NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 412

COURSE TITLE: NORMED LINEAR SPACES

UNIT 1: NORMED LINEAR SPACES

Contents

1 Introduction

In your course of topology, you know that a topological space is a set with a topology defined on it and this topology is a set of open sets of the set which is closed under arbitrary union and finite intersection of its elements. Also in Linear algebra you learnt that a linear space is a space equipped with the operations addition of elements and multiplication of elements by numbers. If you combine this ideas, you will arrive to the notion of a topological linear space, equipped with a topology as well as with the algebraic operations characterizing a linear space. In this unit, you shall understand the concept of a particularly important type, namely normed linear spaces. You will study in general topological linear spaces in subsequent unit.

2 OBJECTIVES.

At the end of this unit, you should be able to;

(i) correctly define a norm and a normed linear space.

(ii) show that a given function is a norm.

(iii) show that every norm defines a metric.

(iv) identify a metric induced by a norm.

(v) define a convex set and give examples.

(vi) state, prove and apply the Holder's and Minkwoski's inequality.

3 Normed Linear Spaces

3.1 Definition and Examples

Definition 3.1 Let X be a linear space over a scalar field $K = (\mathbb{R}$ or $\mathbb{C})$, a function $\|\cdot\| : X \to \mathbb{R}$ is said to be a norm (in X) if it satisfies the following properties:

N1. $\|x\| \geq 0$ for all $x \in X$, where $\|x\| = 0$ if and only if $x = 0$;

N2. $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and all scalar λ;

N3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$. (Triangle inequalilty)

Definition 3.2 A linear space X, equipped with a norm $\|x\|$, is called a normed linear space.

Definition 3.2 tells you that if $\|\cdot\|$ is a norm defined on a linear space X, then the pair $(X, \|\cdot\|)$ is a normed linear space. In what follows, provided no confusion will arise, you shall call X a normed linear space.

Example 3.1 The real line $\mathbb{R}$ becomes a normed linear space if you set $\|x\| = |x|$ for every number $x \in \mathbb{R}$.

Proof. It is enough for you to show that $\|\cdot\|$ is a norm on the set of real numbers $\mathbb{R}$. That is, you have to verify that $\|\cdot\|$ satisfies the three axioms N1-N3 of a norm defined on a linear space. So verify as follows;

N1. Let $x \in \mathbb{R}$ be arbitrary. Since the absolute value function $|\cdot|$ defined on $\mathbb{R}$ is nonnegative, $\|x\| = |x| \geq 0$. Thus, $\|x\| \geq 0$ for all $x \in \mathbb{R}$.
Now, $\|x\| = 0$, if and only if $|x| = 0$ if and only if $x = 0$.

N2. Let $x, \lambda \in \mathbb{R}$.

$$\|\lambda x\| = |\lambda x| = |\lambda||x| = |\lambda|\|x\|$$

Therefore, $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda, x \in \mathbb{R}$.

N3. Let $x, y \in \mathbb{R}$ be arbitrary elements of $\mathbb{R}$.

$$\|x + y\| = |x + y| \leq |x| + |y| = \|x\| + \|y\|$$

Therefore, $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{R}$.

■

Thus, $\|x\| = |x|$, $x \in \mathbb{R}$ defines a norm on $\mathbb{R}$.

Example 3.2 Let $X = \mathbb{R}^2$. For each vector $\bar{x} = (x_1, x_2) \in X$, define $\|\cdot\|_2 : X \to \mathbb{R}$ by

$$\|\bar{x}\|_2 = \left(\sum_{k=1}^{2} x_k^2\right)^{\frac{1}{2}}$$

Then $\|\cdot\|_2$ is a norm on X.

Proof. You should verify the conditions N1 to N3 of Definition 1 as follows;

N1. Let $\bar{x} = (x_1, x_2) \in X$ be arbitrary. For each $k = 1, 2$, $x_k^2 \geq 0$, so that $\sum_{k=1}^{2} x_k^2 \geq 0$, which implies that $\|\bar{x}\|_2 = \left(\sum_{k=1}^{2} x_k^2\right)^{\frac{1}{2}} \geq 0$. Thus $\|\bar{x}\|_2 \geq 0$, for all $\bar{x} \in X$.

Now, $\|\bar{x}\|_2 = 0$, if and only if $\left(\sum_{k=1}^{2} x_k^2\right)^{\frac{1}{2}} = 0$ if and only if $\sum_{k=1}^{2} x_k^2 = 0$ if and only if $x_1^2 = 0$ and $x_2^2 = 0$, if and only if $x_1 = 0$ and $x_2 = 0$, or, $\bar{x} = (x_1, x_2) = (0, 0) = \bar{0}$. Thus, $\|\bar{x}\|_2 = 0$, if and only if $\bar{x} = (0, 0) =$

N2. Let $\bar{x} = (x_1, x_2) \in X$, $\lambda \in K$ be arbitrary, $\lambda\bar{x} = (\lambda x_1, \lambda x_2)$. So

$$\|\lambda\bar{x}\|_2 = \left(\sum_{k=1}^{2} \lambda x_k^2\right)^{\frac{1}{2}} = \left(|\lambda|^2 \sum_{k=1}^{2} x_k^2\right)^{\frac{1}{2}} = |\lambda| \left(\sum_{k=1}^{2} x_k^2\right)^{\frac{1}{2}} = |\lambda|\|\bar{x}\|_2.$$

N3. Let $\bar{x} = (x_1, x_2)$, $\bar{y} = (y_1, y_2) \in X$ be arbitrary.

$$\begin{aligned}
\|\bar{x} + \bar{y}\|_2^2 &= \sum_{k=1}^{2} (x_k + y_k)^2 = \sum_{k=1}^{2} x_k^2 + 2x_k y_k + y_k^2 \leq \sum_{k=1}^{2} x_k^2 + 2|x_k y_k| + y_k^2 \\
&= \sum_{k=1}^{2} x_k^2 + 2\sum_{k=1}^{2} |x_k y_k| + \sum_{k=1}^{2} y_k^2 \\
&\leq \|\bar{x}\|_2^2 + 2\left(\sum_{k=1}^{2} |x_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{2} |y_k|^2\right)^{\frac{1}{2}} + \|\bar{y}\|_2^2 \quad \text{(By Cauchy Schwart Inequality)} \\
&= \|\bar{x}\|_2^2 + 2\|\bar{x}\|_2\|\bar{y}\|_2 + \|\bar{y}\|_2 \\
&= (\|\bar{x}\|_2 + \|\bar{y}\|_2)^2
\end{aligned}$$

i.e., $\|\bar{x} + \bar{y}\|_2 \leq \|\bar{x}\|_2 + \|\bar{y}\|_2$ for all $\bar{x}, \bar{y} \in X$.

Thus $\|\cdot\|_2$ is a norm on X since it satisfies the conditions of a norm as shown above. ■

So far, you have seen that to verify the axioms N1 and N2 of a norm are not very difficult, but the major task is in proving N3, i.e., the triangle inequality. Before you proceed to see other example, see some important inequality that will be useful to you in verifying the triangle inequality. The proof of them will not be given now.

3.2 Holder's and Mikwoski's Inequalities

The following notions are of paramount importance.

Definition 3.3 (l_p- spaces) Let $1 \le p < \infty$. Let $K = (R \text{ or } C)$. The set

$$l_p(K) = \left\{ x = (x_1, x_2, ..., x_k, ...),\ x_k \in K,\ k \ge 1 : \sum_{k=1}^{\infty} |x_k|^p < \infty \right\} \quad (1)$$

is called an l_p- space.

For example, if $p = 1$ and $K = R$ then (1) becomes

$$l_1(R) = \left\{ x = (x_1, x_2, ..., x_k, ...),\ x_k \in R,\ k \ge 1 : \sum_{k=1}^{\infty} |x_k| < \infty \right\}$$

Which is the set of all sequences $\{x_n\}_{n=1}^{\infty}$ of real numbers such that $\sum_{k=1} |x_k|$ is finite (i.e., such that series converges.)

You can see that $x = \left(\frac{1}{n^2}\right)_{n=1}^{\infty} \in l_1$ because

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1} \frac{1}{k^2} < \infty$$

By $p-$ series $(p = 2)$. But the sequence $x = \{\frac{1}{n}\}_{n=1}^{\infty} \notin l_1$ since

$$\sum_{k=1} \frac{1}{k} = \sum_{k=1} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is the harmonic series which diverges.

For $p = 2$ and $K = R$, (1) becomes

$$l_2(K) = \left\{ x = (x_1, x_2, ..., x_k, ...),\ x_k \in R,\ k \ge 1 : \sum_{k=1}^{\infty} |x_k|^2 < \infty \right\}$$

Observe that $x = \{\frac{1}{n}\}_{n=1}^{\infty} \in l_2$. Also is the first example. You can see that all elements of l_1 is also an element of l_2, but not every element of l_2 is an element of l_1. In fact this is a general fact about l_p spaces as stated in the following proposition.

Proposition 3.1 Let $1 \le p, q \le +\infty$. If $p \le q$, then $l_p \subset l_q$.

Proof. There are two cases you can consider.

(i) $q = \infty$ and $p < \infty$. If $\{x_n\}_{n\ge 1} \in l_p$ then $\sum |x_n|^p < \infty$, this implies that $|x_n| \to 0$ as $n \to \infty$, so that $\sup |x_n| < \infty$. This in turn, implies, $\{x_n\} \in l_q (= l_\infty)$.

$n \geq 1$

4

(ii) $p < q < \infty$. Let $\{x_n\} \in l_p$. Then $\sum_{n=1}^{\infty} |x_n|^p < \infty$. As in part (i), $M := \sup_{n\geq 1} |x_n| < \infty$. Hence,

$$\sum_{n=1} |x_n|^q \leq M^{q-p} \sum_{n=1} |x_n|^p < \infty$$

and this implies $\{x_n\} \in l_q$. Hence $l_p \subset l_q$.

∎

Definition 3.4 ($l_\infty -$ space). The space of all bounded sequence of real(or complex) i.e.,

$$l_\infty = \{x = (x_1, x_2, ..., x_k, ...) : x \text{ is bounded}\}$$

is called the $l_\infty -$ space.

Definition 3.5 Let $1 \leq p \leq +\infty$. If for arbitrary $x = \{x_k\}$, $y = \{y_k\}$ in l_p and $\lambda \in K$, you define vector addition and scalar multiplication respectively as

$$x + y = (x_1, x_2, ..., x_k, ...) + (y_1, y_2, ..., y_k, ...) = (x_1 + y_1, x_2 + y_2, ..., x_k + y_k, ...)$$

and

$$\lambda x = \lambda(x_1, x_2, ..., x_k, ...) = (\lambda x_1, \lambda x_2, ..., \lambda x_k, ...),$$

(i.e. componentwise,) then l_p is a linear space.

The proof of the following inequalites are given at the end of this unit.

Proposition 3.2 (Holder's Inequality). Let $1 \leq p, q \leq \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$. If $x = \{x_k\} \in l_p$ and $y = \{y_k\} \in l_q$, then,

$$\sum_{k=1} |x_k y_k| \leq \left(\sum_{k=1} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1} |y_k|^q\right)^{\frac{1}{q}} \tag{2}$$

Proposition 3.3 (Cauchy-Schwartz inequality). $p = 2$ gives the Cauchy-Schwartz inequality. i.e., for $x = \{x_k\}$ and $y = \{y_k\}$ in l_2,

$$\sum_{k=1} |x_k y_k| \leq \left(\sum_{k=1} |x_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1} |y_k|^2\right)^{\frac{1}{2}} \tag{3}$$

You used this inequality in the proof of triangle inequality of example (3.2), for a finite sum (n=2), and you need the same inequality while proving the general case given in example (3.3).

Proposition 3.4 (Minkwoski Inequality). Let $1 \leq p < \infty$. For arbitrary $x = \{x_k\}, y = \{y_k\} \in l_p$,

$$\left(\sum_{k=1}^{\infty} |x_k + y_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{\frac{1}{p}} \tag{4}$$

Example 3.3 Let $X=\mathbb{R}^n$ (the real n-space). You can make it a normed linear space, by setting

$$\|x\|_2 = \left(\sum_{k=1}^{n} |x_k|^2\right)^{\frac{1}{2}} \tag{5}$$

for every element $x = (x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$.

The verification of the fact that $\|\cdot\|_2$ is a norm on $\mathbb{R}^n$ is a generality of the proof of example (3.2), where the proof is given for $n = 2$.

Example 3.4 You can also equip $\mathbb{R}^n$, the real n-space, with the norm

$$\|x\|_1 = \sum_{k=1} |x_k| \tag{6}$$

or the norm

$$\|x\|_\infty = \max_{1\leq k\leq n} |x_k| \tag{7}$$

You can see that the function $\|\cdot\|_\infty$ in (7) is well defined since maximum is taken over a finite set of points. You can verify that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ as defined in equations (6) and (7) is a norm on $\mathbb{R}^n$.

Example 3.5 The function

$$\|z\| = \left(\sum_{k=1}^{n} |z_k|^2\right)^{\frac{1}{2}}$$

introduces a norm in the complex n-space $\mathbb{C}^n$. Other possible norms in $\mathbb{C}^n$ are given by (6) and (7).

Example 3.6 The space $C[a, b]$ of all functions continuous on the interval can be equipped with the norm

$$\|f\|_\infty = \max_{a\leq t\leq b} |f(t)|.$$

Other norms that can be defined on $C[a, b]$ for arbitrary $f \in C[a, b]$ are as follows;

$$\|f\|_1 = \int_a^b |f(t)|\, dt \tag{8}$$

and

$$\|f\|_2 = \left(\int_a^b |f(t)|^2\, dt\right)^{\frac{1}{2}} \tag{9}$$

To verify the triangle inequality for (9), you will use the integral version of the Schwartz inequality stated in the following proposition.

Proposition 3.5 If $f, g \in C[a, b]$ are arbitrary, then

$$\int_a^b |f(t)g(t)|\, dt \leq \left(\int_a^b |f(t)|^2\, dt\right)^{\frac{1}{2}} \left(\int_a^b |g(t)|^2\, dt\right)^{\frac{1}{2}} \tag{10}$$

Example 3.7 Let l_∞ be the space of all bounded real sequences

$$x = (x_1, x_2, ..., x_k, ...),$$

and let

$$\|x\| = \sup_{k\geq 1} |x_k| \tag{11}$$

Then (1.3) obviously has all the properties of a norm.

Example 3.8 Let $1 \leq p < +\infty$. For arbitrary $x = \{x_k\}_{k=1}^{\infty} \in l_p$, the function

$$\|x\|_{l_p} = \left(\sum_{k=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} \tag{12}$$

defines a norm on l_p. So that $(l_p, \|\cdot\|_{l_p})$ is a normed linear space.

Proof. You can easily verify that the function defined above satisfies axioms N1 and N2 of a norm. Now, to verify N3, you shall be needing the Holder's Inequality as stated in proposition (3.2). Study the following carefully and learn the trick applied in the proof.

Let $x = (x_1, x_2, ..., x_k, ...)$, $y = (y_1, y_2, ..., y_k, ...) \in l_p$. Consider two cases for p.
Case 1: For $p = 1$, (12) and the triangle inequality for absolute value gives you that

$$\sum_{k=1}^{\infty} |x_k + y_k| \leq \sum_{k=1}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k|$$

i.e. $\|x + y\|_{l_1} \leq \|x\|_{l_1} + \|y\|_{l_1}$ as required
Case 2. For $1 < p < \infty$,

$$\begin{aligned}
\|x + y\|^p_{l_p} &= \sum_{k=1}^{\infty} |x_k + y_k|^p = \sum_{k=1}^{\infty} |x_k + y_k||x_k + y_k|^{p-1} \\
&\leq \sum_{k=1}^{\infty} (|x_k| + |y_k|)|x_k + y_k|^{p-1} = \sum_{k=1}^{\infty} |x_k||x_k + y_k|^{p-1} + \sum_{k=1}^{\infty} |y_k||x_k + y_k|^{p-1} \\
&\leq \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |x_k + y_k|^{(p-1)q} \right)^{\frac{1}{q}} + \left(\sum_{k=1}^{\infty} |y_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |x_k + y_k|^{(p-1)q} \right)^{\frac{1}{q}} \quad (*)
\end{aligned}$$

(By Holder's Inequality)

$$= \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} \|x + y\|^{\frac{p}{q}}_{l_p} + \left(\sum_{k=1}^{\infty} |y_k|^p \right)^{\frac{1}{p}} \|x + y\|^{\frac{p}{q}}_{l_p}$$

(*) is true because if $1 < p, q, < \infty$, satisfies $\frac{1}{p} + \frac{1}{q} = 1$, then $p = (p - 1)q$. So, the sequence defined for each $k \geq 1$ by $z_k = |x_k + y_k|^{p-1}$ is in l_q. Thus,

$$\|x+y\|^p_{l_p} \le \left[\left(\sum |x_k|^p \right)^{\frac{1}{p}} + \left(\sum |y_k|^p \right)^{\frac{1}{p}} \right] \|x + y\|^{\frac{p}{q}}$$

$$\|x + y\|_{l_p} \leq \quad \sum_{k=1} |x_k| \quad + \quad \sum_{k=1} |y_k| \quad \| + \quad l_p$$

or

$$\|x+y\|_{l_p}^{p-\frac{p}{q}} \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{\frac{1}{p}} = \|x\|_{l_p} + \|y\|_{l_p}$$

■

i.e., $\|x+y\|_{l_p} \leq \|x\|_{l_p} + \|y\|_{l_p}$, for arbitrary $x, y \in l_p$.

Example 3.9 Let $X = l_\infty$, the space of all bounded sequence. The function defined by

$$\|x\|_\infty = \sup_{k \geq 1} |x_k| \tag{13}$$

for arbitrary sequence $x = \{x_k\}_{k=1}^{\infty}$ in l_∞ is a norm on l_∞. Thus $(l_\infty, \|\cdot\|)$ is a normed linear space

3.3 EQUIVALENT NORMS

Definition 3.6 (Equivalent norms) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms defined on a linear space X. You will call $\|\cdot\|_1$ and $\|\cdot\|_2$ equivalent if there exist constants $a, b > 0$ such that

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \quad \text{or} \quad a\|x\|_2 \leq \|x\|_1 \leq b\|x\|_2 \tag{14}$$

for all $x \in X$

Example 3.10 With this definition, you can see that the norms defined on R^n by (6), (7) and (8) are equivalent.

Theorem 3.1 All norms defined on a finite dimensional space are equivalent.

Proof. Let $\dim E = n < \infty$ and let $\{e_i\}_{i=1}^n$ be a basis for E. For arbitrary vector $x \in E$, there exist unique scalars $\{\alpha_i\}_{i=1}^n$ such that $x = \sum_{i=1}^{n} \alpha_i e_i$. Define $\|x\|_0 = \max_{1 \leq i \leq n} |\alpha_i|$. Clearly, $\|\cdot\|_0$ is a norm on E. It suffice now to prove that any norm on E is equivalent to $\|\cdot\|_0$; i.e., that if $\|\cdot\|$ is an arbitrary norm on E, there exists constants $a > 0, b > 0$ such that

$$a\|x\|_0 \leq \|x\| \leq b\|x\|_0 \text{ for all } x \in E.$$

From $x = \sum_{i=1}^{n} \alpha_i e_i$ we obtain,

$$\|x\| = \left\|\sum_{i=1}^{n} \alpha_i e_i\right\| \leq \max_{1 \leq i \leq n} |\alpha_i| \sum_{i=1}^{n} \|e_i\|.$$

Since $\{e_i\}_{i=1}^n$ is a basis, it follows that $\sum \|e_i\|$ is a number. Call it b. The last inequality then yields.

i $=1$

$$\|x\| \leq b\|x\|_0.$$

It now only remains to prove $a\|x\|_0 \leq \|x\|$. Let $S = \{x \in E : \|x\|_0 = 1\}$. Define the map

$$\phi : (E, \|\cdot\|_0) \to \mathbb{R} \text{ by } \phi(x) = \|x\| \text{ for all } x \in E.$$

Observe that $\left\|\frac{x}{\|x\|_0}\right\|_0 = 1$. Moreover, ϕ is continuous. To see this, let ϵ be given. You have to find a $\delta > 0$ such that for arbitrary $x^* \in E$, if $\|x - x^*\|_0 < \delta$ then $|\phi(x) - \phi(x^*)| < \epsilon$. Recall that in part
(a) we proved that $\|x\| \leq b\|x\|_0$ for all $x \in E$. Using this we now obtain that

$$|\phi(x) - \phi(x^*)| = |\|x\| - \|x^*\|| \leq \|x - x^*\| \leq b\|x - x^*\|_0 \leq b\delta < \epsilon$$

where we have chosen $\delta = \frac{\epsilon}{1+b}$. So, ϕ is continuous. Since S is compact, ϕ attains its infimum on S. Let this infimum be denoted by a. Then $0 < a < \phi(x) := \|x\|$, for all $x \in S$. But $\frac{x}{\|x\|_0} \in S$. So, $0 < a \leq \left\|\frac{x}{\|x\|_0}\right\|$ for all $x \in E$ and this implies $a\|x\|_0 \leq \|x\|$ for all $x \in E$. Combining this with $\|x\| \leq b\|x\|_0$, we obtain that $a\|x\|_0 \leq \|x\| \leq b\|x\|_0$ for all $x \in E$ as required. ■

Proposition 3.6 Let $\|\cdot\|$ be a norm defined on a linear space X. If $\rho : X \times X \to \mathbb{R}$ is defined for arbitrary $x, y \in X$ by

$$\rho(x, y) = \|x - y\| \tag{15}$$

then, ρ is a metric on X, and so (X, ρ) is a metric space.

Proof. You have to verify that ρ satisfies all the axioms of a metric.

1. Let $x, y \in X$ be arbitrary. $x - y \in X$ since it is linear space. So by axiom N1 of a norm, we have $\rho(x, y) = \|x - y\| \geq 0$. Which implies that $\rho(x, y) \geq 0$ for all $x, y \in X$.

2. $\rho(x, y) = 0$ if and only if $\|x - y\| = 0$, if and only if $x - y = 0$ if and only if $x = y$.

3. Let $x, y \in X$ be arbitrary. Using axiom N2,

$$\rho(x, y) = \|x - y\| = \|(-1)(y - x)\| = |-1|\|y - x\| = \|y - x\|.$$

   Thus, $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.

4. Let $x, y, z \in X$ be arbitrary. Using axiom N3 of a norm, i.e. the triangle inequality property, ■

$$\rho(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = \rho(x, y) + \rho(y, z)$$

   i.e., $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$, for all $x, y, z \in X$

Since ρ satisfies 1, 2, 3 and 4, as shown demonstrated in the proof above, it is a metric on X Hence, (X, ρ) is a metric space.

You saw that the above proposition is given for an arbitrary norm $\|\cdot\|$ defined on an arbitrary linear space X. Hence, every norm induces a metric but not all metrics are induced by a norm.

Example 3.11 The formula

$$\rho_2(x, y) = \|x - y\|_2 = \left(\sum_{k=1}^{n} |x_k - y_k|^2 \right)^{2}$$

9

defines a metric for arbitrary $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ in R^n. Also $\|\cdot\|_1$ and $\|\cdot\|_\infty$ induces the metrics

$$\rho_1(x, y) = \|x - y\|_1 = \sum_{k=1}^{n} |x_k - y_k|$$

and

$$\rho_\infty(x, y) = \|x - y\|_\infty = \max_{1\leq k \leq n} |x_k|$$

respectively on R^n.

Example 3.12 The formula

$$\rho_p(x, y) = \|x - y\|_{l_p} = \left(\sum_{k=1}^{\infty} |x_k - y_k|^p\right)^{\frac{1}{p}}$$

defines a metric on l_p, $1 \leq p < \infty$

Example 3.13 The formula

$$\rho_\infty(x, y) = \|x - y\|_\infty = \sup_{k\geq 1} |x_k - y_k|$$

induces a metric on the space l_∞ of bounded sequences.

Example 3.14 The formula

$$\rho_2(f, g) = \|f - g\|_2 = \left(\int_a^b |f(t) - g(t)|^2 \, dt\right)^{\frac{1}{2}}$$

induces a metric on the space, $C[a, b]$, of continuous real-valued functions on the closed and bounded interval $[a, b]$.

4 Convex Sets and Convex Functions

The study of convexity is a richly rewarding mathematical experience. Theorems dealing with the convexity are invariably clean and easily understood statement. The notion of convexity is of paramount importance in functional analysis. In this section we shall be dealing with convex sets and convex functions and afterwards, use it to prove the earlier stated Holder's and Minkwoski's Inequality.

4.1 Convex Sets

Here is the basic definition.

Definition 4.1 Let X be a linear space, and $x, y \in X$. The line segment $[x, y]$ joining x and y is defined by

$$[x, y] = \{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1\} \tag{16}$$

Definition 4.2 Let X be a linear space. A subset C of X is convex if for every $x, y \in C$, and $\lambda \in [0, 1]$,

$$\lambda x + (1 - \lambda)y \in C. \tag{17}$$

Definition 4.2 can be restated as, $C \subset X$ is convex if for every elements $x, y \in C$, the line segment joining x and y is in C, i.e, $[x, y] \subset C$, where $[x, y]$ is as defined in (16).

