\},\{\dot{q}_k\})$ in the following manner;

$$f_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

then one may include this function U as a potential energy in the lagrangian and apply the Euler-Lagrangian equation.
If one has non-conservative forces that cannot be written in the above form, one can still write down a generalized Euler-Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = f_k^{no-L},$$

where f_k^{no-L} encompasses all forces that cannot be included in the Lagrangian.

6.0 Tutor Marked Assignment (TMAs)

1. A pendulum of mass m hung on a rigid rod of length *l* whose (upper) end is fixed. The mass is free to move on a circle of radius *l* on the xz-plane. The angle that the rod makes with the vertical axis is θ and the relation between r and θ is $r = f(\theta) = (l\sin\theta, 0, -l\cos\theta)$ hence calculate (i) the Lagrangian and, (ii) the equation of motion.
2. Suppose now that this pendulum is free to move on the sphere defined by the distance *l* from the anchor point (i.e., a two-dimensional manifold) in which *r* is function of θ and ϕ, and are related by $r = f(\theta,\phi) = l(\sin\theta\cos\phi, \sin\theta\sin\phi, -\cos\theta)$. Calculate the Lagrangian and the corresponding equation of motion.

7.0 Further Reading and Other Resources

Classical Mechanics by H.Goldstein, Narosa Publishing Home, New Delhi.

Classical Dynamics of Particles and Systems by Marion and Thomtron, Third Edition, Horoloma Book Jovanovich College Publisher.

Introduction to Classical Mechanics by R.G.Takawale and P.S.Puranik, Tata Mc-Graw Hill Publishing Company Limited, New Delhi.

UNIT 2 Langrange's Mechanics
CONTENTS

1.0 Introduction

It is not possible to develop the classical mechanical approach to statistical mechanics without having some understanding of the principles of classical mechanics. We will review the basic principles of Newton's Laws of Motion. We will also recast Newton's second law into the forms developed by Lagrange and Hamilton. These notes provide some of the details about the Lagrangian and Hamiltonian formulations of classical mechanics.

2.0 Objectives

After studying this unit, you will be able to:

- express the Lagrangian L in Cartesian coordinates;
- transform L to generalized coordinates;
- give Lagrange's equations in generalized coordinates.
- Illustrate how to use Lagrange's equations in generalized coordinates.
- Apply the approach to the free motion of a particle confined to move on the perimeter of a ring.
- Solve related Problems.

3.0 Main Contents

3.1 Newton's Second Law

We consider N particles moving in three-dimensional space, and we describe the location of each particle using Cartesian coordinates. We let m_i be the mass of particle *i*, and we let x_i, y_i and z_i be respectively the x, y and z-coordinates of particle *i*. For time derivatives of the

coordinates (and all other physical observables), we use the "dot" notation first introduced by Isaac Newton

$$\dot{x}_i = \frac{dx_i}{dt} \quad \text{and} \quad \ddot{x}_i = \frac{d^2 x_i}{dt^2}. \tag{2.1}$$

We let F_{x_i} be the x-component of the force on particle *i.* Then, Newton's Second Law takes the form $F_{x_i} = m\ddot{x}_i$.

For example, if we study the motion of a single particle of mass m moving in one dimension in a harmonic potential with associated force $F_x = -kx$ Newton's second law takes the form $-kx = m\ddot{x}$. Solution of this differential equation for the coordinate x as a function of the time t gives a complete description of the motion of the particle; i.e. at any time t one knows the location and velocity of the particle.

3.1.1 Conservative Systems

In quantum mechanics, we often restrict our attention to a class of physical systems that are called conservative. In a qualitative sense conservative systems are those for which the total energy E is the sum of the kinetic and potential energies. For any isolated system E is conserved *(i.e. dE/dt = 0),* and for conservative systems, the sum of the potential energy and kinetic energy is conserved. Explicitly, we have the following definition:

Definition: A classical mechanical system is conservative if there exists a function $U(x_1, y_1, z_1, x_2, ...z_N)$ called the potential energy such that for any coordinate $x_i (or\ y_i or\ z_i)$, we can write

$$F_{x_i} = -\frac{\partial U}{\partial x_i} \left(or\ F_{y_i} = -\frac{\partial U}{\partial_{y_i}}\ or\ F_{z_i} = -\frac{\partial U}{\partial z_i} \right) \tag{2.2}$$

where F_{x_i} $(or\ F_{y_i}\ or\ F_{z_i})$ is the *x* (or *y* or *z*) component of the force on particle *i.*

As an example, we can consider the one dimensional particle moving in the harmonic well with force $F = -kx$. For such a system, a potential energy exists and is given by $U(x) = \frac{1}{2}kx^2$. By differentiating the potential energy with respect to *x*, the force is obtained. We can then be sure that for a harmonic oscillator the total energy is conserved

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2. \tag{2.3}$$

3.1.2 Notation

The notation used for derivatives, we need to distinguish the explicit and implicit dependence of a function on a variable. As an example, the expression for the kinetic energy T of a system of N particles in Cartesian coordinates is given by

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \left[\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right] \tag{2.4}$$

Equation 2.4 above for the kinetic energy is an explicit function of the velocities of each particle; i.e. the $(\dot{x}_i, \dot{y}_i, \dot{z}_i)$, but no other variables. We can write

$$\frac{\partial T}{\partial \dot{x}_j} = m_j \dot{x}_j. \tag{2.5}$$

However, in the expression for T, there is no explicit dependence on the time *t* or the coordinates $\{x_i, y_i, z_i\}$, consequently, we write

$$\frac{\partial T}{\partial t} = 0 \quad \text{and} \quad \frac{\partial T}{\partial z_j} = 0 \tag{2.6}$$

Equations 2.6 do not imply that the kinetic energy is independent of the time or the z-component of the coordinate for particle *j*. Equation 2.6 only state that in the expression for T as written,

there is no explicit dependence of T on t or z_j. If we want the actual (i.e. implicit) dependence of the kinetic energy on time, we write

$\frac{dT}{dt}$ which is not zero in the general case ($^{dT}/_{dt}$ does equal zero for a free particle). We understand the notation ∂ to represent a derivative of the explicit dependence of a function as written on a variable and the notation d to represent a derivative for the actual (i.e. implicit) dependence of a function on a variable. It is important that the differences between ∂ and *d* are clear for the developments that follow.

3.2 Lagrange's Equations

We now derive Lagrange's equations for the special case of a conservative system in Cartesian coordinates. However, using Lagrange's equations in a more general coordinate systems will be discussed after the derivation in Cartesian coordinates.

We begin with equation 1.4 for the kinetic energy. We first differentiate the expression with respect to one of the velocities.

$$\frac{\partial T}{\partial \dot{x}_j} = m_j \dot{x}_j .$$

We next take the implicit derivative of equation 1.4 above with respect to time

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_j} = m_j \ddot{x}_j . \tag{2.7}$$

The right hand side of equation 2.7 is the mass of the particle multiplied by the acceleration of the particle, which by Newton's law must be the force on the particle. For a conservative system, We can express this force by $-\partial U/\partial x_j$, so that

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_j} = -\frac{\partial U}{\partial x_j} . \tag{2.8}$$

We now give a defining relation for the classical lagrangian:

Definition: The Classical Lagrangian is given by

$$\mathbf{L = T - U.} \tag{2.9}$$

The classical Lagrangian is the difference between the kinetic and potential energies of the system. Using this definition in equation 1.7 we obtain

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} = 0 . \tag{2.10}$$

Equations 2.10 above are Lagrange's equations in Cartesian coordinates. We use the plural (equations), because Lagrange's equations are a set of equations. We have a separate equation for each coordinate x_j. A completely analogous set of equations is obtained for the other Cartesian directions y and z.

We emphasize that Lagrange's equations are just a new notation for Newton's second law.

Example 2.1: Consider a one-dimensional harmonic oscillator. The Lagrangian for the system is

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{2.11}$$

and Lagrange's equation is $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}m\dot{x} + kx = 0$, (2.12)

or $m\ddot{x} = -kx$

which is just Newton's Second law. To understand the utility of the new notation, we need to introduce the notion of generalized coordinates.

Self Assessment Exercise A

Derive Lagrange's equation of motion in Cartesian coordinates.

3.2.1 Generalized Coordinates

Much of Lagrange's work was concerned with methods useful for systems subject to external constraints.

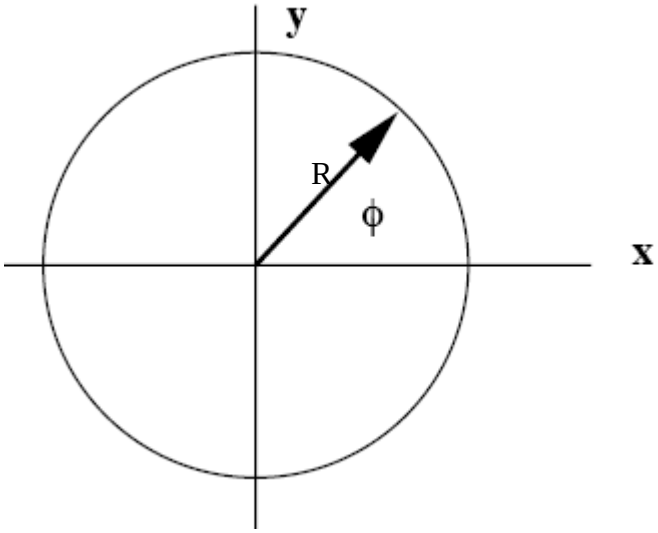


figure 2.1

A simple example of the kind of problem that interested Lagrange is the motion of a free particle of mass m confined to move on the perimeter of a ring of radius R depicted in the figure above. Constraints on a particle's motion arise from some set of unspecified forces.

For the particle on a ring, it is possible to imagine some force of infinite strength that limits the motion of the particle. The exact nature of the force is not important to us. We only need to consider the confined space.

For a particle on a ring, the Cartesian coordinates x and y are not the most convenient to describe the motion of the particle. As a result of the constraint, a single coordinate ϕ is sufficient to locate the particle. The coordinate ϕ is defined to be the angle that a line connecting the current location of the particle with the origin of coordinates makes with the x-axis. The connections between ϕ and the Cartesian coordinates are given by

$$x = R\cos\phi \text{ and } y = R\sin\phi. \qquad (2.13)$$

The angle ϕ is an example of a generalized coordinate. Generalized coordinates are any set of coordinates that are used to describe the motion of a physical system. Cartesian coordinates and spherical polar coordinates are other examples of generalized coordinates.

We may choose any convenient set of generalized coordinates for a particular problem. For the particle in a ring example, the convenient coordinate is ϕ. For systems with spherically symmetric potentials (the motion of the earth about the sun, the hydrogen atom), we can choose spherical polar coordinates. We label the *i'th* generalized coordinates with the symbol q_i, and we let $\dot{q}_i$ represent the time derivative of q_i.

Self Assessment Exercise B

1. Derive Lagrange's equations of motions for a conservative system in Cartesian coordinates.
2. Choose any convenient set of generalized coordinates for the following systems: (i) free particle of mass m confined to move on the perimeter of a ring of radius R depicted in the figure 2.1 above. (ii) the motion of a particle on a ring, (iii) for systems with spherically symmetric potentials (the motion of the earth about the sun, the hydrogen atom).

3.2.1 Lagrange's Equations in Generalized Coordinates

Lagrange has shown that the form of Lagrange's equations is invariant to the particular set of generalized coordinates chosen. For any set of generalized coordinates, Lagrange's equations take the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \tag{2.14}$$

exactly the same form that we derived in Cartesian coordinates. We now illustrate how to use Lagrange's equations in generalized coordinates by applying the approach to the free motion of a particle confined to move on the perimeter of a ring as discussed previously.

The meaning of the expression of "free particle" is the absence of any external forces. We can arbitrarily set the potential energy U to zero. Then, in Cartesian coordinates, the Lagrangian for any free particle in the xy-plane can be expressed as

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2). \tag{2.15}$$

We now transform L to generalized coordinates using equation 2.13. We need the time derivatives of *x* and *y* expressed in terms of the generalized coordinate system.

$$\dot{x} = -R\sin\phi\dot{\phi} + \dot{R}\cos\phi \quad \text{and} \quad \dot{y} = R\cos\phi\dot{\phi} + \dot{R}\sin\phi. \tag{2.16}$$

Owing to the constraint, R is a constant and $\dot{R} = 0$, then

$$\dot{x} = -R\sin\phi\dot{\phi} \quad \text{and} \quad \dot{y} = R\cos\phi\dot{\phi} \tag{2.17}$$

so that L becomes $$L = \frac{1}{2}mR^2\dot{\phi}^2\left[\cos^2\phi + \sin^2\phi\right] = \frac{1}{2}mR^2\dot{\phi}^2. \tag{2.18}$$

Because of the constraint, the Lagrangian is a function of a single coordinate ϕ. We finally give Lagrange's equations

$$\frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi}, \text{ and} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = mR^2\ddot{\phi} \text{ and} \qquad \frac{\partial L}{\partial \phi} = 0. \tag{2.19}$$

So therefore,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt}mR^2\dot{\phi} = mR^2\ddot{\phi} = 0 \tag{2.20}$$

Equation 2.20 above implies the acceleration of the coordinates ϕ is zero so that the particle moves with a constant generalized velocity $\dot{\phi}$

Self Assessment Exercise C

1. Express Lagrange's equation in generalized coordinates.
2. Derive the Lagrange's equation for a free motion of a particle confined to move on the perimeter of a ring in generalized coordinates.

3.3 Generalized Momenta

Equation 2.20 can be interpreted to mean that the quantity $\partial L/\partial\dot{\phi} = mR^2\dot{\phi}$ is conserved. To fully explore the meaning of the conservation of a quantity like $\partial L/\partial\dot{\phi}$, consider the Lagrangian in Cartesian coordinates for a particle of mass in one dimension

$$L = \frac{1}{2}m\dot{x}^2 - U(x). \tag{2.21}$$

By differentiating L with respect to the velocity $\dot{x}$, we obtain the linear momentum

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \tag{2.22}$$

which is conserved in the case of no external forces; i.e. the linear momentum is conserved if *U(x)* is a constant. Using this simple equation, we are lead to the following definition:

Definition: The generalized momentum p_i conjugate to the coordinate q_i is defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{2.23}$$

for example in the particle in a ring, the generalized momentum conjugate ϕ, $\partial L/\partial\dot{\phi} = p_\phi$ can be shown to be angular momentum of the particle.

Self Assessment Exercise D

1. Define the generalized momentum in Cartesian and generalized coordinates

3.4 Lagrangian for Some Physical Systems

3.4.1 Example 1: 1-D motion—the pendulum

One of the simplest nonlinear systems is the one-dimensional physical pendulum (so called to distinguish it from the linearized harmonic oscillator approximation). As depicted in Figure 2.2 below, the pendulum consists of a light rigid rod of length l, making an angle θ with the vertical, swinging from a fixed pivot at one end and with a bob of mass m attached at the other end.

The constraint *l* = const and the assumption of plane motion reduces the system to one degree of freedom, described by the generalized coordinate θ. (This system is also called the simple pendulum to distinguish it from the spherical pendulum and compound pendula, which have more than one degree of freedom.)

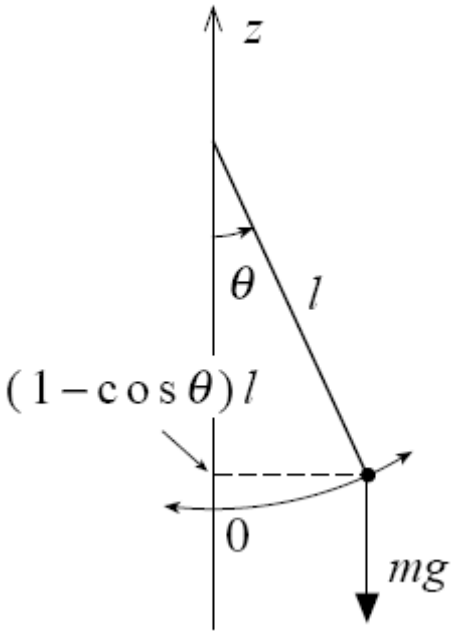


Figure 2.2 Physical Pendulums

The potential energy with respect to the equilibrium position $\theta = 0$ is $U(\theta) = mgl(1-\cos\theta)$, where g is the acceleration due to gravity, and the velocity of the bob is $v_\theta = l\dot{\theta}$, So that the kinetic energy $T = \frac{1}{2}mv_\theta^{\,2} = \frac{1}{2}ml^2\dot{\theta}^2$.

The lagrangian, L= T – U, is thus

$$L(\theta,\dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta). \qquad (2.24)$$

This is also essentially the Lagrangian for a particle moving in a sinusoidal spatial potential, so the physical pendulum provides a paradigm for problems such as the motion of an electron in a crystal lattice or of an ion or electron in a plasma wave.

Lagrangian equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \quad \text{where} \quad \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \quad \text{and} \quad \frac{\partial L}{\partial \theta} = -mgl\sin\theta. \qquad (2.25)$$

So therefore the Lagrangian equation of motion is

$$ml^2\ddot{\theta} = -mgl\sin\theta. \qquad (2.26)$$

3.4.2 Example 2: 2-D motion in a central potential

Using plane polar coordinates, $q = \{r, \theta\}$ such that

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{2.27}$$

so that

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \qquad \dot{y} = \dot{r}\sin\theta - r\dot{\theta}\cos\theta \tag{2.28}$$

hence the kinetic energy $T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$ is found to be $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$. (2.29)

The lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \tag{2.30}$$

Since L is independent of θ then $\partial L/\partial\theta = 0$ while $\partial L/\partial\dot{\theta} = mr^2\dot{\theta}$

Therefore the θ-component of the Lagrangian equation of motion is

$mr^2\ddot{\theta} = 0$ while the r-component of the Lagrangian equation of motion will be $\partial L/\partial\dot{r} = m\dot{r}$ while $\partial L/\partial r = mr\dot{\theta}^2 - U(r)$

so the radial Lagrange's equation of motion is

$\frac{d}{dt}(m\dot{r}) = mr\dot{\theta}^2 - U(r)$ which gives

$$m\ddot{r} - mr\dot{\theta}^2 = -U(r)\ . \tag{2.31}$$

3.4.3 Example 3: 2-D motion with time-varying constraint

Consider a weight rotating about the origin on a frictionless horizontal surface as depicted in the figure 2.3 below and constrained by a thread, initially of length a, that is being pulled steadily downward at speed u through a hole at the origin so that the radius $r = a - ut$.

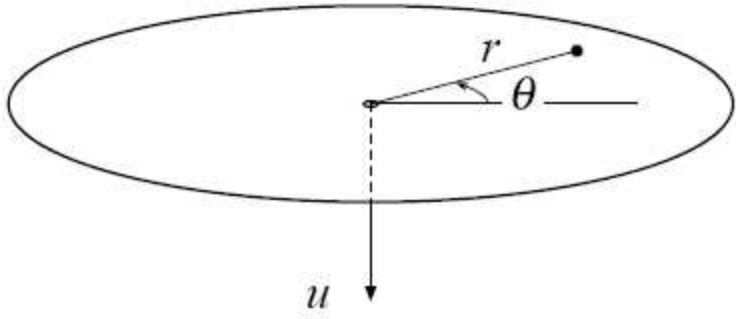


figure 2.3

Then, on substituting for r in equation 2.29, the Lagrangian becomes,

$$L = T = \frac{1}{2}m\left[u^2 + (a-ut)^2\dot{\theta}^2\right]. \tag{2.32}$$

So we again have the conservation of angular momentum

$$m(a-ut)^2\dot{\theta} = l = cons\tan t$$

which can be integrated to give θ as a function of t,

$$\theta = \theta_0 + (l/mu)[1/(a-ut) - 1/a] = \theta_0 + lt/[ma(a-ut)]\,. \tag{2.33}$$

3.4.4 Example 4: Atwood Machine

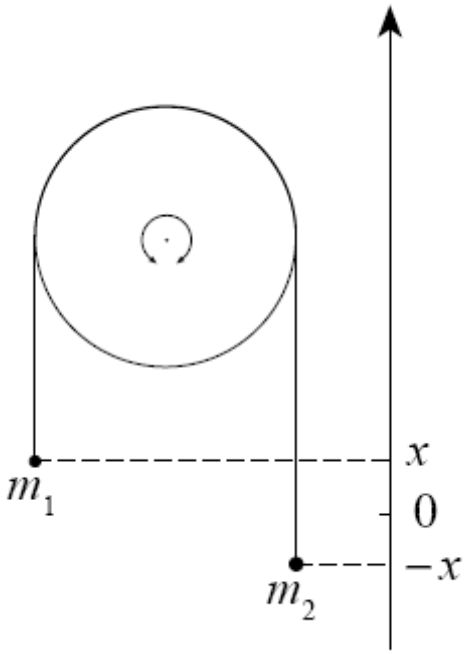


Figure 2.4

Consider two weights of mass m_1 and m_2 suspended from a frictionless, inertialess pulley of radius a by a rope of fixed length, as depicted in Figure 2.4 above. The height of weight 1 is x with respect to the chosen origin and the holonomic constraint provided by the rope allows us to express the height of weight 2 as -*x*, so that there is only one degree of freedom for this system.

The kinetic and potential energy are $T = \frac{1}{2}(m_1 + m_2)\dot{x}^2$ and $U = m_1 g x - m_2 g x$. (2.34)

Thus, $L = T - U$

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 - (m_1 - m_2)gx \tag{2.35}$$

and its derivatives are

$$\frac{\partial L}{\partial x} = -(m_1 - m_2)g \quad \frac{\partial L}{\partial \dot{x}} = -(m_1 + m_2)\dot{x}. \tag{2.36}$$

The Lagrange's equation of motion now becomes

$$(m_1 + m_2)\ddot{x} = -(m_1 - m_2)g$$

$$\Rightarrow \ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g. \tag{2.37}$$

3.4.5 Example 5: Elliptical wire

For the elliptical wire of figure (1.1b), taking *a* and *b* to be constant, the Lagragian is

$$L = T - U = \frac{m}{2}\left[a^2\dot{\alpha}^2 \sin^2\alpha + b^2\dot{\alpha}^2\cos^2\alpha\right] - mgb\sin\alpha. \tag{2.38}$$

The Lagrange equations of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right)$$

$$= \frac{d}{dt}\left[m\dot{\alpha}\left(a^2\sin^2\alpha + b^2\cos^2\alpha\right)\right] \tag{2.39}$$

$$= m\ddot{\alpha}\left(a^2\sin^2\alpha + b^2\cos^2\alpha\right) + 2m\dot{\alpha}^2\left(a^2 - b^2\right)\sin\alpha\cos\alpha$$

$$\frac{\partial L}{\partial \alpha} = \frac{\partial T}{\partial \alpha} - \frac{\partial U}{\partial \alpha} = 2m\dot{\alpha}^2\left(a^2 - b^2\right)\sin\alpha\cos\alpha - mgb\cos\alpha. \tag{2.40}$$

Thus

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) - \frac{\partial L}{\partial \alpha} = 0 \Rightarrow m\ddot{\alpha}\left(a^2\sin^2\alpha + b^2\cos^2\alpha\right) + mgb\cos\alpha = 0\,. \tag{2.41}$$

3.4.6 Example 6: Particle in e.m. field

The fact that Lagrange's equations are the Euler–Lagrange equations for the extraordinarily simple and general Hamilton's principle suggests that Lagrange's equations of motion may have a wider range of validity than simply problems where the force is derivable from a scalar potential.

In particular, it is obviously of great physical importance to find a Lagrangian for which Lagrange's equations of motion equation 2.14 reproduce the equation of motion of a charged particle in an electromagnetic field, under the influence of the Lorentz force,

$$m\ddot{r} = e\mathbf{E}(r,t) + e\dot{r}\times\mathbf{B}(r,t), \tag{2.42}$$

where *e* is the charge on the particle of mass *m*.

We assume the electric and magnetic fields **E** and **B**, respectively, to be given in terms of the scalar potential Φ and vector potential **A** by the standard relations

$$\begin{aligned}\mathbf{E} &= -\nabla\Phi - \partial_t\mathbf{A},\\ \mathbf{B} &= \nabla\times\mathbf{A}.\end{aligned} \tag{2.43}$$

The electrostatic potential energy is eΦ, so we expect part of the Lagrangian to be $\frac{1}{2}m\dot{r}^2 - e\Phi$, but how do we include the vector potential?

Clearly, we need to form a scalar since L is a scalar, so we need to dot **A** with one of the naturally occurring vectors in the problem to create a scalar.

