NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 304

COURSE TITLE: COMPLEX ANALYSIS

MTH 304 COMPLEX ANALYSIS

COURSE GUIDE

NATIONAL OPEN UNIVERSITY OF NIGERIA

Course Code	MTH 304
Course Title	Complex Analysis
Course Writer	ANTNONIO YISA OLUWATOYIN Obafemi Awolowo University O.A.U. Ile-Ife Osun State.
Course Editor	Prof. Ajetunmobi M.O Faculty of Science Lagos State University, Ojo. Lagos
Programme Leader	Dr. Ajibola Saheed O. School of Science and Technology National Open University of Nigeria Victoria Island, Lagos

NATIONAL OPEN UNIVERSITY OF NIGERIA

Contents

INTRODUCTION

Complex number is the study of complex number together with their derivatives, manipulation, and other properties. Complex analysis is an extremely powerful tool with large number of practical application to the solution of physical problems – contour integration which provides a means of computing difficult integrals by investigating the singularities of the functions in the region of complex plain near and between the limits of integration. Complex analysis is also very useful in Taylor series expansion, Laurent series, Bilinear transformations, hydrodynamics, thermodynamics etc.

THE COURSE: MTH 304 Complex Analysis

This course comprises a total of six Units distributed across three modules as follows:

Module 1 comprised of 2Units
Module 2 is comprised of 2 Units
Module 3 is comprised of 1 Unit and
Module 4 is comprised of 1 Unit

In Module 1 we shall commence this course with a visit to the Preliminary Concepts complex numbers in unit 1 and in unit2 we focus on complex functions. Module 2 has two units, while the first unit discussed Analytic functions in complex form and the second unit deals with the ideal of limits and continuity as it relates to complex analysis. Lastly, Module three has only one unit and focused on Taylor and Laurent Series while the last, Module 4 has only one unit which presents the topic Bilinear Transformation.

COURSE AIMS AND OBJECTIVES

The objects of this course is to have your understanding of Complex Analysis whilst acquainting you with the graphical and mathematical significance of Complex numbers and functions and its applications to, “Taylor and Laurent Series and Bilinear Transformation’’ all of the above of which are expected to motivate you towards further enquiry into this very interesting and highly specialized mathematical habitat.

On your part we expect you in turn to conscientiously and diligently work through this course upon completion of which you should be able to:

The objectives of this unit are to:

. look at Cauchy-Riemann Equation

- Appreciate the basic concepts underlying complex numbers.
- Investigate and explain geometry on complex plane
- Treat some polar co-ordinates
- Work through a series of examples of transformations and conversions, and their solutions
- Study derivatives of a function.
- Investigate and study functions of a complex variable.

- Investigate and study Analytic functions
- Study Cauchy's integral formular
- Look at solutions on Liouville's Theorem
- To see how $|f(z)|$ must attain its maximum value somewhere in this domain D.
- Define the Limit and continuous functions
- To study Series
- Know the Power Series.
- Study bilinear transformation
- Design of an IIR low-pass filter by the bilinear transformation method
- Explain higher order IIR digital filters
- Discuss IIR discrete time high-pass band-pass and band-stop filter design
- Comparison of IIR and FIR digital filters.

WORKING THROUGH THE COURSE

This course requires you to spend quality time to read. The course content is presented in clear mathematical language that you can easily relate to and the presentation style adequate and easy to assimilate.
You should take full advantage of the tutorial sessions because this is a veritable forum for you to "rub minds" with your peers – which provides you valuable feedback as you have the opportunity of comparing knowledge and "rubbing minds" with your course mates.

COURSE MATERIAL

You will be provided course material prior to commencement of this course, which will comprise your Course Guide as well as your Study Units. You will receive a list of recommended textbooks which shall be an invaluable asset for your course material. These textbooks are however not compulsory.

STUDY UNITS

You will find listed below the study units which are contained in this course and you will observe that there are four modules. The first and second penultimate modules comprise two Units each, while in the third and the last modules; one Unit each.

Module 1	Unit 1	Complex Numbers
	Unit 2	Complex Functions
Module 2	Unit 1	Analytic Functions
	Unit 2	Limit and Continuity
Module 3	Unit 1	Taylor and Laurent Series
Module 4	Unit 2	Bilinear Transformation

TEXTBOOKS

There are more recent editions of some of the recommended textbooks and you are advised to consult the newer editions for your further reading.

Advance Calculus by Schum series
Engineering Mathematics by K.A.Stroud

George Arfken and Hans Weber. Mathematical Methods for Physicists.
Harcourt/Academic Press, 2000

Andrei D. Polyanin and Alexander V. Manzhirov Handbook of Integral Equations.
CRC Press, Boca Raton, 1998. ISBN 0-8493-2876-4

E. T. Whittaker and G. N. Watson. A Course of Modern Analysis.
Cambridge Mathematical Library

ASSESSMENT

Assessment of your performance is partly trough Tutor Marked Assessment which you can refer to as TMA, and partly through the End of Course Examinations.

TUTOR MARKED ASSIGNMENT

This is basically Continuous Assessment which accounts for 30% of your total score. During this course you will be given 4 Tutor Marked Assignments and you must answer three of them to qualify to sit for the end of year examinations. Tutor Marked Assignments are provided by your Course Facilitator and you must return the answered Tutor Marked Assignments back to your Course Facilitator within the stipulated period.

END OF COURSE EXAMINATION

You must sit for the End of Course Examination which accounts for 70% of your score upon completion of this course. You will be notified in advance of the date, time and the venue for the examinations which may, or may not coincide with National Open University of Nigeria semester examination.

SUMMARY

Each of the four modules of this course has been designed to stimulate your interest in Complex number through associative conceptual building blocks in the study and application of Complex analysis to practical problem solving.

By the time you complete this course, you should have acquired the skills and confidence to solve many Integral Equations more objectively than you might have thought possible at the commencement of this course. This however; is subject to this advise - make sure that you have enough referential and study material available and at your disposal at all times, and – devote sufficient quality time to your study.

I wish you the best in your academic pursuits – good luck.

COURSE MATERIAL

NATIONAL OPEN UNIVERSITY OF NIGERIA

Course Code	MTH 304
Course Title	Complex Analysis
Course Writer	ANTNONIO YISA OLUWATOYIN Obafemi Awolowo University O.A.U. Ile-Ife Osun State.
Course Editor	Prof. Ajetunmobi M.O Faculty of Science Lagos State University, Ojo. Lagos
Programme Leader	Dr. Ajibola Saheed O. School of Science and Technology National Open University of Nigeria Victoria Island, Lagos

NATIONAL OPEN UNIVERSITY OF NIGERIA Contents

MODULE 1

Unit 1 Complex Numbers

CONTENTS

1.0 INTRODUCTION

It was observed that when the only number(s) you known were the ordinary everyday integers. You had no trouble solving problems in which you were, for instance, asked to find a variable x such that $3x = 6$. You are quick to answer"2". Then, find a number x such that $3x = 8$. You were stumped—there was no such "number"! You perhaps explained that 3(2) = 6 and 3(3) = 9, and since 8 is between 6 and 9, you would somehow need a number between 2 and 3, but there isn't any such number. Thus one is introduced to "fractions."

These fractions, or rational or quotient numbers, were defined to be ordered pairs of integers, for instance, (8,3) is a rational number. Two rational numbers (n,m) and (p,q) were defined to be equal whenever $nq = pm$. (More precisely, in other words, a rational number is an equivalence class of ordered pairs, *etc.*) Recall that the arithmetic of these pairs was then introduced: the sum of (n,m) and (p,q) was defined by

$$(n,m) + (p,q) = (nq + pm, mq),$$

and the product by

$$(n,m)(p,q) = (np, mq).$$

Subtraction and division were defined, as usual, simply as the inverses of the two operations.

You probably felt at first like you had thrown away the familiar integers and were starting over. But no, you noticed that $(n,1) + (p,1) = (n + p,1)$ and also $(n,1)(p,1) = (np,1)$. Thus the set of all rational numbers whose second coordinate is one behave just like the integers. If we simply abbreviate the rational number $(n,1)$ by n, there is absolutely no danger of confusion: $2 + 3 = 5$ stands for $(2,1) + (3,1) = (5,1)$. The equation $3x = 8$ that started thisall may the n be interpreted as shorthand for the equation $(3,1)(u, v) = (8,1)$,

and one easily verifies that $x = (u, v) = (8,3)$ is a solution. Now, if someone runs at you in the night and hands you a note with 5 written on it, you do not know whether this is simply the integer 5 or whether it is shorthand for the rational number (5,1).

What we see is that it really doesn't matter. What we have "really" done is expanded the collection of integers to the collection of rational numbers. In other words, we can think of the set of all rational numbers as including the integers–they are simply the rationals with second Coordinate 1.

One last observation about rational numbers. It is, as everyone must know, traditional to write the ordered pair (n,m) as nm. Thus n stands simply for the rational number $n1$, *etc.*

Now why have we spent this time on something everyone learned in the grade ? Because this is almost a paradigm for what we do in constructing or defining the so-called complex numbers.

Euclid showed us there is no rational solution to the equation $x^2 = 2$. We were thus led to defining even more new numbers, the so-called real numbers, which, of course, include the rationals. This is hard, and you likely did not see it done in elementary school, but we shall assume you know all about it and move along to the equation $x^2 = -1$.

2.0 OBJECTIVE

The objectives of this unit are to

- investigate complex numbers.
- explain geometry on complex plane.
- study some polar co-ordinates.

3.0 MAIN CONTENT

3.1 Complex Numbers

We define complex numbers. These are simply ordered pairs (x, y) of real numbers, just as the rationals are ordered pairs of integers. Two complex numbers are equal only when there are actually the same– that is $(x, y) = (u, v)$ precisely when $x = u$ and $y = v$. We define the sum and product of two complex numbers:

$$(x, y) + (u, v) = (x + u, y + v)$$

and

$$(x, y)(u, v) = (xu - yv, xv + yu)$$

As always, subtraction and division are the inverses of these operations.

Now let's consider the arithmetic of the complex numbers with second coordinate 0:

$$(x,0) + (u,0) = (x + u,0),$$

and

$$(x,0)(u,0) = (xu,0).$$

Note that what happens is completely analogous to what happens with rationals with second coordinate 1. We simply use x as an abbreviation for $(x,0)$ and there is no danger of confusion: $x + u$ is short-hand for $(x,0) + (u,0) = (x + u,0)$ and xu is short-hand for $(x,0)(u,0)$. We see that our new complex numbers include a copy of the real numbers, just
as the rational numbers include a copy of the integers.

Notice that $x(u, v) = (u, v)x = (x,0)(u, v) = (xu, xv)$. Now then, any complex number $z = (x, y)$ may be written

$$z = (x, y)=(x, 0) + (0,y)$$

$$z = x +y(x, 0)$$

When, we let α=(0,1), then we have

$$z =(x, y) = x +\alpha y$$

Now, suppose $z = (x, y) = x +\alpha y$ *and* $w= (u, v) = u +\alpha v$. *Then we have*

$$z\, w=(x +\alpha y)\ (u +\alpha v)$$

$$= xu+\alpha(xv+yu)+ \alpha^2 yv$$

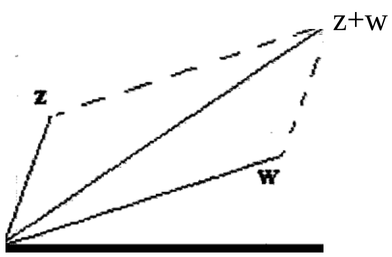
We need only see what α^2 *is:* $\alpha^2=(0, 1)(0, 1)=(-1,0)$, *and we have agreed that we can safely abbreviate* $(-1, 0)$ *as* -1. *Thus,* $\alpha^2= -1$ *and so*

$$zw=(xu-yv)+\alpha(xv+yu)+ \alpha^2 yv$$

and we have reduced the fairly complicated definition of complex number arithmetic simply to ordinary real arithmetic together with the fact that $\alpha^2= -1$.

3. 2 GEOMETRY

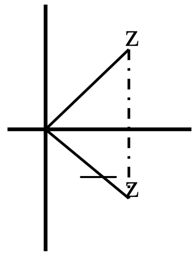
We now have this collection of all ordered pairs of real numbers, and so there is an uncontrollable urge to plot them on the usual coordinate axes. We see at once then there is a one-to-one correspondence between the complex numbers and the points in the plane. In the usual way, we can think of the sum of two complex numbers, the point in the plane corresponding to z + w is the diagonal of the parallelogram having z and w as sides:



The geometric interpretation of the product of two complex numbers. The modulus of a complex number $z = x + iy$ is defined to be the nonnegative real number

$\sqrt{x^2+y^2}$, which is, of course, the length of the vector interpretation of z. This modulus is traditionally denoted $|z|$, and is sometimes called the length of z. Note that $|(x,0)| = \sqrt{x^2} = |x|$, and so $|\bullet|$ is an excellent choice of notation for the modulus.

The conjugate $\bar{z}$ of a complex number $z = x + iy$ is defined by $\bar{z} = x - iy$. Thus $|z|^2 = z\bar{z}$. Geometrically, the conjugate of z is simply the reflection of z in the horizontal axis:



Observe that if $z = x + iy$ and $w = u + iv$, then

$$\overline{(z+w)} = (x+u) - i(y+v)$$
$$= (x - iy) + (u - iv)$$
$$= \bar{z} + \bar{w}.$$

In other words, the conjugate of the sum is the sum of the conjugates. It is also true that $\overline{zw} = \overline{zw}$. If $z = x + iy$, then x is called the real part of z, and y is called the imaginary part of z. These are usually denoted Rez and Imz, respectively. Observe then that $z + \bar{z} = 2\mathrm{Re}z$ and $z - \bar{z} = 21mz$.

Now, for any two complex numbers z and w consider

$$|z + w|^2 = (z+w)\,(\overline{z+w}) = (z+w)\,(\bar{z} + \bar{w})$$
$$= z\bar{z} + (w\bar{z} + \bar{w}z) + w\bar{w}$$
$$= |z|^2 + 2\mathrm{Re}\,(w\bar{z}) + |w|^2$$
$$\leq |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2$$

In other words,

$$|z + w| \leq |z| + |w|$$

the so-called triangle inequality.