Example 4.1 Let x and v be vectors in R^n the line L through x in the direction of v given by

$$L = \{x + \alpha v : \alpha \in R\}$$

is a convex set.

Proof. Let $p, q \in L$. This implies that there exists $\alpha_1, \alpha_2 \in [0, 1]$ such that $p = x + \alpha_1 v$ and $q = x + \alpha_2 v$. Now, for arbitrary $\lambda \in [0, 1]$,

$$\begin{aligned} \lambda p + (1 - \lambda)q &= \lambda(x + \alpha_1 v) + (1 - \lambda)(x + \alpha_2 v) \\ &= x + (\lambda\alpha_1 + (1 - \lambda)\alpha_2)v \end{aligned}$$

Let $\alpha_3 = \lambda\alpha_1 + (1 - \lambda)\alpha_2 \in R$, then $\lambda p + (1 - \lambda)q = x + \alpha_3 v \in L$. Since $p, q \in L$ and $\lambda \in [0, 1]$ are arbitrary then L is convex. ■

Example 4.2 Any linear subspace M of R^n is a convex set since linear subspaces are closed under addition and scalar multiplication. This also entails that every linear space is a convex space since a linear space is also a linear subspace of itself.

Example 4.3 If $x^* \in R^n$ and if $\alpha \in R$, then the closed half-spaces

$$F^+ = \{y \in R^n : \langle x^*, y\rangle \geq \alpha\}$$

$$F^- = \{y \in R^n : \langle x^*, y\rangle \leq \alpha\}$$

and the open half-spaces

$G^{+} = \{y \in R^{n} : \langle x^{*}, y\rangle > \alpha\}$ $\quad$ $G^{-} = \{y \in R^{n} : \langle x^{*}, y\rangle < \alpha\}$

determined by x^* and α are all convex sets. For example, if u, w are in F^+ and if $0 \leq \lambda \leq 1$, then

$$\langle x^*, [\lambda u + (1-\lambda)w]\rangle = \lambda\langle x^*, u\rangle + (1-\lambda)\langle x^*, w\rangle \geq \lambda\alpha + (1-\lambda)\alpha = \alpha$$

so

$$\lambda u + (1-\lambda)w \in F^+$$

Example 4.4 If $x^* \in R^n$ and if $r > 0$, then the ball

$$B(x^*, r) = \{y \in R^n : \|y - x^*\| < r\}$$

centered at x^* of radius r is a convex set. In fact, if y, z are in $B(x^*, r)$, then

$$\|y - x^*\| < r, \qquad \|z - x^*\| < r.$$

For $0 \leq \lambda \leq 1$, you can apply the Triangle inequality as follows:

$$\begin{aligned} \|\lambda y + (1-\lambda)z - x^*\| &= \|\lambda(y - x^*) + (1-\lambda)(z - x^*)\| \\ &\leq \lambda\|y - x^*\| + (1-\lambda)\|z - x^*\| \\ &< \lambda r + (1-\lambda)r \\ &= r \end{aligned}$$

to conclude that $\lambda y + (1-\lambda)z \in B(x^*, r)$.

Example 4.5 Let X be a linear space and $\alpha \in R$. Let $f : X \to R$ be a real linear functional. The following sets

1. The Closed half-spaces;

$$H_{f,\alpha} = \{x \in X : f(x) \leq \alpha\}$$

$$H^+_{f,\alpha} = \{x \in X : f(x) \geq \alpha\}$$

2. The open half-spaces

$$H^-_{f,\alpha} = \{x \in X : f(x) < \alpha\}$$

$$H_{f,\alpha} = \{x \in X : f(x) > \alpha\}$$

3. The hyperplane

$$H = \{x \in X : \mathbf{f}(x) = \alpha\}$$

are convex subsets of X, when $\mathbf{f}$ is not the zero linear functional.

Example 4.6 If $\{C_\alpha\}_{\alpha\in\Delta}$ is a collection of convex subsets of a linear space X, then the intersection

$$C = \bigcap_{\alpha\in\Delta} C_\alpha$$

is also convex. For if $y, z \in C$ and if $0 \leq \lambda \leq 1$, then $y, z \in C_\alpha$ for each $\alpha \in \Delta$, so $\lambda y+(1-\lambda)z \in C_\alpha$
for each α since C_α is convex. But $\lambda y + (1 - \lambda)z \in \bigcap_{\alpha\in\Delta} C_\alpha =: C$, so that C is convex.

In the definition of convexity above, you used above merely two points. You can also deal with the case of more than 2 points. This leads to the introduction of the concept of convex combination.

Definition 4.3 Convex Combination. Let $x_1, ..., x_n$ be n points of the vector space X. Any elements of the form

$$x = \sum_{i=1}^{n} \lambda_i x_i \tag{18}$$

with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, is called convex combination of the elements $x_1, ..., x_n$.

The above definition gives you the following other characterization of convex sets.

Proposition 4.1 A nonempty subset C of the vector space X is convex if and only if C contains all convex combinations of all its elements.

Proof. $(\Leftarrow)$: This implication is obvious.

$(\Rightarrow)$: Assume that the nonempty set C is convex, you have to show that C contains all its convex combinations. You can proceed by induction as follows. Define the property P_n as follows;

$$P_n : \sum_{i=1}^{n} \lambda_i x_i \in C \text{ for all } x_1, ..., x_n \in C, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1$$

1. The property obviously hold for $n = 1$, i.e., (P_1) is fulfilled.
2. Assume that properties $(P_1), ..., (P_n)$ holds. Let $x_1, ..., x_n, x_{n+1} \in C$, $\lambda_1 \geq , \lambda_2 \geq 0, ..., \lambda_n \geq 0, \lambda_{n+1} \geq 0$with

   $$\sum_{i=1}^{n+1} \lambda_i = 1$$

   Of course, if $\lambda_{n+1} = 1$, then

   $$\sum_{i=1}^{n+1} \lambda_i x_i = x_{n+1} \in C,$$

because $\lambda_1 = \cdots = \lambda_n = 0$ in this case. And so

$$\sum_{i=1}^{n+1} \lambda_i x_i \in C.$$

Assume that $\lambda_{n+1} = 1$. This allows you to write

$$\begin{aligned} \sum_{i=1}^{n+1} \lambda_i x_i &= \sum_{i=1}^{n} \lambda_i x_i + \lambda_{n+1} x_{n+1} \\ &= (1 - \lambda_{n+1}) \left(\sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} x_i \right) + \lambda_{n+1} x_{n+1}. \end{aligned} \tag{19}$$

You have

$$\sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} = \frac{1}{1 - \lambda_{n+1}} \sum_{i=1}^{n} \lambda_i = \frac{1}{1 - \lambda_{n+1}} (1 - \lambda_{n+1}) \text{ since } \sum_{i=1}^{n+1} \lambda_i = 1$$

and

$$\frac{\lambda_i}{1 - \lambda_n} \geq 0 \text{ and } x_1, ..., x_n \in C,$$

hence by induction assumption,

$$x^0 := \sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} x_i \in C$$

Since $y^0 := x_{n+1} \in C$ by assumption you get that

$$(1 - \lambda_{n+1}) x^0 + \lambda_{n+1} y^0 \in C \tag{20}$$

because $\lambda_{n+1} \in [0, 1]$. Combining (2) and (3) you can conclued that

$$\sum_{i=1}^{n+1} \lambda_i x_i \in C$$

This completes the proof. ■

Definition 4.4 Convex hull. Let A be a subset of a vector space X. The intersection of the (nonempty) family of all convex sets of X containing A is a convex set containing A and is obviously the smallest convex set of X containing A. This is called the convex hull of A, and it is denoted by

co A or co (A) or cov A or cov (A)

Proposition 4.2 Let A be a nonempty subset of the vector space X. Then co A conicides with the set of all convex combinations of elements of A.

The following result follows from the above proposition.

Corollary 4.1 A subset C of a vector space X is convex if and only if

$$C = \text{cov } C$$

4.2 Convex Functions

Linear functions are very appealing because they are easy to manipulate and their graphs are especially simple (line in the plane for one independent variable, planes in space for two independent variables, and so on). Anyone who has ever used linear programming knows that linear functions are important in applied mathematics.

In this section, we will begin study of a class of functions, called convex function, which includes the class of linear functions but which has much wider range of applications than the class of linear functions.

Definition 4.5 Let D be a convex subset of a real vector space X and $f : D \to R$. Then,

(i) f is said to be convex (respectively strictly convex) if for each $x, y \in D$ and $\lambda \in [0, 1]$, you have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

respectively

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$$

(ii) f is concave (resp. strictly concave) if for each $x, y \in D$, and $\lambda \in [0, 1]$, you have

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

respectively

$$f(\lambda x + (1 - \lambda)y) > \lambda f(x) + (1 - \lambda)f(y)$$

Proposition 4.3 f is convex if and only if $(-f)$ is concave, and f is concave if and only if $(-f)$ if convex.

Example 4.7 If $f : X \to R$ be a linear functional defined on a linear space X, then f is a convex function. For if $0 \leq \lambda \leq 1$ and $x, y \in X$, then $\lambda x + (1 - \lambda)y \in X$, since X is convex, and linearity of f gives you that,

$$f(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y) \tag{21}$$

Which implies that $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$. Also, all linear functional are concave but neither strictly convex nor strictly concave.

Example 4.8 Every norm $\|\cdot\|$ defined on a linear space X is a convex function.

For If $x, y \in X$ and $0 \leq \lambda \leq 1$, then by triangle inequality

$$\|\lambda x + (1 - \lambda)y\| \leq |\lambda|\|x\| + |(1 - \lambda)|\|y\| = \lambda\|x\| + (1 - \lambda)\|y\|$$

5 CONCLUSION

In this unit, you have defined a norm on a linear space to make it a normed linear space. You have seen and verified different kinds of norms defined on given linear spaces. You now know how to apply properly the Holder's, Minkwoski's and Cauchy Schwartz's inequalities to verify the third property of a norm (i.e the triangle inequality) for some norms. And you have also seen how to make a linear space a metric space.

6 SUMMARY

Having gone through this unit, you now know that

(i) a norm $\|\cdot\|$ defined on a linear space X is a real valued function satisiying

(a) $\|x\| \geq 0$ for all $x \in X$, and $\|x\| = 0$ if and only if $x = 0$;

(b) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and all scalar λ;

(c) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

(ii) the pair $(X, \|\cdot\|)$ comprising of the linear space X and a norm $\|\cdot\|$ defined on it is call a normed linear space.

(iii) if $1 \leq p < \infty$, and $K = (R$ or $C)$, then the set

$$l_p(K) = \left\{ x = (x_1, x_2, ..., x_k, ...),\ x_k \in K,\ k \geq 1 : \sum_{k=1}^{\infty} |x_k|^p < \infty \right\}$$

is called an l_p space.

(iv) the l_∞ space is the space of all bounded sequences of real or complex numbers, given by

$$l_\infty = \{x = (x_1, x_2, ..., x_k, ...) : x \text{ is bounded}\}$$

(v) if $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $x = \{x_k\} \in l_p$, $y = \{y_k\} \in l_q$, then,

$$\sum_{k=1} |x_k y_k| \leq \left(\sum_{k=1} \right)^{\frac{1}{p}} \left(\sum_{k=1} |y_k|^q\right)^{\frac{1}{q}}$$

This is the Holder's Inequality. If $p = q = 2$, then the Holder's Inequality becomes the Cauchy-Schwartz inequality.

(vi) if $1 \leq p < \infty$ and $x = \{x_k\}$, $y = \{y_k\}$ are elements of l_p, then

$$\left(\sum_{k=1} |x_k + y_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1} |y_k|^p\right)^{\frac{1}{p}}$$

(vii) two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ defined on a linear space X are called equivalent if there exist constants $a, b > 0$ such that

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \text{ or } a\|x\|_2 \leq \|x\|_1 \leq b\|x\|_2$$

for all $x \in X$.

(viii) all norms defined on a finite dimensional space are equivalent.

(ix) every norm defined on a linear space X defines a metric $d\rho : X \times X \to R$ on X by

$$\rho(x, y) = \|x - y\| \text{ for all } x, y \in X$$

there by making X a metric space.

(x) a subset C of a linear space X is convex if for every $x, y \in C$ and $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in C$.

(xi) if $x_1, ..., x_n$ are n points of a linear space, then the element

$$x = \sum_{k=1}^{n} \lambda_k x_k$$

with $\lambda_k \geq 0$ for each k and $\sum_{k=1}^{n} \lambda_k = 1$, is called convex combination of the elements $x_1, ..., x_n$.

(xii) a nonempty subset C of a linear space X is convex if and only if it contains the convex combination of all its elements.

(xiii) if A is nonempty subset of a linear space, then the the intersection of all convex sets containing A gives you the convex hull of A denoted by co A

(xiv) a real-valued function $f : D \to R$ defined on a convex subset D of a linear space X is

(a) convex (resp. strictly convex) if for all $x, y \in D$, and $0 \leq \lambda \leq 1$

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

respectively

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$$

(b) concave (resp. strictly concave) if for all $x, y \in D$, and $0 \leq \lambda \leq 1$

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

respectively

$$f(\lambda x + (1 - \lambda)y) > \lambda f(x) + (1 - \lambda)f(y)$$

7 TUTOR MARKED ASSIGNMENTS (TMAs)

1. Let $R^n := \{x = (x_1, x_2, ..., x_n), x_k \in R\}$. For arbitrary element $x = (x_1, x_2, ..., x_n) \in R^n$, show that the following real valued functions;

$$\|x\|_1 = \sum_{k=1} |x_k|,$$

$$\|x\|_2 = \left(\sum_{k=1}^{n} |x_k|^2 \right)^{\frac{1}{2}}$$

and

$$\|x\|_\infty = \max_{1 \le k \le n} |x_k|$$

are norms on $\mathbb{R}^n$. Therefore, $(\mathbb{R}^n, \|\cdot\|_\infty)$, $(\mathbb{R}^n, \|\cdot\|_1)$ and $(\mathbb{R}^n, \|\cdot\|_2)$ are normed linear spaces. $(\mathbb{R}^n, \|\cdot\|_2)$ is called the Euclidean space.

2. Show that the norms in Question 1 are equivalent.

3. For arbitrary $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, define $\|\cdot\|_p : \mathbb{R}^n \to \mathbb{R}$, $p \ge 1$ as follows,

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}$$

Verify that $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$. The space $(\mathbb{R}^n, \|\cdot\|_p)$ is usually denoted by l_p^n.

4. Let $\mathbb{C}^n := \{z = (z_1, z_2, ..., z_n), z_k \in \mathbb{C}\}$ be the complex n-space, with vector addition and scalar multiplication are defined componentwise. Let $\|\cdot\|_2 : \mathbb{C}^n \to \mathbb{R}$ be defined as follows

$$\|z\|_2 = \left(\sum_{k=1}^{n} |z_k|^2 \right).$$

Verify that $(\mathbb{C}^n, \|\cdot\|_2)$ is a normed linear space. This space is usually referred to as the unitary space.

5. Let $X = C^1[a, b]$ be the space of all continuous real valued functions on $[a, b]$ which also have continous derivatives on $[a, b]$. For arbitrary $f \in C^1[a, b]$, define

$$\|f\| = \max_{a \le t \le b} |f(t)| + \max_{a \le t \le b} \frac{df(t)}{dt}.$$

Verify that $(X, \|\cdot\|)$ is a normed linear space.

6. Let $X = C[a, b]$ be the space of all continuous real-valued functions on $[a, b]$

$$\|f\|_1 = \int_a^b |f(t)|\, dt$$

Verify that $(X, \|\cdot\|_1)$ is a normed linear space.

7. Let $X = C[a, b]$ be the space of all continuous real-valued functions on $[a, b]$

$$\|f\|_2 = \left(\int_a^b |f(t)|^2 \right)^{\frac{1}{2}} dt$$

Verify that $(X, \|\cdot\|_2)$ is a normed linear space.

8. Let X be a normed linear space. Prove that for arbitrary $x, y \in X$,

(a) $|\|x\| - \|y\|| \le \|x - y\|$;

(b) The mapping $x \to \|x\|$ is continuous (in the sense that if $x_n \to x$ then $\|x\| \to \|x\|$);

(c) Addition and scalar multiplication are jointly continous, i.e., if $x_n \to x$ and $y_n \to y$ then $x_n + y_n \to x + y$; and if $x_n \to x$ and $a_n \to a$ then $a_n x_n \to ax$ as $n \to \infty$, where a_n and a are real scalars.

9. Consider vectors $x = (1, -3, 4, 1, -2)$ and $y = (3, 1, -2, -3, 1)$ in R^5. Find

   (a) $\|x\|_\infty$ and $\|y\|_\infty$,

   (b) $\|x\|_1$ and $\|y\|_1$,

   (c) $\|x\|_2$ and $\|y\|_2$,

   (d) $\rho_\infty(x, y)$, $\rho_1(x, y)$, and $\rho_2(u, v)$

10. Repeat Problem 9 for $z = (1 + i, 2 - 4i)$ and $\omega = (1 - i, 2 + 3i)$ in C^2.

11. Consider the function $f(t) = 5t - t^2$ and $g(t) = 3t - t^2$ in $C[0, 4]$.

   Find: (a) $d_\infty(f, g)$,

   (b) $d_1(f, g)$,

   (c) $d_2(f, g)$.

UNIT 2: BANACH SPACES

Contents

1 Introduction

In your course of metric spaces, you learnt that a metric space is a space with a metric (i.e., a function that measures the distance between any two point of a space) defined on it. And you also know that a complete metric space is one in which every Cauchy sequence of its element converges in it. You also learnt from the last unit that every normed linear space is a metric space. In this unit, you will be introduced to normed linear spaces in which every cauchy sequence of its element converges in it. It is this kind of normed linear spaces that you shall call complete normed linear space or Banach Spaces.

2 OBJECTIVES.

At the end of this unit, you should be able to;

(i) identify normed linear spaces that are complete and prove them.

(ii) identify incomplete normed linear spaces.

(ii) use the fact that every subspace of a complete space is complete to prove that some normed linear spaces are complete.

3 MAIN CONTENT

The following serves as a reminder of the things you know already, which will be of paramount importance to you in this unit.

Definition 3.1 Let (X, ρ) be a metric space. A sequence $\{x_n\}$ of elements of X is said to converge to a point $x \in X$, if given any > 0, there exists $N := N(\ ) \in \mathbb{N}$ such that $n \geq N$ implies that

$$\rho(x_n, x) <$$

Definition 3.2 Let (X, ρ) be a metric space. A sequence $\{x_n\}$ of elements of X is called Cauchy if given any > 0, there exists $N := N(\ ) \in \mathbb{N}$ such that $n, m > N$ implies that

$$\rho(x_n, x_m) < \ .$$

Definition 3.3 Let (X, ρ) be a metric space. Let $E \subset X$. E said to be closed if and only if every sequence $\{x_n\}$ of elements of E converges in E.

That is E is closed if and only if given an arbitrary sequence $\{x_n\}$ of elements of E such that $x_n \to x$ in X, then $x \in E$.

Definition 3.4 Let E be a subset of a metric space (X, ρ). E is called complete if every Cauchy sequence of elements of E converges in E.

Theorem 3.1 Let (X, ρ) be a complete metric space, and let $E \subset X$. (E, ρ_E) is complete if and only if it is closed. Where ρ_E is the subspace metric induced by ρ.

Proof. $(\Rightarrow)$ Assume that E is complete, Let $\{x_n\}$ be a sequence of elements of E such that $x_n \to x \in X$ as $n \to \infty$. $\{x_n\}$ is a Cauchy sequence in X as well as in E since every convergent sequence of a metric space is Cauchy. E is complete gives you that $x \in E$.

$(\Leftarrow)$ For this direction, assume that E is closed, you have to show that E is complete. Now take any Cauchy sequence $\{x_n\}$ of elements of E. You have that $\{x_n\}$ is also Cauchy in X, thus, $x_n \to x \in X$ as $n \to \infty$. But by your assumption that E is closed, you have that $x \in E$. $\blacksquare$

4 COMPLETE NORMED LINEAR SPACE.

You have taken time to go through the preamble, you can now deduce that as a metric space (X, ρ), where ρ is the metric induced by the norm $\|\cdot\|$, defined on X, a complete normed linear space is one which contains the limit of any Cauchy sequence of its elements.

Now, when you are confronted with the problem of verifying that the metric space (X, ρ) is complete, you have to take an arbitrary Cauchy sequence $\{x_n\}$ of elements of X and show that it converges to a point of X. The general pattern is the following:

(a) Construct an element x^* which you will use as the limit of the Cauchy sequence,

(b) prove that x^* is in the space under consideration,

(c) prove that $x_n \to x^*$ (in the sense of the metric under consideration).

For you to construct x^* as mentioned in (a), you will use the fact that given sequence $\{x_n\}$ is Cauchy to generate a Cauchy sequence in a complete space that is associated with the normed linear space under consideration. (This complete space is usually the real, R or the complex numbers, C). Once you have successfully constructed x^*, steps (b) and (c) are generally not too difficult to complete.

Take a little step further and you will see some examples, they are rigorously simple and interesting.

4.1 COMPLETENESS OF R^n AND C^n

Theorem 4.1 The real n-space R^n equipped with the Euclidean norm,

$$\|x\|_2 = \left(\sum_{k=1}^{n} |x_k|^2\right)^{\frac{1}{2}}$$

for arbitrary $x = (x_1, x_2, ..., x_n) \in R^n$, is complete.

The metric induced by $\|\cdot\|_2$ is

$$\rho(x^{(r)}, x^{(s)}) = \|x^{(r)} - x^{(s)}\|_2 = \left(\sum_{k=1} |x_k^{(r)} - x_k^{(s)}|^2\right)^{\frac{1}{2}} \qquad (1)$$

Proof. You know that $R^n = \{x = (x_1, x_2, ..., x_n), x_k \in R^n\}$. To proceed with this proof, you have to first of all take an arbitrary Cauchy sequence of elements of R^n. So let $x^{(m)} =$

$x^{(1)}, x^{(2)}, ..., x^{(m)}, ...$ be an arbitrary Cauchy sequence in R^n where

$$
\begin{aligned}
x^{(1)} &= (x_1^{(1)},\ x_2^{(1)},\ x_3^{(1)},\ ...,\ x_n^{(1)}) \\
x^{(2)} &= (x_1^{(2)},\ x_2^{(2)},\ x_3^{(2)},\ ...,\ x_n^{(2)}) \\
&\ \ .\qquad .\qquad .\qquad .\qquad . \\
x^{(m)} &= (x_1^{(m)},\ x_2^{(m)},\ x_3^{(m)},\ ...,\ x_n^{(m)}) \\
&\ \ .\qquad .\qquad .\qquad .\qquad .
\end{aligned}
$$

$\{x^{(m)}\}$ being a Cauchy sequence implies that given > 0, there exists an integer N such that for all $r, s \geq N$, we have

$$\rho(x^{(r)}, x^{(s)}) = \|x^{(r)} - x^{(s)}\|_2 = \left(\sum_{k=1}^{n} |x_k^{(r)} - x_k^{(s)}|^2\right)^{\frac{1}{2}} <$$

This implies that

$$\sum_{k=1}^{n} \left| x_k^{(r)} - x_k^{(s)} \right|^2 <$$

i.e. for each $k \in \{1, 2, ..., n\}$

$$\left| x_k^{(r)} - x_k^{(s)} \right|^2 < , \qquad \text{for all } r, s \geq N$$

This means that for each $k \in \{1, 2, ..., n\}$, $\{x_k^{(m)}\}$ is a Cauchy sequence in R. and since R is complete, it follows that $x_k^{(m)} \to x_k^* \in R$ as $m \to \infty$, i.e.,

$$
\begin{array}{ccccccc}
x^{(m)} = & (x_1^{(m)}, & x_2^{(m)}, & x_3^{(m)}, & ..., & x_n^{(m)}) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
 & (x^*_1, & x^*_2, & x^*_3, & ..., & x^*_n)
\end{array}
$$

Now define $x^* = (x_1^*, x_2^*, x_3^*, ..., x_n^*)$ and you have completed step (a).

Observe that you have used the completeness of R to get x_k; $k = 1, 2, ..., n$ and this helped you to define x^*. Proceed now to step (b).

Step (b)

You have to show that x^* is in R^n. This is simply obvious because each x^*_k's are in R, making, $x^* = (x_1^*, x_2^*, ..., x_n^*) \in R^n$, and that's it you have completed step (b). You can now proceed to the final step, i.e., step (c).

