NATIONAL OPEN UNIVERSITY OF NIGERIA

COURSE CODE: BHM 109

COURSE TITLE: BUSINESS MATHEMATICS

COURSE GUIDE

BHM 109
BUSINESS MATHEMATICS

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Introduction

BHM 109 - Business Mathematics is designed to teach you how mathematics could be used in solving problems in the contemporary business world. Therefore, the course is structured to expose you to the skills required in other to attain a level of proficiency in business management.

What you will learn in this course

You will be taught the basis of mathematics required in solving problems in business.

Course Aims

There are ten study units in the course and each unit has its objectives. You should read the objectives of each unit and bear them in mind as you go through the unit. In addition to the objectives of each unit, the overall aims of this course include:

(i) To introduce you to the words and concepts in business mathematics
(ii) To familiarize you with the peculiar characteristics in business mathematics.
(iii) To expose you to the need for and the demands of mathematics in the business world.
(iv) To prepare you for the contemporary business world.

Course Objectives

The objectives of this course are:

- To inculcate appropriate mathematical skills required in business

- Educate learners on how to use mathematical Techniques in solving problems.

- Educate the learners on how to integrate mathematical models in business.

Working through This Course

You have to work through all the study units in the course. There are two modules and ten study units in all.

Course Materials

Major components of the course are:

1. Course Guide
2. Study Units
3. Textbooks
4. CDs
5. Assignments File
6. Presentation Schedule

Study Units

The breakdown of the two modules and ten study units are as follows:

Module 1

Unit 1 Mathematics and Symbolic Logic
Unit 2 Matrices and Determinants
Unit 3 Vectors and Complex Numbers
Unit 4 Introduction to Straight Lines
Unit 5 Introduction to Circle

Module 2
Unit 1 Simple Sequence and Series
Unit 2 Limits
Unit 3 Differentiation and Integration
Unit 4 Maximum and Minimum Points and Value
Unit 5 Linear Programming (Inequalities and Constraints)

References and Other Resources

Every unit contains a list of references and further reading. Try to get as many as possible of those textbooks and materials listed. The textbooks and materials are meant to deepen your knowledge of the course.

Assignment File

In this file, you will find all the details of the work you must submit to your tutor for marking. The marks you obtain from these assignments will count towards the final mark you obtain for this course. Further information on assignments will be found in the Assignment File itself and later in this *Course Guide* in the section on assessment.

Presentation Schedule

The Presentation Schedule included in your course materials gives you the important dates for the completion of tutor-marked assignments and attending tutorials. Remember, you are required to submit all your assignments by the due date. You should guard against falling behind in your work.

Assessment

Your assessment will be based on tutor-marked assignments (TMAs) and a final examination which you will write at the end of the course.

Tutor Marked Assignments (TMA)

Every unit contains at least one or two assignments. You are advised to work through all the assignments and submit them for assessment. Your tutor will assess the assignments and select four which will constitute the 30% of your final grade. The tutor-marked assignments may be presented to you in a separate file. Just know that for every unit there are some tutor-marked assignments for you. It is important you do them and submit for assessment.

Final Examination and Grading

At the end of the course, you will write a final examination which will constitute 70% of your final grade. In the examination which shall last for two hours, you will be requested to answer three questions out of at least five questions.

Course Marking Scheme

This table shows how the actual course marking is broken down.

Assessment	Marks
Assignments	Four assignments, best three marks of the four count at 30% of course marks
Final Examination	70% of overall course marks
Total	100% of course marks

How to Get the Most from This Course

In distance learning, the study units replace the university lecture. This is one of the great advantages of distance learning; you can read and work through specially designed study materials at your own pace, and at a time and place that suits you best. Think of it as reading the lecture instead of listening to the lecturer. In the same way a lecturer might give you some reading to do, the study units tell you when to read, and which are your text materials or set books. You are provided exercises to do at appropriate points, just as a lecturer might give you an in-class exercise. Each of the study units follows a common format. The first item is an introduction to the subject matter of the unit, and how a particular unit is integrated with the other units and the course as a whole. Next to this is a set of learning objectives. These objectives let you know what you should be able to do by the time you have completed the unit. These learning objectives are meant to guide your study. The moment a unit is finished, you must go back and check whether you have achieved the objectives. If this is made a habit, then you will significantly improve your chances of passing the course. The main body of the unit guides you through the required reading from other sources. This will usually be either from your set books or from a Reading section. The following is a practical strategy for working through the course. If you run into any trouble, telephone your tutor. Remember that your tutor's job is to help you. When you need assistance, do not hesitate to call and ask your tutor to provide it.

In addition do the following:

1. Read this Course Guide thoroughly, it is your first assignment.

2. Organise a Study Schedule. Design a 'Course Overview' to guide you through the Course. Note the time you are expected to spend on each unit and how the assignments relate to the units. Important information, e.g. details of your tutorials, and the date of the first day of the Semester is available from the study centre. You need to gather all the information into one place, such as your diary or a wall calendar. Whatever method you choose to use, you should decide on and write in your own dates and schedule of work for each unit.

3. Once you have created your own study schedule, do everything to stay faithful to it. The major reason that students fail is that they get behind with their course work. If you get into difficulties with your schedule, please, let your tutor know before it is too late for help.

4. Turn to Unit 1, and read the introduction and the objectives for the unit.

5. Assemble the study materials. You will need your set books and the unit you are studying at any point in time.

6. Work through the unit. As you work through the unit, you will know what sources to consult for further information.

7. Keep in touch with your study centre. Up-to-date course information will be continuously available there.

8. Well before the relevant due dates (about 4 weeks before due dates), keep in mind that you will learn a lot by doing the assignment carefully. They have been designed to help you meet the objectives of the course and, therefore, will help you pass the examination. Submit all assignments not later than the due date.

9. Review the objectives for each study unit to confirm that you have achieved them. If you feel unsure about any of the objectives, review the study materials or consult your tutor.

10. When you are confident that you have achieved a unit's objectives, you can start on the next unit. Proceed unit by unit through the course and try to pace your study so that you keep yourself on schedule.

11. When you have submitted an assignment to your tutor for marking, do not wait for its return before starting on the next unit. Keep to your schedule. When the Assignment is returned, pay particular attention to your tutor's comments, both on the tutor-marked assignment form and also the written comments on the ordinary assignments.

12. After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives (listed in the Course Guide).

Tutors and Tutorials

The dates, times and locations of these tutorials will be made available to you, together with the name, telephone number and the address of your tutor. Each assignment will be marked by your tutor. Pay close attention to the comments your tutor might make on your assignments as these will help in your progress. Make sure that assignments reach your tutor on or before the due date.

Your tutorials are important therefore try not to skip any. It is an opportunity to meet your tutor and your fellow students. It is also an opportunity to get the help of your tutor and discuss any difficulties encountered on your reading.

Summary

This course would train you on the concept of multimedia, production and utilization of it.

Wish you the best of luck as you read through this course

Course Code	BHM 109
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MODULE 1

UNIT 1 MATHEMATICS AND SYMBOLIC LOGIC

CONTENTS

1.0 INTRODUCTION

One important instrument for communication, but not easily mentioned is the use of figures. Figures are major instrument used for expression, especially where large data are involved. It makes expression very concise in explanation and interpretation. This makes it important for us to know how mathematical symbols and logic could be used.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Solve simple problems using mathematical symbols and logic.

3.0 MAIN CONTENT

3.1 Definition

Mathematics is the science of numbers and shape. It can also be applied to some real life happening.

Example includes Arithmetic algebra (using of symbols or letters). Symbol in this context is a letter that is made to represent a statement or an argument.

Example, let 'p' represent the statement "She is tall and beautiful" and the letter 'q' represent the statement "She is intelligent".
Therefore p and q can be regarded as a statement also meaning "She is tall, beautiful and intelligent."
Hence, Logic is the science of reasoning or explaining events. In other words, it could still be defined as the study and analysis of a mathematical proposition as to ascertain the Truth Value of the proposition.
The TRUTH VALUE of a proposition is termed "T" if it is true and "F" if it is false.

Example 1

Given that 2 + 3 = 4, the Truth Value of this statement is False i.e. 'F'.

Example 2

Lagos is a cosmopolitan state.
The Truth Value of this is True 'T'
Te Truth Value Table can be used to depict or analyze the Truth Value of any mathematical statement.

Example 3

Find the Truth Table of $p \wedge q$, meaning (p and q).

p	Q	$p \wedge q$
T	T	T
T	F	F

Meaning, “if p is true, q is true; then p and q must be true”. Also, “if p is true, q is false; then p and q must be false”.
This shall be expansible in this further study.

3.1 Some Objectives of Symbolic Logic Reasoning

1. Logical reasoning could be used to ascertain or analyze complicated business argument.

2. It could be used to decode equivalent, negation or contradictory propositions.

3. It could be used to analyze a statement.

4. It could be used to form composite statements from simple ones using logical connectives.

5. It could be used to ascertain the Truth Value of a statement.

3.2 Definition of Basic Logical Terms

A statement or (proposition) in the context of logic is a declarative sentences or an expression of words which are either true or false, but cannot assume both.

3.3 Types of Statement

There are two types of statements: simple and composite (compound) statements respectively.

Example of Simple Statement

1. Mr. Alakija is the H.O.D
2. I am an undergraduate
3. Rose flower is red
4. $2 + 5 = 4$
5. Accra is in Nigeria

The proposition/statement/ascertain (1 – 5) above are simple ones and can either be true or false.

However, a composite statement is the use of connectives to combine two or more simple statements to form just one.

Example of Composite Statements

1. Mr. Alakija is the H.O.D and I am an undergraduate
2. Accra is in Nigeria implies that $2 + 5 = 4$

The Truth Value of a composite statement is determined by the Truth Value of each of the simple statements and the way they are connected.

3.4 Connectives

These are expressions; phrases or symbols that are made used of, to join (combine) two or more simple statements together in order to form a composite statement.

Examples of connectives used in logic are:

	Name	Meaning	Symbol
i.	Conjunction	"and"	$\wedge$
ii.	Disjunction	"OR"	$\vee$
iii.	Negation	"NOT"	$\sim$
iv.	Conditional	"If …. Implies that"	$\rightarrow$ or $\Rightarrow$
v.	Bi – implication	"if and only if"	$\Leftrightarrow$

Other connectives could be used but must be defined properly.

3.5 Tautology and Contradiction

The negation of contradiction is a tautology, while the negation of tautology is a contradiction.

3.6 An Argument

It is said to be formed when the conjunction of a set of simple mathematical statements gives rise to another mathematical statement (conclusion).

An argument may be valid or invalid.
If the conclusion derives its support from its support from its premises, the argument is said to be valid, otherwise invalid.

Example 1

Let p: Lagos is in Nigeria
q: Nigeria is in West Africa
r: It means that Lagos is in West Africa.

The above argument is valid i.e. r derives its supports from p and q respectively.

3.7 Algebra of Propositions

Any logical statement can be represented by the letter p, q, r…

The fundamental property of a logical statement is that it is either true or false but not both.

3.8 Conjunction (p∧q)

$p \wedge q$ represent the conjunction of p and q by 'and'.

Example 2

Let p be "It is sunning" and q be "The day is windy".

Then $p \wedge q$ means "it is sunning and the day is windy".

Recall the symbol (∧) represents intersection of two sets.

$p \cap q = \{x / x\ p \wedge\in x \in q\}$

The Truth Value of the composite statement $p \cap q$ satisfies the following property.

If p is true and q is true, then $p \cap q$ is true; otherwise $p \cap q$ is false.

Example 3

Consider the following four propositions

i. Garri is from cassava and 5 + 2 = 3
ii. Garri is from yam and 5 + 2 = 7
iii. Garri is from yam and 5 + 2 = 6
iv. Garri is from cassava and 5 + 2 = 7

Considering the statements (i) to (iv), only proposition (iv) is true, and (i), (ii), (iii) are false because in each case, one of the simple statement is false.

The above proposition can be written in tabular form thus:

Statements	p	q	$(p \cap q)$
i)	T	F	F
ii)	F	T	F
iii)	F	F	F
iv)	T	T	T

Remark

This is the exact order the statement above follows, but in the final presentation, the Truth Value Table must be formulated considering Truth first before False.

Please see the acceptable way below:

Statements	p	q	$(p \cap q)$
iv)	T	T	T
i)	T	F	F
ii)	F	T	F
iii)	F	F	F

3.9 Disjunction ($p \vee q$)

Any two statements can be combined by the word "or" ($\vee$), to form another statement. The disjunction of p and q is denoted by $p \vee q$.

Example 4

Let p be "It is raining" and q be "The weather is cold".

Then, $\underline{p \vee q}$ means that "It is raining or the weather is cold".

The symbol ($\vee$) can be used to define union of two sets.
Thus: $p \cup q = \{x / x \in p \vee x \in q\}$

Note that, the ($p \vee q$) satisfies the following property;
If p is true or q is true or both p and q are true, then ($p \vee q$) is true: otherwise ($p \vee q$) is false.

Hence, consider the Truth Table below:

p	q	$(p \cup q)$
T	T	T
T	F	F
F	T	T
F	F	F

Example 5

Consider the following four statements:

i. Garri is from cassava or 5 + 2 = 4
ii. Garri is from yam or 5 + 2 = 7
iii. Garri is from cassava or 5 + 2 = 7
iv. Garri is from yam or 5 + 2 = 6

Only (iv) is false, every other proposition is true; since at least one of the propositions is true.

3.10 Negation

If (p) is a given proposition, its negation is denoted by *(~ p)* meaning it is not (p), or it is false that (p).

Example 6

Let (p) be Garri is from yam; therefore *(~ p)* means, "It is false that Garri is from yam".

Then, the Truth Value of the negation of statement satisfies the property. If (p) is true, (~ *p)* is false.

Consider the Truth Value of negation below:

P	*(~ p)*
T	F
F	T

Further Examples

If (p) is the statement 5 + 2 = 6, then *(~ p)* is the statement 5 + 2 ≠ 6

3.10.1 Conditional (p → q)

Conditional statement is of the form "if p then q"

Moreso, *(p → q)* can also be read as:

i. p implies w
ii. p is sufficient for q
iii. q is necessary for p
iv. p only if q
v. q if p or
vi. if p then q or
vii. q follows from p or *(p → q)*satisfies the following property:

The conditional (p → q) is true unless (p) is true and (q) is false.

In other words, a true statement cannot imply a false statement.
His can be stated in tabular form, thus:

p	q	(p → q)
T	T	T
T	F	F
F	T	T
F	F	F

Example 7

Consider the following statements:

i. IF Garri is from cassava then 5 + 2 = 6
ii. If Garri is from Yam then 5 + 2 = 7
iii. If Garri is from cassava then 5 + 2 = 7
iv. If Garri is from yam then 5 + 2 = 7

Here, only (i) is false, because true statement cannot imply false statement, Garri is from cassava is true, 5 + 2 =6: while all other statements are true.

3.10.2 Bi-conditional (p $\Leftrightarrow$ q)

Consider the statement of the form "p if and only if q" or (p if q) and can be denoted by ($p \Leftrightarrow q$).

($p \Leftrightarrow q$) satisfies the following property:

If (p) and (q) have the same Truth Value, then ($p \Leftrightarrow q$) is true.

If (p) and (q) have opposite Truth Value, then ($p \Leftrightarrow q$) is false.

Consider the True value of Bi-conditional statement

p	Q	($p \Leftrightarrow q$)
T	T	T
T	F	F
F	T	F
F	F	T

Example 8

Consider the following statements:

i. Garri is from cassava if and only if 5 + 2 = 6
ii. Garri is from yam if and only if 5 + 2 = 7
iii. Garri is from cassava if and only if 5 + 2 = 7
iv. Garri is from yam if and only if 5 + 2 = 7

Hence, statements (iii) and (iv) are true and statements (i) and (ii) are false.

3.10.3 Laws of Algebra of Proposition

IdempotOtent Laws

1. (a) $(p \wedge q) = p$ (b) $(p \vee q) = p$

The intersection or the union of two similar sets is the same set.

Associate Laws

2. (a) $(p \vee q) \vee \mathrm{r} = p \vee (q \vee r)$ (b) $(p \wedge q) \wedge \mathrm{r} = p \wedge (q \wedge r)$

Commutative Laws

3. (a) $(p \wedge q) = (q \wedge p)$ (b) $(p \vee q) = (q \vee p)$

Distributive Laws

4. (a) $p \wedge (q \vee \text{r}) = (p \wedge q) \vee (p \wedge r)$

(b) $p \vee (q \wedge \text{r}) = (p \vee q) \wedge (p \vee r)$

Identity Laws

5. (a) $(P \wedge T) = P$ (b) $(P \wedge F) = F$

(c) $(P \vee F) = P$ (d) $(P \vee T) = T$

Complement Laws

6. (a) $(P \wedge \sim P) = F$ (b) $(\sim T) = F, (\sim F) = T$

(c) $(P \vee \sim T) = T$ (d) $(\sim P) = P$

De Morgan's Laws

7. (a) $\sim (p \wedge q) = (\sim p \vee \sim q)$ (b) $\sim (p \vee q) = (\sim p \wedge \sim q)$

Example 1.1

Given the following statements:

Let p be "The goods are standard" and q be "The goods are expensive".

Write a proposition, which describes each of the following statements:

1. $(p \wedge q)$
2. $(p \vee q)$
3. $(\sim\sim q)$
4. $(p \leftrightarrow q)$
5. $(\sim p \wedge \sim q)$
6. $\sim (\sim p \wedge \sim q)$

Solution

1. $(p \wedge q)$ Reads "The goods are standard and The goods are expensive"

2. $(p \vee q)$ Reads "The goods are standard or The goods are expensive"

3. $(\sim\sim q)$ Reads "It is not false that the goods are expensive"

4. $(p \leftrightarrow q)$ Reads "The goods are standard if and only if the goods are expensive"
5. $\sim (\sim p \wedge \sim q)$ Reads "It is not true that the goods are standard or the goods are expensive"
6. $\sim (\sim p \wedge \sim q)$ Is equivalent to $(p \vee q)$

Example 1.2

Let (p) be "she is tall" and (q) be "She is dark"

Write each of the following statements in symbolic form using p and q:

a) "She is tall and dark"
b) "She is tall or dark"
c) "She is neither tall nor dark"
d) "It is false that she is tall or dark"
e) "She is tall or she is short and dark"
f) "She is tall but not dark"

Solution

a) $(p \wedge q)$
b) $(p \vee q)$
c) $(\sim p \wedge \sim q) = \sim (p \vee q)$
d) $(\sim p \vee q)$
e) $p \vee (\sim p \wedge q)$
f) $(p \wedge \sim q)$

Example 1.3

Use the Truth Value Table to express the following statements:

a) $(p \wedge q) \rightarrow (p \vee q)$

b) $p \vee \sim (p \wedge q)$

Solution

a) $(p \wedge q) \rightarrow (p \vee q)$

P	Q	$(p \wedge q)$	$(p \vee q)$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	F

b) $p \vee \sim (p \wedge q)$

p	q	$(p \wedge q)$	$\sim (p \vee q)$	$p \vee \sim (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Example 1.4

By using the Truth Value Table, prove the following:

a. $(p \vee q) \wedge (p \vee \text{r}) = p \vee (q \wedge r) =$, Distributive Law

b. $(\sim (p \vee \sim q) = \sim (p \wedge \sim q)$, De Morgan's Law

Solution

a. $(p \vee q) \wedge (p \vee \text{r}) = p \vee (q \wedge r)$

p	Q	T	$(p \vee q)$	$(p \vee \text{r})$	$(p \vee q) \wedge (p \wedge q)$	$(q \wedge r)$	$p \vee (q \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	T	F	T
T	F	F	T	T	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F

Therefore, its clearly from the table that $(p \vee q) \wedge (p \wedge q) = p \vee (q \wedge r)$.

b. $(\sim (p \vee \sim q) = \sim (p \wedge \sim q)$

p	Q	$(\sim p)$	$(\sim q)$	$(\sim (p \vee \sim q)$	$(p \wedge q)$	$\sim(q \wedge r)$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

Also, it can be seen that $(\sim (p \vee \sim q) = \sim (p \wedge \sim q)$

Example 1.5

Use the Truth Value Table to show that $(\sim\sim q) = (p)$

Solution

$(\sim\sim q) = (p)$

Q	(~ q)	$(\sim\sim q)$
T	F	T
F	T	F

Therefore, $(\sim\sim q) = (p)$ as shown above.

10.1.4 Converse/Inverse and Contra-Positive Statements

If $p \rightarrow q$ is a conditional statement, then (if q then p) is called the converse statement of the conditional statement.

Example 1.6

Let p be the statement "2 + 2 = 5" and, q be the statement "Abuja is in U.K"
Then p implies q, i.e. $p \rightarrow q$, i .e. 2 + 2 =5 implies that Abuja is in U.K.
The converse is that Abuja is in U.K. then 2 + 2 =5.

Example 1.7

Let p be the statement "It is born again" and, q be the statement "He can see the kingdom of God
Then, (~ p) Is "He is not born again"
(~ q) Is "He cannot see the kingdom of God".
$\therefore \sim q \rightarrow \sim p$ is called the contra-positive of p → q.

He cannot see the kingdom of God implies that he is not born again.

Where ~ p → ~ q *is* the inverse of p → q.

He is born again implies that He can see the kingdom of God.

4.0 CONCLUSION

In conclusion, Logic as science of reasoning has been thoroughly dealt with in this section. However, some real life happenings have been demonstrated in the examples and even in the summary below.

5.0 SUMMARY

Consider proposition p and q respectively.
The converse of p → q and q is q → p
The inverse of p → q is ~ p → ~ q
While, the contra-positive of p → q is ~ q → ~ p
The Truth Table of the above propositions, converse, inverse and contra-positive are show below:

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$	$q \rightarrow p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	F	T	F	T	F
F	F	T	T	T	T	T	T

The analysis of the table above is:
p → q ≡ ~ q → ~ p, Contra-positive
p → q ≡ ~ p → ~ q, Inverse.
Where, the symbol ≡ means "Logical Equivalence".

6.0 TUTOR-MARKED ASSIGNMENT

1. Let the following statements be defined as stated below: (p) be "3 is greater than 2", and (q) be "The sum of 2 and 6 is 30".

Give the verbal expression of the following:

i. $p \wedge q$ ii. $p \to q$ iii. $(p \vee q) \to p$

iv. $(p \vee \sim q)$ v. $(\sim p \wedge \sim q)$ vi. $(p \leftrightarrow \sim q)$

2. Give the contra-positive statement of $p \to q$ above in symbols and words.
3. Verify that implication and its contra-positive are equivalent.
4. Is the following argument valid?
   "If the temperature is set for 20^0C, the chemical will not explode. The temperature is set for 20^0C; therefore the chemical did not explode.
5. Let the connective (Δ) be defined as follows:

	Q	$p \Delta q$
T	T	T
T	F	F
F	T	F
F	F	T

Construct the Truth Value Table for the following:

i. $p \Delta (p \to q)$ ii. $(p\Delta q) \Delta q$ iii. $(p\Delta q) \to p$

iv. $(p\Delta q) \Delta (p \to q)$ v. $(p\Delta \sim p)$

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7) Introduction to Mathematical Economics SCHAUM'S Out lines

UNIT 2 MATRICES AND DETERMINANT

CONTENTS

1.0 INTRODUCTION

Matrix algebra is a mathematical model which allows the enumeration, display, manipulation and processing of both small and large volumes of vectors (numbers in this context) easily; in order to achieve a desired result.

2.0 OBJECTIVES

At the end of this unit, you should be able to solve physical problem via matrix algebra as a mathematical model.

3.0 MAIN CONTENT

3.1 Definition

A matrix is a rectangular array of numbers with reference to specific rules governing the array. The rule can be mathematically expressed.

Thus, let the entries or numbers be denoted by a_{ij} Where i infers row arrangement i.e. $\rightarrow$ and j infers column arrangement i.e. $\downarrow$.

Example 1

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{array}{l} \longrightarrow \textit{row 1} \\ \longrightarrow \textit{row 2} \\ \longrightarrow \textit{row 3} \end{array}$$

$$\uparrow \quad \uparrow \quad \uparrow$$

Column 1, 2, 3

i represent elements in the row while j represent elements in the column for better understanding of the row-column concept, let the elements of the

i^{th} row and j^{th} column of matrix be A be a_{ij} i.e.

a_{11} means *row* 1, *column* 1;

a_{12} means *row* 1, *column* 2;

a_{13} means *row* 1, *column* 3;

a_{21} means *row* 2, *column* 1;

a_{22} means *row* 2, *column* 2;

a_{23} means *row* 2, *column* 3;

a_{31} means *row* 3, *column* 1;

a_{32} means *row* 3, *column* 2; and

a_{33} means *row* 3, *column* 3;

By now, the row-column concept should be clearer.

The dimension of a matrix is the size of the matrix, which is denoted by, (*m* x *n*) called *m* by *n* matrix, where *m* is the number of rows and *n* is the number of columns.

Example 2

1. $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$, This is a 2 by 3 matrix i.e. 2 *rows*, and 3 *columns*

2. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, This is a 3 by 2 matrix i.e. 3 *rows*, and 2 *columns*

3.2 Square Matrix

Any matrix, which has the same number of rows and columns, is called "a square matrix".

Example 3

$\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$, The dimension of this matrix is 3 by 3 or (3 x 3) i.e. 3 *rows*, and 3 *columns*

3.3 Types of Matrix

i. **Zero /Null / Void Matrix:** Any matrix of dimension (*m* x *n*) with all its elements equal to zero.

E.g. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a (2 x 2) Null Matrix.

ii. **Identify Matrix:** this is a matrix in which all its diagonal elements are one (1), where all other elements are zero.

E.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a (2 x 2) Identity Matrix, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a (3 x 3) Identity Matrix.

iii. **Diagonal Matrix:** This is a matrix that has elements only on its diagonal.

E.g. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is a (2 x 2) diagonal matrix, $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a (3 x 3) diagonal matrix.

iv. **Transpose of a Matrix:** The transpose of a matrix is the inter-changing of its row with the column.

Let matrix A be (m x n), then matrix A^T is (n x m) called "the transpose of A". E.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

row 1 of A = *column* 1 of A^T .

row 2 of A = *column* 2 of A^T.

v. **Symmetric Matrix:** This is a matrix in which its transpose is equal to itself.

i.e. if A = A^T e.g = $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$, $A^T = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

vi. **Idempotent Matrix:** Let A be a matrix, A is said to be Idempotent if $A = A^2$ e.g. Identify Matrix.

vii. **Upper Triangular Matrix:** Let A be a square matrix where the element $a_{ij} = 0$. For $i > j$, then A is called "An Upper Triangular Matrix" e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ $a_{ij} = 0$, For $i > j$.

$a_{21} = a_{31} = a_{32} = 0$ i.e.
$2 > 1$, $3 > 1$ and $3 > 2$.

3.4 Equality of Matrix

Two matrices are said to be equal if their corresponding elements are the same.

e.g. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

A = B. If and only if a = e, b = f, c = g, d = h, corresponding elements.

Example 4

Find the value of 2a + 3b if $\begin{bmatrix} 0 & a \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} b & 3 \\ 5 & 4 \end{bmatrix}$

Solution

Comparing the two matrices to be equal.

It means that a = 3, b = 0;

Corresponding elements.

$\therefore$ 2a + 3b = 2(3) + 3(0) = 6 + 0 = 6

3.5 Algebra of Matrices

Two or more matrices can be added or subtracted if they are of same dimension.

e.g. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$\therefore A + B \begin{bmatrix} a + g & b + h \\ c + e & d + f \end{bmatrix}$

Similarly,

$A - B \begin{bmatrix} a - g & b - h \\ c - e & d - f \end{bmatrix}$

Example 5

Let $A = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 6 \\ 2 & 1 & 4 \end{bmatrix}$ Find $A + B$ and $A - B$.