3.3 POLAR COORDINATES

Now let's look at polar coordinates (r, 8) of complex numbers.
Then we may write $z = r(\cos\theta + i\sin\theta)$. In complex analysis, we do not allow r to be negative; thus r is simply the modulus of z. The number 9 is called an argument of z, and there are, of course, many different possibilities for 9. Thus a complex numbers has an infinite number of arguments, any two of which differ by an integral multiple of 2π. We usually write $\theta = \arg z$. The principal argument of z is the unique argument that lies on the interval $(-\pi, \pi)$.

SELF ASSSIGMENT EXERCISE

If 1- i, we have

$$1-i = \sqrt{2}\,(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right))$$
$$= \sqrt{2}\,(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right))$$
$$= \sqrt{2}\,(\cos\left(\frac{399\pi}{4}\right) + i\sin\left(\frac{399\pi}{4}\right))$$

Each of the numbers $\frac{7\pi}{4}$, $-\frac{\pi}{4}$, and $\frac{399\pi}{4}$ is an argument of $1 - i$, but the principal argument is $-\frac{\pi}{4}$.

Suppose

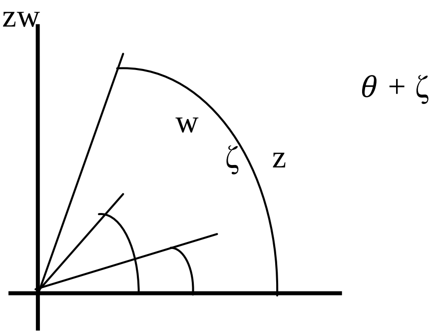
$z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\zeta + i\sin\zeta)$.Th

$zw = r(\cos\theta + i\sin\theta)s(\cos\zeta + i\sin\zeta)$

$= rs[(\cos\theta\cos\zeta - \sin\theta\sin\sim) + i(\sin\theta\cos\zeta + \sin\zeta\cos\theta)]$

$= rs(\cos(\theta + \zeta) + i\sin(\theta + \zeta))$

We have the nice result that the product of two complex numbers is the complex number whose modulus is the product of the moduli of the two factors and an argument is the sum of arguments of the factors. A picture:



We now define $\exp(i\theta)$, or $e^{i\theta}$ by

$$e^{i\theta} = \cos\theta + i\sin\theta$$

We shall see later as the drama of the term unfolds that this very suggestive notation is an excellent choice. Now, we have in polar form

$$z = re^{i\theta},$$

where $r = |z|$ and θ is any argument of z. Observe we have just shown that

$$e^{i\theta}\, e^{i\zeta} = e^{i(\theta+\zeta)}.$$

It follows from this that $e^{i\theta}\, e^{-i\theta} = 1$. Thus

$$\frac{1}{e^{i\theta}} = e^{-i\theta}$$

It is easy to see that

$$\frac{z}{w} = \frac{re^{i\theta}}{se^{i\zeta}} = \frac{r}{s}(\cos(\theta - \zeta) + i\sin(\theta - \zeta))$$

4.0 CONCLUSION

In this closing unit, the achievement resulting from this unit are highlighted in the summary.

5.0 SUMMARY

The summary of the work carried out in this unit given most importantly, the achievement resulting from this unit are highlighted below.

we introduce to fraction or rational or quotient numbers, which was defined to be ordered pairs of integers.

We showed that there is no rational solution to the equation $x^2 = 2$. We were thus led to defining even more new numbers, the so-called real numbers, which, of course, include the rationals.

Complex numbers were also defined on modulus, length conjugate, triangle inequality, argument and principal argument using examples to illustrate these definitions.

TUTOR MARKED ASSIGNMENT

1. Find the following complex numbers in the form x + iy:

a) $(4-7i)(-2+3i)$ b) $(1-i)^3$

b) $\frac{(5+2i)}{(1+i)}$ c) $\frac{1}{i}$

2. Find all complex z = (x,y) such that

$$z^2 + z + 1 = 0$$

3. Prove that if wz = 0, then w = 0 or z = 0.

4. a) Prove that for any two complex numbers, $\overline{zw} = \overline{z}\,\overline{w}$.

b) Prove that $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$.

c) Prove that $||z| - |w|| \leq |w|$.

5. Prove that $|zw| = |z||w|$ and that $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$.

6. Sketch the set of points satisfying

a) $|z - 2 + 3i| = 2$ b) $|z + 2i| \leq 1$

c) $\mathrm{Re}(\overline{z} + i) = 4$ d) $|z - 1 + 2i| = |z + 3 + i|$

e) $|z + 1| + |z - 1| = 4$ f) $|z + 1| - |z - 1| = 4$

7. Write in polar form $re^{i\theta}$:

a) i b) $1 + i$

c) -2 d) $-3i$

e) $\sqrt{3} + 3i$

8. Write in rectangular form-no decimal approximations, no trig functions:

a) $2e^{i3\pi}$ b) $e^{i100\pi}$

c) $10e^{3\pi/6}$ d) $\sqrt{2}\, e^{i5\pi/4}$

9. a) Find a polar form of $(1 + i)(1 + i\sqrt{3})$.

b) Use the result of a) to find $\cos\left(\frac{7\pi}{12}\right)$ and $\sin\left(\frac{7\pi}{12}\right)$.

10. Find the rectangular form of $(-1 + i)^{100}$

11. Find all z such that $z^3 = 1$. (Again, rectangular form, no trig functions.)
12. Find all z such that $z^4 = 16i$. (Rectangular form, etc.).

8.0 REFERENCES/FURTHER READINGS

Advance Calculus by Schum series
Engineering Mathematics by K.A.Stroud

UNIT 2

Complex Functions

CONTENTS

14.0 Reference/further Readings

1.0 INTRODUCTION

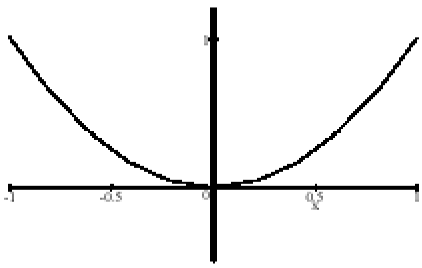
A function y: I → **C** from a set I of real's into the complex numbers C is actually a familiar concept from elementary calculus. It is simply a function from a subset of the reals into the plane, what we sometimes call a vector-valued function.

Assuming the function y is nice, it provides a vector, or parametric, description of a curve. Thus, the set of all $\{y(t) : y(t) = e^{it} = \cos t + i \sin t = (\cos t, \sin t),\ 0 \le t \le 2\pi\}$ is the circle of radius one, centered at the origin.

We also already know about the derivative of such functions. If $y(t) = x(t) + iy(t)$, then the derivative of y is simply $y^1(t) = x'(t) + iy'(t)$, interpreted as a vector in the plane, it is tangent to the curve described by y at the point y(t)

SELF ASSIGNMENT EXERCISE 1

Let $y(t) = t + it^2, -1 < t < 1$. One easily sees that this function describes that part of the curve $y = x^2$ between x = -1



SELF ASSIGNMENT EXERCISE 2

Suppose there is a body of mass "fixed" at the origin-perhaps the sun-end there is a body of mass m which is free to move-perhaps a planet.

Let the location of this second body at time t be given

by the complex-valued function z(t). We assume the only force on this mass is the gravitational force of the fixed body. This force f is thus

$$f = \frac{GMm}{|z(t)|^2}\left(-\frac{z(t)}{|z(t)|}\right)$$

Where G is the universal gravitati s us that

\

$$mz''(t) = f = \frac{GMm}{|z(t)|^2}\left(-\frac{z(t)}{|z(t)|}\right)$$

Hence,

$$z'' = -\frac{GM}{|z|^3}z$$

Next, let's write this in polar forr

$$\frac{d^2}{dt^2}(re^{i\theta}) = -\frac{k}{r^2}e^{i\theta}$$

Where we have GM = k. now, let's see wnat we nave

$$\frac{d}{dt}(re^{i\theta}) = r\frac{d}{dt}(e^{i\theta}) + \frac{dr}{dt}e^{i\theta}$$

Now,

$$\begin{aligned}\frac{d}{dt}(e^{i\theta}) &= \frac{d}{dt}(\cos\theta + i\sin\theta)\\ &= (-\sin\theta + i\cos\theta)\frac{d\theta}{dt}\\ &= i(\cos\theta + i\sin\theta)\frac{d\theta}{dt}\\ &= i\frac{d\theta}{dt}e^{i\theta}.\end{aligned}$$

(additional evidence that our notation $e^{i0} = \cos\theta + i\sin\theta$ is reasonable.)
Thus,

$$\begin{aligned}\frac{d}{dt}(re^{i\theta}) &= r\frac{d}{dt}(e^{i\theta}) + \frac{dr}{dt}e^{i\theta}\\ &= r\left(i\frac{d\theta}{dt}e^{i\theta}\right) + \frac{dr}{dt}e^{i\theta}\\ &= \left(\frac{dr}{dt} + ir\frac{d\theta}{dt}\right)e^{i\theta}.\end{aligned}$$

Now,

$$\frac{d^2}{dt^2}(re^{i\theta}) = \left(\right.$$

$$= \left[\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) + i\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\right]e^{i\theta}$$

Now the

$$\frac{d^2}{dt^2}(re^{i\theta}) = -\frac{k}{r^2}e^{i\theta}$$

This gives us th

$$\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) + i\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) = -\frac{k}{r^2}.$$

And,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{k}{r^2},$$

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0.$$

Multiply by r and thus second equation becomes

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0.$$

tells us that

$$\alpha = r^2\frac{d\theta}{dt}$$

is a constant. (This constant is cai... **...nentum.**) This result allows us to get rid of "' in the first of the two differential equations above:

or,

$$\frac{d^2r}{dt^2} - r\left(\frac{\alpha}{r^2}\right)^2 = -\frac{k}{r^2}$$

$$\frac{d^2r}{dt^2} - \frac{\alpha^2}{r^3} = -\frac{k}{r^2}.$$

Although this now involves ... :tion r, as

it stands it is tough to solve. Let's change variables and think of r as a function of *o*. Let's also write things in terms of the function s $= \frac{1}{R}$ Then

$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{\alpha}{r^2}\frac{d}{d\theta}.$$

Hence

$$\frac{dr}{dt} = \frac{\alpha}{r^2}\frac{dr}{d\theta} = -\alpha\frac{ds}{d\theta},$$

And so

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(-\alpha\frac{ds}{d\theta}\right) = \alpha s^2\frac{d}{d\theta}\left(-\alpha\frac{ds}{d\theta}\right)$$

and our differential equation looks like $-\alpha^2 s^2\frac{d^2s}{d\theta^2}$,

or,

$$\frac{d^2r}{dt^2} - \frac{\alpha^2}{r^3} = -\alpha^2 s^2\frac{d^2s}{d\theta^2} - \alpha^2 s^3 = -ks^2,$$

This one is easy. From high school differential equations class, we remember that

$$s = \frac{1}{r} = A\cos(\theta + \varphi) + \frac{k}{\alpha^2},$$

where A and (p are constants which depend on the initial conditions. At long last,

$$r = \frac{\alpha^2/k}{1 + \varepsilon\cos(\theta + \varphi)},$$

where we have set $\epsilon = A\alpha^2/k$. The graph of this equation is, of course, a conic section of eccentricity ϵ.

2.0 OBJECTIVES

The objectives of this unit are to:

- investigate and study functions of a complex variable.
- study derivatives of a function.
- look at Cauch-Riemann Equation

3.0 MAIN CONTENT

3.1 Functions of a complex variable

The real excitement begins when we consider function $f : D$ **-** **C** in which the domain *D* is a subset of the complex numbers. In some sense, these too are familiar to us from elementary calculus they are simply functions from a subset of the plane into the plane:

$$f(z) = f(x, y)= u(x, y)+iv(x, y)= ((x,y),v(x, y))$$

Thus $f(z) = z^2$ looks like $f(z) = z^2 = (x + iy)^2 = z^2 - y^2\ 2xyi$. In other words, $u(x, y) = x^2 - y^2$ and $v(x,y) = 2xy$. The complex perspective, as we shall see, generally provides richer and more profitable insights into these functions.

The definition of the limit of a function *f* at a point $z = z_0$ is essentially the same as that which we learned in elementary calculus:

$$\lim_{z\to z_0} f(z) = L$$

means that given an $\varepsilon > 0$, there is a δ so that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$. As you could guess, we say that *f* is **continuous** at z_0 if it is true that $\lim_{z\to z_0} f(z) = f(z_0)$. If *f* is continuous at each point of its domain, we say simply that *f* is **continuous.**

Suppose both $\lim_{z \to z_0} f(z)$ and $\lim_{z \to z_0} g(z)$ exist. Then the following properties are easy to establish:

$$\lim_{z \to z_0} [f(z) \pm g(z)] = \lim_{z \to z_0} f(z) \pm \lim_{z \to z_0} g(z)$$

And,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}$$

Provided, of course, th

It now follows at once from these properties that the sum, difference, product, and quotient of two functions continuous at z_0 are also continuous at z_0. (We must, as usual, except the dreaded 0 in the denominator.)

It should not be too difficult to convince yourself that if

$z = (x,y)$, $z_0 = (x_0,y_0)$

and

$f(z) = u(x,y) + iv(x,y)$, then

$$\lim_{z \to z_0} f(z) = \lim_{(x,y) \to (x_0,y_0)} u(x,y) + i \lim_{(x,y) \to (x_0,y_0)} v(x,y)$$

Thus f is continuous at $z_0 = (x_0,y_0)$ precisely when u and v are.