The three vectors available are **A** itself, r and $\dot{r}$. However we do not wish to use **A**, since **A**·**A** in the Lagrangian would give an equation of motion that is nonlinear in the electromagnetic field, contrary to equation 2.42. Thus, we can only use *r* and $\dot{r}$. Comparing equations 2.42 and 2.43 we see that $\dot{r}\cdot\mathbf{A}$ has the same dimensions as Φ, so let us try adding that to form the total Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - e\Phi + e\dot{r}\cdot\mathbf{A}\ . \tag{2.44}$$

Taking $q \equiv \{q_1, q_2, q_3\} = \{x, y, z\}$, we have

$$\frac{\partial L}{\partial q_i} = -e\frac{\partial\Phi}{\partial q_i} + \sum_{j=1}^{3}\dot{q}_j\frac{\partial \mathbf{A}_j}{\partial q_i}$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = m\ddot{q}_i + e\frac{d\mathbf{A}}{dt} = m\ddot{q}_i + e\left[\frac{\partial\mathbf{A}}{\partial t} + \sum_{j=1}^{3}\dot{q}_j\frac{\partial\mathbf{A}_i}{\partial q_j}\right].$$

The lagrange's equation of motion then becomes

$$m\ddot{q} = e\left[-\frac{\partial\Phi}{\partial q_i} - \frac{\partial\mathbf{A}_i}{\partial t}\right] + e\sum_{j=1}^{e}\dot{q}_j\left[\frac{\partial\mathbf{A}_j}{\partial q_i} - \frac{\partial\mathbf{A}_i}{\partial q_j}\right]. \tag{2.45}$$

This is simply equation 2.42 in Cartesian component form, so our guessed Lagrangian is indeed correct.

Self Assessment Exercise E

1. Make sure you study and understand the above examples very well.

3.5 Transformations of the Lagrangian

3.5.1 Point transformations

Given an arbitrary Lagrangian $L(q,\dot{q},t)$ in one generalized coordinate system, $q \equiv \{q_i | i = 1,n\}$(e.g. a Cartesian frame), we often want to know the Lagrangian $L'(Q,\dot{Q},t)$ in another coordinate system, $Q \equiv \{Q_i | i = 1,n\}$ (e.g. polar coordinates). Thus, suppose there exists a set $g \equiv \{g_i | i = 1,n\}$ of twice differentiable functions g_i such that $q_i = g_i(Q,t)$, i = 1,…, n.

We require the inverse function of g also to be twice differentiable, in which case $g : Q \to q$ is said to be C^2 diffeomorphism. Note that we have allowed the transformation to be time dependent, so transformations to a moving frame are allowed. The transformation g maps a path in Q-space to a path in q-space. However, it is physically the same path; all we have changed is its representation. What we need in order to discuss how the Lagrangian transforms is a coordinate-free formulation of Lagrangian dynamics. This is another virtue of Hamilton's principle i.e. $\partial S = 0$ on a physical path for all variations with fixed endpoints and also since the action integral S given by $S \equiv \int_{t_1}^{t_2} dt\, L(\dot{q},q,t)$ is an integral over time only. Thus, if we can define L' so that

$$L'(Q(t),\dot{Q}(t),t) = L(q(t),\dot{q}(t),t) \tag{2.46}$$

for any path, then S is automatically invariant under the coordinate change and will be stationary for the same physical paths, irrespective of what coordinates they are represented in.

We can guarantee this trivially, simply by choosing the new Lagrangian to be the old one in the new coordinates:

$$L'(Q,\dot{Q},t) = L(g(Q,t),\dot{g}(Q,\dot{Q},t),t), \tag{2.47}$$

where $$\dot{g}_i \equiv \frac{\partial g_i}{\partial t} + \sum_{i=1}^{n} \dot{Q}_j \frac{\partial g_i}{\partial Q_j}, \qquad i = 1, \ldots, n.$$

One can prove that eq. (2.47) gives the correct dynamics by calculating the transformation of the Euler–Lagrange equations explicitly and showing that eq. 2.14 implies

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{Q}_i}\right) - \frac{\partial L'}{\partial Q_i} = 0, \tag{2.48}$$

but clearly Hamilton's principle provides a much simpler and more elegant way of arriving at the same result, since equation 2.48 are simply the Euler–Lagrange equations for S to be stationary in the new variables.

3.4.7 Gauge transformations

It must be noted that we cannot always expect that the Lagrangian is of the form T − V, but nevertheless the lagrangian gives a function for which Lagrange's equations represent the correct dynamical equations of motion. Thus it is the requirement that the equations of motion be in the form of Lagrange's equations (or, equivalently, that Hamilton's principle apply) that is fundamental, rather than the specific form of L. This naturally raises the question: for a given system, is there only one Lagrangian giving the correct equations of motion, or are there many?

Clearly there is a trivial way to generate multiple physically equivalent Lagrangians, and that is to multiply L by a constant factor. However, we usually normalize L in a natural way, e.g. by requiring that the part linear in the mass be equal to the kinetic energy, so this freedom is not encountered much in practice.

However, there is a more important source of non-uniqueness, known as a gauge transformation of the Lagrangian in which L is replaced by L', defined by

$$L'(q,\dot{q},t) = L(q,\dot{q},t) + \frac{\partial M}{\partial t} + \dot{q} \cdot \frac{\partial M}{\partial q}. \tag{2.49}$$

Example 2.2: Harmonic oscillator
Consider the harmonic oscillator Lagrangian, for a particle of unit mass

$$L = \frac{1}{2}\left(\dot{x}^2 - \omega_0^2 x^2\right) \tag{2.50}$$

where we take ω_0 to be constant. Since $\frac{\partial L}{\partial \dot{x}} = \dot{x}$ and $\frac{\partial L}{\partial x} = -\omega_0^2 x$, we immediately verify that Lagrange's equation, equation 2.14 gives the harmonic oscillator equation

$$\ddot{x} = -\omega_0^2 x \tag{2.51}$$

Now add a gauge term, taking $M = \frac{1}{2}\omega_0^2 x$. Then, from equation 2.49, the new Lagrangian is

$$L' = \frac{1}{2}\left(\dot{x}^2 + 2\omega x\dot{x} - \omega_0^2 x^2\right). \tag{2.52}$$

Calculating $\frac{\partial L'}{\partial \dot{x}} = \dot{x} + \omega_0 x,$ and $\frac{\partial L'}{\partial x} = -\omega_0^2 x + \omega_0 \dot{x},$ and substituting into the Lagragian equation of motion, we do indeed recover the harmonic oscillator, equation, equation 2.51 and the new Lagrangian is a perfectly valid one despite the fact that it is no longer in the natural form,T – U

Example 2.3: Electromagnetic gauge transformation
It is well known that the scalar and vector potentials in equation 2.43 are not unique, since the electric and magnetic fields are left unchanged by the gauge transformation

$$\begin{aligned} \mathbf{A}(r,t) \to (r,t) &\equiv \mathbf{A}'(r,t) + \nabla\chi(r,t) \\ \Phi(r,t) \to \Phi'(r,t) &\equiv \Phi(r,t) - \partial_t \chi(r,t), \end{aligned} \tag{2.53}$$

where χ is an arbitrary scalar function (a gauge potential).
Using the gauge-transformed potentials Φ' and $\mathbf{A}'$ to define new Lagrangian L' in the same way as L was defined by equation 2.44 we have

$$L' = \frac{1}{2}m\dot{x}^2 - e\Phi' + e\dot{r}\cdot A'. \tag{2.54}$$

Substituting equation 2.53 in equation 2.54 we find

$$L' = L + \frac{\partial(e\chi)}{\partial t} + \dot{r}\cdot\nabla(e\chi) \tag{2.55}$$

which is exactly of the form equation 2.49 with $M = e\chi$. Thus electromagnetic and Lagrangian gauge transformations are closely related.

Self Assessment Exercise F
1. State what is meant by point and gauge transformation.
2. Find the gauge transformation of Lagrangian of harmonic oscillator and electromagnetic fields.

4.0 Conclusion

The Classical Lagrangian is given by

$$\mathbf{L = T - U.} \tag{2.9}$$

The classical Lagrangian is the difference between the kinetic and potential energies of the system. Lagrange has shown that the form of Lagrange's equations is invariant to the particular set of generalized coordinates chosen. For any set of generalized coordinates, Lagrange's equations take the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

5.0 Summary

- Using the "dot" notation introduced by Isaac Newton

$$\dot{x}_i = \frac{d\dot{x}_i}{dt} \quad \text{and} \quad \ddot{x}_i = \frac{d^2 x_i}{dt^2}$$

- For motion of a single particle of mass m moving in one dimension in a harmonic potential with associated force $F_x = -kx$, Newton's second law takes the form $-kx = m\ddot{x}$
- A classical mechanical system is conservative if there exists a function $U(x_1, y_1, z_1, x_2, ...z_N)$ called the potential energy such that for any coordinate x_i *or* $(y_i$ *or* $z_i)$ we can write

$$F_{x_i} = -\frac{\partial U}{\partial x_i}\left(or\ F_{y_i} = -\frac{\partial U}{\partial_{y_i}}\ or\ F_{z_i} = -\frac{\partial U}{\partial z_i}\right) \quad \text{where} \quad F_{x_i}\ (or\ F_{y_i}\ or\ F_{z_i}) \quad \text{is}$$

the *x* (or *y* or *z*) component of the force on particle *i.*

- The Classical Lagrangian is given by

L = T – U

- The classical Lagrangian is the difference between the kinetic and potential energies of the system. Using equation 1.7 we obtain

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} = 0\,.$$

- For any set of generalized coordinates, Lagrange's equations take the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0,\,.$$

- The generalized momentum p_i conjugate to the coordinate q_i is defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}\,.$$

- Given an arbitrary Lagrangian $L(q, \dot{q}, t)$ in one generalized coordinate system, $q \equiv \{q_i | i = 1, n\}$(e.g. a Cartesian frame), we often want to know the Lagrangian $L'(Q, \dot{Q}, t)$ in another coordinate system, $Q \equiv \{Q_i | i = 1, n\}$ (e.g. polar coordinates).

  However, there is a more important source of non-uniqueness, known as a gauge transformation of the Lagrangian in which L is replaced by L', defined by

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{\partial M}{\partial t} + \dot{q} \cdot \frac{\partial M}{\partial q}\,.$$

6.0 Tutor Marked Assignments (TMAs)

1. As a model of the motion of a fluid element or dust particle in a planetary (e.g. Earth's) atmosphere, consider the motion of particle of unit mass constrained to move on the surface of a perfectly smooth sphere of radius R rotating with angular velocity ω about the z-axis. Suppose the force on the particle is given by an effective potential $U(\theta, \phi)$ *,)*, where θ and ϕ are the latitude and longitude respectively.

   (a) Write down the Lagrangian in a frame rotating with the planet, taking the generalized coordinates to be the latitude and longitude so that $z = R\sin\theta$, $x = R\cos\theta\cos(\phi + \omega t)$, $y = R\cos\theta\sin(\phi + \omega t)$, where *x, y, z* is a non-rotating Cartesian frame. (b) Write down the equations of motion, and find a first integral (i.e. constant of the motion) in the case where U is independent of longitude. Among this class of potentials find the special case $U = U(\theta)$ required to make equilibrium possible (i.e. so that the equations of motion admit the solution $\dot{\theta} = \dot{\phi} = 0$ at each latitude).

Answers

$\dot{x} = -R\dot{\theta}\sin\theta\cos(\phi+\omega t) - R(\omega+\dot{\phi})\cos\theta\sin(\phi+\omega t)$

$y = -R\dot{\theta}\sin\theta\sin(\phi+\omega t) + R(\omega+\dot{\phi})\cos\theta\cos(\phi+\omega t)$

$\dot{z} = R\dot{\theta}\cos\theta$

Thus the kinetic energy is given by

$$T = \frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$$

$$= \frac{1}{2}R^2\left[\dot{\theta}^2\sin^2\theta\left(\cos^2(\phi+\omega t)+\sin^2(\phi+\omega t)\right)+(\omega+\dot{\phi})^2\cos^2\theta\left(\sin^2(\phi+\omega t)+\cos^2(\phi+\omega t)\right)+\dot{\theta}^2\cos^2\theta\right]$$

$$= \frac{1}{2}R^2\left[\dot{\theta}^2+(\omega+\dot{\phi})^2\cos^2\theta\right]$$

and the Lagrangian, L = T – U, by

$L = \frac{1}{2}R^2\left[\dot{\theta}^2+(\omega+\dot{\phi})^2\cos^2\theta\right] - U(\theta,\phi)$.

$\frac{\partial L}{\partial\theta} = -R^2(\omega+\dot{\phi})^2\sin\theta\cos\theta - \frac{\partial U}{\partial\theta}$, and $\frac{\partial L}{\partial\phi} = -\frac{\partial U}{\partial\phi}$

Thus $\frac{\partial L}{\partial\dot{\theta}} = R^2\dot{\theta}$ and $\frac{\partial L}{\partial\dot{\phi}} = R^2(\omega+\dot{\phi})\cos^2\theta$

and thus the lagrange equations of motion $\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\theta}$ and $\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = \frac{\partial L}{\partial\phi}$ become

$R^2\ddot{\theta} = -R^2(\omega+\dot{\phi})^2\sin\theta\cos\theta - \frac{\partial U}{\partial\theta}$ and

$R^2\ddot{\phi}\cos^2\theta - 2R^2\dot{\theta}(\omega+\dot{\phi})\sin\theta\cos\theta = -\frac{\partial U}{\partial\phi}$.

In the case where $U = U(\theta)$, ϕ is an ignorable coordinate and thus $\frac{\partial L}{\partial\dot{\phi}}$ given by $\frac{\partial L}{\partial\dot{\phi}} = R^2(\omega+\dot{\phi})\cos^2\theta$ is an integral of the motion.

If U_0 be such that $\dot{\theta} = \dot{\phi} = 0$ for all θ and ϕ, and use the identity $\sin 2\theta \equiv 2\sin\theta\cos\theta$, then, $\frac{\partial U_0}{\partial\theta} = \frac{1}{2}R^2\omega^2\sin 2\theta$

Or integrating, we have

$U_0(\theta) = \frac{1}{4}R^2\omega^2\cos 2\theta + \cos 2\theta + const = \frac{1}{2}R^2\omega^2\cos^2\theta$

where the second equality follows fom the identity $\cos 2\theta \equiv 2\cos^2\theta - 1$ and the choice $const = \frac{1}{4}R^2$.

2. Consider a charged particle constrained to move on a non-rotating smooth insulating sphere, immersed in a uniform magnetic field $\boldsymbol{B} = B\boldsymbol{e}_z$, on which the electrostatic potential is a function of latitude and longitude. Write down the Lagrangian in the same generalized coordinates as above and show that it is the same as for the particle on the rotating planet with appropriate identifications of ω and U.

The Lagrangian is $L = T + e\dot{\mathbf{r}} \cdot \mathbf{A} - e\Phi$, with ω set o zero. We have now need a vector potential **A** such that $\mathrm{B}_z \equiv \partial_x \mathrm{A}_y - \partial_y \mathrm{A}_x = \mathrm{B}$. A suitable choice is $\mathrm{A}_x = -\frac{1}{2}\mathrm{B}y,\ \mathrm{A}_y = -\frac{1}{2}\mathrm{B}x,\ \mathrm{A}_z = 0$.

Thus, $e\dot{\mathbf{r}} \cdot \mathbf{A} = \frac{1}{2}e\mathbf{B}(x\dot{y} - y\dot{x})$

$$e\dot{\mathbf{r}} \cdot A = \frac{1}{2}e\mathrm{B}R^2\cos\theta\left[\cos\phi\left(-\dot{\theta}\sin\theta\sin\phi + \dot{\phi}\cos\theta\cos\phi\right) + \sin\phi\left(\dot{\theta}\sin\theta\cos\phi + \dot{\phi}\cos\theta\sin\phi\right)\right]$$

$$= \frac{1}{2}e\mathrm{B}R^2\dot{\phi}\cos^2\theta$$

Thus, the Lagrangian is

$$L = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \dot{\phi}^2\cos^2\theta\right) + \frac{1}{2}e\mathrm{B}R^2\dot{\phi}\cos^2\theta - e\Phi(\theta,\phi).$$

Completing the square we can write the Lagrangian in the form

$$L = \frac{1}{2}mR^2\left[\dot{\theta}^2 + \left(\dot{\phi} + \frac{e\mathrm{B}}{2m}\right)^2\cos^2\theta\right] - \frac{e^2\mathrm{B}^2R^2}{8m}\cos^2\theta - e\Phi(\theta,\phi).$$

This is the same as the Lagrangian for particle on the rotating planet where

$\omega = \frac{e\mathrm{B}}{2m}$ and $U(\theta,\phi) = \frac{e^2\mathrm{B}^2R^2}{8m}\cos^2\theta + e\phi(\theta,\phi)$ Note that the equation above can also be written as $\omega = \frac{1}{2}\omega_0$, where $\omega_0 = \frac{e\mathrm{B}}{m}$ is the cyclotron frequency.

3. Write down a Lagrangian for the problem of two particles of mass m_1 and m_2 connected by a light rigid rod of length l in a gravitational field g. Take the generalized coordintes of the system to be $q = \{x, y, z, \theta, \phi\}$, with the coordinates of two particles being given by

$$x_1 = x - \alpha_1 l\sin\theta\cos\phi,$$
$$y_1 = y - \alpha_1 l\sin\theta\sin\phi,$$
$$z_1 = z - \alpha_1 l\cos\theta,$$
$$x_2 = x + \alpha_2 l\sin\theta\cos\phi,$$
$$y_2 = y - \alpha_2 l\sin\theta\sin\phi,$$
$$z_2 = z + \alpha_2 l\cos\theta,$$

where $\alpha_1 \equiv \frac{m_2}{m_1 + m_2}$ and $\alpha_2 \equiv \frac{m_1}{m_1 + m_2}$ (so that (x, y, z) is the centre of mass).

Answer

$$\dot{x}_1 = \dot{x} + \alpha_1 l\dot{\phi}\sin\theta\sin\phi - \alpha_1 l\dot{\theta}\cos\theta\cos\phi$$
$$\dot{x}_2 = \dot{x} + \alpha_2 l\dot{\phi}\sin\theta\sin\phi - \alpha_2 l\dot{\theta}\cos\theta\cos\phi.$$

Because $m_1\alpha_1 = m_2\alpha_2$ the cross terms cancel when we expand the x-contribution to the kinetic energy.

$$\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 = \frac{m_1 + m_2}{2}\dot{x}^2 + \frac{m_1m_2l^2}{2(m_1 + m_2)}\left(\dot{\phi}\sin\theta\sin\phi - \dot{\theta}\cos\theta\cos\phi\right)^2$$

Similarly

$$\frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 = \frac{m_1 + m_2}{2}\dot{y}^2 + \frac{m_1m_2l^2}{2(m_1 + m_2)}\left(\dot{\phi}\sin\theta\cos\phi - \dot{\theta}\cos\theta\sin\phi\right)^2$$

$$\frac{1}{2}m_1\dot{z}_1^2 + \frac{1}{2}m_2\dot{z}_2^2 = \frac{m_1 + m_2}{2}\dot{z}^2 + \frac{m_1m_2l^2}{2(m_1 + m_2)}\dot{\theta}^2\sin^2\theta.$$

Thus, adding the kinetic energy, we have

$$T = \frac{m_1 + m_2}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{m_1 m_2}{2(m_1 + m_2)}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)$$

The potential energy is given by

$$U = (m_1 z_1 + m_2 z_2)g = (m_1 + m_2)zg$$

and the Lagrangian is L = T – U

$$L = \frac{m_1 + m_2}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{m_1 m_2}{2(m_1 + m_2)}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - (m_1 + m_2)zg\ .$$

7.0 Further Reading and Other Resources

Classical Mechanics by H.Goldstein, Narosa Publishing Home, New Delhi.

Classical Dynamics of Particles and Systems by Marion and Thomtron, Third Edition, Horoloma Book Jovanovich College Publisher.

Introduction to Classical Mechanics by R.G.Takawale and P.S.Puranik, Tata Mc-Graw Hill Publishing Company Limited, New Delhi.

UNIT 3 HAMILTONIAN MECHANICS
CONTENTS

1.0 Introduction

Hamiltonian mechanics is a reformulation of classical mechanics that was introduced in 1833 by Irish mathematician William Rowan Hamilton. It arose from Lagrangian Mehanics, a previous reformulation of classical mechanics introduced by Joseph Louis Lagrange in 1788, but can be formulated *without* recourse to Lagrangian mechanics using symplectic spaces. [The Hamiltonian method differs from the Lagrangian method, in that, instead of expressing second-order differential constraints on an n-dimensional coordinate space (where n is the number of degrees of freedom of the system)], it expresses first-order constraints on a $2n$-dimensional phase space.
As with Lagrangian mechanics, Hamilton's equations provide a new and equivalent way of looking at classical mechanics. Generally, these equations do not provide a more convenient way of solving a particular problem. Rather, they provide deeper insights into both the general structure of classical mechanics and its connection to quantum mechanics as understood through Hamiltonian mechanics, as well as its connection to other areas of science.

2.0 Objectives

After studying this unit, you will be able to:

- Explain Legendre transform.
- Understand the application of legendre transform in thermodynamics.
- Find the Legendre transform of any function.
- Derive the Hamiltonian from Legendre transform and the corresponding Hamilton's equation of motion.
- Derive the Hamiltonian and Hamilton's equation of motion in spherical polar coordinates.
- Solve related Problems.

3.0 Main Contents

3.1 Legendre Transform

The Legendre transform is a method of changing the dependence of a function of one set of variables to another set of variables.

In mathematics, it is often desirable to express a functional relationship *f(x)* as a different function, whose argument is the derivative of f, rather than x. If we let $p = \frac{df}{dx}$ be the argument of this new function, then this new function is written $f^*(p)$ and is called the Legendre transform of the original function, named after Adrien-Marie Legendre. The Legendre transform f^* of a function f is defined as follows: $f^*(p) = \max_x (px - f(x))$

The notation $\max_x$ indicates the maximization of the expression with respect to the variable x while p is held constant. The Legendre transform is its own inverse. Like the familiar Fourier transform, the Legendre transform takes a function *f(x)* and produces a function of a different variable *p*. However, while the Fourier transform consists of an integration with a kernel, the Legendre transform uses maximization as the transformation procedure. The transform is well behaved only if is a convex function.

The Legendre transformation is an application of the duality relationship between points and lines. The functional relationship specified by *f*(*x*) can be represented equally well as a set of (*x*, *y*) points, or as a set of tangent lines specified by their slope and intercept values.

The Legendre transformation can be generalized to the Legendre-Fenchel transformation. It is commonly used in thermodynamics and in the Hamiltonian formulation of classical mechanics.

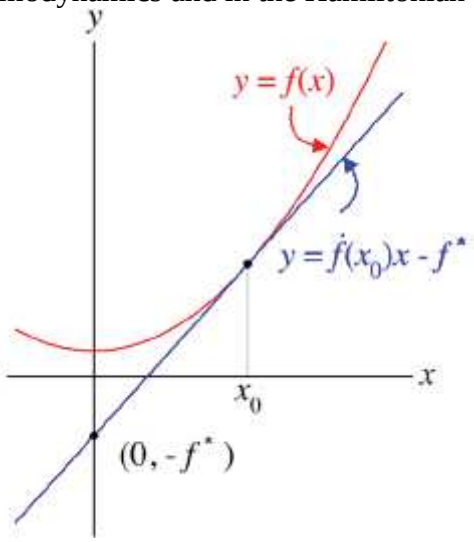


figure 1.1 The Legendre transformation: the cuve graph is f(x). Its tangent with slope P intersects the y-axis at the point *–f*(p).*

The Legendre transform is most often used in the study of thermodynamics. Recall from thermodynamics, there are two free energy functions called the Gibbs free energy and the Helmholtz free energy. The total differential of the Helmholtz free energy is given by

$$dA = -SdT - PdV, \qquad (3.1)$$

where S is the entropy, T is the temperature, V is the volume and p is the pressure. From the total differential, it is evident that the Helmholtz free energy is expressed as a function of the temperature and the volume; i.e. *A = A(T, V)*. Now suppose we prefer to express the state of our system in terms of temperature and pressure rather than temperature and volume. We can define a new function G by

$$G = A + PV \qquad (3.2)$$

where we have added to A the product of the variable we want (p) and the variable we want to eliminate (V). The algebraic sign of the included *PV* product is chosen to be the opposite of the algebraic sign of the *PdV* term in Equation 3.1. Taking the differential of the expression for G we obtain

$$dG = dA + PdV + V\,dP = -SdT - PdV + PdV + V\,dP = -SdT + V\,dP. \qquad (3.3)$$

It is evident that G is a function of T and P as desired. Of course, G is the Gibbs free energy, and G is said to be the Legendre transform of A. As previously mentioned, in Equation 3.2 the

product *PV* is included with an algebraic sign opposite to the sign of *PdV* in Equation 3.1, so that the cancellation of the two *PdV* terms is assured.