Solution

$$A + B = \begin{bmatrix} 4 + 1 & 5 + 0 & 6 + 6 \\ 7 + 2 & 8 + 1 & 9 + 4 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 12 \\ 9 & 9 & 13 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 4 - 1 & 5 - 0 & 6 - 6 \\ 7 - 2 & 8 - 1 & 9 - 4 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 0 \\ 5 & 7 & 5 \end{bmatrix}$$

3.6 Scalar Multiplication of Matrices

Let k be a scalar quantity and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\therefore$ The product of k and A denoted by kA is:

$$kA = k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

Example 6

Given that $A = \begin{bmatrix} 2 & 4 & 7 & 8 \\ 1 & 0 & 2 & 3 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix}$

Find (i) $3A + 2B$ (ii) $4A - 7B$

Solution

i. $3A + 2B = 3\begin{bmatrix} 2 & 4 & 7 & 8 \\ 1 & 0 & 2 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 16 & 12 & 21 & 24 \\ 3 & 0 & 6 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 + 2 & 12 + 2 & 21 + 2 & 24 + 2 \\ 3 + 4 & 0 + 4 & 6 + 4 & 9 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 14 & 23 & 26 \\ 7 & 4 & 10 & 13 \end{bmatrix}$$

ii. $4A - 7B = 4 \begin{bmatrix} 2 & 4 & 7 & 8 \\ 1 & 0 & 2 & 3 \end{bmatrix} -7 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 8 & 16 & 28 & 32 \\ 4 & 0 & 8 & 12 \end{bmatrix} - \begin{bmatrix} 7 & 7 & 7 & 7 \\ 14 & 14 & 14 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 8 - 7 & 16 - 7 & 28 - 7 & 32 - 7 \\ 4 - 14 & 0 - 14 & 8 - 14 & 12 - 14 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 9 & 21 & 25 \\ -10 & -14 & -6 & -2 \end{bmatrix}$$

3.6.1 Multiplication of Matrix

For Multiplication of two matrices to be possible, they must be conformable i.e. matrices *A* and *B* are conformable if the column of *A* has the same dimension as the row of *B*.

Note that $AB \neq BA$.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow$ (2 x 3) matrix.

B = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \Rightarrow$ (3 x 2) matrix.

The column of A = 3, the row of B = 3

$\therefore$ Matrices A and B are said to be conformable and can be multiplied by one another.

For better understanding of multiplication of two matrices:

Let $A = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ and $B = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$

Also, let *AB* = *C* another matrix;

$$C = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

Since *A* and *B* are conformable,

$$\therefore \; AB = C \Rightarrow \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

$\Rightarrow z_{11} = x_{11}y_{11} + x_{12}y_{21}$

$\Rightarrow z_{12} = x_{11}y_{12} + x_{22}y_{22}$

$\Rightarrow z_{21} = x_{21}y_{11} + x_{22}y_{21}$

$\Rightarrow z_{22} = x_{21}y_{12} + x_{22}y_{22}$

We can say generally that: $z_{ij} = x_{ij}y_{ij} + x_{i2}y_{2j} + \ldots\ldots + x_{ik}y_{kj}$

Example 7

Given that $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow$ (2 x 3) matrix,

$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \Rightarrow$ (3 x 2) matrix.

Find (i) *AB* (ii) *BA*.

Solution

i. $AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$= \begin{bmatrix} \{(1 \text{ x } 1)\} + \{(2 \text{ x } 3)\} + \{(3 \text{ x } 5)\} & \{(1 \text{ x } 2)\} + \{(2 \text{ x } 4)\} + \{(3 \text{ x } 6)\} \\ \{(4 \text{ x } 1)\} + \{(5 \text{ x } 3)\} + \{(6 \text{ x } 5)\} & \{(4 \text{ x } 2)\} + \{(5 \text{ x } 4)\} + \{(6 \text{ x } 6)\} \end{bmatrix}$$

$$= \begin{bmatrix} (1+6+15) & (2+8+18) \\ (4+15+30) & (8+20+36) \end{bmatrix} = \begin{bmatrix} 22 & 28 \\ 49 & 64 \end{bmatrix} \Rightarrow \text{(2 x 2) matrix.}$$

ii. $BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

Let $BA = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

a_{11}= (1x1) + (1x4) = 1+4 = 5
a_{12}= (1x2) + (2x5) = 2+10 = 12
a_{13}= (1x3) + (2x6) = 3+12 = 15
a_{21}= (3x1) + (4x4) = 3+16 = 19
a_{22}= (3x2) + (4x5) = 6+20 = 26
a_{23}= (3x3) + (4x6) = 9+24 = 31
a_{31}= (5x1) + (6x4) = 5+24 = 29
a_{32}= (5x2) + (6x5) = 10+30 = 40
a_{33}= (5x3) + (6x6) = 15+6 = 51

$BA = \begin{bmatrix} 5 & 12 & 15 \\ 19 & 26 & 31 \\ 29 & 40 & 51 \end{bmatrix} \Rightarrow$ (3 x 3) matrix.

3.7 Determinant of A Matrix

To find the determinant of a matrix, the matrix must be a square matrix.

1. Consider a (2x2) square matrix *A*, the determinant of *A* denoted by det *A*.

   Let = $\begin{bmatrix} a & b \\ c & c \end{bmatrix}$ Or $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

   det $|A|\,|$ = ad – cb Or det A = $|A|$ = $a_{11}a_{22} - a_{21}a_{22}$

Example 8

Find the determinant of the following matrices:

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -4 \\ 5 & 6 \end{bmatrix}$

Solution

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow$ det A = $|A|$ = (1 x 4) – (3 x 2) = 4 - 6 = -2

$B = \begin{bmatrix} 2 & -4 \\ 5 & 6 \end{bmatrix} \Rightarrow$ det B = $|B|$ = (2 x 6) + (5 x 4) = 12 + 20 = 32

2. Also, consider a (3x3) square matrix A.

$$\text{Let } A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

Assign +ve and –ve sign to the entries in the rows and columns and pick

a_{11}, - a_{12} and a_{13} as co-factor i.e.

$$+a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, -a_{11}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \text{ and } + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Rightarrow \text{det } A = |A| = +a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, -a_{11}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \text{ and } + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$= a_{11}(a_{22}a_{33} – a_{21}a_{22}) – a_{12}(a_{21}a_{33} – a_{31}a_{23}) + a_{13}(a_{21}a_{32} – a_{31}a_{22})$

Example 9

Find the det A, given that $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$

Solution

$$\text{Let } A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array} \Rightarrow |A| = 1\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} -2\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} +3\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

$= 1(4-3) - 2(6-1) + 3(9-2) = 1(1) - 2(5) + 3(7)$ = 1 – 10 + 21 = 22

Another method can be used to evaluate the determinant of (3 x 3) square matrix A:

Suppose a = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, columns 1 and 2 can be repeated thus:

$$\begin{array}{cccccc} & (1) & (2) & (3)(4) & (5) & (6) \\ & a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ \Rightarrow & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ & a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$ This is SARUS CHART

Hence, det A = $|A|$ = $[(1) + (2) + (3)] - [(4) + (5) + (6)]$

$[(a_{11}a_{22}a_{33}) + (a_{12}a_{23}a_{31}) + (a_{13}a_{21}a_{33})] - [(a_{31}a_{22}a_{13}) + (a_{32}a_{23}a_{11}) + (a_{33}a_{21}a_{12})]$

Considering the **example 9** above:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \Rightarrow \begin{array}{ccccc} (1) & (2) & (3)(4) & (5) & (6) \\ 1 & 2 & 3 & 1 & 2 \\ 3 & 2 & 1 & 3 & 2 \\ 1 & 3 & 2 & 1 & 2 \end{array}$$

$\Rightarrow |A| = [(1) + (2) + (3)] - [(4) + (5) + (6)]$

$= [(1 \text{ x } 2 \text{ x } 2) + (2 \text{ x } 1 \text{ x } 1) + (3 \text{ x } 3 \text{ x } 3)] - [(1 \text{ x } 2 \text{ x } 3) + (3 \text{ x } 1 \text{ x } 1) + (2 \text{ x } 3 \text{ x } 2)]$

$= [(4) + (2) + (27)] - [(6) + (3) + (12)] = 33 - 21 = 12$

You can see that the answers are the same.

3.7.1 Singularity in Matrix

A matrix is said to be singular if the determinant is equal to zero.

Example 10

Given that $\begin{bmatrix} 1 & 3 \\ k & 4 \end{bmatrix}$ is a singular matrix. Find k

Solution

For a matrix to be singular, it's determinant equal to zero.

$\therefore$ (1 x 4) – 3k = 0 $\Rightarrow$ 4 – 3k = 0 $\Rightarrow$ 3k = 4 $\Rightarrow$ k = $\frac{4}{3}$

Example 11

Given that A = $\begin{bmatrix} 2 & 0 & 1 \\ k & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$, Find k for A to be singular.

Solution

For A to be singular, det A = 0.

$\therefore$ $|A|$ = 2$\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$ -0$\begin{vmatrix} k & 3 \\ 2 & 4 \end{vmatrix}$+1$\begin{vmatrix} k & 2 \\ 2 & 1 \end{vmatrix}$= 0 $\Rightarrow$ 2(8 – 3) -0(4k - 6) +1(k - 4) = 0

$\Rightarrow$ 2(5) -0 + k – 4 = 0 $\Rightarrow$ 10 + k – 4 = 0 $\Rightarrow$ 6 + k = 0 $\Rightarrow$ k = -6.

3.7.2 Application of Determinant

It could be used to solve linear equations.

Consider the system of two linear equations.

$a_1x + b_1y = c_1$........(1)

$a_2x + b_2y = c_2$........(2)

Where $a_1, a_2, b_1, b_2, c_1, c_2$ are constants?

This can be stated in matrix form thus:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

We are interested in finding the values of x and y.

Using Cramer's rule, we have:

$$x = \begin{bmatrix} \dfrac{\begin{matrix} c_1 & b_1 \\ c_2 & b_2 \end{matrix}}{\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix}} \end{bmatrix} = \frac{(b_2c_1 - c_2b_1)}{(a_1b_2 - a_2b_1)} \text{ and } y = \begin{bmatrix} \dfrac{\begin{matrix} c_1 & a_1 \\ c_2 & a_2 \end{matrix}}{\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix}} \end{bmatrix} = \frac{(c_1a_2 - c_2a_1)}{(a_1b_2 - a_2b_1)}$$

Example 12

Use Cramer's rule to solve the following:

i. $x + 2y = 3$
$3x + 4y = 1$

ii. $2x + 3y = 1$
$5x + 6y = 0$

Solution

i. $x + 2y = 3$
$3x + 4y = 1$

It can be written in matrix form, thus:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} \dfrac{\begin{matrix} 3 & 2 \\ 1 & 4 \end{matrix}}{\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}} \end{bmatrix} = \frac{(3x4)-(1x2)}{(1x4)-(3x2)} = \frac{12-2}{4-6} = \frac{10}{-2} = -5$$

$$\text{Similarly } y = \begin{bmatrix} \dfrac{\begin{matrix} 1 & 3 \\ 3 & 1 \end{matrix}}{\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}} \end{bmatrix} = \frac{(1x1)-(3x3)}{(1x4)-(3x2)} = \frac{1-9}{4-6} = \frac{-8}{-2} = 4$$

$\therefore x = -5$ and $y = 4$.

Let us cross-check by substituting x and y into the given equation:

$x + 2y = 3 \Rightarrow -5 + (4) = -5 + 8 = 3$

ii. $2x + 3y = 1$
$5x + 6y = 0$

In matrix form:

$$\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\therefore \; x = \left[\frac{\begin{matrix} 1 & 3 \\ 0 & 6 \end{matrix}}{\begin{matrix} 2 & 3 \\ 5 & 6 \end{matrix}} \right] = \frac{(1x6)-(0x3)}{(2x6)-(5x3)} = \frac{6}{12-15} = \frac{6}{3} = -2$$

$$\text{Similarly} \quad y = \left[\frac{\begin{matrix} 2 & 1 \\ 5 & 0 \end{matrix}}{\begin{matrix} 2 & 3 \\ 5 & 6 \end{matrix}} \right] = \frac{(2x0)-(5x1)}{(2x6)-(5x3)} = \frac{-5}{12-15} = \frac{-5}{-3} = \frac{5}{3}$$

$\therefore x = -2$ and $y = \frac{5}{3}$.

Cross checking by substituting x and y into the given equation:

$$2x + 3y = 1 \Rightarrow 2(-2) + 3(\frac{5}{3}) = -4 + 5 = 1.$$

Cramer's rule can be extended to systems of three equations with three unknowns.

Consider the system of three equations:

$$a_{11}x + a_{12}y + a_{13}z = c_1$$
$$a_{21}x + a_{22}y + a_{23}z = c_2$$
$$a_{31}x + a_{32}y + a_{33}z = c_3$$

These can be written as in matrix form:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Let the determinant of $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \Delta$

Replacing column 1 by $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$:

$$\Rightarrow x = \frac{\begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}}{\Delta} = \frac{c_1\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} c_2 & a_{23} \\ c_3 & a_{33} \end{vmatrix} - a_{13}\begin{vmatrix} c_2 & a_{22} \\ c_3 & a_{32} \end{vmatrix}}{\Delta}$$

Replacing column 2 by $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$:

$$\Rightarrow y = \frac{\begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}}{\Delta} = \frac{a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - c_1\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & c_2 \\ a_{31} & c_3 \end{vmatrix}}{\Delta}$$

Replacing column 3 by $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$:

$$\Rightarrow z = \frac{\begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}}{\Delta} = \frac{a_{11}\begin{vmatrix} a_{22} & c_2 \\ a_{32} & c_3 \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & c_2 \\ a_{31} & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}{\Delta}$$

Example 13

$$x + 2y + 3z = 1$$
$$3x + 2y + z = 4$$
$$x + 3y + 2z = 0$$

Solution

Writing the above equation in matrix form:

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix} = \Delta = 12$$

$$\therefore x = \frac{\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 0 & 3 & 2 \end{vmatrix}}{12} = \frac{1\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ 0 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & 2 \\ 0 & 3 \end{vmatrix}}{12} = \frac{1(1) - 2(8) + 3(12)}{12}$$

$$= \frac{21}{12} = \frac{7}{4}$$

$$\therefore y = \frac{\begin{vmatrix} 1 & 1 & 3 \\ 3 & 4 & 1 \\ 1 & 0 & 2 \end{vmatrix}}{12} = \frac{1\begin{vmatrix} 4 & 1 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix}}{12} = \frac{1(8) - 1(5) + 3(-4)}{12}$$

$$= \frac{-9}{12} = \frac{-3}{4}$$

$$\therefore z = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 4 \\ 1 & 3 & 0 \end{vmatrix}}{12} = \frac{1\begin{vmatrix} 2 & 4 \\ 3 & 0 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}}{12} = \frac{1(-12) - 2(-4) + 1(7)}{12}$$

$$= \frac{3}{12} = \frac{1}{4}$$

Consequently, $x = \frac{7}{4}$, $y = \frac{-3}{4}$ and $z = \frac{1}{4}$.

Cross-checking by substituting x, y and z into equation (3)

$$x + 3y + 2z = 0 \Rightarrow \frac{7}{4} + 3(-\frac{3}{4}) + 2(\frac{1}{4}) \Rightarrow \frac{7}{4} - \frac{9}{4} + \frac{2}{4} \Rightarrow \frac{9}{4} - \frac{9}{4} = 0.$$

3.7.3 Matrix Inversion

Cofactor Matrix: It is defined to be the matrix obtained by replacing every number a_{ij} of the given matrix A by its cofactor in the determinant of A.

Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 2 & 1 & 4 \end{bmatrix}$, then the cofactor of matrix A is the matrix:

$$\begin{bmatrix} \begin{vmatrix} 5 & 0 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 4 & 0 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 4 & 5 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 20 & -16 & -6 \\ -5 & -2 & 3 \\ -15 & 12 & -3 \end{bmatrix}$$

Adjoint of a Matrix: Let A be a matrix and let C be its cofactor matrix, then the transpose CT of C is called the Adjoint of A or Adj A.

Thus, from the above example:

$$\text{Adj A} = \begin{bmatrix} 20 & -5 & -15 \\ -16 & -2 & 12 \\ -6 & 3 & -3 \end{bmatrix}$$

Thus, we state an important result for matrix A, i.e.

A (Adj A) = (Adj A) A = $|A|$ I.

The inverse of matrix A is given as: $A^{-1} = \frac{1}{|A|}\text{Adj A}$.

Example 14

Find the inverse of the following matrices:

a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 2 & 1 & 4 \end{bmatrix}$ b) $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{bmatrix}$

Solution

a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 2 & 1 & 4 \end{bmatrix}$ is a (3 x 3) square matrix.

$$|A| = 1\begin{vmatrix} 5 & 0 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 4 & 0 \\ 2 & 4 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 2 & 1 \end{vmatrix} = 1(20 - 0) -2(16 - 0) +3(4 -10) = -30 \neq 0$$

The cofactor matrix of A is:

$$\begin{bmatrix} \begin{vmatrix} 5 & 0 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 4 & 0 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 4 & 5 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 20 & -16 & -6 \\ -5 & -2 & 3 \\ -15 & 12 & -3 \end{bmatrix}$$

$$\therefore \text{ Adj } A = \begin{bmatrix} 20 & -5 & -15 \\ -16 & -2 & 12 \\ -6 & 3 & -3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{Adj } A = -\frac{1}{30}\begin{bmatrix} 20 & -5 & -15 \\ -16 & -2 & 12 \\ -6 & 3 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{8}{15} & \frac{1}{15} & -\frac{2}{5} \\ \frac{1}{5} & -\frac{1}{10} & \frac{1}{10} \end{bmatrix}$$

b) $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{bmatrix}$

$$|B| = 1\begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 1(16 - 4) -2(4 - 6) +1(4 -9) = 1 \neq 0$$

The cofactor matrix of B is:

$$\begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 2 & -5 \\ -2 & -1 & 4 \\ 1 & 0 & -1 \end{bmatrix}$$

The Adjoint of B is $\begin{bmatrix} 2 & -2 & 1 \\ 2 & -1 & 0 \\ -5 & 4 & -1 \end{bmatrix}$

$$\therefore\ B^{-1} = \frac{1}{|B|}\text{Adj B} = 1\begin{bmatrix} 2 & -2 & 1 \\ 2 & -1 & 0 \\ -5 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 2 & -1 & 0 \\ -5 & 4 & -1 \end{bmatrix}$$

We can also get the solution of equations using the inverse.

Suppose, we have a matrix equation $A X = B$.

If $A^{-1} A X = A^{-1} B$
$\Rightarrow I X = A^{-1} B$ where I is unit matrix
$\Rightarrow A^{-1} B$.

Example 15

Solve the equations

x + 3y + 3z = 1

x + 4y + 3z = 2

x + 3y + 4z = 3

Solution

In matrix form, we have:

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow A\,X = B$$

$$|A| = 1\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1(16 - 9) - 3(4 - 3) + 3(3 - 4) = 1 \neq 0$$

The cofactor matrix of A is:

$$\begin{bmatrix} \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

The Adjoint of A is $\begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$\therefore\ A^{-1} = \frac{1}{|A|}\text{Adj A} = 1\begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7(1) & -3(2) & -3(3) \\ -1(1) & +1(2) & +0(3) \\ -1(1) & +0(2) & +1(3) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & -9 \\ -1 & +2 & +0 \\ -1 & +0 & +3 \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \\ 2 \end{bmatrix}$$

$\Rightarrow$ x -8, y = 1, z = 2.

Example 6

To control a crop disease, it is necessary to use 8units of chemical *A*, *14units* of chemical *B* and *13units* of chemical *C*.

One barrel of spray *P* contains *1unit* of *A*, *2units* of *B* and *3units* of *C*.
One barrel of spray *Q* contains *2unit* of *A*, 3*2units* of *B* and *2units* of *C*.
One barrel of spray *R* contains *1unit* of *A*, *2units* of *B* and *2units* of *C*.

Find how many barrels of each type of spray will be used to just meet the requirements.

Solution

The matrix of the problem is: $$\begin{array}{c} \begin{matrix} P & Q & R \end{matrix} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \\ 13 \end{bmatrix}$$

Where x, y, z are the number of barrels (to be used) of spray *P, Q, R* respectively.

$$|A| = 1\begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 1(6 - 4) -2(4 - 6) +1(4 - 9) = 1 \neq 0$$

The cofactor matrix of A is:

$$\begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 2 & -5 \\ -2 & -1 & 4 \\ 1 & 0 & -1 \end{bmatrix}$$

The Adjoint of *A* is $\begin{bmatrix} 2 & -2 & 1 \\ 2 & -1 & 0 \\ -5 & 4 & -1 \end{bmatrix}$

$$\therefore\ A^{-1} = \frac{1}{|A|}\text{Adj } A = 1\begin{bmatrix} 2 & -2 & 1 \\ 2 & -1 & 0 \\ -5 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 2 & -1 & 0 \\ -5 & 4 & -1 \end{bmatrix}$$

Thus, the matrix equation becomes:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 2 & -1 & 0 \\ -5 & 4 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 14 \\ 13 \end{bmatrix} = \begin{bmatrix} 2(8) & -2(14) & +1(13) \\ 2(8) & -1(14) & +0(13) \\ -5(8) & +4(14) & -1(13) \end{bmatrix}$$

$$\begin{bmatrix} 16 & -28 & +13 \\ 16 & -14 & +0 \\ -40 & +56 & -13 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow$ We need 1 barrel of spray *P*, 2 barrels of spray *Q* and 3 barrels of spray *R* to meet the requirement.

Example 17

In a market survey, three commodities *A, B, C* were considered. In finding out the index number, some fixed weights were assigned to three varieties in each of the commodities. The table below provides the information regarding the consumption of three commodities according to three varieties and also the total weight received by the commodity:

Commodity	**Variety**			**Total weight**
	I	**II**	**III**	
A	1	2	3	11
B	2	4	5	21
C	3	5	6	27

Find the weights assigned to the three varieties by using matrix –inverse method, given that the weights assigned to commodities are equal to the sum of weights of the various varieties multiplied by the corresponding consumption.

Solution

Let *x, y, z* be the weights assigned to *A, B, C* respectively.

We have:

x + 2y + 3z = 11
2x + 4y + 5z = 21

x + 2y + 3z = 11

In matrix form:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 21 \\ 27 \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 1(24 - 25)-2(12 – 15 + 3(10-12) = -1 \neq 0$$

The cofactor matrix of A is:

$$\begin{bmatrix} \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

Thus:

$$\therefore \text{ Adj A} = \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

$$\therefore \text{ A}^{-1} = \frac{1}{|A|}\text{Adj A} = \frac{1}{-1}\begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 11 \\ 21 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 11 \\ 21 \\ 27 \end{bmatrix} = \begin{bmatrix} 11 & -63 & +54 \\ -3 & +63 & -273 \\ & 22 & -21 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$\Rightarrow x = 2, y = 3, z = 1$ are the required weights assigned to *A*, *B*, *C*.

Example 18

Given the following equations for two related markets *(A)* and *(B)*. Find the equilibrium conditions for each market. Also find the equilibrium price for each market.

$x_d\ (A) = 82 - 3P_A + P_B$ $\qquad$ $x_d\ (B) = 92 + 2P_A - 4P_B$

$x_s\ (A) = -5 + 15P_A$ $\qquad$ $x_s\ (A) = -6 + 32P_B$

Where x_d and x_s denote quantity demanded and quantity supplied respectively.

Solution

Equilibrium is supply and demand occurs when $x_s = x_d$

$\therefore$ For market (A):

$82 - 3P_A + P_B = 82 - 3P_A + P_B \Rightarrow 18P_A - P_B = 87$ ………………… (1)

$\therefore$ For market (B):

$92 + 2P_A - 4P_B = -6 + 32P_A \Rightarrow 36P_A - 2P_B = 98$ ……………………… (2)

$\therefore$ Solving equations (1) and (2) will give us the equilibrium price for each market:

In matrix form, we have: $\begin{bmatrix} 18 & -1 \\ -2 & 36 \end{bmatrix} \begin{bmatrix} P_A \\ P_B \end{bmatrix} = \begin{bmatrix} 87 \\ 98 \end{bmatrix} \Rightarrow A\ X = B$

Thus:

$|A|$ = (18 x 36) – {(-1) x (-2)} = (648 – 2) = 646

The cofactor matrix of A is:

$$\begin{bmatrix} 36 & -(-2) \\ -(-1) & 18 \end{bmatrix} = \begin{bmatrix} 36 & 2 \\ 1 & 18 \end{bmatrix}$$

Adj A = $\begin{bmatrix} 36 & 1 \\ 2 & 18 \end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{|A|} \text{ Adj} = \frac{1}{646}\begin{bmatrix} 36 & 1 \\ 2 & 18 \end{bmatrix} = \begin{bmatrix} \frac{36}{646} & \frac{1}{646} \\ \frac{2}{646} & \frac{18}{646} \end{bmatrix} = \begin{bmatrix} \frac{18}{323} & \frac{1}{646} \\ \frac{1}{323} & \frac{9}{323} \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} \frac{18}{323} & \frac{1}{646} \\ \frac{1}{323} & \frac{9}{323} \end{bmatrix}\begin{bmatrix} 87 \\ 98 \end{bmatrix} = \begin{bmatrix} \left(\frac{18}{323} \text{ x } 87\right) + \left(\frac{1}{646} \text{ x } 98\right) \\ \left(\frac{1}{323} \text{ x } 87\right) + \left(\frac{9}{323} \text{ x } 98\right) \end{bmatrix} = \begin{bmatrix} \frac{3230}{646} \\ \frac{969}{323} \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$\Rightarrow P_A = 5$ $P_B = 3$ are the equilibrium prices for each market.

3.8 Theorem

A necessary and sufficient condition for a matrix (square) A to be invertible is that $|A| \neq 0$

Proof

Suppose A is an invertible matrix and suppose B is its inverse.

Then, $A B = I$

$\Rightarrow |AB| = |I|$

$\Rightarrow |A| \; |B| = I$

$\Rightarrow |A| \neq 0$

Thus, necessity follows.

Sufficiency; Let $|A| \neq 0$

Take $B = \frac{1}{|A|} \, Adj \, A$

Thus: $A B = (\frac{1}{|A|} \, Adj \, A)$

$= (\frac{1}{|A|} \, Adj \, A)$

$= (\frac{1}{|A|}\ |A|\ I = I$

Again

$A\ B = (\frac{1}{|A|}\ Adj\ A)\ A$

$A = [\,(\frac{1}{|A|}\ Adj\ A)]$

$A = \frac{1}{|A|}.\ |A|\ I = I.$

Hence, *B* is the inverse of *A* and the assertion follows.

4.0 CONCLUSION

In conclusion, matrix algebra forms the basis of various techniques used in solving cumbersome business and technical related situations. As mentioned earlier, it is mostly used in handling small and large volumes of events (vectors) that cannot yield to other known mathematical methods.

5.0 SUMMARY

Basic definitions, examples and various applications of matrix algebra were discussed. And tutor- marked assignments were given below for further reading.

6.0 TUTOR-MARKED ASSIGNMENT

1. Given the matrices $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 9 \\ 1 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 1 & 1 \\ 6 & 3 & 1 \end{bmatrix}$

   Evaluate (i) $3A - 2B$ (ii) $4A\ \frac{3}{2}B$

2. Given that: $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$,

   Evaluate (i) AB (ii) BA

3. Given that: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

Evaluate (i) $A^2 - 2A + 4I$ (ii) $A^3 - 3A + 3I$

4. Given the matrices $A = \begin{bmatrix} 2 & 1 & 8 \\ 4 & 2 & 0 \\ 6 & 7 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 4 & 3 \end{bmatrix}$

Evaluate (i) $A^T B^T$ (ii) $(AB)^T$

5. Find what value of k would the matrix $\begin{bmatrix} 2k+1 & 5 \\ 4 & 6 \end{bmatrix}$ be singular

6. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, show that $A^{-1}A = I$

7. Use Cramer's rule to solve the following systems of two and three equations:

(i) $x + 2y = 5$
$2x + 3y = 2$

(ii) $5x + 9y = 16$
$12x + y = 11$

(iii) $-x + y + 3z = -1$
$2x + 3y - z = 2$
$X + 2y - z = 4$

(iv) $x + 2y - 6z = 2$
$4y + 3z = 1$
$2x + y + z = 1$

8. If $\begin{vmatrix} 3 & 5 & 1 \\ 0 & 4 & k \\ k & -1 & 6 \end{vmatrix} = 90$, find the value of k

9. Evaluate the determinant of $\begin{bmatrix} 9 & 4 & 7 \\ 1 & 3 & -5 \\ 4 & 0 & 3 \end{bmatrix}$

10. Suppose $A = \begin{bmatrix} 1 & 4 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ k & 2 \end{bmatrix}$,

Evaluate (i) $|AB|$ (ii) Then, if $|AB| = 112$, what is the value of k

11. Compute the inverse of the following matrices:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 4 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix}$$

12. Solve the equations:

x + 2y + 3z = 14

3x + y + 2z = 11

2x + 3y + z = 11

13. An amount of ₦500.00 is put into three investments at the rates of 6, 7 and 8 percent per annum respectively. The total annual income is ₦358. The combined income for the first two investments is ₦70 more that the income from the third investment. Find the amount of each investment.

14. The equilibrium conditions for three related markets are given by the equations:

$3P_1 - P_2 + P_3 = 2$
$-15P_1 + 6P_2 - 5P_3 = 5$
$5P_1 - 2P_2 + 2P_3 = 3$

Find the equilibrium price for each market.