Our next step is the definition of the derivative of a complex function f It is the obvious thing. Suppose f is a function and z_o is an interior point of the domain of f the derivative f (z_o) off is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

SELF ASSIGNMENT E

Suppose $f(z) = z^2$. Then, letting $\Delta z = z - z_0$, we have

$$\begin{aligned}
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{2z_0 \Delta z + (\Delta z)^2}{\Delta z} \\
&= \lim_{\Delta z \to 0} (2z_0 + \Delta z) \\
&= 2z_0
\end{aligned}$$

No surprise he ... 2z.

SELF ASSIG

Let $f(z) = z\overline{z}$. Then,

$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)\overline{(z_0 + \Delta z)} - z_0 \overline{z}_0}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{z_0 \overline{\Delta z} + \overline{z}_0 \Delta z + \Delta z \overline{\Delta z}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \left(\overline{z}_0 + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z} \right)
\end{aligned}$$

Su

$\Delta z \to 0$ $\Delta x \to 0$

$$= \overline{z}_0 + z_0$$

Now, choose $\Delta z = (0, \ y)$. Then,

$$\lim_{\Delta z\to 0}\left(\overline{z}_0+\overline{\Delta z}+z_0\frac{\overline{\Delta z}}{\Delta z}\right)=\lim_{\Delta y\to 0}\left(\overline{z}_0-i\Delta y-z_0\frac{i\Delta y}{i\Delta y}\right)$$

Thus, we must have $\overline{z}_0 + z_0 = \overline{z}_0 - z_0 = 0$. In other words, there is no chance of this limit's existing, except possibly at $z_0 = 0$. So, this function does not have a derivative at most places.

Now, take another look at the first of these two examples. Meditate on this and you will be convinced that all the "usual" results for real-valued functions also hold for these new complex functions: the derivative of a constant is zero, the derivative of the sum of two functions is the sum of the derivatives, the "product" and "quotient" rules for derivatives are valid, the chain rule for the composition of functions holds, etc., etc. For proofs, you need only go back to your elementary calculus book and change x's to z's.

If f has a derivative at z_o, we say that f is **differentiable** at z_o. If f is differentiable at every point of a neighborhood of z_o, we say that is **analytic** at z_0. (A set S is a neighborhood of z_0 if there is a disk D = { : $|z—z_0| < r, r > 0$} so that D S. (If f is analytic at every point of some set S, we say that f is analytic on S. A function that is analytic on the set of all complex numbers is said to be an entire function.

3.3. DERIVATIVES

Suppose the function f *given* by $f(z) = u(x, y) + iv(x, y)$ has a

derivative at $z = z_0 = (x_0,y_0)$.

We know this means there is a number $f(z_0)$ so that

$$f'(z_0)=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}.$$

Choose $\Delta z = (\Delta x, 0) = Ax$. Then,

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\\
&= \lim_{\Delta x\to 0}\frac{u(x_0+\Delta x,y_0)+iv(x_0+\Delta x,y_0)-u(x_0,y_0)-iv(x_0,y_0)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\left[\frac{u(x_0+\Delta x,y_0)-u(x_0,y_0)}{\Delta x}+i\frac{v(x_0+\Delta x,y_0)-v(x_0,y_0)}{\Delta x}\right]\\
&= \frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)
\end{aligned}$$

Next, choose $\Delta z=(0, \Delta y) = i\Delta y$. Then,

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\\
&= \lim_{\Delta y\to 0}\frac{u(x_0,y_0+\Delta y)+iv(x_0,y_0+\Delta y)-u(x_0,y_0)-iv(x_0,y_0)}{i\Delta y}\\
&= \lim_{\Delta y\to 0}\left[\frac{v(x_0,y_0+\Delta y)-v(x_0,y_0)}{\Delta y}-i\frac{u(x_0,y_0+\Delta y)-u(x_0,y_0)}{\Delta y}\right]
\end{aligned}$$

We have two different expressions for the derivative $f(z_0)$, and so

$$\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)=\frac{\partial v}{\partial y}(x_0,y_0)-i\frac{\partial u}{\partial y}(x_0,y_0)$$

Or

$$\frac{\partial u}{\partial x}(x_0,y_0)=\frac{\partial v}{\partial y}(x_0,y_0),$$

$$\frac{\partial u}{\partial y}(x_0,y_0)=-\frac{\partial v}{\partial x}(x_0,y_0)$$

These equatior

We have shown that if f has a derivative at a point z_0, then its real and imaginary parts satisfy these equations. Even more exciting is the fact that if the real and imaginary parts of *f* satisfy these equations and if in addition, they have continuous first partial derivatives, then the function *f* has a derivative. Specifically, suppose $u(x, y)$ and $v(x, y)$ have partial derivatives in a neighborhood of $z_0 = (x_0,y_0)$, suppose these derivatives are continuous at z_0, and suppose

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0),$$

$$\frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0).$$

We shall see that *f* is differentiable at z_0.

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{[u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0)]+i[v(x_0+\Delta x,y_0+\Delta y)-v(x_0,y_0)]}{\Delta x+i\Delta y}.$$

Observe that

$$u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0) = [u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0+\Delta y)]+[u(x_0,y_0+\Delta y)-u(x_0,y_0)].$$

Thus,

$$u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0+\Delta y) = \Delta x\frac{\partial u}{\partial x}(\xi,y_0+\Delta y),$$

and,

$$\frac{\partial u}{\partial x}(\xi,y_0+\Delta y) = \frac{\partial u}{\partial x}(x_0,y_0)+\varepsilon_1,$$

Where

$$\lim_{\Delta z\to 0}\varepsilon_1 = 0.$$

Thus

$$u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0+\Delta y) = \Delta x\left[\frac{\partial u}{\partial x}(x_0,y_0)+\varepsilon_1\right].$$

Proceeding similarly, we get

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{[u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0)]+i[v(x_0+\Delta x,y_0+\Delta y)-v(x_0,y_0)]}{\Delta x+i\Delta y}$$

$$= \frac{\Delta x\left[\cdots\right] \cdots + i\varepsilon_4\right]}{\cdots},$$

Where $\varepsilon \to 0$ $\Delta z \to 0$. Now, unleash the Cauchy-Riemann equation on this quotient and obtain,

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{\Delta x\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]+i\Delta y\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]}{\Delta x+i\Delta y}+\frac{\text{stuff}}{\Delta x+i\Delta y}$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + \frac{\text{stuff}}{\Delta x+i\Delta y}.$$

Here,

$$\text{stuff} = \Delta x(\varepsilon_1+i\varepsilon_2)+\Delta y(\varepsilon_3+i\varepsilon_4).$$

It's easy to show that

$$\lim_{\Delta z\to 0}\frac{\text{stuff}}{\Delta z} = 0,$$

and so,

$$\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}.$$

In particular we have,

SELF ASSIGNMENT EXERCISE 3

Let's find all points at which the function *f* given by

f (z) = x^3 - *i*(I ˉy)^3 is differentiable. Here we have u = x^3

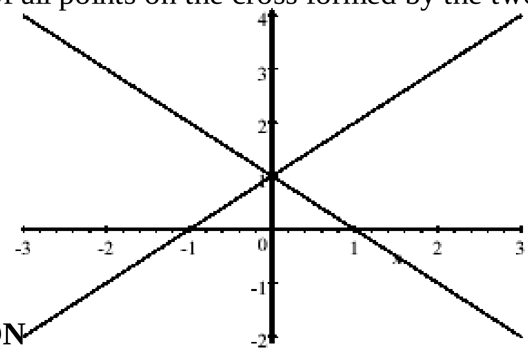
and v = -(1 -y)^3. The **Cauchy-Riemann** equations thus look like

$$3x^2 = 3(1 - y)^2, \text{ and}$$

$$0 = 0.$$

The partial derivatives of *u* a[illegible]ere, so *f* will be differentiable everywhere the C-R equations are satisfied That is, everywhere

This is simply the set of all points on the cross formed by the two straight lines.



4.0 CONCLUSION

To end the unit we now give the summary of what we have covered in it.

5.0 SUMMARY

The unit can be summarized as follows :
We discuss complex number on functions of a real variable as a function f : I → C from a set of real numbers into the complex number C while functions of a complex variable was defined as a function f : D → C in which the domain D is a subset of the complex number.
We also showed that if f has a derivative at a point z_0, then its real and imaginary parts satisfied the following equations

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0),$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

These equations are called the Cauchy-Riemann equations.
If the real and imaginary parts of *f* satisfy these equations and if in addition, they have continuous first partial derivatives, then the function *f* has a derivative.

6.0 TUTOR ASSIGNMENT EXERCISES

l. a)What curve is described by the function

$y(t) = (3t + 4) + i(t - 6), 0 \le t \le 1$?

b)Suppose z and w are complex numbers. What is the curve described by

$y(t) = (1 - t)w + tz, 0 \le t \le 1$

2. Find a function y that describes that part of the curve y = $4x^3 + 1$ between x = 0 and x = 10.

3. Find a function y that describes the circle of radius 2 centered at z = 3 - 2*i*.

4. Note that in the discussion of the motion of a body in a central gravitational force field, it was assumed that the angular momentum a is nonzero. Explain what happens in case α =0

5. Suppose f (z_0 = $3xy + i(x\text{-}y2)$. Find $\lim_{Z\text{-}3+2i} f(z)$. Or explain carefully why it does not exist

6. Prove that if f has a derivative at z, then f is continuous at z

7. Find all points at which the valued function f defined by $f(z) = \overline{z}$ has a derivative.

8. Find all points at which the valued function f defined by f

$$f(z) = (2+i)z^3 - iz^2 + 4z - (1+7i)$$

has a derivative

9. Is the function f given by

$$f(z) = \begin{cases} \dfrac{(\overline{z})^2}{z} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$$

differentiable at z = 0? Explain.

10. At what points is the function f given by $f(z) = x^3 + i(1 - y)3$ analytic? Explain

11. Do the real and imaginary parts of the function f in Exercise **9** satisfy the Cauchy-Riemann equations at z = 0? What do you make of your answer?

12. Find all points at which $f(z) = 2y - ix$ is differentiable.

13. Suppose f *is* analytic on a connected open set D, and $f(z) = 0$ for all zϵD. Prove that f is constant.

14. Find all points at which

$$f(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

is differentiable. At what point

15. Suppose f is analytic on the set D, and supposes Re f is constant on D. *Is* f necessarily Constant on D*?* Explain.

16. Suppose f *is* analytic on the set D, and suppose f (z) I is constant on D. *Is* f necessarily constant on D*?* Explain.

REFERENCES/FURTHER READINGS

Advance calculus by Schum series
Engineering Mathematics by K.A.Stroud

MODULE 2

Unit: 1 ANALYTIC FUNCTION

CONTENTS

3.0 . INTRODUCTION

A function f(z) is analytic at a point z_0 if its derivatives f'(z) exist not only at z_0 but atevery point z in a neighborhood of z_0.

Suppose *f* is entire and bounded; that is, *f* is analytic in the entire plane and there is a constant M such that $|f(z)| \leq M$ for all z.

They say that the derivative of an analytic function is also analytic. Now suppose f is continuous on a domain D in which every point of D is an interior point and suppose that $\oint_C f(z)dz = 0$ for every close curve in D.

Even more exciting is the fact that if the real and imaginary parts of f satisfy these equations and if in addition, they have continuous first partial derivatives, then the function f has a derivative.

2.0 OBJECTIVES

The objectives of this unit are to:

- investigate and study Analytic functions
- study Cauchy's integral formular
- look at solutions on Liouville's Theorem
- to see how $|f(z)|$ must attain its maximum value somewhere in this domain D.

3.0 MAIN CONTENT

3.1 CAUCHY'S INTEGRAL FORMULAR

Suppose f is analytic in a region containing a simple closed contour C with the usual positive orientation and its 'inside, and suppose zo is inside C. Then it turns out that

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz.$$

This is **the famous Cauchy Integral Formula.**

Let $\epsilon > 0$ be any positive number. We know that f is continuous at zo and so there is a number δ such that $|f(z) - f(zo)| < \epsilon$ whenever $|z - zo| < \delta$. Now let $p > 0$ be a number such that $p < \delta$ and the circle $Co = \{z : |z - zo| = p\}$ is also inside C. Now, the function $\frac{f(z)}{}$ is analytic in the region between C and C_o; thus

$$\int_C \frac{f(z)}{z - z_0} dz = \int_{C_0} \frac{f(z)}{z - z_0} dz.$$

We know that $\frac{1}{z - zo}$ dz = $2\pi i$, so we can write

$$\begin{aligned} \int_{C_0} \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) &= \int_{C_0} \frac{f(z)}{z - z_0} dz - f(z_0) \int_{C_0} \frac{1}{z - z_0} dz \\ &= \int_{C_0} \frac{f(z) - f(z_0)}{z - z_0} dz. \end{aligned}$$

For $z \epsilon C_0$ we have

$$\begin{aligned} \left| \int_{C_0} \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \right| &= \left| \int_{C_0} \frac{f(z) - f(z_0)}{z - z_0} dz \right|. \\ &\leq \frac{\varepsilon}{\rho} 2\pi\rho = 2\pi\varepsilon. \end{aligned}$$

Thus,

Buts is any positive number, and so

$$\left| \int_{C_0} \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \right| = 0,$$

Or,

$$f(z_0) = \frac{1}{2\pi i} \int_{C_0} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz,$$

Which is exactly what we set out to show.
It says that if *f* is analytic on and inside a simple closed curve and we know the values *f*(z) for every z on the stipple closed curve, then we know the value for the function at every point inside the curve.

SELF ASSIGNMENT EXERCISE

Let C be the circle |z| = 4 traversed once in the counterclockwise direction.