Self Assessment Exercise A
Find the Legendre transform of the function e^x.
Show that the Legendre transform satisfies the following algebraic properties.

1. $f(x) = ag(x) \Rightarrow f^*(p) = ag^*(p/a)$
2. $f(x) = g(ax) \Rightarrow f^*(p) = g^*(p/a)$
3. $f(x) = g(x+b) \Rightarrow f^*(p) = g^*(p) - b$
4. $f(x) = g^{-1}(x) \Rightarrow f^*(p) = -pg^*(1/p)$.

where $f^*(p)$ is the Legendre transform

3.2 The Classical Hamiltonian and Hamilton's Equations

We now apply the notion of the Legendre transform to the classical Lagrangian. In our previous developments, we have taken L to be a function of all the generalized coordinates and their respective time derivatives; i.e. $L = L(\{q_i\},\{\dot{q}_i\},t)$. For generality, we have also included the possibility that the Lagrangian has explicit time dependence. Such explicit time dependence can occur when the external forces acting on a system are time-dependent.
The resulting time-dependent potentials can be important in systems, as for example, the study of the interaction of radiation with matter. Light is composed of electric and magnetic fields that oscillate in time, and when light interacts with matter, the electrons are subjected to time-dependent potentials. The quantum treatment of spectroscopy includes time dependent potentials, and we generalize the Lagrangian to admit such time dependences.
We now use the Legendre transform to define a new function where we replace the velocity (the $\dot{q}_i$) dependence by a dependence on the generalized momenta. The transformed function is as follows:
Definition: For a system of particles each having masses m_i described by a set of generalized coordinates q_i, the classical Hamiltonian is defined by

$$H = \sum_i p_i \dot{q}_i - L(\{q_i\},\{\dot{q}_i\},t). \tag{3.4}$$

As we now show, the particular choice of the relative signs of the first and second terms in Equation 3.4 above makes the classical Hamiltonian a natural function of the generalized coordinates and momenta rather than the generalized coordinates and the velocities. The reason that the sum is included with a positive sign and the Lagrangian is included with a negative sign (rather than the opposite) is made clear shortly when we identify the meaning of the classical Hamiltonian.
We now take the total differential of equation 3.4

$$dH = \sum_i \left(p_i d\dot{q}_i + \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right) - \frac{\partial L}{\partial t} dt \tag{3.5}$$

The derivative $\partial L/\partial \dot{q}_i$ is the definition of the generalized momenta p_i. From Lagrange's equation, we can write

$$\frac{d}{dt} p_i - \frac{\partial L}{\partial q_i} = 0 \quad \text{or} \quad \dot{p}_i = \frac{\partial L}{\partial q_i}. \tag{3.6}$$

Then the total differential of the classical Hamiltonian becomes

$$\begin{aligned} dH &= \sum_i (p_i d\dot{q}_i + \dot{q}_i dp_i - \dot{p}_i dq_i - p_i d\dot{q}) - \frac{\partial L}{\partial t} dt \\ &= \sum (\dot{q}_i dp_i - \dot{p}_i dq_i) - \frac{\partial L}{\partial t} dt. \end{aligned} \tag{3.7}$$

The classical Hamiltonian is manifestly a function of the generalized coordinates and momenta rather than the generalized coordinates and velocities from equation 3.7, we have

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i \quad \text{and} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{3.8}$$

Equations 3.8 are called Hamilton's equation of motion.

Consider the motion of a particle of mass in one dimension with the Lagrangian given in equation 2.21. The generalized momentum is given as $p_x = \partial L/\partial \dot{x} = m\dot{x},$ and from the definition of the Hamiltonian, we have

$$H = p_x\dot{x} - L = m\dot{x}\dot{x} - \left(\frac{1}{2}m\dot{x}^2 - U(x)\right) = \frac{1}{2}m\dot{x} + U(x)\ , \tag{3.9}$$

the Hamiltonian is seen to be the sum of the kinetic and potential energies of the system.

For conservative system, H is the total energy. But the equation above is not acceptable because H is not written explicitly as a function of the generalized momentum. We must substitute $\dot{x} = p_x/m$ then equation 3.9 above becomes

$$H = \frac{p_x^2}{2m} + U(x). \tag{3.10}$$

The Hamiltonian is then seen to be an expression for the total energy of a conservative system in terms of the generalized coordinates and momenta. With the Hamiltonian expressed in terms of the proper variables, we can give Hamilton's equations of the system as

$$\frac{\partial H}{\partial p_x} = \frac{p_x}{m} = \dot{x} \qquad \text{and} \quad \frac{\partial H}{\partial x} = \frac{\partial U}{\partial x} = -\dot{p}_x. \tag{3.11}$$

3.2.1 Derivation of Hamiltonian and Hamilton's equation of motion if the Lagrangian is a function of two variables q and z (and possibly time).

Consider the Legendre transformation of the Lagragian with respect to the variables z.

$$L^*(q, p, t) = pz(q, p, t) - L(q, z(q, p, t), t),$$

where $z(q, p, t)$ is the inverse relation to $p = \frac{\partial L}{\partial z}(q, z, t),$ i.e. it is our definition of the conjugate momentum. The Legendre transformation of the Lagrangian is called the Hamiltonian, and it is commonly denoted by the letter H,

$$H(q, p, t) = pz(q, p, t) - L(q, z(q, p, t), t).$$

We now examine the derivatives of H with respect to its arguments. Firstly

$$\frac{\partial H}{\partial q}(q, p, t) = p\frac{\partial z}{\partial q}(q, p, t) - \frac{\partial L}{\partial q}(q, z(q, p, t), t) - \frac{\partial L}{\partial z}(q, z(q, p, t), t)\frac{\partial z}{\partial q}(q, p, t).$$

However, the first and third term cancel, since $p = \frac{\partial L}{\partial z}(q, z, t),$ so that

$$\frac{\partial H}{\partial q}(q, p, t) = -\frac{\partial L}{\partial q}(q, z(q, p, t), t)$$

Secondly

$$\frac{\partial H}{\partial p}(q, p, t) = z(q, p, t) + p\frac{\partial L}{\partial p}(q, z(q, p, t), t)\frac{\partial z}{\partial p}(q, p, t).$$

Also, the second and the third term cancel to become

$$\frac{\partial H}{\partial p}(q, p, t) = z(q, p, t)$$

The Hamilton's equations are

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}(q, p, t), \quad \frac{dp}{dt} = -\frac{\partial H}{\partial p}(q, p, t),$$

this procedure can be extended to vector-valued *q*. Given the Lagrangian $L(q,z,t)$, we define the Hamiltonian as

$$H(q,p,t)=\sum_{i=1}^{n} p_i z_i(q,p,t)-L(q,z(q,p,t),t).$$

Where the transformation between *(q,z,t)* and *(q,p,t)* is

$$p_i=\frac{\partial L}{\partial z_i}(q,z,t).$$

Hamilton's equations are system of 2n first-order equations

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i}(q,p,t),\ \frac{dp}{dt}=-\frac{\partial H}{\partial q_i}(q,p,t)$$

3.2.2 Example 1: Scalar potential

Consider the Lagrangian for a particle in Cartesian coordinates, so $q=\{x, y, z\}$ may be replaced by $r = xe_x + ye_y + ze_z$. Also assume that it moves under the influence of a scalar potential $U(r, t)$ so that the natural form of the Lagrangian is

$$L=T-U=\frac{1}{2}m|\dot{r}|^2-U(r,t) \tag{3.12}$$

General momentum $$p\equiv\frac{\partial L}{\partial \dot{r}}=m\dot{r}, \tag{3.13}$$

so that in this case the canonical momentum is the same as the ordinary kinematics' momentum. Equation.3.8 is solved trivially to give $\dot{q}=u(p)$ where $u(p)=p/m$. Thus, from equation 3.9 we have

$$\begin{aligned} H&=\frac{|p|^2}{m}-\left(\frac{|p|^2}{2m}-U(r,t)\right)=\frac{|p|^2}{2m}+U(r,t),\\ &=T+U. \end{aligned} \tag{3.14}$$

That is, the Hamiltonian is equal to the total energy of the system, kinetic plus potential. The fact that the Hamiltonian is an important physical quantity, whereas the physical meaning of the Lagrangian is more obscure, is one of the appealing features of the Hamiltonian approach. Both the Lagrangian and Hamiltonian have the dimensions of energy and both approaches can be called energy methods. They are characterized by the use of scalar quantities rather than the vectors encountered in the direct use of Newton's second law.

An example is the harmonic oscillator, Hamiltonian corresponding to the Lagrangian equation 3.9 is

$$H=\frac{p^2}{2m}+\frac{m\omega_0 x^2}{2}. \tag{3.15}$$

From equation 3.9 the Hamiltonian equations of motion are

$$\dot{x}=\frac{p}{m},\qquad \dot{p}=-m\omega_0 x. \tag{3.16}$$

Gauge-transformed Harmonic Oscillator.

Now consider the gauge-transformed harmonic oscillator Lagrangian

$$L'=\frac{1}{2}m\left(\dot{x}^2+2\omega_0 x\dot{x}-\omega_0^2 x^2\right) \tag{3.17}$$

The canonical momentum is thus

$$p=\frac{\partial L'}{\partial \dot{x}}=m(\dot{x}+\omega_0 x) \tag{3.18}$$

and we see that the gauge transformation has affected a transformation of the canonical momentum. Even though the generalized coordinate x remains the same.

Solving equation 3.9 for $\dot{x}$, we find $u(p) = (p - m\omega_0 x)/m$. Hence

$$H = p\frac{(p - m\omega_0 x)^2}{m} + \frac{1}{2}m\omega_0^2 x^2 = T + U \tag{3.19}$$

Thus, even though L was not of the natural form T – U in this case, the Hamiltonian remains equal to the total energy, thus confirming that it is a quantity with a more direct physical significance than the Lagrangian. (The functional form of the Hamiltonian changes under the gauge transformation because the meaning of *p* changes).

3.2.3 Example 2: Physical pendulum

A nonlinear one-dimensional case is provided by the physical pendulum was introduced in module 2 The Hamiltonian is

$$H = p_\theta\theta - L \tag{3.20}$$

becomes

$$H(\theta, p_\theta) = \frac{p_\theta^2}{2ml^2} + mgl(1 - \cos\theta) \tag{3.21}$$

which again is of the form T + U Thus the Hamiltons equations are

$$p_\theta = ml^2\dot{\theta}, \text{ and } \dot{\theta} = \frac{p_\theta}{ml^2}. \tag{3.22}$$

Self Assessment Exercise B

1. Give the expression for the Hamiltonian of a system in generalized coordinates.
2. Write the Hamiltonian equation of motion.
3. Derive the Hamiltonian of Scalar potential and physical pendulum and their corresponding Hamilton's equation of motion.

3.3 Construction of the Hamiltonian in Spherical Polar Coordinates - Central Force Motion

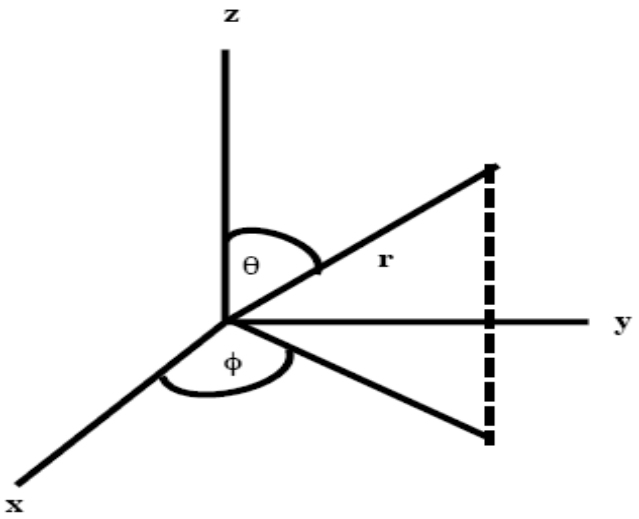


figure 3.1: systems with spherical symmetry (spherical polar coordinates)

For systems with spherical symmetry (e.g. the rigid rotator), spherical polar coordinates are the most convenient set of generalized coordinates. As depicted above, the spherical polar coordinates are r, θ and ϕ. The coordinate *r* is the distance from the origin of coordinates to the particle, θ is the angle the line connecting the origin of the coordinates to the particle position makes with the z-axis and ϕ is the angle the projection of the line defining θ onto the xy-plane makes with the x-axis. The connections between Cartesian coordinates and spherical polar coordinates can be derived readily using trigonometry. The result is

$$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta . \tag{3.23}$$

Another important relation is the direct result of the Pythagoras theorem

$$r = \sqrt{x^2 + y^2 + z^2} . \tag{3.24}$$

Consider a particle of mass m moving in three-dimensional space to a conservative central force with associated potential energy U(r). The meaning of a central force is the potential energy is a function only of the coordinate and independent of θ and ϕ. In Cartesian coordinates, the Lagrangian for the system is

$$L = \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right] - U . \tag{3.25}$$

To transform L from Cartesian to spherical polar coordinates, we need expression for the time derivatives of each Cartesian coordinates in terms of spherical polar coordinates.
Using equation 3.23 we have

$$\dot{x} = \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi \tag{3.26}$$

$$\dot{y} = \dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi \tag{3.27}$$

and $$\dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \tag{3.28}$$

We then substitute equations 3.26-3.28 into equation 3.25. After some algebra, the result is

$$L = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 + \right] - U(r) . \tag{3.29}$$

The lagrange equation is

$$\frac{d}{dt}m\dot{r} - mr\dot{\theta}^2 - mr\sin^2\theta\dot{\phi}^2 + \frac{dU}{dr} = 0, \tag{3.30}$$

for the θ- coordinate, we have

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) - mr^2\sin\theta\cos\theta\dot{\phi}^2 = 0 \tag{3.31}$$

and for the ϕ-coordinate we have

$$\frac{d}{dt}\left(mr^2\sin^2\theta\phi\right) = 0 . \tag{3.32}$$

The equation for the ϕ-coordinate is an expression of the conservative of the momentum conjugate to the coordinate ϕ (the angular momentum).

To construct the classical Hamiltonian, we need expressions for the generalized momenta conjugate to each of the spherical polar coordinates. These expressions have already been obtained when constructing Lagrange's equations. In particular,

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \text{and} \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi}. \tag{3.33}$$

Then using the definition of the classical Hamiltonian

$$\begin{aligned} H &= p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - L \\ &= \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right] + U(r). \end{aligned} \tag{3.34}$$

Finally equation 3.34 above must be transformed to replace the velocities with generalized momenta. We finally obtain

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + U(r) . \tag{3.35}$$

If needed the equations of motion can be obtained by applying Hamilton's equations to the constructed Hamiltonian.

Self Assessment Exercise C

1. Derive the Hamiltonian in spherical polar coordinates.
2. Derive the corresponding equation of motion in spherical polar coordinates.

4.0 Conclusion

The Legendre transform is a method of changing the dependence of a function of one set of variables to another set of variables. It is commonly used in thermodynamics and in the Hamiltonian formulation of classical mechanics.
Using Legendre transform to classical mechanics then, for a system of particles each having masses m_i described by a set of generalized coordinate's q_i, the classical Hamiltonian is defined by

$$H = \sum_i p_i \dot{q}_i - L\left(\{q_i\},\{\dot{q}_i\},t\right).$$

The classical Hamiltonian is manifestly a function of the generalized coordinates and momenta rather than the generalized coordinates and velocities.
Hamilton's equation of motion are.

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i \quad \text{and} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

The Hamiltonian is then seen to be an expression for the total energy of a conservative system in terms of the generalized coordinates and momenta. With the Hamiltonian expressed in terms of the proper variables, we can give Hamilton's equations of the system

$$\frac{\partial H}{\partial p_x} = \frac{p_x}{m} = \dot{x} \qquad \text{and} \quad \frac{\partial H}{\partial x} = \frac{\partial U}{\partial x} = -\dot{p}_x$$

5.0 Summary

- The Hamiltonian and Hamilton's equations, the Hamiltonian function is derived from the Lagrangian function via the lengendre transformation

$$H = \sum_k p_k \dot{q}_k - L,$$
$$H = H\left(\{q_k\},\{p_k\},t\right).$$

- The Hamiltons equation of motion are

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad -\dot{p}_k = \frac{\partial H}{\partial q_k}, \quad \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

6.0 Tutor Marked Assignments (TMAs)

1. Use the Lagrangian to construct the Hamiltonian for the system.
   Answer

   Let $M \equiv m_1 + m_2$ be the total mass and $\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$ that is, the reduced mass. Then,

$$H = \frac{p_x^2}{2M} + \frac{p_Y^2}{2M} + \frac{p_z^2}{2M} + \frac{p_\theta^2}{2\mu l^2} + \frac{p_\phi^2}{2\mu l^2 \sin^2\theta} + Mgz\,.$$

2. Find the Hamiltonian corresponding to the Coriolis Lagrangian

$$L = \frac{1}{2}R^2\left[\dot{\theta} + \cos^2\theta\left(\omega + \dot{\phi}\right)^2\right] - U(\theta,\phi)\,.$$

   Answer
   From the above,

$$p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = R^2\dot{\theta}, \text{ so that} \quad \dot{\theta} = \frac{p_\theta}{R^2}\,.$$

Also $p_\phi \equiv \dfrac{\partial L}{\partial \dot{\phi}} = R^2 \cos^2\theta\left(\omega + \dot{\phi}\right)$ so $\dot{\phi} = \dfrac{p_\phi}{R^2\cos^2\theta} - \omega$

$$H \equiv p_\theta \dot{\theta} + p_\phi \dot{\phi} - L$$

$$= \frac{p_\theta^2}{R^2} + \frac{p_\phi^2}{R^2\cos^2\theta} - \omega p_\phi - \frac{p_\theta^2}{2R^2} - \frac{p_\phi^2}{2R^2\cos^2\theta} + U(\theta,\phi)$$

$$= \frac{p_\theta^2}{2R^2} + \frac{p_\phi^2}{2R^2\cos^2\theta} - \omega p_\phi + U(\theta,\phi)$$

3. Find the Hamiltonian corresponding to the Lagrangian of an harmonic oscillator of problem Answer

The Lagragian $L = \dfrac{1}{2}m\dot{x}^2 + U(x)$, which is the standard form treated. Thus, $\dot{x} = \dfrac{p}{m}$ and the Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{m\omega_0^2}{2}\left(x^2 - \frac{x^4}{l_0^2}\right).$$

4. Consider the motion of a particle of charge e and mass m in a straight infinitely long magnetic confinement system with vector potential $\mathbf{A} = \psi(x,y)e_z$.

If the Hamiltonian is

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{[p_x - e\psi(x,y)]^2}{2m},$$

write down the Hamiltonian equation of motion

Answer

The Hamiltonian equation of motion are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_z}{m},$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m},$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{(p_z - e\psi)}{m},$$

and

$$\dot{p}_x = -\frac{\partial H}{\partial x} = \frac{e(p_z - e\psi)}{m}\frac{\partial \psi}{\partial x}.$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = \frac{e(p_z - e\psi)}{m}\frac{\partial \psi}{\partial y}.$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = 0.$$

5. Suppose a system has a Lagrangian

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \dot{q}_1^2 + \frac{\dot{q}_2^2}{a + bq_1^2} + k_1 q_1^2 + k_2 \dot{q}_1 \dot{q}_2.$$

Find the equations of motion using the Hamiltonian formulation.

7.0 Further Reading and Other Resources

Classical Mechanics by H.Goldstein, Narosa Publishing Home, New Delhi.

Classical Dynamics of Particles and Systems by Marion and Thomtron, Third Edition, Horoloma Book Jovanovich College Publisher.

Introduction to Classical Mechanics by R.G.Takawale and P.S.Puranik, Tata Mc-Graw Hill Publishing Company Limited, New Delhi.

Module 3 Central force and Scattering

Unit 1 The Generic Central Force Problem
Unit 2 Kepler's Problem
Unit 3 Scattering Cross Section

UNIT 1 THE GENERIC CENTRAL FORCE PROBLEM
CONTENTS

1.0 Introduction

The problem of the motion of two bodies interacting via a central force is an important application of Lagrangian dynamics, and the conservation theorems we have learned about. Central forces describe a large variety of classical systems, ranging from gravitationally interacting celestial bodies to electrostatic and nuclear interactions of fundamental particles. The central force problem provides one of the few exactly solvable problems in mechanics. And central forces underlying most scattering phenomena, again ranging from gravitational to electrostatics to nuclear.

2.0 Objectives

After studying this unit, you will be able to:

- Define central force and know the properties of an isolated two body central force system.
- Discuss the reduction of the two body problem to a mathematically equivalent problem of a single particle moving in one direction.
- Discuss the reduction of a two-body three-dimensional problem to one with single degree of freedom.
- Explain different shape of the effective potential energy function and its implications for the motion of the system.
- Give the differential relation between r and θ.
- Solve related Problems.

3.0 Main Contents

3.1 The Generic Central Force Problem

We first discuss the central force problem in general terms, considering arbitrary radially dependent potential energy functions.

3.1.1 The Equation of Motion

Review of Central Forces.

Central force is defined as one that satisfies the strong form of Newton's third law. That is, given two particles a and b, the force exerted by particle a on b is equal and opposite to that exerted by particle b on particle a, and, moreover, the force depends only on the separation of the two particles and points along the vector between the two particles. Mathematically, this means,

$$\vec{f}_{ab} = -\vec{f}_{ba}, \qquad \vec{f}_{ab} = f_{ab}(r_{ab})\hat{r}_{ab}$$
$$\text{where} \qquad \vec{r}_{ab} = \vec{r}_a - \vec{r}_b, \qquad r_{ab} = |\vec{r}_{ab}|, \qquad \hat{r}_{ab} = \frac{\vec{r}_{ab}}{r_{ab}}. \tag{1.1}$$

Properties of an Isolated Two-Body Central-Force System

Let us now consider an isolated two-body system interacting via a conservative central force. There are no other forces acting on the bodies.

From the concepts of force, momentum, and energy for systems of particles. One can abstract the following facts about isolated two-body system:

• Since no external forces act on the system, Newton's second law for systems of particles tells us that the total linear momentum is conserved.

$$0 = \frac{d}{dt}\vec{p} = \frac{d}{dt}(m_a\dot{\vec{r}}_a + m_b\dot{\vec{r}}_b). \tag{1.2}$$

Since $\vec{P}$ is constant, the velocity of the center-of-mass $\vec{R} = \vec{P}/M$ is fixed and thus the center-of-mass system is inertial. We may therefore assume, without loss of generality, that $\vec{r}_a$ and $\vec{r}_b$ are coordinates in the center of mass system, where $\vec{P}$ vanishes and the center of mass is at the origin:

$$0 = \vec{R} = m_a\vec{r}_a + m_b\vec{r}_b.$$
$$0 = \vec{P} = m_a\dot{\vec{r}}_a + m_b\dot{\vec{r}}_b. \tag{1.3}$$

This eliminates three of the six degrees of freedom in the problem. The difference coordinate $\vec{r}_{ab}$ is now

$$\vec{r}_{ab} = \vec{r}_a - \vec{r}_b = \vec{r}_a\left(1+\frac{m_a}{m_b}\right) = -\vec{r}_b\left(1+\frac{m_b}{m_a}\right). \tag{1.4}$$

Defining the reduced mass

$$\mu = \frac{m_a m_b}{m_a + m_b} \tag{1.5}$$

gives us

$$\vec{r}_a = \frac{\mu}{m_a}\vec{r}_{ab} \qquad \vec{r}_b = \frac{\mu}{m_b}\vec{r}_{ab}. \tag{1.6}$$

We shall see that the dynamics of the two-particle system will be reduced to that of a single particle with mass μ moving in the potential $U(r_{ab})$. We may consider two simple limits immediately:

- In the limit when m_b is far greater than m_a we have $\mu \to m_a$, $\vec{r}_b \to o$, and $\vec{r}_{ab} \to \vec{r}_a$. That is, the center of mass is fixed on the heavier mass and the motion is entirely of the smaller mass.