Step (c)

Here, you have to show that $x^{(m)} \to x^*$ as $m \to \infty$. Already, you have that for each $k \in \{1, 2, ..., n\}$ fixed, $x_k^{(m)} \to x_k^*$ as $m \to \infty$ which means that given any > 0, there exists $N_k \in N$ such that for all $m \geq N_1$

$$x_k^{(m)} - x_k^* < \frac{}{n^{\frac{1}{2}}} \quad \text{(where n is the one appearing in } R^n \text{ and is fixed).}$$

Now, you can choose $N = \max_{1 \leq k \leq n} N_k$, then for all $m > N$,

$$\rho(x^{(m)}, x^*) = \left(\sum_{k=1}^{n} |x_k^{(m)} - x_k^*|^2 \right)^{\frac{1}{2}}$$

$$< \left(\sum_{k=1}^{n} \frac{2}{n} \right)^{\frac{1}{2}} = n \cdot \left(\frac{2}{n} \right)^{\frac{1}{2}} =$$

i.e.,

$$\rho(x^{(m)}, x^*) < \quad \text{for all} \quad m \geq N.$$

Hence you have that $x^{(m)} \to x^*$ as $m \to \infty$.

Since $x^* \in R^n$ (you got this in step (b)) and the Cauchy sequence $\{x^{(m)}\}$ of elements of R^n is arbitrary, you have that R^n is complete.

Following this same approach, you can prove that the complex n-space C^n with the norm

$$\|z\| = \left(\sum_{k=1}^{n} |z_k|^2 \right)^{\frac{1}{2}}$$

is complete

4.2 Completeness of l_∞ .

Theorem 4.2 The space l_∞ is complete.

Proof. Let $\{x^{(m)}\}_{m=1}^{\infty}$ be a Cauchy sequence in l_∞ , where

$$x^{(1)} = (x_1^{(1)}, \; x_2^{(1)}, \; x_3^{(1)}, \; ..., \; x_n^{(1)}, \; ...)$$

$$x^{(2)} = (x_1^{(2)}, \; x_2^{(2)}, \; x_3^{(2)}, \; ..., \; x_n^{(2)}, \; ...)$$

$$. \qquad . \qquad . \qquad . \qquad .$$

$$x^{(m)} = (x_1^{(m)}, \; x_2^{(m)}, \; x_3^{(m)}, \; ..., \; x_n^{(m)}, \; ...)$$

$$. \qquad . \qquad . \qquad . \qquad .$$

$\{x^{(m)}\}_{m=1}^{\infty}$ is Cauchy implies that given any > 0, there exists N such that for all $r, s \geq N$,

$$\rho(x^{(r)}, x^{(s)}) = \sup_{k \geq 1} |x_k^{(r)} - x_k^{(s)}| < .$$

i.e., for each k fixed,

$$|x_k^{(r)} - x_k^{(s)}| \leq \sup_{k \geq 1} |x_k^{(r)} - x_k^{(s)}| < \quad \text{for all } r, s \geq N$$

This implies that for each $k \geq 1$ fixed, $\{x_k^{(n)}\}^{\infty}$ is a Cauchy sequence of elements of R and by the completeness of R, $x_k^{(m)} \to x_k^* \in R$ as $m \to \infty$ (i.e., $\{x_k^{(m)}\}$ converges to the point x_k^* in R.). This is illustrated by the array below

$$\begin{array}{ccccccccc} x^{(m)} & = & (x_1^{(m)}, & x_2^{(m)}, & x_3^{(m)}, & \ldots, & x_n^{(m)}, & \ldots) \\ & & \downarrow & \downarrow & \downarrow & & \downarrow & \\ & & (x_1^*, & x_2^*, & x_3^*, & \ldots, & x_n^*, & \ldots) \end{array}$$

Define $x^* = (x_1^*, x_2^*, x_3^*, \ldots, x_n^*, \ldots)$.

Step (b)

For this step, you have to show that $x^* \in l_\infty$. Now $\{x^{(m)}\}_{m=1}^{\infty} \in l_\infty$ implies that $\|x^{(m)}\|_\infty \leq t_m$ for each m, which implies that $|x_k^{(m)}| \leq t_m$ for each k. Also $x_k^{(m)} \to x_k^*$ as $m \to \infty$, so that

$$|x_k^*| \leq |x_k^* - x_k^{(m)}| + |x_k^{(m)}| < + t_m$$

This inequality holds for every k and the right-hand side is independent of k. Hence $\{x_k^*\}_{k=1}^{\infty}$ is a bounded sequence of numbers. This implies $x^* \in l_\infty$. So you have completed step (b).

Step (c)

From the convergence of $\{x_k^{(m)}\}$ you have that given any > 0, there exist $N \in \mathbb{N}$ such that

$$|x_k^{(m)} - x_k^*| <$$

for all $m \geq N$. This implies that

$$\rho(x^{(m)}, x^*) = \sup_{k \geq 1} |x_k^{(m)} - x_k^*| \leq \quad \text{for all } m \geq N.$$

This shows that $x^{(m)} \to x^*$ in l_∞. Since $\{x^{(m)}\}_{m=1}^{\infty}$ is arbitrary, l_∞ is complete ▮

4.3 Completeness of C[a, b]

Theorem 4.3 The space $C[a,b]$ of continous real valued function defined on [a,b] is complete if endowed with the sup norm

$$\|f\|_0 = \max_{a \leq t \leq b} |f(t)|$$

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $C[a,b]$. For each $t \in [a,b]$, there exists a positive integer N such that given > 0,

$$\|f_n - f_m\|_0 = \sup_{a \le t \le b} |f_n(t) - f_m(t)| < \text{ , for all } n, m \ge N.$$

Hence, for any fixed $t_0 \in [a,b]$, $|f_n(t_0) - f_m(t_0)| <$ for all $n, m \ge N$. This shows that $\{f_n(t_0)\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers. Since R is complete, $\{f_n(t_0)\}_{n=1}^{\infty}$ converges to a real number, say $f(t_0)$ as $n \to \infty$, i.e.,

$$f_n(t_0) \to f(t_0) \text{ as } n \to \infty.$$

This is the same as saying that the function f_n converges pointwise to the function f. You have to prove next that this pointwise convergence is actually uniform in $t \in [a,b]$, i.e., given any > 0, there is an integer N^* such that $\sup_{a \le t \le b} |f_n(t) - f(t)| <$ for all $n \ge N^*$. So given any > 0, choose N such that $\|f_n - f_m\| < \frac{}{2}$ for $n, m \ge N$. Then for $n > N$,

$$\begin{aligned} |f_n(t) - f(t)| &\le |f_n(t) - f_m(t)| + |f_m(t) - f(t)| \\ &\le \sup_{a \le t \le b} |f_n(t) - f_m(t)| + |f_m(t) - f(t)| \\ &= \|f_n - f_m\| + |f_m(t) - f(t)| \end{aligned}$$

By choosing m sufficiently large (m may depend on t) each term on the right-hand side can be make less than $\frac{}{2}$ so that $\sup_{a \le t \le b} |f_n(t) - f(t)| <$ for all $n \ge N$. Hence, the convergence is uniform, i.e.,

$f_n \to f$ uniformly on $[a,b]$. Hence, $C[a,b]$ is complete. ∎

5 INCOMPLETE NORMED LINEAR SPACES.

In the last section, you saw that a normed linear space $(X, \|\cdot\|)$ is complete if every Cauchy sequence of its elements converges in the space. This means that you will call $(X, \|\cdot\|)$ if you can get a Cauchy sequence of elements of X the a does not converge in X. Below are examples of some normed linear space that are incomplete.

Example 5.1 The space $C[a,b]$ of continuous real valued functions defined on $[a,b]$ endowed with any of the integral norms

$$\|f\|_1 = \int_a^b |f(t)|dt$$

and

$$\|f\|_2 = \left(\int_a^b |f(t)|^2 dt\right)^{\frac{1}{2}}$$

is incomplete.

You can illustrate this examples by taken particular cases for $[a,b]$, as in the examples below.

Example 5.2 Let $X = C[-1,1]$ be the space of all continuous real-valued functions defined on the closed and bounded interval $[-1,1]$ with norm $\|\cdot\|_1$ given by

$$\|f\|_1 = \int_{-1}^{1} |f(t)|dt,\ t \in [-1,1],\ f \in C[-1,1]$$

Then the space $(X, \|\cdot\|_1)$ is not complete.

Solution. Following the comment at the begining of this section, you see that it is enough for you to produce a Cauchy sequence of elements of $C[-1,1]$ that does not converge to an element of $C[-1,1]$. Consider the function f_n given by

$$f_n(t) = \begin{cases} 0, & \text{if } -1 \le t \le 0; \\ nt, & \text{if } 0 \le t \le \frac{1}{n}; \\ 1, & \text{if } \frac{1}{n} \le t \le 1 \end{cases}$$

for each $n \in N$. The function is sketched in figure 1. First of all you have to check $f_n \in C[-1,1]$ for each $n \in N$. That is you must show that $\{f_n\}$ is a sequence of continuous real valued function defined on $[-1,1]$ The proof that f_n is continuous for each $n \in N$ is done by a very important lemma you studied in metric spaces called pasting lemma.

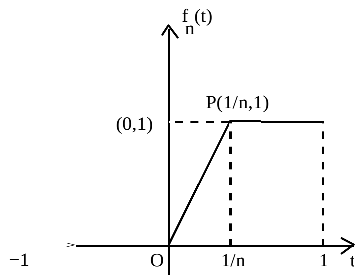

Figure 1:

To see this, you can set $A_1 = [-1,0]$, $A_2 = [0,\frac{1}{n}]$ and $A_3 = [\frac{1}{n},1]$ $n \in N$. Then for all $t \in A_1$, $f_n(t) = 0$, implies that f_n is continuous on A_1 as a constant function; also for all $t \in A_2$, $f_n(t) = nt$, is continuous; and for all $t \in [\frac{1}{n},1]$, $f_n(t) = 1$ is also continuous. A_1, A_2 and A_3 are closed sets in R, $A_1 \cap A_2 = \{0\}$ and $f_n(0) = 0$, $A_2 \cap A_3 = \{\frac{1}{n}\}$ and $f_n(\frac{1}{n}) = 1$, and $[-1,1] = [-1,0] \cup [0,\frac{1}{n}] \cup [\frac{1}{n},1]$.
Therefore by pasting lemma, f_n is continuous on $[-1,1]$ for each $n \in N$. Hence, $f_n \in C[-1,1]$ for
each $n \in N$.

Next is to show that $\{f_n\}$ is a Cauchy sequence in $C[-1,1]$. So let $m > n$ so that $\frac{1}{m} < \frac{1}{n}$ You have to show that $\|f_n - f_m\|_1 \to 0$ as $n, m \to \infty$

$$0 \le \|f_n - f_m\|_1 = \int_{-1}^{1} |f_n(t) - f_m(t)|dt$$

But this integral represents the area between f_n and f_m. So from figure 2, you have that

$$\|f_n - f_m\|_1 = \int_{-1}^{1} |f_n(t) - f_m(t)|dt = \text{Area of } \triangle OCD = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{m}\right) \to 0 \text{ as } n, m \to \infty$$

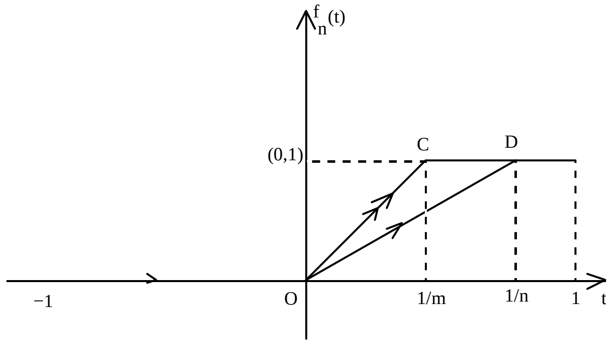

Figure 2:

You can also compute $\int_{-1}^{1} |f_n(t) - f_m(t)|dt$ directly as follows

$$
\begin{aligned}
\|f_n(t) - f_m(t)\|_1 &= \int_{-1}^{1} |f_n(t) - f_m(t)|dt \\
&= \int_{-1}^{0} |0 - 0|dt + \int_{0}^{\frac{1}{m}} |mt - nt|dt + \int_{\frac{1}{m}}^{\frac{1}{n}} |1 - nt|dt + \int_{\frac{1}{n}}^{1} |1 - 1|dt \\
&= \int_{0}^{\frac{1}{m}} (m - n)tdt + \int_{\frac{1}{m}}^{\frac{1}{n}} (1 - nt)dt \\
&= \left[\frac{(m-n)}{2}t^2\right]_0^{\frac{1}{m}} \pm \left[t - \frac{n}{2}t^2\right]_{\frac{1}{n}}^{\frac{1}{m}} \\
&= \frac{(m-n)}{2}\frac{1}{m^2} \pm \frac{1}{m} \; \frac{n}{2}\frac{1}{m^2} - \frac{1}{n} + \frac{n}{2}\frac{1}{n^2} \to 0 \text{ as } n, m \to \infty
\end{aligned}
$$

Thus $\{f_n\}$ is a Cauchy sequence of elements of $C[-1, 1]$.

Next is for you to show that $\{f_n\}$ converges to an element that is not in $C[-1, 1]$. If $\{f_m\}_{m=1}^{\infty}$ is convergent, in order to find the candidate for its limit, you should examine $f_n(t)$ defined above. If $n \to \infty$, the interval $0 \leq t \leq \frac{1}{n}$ reduces to $t = 0$ and the interval $\frac{1}{n} \leq t \leq 1$ becomes $0 < t \leq 1$. You can there conclude that $f : [-1, 1] \to [0, 1]$ defined by

$$
f(t) = \begin{cases} 0, & \text{if } -1 \leq t \leq 0; \\ 1, & \text{if } 0 < t \leq 1. \end{cases}
$$

is a candidate for the limit.

Clearly, $\|f_n - f\|_1 = \int_{-1}^{1} |f_n(t) - f(t)|dt =$ area of $\triangle OAB$ $\frac{1}{2}\frac{1}{n} \to 0$ as $n \to \infty$.

Thus the sequence $\{f_n\}_{n=1}^{\infty}$ defined above is a Cauchy sequence in $C[-1, 1]$ which converges to f. However, f is not in $C[-1, 1]$ since f is not continuous at 0, you can check this. Thus $C[-1, 1]$ is not complete.

Example 5.3 Let $X = C[-3, 3]$ with $\|f\|_2 = \left(\int_{-3}^{3} |f(t)|^2\right)^{\frac{1}{2}}$. Then X is not complete.

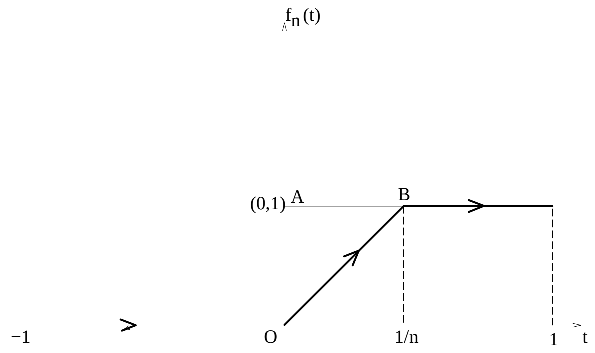

Figure 3: Ã

Solution. You can easily verify, using Cauchy Schwartz inequality for integrals, that

$$\int_{-3}^{3}|f(t)|dt \leq \sqrt{6}\left(\int_{-3}^{3}|f(t)|^2dt\right)^{\frac{1}{2}}$$

It follows then from this inequality and the sequence $\{f_n\}$ in last example that $X = C[-3, 3]$ is not complete. This is because the above inequality shows that if $\{f_n\}$ does not converge in X with $\|\cdot\|_1$, then it will not converge with $\|\cdot\|_2$. And you have seen in the last example that $\{f_n\}$ does not converge with $\|\cdot\|_1$.

6 CONCLUSION

In this unit, you have seen some examples of a Banach spaces and some examples of incomplete normed linear spaces. You have also seen that a closed subspace of a complete normed linear space is also complete.

7 SUMMARY

Having gone through this unit, you now know that;

(i) a normed linear space $(X, \|\cdot\|)$ is complete if every Cauchy sequence converges in it.

(ii) the real $n-$ space, R^n equipped with the euclidean norm,

$$\|x\|_2 = \left(\sum_{k=1}^{n}|x_k|^2\right)^{\frac{1}{2}}$$

is complete.

(iii) the complex $n-$ space, C^n equipped with the norm

$$\|z\|_2 = \left(\sum_{k=1}^{n}|z_k|^2\right)^{\frac{1}{2}}$$

is complete.

(iv) The space l_∞ endowed with the sup − norm is complete.

(v) The space $C[a,b]$ of continuous real valued function defined on $[a,b]$ is complete when endowed with the sup − norm

$$\|f\|_\infty = \sup_{a\leq t\leq b} |f(t)|$$

(vi) The normed linear space $X = C[a,b]$ of all continuous real valued function is not complete when endowed with any of the integral norms

$$\|f\|_1 = \int_a^b |f(t)|\, dt$$

and/or

$$\|f\|_2 = \left(\int_a^b |f(t)|^2\, dt\right)^{\frac{1}{2}}$$

8 EXERCISES TMAs

1. Let $X = C[-2,2]$ be the space of real-valued continuous functions on the closed and bounded interval $[-2,2]$, and let X be endowed with the norms

   (a) $\|f\|_1 = \int_{-2}^{2} |f(t)|dt$, and

   (b) $\|f\|_2 = \left(\int_{-2}^{2} |f(t)|^2 dt\right)^{\frac{1}{2}}$

   for arbitrary f in X. Justify the following statements:

   (a) $(X, \|\cdot\|_1)$ is not complete;

   (b) $(X, \|\cdot\|_2)$ is not complete.

2. Let $X = C[0,4]$, be the space of real-valued functions defined on a closed and bounded interval $[0,4]$ and suppose X is endowed with the norm,

$$\|f\|_2 = \left(\int_0^4 |f(t)|^2 dt\right)^{\frac{1}{2}}$$

   Consider the sequence $\{g_n\}_{n=1}^\infty$ defined by the following graph

   (a) Write down the explicit expression for $g_n(t)$, $0 \leq t \leq 4$.

   (b) Verify that

      i. $g_n \in C[0,4]$ for each n;

      ii. $\{g_n\}_{n=1}^\infty$ is a Cauchy sequence;

      iii. $g_n \to g$ as $n \to \infty$, where

$$g(t) = \begin{cases} 0, & 0 \leq t \leq 2 \\ 1, & 2 < t \leq 4. \end{cases}$$

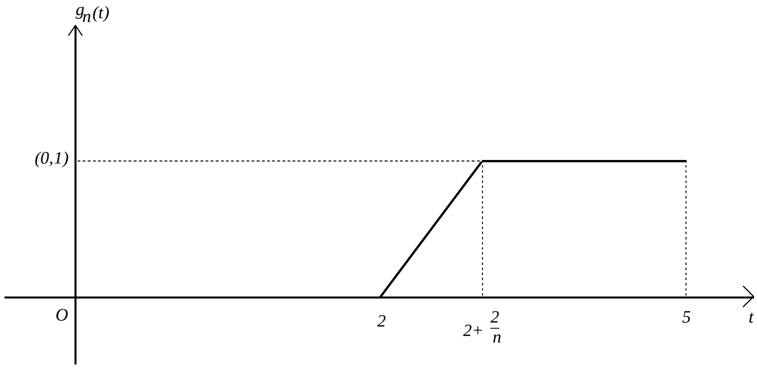

Figure 4:

Conclude that $C[0,4]$ with the norm is not a complete space. Give another proof (similar to Example 1.19) that $C[0,4]$ with $\|f\|_2 := \left(\int_0^4 |f(t)|^2 dt\right)^{\frac{1}{2}}$ is not complete.

3. Let $X = C[0,5]$ be the space of real-valued continuous functions on the closed and bounded interval $[0,5]$. Let X be endowed with norms,

$$\|f\|_1 = \int_0^5 |f(t)| dt,$$

$$\|f\|_2 = \left(\int_0^5 |f(t)|^2 dt\right)^{\frac{1}{2}}$$

for arbitrary f in X. Show that

(a) $(X, \|\cdot\|_1)$ is not complete.

(b) $(X, \|\cdot\|_2)$ is not complete.

(Hint: You may use Fig. 5).

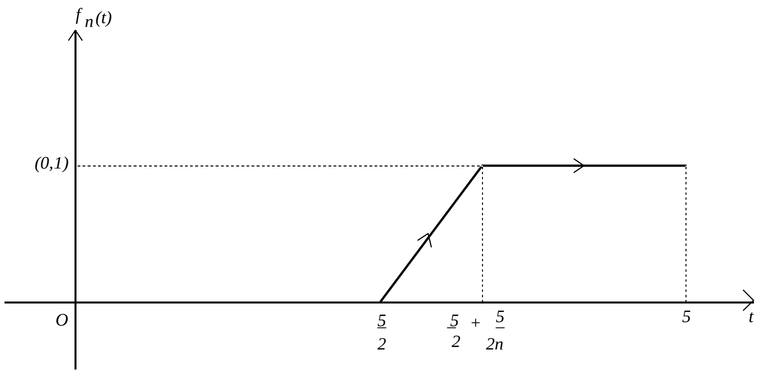

Figure 5:

4. Let S be the set of real sequences having only a finite number of nonzero terms.

(a) Show that $S \subset l_\infty$.

(b) Take a sequence $\{x^n\}$ in S, where $x^n = (1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, 0, 0, ...)$. Prove that:

i. $\{x^n\}$ is a Cauchy sequence in S;
ii. $x^n \to x$ where $x := (1, \frac{1}{2}, \frac{1}{3}, ...)$.

Conclude that S is not complete.

5. Prove the following statements;

(a) C^n the complex n-space endowed with the norm

$$\|z\|_2 = \left(\sum_{k=1}^{n} |z_k|^2\right)^{\frac{1}{2}}$$

for arbitrary $z = (z_1, z_2, ..., z_n) \in C^n$ is complete.

(b) $l_p (1 \leq p < \infty)$ is complete.

(c) c is complete, where c is the space of all convergent sequences endowed with the norm

$$\|x\| = \sup_{k\geq 1} |x_k|$$

for arbitrary sequence $x = (x_1, x_2, ..., x_n, ...) \in c_0$ [Hint, you can show that c is a closed supspace of l_∞ and then apply theorem 3.1.]

(d) c_0 is complete; where c_0 is the space of all sequences that converges to 0 endowed with the norm

$$\|x\| = \sup_{k\geq 1} |x_k|$$

for arbitrary sequence $x = (x_1, x_2, ..., x_n, ...) \in c_0$ [Hint, you can show that c_0 is a closed supspace of c and then apply theorem 3.1.]

6. Let $X = R^5$. X is not complete when endowed with the norm

(a) $\|x\|_1 = \sum_{k=1}^{5} |x_k|$

(b) $\|x\|_2 = \left(\sum_{k=1}^{5} |x_k|^2\right)^{\frac{1}{2}}$

(c) $\|x\|_\infty = \max_{1\leq k\leq 5} |x_k|$

(d) None

7. Let $X = C[0, 1]$, the space of contiuous functions on $[0, 1]$ Consider the sequence of functions $\{u_n\} \subset X$, given by

$$u_n(t) = \begin{cases} 0, & 0 \leq t \leq \frac{1}{2} - \frac{1}{n} \\ \frac{nt}{2} - \frac{n}{4} + \frac{1}{2}, & \frac{1}{2} - \frac{1}{n} \leq t \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} \leq t \leq 1. \end{cases}$$

This sequence is converges in X when endowed with the norm

8 Tutor Marked Assignments (TMAs)

(a) $\|f\|_1 = \int_a^b |f(t)|\,dt.$

(b) $\|f\|_2 = \left(\int_0^1 |f(t)|^2\,dt\right)^{\frac{1}{2}}$

(c) $\|f\|_\infty = \max_{0\le t\le 1} |f(t)|$

(d) none of the above.

UNIT 3: LINEAR FUNCTIONAL SPACES

Contents

1 INTRODUCTION

Linear maps play a crucial role in Functional Analysis. In this unit, you shall be introduced with the concept of linear maps, Bounded linear maps, linear functionals and a special class of Banach spaces called linear functional space. You shall also be presented with some of their basic properties.

2 OBJECTIVES.

At the end of this unit, you should be able to;

1. Identify maps that are linear.
2. Show that a given linear is bounded or not.
3. Identify bounded linear functionals on finite dimensional space
4. Identify bounded linear functionals on infinite dimensional space.
5. Compute the norms of bounded linear maps.

3 LINEAR MAPS

3.1 Definition and some examples.

Definition 3.1 Let X and Y be linear spaces over the scalar field K. A mapping $T : X \to Y$ is said to be a linear map if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y). \tag{1}$$

for arbitrary $x, y \in X$ and arbitrary scalars $\alpha, \beta \in K$.

Linear maps can also be called Linear transformations or linear operators.

Condition (1) is equivalent to the following two conditions

1. $T(x + y) = T(x) + T(y)$ for all $x, y \in X$;
2. $T(\alpha x) = \alpha T(x)$ for all $x \in X$ and $\alpha \in K$.

Definition 3.2 (Linear functionals). The linear map $T : X \to K$ from the vector space X to the scalar field K is called a linear functional.

In the definition (3.2), if T is called real or complex linear functional according as K is either R or C.

Example 3.1 Let $X = l_2$. For each $\overline{x} = (x_1, x_2, ..., x_k, ...)$ in l_2 define

$$T\overline{x} = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, ...).$$

Then T is a linear map on l_2.