Let's evaluate the integral

$$\int_C \frac{\cos z}{z^2 - 6z + 5}\,dz.$$

We simply write the integrand as

$$\frac{\cos z}{z^2 - 6z + 5} = \frac{\cos z}{(z-5)(z-1)} = \frac{f(z)}{z-1},$$

where $$f(z) = \frac{\cos z}{z-5}.$$

Observe that *f* is analytic on and inside C, and so,

$$\int_C \frac{\cos z}{z^2-6z+5}dz = \int_C \frac{f(z)}{z-1}dz = 2\pi i f(1)$$

$$= 2\pi i \frac{\cos 1}{1-5} = -\frac{i\pi}{2}\cos 1$$

3.2 FUNCTIONS DEFINED BY INTEGRAL

Suppose C is a curve (not necessarily a simple closed curve, just a curve) and suppose the function g is continuous on C (not necessarily analytic, just continuous). Let the function G be defined by

$$G(z) = \int_C \frac{g(s)}{s-z}ds$$

For all z ϵ C. We shall show that G is analytic. Here we go. C

Consider,

$$\frac{G(z+\Delta z)-G(z)}{\Delta z} = \frac{1}{\Delta z}\int_C \left[\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right]g(s)ds$$

$$= \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)}ds.$$

$$\frac{G(z+\Delta z)-G(z)}{\Delta z} - \int_C \frac{g(s)}{(s-z)^2}ds = \int_C \left[\frac{1}{(s-z-\Delta z)(s-z)} - \frac{1}{(s-z)^2}\right]g(s)ds$$

$$= \int_C \left[\frac{(s-z)-(s-z-\Delta z)}{(s-z-\Delta z)(s-z)^2}\right]g(s)ds$$

$$= \Delta z \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)^2}ds.$$

Now we want to show that

$$\lim_{\Delta z \to 0}\left[\Delta z \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)^2}ds\right] = 0.$$

To that end, let M = max{|g(s)| : s ϵ C}, and let d be the shortest distance from z to C.
Thus, for s ϵ C, we have |s-z|≥ d > 0 and also

$$|s-z-\Delta z| \geq |s-z| - |\Delta z| \geq d - |\Delta z|.$$

Putting this all together, we can estimate the integrand above:

$$\left|\frac{g(s)}{(s-z-\Delta z)(s-z)^2}\right| \leq \frac{M}{(d-|\Delta z|)d^2}$$

For all *s* ϵ C. Finally,

$$\left|\Delta z \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)^2}ds\right| \leq |\Delta z|\frac{M}{(d-|\Delta z|)d^2}\text{length}(C),$$

And it is clear that

$$\lim_{\Delta z \to 0}\left[\Delta z \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)^2}ds\right] = 0,$$

Just as we set out to show. Hence G has a derivative at z, and

$$G'(z) = \int_C \frac{g(s)}{(s-z)^2}ds.$$

we see that G` has a derivative and it is just what you think it should be. Consider

$$\begin{aligned}
\frac{G'(z+\Delta z)-G'(z)}{\Delta z} &= \frac{1}{\Delta z}\int_C\left[\frac{1}{(s-z-\Delta z)^2} - \frac{1}{(s-z)^2}\right]g(s)ds \\
&= \frac{1}{\Delta z}\int_C\left[\frac{(s-z)^2-(s-z-\Delta z)^2}{(s-z-\Delta z)^2(s-z)^2}\right]g(s)ds \\
&= \frac{1}{\Delta z}\int_C\left[\frac{2(s-z)\Delta z-(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^2}\right]g(s)ds \\
&= \int_C\left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}\right]g(s)ds
\end{aligned}$$

Next,

$$
\begin{aligned}
&\frac{G'(z+\Delta z)-G'(z)}{\Delta z}-2\int_C \frac{g(s)}{(s-z)^3}ds \\
&=\int_C\left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}-\frac{2}{(s-z)^3}\right]g(s)ds \\
&=\int_C\left[\frac{2(s-z)^2-\Delta z(s-z)-2(s-z-\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\right]g(s)ds \\
&=\int_C\left[\frac{2(s-z)^2-\Delta z(s-z)-2(s-z)^2+4\Delta z(s-z)-2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\right]g(s)ds \\
&=\int_C\left[\frac{3\Delta z(s-z)-2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\right]g(s)ds
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left|\frac{G'(z+\Delta z)-G'(z)}{\Delta z}-2\int_C \frac{g(s)}{(s-z)^3}ds\right| &= \left|\int_C\left[\frac{3\Delta z(s-z)-2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\right]g(s)ds\right| \\
&\leq |\Delta z|\frac{(|3m|+2|\Delta z|)M}{(d-\Delta z)^2 d^3},
\end{aligned}
$$

Where m = max{|s – z| : s ϵ C}. It should be clear then that

$$\lim_{\Delta z\to 0}\left|\frac{G'(z+\Delta z)-G'(z)}{\Delta z}-2\int_C \frac{g(s)}{(s-z)^3}ds\right| = 0,$$

Or in other words,

$$G''(z) = 2\int_C \frac{g(s)}{(s-z)^3}ds.$$

Suppose f *is* analytic in a region *D* and suppose C is a positively oriented simple closed curve in D.

Suppose also the inside of C is in *D.* Then from the Cauchy Integral formula, we know that

$$2\pi i f(z) = \int_C \frac{f(s)}{s-z} ds$$

and so with g = f in the formulas just derived, we have

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2} ds, \text{ and } f''(z) = \frac{2}{2\pi i} \int_C \frac{f(s)}{(s-z)^3} ds$$

For all z inside the closed curve C. They say that the derivative of an analytic function is also analytic. Now suppose f is continuous on a domain D in which every point of D is an interior point and suppose that $\int_C f(z)$dz = 0 for every closed curve in D. Then we know that f has an antiderivative in D—in other words f is the derivative of an analytic function. We now know this means that f *is* itself analytic. We thus have the celebrated **Morera's Theorem:**

If $f.D \rightarrow$ C is continuous and such that $\int_C f(z)$dz = 0 for every closed curve in *D,* then f *is* analytic in *D.*

SELF ASSIGNMENT EXERCISE

Let's evaluate the integral

$$\int_C \frac{e^z}{z^3}dz,$$

Where C is any positively oriented closed curve around the origin. We simply use the equation

$$f''(z) = \frac{2}{2\pi i}\int_C \frac{f(s)}{(s-z)^3}ds$$

With z = 0 and *f*(s) = e^s. Thus,

$$\pi i e^0 = \pi i = \int_C \frac{e^z}{z^3}dz.$$

3.3 LIOUVILLE'S THEOREM

Suppose *f* is entire and bounded; that is, *f* is analytic in the entire plane and there is a constant M such that |*f* (z)| ≤ M for all z. Then it must be true that *f* (z) = 0 identically. To see this, suppose that *f* (w): ≠ 0 for some w. Choose R large enough to insure that $\frac{M}{R}$ < |*f* (w)|. Now let C be a circle centered at 0 and with radius p > max{R, 1w 1}.

Then we have:

$$\frac{M}{\rho} < |f'(w)| \leq \left| \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-w)^2}ds \right|$$

$$\leq \frac{1}{2\pi}\frac{M}{\rho^2}2\pi\rho = \frac{M}{\rho},$$

a contradiction. It must therefore be true that there is no w for which, f' (w) * 0; or, in other words, f (z) = 0 for all z. This, of course, means that f is a constant function. We have shown **Liouville's Theorem:** The only bounded entire functions are the constant function.
Let's put this theorem to some good use.

Let p(z) = $a_n z^n + a_n\text{-}iz^{n\text{-}l} + \ldots + a_1 z + a_o$ be a polynomial. Then

$$p(z) = \left(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n}\right) z^n.$$

Now choose R large enough to insure that for each j = 1, 2,...., n, we have $\left|\frac{an-j}{zj}\right| < \frac{an}{2n}$ whenever |z| > R. (We are assuming that $a_n \neq$ 0.) Hence, for |z| > R,

we know that

$$\begin{aligned} |p(z)| &\geq \left| |a_n| - \left| \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n} \right| \right| |z|^n \\ &\geq \left| |a_n| - \left| \frac{a_{n-1}}{z} \right| - \left| \frac{a_{n-2}}{z^2} \right| - \ldots - \left| \frac{a_0}{z^n} \right| \right| |z|^n \\ &> \left| |a_n| - \frac{|a_n|}{2n} - \frac{|a_n|}{2n} - \ldots - \frac{|a_n|}{2n} \right| |z|^n \\ &> \frac{|a_n|}{2} |z|^n. \end{aligned}$$

Hence, for $|z| > R$,

$$\frac{1}{|p(z)|} < \frac{2}{|a_n||z|^n} \leq \frac{2}{|a_n|R^n}.$$

Now suppose-p(z) $\neq$ 0 for all z. Then r $\frac{1}{|p(z)|}$ is also bounded on the disk IzI <_ R. Thus, $\frac{1}{p(z)}$
is a bounded entire function, and hence, by Liouville's Theorem, constant! Hence the polynomial is constant if it has no zeros. In other words, if p(z) is of degree at least one, there must be at least one z_o for which $p(z_o) = 0$. This is, of course, the celebrated Fundamental Theorem of Algebra.

3.4 MAXIMUM MODULI

Suppose f is analytic on a closed domain D. Then, being continuous, $|f(z)|$ must attain its maximum value somewhere in this domain. Suppose this happens at an interior point. That is, suppose $|f(z)| \leq M$ for all z ϵ *D* and suppose that $|f(z_o)| = M$ for some z_o in the interior of D. Now z_o is an interior point of D, so there is a number R such that the disk Λ centered at z_o having radius R is included in D. Let C be a positively oriented circle of radius p ≤ R centered at z_o. From Cauchy's formula, we know

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s - z_0}ds.$$

Hence,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{it})dt,$$

and so,

$$M = |f(z_0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})|dt \le M.$$

Since $|f(z_o + pe^{it})| \le M$. This means

$$M = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})|dt.$$

Thus,

$$M - \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})|dt = \frac{1}{2\pi} \int_0^{2\pi} [M - |f(z_0 + \rho e^{it})|]dt = 0.$$

This integrand is continuous and non-negative, and so must be zero. In other words, $|f(z)| = M$ for all $z \in C$. There was nothing special about C except its radius $p \le R$, and so we have shown that f must be constant on the disk Λ.

I hope it is easy to see that if D is a region (=connected and open), then the only way in which the modulus $|f(z)|$ of the analytic function f can attain a maximum on D is for f to be constant.

4.0 CONCLUSION

In this closing unit, the achievement resulting from this unit are highlighted in the summary.

5.0 SUMMARY

The famous Cauchy integral formula was well defined in the beginning of the unit.

We have observed that if *f* is analytic on and inside a simple closed curve and we know the values *f*(z) for every z on the stipple closed curve, then we know the value for the function at every point inside the curve.

We also knew that the derivative of an analytic function is also analytic.

Suppose *f* is continuous on a domain D in which every point of D is an interior point and suppose that

$\int f(z)$dz = 0 for every closed curve in D. Then we knew that f has an antiderivative in D—in other words *f* is the derivative of an analytic function. We said that *f is* itself analytic.

We thus have the celebrated **Morera's Theorem.**

If *f* is entire and bounded; that is, *f* is analytic in the entire plane and there is a constant M such that $|f(z)| \leq$ M for all z. Then it must be true that f'(z)=0 identically.

Suppose f is analytic on a closed domain D. Then, being continuous, $|f(z)|$ must attain its maximum value somewhere in the domain. Suppose this happens at an interior point. That is, suppose $|f(z)| \leq M$ for all $z \in D$ and suppose that $|f(z_o)| = M$ for some z_o in the interior of D. Now z_o is an interior point of D, so there is a number R such that the disk Λ centered at z_o having radius R is included in D. Let C be a positively oriented circle of radius $p \leq R$ centered at z_o.

6.0 TUTOR MARKED ASSIGNMENT

l. Suppose f and g are analytic on and inside the simple closed curve C, and suppose moreover that $f(z) = g(z)$ for all z on C. Prove that $f(z) = g(z)$ for all z inside C.

2. Let C be the ellipse $9x^2 + 4y^2 = 36$ traversed once in the counterclockwise direction. Define the function g by

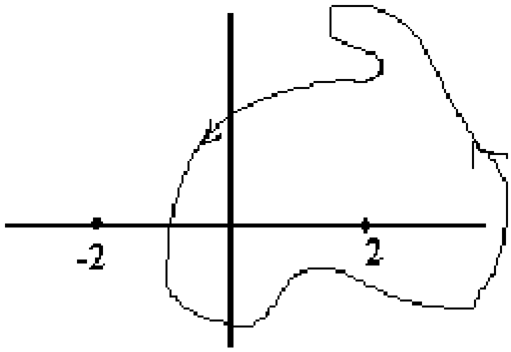
$$g(z) = \int_C \frac{s^2 + s + 1}{s - z} ds.$$

Find a) $g(i)$ b) $g(4i)$

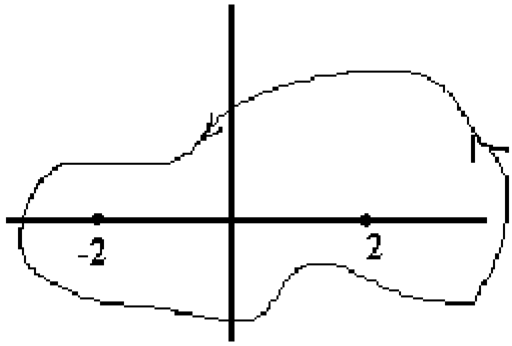
3. Find

$$\int_C \frac{e^{2z}}{z^2 - 4} dz,$$

Where C is the closed curve in the picture:



4. Find $\int_{\Gamma} \frac{e^{2z}}{z^2-4} dz,$ where Γ is the contour in the picture:



5. Evaluate

$$\int_C \frac{\sin z}{z^2} dz$$

Where C is a positively oriented closed curve around the origin.

6. Let C be the circle $|z - i| = 2$ with the positive orientation.

Evaluate a) $\int_C \frac{1}{z^2+4} dz$ b) $\int_C \frac{1}{(z^2+4)^2} dz$

7. Suppose f is analytic inside and on the simple closed curve C. Show that

$$\int_C \frac{f'(z)}{z-w} dz = \int_C \frac{f(z)}{(z-w)^2} dz$$

for every w ϵ C.