- In the limit $m_a = m_b = m,$ we have $\mu = \frac{m}{2}, \vec{r}_a = \frac{\vec{r}_{ab}}{2} and\ \vec{r}_b = \frac{\vec{r}_{ab}}{2},$ [in this case the motion

Since there are no external forces, there is no external torques either]. This angular momentum of the system is conserved.

Since the system's center-of-mass has been taken to be at rest at the origin, the angular momentum consists only of the internal angular momentum due to motion of the two particles about the center of mass. This angular momentum is

$$\vec{L} = m_a \vec{r}_a \times \dot{\vec{r}}_a + m_b \vec{r}_b \times \dot{\vec{r}}_b , \tag{1.7}$$

let's re-write this in terms of $\vec{r}_{ab}$

$$\vec{L} = \mu \vec{r}_{ab} \times \dot{\vec{r}}_a - \mu \vec{r}_{ab} \times \dot{\vec{r}}_b = r_{ab} \times \mu \dot{\vec{r}}_{ab} = \vec{r}_{ab} \times \vec{\rho}_{ab} \tag{1.8}$$

The two-body system begins to look like a single particle of mass μ and coordinate $\vec{r}_{ab}$.

The kinetic and potential energies of the system are

$$T = \frac{1}{2} m_a \dot{\vec{r}}_a^{\,2} + \frac{1}{2} m_b \dot{\vec{r}}_b^{\,2} = \frac{1}{2} \mu^2 \dot{\vec{r}}_{ab} \left(\frac{1}{m_a} + \frac{1}{m_b} \right) = \frac{1}{2} \mu \dot{\vec{r}}_{ab}^{\,2} , \tag{1.9}$$

$$U = U(r_{ab})$$

The Lagrangian is

$$L = \frac{1}{2} \mu \dot{\vec{r}}_{ab}^{\,2} + U(r_{ab}) . \tag{1.10}$$

The Lagrangian is identical to that of a single particle system with mass μ and coordinate $\vec{r}_{ab}$. Since $\vec{L}$ is conserved, we know the motion is restricted to the plane defined by $\vec{r}_{ab}$ and $\vec{\rho}_{ab}$. Let this plane define a spherical polar coordinate system ($r_{ab}, \theta_{ab}, \phi_{ab}$), where ϕ_{ab}, is the azimuthal angle of the plane and θ_{ab} is the polar angle of the position vector $\vec{r}_{ab}$ relative to the z-axis in the plane. Rewriting L in this system gives

$$L = \frac{1}{2} \mu \left(\dot{r}_{ab}^{\,2} + r_{ab}^{\,2} \dot{\theta}_{ab}^{\,2} + r_{ab}^{\,2} \sin^2 \theta_{ab} \dot{\phi}_{ab} \right) - U(r_{ab}) . \tag{1.11}$$

We may choose $\phi_{ab} = 0$ without loss of generality. The angular momentum vector points out of this plane (in the y direction) and the motion remains in this plane at all time by conservation of angular momentum.

3.1.2 Reduction to a one body problem

For simplicity and better understanding of the above equation of motion we discuss the reduction of the two body problem to a mathematically equivalent problem of a single particle moving in one direction. First we reduce it to a one-body problem, and then we reduce the dimensionality. Our original problem has six degrees of freedom, but because of the symmetries in the problem, many of these can be simply separated and solved for. As there are no external forces, we expect the center of mass coordinate to be in uniform motion, and it allows us to use

$$\vec{R} = \frac{\sum_\alpha m_\alpha \vec{r}_\alpha}{\sum_\alpha m_\alpha} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \tag{1.12}$$

as three of our generalized coordinates. For the other three, we first use the Cartesian components of the relative coordinate

$$\vec{r} = \vec{r}_1 + \vec{r}_2 \tag{1.13}$$

although, we will soon change to spherical coordinates for this vector. In terms of $\vec{R}$ and $\vec{r}$ the particle positions are:

$$\vec{r}_1 = \vec{R} - \frac{m_2}{M} \vec{r} \qquad \vec{r}_2 = \vec{R} - \frac{m_2}{M} \vec{r} \tag{1.14}$$

where $M = m_1 + m_2$.

The kinetic energy is

$$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2},$$

$$T = \frac{1}{2} m_1 \left(\dot{\vec{R}} - \frac{m_2}{M} \dot{\vec{r}} \right)^2 + \frac{1}{2} m_2 \left(\dot{\vec{R}} + \frac{m_1}{M} \dot{\vec{r}} \right)^2 = \frac{1}{2} (m_1 + m_2) \dot{\vec{R}}^2 + \frac{1}{2} \frac{m_1 m_2}{M} \dot{\vec{r}}^{\,2}, \qquad (1.15)$$

$$T = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^{\,2} \quad ,$$

where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ is called the **reduced mass**. Thus the kinetic energy is transformed to the form for two effective particles of mass *M* and μ which is neither simpler nor more complicated than it was in the original variables.

For the potential energy, however, the new variables are to be preferred, $U\left(\left|\vec{r}_1 + \vec{r}_2\right|\right) = U\left(\left|\vec{r}\right|\right)$ is independent of $\vec{R}$, whose three components are therefore ignorable coordinates, and their conjugate momenta.

$$\left(\vec{P}_{cm}\right)_i = \frac{\partial (T - U)}{\partial \dot{R}_i} = M \dot{R}_i \qquad (1.16)$$

are conserved. This reduces half of the motion to triviality, leaving an effective one-body problem with $T = \dfrac{1}{2} \mu \dot{\vec{r}}^{\,2}$ and the given potential to $U(\vec{r})$. We have not yet made use of the fact that *U* only depends on the *magnitude* of $\vec{r}$. In fact, the above reduction applies to any two-body system without external forces, as long as Newton's third Law holds.

Self Assessment Exercise A

1. Use momentum conservation to reduce the two-body problem to the problem of one body motion in a central force field.

3.2 Dynamics of an Isolated Two-Body Central-Force System

Now, let's explore the dynamics using the Lagrangian. Conservation of $\vec{L}$ already leads us to expect that one of the Euler-Lagrange equations will be trivial. Explicitly, we have (dropping the $_{ab}$ subscripts now):

$$\mu \ddot{r} = -\frac{dU}{dr} + \mu r \theta^2 + \mu r \sin^2 \theta \phi^2 \qquad (1.17)$$

$$\frac{d}{dt}\left[\mu r^2 \theta\right] = \mu r^2 \sin\theta \cos\theta \phi^2$$

$$\frac{d}{dt}\left[\mu r^2 \sin^2 \theta \phi\right] = 0 \qquad (1.18)$$

We make the general point that, when writing the Euler-Lagrange equations for a multidimensional system, it is a good idea to start with the time derivatives on the left side unexpanded until the rest has been simplified because they are just the time derivatives of the canonical momenta and some of them may end up being conserved if the right side of the corresponding equation vanishes, either explicitly or by appropriate choice of initial conditions. We see in the above case that the ϕ equation of motion tells us $l_\phi = cons\tan t$, which would have become very unobvious if the derivative had been expanded into its three terms.

For initial conditions, we take $\vec{r}$ and $\vec{p}$ to be in the plane $\phi = 0$. That $\vec{p}$ is in the plane $\phi = 0$ also implies $\dot{\phi} = 0$ initially. With these initial conditions, the ϕ equation of motion implies that $\dot{\phi} = 0$ for all time. Thus, $l_\theta = \mu r^2 \sin^2 \theta \dot{\phi} = 0$, for all time. The ϕ equation then tells us we have $l_\theta = \mu r^2 \dot{\theta} = cons\tan t$. The angular momentum vector has length $\left|\vec{L}\right| = l_\theta$ and points

perpendicular to the plane $\phi = 0$ in which the motion occurs: $\vec{L}$ is along the *y*-axis. The *r* equation of motion simplifies to

$$\mu\ddot{\vec{r}} = \frac{dU}{dr} + \frac{l_\theta^{\,2}}{\mu r^3}. \tag{1.19}$$

The equation of motion is now that of a single particle in one dimension with the effective potential function

$$U_{eff}(r) = U(r) + \frac{l_\theta^{\,2}}{2\mu r^2}. \tag{1.20}$$

We acquire a new "centrifugal potential" that arises due to conservation of angular momentum. It is a repulsive potential, reflecting the fact that, with l_ϕ constant, the kinetic energy must increase as r^{-2} if r decreases then more energy is needed to go to small radii. We may use the effective potential to write an effective one-dimensional Lagrangian:

$$L_{1D} = \frac{1}{2}\mu\dot{r}^2 - \frac{l_\theta}{2\mu r^2} - U(r). \tag{1.21}$$

Note that the 1D Lagrangian is not obtained simply by rewriting the 3D Lagrangian using $\phi = 0$, $\dot{\phi}^2 = 0$ and $\mu r^2\dot{\theta} = l_\theta$ one would have gotten the wrong sign for the centrifugal term.
This difficulty occurs because the Lagrangian formalism assumes independent variations of the different coordinates. Instead, we must simply start with the effective potential.
The effective total energy is

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l_\theta^{\,2}}{2\mu r^2} + U(r) \tag{1.22}$$

which is conserved because the effective potential is conservative. This effective total energy turns out to be equal to the true total energy because the θ kinetic energy is included via the l_θ term.

3.2.1 Reduction to one dimension

In the problem under discussion, however, there is the additional restriction that the potential depends only on the magnitude of $\vec{r}$, that is, on the distance between the two particles, and not on the direction of $\vec{r}$. Thus we now convert from cartesian to spherical coordinates $(r;\theta;\phi)$, for $\vec{r}$. In terms of the cartesian coordinates (*x; y; z*)

$$\begin{aligned} &r = (x^2 + y^2 + z^2)^{1/2}; && x = r\sin\theta\cos\phi, \\ &\theta = \cos^{-1}\left(z/r\right); && y = r\sin\theta\sin\phi, \\ &\phi = \tan^{-1}\left(y/z\right); && z = r\cos\theta, \end{aligned} \tag{1.23}$$

Plugging into the kinetic energy is messy but eventually reduces to a rather simple form

$$\begin{aligned} T &= \frac{1}{2}\mu\left[\dot{x}_1^{\,2} + \dot{y}^2 + \dot{z}^2\right] \\ &= \frac{1}{2}[(\dot{r}\sin\theta\cos\phi + \dot{\theta}r\cos\theta\cos\phi - \dot{\phi}r\sin\theta\sin\phi)^2 \\ &= +(\dot{r}\sin\theta\sin\phi + \dot{\theta}r\cos\theta\sin\phi + \dot{\phi}r\sin\theta\cos\phi)^2 + (\dot{r}\cos\theta - \dot{\theta}r\sin\theta)^2] \\ &= \frac{1}{2}\mu\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right] \end{aligned} \tag{1.24}$$

Notice that in spherical coordinates T is a function of r and θ as well as $\dot{r}, \dot{\theta}$ *and* $\dot{\phi}$, but it is not a function of ϕ, which is therefore an ignorable coordinate and

$$P_\phi = \frac{\partial L}{\partial \phi} = \frac{\partial l_\theta}{\partial \phi} = \mu r^2 \sin^2\theta\dot{\phi} = cons\tan t\,. \tag{1.25}$$

Note that $r\sin\theta$ is the distance of the particle from the *z*-axis, so P_ϕ is just the *z*-component of the angular momentum, l_z Of course all component of the angular momentum $\vec{L} = l_\theta = \vec{r}\times\vec{p}$ is conserved, because in our effective one body problem there is no torque about the origin. Thus l_θ is a constant and the motion must remain in a plane perpendicular to l_θ and passing through the origin as a consequence of the fact that $\vec{r}$ is perpendular to l_θ. It simplifies things if we choose our coordinates so that $\vec{L}$ is in the z-direction. Then $\theta = \pi/2$, $\dot{\theta} = 0$, $l_\theta = \mu r^2\dot{\phi}$. The r equation of motion is then

$$\mu\ddot{r} = \mu r\dot{\phi}^2 + dU/dr = 0 = \mu r^2 - \frac{l_\theta^{\,2}}{\mu r^3} + dU/dr\,. \tag{1.26}$$

This is the one-dimensional motion of body in an effective potential

$$U_{eff}(r) = U(r) + \frac{l_\theta^{\,2}}{2\mu r^2}\,. \tag{1.27}$$

Thus we have reduced a two-body three-dimensional problem to one with single degree of freedom, without any addition of a centrifugal barrier term $\frac{l_\theta^{\,2}}{2\mu r^2}$ to the potential.

Self Assessment Exercise B

1. Use total angular momentum conservation to show that one body in a central force field moves in a plane. Write corresponding Lagrangian in polar coordinates.
2. Use angular momentum conservation to reduce the problem to an analysis of a one-dimensional motion.
3. Give the expression for the centrifugal potential energy.

3.3 The formal Solution to the Equation of Motion

We have obtained two equations of motion

$$\begin{aligned} \mu\ddot{r} &= \frac{d}{dr}U_{eff}(r, l_\theta), \\ \frac{d}{dt}\left[\mu r^2\dot{\theta}\right] &= 0. \end{aligned} \tag{1.28}$$

Let's attempt to integrate these two equations. Integrating the *r* equation most obviously yields as equation for $\dot{r}$. But we already know what equation will be, by energy conservation.

3.3.1 Integration of Equation of motion

We can simplify the problem even more by using the conservation law, that of energy. Because the energy of the effective motion is a constant,

$$E = \frac{1}{2}\mu r^2 + U_{eff} = cons\tan t\,, \tag{1.29}$$

we can immediately solve for

$$\dot{r} = \frac{dr}{dt} = \pm\left\{\frac{2}{\mu}(E - U_{eff}(r))\right\}^{1/2}. \tag{1.30}$$

This can be inverted and integrated over *r*, to give

$$t = t_0 \pm \int \frac{dr}{\sqrt{2(E - U_{eff}(r))/\mu}} \tag{1.31}$$

which is the inverse function of the solution to the radial motion problem *r*(*t*). We can also find the orbit since

$$\frac{d\phi}{dr} = \frac{\phi}{dr/dt} = \frac{l_\theta}{\mu r^2}\frac{dt}{dr} \tag{1.32}$$

$$\phi = \phi_0 \pm l_\theta \int_{r_0}^{r} \frac{dr}{r^2\sqrt{2\mu(E - U_{eff}(r)}}. \tag{1.33}$$

The sign ambiguity from the square root is only because *r* may be increasing or decreasing, but time, and usually ϕ/L are always increasing. Qualitative features of the motion are largely determined by the range over which the argument of the square root is positive, as for other values of *r* we would have imaginary velocities. Thus the motion is restricted to this allowed region. Unless $L = 0$ or the potential $U(r)$ is very strongly attractive for small *r*, the centrifugal barrier will dominate, so $U_{eff} \to +\infty$ and there must be a smallest radius $rp > 0$ for which $E \geq U_{eff}$.

3.3.2 A Differential Relation between r and θ

While we have formally eliminated *t* and obtained an integral relationship between θ and *r*, the fact that the integral is not in general analytic limits its usefulness. It may be more useful to have a differential, rather than integral, relation between r and θ. We need to eliminate $d/d\theta$ from our original differential equations. The θ equation tells us that

$$dt = \frac{\mu r^2}{l_\theta} d\theta, \qquad \frac{d}{dt} = \frac{l_\theta}{\mu r^2}\frac{d}{d\theta}. \tag{1.34}$$

Our equation of motion for *r* can then be re-written:

$$\begin{aligned}
\mu\frac{l_\theta}{\mu r^2}\frac{d}{d\theta}\left[\frac{l_\theta}{\mu r^2}\frac{dr}{d\theta}\right] - \frac{l_\theta^2}{\mu r^2} &= F(r) \\
\frac{d}{d\theta}\left[\frac{1}{r^2}\frac{dr}{d\theta}\right] - \frac{1}{r} &= \frac{\mu r^2}{l_\theta^2}F(r) \\
-\frac{d}{d\theta}\left[\frac{d}{d\theta}\frac{1}{r}\right] - \frac{1}{r} &= \frac{\mu r^2}{l_\theta^2}F(r) \\
\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} &= -\frac{\mu r^2}{l_\theta^2}F(r).
\end{aligned} \tag{1.35}$$

We now have a differential equation with θ as the independent variable and *r* as the dependent variable. The equation may be useful for obtaining the shapes of the orbits, and frequently written via a change of variables to *u=1/r* in the form

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l_\theta^2 u^2}F\left(\frac{1}{u}\right). \tag{1.36}$$

The constant of the motion, the energy, can be rewritten in terms of u and θ alone (i.e, eliminating explicit time derivatives):

$$\begin{aligned}
E &= \frac{1}{2}\mu\dot{r}^2 + \frac{l_\theta^2}{2\mu r^2} + U(r) \\
&= \frac{l_\theta^2}{2\mu}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 + \frac{l_\theta^2}{2\mu r^2} + U(r) \\
E &= \frac{l_\theta^2}{2\mu}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + U\left(\frac{1}{u}\right).
\end{aligned} \tag{1.37}$$

Self Assessment Exercise C

1. Use energy conservation to find formally an orbit of a body moving in central field in coordinates r, θ. Also Derive the relationship between r and θ

3.3.3 Qualitative Dynamics

It is instructive to consider the shape of the effective potential energy function and its implications for the motion of the system. The effective potential consists of two terms. The first is the true central force potential. The second is a "centrifugal" term: it is a repulsive potential arising from angular momentum conservation, which requires the kinetic energy to increase as r is reduced. The relative sizes of the two terms determine if or whether the effective potential is attractive or repulsive. The shape of the effective potential and the total energy of the system determine whether the orbits are unbounded, bounded, or bounded and circular, and whether bounded orbits are periodic (closed). To be clear: bounded and unbounded refers to whether there is an upper limit on r or not; open and closed refer to whether the orbit repeats itself after some period. All unbounded orbits are open, but not all bounded orbits are closed.
The effective potential is:

$$U_{eff}(r) = U(r) + \frac{l_\theta}{2\mu r^2}. \tag{1.38}$$

Consider different cases for the shape of $U(r)$. Some of these are illustrated in the Figure 1.0.

- Repulsive Potentials: If $U(r)$ has no attractive regions (no regions with positive slope), then both terms are repulsive at all r and all orbits are unbounded and open. This occurs, for example, for the Coulomb force between particles of like charge.
- Small r Behaviour: The small r behaviour of the effective potential determines whether r is bounded below or whether there are "small r" bounded orbits with r bounded above.

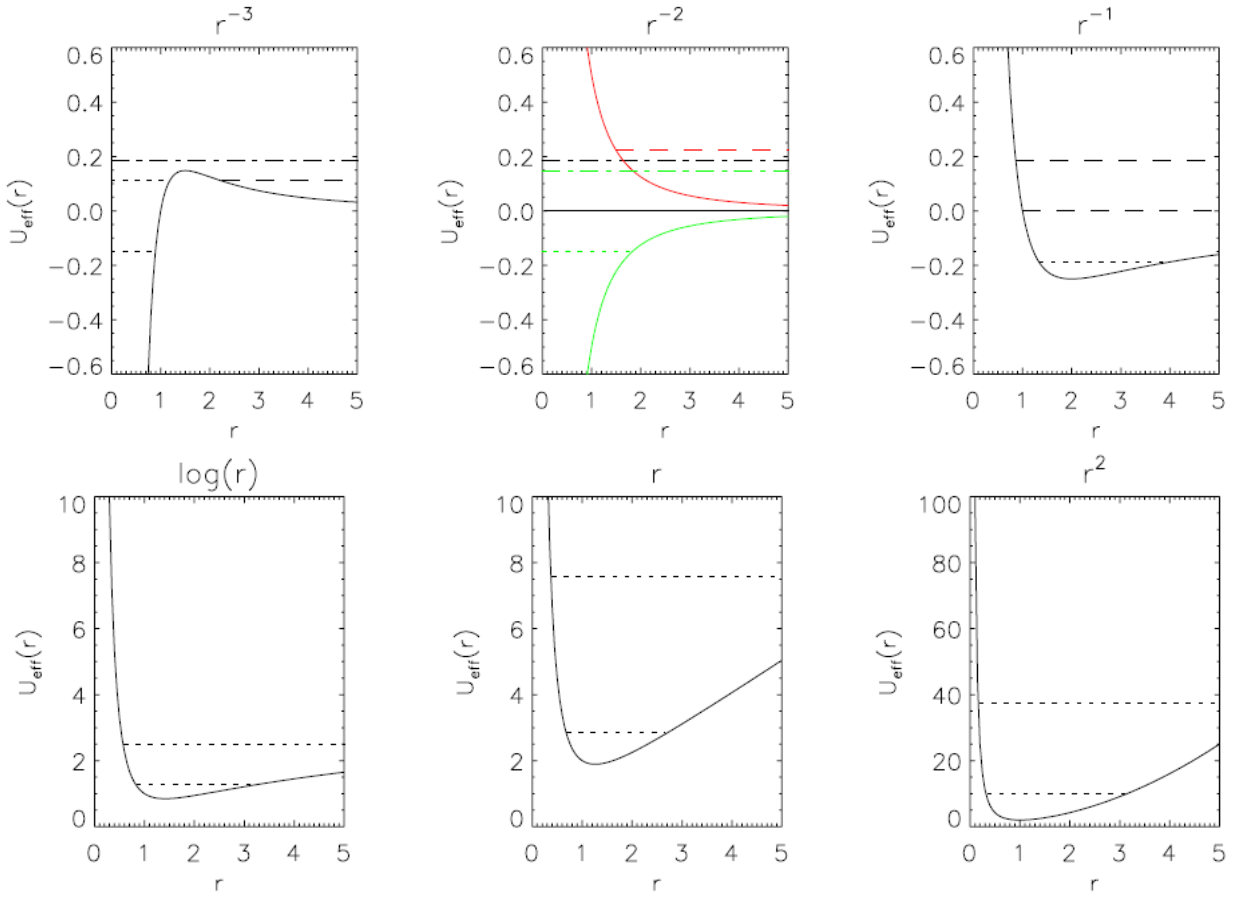


Figure 1.0: Effective potential for various true potentials.
The potentials are $U(r) = r^n$ or $U(r) = \log r$ with unity coefficient except for the r^{-2} case. For all cases except r^{-2}, the behaviour is qualitatively independent of the factor $\frac{l_\theta^2}{2\mu}$, so it has been set to 1. For the r^{-2} potential, we show the three cases $\frac{l_\theta^2}{2\mu}$ = 0.5, 1, 1.5. The locus of r values filled by example bound orbits are indicated by dotted lines, by example

orbits bounded below but unbounded above by dashed lines, and by completely unbounded example orbits by dash-dot lines, respectively.

- Large *r* Behavior: The large *r* behavior of the effective potential determines whether *r* is bounded above at large r, which is a necessary condition for bounded orbits.
- Circular orbits are obtained when there is a point in the effective potential where the gradient (effective force) vanishes. For potentials steeper than r^{-2}, there is one unstable circular orbit. For potentials shallower than r^{-2} and for log or positive power law potentials, there is one stable circular orbit.
- Of course, if the true potential is a more complicated function of *r* than simple power law or log functions, there may be additional bound orbits and circular orbits.
- Whether an orbit is periodic (that is, "closed" the orbit repeats on itself after some time) is a nontrivial problem. Clearly, all bound orbits show periodic motion in *r* as a function of time. But periodicity in space that is, the periodicity of $r(\theta)$ is a more complicated problem. All circular orbits are periodic in space because r is constant. More generally, periodicity of $r(\theta)$ requires that the time periods for radial motion and angular motion be commensurate that is, their ratio must be a rational fraction. It is not obvious what the relation is between the form of the potential and whether this condition is satisfied. Bertrand's theorem tells us that bound orbits are closed only if the potential in the vicinity of the bound orbits follows r^{-1} or r^2. Details on Bertrand's theorem can be found in Goldstein.

When orbits are bounded, the two turning points are called the apsides or apsidal distances.
When orbits are not closed, the apsides precess in (r,θ) space. The angle between two consecutive apsides is called the apsidal angle. Precession of the apsides occurs whenever the conditions of Bertrand's theorem are not satisfied, including small perturbations of Bertrand's theorem potentials by non-Bertrand's theorem terms.