15. A salesman has the following record of sales during three months for three items A, B and C which has different rates of commission.

Months	Sales of Units			Total Commission drawn (₦)	
	A	B	C		
January	90	100	20	800	200
February	130	50	40	900	300
March	60	100	30	850	400

Find out the rates of commission on items *A*, *B* and *C*.

16. Find the equilibrium prices and quantities for two commodity market models:

$$x_{d1} = -2 - p + q; \qquad x_{s1} = -2 - q$$

$$x_{d2} = -3p - q; \qquad x_{s2} = -q + p + q$$

Where p is the price and q is the quantity.

7.0 REFERENCES/FURTHER READING

1) Pure Mathematics for Advanced Level By B.D Bunday H Mulholland 1970.
2) Introduction to Mathematical Economics By Edward T. Dowling.
3) Mathematics and Quantitative Methods for Business and Economics.By Stephen P. Shao. 1976.
4) Mathematics for Commerce & Economics By Qazi Zameeruddin & V.K. Khanne 1995.
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7) Introduction to Mathematical Economics SCHAUM'S Out lines

UNIT 3 VECTORS AND COMPLEX NUMBERS

CONTENTS

1.0 INTRODUCTION

A vector is a single row or column matrix. Say , let A = 3i+4j-5k be a vector. Then, A = [3, 4, -5] is a row matrix.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Inter-marry science and economic words using vector as a mathematical model of simplifying and solving problems.

3.0 MAIN CONTENT

3.1 Vectors

DEFINITION

SCALARS: Scalars are quantities, which are completely specified by numbers, which measure their magnitude in terms of some chosen units, but have no definite direction in space.
Examples are: speed, energy, mass, density, time, length, work, temperature e.t.c.

Other quantities exist in engineering and science; however which have the property of direction, in addition to magnitude and these are called **VECTORS**"

DEFINITION

VECTOR: A vector is a physical quantity, which has magnitude and direction. It can also be defined as a directed line segments.

Examples are: velocity, acceleration, displacement, force, e.t.c.

3.1.1 Types of Vectors

These are:

- Free Vectors,
- Time and related Vectors,
- Point located Vectors and
- Position Vector.

1. **Free Vectors**: These have magnitude and direction but have no particular position associated with them e.g. Displacement.

   A displacement vector 20km north is the same in Lagos, Ojo, or anywhere.

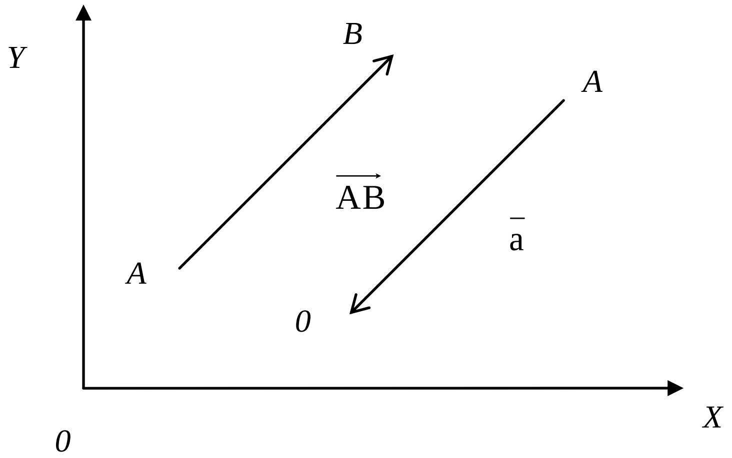
2. **Time Related Vectors**: These are located along a straight line e.g. Force. Force acting on a rigid body always moves along their lines of action without changing its effect on the body.

3. **Point Located Vectors**: this is a vector whose position in space is fixed in addition to its magnitude and direction. e.g a position vector.

4. Position Vector: this is a vector say $\overline{PQ}$ which is drawn from a fixed point P which gives the displacement of the point Q from the point P. The point P is called the initial point while the point Q is called the end point of the position vector $\overline{PQ}$.

3.1.2 Vector Representation

A vector can be represented by directed line segment, where the length and direction of the segment corresponds to the magnitude and direction of the vectors.



As can be seen above, a vector can be represented by $\overrightarrow{AB}$ AB or $\bar{a}$

Thus, the modulus (absolute value) of a vector $\bar{a}$ or $\overrightarrow{AB}$ is that positive number which is a measure of the length of the directed segment.

The modulus of $\bar{a}$ is denoted by $|\bar{a}|$ or $|\overrightarrow{AB}|$

3.1.3 Definitions

i. $|\bar{a}|$ is equal to 'a' if 'a' (the measure) is greater than zero i.e. positive.

ii. $|\bar{a}|$ is equal to zero if 'a' = 0.

iii. $|\bar{a}|$ is equal to (- a) if 'a' is less than zero.

Mathematically, $|\bar{a}|$ can be stated thus:

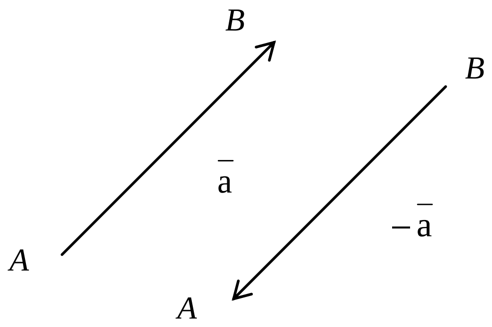
$$|\bar{a}| = \begin{cases} a \text{ If } a > 0 \\ 0 \text{ If } a = 0 \\ -a \text{ If } a < 0 \end{cases}$$

Special Cases

i. If $|\overrightarrow{AB}|$ = 0, then A = B and this is called a zero vector.

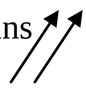
ii. If $|\overrightarrow{AB}|$ = 1, then $\overrightarrow{AB}$ and this is called a unit vector. This is represented as $\ell_{\vec{a}}$, or $\ell_{\vec{b}}$, respectively.

iii. $\overrightarrow{AB} = -\overrightarrow{BA}$ are vectors of the same magnitude but in opposite direction.



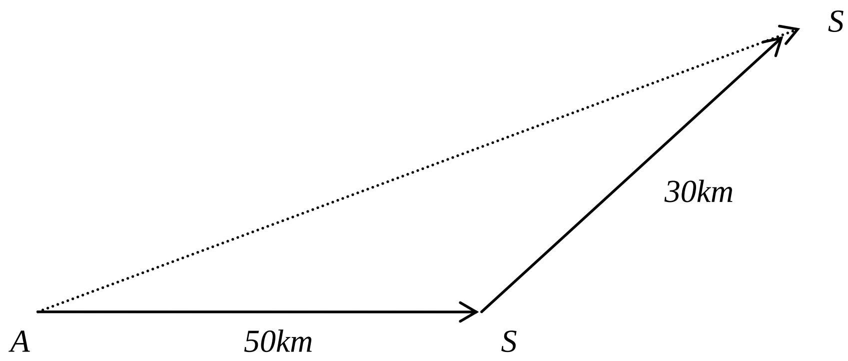
iv. Equal Vectors: Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have the same magnitude and direction.

If $\vec{a}$ and $\vec{b}$, then $= |\vec{a}|$ and $|\vec{b}|$.

Equal direction means  that they are parallel

Example

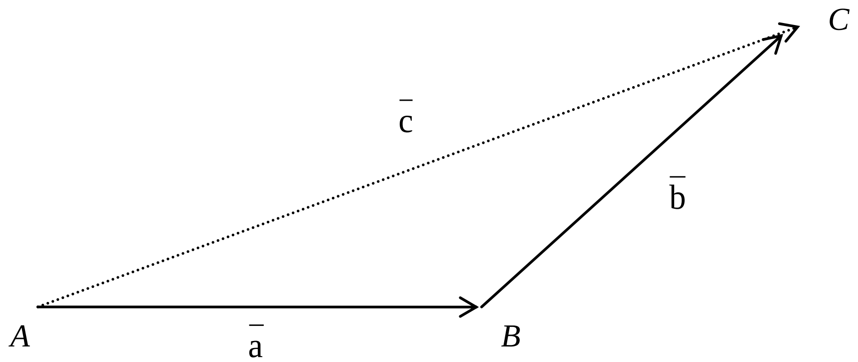
Suppose that from Anthony Campus (A), a car moves 50km due east Maryland (M) and then 30km North east to Sheraton (S). It would seem reasonable to say that the car has moved the distance represented by the length 'AS' in the direction *A* to *S*.



3.1.4 Vectors Addition

The addition of two vectors is defined by the triangular rule

Let $\overrightarrow{AB}$ and $\overrightarrow{BC}$ represent $\vec{a}$ and $\vec{b}$, therefore the addition of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is $\vec{a} + \vec{b}$ and is defined by $\vec{a} + \vec{b} = \vec{c}$.



$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, called resultant vector.

To find the sum of two vectors $\vec{a}$ and $\vec{b}$, we draw them as a chain; starting from the second where the first vector ends. The sum $\bar{c}$ is then given by the single vector joining the start of the first to the end of the second vector.

If the point A, B and C are collinear (i.e. Lie on the same straight line).

The law of vector addition still requires that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

Thus,

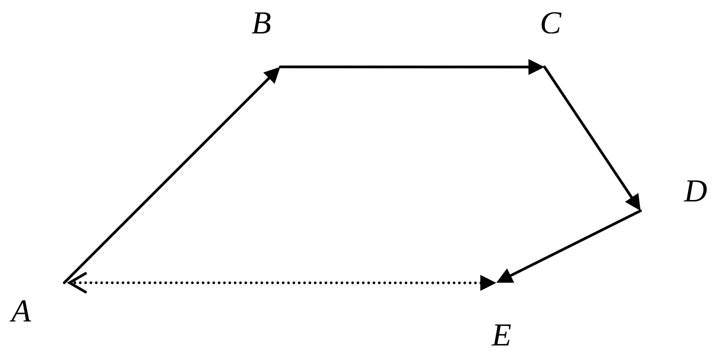
Although the ΔABC has now vanished

The sum of a number of vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DE}$ can be obtained by drawing the vectors as chain and then systematically apply the triangular law of addition i.e.:

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AD} + \overrightarrow{DE} = \overrightarrow{AE}$



Example 1

Find the sum of $\overrightarrow{AB}, -\overrightarrow{CB}, \overrightarrow{CD}, -\overrightarrow{ED}$.

Solution

Note that $-\overrightarrow{CB}, = \overrightarrow{BC}$ and $-\overrightarrow{ED} = \overrightarrow{DE}$

$\therefore\ \overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CD} - \overrightarrow{ED} \Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{DE}$

Note also that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{CD} \Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

$\therefore\ \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AD} + \overrightarrow{DE} = \overrightarrow{AE}$.

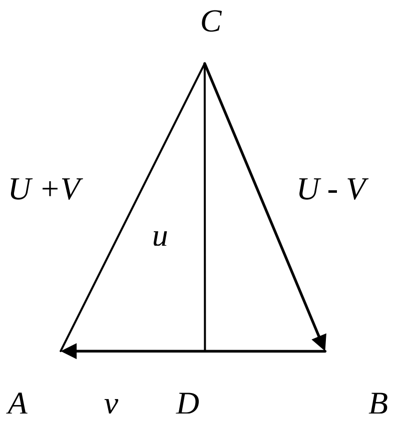
3.1.5 Subtraction of Vectors

The subtraction of vector can be considered as the addition of its negative.

Consider the figure below:

$\underline{CB} = \underline{u}$, $BA = \underline{v}$

Therefore $AC = \underline{u} + \underline{v}$



Let $\overrightarrow{DA}$ be $\vec{v}$ and $\overrightarrow{DB}$ be $-\vec{v}$; Note that $\overrightarrow{DA} = -\overrightarrow{DB}$;

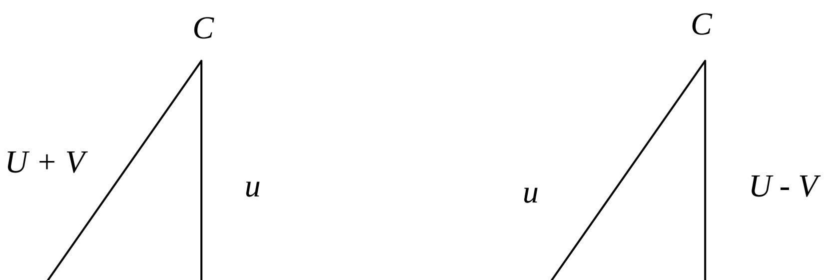
$\therefore \overrightarrow{CD} - \overrightarrow{DA} = \overrightarrow{CD} + (-\overrightarrow{DA}) = \overrightarrow{CD} + \overrightarrow{AD}$, recall that $\overrightarrow{DA} = -\overrightarrow{DB}$; $= \overrightarrow{CD} + \overrightarrow{DB} = \overrightarrow{CB}$.

Since $\overrightarrow{CD}$ = u, $\overrightarrow{DA}$ = v and $\overrightarrow{DB}$ = -v.

From triangular law of addition, we have:

$\overline{AC}$ = u + v

$\overrightarrow{CB}$ = u – v



3.1.6 Multiplication of Vectors

Going by the laws of addition of vectors, $\vec{a}+\vec{a} = 2\vec{a}$ is twice the magnitude of $\vec{a}$.
If we decide to add continuous to nth term of the same vector, $\Rightarrow \vec{a} + \vec{a} + \vec{a} + \ldots\ldots = n\vec{a}$.

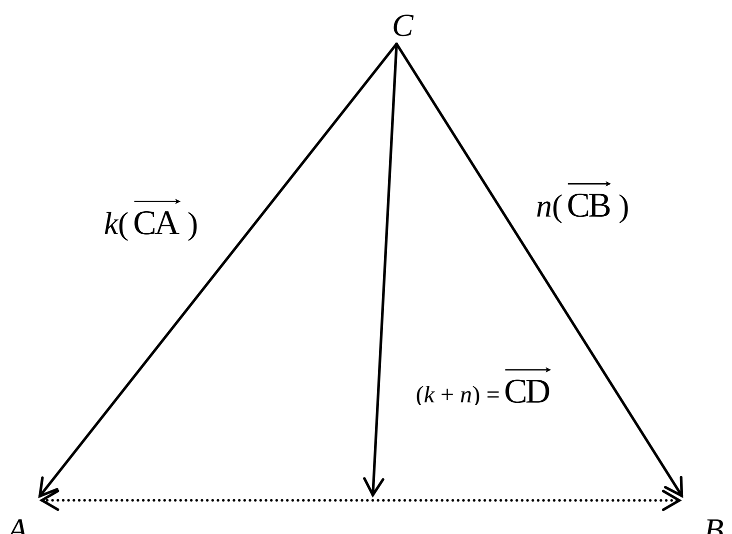
The product of a scalar 'n' and a vector $\vec{a}$ can be written as $(n\vec{a})$ or $(\vec{a}n)$ is a vector in the direction of $\vec{a}$.

If k is also another scalar, then:

i. $k(n\vec{a}) = (kn)\vec{a} = n(k\vec{a})$
ii. $(k + n)\vec{a} = k\vec{a} + n\vec{a}$
iii. $n(\vec{a} + \vec{b}) = n\vec{a} + n\vec{b}$

Theorem

If two vectors are presented by $k(\overrightarrow{CA})$ and $n(\overrightarrow{CB})$, then their resultant is represented by $(k + n)\overrightarrow{CD}$, where D is the pair, which divides AB in the ratio n: k.



From the above, we have:

$\overrightarrow{CA} = \overrightarrow{CD} + \overrightarrow{DA} \Rightarrow$ Vector addition.

$k\overrightarrow{CA} = k\overrightarrow{CD} + k\overrightarrow{DA} \Rightarrow$ Scalar multiplication …………..………..…… (1)

Similarly,

$\overrightarrow{CB} = \overrightarrow{CD} + \overrightarrow{DB} \Rightarrow$ Vector addition.

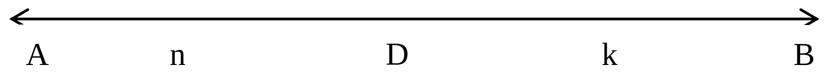
$n\overrightarrow{CB} = n\overrightarrow{CD} + n\overrightarrow{DB} \Rightarrow$ Scalar multiplication ………...……………… (2)

Equation (1) + (2):

$\Rightarrow k\overrightarrow{CA} + n\overrightarrow{CB} = (k\overrightarrow{CD} + k\overrightarrow{DA}) + (n\overrightarrow{CD} + n\overrightarrow{DB})$

$= (k\overrightarrow{CD} + n\overrightarrow{CD}) + k\overrightarrow{DA} + n\overrightarrow{DB} = (\text{k + n})\overrightarrow{CD} + k\overrightarrow{DA} + n\overrightarrow{DB}$ ……..…. (3)

Recall that $\overrightarrow{DA}$ and $\overrightarrow{DB}$ are collinear and directed in opposite sides i.e.



Also, $\overrightarrow{DA} : \overrightarrow{DB}$ = n: k $\Rightarrow \dfrac{\overline{DA}}{\overrightarrow{DB}} = \dfrac{n}{k} \Rightarrow k\overrightarrow{DA} = n\overrightarrow{DB}$

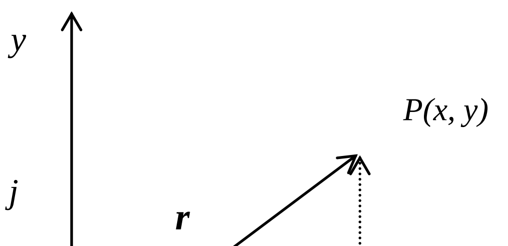
$\therefore\ k\overrightarrow{DA} + n\overrightarrow{DB} = 0$ ……………………………………….…………. (4)

Equation (4) in (3);

$\Rightarrow k\overrightarrow{CA} + n\overrightarrow{CB} = (\text{k + n})\overrightarrow{CD} + k\overrightarrow{DA} + n\overrightarrow{DB} = (\text{k + n})\overrightarrow{CD} + 0 = (\text{k + n})\overrightarrow{CD}$

Where (k + n) is a scalar quantity multiplying $\overrightarrow{CD}$, a vector quantity.

3.1.7 Position Vectors



The position of the point p(x,y) in a rectangular plane with reference to origin can be called " a vector".

The unit vector along the x-axis is i, while the unit vector along the y-axis is j.

$\therefore \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$ = xi + yj, $\overrightarrow{MP}$ is parallel to y-axis

$\therefore \overrightarrow{OP} = \vec{r} = xi + yj$

The vector $\overrightarrow{OP} = \vec{r}$ is called the position of point 'p' in the plane with reference to the origin.

Consider the elementary Pythagoras theorem:

$$OP^2 = OM^2 + MP^2 \qquad \Rightarrow r^2 = x^2 + y^2$$
$$\therefore |r|^2 = x^2 + y^2 \qquad \Rightarrow |r| = \sqrt{x^2 + y^2}.$$

Where $|r|$ is called the magnitude or the modulus of the vector, r.

The expression of vectors in components form also obeys the laws of addition, subtraction and scalar multiplication.

Example

Consider the vector r_1 and r_2:

Where $r_1 = x_i j + y_i j,\ r_2 = x_2 i + y_2 j$

i. $r_1 + r_2 = x_1 i + y_1 j + x_2 i + y_2 j = x_1 i + x_2 i + y_1 j + y_2 j$
$= (x_1 + x_2)i + (y_1 + y_2)j$.......(1)

ii. $r_1 - r_2 = (x_1 i + y_1 j) - (x_2 i + y_2 j) = (x_1 - x_2)i + (y_1 - y_2)j$ ....(2)

iii. $kr_1 = kx_1 i + ky_1 j.$

Example 2

Find the magnitude of the following vectors:

i. $3i + 4j$
ii. $5i \ \ + 7j$

Solution

i. $r = xi + yj$

$\therefore \ |r|^2 = x^2 + y^2$

$\Rightarrow |r| \ = \sqrt{x^2 + y^2}$

$\therefore r = 3i = 4j; \ x = 3 \ \ and \ \ y = 4$

$\Rightarrow |r| \ = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$

ii. $r = 5i - 7j; \ x = 5 \ \ and \ \ y = -7$

$\Rightarrow |r| \ = \sqrt{5^2 + (-7)^2} = \sqrt{25} + 49 = \sqrt{74}$.

Example 3

Given that $r_1 = 3i + 5j$ and $r_2 = -4i + 19j$. Find the modulus of the following:

i. $3r_1 + 4r_2$
ii. $5r_1 - r_2$

Solution

i. $3r_1 + 4r_2 = 3(3i + 5j) + 4(-4i + 19j) = 9i + 15j - 16i + 76j$

$= -7i + 19j$

Let R $= -7i + 19j$

$\therefore$ The modulus of R = $|R|$

$\Rightarrow |R| = \sqrt{(-7)^2 + 91^2} = \sqrt{45 + 8281 = \sqrt{8326.}}$

ii. $5r_1 - r_2 = 3(3i + 5j) - (-4i + 19j) = 15i + 25j + 4i + 19j$
$= 19i + 6j$
Let R $= 19i + 6j$
$\therefore |R| = \sqrt{19^2 + 6^2} = \sqrt{361 + 36} = \sqrt{397.}$
=19.92

3.1.8 Unit Vectors

Let $\hat{a}$ be the unit vector in the direction of vector then:

Let a = $= xi + yj \Rightarrow \hat{a} = \dfrac{a}{|a|}$, where $|a| = \sqrt{x^2 + y^2}$

Example 4

Find the unit vectors in the direction of the vector:

i. a = $4i + 3j$
ii. b = $-3i + 7j$
iii. c = $5i + 3j$

Solution

i. $a = 4i + 3j \Rightarrow |a| = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$

$$\therefore \hat{a} = \frac{a}{|a|} \Rightarrow \hat{a} = \frac{4i + 3j}{|5|} = \frac{4}{5}i + \frac{3}{5}j \;\; Or \;\; \hat{a} = \frac{1}{5}(4i + 3j).$$

ii. $b = -3i + 7j \Rightarrow |b| = \sqrt{(-3)^2 + 7^2} = \sqrt{9 + 49} = \sqrt{58}.$

$$\therefore \hat{b} = \frac{b}{|a|} \Rightarrow \hat{b} = \frac{-3i + 7j}{|\sqrt{58}|} = \frac{1}{\sqrt{58}}(3i + 7j).$$

iii. $c = -5i + 3j \Rightarrow |c| = \sqrt{5^2 + (-3)^2} = \sqrt{25 + 9} = \sqrt{34}.$

$$\therefore \hat{c} = \frac{c}{|c|} \Rightarrow \hat{c} = \frac{5i - 3j}{|\sqrt{34}|} = \frac{1}{\sqrt{34}}(5i + 3j).$$

3.2 Complex Numbers

DEFINITION

A complex variable z is of the form a + bi, where a and b are real numbers and i is called the imaginary number.

Mathematically, z = a + bi

Two complex numbers z_1 and z_2 are equal if their real parts are equal.

If $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$, then $z_1 = z_2$, if $a_1 = a_2$ and $b_1 = b_2$

Suppose a complex number is z = a + bi, then z = a – bi is called the conjugate of the complex number,
$(a+bi)\,(a-bi) = a^2 - b^2i^2 = a^2 + b^2$ (Real number), where $i^2 = -1$.

3.2.1 Properties of Imaginary Numbers

Consider the solution of the polynomial:

$$x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow \sqrt{x^2} = \sqrt{-1} \Rightarrow x = \sqrt{-1} = i \text{ (Imaginary)}$$
$$\Rightarrow i = \sqrt{-1}$$
$$\Rightarrow i^2 = (\sqrt{-1})^2 \Rightarrow -1$$
$$\Rightarrow i^3 = i\,x\,i\,(\sqrt{-1})(-1) = -\sqrt{-1} = -1$$
$$\Rightarrow i^4 = i^2\;x\;i^2(-1)x\,(-1) = 1$$
$$\Rightarrow i^5 = i^4x\,i\;\;(-1)^2\;x\;\sqrt{-1} = 1\,x\;\sqrt{-1} = \sqrt{-1}.$$

Thus:
$$\Rightarrow i \;\; = i^5 \;\; = \sqrt{-1};$$
$$\Rightarrow i^2 = i^6 \;\; = -1;$$
$$\Rightarrow i^3 = i^7 \;\; = -i;$$
$$\Rightarrow i^4 = i^8 \;\; = 1.$$

3.2.2 Algebra of Complex Numbers

Let $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$

$\therefore\ z_1 + z_2 = (a_1 + b_1i) + (a_2 + b_2i) = (a_1 + a_2) + (b_1 - b_2)i$

$\therefore\ z_1 - z_2 = (a_1 + b_1i) - (a_2 + b_2i) = (a_1 - a_2) + (b_1 - b_2)i$

Example 1

If $\therefore\ z_1 + 3 + 2i$ and $z_2 = 4 + 3i$. Find (i) $z_1 + z_2$ (ii) $z_1 - z_2$

Solution

i. $z_1 + z_2 = (3 + 2i) + (4 + 3i) = (3 + 4) + (2 + 3)i = 7 + 5i$

ii. $z_1 - z_2 = (3 + 2i) - (4 + 3i) = (3 - 4) + (2 - 3)i = -1-i$

3.2.3 Scalar Multiplication in Complex Numbers

Let k be a constant or a real number and $z_1 = a + bi$,

$\therefore\ kz_1 = k\,(a + bi) = ka + kbi$

Example 2

Find 5 z_1 if $z_1 = 3 + 2i$

Solution

$5\ z_1 = 5(3 + 2i) = 15 + 10i$

3.2.4 Multiplication of Complex Numbers

$z_1\ z_2 = (a_1 + b_1i)\ (a_2 + b_2i)$

$= a_1\ (a_2 + b_2i) + b_1i\ (a_2 + b_2i)$

$= a_1a_2 + a_1b_2i + a_2b_1i + b_1b_2(i)^2$

$= a_1a_2 + (a_1b_2i + a_2b_1)i - b_1b_2,$ where $i^2 = -1$

$= (a_1a_2 + b_1b_2) + (a_1b_2 + a_2b_1)i,$

When $(a_1a_2 + b_1b_2)$ and $(a_1b_2 + a_2b_1)i$ are real numbers.

Example 3

If $z_1 = 3 + 2i$ and $z_2 = 4 + 3i$. Find $z_1\,z_2$

Solution

$z_1\ z_2 = (3 + 2i)\ (4 + 3i) = 3(3 + 4) + 2i(4 + 3i) = 12 + 9i + 8i + 6(i)^2$

$= 12 + (9 + 8)i\ 6(-1) = (12 – 6) + 17i = 6 + 17i.$

3.2.5 Division of Complex Numbers

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i}$$

Multiply and divide through by the conjugate of the denominator, $a_2 - b_2 i$

$$\therefore \frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i} \text{ x } \frac{a_2 - b_2 i}{a_2 - b_2 i} = \frac{(a_1 + b_1 i)(a_2 - b_2 i)}{(a_2 + b_2 i)(a_2 - b_2 i)}$$

$$= \frac{(a_1 + b_1 i)(a_2 - b_2 i)}{(a^2 - b^2 2i^2)}$$

$$= \frac{(\mathrm{a}_1 + \mathrm{b}_1 \mathrm{i})(\mathrm{a}_2 - \mathrm{b}_2 \mathrm{i})}{\mathrm{a}_2^2 + \mathrm{b}_2^2}$$

Example 4

If $z_1 = 3 + 2i$ and $z_2 = 4 + 3i$. Evaluate $\frac{z_1}{z_2}$

Solution

$$\therefore \frac{z_1}{z_2} = \frac{3 + 2i}{4 + 3i} \text{ x } \frac{4 - 3i}{4 - 3i} = \frac{(3 + 2i)(4 - 3i)}{4^2 - 3^2 (i)^2}$$

$$= \frac{(3 + 2i)(4 - 3i)}{16 + 9}$$

$$= \frac{1}{25}(3 + 2i)(4 - 3i) .$$

3.2.6 Absolute Value or Modulus of a Complex Number

The modulus of $z_1 = |z_1|$.

If $z_1 = a + bi$,

$$|z_1| = \sqrt{a^2 + b^2}$$

Example 5

If $z_1 = 3 + 2i$. Evaluate $|z_1|$

Solution

$$|z_1| = \sqrt{a^2+b^2} = \sqrt{3^2+2^2} = \sqrt{9+4} = \sqrt{13}$$

Also, if z_1, z_2 and z_3 are complex numbers; then for absoluteness, the following axiom holds:

- $|z_1 z_2| = |z_1||z_2|$
- $|z_1 + z_2| < |z_1| + |z_2|$
- $|z_1 + z_2| > |z_1| - |z_2|$

3.2.7 Working Distances in Complex Numbers

The distance between two points $z_1 = (a_1 + b_1 i)$ and $z_2 = (a_2 + b_2 i)$ in complex plane is given by:

$$|z_1 - z_2| = |(a_1 + b_1 i) - (a_2 + b_2 i)| = |(a_1 - a_2) + (b_1 - b_2)i|$$

$$= \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$$

Example 6

Find the distance between the points z_1 and z_2, given that $z_1 = 3 + 2i$ and $z_2 = 4 + 3i$.