8. a) Let a be a real constant, and let C be the circle $y(t) = e^{it}$, $-\pi \leq t \leq \pi$. Evaluate

$$\int_C \frac{e^{az}}{z} dz.$$

b) Use your answer in part a) to show that

$$\int_0^{\pi} e^{\alpha \cos t} \cos(\alpha \sin t)dt = \pi.$$

9. Suppose *f* is an entire function, and suppose there is an M such that Re*f* (z) ≤ M for all z. Prove that *f* is a constant function.

10. Suppose w is a solution of 5z$^{\&}$z^{2} - 7z + 14 = 0. Prove that (w J < 3.

11. Prove that if p is a polynomial of degree n, and if p(a) = 0, then p(z) = (z - a)q(z), where q is a polynomial of degree n - 1.

12. Prove that if p is a polynomial of degree n > 1, then

$$p(z) = c(z - z_1)^{k_1}(z - z_2)^{k_2} \ldots (z - z_j)^{k_j},$$

13. Suppose p is a polynomial with real coefficients. Prove that p can be expressed as a product of linear and quadratic factors, each with real coefficients.

14. Suppose *f* is analytic and not constant on a region D and suppose *f* (z) ≠ 0 for all z ϵ D. Explain why |*f*(z)| does not have a minimum in D.

15. Suppose *f* (z) = u(x, y) + *iv*(x, y) is analytic on a region D. Prove that if u(x, y) attains a maximum value in D, then u must be constant.

7.0 REFERENCE/FURTHER READINGS

Advance Calculus by Schaum's outline Series

Advance Engineering Mathematics by Stroud

MODULE 2

Unit 2 LIMIT AND ONTINUITY

CONTENTS

UNIT 2 LIMT AND CONTINUITY

1.0 INTRODUCTION

The basic definitions for complex sequences and series are essentially the same as for the real case. **A sequence** of complex numbers is a function g: $Z_+ \to C$ from the positive integers into the complex numbers. It is traditional to use subscripts to indicate the values of the function. Thus we write $g(n) = z_n$ and an explicit name for the sequence is seldom used; we write simply (z_n) to stand for the sequence g which is such that $g(n) = z_n$.

2.0 OBJECTIVES

The objectives of this unit are to :

. Define the Limit and continuous functions

. To study Series

. Know the Power Series.

3.0 MAIN CONTENT

3.1 LIMIT

The number L is a **limit** of the sequence (z_n) if given an $\epsilon > 0$, there is an integer N_ϵ such that $|z_n - L| < \epsilon$ for all $n \geq N_\epsilon$. If L *is a limit o*f (z_n), we sometimes say that (z_n) converges to L. We frequently write $\lim (z_n) = L$. It is relatively easy to see that if the complex sequence $(z_n) = (u_n + iv_n)$ converges to L, then the two real sequences (u_n) and (v_n) each have a *limit:* (u_n) converges to ReL and (v_n) converges to ImL. Conversely, if the two real sequences (u_n) and (v_n) each have a limit, then so *also* does the complex sequence $(u_n + iv_n)$.

All the usual properties of *limits of* sequences are

$$\mathrm{Lim}(z_n \pm w_n) = \mathrm{Lim}(z_n) \pm \lim(w_n);$$

$$\mathrm{Lim}(z_n w_n) = \mathrm{Lim}(z_n) \lim(w_n); \text{ and}$$

$$\lim\left(\frac{z_n}{w_n}\right) = \frac{\lim(z_n)}{\lim(w_n)}.$$

Provided that $\lim(z_n)$ and $\lim(w_n)$exist. (And in the last equation, we must, of course, insist that $\lim(w_n) \neq 0$.)

A necessary and sufficient condition for the convergence of a

sequence (a_n) is the celebrated

Cauchy criterion: given $\epsilon > 0$, there is an integer N_ϵ so that

$|a_n - a_n| < \epsilon$ whenever n, m > $N\epsilon$.

A sequence (f_n) of functions on a domain D is the obvious thing a function from the positive integers into the set of complex functions on D. Thus, for each $z \epsilon D$, we have an ordinary sequence $(f_n(z))$. If each of the sequences $(f_n(z))$ converges, then we say the sequence of functions (f_n) converges to the function f defined by $f(z) = \lim(f_n(z))$. The sequence (f_n) is said to converge to f uniformly on a set S if given an $\epsilon > 0$, there is an integer N_ϵ so that $|f_n(z) - f(z)| < \epsilon$ for all $n \geq N_\epsilon$ and all $z \in S$.

Note that it is possible for a sequence of continuous functions to have a limit function that is not continuous. This cannot happen if the convergence is uniform. To see this, suppose the sequence (f_n) of continuous functions converges uniformly to f on a domain D, let $z_0 \epsilon D$, and let $\epsilon > 0$. We need to show there is a δ so that $|f(z_0)-f(z)| < \epsilon$ whenever $|z_0 - z)| < \delta$.

Choose N so that $|f_N(z) - f(z)| < \frac{\epsilon}{3}$. We can do this because of the uniform convergence of the sequence (f_n). Next, choose δ so that $|f_N(z_0) - f_N(z)| < \frac{\epsilon}{3}$ whenever $|z_0 - z| < \delta$. This is possible because, f_N is continuous.

Now then, when $|z_0 - z| < \delta$, we have

$$\begin{aligned} |f(z_0) - f(z)| &= |f(z_0) - f_N(z_0) + f_N(z_0) - f_N(z) + f_N(z) - f(z)| \\ &\leq |f(z_0) - f_N(z_0)| + |f_N(z_0) - f_N(z)| + |f_N(z) - f(z)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

Now suppose we have a sequence (f_n) of continuous functions

which converges uniformly on a contour C to the function f.

Then the sequence $\left(\int_C f_n(z)dz\right)$ converges to $\int_C f(z)dz$. This

is easy to see. Let $\epsilon > 0$. Now let N be so that $|f_n(z) - f(z)| < \frac{\epsilon}{A}$ for $n > N$, where A is the length of C. Then,

$$\left|\int_C f_n(z)dz - \int_C f(z)dz\right| = \left|\int_C (f_n(z) - f(z))dz\right|$$
$$< \frac{\varepsilon}{A}A = \varepsilon$$

whenever n > N.

Now suppose *(f_n)* is a sequence of functions each analytic on some region D, and suppose the sequence converges uniformly on D to the function f. Then f is analytic. This result is in marked contrast to what happens with real functions ─ examples of uniformly convergent sequences of differentiable functions with a non-differentiable limit abound in the real case. To see that this uniform limit is analytic, let $z_0 \epsilon D$, and let $S = \{z: |z - z_0| < r\} \subset D$. Now consider any simple closed curve $C \subset S$. Each f_n, is analytic, and so $\int_C f_n(z)dz$ = 0 for every n.

From the uniform convergence of (f_n), we know that $\int_C f(z)dz$ is the limit of the $\int_C$ $\left(\int_C f_n(z)dz\right)$ and so f(z)dz = 0.

Morera's theorem now tells us that f is analytic on S, and

hence at z_0.

3.2 SERIES

A series is simply a sequence (s_n) in which $s_n = a_1 + a_2 + \ldots\ldots + a_n$, In other words, there is sequence (a_n) so that $s_n = s_n + a_n$. The s_n are usually called the partial Sums.

if the series $\left(\sum_{j=1}^{n} a_j\right)$ has a limit, then it must be true that

$$\lim_{n\to\infty}(a_n) = 0.$$ Consider a

$\left(\sum_{j=1}^{n} f_j(z)\right)$ of functions. Chances are this series will converge for some values

of z and not converge for others. A useful result is the celebrated.

Weierstrass M-test:

Suppose (M_j) is a sequence of real numbers such that $M_j \geq 0$ *for all* $j > J$, where J is some number, and suppose that the series

$\left(\sum_{j=1}^{n} f_j(z)\right)$ $\left(\sum_{j=1}^{n} M_j\right)$ converges. If for all $z \epsilon D$, we have $|f_j(z)| \leq M_j$ for all $j > J$, then,the series

converges uniformly on D

To prove this, begin by letting $\epsilon > 0$ and choosing $N > J$ so that

$$\sum_{j=m}^{n} M_j < \varepsilon$$

For all n, m > N (We can do this because of the famous Cauchy criterion.) Next, we observe that

$$\left|\sum_{j=m}^{n} f_j(z)\right| \leq \sum_{j=m}^{n} |f_j(z)| \leq \sum_{j=m}^{n} M_j < \varepsilon.$$

This shows that $\left(\sum_{j=1}^{n} f_j(z)\right)$ converges. To see the uniform convergence, observe that

$$\left|\sum_{j=m}^{n} f_j(z)\right| = \left|\sum_{j=0}^{n} f_j(z) - \sum_{j=0}^{m-1} f_j(z)\right| < \varepsilon$$

for all $z \epsilon D$ and $n > m > N$. Thus,

$$\lim_{n\to\infty}\left|\sum_{j=0}^{n} f_j(z) - \sum_{j=0}^{m-1} f_j(z)\right| = \left|\sum_{j=0}^{\infty} f_j(z) - \sum_{j=0}^{m-1} f_j(z)\right| \leq \varepsilon$$

for $m > N$.(The limit of a series $\left(\sum_{j=0}^{n} a_j\right)$ is almost always written as $\sum_{j=0}^{\infty} a_j$.)

3.3 POWER SERIES

We are particularly interested in series of functions in which the partial sums are polynomials of increasing degree:

$$s_n(z) = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \ldots + c_n(z-z_0)^n.$$

(We start with n = 0 for esthetic reasons.)
These are the so-called **power series.** Thus,

$$\left(\sum_{j=0}^{n} c_j(z-z_0)^j\right)$$

a power series is a series of functions of the form

Let's look first at a very special power series, the so-called **Geometric series.**

$$\left(\sum_{j=0}^{n} z^j\right)$$

Here

$$s_n = 1 + z + z^2 + \ldots + z^n, \text{ and}$$
$$zs_n = z + z^2 + z^3 + \ldots + z^{n+1}.$$

Subtracting the second of these from the first gives us

$$(1 - z)s_n = 1 - z^{n+1}$$

If z = 1, then we can't go any further with this, but I hope it's clear that the series does not have a limit in case z = 1. Suppose now $z \neq 1$. Then we have

$$s_n = \frac{1}{1-z} - \frac{z^{n+1}}{1-z}$$

Now if $|z| < 1$, it should be clear that $\lim(z^{n+1}) = 0$, and so

$$\lim\left(\sum_{j=0}^{n} z^j\right) = \lim s_n = \frac{1}{1-z}$$

Or,

$$\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}, \text{ for } |z| < 1.$$

Note that if $|z| > 1$, then the Geometric series does not have a limit . Next, note that if $|z| \leq p < 1$, then the Geometric series converges uniformly to $\frac{1}{1-z}$. To see this, note that

$$\left(\sum_{j=0}^{n} \rho^j\right)$$

has a limit and appeal to the Weierstrass M-test.

Clearly a power series will have a limit for some values of z and perhaps not for others.
First, note that any power series has a limit when $z = z_0$. Let's see what else we can say. Consider a power series

$$\cdot \quad \left(\sum_{j=0}^{n} c_j(z-z_0)^j\right)$$

Let $\lambda = \limsup\left(\sqrt[j]{|c_j|}\right).$

(Recall that $\limsup(a_k) = \lim(\sup\{a_k : k \geq n\}.$) Now let $R = \frac{1}{\lambda}$. (We shall say R = 0 if $\lambda = \infty$, and $R = \infty$, if $\lambda = 0$.) We are going to show that the series converges uniformly for all $(z - z_0| \leq p < R$ and diverges for all $|z - z_0| > R$.

First, let's show the series does not converge for $|z - z_0| > R$. To begin, let k be so that

$$\frac{1}{|z - z_0|} < k < \frac{1}{R} = \lambda.$$

There are an infinite number of c_j. For which $\sqrt[j]{|c_j|} > k$, otherwise $\limsup\left(\sqrt[j]{|c_j|}\right) < k$. For each of these c_j we have

$$|c_j(z - z_0)^j| = \left(\sqrt[j]{|c_j|}\,|z - z_0|\right)^j > (k|z - z_0|)^j > 1$$

It is thus not possible for $\lim |c^n(z - z^0)^n| = 0$, and so the series does not converge.

we show that the series does converge uniformly for $|z - z_0| \leq p < R$. Let k be so that

$$\lambda = \frac{1}{R} < k < \frac{1}{\rho}.$$

Now, for *j* large enough, we have $\sqrt[j]{|c_j|} < k$. Thus for $|z - z_0| \leq p$, we have

$$|c_j(z-z_0)^j| = \left(\sqrt[j]{|c_j|}\,|z-z_0|\right)^j < (k|z-z_0|)^j < (k\rho)^j$$

The geometric series $\left(\sum_{j=0}^{n}(k\rho)^j\right)$ converges because kp < 1 *and* the uniform convergence

of $\left(\sum_{j=0}^{n}c_j(z-z_0)^j\right)$ follows from the M-test.

SELF ASSIGNMENT EXERCISE

Consider the series $\left(\sum_{j=0}^{n}\frac{1}{j!}z^j\right)$. Let's compute

R = 1/lim sup $\left(\sqrt[j]{|c_j|}\right)$ = lim sup $\left(\sqrt[j]{j!}\right)$. Let

K be any positive integer and choose an integer *m* large enough

to insure that $2^m > \frac{K^{2K}}{(2K)!}$

$\frac{n!}{K^n}$

Now consider , where $n = 2K + m$:

$$\frac{n!}{K^n} = \frac{(2K+m)!}{K^{2K+m}} = \frac{(2K+m)(2K+m-1)\ldots(2K+1)(2K)!}{K^m K^{2K}}$$
$$> 2^m\frac{(2K)!}{K^{2K}} > 1$$

Thus $\sqrt[n]{n!}$ > K. Reflect on what we have just shown: given any number K, there is a number *n*

such that $\sqrt[n]{n!}$ is bigger than it. In other words,
R = lim sup $\left(\sqrt[j]{j!}\right)$ = ∞, and so the series $\left(\sum_{j=0}^{n}\frac{1}{j!}z^j\right)$

converges for all z.

