NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 411

COURSE TITLE: MEASURE OF THEORY AND INTEGRATION

COURSE GUIDE

MTH 411

MEASURE THEORY AND INTEGRATION

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Introduction

MTH 411 – Measure Theory and Integration is a three [3] credit unit course of thirteen units. This course is an introduction to measure theory and integration.

Measure theory is the axiomatized study of areas and volumes. It is the basis of integration theory and provides the conceptual framework for probability.

Towards the end of the nineteenth century it became clear to many mathematicians that the Riemann integral (about which one learns in calculus courses) should be replaced by some other type of integral, more general and more flexible, better suited for dealing with limit processes. Among the attempts made in this direction, the most notable ones were due to Borel and Lebesgue. It was Lebesgue's construction which turned out to be the most successful.

The Riemann integral of a function f over an interval [a, b] can be approximated by sums of the form

$$\sum_{i=1}^{n} f(t_i)m(E_i)$$

where $E_1, E_2, \dots, E_n$ are disjoint intervals whose union is [a, b], $m(E_i)$ denotes the length of $E_i, and\ \ t_i \in\ E_i$ for i = 1, 2, …, n.

Lebesgue discovered that a completely satisfactory theory of integration results if the sets E_i in the above sum are allowed to belong to a larger class of subsets of the line, the so called measurable sets and if the class of functions under consideration is enlarged to the class of measurable functions. The crucial set – theoretical properties involved is the following: The union and the intersection of any countable family of measurable sets are measurable; so is the complement of every measurable set; and most important, notion of length is now replace with measure.

In this note, we shall present an abstract (axiomatic) version of the Lebesgue integral, relative to any countably additive measure on any set. This shows that a large part of integration theory is independent of any geometry (or topology) of the underlying space, and it gives us a tool of much wider application. We shall also consider the special case of the real line.

The course is divided into five modules. The first module deals with the measure of bounded open and closed sets, the outer and inner measures of bounded sets.

The second module deals with general measure space, algebra and sigma-algebra, measures and measurable functions.

The third module deals with the theory of integration, and Lebesgue integration of real – valued functions defined on the real line.

The fourth module deals with the sets of measure zero and the L^p-spaces.

The fifth module deals with the product space, product measure, the evaluation of integrals over product measures and the Fubini's theorem.

This Course Guide gives you a brief overview of the course content, course duration, and course materials.

What you will learn in this course

The main purpose of this course is to introduce the concepts of measure theory and integration which provide the conceptual framework for probability and other areas of analysis. Thus, we intend to achieve this through the following:

Course Aims

I. Introduce the concepts associated with measurable space, measure space and measurable functions;
II. Provide some of the properties of measures, and measurable functions necessary for the theory of integration;
III. Introduce the concept of Lebesgue integration for the general measure space and for the real line;
IV. Provide some of the properties and theory of Lebesgue integrals; and
V. Provide you with the necessary foundation on the Lebesgue integration on the real line with examples.

Course Objectives

Certain objectives have been set out to ensure that the course achieves its aims. Apart from the course objectives, every unit of this course has set objectives. In the course of the study, you will need to confirm, at the end of each unit, if you have met the objectives set at the beginning of each unit. By the end of this course you should be able to:

(i) Explain the term measure of bounded open and closed sets and also know their basic properties.

(ii) Know the meaning of outer and inner measures with their basic properties.

(iii) Know the meaning with examples of algebras, sigma-algebras, measurable sets, measurable space and measure space.

(iv) Understand the concept of measurable functions, with examples and some basic theorems on measurable functions.

(v) Understand the concept of Lebesgue integration both on the general measure space and the real line.

(vi) Understand the basic theory of integration and convergence, with the application in evaluating integrals.

(vii) Know the space of integrable functions as a Banach space with the norm defined on it.

(viii) Know the product measures and product spaces and how integrals are evaluated on them with Fubini's theorem.

Working Through This Course

In order to have a thorough understanding of the course units, you will need to read and understand the contents, practise the exercise provided and be committed to learning and group discussion of the note.

This course is designed to cover approximately sixteen weeks, and it will require your devoted attention. You should do the exercises in the Tutor-Marked Assignments and submit to your tutors.

Course Materials

These include:

1. Course Guide
2. Study Units
3. Recommended Texts
4. Tutor Marked Assignments.
5. Presentation Schedule

Study Units

There are thirteen study units in this course:

MODULE ONE: Lebesgue Measure of Subset of $\mathbb{R}$

UNIT 1: Measure of a Bounded Open Set

UNIT 2: Measure of a Bounded Closed Set

UNIT 3: The Outer and Inner Measures of Bounded Sets

MODULE TWO: General Measure Space ($\mathbf{X}$, $\boldsymbol{\mathcal{M}}, \boldsymbol{\mu}$)

UNIT 1: Algebras and Sigma-Algebras

UNIT 2: Measures

UNIT 3: Measurable Functions

MODULE THREE: Theory of Integration

UNIT 1: Integration of Positive Functions

UNIT 2: Integration of Complex Functions

UNIT 3: Lebesgue Integration of Real – Valued Functions Defined on $\mathbb{R}^n$

MODULE FOUR: Classical Banach Spaces

UNIT 1: Sets of Measure Zero

UNIT 2: L^p – Spaces

MODULE FIVE: Product Measures and Fubini's Theorem

UNIT 1: Product Measures and Product Spaces

UNIT2: Fubini's Theorem

Make use of the course materials, do the exercises to enhance your learning.

Textbooks and References

D. L. Cohn, Measure Theory, Birkhauser, 1985.

P.R. Halmos, Measure Theorey, Springer Verlag, 1974.

Walter Rudin, Principle of Mathematical Analysis, Second Edition, McGraw Hill, New York, 1976.

Walter Rudin, Real and Complex Analysis, Third Edition, McGraw – Hill, New York, 1987.

S. Saks, Theory of the Integrals, Second Edition, Dover, New York, 1964.

Presentation Schedule

The Presentation Schedule included in your course materials gives you the important dates for the completion of tutor marked assignments and attending tutorials. Remember, you are required to submit all your assignments by the due date. You should guard against lagging behind in your work.

Assessment

There are two aspects to the assessment of the course. First are the tutor marked assignments; second, is a written examination.

In tackling the assignments, you are expected to apply information and knowledge acquired during this course. The assignments must be submitted to your tutor for formal assessment in accordance with the deadlines stated by your tutor. The work you submit to your tutor for assessment will count for 30% of your total course mark.

At the end of the course, you will need to sit for a final three-hour examination. This will also count for 70% of your total course mark.

Tutor Marked Assignments (TMAS)

There are some tutor marked assignments in this course. You need to submit all the assignments. The total marks for the assignments will be 30% of your total course mark.

You should be able to complete your assignments from the information and materials contained in your set textbooks, reading and study units. However, you may wish to use other references to broaden your viewpoint and provide a deeper understanding of the subject.

When you have completed each assignment, send it to your tutor. Make sure that each assignment reaches your tutor on or before the deadline given. If, however, you cannot complete your work on time, contact your tutor before the assignment is done to discuss the possibility of an extension.

Examination and Grading

The final examination for the course will carry 70% percentage of the total marks available for this course. The examination will cover every aspect of the course, so you are advised to revise all your corrected assignments before the examination.

This course endows you with the status of a teacher and that of a learner. This means that you teach yourself and that you learn, as your learning capabilities would allow.

The course units are similarly designed with the introduction following the table of contents, then a set of objectives and then the dialogue and so on.

The objectives guide you as you go through the units to ascertain your knowledge of the required terms and expressions.

Course Marking Scheme

This table shows how the actual course marking is broken down.

Assessment	Marks
Assignments	Several assignments, best three marks of the assignments count at 10% each -30% of course marks
Final Examination	70% of overall course marks
Total	100% of course marks

Table 1: Course Marking Scheme

Course Overview

Unit	Title of Work	Weeks Activity	Assessment (End of Unit)
	Course Guide	Week 1	
	Module 1		
1	Measure of a Bounded Open Set	Week 1	Assignment 1
2	Measure of a Bounded Closed Set	Week 2	
3	The Outer and Inner Measure of Bounded Sets	Week 3	Assignment 2
	Module 2		
1	Algebras and sigma-algebras	Week 4-5	

2	Measures	Week 6	Assignment 3
3	Measurable Functions	Week 7	Assignment 4
	Module 3		
1	Integration of Positive Functions	Week 8-9	Assignment 5
2	Integration of Complex Functions	Week 10	
3	Lebesgue Integration of Real – Valued Function defined	Week 11	Assignment 6
	Module 4		
1	Sets of Measure Zero	Week 12	Assignment 7
2	L^p – Spaces	Week 13	Assignment 8
	Module 5		
1	Product Measures and Product Spaces	Week 14	
2	Fubini's Theorem	Week 15	
	Revision	Week 16	
	Examination	Week 17	

How to get the best from this course

In distance learning the study units replace the university lecturer. This is one of the great advantages of distance learning; you can read and work through specially designed study materials at your own pace, and at a time and place that suit you best. Think of it as reading the lecture instead of listening to a lecturer. In the same way that a lecturer might set you some reading to do, the study units tell you when to read your set books or other material. Just as a lecturer might give you an in-class exercise, your study units provide exercises for you to do at appropriate points.

Each of the study units follows a common format. The first item is an introduction to the subject matter of the unit and how a particular unit is integrated with the other units and the course as a whole. Next is a set of learning objectives. These objectives enable you know what you should be able to do by the time you have completed the unit. You should use these objectives to guide your study. When you have finished the units you must go back and check whether you have achieved the objectives. If you make a habit of doing this you will significantly improve your chances of passing the course.