Verification. Let $\overline{x} = (x_1, x_2, ..., x_k, ..)$ and $\overline{y} = (y_1, y_2, ..., y_k, ...)$ be arbitrary elements of l_2 and let α, β be scalars. Then

$$\begin{aligned}\alpha\overline{x} + \beta\overline{y} &= (\alpha x_1, \alpha x_2, ..., \alpha x_k, ...) + (\beta y_1, \beta y_2, ..., \beta y_k, ...)\\ &= (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, ..., \alpha x_k + \beta y_k, ...)\end{aligned}$$

and

$$\begin{aligned} T(\alpha\bar{x} + \beta\bar{y}) &= \left(0, \tfrac{\alpha x_1+\beta y_1}{1}, \tfrac{\alpha x_2+\beta y_2}{2}, \ldots, \tfrac{\alpha x_k+\beta y_k}{k}, \ldots\right) \\ &= \left(0, \alpha x_1, \tfrac{\alpha x_2}{2}, \ldots, \tfrac{\alpha x_k}{k}, \ldots\right) + \left(0, \beta y_1, \tfrac{\beta y_2}{2}, \ldots, \tfrac{\beta y_k}{k}, \ldots\right) \\ &= \alpha\left(0, x_1, \tfrac{x_2}{2}, \ldots, \tfrac{x_k}{k}, \ldots\right) + \beta\left(0, y_1, \tfrac{y_2}{2}, \ldots, \tfrac{y_k}{k}, \ldots\right) \\ &\alpha T(\bar{x}) + \beta T(\bar{y}). \end{aligned}$$

and so T is a linear map.

Example 3.2 Differentiation, Integration and limits are examples of linear maps.

Example 3.3 Let C denote the linear space of complex numbers over C, and define the map $T : C \to C$ by $Tz = \bar{z}$, where $\bar{z}$ denotes the conjugate of z. Then T isnot a linear map.

Remark 3.1 Note that since linear functionals are special forms of linear maps, any result proved for linear maps also holds for linear functionals.

Proposition 3.1 Let X and Y be two linear spaces over scalar field, K, and let $T : X \to Y$ be a linear map. Then

(i) $T(0) = 0$;

(ii) The range of T, $R(T) = \{y \in Y : Tx = y \text{ for some } x \in X\}$ is a linear subspace of Y;

(iii) T is one-to-one if and only if $T(x) = 0$ implies $x = 0$;

(iv) If T is one-to-one, then T^{-1} exists on $R(T)$ and $T^{-1} : R(T) \to X$ is also a linear map.

Proof.

(i) Since T is linear, you have $T(\alpha x) = \alpha T(x)$ for each $x \in X$ and each scalar α. Take $\alpha = 0$ and (i) follows immediately.

(ii) You need to show that for $y_1, y_2 \in R(T)$ and α, β scalars, $\alpha y_1 + \beta y_2 \in R(T)$. Now let $y_1, y_2 \in R(T)$. This implies that there exist $x_1, x_2 \in X$ such that $T(x_1) = y_1, T(x_2) = y_2$. Moreover, $\alpha x_1 + \beta x_2 \in X$ (since X is a linear space). Furthermore, by the linearity of T,

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2) = \alpha y_1 + \beta y_2.$$

Hence $\alpha y_1 + \beta y_2 \in R(T)$, and so $R(T)$ is a linear subspace of Y.

(iii) ($\Rightarrow$) Assume T is one-to-one. Clearly, $Tx = 0$ implies $T(x) = T(0)$ since T is linear (and so $T(0) = 0$). But T is one-to-one. So, $x = 0$.

($\Leftarrow$) Assume that whenever $Tu = 0$, then u must be 0. You have to prove that T is one- to-one. So, Let $Tx = Ty$. Then, $Tx - Ty = 0$ and by the linearity of T, $T(x - y) = 0$. By hypothesis, $x - y = 0$ which implies that $x = y$. Hence T is one-to-one.

(iv) Without loss of Generality, assume that X is the domain of T, otherwise we can take the restriction of T on $D(T)$, the domain of T. Suppose $T : X \to Y$ is one-to-one, then $T : X \to R(T)$ is both one-to-one and onto. Thus, for every $y \in R(T)$, there exists a unique $x^* \in X$ such that $T(x^*) = y$. Thus if $Tx = y$ then $x = x^*$ So let $S : R(T) \to X$ be defined by $S(y) = x^*$. Now,

(a) $(T \circ S)(y) = T(S(y)) = T(x^*) = y$ for every $y \in R(T)$, hence $T \circ S = 1_{R(T)}$.

(b) $(S \circ T)(x) = S(T(x)) = S(y) = x^* = x$ for every $x \in X$, hence $S \circ T = 1_X$.

Accordingly T has an inverse and $T^{-1} = S$.

Now to show that $T^{-1} : R(T) \to X$ is a linear map, let $y_1, y_2 \in R(T)$ and $\alpha, \beta \in K$, $\alpha y_1 + \beta y_2 \in R(T)$ (by (ii)). Since $T : X \to R(T)$ is one-to-one and onto, there exists unique vectors $x_1, x_2 \in X$ such that $y_1 = Tx_1$ and $y_2 = Tx_2$. Linearity of T gives you that

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = \alpha y_1 + \beta y_2.$$

By the definition of inverse mapping,

$$T^{-1} y_1 = x_1, T^{-1} y_2 = x_2, \text{ so that } T^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2 = \alpha T^{-1} y_1 + \beta T^{-1} y_2$$

Thus

$$T^{-1}(\alpha y_1 + \beta y_2) = \alpha T^{-1} y_1 + \beta T^{-1} y_2$$

Hence T^{-1} is linear. ■

Example 3.4 Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = ax + b$; $a, b \in \mathbb{R}$, $b = 0$. Then T is not a linear map. It suffices to observe that $T(0) = b = 0$.

4 BOUNDED LINEAR MAPS

Definition 4.1 Let X and Y be normed linear spaces over a scalar field K, and let $T : X \to Y$ be a linear map. Then T is said to be bounded if there exist some constant, $K \geq 0$ such that for each $x \in X$,

$$\|T(x)\| \leq K\|x\|, \tag{2}$$

the constant K is called a bound for T and in this case, T is called a bounded linear map.

The next thing to discuss is linear maps that are continuous. The following theorem is very useful in identifying continuous linear maps.

Theorem 4.1 Let X and Y be normed linear spaces and let $T : X \to Y$ be a linear map. Then the following are equivalent:

(i) T is continous;

(ii) T is continous at the origin 0 (in the sense that if $\{x_n\}$ is a sequence in X such that $x_n \to 0$ as $n \to \infty$, then $Tx_n \to 0$ in Y as $n \to \infty$);

(iii) T is Lipschitz, i.e., there exists a constant $K \geq 0$ such that, for each $x \in X$,

$$\|Tx\| \leq K\|x\|;$$

(iv) If $D := \{x \in X : \|x\| \leq 1\}$ is the closed unit disc in X, then $T(D)$ is bounded (in the sense that there exists a constant $M \geq 0$ such that $\|Tx\| \leq M$ for all $x \in D$).

Proof. (i) $\Rightarrow$ (ii). Let $T : X \to Y$ be a linear and continuous. You have to prove that T is continuous at 0. So, let $\{x_n\}$ be a sequence in X such that $x_n \to 0$. By continuity of T, we have, $Tx_n \to T(0)$. But T is a linear map so that $T(0) = 0$. Hence, the result follows.

(ii) $\Rightarrow$ (iii) You have that if $\{x_n\}$ is any sequence in X such that $x_n \to 0$ as $n \to \infty$, then $T(x_n) \to 0$. You have to prove that there exists a constant $K \geq 0$ such that $\|Tx\| \leq K\|x\|$ for each $x \in X$. Suppose for contradiction that there is no such K. Then for any positive integer n, there exists some $x = x(n) \in X$, $x(n)$ different from 0, call it x_n (since it depends on n) such that

$$\|Tx_n\| > n\|x_n\|.$$

This implies that

$$\frac{\|Tx_n\|}{n\|x_n\|} > 1.$$

Now define the sequence $u_n := \frac{x_n}{n\|x_n\|}$. Clearly, $u_n \to 0$. For,

$$\|u_n - 0\| = \left\|\frac{x_n}{n\|x_n\|}\right\| = \frac{1}{n}\frac{\|x_n\|}{\|x_n\|} = \frac{1}{n} \to 0.$$

However, $Tu_n \nrightarrow 0$. For,

$$\|Tu_n - 0\| = \left\|\frac{T(x_n)}{n\|x_n\|}\right\| = \frac{\|T(x_n)\|}{n\|x_n\|} > 1,$$

and so $Tu_n \nrightarrow 0$ (even $u_n \to 0$). This contradicts the hypothesis that T is continous at the origin. Thus our supposition is false and so, (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv) Given that a linear map T is Lipschitz we want to prove that $T(D)$ is bounded, (i.e., $\|T(x)\| \leq M$ for all $x \in D$, and some constant $M \geq 0$). Take $y \in D$ arbitrary. Then, $\|y\| \leq 1$. By (iii), $\|T(y)\| \leq K\|y\|$ for some constant $K \geq 0$. But $\|y\| \leq 1$. So,

$$\|Ty\| \leq K\|y\| \leq K.$$

Since y was arbitrary chosen in D, it follows that for all $y \in D$, $\|Ty\| \leq K$. Now take $K = M$ and the proof is complete.

(iv) $\Rightarrow$ (i) Let x_1, x_2 be arbitrary elements of X. Assume first that $x_1 - x_2 = 0$. Consider the vector $u := \frac{x_1 - x_2}{\|x_1 - x_2\|}$. Clearly, $u \quad D$, and so, by condition (iv), (i.e., $T(D)$ is bounded), there exists a constant $K \geq \quad$ such that $\|T(u)\| \leq \quad K$ i.e.,

$$T\left(\frac{x_1 - x_2}{\|x_1 - x_2\|}\right) \leq K, \text{ or } \|Tx_1 - Tx_2\| \leq K\|x_1 - x_2\|.$$

If, on the other hand, $x_1 - x_2 = 0$, this inequality clearly holds, Thus, the inequality holds for all $x_1, x_2 \in X$. Now, given any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{K+1}$. So, if $\|x_1 - x_2\| < \delta$, then

$$\|Tx_1 - Tx_2\| \leq K\|x_1 - x_2\| \leq K\delta = \frac{K}{K+1}\epsilon < \epsilon,$$

■

and hence T is (uniformly) continuous on X.

Property (iii) of the last theorem is a very important one. In fact, you have the following definition:

Definition 4.2 A linear map $T : X \to Y$ is continuous if and only if it is bounded.

Thus, for linear maps, continuity and boundedness are equivalent.

Definition 4.3 If $T : X \to Y$ is a bounded linear map from a normed linear space X into a normed linear space Y, the norm of T can be defined by

$$\|T\| := \inf\{K : \|Tx\| \leq K\|x\|\} \text{ for each } x \in X.$$

The above definition gives you immediately that $\|Tx\| \leq \|T\|\|x\|$ for each $x \in X$ and that for every $\epsilon > 0$, there exists $x_\epsilon \in X$, $x_\epsilon \neq 0$, such that $\|Tx_\epsilon\| > (\|T\| - \epsilon)\|x_\epsilon\|$

Definition 4.4 Let X and Y be normed linear spaces and let B(X, Y) denote the family of all bounded linear maps from X to Y. Define

$$(T + L)(x) = T(x) + L(x);$$

$$(\alpha T)(x) = \alpha T(x),$$

for all $T, L \in B(X, Y)$ and scalar α. Then, clearly $B(X, Y)$ is a vector space.

Theorem 4.2 Let $B(X, Y)$ be the family of all bounded linear maps from X to Y. Then we have the following: For arbitrary $T \in B(X, Y)$,

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}. \tag{3}$$

Proof. Since T is bounded and linear, there exists $K \geq 0$ such that for all $x \in X$, $\|Tx\| \leq K\|x\|$. If $\|x\| \leq 1$, then $\|Tx\| \leq K\|x\| \leq K$ and so $\{\|Tx\| : \|x\| \leq 1\}$ is a bounded set in R so its "sup" exists and $\sup_{\|x\|\leq 1} \|Tx\| \leq K$ for any K such that $\|Tx\| \leq K\|x\|$ for all $x \in X$. Taking the infimum over all such K's gives

$$\sup_{\|x\|\leq 1} \leq \inf\{K \geq 0 : \|Tx\| \leq K\|x\| \text{ for all } x \in K\} := \|T\|.$$

Hence,

$$\sup_{\|x\|\leq 1} \|Tx\| \leq \|T\|. \tag{4}$$

Conversely, from Definition (4.3), we get that for every $\varepsilon > 0$, there exists $x_\varepsilon \in X$, $x_\varepsilon \neq 0$, such that

$$\|Tx_\varepsilon\| > (\|T\| - \varepsilon)\|x_\varepsilon\|.$$

(Otherwise $\|Tx_\varepsilon\| \leq (\|T\| - \varepsilon)\|x_\varepsilon\|$ and $\|T\|$ would no longer be the infimum). Let $u_\varepsilon = \frac{x_\varepsilon}{\|x_\varepsilon\|}$ then $\|u_\varepsilon\| = 1$ and $\|Tu_\varepsilon\| > \|T\| - \varepsilon$.

Consequently, we obtain from inequality (4) that

$$\|T\| \geq \sup_{\|x\|\leq 1} \|Tx\| \geq \sup_{\|x\|=1} \|Tx\| \geq \sup_{\|x_\varepsilon\|\neq 0} \left\| T\left(\frac{x_\varepsilon}{\|x_\varepsilon\|}\right)\right\| > \|T\| - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get that

$$\|T\| \geq \sup_{\|x\|\leq 1} \|Tx\| \geq \sup_{\|x\|=1} \|Tx\| \geq \sup_{\|x\|\neq 0} \frac{\|Tx\|}{\|x\|} \geq \|T\|$$

■

so that all this quantities are equal. The proof is complete.

It important to remark here that since $\|T\|$ is the smallest M such that

$$\|Tx\| \leq M\|x\| \text{ for each } x \in X,$$

it follows that whatever value M has, we must always have

$$\|T\| \leq M. \tag{5}$$

Example 4.1 Let $T : l_2 \to l_2$ be defined by

$$T(x_1, x_2, ..., x_k, ...) = (0, x_1, \frac{x_2}{2}, ..., \frac{x_k}{k}, ...)$$

Then $\|Tx\| \leq \|x\|$ so that $\|T\| \leq 1$. In this example, we actually have $\|T\| = 1$ since $T(1, 0, 0, ..., 0, ...) = (0, 1, 0, 0, ..., 0, ...)$.

Proposition 4.1 Let X and Y be normed linear spaces. Then, $\|\cdot\|$ defined by

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\|$$

for each $T \in B(X, Y)$ is a norm on $B(X, Y)$. Hence $B(X, Y)$ is a normed linear space with this norm.

Proposition 4.2 Let X, Y, Z be normed linear spaces and let $P \in B(Y, Z)$, $Q \in B(X, Y)$. Define $(PQ)(x) = P(Qx)$. Then

(a) $PQ \in B(X, Z)$, and

(b) $\|PQ\| \leq \|P\|\|Q\|$.

---

5 Linear functionals

Definition 5.1 A linear functional on a normed linear space X over K, is a linear transformation

$$f : X \to K.$$

In the sequel, the linear functionals of interest would be bounded linear functionals i.e., those functionals that are elements of the dual space X^*, but you shall see some examples of unbounded linear functionals now.

Example 5.1 Let $X = l_\infty$ and for any $x = (x_1, x_2, ..., x_k, ...) \in l_\infty$ consider the linear functional

$$f(x) = \sum_{k=1}^{\infty} x_k.$$

For $\bar{x} = (1, 1, 1, ...)$, $\bar{x} \in X$ because $\|\bar{x}\|_\infty$, but $f(\bar{x})$ does not have finite norm, and f is unbounded on l_∞. If instead we take the above linear functional on $X = l_1$ we have

$$|f(x)| \leq \sum_{k=1}^{\infty} |x_k| = \|x\|_1$$

Hence f bounded on l_1 and $\|f\| \leq 1$ in fact $\|f\| = 1$.

Example 5.2 Let $X = C[a, b]$ be the space of continuous real valued functions on the closed and bounded interval $[a, b]$. Suppose X is endowed with the norm

$$\|u\|_X = \max_{a \leq t \leq b} |u(t)|$$

for arbitrary $u \in C[a, b]$, and consider the linear functional $f : X \to R$ defined by

$$f(u) = \int_a^b u(t)dt.$$

Then $|f(u)| \leq \max_{a \leq t \leq b} |u(t)|(b - a) = (b - a)\|u\|_X$ and so f is continuous and $\|f\| \leq (b - a)$.

Now for $u_0(t) = 1$, $t \in [a, b]$, you have

$$|f(u)| = (b - a)$$

Hence $\|f\| = (b - a)$.

Remark 5.1 Computation of norm of a bounded linear functional:

The general idea in the computation of norms of bounded linear functional $f : X \to R$ is that you have to first of all show that f is bounded. i.e., there exists $M \geq 0$

$$|f(x)| \leq K\|x\|_X$$

for all $x \in X$. So by (5) you have Now, if you can find $x_0 \in X$ such that

Then $\|f\| = M$.

$\|f\| \leq$

M

$|f(x)| = M$

5.1 DUAL OR CONJUGATE SPACE

Definition 5.2 Dual (Conjugate) space.

The topological dual or the conjugate space of a normed linear space is the normed linear space of all bounded linear functionals $f : X \to K$, with norm

$$\|f\| = \sup_{\|x\|_X = 1} |f(x)| = \sup_{\|x\|_X \leq 1} |f(x)|$$

The dual of a normed linear space is denoted by X^* $(= B(X, K))$ Thus $f \in X^*$ means that f is mapping of X into K, which is linear and bounded. Recall that any map from a linear space into a scalar field is called a functional. The members of X^* are therefore bounded linear functionals.

Proposition 5.1 Let $X (= R^n$ or $C^n)$ (finite dimensional normed linear spaces) and Y be a normed linear space. Let $T : X \to Y$ be a linear map. Then T is continuous. In particular, for $Y = R$ or C, the topological and algebraic duals of R^n and C^n are the same.

Example 5.3 Consider $X = l_p^n$ over C, i.e.,

$$l_p^n(C) = \{(x_1, x_2, ..., x_n), x_k \in C\}$$

and define

$$f_a(x) = a_1x_1 + \cdots + a_nx_n$$

for a fixed $a \in l_q^n$ with $a = 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, $p \geq 1$.

Clearly $f_a : X \to C$ and is linear. It is also bounded since

$$\begin{aligned} |f_a(x)| &= |a_1 x_1 + \cdots + a_nx_n| \\ &\leq (|a_1|^q + \cdots + |a_n|^q)^{\frac{1}{q}} (|x_1|^q + \cdots + |x_n|^p)^{\frac{1}{p}} \\ &= \|a\|_q \|x\|_p . \end{aligned}$$

Here you have used Holders inequality, series version. For a remainder

$$\sum_{k=1} |a_k x_k| \leq \left(\sum_{k=1} |a_k|^q\right)^{\frac{1}{q}} \left(\sum_{k=1} |a_k|^p\right)^{\frac{1}{p}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p < \infty$.

Hence f_a is bounded and $\|f_a\|_{X^*} \leq \|a\|_q$. If we set

$$x_k = \begin{cases} \dfrac{|a_k|^q}{a_k} & \text{if } a_k = 0 \\ 0 & \text{if } a_k = 0. \end{cases}$$

then

$$\|x\|_p = \left(\sum^{n} |a_k|^{qp-p}\right)^{\frac{1}{p}} .$$

$k=1$

But

$$qp - p = p(q - 1) = q$$

and therefore

$$\|x\|_p = \left(\sum_{k=1}^{n} |a_k|^q\right)^{\frac{1}{p}}$$

But

$$|f_a(x)| = \sum_{k=1}^{n} |a_k|^q = \|x\|_p \left(\sum_{k=1}^{n} |a_k|^q\right)^{1-\frac{1}{p}}$$

$$= \|x\|_p \|a\|_q .$$

Hence

$$\|f_a\|_{X^*} = \|a\|_q .$$

In fact all continuous linear functionals on l_p^n have this form, so that

$$(l_p^n)^* \cong l_q^n .$$

Similarly, it can be shown that

$$(l_p)^* \cong l_q \text{ for } 1 \leq p < \infty .$$

The detail is in the following proposition.

Proposition 5.2 For $1 < p < \infty$, $l^*_p = l_q$ where $\frac{1}{p} + \frac{1}{q} = 1$, (i.e., the dual of l_p is l_q, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$).

Proof. The proof is divided into 2 parts.

Part 1. Is to prove that every element $y \in l_q$ defines an element f_y in l_p^* with $\|y\| = \|f_y\|$.

Part 2. Is to prove that every element $h \in l_p^*$ defines an element z_h in l_q with $\|h\| = \|z_h\|$. l_p^* and l_q are isometric.

Part 1. Let $y \in \{y_k\}_{k=1}^{\infty} \in l_q$. For every $x = \{x_k\}_{k=1}^{\infty}$ in l_p, define $f_y : l_p \to R$ by

$$f_y(x) = \sum_{k=1}^{\infty} x_k y_k$$

Clearly, f_y is well defined. For,

$$|f_y(x)| \leq \sum_{k=1}^{\infty} |x_k y_k| \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |y_k|^q\right)^{\frac{1}{q}} = \|x\|\|y\| < \infty .$$

This inequality also implies f_y is bounded and

$$\|f_y\| \leq \|y\|.$$

Linearity of f_y is obvious. Hence, f_y is a bounded linear functional on l_p and so $f_y \in l_p^*$.

For Part 2. Let $h \in l_p^*$. Let $\{e_k\}_{k=1}^{\infty}$ be the canonical basis in l_p, i.e.,

$$e_k = (0, 0, ..., 0, 1, 0, ...)$$

with 1 at the kth position. Then for every element $x = \{x_k\}_{k=1}^{\infty}$ in l_p we have that

$$x = \sum_{k=1}^{\infty} x_k e_k = \lim_{n\to\infty} \sum_{k=1}^{n} x_k e_k.$$

By the linearity and continuity (boundedness) of h,

$$h(x) = h\left(\lim_{n\to\infty} \sum_{k=1}^{n} x_k e_k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} x_k h(e_k) = \sum_{k=1}^{\infty} x_k h(e_k).$$

Define

$$z_h := \{h(e_k)\}_{k=1}^{\infty} = \{\alpha_k\}_{k=1}^{\infty}.$$

It suffices now to prove that $z_h \in l_q$. Let $x_n \in l_p$ be defined as follows:

$$x_n : \{|\alpha_1|^{q-1} \operatorname{sign} \alpha_1, |\alpha_2|^{q-1} \operatorname{sign} \alpha_2, ..., |\alpha_n|^{q-1} \operatorname{sign} \alpha_n, 0, 0, ...\}.$$

Then

$$|h(x_n)| = \sum_{k=1}^{n} |\alpha_k|^{q-1}(\operatorname{sign} \alpha_k)\alpha_k = \sum_{k=1}^{n} |\alpha_k|^q \leq \|h\| \cdot \|x_n\|$$

$$= \|h\| \cdot \left(\sum_{k=1}^{n} |\alpha_k|^{(q-1)p}\right)^{\frac{1}{p}} = \|h\| \cdot \left\{\sum_{k=1}^{n} |\alpha_k|^q\right\}^{\frac{1}{p}}$$

Hence, $\left\{\sum_{k=1}^{n} |\alpha_k|^q\right\}^{1-\frac{1}{p}} \leq \|h\|$ for all $n = 1, 2, 3, ...$ Hence $\sum_{k=1}^{\infty} |\alpha_k|^q < \infty$, i.e.,

$$z_h := \{\alpha_k\}_{k=1}^{\infty} \in l_q \text{ and } \|z_h\| \leq \|h\|.$$

Observe that since $h \in l_p^*$ is arbitrary, if you now take $h = f_y$ (the functional induced by y) then $z_h = y$ and we get that

$$\|y\| = \|z_h\| = \left(\sum_{k=1}^{\infty} |\alpha_k|^q\right)^{\frac{1}{q}} \leq \|h\| = \|f_y\|.$$

Hence, $\|y\| \leq \|f_y\|$. But you already have that $\|f_y\| \leq \|y\|$, so that $\|f_y\| = \|y\|$. This proves that l_p^* is isometrically isomorphic to l_q. This proof is complete. ■

Example 5.4 Consider l_∞^n over R, i.e.,

$$l_\infty^n(R) = \{(x_1, x_2, ..., x_n) : x_k \in R\}$$

Recall that

$$\|x\|_\infty = \max_{1\leq k\leq n} |x_k|$$

for arbitrary $x = (x_1, x_2, ..., x_n) \in l_\infty^n$, is a norm on l_∞^n.