$\left(\sum_{j=0}^{n}c_j(z-z_0)^j\right)$

Let's summarize what we have. For any power series
there is a number

R = $\frac{1}{\limsup\left(\sqrt[j]{|c_j|}\right)}$ such that the series converges uniformly for $|z - z_0| \leq p < R$ and does not

converge for $|z - z_0| > R$. (Note that we may have R = 0 or R = ∞.) The number R is called the **radius of convergence** of the series, and the set $|z - z_0| = R$ is called the **circle of convergence**. Observe also that the limit of a power series is a function analytic inside the circle of convergence.

3.4 INTEGRATION OF POWER SERIES

Inside the circle of convergence, the limit

$$S(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j$$

is an analytic function. We shall show that this series may be integrated "term-by-term"—that is, the integral of the limit is the limit of the integrals. Specifically, if *C* is any contour inside the circle of convergence, and the function g is continuous on *C*, then

$$\int_C g(z) S(z) dz = \sum_{j=0}^{\infty} c_j \int_C g(z)(z - z_0)^j dz.$$

If $\epsilon > 0$. Let M be the maximum of $|g(z)|$ on *C* and let *L* be the length of *C*. Then there is an integer N so that

$$\left| \sum_{j=n}^{\infty} c_j (z - z_0)^j \right| < \frac{\varepsilon}{ML}$$

For all n > *N*. Thus,

$$\left|\int_C \left(g(z) \sum_{j=n}^{\infty} c_j (z-z_0)^j \right) dz \right| < ML \frac{\varepsilon}{ML} = \varepsilon,$$

Hence,

$$\left|\int_C g(z)S(z)dz - \sum_{j=0}^{n-1} c_j \int_C g(z)(z-z_0)^j dz \right| = \left|\int_C \left(g(z) \sum_{j=n}^{\infty} c_j (z-z_0)^j \right) dz \right|$$
$$< \varepsilon,$$

and we have shown what we promised.

3.5 DIFFERENTIATION OF POWER SERIES

Let

$$S(z) = \sum_{j=0}^{\infty} c_j (z-z_0)^j.$$

Now we are ready to show that inside the circle of convergence,

$$S'(z) = \sum_{j=1}^{\infty} j c_j (z-z_0)^{j-1}.$$

Let z be a point inside the circle of convergence and let *C* be a positive oriented circle centered at z and inside the circle of convergence. Define

$$g(s) = \frac{1}{2\pi i (s-z)^2},$$

and apply the result of the previous section to conclude that

$$\int_C g(s)S(s)ds = \sum_{j=0}^{\infty} c_j \int_C g(s)(s-z_0)^j ds, \text{ or}$$

$$\frac{1}{2\pi i}\int_C \frac{S(s)}{(s-z)^2}ds = \sum_{j=0}^{\infty} c_j \frac{1}{2\pi i}\int_C \frac{(s-z_0)^j}{(s-z)^2}ds. \text{ Thus}$$

$$S'(z) = \sum_{j=0}^{\infty} jc_j(z-z_0)^{j-1},$$

4.0 CONCLUSION

We now end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

A sequence (f_n) of functions on a domain D is a function from the positive integers into the set of complex functions on D. Thus, for each $z \in D$, we have an ordinary sequence $(f_n(z))$. If each of the sequences $(f_n(z))$ converges, then we say the sequence of functions (f_n) converges to the function f defined by $f(z) = \lim(f_n(z))$. The sequence (f_n) is said to converge to f uniformly on a set S if given an $\epsilon > 0$, there is an integer N_ϵ so that $|f_n(z) - f(z)| < \epsilon$ for all $n \geq N_\epsilon$ and all $z \in S$.
Note that it is possible for a sequence of continuous functions to have a limit function that is not continuous.
If (f_n) a sequence of functions, each analytic on some region D, and suppose the sequence converges uniformly on D to the function f. Then f is analytic.
The number R is called the **radius of convergence** of the series, and the set $|z - z_0| = R$ is called the **circle of convergence**. We

observed that the limit of a power series is a function analytic inside the circle of convergence.

We showed that the series may be integrated "term-by-term"—that is, the integral of the limit is the limit of the integrals. Specifically, if *C* is any contour inside the circle of convergence, and the function g is continuous on *C*,
then

$$\int_C g(z)S(z)dz = \sum_{j=0}^{\infty} c_j \int_C g(z)(z-z_0)^j dz.$$

we showed that inside the circle of convergence,

$$S'(z) = \sum_{j=1}^{\infty} jc_j(z-z_0)^{j-1}.$$

if z be a point inside the circle of convergence and let *C* be a positive oriented circle centered at z and inside the circle of convergence.

6.0 TUTOR MARKED ASSIGNMENT

1. Prove that a sequence cannot have more than one limit (We thus speak of the limit of a sequence.)
2. Give an example of a sequence that does not have a limit, or explain carefully why there is no such sequence.
3. Give an example of a bounded sequence that does not have a limit, or explain carefully why there is no such sequence.
4. Give a sequence (f_n) of functions continuous on a set D with a limit that is not continuous.
5. Give a sequence of real functions differentiable on an interval which converges uniformly to a non differentiable function

6. Find the set D of all z for which the sequence $\left(\frac{z^n}{z^n-3^n}\right)$ has a limit. Find the limit.

7. Prove that the series $\left(\sum_{j=1}^{n} a_j\right)$ converges if and only if both the series $\left(\sum_{j=1}^{n} \mathrm{Re}\, a_j\right)$ and $\left(\sum_{j=1}^{n} \mathrm{Im}\, a_j\right)$ converge.

8. Explain how you know that the series $\left(\sum_{j=1}^{n} (\frac{1}{z})^j\right)$ converges uniformly on the set $|z| \geq 5$.

9. Suppose the sequence of real number (a_j) has a limit. Prone that

$$\limsup(a_j) = \lim(a_j).$$

For each of the following, find the set D of points at which the series converges:

10 . $\left(\sum_{j=0}^{n} j!z^j\right)$

11. $\left(\sum_{j=0}^{n} jz^j\right)$

12. $\left(\sum_{j=0}^{n} \frac{j^2}{3^j} z^j\right)$

13. $\left(\sum_{j=0}^{n} \frac{(-1)^j}{2^{2j}(j!)^2} z^{2j}\right)$

14. Find the limit of

$$\left(\sum_{j=0}^{n} (j+1)z^j\right).$$

For what values of z does the series converge?

15. Find the limit of

$$\left(\sum_{j=1}^{n} \frac{z^j}{j}\right).$$

For what values of z does the series converge?

16. Find a power series $\left(\sum_{j=0}^{n} c_j(z-1)^j\right)$ such that

$$\frac{1}{z} = \sum_{j=0}^{\infty} c_j(z-1)^j, \text{ for } |z-1| < 1.$$

17. Find a power series $\left(\sum_{j=0}^{n} c_j(z-1)^j\right)$ such that

$$\operatorname{Log} z = \sum_{j=0}^{\infty} c_j(z-1)^j, \text{ for } |z-1| < 1.$$

6.0 REFERENCE/FURTHER READINGS

Advance Calculus by Schaum's outline Series

Advance Engineering Mathematics by Stroud

MODULE 3

UNIT 1 TALOR AND LAURENT SERIES

CONTENTS

1.0 INTRODUCTION

2.0 OBJECTIVES

The objective in this unit are to :

- study Taylor series
- explain Laurent Series

3.0 MAIN CONTENT

3.1 Taylor Series

Suppose f is analytic on the open disk $|z - z_0| < r$. Let z be any point in this disk and choose C to be the positively oriented circle of radius p, Where $|z - z_0| < p < r$. Then for $s \epsilon C$ we have

$$\frac{1}{s-z} = \frac{1}{(s-z_0)-(z-z_0)} = \frac{1}{s-z_0}\left[\frac{1}{1-\frac{z-z_0}{s-z_0}}\right] = \sum_{j=0}^{\infty}\frac{(z-z_0)^j}{(s-z_0)^{j+1}}$$

Since $\left|\frac{z-zo}{s-zo}\right| < 1$. The convergence is uniform, so we may integrate

$$\int_C \frac{f(s)}{s-z}ds = \sum_{j=0}^{\infty}\left(\int_C \frac{f(s)}{(s-z_0)^{j+1}}ds\right)(z-z_0)^j, \text{ or}$$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}ds = \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{j+1}}ds\right)(z-z_0)^j.$$

We have thus produced a power series having the given analytic function as a limit:

$$f(z) = \sum_{j=0}^{\infty} c_j(z-z_0)^j, \ |z-z_0| < r,$$

where

$$c_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{j+1}} ds.$$

This is the celebrated **Taylor Series** for *f* at $z = z_0$.

We know we may differentiate the series to get and this one converges uniformly where the series for *f* does. We can thus differentiate again and again to obtain

$$f'(z) = \sum_{j=1}^{\infty} j c_j (z - z_0)^{j-1}$$

$$f^{(n)}(z) = \sum_{j=n}^{\infty} j(j-1)(j-2)\ldots(j-n+1) c_j (z - z_0)^{j-n}.$$

Hence,

$$f^{(n)}(z_0) = n! c_n, \text{ or}$$

$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

But we also know that

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{n+1}} ds.$$

This gives us

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{n+1}} ds, \text{ for } n = 0, 1, 2, \ldots .$$

This is the famous **Generalized Cauchy Integral Formula**.

Recall that we previously derived this formula for $n = 0$ and 1.

What does all this tell us about the radius of convergence of a power series'

$$f(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j,$$

and the radius of convergence is *R*. Then we know, of course, that the limit function f is analytic for $1z - z_0 \; I < R$. We showed that if f is analytic in $|z - z_0| < r$, then the series converges for $|z - z_0|$ $f < \text{r}$. Thus $r \leq R$, and so f cannot be analytic at any point z for which $|z - zp| > R$. In other words, the circle of convergence is the largest circle centered at z_0 inside of which the limit f' is analytic.

SELF ASSIGNMENT EXERCISE

Let $f(z) = \exp(z) = e^z$. Then

$f(0) = f^{(n)}(0) = \ldots = f^{(n)}(0) = \ldots = 1$. and the Talor series for f at $z_0 = 0$ is

$$e^z = \sum_{j=0}^{\infty} \frac{1}{j!} z^j$$

and this is valid for all values *of* z since f is entire. (We also showed earlier that this particular series has an infinite radius of convergence.)

3.2 LAURENT SERIES

Suppose f is analytic in the region $R_1 < |z - z_0|$ ($< R_2$, and let C be a positively oriented simple closed curve around z_0 in this region. (Note: we include the possibilities that R_1 can be 0, and $R_2 = \infty$.) We shall show that for $z \in C$ in this region

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j},$$

where

$$a_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{j+1}} ds, \text{ for } j = 0, 1, 2, \ldots$$

and

$$b_j = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{-j+1}}ds, \text{ for } j = 1,2,\ldots.$$

The sum of the limits of these two series is frequently written

$$f(z) = \sum_{j=-\infty}^{\infty} c_j(z-z_0)^j,$$

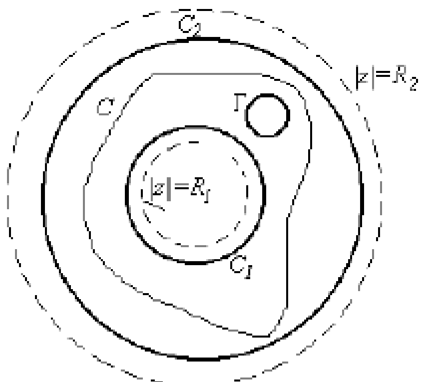
where

$$c_j = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{j+1}}ds, j = 0, \pm 1, \pm 2, \ldots.$$

This recipe for f(z) is called a **Laurent series**, although it is important to keep in mind that it is really two series.

Let's derive the above formula.
First, let r_i and r_2 be so that $R_1 < r_1 \le |z - z_0| \le r_2 < R_2$ and so that the point z and the curve C are included in the region $r_1 \le |z-z_0| \le r_2$.

Also, let Γ be a circle centered at z and such that Γ is included in this region.



Then $\frac{f(s)}{s-z}$ is an analytic function (of s) on the region bounder by C_1, C_2, and Γ, where C_1 is the circle $|z| = r_1$ and C_2 is the circle $|z| = r_2$. Thus,

$$\int_{C_2} \frac{f(s)}{s-z} ds = \int_{C_1} \frac{f(s)}{s-z} ds + \int_{\Gamma} \frac{f(s)}{s-z} ds.$$

$$\int_{\Gamma} \frac{f(s)}{s-z} ds = 2\pi i f(z),$$

(All three circles are positively oriented, of course.) But
and so we have

$$2\pi i f(z) = \int_{C_2} \frac{f(s)}{s-z} ds - \int_{C_1} \frac{f(s)}{s-z} ds.$$

Look at the first of the two integrals on the right-hand side of this equation. For $s \in C_2$,

we have $|z - z_0| < |s - z_0|$, and so

$$\begin{aligned} \frac{1}{s-z} &= \frac{1}{(s-z_0)-(z-z_0)} \\ &= \frac{1}{s-z_0}\left[\frac{1}{1-(\frac{z-z_0}{s-z_0})}\right] \\ &= \frac{1}{s-z_0}\sum_{j=0}^{\infty}\left(\frac{z-z_0}{s-z_0}\right)^j \\ &= \sum_{j=0}^{\infty}\frac{1}{(s-z_0)^{j+1}}(z-z_0)^j. \end{aligned}$$

Hence,

$$\int_{C_2} \frac{f(s)}{s-z}ds = \sum_{j=0}^{\infty}\left(\int_{C_2} \frac{f(s)}{(s-z_0)^{j+1}}ds\right)(z-z_0)^j$$
$$= .\sum_{j=0}^{\infty}\left(\int_{C} \frac{f(s)}{(s-z_0)^{j+1}}ds\right)(z-z_0)^j$$

For the second of these two integrals, note that for $s \in C_1$ we have $|s - z_0| < |z - z_0|$, and so

$$\frac{1}{s-z} = \frac{-1}{(z-z_0)-(s-z_0)} = \frac{-1}{z-z_0}\left[\frac{1}{1-\left(\frac{s-z_0}{z-z_0}\right)}\right]$$
$$= \frac{-1}{z-z_0}\sum_{j=0}^{\infty}\left(\frac{s-z_0}{z-z_0}\right)^j = -\sum_{j=0}^{\infty}(s-z_0)^j\frac{1}{(z-z_0)^{j+1}}$$
$$= -\sum_{j=1}^{\infty}(s-z_0)^{j-1}\frac{1}{(z-z_0)^j} = -\sum_{j=1}^{\infty}\left(\frac{1}{(s-z_0)^{-j+1}}\right)\frac{1}{(z-z_0)^j}$$

As before,

$$\int_{C_1} \frac{f(s)}{s-z}ds = -\sum_{j=1}^{\infty}\left(\int_{C_1} \frac{f(s)}{(s-z_0)^{-j+1}}ds\right)\frac{1}{(z-z_0)^j}$$
$$= -\sum_{j=1}^{\infty}\left(\int_{C} \frac{f(s)}{(s-z_0)^{-j+1}}ds\right)\frac{1}{(z-z_0)^j}$$

Putting this altogether, we have the Laurent series:

$$f(z) = \frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s-z}ds - \frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s-z}ds$$
$$= \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C}\frac{f(s)}{(s-z_0)^{j+1}}ds\right)(z-z_0)^j + \sum_{j=1}^{\infty}\left(\frac{1}{2\pi i}\int_{C}\frac{f(s)}{(s-z_0)^{-j+1}}ds\right)\frac{1}{(z-z_0)^j}.$$

SELF ASSIGNMENT EXERCISE

Let *f* be defined by

$$f(z) = \frac{1}{z(z-1)}.$$

First, observe that *f* is analytic in the region 0 < |z| < l.
Let's find the Laurent series for *f* valid in this region.