Remember that your tutor's job is to assist you. When you need help, don't hesitate to call and ask your tutor to provide it.

1. Read this *Course Guide* thoroughly.

2. Organize a study schedule. Refer to the 'Course Overview' for more details. Note the time you are expected to spend on each unit and how the assignments relate to the units. Whatever method you chose to use, you should decide on it and write in your own dates for working on each unit.

3. Once you have created your own study schedule, do everything you can to stick to it. The major reason that students fail is that they lag behind in their course work.

4. Turn to *Unit 1* and read the introduction and the objectives for the unit.

5. Assemble the study materials. Information about what you need for a unit is given in the 'Overview' at the beginning of each unit. You will almost

always need both the study unit you are working on and one of your set of books on your desk at the same time.

6. Work through the unit. The content of the unit itself has been arranged to provide a sequence for you to follow.

7. Review the objectives for each study unit to confirm that you have achieved them. If you feel unsure about any of the objectives, review the study material or consult your tutor.

8. When you are confident that you have achieved a unit's objectives, you can then start on the next unit. Proceed unit by unit through the course and try to pace your study so that you keep yourself on schedule.

9. When you have submitted an assignment to your tutor for marking, do not wait for its return before starting on the next unit. Keep to your schedule. When the assignment is returned, pay particular attention to your tutor's comments, both on the tutor-marked assignment form and also written on the assignment. Consult your tutor as soon as possible if you have any questions or problems.

10. After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives (listed in this *Course Guide*).

Tutors and Tutorials

There are 15 hours of tutorials provided in support of this course. You will be notified of the dates, times and location of these tutorials, together with the name and phone number of your tutor, as soon as you are allocated a tutorial group.

Your tutor will mark and comment on your assignments, keep a close watch on your progress and on any difficulties you might encounter and provide assistance to you during the course. You must mail or submit your tutor-marked assignments to your tutor well before the due date (at least two working days are required). They will be marked by your tutor and returned to you as soon as possible.

Do not hesitate to contact your tutor by telephone, or e-mail if you need help. The following might be circumstances in which you would find help necessary. Contact your tutor if:

- you do not understand any part of the study units or the assigned readings,
- you have difficulty with the self-tests or exercises,
- you have a question or problem with an assignment, with your tutor's comments on an assignment or with the grading of an assignment.

You should try your best to attend the tutorials. This is the only chance to have face to face contact with your tutor and to ask questions which are answered instantly. You can raise any problem encountered in the course of your study. To gain the maximum benefit from course tutorials, prepare a question list before attending them. You will learn a lot from participating in discussions actively.

Summary

Measure theory introduces you to the axiomatized study of areas and volumes which is the basis of integration theory and provides you the conceptual framework for probability and modern analysis.

We hope that by the end of this course you would have acquired the required knowledge to view Integration in a new way.

I wish you success with the course and hope that you will find it both interesting and useful.

MODULE 1: LEBESGUE MEASURE OF SUBSET OF $\mathbb{R}$ (UNEDITED)

Unit 1: Measure of a Bounded Open Set

1.0 Introduction

The concept of measure m(E) of a set E is a generalization of such concept as

(i) The length L(Δ) of a line segment Δ.

(ii) The area A(f) of a plane figure f.

(iii) The volume V(G) of a space figure G.

(iv) The increment f(b) – f(a) of a non-decreasing function f(t) over the close and bounded interval [a, b].

(v) The integral of non negative functions over a set on the real line.

We shall extend the notion of set to more complicated set rather than an interval. Thus, we want a function m define on a subset E of $\mathbb{R}$, if possible to have the following properties:

(i) m($E_1 \cup E_2$ $) = m(E_1) + m(E_1)$ for $E_1 \cap E_2$ = ∅

(ii) m(∅) = 0

(iii) m(E + x) = m(E), where E + x means the translation of E distance x

(iv) if $E_1 \subseteq E_2$, then $m(E_1) \leq m(E_1)$

(v) m(∪ $_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$, $E_i \cap E_j$ = ∅ for i ≠ j

If we have such an m, then we will be able to integrate more functions.

2.0 Objectives

By the end of this unit, you should be able to:

(i) define what the measure of a bounded open set is;

(ii) understand some basic properties of measure of bounded open sets;

(iii) prove some basic results on measures of bounded open sets.

3.0 Measure of a Bounded Open Set

We recall that a set is bounded if it is contained within a ball. Since open sets possesses a very simple structure, it is customary to begin the study of measure with open sets.

Definition 3.1.1: The measure m(G) of a non empty bounded open set G is the sum of the lengths of all its component intervals. i.e. m(G) = $\sum_k m(I_k)$, where I_k are component intervals in G.

Clearly m(G) < ∞.

Lemma 3.1.2: If a finite number of pairwise disjoint open intervals $I_1, I_2, I_3, \ldots I_n$are contained in an open interval G, then

$$m(G) \geq \sum_{k=1}^{n} m(I_k).$$

Proof: Let G = (A, B) and let I_k= (a_k, b_k), k = 1, 2,…, n, with a_1 < a_2 < … < a_n. Then b_k < b_{k+1},

k = 1, 2,…, n-1 (since otherwise, I_k and I_{k+1}will have some common points). Hence the representation

$$Q = (B - b_n) \cup (a_n - b_{n-1}) \cup (a_{n-1} - b_{n-2}) \cup \ldots \cup (a_2 - b_1) \cup (a_1 - A)$$

is non empty. It follows that

$$B - A = m(G) = \sum_{k=1}^{n} m(I_k) + m(Q)$$

And since Q ≠ ∅, then m(Q) is non - zero and so m(G) ≥ $\sum_{k=1}^{n} m(I_k)$.

Corollary 3.1.3: If a denumerable (countable) family of disjoint open intervals I_k ,

$k = 1,2,3, \ldots$ are contained in an open interval G, then m(G) ≥ $\sum_{k=1}^{\infty} m(I_k)$.

Proof: This follows from Lemma 3.1.2 as n tends to infinity.

Theorem 3.1.4: Let G_1, G_2 be open sets such that$G_1 \subseteq G_2$, then $m(G_1) \leq m(G_2)$.

Proof: Let I_i, i = 1, 2, 3,… and $J_k, k = 1,2,3, \ldots$

be the components intervals of G_1 and G_2 respectively. Then we know that each of the interval I_i is contained in one and only one of the interval J_k, hence the family of I_i can be divided into pairwise disjoint sub – families A_k, where we put I_i in A_k if $A_i \subseteq J_k$. Then by Definition 3.1.1, we have

$$\text{m}(G_1) = \sum_i m(I_i) = \sum_k \sum_{i:\, I_i \subseteq\, A_k} m(I_i)$$

$$\leq \sum_k m(J_k) = \text{m}(G_1).$$

Corollary 3.1.5: The measure of a bounded open set G is the greatest lower bound of the measures of all bounded open sets containing G.

That is

$$\text{m(G)} = \inf\{m(E): G \subseteq E, E \text{ is open and bounded}\}.$$

Proof: Exercise

Theorem 3.1.6: If the bounded open set G is the union of finite or denumerable family of pairwise disjoint open sets (that is, G = $\bigcup_k G_k$, $G_i \cap G_j = \emptyset$ for i ≠ j), then m(G) = $\sum_k m(G_k)$.

(This is called the countable additive property of m).

Proof: Exercise

Lemma 3.1.7: Let the open interval J be the union of finite or denumerable family of open sets (that is, J = $\bigcup_k G_k$). Then

$$\text{m(J)} \leq \sum_k m(G_k).$$

Proof: Exercise

Theorem 3.1.8: Let the bounded open set G be the union of finite or denumerable number of open sets G_k (that is, G = $\bigcup_k G_k$). Then

$$m(G) \leq \sum_k m(G_k).$$

Proof: Let I_i, i = 1, 2, 3, ... be the component intervals of the open set G. Then

$m(G) = \sum_i m(I_i)$. However $I_i = I_i \cap (\bigcup_k G_k) = \bigcup_k (I_i \cap G_k)$. Hence by Lemma3.1.7, we have

$m(I_i) \leq \sum_k m(I_i \cap G_k)$, and so, we have

$$m(G) \leq \sum_i \left[\sum_k m(I_i \cap G_k) \right] \qquad (3.1.1).$$

On the other hand, $G_k = G_k \cap (\bigcup_i I_i) = \bigcup_i (I_i \cap G_k)$. Each terms of the right hand side of the last equality are pairwise disjoint (since $I_i \cap I_j = \emptyset$ for i ≠ j). Hence, by applying Theorem 3.1.6, we have

$$m(G_k) = \sum_i m(I_i \cap G_k) \qquad (3.1.2).$$

From (3.1.1) and (3.1.2), we have

$$m(G) \leq \sum_i \left[\sum_k m(I_i \cap G_k) \right]$$

$$= \sum_k \left[\sum_i m(I_i \cap G_k) \right] = \sum_k m(G_k).$$

4.0 Conclusion

5.0 Summary

In this unit we have learnt:

(i) the definition of measure of a bounded open set;

(ii) the basic properties of measure of bounded open sets;

(iii) how to prove some basic results on measures of bounded open sets.

6.0 Tutor Marked Assignment

(1) Find the length (measure) of the set

$$\bigcup_{k=1}^{\infty}\{x : \frac{1}{k+1} \leq x < \frac{1}{k}\}$$

(2) Find the length of the set

$$\bigcup_{k=1}^{\infty}\{x : 0 \leq x < \frac{1}{3^k}\}$$

7.0 Further Reading and Other Resource

H.L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

Walter Rudin, Principle of Mathematical Analysis, Second Edition, McGraw Hill, New York, 1976.

Walter Rudin, Real and Complex Analysis, Third Edition, McGraw – Hill, New York, 1987.