Let $f_a(x) = \sum_{k=1} a_k x_k$, then f_a is a linear functional on l_∞^n and

$$|f_a(x)| \le \sum_{k=1}^{n} |a_k| \|x_k\| \le \|x\|_\infty \|a\|_1 .$$

For $x_k = \text{sign}(a_k)$, where

$$\text{sign}(a_k) = \begin{cases} +1 & \text{for } a_k > 0 \\ 0 & \text{for } a_k = 0 \\ -1 & \text{for } a_k < 0 \end{cases}$$

you have that $\|f_a\| = \|a\|_1$. You can show that all $f \in (l_\infty^n)^*$ have this form, so that

$$(l_\infty^n)^* \cong l_1^n .$$

It is also true that $(l_1^n)^* \cong l_\infty^n$ and $l_1^* \cong l_\infty$; however it is not true that $l_\infty^* \cong l_1$. In fact we have $c_0^* \cong l_1$, where c_0 is the normed linear space of infinite sequences $(x_1, x_2, ..., x_k, ...)$ with $\lim_{k=1} x_k = 0$ and $\|x\|_{x_0} = \sup_{k=1} |x_k|$. Similar results hold when l_∞^n, l_1^n, l_1, l_∞ are defined over C.

The above results have a special interpretation when $p = q = 2$. In such case, you find that

$$(l_2^n)^* \cong l_2^n, \text{ and } l_2^* \cong l_2 .$$

Such spaces are called self dual.

5.2 Dual Basis

Suppose X is a vector space of dimension n over K. The theorem that follows shows that the dual space X^* is also n (since K is of dimension 1 over itself). in fact, each basis of X determines a basis of X^* as follows.

Theorem 5.1 Suppose $\{x_1, ..., x_n\}$ is a basis of X over K. Let $f_1, ..., f_n \in X^*$ be the linear functionals defined by

$$f_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i = j \end{cases}$$

then $\{f_1, ..., f_k\}$ is a basis of X^*.

Proof. You can first show that $\{f_1, ..., f_k\}$ spans X^*. So let f be an arbitrary element of X^* and suppose

$$\mathbf{f}(x_1) = k_1, \qquad \mathbf{f}(x_2) = k_2, \qquad \cdots, \quad , \mathbf{f}(x_n) = k_n.$$

Set $\sigma = k_1\mathbf{f}_1 + ... + k_n\mathbf{f}_n$. Then

$$\sigma(x_1) = (k_1\mathbf{f}_1 + \cdots k_n\mathbf{f}_n)(x_1) = k_1\mathbf{f}_1(x_1) + k_2\mathbf{f}_2(x_1) + \cdots k_n\mathbf{f}_n(x_1)$$

$$= k_1 \cdot 1 + k_2 \cdot 0 + k_n \cdot 0 = k_1$$

Similarly, for $i = 1, ..., n$,

$$\sigma(x_i) = (k_1\mathbf{f}_1 + \cdots + k_n\mathbf{f}_n)(x_i) = k_1\mathbf{f}_1(x_i) + \cdots + k_i\mathbf{f}_i(x_i)\cdots + k_n\mathbf{f}_n(x_i) = k_i$$

Thus $\mathbf{f}(x_i) = \sigma(x_i)$ for $i = 1, ..., n$. Since $\mathbf{f}$ and σ agree on the basis vectors, $\mathbf{f} = \sigma = k_1\mathbf{f}_1 + \cdots + k_n\mathbf{f}_n$. Accordingly, $\{\mathbf{f}_1, ..., \mathbf{f}_n\}$ spans X^* .

It remains to be shown that $\{x_1, ..., x_n\}$ is linearly independent. Suppose

$$a_1\mathbf{f}_1 + a_2\mathbf{f}_2 + \cdots + a_n\mathbf{f}_n = \bar{0}$$

Applying both sides to x_1, you obtain

$$0 = 0(x_1) = (a_1\mathbf{f}_1 + \cdots + a_n\mathbf{f}_n)(x_1)a_1\mathbf{f}_1(x_1) + a_2\mathbf{f}_2(x_1) + \cdots + a_n\mathbf{f}_n(x_1)$$

$$= a_1 \cdot 1 + a_2 \cdot 0 + \cdots + a_n \cdot 0 = a_1$$

Similarly, for $i = 2, ..., n$

$$0 = 0(x_i) = (a_1\mathbf{f}_1 + \cdots + a_n\mathbf{f}_n)(x_i) = a_1\mathbf{f}_1(x_i) + \cdots + a_i\mathbf{f}_i(x_i) + \cdots + a_n\mathbf{f}_n(x_i) = a_i$$

That is, $a_1 = 0, ..., a_n = 0$. Hence $\{\mathbf{f}_1, ..., \mathbf{f}_n\}$ is linearly independent, and so it is a basis of X^*. ■

Example 5.5 Consider the basis $\{v_1 = (2, 1), v_2 = (3, 1)\}$ of R^2. Find the dual basis $\{\mathbf{f}_1, \mathbf{f}_2\}$ of $(R^2)^*$.

Solution. Your interest is to look for linear functionals $\mathbf{f}_1(x, y) = ax + by$ and $\mathbf{f}_2 = (x, y) = cx + dy$ such that

$$\mathbf{f}_1(v_1) = 1, \quad \mathbf{f}_1(v2) = 0, \quad \mathbf{f}_2(v_1) = 0, \quad \mathbf{f}_2(v_1) = 1$$

These four conditions lead to the following two systems of linear equations:

$$\left.\begin{array}{l} \mathbf{f}_1(v_1) = \mathbf{f}_1(2, 1) = 2a + b = 1 \\ \mathbf{f}_1(v_2) = \mathbf{f}_1(3, 1) = 3a + b = 0 \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} \mathbf{f}_2(v_1) = \mathbf{f}_2(2, 1) = 2c + d = 0 \\ \mathbf{f}_2(v_2) = \mathbf{f}_1(3, 1) = 3c + d = 0 \end{array}\right\}$$ ■

The solutions yeild $a = -1$, $b = 3$ and $c = 1$, $d = -2$. Hence $\mathbf{f}_1(x, y) = -x + y$ and $\mathbf{f}_2(x, y) = x - 2y$ form the dual basis.

Example 5.6 Let $P_1 = \{a + bt : a, b \in R\}$, be the vector space of real polynomials of degree$\leq$

1. Find the basis $\{v_1, v_2\}$ of P_1 that is dual to the basis $\{\phi_1, \phi_2\}$ of $(P_1)^*$ defined by

$$\phi_1(f(t)) = \int_0^1 f(t)dt \quad \text{and} \quad \phi_2(f(t)) = \int_0^2 f(t)dt$$

Solution. Let $v_1 = a + bt$ and $v_2 = c + dt$. By definition of the dual basis,

$$\phi_1(v_1) = 1, \quad \phi_1(v_2) = 0 \quad \text{and} \quad \phi_2(v_1) = 0, \phi_2(v_2) = 1$$

Thus

$$\begin{cases} \phi_1(v_1) = \int_0^1 (a + bt)dt = a + \frac{1}{2}b = 1 \\ \phi_2(v_1) = \int_0^2 (a + bt)dt = 2a + 2b = 0 \end{cases} \quad \text{and} \quad \begin{cases} \phi_1(v_2) = \int_0^1 (c + dt)dt = c + \frac{1}{2}d = 0 \\ \phi_2(v_1) = \int_0^2 (c + dt)dt = 2c + 2d = 0 \end{cases}$$

Solving each system yields $a = 2$, $b = -2$ and $c = -\frac{1}{2}$, $d = 1$. Thus $\{v_1 = 2 - 2t, v_2 = -\frac{1}{2} + t\}$ is the basis of P_1 that is dual to $\{\phi_1, \phi_2\}$. ■

The next theorem give a relationship between bases and their duals.

Theorem 5.2 Let $\{u_1, ..., u_n\}$ be a basis of a finite dimensional space X and let $\{\phi_1, ..., \phi_n\}$ be the dual basis in X^* . Then:

(a) For any vector $x \in X$, $x = \sum_{i=1}^{n} \phi_k(x)u_k$.

(b) For any linear functional $\sigma \in X^*$, $\sigma = \sum_{i=1}^{n} \sigma(u_k)\phi_k$

Proof. Suppose

$$x = a_1u_1 + a_2u_2 + \cdots + a_nu_n \tag{6}$$

Then

$$\phi_1(x) = a_1\phi_1(u_1) + a_2\phi_1(u_2) + \cdots + a_n\phi_1(u_n) = a_1 \cdot 1 + a_2 \cdot 0 + \cdots + a_n \cdot 0 = a_1$$

Similarly, for $i = 2, ..., n$,

$$\phi_i(x) = a_1\phi_i(u_1) + \cdots + a_i\phi_i(v_i) + \cdots + a_n\phi_i(v_n) = a_i$$

That is, $\phi_1(x) = a_1$, $\phi_2(x) = a_2$, ..., $\phi_n(x) = a_n$. Substituting these results into (6), gives you (a).

It is now left for you to prove (b). Applying the linear functional to both sides of (a),

$$\begin{aligned}
\sigma(x) &= \phi(x)\sigma(u_1) + \phi_2(x)\sigma(u_2) + \cdots + \phi_n(x)\sigma(u_n) \\
&= \sigma(u_1)\phi_1(x) + \sigma(u_2)\phi(x) + \cdots + \sigma(u_n)\phi_n(x) \\
&= (\sigma(u_1)\phi_1 + \sigma(u_2)\phi_2 + \cdots \sigma(u_n)\phi_n)(x)
\end{aligned}$$

Since the above holds for every $x \in X$, $\sigma = \sigma(u_1)\phi_1 + \sigma(u_2)\phi_2 + \cdots \sigma(u_n)\phi_n$ ■

5.3 Hahn-Banach Theorem

Theorem 5.3 The Hahn-Banach Theorem

Every continuous linear functional $f : M \to K$ defined on a linear subspace M of a normed linear space X can be extended to a continuous linear functional F on all of X with preservation of norm.

To see that this guarantees the existence of non-trivial continuous linear functionals, consider the subspace $M = \{\alpha x_0\}$ where $x_0 \in X$, $x_0 = 0$ and α is any scalar. A linear functional f defined on M is

$$f(y) = \alpha \|x_0\| \text{ for } y = \alpha x_0 .$$

Then

$$|f(y)| = \|y\|; \text{ hence } \|f\| = 1,$$

and so the Hahn-Banach theorem says there is an F defined on X with norm 1.

A useful corollary of the Hahn-Banach Theorem is

Corollary 5.1 Let E be an arbitrary subset of a normed linear space X. Then $\overline{\text{span} E} \supseteq X$ if and only if the zero functional is the only bounded linear functional which vanishes on all of E.

5.4 Convergence and Continuity $B(X, Y)$.

Definition 5.3 Uniform or norm convergence.

Let $\{T_n\}$ is a sequence of bounded linear operators in $B(X, Y)$. T_n converges uniformly to $T \in B(X, Y)$ as $n \to \infty$, if

$$\|T_n - T\|_{B(X,Y)} \to 0 \text{ as } n \to \infty .$$

Frequently in applications this kind of convergence is useful.

Definition 5.4 Strong convergence.

Let $\{T_n\}$ be a sequence of bounded linear operators in $B(X, Y)$. T_n converges strongly to $T \in B(X, Y)$ if

$$\|T_n x - Tx\|_Y \to 0 \text{ as } n \to \infty \quad \text{for all } x \in X.$$

If a bounded linear operator depends on a parameter t from some interval of R, you can define strong continuity, and uniform continuity with respect to t in an analogous manner.

Definition 5.5 Uniform continuity

Let $T(t) \in B(X, Y)$ for every $t \in [a, b]$, then $T(t)$ is uniformly continuous at t_0, if

$$\|T(t) - T(t_0)\|_{B(X,Y)} \to 0 \text{ as } t \to t_0.$$

Definition 5.6 Strong continuity

Let $T(t) \in B(X, Y)$ for every $t \in [a, b]$, then $T(t)$ is strongly continuous at t_0 if

$$\|T(t)x - T(t_0)x\|_Y \to 0 \text{ for all } x \in X \text{ as } t \to t_0.$$

The following are very important theorems on linear operators which are used in application.

5.5 The Uniform Boundedness Principle

Theorem 5.4 The Uniform Boundedness Theorem

Let X, Y be Banach spaces, let $\{T_\alpha\}_{\alpha\in I} \subset B(X, Y)$. Suppose that

$$\sup_{\alpha\in I} \|T_\alpha(x)\| < \infty \quad \text{for each } x \in X.$$

Then

$$\sup_{\alpha\in I} \|T_\alpha\| < \infty .$$

An immediate consequence of the Uniform Boundedness Principle is the following theorem;

Theorem 5.5 Banach-Steinhaus Theorem

Let X, Y be a Banach space and $\{T_n\}$ be a family of bounded linear operators in $B(X, Y)$. If the family $\{T_n x\}$ converges to a limit defined by Tx. Then

(a) $\sup_{n\geq 1} \|T_n\| < \infty$

(b) $T \in B(X, Y)$.

(c) $\|T\| \leq \liminf_{n\to\infty} \|T_n\|$

5.6 The Open Mapping Theorem

Theorem 5.6 Open Mapping Theorem

Let X and Y be Banach spaces and $T : X \to Y$. Suppose

(i) $T \in B(X, Y)$,

(ii) T is surjective (i.e., onto)

then T is an open map (i.e., T maps every open set of X onto an open set of Y.)

The following is a corollary of this theorem.

Corollary 5.2 Let X and Y be Banach spaces and $T : X \to Y$. Suppose

(i) $T \in B(X, Y)$,

(ii) T is bijective (i.e., one-to-one and onto)

Then T^{-1} is also a bounded linear operator.

5.7 The Closed Graph Theorem

Definition 5.7 Graph of a linear operator.

Let X and Y be normed linear spaces and $T : X \to Y$ be any map. Then, the graph of T denoted by $G(T)$, is defined by

$$G(T) := \{(x, Tx) : x \in X\}$$

Observe that $G(T)$ is a subset of $X \times Y$ and that

$$(x, y) \in G(T) \text{ if and only if } Tx = y.$$

Example 5.7 Let $X = [0, 1]$, $Y = \mathbb{R}$, and $T : [0, 1] \to \mathbb{R}$ be defined by

$$Tx = x^2, \; x \in [0, 1].$$

Then, the graph of T, $G(T)$, is given by

$$G(T) = \{(x, Tx) : x \in [0, 1]\} = \{(x, x^2) : x \in [0, 1]\}$$

Example 5.8 Let $X = [-1, 1]$, $Y = [0, 1]$ and $S : X \to Y$ be defined by

$$Sx = \begin{cases} 0, & -1 \leq x \leq 0; \\ 1 & 0 \leq x \leq 1. \end{cases}$$

Then,

$$G(S) = \{(x, Sx) : x \in [0, 1]\} = \{(x, 0) : x \in [-1, 0]\} \bigcup \{(x, 1) : x \in (0, 1]\}.$$

Definition 5.8 Closed operator.

Let X and Y be normed linear spaces and $T : X \to Y$ be linear operator. T is said to be closed if its graph $G(T)$ is a closed subset of $X \times Y$.

Alternatively, T is closed if whenever

$$x_n \in D(T), \; n = 1, 2, \ldots \text{ and } \lim_{n\to\infty} x_n = x, \; \lim_{n\to\infty} T x_n = y,$$

it follows that $x \in D(T)$ and $Tx = y$.

That is if $\{x_n\}$ is a sequence of elements of $D(T)$ (the domain of T) such that $x_n \to x \in X$ and $Tx_n \to y \in Y$ then $x \in D(T)$ and $Tx = y$.

The essential difference between bounded linear operators and closed linear operators is their domain of definition.

To apperciate the importance of our next theorem, you will be presented with an example of a map which is

(i) linear;

(ii) closed; and

(iii) not bounded.

Example 5.9 Let $X = C[0,1] = Y$, where $C[0,1]$ is endowed with the sup norm. Let

$$D = \{f \in C^1[0,1] : f' \in C[0,1]\}$$

where the prime denotes differentiation. Let $T : C[0,1] \to C[0,1]$ be a map with domain D defined by

$$Tf = f',$$

(i.e., T is the differentiation operator). Then,

(i) T is linear

(ii) T is closed;

(iii) T is not bounded.

The proof of (i) is obvious. To verify (ii), Let $\{f_n\}$ be a sequence in D such that

$$f_n \to f \text{ and } Tf_n := f_n' \to y.$$

Observe that $Tf_n \to y$, implies that $\|Tf_n - y\|_{C[0,1]} = \sup_{t\in[0,1]} |(Tf_n)(t) - y(t)| = \sup_{t\in[0,1]} |f_n' - y(t)| \to 0$ as $n \to \infty$. This convergence is uniform and $y(t) = \lim_{n\to\infty} f_n'(t)$. Since the convergence is uniform, we can interchange limit and integral, so we have:

$$\int_0^t y(s)ds = \int_0^t \lim_{n\to\infty} f_n'(s)ds = \lim_{n\to\infty} \int_0^t f_n'(s)ds = f(t) - f(0),$$

so that

$$f(t) = f(0) + \int_0^t y(s)ds.$$

It is now easy to see that $f'(t)$ exists and $f'(t) = y(t)$ for all $t \in [0,1]$. Thus, $f \in D$ and $Tf = y$ so T is closed. It now remains to show (iii) that T is not bounded. Take $f_n(t) = t^n$. Then $\|f_n\| = \sup_{t\in[0,1]} |t^n| = 1$ and $Tf_n = f_n'(t) = nt^{n-1}$ so that

$$\|Tf_n\| = \sup_{t\in[0,1]} |nt^{n-1}| = n, \; n = 1, 2, \ldots.$$

Thus T is not bounded

The next theorem now tells us when a closed linear operator is bounded.

Theorem 5.7 The Closed Graph Theorem

Let X and Y be real Banach spaces. Let

(i) $T : X \to Y$ be a linear map;

(ii) The graph of T, $G(T)$ be closed.

Then T is continuous.

6 Conclusion

In this unit, you were introduced to a special class of linear maps called Bounded linear maps. You were also introduced to the concept of Dual space of a given normed linear space, which is the space of all bounded linear functionals defined on a normed linear space. You also learnt how to compute the norms of a bounded linear map of functional. You also saw some theorems that you can use to determine when a given linear map or a sequence of linear maps is bounded. You also learnt how to extend a linear functional defined on a subspace of a given vector space to the whole space, preserving norm.

7 Summary

Having gone through this unit, you know that;

(i) If X and Y are linear spaces then the mapping $T : X \to Y$ is called a linear map if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for arbitrary $x, y \in X$ and arbitrary scalars $\alpha, \beta \in K$.

(ii) A linear map $T : X \to Y$ is bounded if there exists some constant $K \geq 0$ such that for each $x \in X$,4

$$\|T(x)\| \leq K \|x\|.$$

(iii) A linear map is continuous if and only if it is bounded.

(iv) the family $B(X, Y)$ of bounded linear maps from normed linear space X into the normed linear space Y is a linear space if vector addition and scalar multiplication is defined by

$$(T + L)(x) = T(x) + L(x)$$

and

$$(\alpha T)(x) = \alpha T(x)$$

respectively for arbitrary $T, L \in B(X, Y)$ and scalar α.

(v) The space $B(X, Y)$ of bounded linear maps between X and Y becomes a normed linear space with the norm

$$\|T\| = \sup_{\|x\| \leq 1}$$

for arbitrary $T \in B(X, Y)$

(vi) a linear map from a vector space to a scalar field is called a linear functional

(vii) the space of all bounded linear functionals is called the dual space.

(viii) the dual space of the $n-$ real space R^n and the $n-$ complex space is the same.

(ix) if $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ then the dual of l_p is l_q.

(x) if $f : M \to K$ is a linear functional defined on a linear subspace M of a normed linear space X, then there exists a linear functional F defined on X such that $\|F\| = \|f\|$ (Hahn-Banach theorem).

(xi) (The Uniform Boundedness Theorem) if X, Y be Banach spaces, let $\{T_\alpha\}_{\alpha \in I} \subset B(X, Y)$. Suppose that

$$\sup_{\alpha \in I} \|T_\alpha(x)\| < \infty \quad \text{for each } x \in X,$$

then

$$\sup_{\alpha \in I} < \infty .$$

(xii) (Banach-Steinhaus) if X, Y be a Banach space and $\{T_n\}$ be a family of bounded linear operators in $B(X, Y)$ and suppose that the family $\{T_n x\}$ converges to a limit defined by Tx. Then

(a) $\sup_{n \geq 1} \|T_n\| < \infty$

(b) $T \in B(X, Y)$

(c) $\|T\| \leq \liminf_{n \to \infty} \|T_n\|$

(xiii) a linear map $T : X \to Y$ is called an open map if T maps open set of X onto an open set of Y.

(xiv) Open Mapping theorem if X and Y are Banach spaces and $T : X \to Y$, and suppose that

(a) $T \in B(X, Y)$,

(b) T is surjective (i.e., onto)

Then T is an open map (i.e., T maps open set of X onto an open set of Y.)

(xv) Closed graph theorem If X and Y are real Banach spaces, and suppose

(a) $T : X \to Y$ is a linear map and

(b) the graph of T, $G(T)$ is close,

Then T is continuous.

8 TMAs

1. Which of the following mappings is not linear?

   (a) $f : R^2 \to R^3$ defined by $f(x, y) = (x + 3, 2y, x + y)$

   (b) $f : R^3 \to R^2$ defined by $f(x, y, z) = (x + y + z, 2x - 3y + 4z)$

   (c) $f : R^2 \to R^2$ defined by $f(x, y) = (x + y, x)$

   (d) None

2. The function $f : R^3 \to R^2$ defined by $f(x, y, z) = (|x|, y + z)$ is linear. (True/ False)

3. The mapping $f : R^2 \to R^2$ defined by $f(x, y) = (xy, x)$ is linear. (True/ false)

4. Let X be a finite dimensional space and $x_1, x_2, ..., x_n$ be a basis. If $x = \sum_{i=1}^{n} \alpha_i x_i$, define

$$\|x\| = \left(\sum_{i=1}^{n} |\alpha_i|^2 \right)^{\frac{1}{2}}$$

   If f is a bounded linear functional on X, find $\|f\|$. What will $\|f\|$ be if $\|x\| = \max_i |\alpha_i|$

5. If X is a finite-dimensional space, then all linear functionals are bounded. (True/False).

6. If X is an infinite-dimensional space, are all linear functionals bounded? (Yes/No).

7. Let f be a bounded linear functional defined on $C[a, b]$ (with the sup norm) by

$$(fx)(t) = \int_a^b x(t)dt$$

   for all $x \in C[a, b]$. Find $\|f\|$.

8. On the space l_1, for $x = (\alpha_1, \alpha_2, ...)$, define

$$f(x) = \sum_{n=1}^{\infty} \alpha_n$$

   and introduce a new norm $\|x\|_\infty = \sup |\alpha_n|$. Then with respect to this norm,

   (a) f is linear and continuous.

   (b) f is bounded and continuous.

   (c) f is linear and not continuous.

   (d) f is linear and bounded.

9. Let R^n be the real n-space, and let a be any fixed nonzero vector in R^n. Define $f : R^n \to R$ by

$$f(x) = \langle x, a \rangle \text{ for all } x \in R^n$$

   (where $\langle , \rangle$ denotes the inner product, or scalar product). Compute $\|f\|$.

(a) $\|a\|^2$

(b) $\|a\|$

(c) $\|x\|$

(d) $\|x\|\|a\|$

10. Let $g_0(t)$ be a fixed function in $C[a, b]$, and let

$$Tf = \int_a^b f(t)g_0(t)dt.$$

Then T a linear functional on $C[a, b]$. (True/ False)

11. If your answer is Yes in problem 7 above, what is $\|T\|$?

(a) $\|T\| = 1$

(b) $\|T\| = |g_0(t)|$

(c) $\|T\| = \int_a^b |g_0(t)|\, dt$

(d) $\|T\| = K$, Where K is an arbitrary constant.

12. Consider the functions defined on the space $C[0, 1]$,

$$\text{(I) } f(x) = ax(0) + bx(1) \qquad \text{(II) } g(x) = \int_0^{1/2} x(t)dt - \int_{1/2}^{1} x(t)dt$$

which one(s) is/are bounded linear functional(s) on the space $C[0, 1]$?

(a) I only

(b) II only

(c) Both I and II

(d) None

13. Find

(a) $\phi + \sigma$,

(b) 3ϕ

(c) $2\phi - 5\sigma$,

where $\phi : R^3 \to R$ and $\sigma : R^3 \to R$ and define by

$$\phi(x, y, z) = 2x - 3y + z \text{ and } \sigma(x, y, z) = 4x - 2y + 3z$$

14. Find the dual basis of each of the following bases of R^3 :

(a) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

(b) $\{(1, -2, 3), (1, -1, 1), (2, -4, 7)\}$

15. Let V be the space of polynomials over R of degree $\leq$ 2. Let ϕ_1, ϕ_2, ϕ_3 be the linear functionals on V defined by

$$\phi_1(f(t)) = \int_0^1 f(t)dt \qquad \phi_2(f(t)) = f'(1), \qquad \phi_3(f(t)) = f(0)$$

Here $f(t) = a + bt + ct^2$ and $f^{0}(t)$ denotes the derivatives of $f(t)$. Find the basis $\{f_1(t), f_2(t), f_3(t)\}$ of V that is dual to $\{\phi_1, \phi_2, \phi_2\}$

UNIT 4: INNER PRODUCT SPACES

Contents

1 INTRODUCTION

In this unit, you shall be introduced with inner product spaces i.e., spaces with a real valued fuction called an inner product or scalar product defined on it and some special properties of such spaces. You shall discorver that an inner product space is a normed linear space, where the norm is induced by the inner product defined on the space. Therefore, being a normed linear space, it is also a metric space as you have studied in the previous units. Hence you see that there is need to talk about completeness. This will now lead you to some some special kinds of inner product space called Hilbert spaces which turn out to be complete inner product spaces.