Solution

$$|z_1 - z_2| = |(3+2i)-(4+3i)| = |(3-4)+(2-3)i| = |-1-1i| = \sqrt{(-1)^2+(-1)^2} = \sqrt{1+1} = \sqrt{2}.$$

Example 7

Evaluate $\left|\frac{z_1}{z_2}\right| = \left|\frac{(3+2i)}{(4+3i)}\right| = \left|\frac{(3+2i)}{(4+3i)} x \frac{(4-3i)}{(4-3i)}\right| = \left|\frac{(3+2i)(4-3i)}{4^2+3^2}\right|$

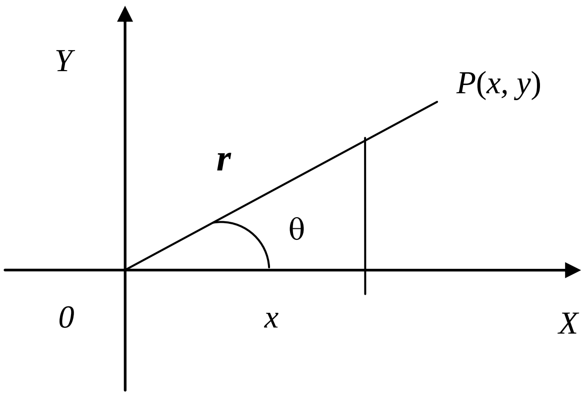
$$= \left|\frac{1}{25}(3+2i)(4-3i)\right| = \left|\frac{3(4-3i)+2i(4-3i)}{25}\right| = \left|\frac{12-9i+8i+6}{25}\right| = \left|\frac{18-i}{25}\right|$$

$$= \sqrt{(\frac{18}{25})^2 - (\frac{1}{25})^2} = \sqrt{\frac{(18^2-1^2)}{25^2}}$$

$$= \sqrt{\frac{18^2-1^2}{25}} = \sqrt{\frac{323}{25}} = 3.59$$

3.2.8 Polar Form of Complex Numbers

If p is a point in the complex plane corresponding to the complex number (x, y) or (x + iy), then:



x = r cosθ and y = r sinθ ………….……………...…………………. (1)

Using Pythagoras theorem:

$x^2 + y^2 = r^2$ ……………………………...………………………. (2)

$\sqrt{x^2+y^2}$ = r …………………………………...………………….. (3)

In complex numbers, we have distance r = $|x+iy|$, where $|x+iy|$ is called the modulus or absolute value of the complex number.

Similarly, $x^2 + y^2 = r^2$ ……………. (1).
i.e. substituting equations (1) in (2)
$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta \Rightarrow r^2(\cos^2\theta + \sin^2\theta) \Rightarrow r^2(1) = r^2$

If z = x + iy, $\Rightarrow$ z = r cosθ + r i sinθ = (cosθ + isin$^2\theta$) is called the polar form of z.

De Moivre's Theorem

Given the following:

x = r cosθ and y = r sinθ …………………………………...………. (1)

$x^2 + y^2 = r^2$ ……………………..……………………...……..…. (2)

$\sqrt{x2+y2}$ = r………………..…………………….………..……..…. (3)

If $z_1 = (x_1 + iy_1) = r_1 (\cos\theta_1 + i\sin\theta_1)$ ……………………… ……..(4)
and

$z_2 = (x_2 + iy_2) = r_2 (\cos\theta_2 + i\sin\theta_2)$ …………………………………. (5)

$\therefore\ z_1z_2 = r_1r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$ …………………..……. (6)

Also, $\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$ …………………….……. (7)

$\therefore\ z_1z_2 =\ldots z_n = r_1r_2\ldots r_n [\cos(\theta_1 + \theta_2 + \ldots\theta_n) +$
$i\sin(\theta_1 + \theta_2 + \ldots\theta_n)]$ ………………………………………...…. (8)

If = $z_1 = z_2\ldots z_n = z$, then, $r_1 = r_2\ldots r_n = r$

Also, if $z_1z_2\ldots z_n = z^n$, then, $r_1r_2\ldots r_n = r^n$ ……………………...….. (9)

$\therefore$ Equations (8) and (9) becomes:

$z^n = r^n (\cos\theta + i\sin\theta) = [r(\cos\theta + i\sin\theta)]^n$

$\therefore$ Equation (10) is called **"De Moivre's Theorem"**

Example 8

Express the complex number in polar form:

$3 = 2\sqrt{3i}$

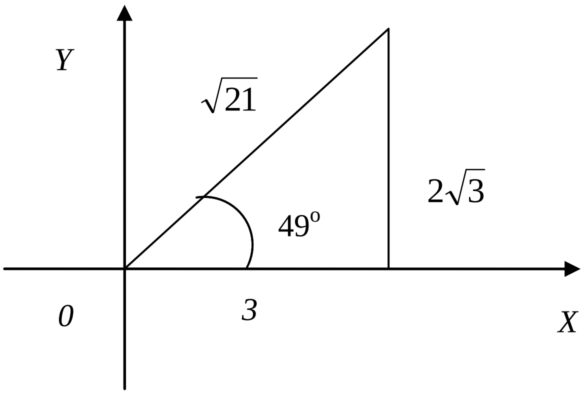
Solution

The absolute value, r = $\left|3+2\sqrt{3i}\right| = \sqrt{3^2+(2\sqrt{3^2})}$
$= \sqrt{9+12} = \sqrt{21}$

Argument θ = sin $-1\left(\frac{2\sqrt{3}}{\sqrt{21}}\right)$ = $\sin^{-1}$ (0.7559) = 49°

Hence, 3 + 2$\sqrt{3i}$ = r $(\cos\theta + i\sin\theta) = \sqrt{9+12}$
$=\sqrt{21}$ $(\cos49^\circ + i\sin49^\circ)$.

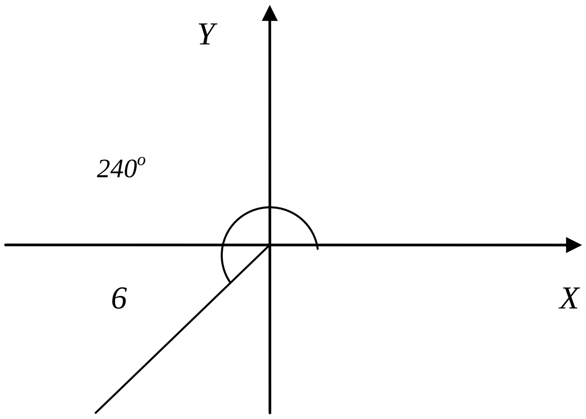
Graphically:



Example 9

Graphically, represent 6 (cos240° + isin240°)

Solution



4.0 CONCLUSION

We shall conclude as in the summary below.

5.0 SUMMARY

In summary, we defined vector, types of vector, manipulation of vectors vis a vis addition, subtraction, multiplication and division of vectors were carried out and relevant examples listed out and solved.

6.0 TUTOR-MARKED ASSIGNMENT

1. Find the sum of the following vectors:

   i. $\overrightarrow{AB}, \overrightarrow{BC}, -\overrightarrow{DC}, -\overrightarrow{AD}.$

   ii. $\overrightarrow{PQ}, \overrightarrow{QR}, \overrightarrow{RS}, \overrightarrow{SQ}.$

   iii. $\overrightarrow{AC}, \overrightarrow{CL}, -\overrightarrow{ML}.$

   iv. $\overrightarrow{GH}, \overrightarrow{HJ}, \overrightarrow{JH}, \overrightarrow{KL}, \overrightarrow{LE}.$

   v. $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DB}.$

2. $\overrightarrow{ABCD}$ is a rectangle with $\overrightarrow{AB}$ = 3a and $\overrightarrow{AD}$ = 4b. Find the magnitude and direction of the vector $\overrightarrow{AB} + \overrightarrow{AD}$.

3. If $\vec{a}$ = (3i + 4j) and $\vec{b}$ = (-i + j), evaluate the following:

   (i) $|a+4b|$ (ii) $|2a-3b|$

4. The position vectors of points A, B and C are:
   (i) a = 3i + 2j (ii) b = -4i + 3j (iii) c = 71 + 5j respectively:
   Find k if $\overrightarrow{BC} = k\overrightarrow{AB},$ where k is a scalar.

5. Prove De Moivre's Theorem (cosnθ + isinnθ), where n is any positive integer.

6. Prove $\overline{z_1 + z_2}$ = $z_1 + z_2$, where z_1 and z_2 are conjugates.

7. If z_1 = (2 + 1), z_2 = (3 + i) and z_3 = (-1-1),
   Find:
   i. $z_1z_2z_3$
   ii. $|z_3 - z_2|$
   iii. $|z_2 - z_1z_3|$
   iv. $\frac{z_1}{z_2}$
   v. $|2z_1z_3|$

8. Construct graphically:

i. $(3z_1 - z_2)$
ii. $7(\cos 240^\circ + i\sin 240^\circ)$

9. Express the complex numbers in polar form:
i. $(-7 + 7i)$
ii. $(2 - 3i)$

7.0 REFERENCES/FURTHER READINGS

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2) Introduction to Mathematical Economics By Edward T. Dowling.
3) Mathematics and Quantitative Methods for Business and Economics.By Stephen P. Shao. 1976.
4) Mathematics for Commerce & Economics By Qazi Zameeruddin & V.K. Khanne 1995.
5) Engineering Mathematics By K. A Stroad.
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7) Introduction to Mathematical Economics SCHAUM'S Out lines

UNIT 4 INTRODUCTION TO STRAIGHT LINES

CONTENTS

1.0 INTRODUCTION

This is an equation of the first degree, it is an equation containing no higher powers than the first of x and y, and is of the type y = mx+c where m and c are both constants.

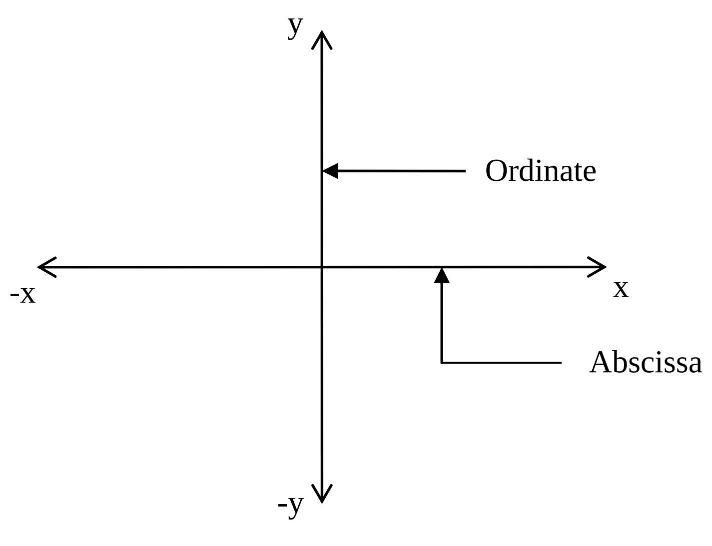
2.0 OBJECTIVES

To simplify and interpret physical, business and scientific situations in graphical form.(i.e ordinate and abscissa).

3.0 MAIN CONTENT

3.1 Definition

A point *P*(x, y) means that *P* is in the (x, y)-plane, where x is called the abscissa and y is the ordinate. Therefore the x and y coordinates of a point are called the rectangular Cartesian coordinates of the point.



3.2 Determination of a Point in (x, y) – Plane

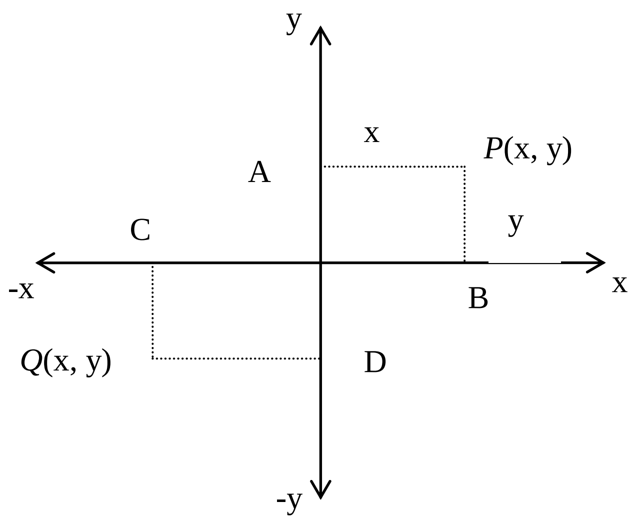


Fig. 2

P(x, y) and *Q*(x, y) are two points on (x, y)-plane.

Note here that:

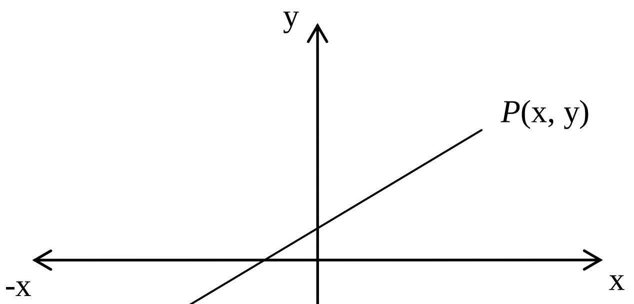
AP is parallel to x-axis, (+ve).

BP is parallel to y-axis, (+ve).
CQ is parallel to y-axis, (-ve).

DQ is parallel to x-axis, (-ve).

A straight line is formed when two points are joined together.

Here *PQ* is a straight line.



3.3 Distance Between Two Points

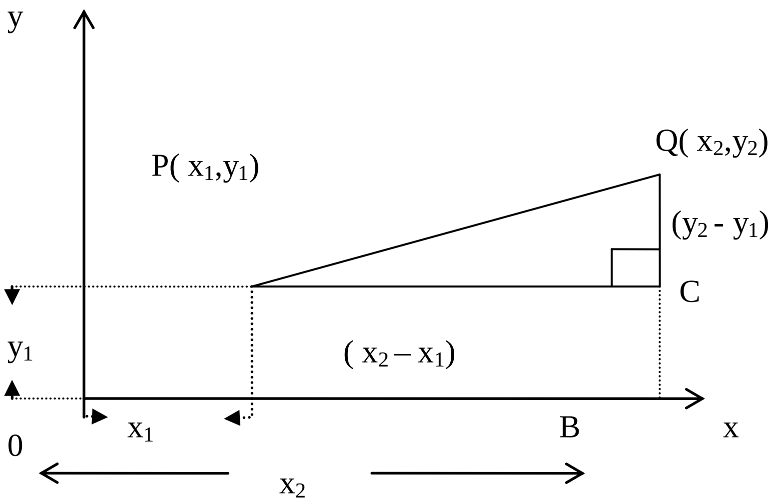


Fig. 4

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the (x, y)-plane. We calculate the distance between points *P and Q*.

$\therefore$ *PCQ* is a right angle triangle from Fig. 4 i.e.

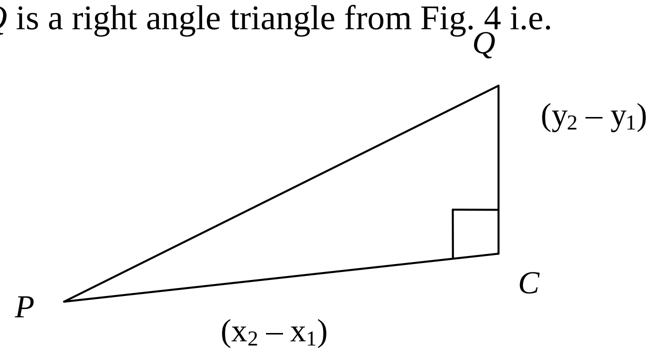


Fig. 5

The distance $QC = (y_2 - y_1)$, $PC = (x_2 - x_1)$

$\therefore PQ = ?$ is the required distance.

Therefore, using elementary Pythagoras theorem, we can see that:

$$\therefore PQ^2 = QC^2 + PC^2 = ((y_2 - y_1)^2 + (x_2 - x_1)^2$$

$$\Rightarrow PQ = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

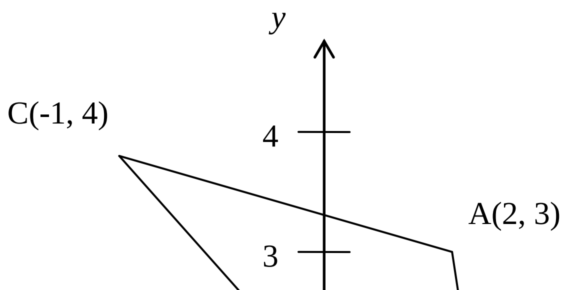
This is the formula for finding the distance between two points.

Example 1

Let A, B, C be the points (2, 3), (3, -2) and (-1, 4).

Find the length of the following: (i) AB (ii) BC and (iii) AC.

Solution



(i) Distance $AB = \sqrt{(-2-3)^2 + (3-2)^2} = \sqrt{(-5)^2 + (1)^2} = \sqrt{26}.$

(ii) Distance $BC = \sqrt{\{-2-4\}^2 + \{3-(-1)\}^2} = \sqrt{(-6)^2 + (4)^2} = \sqrt{52}.$

(iii) Distance $AC = \sqrt{\{3-4\}^2 + \{2-(-1)\}^2} = \sqrt{(-1)^2 + (3)^2} = \sqrt{10}.$

Example 2

Find the distance between the following pairs of points:

a) $A(5, -3)$ and $B(3, -1)$

b) $A(0, 1)$ and $B(6, 9)$

c) $A(6, 3)$ and $B(11, 15)$

Solution

(i) Distance $AB = \sqrt{\{1-(-3)\}^2 + \{3-5\}^2} = \sqrt{(4)^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}.$

(ii) Distance A$B = \sqrt{(9-1\}^2 + (6-0)^2} = \sqrt{(8)^2 + (6)^2} = \sqrt{100} = 10.$

(iii) Distance $AB = \sqrt{(15-3)^2 + (11-6)^2} = \sqrt{(12)^2 + (5)^2} = \sqrt{169} = 13.$

3.4 Gradient or Slope of a Straight Line

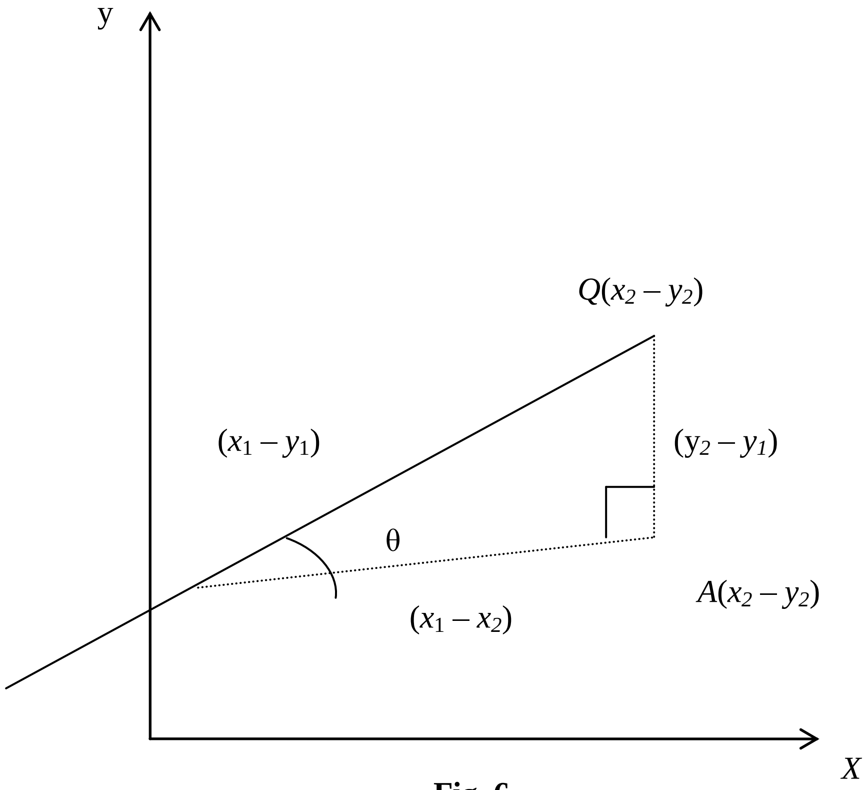


Fig. 6

The slope of a straight line can be defied in respect to the angle of inclination of the line with the x -axis i.e.

Slope = tangent of angle of inclination with x-axis = tanθ, where θ is the angle of inclination.

$\therefore$ From Fig. 6 above, we have:

$\tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = m,$ where m is the gradient/slope or rate of change

$\therefore \frac{y_2 - y_1}{x_2 - x_1} = m \Rightarrow (y_2 - y_1) = m(x_2 - x_1).$

Let $(y_2 - y_1) = \Delta y$ i.e. change in y, and

$(x_2 - x_1) = \Delta x \Rightarrow$ *i.e.* change in x.
$\therefore \Delta y = m\,\Delta x \Rightarrow \frac{\Delta y}{\Delta x} = m.$

Example 3

Find the gradient of the straight lines and their angles of inclination.

a) $A(2, -3)$ and $B(4, 5)$

b) $A(-2, 0)$ and $B(6, -4)$

Solution

a) Slope = $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{5-3}{4-2} = \dfrac{2}{2} = 1.$

Angle of inclination $\theta \Rightarrow \tan\theta = m = 1$

$\Rightarrow \theta = \tan^{-1} 1 \Rightarrow \theta = 45^{o}$

At angle of 45^{o}, $\sin 45^{o} = \cos 45^{o}$ $\Rightarrow \tan 45^{o} = \dfrac{\sin 45^{o}}{\cos 45^{o}} = \dfrac{\sin 45^{o}}{\cos 45^{o}} =$ 1.

$\therefore \tan\theta = 1 \Rightarrow \theta = 45^{o}$.

b) Slope = $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-4-0}{6-(-2)} = \dfrac{-4}{8} = -\dfrac{1}{2}.$

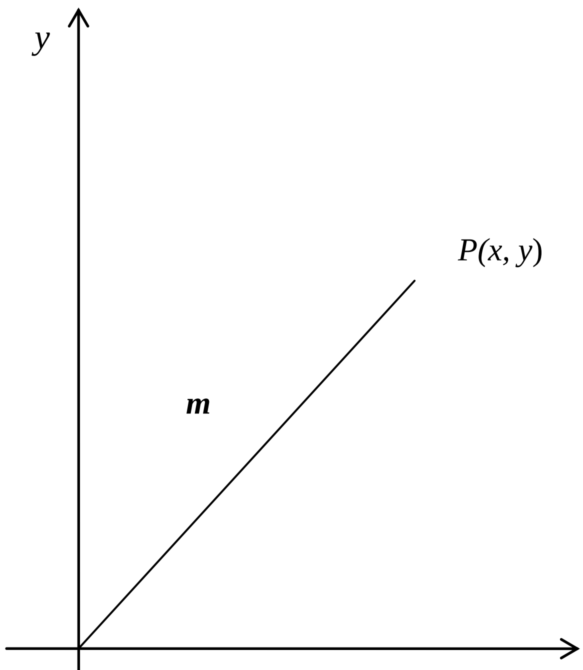
Angle of inclination $\theta \Rightarrow \tan\theta = m = \dfrac{1}{2}$

$\Rightarrow \theta = \tan^{-1} 1 \Rightarrow \theta = 45^{o}$.

3.5 Equations of a Straight Line with Slope "m"

He equation of a straight line is of the form $y – mx + c$; where x and y are the coordinates, m is the slope and, c is the intercept on the y-axis.

Example



The equation of the line is: $\frac{(y-a)}{(x-a)} = m \Rightarrow y - b = m\,(x - a)$

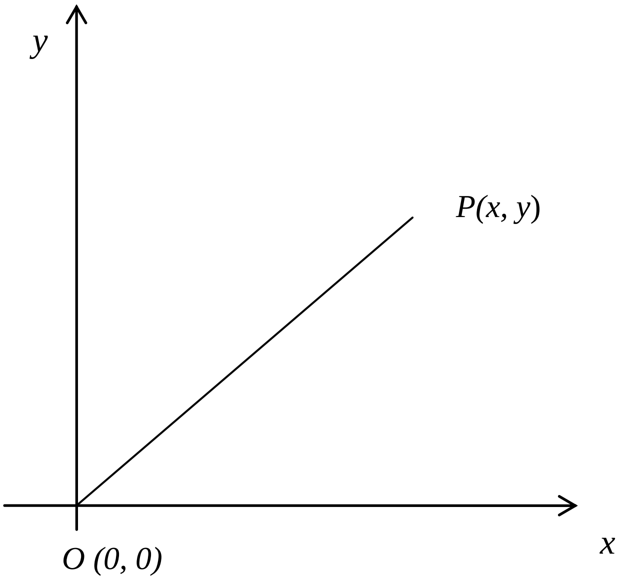
But if the line passes through the origin i.e. *O(a, b) = O(0, 0),* where a = 0 and b = 0.

$\therefore\ y - 0 = m\,(x - 0) \Rightarrow y = m\,x.$

This is the equation of a line passing through the origin.

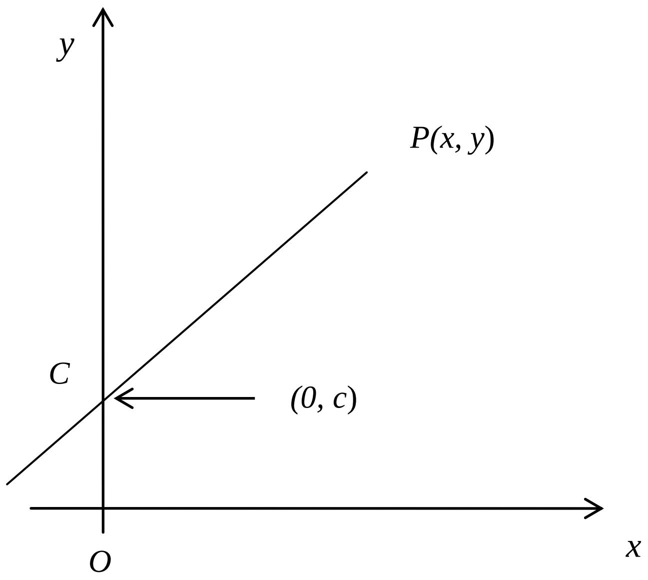
3.5.1 Graphical Illustrations of Equation of a Straight Line

a)



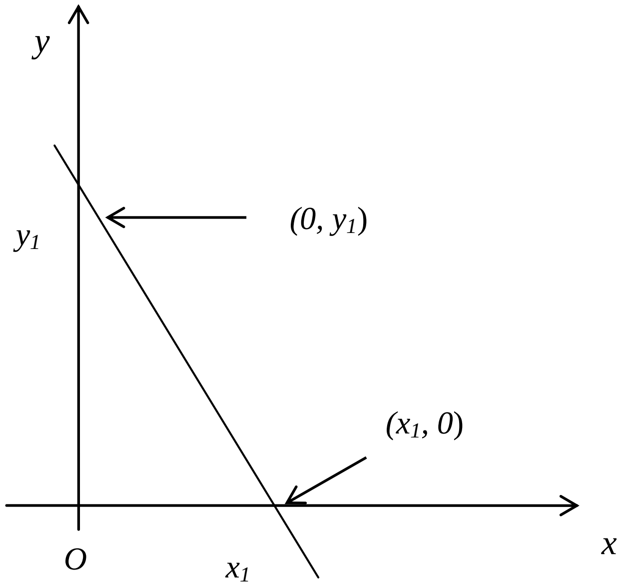
The equation of the line is: $y = m\,x$.

b)



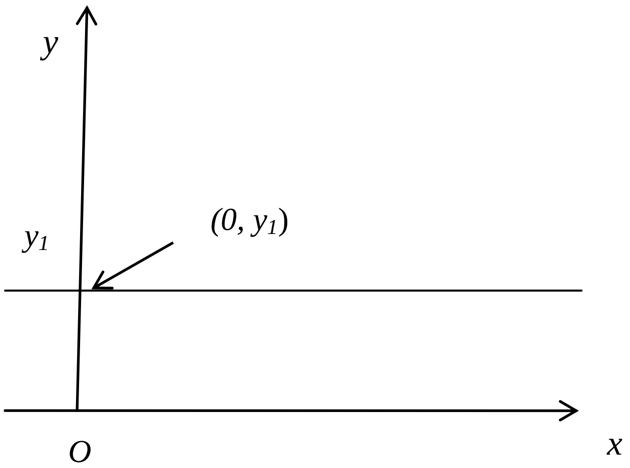
The equation of the line is: $y = mx + c$

c)



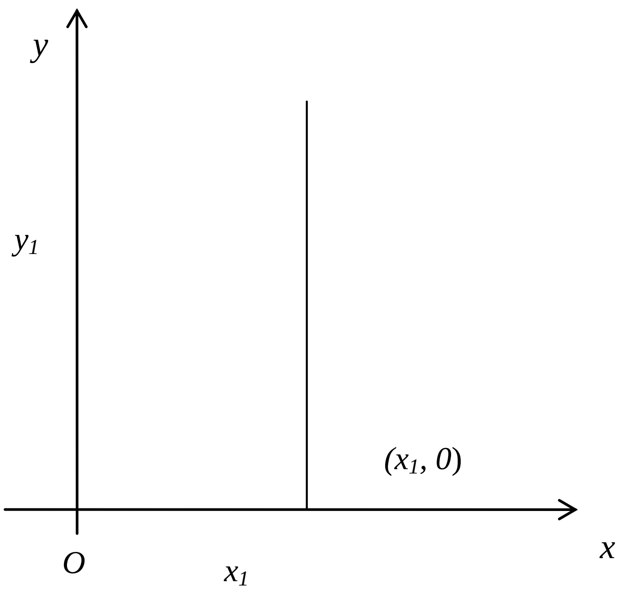
The equation of the line is: $y_1 = -m\ x_1$.

d)



This line is parallel to the x-axis.

e)



This line is parallel to the *y*-axis.