Self Assessment Exercise D

1. Discuss briefly the shape of the effective potential energy function and its implications for the motion of the system

4.0 Conclusion

Central force is defined as one that satisfies the strong form of Newton's third law. That is, given two particles a and b, the force exerted by particle *a* on *b* is equal and opposite to that exerted by particle b on particle a, and, moreover, the force depends only on the separation of the two particles and points along the vector between the two particles.
Considering an isolated two-body system interacting via a conservative central force. The dynamics of the two-particle system is equivalent to that of a single particle with mass μ moving in the potential $U(r_{ab})$ and likewise a two-body three-dimensional problem is equivalent to one with single degree of freedom, without any addition of a centrifugal barrier term $\frac{l_\theta^{\,2}}{2\mu r^2}$ to the potential. The shape of the effective potential energy function and its implications for the motion of the system was also considered for various true potential and it's depicted in figure 1.0 above

5.0 Summary

- Generic Central Forces.
  The problem of two particles interacting via a strong-form third law central force can be reduced to translational motion of the center-of-mass system combined with one particle in a central force.

If the two particles have masses m_a and m_b and position $\vec{r}_a$ and $\vec{r}_b$, then the relative coordinate and reduced mass are

$$\vec{r}_{ab} = \vec{r}_a - \vec{r}_b \qquad \mu \equiv \frac{m_a m_b}{m_a + m_b}.$$

- The original coordinates can be rewritten as

$$\vec{r}_a = \frac{\mu}{m_a}\vec{r}_{ab} \qquad \vec{r}_b = \frac{\mu}{m_b}\vec{r}_{ab}.$$

- The lagrangian can be rewritten in the following ways

$$L = \frac{1}{2}\mu\dot{\vec{r}}_{ab}^{\,2} - U(r_{ab}) = \frac{1}{2}\mu\left(\dot{\vec{r}}_{ab}^{\,2} + r_{ab}^2\dot{\theta}_{ab}^2 + r_{ab}^2\sin^2\theta_{ab}\dot{\phi}_{ab}^2\right) - U(r_{ab}).$$

- We obtain equations of motion (after eliminating the constant ϕ coordinate)

$$\mu\ddot{r} = -\frac{dU}{dr} + \frac{l_\theta^2}{\mu r^3} \qquad \mu r^2\dot{\theta} = l_\theta = cons\tan t\,.$$

- We define the effective potential

$$U_{eff}(r) = U(r) + \frac{l_\theta^2}{2\mu r^2}.$$

- The Lagrangian and energy can be rewritten in one dimensional form

$$L_{1D} = \frac{1}{2}\mu\dot{r}^2 - \frac{l_\theta^2}{2\mu r^2} - U(r) = \frac{1}{2}\mu\dot{r}^2 - U_{eff}(r),$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l_\theta^2}{2\mu r^2} + U(r) = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r),$$

where $\frac{l_\theta^2}{2\mu r^2}$ is the repulsive "centrifugal potential" The quantitative behaviour of the system can be obtained by examining the shape of the effective potential: where it is repulsive, attractive, where its slope vanishes e.t.c The constancy of l_θ gives us kepler's second law

$$\frac{dA}{dt} = \frac{1}{2}\frac{l_\theta}{\mu} = cons\tan t\,.$$

- The generic quadrature solution to the central force problem is given by

$$t = \pm\int_{r(0)}^{r} dr'\left[\frac{2}{\mu}\left(E - U(r)\right) - \frac{l_\theta^2}{\mu^2 r'^2}\right]^{-1/2}$$

$$\theta - \theta(0) = \pm\frac{l_\theta}{\mu}\int_0^t \frac{dt'}{[r(t')]^2}.$$

Elimination of t from the original differential relations also allows us to obtain the quadrature solution for θ in terms of r.

$$\theta(r) - \theta(0) = \pm l_\theta\int_{r(0)}^{r}\frac{dr'}{[r(t)]^2}\left[2\mu\left(E - U(r)\right) - \frac{l_\theta^2}{r'^2}\right]^{-1/2}.$$

- We may obtain a generic differential equation relating r and θ

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l_\theta^2}F(r).$$

Defining $u = \frac{1}{r}$, this may be re-written

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l_\theta^2 u^2} F\left(\frac{1}{u}\right).$$

- The total energy of the system is constant and can be written as

$$E = \frac{l_\theta^2}{2\mu}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) + \frac{l_\theta^2}{2\mu r^2} + U(r) = \frac{l_\theta^2}{2\mu}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + u\left(\frac{1}{u}\right).$$

6.0 Tutor Marked Assignments (TMAs)

1. Consider a particle constrained to move on the surface described in cylindrical coordinates by $z = \alpha r^3$, subject to a constant gravitational force $\vec{F} = -mg\hat{e}_z$. Find the Lagrangian, two conserved quantities, and reduce the problem to a one dimensional problem. What is the condition for circular motion at constant *r*?
2. From the general expression for ϕ as an integral over *r*, applied to a three dimensional symmetrical harmonic oscillator $V(\vec{r}) = \frac{1}{2}kr^2$, integrate the equation, and show that the motion is an ellipse, with the center of force at the center of the ellipse. Consider the three complex quantities $Q_i = p_i - i\sqrt{km}r_i$ and show that each has a very simple equation of motion, as a consequence of which the nine quantities $Q_i^* Q_k$ are conserved. Identify as many as possible of these with previously known conserved quantities.
3. Show that if a particle under the influence of a central force has an orbit which is a circle passing *through* the point of attraction, then the force is a power law with $|F| \propto r^{-5}$. Assuming the potential is defined so that $U(\infty) = 0$, show that for this particular orbit $E = 0$, find the period, and by expressing $\dot{x}$, $\dot{y}$ and the speed as a function of the angle measured from the center of the circle, and its derivative, show that $\dot{x}$, $\dot{y}$ and the speed all go to infinity as the particle passes through the center of force.

7.0 Further Reading and Other Resources

Classical Mechanics by H.Goldstein, Narosa Publishing Home, New Delhi.

Classical Dynamics of Particles and Systems by Marion and Thomtron, Third Edition, Horoloma Book Jovanovich College Publisher.

Introduction to Classical Mechanics by R.G.Takawale and P.S.Puranik, Tata Mc-Graw Hill Publishing Company Limited, New Delhi.

UNIT 2 KEPLER"S PROBLEM
CONTENTS

1.0 Introduction

The Kepler problem is named after Johannes Kepler, who proposed Kepler's laws of planetary motion (which are part of classical mechanics and solved the problem for the orbits of the planets) and investigated the types of forces that would result in orbits obeying those laws (called *Kepler's inverse problem*)
In classical mechanics, **Kepler's problem** is a special case of the two-body problem, in which the two bodies interact by a central force **F** that varies in strength as the inverse square of the distance *r* between them. The force may be either attractive or repulsive. The "problem" to be solved is to find the position or speed of the two bodies over time given their masses and initial positions and velocities. Using classical mechanics, the solution can be expressed as a Kepler orbit using six orbital elements.

2.0 Objectives

After studying this unit, you will be able to:

- State the three Kepler's Laws.
- Prove the three Keplers laws.
- Define an orbit.
- Derive and explain the conic equation of an orbit.
- Explain Keplerian orbit.
- Solve related problems.

3.0 Main Contents

3.1 Kepler's Law

Practical integration of equation $\phi = \phi_0 \pm l_\theta \int_{r_0}^{r} \frac{dr}{r^2 \sqrt{2\mu(E - U_{eff}(r)}}$ is in general a formidable task.

The problem can be solved explicitly, however, for the potential of the form $U(r) = -\frac{k}{r}$ corresponding for example to the gravitation field. Consider motion of a planet of mass m_p

around the sun of mass m_s. It was shown that this problem is equivalent to an analysis of a motion of one body of reduced mass

$\frac{1}{\mu} = \frac{1}{m_p} + \frac{1}{m_s}$ in central field *U(r)* around the center of mass $R = \frac{m_p r_p + m_s r_s}{m_p + m_s}$,

where r_p and r_s are the position of the planet and of the sun and $r = |r_p - r_s|$. The corresponding effective Lagrangian, $L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$

Lagrange's equations in polar coordinates are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0. \qquad \text{and} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0,$$

Using the second equation above to prove kepler's second law, note that L does not depend on θ, therefore

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} = 0, \Rightarrow p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} = l_\theta \tag{2.1}$$

where l_θ is a constant. The radius vector r sweeps out an area $dA = \frac{1}{2}r^2\theta$ in time *dt*. The rate at which the radius vector sweeps out area is $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta}$ Comparing this rate with the momentum p_θ we proved that

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2\mu} = cons\tan t \tag{2.2}$$

which is **Kepler's second law** which state that the radius vector drawn from the sun to a planet describes equal areas in equal times.

Using first Lagrange's equation we can prove Kepler's first law which states that each planet moves in elliptical orbit with the sun at one focus

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow \mu\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\partial U(r)}{\partial r} = -\frac{k}{r^2}. \tag{2.3}$$

Proof:

$$\mu r^2\dot{\theta} = l_\theta \Rightarrow \dot{\theta} = \frac{l_\theta u^2}{\mu} \Rightarrow dt = d\theta\frac{\mu}{lu^2}, \tag{2.4}$$

since $u = \frac{1}{r}$.

The equation above can rewritten as follows: $\frac{d}{dt} = \frac{l_\theta u^2}{\mu}\frac{d}{d\theta}$, also, $\dot{r}$ can be re-written as

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{\mu}{l_\theta}\frac{du}{d\theta}$$

therefore $\frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{l_\theta u^2}{\mu}\frac{d}{d\theta}\left(-\frac{1}{\mu}\frac{du}{d\theta}\right)$ on substituting for $\ddot{r}$ and $\dot{\theta}$ in the equation, we have

$$\mu\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{k}{r^2} \Rightarrow -\frac{l_\theta{}^2 u^2}{\mu}\frac{d^2u}{d\theta^2} - \frac{l_\theta{}^2 u^3}{\mu} = -ku^2 \Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{k\mu}{l_\theta{}^2}. \tag{2.5}$$

Thus, equation for the planet motion for linear oscillator in a constant field. Its solution is a sum of a general solution $b\cos(\theta - \theta_0)$ and a particular solution $\frac{k\mu}{l_\theta}$. i.e

$$u = \frac{k\mu}{l_\theta{}^2} + b\cos(\theta - \theta_0). \tag{2.6}$$

Or, in polar coordinates, we have

$$r = \frac{\alpha}{1 + p\cos(\theta - \theta_0)} \tag{2.7}$$

where $\alpha = \frac{l_\theta^{\,2}}{k\mu}$ and $p = \sqrt{1 + \frac{qEl_\theta^{\,2}}{k^2\mu}}$. (2.8)

This is an equation for conic sections which describes

$$\begin{cases} p > 1 & \textit{hyperbola} \\ p = 0 & \textit{parabola} \\ p < 1 & \textit{ellipse.} \end{cases} \tag{2.9}$$

Motion of the planet is bounded to the sun and therefore corresponds to the case $p<1$. Futhermore, since m_s is far less than m_p (m_s=333,500 x m_{earth} and m_{earth}=5,977x10^{24}kg), we have

$$\frac{1}{\mu} = \frac{1}{m_p} + \frac{1}{m_s} \approx \frac{1}{m_p}$$

and $R = \frac{m_p r_p + m_s r_s}{m_p + m_s} \approx \frac{m_p}{m_p + m_s} r_p + \frac{m_s}{m_p + m_s} r_s \approx r_s$

i.e. centre of mass approximately coincides with the position of the sun and r is approximately distance from the sun to the planet. Therefore, we have proved **Kepler's first law** which states that each planet moves in elliptical orbit with the sun at one focus.

Using the second Kepler,s law $\frac{dA}{dt} = \frac{1}{2} r^2\theta$, and the expression for the angular momentum $p_\theta = \mu r^2\dot{\theta}$ to write $dA = \frac{1}{2\mu} dt$ Integrating this expression over the whole area of the ellipse, we obtained $A = \pi ab = \frac{1}{2\mu} T$, where T is the period of planetary motion and πab is the area of the ellipse. Using the following relation $a = \alpha^{-1} b^2$ to obtain $\pi\alpha^{1/2} a^{3/2} = \frac{1}{2\mu} T \Rightarrow a^3 = \frac{l^2\alpha^{-1}}{(2\pi\mu)^2} T^2$.

Using the definition $\alpha = \frac{l^2}{k\mu}$ to finally get $a^3 = \frac{k}{4\pi^2\mu} T^2$ which is the Kepler's third law which states that the square of the period of revolution about the sun is proportional to the cube of the major axis of the orbit.

Self Assessment Exercise A

1. Use θ-component of Lagrangian equation to prove Kepler's second law.
2. Use r-component of Lagrange's equation to prove Kepler's first law.
3. Use the second Kepler's law and the expression for the angular momentum to prove Kepler's third Law
4. Derive the equation of a conic section and describe the shape for $p > 0$, $p = 0$ and $p < 0$

3.2 The Kepler Problem: The Special Case of Gravity

We now specialize to gravity, which allows us to fix the form of the central-force potential energy function. We solve the equation of motion and study the various solutions with $k=G\mu M$.

3.2.1 The Shape of Solution of The Kepler's Problem

The General Solution

In our generic study of central forces, we obtained the differential relation in equation 1.36

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{\mu r^2}{l_\theta}F(r).$$

For the gravitational force, we have, $F(r)=-G\frac{m_a m_b}{r^2}$, (2.10)

so the above reduces to

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{G\mu^2(m_a+m_b)}{l_\theta^{\,2}}. \tag{2.11}$$

Re-writing using $u=\frac{1}{r}$, we have

$$\frac{d^2u}{d\theta^2}+u=-\frac{G\mu^2 M}{l_\theta^{\,2}}, \tag{2.12}$$

where $M=m_a+m_b=m_a m_b\big/\mu$.

This is now a simple harmonic oscillator equation with a constant driving force. The solution is the sum of the generic solution to the homogeneous equation and a particular solution to the inhomogeneous equation:

$$u(\theta)=A\cos(\theta-\theta_0)+\frac{G\mu^2 M}{l_\theta^{\,2}}. \tag{2.13}$$

We can relate the coefficient A in the solution to the constants of the motion, the energy and angular momentum by using equation 1.37

$$\begin{aligned}
E&=\frac{l_\theta^{\,2}}{2\mu}\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]+U\left(\frac{1}{u}\right),\\
&=\frac{l_\theta^{\,2}}{2\mu}\left[A^2\sin^2(\theta-\theta_0)+A^2\cos^2(\theta-\theta_0)+2A\cos(\theta-\theta_0)\frac{G\mu^2M}{l_\theta^{\,2}}+\left(\frac{G\mu^2M}{l_\theta^{\,2}}\right)^2\right]\\
&\quad -G\mu M\left(A\cos(\theta-\theta_0)+\frac{G\mu^2M}{l_\theta^{\,2}}\right),\\
&=\frac{l_\theta^{\,2}A^2}{2\mu}+AG\mu M\cos(\theta-\theta_0)+\frac{G^2\mu^2M^2}{2l_\theta^{\,2}}-G\mu MA\cos(\theta-\theta_0)-\frac{G^2\mu^2M^2}{l_\theta^{\,2}},\\
&=\frac{l_\theta^{\,2}A^2}{2\mu}-\frac{G^2\mu^2M^2}{2l_\theta^{\,2}},\\
&=\frac{l_\theta^{\,2}}{2\mu}\left[A^2-\left(\frac{G\mu^2M}{l_\theta^{\,2}}\right)^2\right].
\end{aligned} \tag{2.14}$$

Let's us write the orbit in terms of *r* instead of *u* and also drop the offset phase θ_0,

$$\frac{p}{r}=1+\varepsilon\cos\theta\Rightarrow p=r+\varepsilon r\cos\theta \tag{2.15}$$

where $p=\frac{l_\theta^{\,2}}{G\mu^2M}$ and $\varepsilon=pA$.

If we write in Cartesian coordinates, with $x=r\cos\theta$ and $y=r\sin\theta$, we have

$$p=\sqrt{x^2+y^2}+\varepsilon x\Rightarrow\left(p-\varepsilon x\right)^2=x^2+y^2 \tag{2.16}$$

$$\left(1-\varepsilon^2\right)x^2+2\varepsilon px+y^2-p^2=0. \tag{2.17}$$

In terms of p and ε, the total energy is

$$E = \frac{l_\theta^2}{2\mu}\left[\frac{\varepsilon^2}{p^2} - \frac{1}{p^2}\right], \tag{2.18}$$

$$\begin{aligned} E &= \frac{G^2\mu^3 M^2}{2l_\theta^2}\left(\varepsilon^2 - 1\right) \\ &= \frac{G\mu M}{2p}\left(\varepsilon^2 - 1\right). \end{aligned} \tag{2.19}$$

Let's re-write in a more obvious form: complete the square on *x* to obtain

$$\begin{aligned} \left(1-\varepsilon^2\right)\left(x + \frac{\varepsilon p}{1-\varepsilon^2}\right) + y^2 &= \frac{p^2}{1-\varepsilon^2} \\ \frac{\left(x - x_c\right)^2}{a^2} \pm \frac{y^2}{b^2} &= 1, \end{aligned} \tag{2.20}$$

which is the equation for a conic section with

$$x_c = \frac{\varepsilon p}{1-\varepsilon^2},\ a = \frac{p}{1-\varepsilon^2},\ b = \frac{p}{\sqrt{\pm\left(1-\varepsilon^2\right)}} \quad f = x_c \pm \varepsilon a = 0 \quad and \quad 2x_c$$

where *f* denotes the *x* coordinates of the foci of the conic section. Recall that the center of mass of the system is at the origin, so one of the foci coincides with the center of mass. The $\pm$ sign is picked depending on the sign of $1-\varepsilon^2$ to ensure that b is real. The turning points of the motion are given by the maximum and minimum values of *r*. Our polar form for the orbit, equation 2.15 provides the easiest means to obtain these: they are $\cos\theta = \pm 1$ they are therefore

$$r_1 = \frac{p}{1+\varepsilon} \quad r_2 = \frac{p}{1+\varepsilon} \tag{2.21}$$

$$\begin{aligned} x_1 = \frac{p}{1+\varepsilon} \quad x_2 = \frac{p}{1+\varepsilon} \\ y_1 = 0 \qquad y_2 = 0 \end{aligned} \tag{2.22}$$

Where $x_{1,2} = r_{1,2}\cos\theta$ so *x* pick up a sign and $y_{1,2} = r_{1,2}\sin\theta$, so *y* vanishes in both cases. The energy is now

$$E = \frac{G\mu M}{2p}(\varepsilon^2 - 1) = -\frac{G\mu M}{2a}. \tag{2.23}$$

The sign and magnitude of the energy thus scales inversely as the semi major axis *a*.

So what we have is the equation of a conic section. Being a circle, ellipse, or hyperbola depending on the sign of $1-\varepsilon^2$. From the qualitative discussion of central force orbits, E=0 is the dividing line between bound and unbound orbits. The implication then is that bound orbits with E<0 have positive *a* and $\varepsilon^2 < 1$ while unbound orbits with *E>0* have negative *a* and $\varepsilon^2 > 1$. The dividing case is E=0, $a = \infty$, $\varepsilon = 1$. The conic section formula is undefined there, but if we go back before we completed the square, we see that $\varepsilon^2 = 1$ causes x^2 term to vanish leaving us with the equation n of a parabola (*x* a quadratic function of *y*).

Self Assessment Exercise B

1. Derive the equation for a conic section of an orbit and discuss he nature of the curve for every possible value of *e.*

3.2.2 Detailed Study of the Different Solutions

Let's study these various solutions in some details. First, let's obtain a dimensionless parameterization of the solutions. The shape of the effective potential is set by l_θ and μ. The effective potential is minimized when the effective force vanishes, equation 1.19 we have

$$\frac{G\mu M}{r^2} = \frac{l_\theta^{\,2}}{\mu r^3} \Rightarrow r = \frac{l_\theta^{\,2}}{G\mu^2 M} = p\,. \tag{2.24}$$

The value of the effective potential at this point, which gives the minimum physically allowed value of the total energy is

$$\begin{aligned} E_{\min} = U_{eff}(r=p) &= \frac{l_\theta^{\,2}}{2\mu p^2} - \frac{G\mu M}{p} = -\frac{l_\theta^{\,2}}{2\mu p^2}, \\ &= \frac{1}{2}\frac{G\mu M}{p} \equiv -E_{scale}, \end{aligned} \tag{2.25}$$

where we have defined a scale energy that is the absolute value of the minimum energy. Referring back to our equation for E in terms of ε and p. Equation 2.19, we see that

$$E = E_{Scale}(\varepsilon^2 - 1) \tag{2.26}$$

With E_{min} and E_{scale} in hand, let's consider the various cases. Examples are illustrated in figure 2.1 below

- $E/E_{scale} < -1$: not physically allowed.
- $E/E_{scale} = -1$: Equation 2.26 tells us that E=-E_{scale} corresponds to $\varepsilon^2 = 0$. Since the eccentricity vanishes. The solution from Equation 2.15 is $p = r$ for all θ; i.e. the orbit is a circle. This is as one would expect from the effective potential that if the solution is at minimum of the effective potential then, there is no radial force. The conic section solution is elliptical (because $\varepsilon^2 < 1$) and the semimajor axes are equal a=b=p as one would expect for a circle.
- $-1 < E/E_{scale} < 0$: Because the energy remains negative, Equation 2.26 implies that $0 < \varepsilon^2 < 1$ and the conic section solution is an ellipse. As the energy increases, the eccentricity of the ellipse increases. Remember that the center of mass coincides with one of the foci of the ellipse. The center of the ellipse x_c and the second focus $2x_c$ move off to $-\infty$ as $\varepsilon \to 1$.
- $E/E_{scale} = 0$: For this solution, equation 2.26 tells us that $\varepsilon^2 = 1$. Our derivation of the conic section form fails here because the coefficient of the x^2 term vanishes, but we can return to the Cartesian form of the solution to find

$$\begin{aligned} 2px + y^2 - p^2 &= 0 \\ x &= \frac{p}{2} - \frac{y^2}{2p} \end{aligned} \tag{2.27}$$

This is a parabola whose vertex is at $p/2$, whose focus is at the origin, and whose directrix is at p. Recall that the directrix is the line perpendicular to the axis of the parabola such that the parabola consists of the set of points equidistant from the focus and the directrix. The system is just barely not bound, with the radial kinetic energy and velocity approaching zero as $r \to \infty$, because the total energy vanishes and the effective potential energy approaches zero as $r \to \infty$.

- $E/E_{scale} > 0$: For this solution, equation 2.26 gives $\varepsilon^2 > 1$. The conic section is a hyperbola. A hyperbola has two branches. Because the polar form of the solution, Equation 2.15 implies a one-to-one relationship between r and θ, only one branch of the hyperbola can be a valid solution. Intuitively, based on continuously transforming the eccentricity, we expect this to be the left branch. We can see this explicitly as follows. The left and right branches are distinguished by the fact that the former has regions with $x < 0$, while the latter does not. In order to have negative values of x, θ must be allowed to go outside the range $(-\pi/2, +\pi/2)$. A restriction on θ is placed by the requirement that r be positive, which translates to the requirement $\cos\theta \geq -\frac{1}{\varepsilon}$. Since $\varepsilon^2 > 1$ (and ε is taken to be positive always), this defines a maximum value of $|\theta|$ that is between π/2 and π.

Hence, x is allowed to be negative, and so the left branch solution is the appropriate one. For the hyperbolic solution, there is only one turning point, which is the x_1 turning point. Let's consider the evolution of the solution with ε. One focus of the hyperbola always remains at the origin. The "center" of the hyperbola, x_c, starts out at +1 and moves in toward the origin as ε gets larger, with $x_c \to 0$ in the limit $\varepsilon \to \infty$.Thus, the hyperbola continuously transforms from a parabolic-like orbit to a straight vertical line, with the turning point x_1 moving closer to the origin as ε increases. These solutions are definitely not bound. They in fact have excess kinetic energy so that, as $r \to \infty$, the radial kinetic energy (and hence radial velocity) remains non zero.

A Note on Repulsive Potentials.

While we have so far considered only attractive potentials, it is straightforward to translate the above solution to the case of repulsive potentials. We will see that, for a given energy E, we obtain the same hyperbola as for the attractive potential, but we must choose the right branch, not the left branch.