2 OBJECTIVES

At the end of this unit, you should be able to;

1. define an inner product space,

2. identify and show that a function is an inner product.
3. prove that an inner product is complete.
4. identify Hilbert spaces.

3 INNER PRODUCT SPACES

Definition 3.1 Let E be a linear space over the scalar field $K(= R$ or $C)$. A complex-valued function

$$\langle , \rangle : E \times E \to K$$

defined for arbitrary pair of elements $x, y \in E$ is said to be a scalar product or inner product if it satisfies the following properties:

I_1. $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$;

I_2. $\langle x, y\rangle = \overline{\langle y, x\rangle}$; where "bar" is the complex conjugation.

I_3. $\langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle y, z\rangle$

(valid for all $x, y, z \in E$ and all complex λ and μ)

Definition 3.2 A linear space E with an inner product $\langle , \rangle$ defined on it is called an inner product space.

3.1 BASIC PROPERTIES

Here are some basic properties of inner product spaces.

Remark 3.1 The inner product defined on the linear space E is a linear function in the first variable except for the case that E is a linear space over R, the field of real numbers. You can observe this as an immediate consequence of I_1 and I_2 that for arbitrary elements $x, y, z \in E$, and $\lambda, \mu \in C$,

$$\langle z, \lambda x + \mu y\rangle = \bar{\lambda}\langle z, x\rangle + \mu\langle z, y\rangle.$$

Lemma 3.1 (Cauchy Schwartz Inequality) Let $(E, \langle , \rangle)$ be an inner product space. If $x, y \in E$ are arbitrary then

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle \tag{1}$$

$|\langle x, y\rangle|^2 = \langle x, x\rangle\langle y, y\rangle$ if and only if x and y are linearly independent.

Proof. Let $x, y \in E$ be arbitrary. Let $z \in C$ such that $|z| = 1$ and $z\langle x, y\rangle = |\langle x, y\rangle|$. Then for all $t \in R$,

$$\begin{aligned} 0 \leq \varphi(t) &= \langle tzx + y, tzx + y\rangle \\ &= t^2 z\bar{z}\langle x, x\rangle + tz\langle x, y\rangle + t\bar{z}\langle y, x\rangle + \langle y, y\rangle \\ &= \langle x, x\rangle t^2 + 2t|\langle x, y\rangle| + \langle y, y\rangle. \end{aligned}$$

Therefore,

$$|\langle x, y\rangle|^2 - \langle x, x\rangle\langle y, y\rangle \leq 0 \tag{2}$$

Otherwise there exist $t_0 \in \mathbb{R}$ such that $\phi(t_0) < 0$ and would contradict the fact that $\phi(t) \geq 0$ for all $t \in \mathbb{R}$. Thus from (2),

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$$

as required. ∎

Theorem 3.1 An inner product space E becomes an normed linear space when equipped with the norm

$$\|x\| = \sqrt{\langle x, x\rangle}. \tag{3}$$

for all $x \in E$.

Proof. The proof that $\|\cdot\|$ satisfies N_1 and N_2 from the definition of $\|\cdot\|$ and conditions I_1 and I_2. It is now left for you to verify that $\|\cdot\|$ satisfy N_3. So take an arbitrary elements $x, y \in E$.

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \|x\|^2 + \langle x, y\rangle + \overline{\langle y, x\rangle} + \|y\|^2 \\
&= \|x\|^2 + 2\mathrm{Re}\,\langle x, y\rangle + \|y\|^2 \\
&\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \\
&\leq \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2
\end{aligned}$$

Which implies that $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in E$. The proof in now complete, hence, $(E, \|\cdot\|)$ is a normed linear space. ∎

Remark 3.2 As a consequence of Lemma 3.1, the Cauchy-Schwartz inequality is generally written as follows:

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \qquad \text{for arbitrary } x, y \in E.$$

Lemma 3.2 The inner product $\langle , \rangle$ is a continuous function on $E \times E$.

Proof. The proof follows from the Cauchy-Schwartz inequality. ∎

Proposition 3.1 (The Parallelogram Law) Let E be an inner product space. the for arbitrary $x, y \in E$,

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \tag{4}$$

Proof. Take arbitrary elements $x, y \in E$. Since E is an inner product space,

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2
\end{aligned}$$

and

$$\begin{aligned}\|x-y\|^2 &= \langle x-y, x-y\rangle = \langle x, x-y\rangle - \langle y, x-y\rangle \\ &= \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 - \langle x, y\rangle - \langle y, x\rangle + \|y\|^2\end{aligned}$$

From the expansion above, you have

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

for all $x, y \in E$. ▮

Proposition 3.2 (The Polarization Identity) Let E be an inner product space. Then for arbitrary $x, y \in E$,

$$\langle x, y\rangle = \tfrac{1}{4}\{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\}, \text{ where } i^2 = -1$$

Proof. You can easily prove this by expanding the right hand side to get the left hand side. ▮

3.2 Examples of inner product spaces

Example 3.1 The linear space R^n, with the function $\langle , \rangle$ defined, for arbitrary vectors $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ in R^n, by

$$\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k \tag{5}$$

is an inner product space. The norm induced by (5) is given by

$$\|x\| = \sqrt{\langle x, x\rangle} = \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}}$$

for arbitrary $x = (x_1, x_2, ..., x_n)$ i.e., the Euclidean norm.

Verification.

To verify that $(R^n, \langle , \rangle)$ is an inner product space, you have to first show that $\langle , \rangle$ is an inner product on R^n. That is to say that $\langle , \rangle$ must satisfy axioms I_1, I_2 and I_3 of an inner product. So you can verify as follows.

I_1. Let $x \in R^n$, $\langle x, x\rangle = \sum_{k=1}^{n} x_k^2 \geq 0$ as a finite sum of nonnegative real numbers. Which implies that $\langle x, x\rangle \geq 0$ for all $x \in R^n$.

Now, $\langle x, x\rangle = 0$ if and only if $\sum_{k=1}^{n} x_k^2 = 0$ if and only if $x_k^2 = 0$ for each $k \in \{1, 2, ..., k\}$ (since they are nonnegative) if and only if all $x_k's = 0$, if and only if $x = (x_1, x_2, ..., x_n) = (0, 0, ..., 0) = 0$ in R^n.

I_2. Let $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ in R^n be arbitrary.

$$\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n} y_k x_k = \langle y, x\rangle$$

since the complex conjugatation does not have any effect on the set of real numbers.

I_3. Let $x, y, z \in R^n$, $x = (x_1, x_2, ...,_n)$, $y = (y_1, y_2, ..., y_n)$ and $z = (z_1, z_2, ..., z_n)$ and $\lambda, \mu \in R$ be arbitrary.

$$\begin{aligned}
\langle \lambda x + \mu y, z\rangle &= \sum_{k=1}^{n} (\lambda x_k + \mu y_k) z_k = \sum_{k=1}^{n} \lambda x_k z_k + \sum_{k=1}^{n} \mu y_k z_k \\
&= \lambda \sum_{k=1}^{n} x_k z_k + \mu \sum_{k=1}^{n} y_k z_k \\
&= \lambda\langle x, y\rangle + \mu\langle y, z\rangle
\end{aligned}$$

Therefore, $\langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle y, z\rangle$ for all $x, y, z \in R^n$ and $\lambda, \mu \in R$.

Thus, since $\langle , \rangle$ satisfies I_1, I_2 and I_3 as demonstrated above, then it is an inner product on R^n. Hence, $(R^n, \langle , \rangle)$ is an inner product space.

Other examples of inner product spaces are as given below.

Example 3.2 The linear space C^n, i.e. the n-complex space, with the function $\langle , \rangle$ defined for arbitrary $z = (z_1, z_2, ..., z_n)$, $\omega = (\omega_1, \omega_2, ..., \omega_n)$ in C^n, by

$$\langle z, \omega\rangle = \sum_{k=1}^{n} z_k \overline{\omega}_k \tag{6}$$

is an inner product space.

The norm induced by (6) is given by

$$\|z\|_2 = \left(\sum_{k=1}^{n} |z_k|^2\right)^{\frac{1}{2}}$$

for arbitrary $z = (z_1, z_2, ..., z_n) \in C^n$, i.e., the unitary norm.

Example 3.3 The linear space $l_2(C)$ with the function $\langle , \rangle$ defined, for arbitrary vectors $x = (x_1, x_2, ..., x_k, ...)$, $y = (y_1, y_2, ..., y_k, ...)$ in l_2, by

$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_k \overline{y}_k \tag{7}$$

is an inner product space.

The norm induced by (7) is given by

$$\|x\|_{l_2} = \left(\sum_{k=1}^{\infty} |x_k|^2\right)^{\frac{1}{2}}$$

for arbitrary sequence $x = (x_1, x_2, ..., x_k, ...)$

Observe that the series in (7) converges follows from the Cauchy-Schwartz inequality applied to partial sums (Lemma 4.1).

Example 3.4 The linear space $C_{[0,1]}$ of continuous real valued function define on the closed and bounded interval [0, 1], with the function $\langle , \rangle$ defined for arbitrary $f, g \in C_{[0,1]}$ by

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt \tag{8}$$

is an inner product space.

The induced by (8) is given by

$$\|f\|_2 = \left(\int_0^1 |f(t)|^2\right)^{\frac{1}{2}} \tag{9}$$

for arbitrary $f \in C_{[0,1]}$.

Example 3.5 The linear space $L_2[0, T]$ of all Lebesgue integrable functions on $[0, T]$, with $\langle , \rangle$ defined for arbitrary $f, g \in L_2[0, T]$ by

$$\langle f, g\rangle = \int_0^T f(t)\overline{g(t)}\,dt \tag{10}$$

is an inner product space.

The norm induced by (10) is similar to that given in (9).

You can easily verify that the above examples are inner product space by properly applying the properties of complex conjugation.

3.3 JORDAN VON NEUMANN THEOREM

You have seen that in an inner product space E, the inner product induces a norm which is given by (3) in theorem 4.1. Thus showing us that every inner product space is also a normed linear space.

Question.

The question now is, given a norm on a linear space, how can we know that it is induced by an inner product?

The answer to the above question is given in the theorem.

Theorem 3.2 (Jordan Von Neumann) The norm on a normed linear space E is given by an inner product if and only if the norm satisfies the parallelogram law. i.e., if and only if for arbitrary $x, y \in E$,

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|)^2$$

Thus, a normed linear space is an inner product space if and only if the norm on E satisfies the parallelogram law.

For example, the norm on $\mathbb{R}^n$ given by

$$\|x\|_0 = \max_{1\le k\le n} |x_k|$$

is not given by an inner product. Hence, $\mathbb{R}^n$ with this norm is not an inner product space.

You can apply theorem 4.2 to check this. That is, it is enough for you to check that $\|\cdot\|_0$ does not satisfy the parallelogram law. But observe that the parallelogram law states for all elements x and y in an inner product space. Hence, if you can find two elements of $\mathbb{R}^n$ for which parallelogram law fails for the norm $\|\cdot\|_0$, then you are done.

4 CONCLUSION

In this unit you have studied inner product spaces, that is, linear spaces with an inner product defined on it. You have seen some examples inner product spaces and how to verify that a given fuction defined on a linear space is an inner product. You saw that every inner product induces a norm thereby making the inner product a normed linear space. You have also seen that you can actually check, by use of parallelogram law that some norms are not induced by an inner product.

5 SUMMARY

Having gone through this unit, you now know that;

(i) If E is a linear space over a scalar field $K(= \mathbb{R}$ or $\mathbb{C})$, then an inner product or scalar product on E is a complex valued function $\langle\, ,\rangle : E \times E \to K$ defined for arbitrary pairs of elements $x, y \in E$ and satisfies

I_1. $\langle x, x\rangle \ge 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$;

I_2. $\langle x, y\rangle = \overline{\langle y, x\rangle}$; where "bar" is the complex conjugation.

I_3. $\langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle y, z\rangle$.

(valid for all $x, y, z \in E$ and all complex λ and μ)

(ii) A linear space E with an inner product $\langle\, ,\rangle$ defined on it is called an inner product space

(iii) (Cauchy-Schwartz Inequality) if $(E, \langle\, ,\rangle)$ is an inner product space, and if $x, y \in E$ are arbitrary, then

$$|\langle x, y\rangle| \le \langle x, x\rangle\langle y, y\rangle.$$

Where equality holds if x and y are linearly independent.

(iv) An inner product defined on an inner product space E induces a norm on E defined by

$$\|x\| = \sqrt{\langle x, x\rangle}$$

(v) (Parallelogram Law) If E is an inner product space then for arbitrary $x, y \in E$

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

(vi) (The Polarization Identity) If E is an inner product space, then for arbitrary $x, y \in E$,

$$\langle x, y\rangle = \tfrac{1}{4}\{\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2\}, \text{ where } i^2 = 1$$

6 EXERCISE TMAs

1. Verify that the spaces defined in Examples 3.2 to 3.6 are inner product spaces.

2. Compute the l_2 inner product of x and y where

$$x = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right) \quad \text{and } y = \left(\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right)$$

3. Let $E = l_2$. Check if $\|x + y\| = \|x\| + \|y\|$ where

   (a) $x = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$, $y = (1, 0, 0, \ldots)$;

   (b) $x = (1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots)$, $y = 3x$

4. Consider the sequences $x = \{\frac{1}{n}\}_{n=1}^{\infty}$ and $y = \{\frac{1}{n+1}\}_{n=1}^{\infty}$

   (a) Verify that x and y are in l_2.

   (b) Compute l_2 inner product of x and y.

5. Consider the fucntions,

   $f(t) = \sin 2t; \quad g(t) = \cos 3t;$

   $h(t) = \sin^4 t; \quad m(t) = \sec^2 t.$

   (a) Verify that f, g, h and m are elements of $L_2[0, 2\pi]$

   (b) Compute the $L_2[0, 2\pi]$ inner products:

      i. $\langle f, g\rangle$ and

      ii. $\langle h, m\rangle$.

6. Let $P_n = \{p = p(t)$, polynomial of degree less than or equal to n over the interval $[a, b]\}$. Define the function $\langle , \rangle$ on $P_n \times P_n$ by

$$\langle p, q\rangle = \int_a^b p(t)q(t)dt$$

   (a) Prove that $\langle , \rangle$ is inner product on P_n.

   (b) Compute $\langle f, g\rangle$ where $f(t) = 1 - \frac{3}{2}t^2$, $g(t) = 3t^2$.

7. Let

   (i) $x(t) = t^2$, $y(t) = -t^2$.

   (ii) $x(t) = \begin{cases} 2, & 0 \le t \le \frac{1}{4} \\ 0, & \frac{1}{4} < t \le 1 \end{cases}, \qquad y(t) = \begin{cases} 0, & 0 \le t \le \frac{1}{4} \\ 2, & \frac{1}{4} < t \le 1 \end{cases}$

   (a) Verify that x and y are in $L_2[0, 1]$.

   (b) Compute $\langle x, y\rangle$ in each of (i) and (ii).

   (c) Check whether or not $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

8. Verify that $C_{[0,1]}$ with the norm defined for arbitrary $\mathbf{f} \in C_{[0,1]}$ by

$$\|f\|_0 = \sup_{t\in[0,1]} |f(t)|$$

is not an inner product space.

9. Verify that l_p, $1 < p < \infty$, $p = 2$, is not an inner product space.

10. Is R^n with norm defined for arbitrary $x = (x_1, x_2, ..., x_n) \in R^n$ by

$$\|x\|_\infty = \max_{1 \leq k \leq n} |x_k|$$

an inner product space? Justify your answer.

UNIT 5: HILBERT SPACES

Contents

1 INTRODUCTION

In the preceeding units you have studied Banach spaces and Inner product spaces. You were able to prove that an inner product induces a norm on the space, making an inner product space a normed linear space. As a result, there is need to discuss completeness in an inner product space which would give you some special class Banach spaces called Hilbert spaces. In what follows, you shall be introduced with this Hilbert spaces and its properties. These spaces are extremely useful in applications.

2 OBJECTIVES

After studying this unit, you should be able to

(i) identify and show that an inner product space is a Hilbert space.

(ii)

3 HILBERT SPACES

Recall that you defined a norm $\|\cdot\|$ on an inner product space E, by

$$\|x\| = \sqrt{\langle x, x\rangle}$$

for arbitrary $x \in E$. With respect to this norm you can also define a Cauchy sequence in E.

Definition 3.1 Let E be an inner product space. A sequence $\{x_n\}_{n=1}^{\infty}$ in E is Cauchy if and only if

$$\langle x_n - x_m, x_n - x_m\rangle^{\frac{1}{2}} := \|x_n - x_m\| \to 0 \text{ as } n, m \to \infty .$$

Consequently, an inner product space E is called complete if every Cauchy sequence in E converges to a point of E.

Definition 3.2 A complete inner product space is called a Hilbert space.

In what follows you shall call H a Hilbert space. Note that a Hilbert space H could be a complex Hilbert space or a real Hilbert space according as the underlying linear space is complex or real respectively.

3.1 EXAMPLES OF HILBERT SPACES.

Example 3.1 $L_2[a, b]$, where the functions are complex valued is a Hilbert spaces with inner product

$$\langle u, v\rangle = \int_a^b u(t)\overline{v(t)}dt \tag{1}$$

and norm

$$\|u\| = \left(\int_a^b |u(t)|^2 dt\right)^{\frac{1}{2}}$$

Example 3.2 The linear space R^n, with inner product $\langle , \rangle$ defined, for arbitrary vectors $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ in R^n, by

$$\langle x, y\rangle = \sum_{k=1} x_k y_k \tag{2}$$

is a Hilbert space.

The norm is given by

$$\|x\| = \sqrt{\langle x, x\rangle} = \left(\sum_{k=1} x_k^2\right)^{\frac{1}{2}}$$

for arbitrary $x = (x_1, x_2, ..., x_n)$ i.e., the Euclidean norm.

Example 3.3 The linear space C^n, i.e. the n-complex space, with the inner product, $\langle , \rangle$ defined for arbitrary $z = (z_1, z_2, ..., z_n)$, $\omega = (\omega_1, \omega_2, ..., \omega_n)$ in C^n, by

$$\langle z, \omega \rangle = \sum_{k=1} z_k \overline{\omega}_k \tag{3}$$

is a Hilbert space.

The norm is given by

$$\|z\|_2 = \left(\sum_{k=1}^{n} |z_k|^2 \right)^{\frac{1}{2}}$$

for arbitrary $z = (z_1, z_2, ..., z_n) \in C^n$, i.e., the unitary norm.

Example 3.4 The linear space $l_2(C)$ with the function $\langle , \rangle$ defined, for arbitrary vectors $x = (x_1, x_2, ..., x_k, ...)$, $y = (y_1, y_2, ..., y_k, ...)$ in l_2, by

$$\langle x, y \rangle = \sum_{k=1} x_k \overline{y}_k \tag{4}$$

is a Hilbert space.

The norm is given by

$$\|x\|_{l_2} = \left(\sum_{k=1} |x_k|^2 \right)^{\frac{1}{2}}$$

for arbitrary sequence $x = (x_1, x_2, ..., x_k, ...)$

Example 3.5 The space $C[a, b]$ of continuous complex or real valued function defined on $[a, b]$ with inner product

$$\langle f, g \rangle = \int_a^b f(t)\overline{g(t)}dt$$

is not a Hilbert space, because the norm induced by this inner product is

$$\|f\|_2 = \left(\int_a^b |f(t)|^2 dt \right)^{\frac{1}{2}}$$

and you proved in previous units that the space $C[a, b]$ with this norm is not complete. Or you can use the fact that $C[a, b]$ is not a closed subspace of $L_2[a, b]$ the space of Lebegue integrable functions on $[a, b]$

3.2 Orthogonality, Orthonormal sets and Orthogonal complements

Definition 3.3 Orthogonality.

Two vectors x and y in an inner product space E are orthogonal if

$$\langle x, y \rangle = 0$$

in which case we write $x \perp y$. If M is a subset of E, then we write $x \perp M$ if $x \perp y$ for every $y \in M$.

If $x \perp y$, then the parallelogram law, reduces to a generalized statement of Pythagoras' theorem, namely

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Definition 3.4 Orthogonal and Orthonormal sets. A set S in an inner product space E is called an orthogonal set if $\langle x, y\rangle = 0$ for all $x, y \in S$, $x = y$. The set S is called orthonormal if it is an orthogonal set and $\|x\| = 1$ for each $x \in S$.

Definition 3.5 Orthogonal and Orthonormal systems (basis)

A set of nonzero vectors $\{x_\alpha\}$ in E is said to be an Orthogonal basis if

$$\langle x_\alpha, x_\beta\rangle = 0 \text{ for } \alpha = \beta.$$

and an orthonormal basis if

$$\langle x_\alpha, x_\beta\rangle = \begin{cases} 0 & \text{for } \alpha = \beta, \\ 1 & \text{for } \alpha = \beta. \end{cases}$$

If $\{x_\alpha\}$ is an orthogonal basis, then clearly,

$$\frac{x_\alpha}{\|x_\alpha\|}$$

is an orthonormal basis. This is called normalizing an orthogonal set.

Example 3.6 In l_2, $\{e_n\}_{n=1}^{\infty}$ given by

$$\begin{aligned} e_1 &= (1, 0, 0, ...) \\ e_2 &= (0, 1, 0, ...) \\ . &\quad . \\ e_n &= (0, 0, ..., 0, 1, 0, ...) \\ . &\quad . \end{aligned}$$

with 1 at the n^{th} position is an orthonormal system.

Example 3.7 In $L_2[0, 2\pi]$, the set $S = \left\{\frac{1}{\sqrt{2\pi}} e^{int}\right\}_{n=1}^{\infty}$ is an orthonormal set.

Theorem 3.1 The vectors in an orthonormal system $\{x_\alpha\}$ are linearly independent.

Proof. Suppose

$$c_1 x_{\alpha_1} + c_2 x_{\alpha_2} + \cdots + c_n x_{\alpha_n} = 0$$

Then, taking the scalar product with x_{α_k}, we get

$$\langle x_{\alpha_k}, c_1 x_{\alpha_1} + c_2 x_{\alpha_2} + \cdots + c_n x_{\alpha_n} \rangle = c_k \langle x_{\alpha_k}, x_{\alpha_k} \rangle = 0$$

by the orthogonality of $\{x_\alpha\}$. But $\langle x_{\alpha_k}, x_{\alpha_k} \rangle = 0$, hence

$$c_k = 0 \quad (k = 1, 2, ..., n).$$

■

Proposition 3.1 Let S be an orthonormal set in an inner product space E. Let $\{v_1, v_2, ..., v_n\}$ be a finite subset of S. Then for any $x \in E$,

$$\sum_{k=1}^{n} |\langle x, v_k \rangle|^2 \le \|x\|^2. \tag{5}$$

Proof. Let $\alpha_k = \langle x, v_k \rangle$. Then,

$$\left\| x - \sum_{k=1}^{n} \alpha_k v_k \right\|^2 \ge 0$$

so that,

$$\left\langle x - \sum_{k=1}^{n} \alpha_k v_k , x - \sum_{j=1}^{n} \alpha_j v_j \right\rangle \ge 0,$$

i.e.,

$$\langle x, x \rangle - \left\langle x, \sum_{j=1}^{n} \alpha_j v_j \right\rangle - \left\langle \sum_{k=1}^{n} a_k v_k, x \right\rangle + \left\langle \sum_{k=1}^{n} \alpha_k v_k, \sum_{j=1}^{n} \alpha_j v_j \right\rangle \ge 0$$

or

$$\|x\|^2 - \sum_{j=1}^{n} \bar{\alpha}_j \langle x, v_j \rangle - \sum_{k=1}^{n} \alpha_k \langle v_k, x \rangle + \sum_{k=1}^{n} \sum_{j=1}^{n} \alpha_k \bar{\alpha}_j \langle v_k, v_j \rangle \ge 0,$$

By definition, $\alpha_k = \langle x, v_k \rangle$, $\alpha_j = \langle x, v_j \rangle$. Therefore,

$$\|x\|^2 - \sum_{j=1}^{n} \bar{\alpha}_j \alpha_j - \sum_{k=1}^{n} \alpha_k \bar{\alpha}_k + \sum_{k=1}^{n} |\alpha_k|^2 \ge 0,$$

using $\langle v_k, v_j \rangle = 0$, $k = j$, so that

$$\|x\|^2 - \sum_{j=1}^{n} |\alpha_j|^2 + \sum_{k=1}^{n} |\alpha_k|^2 - \sum_{k=1}^{n} |\alpha_k|^2 \ge 0$$

which yields

j=1 k=1 k=1

$$\sum_{j=1}^{} |\alpha_j|^2 := \sum_{j=1}^{} |\langle x, v_j \rangle|^2 \leq \|x\|^2 \text{, as required.}$$

∎

The above theorem can be generalize to yeild the following theorem.