3.5.2 Equation of a Line given One Point and the Slope

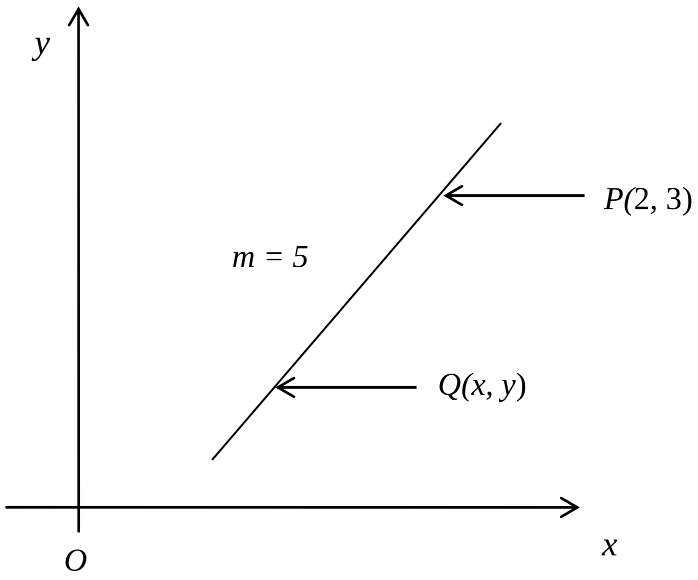


Fig. 8

The equation of the line is: $y - y_1 = m(x - x_1)$.

$\therefore\ y - 3 = 5(x - 2) \Rightarrow y - 3 = 5x - 10$

$\Rightarrow y = 5x - 7.$

Example 4

Find the equation of a line (-1, -4) whose gradient is 1.

Solution

$y - y_1 = m(x - x_1) \Rightarrow y - (-4) = 1\{x - (-0)\} \Rightarrow y + 4 = x + 1$

$\Rightarrow y = x - 3.$

3.5.3 Equation of a Line given Two Points

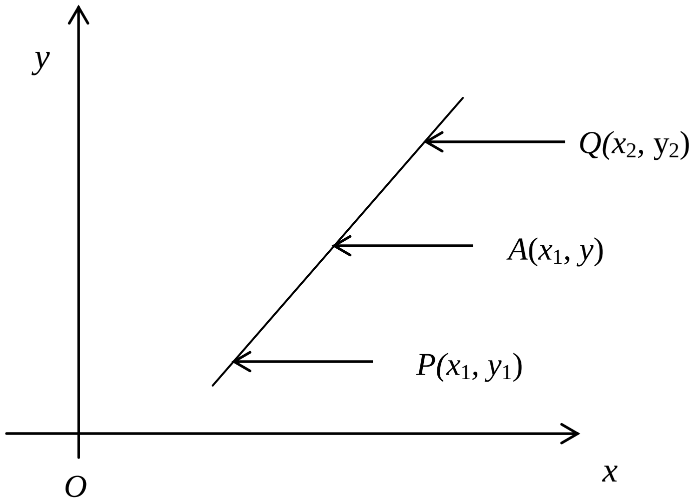


Fig. 9

PA = AQ = PQ.

$$\therefore\ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1} = \frac{y - y_2}{x - x_2}$$

Example 5

Find the equation of lines of the following given points:

a) *A(2, 3) and B*(4, 6)

b) *A(-3, -2) and B*(-1, 0)

Solution

a. $m = \frac{6-3}{4-2} = \frac{y-3}{x-2} = \frac{y-6}{x-4} \Rightarrow \frac{3}{2} = \frac{y-3}{x-2}$ Or $\frac{3}{2} = \frac{y-6}{x-4}$

$\Rightarrow y - 3 = \frac{3}{2}(x - 2) \Rightarrow y - 3 = \frac{3x}{2} - 3 \Rightarrow y = \frac{3x}{2} - 3 + 3$

$\Rightarrow y = \frac{3x}{2}$.

Or $\frac{3x}{2} = \frac{y-6}{x-4} \Rightarrow y - 6 = \frac{3}{2}(x-4) \Rightarrow y - 6 = \frac{3x}{2} - 6$

$\Rightarrow y = \frac{3x}{2} - 6 + 6$

$\Rightarrow y = \frac{3x}{2}.$

SELF ASSESSMENT EXERCISE

Solve for (b) using the same method.

3.5.4 Equation of a Line given Intercepts on 'x' and 'y" axis only

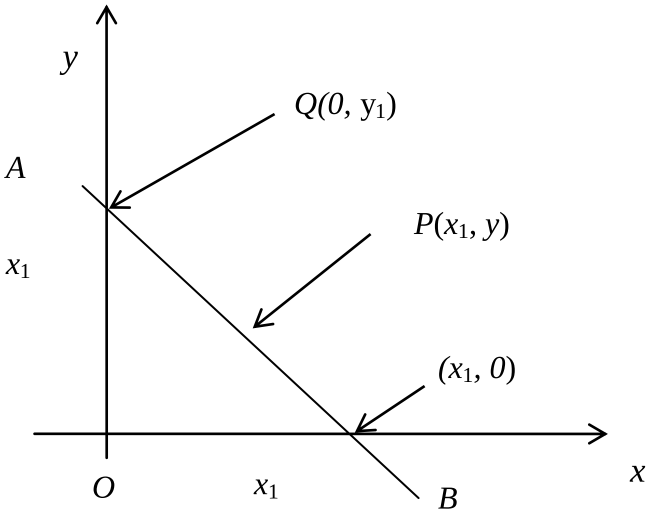


Fig. 10

$m = \frac{y - y_1}{x - 0},\ m = \frac{y - 0}{x - x_1}$

$\therefore \frac{y - y_1}{x - 0} = \frac{y - 0}{x - x_1} \Rightarrow (y - y_1)(x - x_1) = (y - 0)(x - 0)$

$\Rightarrow xy - x_1y - y_1x + x_1y_1 = xy \Rightarrow -x_1y - y_1x + x_1y_1 = 0$

$\Rightarrow x_1y_1 = x_1y + y_1x$

Dividing through by x_1y_1,

$\Rightarrow \frac{x_1y_1}{x_1y_1} = \frac{x_1y}{x_1y_1} + \frac{y_1x}{x_1y_1}$

$\Rightarrow 1 = \frac{y}{y_1} + \frac{x}{x_1}$

Example 6

Find the equation of lines, which passes through the points:

a) (0, 3) and (4, 0)

b) (0, 1) and (-3, 0)

c) (0, -6) and (5, 0)

Solution

a) (0, 3) and (4, 0) $\Rightarrow x_1 = 4,\ y_1 = 3$

$\Rightarrow 1 = \dfrac{x}{x_1} + \dfrac{y}{y_1} \Rightarrow 1 = \dfrac{x}{4} + \dfrac{y}{3}$

$\Rightarrow 12 = 3x + 4y$

b) (0, 1) and (-3, 0) $\Rightarrow x_1 = -3,\ y_1 = 1$

$\Rightarrow 1 = \dfrac{y}{y_1} + \dfrac{x}{x_1} \Rightarrow 1 = \dfrac{y}{1} + \dfrac{x}{-3}$

$\Rightarrow -3 = -3y + x$

c) (0, -6) and (5, 0) $\Rightarrow x_1 = 5,\ y_1 = -6$

$\Rightarrow 1 = \dfrac{x}{x_1} + \dfrac{y}{y_1} \Rightarrow 1 = \dfrac{x}{5} + \dfrac{y}{-6}$

$\Rightarrow -30 = -6x + 5y$

3.6 Intersection of Two Lines

The necessary and sufficient conditions for two lines to intercept are that they must be consistent and independent. A good application of intercept of two lines is the "**Market Equilibrium**".

Consider a linear demand and supply functions respectively. At a point when the demand equals the supply, it simply means that intercepts of demand and supply functions.

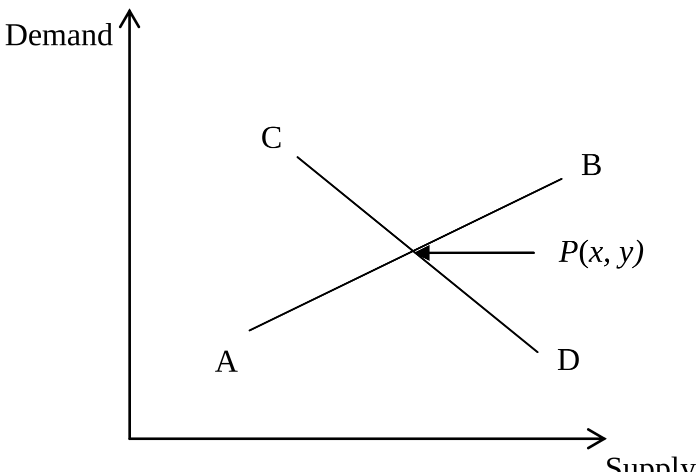


Fig. 11

$P(x, y)$ represents the quantity and the price of commodity at point of equilibrium.

Lines AB and CD can be solved simultaneously to evaluate the values of x and y respectively.

Example 7

Find the coordinate of the point of interception of the lines:

(a) $2x + 3y = 5$ (b) $x + 2y = 3$

Solution

$2x + 3y = 5$ …………………….… …………………………. (1)
$x + 2y = 3$ …………………….. ……………..……………. (2)

Equation (2) multiplies by 2 and subtract from Equation (1):

$2x + 3y = 5$ ……………… …………………………..……. (3)
$2x + 4y = 6$ ……………… …………………..……………. (4)

$\Rightarrow -y = 1 \Rightarrow \quad y = 1$

Substitute $y = 1$ into eqn. (1) or (2):

$2x + 3y = 5 \Rightarrow 2x + 3(1) = 5 \Rightarrow 2x + 5 - 3 = 2$
$\Rightarrow x = 1.$

Thus, the coordinate of the point of interception is (1, 1).

Example 8

If *UAC* Foods use 10 bags of flour and 5 tins of butter to produce 100 raps of gala and *UTC* uses 15 bags of flour and 8 tins of butter to produce 120 raps of gala. Find how many raps of gala that can be produced from 30 bags of flour and 15 tins of butter.

Solution

Let F represent bag of Flour and T represent tin of Butter,

$\therefore$ *UAC*: $10F + 5T = 100$ …………………………………………….. (1)
$\therefore$ *UTC*: $15F + 8T = 120$ …………………………………………….. (2)

Equations (1) and (2) can be solved simultaneously,

Equation (1) divided by 5 and multiplies by 8, then subtract from equation (2):
$16F + 8T = 160$ ………………………………………..……….. (3)
$15F + 8T = 120$ ………………………………………..……….. (2)

$(3) – (2) \Rightarrow F = 40$ …………………………………………….. (4)

Substitute $F = 40$ into eqn. (1):

$\Rightarrow 10F + 15T = 100 \Rightarrow 10(40) + 5T = 100 \Rightarrow 400 + 5T = 100$
$\Rightarrow 5T = 100 – 400 = -300$
$\Rightarrow T = \dfrac{-300}{5} = -60$

$\Rightarrow T = -60$ ……………………………………………………. (5)

$\therefore$ The required equation is:

$30F + 15T =?$ ………………………………………..……….. (6)

Substitute equations (4) and (5) into eqn. (6):

$\Rightarrow 30F + 15T =? \Rightarrow 30(40) + 15(-60) = 1200 – 900 = 300$ raps of gala.

Example 9

Find the equation of the straight line, which passes through the interception of $x + y = 3$ and $x +2y = 5$ and has a gradient of 2.

Solution

$x + y = 3$………………………..……………………………….. (1)

$x +2y = 5$ ……………………..……………………...………….. (2)

Equation (1) – (2):

$\Rightarrow -y = -2 \Rightarrow y = 2.$

Substitute $y = 2$ into eqn. (1):

$x + y = 3 \Rightarrow x + 2 = 3 \ \Rightarrow x = 1.$

$\therefore$ The coordinate of the point of interception is (1, 2)

$\therefore$ The equation of the line of P(1, 2) and slope2 is given by: $m = \frac{y - y_1}{x - x_1}$

$\Rightarrow 2 = \frac{y-2}{x-1} \Rightarrow y - 2 = 2(x - 1) \Rightarrow y - 2 = 2x - 2 \Rightarrow y = 2x.$

3.7 Parallel and Perpendicular Lines

a. Parallel Lines

Two lines are sad to be parallel if their gradient is the same i.e. they have equal rate of change.

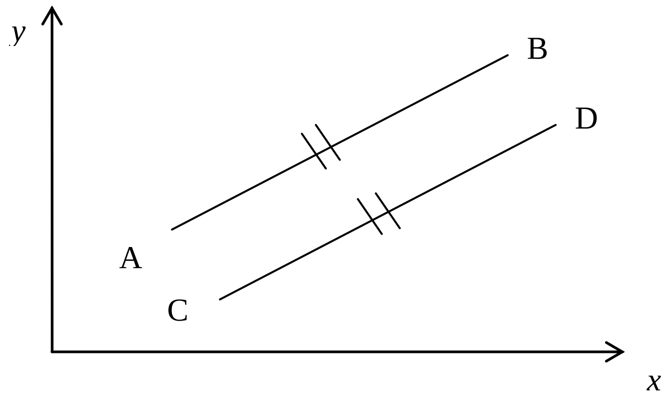


Fig. 12

Alternatively, if the angle between two lines is zero, it means that they are parallel.

Recall that slope = $\tan\theta = 0$.

Let m_1 and m_2 be the slope of lines AB and CD

$\therefore$ Lines AB *and* CD are parallel if and only if $m_1 = m_2$.

b. Perpendicular Lines

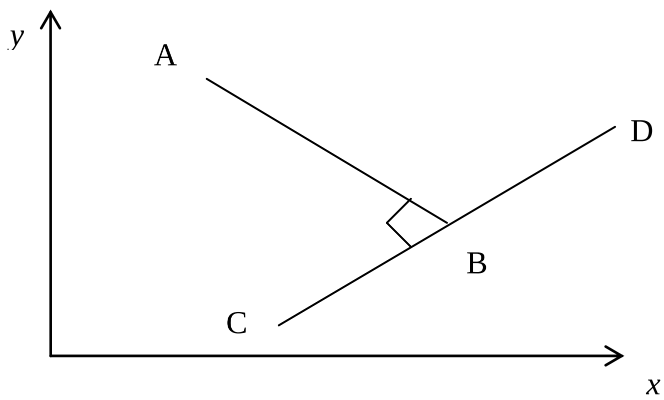


Fig. 13

The product of their gradient must be equal to -1.

$\therefore\ m_1 \text{ x } m_2 = -1 \Rightarrow m_1 = \dfrac{-1}{m_2}$.

Example 10

Find the equation of the lines, which are (a) parallel (b) perpendicular o

$3x + 2y = 5$ and passes through the point (2, -4).

Solution

To find the slope of $3x + 2y = 5$:

$\Rightarrow 2y = -3x + 5 \Rightarrow y = \dfrac{3}{2}x + \dfrac{5}{2}$

$\Rightarrow y = mx + c$

$\Rightarrow m = \dfrac{3}{2}$.

a. The equation of the line parallel to $\dfrac{3}{2}$ would have gradient of $-\dfrac{3}{2}$ i.e. $m_1 = m_2$. The given point is (2, -4).

$$\therefore\ m = \frac{y - y_1}{x - x_1} \Rightarrow -\frac{3}{2} = \frac{y-(-4)}{x-2} \Rightarrow -\frac{3}{2} = \frac{y+4}{x-2} = y + 4 = -\frac{3}{2}(x-2)$$

$$\Rightarrow y = -\frac{3}{2}x + 3 - 4$$

$\Rightarrow y = -\frac{3}{2}x - 1.$

b. The equation of the line perpendicular to $\frac{3}{2}$ would have gradient of $-\frac{3}{2}$ i.e. $m_1 = m_2$. The given point is (2, -4).

$\Rightarrow$ If $m_1 = -\frac{3}{2} \Rightarrow m_2 \frac{-1}{m_1} = \frac{-1}{\frac{3}{2}} = \frac{2}{3}$

$\therefore \ m = \frac{y - y_1}{x - x_1} \Rightarrow \frac{2}{3} = \frac{y-(-4)}{x-2} \Rightarrow \frac{2}{3} = \frac{y+4}{x-2}$

$\Rightarrow y + 4 = \frac{2}{3}(x-2).$

3.8 Application of Straight Lines to Business

Demand: This is the amount of goods requested for by customers at any point in time. The demand curve is the relationship between quantity and price of goods

The demand curve can be represented in many forms such as Linear, Quadratic, Cubic etc. Since we are concerned with straight lines in this section, we would be giving examples of linear demand curves.

Example 11

11 boxes of Lux soap are demanded for, each box costs ₦300. 20 boxes of Premier soap are also required by customers, each box costs ₦350. Draw up a demand curve showing the relationship between demand for both products.

Solution

The general demand function is given as $q = mp + c$; where:

q is the quantity of goods demanded for;
m is the gradient of demand function;
p is the price of goods; and
c is the constraint or intercept on demand function.

For Lux soap:

When price (p) = ₦300, demand (q) = 11.
The demand equation is given as;

$11 = 300m + c$ …………………………………………………… (1)

For Premier soap:

When price (p) = ₦350, demand (q) = 20.
The demand equation is given as;
$20 = 350m + c$ …………………………………………………… (2)

Solving equations (1) and (2) simultaneously, we have:
$11 = 300m + c$…. $\Rightarrow c = 11 - 300m$.
$20 = 350m + c$ ……………………………………………………. (2).

Substitute c in equation (2), we have:

$20 = 350m + 11 - 300m$

$\Rightarrow 20 = 50m + 11$

$\Rightarrow 50m = 9$
$\Rightarrow m = 0.18$.

Substitute m in equation (2), we have:

$20 = 350(0.18) + c$
$\Rightarrow 20 + 63 = c$
$\Rightarrow c = -43$.

From the general demand function is $q = pm + c$.

Hence, the demand function of customers for Lux and Premier soap is given as: $q = 0.18p - 43$.

This shows the functional relationship between demands for both products.

3.9 Inequalities

With inequalities, we can define intervals on the number scale. This will enable graphs of functions to be easily defined. Functions of inequalities can be Linear, Quadratic, etc.

Example 12

Solve and graph the following functions:

i. $18x - 3x^2 > 0$.
ii. $(x + 3)(x - 2)(x - 4) < 0$.

Solution

i. $18x - 3x^2 > 0$.

Consider $f(x) = 18x - 3x^2 = 0$.
$\Rightarrow 3x\,(6 - x) = 0$.
$\Rightarrow x = 0$ or x = 6.

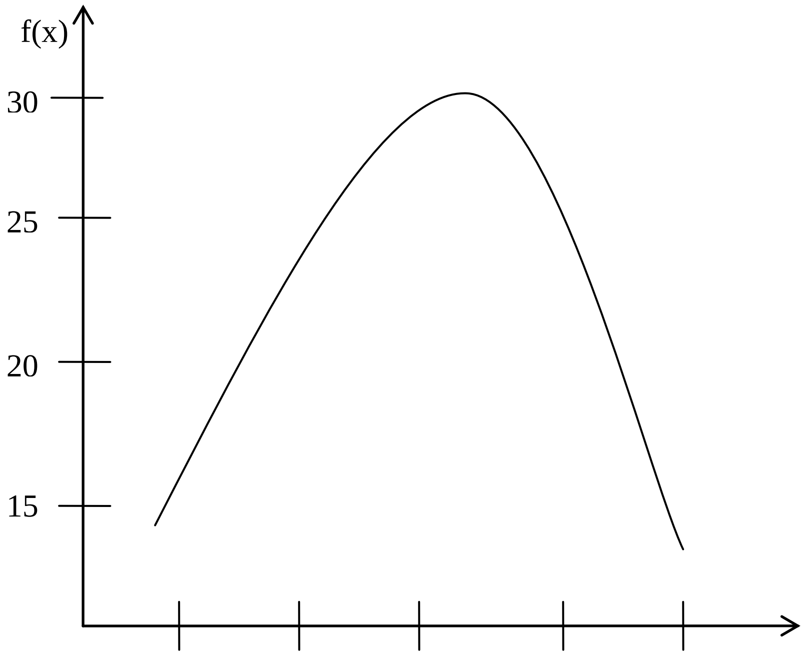
We determine the sign of $18x - 3x^2 > 0$ for the following intervals:

$x < 0$;
$0 < x < 6$; and
$x > 6$.

Picking numbers arbitrary for the above intervals, we fins out that x is satisfied for all x on the interval $0 < x < 6$.

x	$f(x) = 18x - 3x^2$
1	15
2	24
3	27
4	24
5	15

Graph of $18x - 3x^2 > 0$



ii. $(x + 3)(x - 2)(x - 4) < 0$.

Consider $f(x) = (x + 3)(x - 2)(x - 4) = 0$.
$\Rightarrow x$ -3, 2, 4 respectively.

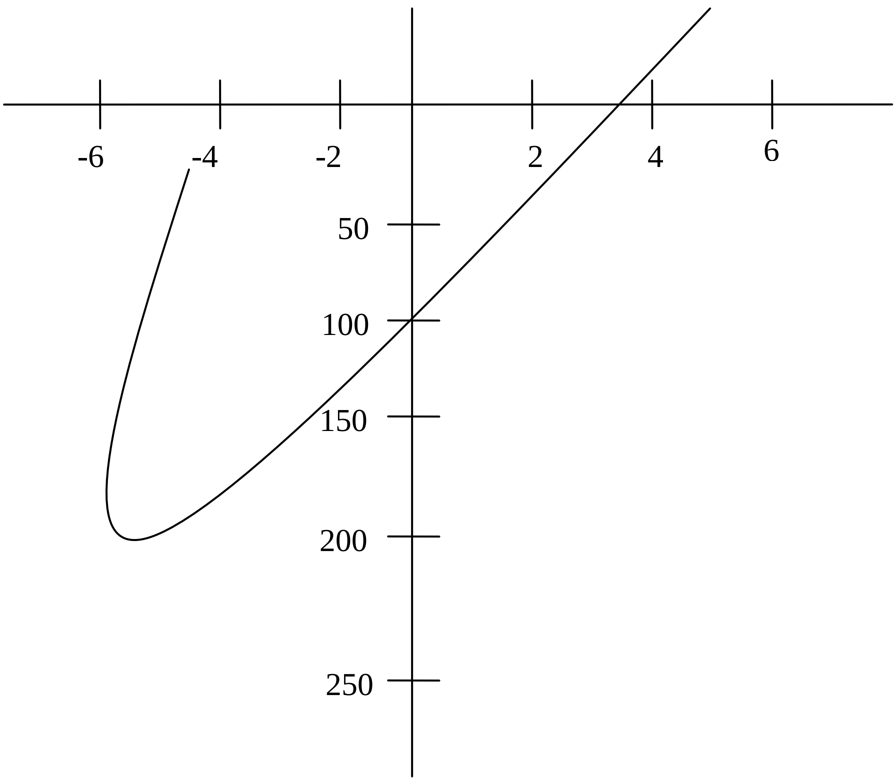
We consider the following intervals:

$x < -3$;
$-3 < x < 2$;
$2 < x > 4$; and
$x > 4$.

Picking numbers arbitrary for the above intervals, we fins out that x is satisfies the following intervals $x < -3$ and $2 < x < 4$.

x	$f(x) = (x + 3)(x – 2)(x – 4)$
-4	-48
-5	-126
-6	-240
3	-6

Graph of (*x* + 3)(*x* – 2)(*x* -4)



4.0 CONCLUSION

As in the summary

5.0 SUMMARY

Straight line graph is dealt with, both parallel and perpendicular lines and their relationship to business problems. Demand, supply and equilibrium conditions are also discussed. Some illustrative examples are shown in examples 8,9and 10 above.

6.0 TUTOR-MARKED ASSIGNMENT

1. A Market woman sells 10 bales of Ankara for ₦500 per bale. She also sells 12 bales of George's material for ₦800 per bale. Assuming that both sales are at a point of equilibrium, determine the equilibrium price and quantity.

2. Draw the graphs of the following inequalities:

   i. $(x+1)^2 (x-3) > 0$.

   ii. $\frac{x-1}{x+1} < 0$.

7.0 REFERENCES/FURTHER READINGS

1. Pure Mathematics for Advanced Level By B.D Bunday H Mulholland 1970.

2 Introduction to Mathematical Economics By Edward T. Dowling.

3 Mathematics and Quantitative Methods for Business and Economics.By Stephen P. Shao. 1976.

4 Mathematics for Commerce & Economics By Qazi Zameeruddin & V.K. Khanne 1995.

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UNIT 5 INTRODUCTION TO CIRCLE

CONTENTS

1.0 INTRODUCTION

This unit will introduce you to the properties of equation of a circle, detailing how these properties could be used to solve day-day problems.

2.0 OBJECTIVES

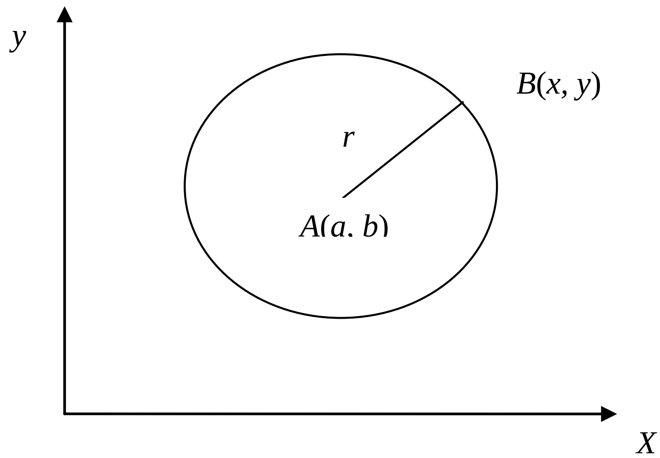
At the end of this unit, you should be able to solve simple problems involving equation of a circle.

3.0 MAIN CONTENT

3.1 Definition

A circle is the locus of a curve (equidistant from a point).

A circle could be described by its centre (fixed) and its radius. The radius is the distance between the centre of the circle and the circumference.



To form the equation of the circle, whose centre is the point *A* (*a*, *b*) and radius, 'r' joined to point *B*(*x*, *y*) on the circumference.

Distance AB = r = $\sqrt{(y-a)^2 + (x-b)^2}$ …………………………… (1)

Squaring both sides, we have:

$r^2 = (y - a)^2 + (x - b)^2$……………………………………………… (2)

This is the required equation.

Suppose the centre of the circle is located at the origin meaning that a = 0, b = 0; then the equation becomes:

$r^2 = (y - 0)^2 + (x - 0)^2 = y^2 + x^2$ ……………………………...………… (3)

This is equation of the circle with centre at the origin

In gradient the equation (2) can be expanded thus:

$r^2 = (y - a)^2 + (x - b)^2 = y^2 - 2ay + a^2 + x^2 - 2bx + b^2$

$\Rightarrow y^2 + x^2 - 2ay - 2bx + a^2 + b^2 - r^2 = 0$ …………………...………… (4)

$\Rightarrow y^2 + x^2 - 2gx - 2fy + (a^2 + b^2 - r^2) = 0$ …………………………… (5)

Where $-a = f$, $-b = g$ and let $a^2 + b^2 - r^2 = c$

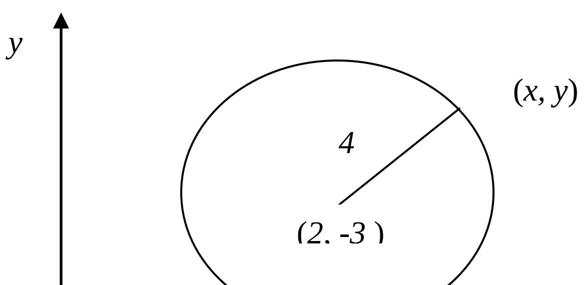
$\Rightarrow y^2 + x^2 - 2gx - 2fy + c = 0$ …………………...…………………… (6)

Example

Find the equation of the circle centre (2, -3) and radius 4.