MODULE 1: LEBESGUE MEASURE OF SUBSET OF $\mathbb{R}$

Unit 2: Measure of a Bounded Closed Set

1.0 Introduction

In this unit, we discuss the measure of a bounded close set and some of its basic properties.

2.0 Objectives

By the end of this unit, you should be able to:

(i) define what the measure of a bounded closed set is;

(ii) understand some basic properties of measure of bounded closed sets;

(iii) prove some basic results on measures of bounded closed sets.

3.0 Measure of a Bounded Closed Set

Let F be a non – empty closed set and let S be the smallest closed interval containing the set F. The set C_SF = [a, b] – F is open and hence has a definite measure m(C_SF). S = [a, b], C_SF is the complement of F with respect to S, and so is open since F is closed. This leads to the following definition.

Definition 3.2.1: The measure of a non – empty bounded closed set F is the number

m(F) = b – a – m(C_SF), where S = [a, b] is the smallest closed interval containing the set F.

Example 3.2.1:

(i) If F = [a, b], then S = [a, b] and C_SF = ∅. Hence

$$m(F) = b - a - m(C_S F) = b - a - 0 = b - a.$$

Thus, the measure of a closed interval is equal to its length.

(ii) If F is the union of a finite number of pairwise disjoint closed interval.

F = $[a_1,b_1] \cup [a_2,b_2] \cup \ldots \cup [a_n,b_n]$. We may consider the closed intervals as being enumerated in the order of increasing left end points, then $b_k < a_{k+1}$, k = 1,2, …, n-1. It follows that S = $[a_1,b_n]$ and C_SF = $(b_1,a_2) \cup (b_2,a_3) \cup \ldots \cup (b_{n-1},a_n)$. Hence

$$m(F) = b_n - a_1 - \sum_{k=1}^{n-1}(a_{k+1} - b_k) = \sum_{k=1}^{n}(b_k - a_k).$$

That is, the measure of a union of a finite number of pairwise disjoint closed intervals equals the sum of the length of these intervals.

Theorem 3.2.3: The measure of a bounded closed set F is non – negative.

Proof: Clearly C_SF ⊆ (a, b) and so m(C_SF) ≤ m(a, b) = b – a.

But m(F) = b – a – m(C_SF). Hence m(F) ≥ b – a – (b – a) = 0.

Theorem 3.2.4: The measure of a bounded closed set F is the least upper bound of the measures of the closed sets contained in F.

Proof: Exercise.

Lemma 3.2.5: Let F be a bounded closed set contained in the open interval I. Then

$$m(F) = m(I) - m(I - F).$$

Proof: Exercise

Theorem 3.2.6: Let F be a closed set and let G be a bounded open set. If F ⊆ G, then

$$m(F) \leq m(G).$$

Proof: Let I be an interval containing G. Then I = G ∪ (I – F) and applying Theorem 3.1.8, we have

$$m(F) = m(I) - m(I - F) \leq m(G) + m(I - F) - m(I - F) = m(G).$$

Theorem 3.2.7: The measure of a bounded open set G is the least upper bound of the measures of all closed sets contained in G.

Proof: Exercise

Theorem 3.2.8: The measure of a bounded closed set F is the greatest lower bound of the measures of all possible bounded open sets containing F.

Proof: Exercise

Theorem 3.2.9: Let the bounded closed set F be the union of a finite number of pairwise disjoint closed sets, F = $\bigcup_{i=1}^{n} F_i$, $F_i \cap F_j$ = ∅ for i ≠ j. Then

$$m(F) = \sum_{i=1}^{n} m(F_i).$$

Proof: Exercise

4.0 Conclusion

5.0 Summary

In this unit we have learnt:

(i) the definition of measure of a bounded closed set;

(ii) the basic properties of measure of bounded closed sets;

(iii) how to prove some basic results on measures of bounded closed sets.

6.0 Tutor Marked Assignment

Prove Theorem 3.2.4, Theorem 3.2.7, Theorem 3.2.8 and Theorem 3.2.9

7.0 Further Reading and Other Resource

H.L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

Walter Rudin, Principle of Mathematical Analysis, Second Edition, McGraw Hill, New York, 1976.

Walter Rudin, Real and Complex Analysis, Third Edition, McGraw – Hill, New York, 1987.

MODULE 1: LEBESGUE MEASURE OF SUBSET OF $\mathbb{R}$

Unit 3: The Outer and Inner Measures of Bounded Sets

1.0 Introduction

In this unit, we shall study the outer and inner measures of bounded sets and also establish some basic results on outer and inner measures.

2.0 Objectives

By the end of this unit, you should be able to:

(i) define what the outer and inner measures of bounded sets are;

(ii) know what a measurable set is;

(iii) understand some basic properties of outer and inner measures of bounded sets;

(iv) prove some basic results on outer and inner measures of bounded sets.

3.0 The Outer and Inner measures of Bounded Sets

Definition 3.3.1: The outer measure $m^*(E)$ of a bounded set E is the greatest lower bound of the measures of all bounded open sets containing the set E.

That is,

$$m^*(E) = \inf\{m(G) : E \subseteq G, G \text{ is open and bounded}\}.$$

Clearly for any bounded set E, $m^*(E)$ is well defined and $0 \leq m^*(E) < +\infty$.

Definition 3.3.2: The inner measure $m_*(E)$ of a bounded set E is the least upper bound of the measures of all closed sets contained in the set E.

That is,

$$m_*(E) = \sup\{m(F) : F \subseteq E \text{ and } F \text{ is closed}\}.$$

Also, for any bounded set E, $m_*(E)$ is well defined and $0 \leq m_*(E) < +\infty$.

Theorem 3.3.3: Let G be a bounded open set. Then $m^*(G) = m_*(G) = m(G)$.

Proof: This follows from Corollary 3.1.5 and Theorem 3.2.7.

Theorem 3.3.4: Let F be a bounded closed set. Then $m^*(F) = m_*(F) = m(F)$.

Proof: This follows from Theorem 3.2.4 and Theorem 3.2.8.

We recall that a set is measurable if its outer measure equals its inner measure.

Theorem 3.3.5: For any bounded set E, $m_*(E) \leq m^*(E)$.

Proof: Let G be a bounded open set containing the set E, for any closed subset F of the set E, we have F ⊆ G, and so, m(F) ≤ m(G). Hence $m_*(F)$ ≤ m(G). Since this is true for every bounded open set G containing E, we have that $m_*(E) \leq m^*(E)$.

Theorem 3.3.6: Let A and B be bounded sets such that A ⊆ B. Then

$$m_*(A) \leq m_*(B) \text{ and } m^*(A) \leq m^*(B).$$

Proof: We prove the first part and leave the second part as assignment. Let S be the set of numbers consisting of the measures of all closed subsets of the set A and let T be the analogous set for the set B. Then $m_*(A)$ = sup S and $m_*(B)$ = sup T. If F is a closed subset of A, then F is necessarily a subset of B since A ⊆ B. It follows thus that S ⊆ T and so $m_*(A) \leq m_*(B)$.

Theorem 3.3.7: If a bounded set E is the union of a finite or denumerable number of sets E_k,

E = $\bigcup_k E_k$. Then

$$m^*(E) \leq \sum_k m^*(E_k).$$

Proof: If the series $\sum_k m^*(E_k)$ diverges the result is trivial. Suppose the series $\sum_k m^*(E_k)$ converges. Pick an arbitrary $\epsilon > 0$, we can find bounded open set G_k such that $E_k \subseteq G$,

$$m(G_k) < m^*(E_k) + \frac{\epsilon}{2^k}, \quad k = 1, 2, \ldots .$$

Denote by I an open interval containing the set E. Then

E ⊆ I ∩ ($\bigcup_k G_k$), and so, we have

$$m^*(E) \leq m(I \cap (\bigcup_k G_k)) = m(\bigcup_k (G_k \cap I))$$

$$\leq \sum_k m(G_k \cap I) \leq \sum_k m(G_k) \leq \sum_k m^*(E_k) + \epsilon.$$

The result follows since ϵ is arbitrary.

Theorem 3.3.8: If a bounded set E is the union of a finite or denumerable number of pairwise disjoint sets E_k, (that is E = $\bigcup_k E_k$, $E_i \cap E_j = \emptyset$ for i ≠ j). Then

$$m_*(E) \geq \sum_k m_*(E_k).$$

Proof: Consider the first n sets $E_1, E_2, \ldots, E_n$. For an arbitrary $\epsilon > 0$, there exist closed sets F_k such that $F_k \subseteq E_k$, m(F_k) > $m_*(E_k) - \frac{\epsilon}{n}$, k = 1,2,…n. The sets F_k are pairwise disjoint and their union $\bigcup_k F_k$ is closed. Hence, by applying Theorem 3.2.9, we have

$$m_*(E) \geq m(\bigcup_{k=1}^n F_k) = \sum_{k=1}^n m(F_k)$$

$$> \sum_{k=1}^n m_*(E_k) - \epsilon.$$

Since ϵ is arbitrary, it follows that

$$\sum_{k=1}^n m_*(E_k) \leq m_*(E).$$

This proves the theorem for finite case. For the denumerable case, by noting that the number n is arbitrary, we can establish the convergence of the series $\sum_k m_*(E_k)$ and the inequality

$$\sum_{k=1}^{\infty} m_*(E_k) \leq m_*(E).$$

Remark: The above theorem is not generally true if we omit the condition that the sets E_k have no common points (It may be false if E_k have some points in common).

For example, if $E_1 = [0,1]$, $E_2 = [0,1]$ and E = $E_1 \cup E_2$.

Then $m_*(E_1)$ = 1, while $m_*(E_1 \cup E_2) = 1 + 1$ = 2.