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}.$$

From our vast knowledge of the Geometric series, we have

$$f(z) = -\frac{1}{z} - \sum_{j=0}^{\infty} z^j.$$

Now let's find another Laurent series for *f*, the one valid for the region 1 < |z| < ∞.

$$\frac{1}{z-1} = \frac{1}{z}\left[\frac{1}{1-\frac{1}{z}}\right].$$

Now since $|\frac{1}{z}| < l$, we have

$$\frac{1}{z-1} = \frac{1}{z}\left[\frac{1}{1-\frac{1}{z}}\right] = \frac{1}{z}\sum_{j=0}^{\infty} z^{-j} = \sum_{j=1}^{\infty} z^{-j},$$

and so

$$f(z) = -\frac{1}{z} + \frac{1}{z-1} = -\frac{1}{z} + \sum_{j=1}^{\infty} z^{-j}$$

$$f(z) = \sum_{j=2}^{\infty} z^{-j}.$$

4.0 CONCLUSION

We now end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

We have thus produced a power series having the given analytic function as a limit.
We have differentiated the series to get and this one converges uniformly where the series for f does.
We showed that if f is analytic in $|z - z_0| < r$, then the series converges for $|z - z_0| < r$. Thus $r \leq R$, and so f cannot be analytic at any point z for which $|z - zp| > R$. In other words, the circle of convergence is the largest circle centered at z_0 inside of which the limit f' is analytic.
We find another Laurent series for f, the one valid for the region $1 < |z| < \infty$.

6.0 TUTOR MARKED ASSIGNMENT

l. Show that for all z,

$$e^z = e\sum_{j=0}^{\infty} \frac{1}{j!}(z-1)^j.$$

2. What is the radius of convergence o $\left(\sum_{j=0}^{n} c_j z^j\right)$ taylor series for tanhz ?

3. Show that

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} \frac{(z-i)^j}{(1-i)^{j+1}}$$

For $|z-i| < \sqrt{2}$.

4. if $f(z) = \frac{1}{1-z,}$ what is $f^{10n)}(i)$?

5. Suppose f is analytic at z = 0 and $f(0) = f(0) = f(0) = 0$. Prove there is a function g analytic at 0 such that $f(z) = z^3 g(z)$ in a neighborhood of 0.

6. Find the Taylor series for $f(z) = \sin z$ at $z_0 = 0$.

7. Show that the function f defined by

$$f(z) = \begin{cases} \frac{\sin z}{z} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}$$

is analytic at z = 0, and find $f(0)$.

8. Find two Laurent series in powers of z for the function f defined by

$$f(z) = \frac{1}{z^2(1-z)}$$

and specify the regions in which the series converge to $f(z)$.

9. Find two Laurent series in powers of z for the function f defined by

$$f(z) = \frac{1}{z(1+z^2)}$$

and specify the regions in which the series converge to *f* (z).

10. Find the Laurent series in powers of z - 1 for *f*(z) = $\frac{1}{z}$ in the region $1 < |z - 1| < \infty$.

8.0 REFERENCE/FURTHER READINGS

Advance Calculus by Schaum's outline Series

Advance Engineering Mathematics by Stroud

MODULE 4

BILEAR TRANFORMATION

CONTENTS

1.0 INTRODUCTION

This is the most common method for transforming the system function $H_a(s)$ of an analogue filter to the system function H(z) of an IIR discrete time filter. It is not the only possible transformation, but a very useful and reliable one.

Consider derivative approximation technique:

D(y[n]) = dy(t) /dt at t=nT ≈ (y[n] - y[n-1]) / T.
D(x[n]) = dx(t) /dt at t=nT ≈ (x[n] - x[n-1]) / T.
D'(y[n]) = $d^2y(t)/dt^2$ at t=nT ≈ D(D(y[n])) = (y[n] - 2y[n-1]+y[n-2])/T^2
D''(y[n]) = $d^3y(t)/dt^3$ at t=nT ≈ D(D'(y[n])) = (y[n]-3y[n-1]+3y[n-2]-y[n-3])/T^3

“Backward difference” approximation introduces delay which becomes greater for higher orders.
Try "forward differences" : D[n] ≈ [y[n+1] - y[n]] / T, etc.
But this does not make matters any better.

Bilinear approximation:

$0.5(D[n] + D[n-1]) \approx (y[n] - y[n-1]) / T$

and similarly for dx(t)/dt at t=nT.

Similar formulae may be derived for $d^2y(t)/dt^2$, and so on.

If D(z) is the z-transform of D[n] :

$0.5(D(z) + z^{-1}D(z)) = (Y(z) - z^{-1}Y(z)) / T$

$\therefore D(z) = [2 (1 - z^{-1})/ [T(1+z^{-1})] Y(z)$

$= [(2/T) (z-1)/(z+1)] Y(z).$

Applying y[n] to [(2/T) (z-1)/(z+1)] produces an approximation to dy(t)/dt at t=nT.

In an analogue circuit, applying y(t) to an LTI circuit with system function H(s) = s produces dy(t)/dt since the Laplace Transform of dy(t)/dt is sY(s) .

Therefore, replacing s by [(2/T) (z-1)/(z+1)] is the bilinear approximation.

2.0 OBJECTIVES

The objectives of this unit are to

. study bilinear transformation

. design of an IIR low-pass filter by the bilinear transformation method

. explain higher order IIR digital filters

. discuss IIR discrete time high-pass band-pass and band-stop filter design

. comparison of IIR and FIR digital filters.

3.0 MAIN CONTENT

3.1 Bilinear transformation technique

Definition: Given analogue transfer function $H_a(s)$, replace s by :

$$\frac{2}{T}\left[\frac{z-1}{z+1}\right]$$

to obtain H(z). For convenience we can take T=1.

SELF ASSIGNMENT EXERCISE

If $H_a(s) = \dfrac{1}{(1+RCs)}$ then,

$$H(z) = \frac{z+1}{(1+2RC)z+(1-2RC)} = K\frac{1+z^{-1}}{1+b_1z^{-1}}$$

where $k = \dfrac{1}{(1+2RC)}$ and $b_1 = \dfrac{(1-2RC)}{(1+2RC)}$

3.2 Properties of Bilinear transformation

(i) This transformation produces a function H(z) such that given any complex number z,

$H(z) = H_a(s)$ where $s = 2(z - 1) / (z + 1)$

(ii) The order of H(z) is equal to the order of $H_a(s)$

(iii) If $H_a(s)$ is causal and stable, then so is H(z).

(iv) $H(\exp(j\Omega)) = H_a(j\omega)$ where $\omega = 2\tan(\Omega/2)$

Proofs of properties (ii) and (ii) are straightforward but are omitted here.

<u>Proof of property (iv)</u>: When $z = \exp(j\Omega)$, then

$$s = 2\frac{e^{j\Omega}-1}{e^{j\Omega}+1} = \frac{2\left(e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}\right)}{e^{j\frac{\Omega}{2}} + e^{-j\frac{\Omega}{2}}} = 2j\tan\left(\frac{\Omega}{2}\right)$$

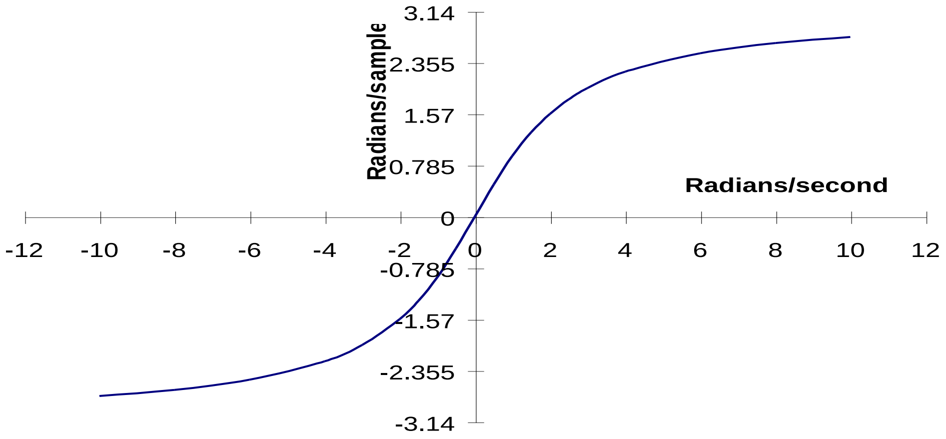


Fig 6.1 Frequency Warping

Frequency warping:

By property (iv) the discrete time filter's frequency response $H(\exp(j\Omega))$ at relative frequency Ω will be equal to the analogue frequency response $H_a(j\omega)$ with $\omega = 2\tan(\Omega/2)$. The graph of Ω against ω in fig 6.1, shows how ω in the range $-\infty$ to ∞ is mapped to Ω in the range $-\pi$ to π. The mapping is reasonably linear for ω in the range -2 to 2 (giving Ω in the range $-\pi/2$ to $\pi/2$), but as ω increases beyond this range, a given increase in ω produces smaller and smaller increases in Ω. Comparing the analogue gain response shown in fig 6.2(a) with the discrete time one in fig. 6.2(b) produced by the transformation, the latter becomes more and more compressed as $\Omega \rightarrow \pm\pi$. This "frequency warping" effect must be taken into account when determining a suitable $H_a(s)$ prior to the bilinear transformation.

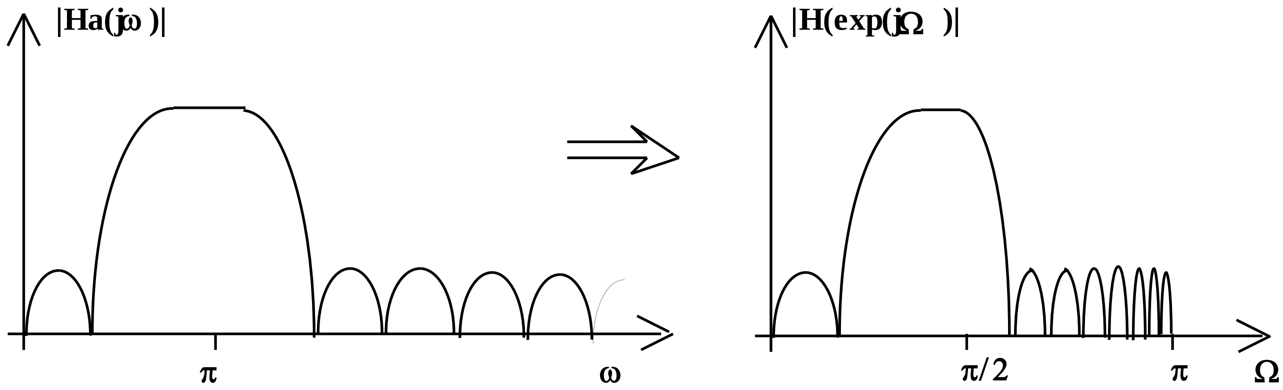


Fig. 6.2(a): Analogue Gain Response **Fig. 6.2(b): Effect Of Bilinear Transformation**

3.3 Design of an IIR low-pass filter by the bilinear transformation method

Given the required cut-off frequency Ω_c in radians/sample:-

(i) Find $H_a(s)$ for an analogue low-pass filter with cut-off $\omega_c = 2\tan(\Omega_c/2)$ radians/sec.
(ω_c is said to be the "pre-warped" cut-off frequency).
(ii) Replace s by 2(z - 1)/(z + 1) to obtain H(z).
(iii) Rearrange the expression for H(z) and realise by bi-quadratic sections.

SELF ASSIGNMENT EXERCISE

Design a second order Butterworth-type IIR lowpass filter with $\Omega_c = \pi / 4$.
Solution: Pre-warped frequency $\omega_c = 2\tan(\pi / 8) = 0.828$
For an analogue Butterworth low-pass filter with cut-off frequency 1 radian/second:

$H_a(s) = 1 / (1 + \sqrt{2}\, s + s^2)$

Replace s by s / 0.828, then replace s by 2(z - 1)/(z + 1) to obtain:

$$H(z) = \frac{z^2 + 2Z + 1}{10.3z^2 - 9.7z + 3.4} = 0.093\left(\frac{1 + 2z^{-1} + z^{-2}}{1 - 0.94z^{-1} + 0.33z^{-2}}\right)$$

which may be realised by the signal flow graph in fig 6.5. Note the extra multiplier scaling the input by 0.097.