Solution

The equation of the circle centre (2, -3) and radius 4 is:



$\Rightarrow y^2 + 6y + 9 + x^2 – 4x + 4 = 16$
$[y – (-3)]^2 + [x – 2]^2 = 4^2$
$\Rightarrow y^2 + 6y + 9 + x^2 – 4x + 4 = 16$
$\Rightarrow y^2 + x^2 – 4x + 6y - 3 = 0.$

3.2 Properties of Equation of a Circle

1. The coefficient of the two variables x and y must be the same;
2. There must be no term in xy and;
3. The equation must be second degree or simply in second degree of x and y respectively.

Example

Find the coordinate of the centre and the radius of the circle

$y^2 + x^2 – 14x -8y + 56 = 0$

Solution

The circle $x^2 + y^2 – 14x -8y + 56 = 0$ ………………………………….. (1)

Collecting like terms:

$\Rightarrow x^2 -14x + [\ ]\ y^2 – 8y + [\ ] = -56$ ……………………….………… (2)

Fill-up the bracket to make each equation to be perfect square.

Find the half of the coefficients of x and y and square them and add to both sides:

$\Rightarrow \{\frac{1}{2}(-14)\}^2 = [-7]^2 = 49$ ……………………………...…………… (3)

$\Rightarrow \{\frac{1}{2}(-8)\}^2 = [-4]^2 = 16$ …………………………………………… (4)

Equation (2) becomes:

$\Rightarrow x^2 - 14x + 49 + y^2 - 8y + 16 = -56 + 49 + 16$
$\Rightarrow x^2 - 14x + 7^2 + y^2 - 8y + 4^2 = 9$
$\Rightarrow (x - 7)^2 + (y - 4)^2 = 3^2$

$\therefore$ (7, 4) are the coordinates or centre of the circle, while 3 is the radius.

Alternative Solution

Comparing $x^2 + y^2 - 14x - 8y + 56 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$, which is the general form.

$\Rightarrow 2g - 14 \Rightarrow g = -7$, and $2f = -8 \Rightarrow f = -4$

Where a and b are the coordinates, recall that $-a = g$ and $-b = f$

$\therefore$ $a = 7$, $b = 4$ (the coordinates).

Also, $c = g^2 + f^2 - r^2$

$\Rightarrow 56 = (-7)^2 + (-4)^2 - r^2$
$\Rightarrow 56 = 49 + 16 - r^2$
$\Rightarrow r^2 = 65 - 56$
$\Rightarrow r^2 = 9$
$\Rightarrow r = 3$ (the radius)

Hence, the circle has centre (7, 4), radius 3.

4.0 CONCLUSION

As in the summary

5.0 SUMMARY

In summary, this unit discourses equation of circle. Simple problems involving equation of circle and some properties of circle is also discoursed.

6.0 TUTOR-MARKED ASSIGNMENT

1. Find the equation of the circle giving the following coordinates and radii:

   i. Centre (-1, 2), radius 4
   ii. Centre (6, 2), radius 1
   iii. Centre (-2, -3), radius 5

2. Write down the centre and radius of the following circles:

   i. $x^2 + y^2 - 10x + 6y + 30 = 0$
   ii. $9x^2 + 9y^2 - 12x - 6y + 4 = 0$
   iii. $4x^2 + 4y^2 - 12x + 4y - 71 = 0$

7.0 REFERENCES/FURTHER READINGS

1) Pure Mathematics for Advanced Level By B.D Bunday H Mulholland 1970.

2) Introduction to Mathematical Economics By Edward T. Dowling.

3) Mathematics and Quantitative Methods for Business and Economics.By Stephen P. Shao. 1976.

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MODULE 2

UNIT 1 SIMPLE SEQUENCE AND SERIES

CONTENTS

1.0 INTRODUCTION

A series is a succession of numbers, of which each number is formed according to a definite law which is the same throughout the series.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- find the sum of an arithmetic progressions,
- find the sum of a geometric progressions,
- apply it to simple and compound interest problems,
- commerce related problems.

3.0 MAIN CONTENT

3.1 Definition

A sequence is a succession of terms spanned by a rule or formula.

Examples

1. 1, 2, 3, 4, 5…
2. 2, 4, 6, 8, 10…
3. $a, a^2, a^4, a^8, a^{16} \ldots$

A sequence may be finite or infinite.

A finite sequence is one whose first and last element are known, while an infinite sequence is one whose terms are uncountable.

The general term of a sequence (formula) can be written as: $U_n = \frac{n+1}{2n-1}$

We can now generate the sequence by substituting n = 1, 2, 3 …

- When n = 1, $\Rightarrow U_1 = \frac{1+1}{2(1)-1} = \frac{2}{1} = 2$
- When n = 2, $\Rightarrow U_2 = \frac{2+1}{2(2)-1} = \frac{3}{3} = 1$
- When n = 3, $\Rightarrow U_3 = \frac{3+1}{2(3)-1} = \frac{4}{5} = 0.8$
- When n = 4, $\Rightarrow U_2 = \frac{4+1}{2(4)-1} = \frac{5}{7} = 0.7143$

Hence, the sequence is: 2, 1, $\frac{4}{5}, \frac{5}{7}$........

3.2 Arithmetic Sequence (A. P)

This is a sequence in which each term differs by a common difference.

Let ‘a’ be the first term,

Let ‘s’ be the common difference.

The sequence is of the form:

- a, a + d, a + d +d, a + d + d+ d, ………
- a, a+ d, a + 2d, a + 3d…..

If u_1, u_2, u_3 ……..u_n are the n^{th} terms of an AP.

Then:

U_1 = a
U_2 = a + 1d
U_3 = a+ 2d
U_4 = a + 3d
U_5 = a + 4d
:   :

$U_n = a + (n - 1)d$. Following the same pattern

The last term or general term for an Arithmetic Sequence is

$$U_n = a + (n - 1)d.$$

Example

Find the common difference in the following sequence

i. 3, 5, 7, 9, 11…
ii. 102, 99, 96, 93 …

Solution

i. $d = u_2 - u_1 = u_3 - u_2$
$d = 5 - 3 = 7 - 5 = 2$

$\therefore$ The common difference is d = 2

ii. $d = u_2 - u_1 = u_3 - u_2$
$d = 99 - 102 = 96 - 99 = -3$

$\therefore$ The common difference is d = -3

Example

Find the 7th term of an A.P whose first term is 102 and common difference is -3

Solution

Let 'a' be the first term

Let'd' be the common difference

$\therefore$ The *nth* term is $U_n = a + (n - 1)\, d$

$\therefore$ *7th* term, n = 7; d =-3; a = 102
$\Rightarrow U_7 = a + (7 - 1)d = a + 6d = 102 + 6(-3) = 102 - 18 = 84$

Example

The 7th term of an A.P is 15 and the fourth term is 9. Find the sequence, first term and the common difference.

Solution

$\therefore$ The *nth* term of an A.P is $U_n = a + (n - 1)d$

$$\Rightarrow U_7 = a + (7-1)d = a + 6d = 15 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$\Rightarrow U_4 = a + (4-1)d = a + 3d = 9 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Equation (1) and (2) can be solved simultaneously.

To find ‘a’ and‘d’, equation (2) minus equation (1),

$$\Rightarrow (a+6d) - (a+3d) = 15 - 9$$
$$\Rightarrow 3d = 6 \Rightarrow d = 2$$

Substitute d = 2 into equation (2).

$$\Rightarrow a + 3(2) = 9 \Rightarrow a + 6 = 9 \Rightarrow a = 9 - 6 \Rightarrow a = 3$$
$$\therefore a = 3, d = 2$$

$\therefore$ The sequence is: 3, 5, 7, 9, 11, 13, 15, …

Sum of the First n-terms of an A.P

The sum of the sequence $a, a+d, a+2d \ldots a+(n-1)d$ is:

$$S_n = a + (a+d) + (a+2d) + \ldots + a + (n-1)d \ldots\ldots(1)$$
$$S_n = a + (n-1)d + \ldots + (a+2d) + a \ldots\ldots\ldots\ldots(2)$$

Summing equation (1) and (2), we have:

$$2a + (n-1)d + 2a + (n-1)d + \ldots + 2a + (n-1)d$$
$$2S_n = n[2a + (n-1)d]$$
$$2S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + u_n]$$

Where *S*, is the sum of the first n –terms of the A.P sequence.

3.3 Geometric Sequence (G. P)

A geometric sequence is a sequence in which each successive terms of the sequence are in equal ratio.

Let ‘a’ be the first term
Let ‘r’ be the common ratio.
The sequence becomes: a, ar, ar^2, ar^3, …..

If $U_1 = a$

$U_2 = ar^{2-1}$

$U_3 = ar^{3-1}$

$U_4 = ar^{4-1}$

$\vdots \qquad \vdots$

$U_n = ar^{n-1}$

The *nth* term is $U_n = ar^{n-1}$

Example

Find the common ration in each the following:

i. 2, 6, 18, 54, 162,…

ii. 1, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Solution

Let r be the common ratio

i. r = $\frac{6}{2} = \frac{18}{6} = 3$

ii. r = $\frac{\frac{1}{2}}{1} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4} x \frac{2}{1} = \frac{1}{2}$

Example

Find the value P in the sequence of G.P:
$\left(\sqrt{2}-1\right), \ \left(3+2\sqrt{2}\right), \left(5\sqrt{2}-7\right), P.$

Solution

Let r be the common ratio.

$\mathrm{r} = \dfrac{\mathrm{u}_2}{\mathrm{u}_1} = \dfrac{\mathrm{u}_3}{\mathrm{u}_2}$ ……………………………………………..……... (1)

$= \dfrac{3+2\sqrt{2}}{\sqrt{2-1}} = \dfrac{5\sqrt{2}-7}{\mathrm{P}}$ …………………………………… (2)

Now, solve (2) for P:

$$\mathrm{P}(3+2\sqrt{2} = (\sqrt{2}-1)(5\sqrt{2}-7)$$

$$\mathrm{P} = \frac{(\sqrt{2}-1)(5\sqrt{2}-7)}{3+2\sqrt{2}} = \frac{\sqrt{2}(5\sqrt{2}-7)-1(5\sqrt{2}-7)}{3+2\sqrt{2}}$$

$$= \frac{10-7\sqrt{2}-5\sqrt{2}+7)}{3+2\sqrt{2}}$$

$$= \frac{17-12\sqrt{2}}{3+2\sqrt{2}},$$

Multiplying the numerator and the denominator by the conjugate of the denominator, we have

$$= \left(\frac{17-12\sqrt{2}}{3+2\sqrt{2}}\right)x\left(\frac{3-2\sqrt{2}}{3-2\sqrt{2}}\right) = \frac{(17-12\sqrt{2})(3-2\sqrt{2})}{3^2-(2\sqrt{2})^2}$$

$$= \frac{1793-2\sqrt{2})-12\sqrt{2}(3-2\sqrt{2})}{9-8} = 51-34\sqrt{2}-36\sqrt{2}+48 = 99-70\sqrt{2}.$$

Hence, the sequence is: $\left(\sqrt{2}-1\right),\left(3+2\sqrt{2}\right),\left(5\sqrt{2}-7\right),\left(99-70\sqrt{2}\right)$

Series

A series is the term wise summation of a sequence.

Let $\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3 \ldots\mathrm{u}_\mathrm{n}$ be a sequence

$$\therefore u_1 + u_2 + u_3 + \ldots + u_n = \sum_{k=1}^{n} u_k$$

4.0 CONCLUSION

As in the summary

5.0 SUMMARY

This section introduces student to arithmetic and geometric progressions which is a good foundation for various methods of investment appraisal, with particular emphasis on discount, simple and compound interests ,commission and depreciation,

6.0 TUTOR-MARKED ASSIGNMENT

1) At what rate of interest will a single investment triple its value in 5years?
2) Calculate the present value of #5,000 at 10% p.a. for 4years.
3) In how many years will #2,000 amount to #10,200 at 5% p.a.compound intrest?

7.0 REFERENCES/FURTHER READINGS

1) Pure Mathematics for Advanced Level By B.D Bunday H Mulholland 1970.

2)Introduction to Mathematical Economics By Edward T. Dowling.

3)Mathematics and Quantitative Methods for Business and Economics.By Stephen P. Shao. 1976.

4) Mathematics for Commerce & Economics By Qazi Zameeruddin & V.K. Khanne 1995.

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UNIT 2 LIMITS

CONTENTS

1.0 INTRODUCTION

In this unit, you will learn about properties on limits, limits at infinity, discrete and continuous variable and how each of these could be used to solve problems.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- State the properties on limits,
- obtain limits at infinity,
- identify the variables that could affect limits,
- solve problems involving limits.

3.0 MAIN CONTENT

3.1 Definition

Limits describe what happen to a function *f(x)*, as its variable *x* approaches a particular number, say *c*. If the value *f(x)* is *k as x approaches c*, then, the number *k* is said to be the limit of the function *f(x)* as *x* approaches *c*.

This statement can be written mathematically as:

$\lim_{x \to c} = k$, or $\lim_{x \to c} f(x) = f(c)$

$\therefore f(c) = k.$

The symbol $x \to c$ means as *x* tends to *c*.

It should be note that, if the function *f(x)* does not get closer and closer to a number as *x* gets closer to *c*, then, it means that the function *f(x)* has no limit as $x \to c$.

Example 1

Consider the function: $f(x) = \left(\dfrac{x^2 - 9}{x - 3}\right)$ as $x \to 3$.

$$\therefore \lim_{x\to 3} = \left(\frac{x^2-9}{x-3}\right) = \left(\frac{3^2-9}{3-3}\right) = \frac{0}{0}.$$

Here the function *f(x)* is undefined as x = 3.

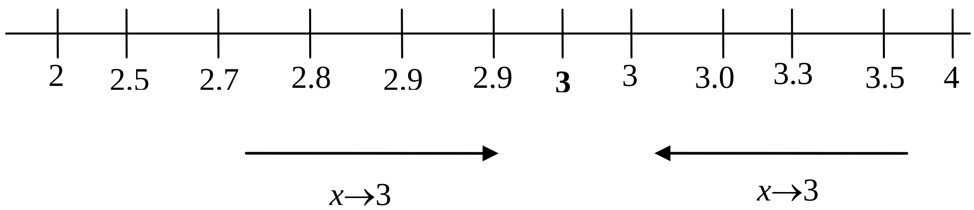
Mathematically speaking, the function is not defined as x = 3, but defined as x ≠ 3.
To obtain the limit, we go thus:

$$f(x) = \left(\frac{x^2-9}{x-3}\right) = \left(\frac{(x-3)(x+3)}{(x-3)}\right) = (x+3)$$

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} (x + 3) = 3 + 3 = 6$$

$$\therefore \lim_{x\to 3} \left(\frac{x^2-9}{x-3}\right) = 6$$

Considering values < 3 and > 3:



From Left **From Right**

x → 3	*F*(*x*)	x → 3	*F*(*x*)
2	5	4	7
2.5	5.5	3.5	6.5
2.8	5.8	3.3	6.3
2.9	5.9	3.001	6.001

2.9999	5.9999	3.00001	6.00001

5.9999 ≈ 6 and 6.00001 ≈ 6.

In conclusion $\lim_{x \to 3} (x + 3) = 6$.

Example 2

Find the $\lim_{x \to 0} \frac{|x|}{x}$

$|x|$ is called modulus of x or absolute value of x.

Definition $|x| = \begin{cases} -x \text{ for } x < 0 \\ 0 \text{ for } x = 0 \\ 0 \text{ for } x \geq 0 \end{cases}$

If x > 0, that is positive x.

$\therefore \quad f(x) = \frac{|x|}{x} = \frac{x}{x} = 1,$ positive sense

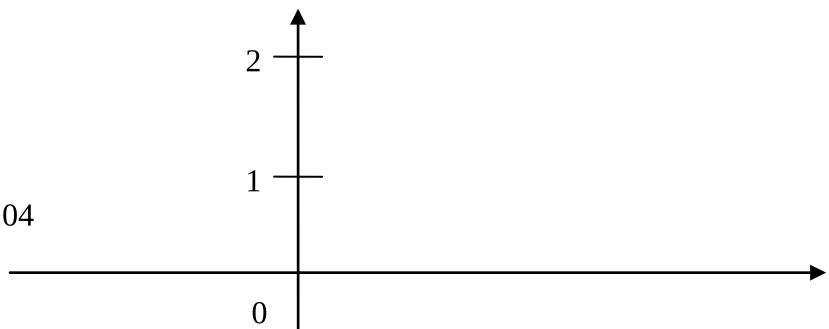
If x < 0, that is negative x.

$\therefore \quad f(x) = \frac{|x|}{x} = \frac{-x}{x} = -1,$ negative sense

If x = 0

$\therefore \quad f(x) = \frac{|x|}{x} = \frac{0}{x} = 0,$ zero.

Consider the graph of $f(x) = \frac{|x|}{x}$



3.2 Properties on Limits

Let $f(x)$, $g(x)$ and $h(x)$ be functions of x such that f, g and h are defined.

Let:

$\lim\limits_{x \to c} f(x) = L$

$\lim\limits_{x \to c} g(x) = K$

$\lim\limits_{x \to c} h(x) = P$

Then:

1. $\lim\limits_{x \to c} [f(x) + g(x) + h(x)] = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x) + \lim\limits_{x \to c} h(x)$

$$= \quad L \quad + \quad K \quad + \quad P$$

2. $\lim\limits_{x \to c} [Mf(x)] = M \lim\limits_{x \to c} f(x) = ML$, where M is a constant.

3. $\lim\limits_{x \to c} [f(x) + g(x) + h(x)] = \lim\limits_{x \to c} f(x) \, . \, \lim\limits_{x \to c} g(x) \, . \, \lim\limits_{x \to c} h(x) = L. \, K. \, P$

4. $\lim\limits_{x \to c} \left(\dfrac{f(x)}{g(x)} \right) = \dfrac{\lim\limits_{x \to c} f(x)}{\underset{x \to c}{imf}(x)} = \dfrac{L}{K}$, Provided $K \neq 0$

Example 1

Evaluate the following limits:

1) $\lim\limits_{x \to 2} (3x^2 + 1)$

2) $\lim_{x\to -3}\left(\dfrac{x-3}{x^2+5x+7}\right)$

3) $\lim_{x\to 6}\left(\sqrt{2(x^2-4)}\right)$

4) $\lim_{x\to -1}\left\{(x^2+1)(x-1)\right\}$

Solution

1) $\lim_{x\to 2}$ $(3x^2 + 1) = 3(2)^2 + 1 = 3(4) + 1 = 13.$

2) $\lim_{x\to -3}\left(\dfrac{x-3}{x^2+5x+7}\right) = \dfrac{-3-3}{(-3)^2+5(-3)+7} = \dfrac{-6}{9-15+7} = \dfrac{-6}{1} = -6$

3) $\lim_{x\to 6}\left(\sqrt{2(x^2-4)}\right) = \sqrt{2(6^2-4)} = \sqrt{2(36-4)} = \sqrt{2(32)} = \sqrt{68} = 8.$

4) $\lim_{x\to -1}\left\{(x^2+1)(x-1)\right\} = (-1^2+1)(-1-1) = (1+1)(-1-1) = 2(-2) = -4.$

Example 2

Find the limit as x $\to$ to zero o the following:

1. $\dfrac{x^4-1}{x^2+1}$

2. $\dfrac{(\mathrm{x}-1)^3-3\mathrm{x}+1}{\mathrm{x}^2}$

3. $\dfrac{(x+2)^3-8}{x}$

4. $\dfrac{(1-\mathrm{x})^2-1}{\mathrm{x}}$

5. $\dfrac{\sqrt{9+2\mathrm{x}-3}}{\mathrm{x}}$

Solution

1. $\lim_{x\to 0}\left(\dfrac{x^4-1}{x^2+1}\right) = \left(\dfrac{0-1}{0+1}\right) = \dfrac{-1}{1} = -1$

Alternatively, it could be solve analytically:

$$\lim_{x\to 0}\left(\frac{x^4-1}{x^2+1}\right)=\lim_{x\to 0}\left(\frac{(x^2)^2-(1)^2}{x^2+1}\right)=\lim_{x\to 0}\left(\frac{(x^2-1)(x^2+1)}{x^2+1}\right)=\lim_{x\to 0}(x^2-1)=0-1=-1$$

2. $$\lim_{x\to 0}\left(\frac{(x-1)^3-3x+1}{x^2}\right)=\left(\frac{(0-1)^3-3(0)+1}{0}\right)=\frac{-1+1}{0}=\frac{0}{0},\ \text{Undefined}$$

$$\therefore \lim_{x\to 0}\left(\frac{(x-1)^3-3x+1}{x^2}\right)=\lim_{x\to 0}\left(\frac{x^3-3x^3+3x-1-3x+1}{x^2}\right)=\lim_{x\to 0}\left(\frac{x^3-3x^2}{x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{x^2(x-3}{x^2}\right)=\lim_{x\to 0}(x-3)=0-3=-3$$

3. $$\lim_{x\to 0}\left(\frac{(x+2)^3-8}{x}\right)=\left(\frac{(0+2)^3-8}{0}\right)=\frac{0}{0}$$

$$\therefore \lim_{x\to 0}\left(\frac{(x+2)^3-8}{x}\right)=\lim_{x\to 0}\left(\frac{x^3-6x^3+12x+8-8}{x}\right)$$

$$=\lim_{x\to 0}\left(\frac{x(x^2-6x+12}{x}\right)=\lim_{x\to 0}(x^2-6x+12)=\{0-6(0)+12\}=12$$

4. $$\lim_{x\to 0}\left(\frac{(1-x)^2-1}{x}\right)=\left(\frac{(1-0)^2-8}{0}\right)=\frac{0}{0}$$

$$\therefore \lim_{x\to 0}\left(\frac{(1-x)^2-1}{x}\right)=\lim_{x\to 0}\left(\frac{1-2x+x^2-1}{x}\right)$$

$$=\lim_{x\to 0}\left(\frac{x(x-2}{x}\right)=\lim_{x\to 0}(x-2=0-2=-2$$

3.3 Limits at Infinity

Consider the function $f(x)$ = $\frac{1}{2^x}$ as x→ ∞:

- When x = 0, ⇒ $\frac{1}{2^0}=\frac{1}{1}=1$
- When x = 1, ⇒ $\frac{1}{2^1}=\frac{1}{2}=0.5$

- When x = 2, $\Rightarrow \frac{1}{2^2} = \frac{1}{4} = 0.25$
- When x = 3, $\Rightarrow \frac{1}{2^3} = \frac{1}{8} = 0.125$
- When x = 4, $\Rightarrow \frac{1}{2^4} = \frac{1}{16} = 0.0625$
- When x = 5, $\Rightarrow \frac{1}{2^5} = \frac{1}{32} = 0.03125$
- When x = 10, $\Rightarrow \frac{1}{2^{10}} = \frac{1}{1024} = 0.0009766$

We can see from above that as x tends to infinity, *f*(*x*) is getting to zero.

$$\therefore \lim_{x \to 0} \left(\frac{1}{2^x} \right) = \frac{1}{2^\infty} = \frac{1}{\infty} = 0$$

Because $2^\infty = \infty$

Example 1

Find $\lim_{x \to \infty} \left(\frac{6x}{x+1} \right)$

To evaluate this, divide by the highest power of x, the numerator and denominator respectively, then carry out the limit operation.

Solution

$$\lim_{x \to \infty} \left(\frac{6x}{x+1} \right) = \lim_{x \to \infty} \left(\frac{\frac{6x}{x}}{\frac{x}{x} + \frac{1}{x}} \right) = \lim_{x \to \infty} \left(\frac{6}{1 + \frac{1}{x}} \right)$$

$$= \left(\frac{6}{1 + \frac{1}{\infty}} \right) = \left(\frac{6}{(1+0)} \right) = 6$$

Example 2

Evaluate the limit as x$\rightarrow \infty$ of the following functions:

1. $\dfrac{3x^4 - 4x - 1}{5x^2 - 3x + 2}$

2. 7 - $\dfrac{2}{x}$

3. $\dfrac{5x^4 + 4x^3 - 2x^2 + 6}{2x - 15x^2 + 7x^5}$

4. $\dfrac{15x^4 + 12x^2 + 7}{6x^4 + 3x2}$

Solution

1. $$\lim_{x\to\infty}\left(\frac{3x^4 - 4x - 1}{5x^2 - 3x + 2}\right) = \lim_{x\to\infty}\left(\frac{\frac{3x^4}{x^4} - \frac{4x}{x^4} - \frac{1}{x^4}}{\frac{5x^2}{x^4} - \frac{3x}{x^4} + \frac{2}{x^4}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{3 - \frac{4x}{x^3} - \frac{1}{x^4}}{\frac{5}{x} - \frac{3}{x^3} + \frac{2}{x^4}}\right) = \left(\frac{3 - \frac{4x}{x^3} - \frac{1}{x^4}}{\frac{5}{\infty} - \frac{3}{\infty^3} + \frac{2}{\infty^4}}\right) = \frac{3-0-0}{0-0+0} = \frac{3}{0} = \infty$$

2. $$\lim_{x\to\infty}\left(7 - \frac{2}{x}\right) = \left(7 - \frac{2}{\infty}\right) = 7 - 0 = 7$$

3. $$\lim_{x\to\infty}\left(\frac{5x^4 + 4x^3 - 2x^2 + 6}{2x - 15x^2 + 7x^5}\right) = \lim_{x\to\infty} = \lim_{x\to\infty}\left(\frac{\frac{5x^4}{x^5} + \frac{4x^3}{x^5} - \frac{2x^2}{x^5} + \frac{6}{x^5}}{\frac{2x}{x^5} - \frac{15x^2}{x^5} + \frac{7x^5}{x^5}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\frac{5}{\infty} + \frac{4}{\infty^2} - \frac{2}{\infty^3} + \frac{6}{\infty^5}}{\frac{2}{\infty^4} - \frac{15}{\infty^3} + 7}\right) = \left(\frac{\frac{5}{x} + \frac{4}{x^2} - \frac{2}{x^3} + \frac{6}{x^5}}{\frac{2}{x^4} - \frac{15}{x^3} + 7}\right) =$$

$$\left(\frac{0+0-0+0}{0-0+7}\right) = \frac{0}{7} = 0$$

4. $\lim_{x\to\infty}\left(\dfrac{15x^4+12x^2+7}{6x^4+3x2}\right) = \lim_{x\to\infty}\left(\dfrac{\dfrac{15x^4}{x^4}+\dfrac{12x^2}{x^4}+\dfrac{7}{x^4}}{\dfrac{6x^4}{x^4}+\dfrac{3x}{x^4}}\right)$

$$= \lim_{x\to\infty}\left(\frac{15+\dfrac{12}{x^2}+\dfrac{7}{x^4}}{6+\dfrac{3}{x3}}\right) = \left(\frac{15+\dfrac{12}{x^2}+\dfrac{7}{x^4}}{6+\dfrac{3}{\infty^3}}\right)$$

$$= \left(\frac{15+0+0}{6+0}\right) = \frac{15}{6} = \frac{5}{2}$$

3.4 Discrete and Continuous Variable

We shall consider the idea of limit and continuity to the concept of a function $y = f(x)$.

Consider a function $f(x)$; $f(x)$is said to be continuous at the point $x = c$ if the following holds:

1. $f(x)$ exists as $x \to c$
2. If $\lim_{x\to c} f(x)$ exists
3. $\lim_{x\to c} f(x) = f(c)$

Example

Show that $f(x) = \dfrac{x^2-25}{x-5}$ is discontinuous at $x = 5$

Solution

$$\lim_{x\to 5}\left(\frac{x^2-25}{x-5}\right) = \lim_{x\to 5}\left(\frac{(x-5)(x+5)}{(x-5)}\right)\lim_{x\to 5}(x+5) = 5+5 = 10$$

But $f(5) = \dfrac{5^2-25}{5-5} = \dfrac{0}{0}$,undefined

Recall $\lim_{x\to 5} f(x) = f(5)$

$\therefore$ $f(x)$ is not continuous at $x = 5$.