4.0 Conclusion

5.0 Summary

In this unit we have learnt:

(i) the definition of outer and innner measures of a bounded sets;

(ii) the definition of measurable sets;

(iii) the basic properties of outer and innner measures of bounded sets;

(iv) how to prove some basic results on outer and inner measues of bounded sets.

6.0 Tutor marked Assignment

(1) Let A and B be bounded sets such that A ⊆ B. Show that

$$m^*(A) \leq m^*(B).$$

(2) Show that the outer measure of an interval is its length.

(3) Show that outer measure is translation invariant.

7.0 Further Reading and Other Resources

H.L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

D. L. Cohn, Measure Theory, Birkhauser, 1985.

Walter Rudin, Principle of Mathematical Analysis, Second Edition, McGraw Hill, New York, 1976.

Walter Rudin, Real and Complex Analysis, Third Edition, McGraw – Hill, New York, 1987.

MODULE2: GENERAL MEASURE SPACE (X, $\mathcal{M}, \mu$)

Unit 1: Algebras and Sigma – Algebras

1.0 Introduction

8.0 In this unit, we shall discuss Algebras and Sigma – Algebras with examples and some basic results on them.

2.0 Objectives

By the end of this unit, you should be able to:

(i) define what the algebra and sigma - algebras are;

(ii) know what measurable space and measurable set are;

(iii) understand the concept of Borel sigma – algebra on the real line;

(iv) prove some basic results on measurable space.

3.0 Algebras and Sigma – Algebras

Definition 3.1.1: Let X be an arbitrary set. A collection **Ω** of subsets of X is called an algebra if

(i) $X \in \mathcal{M}$

(ii) for any set $A \in \mathcal{M}$, the set $A^c \in \mathcal{M}$

(iii) for each finite sequence $A_1, A_2, \ldots, A_n$ of sets that belong to $\mathcal{M}$, the set $\bigcup_{i=1}^{n} A_i \in \mathcal{M}$

(iv) for each finite sequence $A_1, A_2, \ldots, A_n$ of sets that belong to $\mathcal{M}$, the set $\bigcap_{i=1}^{n} A_i \in \mathcal{M}$

Of course, in conditions (ii), (iii) and (iv), we have required $\mathcal{M}$ be closed under complementation, under the formation of finite unions and under the formation of finite intersections.

Remark 3.1.2:

(i) It is easy to check that closure under complementation and closure under the formation of finite unions together imply closure under the formation of finite intersection (using the fact that $\bigcap_{i=1}^{n} A_i$ = $(\bigcup_{i=1}^{n} A_i^c)^c$). Thus, we could have defined an algebra (Definition 3.1.1) using only conditions (i), (ii) and (iii).

(ii) $\emptyset \in \mathcal{M}$, since $\emptyset = X^c$ and X ∈ $\mathcal{M}$

Definition 3.1.3: Let X be an arbitrary set. A collection $\mathcal{M}$ of subsets of X is called a $\sigma -$ algebra if

(i) X ∈ $\mathcal{M}$

(ii) for any set A ∈ $\mathcal{M}$**,** the set $A^c \in \mathcal{M}$

(iii) for each infinite sequence A_1, A_2, … of sets that belong to $\mathcal{M}$**,** the set $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$

(iv) for each infinite sequence A_1, A_2, … of sets that belong to $\mathcal{M}$**,** the set $\bigcap_{i=1}^{\infty} A_i \in \mathcal{M}$

Thus a $\sigma -$ algebra on X is a family of subsets of X that contains and is closed under complementation, under the formation of countable unions and under the formation of countable intersections.

Remark 3.1.4:

(i) As in the case of algebra (Definition 3.1.1), we could have used only conditions (i), (ii) and (iii) or only conditions (i), (ii) and (iv) in our Definition 3.1.3, since $\bigcap_{i=1}^{\infty} A_i$ = $(\bigcup_{i=1}^{\infty} A_i^c)^c$.

(ii) $\emptyset \in \mathcal{M}$, since $\emptyset = X^c$ and X ∈ $\mathcal{M}$**.**

(iii) Let $A_1, A_2, \ldots, A_n$ be sets that belong to $\mathcal{M}$**,** the set $\bigcup_{i=1}^{n} A_i \in \mathcal{M}$**.**

We take $A_{n+1} = A_{n+2} = A_{n+3} = \ldots = \emptyset$ in (ii) of Definition2.1.3.

(iv) Let A, B ∈ $\mathcal{M}$ then $A \cap B^c \in \mathcal{M}$ since $A \cap B^c = (A^c \cup B)^c$.

(v) Each $\sigma -$ algebra on X is an algebra on X.

Example 3.1.5:

(i) Let X be any set and let $\mathcal{M}$ = P(X), the power set of X. Then $\mathcal{M}$ is a $\sigma -$ algebra on X.

(ii) Let X be any set, then $\mathcal{M}$ = {∅, X} is a $\sigma -$ algebra on X.

(iii) Let X = **N** = [1, 2, 3, 4,…}. Then $\mathcal{M}$ = {∅, X, {1, 3, 5, 7,…}, {2, 4, 6, 8,…}} is a $\sigma -$ algebra on X.

(iv) Let X be an infinite set, and let $\mathcal{M}$ be the collection of all subsets A of X such that either A or A^c is finite. Then $\mathcal{M}$ is an algebra on X, but it is not a $\sigma -$ algebra on X, since it is not closed under the formation of countable unions.

(v) Let X be any set, and let $\mathcal{M}$ be the collection of all subsets A of X such that either A or A^c is countable. Then $\mathcal{M}$ is a $\sigma -$ algebra on X.

(vi) Let X = {a, b, c, d, e, f} and $\mathcal{M}$ = {∅, X, {a, b, c}, {f}}. Then $\mathcal{M}$ is not a $\sigma -$ algebra on X.

(vii) Let X be an infinite set, and let $\mathcal{M}$ be the collection of all finite subsets of X. Then $\mathcal{M}$ does not contain X and is not closed under complementation, and so is not an algebra (or a $\sigma -$ algebra) on X.

(viii) Let $\mathcal{M}$ be the collection of all subsets of $\mathbb{R}$ that are unions of finitely many intervals of the form (a, b], (a, +∞) or (-∞ , b]. It is easy to check that each set that belongs to $\mathcal{M}$ is the union of a finite disjoint collection of intervals of the type listed above, and then to check that $\mathcal{M}$ is an algebra on $\mathbb{R}$. $\mathcal{M}$ is not a $\sigma -$ algebra on $\mathbb{R}$, for example, the bounded open subintervals of $\mathbb{R}$ are unions of sequence of sets in $\mathcal{M}$, but do not themselves belong to $\mathcal{M}$.

Definition 3.1.6: Let X be any non – empty set and let $\mathcal{M}$ be a $\sigma -$ algebra on X. The pair (X, $\mathcal{M}$) is called a measurable space. The members of $\mathcal{M}$ are called measurable sets

Proposition 3.1.7: Let (X, $\mathcal{M}$) be a measurable space. Then

(i) $\emptyset \in \mathcal{M}$

(ii) $\mathcal{M}$ is an algebra of sets

(iii) If $\{A_i\}_{i=1}^{\infty} \in \mathcal{M}$, then $\bigcap_{i=1}^{n} A_i \in \mathcal{M}$

(iv) If A, B $\in \mathcal{M}$, then A $\cap$ B^c $\in \mathcal{M}$

Proof: See Remarks 3.1.2 and 3.1.4

Proposition 3.1.8: Let X be any non – empty set. Then the intersection of an arbitrary non – empty collection of $\sigma -$ algebras on X is a $\sigma -$ algebra on X.

Proof: Denote $\{\mathcal{M}_\alpha\}_{\alpha \in \Omega}$ the non – empty collection of $\sigma -$ algebra on X. We show that $\bigcap_{\alpha \in \Omega} \mathcal{M}_\alpha$ is a $\sigma -$ algebra on X. It is enough to check that $\bigcap_{\alpha \in \Omega} \mathcal{M}_\alpha$ contains X, is closed under complementation, and is closed under the formation of countable unions.

X $\in$ $\bigcap_{\alpha \in \Omega} \mathcal{M}_\alpha$, since X $\in \mathcal{M}_\alpha \ \forall \alpha \in \Omega$. Now suppose A $\in$ $\bigcap_{\alpha \in \Omega} \mathcal{M}_\alpha$, then $A^c \in \mathcal{M}_\alpha$ $\forall \alpha \in \Omega$ and so $A^c \in \bigcap_{\alpha \in \Omega} \mathcal{M}_\alpha$. Finally, suppose $\{A_i\}$ is a sequence of sets that belong to $\bigcap_{\alpha \in \Omega} \mathcal{M}_\alpha$, then $\{A_i\} \in \mathcal{M}_\alpha \ \forall \alpha \in \Omega$, and so $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}_\alpha \ \forall \alpha \in \Omega$.

Thus, $\bigcup_{i=1}^{\infty} A_i \in \bigcap_{\alpha \in \Omega} \mathcal{M}_\alpha$.

Corollary 3.1.9: Let X be any non – empty set, and let Ω be a family of subsets of X. Then there is a smallest $\sigma -$ algebra on X that includes Ω.

Proof: Let $\{\mathcal{M}_\alpha\}_{\alpha \in \Omega}$ be the collection of all $\sigma -$ algebras on X that includes Ω. Then $\{\mathcal{M}_\alpha\}_{\alpha \in \Omega}$ is non – empty since P(X) ∈ $\{\mathcal{M}_\alpha\}_{\alpha \in \Omega}$. Then the intersection of all $\sigma -$ algebras on X that belong to $\{\mathcal{M}_\alpha\}_{\alpha \in \Omega}$ is a $\sigma -$ algebra on X by Proposition2.1.6, that is $\bigcap_{\alpha \in \Omega} \mathcal{M}_\alpha$ is a

$\sigma -$ algebra on X and $\bigcap_{\alpha \in \Omega} \mathcal{M}_\alpha$ includes Ω and is included in every $\sigma -$ algebra on X that included Ω.