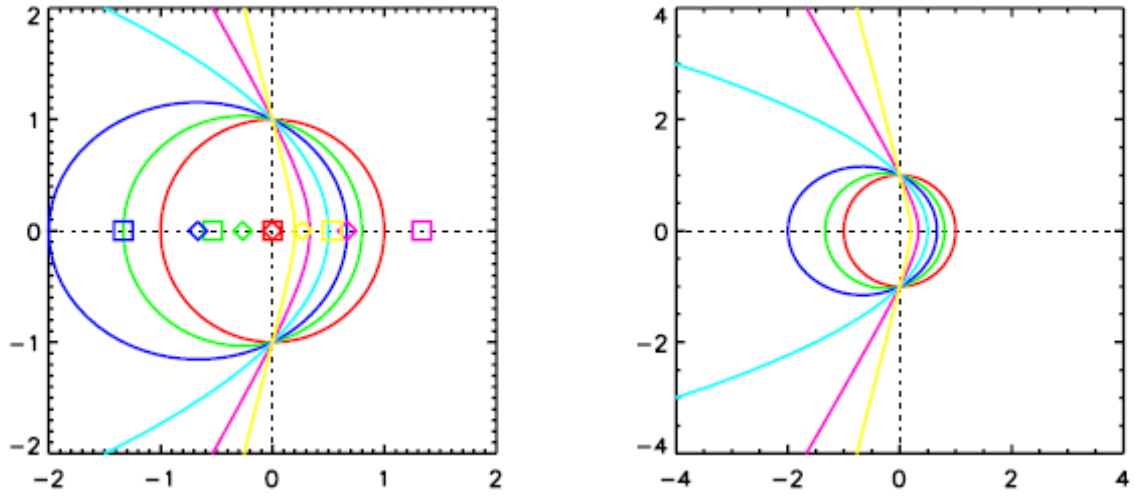


Figure 2.1: Example Keplerian orbits. The left and right figures have identical orbits; only the axis range is different. All these orbits have $\frac{l_\theta^{\,2}}{2\mu}$ and p = 1, so they have the same angular momentum (same centrifugal barrier) but different total energies. The scale factor for the energy $\frac{l_\theta^{\,2}}{2\mu p}$ is therefore also 1, so $E_{scale} = 1$ and the various orbits have energy $E = \varepsilon^2 - 1$. The legend is, in order of increasing eccentricity: ε = 0 (red), ε = 0.25 (green), ε = 0.5 (blue), ε = 1 (cyan), ε = 2 (magenta), ε = 4 (yellow). The center of each orbit (x_c) is shown by the diamond of the same color, and the second focus by the squares. The first focus of all orbits is at the origin. The second branch of the hyperbolic orbits is not shown.

The repulsive potential solution can be obtained from the attractive potential solution by simply taking $G\mu M < 0$ and making use of the physical fact that only E > 0 is allowed for a repulsive potential. Let's go through the derivation with these changes. First, the solution for u becomes

$$u(\theta) = A\cos\theta - \frac{\left|G\mu^2 M\right|}{l_\theta^{\,2}}. \tag{2.28}$$

Since $u \geq 0$ is required, the solution must have A > 0 and is only valid for some range of θ. Keeping our original definition of p (which now implies p < 0), our polar form of the solution is

$$\frac{p}{r} = 1 + \varepsilon\cos\theta. \tag{2.29}$$

Since $p < 0$ and $A > 0$, we have $\varepsilon = pA < 0$ also. Since $p < 0$, the solution is valid only when the right side is less than zero. (Originally, our requirement was that the right side be greater than zero because both p and r were positive.) The region of validity is given by $\cos\theta \geq -\frac{1}{\varepsilon} = \frac{1}{|\varepsilon|}$. For there to be any region of validity, we must have $|\varepsilon| > 1$, which implies that only hyperbolic solutions will be allowed. Furthermore, the region of validity has $\cos\theta > 0$, so we must use the right branch of the hyperbolic solution we obtained. The conversion from polar to Cartesian coordinates goes as before, as does the completion of the square. So the hyperbolic solution is still valid, as are the formulae for x_c, a, b, and f. x_c has the same sign as in the attractive hyperbolic case (since the sign flips in ε and p cancel each other), a and b change sign ($a > 0$, $b < 0$ for all repulsive solutions). Because x_c does not change sign, the foci are in the same place as the attractive hyperbolic case. The sign flips in a and b do not affect the shape of the hyperbola since only a^2 and b^2 enter in the conic section formula. Thus, we have the exact same hyperbola as in the attractive case, except that the restriction on θ implies that we must now take the right branch, not the left branch. This also means that the turning point is now x_2, not x_1.

The energy expressions, equations 2.19 and 2.23, hold without change. The starting point for the energy equation, equation 2.18 is insensitive to the sign of ε and p. Equation 2.19 does not change meaning that because the sign flips in $G\mu M$ and p cancel each other, so $|\varepsilon| > 1$ still gives $E > 0$ for all repulsive potential solutions. Equation 2.23 also keeps its same sign because both $G\mu M$ and a change sign.

Intuitively, the change from the left branch to the right branch reflects the fact that an attractive potential turns the trajectory inward toward the center of force while a repulsive potential turns it outward.

3.2.3 Summary of Quantities

The various quantities involved in Keplerian orbits are summarized in Table below.

Quantity	Symbol	Formula(e)	Sign	Significance
Angular momentum	L	$\mu r^2 \dot{\theta}$	≥ 0 $= 0$ gives trivial orbit	Centrifugal potential (brings in effect of θ motion)
Scale energy	E_{scale}	$\frac{1}{2}\frac{G\mu M}{p}$	> 0	Scale energy $= \lvert E_{min} \rvert$ for attractive potential
Scale radius	P	$\frac{l_\theta^2}{G\mu^2 M}$	> 0 for attractive pot < 0 for repulsive pot	Sets scale of orbit
eccentricity	E	$\sqrt{1 + \frac{E}{E_{scale}}}$ $-\sqrt{1 + \frac{E}{E_{scale}}}$	≥ 0 attractive pot < -1 repulsive pot.	Sets shape of conic section, related to ratio of energy to scale energy
Orbit center	x_c	$-\frac{\varepsilon p}{1-\varepsilon^2}$ $= \varepsilon a$	$= 0$ circular < 0 elliptical > 0 hyperbolic	
Semimajor axis	A	$\frac{p}{1-\varepsilon^2}$	> 0 circle/ elliptical < 0 hyperbolic attractive	Distance from x_c to vertices along major axis
Semimajor axis	B	$\frac{p}{\sqrt{\pm 1-\varepsilon^2}}$	> 0 attractive pot. < 0 repulsive Pot	Distance from x_c to vertices along major axis
Turning points	X_1	$\frac{p}{1+\varepsilon^2}$	> 0 < 0 circle/elliptical	Turning points of motion relative to CM = focus 1 apsides for circ./ellip. orbits
	x_2	$-\frac{p}{1-\varepsilon^2}$	> 0 hyperbolic	

For hyperbolic orbits, x_1 is the turning point for attractive potentials, x_2 the turning point for repulsive potentials

Table 2.1: Parameters for Keplerian orbits.

3.2.4 Time Dependence of the Kepler Problem Solutions

So far we have only found the orbit solutions as functions $r(\theta)$. This of course describes much of the dynamics of the problem. But one does indeed frequently want the orbit as a function of time, so we obtain that result here.

Period of Elliptical Orbits

We can quickly obtain the period of elliptical orbits by using Kepler's second law, Kepler's second law tells us that

$$\frac{dA}{dt} = \frac{l_\theta}{2\mu}. \tag{2.30}$$

The area of the ellipse is $A = \pi ab$, so the period is the time required to sweep out the area of the ellipse,

$$T = \frac{A}{\frac{dA}{dt}} = \frac{\pi ab}{\frac{l_\theta}{2\mu}}. \tag{2.31}$$

Let's write this in terms of the parameters of the orbit p and ε to obtain Kepler's third law:

$$\begin{aligned} T &= 2\pi\mu \frac{p^2}{\left(1-\varepsilon^2\right)^{3/2}} \frac{1}{\mu\sqrt{GMp}} \\ &= 2\pi\sqrt{\frac{a^3}{GM}} \end{aligned} \tag{2.32}$$

The period depends only on *a*. Of course, a encodes information about the total energy E and the angular momentum l_θ. The implication of Kepler's third law for the solar system is that all orbits should lie on a single $T^2 \alpha\, a^3$ curve because M, dominated by the sun, is almost the same for all planets.

Self Assessment Exercise C

1. Prove that the scale energy is absolute value of minimum energy and that $E = E_{Scale}(\varepsilon^2 - 1)$
2. Prove that the Kepler's law asserts that the period T of the motion and the main radius *a* of the ellipse satisfy a relation $T = Ca^{3/2}$, where C is a constant independent of the initial data.

4.0 Conclusion

Kepler tells us not only that the orbit is an ellipse, but also that the sun is at one focus. To verify that, note the other focus of an ellipse is symmetrically located, at (−2ε*a*, 0), and work out the sum of the distances of any point on the ellipse from the two foci. This will verify that $d+r = 2a$ is a constant, showing that the orbit is indeed an ellipse with the sun at one focus.

5.0 Summary

- The Kepler's problem

When we specialize to $U(r)\alpha\left(\frac{1}{r}\right)$, we may obtain more specific results. The differential equation relating r and θ or u and θ is

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{G\mu^2M}{l_\theta^2} \qquad \frac{d^2u}{d\theta^2}+u=\frac{G\mu^2M}{l_\theta^2}.$$

- The generic solution may be written

$$u(\theta)=A\cos(\theta-\theta_0)+\frac{G\mu^2M}{l_\theta^2} \qquad p=r+\varepsilon r\cos\theta \qquad p=\frac{l_\theta^2}{G\mu^2M} \quad \varepsilon=pA.$$

We specify two initial conditions (neglecting θ_0): and the total energy E. The constant A and ε are related to the total energy by

$$E=\frac{l_\theta^2}{2\mu}\left[A^2-\frac{G\mu^2M}{l_\theta^2}\right]=\frac{G\mu M}{2p}\left(\varepsilon^2-1\right)\equiv E_{scale}\left(\varepsilon^2-1\right).$$

- The energy may also be written as

$$E=-\frac{G\mu M}{2a}$$

- The assorted orbital parameters are summarized in table 2.1
- Kepler's third law, which tells us that the period of elliptical orbits, is obtained from Kepler's second law and the area of an elliptical orbit, giving as

$$T=2\pi\sqrt{\frac{a^3}{GM}}.$$

- The full time dependence of elliptical orbits can be obtained through use of the eccentric anomaly, ε, which is defined implicitly in terms of the true anomaly, θ. One begins with

$$r(\theta)\cos\theta=x=x_c+a\cos\varepsilon$$

and one obtains

$$r(\varepsilon)=a(1-\varepsilon\cos\varepsilon) \qquad t(\varepsilon)=\sqrt{GMa^3}\left(\varepsilon-\varepsilon\sin\varepsilon\right)$$

$$x(\varepsilon)=a(\cos\varepsilon-\varepsilon) \qquad y(\varepsilon)=a\sqrt{1-\varepsilon^2}\sin\varepsilon.$$

- An analogous parametization can be done for parabolic orbits, though the geometrical interpretation of the eccentric anomaly is no longer valid. Beginning with

$$r(\varepsilon)=a(\varepsilon\cosh\varepsilon\mp1) \qquad t(\varepsilon)=\sqrt{GMa^3}\left(\varepsilon\sin\varepsilon\mp\varepsilon\right)$$

$$x(\varepsilon)=a(\varepsilon\mp\cosh\varepsilon) \qquad y(\varepsilon)=a\sqrt{1-\varepsilon^2}\sin\varepsilon$$

where the upper signs are for an attractive orbit and the lower signs for a repulsive one.

6.0 Tutor Marked Assignments (TMAs)

1. Use θ-component of Lagrangian equation to prove Kepler's second law.
2. Use r-component of Lagrange's equation to prove Kepler's first law.
3. Use the second Kepler's law and the expression for the angular momentum to prove Kepler's third Law.
4. Derive the equation of a conic section and describe the shape for $p > 0$, $p = 0$ and $p < 0$.
5. Derive the equation for a conic section of an orbit and discuss the nature of the curve for every possible value of *e.*
6. Prove that the scale energy is absolute value of minimum energy and that $E=E_{Scale}(\varepsilon^2-1)$.
7. Prove that the Kepler's law asserts that the period T of the motion and the main radius a of the ellipse satisfy a relation $T=Ca^{3/2}$ where C is a constant independent of the initial data.

7.0 Further Reading and Other Resources

Classical Mechanics by H.Goldstein, Narosa Publishing Home, New Delhi.

Classical Dynamics of Particles and Systems by Marion and Thomtron, Third Edition, Horoloma Book Jovanovich College Publisher.

Introduction to Classical Mechanics by R.G.Takawale and P.S.Puranik, Tata Mc-Graw Hill Publishing Company Limited, New Delhi.

UNIT 3 SCATTERING CROSS SECTIONS

CONTENTS

1.0 Introduction

Now that we have studied central force motion, obtaining qualitative results for arbitrary potentials and specific results for $1/r$ potentials, we have information about the dynamics of collisional interactions. We can use this to develop the concept of scattering cross section, which intimately uses the kinematics of collisions and the orbit information from central force motion. The archetypal scattering problem, we will consider is one involving a particle incident on a force center that scatters the incident particle via a conservative central force with potential *U(r).* We have demonstrated that, a two-particle system interacting via a conservative central force is equivalent to such a system when considered in its center-of-mass frame.

2.0 Objectives

After studying this unit, you will be able to:

- State the expression for energy of a system of particle incident on a force center subject to a scattering potential.
- Calculate the differential cross section.
- Calculate the total cross section.

3.0 Main Contents

3.1 Setting up the Problem

3.1.1 Initial Conditions

As noted, we consider a particle incident on a force center subject to a scattering potential. It is assumed that the particle is asymptotically free, having enough energy to be in an unbound orbit. In order to have unbound orbits, the potential energy must go to zero at large *r*. The energy of the system is therefore

$$E = \frac{1}{2}\mu v_\infty^{\,2}, \tag{3.1}$$

where v_∞ is the asymptotic particle speed as $r \to \infty$. We will use v_∞ to parametize the initial energy of the system, v_∞ only specifies an energy. We must also specify the geometry. The only free parameter left is l_θ, which we can see specifies the geometry as follows. Since the system is unbound, the trajectory must become straight lines at large radii, reflecting the incoming and outgoing velocity vectors. For example, for a $1/r$ potential, we know that the ingoing and outgoing velocity vectors define the asymptotes of the hyperbolic orbit. We can calculate the distance between the scattering center and these straight lines where they come closest to the scattering center. This distance is defined to be the impact parameter, usually associated with the symbol *b*. This is displayed in Figure 3.1 below. The small line from the center to the dotted line is the impact parameter. The impact parameter is related to l_θ. We know that $l_\theta = \vec{r} \times \mu\dot{\vec{r}}$. At $r = \infty$, we know that $\dot{\vec{r}}$ points in the direction along the asymptote and $\vec{r}$ points to the origin, so the angle between $\vec{r}$ and $\dot{\vec{r}}$ which we call γ is subtended by the impact parameter *b*. $\dot{\vec{r}}$ is shown approximately in Figure 3.1 by the line that extend from the center to the -10 point. So,

$$l_\theta = r\mu v_\infty \sin\gamma = \mu v_\infty b \tag{3.2}$$

l_θ specifies the geometry through *b*, which fixes the position of the asymptote (v_∞) relative to the scattering center.

3.1.2 Scattering Angle

In Figure 3.1, the scattering angle, θ_s, is the angle between the incoming and outgoing velocity vectors. θ_s is c form of the potential function.

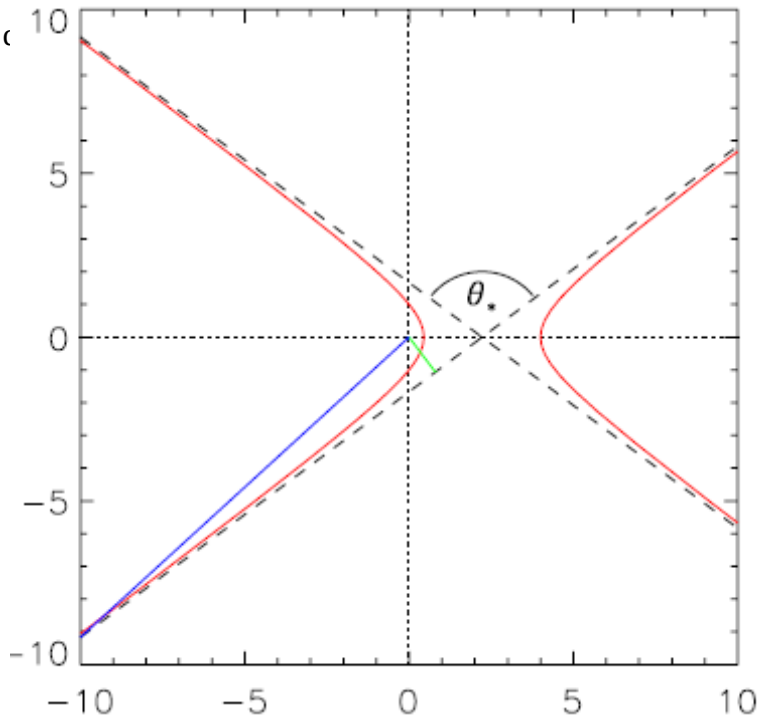


Figure 3.1: Scattering impact parameter illustration
(where θ_* is the Scattering angle herein represented as θ_s)

Self Assessment Exercise A

1. Derive the expression of the angular momentum of the particle incident on a force center subject to a scattering potential in terms of μ, E and b

3.2 The Generic Cross Section

3.2.1 Incident Beam

Now that we have made our definitions, let us bring in the concept of differential cross section. Suppose we have a beam of incoming particles, all with same velocity $\vec{v}_\infty$. Let the flux of particles F be the number of particles passing through a unit area in a unit time. If the beam has particle number density n, then the flux is

$$F = nv_\infty \tag{3.3}$$

Assume that the beam has a circular cross section and the axis of the beam points directly at the scattering center. The incoming particles will thus have a range of impact parameter values, ranging from $b = 0$ (along the beam axis) to $b = b_{max}$ (at the outer edge of the beam). The beam radius is b_{max} and its cross-sectional area is $A = \pi b_{\max}^2$.

3.2.2 Differential Cross Section

The incident particles in the beam will be scattered into a range of angles depending on their input impact parameters (and the beam velocity). We define the differential scattering cross section, $\frac{d\sigma}{d\Omega}(\theta_s, \phi_s)$, via the probability of an incident particle being scattered into the solid angle dΩ in the direction (θ_s, ϕ_s) where θ_s is the polar angle measured from the beam axis and ϕ_s is the azimuthal angle around the beam axis. If $dN(\theta_s, \phi_s)$ is the number of particles per unit time scattered into the solid angle $d\Omega$ at (θ_s, ϕ_s), we define the differential cross section via the relation

$$\frac{1}{A}\frac{d\sigma}{d\Omega}(\theta_s,\phi_s)d\Omega = \frac{dN(\theta_s,\phi_s)}{FA}. \tag{3.4}$$

Let us explain the above. On the right hand side of the first line, the denominator is the number of particles per unit time incident on the target from a beam of flux F and cross-sectional area A. Since the numerator is the number of particles per unit time that scatter into $d\Omega$ at (θ_s, ϕ_s), the right hand side is thus the fraction of particles that scatter into $d\Omega$ at (θ_s, ϕ_s); it is a probability. We include additional factors so that $\frac{d\sigma}{d\Omega}$ is defined only by the scattering force, not by parameters of the experiment. We have 1/F on the right hand side but none on the left side because including the *1/F* makes the ratio *dN/F* independent of *F*: if *F* goes up, *dN* goes up proportionally. However, we include *1/A* on the left hand side to cancel the *1/A* on the right side because $dN(\theta_s, \phi_s)$ may not scale with A: if one adds cross-sectional area at a radius from which particles do not scatter into the particular solid angle $d\Omega$ at (θ_s, ϕ_s), $dN(\theta_s, \phi_s)$ will not increase when A increases. Hence the different treatment of F and A. Solving for the differential cross section gives

$$\frac{d\sigma}{d\Omega}(\theta_s,\phi_s) = \frac{1}{F}\frac{dN(\theta_s,\phi_s)}{d\Omega}. \tag{3.5}$$

The beam area A has dropped out, $\frac{d\sigma}{d\Omega}$ has units of area per steradian; hence the name cross section. If we assume central force scattering, the problem is azimuthally symmetric about the beam axis and we may integrate over ϕ_s so that $d\Omega = 2\pi \sin\theta_s d\theta_s$. Furthermore, we know for central force scattering that there is a one-to-one correspondence between b and θ_s for a given v_∞. Therefore, the particles scattering into the interval $d\theta_s$ in polar angle come from some range db

in impact parameter. Azimuthal symmetry of the problem allows us to integrate over azimuthal angle in the beam also. We may relate *db* and $d\theta_s$ by requiring conservation of particle number:

$$F\,2\pi b db = -F\frac{d\sigma}{d\Omega}2\pi\sin\theta_s d\theta_s . \tag{3.6}$$

A negative sign has been inserted under the assumption that the potential decreases in strength monotonically with radius: if you increase the impact parameter a little bit, the scattering angle should decrease, so a positive *db* implies a negative $d\theta_s$. Re-writing, we have

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta_s}\left|\frac{db}{d\theta_s}\right| . \tag{3.7}$$

That is, if we know the function $b(\theta_s, v_\infty)$, then, we can determine the distribution of particles in scattering angle given a uniform incoming beam.

3.2.3 Total Cross Section

Once one has calculated $\frac{d\sigma}{d\Omega}$, it is formally a straightforward thing to calculate the total cross section:

$$\sigma = \int d\Omega\frac{d\sigma}{d\Omega} = 2\pi\int_0^\pi d\theta_s\sin\theta_s\frac{d\sigma}{d\Omega} = 2\pi\int_0^\infty dbb . \tag{3.8}$$

As one would expect, the total cross section is related to the probability that an incoming particle will be scattered to any angle. It can be viewed as a total "effective area" of the scattering center; the number of particles in the incoming beam that will be scattered is the same as if every particle within the central σ of the beam were scattered and all others left untouched.

3.2.4 Calculating $b(\theta_s)$

For a generic central potential, we can obtain a formula relating *b* and θ_s by returning to the integral relations that define the orbit. Recall equation 1.34:

$$\theta(r)-\theta(0) = \pm l_\theta\int_{r(0)}^{r}\frac{dr'}{|r|^2}\left[2\mu(E-U(r)-\frac{l_\theta^{\,2}}{r'^2}\right]^{-1/2} \tag{3.9}$$

θ is not θ_s, but it is related to it. Referring back to our scattering picture above, the scattering angle θ_s is the complement of the angle between the incoming and outgoing asymptotes. The angle between the asymptotes is related to the orbit angle as $r\to\infty$. For repulsive scattering, the angle between the asymptotes is just $|\theta_{out}-\theta_{in}|$, where these are the asymptotic orbit angles for the incoming and outgoing particles. For attractive scattering, since θ is measured from the +x-axis, the angle between the asymptotes is $2\pi-|\theta_{out}-\theta_{in}|$. So the scattering angle is

$$\theta_s = \mp\left(2\pi-|\theta_{out}-\theta_{in}|\right) \tag{3.10}$$

where – (negative) goes with the attractive potential. Next, $|\theta_{out}-\theta_{in}|$ is twice the angle between θ_{out} or θ_{in} and $\theta=0$. $\theta=0$ is obtained when $r=r_{\min}$, the turning point. So we have

$$\begin{aligned}\theta_s &= \mp\left(\pi-2\left|l_\theta\int_{r_{\min}}^{\infty}\frac{dr'}{[r'(t)]^2}\left[2\mu(E-U(r)-\frac{l_\theta^{\,2}}{r'^2}\right]^{-1/2}\right|\right)\\ &= \mp\left(\pi-2\left|\int_{r_{\min}}^{\infty}\frac{bdr'}{r'^2}\left[1-\frac{U(r)}{E}-\frac{b^2}{r'^2}\right]^{-1/2}\right|\right).\end{aligned} \tag{3.11}$$

Self Assessment Exercise B

1. Given the differential cross section $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta_s}\left|\frac{db}{d\theta_s}\right|$ Calculate the total cross section and the $b(\theta_s)$.