Theorem 3.2 (Bessel's Inequality) If $\{v_k\}_{k=1}^{\infty}$ is an orthonormal set in an inner product space E, then for arbitrary $x \in E$,

$$\sum_{k=1}^{\infty} |\langle x, v_k\rangle|^2 \leq \|x\|^2.$$

Furthermore, $\left(x - \sum_{k=1}^{\infty} \langle x, v_k\rangle v_k\right)$ is orthogonal to v_k for each k.

Theorem 3.3 (Pythagoras) Suppose $\{u_1, u_2, ..., u_r\}$ is an orthogonal set of vectors. Then

$$\|u_1 + u_2 + \cdots + u_r\|^2 = \|u_1\|^2 + \|u_2\|^2 + \cdots + \|u_r\|^2.$$

Proof. Expanding the inner product, gives you

$$\begin{aligned}\|u_1 + u_2 + \cdots + u_r\|^2 &= \langle u_1 + u_2 + \cdots + u_r, u_1 + u_2 + \cdots + u_r\rangle \\ &= \langle u_1, u_1\rangle + \langle u_2, u_2\rangle + \cdots \langle u_r, u_r\rangle + \textstyle\sum_{i \neq j} \langle u_i, u_j\rangle\end{aligned}$$

The theorem follows from the fact that $\langle u_i, u_i\rangle = \|u_i\|^2$ and $\langle u_i, u_j\rangle = 0$ for $i \neq j$ ∎

3.2.1 Orthogonal Basis and Linear Combination, Fourier Coefficients

Definition 3.6 (Fourier Coefficients) Let $\{u_k\}$ be an orthogonal system in an inner product space E which is not orthonormal. Let

$$v_k = \frac{u_k}{\|u_k\|}$$

Then the system $\{v_k\}$ is orthonormal. Given $\mathbf{f} \in E$, let

$$c_k = \langle \mathbf{f}, v_k\rangle = \frac{1}{\|u_k\|}\langle \mathbf{f}, u_k\rangle,$$

and consider the series

$$\sum_{k=1}^{\infty} c_k v_k = \sum_{k=1}^{\infty} \frac{c_k}{\|u_k\|} u_k = \sum_{k=1}^{\infty} a_k u_k,$$

where

$$a_k = \frac{c_k}{\|u_k\|} = \frac{\langle \mathbf{f}, u_k\rangle}{\|u_k\|^2}. \tag{6}$$

Then the coefficients (6) are called the Fourier coefficients of the elements $\mathbf{f} \in E$ with respect to the orthogonal (but nor orthonormal) system $\{u_k\}$.

Example 3.8 Let S consist of the following three vectors in R^3 :

$$u_1 = (1, 2, 1), \quad u_2 = (2, 1, -4), \quad u_3 = (3, -2, 1)$$

You can verify that the vectors are orthogonal; hence they are linearly independent. Thus S is an orthogonal basis of R^3.

Suppose you want to write $v = (7, 1, 9)$ as a linear combination of u_1, u_2, u_3. First you have to set v as a linear combination of u_1, u_2, u_3 using unknowns x_1, x_2, x_3 as follows:

$$v = x_1u_1 + x_2u_2 + x_3u_3 \quad \text{or} \quad (7, 1, 9) = x_1(1, 2, 1) + x_2(2, 1, -4) + x_3(3, -2, 1) \ (*)$$

You can proceed in two ways.

Method 1: Expand (*) to obtain the system

$$x_1 + 2x_2 + 3x_3 = 7, \quad 2x_1 + x_2 - 2x_3 = 1, \quad x_1 - 4x_2 + x_3 = 9$$

Solving the above system of linear equations give you $x_1 = 3$, $x_2 = -1$ and $x_3 = 2$. Thus $v = 3u_1 - u_2 + 2u_3$.

Method 2: (This method uses the fact that the basis vectors are orthogonal, and the arithmetic is much simpler.) If we take the inner product of each side of (*) with respect to u_i, we get

$$\langle v, u_i\rangle = \langle x_1u_1 + x_2u_2 + x_3u_3, u_i\rangle \quad \text{or} \quad \langle v, u_i\rangle = x_i\langle u_i, u_i\rangle \quad \text{or} \quad x_i = \frac{\langle v, u_i\rangle}{\langle u_i, u_i\rangle}$$

Here two therms drop out, since u_1, u_2, u_3 are orthogonal. Accordingly,

$$x_1 = \frac{\langle v, u_1\rangle}{\langle u_1, u_1\rangle} = \frac{7+2+9}{1+4+1} = \frac{18}{6} = 3, \quad x_2 = \frac{\langle v, u_2\rangle}{\langle u_2, u_2\rangle} = \frac{14+1-36}{4+1+16} = \frac{-21}{21} = -1$$

$$x_3 = \frac{\langle v, u_3\rangle}{\langle u_3, u_3\rangle} = \frac{21-2+9}{9+4+1} = \frac{28}{14} = 2$$

Thus, again, you get $v = 3u_1 - u_2 + 2u_3$.

The procedure in method 2 is true in general. Namely, we have the following theorem.

Theorem 3.4 Let $\{u_1, u_2, ..., u_n\}$ be an orthogonal basis of an inner product E. Then for any $v \in E$,

$$v = \frac{\langle v, u_1\rangle}{\langle u_1, u_1\rangle}u_1 + \frac{\langle v, u_2\rangle}{\langle u_2, u_2\rangle}u_2 + \cdots \frac{\langle v, u_n\rangle}{\langle u_n, u_n\rangle}u_n$$

Proof. Suppos $v = k_1u_1 + k_2u_2 + \cdots + k_nu_n$. Taking the inner product of both sides with u_1 yields

$$\begin{aligned}
\langle v, u_1\rangle &= \langle k_1u_1 + k_2u_2 + \cdots + k_nu_n, u_1\rangle \\
&= k_1\langle u_1, u_1\rangle + k_2\langle u_2, u_1\rangle + \cdots k_n\langle u_n, u_1\rangle \\
&= k_1\langle u_1, u_1\rangle + k_2 \cdot 0 + \cdots + k_n \cdot 0 = k_1\langle u_1, u_1\rangle
\end{aligned}$$

Thus $k_1 = \dfrac{\langle v, u_1\rangle}{\langle u_1, u_1\rangle}$. Similarly, for $i = 2, ..., n$,

$$\begin{aligned}
\langle v, u_i\rangle &= \langle k_1u_1 + k_2u_2 + \cdots + k_nu_n, u_i \\
&= k_1\langle u_1, u_i\rangle + k_2\langle u_2, u_i\rangle + \cdots k_n\langle u_n, u_i\rangle \\
&= k_1 \cdot 0 + \cdots + k_i\langle u_i, u_i\rangle + \cdots + k_n \cdot 0 = k_i\langle u_i, u_1\rangle
\end{aligned}$$

Thus $k_i = \dfrac{\langle v, u_i\rangle}{\langle u_i, u_i\rangle}$. Substituting for k_i in the equation $v = k_1u_1 + \cdots + k_nu_n$, we obtain the desired result. ■

Theorem 3.5 (Riesz-Fischer theorem). Given an orthonomal system $\{u_k\}$ in complete inner product space E, let the number the numbers $c_1, c_2, ..., c_k, ...$ be such that

$$\sum_{k=1} |c_k|^2 \tag{7}$$

converges. Then there exists an element $f \in E$ with $c_1, c_2, ..., c_k, ...$ as its Fourier coefficients, i.e.m such that

$$\sum_{k=1} |c_k^2| = \|f\|^2$$

where

$$c_k = \langle f, u_k\rangle \quad (k = 1, 2, ...).$$

Proof. Writing

$$f_n = \sum_{k=1} c_k u_k,$$

we have

$$\|f_{n+p} - f_n\|^2 = \|c_{n+1}u_{n+1} + \cdots + c_{n+p}u_{n+p}\|^2 = \sum_{k=n+1}^{n+p} |c_k|^2.$$

Hence f converges to some element $f \in E$, by the convergence of (7) and the completeness of E. Moreover,

$$\langle f, u_k\rangle = \langle f_n, u_k\rangle + \langle f - f_n, u_k\rangle \tag{8}$$

where the first on the right equals c_k if $n \geq k$ and the second term approaches zero as $n \to \infty$, since

$$|\langle f - f_n, u_k\rangle| \leq \|f - f_n\|\|u_k\|.$$

Taking the limit as $n \to \infty$ in (8), we get

$$\langle f, u_k\rangle = c_k$$

since the left-hand side is independent of n. Moreover,

$$\|f - f_n\| \to 0$$

as $n \to \infty$, and hence

$$\left\langle f - \sum_{k=1}^{n} c_k u_k, f - \sum_{k=1}^{n} c_k u_k \right\rangle = \langle f, f \rangle - \sum_{k=1}^{n} |c_k|^2 \to 0$$

as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \sum_{k=1}^{n} |c_k|^2 = \sum_{k=1}^{\infty} |c_k|^2 = \|f\|^2.$$

■

3.3 The Projection Theorem

Theorem 3.6 Let E be an inner product space, let U be a subspace of E, and x an arbitrary vector in E. If there exists a vector u^* be a vector in U such that

$$\|x - u^*\| = \inf_{u\in U} \|x - u\|$$

then u^* is unique. In fact, $u^* \in U$ is a unique minimizing vector if and only if $(x - u^*) \perp U$.

Proof. ($\Rightarrow$) Let u^* be the unique minimizing vector in U. Then you have to show that $(x - u^*) \perp U$. Suppose for contradiction that this is not the case. Then there exists $0 = u_0 \in U$ which is not orthogonal to $(x - u^*)$. Without loss of generality you may assume $\|u_0\| = 1$ (otherwise, normalize u_0 by dividing by $\|u_0\|$). Since u_0 is not orthogornal to $(x - u^*)$, let $\langle x - u^*, u_0 \rangle = \delta = 0$. Define a vector $u_\delta \in U$ by $u_\delta = u^* + \delta u_0$. Then

$$\begin{aligned}
\|x - m_\delta\|^2 &= \|x - u^* - \delta u_0\|^2 \\
&= \|x - u^*\|^2 - \langle x - u^*, \delta u_0 \rangle - \langle \delta u_0, x - u^* \rangle + |\delta|^2 \\
&= \|x - u^*\|^2 - |\delta|^2 < \|x - u^*\|^2,
\end{aligned}$$

contradicting the hypothesis that u^* is the unique minimizing vector. Hence $(x - u^*) \perp U$.

($\Leftarrow$) Let $(x - u^*) \perp U$. You have to show that u^* is the unique minimizing vector. For arbitrary $u \in U, u = u^*$, we compute:

$$\|x - u\|^2 = \|(x - u^*) + (u^* - u)\|^2 = \|x - u^*\|^2 + \|u^* - u\|^2.$$

(by pythagoras theorem). Thus

$$\|x - u\| > \|x - u^*\| \text{ for } u = u^*.$$

which shows that u^* is the minimizing vector. Uniqueness follows trivially, completing the proof of this theorem. ■

Observe that the above theorem does not guarantee existence of the minimizing vector. It only asserts that if it exists, then it is unique and $(x - u^*) \perp U$. But if instead of an arbitrary inner product space we consider a Hilbert space H, and a closed subspace U of H, the following theorem guarantees the existence of minimizing vector.

Theorem 3.7 (The Projection Theorem) Let H be a Hilbert space, let U be a closed subspace of H. For arbitrary vector in H, there exists a unique $u^* \in U$ such that $\|x - u^*\| \leq \|x - u\|$ for all $u \in U$. Furthermore, $u^* \in U$ is the unique vector if and only if $(x - u^*) \perp U$.

Proof. You need only to establish the existence of a minimizing vector u^*. The uniqueness follows from the preceeding theorem. Now let $x \in \mathsf{H}$. If $x \in U$, then choose $u^* = x$ and you have nothing left to prove. So you assume that $x \notin U$ and define

$$\delta := \{\|x - u\| : u \in U\}.$$

It is enough now for you to produce a $u^* \in U$ with $\|x - u^*\| = \delta$. So let $\{u_j\}_{j=1}^{\infty}$ be a sequence of in U such that $\|x - u_j\| \to \delta$ as $j \to \infty$ (This follows from the definition of "inf"). By the parallelogram law,

$$\|(u_k - x) + (x - u_j)\|^2 + \|(u_k - x) - (x - u_j)\|^2 = 2\|u_k - x\|^2 + 2\|x - u_k\|^2.$$

Rearrangement yields,

$$\|u_k - u_j\|^2 = 2\|u_k - x\|^2 + 2\|x - u_j\|^2 - 4\left\|x - \frac{u_k - u_j}{2}\right\|^2$$

For k and j, the vector $\frac{u_k + u_j}{2}$ is in U since U is a linear subspace. Hence, by the definition of δ,

$$\left\|x - \frac{u_k + u_j}{2}\right\| \geq \delta$$

so that,

$$\|u_k - u_j\|^2 \leq 2\|u_k - x\|^2 + 2\|x - u_j\|^2 - 4\delta^2.$$

Since $\|u_j - x\| \to \delta$ as $j \to \infty$, you have that $\|u_k - u_j\| \to 0$ as $k, j \to \infty$. Hence $\{u_j\}_{j=1}^{\infty}$ is a Cauchy sequence in U and since U is complete (as a closed subset of Hilbert space), it follows that $\{u_j\}_{j=1}^{\infty}$ has a limit u^* in U. This implies, $x - u_j \to x - u^*$ as $j \to \infty$ and so (by the continuity of the norm) $\|x - u_j\| \to \|x - u^*\|$ as $j \to \infty$. But $\|x - u_j\| \to \delta$ as $j \to \infty$, so that by the uniqueness of limit, you obtain,

$$\|x - u^*\| = \delta = \inf\{\|x - u\| : u \in U\}.$$

■

This completes the proof.

As an immediate application of the Projection Theorem, the next theorem gives you that a Hilbert space H can be represented as a "direct sum" of two of its closed subspaces.

3.3.1 DIRECT SUM DECOMPOSITION

Definition 3.7 Direct sums

Let E be a vector space. E is said to be the direct sums of two subspaces U and V of E, written $E = U \oplus V$ if each $x \in E$ can be represented uniquely as $x = u + v$ with $u \in U$ and $v \in V$. In this case, U is called the algebraic complement of V in E (and vice versa). The subspaces U and V are called complementary pair of subspaces in E.

Definition 3.8 Orthogonal complement.

If U is a subspace of a Hilbert space H, then the orthogonal complement $U^{\perp}$ is defined by

$$U^{\perp} = \{x \in H : \langle x, y \rangle = 0 \text{ for all } y \in U\}.$$

In particular, for any given vector $x \in H$, we have

$$x^{\perp} = \{y \in H : \langle x, y \rangle = 0\}$$

that is, $x^{\perp}$ consists of all vectors in H that are orthogonal to the given vector y.

Proposition 3.2 Let U and V be arbitrary subspaces of a Hilbert space H. Then,

1. $U^{\perp}$ is a closed subspace of H;
2. $U \subset U^{\perp\perp}$;
3. If $U \subset V$ then $V^{\perp} \subset U^{\perp}$;
4. $V^{\perp\perp\perp} = U^{\perp}$.

Proof.

1. Let $\{x_n\}$ be a sequence of elements of $U^{\perp}$ such that $x_n \to x \in H$. You have to show that $x \in U^{\perp}$. Let $y \in U$ be fixed, $x_n \in U^{\perp}$ implies that $\langle x_n, y \rangle = 0$ for all $n \in N$. Now,

$$\begin{aligned} 0 \le |\langle x, y \rangle| &= |\langle x - x_n + x_n, y \rangle| \\ &= |\langle x - x_n, y \rangle + \langle x_n, y \rangle| \\ &= |\langle x - x_n, y \rangle| \le \|x - x_n\|\|y\| \to 0 \text{ as } n \to \infty \end{aligned}$$

   This implies that $\langle x, y \rangle = 0$, so that $x \in U^{\perp}$. Hence $U^{\perp}$ is a closed subset of H.

2. If $x \in U$, then $x \perp y$ for all $y \in U^{\perp}$. Therefore, $x \in (U^{\perp})^{\perp} = U^{\perp\perp}$.
3. $y \in V^{\perp}$ implies that $y \perp x$ for each $x \in U$. Therefore, $y \in U^{\perp}$.
4. Using (2), $U^{\perp} \subset U^{\perp\perp\perp}$ and $U \subset U^{\perp\perp}$. From (3), $(U^{\perp\perp})^{\perp} \subset U^{\perp} \subset U^{\perp\perp\perp}$ and so $U^{\perp\perp\perp} = U^{\perp}$.

■

Theorem 3.8 (Direct Sum Decomposition) Let U be a closed subspace of a Hilbert space H. Then $H = U \oplus U^{\perp}$

Proof. The proof is an application of the Projection Theorem. Let $x \in H$ be arbitrary. By the projection Theorem, there exists a unique vector $u^* \in U$ such that

$$\|x - u^*\| \leq \|x - u\| \text{ for all } u \in U \text{ and } v^* := x - u^* \in U^\perp .$$

Consequently, we can write

$$x = u^* + (x - u^*) = u^* + v^*, \text{ where } v^* := x - u^*,$$

with $u^* \in U$ and $v^* \in U^\perp$. It remains now to show that this representation is Unique. Suppose that
$x = u_1 + v_1$ with $u_1 \in U$ and $v_1 \in U^\perp$ is another representation of x. Then,

$$u^* + v^* = u_1 + v_1 \text{ so that } u_1 - u^* + (v_1 - v^*) = 0.$$

But $(u_1 - u^*)$ and $(v_1 - v_*)$ are orthogonal. Therefore by Pythagoras Theorem,

$$0 = \|(u_1 - u^*) + (u_1 - u^*)\|^2 = \|u_1 - u^*\|^2 + \|v_1 - v^*\|^2.$$

This implies, $\|u_1 - u^*\| = 0$ and $\|u_1 - u^*\|^2 = 0$, i.e., $u_1 = u^*$ and $v_1 = v^*$, establishing the uniqueness of the representation. Thus $H = U \oplus U^\perp$ ■

Proposition 3.3 Let H be a Hilbert space. Then $H^* = H$, where H^* denotes the dual of H.

Proof. The proof of this proposition is divided into 2 parts.

Part 1. You have to prove that every element $y \in H$ defines an element $f_y \in H^*$ with the same norm (i.e., $\|f_y\| = \|y\|$)

Part 2. Prove that every $f \in H^*$ defines a unique vector $y \in H$ with the same norm ($\|f\| = \|y\|$). Hence H and H^* are isometric.

Begining with Part 1. Let $y \in H$ be given. For arbitrary $x \in H$, define $f_y : H \to K$ by $f_y(x) = \langle x, y\rangle$. Clearly, f_y is linear. Moreover, $|f_y(x)| = \langle x, y\rangle| \leq \|x\| \cdot \|y\|$ so that f_y is bounded. Hence $f_y \in H^*$. Furthermore, the last inequality also yields that

$$\|f_y\| \leq \|y\| \tag{9}$$

If $y = 0$, then from (9) we have $\|f_y\| \leq \|y\| = 0$. In this case, $f_y = 0$ and you are done. So you may suppose that $y = 0$. Take $x = \frac{y}{\|y\|}$ then we have $\|x\| = 1$, and $f_y(x) = \|y\|$. This shows that

$$|f_y(x)| = \|y\| \leq \|f_y\| \cdot \|x\| = \|f_y\|, \text{ i.e., } \|y\| \leq \|f_y\|. \tag{10}$$

From (9) and (10), we have that $\|f_y\| = \|y\|$.

Part 2. Let $f \in H^*$. Consider the kernel of f, $\ker f$ defined by

$$\ker f = \{u \in H : f(u) = 0\}.$$

Set $K : \ker f$. K is closed subspace of H. By theorem 3.6, every element $x \in H$ can be written uniquely as $x = w + z$ where $w \in K$ and $z \perp K$. Let $x \in K$, then $x = w^* + z$ where $z = 0$ and $f(z) = \delta = 0$. Let $x_1 = (\frac{x}{\delta})$. Then, $f(x_1) = 1$. Hence for arbitrary $u \in H$, $u \notin K$, therefore $f(u) = \alpha$ implies that $f(u) = \alpha f(x_1)$ i.e., $f(u - \alpha x_1) = 0$. Let $u - \alpha x_1 = w^0$. Hence, $u = w^0 + \alpha x_1$, $w^0 + \alpha x_1$, $w^0 \in K$ and $\alpha x_1 \perp K$ so that

$$\langle u, x_1\rangle = \langle w^0 + \alpha x_1, x_1\rangle = \langle w^0, x_1\rangle + \alpha\langle x_1, x_1\rangle = \alpha\|x_1\|^2.$$

Hence $f(u) = \alpha = \langle u, \frac{x}{\|x_1\|^2}\rangle$ so that $f(u) = \langle u, y_0\rangle$ for all $u \in H \setminus K$ where $y_0 = \frac{x_1}{\|x_1\|^2}$. But this also holds if $u \in K$. Hence it holds for all $u \in H$. Moreover by part 1, you have that $\|f\| = \|y_0\|$. Finally, it only remains to show that y_0 is unique. But, if $f(x) = \langle x, y^*\rangle$ for all $x \in H$ then

$$\langle x, y_0\rangle = \langle x, y^*\rangle \text{ for all } x \in H \text{ so } \langle x, y_0 - y^*\rangle = 0 \text{ for all } x \in H.$$

Take $x = y_0 - y^*$. Then $\langle y_0 - y^*, y_0 - y^*\rangle = \|y_0 - y^*\|^2 = 0$ implies that $y_0 = y^*$. The proof is complete.

∎

4 The Riesz Representation Theorem

Here is an important theorem in Hilbert spaces. It states that any bounded linear functional on a Hilbert space can be represented as an inner product with a unique vector in H.

Theorem 4.1 (The Riesz Representation Theorem.) Let H be a Hilbert space and let f be a bounded linear functional on H. Then

1. There exists a unique vector $y_0 \in H$ such that

$$f(x) = \langle x, y_0 \rangle \text{ for each } x \in H;$$

2. Moreover, $\|f\| = \|y_0\|$.

5 CONCLUSION

In this unit, you have learned Hilbert space and have seen some examples. You were also introduced to some important notions such as orthogonality, orthonormal systems and orthorgonal complement of a set. You learnt the projection theorem which guarantees you the existence of a unique vector of minimum norm in a closed subspace of a Hilbert space. You also learnt that a Hilbert space can be decomposed into the direct sums of its closed subspace and the orthogonal complement of the subspace. Finally, the Rieze representation theorem gives you that every bounded linear functional in a Hilbert space can be represented as an inner product with a unique vector in the Hilbert space.

6 SUMMARY

Having studied this unit, you now know that;

(i) A complete inner product space is called a Hilbert space.

(ii) Two vectors x and y in an inner product space E are orthogonal (denoted $x \perp y$) if

$$\langle x, y \rangle = 0$$

(iii) $x \perp M$ if and only if $x \perp y$ for every $y \in M \subset E$.

(iv) A set S in an inner product space E is called an orthogonal set if $\langle x, y \rangle = 0$ for all $x, y \in X$, $x = y$. And is called an orthonormal set if it is an orthogonal set and $\|x\| = 1$ for each $x \in X$.

(v) A set of nonzero vectors $\{x_\alpha\}$ in E is said to be an orthogonal system if

$$\langle x_\alpha, x_\beta \rangle = 0 \text{ for } \alpha = \beta.$$

(vi) $\{x_\alpha\}$ is called an orthonormal system if

$$\langle x_\alpha, x_\beta \rangle = \begin{cases} 0 & \text{for} \quad \alpha = \beta \\ 1 & \text{for} \quad \alpha = \beta. \end{cases}$$

(vii) If $\{x_\alpha\}$ is an orthogonal system, then clearly $\frac{x_\alpha}{\|x_\alpha\|}$ is an orthonormal system.

(viii) (The Projection Theorem) If H is a Hilbert space, and U is a closed subspace of H, then for arbitrary vector $x \in \mathsf{H}$, there exists a unique vector $u^* \in U$ such that

$$\|x - u^*\| \leq \|x - u\| \text{ for all } u \in U.$$

Furthermore, $u^* \in U$ is the unique vector if and only if $(x - u^*) \perp U$.

(ix) If E is a vector space, then E is said to be the direct sum of two subspace U and V of E, written $E = U \oplus V$ if each $x \in E$ can be represented uniquely as $x = u + v$ with $u \in U$ and $v \in V$.

(x) If U is a subspace of a Hilbert space H, then the orthogonal complement $U^\perp$ is defined by

$$U^\perp = \{x \in \mathsf{H} : \langle x, y\rangle = 0 \text{ for all } y \in U\}.$$

(xi) If U is a closed subspace of a Hilbert space H. Then $\mathsf{H} = U \oplus U^\perp$.