Remark 3.1.10: To say that $\mathcal{M}$ is the smallest $\sigma -$ algebra on X that includes Ω is to say that $\mathcal{M}$ is a $\sigma -$ algebra on X that includes Ω, and that every $\sigma -$ algebra on X that includes Ω also includes $\mathcal{M}$. This smallest $\sigma -$ algebra on X that includes Ω is clearly unique, it is called the

$\sigma -$ algebra generated by Ω, and it is often denoted by $\sigma(\Omega)$.

We now use the preceding corollary to define an important family of $\sigma -$ algebras.

Definition 3.1.11: The Borel $\sigma -$ algebra on $\mathbb{R}^n$ is the $\sigma -$ algebra on $\mathbb{R}^n$ generated by the collection of open subsets of $\mathbb{R}^n$, and it is denoted by **B(**$\mathbb{R}^n$). The Borel subsets of $\mathbb{R}^n$ are those that belong to **B(**$\mathbb{R}^n$). In the case n = 1, we generally writes **B**($\mathbb{R}$).

Proposition 3.1.12: The $\sigma -$ algebra **B**($\mathbb{R}$) of Borel subsets of $\mathbb{R}$ is generated by each of the following collections of sets;

(i) the collection of all closed subsets of $\mathbb{R}$

(ii) the collection of all subintervals of $\mathbb{R}$ of the form (-∞ , b]

(iii) the collection of all subintervals of $\mathbb{R}$ of the form (a, b]

Proof: Exercise

For general Borel sets, we have the following.

Definition 3.1.13: Let (X,τ) be a topological space. (By Corollary3.1.9), there exists a smallest $\sigma -$ algebra **B** on X which contain the family of open subsets of X. This is called the Borel $\sigma -$ algebra on X. We shall denote this by **B**(X). For any U ∈ **B**(X), U is called a Borelian or a Borel set.

In particular;

(i) All the closed subsets are Borelian (Since U ∈ **B**(X) imply U^c ∈ **B**(X) and U^c is closed.

(ii) (X, **B**(X)) is a measurable space.

4.0 Conclusion

5.0 Summary

In this unit we have learnt:

(i) the definition of algebra and sigma algebra with examples;

(ii) the concept of measurable space and measurable sets;

(iii) some results on Borel sigma algebra;

(iv) how to prove some results on measurable space.

6.0 Tutor marked Assignment

(1) Show by example that the union of a collection of $\sigma-$ algebras on a set X can fail to be a $\sigma-$ algebra. (Hint: There are examples in which X is a small finite set.)

(2) Find an infinite collection of subsets of $\mathbb{R}$ that contains $\mathbb{R}$, is closed under the formation of countable unions, and is closed under the formation of countable intersections, but is not a $\sigma-$ algebra.

(3) Find the $\sigma-$ algebra on $\mathbb{R}$ that is generated by the collection of all one – point subsets of $\mathbb{R}$.

(4) Show that **B**($\mathbb{R}$) is generated by the collection of all compact subsets of $\mathbb{R}$.

(5) Let (X, $\mathcal{M}$) be a measurable space and let Y be a topological space. Let f: X → Y. If

$$\Omega = \{E \subseteq Y: f^{-1}(E) \in \mathcal{M}\},$$

show that Ω is a σ-algebra on Y.

(6) Let (X, $\mathcal{M}$, μ) be a measure space and

$$\mathcal{M}^* = \{E \subseteq X: \exists\ A, B \in \mathcal{M}, \text{ with } A \subseteq E \subseteq B \text{ and } \mu(B - A) = 0\}.$$

Show that $\mathcal{M}^*$ is a σ-algebra on X.

7.0 Further Reading and Other Resources

H.L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

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MODULE2: GENERAL MEASURE SPACE (X, $\mathcal{M}, \mu$)

Unit 2: Measures

1.0 Introduction

In this unit, we shall discuss measures and some of their basic properties. We start by discussing set functions, additive and countable additive set functions with some examples.

2.0 Objectives

By the end of this unit, you should be able to:

(i) define what the additive and countably additive set functions are;

(ii) know what a measure is and also give some examples of measures;

(iii) know some basic results on measures;

(iv) prove some basic results on measures.

3.0 Measures

Definition 3.2.1: A set function μ is a function whose domain is a collection of sets.

Definition 3.2.2: Let (X, $\mathcal{M}$) be a measurable space. A set function μ whose domain is the $\sigma -$ algebra $\mathcal{M}$ is called

(i) Additive if whenever A, B ∈ $\mathcal{M}$ and A ∩ B = ∅, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

(ii) Count ably additive if for and $\{A_n\}$ which are members of $\mathcal{M}$ and $A_i \cap A_j$ = ∅ whenever i ≠ j, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

In the case (ii), we take the values of μ to belong to the extended half – line [0, +∞].

Remark: Since $\mu(A_n)$ is non – negative for each n, the sum $\sum_{n=1}^{\infty} \mu(A_n)$ always exists, either as a real number as +∞.

Definition 3.2.3: Let (X, $\mathcal{M}$) be a measurable space. A measure (or a countable additive measure) on $\mathcal{M}$ is a function : $\mathcal{M}$ → [0 , +∞] that satisfy μ(∅) = 0 and is countable additive.

Definition 3.2.4: Let $\mathcal{M}$ be an algebra (not necessarily a $\sigma -$ algebra) on the set X. A function μ whose domain is $\mathcal{M}$ is called finitely additive if $\mu(\bigcup_{n=1}^{k} A_k) = \sum_{n=1}^{k} \mu(A_k)$ for each finite sequence $A_1, A_2, \ldots, A_n$ of disjoint sets that belong to $\mathcal{M}$.

A finitely additive measure on the algebra $\mathcal{M}$ is a function $: \mathcal{M} \to [0, +\infty]$ that satisfy $\mu(\emptyset) = 0$ and is finitely additive.

It is easy to check that every countably additive measure is finitely additive.

Remark: Countably additive measures seem to be sufficient for almost all applications, and support a much powerful theory of integration than do finitely additive measures. Thus we shall devote all our attention to countably additive measures. We shall emphasize that a measure will always be a countably additive measure.

If (X, $\mathcal{M}$) be a measurable space and μ is a measure on $\mathcal{M}$, then the triple (X, $\mathcal{M}$, μ) is called a measure space. Thus, a measure space is a measurable space in which a measure is defined.

We shall assume that there exists at least one A ∈ $\mathcal{M}$, such that μ(A) < ∞.

Examples 3.2.5:

(i) Let (X, $\mathcal{M}$) be a measurable space. Define a function $: \mathcal{M} \to [0, +\infty]$ by letting $\mu(A)$ be n if A is a finite set with n elements, and letting $\mu(A) = +\infty$ if A is an infinite set. Then μ is a measure, it is often called counting measure on (X, $\mathcal{M}$).

(ii) Let (X, $\mathcal{M}$) be a measurable space. Let x ∈ X.

Define

$$\delta_x(A) = \begin{cases} 1 \text{ if } x \text{ is an element of } A \\ 0 \text{ if } x \text{ is not an element of } A \end{cases}$$

Then δ_x is a measure, it is called a point mass concentrated at x.

(iii) Let X = $\mathbb{R}$ and $\mathcal{M}$ = **B**($\mathbb{R}$) be the Borel $\sigma -$ algebra on $\mathbb{R}$. Let $: \boldsymbol{B}(\mathbb{R}) \to$ [0 , +∞] be defined by $\mu(A) =$ length of A , where A is a subinterval of $\mathbb{R}$. Then μ is a measure, it is known as the Labesgue measure.

(iv) Let X = ℵ the set of natural numbers and let $\mathcal{M}$ be the collection of all subsets A of X such that either A or A^c is finite. Then $\mathcal{M}$ is an algebra, but not a $\sigma -$ algebra.

Define $\mu : \mathcal{M} \to$ [0 , +∞] as

$$\mu(A) = \begin{cases} 1 & if\ A\ is\ infinite \\ 0 & if\ A\ is\ finite \end{cases}$$

Then μ is a finitely additive measure.

(v) Let (X, $\mathcal{M}$) be a measurable space. Define $\mu : \mathcal{M} \to$ [0 , +∞] by

$$\mu(A) = \begin{cases} 1 & if\ A \neq \emptyset \\ 0 & if\ A = \emptyset \end{cases}$$

Then μ is not a measure, nor even a finitely additive measure, for if A, B ∈ $\mathcal{M}$ and A ∩ B = ∅, then $\mu(A \cup B) = 1$, while $\mu(A) + \mu(B) = 1 + 1 = 2$.

Proposition 3.2.6: Let (X, $\mathcal{M}, \mu$) be a measure space, and let A and B be subsets of X that belong to $\mathcal{M}$ and satisfy A⊆ B. Then $\mu(A) \leq \mu(B)$. If in addition A satisfies $\mu(A) < +\infty$, then

$$\mu(B - A) = \mu(B) - \mu(A).$$

Proof: Since A ∩ (B – A) = ∅ and B = A ∪ (B – A), then by additivity of μ, we have

$$\mu(B) = \mu(A) + \mu(B - A) \qquad (3.2.1)$$

And since $\mu(B - A) \geq 0$, then

$$\mu(A) \leq \mu(B).$$

In the case $\mu(A) < +\infty$, then we have

$$\mu(B) - \mu(A) = \mu(B - A),$$

from equation (3.2.1).