3.3 $\frac{1}{r}$ Potential

For the $\frac{1}{r}$ potential, we can find the differential cross section explicitly because we have explicit relationships between θ and r

3.3.1 Finding $b(\theta_s)$

We demonstrated earlier that the azimuthal angle θ of the orbit relative to the center of mass is limited to be $\cos\theta > \frac{1}{\varepsilon}$ this gives us θ_{out} and θ_{in}

$$\theta_{out,in} = \pm\arccos\left(-\frac{1}{\varepsilon}\right) \tag{3.12}$$

where $\theta_{out} = -\theta_{in}$ and the choice of sign for θ_{in} depends on initial conditions. So we have

$$\begin{aligned}\left|\theta_{out} - \theta_{in}\right| &= 2\left|\arccos\left(-\frac{1}{\varepsilon}\right)\right| \\ &= 2\arccos\left(\mp\frac{1}{|\varepsilon|}\right)\end{aligned} \tag{3.13}$$

where the – (negative) is for an attractive potential. So we have

$$\theta_s = \mp\left[\pi - 2\arccos\left(\mp\frac{1}{|\varepsilon|}\right)\right] = \pi - 2\arccos\frac{1}{|\varepsilon|}\frac{1}{|\varepsilon|} = \sin\frac{\theta_s}{2} \tag{3.14}$$

where the two different potentials have yielded the same result. With some works, we can write b in terms of ε. Starting with equation 2.19, we have

this can be re-written $E = \frac{l_\theta^{\,2}}{2\mu p^2}(\varepsilon^2 - 1)$

$$\frac{\mu p}{l_\theta} = \sqrt{\frac{\mu}{2E}(\varepsilon^2 - 1)}\,. \tag{3.15}$$

Then, $\frac{l_\theta}{G\mu M} = \sqrt{\frac{\mu}{2E}}\sqrt{\varepsilon^2 - 1}$

We have $\mu v_\infty b = \frac{G\mu M}{\sqrt{2E/\mu}}\sqrt{\varepsilon^2 - 1}\,.$ (3.16)

that is $b = \frac{G\mu M}{2E}\sqrt{\varepsilon^2 - 1}\,.$ (3.17)

So we may now find $b(\theta_s)$:

$$b = \frac{G\mu M}{2E}\sqrt{\csc^2\frac{\theta_s}{2} - 1} = \frac{G\mu M}{2E}\cot\frac{\theta_s}{2}, \tag{3.18}$$

where we take positive square root because $0 < \frac{\theta_s}{2} < \frac{\pi}{2}$.

3.3.2 Calculating the Differential Cross Section

We will need $\frac{db}{d\theta_s}$ to calculate the differential cross section, so, taking the derivative:

$$\left|\frac{db}{d\theta_s}\right| = \frac{1}{2}\frac{G\mu M}{2E}\csc^2\frac{\theta_s}{2} = \frac{b}{2}\csc\frac{\theta_s}{2}\sec\frac{\theta_s}{2}\,. \tag{3.19}$$

Finally, using our formula for the differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta_s}\left|\frac{db}{d\theta_s}\right| = \frac{b^2}{4}\csc^2\frac{\theta_s}{2}\sec^2\frac{\theta_s}{2} = \left(\frac{G\mu M}{4E}\right)^2\frac{1}{\sin^4\frac{\theta_s}{2}}\,. \tag{3.20}$$

Note that the result is independent of whether the scattering is attractive or repulsive. This result is the well known Rutherford scattering formula, first observed in the scattering of α particles (^{4}He nuclei) from gold nuclei in a foil. The observation that the scattering obeyed the simple expectation of a $\frac{1}{r}$ potential was an important piece of evidence that the atom consists of a charged nucleus surrounded mostly by empty space, as opposed to a "plum pudding" type model with electrons and protons intermixed uniformly in the atom.

3.3.3 The Total Cross Section

If one tries to calculate the total cross section from the Rutherford formula, one will end up with an infinite result. This is because, for a $\frac{1}{r}$ potential, the probability of scattering does not decrease sufficiently quickly with increasing *b* because the effective area of the scattering center is infinite. If the potential is made to converge more quickly (e.g., by multiplying be an exponential decay), then a finite total cross section is obtained.

Self Assessment Exercise C

1. Derive the $b(\theta_s)$, the differential cross section and total cross section for the $\frac{1}{r}$ potential

4.0 Conclusion

We have discussed the 1/r potential in terms of Newtonian gravity but of course it is equally applicable to Coulomb's law of electrostatic forces.

5.0 at infinity **SUMMARY**

- For central-force scattering problems, we generally considered unbound central-force orbits, rather than parametizing in terms of energy E and angular momentum l_θ, we use the velocity v_∞ and the impact parameter b (distance of closest approach to scattering center). These parameter are related by
$$E = \frac{1}{2}\mu v_\infty^2 \qquad l_\theta = \mu v_\infty b\,,$$
- We usually phrase scattering in terms of an incoming beam of particles of number flux
$$F = nv_\infty\,.$$
- The differential scattering cross section gives us the area of the beam that will be scattered into a solid angle $d\Omega$:
$$\frac{d\sigma}{d\Omega} = \frac{1}{F}\frac{dN}{d\Omega}\,.$$
- The differential cross section can be found if the relationship between the input impact parameter and the scattering angle θ_s (the angle between the incoming and outgoing trajectories) is known

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta_s}\left|\frac{db}{d\theta_s}\right|.$$

- The total scattering cross section, which gives us the effective area of the scattering center, is

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = 2\pi\int_0^{\pi} d\theta_s \sin\theta_s \frac{d\sigma}{d\Omega} = 2\pi\int_0^{\infty} dbb\,.$$

- The scattering angle is calculated from the potential function via

$$\theta_s = \mp\left(\pi - 2\left|\int \frac{bdr'}{r'^2}\left[1 - \frac{U(r)}{E} - \frac{b^2}{r'^2}\right]^{-1/2}\right|\right)$$

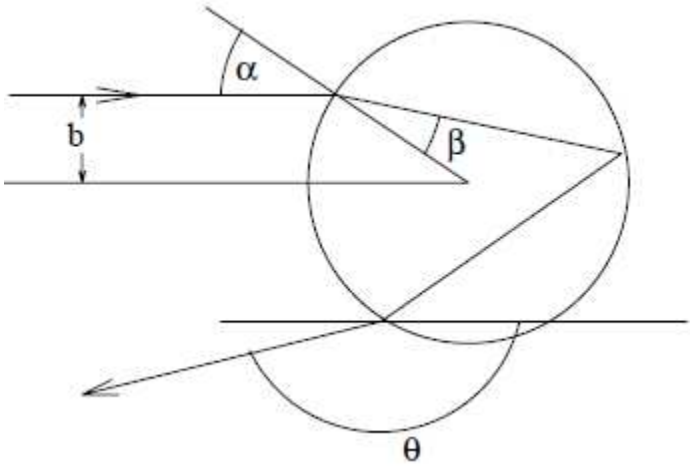
- For a potential $U(r) = \pm\frac{k}{r}$, the relationship between impact parameter and scattering angle and the differential cross section are given by

$$|b| = \frac{k}{2E}\left|\cot\frac{\theta_s}{2}\right| \quad \frac{d\sigma}{d\Omega} = \left(\frac{k}{4E}\right)^2 \frac{1}{\sin^4\frac{\theta_s}{2}}.$$

The relative sign between b and θ_s is chosen based on whether the potential is attractive or repulsive (E > 0 always for unbound orbits)

6.0 Tutor Marked Assignments (TMAs)

1. Consider a spherical droplet of water in the sunlight. A ray of light with impact parameter *b* is refracted, so by Snell's Law $n \sin\beta = \sin\alpha$. It is then internally reflected once and refracted again on the way out.



a) Express the scattering angle θ in terms of α and β.

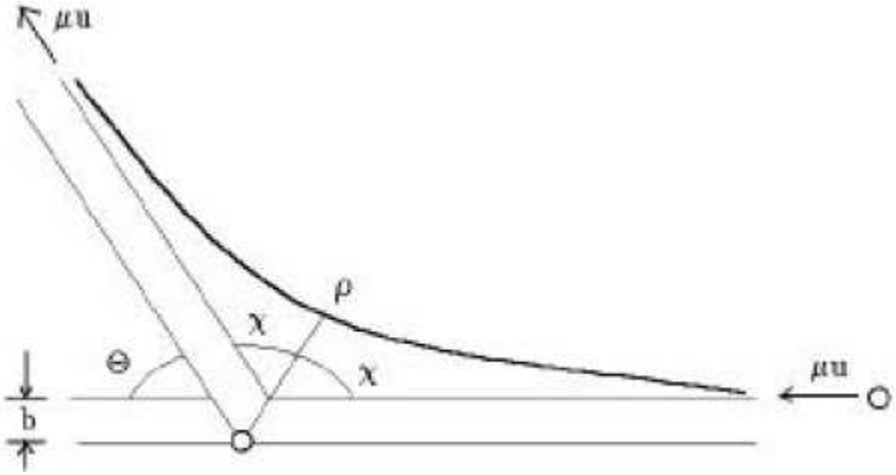
b) Find the scattering cross section $d\sigma = d\Omega$ as a function of θ, α and β (which is implicitly a function of θ *from* (a) and Snell's Law).

c) The smallest value of θ is called the rainbow scattering angle. Why? Find it numerically to first order in δ *if* the index of refraction is $n = 1{:}333 + \delta$.

d) The visual spectrum runs from violet, where $n = 1{:}343$, to red, where $n = 1{:}331$. Find the angular radius of the rainbow's circle, and the angular width of the rainbow, and explain whether the red or blue is on the outside.

2. Consider the case of a projectile particle of mass μ being deflected by a repulsive central force potential U(r) > 0. As the projectile particle approaches from the right (at r = ∞, □ = 0) moving with speed u, it is progressively deflected until it reaches a minimum radius ρ at $\theta = \chi$ after which the projectile particle moves away from the repulsion center until it reaches $r = \infty$ at a deflection angle θ and again moving with speed u. From the Figure

shown below, we can see that the scattering process is symmetric about the line of closest approach.



Define the angle χ in terms of impact parameter *b*, μ and E (total energy).

7.0 Further Reading and Other Resources

Classical Mechanics by H.Goldstein, Narosa Publishing Home, New Delhi.

Classical Dynamics of Particles and Systems by Marion and Thomtron, Third Edition, Horoloma Book Jovanovich College Publisher.

Introduction to Classical Mechanics by R.G.Takawale and P.S.Puranik, Tata Mc-Graw Hill Publishing Company Limited, New Delhi.

Module 4 Motion in Non-Inertial Reference Frame

Unit 1 Time Derivative in Fixed and Rotating Frames
Unit 2 Motion Relative to Earth

UNIT 1 TIME DERIVATIVES IN FIXED AND ROTATING FRAMES

CONTENTS

1.0 Introduction

A non-inertial reference frame is a reference frame that is not tied to the state of motion of an observer, with the property that each physical law portrays itself in the same form in every inertial frame. As such, the laws of physics in such a frame do not take on their most simple form, as required by the special principle of relativity. To explain the motion of bodies *entirely within the viewpoint* of non-inertial reference frames, fictitious forces (also called *inertial forces*, *pseudo-forces* and *d'Alembert forces*) must be introduced to account for the observed motion, such as the Coriolis force or the centrifugal force, as derived from the acceleration of the non-inertial frame

2.0 Objectives

After studying this unit, you will be able to:

- Derive the time derivatives of vector A in fixed and rotating reference frame.
- State the expressions for translational velocity and acceleration, V and A respectively.
- Determine the relationship between the velocities of fixed and rotating reference frame.
- Show the relationship between the acceleration of fixed and rotating reference frame.

3.0 Main Contents

3.1 Time Derivatives in Fixed and Rotating Frames

Let us consider the time derivative of an arbitrary vector **A** in two reference frames. The first reference frame is called the fixed frame and is expressed in terms of the Cartesian coordinates $r' = (x'; y'; z')$. The second reference frame is called the rotating frame and is expressed in terms of the Cartesian coordinates *r = (x; y; z).* In the Figure below, the rotating frame shares the same origin as the fixed frame and the rotation angular velocity ω of the rotating frame (with respect to the fixed frame) has components (ω_x, ω_y, ω_z).

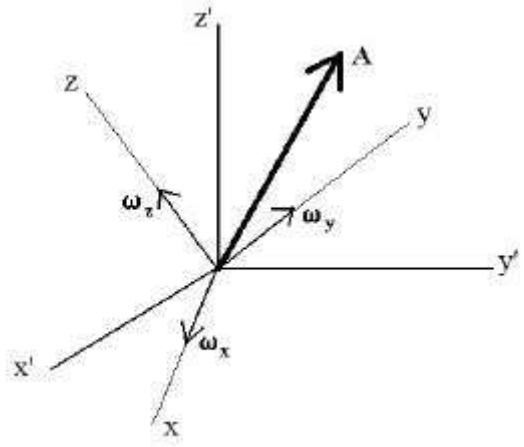


figure 1.1

Since observations are often made in a rotating frame of reference, we decompose the vector *A* in terms of components $\boldsymbol{A}_i$ in the rotating frame (with unit vectors $\hat{x}^i$). Thus, $\mathbf{A} = \mathbf{A}_i \hat{x}^i$ (using the summation rule) and the time derivative of *A* as observed in the fixed frame is

$$\frac{d\mathbf{A}}{dt} = \frac{d\mathbf{A}_i}{dt}\hat{x}^i + \mathbf{A}_i \frac{d\hat{x}^i}{dt}. \qquad (1.1)$$

The interpretation of the first term is that of the time derivative of *A* as observed in the rotating frame (where the unit vector $\hat{x}^i$ are constant) while the second term involves the time dependence of the relation between the fixed and rotating frames. We now express $d\hat{x}^i / dt$ as a vector in rotating frame as

$$\frac{d\hat{x}^i}{dt} = R^{ij}\hat{x}_j = \varepsilon^{ijk}\omega_k \hat{x}_j, \qquad (1.2)$$

where *R* represents the rotation matrix associated with the rotating frame of reference; this rotation matrix associated is anti-symmetric *(R^{ij} =-R^{ji})* and can be written in terms of the anti-symmetric tensor ε^{ijk} (defined in terms of the vector product $\mathbf{A}\times\mathbf{B} = \mathbf{A}_i\mathbf{B}_j\varepsilon^{ijk}\hat{x}_k$ for two arbitrary vectors **A** and **B**) as $R^{ij} = \varepsilon^{ijk}\omega_k$ (see appendix) where ω_k denotes the components of the angular velocity ω in the rotating frame. Hence, the second term in equation 1.1 above becomes

$$\mathbf{A}_i \frac{d\hat{x}^i}{dt} = \mathbf{A}_i \varepsilon^{ijk}\omega_k \hat{x}_j = \omega\times\mathbf{A}. \qquad (1.3)$$

This time derivative of an arbitrary rotating frame vector A in a fixed frame is, therefore expressed as

$$\left(\frac{d\mathbf{A}}{dt}\right)_f = \left(\frac{d\mathbf{A}}{dt}\right)_r + \omega\times\mathbf{A}, \qquad (1.4)$$

where $\left(d/dt\right)_f$ denotes the time derivative as observed in the fixed (f) frame while $\left(d/dt\right)_r$ denotes the time derivative as observed in the rotating *(R)* frame. An application of this formular relates to the time derivative of the rotation angular velocity ω itself. One can easily see that

$$\left(\frac{d\omega}{dt}\right)_f = \dot{\omega} = \left(\frac{d\omega}{dt}\right)_r. \qquad (1.5)$$

Since the second term of equation 1.4 above vanishes for $\boldsymbol{A} = \omega$; the time derivative of ω is, therefore, the same in both frames of reference and is denoted $\dot{\omega}$ in what follows.

Self Assessment Exercise A

1. Show that the time derivative of ω is the same in both fixed and rotating frame of reference

3.2 Accelerations in Rotating Frames

We now consider the general case of a rotating frame and fixed frame being related by translation and rotation.

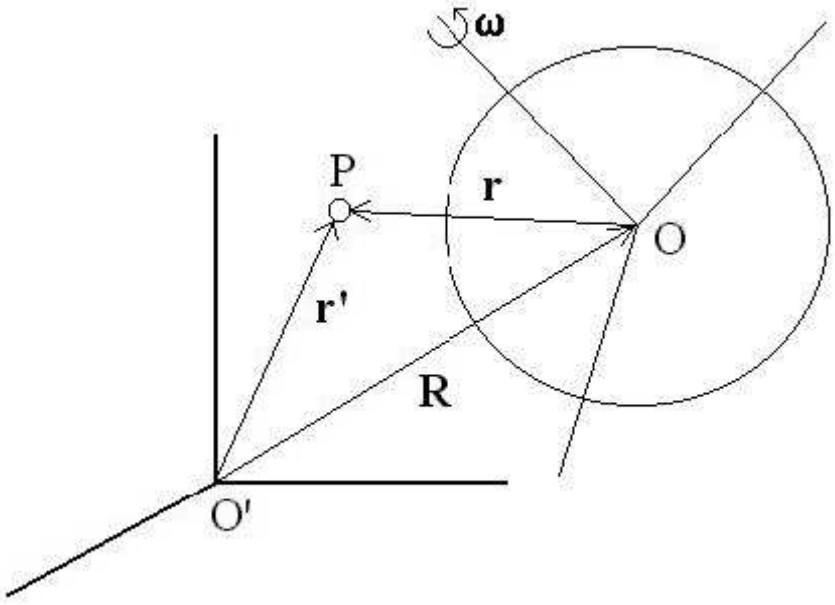


figure 1.2

In the figure 1.2 above, the position of a point P according to the fixed frame of reference is labeled r', while the position of the same point according to the rotating frame of reference is labeled r, and $r' = R + r$, where R denotes the position of the origin of the rotating frame according to the fixed frame. Since the velocity of the point P involves the rate of change of position, we must now be careful in defining which time-derivatives operator, $\left(d/dt\right)_f$ or $\left(d/dt\right)_r$ that is to be used.

The velocities of point P as observed in the fixed and rotating frames are defined as

$$v_f = \left(\frac{dr'}{dt}\right)_f \quad \text{and} \quad v_r = \left(\frac{dr}{dt}\right)_r \tag{1.6}$$

respectively. Using equation 1.4 the relationship between the fixed frame and rotating frame velocity is expressed as

$$v_f = \left(\frac{dR}{dt}\right)_f + \left(\frac{dr}{dt}\right)_f = V + v_r + \omega \times r, \tag{1.7}$$

where $V = \left(\frac{dR}{dt}\right)_f$ denotes the translation velocity of the rotating-frame origin (as observed in the fixed frame).

Using equation 1.7 above, we are now in a position to evaluate expressions for the acceleration of point P as observed in the fixed and rotating frames of reference

$$a_f = \left(\frac{dv_f}{dt}\right)_f \quad \text{and} \quad a_r = \left(\frac{dv_r}{dt}\right)_r \tag{1.8}$$

respectively. Hence, using equation 1.7 we find

$$\begin{aligned} a_f &= \left(\frac{dV}{dt}\right)_f + \left(\frac{dv_r}{dt}\right)_f + \left(\frac{d\omega}{dt}\right)_f \times r + \omega \times \left(\frac{dr}{dt}\right)_f \\ &= A + (a_r + \omega \times v_r) + \dot{\omega} \times r + \omega \times (v_r + \omega \times r), \end{aligned} \tag{1.9}$$

or

$$a_f = A + a_r + 2\omega \times v_r + \dot{\omega} \times r + \omega \times (\omega \times r), \tag{1.10}$$

where $A = \left(\frac{dV}{dt}\right)_f$ denotes the translational acceleration of the rotating frame origin (as observed in the fixed frame of reference). We can now write an expression for the acceleration of point P as observed in the rotating frame as

$$a_r = a_f - A - \omega\times(\omega\times r) - 2\omega\times v_r - \dot{\omega}\times r, \tag{1.11}$$

which represent the sum of the net inertial acceleration *(a_f –A)*, the centrifugal acceleration $-\omega\times(\omega\times r)$ (see figure below)

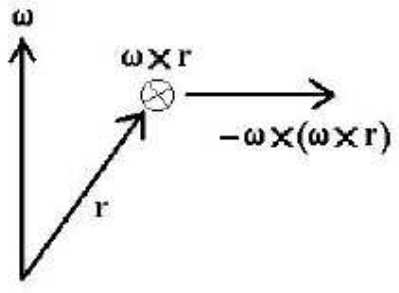


figure 1.3

the coriolis acceleration $-2\omega\times v_r$

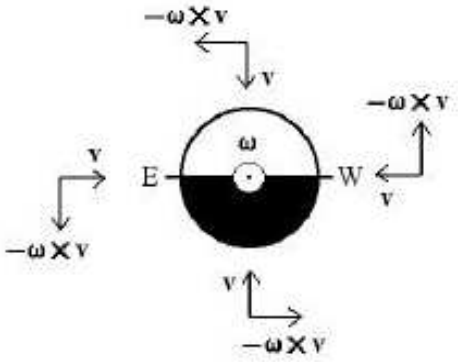


figure 1.4

and an angular acceleration term $-\dot{\omega}\times r\omega$ which depends explicitly on the time dependence of the rotation angular velocity ω.

The centrifugal acceleration (which is directed outwardly from the rotation axis) represents a familiar non-inertial effect in Physics. A less familiar non-inertial effect is the Coriolis acceleration. The Figure 1.4 above shows that an object falling inwardly also experiences an eastward acceleration.

Self Assessment Exercise B

1. Given the figure 2.1 above determine the relationship between acceleration between fixed and rotating reference frame.
2. Make sure you have a good understanding of differentiation and integration

3.3 Lagrangian Formulation Of Non-Inertia Motion

The Lagrangian for a particle of mass *m* moving in a non-inertial rotating frame (with its origin coinciding with the fixed-frame origin) in the presence of the potential *U(r)* is expressed as

$$L(r,\dot{r}) = \frac{m}{2}\left|\dot{r} + \omega\times r\right|^2 - U(r), \tag{1.12}$$

where ω is the angular velocity vector and we use the formula

$$\left|\dot{r} + \omega\times r\right|^2 = \left|\dot{r}\right|^2 + 2\omega\cdot(r\times\dot{r}) + \left[\omega^2 r^2 - (\omega\cdot r)^2\right]. \tag{1.13}$$

Using the Lagrangian (equation 2.1), we now derive the general Euler-Lagrange equation for *r*. First, we derive an expression for the canonical momentum

$$P = \frac{\partial L}{d\dot{r}} = m(\dot{r} + \omega \times r), \tag{1.14}$$

and $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m(\ddot{r} + \dot{\omega} \times r + \omega \times \dot{r}) . \tag{1.15}$$

Next we derive the partial derivative

$$\frac{\partial L}{\partial r} = -\nabla U(r) - m[\omega \times \dot{r} + \omega \times (\omega \times r)], \tag{1.16}$$

so that the Euler-lagrange equation are

$$m\ddot{r} = -\nabla U(r) - m[\dot{\omega} \times r + 2\omega \times \dot{r} + \omega \times (\omega \times r)]. \tag{1.17}$$

Here, the potential energy term generates the fixed-frame acceleration, $-\nabla U = ma_f$ and the Euler-Lagrange equation 2.7 yields equation 1.11

Self Assessment Exercise C

1. State the Lagrangian for a particle moving in a non-inertia rotating frame and the corresponding Lagrange's equation of motion

4.0 Conclusion

This time derivative of an arbitrary rotating frame vector A in a fixed frame is, therefore expressed as

$\left(\frac{dA}{dt}\right)_f = \left(\frac{dA}{dt}\right)_r + \omega \times A$, where $\left(d/dt\right)_f$ denotes the time derivative as observed in the fixed (f) frame, while $\left(d/dt\right)_r$ denotes the time derivative as observed in the rotating *(R)* frame. An application of this formular relates to the time derivative of the rotation angular velocity ω itself. One can easily see that

$\left(\frac{d\omega}{dt}\right)_f = \dot{\omega} = \left(\frac{d\omega}{dt}\right)_r$. Since the second term of equation above vanishes for $A = \omega$; the time derivative of ω is, therefore, the same in both frames of reference and is denoted as $\dot{\omega}$.

5.0 Summary

- **A non-inertial reference frame** is a reference frame that is not tied to the state of motion of an observer, with the property that each physical law portrays itself in the same form in every inertial frame.
- The first reference frame is called the fixed frame and is expressed in terms of the Cartesian coordinates $r' = (x'; y'; z')$.
- The second reference frame is called the rotating frame and is expressed in terms of the Cartesian coordinates *r = (x; y; z).*
- This time derivative of an arbitrary rotating frame vector A in a fixed frame is, therefore expressed as

$$\left(\frac{dA}{dt}\right)_f = \left(\frac{dA}{dt}\right)_r + \omega \times A$$

where $\left(d/dt\right)_f$ denotes the time derivative as observed in the fixed (f) frame while $\left(d/dt\right)_r$ denotes the time derivative as observed in the rotating *(R)* frame.