Because $= f(s) \neq \lim_{x\to 5} f(x)$

3.4.1 Property of Continuity

If $f(x)$ and $g(x)$ are continuous at a point $x = c$, then:

1. $f(x) \pm g(x)$;
2. $f(x).\ g(x)$ are continuous at $x = c$;
3. Also $\frac{f(x)}{g(x)}$, quotient function is continuous at $x = c$, provided $g(x) \neq 0$ at $x = c$.

Example 1

Given the function $H(x) = \begin{cases} kx + 3 & \text{for } x < 2 \\ 3x^2 - x + 3 & \text{for } x > 2 \end{cases}$

Find the values of 'k' for which $H(x)$ is continuous for all real values of k.

Solution

$(kx + 3)$, $(3x^2 - x + 3)$ are polynomials, then $H(x)$ is continuous every where except at $x = 2$.

$\therefore \lim_{x\to 2} H(x) = kx + 3 = 2k + 3$

$\therefore \lim_{x\to 2} |x)| = 3x^2 - x + 3 = 3(2)^2 - 2 + 3 = 13$

$\lim_{x\to 2} |x)|$ exists if and only if $\lim_{x\to 2} H(x) = \lim_{x\to 2} |x)|$

$\Rightarrow 2k + 3 = 13 \Rightarrow 2k = 13 - 3 \Rightarrow 2k = 10 \Rightarrow k = 5$

$\therefore$ $H(x)$ is continuous at $x = 2$ if $k = 5$.

Example 2

OFN pig farms bid for the production and supply of 550kg piglet to U.A.C Nigeria Limited as follows:

The fixed cost of producing the first 50 piglets is ₦25,000.00 and the variable cost, each is ₦500.00.

However, the variable cost per piglet of producing the next 50 piglets is ₦250.00 with a fixed cost of ₦30,000.00, while the variable cost per piglet for producing between 100 and 200 piglets is ₦ 20.00 with a fixed cost of ₦35,000.00.

If a provision of ₦5,000.00 is made fro casualties and other necessities, find:

i. A functional model that describes the cost of producing the piglets;
ii. The points of discontinuity of the cost function; and
iii. The average cost of producing 50, 100 and 200 copies respectively.

Solution

You will recall that the fixed cost is the cost that does not vary with the level of production.

Now, let *y* be the number of the piglets to be produced and supplied. The cost of producing / supply of the first 50 piglets is:

300y + 25,000 ………………………………………………… (1)

The cost of producing / supply of the next 50 piglets is:

250y + 30,000 …………………………………………………… (2)

The cost of producing between 100 and 200 piglets is:

120y + 35,000 ……………………………………………………. (3)

$$f(xy) = \begin{cases} 300y + 25{,}000 & \text{for } 0 \le y \le 5\ 0 \\ 250y + 30{,}000 & \text{for } 50 \le y \le 100 \\ 120y + 35{,}000 & \text{for } 100 \le y \le 200 \end{cases}$$

Therefore, the points of discontinuity are 50 and 100, since the points break the domain of the C(y) into three (3) intervals.

Thus (i):

- $\lim_{y \to 50} C(y)$ =#300(50)+#25,000 =#40,000

- $\lim_{y \to 50} C(y)$ =#2500(50)+#30,000 =#42,500
- $\lim_{y \to 50} C(y)$ =#250(100)+#30,000 =#55,000
- $\lim_{y \to 50} C(y)$ =#120(100)+#35,000 =#47,000

Note that $\lim_{y \to 50^-} C(y) \neq \lim_{y \to 50^+} C(y)$, also $\lim_{y \to 100^-} C(y) \neq \lim_{y \to 100^+} C(y)$,

(ii) Therefore, the function is discontinuous at y = 50 and y = 100.

(iii) Average cost = $\dfrac{Total\ Cost}{Number\ of\ Piglets}$

- Average cost of producing 50 piglets

$$= \frac{Total\ Cost\ of\ producing\ 50\ piglets}{50}$$
$$= \frac{300(50) + 25{,}000}{50} + \frac{40{,}000}{50}$$
$$= ₦800$$

- Average cost of producing 100 piglets

$$= \frac{\text{Total Cost of producing 100 piglets}}{100}$$
$$= \frac{250(100) + 30{,}000}{100} + \frac{55{,}000}{100}$$
$$= ₦550$$

- Average cost of producing 50 piglets

$$= \frac{Total\ Cost\ of\ producing\ 200\ piglets}{200}$$
$$= \frac{120(200) + 35{,}000}{200} + \frac{59{,}000}{200}$$
$$= ₦295$$

4.0 CONCLUSION

In conclusion, all polynomial functions are continuous, as are all rational functions, except where undefined, i.e. where their denominators are zero. Consequently, for any function to be continuous, the limit must exist.

5.0 SUMMARY

We discussed about limit, if the functional values f(x) of a function f draws closer to one and only one finite real number L for values of x as x draws closer to a from both sides, but does not equal a, L is defined as the limit of f(x) as x approaches a is written

$$\lim_{x \to a} f(x) = L$$

Furthermore, we discussed continuity. A continuous function is one which has no breaks in its curve. It can be drawn without lifting the pencil from the paper. A function is continuous at x = a if the three properties mentioned above exists.

Practical examples related to business were illustrated.

6.0 TUTOR-MARKED ASSIGNMENT

Evaluate the following limits:

1) $\lim_{x \to 2} (5x^2 - 21)$

2) $\lim_{x \to 7} \left(\dfrac{x+7}{x^2 + 4x - 21} \right)$

3) $\lim_{x \to 7} \left(\sqrt{2(x^2 - 17)} \right)$

4) $\lim_{x \to -1} \{(x^4 - 1)\}$

Evaluate the limit as $x \to \infty$ of the following functions:

1. $\dfrac{3x^5 - 4x^2 - x + 1}{5x^4 - 3x + 2}$

2. $\dfrac{5x^7 + 4x^3 - 2x^2 + 6}{2x - 15x^7 + 7x^5}$

3. Show that $f(x) = \dfrac{x^2 - 81}{x - 9}$ is discontinuous at $x = 9$

7.0 REFERENCES/FURTHER READINGS

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UNIT 3 DIFFERENTIATION AND INTEGRATION

CONTENTS

1.0 INTRODUCTION

Differentiation could be said to be the inverse of integration .this shall be seen as we continue in this section.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- set out the principles of differentiation and integration
- explain their application to practical and business situations respectively.

3.0 MAIN CONTENT

3.1 Definition of Differentiation

$\lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}\right)$, If it exists, is called the derivative of f with respect to x.

We shall denote $\lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}\right)$ by $f'(x)$ called f prime of x.

Therefore, the process of finding $f'(x)$ is called differentiation.

It should be noted that $\lim_{\partial x \to 0}\frac{\delta y}{\delta x} = f'(x)$.

Example

Find $f'(x)$ for the following functions from the first principle.

1. $f(x) = x^2$
2. $f(x) = x^3$

Solution

1. $f(x) = x^2$

$$f'(x)\ \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}\right)$$

$$\therefore\ f(x+\Delta x) = (x+\Delta x)^2 = x^2 + 2x\Delta\bar{x} + \Delta\bar{x}^2,\ f(x) = x^2$$

$$f(x+\Delta x) - f(x) = x^2 + 2x\Delta\bar{x} + \Delta\bar{x}^2 - x^2 = 2x\Delta\bar{x} + \Delta\bar{x}^2$$

$$\therefore f'(x)\ \lim_{\Delta x \to 0}\left(\frac{2x\Delta\bar{x} + \Delta\bar{x}^2}{\Delta\bar{x}}\right) = \lim_{\Delta x \to 0}\left(\frac{2x\Delta\bar{x}}{\Delta\bar{x}} + \frac{\Delta\bar{x}^2}{\Delta\bar{x}}\right) = \lim_{\Delta x \to 0}(2x+\Delta x)$$

$= 2x + 0 = 2x$

2. $f(x) = x^3$

$f'(x) \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right)$

$\therefore\ f(x+\Delta x) = (x+\Delta x)^3 = x^3 + 3x\Delta \bar{x}^2 + \Delta \bar{x}^3,\ f(x) = x^3$

$f(x+\Delta x) - f(x) = x^3 + 3x^2\Delta \bar{x} + 3x\Delta \bar{x}^2 + \Delta \bar{x}^3 - x^3 = 3x^2\Delta \bar{x} + 3x\Delta \bar{x}^2 + \Delta \bar{x}^3$

$\therefore f'(x) \lim_{\Delta x \to 0} \left(\frac{3x^2\Delta \bar{x} + 3x\Delta \bar{x}^2 + \Delta \bar{x}^3}{\Delta \bar{x}} \right) = \lim_{\Delta x \to 0} \frac{3x^2\Delta \bar{x}}{\Delta \bar{x}} + \frac{3x\Delta \bar{x}^2}{\Delta \bar{x}} + \frac{\Delta \bar{x}^3}{\Delta \bar{x}}$

$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta \bar{x} + \Delta \bar{x}^2) = 3x^2 + 0 + 0 = 3x^2$

Remarks

$f(x) = x^2 \Rightarrow f'(x) = 2x$

$f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$f(x) = x^4 \Rightarrow f'(x) = 4x^3$

$\vdots \qquad\qquad \vdots$

$f(x) = ax^n \Rightarrow f'(x) = anx^{n-1}$

This is called the power rule of differentiation.

3.2 Techniques of Differentiation

- <u>Power Rule</u>

Let $f(x) = ax^n$, where a, and n are constants.

$\therefore\ f'(x) = anx^{n-1}$

Example

If $f(x) = 5x^4$,

Then $f'(x) = \frac{\delta y}{\delta x} = 5(4)\, x^{4-1} = 20x^3$

- <u>Sum of Functions</u>

$f(x) = y = v(x) \pm u(x)$

$$f'(x) = \frac{\delta y}{\delta x} = \frac{\delta}{\delta x}v(x) \pm \frac{\delta}{\delta x}u(x)$$

Example 1

If $y = 3x^5 + 4x^2 - 5x + 3$

$\therefore \frac{\delta y}{\delta x} = \frac{\delta}{\delta x}3x^5 + \frac{\delta}{\delta x}4x^2 - \frac{\delta}{\delta x}5x + \frac{\delta}{\delta x}3$

$= 3(5)x^{5-1} + 4(2)x^{2-1} - 5(1)x^{1-1} + 0$

$= 15x^4 + 8x - 5$

Note that the derivative of a constant is equal to zero.

Example 2

If $y = c,$ where c is a constant

To find $\frac{\delta y}{\delta x}, \Rightarrow y = cx^0$

$\therefore \frac{\delta y}{\delta x} = c(0)x^{0-1} = 0$

Example 3

If $y = 32 \Rightarrow y = 32x^0$

$\therefore \frac{\delta y}{\delta x} = 32(0)x^{0-1} = 0$

- <u>Product Rule</u>

  Suppose that $u(x)$ and $v(x)$ are differentiable functions of x.

  $\therefore y(x) = v(x)u(x)$

  $\therefore \frac{\delta y}{\delta x} = \frac{\delta}{\delta x}(vu) = v(x)\frac{\delta u}{\delta x} + u(x)\frac{\delta v}{\delta x}$

Example

Find the derivative of the following:

i. $(x^3 + 2)(x^2 - 1)$

ii. $\sqrt{x}(x+1)$

Solution

i. If $y = (x^3 + 2)(x^2 - 1)$.

Let $u = (x^3 + 2) \Rightarrow \frac{\delta u}{\delta x} = 3x^2, v = (x^2 - 1) \Rightarrow \frac{\delta v}{\delta x} = 2x$

$$\therefore \frac{\delta y}{\delta x} = \frac{\delta}{\delta x}(uv) \Rightarrow u(x)\frac{\delta v}{\delta x} + v(x)\frac{\delta u}{\delta x} = (x^3 + 2).2x + (x^2 - 1).3x^2$$

$$= 2x^4 + 4x + 3x^4 - 3x^2 = 5x^4 - 3x^2 + 4x$$

Alternatively, we can expand the function and use the rule of sum i.e.

$$y = (x^3 + 2)(x^2 - 1)$$

$$= x^3(x^2 - 1) + 2(x^2 - 1) = x^5 - x^3 + 2x^2 - 2$$

$$\therefore \frac{\delta y}{\delta x} = \frac{\delta}{\delta x}x^5 - \frac{\delta}{\delta x}x^3 + \frac{\delta}{\delta x}2x^2 - \frac{\delta}{\delta x}2 = 5x^4 - 3x^2 + 4 - 0$$

$$= 5x^4 - 3x^2 + 4$$

ii. If $y = \sqrt{x}(x+1)$

Let $u = \sqrt{x} \Rightarrow \frac{\delta u}{\delta x} = \frac{1}{2}x^{\frac{1}{2}} = \frac{1}{2\sqrt{x}}, v = (x+1) \Rightarrow \frac{\delta v}{\delta x} = 1$

$$\therefore \frac{\delta y}{\delta x} = \frac{\delta}{\delta x}(uv) \Rightarrow u(x)\frac{\delta v}{\delta x} + v(x)\frac{\delta u}{\delta x} = \sqrt{x}.(1). + (x+1).\frac{1}{2\sqrt{x}}$$

$$= \sqrt{x} + \frac{(x+1)}{2\sqrt{x}} = \frac{2x + (x+1)}{2\sqrt{x}} = \frac{3x+1}{2\sqrt{x}}$$

- <u>Quotient Rule</u>

Suppose that *u* and *v* are differentiable functions of *x* such that.

$$y(x) = \frac{u(x),}{v(x)} \text{ Where } v(x) \neq 0$$

$$\therefore \frac{\delta y}{\delta x} = \frac{vx\frac{\delta u}{\delta x} - u(x)\frac{\delta v}{\delta x}}{[v(x)]^2}$$

Example 1

Find the derivative of the following

1. $\dfrac{3x^2 + 2x}{x^3 - 2x^2}$

2. $\dfrac{x^3 + 27}{(x+3)}$

Solution

1. If $y = \dfrac{3x^2 + 2x}{x^3 - 2x^2}$

Let $u = 3x^2 + 2x \Rightarrow \dfrac{\delta u}{\delta x} = 6x + 2, v = x^3 - 2x^2 \Rightarrow \dfrac{\delta v}{\delta x} = 3x^2 - 4x$

$$\therefore \frac{\delta y}{\delta x} = \frac{[(x^3 - 2x^2).(6x+2)] - [3x^2 + 2x).(3x^2 - 4x)\}}{[x^3 - 2x^2]^2}$$

$$= \frac{-3x^4 - 4x^3 + 8x^2}{[x^3 - 2x^2]^2} = \frac{-x^2[3x2 + 4x - 8]}{x^2[x^2 - 2x^2]^2} = \frac{-[3x^2 + 4x - 8]}{[x^2 - 2x^2]^2}$$

2. If $y = \dfrac{(x^3 + 27)}{(x+3)}$

Let $u = (x^3 - 27) + 2x \Rightarrow \dfrac{\delta u}{\delta x} = 3x^2, v = (x+3) \Rightarrow \dfrac{\delta v}{\delta x} = 1$

$$\therefore \frac{\delta y}{\delta x} = \frac{(x+3).3x^2 - (x^3 - 27).(1)}{[x+3]^2} = \frac{3x^3 + 9x^2 - x^3 + 27}{[x+3)^2}$$

$$= \frac{2x^3 + 9x^2 + 27}{x^2 + 6x + 9}$$

- Functions of a Function

Consider the functions:

1. $y = (x^3 - 1)^2$
2. $y = (x^2 - 3x + 2)^{40}$

Find $\dfrac{\delta y}{\delta x}$

Solution

1. If $y = (x^3 - 1)^2$

Let $u = x^3 - 1 \Rightarrow \frac{\delta u}{\delta x} = 3x^2$

$\therefore y = u^2, \frac{\delta y}{\delta x} = 2u$

$\therefore \frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} . \frac{\delta u}{\delta x} = 2u.3x^2 = 6x^2 u$

Now, substitute *u*:

$\Rightarrow 6x^2(x^3 - 1)$

2. If $y = (x^2 - 3x + 2)^{40}$

Let $u = x^2 - 3x + 2x \Rightarrow \frac{\delta u}{\delta x} = 2x - 3$

$\therefore y = u^{40}, \frac{\delta y}{\delta u} = 40u^{39}$

$\therefore \frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} . \frac{\delta u}{\delta x} = 40u^{39}.(2x - 3) = 40(2x - 3)(x^2 - 3x + 2)^{39}$

3.2.1 Differentiation of Trigonometric Functions

Find the derivative of the following functions:

i. sin *x*
ii. cos *x*
iii. tan *x*

Solution

i. If $y = \sin x, then \frac{\delta y}{\delta x} = \cos x$

ii. If $y = \cos x, then \frac{\delta y}{\delta x} = \sin x$

iii. If $y = \tan x, \Rightarrow y = \frac{\sin x}{\cos x}$ (using Quotient rule).

$$\therefore \frac{\delta y}{\delta x} = \frac{\cos x \frac{\delta}{\delta x} \sin x - \sin x \frac{\delta}{\delta x} \cos x}{[\cos x]^2} = \frac{\cos x \cos x - \sin x(-\sin x)}{[\cos x]^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 = \sec^2 x$$

Example

Find $\frac{\delta y}{\delta x}$ of the function $y = \cos ecx$

Solution

$$y = \cos ecx = \frac{1}{\sin x}$$

Let $u = 1 \Rightarrow \frac{\delta u}{\delta x} = 0, v = \sin x \Rightarrow \frac{\delta v}{\delta x} = \cos x$

$$\therefore \frac{\delta y}{\delta x} = \frac{\sin x, (0) - (1), \cos x}{[\sin x]^2} = \frac{0 - \cos x}{\sin^2 x} = \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$$= -\cot anx.\cos ecx$$

3.2.2 Differentiation of Inverse Trigonometry Function

The inverse trigonometry function can be differentiated by using the basic trigonometry function and differentiating it term be term.

Example

If $y = \sin^{-1} x$, meaning arc sin *x*. Find $\frac{\delta y}{\delta x}$

Solution

$$y = \sin^{-1} x \Rightarrow \sin y = x$$

$$\therefore \cos y \frac{\delta y}{\delta x} = 1 \Rightarrow \frac{\delta y}{\delta x} = \frac{1}{\cos y}$$

Now write cos *y* in terms of sin *y*

$$\therefore \sin 2x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x \Rightarrow \cos x = \sqrt{1 - \sin^2 x}$$

$$\therefore \frac{\delta y}{\delta x} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

SELF ASSESSMENT EXERCISE

Try $y = \cos^{-1} x$ in the same way

3.2.3 Differentiation of Exponential Functions

If given that $y = \ell^x$, i.e. exponential function of *x*, then $\frac{\delta y}{\delta x} = \ e$

But, if $y = \ell^{ax}$, where '*a*' is a constant

Let $u = ax, \frac{\delta y}{\delta x} = a$:then $y = \ell^u, \frac{\delta y}{\delta x} = \ell^u$

Applying functions of a function:

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} . \frac{\delta u}{\delta x} = \ell^u, a = a\ell^u \ = a\ell^{ax}$$

Example

If $y = \ell^{5x^3}$, find $\frac{\delta y}{\delta x}$

Solution

Let $u = 5x^3, \frac{\delta u}{\delta x} = 15x^2$; then $y = \ell^u, \frac{\delta y}{\delta x} = \ell^u$

$$\therefore \frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} . \frac{\delta u}{\delta x} = \ell^u .15x^2 \ = 15x^u \ell^{5x^3}$$

3.2.4 Differentiation of Logarithmic Functions

Suppose

$$y = \log x$$

Taking the derivatives of both sides

$$\frac{dy}{dx} = \frac{1}{x}$$

Suppose

$$y = \log x^2$$

$$\frac{dy}{dx} = \frac{2x}{x^2} = \frac{2}{x}$$

Example

Find the derivative of the function $\log_\ell x^4$

Solution

Let $y = \log_\ell x^4$

Let $u = x^4 \Rightarrow \dfrac{du}{dx} = 4x^3$; then $y = \log_\ell u \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{u}$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{du}.\dfrac{du}{dx} = \dfrac{1}{u}.4x^3 = \dfrac{4x^3}{u} = \dfrac{4x^3}{x^4}$

3.3 Definition of Integration

Integration is the inverse process of differentiation.

If $f(x)$ is a differentiable function of x such that $\dfrac{d}{dx}[f(x)] = g(x)$

Then $f(x)$ is called the integral of $g(x)$ with respect to x.

Then, $\int df(x) = \int g(x)dx.$

$\Rightarrow f(x) = \int g(x)dx,$ Where $g(x)$ is called the integrand

Example

1. If $\dfrac{d}{dx}x^3 = 3x^2 \Rightarrow x^3 = \int 3x^2 dx$

2. If $\dfrac{d}{dx}x^2 = 2x \Rightarrow x^2 = \int xdx$

Therefore in general:

$\dfrac{d}{dx}(au^n) = anx^{n-1} \Rightarrow ax^n = \int anx^{n-1}dx$

3.3.1 Properties of Integration

1) $\int [f(x) \pm g(x) \pm h(x)]dx = \int f(x)dx \pm \int g(x)dx \pm \int h(x)dx.$
Where $f(x), g(x)$ and $h(x)$ are differentiable of x.

2) If c is a constant or a real number
$\therefore \int cf(x)dx = c\int f(x)dx$
However, an arbitrary constant can always be added to the result.

Hence, $\int cf(x)dx = g(x) + c$
This is known as the indefinite integral of $f(x)$.

Example 1

Find the integral of the following:

a) $\int (2x^4 + 3x^2 + 5x)dx$

b) $\int 2x^4 dx$

Solution

a) $\int \left(2x^4 + 3x^2 + 5x\right)dx = \int 2x^4 dx + \int 3x^2 dx + \int 5x dx$

$= \frac{2x^5}{5} + \frac{3x^3}{3} + \frac{5x^2}{2} + c = \frac{2}{5}x^3 + \frac{5}{2}x^2 + c$

b) $\int 2x^4 dx = 2\int x^4 dx = \frac{2x^5}{5} + c = \frac{2}{5}x^5 + c$

Example 2

If $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c,$ find the intergal of $\int \frac{3\text{x}^2}{x^3}dx$

Solution

Recall that $\frac{d}{dx}x^3 = 3x^2$

$\therefore f(x) = x^3,\ f'(x) = 3x^2$

$\int \frac{3x^2}{x^3}dx = \int \frac{\frac{\delta}{dx}x^3}{x^3}dx = 1n\left|x^3\right| + c$

Also, $\int \frac{1}{x}dx = 1n\ x + 1n\ c = 1n\ xc$

3.3.2 Integration of Functions of a Function

To find the integral of $\int (ax+b)^n dx$

Let $u = (ax+b) \Rightarrow \frac{du}{dx} = a \Rightarrow dx = \frac{du}{a}$

$\therefore \int (ax+b)^n dx = \int u^n \frac{du}{a} = \frac{1}{a}\int u^n du = \frac{1}{a}\frac{u^{n+1}}{n+1} + c$

Example

Evaluate $\int (3x+2)^5\, dx$

Solution

Let $u = (3x+2) \Rightarrow \frac{du}{dx} = 3 \Rightarrow dx = \frac{du}{3}$

$\therefore \int (3x+3)^5\, dx = \int u^5 \frac{du}{3} = \frac{1}{3}\left[\frac{u^6}{6}\right] + c = \frac{1}{18}(3x+2)^6 + c$

3.3.3 Integration as Summation

To find the area between the curve $y = f(x)$ and the x-axis, we first find indefinite integral $\int f(x)dx.$

We then substitute $x = b$ and $x = a$ respectively in the definite integral and subtract the result.

The notation adopted for this definite integral is written as: $\int_{x=a}^{x=b} f(x)dx.$

If $\int_{x=a}^{x=b} f(x)dx = g(x)\Big|_a^b = g(b) - g(a)$

Example

Evaluate the following integral:

1. $\int_2^3 x^3 dx$
2. $\int^3 \{(x+1)(x^2-1)\}dx$

Solution

1. $\int_2^3 x^3 dx = \frac{x^4}{4}\Big|_2^3 = \frac{1}{4}\left[(3)^4 - (2)^4\right] = \frac{1}{4}[81-16] = \frac{1}{4}[65] = \frac{65}{4}$ Unit.

2. $\int_1^3 \{(x+1)(x^2-1)\}dx$

$= \int_1^3 \{x(x^2-1) + 1(x^2-1)\}dx = \int_1^3 (x^3 - x + x^2 - 1)dx$

$= \int_1^3 (x^3 + x^2 - x - 1)dx = \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} - x\Big|_1^3$

$= \left[\frac{3^4}{4} + \frac{3^3}{3} - \frac{3^2}{2} - 3\right] - \left[\frac{1^4}{4} + \frac{1^3}{3} - \frac{1^2}{2} - 1\right]$

$$=\left[\frac{81}{4}+\frac{27}{3}-\frac{9}{2}-\frac{3}{1}\right]-\left[\frac{1}{4}+\frac{1}{3}-\frac{1}{2}-\frac{1}{1}\right]$$

$$=\left[\frac{(81-1)}{4}+\frac{(27-1)}{3}-\frac{(9-1)}{2}-\frac{(3+1)}{1}\right]-\left[\frac{80}{4}+\frac{26}{3}-\frac{8}{2}-2\right]$$

$$=\left[20-6+8.667\right]=\left[22.667\right]\text{ Unit.}$$

3.4 Application of Integration to a Business Problem

The management of LASU is considering the purchase of one of the three annexes of hers in three years time when funds become available. Their mathematician has analyzed each concern's financial position and calculates that the expressions for net profit over the next ten year for the institutions are as follows:

Let y = Net profit (₦'000).
x = Year in question from 1 – 10.

a) Anthony campus, $y=20+12x-x^2$.

b) Jibowu campus, $y=\frac{120}{x^2}+16x-1$.

c) Isolo campus, $y=6x+30$.

Calculate the total net profit that each firm should make from the end of the third year to the end of the tenth of forecasts.

Solution

a) $$\int^{10}(20+12x-x^2)dx=\left[20x-6x^2-\frac{x^3}{2}\right]_3^{10}$$
$$=(200+600-333)-(60+54-9)$$
$$=467-105$$
$$=362.$$

b) $$\int^{10}\left(\frac{120}{x^2}+16x-1\right)dx=\left[-\frac{120}{x}+8x^2-x\right]_3^{10}$$
$$=(-12+800-10)-(-40+72-3)$$
$$=778-29$$
$$=749.$$

$\therefore$ Total net profit for Jibowu campus is ₦749,000.

c) $\int_3^{10}(6x+30)dx = \left[3x^2+30^2\right]_3^{10}$

$$= (300+300)-(27+90)$$
$$= 600-17$$
$$= 483.$$

$\therefore$ Total net profit for Isolo campus is ₦483,000.

4.0 CONCLUSION

As in summary

5.0 SUMMARY

Basic rules of differentiation was discussed, the rules were applied to their sums and differences, products, quotients and function of functions. Basic rule of integration and its application to business was also discussed.

6.0 TUTOR-MARKED ASSIGNMENT

Find the following integral:

1 i. $\int (x^2+5x+2)dx$ ii $\int (x+2)^3\,dx$

iii. $\int \frac{1}{(1-x)}dx$ iv. $\int \frac{1}{(x^2-4)}\,dx$

2. i. $\int_2^6 \left\{\frac{x^2+3}{x^2}\right\}dx$ ii. $\int_0^1 (2x+5)^5\,dx$

iii. $\int^3 \{(2x+1)(x-1)\}dx$ iv. $\int_0^3 \left\{\frac{x^2-9}{x+3}\right\}dx$

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1) Pure Mathematics for Advanced Level By B.D Bunday H Mulholland 1970.

2)Introduction to Mathematical Economics By Edward T. Dowling.

3)Mathematics and Quantitative Methods for Business and Economics.By Stephen P. Shao. 1976.

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7) Introduction to Mathematical Economics SCHAUM'S Out lines

UNIT 4 MAXIMUM AND MINIMUM POINTS AND VALUE

CONTENTS

1.0 INTRODUCTION

This section introduces us to further application of differentiation.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- apply the principles of differentiation,
- explain their application in practical problems.

3.0 MAIN CONTENT

3.1 Stationary Points

These are point on the curve at which $\frac{dy}{dx} = 0$, and the value of the function at this point is called the stationary value. They are also called turning points.

3.2 How to Find Stationary Points

i. Differentiate the function once;
ii. Equate the derivate to zero, and lastly;
iii. Solve for the unknown.