Definition 3.2.7: Let μ be a measure on a measurable space (X, $\mathcal{M}$).

(i) μ is a finite measure if $\mu(X) < +\infty$.

(ii) μ is a σ-finite measure if X is the union of a sequence A_1, A_2, … of sets that belong to $\mathcal{M}$ and satisfy $\mu(A_i) < +\infty$ for each i.

(iii) A set in $\mathcal{M}$ is σ-finite under μ if it is the union of a sequence of sets that belong to $\mathcal{M}$ and have finite measure under μ.

(iv) The measure space (X, $\mathcal{M}$, μ) is called finite if μ is finite.

(v) The measure space (X, $\mathcal{M}$, μ) is called σ-finite if μ is σ-finite.

We note that the measure defined in Example 3.2.5 (i) above is finite if and only if the set X is finite. The measure defined in Example 3.2.5 (ii) is finite. The Lesbesgue measure defined in Example2.2.5 (iii) is σ-finite, since $\mathbb{R}$ is the union of sequence of bounded intervals.

The following propositions give some elementary but useful properties of measures.

Proposition 3.2.8: Let (X, $\mathcal{M}$, μ) be a measure space. If $\{A_k\}$ is an arbitrary sequence of sets that belong to $\mathcal{M}$, then

$$\mu(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mu(A_k)$$

Proof: Define a sequence $\{B_k\}$ of subsets of X by letting $B_1 = A_1$, and $B_k = A_k - (\bigcup_{i=1}^{k-1} A_i)$ for k > 1. Then each $B_k \in \mathcal{M}$ and $B_k \subseteq A_k$ and so $\mu(B_k) \leq \mu(A_k)$.

Since in addition the sets B_k are disjoint and satisfy $\bigcup_k B_k = \bigcup_k A_k$, it follows that

$$\mu(\bigcup_k A_k) = \mu(\bigcup_k B_k) = \sum_k \mu(B_k)$$

$$\leq \sum_k \mu(A_k)$$

Thus,

$$\mu(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mu(A_k)$$

Remark: The above proposition shows that the countable additivity of μ implies the countable subadditivity of μ.

Proposition 3.2.9: Let (X, $\mathcal{M}$, μ) be a measure space.

(i) If $\{A_k\}$ is an increasing sequence of sets that belong to $\mathcal{M}$. Then

$$\mu(\bigcup_k A_k) = \lim_k \mu(A_k).$$

(ii) If $\{A_k\}$ is a decreasing sequence of sets that belong to $\mathcal{M}$, and if $\mu(A_n) < +\infty$ holds for some n, then

$$\mu(\bigcap_k A_k) = \lim_k \mu(A_k).$$

Proof:

(i) Suppose that $\{A_k\}$ is an increasing sequence of sets that belong to $\mathcal{M}$ and define a sequence $\{B_k\}$ of sets by letting $B_1 = A_1$, and $B_j = A_j - A_{j-1}$ for j > 1. The sets B_j are disjoint and belong to $\mathcal{M}$, and satisfy $A_k = \bigcup_{j=1}^{k} B_j$. It follows that $\bigcup_k A_k = \bigcup_j B_j$ and hence that

$$\mu(\bigcup_k A_k) = \mu(\bigcup_j B_j) = \sum_j \mu(B_j)$$

$$= \lim_k \sum_{j=1}^{k} \mu(B_j) = \lim_k \mu(\bigcup_{j=1}^{k} B_j)$$

$$= \lim_k \mu(A_k).$$

(ii) Now suppose that $\{A_k\}$ is a decreasing sequence of sets that belong to $\mathcal{M}$, and let $\mu(A_n) < +\infty$ holds for some n. We can assume n = 1. For each k, let $C_k = A_1 - A_k$. Then $\{C_k\}$ is an increasing sequence of sets that belong to $\mathcal{M}$, and $\bigcup_k C_k = A_1 - (\bigcap_k A_k)$. It follows from (i) that

$$\mu(\bigcup_k C_k) = \lim_k \mu(C_k)$$

and so

$$\mu(A_1 - (\bigcap_k A_k)) = \lim_k \mu(A_1 - A_k) \qquad (3.2.2).$$

By using Proposition 3.2.6 and the assumption that $\mu(A_1) < +\infty$, equation (3.2.2) gives

$$\mu(A_1 - (\bigcap_k A_k)) = \mu(A_1) - \lim_k \mu(A_k).$$

Thus,

$$\mu(\bigcap_k A_k) = \lim_k \mu(A_k).$$

The preceding proposition has the following partial converse, which is sometimes useful for checking when a finitely additive measure is in fact countably additive.

Proposition 3.2.10: Let (X, $\mathcal{M}$) be a measurable space, and let μ be a finitely additive measure on (X, $\mathcal{M}$). Then μ is a measure if either

(i) $\lim_k \mu(A_k) = \mu(\bigcup_k A_k)$ holds for each increasing sequence $\{A_k\}$ of sets that belong to $\mathcal{M}$. Or

(ii) $\lim_k \mu(A_k) = 0$ holds for each decreasing sequence $\{A_k\}$ of sets that belong to $\mathcal{M}$ and satisfy $\bigcap_k A_k = \emptyset$.

Proof:

We need to verify the countable additivity of μ. Let $\{B_j\}$ be a disjoint sequence of sets that belong to $\mathcal{M}$, we shall prove that $\mu(\bigcup_j B_j) = \sum_j \mu(B_j)$.

First assume that condition (i) holds, and for each k, let $A_k = \bigcup_{j=1}^{k} B_j$. Then the finite additivity of μ implies that

$$\mu(A_k) = \sum_{j=1}^{k} \mu(B_j)$$

While condition (i) implies that

$$\mu(\bigcup_{k=1}^{\infty} A_k) = \lim_k \mu(A_k),$$

since $\bigcup_{j=1}^{\infty} B_j = \bigcup_{k=1}^{\infty} A_k$, it follows that

$$\mu(\bigcup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} \mu(B_j).$$

Now assume that condition (ii) holds, and for each k, let $A_k = \bigcup_{j=k}^{\infty} B_j$. Then the finite additivity of μ implies that

$$\mu(\bigcup_{j=1}^{\infty} B_j) = \sum_{j=1}^{k} \mu(B_j) + \mu(A_{k+1}) \qquad (3.2.3).$$

While (ii) implies that $\lim_k \mu((A_{k+1}) = 0$, hence

$$\mu(\bigcup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} \mu(B_j)$$

by taking limit over k in equation (3.2.3).

4.0 Conclusion

5.0 Summary

In this unit we have learnt:

(i) the definition of additive and countably additive set functions;

(ii) the definition of measure with examples;

(iii) some basic results on measures;

(iv) how to prove some basic results on measures.

6.0 Tutor marked Assignment

(1) Suppose that μ is a finite measure on (X, $\mathcal{M}$).

(i) Show that if A, B ∈ $\mathcal{M}$, then

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

(ii) Show that if A, B, C ∈ $\mathcal{M}$, then

$$\mu(A \cup B \cup C) = \mu(A) + \mu(B) + \mu(C) - \mu(A \cap B)$$

$$- \mu(A \cap C) - \mu(B \cap C) + \mu(A \cap B \cap C)$$

(iii) Find and prove a corresponding formula for the measure of the union of n sets

(2) Let (X, $\mathcal{M}$) be a measurable space. Suppose μ is a non – negative countably additive function on $\mathcal{M}$. Show that if $\mu(A)$ is finite for some A ∈ $\mathcal{M}$, then $\mu(\emptyset) = 0$. (Thus μ is a measure).

(3) Let (X, $\mathcal{M}$) be a measurable space, and let x, y ∈ X. Show that the point masses δ_x and δ_y are equal if and only if x and y belong to exactly the same sets in $\mathcal{M}$.

(4) Let (X, $\mathcal{M}, \mu$) be a measure space, and define $\mu^*: \mathcal{M} \rightarrow [0, +\infty]$ by

$$\mu^*(A) = \sup\{\mu(B): B \subseteq A, \mathrm{B} \in \mathcal{M} \text{ and } \mu(B) < +\infty\}.$$

(i) Show that μ^* is a measure on (X, $\mathcal{M}$).

(ii) Show that if μ is σ - finite, then μ^* = μ.

(iii) Find μ^* if X is non – empty and μ is a measure defined by

$$\mu(A)\begin{cases} +\infty \ \ if \ \ A \ \in \mathcal{M} \ and \ A \ \neq \ \emptyset \\ 0 \ \ if \ A = \ \emptyset \end{cases}$$

7.0 Further Reading and Other Resources

H.L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

D. L. Cohn, Measure Theory, Birkhauser, 1985.

P.R. Halmos, Measure Theorey, Springer Verlag, 1974.

Walter Rudin, Principle of Mathematical Analysis, Second Edition, McGraw Hill, New York, 1976.

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MODULE2: GENERAL MEASURE SPACE (X, $\mathcal{M}, \mu$)

Unit 3: Measurable Functions

1.0 Introduction

In this section we introduce measurable functions and study some of their basic properties.

2.0 Objectives

By the end of this unit, you should be able to:

(i) define what a measurable function is with examples;

(ii) know some basic results on measurable functions;

(iii) prove some basic results on measurable functions.

3.0 Measurable Functions

We begin with the following result.