- The velocities of point P as observed in the fixed and rotating frames are defined as
$$v_f = \left(\frac{dr'}{dt}\right)_f \quad \text{and} \quad v_r = \left(\frac{dr}{dt}\right)_r .$$
- Expressions for the acceleration of point P as observed in the fixed and rotating frames of reference is
$$a_f = A + a_r + 2\omega \times v_r + \dot{\omega} \times r + \omega \times (\omega \times r), \quad \text{where} \quad A = \left(\frac{dV}{dt}\right)_f$$ denotes the translational acceleration of the rotating frame origin.
- The Lagrangian for a particle of mass *m* moving in a non-inertial rotating frame (with its origin coinciding with the fixed-frame origin) in the presence of the potential *U(r)* is expressed as
$$L(r, \dot{r}) = \frac{m}{2}\left|\dot{r} + \omega \times r\right|^2 - U(r).$$
- The Euler-Lagrange's equation are
$$m\ddot{r} = -\nabla U(r) - m\left[\dot{\omega} \times r + 2\omega \times \dot{r} + \omega \times (\omega \times r)\right].$$

6.0 Tutor Marked Assignments (TMAs)

1. Given the following parameter of a rotating reference frame, $r=10t+3$, $\omega=6t^2+4t+5$. Determine the following: i) Centrifugal acceleration, ii) Coriolis acceleration, iii) Centripetal acceleration.
2. For a fixed and rotating reference frame associated with translation and rotation as depicted in fig 1.2 if $R=5t^3+2$ Determine the velocity and acceleration of the fixed reference frame?
3. Find the Lagrangian and the corresponding Lagrange's equation of motion for a particle of mass m moving in non-inertia reference frame.

7.0 Further Reading and Other Resources

Classical Mechanics by H.Goldstein, Narosa Publishing Home, New Delhi.

Classical Dynamics of Particles and Systems by Marion and Thomtron, Third Edition, Horoloma Book Jovanovich College Publisher.

Introduction to Classical Mechanics by R.G.Takawale and P.S.Puranik, Tata Mc-Graw Hill Publishing Company Limited, New Delhi.

UNIT 2 MOTION RELATIVE TO EARTH

CONTENTS

1.0 Introduction

We can now apply the expressions developed in unit 1 of this module above to the important case of the fixed frame of reference having its origin at the center of Earth (point O' in the figure 2.1 below) and the rotating frame of reference having its origin at latitude λ and the longitude ψ (point O in the figure below). We notes that the rotation of the Earth is now represented as ψ = ω and that $\dot{\omega} = 0$.

2.0 Objectives

After studying this unit you will be able to:

- Represent the rotation of the earth in terms of fixed frame of latitude angle λ and the azimuthal angle ψ.
- State the expression for acceleration of a point as observed in the rotating frame O.
- State the expression for pure gravitational acceleration in the rotating frame of the Earth.
- Solve related problems.

3.0 Main Contents

3.1 Motion relative to earth

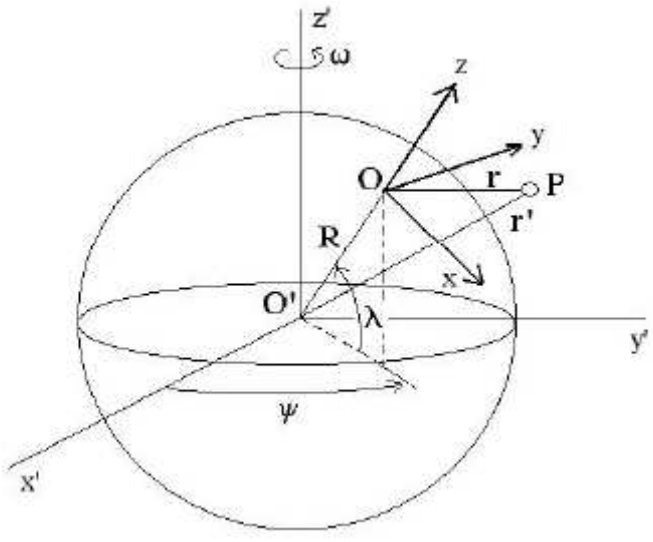


figure 2.1

We arrange the (x, y, z) axis of the rotating frame so that the z-axis is a continuation of the position vector R of the rotating-frame origin, i.e. $\vec{R} = R\hat{z}$ in the rotating frame (where *R=6378km* is the Earth radius assuming a spherical Earth). When expressed in terms of the fixed-frame latitude angle λ and the azimuthal angle ψ, the unit vector $\hat{z}$ is

$$\hat{z} = \cos\lambda(\cos\psi\hat{x}' + \sin\psi\hat{y}') + \sin\lambda\hat{z}'. \tag{2.1}$$

Likewise, we choose the axis to be tangent to a great circle passing through the North and South poles, So that

$$\hat{x} = \sin\lambda(\cos\psi\hat{x}' + \sin\psi\hat{y}') + \cos\lambda\hat{z}'. \tag{2.2}$$

Lastly, the y axis is chosen such that

$$\hat{y} = \hat{z}\times\hat{x} = -\sin\psi\hat{x}' + \cos\psi\hat{y}'. \tag{2.3}$$

We now consider the acceleration of a point as observed in the rotating frame O by writing equation 1.11 as

$$\frac{d^2r}{dt^2} = g_{0f} - \ddot{R}_f - \omega\times(\omega\times r) - 2\omega\times\frac{dr}{dt}. \tag{2.4}$$

The first term g_{0f} represent the pure gravitational acceleration due to the gravitational pull of the Earth on point P (as observed in the fixed located at Earth's center) and is given as

$$g_{0f} = -\frac{GM}{|r'|^3}r' \tag{2.5}$$

where $r' = R + r$ is the position of point P in the fixed frame and r is the location of P in the rotating frame. When expressed in terms of rotating-frame spherical coordinates (τ,θ,ϕ):

$$r = \tau[\sin\theta(\cos\phi\hat{x} + \sin\phi\hat{y}) + \cos\theta\hat{z}], \tag{2.6}$$

the vector r' is written as

$$r' = (R + \tau\cos\theta)\hat{z} + \tau\sin\theta(\cos\phi\hat{x} + \sin\phi\hat{y}), \tag{2.7}$$

and thus,

$$|r'|^3 = \left(R^2 + 2R\tau\cos\theta + \tau^2\right)^{3/2}. \tag{2.8}$$

The pure gravitational acceleration is, therefore, expressed in the rotating frame of the Earth as

$$g_{0f} = -g_{0r}\left[\frac{(1+\varepsilon\cos\theta)\hat{z} + \varepsilon\sin\theta(\cos\phi\hat{x} + \sin\phi\hat{y})}{\left(1 + 2\varepsilon\cos\theta + \varepsilon^2\right)^{3/2}}\right] \tag{2.9}$$

where g_{0r} acceleration due to gravitational pull on the earth as observed in the rotating frame of the earth $g_{0r} = GM/R^2 = 9.789\ m/s^2$ and $\varepsilon = r/R$ which is much lesser than 1.

The angular velocity in the fixed frame is $\omega = \omega\hat{z}$ where

$$\omega = \frac{2\pi rad}{24\times3600\sec} = 7.27\times10^{-5}\,rad/s \tag{2.10}$$

is the rotation speed of Earth about its axis. In the rotating frame, we find

$$\omega = \omega(\sin\lambda\hat{z} - \cos\lambda\hat{x}). \tag{2.11}$$

Because the position vector R rotates with the origin of the rotating frame, its time derivative yield

$$\dot{R}_f = \omega\times R = (\omega R\cos\lambda)\hat{y} \tag{2.12}$$

$$\ddot{R}_f = \omega\times\dot{R}_f = \omega\times(\omega\times R) = -\omega^2 R\cos\lambda(\cos\lambda\hat{z} + \sin\lambda\hat{x}) \tag{2.13}$$

and thus the centrifugal acceleration due to R is

$$\ddot{R} = -\omega\times(\omega\times R) = \alpha g_{0r}\cos\lambda(\cos\lambda\hat{z} + \sin\lambda\hat{x}), \tag{2.14}$$

where $\omega^2R = 0.0337 m/s^2$ can be expressed in terms of pure gravitational acceleration g_{0r} as $\omega^2 R = \alpha g_{0r}$, where α =3.4 x 10^{-3}. We now define the physical gravitational acceleration as

$$\begin{aligned} g &= g_{0f} - \omega\times[\omega\times(R + r)] \\ &= g_{0r}[-(1 - \alpha\cos^2\lambda)\hat{z} + (\alpha\cos\lambda\sin\lambda)\hat{x}], \end{aligned} \tag{2.15}$$

where terms of order ε have been neglected. A plumb line experiences a small angular deviation $\delta(\lambda)$ from the true vertical given as

$$\tan\delta(\lambda) = \frac{g_{rx}}{|g_{rz}|} = \frac{\alpha \sin 2\lambda}{(2-\alpha)+\alpha\cos 2\lambda}. \tag{2.16}$$

The function exhibits a maximum at a latitude $\bar{\lambda}$ defined as $\cos 2\bar{\lambda} = -\alpha/(2-\alpha)$, so that

$$\tan\bar{\delta} = \frac{\alpha \sin 2\bar{\lambda}}{(2-\alpha)+\alpha\cos 2\bar{\lambda}} = \frac{\alpha}{2\sqrt{1-\alpha}} \approx 1.7\times 10^{-3} \tag{2.17}$$

or $\bar{\delta} \approx 5.86 arc\min$ at $\bar{\lambda} \approx \left(\frac{\pi}{4}+\frac{\alpha}{4}\right) rad = 45.05^0$, we now return to equation 2.4, which is written to lowest order in ε and α as

$$\frac{d^2 r}{dt^2} = -g_r \hat{z} - 2\omega \times \frac{dr}{dt} \tag{2.19}$$

where $$\omega \times \frac{dr}{dt} = \omega\left[(\dot{x}\sin\lambda + \dot{z}\cos\lambda)\hat{y} - \dot{y}(\sin\lambda\hat{x} + \cos\lambda\hat{z})\right]. \tag{2.20}$$

Thus, we find three component of equation 2.19 written explicitly as

$$\left.\begin{aligned} \ddot{x} &= 2\omega\sin\lambda\dot{y} \\ \ddot{y} &= -2\omega(\sin\lambda\dot{x} + \cos\lambda\dot{z}) \\ \ddot{z} &= -g_r + 2\omega\cos\lambda\dot{y} \end{aligned}\right\} \tag{2.21}$$

A first integration of equation 2.21 yields

$$\left.\begin{aligned} \dot{x} &= 2\omega\sin\lambda y + C_x \\ \dot{y} &= -2\omega(\sin\lambda x + \cos\lambda z) + C_y \\ \dot{z} &= -g_r t + 2\omega\cos\lambda y + C_z \end{aligned}\right\} \tag{2.22}$$

where (C_x, C_y, C_z) are constants defined from initial conditions (x_0, y_0, z_0) and $(\dot{x}_0, \dot{y}_0, \dot{z}_0)$

$$\left.\begin{aligned} C_x &= \dot{x} - 2\omega\sin\lambda y_0 \\ C_y &= \dot{y}_0 + 2\omega(\sin\lambda x_0 + \cos\lambda z_0) \\ C_z &= \dot{z}_0 - 2\omega\cos\lambda y_0 \end{aligned}\right\} \tag{2.23}$$

A second integration of equation 2.22 yields

$$\begin{aligned} x(t) &= x_0 + C_x t + 2\omega\sin\lambda\int_0^t y dt, \\ y(t) &= y_0 + C_y t - 2\omega\sin\lambda\int_0^t x dt - 2\omega\cos\lambda\int_0^t z dt \\ z(t) &= z_0 + C_z t - \frac{1}{2}gt^2 + 2\omega\cos\lambda\int_0^t y dt \end{aligned} \tag{2.24}$$

Which can also be re-written as

$$\left.\begin{aligned} x(t) &= x_0 + C_x t + \delta x(t) \\ y(t) &= y_0 + C_y t + \delta y(t) \\ z(t) &= z_0 + C_z t - \frac{1}{2}g_r t^2 + \delta z(t) \end{aligned}\right\} \tag{2.25}$$

Where the coriolis drift are

$$\delta x(t) = 2\omega\sin\lambda\left(y_0 t + \frac{1}{2}C_y t^2 + \int_0^t \delta y dt\right),$$

$$\delta y(t) = -2\omega\sin\lambda\left(x_0 t + \frac{1}{2}C_x t^2 + \int_0^t \delta x\,dt\right) - 2\omega\cos\lambda\left(z_0 t + \frac{1}{2}C_z t^2 - \frac{1}{6}g_r t^3 + \int_0^t \delta z\,dt\right), \qquad (2.26)$$

$$\delta z(t) = 2\omega\cos\lambda\left(y_0 t + \frac{1}{2}C_y t^2 + \int_0^t \delta y\,dt\right).$$

Note that each Coriolis drift can be expressed as an infinite series in power of ω and that all Coriolis effects vanish when ω = 0.

Self Assessment Exercise A

1. Derive the constant C_x, C_y and C_z and the coriolis drift.

3.2 Free-Fall Problem

As an example of the importance of Coriolis effects, we consider the simple free fall problem, where $(x_0, y_0, z_0) = (0,0,h)$ and $(\dot{x}_0, \dot{y}_0, \dot{z}_0) = (0,0,0)$

So that the constants (equation 2.23) are $C_x = 0 = C_z$ and $C_y = 2\omega h\cos\lambda$

Substituting these constants into equation 2.25 and keeping only terms up to the first order in ω, we find

$$x(t) = 0, \qquad (2.27)$$

$$y(t) = \frac{1}{3}g_r t^3 \omega\cos\lambda, \qquad (2.28)$$

$$z(t) = h - \frac{1}{2}g_r t^2. \qquad (2.29)$$

Hence, a free falling object starting from rest touches the ground z(T) = 0 after a time $T = \sqrt{2h/g_r}$ after which the object has drifted eastward by a distance of

$$y(T) = \frac{1}{3}g_r T^3 \omega\cos\lambda = \frac{\omega\cos\lambda}{3}\sqrt{\frac{8h^3}{g_r}}. \qquad (2.30)$$

Self Assessment Exercise B

1. State one of the importance of the coriolis effects
2. Find the eastward drift of an object falling freely from a height of 100m with a latitude of 45^0

4.0 Conclusion

Expressing x,y,z of a rotating frame of reference having its origin at latitude λ and the longitude ψ (point O in the figure 2.1 above). in terms of the fixed-frame latitude angle λ and the azimuthal angle ψ, we derive the x,y,z which are function of t as follows:

$$\left.\begin{aligned} x(t) &= x_0 + C_z t + \delta x(t) \\ y(t) &= y_0 + C_y t + \delta y(t) \\ z(t) &= z_0 + C_z t - \frac{1}{2}g_r t^2 + \delta z(t) \end{aligned}\right\}$$ where $\delta x(t)$, $\delta y(t)$ and $\delta z(t)$ are referred to as Coriolis drift and this can be applied to get the drift of an object falling freely from a height.

5.0 Summary

- According to figure 2.1 arrange the (x, y, z) axis of the rotating frame so that the z-axis is a continuation of the position vector R of the rotating-frame origin, i.e. $\vec{R} = R\hat{z}$ in the rotating frame (where $R=6378km$ is the Earth radius assuming a spherical Earth).
- When expressed in terms of the fixed-frame latitude angle λ and the azimuthal angle ψ, the unit vector $\hat{z}$ is $\hat{z} = \cos\lambda(\cos\psi\hat{x}' + \sin\psi\hat{y}') + \sin\lambda\hat{z}'$ while that of x and y-axes are

$\hat{x} = \sin\lambda(\cos\psi\hat{x}' + \sin\psi\hat{y}') + \cos\lambda\hat{z}'$, $\hat{y} = \hat{z}\times\hat{x} = -\sin\psi\hat{x}' + \cos\psi\hat{y}'$.

- When expressed in terms of rotating-frame spherical coordinates (τ,θ,ϕ):

$$r = \tau[\sin\theta(\cos\phi\hat{x} + \sin\phi\hat{y}) + \cos\theta\hat{z}].$$

- In the rotating frame, we find

$$\omega = \omega(\sin\lambda\hat{z} - \cos\lambda\hat{x}).$$

- The angular velocity in the fixed frame is $\omega = \omega\hat{z}$ where

$$\omega = \frac{2\pi rad}{24\times 3600\sec} = 7.27\times 10^{-5} rad/s$$

- The pure gravitational acceleration is, therefore, expressed in the rotating frame of the Earth as

$$g_{0f} = -g_{0r}\left[\frac{(1+\varepsilon\cos\theta)\hat{z} + \varepsilon\sin\theta(\cos\phi\hat{x} + \sin\phi\hat{y})}{(1+2\varepsilon\cos\theta+\varepsilon^2)^{3/2}}\right]$$

where $g_{0r} = GM/R^2 = 9.789 m/s^2$ and $\varepsilon = r/R$ which is much lesser than 1.

- Because the position vector R rotates with the origin of the rotating frame, its time derivative yield

$$\dot{R}_f = \omega\times R = (\omega R\cos\lambda)\hat{y}.$$

- The centrifugal acceleration due to R is

$$\ddot{R} = -\omega\times(\omega\times R) = \alpha g_{0r}\cos\lambda(\cos\lambda\hat{z} + \sin\lambda\hat{x})$$

- The physical gravitational acceleration as

$$\begin{aligned} g &= g_{0f} - \omega\times[\omega\times(R+r)] \\ &= g_{0r}[-(1-\alpha\cos^2\lambda)\hat{z} + (\alpha\cos\lambda\sin\lambda)\hat{x}]. \end{aligned}$$

- The coriolis drift are

$$\delta x(t) = 2\omega\sin\lambda\left(y_0 t + \frac{1}{2}C_y t^2 + \int_0^t \delta y\,dt\right).$$

$$\delta y(t) = -2\omega\sin\lambda\left(x_0 t + \frac{1}{2}C_x t^2 + \int_0^t \delta x\,dt\right) - 2\omega\cos\lambda\left(z_0 t + \frac{1}{2}C_z t^2 - \frac{1}{6}gt^3 + \int_0^t \delta z\,dt\right)$$

$$\delta z(t) = 2\omega\cos\lambda\left(y_0 t + \frac{1}{2}C_y t^2 + \int_0^t \delta y\,dt\right)$$

- As an example of the importance of Coriolis effects, we consider the simple free fall problem, where $(x_0, y_0, z_0) = (0,0,h)$.

6.0 Tutor Marked Assignments (TMAs)

1. Find the eastward drift of an object falling freely from a height of 100m with a latitude of 45^0

7.0 Further Reading and Other Resources

Classical Mechanics by H.Goldstein, Narosa Publishing Home, New Delhi.

Classical Dynamics of Particles and Systems by Marion and Thomtron, Third Edition, Horoloma Book Jovanovich College Publisher.

Introduction to Classical Mechanics by R.G.Takawale and P.S.Puranik, Tata Mc-Graw Hill Publishing Company Limited, New Delhi.

Appendix

Category	Typerface	Example
Scalar	Lower case, normal weight, italic	s
Vector	Lower case or upper case, normal weight, italic, with arrow	$\vec{a}$ or $\vec{A}$
Vector coordinate Representation component	Lower case, with arrow	$\vec{a}_i$
Unit vector	Normal weight, italic, with hat	$\hat{a}$
Tensor coordinate representation	Upper case,	A
Tensor coordinate representation component	Upper case, roman, with subscripts	$A_{i_1 \ldots i_n}$

The following special symbols are defined:

δ_{ij} kronecker delta $=1$ when $i=j$.

$=0$ otherwise.

ε_{ijk} Levi-Civita symbol or tensor

$=1$ if i,j,k is an even permutation of 1,2,3

$=0$ otherwise (two or more of i,j,k are equal)

$=-1$ if i,j,k is an odd permutation of 1,2,3

a. even permutation of 1,2,3 are: 123, 231, 312.
b. odd permutation of 1,2,3 are: 321, 213,132.
c. repeated indices: 112, 122 and so on

Coordinate system

Cylindrical

The three coordinate are

$$\rho = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right) \qquad z = z$$

The inverse relations are

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$

The area elements are

$$\rho d\rho d\phi \qquad d\rho dz \qquad \rho d\phi dz$$

The volume element is

$$\rho d\rho d\phi dz$$

The line element is

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

The unit vectors are

$$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi \qquad \hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi \qquad \hat{z} = \hat{z}$$

Spherical

The three coordinates are:

$$r = \sqrt{x^2 + y^2 + y^2} \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right) \qquad \theta = \cos^{-1}\left(\frac{z}{r}\right)$$

The inverse relations are

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

The area elements are

$$r\sin\theta dr d\phi \qquad r dr d\theta \qquad r^2\sin\theta d\theta d\phi$$

The volume element is

$$r^2 \sin\theta dr d\theta d\phi$$

The line element is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

The unit vectors are

$$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta \qquad \hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

$$\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta\,.$$

Vector and Tensor Definitions and Algebraic Identities

Dot and Cross Products

$$\vec{a}\cdot\vec{b} = ab\cos(\vec{a},\vec{b}) = \delta_{ij}a_i b_j = A_i B_j$$

$$\vec{a}\times\vec{b} = \hat{n}AB(\vec{a},\vec{b}) = \varepsilon_{ijk}\vec{e}_i A_j B_k = A_i B_j\,,$$

where $(\vec{a},\vec{b})$ is the angle from $\vec{a}$ to $\vec{b}$ and $\hat{n}$ is the unit vector normal to the plane defined by $\vec{a}$, $\vec{b}$, and the right-hand rule, and the secondary expression using vector components hold in rectangular coordinates.

Algebraic Identities

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \vec{b}\cdot\left(\vec{c}\times\vec{a}\right) = \vec{c}\cdot\left(\vec{a}\times\vec{b}\right) \equiv \vec{a}\vec{b}\vec{c}$$

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \left(\vec{a}\cdot\vec{c}\right)\vec{b} - \left(\vec{a}\cdot\vec{b}\right)\vec{c}$$

$$\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right) = \vec{a}\cdot\left[\vec{b}\times\left(\vec{c}\times\vec{d}\right)\right] = \vec{a}\cdot\left[\left(\vec{b}\cdot\vec{d}\right)\vec{c} - \left(\vec{b}\cdot\vec{c}\right)\vec{d}\right] = \left(\vec{a}\cdot\vec{c}\right)\left(b\cdot\vec{d}\right) - \left(\vec{a}\cdot\vec{d}\right)\left(\vec{b}\cdot\vec{c}\right)$$

$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right) = \left[\left(\vec{a}\times\vec{b}\right)\cdot\vec{d}\right]\vec{c} - \left[\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right]\vec{d} = \left(\vec{a}\vec{b}\vec{d}\right)\vec{c} - \left(\vec{a}\vec{b}\vec{c}\right)\vec{d} = \left(\vec{a}\vec{c}\vec{d}\right)\vec{b} - \left(\vec{b}\vec{c}\vec{d}\right)\vec{a}\,.$$

Note that, if any of the vectors are the gradient vector $\vec{\nabla}$, care must be taken in how the above expressions are written out to ensure $\vec{\nabla}$ acts on the appropriate vectors. Any two quantities that commute in the above should be commuted as necessary to get reasonable behavior of $\vec{\nabla}$. But in some cases, even that may not be sufficient and you will have to keep track of which vector should be acted on by $\vec{\nabla}$. A good example is the second line when $\vec{b} = \vec{\nabla}$. In the simple case of $\vec{a}$ being constant, one simply needs to move the $\vec{b}$ in the first term:

$$\vec{a}\times\left(\vec{\nabla}\times\vec{c}\right) = \vec{\nabla}\left(\vec{a}\cdot\vec{c}\right) - \left(\vec{a}\cdot\vec{\nabla}\right)\vec{c}$$

But if $\vec{a}$ depends on position and does not give zero when acted on by $\vec{\nabla}$, then the above must be read with care. One has to somehow remember that $\vec{\nabla}$ should not be allowed to act on $\vec{a}$ since it does not act on $\vec{a}$ in the original expression. Since the above expression does not correctly convey that meaning, it is better to abandon the vector notation. The completely unambiguous way to write it, using index notation, is

$$\left[\vec{a}\times\left(\vec{\nabla}\times\vec{c}\right)\right]_i = \sum_j a_j \nabla_i c_j - \sum_j a_j \nabla_j c_i\,.$$

The key point is that in the first term, $\vec{a}$ is in a dot product with $\vec{c}$, but $\vec{\nabla}$ must be allowed to act on $\vec{c}$ first, and not as $\vec{\nabla}\cdot\vec{c}$.