Example 1

Find the stationary points of the function $y = x^3 - 12x + 15$

Solution

Differentiate the function $\frac{dy}{dx} = 3x^2 - 12.$

To equate $\frac{dy}{dx}$ to zero, we have $3x^2 - 12 = 0.$

$\Rightarrow x^2 - 4 = 0, \Rightarrow (x - 2(x + 2) = 0.$

$\Rightarrow x = 2$ or $x = -2$

$\therefore$ The stationary points are 2 and -2

Try this in the same way:

Example 2

Find the stationary points of the function $4x^3 + 15x^2 + 15x + 7.$

Solution

$y = 4x^3 + 15x^2 + 15x + 7.$

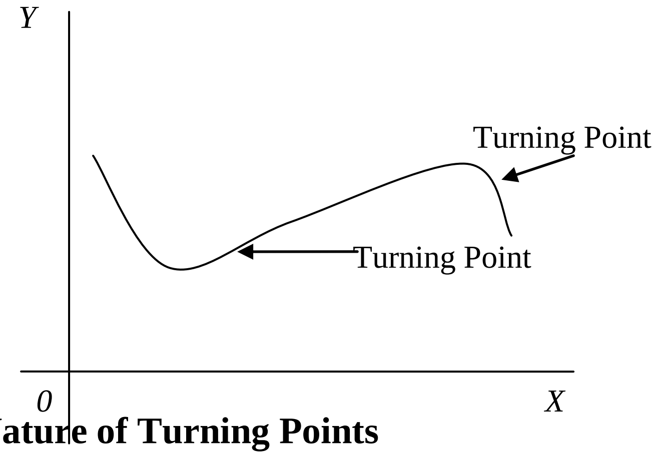
$\frac{dy}{dx} = 12x^2 + 30 + 18.$

At turning point: $\frac{dy}{dx} = 0,$

$\because 12x^2 + 30x + 18 = 0 \Rightarrow 2x^2 + 5x + 3 = 0 \Rightarrow (2x + 2)(x + 1) = 0.$

$\Rightarrow x = -\frac{3}{2}$ or $x = -1$

$\therefore$ The turning points are $-\frac{3}{2}$ or -1 or -1.



3.3 Nature of Turning Points

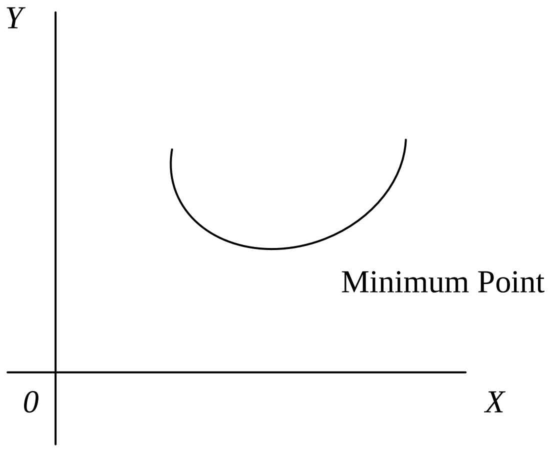
Turning points can be maximum or minimum.The point can be determined by:

i. Find the second derivative of the function; and
ii. Find the value of each of the turning point in the second derivative.

Remark

i. If the value of turning point in the second derivative is negative, then it is a maximum point:
$\therefore \frac{\delta^2 y}{\delta x^2} < 0$

ii. If the value of turning point in the second derivative is positive, then it is a minimum point:
$\therefore \frac{\delta^2 y}{\delta x^2} > 0$



3.4 Maximum and Minimum Value

The value of the original function $y = f(x)$ at the maximum point is the maximum value of the function, while the value of the function at the minimum point gives the minimum value.

Example 3

Find the maximum and minimum value of the following:

1. $y = x^3 - 2x^2 - 4x + 7$
2. $y = 4x^3 - 27x^2 + 60 + 15$

Solution

1. $y = x^3 - 2x^2 - 4x + 7$

$$\frac{dy}{dx} = 3x^2 - 4x - 4$$

At turning point:

$$\frac{dy}{dx} = 0, \Rightarrow 3x^2 - 4x - 4 = 0 \Rightarrow (x-2)(x+\frac{2}{3}) = 0,$$

$$\Rightarrow x = 2 \text{ or } x = -\frac{2}{3}$$

The turning points are 2 or $-\frac{2}{3}$, .

To find the nature of the turning point, the second derivatives i.e. $\frac{d^2y}{dx^2}$

$$\frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(3x^2 - 4x - 4),$$

$$\frac{d^2y}{dx^2} = 6x - 4.$$

Put x = 2 into $\frac{d^2y}{dx^2}$,

$$\therefore \frac{d^2y}{dx^2} = 6x - 4 \Rightarrow 6(2) - 4 = 8.$$

Also, put x = $-\frac{2}{3}$, into $\frac{d^2y}{dx^2}$,

$$\therefore \frac{d^2y}{dx^2} = 6x - 4 \Rightarrow 6(-\frac{2}{3}) - 4 = 8.$$

Remark

- At second derivative, 8> -8,
- At turning point 2, the function is maximum.
- At turning point $-\frac{2}{3}$, the function is minimum.

Turning Points	Value at $\frac{\delta^2 y}{\delta x^2}$,	Conclusion
2	8	Maximum
$-\frac{2}{3}$	-8	Minimum

SELF ASSESSMENT EXERCISE

Try the second example in the same way.

3.5 Point of Inflexion

This is the point at which the curve is neither a maximum nor a minimum value. At point of inflexion, the turning points are equal (a double stationary point).

At point of inflexion, the value of the stationary point at the second derivative is zero(0).

Example 4

Find the maximum and minimum value of the function $y = x^3 + 6x^2 + 12x + 15.$

Solution

$$y = x^3 + 6x^2 + 12x + 15.$$
$$\frac{dy}{dx} = 3x^2 + 12x + 12$$

At turning point:

$$\frac{dy}{dx} = 0, \Rightarrow 3x^2 + 12x + 12 = 0 \Rightarrow x^2 + 4x + 4 = 0$$
$$\Rightarrow (x+2)^2 = 0 \Rightarrow (\mathrm{x}+2)(\mathrm{x}+2) = 0$$
$$\Rightarrow x = -2\,(\text{twice})$$

Then,

$$\frac{d^2y}{dx^2} = 6x + 12,$$

Substitute x = -2, into $\frac{d^2y}{dx^2}$

Since the points (-2, -2)

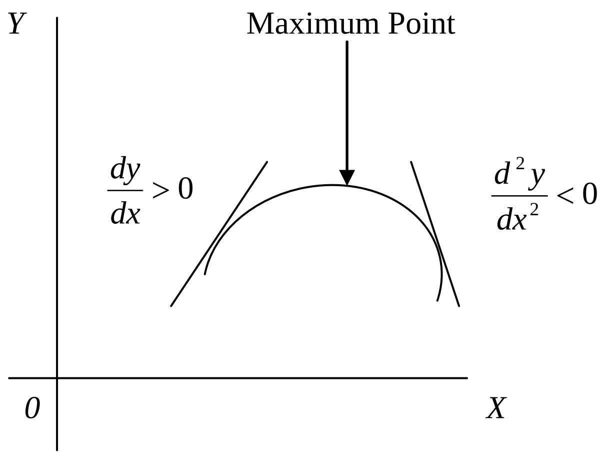
and $\frac{d^2y}{dx^2}(-2) = 0$ (twice).

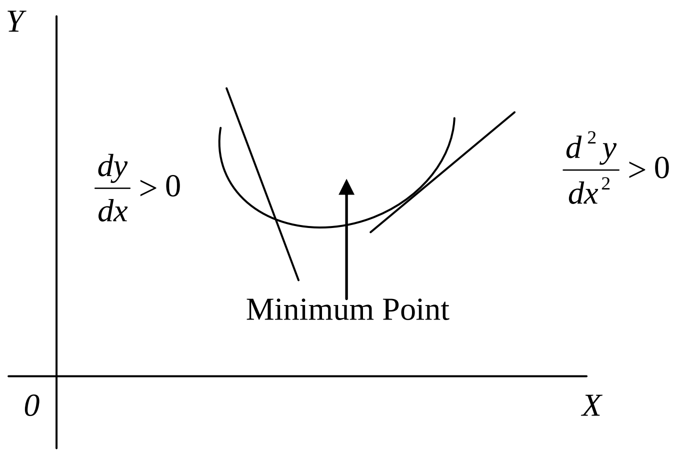
This shows that it is a point of inflexion

SELF ASSESSMENT EXERCISE

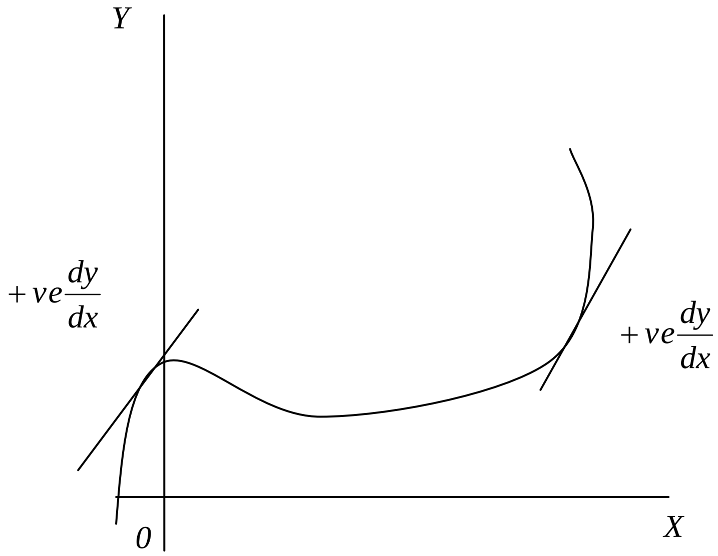
Find the maximum and minimum of the function $y = 4x^3 - 6x^2 + 3x + 14$.

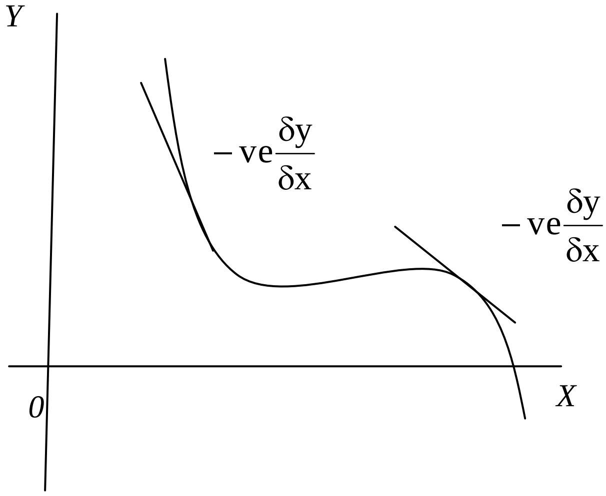
In Summary





But at point of inflexion, $\frac{dy}{dx}$ are same.





3.6 Applications of Maxima and Minima to Business

Given a differentiable function *f*(*x*), the function is said to be differentiable at $x = a$, where *a* is the critical value obtained from $f(x) = 0$.

A function *f*(*x*) is said to have a relative maximum value at $x = a$ if *f*(a) is greater than immediately preceding and succeeding in values of the function.

The function $f(x)$ is said to have a relative minimum value at $x = a$ if $f(a)$ is smaller than immediately preceding and succeeding values of the function.

These definitions would be applied to areas in Business as could be seen in the following examples.

Example 5

The total cost of producing x set of shoes is ₦$\left(\frac{1}{4}x^2 + 35x + 25\right)$and the price set at which they may be sold is ₦$\left(50 - \frac{1}{2}x\right)$.

a) What should be the daily output to obtain a maximum total profit?

b) Show that the cost of producing a set is a relative minimum.

Solution

a) Profit on sales is $p = x\left(50 - \frac{1}{2}x\right)$ - $\left(\frac{1}{4}x^2 + 35x + 24\right)$

$$\therefore \frac{dp}{dx} = 15 - \frac{3x}{2}.$$

When $\frac{dp}{dx} = 0 \Rightarrow 15 - \frac{3x}{2} = 0.$

Critical value is $x = 10$.

Production to give maximum profit is 10 set per day.

b) Cost of producing a set c = ₦ $\left(\frac{\frac{1}{4}x^2 35x + 25}{x}\right) =$

₦ $\left(\frac{1}{4}x^2 + 35x + \frac{25}{x}\right)$

$$\therefore \frac{dp}{dx} = \frac{1}{4} - \frac{25}{x^2}.$$

When

$$\frac{dp}{dx} = 0 \Rightarrow \frac{1}{4} - \frac{25}{x^2} = 0.$$

$\Rightarrow x = 10,$ a minimum.

Example 6

The cost of fuel in running a generator is proportional to the square of the speed and is ₦25 per hour for a speed of 40kmh^{-1}. Other costs amount to ₦100 per hour regardless of the speed. Find the speed which will make the cost per kilometer a minimum.

Solution

Let w be the required speed.
c be the total cost per kilometer

Fuel cost per hour is *kw2,* where k is a constant.
w= 40kmh^{-1}.
$kw^2 = 1600k = 25$.

$$k = \frac{1}{64}$$

$$c = \frac{\cos t}{speed} = \frac{\frac{w^2}{64} + 100}{w}$$

$$\Rightarrow c = \frac{w}{64} + \frac{100}{w^2}.$$

$$\Rightarrow \frac{dc}{dw} = \frac{1}{64} - \frac{100}{w^2}.$$

$$\Rightarrow \frac{dc}{dw} = \frac{(w-80)(w+80)}{64w^2}; w > 0.$$

Hence, $w = 80$.

The most economical speed is 50km per hour.

4.0 CONCLUSION

As in the summary.

5.0 SUMMARY

i) Stationary points are where $\frac{dy}{dx} = 0$.

ii) Maximum and Minimum points

Maximum $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ is negative and

Minimum $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ is positive.

iii) Its application to business was also discussed.

6.0 TUTOR-MARKED ASSIGNMENT

Find the maximum and minimum points and the value of *y* at the turning point:

1. (i) $y = 2x^3 + 3x^2 - 36x$ (ii) $y = x^3 - 3x^2 + 3x + 2$
2. The sum of the base radius and height of a right circular cone is to be 6metres, find the greatest possible volume of the cone.
3. The sum of two numbers is 16. Find their maximum product.
4. The total surface area S of a cone of base radius r and perpendicular height h is given by $S = nr^2 + nr\sqrt{(r^2 + h^2)}$. If r an dh each increases at the rate of 0.25cm/sec., find the rate at which S is increasing at the instant which x = 3cm and h = 4cm.
5. A square sheet of metal has sides of length 8cm. Each square piece is removed from each corner and the remaining piece is bent into the form of an open box. Find the maximum volume of the box.

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2)Introduction to Mathematical Economics By Edward T. Dowling.

3)Mathematics and Quantitative Methods for Business and Economics.By Stephen P. Shao. 1976.

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UNIT 5 LINEAR PROGRAMMING (INEQUALITIES AND CONSTRAINTS)

CONTENTS

1.0 INTRODUCTION

This is the mathematical techniques for finding the optimum mix of products with limited resources. The resource limitations are represented graphically by areas which when combined form a feasibility region. The point on the feasibility region boundary that optimizes the objective function (usually maximizing profit or minimizing costs) is found by plotting an arbitrary value of the objective function on the graph and moving it away from (for maximization) or towards (for minimization) the origin until the last point of the feasibility region is obtained.

In order to apply linear programming, there must be as its title suggests a linear relationship between the factors.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- examine and identify the field of application of linear programming such as :

1. Mixing problems.
2. Limited capacity is allocated to products so as to yield maximum profits.
3. Transportation problem.
4. Purchasing.
6. Production scheduling.
6. Job assignments problem, etc.

3.0 MAIN CONTENT

3.1 Method of Linear Programming

Linear programming reduces the kind of problems outlined above to "series of linear expressions" and then uses these expressions to discover the best solution to achieve a given objective.

3.2 Step in Linear Programming

Step 1: Define the constraints (i.e. the limitations that must be placed on the variables.
Step 2: Graph the constraints.
Step 3: Define the objective function.
Step 4: Manipulate the objective function to find the optimum feasible solution.

Linear programming is best illustrated by means of an example:

Example 1 (Graphical Method)

Bolat Properties Nigeria Limited took the construction of chalets and roads in AJIBOLA ESTATE, LEKKI. Each chalets and road constructed passes through material-purchases process and constructing or assembling process. One chalet which makes a contribution of ₦50T, takes six hours material purchasing time, and four hours construction or assemblage time.

While one road makes a contribution of ₦40T, takes three hours material purchasing time and eight hours construction time. There is a maximum of thirty-six purchasing of materials hours available each week and forty-eight construction hours respectively.

Let x be the number of chalets constructed each week.
y be the number of roads constructed each week.

If 1 chalet requires 6 hours for material purchase,
Then, x chalets requires 8x hours for material purchase.
If a road requires 3 hours for material purchase,
Then y roads require $3y$ hours for material purchase.
Hence, total time to purchase materials is $(6x + 3y)$ hours.

Similarly, the total time required to construct x chalets and y roads is $(4x + 8y)$ hours.

Step 1: Define the Constraints.
The constraints are the amount of time available for material purchases and construction.

The best way of setting out the constraints is to place the units available on the left and those utilized on the right; the inequality sign is the link.

Constraints	**Available**		**Utilized**
Material purchased	36	$\geq$	$(6x + 3y)$
Construction time	48	$\geq$	$(4x + 8y)$

In addition, the logic constraints must be stated i.e $0 \leq x$ and $0 \leq y$.

These simply states that there cannot be negative amounts of chalets and roads. Hence, the constraints are:

i. $6x + 3y \leq 36$.
ii. $4x + 8y \leq 48$.

Step 2: Graphical Constraints.

Thus, x and y are each set to zero in turn and value of y and x computed in those circumstances.

For the equation: $6x + 3y = 36$.

When $x = 0, y = \frac{36}{3} = 12.$

When $y = 0, x = \frac{36}{6} = 6.$

For the equation: $4x + 8y = 48$.

When $x = 0, y = \frac{48}{8} = 6.$

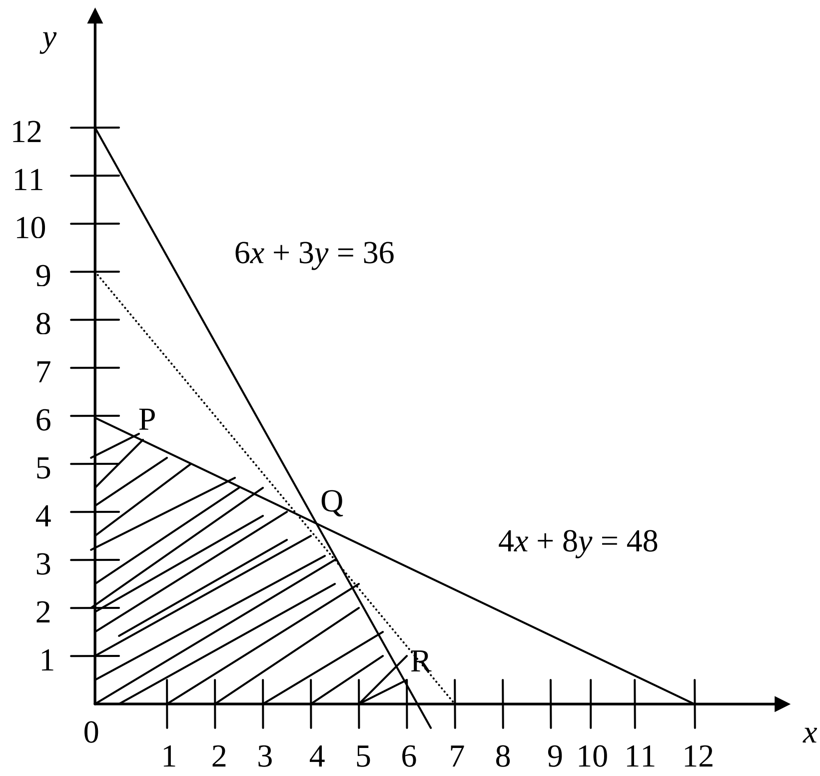
When $y = 0, x = \frac{48}{4} = 12.$

It is then necessary to decide which side of the equality represents the feasible solution space. Where the constraint is of the form $ax + by \leq c$, as in both of the above cases, the values of x and y can lie anywhere on the line or below it.

If the constraints is of the form $ax + by \geq c$, x and y can only lie on the line or above it.

This is the above example, x and y must be on or below the line $6x + 3y = 36$ and on or above the line $4x + 8y = 48$; they must also be ≥ 0.

i.e. Not below the x-axis and not to the left of the y-axis. The constraints can now be represented graphically:



The area OPQR which is shaded represents the feasible solutions i.e. Combinations of the two products which are achievable given the constraints. The problem is to find the optimum solution.

Step 3: Define the Objective Function

The objective is to maximize contribution C, given by: $c = 50x + 40y$

Step 4: Manipulate the Objective (Function)

There are two approaches to this final step:

a) By inspection, it is cleat that the maximum contribution will lie on one of the corners; P, Q, R-! be optimal solution can be leached simply by calculating the Contribution at each; or
b) By drawing an Iso-contribution fine, which is a line where a point represents an equal contribution. This is the recommended approach particularly for more complex problems.

3.3 Slack

Slack is the amount by which a resource is under utilized. It will occur when the "optimum point does not full the given resource line. In the above example, the optimum point Q lies on both the material purchase and construction time lines; therefore, both resources are fully utilized.

This can be checked from the constraint inequalities by solving the two simultaneous equations for the two constraints boundaries.

Point Q is the intersection of the lines.

$6x + 3y = 36$ …………………………………………… (1)
$4x + 8y = 48$ ……………………………………………….. (2)
x (2) – 2 x (1) gives:
$18y = 72$

$$y = \frac{72}{18} = 4.$$

Substituting y into (1) gives:

$\Rightarrow 6x + 3(4) = 36.$
$\Rightarrow 6x = 36 - 12 = 24.$
$x = 24/6 = 4$
$x = y = 4.$

Thus, the maximum contribution is obtained when four chalets and four roads per week are constructed.

The maximum contribution is:

$C = 50x + 40y,$
$\Rightarrow C = 50(4) + 40(4),$
$\Rightarrow C = 200 + 160.$
$\Rightarrow$ C = ₦360T.

From the constraint inequalities, $x = y = 4$,
Material purchasing time available: = 36.
Utilized = $6x + 3y = 6(4) + 3(4) = 24 + 12 = 36$.

Construction time available: equals 48,
(Utilized = $4x + 8y = 4(4) + 8(4) = 16 + 32 = 48$.
Hence, all available time in both department is Utilized.

If, however, the optimum had been at P(x = 0, y = 6), then because P does not lie on the material purchase time line, there would be slack material purchasing time.

Material purchasing time utilized = $6x + 3y = (6 \text{ x } 0) + (3 \text{ x } 6) = 18$;
Slack = 36 – 18 = 18 hours
Slack is important because unused resources can be put to another use e.g. hired out to another construction company?

Example 2

U. A. C Nigeria Limited manufactures and sells Gala and Nourish bread represented as G and N. Each product is processed in two departments; Food & beverages and packaging. Each unit of product G requires 30 minutes in the Food & be department and 10 minutes in the packaging department. The corresponding times for product N are 10 minutes and 20 minutes respectively. Each day, 900 minutes are available in the food/beverages department and 700 minutes are available in the packaging department.

The sales manager wishes to have a daily output of at least 40 products irrespectively of whether they are Gala and Nourish bread are ₦32 and ₦40 respectively.

U. A. C Nigeria Limited wishes to know the output levels in each of the following situations:

i. To maximize sales revenue;
ii. To minimize manufacturing cost;
iii. To maximize profit.
a) Formulates a linear programming problem.
b) Represent graphically and shade the feasible region.
c) Obtain the output levels in each of the situations (i), (ii) and (iii) above.

Solution

a) Let the recommended output levels of Gala and Nourish by g and n respectively.

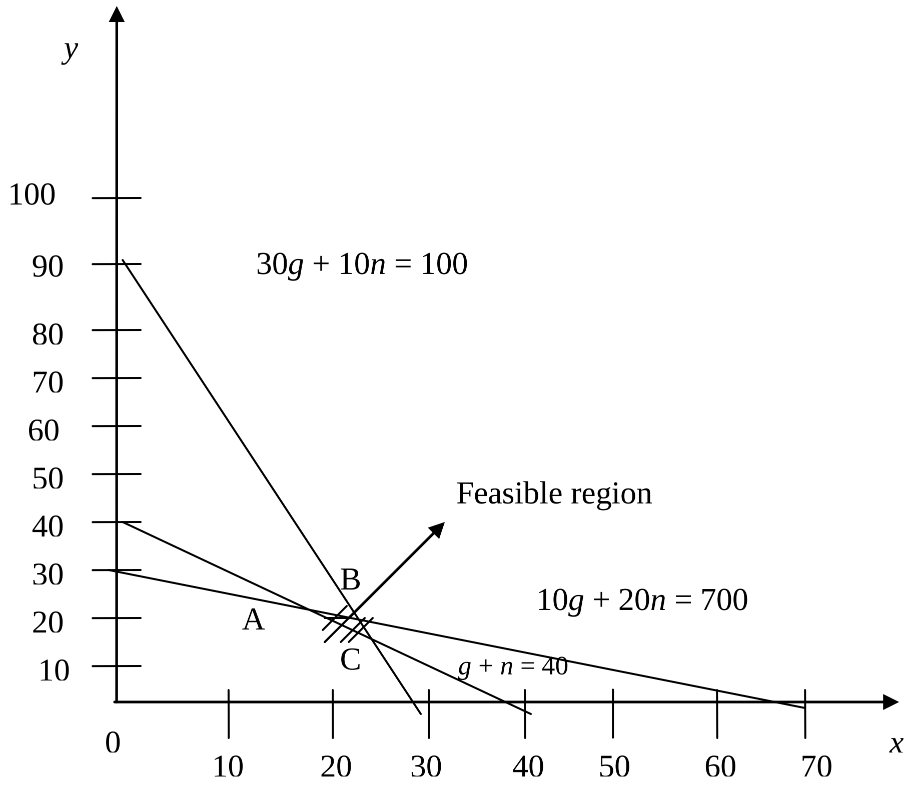
The constraints equations are:

1. $30g + 10n \leq 900$ minutes (Food and beverages department).
2. $10g + 20n \leq 700$ minutes (Packaging department).
3. $g + n \geq 40$ units (Minimum daily output).
4. $g , n \geq 0$ (Non-negatively).

The objectives functions are:

i. Maximize $40g + 60n$.
ii. Minimize $32g + 40n$.
iii. Maximize $8g + 20n$.

b) To plot the graph of the feasible region.



Corner points (g, n)	Revenue (₦)	Cost (₦)	Contribution
A(10,30)	2,200	1,520	680
B(22, 24)	2,320	1,696	624
C(25, 15)	1,900	1,400	500

All these are generated from the graph above.

Revenue contribution = (Selling – cost) price per unit of the product.

c) The optimal output levels are:

i. To maximize sales revenue: 22 units of Gala and 24 units of Nourish bread (Maximize revenue = ₦2,320).
ii. To minimize manufacturing sot: 25 units Gala and 15 units Nourish bread (minimize cost = ₦1,400).

iii. To maximize contribution (profit): 10 units of Gala and 30 units of Nourish bread (Maximize contribution = ₦680).

4.0 CONCLUSION

In conclusion, the basic rule of linear programming have been discussed .

5.0 SUMMARY

From the conclusion, it is important that the student is able to apply the formulae to a practical problem as we have shown in examples 1 and 2 in this section.

6.0 TUTOR-MARKED ASSIGNMENT

In each of the following linear inequalities, describe and graph the feasible region.

1. $3x - 7y \geq -28$
   $x + y \geq 6$
   $9x - 2y \leq 36$
   $x - 6y \leq 48$

2. $3x - 4y \leq 5$
   $x - y \leq -8$

3. Minimize $f = x + y$ subject to $x \geq 0, y \geq 0$ and $x + 2y \leq 0, 2x + y \leq 10, y \leq 4.$

4. A company manufacturers and sells two models of lamps; L_1 and L_2, the profit being ₦15 and ₦10 respectively. The process involves two workers w_1 and w who are available for this kind of work 100 and 80 hours per month respectively. W_1 assembles L_1 in 20 minutes and L_2 in 30 minutes. W_2 assembles L_1 in 20 minutes and L_2 in 10 minutes. Assuming that all lamps made can be sold without difficulty, determine production figures that maximize the profit.

5. Maximize the daily profit in manufacturing two alloys A_1, A_2 that are: different mixtures of two metals m_1, m_2 as shown below:

	Proportion of Metal		
Metal	Alloy	Alloy	Daily supply

	A_1	A_2	(tons)
m_1	0.5	0.25	10
m_2	0.5	0.75	15
Net profit (₦ per ton)	30	25	-

7.0 REFERENCES/FURTHER READINGS

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