Proposition 3.3.1: Let (X, $\mathcal{M}$) be a measurable space, and let A be a subset of X that belongs to $\mathcal{M}$. For a function f : $\mathcal{M} \to [-\infty, +\infty]$ the following conditions are equivalent:

(i) for each real number t the set $\{x \in A : f(x) \leq t\}$ belongs to $\mathcal{M}$

(ii) for each real number t the set $\{x \in A : f(x) < t\}$ belongs to $\mathcal{M}$

(iii) for each real number t the set $\{x \in A : f(x) \geq t\}$ belongs to $\mathcal{M}$

(iv) for each real number t the set $\{x \in A : f(x) > t\}$ belongs to $\mathcal{M}$

Proof: (i) ⇒ (ii). This follows from the identity

$$\{x \in A : f(x) < t\} = \bigcup_n \{x \in A : f(x) \leq t - \frac{1}{n}\}$$

and the fact that arbitrary union of members of $\mathcal{M}$ is in $\mathcal{M}$.

(ii) ⇒ (iii). This follows from the identity

$$\{x \in A : f(x) \geq t\} = A - \{x \in A : f(x) < t\}.$$

(iii) ⇒ (iv). This follows from the identity

$$\{x \in A : f(x) > t\} = \bigcap_n \{x \in A : f(x) \geq t + \frac{1}{n}\}$$

and the fact that arbitrary intersection of members of $\mathcal{M}$ is in $\mathcal{M}$.

(iv) ⇒ (i). This follows from the identity

$$\{x \in A : f(x) \leq t\} = A - \{x \in A : f(x) > t\}.$$

Definition 3.3.2: Let (X, $\mathcal{M}$) be a measurable space and let A be a subset of X that belongs to $\mathcal{M}$. A function f: $A \to$ [-∞, +∞] is said to be measurable with respect to $\mathcal{M}$ if it satisfies one, and hence all of the conditions of Proposition 3.3.1.

In case X = $\mathbb{R}^n$, a function that is measurable with respect to **B**($\mathbb{R}^n$) is called Borel measurable or a Borel function.

Examples 3.3.3:

(i) Let f: $\mathbb{R}^n \to \mathbb{R}$ be continuous. Then for each real number t, the set {x ∈ $\mathbb{R}^n$: $f(x) <$ t} is open, and so is a Borel set. Thus f is Borel measurable.

(ii) Let I be subinterval of $\mathbb{R}$ and let f: I → $\mathbb{R}$ be non – decreasing. Then for each real number t the set {x ∈ $\mathbb{R}^n$: $f(x) <$ t} is a Borel set (it is either an interval, a set consisting of only one point, or the empty set). Thus f is Borel measurable.

(iii) Let (X, $\mathcal{M}$) be a measurable space, and let B be a subset of X. Then χ_B, the characteristic function of B is measurable if and only if B ∈ $\mathcal{M}$.

(iv) Let (X, $\mathcal{M}$) be a measurable space, let f : X → $[-\infty, +\infty]$ be simple and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the values of f. Then f is measurable if and only if {x ∈ X : $f(x) = \alpha_i$} ∈ $\mathcal{M}$ holds of i = 1, 2, …, n.

We recall the following definition

Definition 3.3.4:

(i) A real value function S: X → $\mathbb{R}$ is called a simple function if it has a finite number of values.

(ii) Let X be any set and let A be a subset of X. Denote by χ_A by setting

$$\chi_A(x) = \begin{cases} 1 & if \text{ x } is\ in\ A \\ 0 & if\ \ x\ is\ not\ in\ A \end{cases}$$

Then the function χ_A is called the characteristic function of the set A.

Let X be a measurable space and let S: X → [0, ∞] be a simple function. Let s_1, s_2, …, s_n be the distinct values of S and let $A_i = \{x \in$ X: S(x) = $s_i\}$, i = 1, 2, …, n. Then S can be written as a finite linear combination of characteristic functions of the set A_i. That is

$$S(x) = \sum_{i=1}^{n} s_i\, \chi_{A_i}(x).$$

Let f and g be [-∞, +∞] - valued functions having a common domain A. The maximum and minimum of f and, written as f g and f ⍰ g defined by

(f g)(x) = max(f(x), g(x))

and

(f ⍰ g)(x) = min(f(x), g(x))

If $\{f_n\}$ is a sequence of [-∞, +∞] - valued functions on A, then $sup_n f_n$, $inf_n f_n$, $limsup_n f_n$, $liminf_n f_n$ and $lim_n f_n$ are defined in a similar way.

For example

$$sup_n f_n(x) = \ sup_n\big(f_n(x)\big).$$

The domain of $lim_n f_n$ consists of those points in A at which $limsup_n f_n$ and $liminf f_n$ agree; the domain of each of the other four functions is A. Each of these functions can have infinite values, in particular, $liminf_n f_n$ can be +∞ or -∞.

We also recall the following

$$f^+ = \max(f, 0)$$

$$f^- = \max(-f, 0) = -\min(f, 0)$$

$$|f| = f^+ + f^-$$

$$\max(f, g) = \frac{|f-g| + f+g}{2}$$

$$\min(f, g) = \frac{-|f-g| + f+g}{2}$$

Proposition 3.3.5: Let (X, $\mathcal{M}$) be a measurable space, let A be a subset of X that belongs to $\mathcal{M}$, and let f and g be [-∞, +∞] - valued measurable functions on A. Then f g and f ▯ g are measurable.

Proof: The measurability of f g follows from the identity

$$\{x \in A : (f\ \ g)(x) \le t\} = \{x \in A : f(x) \le t\}\ ▯\ \{x \in A : g(x) \le t\}$$

and the measurability of f ▯ g follows from the identity

$$\{x \in A : (f\ \ g)(x) \le t\} = \{x \in A : f(x) \le t\} \cup \{x \in A : g(x) \le t\}.$$

Proposition 3.3.6: Let (X, $\mathcal{M}$) be a measurable space, let A be a subset of X that belongs to $\mathcal{M}$, and let $\{f_n\}$ be a sequence of [-∞, +∞] - valued measurable functions on A. Then

(i) the functions $sup_n f_n$ and $inf_n f_n$ are measurable

(ii) the functions $limsup_n f_n$ and $liminf_n f_n$ are measurable

(iii) the function $lim_n f_n$ is measurable.

Proof:

(i) The measurability of $sup_n f_n$ follows from the identity

$$\{x \in A : (sup_n f_n)(x) \le t\} = \bigcap_n \{x \in A : f_n(x) \le t\}.$$

The measurability of $inf_n f_n$ follows from the identity

$$\{x \in A : (sup_n f_n)(x) \le t\} = \bigcup_n \{x \in A : f_n(x) \le t\}.$$

(ii) For each positive integer k, define the functions g_k and h_k by

$$g_k = sup_{n \ge k} f_n \text{ and } h_k = inf_{n \ge k} f_n$$

From (i), we have that each g_k is measurable and each h_k is measurable, and that $inf_k g_k$ and $sup_k h_n$ are measurable. Since $limsup_n f_n$ and $liminf_n f_n$ are equal to $inf_k g_k$ and $sup_k h_n$, then they are measurable.

(iii) Let A_0 be the domain of $lim_n f_n$. Then

$$A_0 = \{x \in A : (limsup_n f_n)(x) = (liminf_n f_n)(x)\},$$

and so by Proposition 3.3.1, A_0 belongs to $\mathcal{M}$. Since

$$\{x \in A_0 : lim_n f_n(x) \le t\} = A_0 \cap \{x \in A : limsup_n f_n(x) \le t\},$$

The measurability of $lim_n f_n$ follows.

Proposition 3.3.7: Let (X, $\mathcal{M}$) be a measurable space, let A be a subset of X that belongs to $\mathcal{M}$, let f and g be measurable real – valued functions on A, and let α be a real number. Then αf, f + g, f – g, fg, f/g , $|f|$, f^+ , f^- are measurable.

Proof: Exercise.

Proposition 3.3.8: Let (X, $\mathcal{M}$) be a measurable space, let A be a subset of X that belongs to $\mathcal{M}$. For a function f: A → ℝ, the following are equivalent:

(i) f is measurable

(ii) for each open subset U of ℝ, the set $f^{-1}(U) \in \mathcal{M}$

(iii) for each closed subset C of ℝ, the set $f^{-1}(C) \in \mathcal{M}$

(iv) for each Borel subset B of ℝ, the set $f^{-1}(B) \in \mathcal{M}$.

Proof: Exercise

Lastly in this section, we give a general definition of measurable functions.

Definition 3.3.9: Let (X, $\mathcal{M}_X$) and (Y, $\mathcal{M}_Y$) be measurable spaces. A function f: X → Y is said to measurable with respect to $\mathcal{M}_X$ and $\mathcal{M}_Y$ if for each B ∈ $\mathcal{M}_Y$, the set $f^{-1}(B) \in \mathcal{M}_X$.

Instead of saying that f is measurable with respect to $\mathcal{M}_X$ and $\mathcal{M}_Y$, we shall sometimes say that f is measurable from $\mathcal{M}_X$ to $\mathcal{M}_Y$ or simply f: (X, $\mathcal{M}_X$) → (Y, $\mathcal{M}_Y$) is measurable.

Likewise if A belongs to $\mathcal{M}_X$, a function f: A → Y is measurable if $f^{-1}(B) \in \mathcal{M}_X$ holds whenever B ∈ $\mathcal{M}_Y$.

Proposition 3.3.10: Let (X, $\mathcal{M}_X$), (Y, $\mathcal{M}_Y$) and (Z, $\mathcal{M}_Z$) be measurable spaces and let

f: (Y, $\mathcal{M}_Y$) → (Z, $\mathcal{M}_Z$) and g: (X, $\mathcal{M}_X$) → (Y, $\mathcal{M}_Y$) be measurable. Then the composition

f o g : (X, $\mathcal{M}_X$) → (Z, $\mathcal{M}_Z$) is measurable.

Proof: Exercise

4.0 Conclusion

5.0 Summary

In this unit we have learnt:

(i) the definition of a measurable function with examples;

(ii) the basic properties of measure functions;

(iii) how to prove some basic results on measure functions.