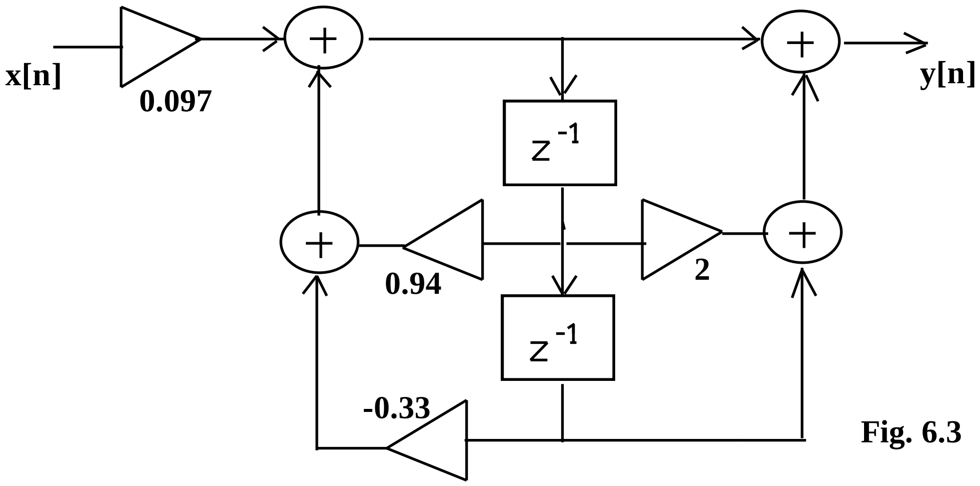


Fig. 6.3

3.4 Higher order IIR digital filters

Recursive filters of order greater than two are highly sensitive to quantisation error and overflow. It is normal, therefore, to design higher order IIR filters as cascades of bi-quadratic sections.

SLEF ASSIGNMENT EXERCISE

Design a 4th order Butterworth-type IIR low-pass digital filter is needed with 3dB cut-off at one sixteenth of the sampling frequency $f_{s.}$

Solution: The relative cut-off frequency is $\Omega_C = \pi/8$ radians/sample
The prewarped cut-off frequency is therefore $\omega_C = 2 \tan(\pi/16) = 0.4$ radians/sec.

Formula for 4th order Butterworth 1 radian/sec low-pass system function:

$$H_a(s) = \left(\frac{1}{1+0.77s+s^2}\right)\left(\frac{1}{1+1.85s+s^2}\right)$$

Scale the analogue cut-off frequency to ω_c by replacing s by s / 0.4.
Then replace s by 2 (z - 1)/(z +1) to obtain:

$$H(z) = 0.033\left(\frac{1+2z^{-1}+z^{-2}}{1-1.6z^{-1}0.74z^{-2}}\right)0.028\left(\frac{1+2z^{-1}+z^{-2}}{1-1.365z^{-1}+0.48z^{-2}}\right)$$

H(z) may be realised in the form of cascaded bi-quadratic sections as shown in fig 1.1.

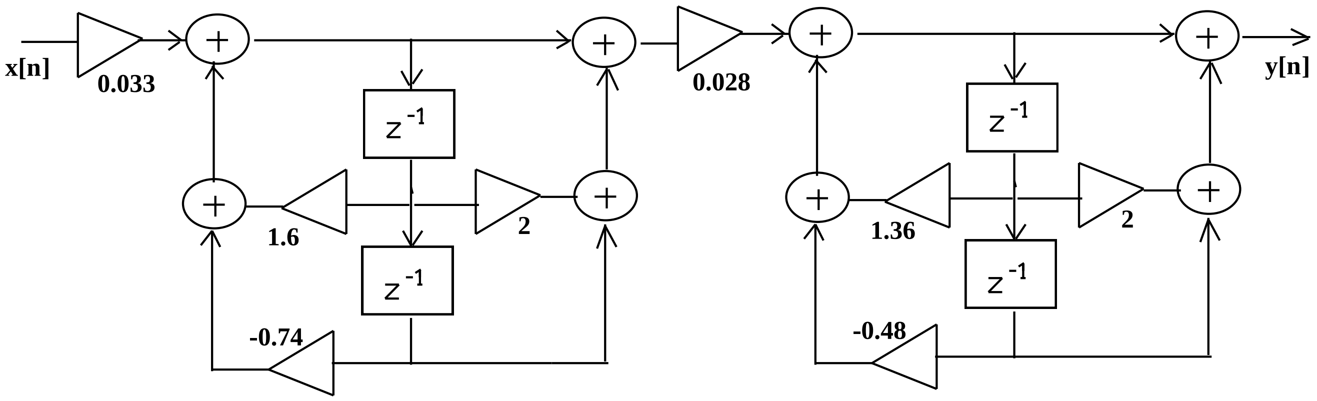


Fig. 6.4 Fourth Order IIR Butterworth Filter With Cut-Off fs/16

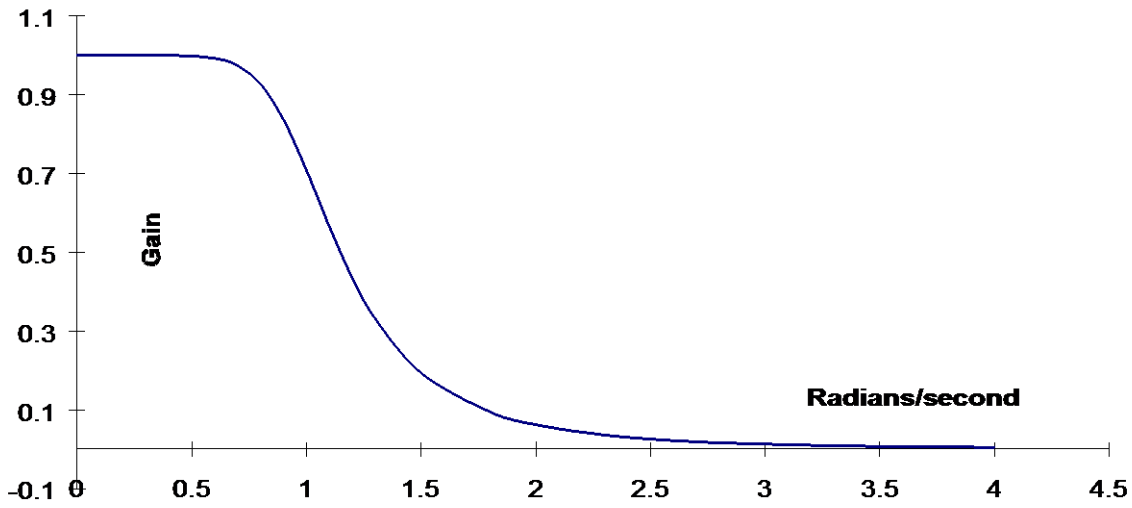


Fig. 6.5(a) Analogue 4th order Butterworth gain response

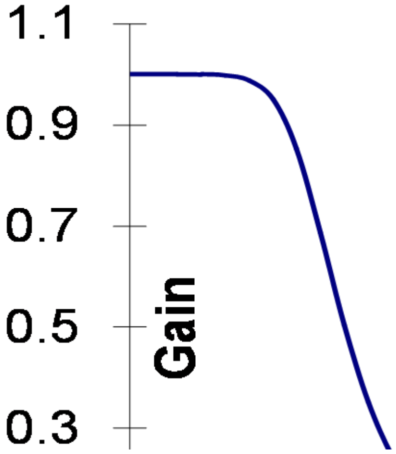


Fig. 6.7(a) shows the gain response for the 4th order Butterworth low-pass filter whose transfer function was used here as a prototype. Fig 6.7(b) shows the gain response of the derived digital filter which, like the analogue filter, is 1 at zero frequency and 0.707 at the cut-off frequency. Note however that the analogue gain approaches 0 as $\omega \to \infty$ whereas the gain of the digital filter becomes exactly zero at $\Omega = \pi$. The shape of the Butterworth gain response is " warped " by the bilinear transformation. However, the 3dB point occurs exactly at Ω_c for the digital filter, and the cut-off rate becomes sharper and sharper as $\Omega \to \pi$ because of the compression as $\omega \to \infty$.

3.5 IIR discrete time high-pass band-pass and band-stop filter design

The bilinear transformation may be applied to analogue system functions which are high-pass, band-pass or band-stop. Such system functions may be obtained from an analogue low-pass 'prototype' system function (with cut-off 1 radian/second) by means of the frequency band transformations introduced in Section 2. Wide-band band-pass and band-stop filters ($f_U >> 2f_L$) may be designed by cascading low-pass and high-pass sections, thus avoiding the need to apply frequency band transformations, but 'narrow band' band-pass/stop filters (f_U not $>> 2f_L$) will not be very accurate if a cascading approach is used.

3.6 Comparison of IIR and FIR digital filters

IIR type digital filters have the advantage of being economical in their use of delays, multipliers and adders. They have the disadvantage of being sensitive to coefficient round-off inaccuracies and the effects of overflow in fixed point arithmetic. These effects can lead to instability or serious distortion. Also, an IIR filter cannot be exactly linear phase.

FIR filters may be realised by non-recursive structures which are simpler and more convenient for programming especially on devices specifically designed for digital signal processing. These structures are always stable, and because there is no recursion, round-off and overflow errors are easily controlled. A FIR filter can be exactly linear phase. The main disadvantage of FIR filters is that large orders can be required to perform fairly simple filtering tasks.

4.0 CONCLUSION

In this closing unit, the achievement resulting from this unit are highlighted in the summary.

5.0 SUMMARY

We defined bilinear transformation and its properties.

We replace s by 2(z - 1)/(z + 1) to obtain H(z) and rearrange the expression for H(z) and realise by bi quadratic sections. Therefore, we design higher order IIR filters as cascades of bi-quadratic sections.

Wide-band band-pass and band-stop filters ($f_U >> 2f_L$) may be designed by cascading low-pass and high-pass sections, thus avoiding the need to apply frequency band transformations, but 'narrow band' band-pass/stop filters (f_U not $>> 2f_L$) will not be very accurate if a cascading approach was used.

These effects can lead to instability or serious distortion. Also, an IIR filter cannot be exactly linear phase.

TUTUR MARKED ASSIGNMENT

1. By referring to the general formula, show that the system function of a third order analogue Butterworth low-pass filter with 3 dB cut-off frequency at 1 radian/second is:

$$H_a(s) = \frac{1}{(s^2 + s + 1)(s + 1)}$$

2. Confirm from the general formula that the system function for a 3$^{\text{rd}}$ order Butterworth type low-

pass analogue filter with cut-off frequency ω_C radians per second is:

$$H_a(s) = \frac{1}{\left[1 + 2\,{}^{s}\!/_{\omega_c} + 2\,{}^{s^2}\!/_{\omega_c^2} + \left({}^{s}\!/_{\omega_c}\right)^3\right]}$$

Give the corresponding differential equation.

Apply the derivative approximation technique to derive from this differential equation a third order IIR Butterworth-type digital filter with cut-off frequency 500 Hz where the sampling frequency is 10 kHz.

3. A third order low-pass IIR discrete time filter is required with a 3dB cut-off frequency of one quarter of the sampling frequency, f_S . If the filter is to be designed by the bilinear transformation applied to a Butterworth low-pass transfer function, design the IIR filter and give its signal flow graph in the form of a second order and a first order section in serial cascade.

4. Give a computer program to implement the third order IIR filter designed above on a processor with floating point arithmetic. How would it be implemented in fixed point arithmetic?

5. A low-pass IIR discrete time filter is required with a cut-off frequency of one quarter of the sampling frequency, f_S , and a stop-band attenuation of at least 20 dB for all frequencies greater than $3f_S/8$ and less than $f_S/2$. If the filter is to be designed by the bilinear transformation applied to a Butterworth low-pass transfer function, show that the minimum order required is three. Design the IIR filter and give its signal flow graph.

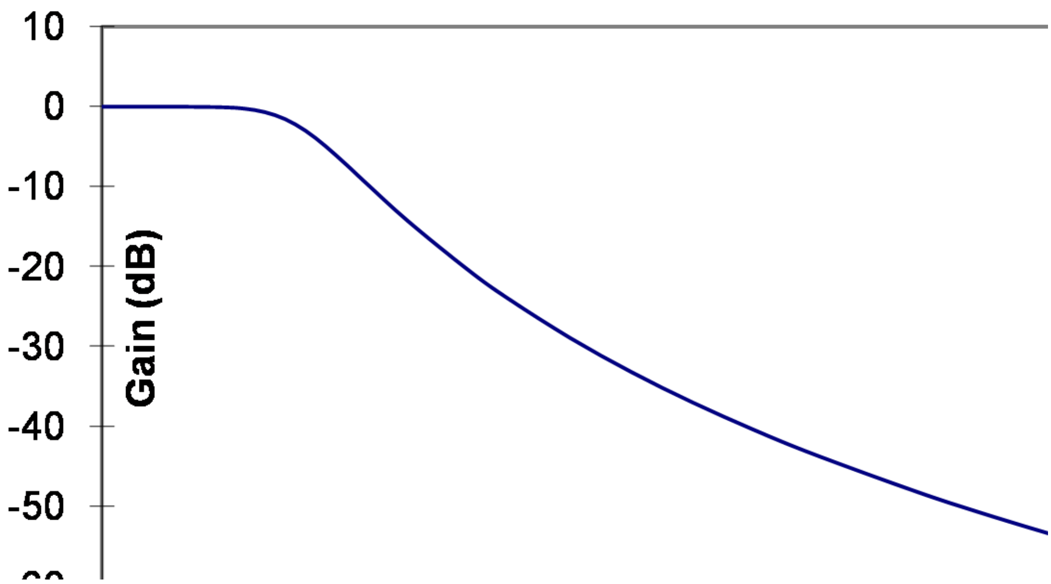


Fig. 6.8(a): Analogue 4th order Butterworth gain response

7.0 REFERENCE/FURTHER READINGS

Advance Calculus by Schaum's outline Series

Advance Engineering Mathematics by Stroud