(xii) $\mathsf{H}^* = \mathsf{H}$ (i.e., the dual of a Hilbert space is itself).

7 TMAs

1. $T_i : l_2 \to \mathbb{R}$ by

$$T_i(x_1, ..., x_n) = x_i \quad \forall\ (x_1, x_2, ...) \in l_2.$$

   (a) Prove that T_i is a bounded linear functional for each i.

   (b) Compute the unique vector y_0 guaranteed by the Riesz representation theorem.

2. Define $T : L_2[0, 2\pi] \to \mathbb{R}$ by

$$(Tf)(t) = \int_0^{2\pi} f(t)dt \quad \forall\ f \in L_2[0, 2\pi]$$

   (a) Prove that T is a bounded linear map.

   (b) Compute the unique vector of minimum norm guaranteed by the Riesz representation theorem.

3. Let $u = (1, 1, 1)$ be a vector in $\mathbb{R}^3$. Which of the following vectors in $\mathbb{R}^3$ is not orthogonal to u?

   (a) $v = (1, 2, -3)$

   (b) $w = (1, -4, 3)$

   (c) $x = (2, 3, 1)$

   (d) $y = (-5, 2, 3)$

4. Find k so that $u = (1, 2, k, 3)$ and $v = (3, k, 7, -5)$ in R^4 are orthogonal.

(a) $k = \frac{4}{3}$

(b) $k = \frac{3}{4}$
(c) $k = \frac{5}{3}$
(d) $k = \frac{3}{5}$

5. Let W be the substance of R^5 spanned by $u = (1, 2, 3, -1, 2)$ and $v = (2, 3, 7, 2, -1)$. Find a basis S of the orthogonal complement $W^\perp$ of W.

   (a) $S = \{w_1 = (2, -1, 0, 0, 0), w_2 = (13, 0, -4, 1, 0), w_3 = (-17, 0, 5, 0, 1)\}$ (b) $S = \{w_1 = (2, -1, 0, 0, 0), w_2 = (13, 0, -4, 1, 0), w_3 = (-17, 0, 0, 0, 5)\}$ (c) $S = \{w_1 = (2, -1, 0, 8, 0), w_2 = (13, 0, 0, 1, 0), w_3 = (-17, 0, 0, 0, 5)\}$ (d) $S = \{w_1 = (2, -1, 0, 0, 9), w_2 = (13, 0, -4, 7, 0), w_3 = (-17, 0, 0, 0, 5)\}$

6. Let $w = (1, 2, 3, 1)$ be a vector in R^4. Find an orthogonal basis S for $w^\perp$.

   (a) $S = \{v_1 = (0, 0, 1, -3), v_2 = (-21, -5, 3, 1), v_3 = (0, -5, 3, 1)\}$
   (b) $S = \{v_1 = (0, 0, 1, -3), v_2 = (0, -5, 3, 1), v_3 = (0, 14, 3, 2)\}$
   (c) $S = \{v_1 = (0, 0, 1, -3), v_2 = (-14, 2, 3, 1), v_3 = (0, -5, 3, 1)\}$ (d) $S = \{v_1 = (1, 4, 1, -3), v_2 = (-21, -5, 3, 1), v_3 = (0, -5, 3, 1)\}$

7. Let V be the vector space of polynomials over R of degree ≤ 2 with inner product defined by

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

   Find the basis of the subspace W orthogonal to $h(t) = 2t + 1$.

   (a) $\{7t^2 - 5t, 12t^2 - 5t\}$ (b) $\{7t^2 - 5t, 12t^2 - 5\}$ (c) $\{7t^2 - 5, 12t^2 - 5t\}$ (d) $\{7t^2 - 5, 12t^2 - 5\}$

8. Find a basis of the subspace W of R^4 orthogonal to $u_1 = (1, -2, 3, 4)$ and $u_2 = (3, -5, 7, 8)$

   (a) $\{(1, 2, 1, 0), (4, 4, 1, 0)\}$
   (b) $\{(1, 2, 1, 0), (4, 4, 0, 1)\}$ (c) $\{(1, 0, 2, 1), (4, 4, 0, 1)\}$ (d) $\{(1, 0, 2, 1), (4, 4, 1, 0)\}$

9. Find a basis for the subspace W of R^5 to the vector $u_1 = (1, 1, 3, 4, 1)$ and $u_2 = (1, 2, 1, 2, 1)$

(a) $\{(-5, 2, 3, 5, 6), (-6, 2, 0, 1, 0), (-5, 2, 1, 0, 0)\}$ (b) $\{(-1, 0, 0, 0, 1), (-6, 2, 0, 1, 0), (-5, 2, 1, 0, 0)\}$ (c) $\{(-1, 0, 0, 0, 1), (-6, 2, 0, 1, 0), (-5, 2, 3, 5, 6)\}$ (d) $\{(-1, 0, 0, 0, 1), (-5, 2, 3, 5, 6), (-5, 2, 1, 0, 0)\}$

Use the following to solve 10 and 11

Let $w = (1, -2, -1, 3)$ be a vector in R^4. Find

10. an orthogonal basis for $w^{\perp}$,

(a) $\{(0, 0, 3, 1), (0, 3, -3, 1), (2, 10, -9, 3)\}$ (b) $\{(0, 0, 3, 1), (0, 3, -3, 1), (1, 5, -6, 7)\}$ (c) $\{(0, 0, 3, 1), (1, 5, -6, 7), (2, 10, -9, 3)\}$

(d) $\{(1, 5, -6, 7), (0, 3, -3, 1), (2, 10, -9, 3)\}$

11. an orthonormal basis for $w^{\perp}$.

(a) $\{\frac{1}{\sqrt{10}}(0, 0, 3, 1), \frac{1}{\sqrt{19}}(0, 3, -3, 1), \frac{1}{\sqrt{194}}(1, 5, -6, 7)\}$

(b) $\{\frac{1}{\sqrt{10}}(0, 0, 3, 1), \frac{1}{\sqrt{19}}(0, 3, -3, 1), \frac{1}{\sqrt{194}}(2, 10, -9, 3)\}$

(c) $\{\frac{1}{\sqrt{10}}(0, 0, 3, 1), \frac{1}{\sqrt{19}}(1, 5, -6, 7), \frac{1}{\sqrt{194}}(2, 10, -9, 3)\}$

(d) $\{\frac{1}{\sqrt{19}}(1, 5, -6, 7), \frac{1}{\sqrt{10}}(0, 3, -3, 1), \frac{1}{\sqrt{194}}(2, 10, -9, 3)\}$

UNIT 6: LINEAR OPERATORS ON HILBERT SPACES

Contents

1 INTRODUCTION

In this unit, you shall be introduced with some important linear operators defined on Hilbert Spaces.

2 OBJECTIVES.

At the end of this unit, you should be able to

1. Define an adjoint operator on a Hilbert space.
2. Define unitary, self adjoint, normal and hermittian operators and give their properties.

3 Adjoint Operators

Definition 3.1 Let $A : \mathsf{H} \to \mathsf{H}$ be a bounded linear operator defined on a Hilbert space H. The Adjoint A^* of A is defined by

$$\langle Ax, y\rangle = \langle x, A^* y\rangle \quad \text{for all} \quad x, y \in \mathsf{H}.$$

That A^* always exists can be seen by considering $\langle Ax, y\rangle = f_y(x)$ as a linear functional on H. Clearly

$$\begin{aligned} |f_y(x)| &\leq \|Ax\|\|y\| \\ &\leq \|A\|\|x\|\|y\| \\ &\leq K\|x\| \end{aligned}$$

for fixed $y \in \mathsf{H}$. Hence f_y is a bounded linear functional, and so by the Riesz representation theorem, there exists a $y^* \in \mathsf{H}$ such that

$$\langle Ax, y\rangle = \langle x, y^*\rangle \text{ for all } x \in \mathsf{H}.$$

Thus A induces a linear map $A^* : \mathsf{H} \to \mathsf{H}$, and we write this

$$y^* = A^* y \text{ where } A^* \text{ is a bounded linear operator on } \mathsf{H}.$$

The following example shows that the adjoint operator has a simple description within the context of matrix mappings.

Example 3.1

(a) Let A be a real $n-$ square matrix viewed as a linear operator on R^n. Then, for every $u, v \in R^n$,

$$\langle Au, v\rangle = (Au)^T v = u^T A^T v = \langle u, A^T v\rangle$$

Thus the transpose A^T of A is the adjoint of A.

(b) Let B be a complex n-spare matrix viewed as a linear operator on C^n. Then, for every $u, v \in C^n$,

$$\langle Bu, v\rangle = (Bu)^T \overline{v} = u^T B^T \overline{v} = \overline{u^T B^*} \overline{v} = \langle u, B^* v\rangle$$

Remark 3.1 B^* may mean either the adjoint of B as a linear operator or the conjugate transpose of B as a matrix.

Example 3.2

1. Find the adjoint of $F : R^3 \to R^3$ defined by

$$F(x, y, z) = (3x + 4y - 5z, 2x - 6y + 7z, 5x - 9y + z)$$

Solution. First find the matrix A that represents F in the usual basis of R^3, that is, the matrix A whose rows are the coefficients of x, y, z, and then form the transpose A^T of A. This yields

$$A = \begin{pmatrix} 3 & 4 & -5 \\ 2 & -6 & -9 \\ 5 & -9 & 1 \end{pmatrix} \text{ and then } A^T = \begin{pmatrix} 3 & 2 & 5 \\ 4 & -6 & -9 \\ -5 & 7 & 1 \end{pmatrix}$$

The adjoint F^* is represented by the transpose of A; hence

$$F^*(x, y, z) = (3x + 2y + 5z, 4x - 6y - 9z, -5x + 7y + z)$$

■

2. Find the adjoint of $G : C^3 \to C^3$ defined by

$$G(x, y, z) = [2x + (1 - i)y, (3 + 2i)x - 4iz, 2ix + (4 - 3i)y - 3z]$$

Solution. Find the matrix B that represent G in the usual basis of C^3, and then form the conjugate transpose B^* of B. This yields

$$B = \begin{pmatrix} 2 & 1-i & 0 \\ 3+2i & 0 & -4i \\ 2i & 4-3i & -3 \end{pmatrix} \text{ and then } B^* = \begin{pmatrix} 2 & 3-2i & -2i \\ 1+i & 0 & 4+3i \\ 0 & 4i & -3 \end{pmatrix}$$

Then $G^*(x, y, z) = [2x + (3 - 2i)y - 2iz, (1 + i)x + (4 + 3i)z, 4iy - 3z]$ ■

Theorem 3.1 Let $A, B : H \to H$ be bounded linear operators on the Hilbert space H. Then

(a) $I^* = I$

(b) $(A + B)^* = A^* + B^*$.

(c) $(\alpha A)^* = \bar{\alpha} A^*$.

(d) $(AB)^* =$

$B^* A^*$. (e) $\|A^*\| =$

$\|A\|$.

(f) $\|A^* A\| = \|A\|^2$.

(g) $(A^*)^* = A$

Proof.

(a) For every $x, y \in H$, $\langle I(x), y\rangle = \langle x, y\rangle = \langle x, I(y)\rangle$; hence $I^* = I$

(b) For any $x, y \in H$,

$$\langle (A + B)(x), y\rangle = \langle A(x) + B(x), y\rangle = \langle A(x), y\rangle + \langle B(x), y\rangle$$

$$= \langle x, A^* y\rangle + \langle x, B^* y\rangle = \langle x, A^*(y) + B^*(y)\rangle$$

$$= \langle x, (A^* + B^*)(y) \rangle$$

The uniqueness of the adjoint implies $(A + B)^* = A^* + B^*$

(c) For any $x, y \in H$,

$$\langle(\alpha A)(x), y\rangle = \langle\alpha A(x), y\rangle = \alpha\langle A(x), y\rangle = \alpha\langle x, A^*(y)\rangle = \langle x, \bar{\alpha}A^*(y)\rangle = \langle x, (\bar{\alpha}A)(y)\rangle$$

The uniqueness of the adjoint implies $(\alpha A)^* = \bar{\alpha}A^*$

(d) For any $x, y \in H$,

$$\begin{aligned}\langle(AB)(x), y\rangle &= \langle A(B(x)), y\rangle = \langle B(x), A^*(y)\rangle \\ &= \langle x, B^*(A^*(y))\rangle = \langle x, (B^*A^*)(y)\rangle\end{aligned}$$

The uniqueness of the adjoint implies $(AB)^* = B^*A^*$

(e) From the relation $\langle x, A^*y\rangle = \langle Ax, y\rangle$, by setting $x = A^*y$, you have

$$\|A^*(y)\|^2 = \langle AA^*y, y\rangle \leq \|AA^*y\|\|y\|$$

and using the boundedness of A, this yields for all $y \in D(A^*)$,

$$\|A^*y\|^2 \leq \|A\| \cdot \|A^*\| \cdot \|y\|$$

so that

$$\|A^*y\| \leq \|A\|\|y\| \tag{1}$$

Observe that if $A^*y = 0$ for all y, then $A^* = 0$ and (e) follows. Inequality (1) yields that A^* is bounded and so

$$\|A^*\| \leq \|A\| \tag{2}$$

Applying (2) to A^* gives you $\|(A^*)^*\| \leq \|A^*\|$. Using (d), we now have

$$\|A\| \leq \|A^*\| \tag{3}$$

Inequalities (2) and (3) yield (e).

(f) Using (e), gives you that

$$\|A^*A\| \leq \|A^*\|\|A\| = \|A\|^2 \tag{4}$$

Moreover, for each $x \in D(A)$,

$$\|Ax\|^2 = \langle Ax, Ax\rangle = \langle A^*Ax, x\rangle \leq \|A^*A\| \cdot \|x\|^2$$

so that $\|Ax\| \leq (\|A^*A\|)^{\frac{1}{2}}\|x\|$ which yields

$$\|A\|^2 \leq \|AA^*\|. \tag{5}$$

(4) and (5) yield the desired result

(g) For any $x, y \in H$,

$$\langle A^*(x), y\rangle = \overline{\langle y, A^*(x)\rangle} = \overline{\langle A(y), x\rangle} = \langle x, T(y)\rangle$$

The uniqueness of the adjoint implies $(A^*)^* = A$ ■

3.1 Self-adjoint, Normal and Unitary Operators

3.1.1 Self-adjoint operators

Definition 3.2 (Self adjoint operators) A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ is said to be self adjoint if

$$A^* = A.$$

Let $B(\mathcal{H})$ denote the space of all bounded linear operators on $\mathcal{H}$. The following theorem for self-adjoint operators.

Theorem 3.2 The collection of self-adjoint operators on $\mathcal{H}$ forms a closed, real linear subspace of $B(\mathcal{H})$

Proof. The set of self-adjoint operators in $B(\mathcal{H})$ is closed under addition and scalar multiplication by real numbers, this is easy to show. Thus they form a subspace. So it is left for you to prove that this subspace is closed. Let $\{T_n\}$ be a sequence of self-adjoint operators on $\mathcal{H}$ such that $T_n \to T$. It suffices to prove that $T = T^*$. But

$$\begin{aligned} \|T - T^*\| &\le \|T - T_n\| + \|T_n - T_n^*\| + \|T_n^* - T^*\| \\ &= \|T - T_n\| + \|T_n^* - T^*\|, \text{ (since } T_n^* = T_n) \\ &= \|T - T_n\| + \|(T_n - T)^*\| = \|T - T_n\| + \|T - T_n\| = 2\|T_n - T\| \to 0 \end{aligned}$$

as $n \to \infty$ and so $T = T^*$. ■

For the next theorem, you shall need the following lemma.

Lemma 3.1 If T is a bounded linear operator on $\mathcal{H}$, then $\langle Tx, x\rangle = 0$ for all $x \in \mathcal{H}$ if and only if $T = 0$.

Proof. If $T = 0$, the result is trivial. Conversely, suppose $\langle Tx, x\rangle = 0$ for all $x \in \mathcal{H}$. Then, for arbitrary scalars α, β; and arbitary vectors $x, y \in \mathcal{H}$,

$$0 = \langle T(\alpha x + \beta y), \alpha x + \beta y\rangle - |\alpha|^2\langle Tx, x\rangle - |\beta|^2\langle Ty, y\rangle = \alpha\bar{\beta}\langle Tx, y\rangle + \beta\bar{\alpha}\langle Ty, x\rangle. \tag{6}$$

Set $\alpha = \beta = 1$ in (6) to obtain, $0 = \langle Tx, y\rangle - \langle Ty, x\rangle$. Thus $2\langle Tx, y\rangle = 0$ i.e., $\langle Tx, y\rangle = 0$ for all y. Now set $y = Tx$ to get $\|Tx\| = 0$ for all x which implies $Tx = 0$ for all x, i.e., $T = 0$ as required. ■

For self-adjoint operators, you also have the following theorem.

Theorem 3.3 Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on a complex Hilbert space, $\mathcal{H}$. Then T is self-adjoint if and only if $\langle Tx, x\rangle$ is real.

Proof. Let T be self-adjoint. It suffices to prove that $\langle Tx, x\rangle = \overline{\langle Tx, x\rangle}$ where the bar indicates complex conjugation. Now, since T is self-adjoint, you have

$$\langle Tx, x\rangle = \langle x, T^* x\rangle = \overline{\langle Tx, x\rangle}$$

and the result follows. Conversely, let

$$\langle Tx, x\rangle = \overline{\langle Tx, x\rangle} = \langle T^* x, x\rangle$$

so that

$$\langle Tx - T^* x, x\rangle = 0 \text{ or } \langle (T - T^*)x, x\rangle = 0$$

for all $x \in H$. By the last lemma, $T - T^* = 0$, i.e., $T = T^*$. ■

Remark 3.2 If $T : H \to H$ is an arbitary bounded linear operator on H, you can always obtain a self-adjoint operator from T in the following way

$$T = \frac{1}{2}(T + T^*) + i[\frac{1}{2i}(T - T^*)] = A + iB$$

where $A = \frac{1}{2}(T + T^*)$ and $B = \frac{1}{2i}(T - T^*)$ are self-adjoint operators.

Remark 3.3 Let $T : H \to H$ be a bounded linear map. You have already proved that $\|T\| = \|T^*\|$. This, however, does not imply, in general, that $\|Tx\| = \|T^* x\|$ for every $x \in H$. To see this, consider the following example:

Example 3.3 Let $T : l_2 \to l_2$ be defined by

$$T(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...)$$

Then, $\|T^*\| = \|T\|$, (already proved for all bounded linear operators on H). Take $x = (1, 0, 0, ...) \in l_2$. Observe that $\|Tx\| = 0$ whereas $\|T^* x\| = 0$.

Definition 3.3 (Positive operators.) A positive operator is a self adjoint bounded linear operator on H such that

$$\langle Ax, x\rangle \geq 0 \text{ for all } x \in H.$$

It is called strictly positive if $\langle Ax, x\rangle = 0$ if and only if $x = 0$.

Example 3.4 If $A = N^* N$, then clearly A is self adjoint and

$$\langle Ax, x\rangle = \langle N^* Nx, x\rangle = \langle Nx, Nx\rangle = \|Nx\|^2.$$

Hence A is a positive operator. If further $N^* = NN^*$, then

$$\langle N^*Nx, x\rangle = \|Nx\|^2 = \|Nx\|^2 = \langle NN^* x, x\rangle = \|N^* x\|^2.$$

so that $\|Nx\| = \|N^* x\|$.

3.1.2 Normal Operators

Definition 3.4 Normal operators A bounded linear operator N on a Hilbert space is normal if

$$NN^* = N^* N.$$

Proposition 3.1 Let $T : H \to H$ be a bounded linear map. If T is normal, then

$$\|Tx\| = \|T^* x\|, \text{ for each } x \in H$$

Proof. Let T be normal. Then

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^* Tx\rangle = \langle x, TT^* x\rangle = \langle T^* x, T^* x\rangle = \|T^* x\|^2$$

and the result follows. ■

For normal operators, you also have the following fundamental result.

Theorem 3.4 The set of all normal operators on H is a closed subset of $B(H)$ which contains the set of all self-adjoint operators, and is closed under scalar multiplication.

Proof. It is obvious that every self-adjoint operator is normal and that if α is a scalar, then αT is normal whenever T is. You can now show that the limit of any convergent sequence $\{T_n\}$ of normal operators is normal. Indeed, if $T_n \to T$, then, $T_n^* \to T^*$. Since $\{T_n\}$ is a sequence of normal operators, you have for each n, $T_n T_n^* - T_n^* T_n = 0$. Hence,

$$\|TT^* - T^* T\| \leq \|TT^* - T_n T_n^*\| + \|T_n T_n^* - T_n^* T_n\| + \|T_n^* T_n - T^* T\|$$

$$= \|TT^* - T_n T_n^*\| + \|T_n^* T_n - T^* T\| \to 0$$

and so $TT^* = T^* T$, completing the proof. ■

3.1.3 Unitary Operators

Definition 3.5 Unitary operators. A bounded linear operator U on a Hilbert space is unitary if

$$UU^* = U^* U = \mathbf{I} \text{ where } \mathbf{I} \text{ is the identity operator.}$$

In this case you have that

$$\|U^* x\| = \|Ux\| = \|x\| \text{ for all } x \in H$$

and so U is an isometry map from H to H. Furthermore, unitary operators have inverses and their adjoints are their inverses.

Theorem 3.5 Let T be an operator on a Hilbert space H. Then, the following are equivalent

(i) $T^* T = \mathbf{I}$

(ii) $\langle Tx, Ty\rangle = \langle x, y\rangle$;

(iii) $\|Tx\| = \|x\|$ for all $x \in H$.

Proof. (i) $\Rightarrow$ (ii) For all $x, y \in H$,

$$\langle x, y\rangle = \langle x, \mathbf{I}y\rangle = \langle x, T^* Ty\rangle = \langle Tx, Ty\rangle.$$

(ii) $\Rightarrow$ (iii)

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, x\rangle = \|x\|^2$$

as required (iii) $\Rightarrow$ (i) For all $x \in H$, $\|x\|^2 = \|Tx\|^2 \Rightarrow \langle x, x\rangle = \langle Tx, Tx\rangle \Rightarrow \langle x, x\rangle = \langle T^* Tx, x\rangle \Rightarrow \langle(\mathbf{I} - T^* T)x, x\rangle = 0$ This implies by lemma 3.1 that $(\mathbf{I} - T^* T) = 0$ i.e., $T^* T = \mathbf{I}$ as required.

4 Conclusion

In this unit, you studied different kinds of Bounded linear operators on a Hilbert space. You also studied adjoint operators, self adjoint maps, unitary operators, and normal operators and Hermittian operators.

5 Summary

Having studied this unit, you know that

1. The adjoint A^* of a bounded linear operator $A : H \to H$ is defined by $\langle Ax, y\rangle = \langle x, A^* y\rangle$ for all x, y in H
2. A bounded linear operator is self adjoint if $A^* = A$
3. A self adjoint bounded linear operator is positive on H if $\langle Ax, x\rangle \geq 0$ for all $x \in H$
4. A bounded linear operator N on a Hilbert space is normal if

$$NN^* = N^*N$$

5. A bounded linear operator U on a Hilbert space is unitary if $UU^* = U^*U = I$ where I is the identity operator

6 TMAs

1. Find the adjoint of

   (a) $A = \begin{pmatrix} 5-2i & 3+7i \\ 4-6i & 8+3i \end{pmatrix}$

   (b) $B = \begin{pmatrix} 3 & 5i \\ i & -2i \end{pmatrix}$

   (c) $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (x + 2y, 3x - 4z, y)$. Find T^*

   (a) $T^*(x, y, z) = (x + 3y, 2x + z, -4y)$

   (b) $T^*(x, y, z) = (x - 4z, 2y + z, y)$

   (c) $T^*(x, y, z) = (x + z, 3y + 2x, -4y)$

   (d) $T^*(x, y, z) = (x + 3y, -4y + z, 2x)$

3. Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be defined by $T(x, y, z) = [ix + (2 + 3i)y, 3x + (3 - i)z, (2 - 5i)y + iz]$. Find $T^*(x, y, z)$.

   (a) $T^*(x, y, z) = [-ix + 3y, (2 + 3i)x + (2 - 5i)z, (3 + i)y - iz]$

(b) $T^*(x, y, z) = [-ix + 3y, (2-3i)x + (2+5i)z, (3+i)y - iz]$

(c) $T^*(x, y, z) = [-ix + 3y, (2 - 3i)x + (2 + 5i)z, (3 - i)y - iz]$

(d) $T^*(x, y, z) = [-ix + 3y, (2 + 3i)x + (2 + 5i)z, (3 + i)y - iz]$

4. If T is an arbitrary bounded linear operator on H and if α and β are scalars such that $|\alpha| = |\beta|$, then $\alpha T + \beta T^*$ is normal (True or False).

5. The set of unitary operators on H is an abelian group. (True or False).

6. If T is a normal operator on a Hilbert space, H, then $(\lambda I - T)$ is also a normal operator, where λ is a scalar and I is the identity operator on H (True or False)