6.0 Tutor Marked Assignment

(1) Let (X, $\mathcal{M}_X$) be a measurable space and let A be a measurable subset of X, show that χ_A the characteristics function of A is a measurable function on A.

(2) Show that the supremum of an uncountable family of $[-\infty, +\infty]$ - valued Borel measurable functions on $\mathbb{R}$ can fail to be Borel measurable.

(3) Show that if f: $\mathbb{R} \to \mathbb{R}$ is differentiable everywhere on $\mathbb{R}$, then its derivative f' is Borel measurable.

(4) (X, $\mathcal{M}$) be a measurable space, and let $\{f_n\}$ be a sequence of $[-\infty, +\infty]$ – valued measurable functions on X. Show that {x ∈ X: $\lim_n f_n(x)$ exists and is finite} belongs to $\mathcal{M}$.

(5) If $|f|$ is a measurable function, is it true to claim that f is measurable? If not give a counter example.

7.0 Further Reading and Other Resources

H.L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

D. L. Cohn, Measure Theory, Birkhauser, 1985.

P.R. Halmos, Measure Theorey, Springer Verlag, 1974.

Walter Rudin, Principle of Mathematical Analysis, Second Edition, McGraw Hill, New York, 1976.

Walter Rudin, Real and Complex Analysis, Third Edition, McGraw – Hill, New York, 1987.

MODULE3: Theory of Integration

Unit 1: Integration of Positive Functions

1.0 Introduction

In this unit, we shall discuss the Lebesgue integrals of positive functions over the general measure space and their basic properties. The monotone convergence and Beppo Levi's theorems with Fatou's lemma will also be discussed.

2.0 Objectives

By the end of this unit, you should be able to:

(i) define the Lebesgue integral of positive function over a general measure space;

(ii) know some basic properties of Lebesgue integrals;

(iii) prove some basic results on Lebesgue integrals;

(iv) know and prove monotone convergence theorem, Beppo Levi's theorem and Fatou's lemma.

3.0 Integration of Positive Functions

Definition 3.1.1: Let (X, $\mathcal{M}$, μ) be a measure space and let S be a simple measurable, non – negative function on X with representation S(x) = $\sum_{i=1}^{n} s_i \, \chi_{A_i}(x)$ where $A_i = \{x \in$ X: S(x) = $s_i\}$. Let E ∈ $\mathcal{M}$, we define

$$\int_E \; S \, d\,\mu = \sum_{i=1}^{n} s_i \;\; \mu(A_i \cap \text{E}), \qquad A_i \;\subseteq X.$$

Let X → $[0, \infty]$ be measurable function, $\int_E \; f \, d\,\mu$ is defined by

$$\int_E \; f \; d\,\mu \;=\; sup_{0\,\leq S\,\leq f} \int_E \; S \; d\,\mu = \sup\int_E \; S \; d\,\mu$$

Where S is a simple measurable function on S such that $0 \leq S \leq f$.

$\int_E f \, d\mu$ is called the Lebesgue integral on f defined on E with respect to μ.

We note that $0 \leq \int_E f \, d\mu \leq \infty$.

Theorem 3.1.2:

(i) If $0 \leq f \leq g$, then $\int_E f \, d\mu \leq \int_E g \, d\mu$.

(ii) If A ⊆ B and f ≥ 0, $\int_A f \, d\mu \leq \int_B g \, d\mu$.

(iii) If f(x) = 0 for every x in E, then $\int_E f \, d\mu = 0$, even if $\mu(E) = +\infty$.

(iv) If f ≥ 0, and $0 \leq c < \infty$, $\int_E cf \, d\mu$ = c$\int_E f \, d\mu$.

(v) If $\mu(E) = 0$, then $\int_E f \, d\mu = 0$ even if f(x) = ∞ for every x in E.

(vi) If f ≥ 0, then $\int_E f \, d\mu = \int_X \chi_E f \, d\mu$.

Proof: (i) – (v) as exercise. For the prove of (vi):

$$\int_X \chi_E f \, d\mu = \int_E \chi_E f \, d\mu + \int_{X \setminus E} \chi_E f \, d\mu$$

$$= \int_E f \, d\mu + 0 = \int_E f \, d\mu$$

Theorem 3.1.3: Let s and t be two simple measurable functions defined on X. For every E in $\mathcal{M}$, let $\varphi(E) = \int_E s \, d\mu$. Then φ is a measure on $\mathcal{M}$ and

$$\int_X (s + t) \, d\mu = \int_X s \, d\mu + \int_X t \, d\mu.$$

Proof: Clearly φ is a set function. We shall show that φ is a measure on $\mathcal{M}$ by showing that $\varphi(E) \geq 0$, and for $E_1, E_2, E_3, \ldots \in \mathcal{M}$ such that $E_i \cap E_j = \emptyset$ for $i \neq j$, $\varphi(\bigcup_{i\geq 1} E_i) = \sum_{i\geq 1} \varphi(E_i)$.

$$\varphi(E) = \int_E s \, d\mu = \sum_{i=1}^{n} s_i \; \mu(A_i \cap E) \geq 0.$$

Also,

$$\varphi(\bigcup_{j\geq 1} E_j) = \int_{\bigcup_{j\geq 1} E_j} s \, d\mu = \sum_{i=1}^{n} s_i \; \mu(A_i \cap (\bigcup_{j\geq 1} E_j)).$$

But

$$A_i \cap (\bigcup_{j\geq 1} E_j) = \bigcup_{j\geq 1}(A_i \cap E_j) = \bigcup_{j\geq 1}(E_{ij}),$$

where $E_{ij} = A_i \cap E_j$ and $E_{ij} \cap E_{ip} = \emptyset$, for $j \neq p$.

Thus,

$$\varphi(\bigcup_{j\geq 1} E_j) = \sum_{i=1}^{n} s_i \; \mu\big(\bigcup_{j\geq 1}(E_{ij})\big) = \sum_{i=1}^{n} s_i \sum_{j\geq 1} \mu(E_{ij})$$

$$\sum_{j\geq 1} \sum_{i=1}^{n} s_i \; \mu\big(E_{ij}\big) = \sum_{j\geq 1} \sum_{i=1}^{n} s_i \; \mu\big(A_i \cap E_j\big)$$

$$= \sum_{i\geq 1} \varphi(E_i).$$

Hence φ is countably additive and so, it is a measure on $\mathcal{M}$.

Let $s = \sum_{i=1}^{n} \alpha_i \, \chi_{A_i}$ and $t \; \sum_{j=1}^{n} \beta_j \, \chi_{B_j}$ and let $E \in \mathcal{M}$. Then

$$\int_E s\, d\mu + \int_E t\, d\mu = \sum_{i=1}^{n} \alpha_i\ \mu(A_i \cap \mathrm{E}) + \sum_{j=1}^{n} \beta_j\ \mu(B_j \cap \mathrm{E})$$

$$= \sum_{i=1}^{n} \alpha_i \left[\int_{E\cap A_i} \chi_{A_i}\, d\mu + \int_{E\cap \mathrm{A_i}^{\mathrm{c}}} \chi_{A_i}\, d\mu\right] +$$

$$\sum_{j=1}^{n} \beta_j \left[\int_{E\cap B_j} \chi_{B_j}\, d\mu + \int_{E\cap \mathrm{B_j}^{\mathrm{c}}} \chi_{B_j}\, d\mu\right]$$

$$= \sum_{i=1}^{n} \alpha_i \int_{E\cap A_i} \chi_{A_i}\, d\mu + \sum_{j=1}^{n} \beta_j \int_{E\cap B_j} \chi_{B_j}\, d\mu$$

$$= \int_E \sum_{i=1}^{n} \alpha_i\, \chi_{A_i}\, d\mu + \int_E \sum_{j=1}^{n} \beta_j\, \chi_{B_j}\, d\mu$$

$$= \int_E \left(\sum_{i=1}^{n} \alpha_i\, \chi_{A_i} + \sum_{j=1}^{n} \beta_j\, \chi_{B_j}\right) d\mu$$

$$= \int_E (s+t)\, d\mu.$$

Theorem 3.1.4: (Lebesgue Monotone Convergence)

Let $\{f_n\}$ be a sequence of extended real valued measurable functions defined on X such that

(i) $0 \le f_1(x) \le f_2(x) \le \dots \le \infty$ for every x in X

(ii) For any x in X, $f_n(x) \to f(x)$ as $\mathrm{n} \to \infty$.

Then f is measurable and $\int_X f_n\, d\mu \to \int_X f\, d\mu$ as $\mathrm{n} \to \infty$. That is

$$\lim_{\mathrm{n}\to\infty} \int_X f_n\, d\mu = \int_X \left(\lim_{\mathrm{n}\to\infty} f_n\right) d\mu = \int_X f\, d\mu.$$

Proof: By hypothesis $\{f_n\}$ is an increasing sequence, and so $f_n \leq f_{n+1}$.

Thus by Theorem3.1.2 (i), we have $\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu$ and there exist $\alpha \in [0, \infty]$ such that $\int_X f_n \, d\mu \rightarrow \alpha$ as n → ∞.

By Theorem 3.3.6, f is measurable since f $= sup_n f_n$.

We prove that

$$\alpha \ \lim_{n \to \infty} \int_X f_n \, d\mu = \int_X (\lim_{n \to \infty} f_n) d\mu \qquad (3.1.1),$$

since if equation (3.1.1) holds, then the result follows.

As $f_n \leq f$, then

$$\int_X f_n \, d\mu \leq \int_X f \, d\mu = \int_X (\lim_{n \to \infty} f_n) d\mu,$$

and this implies,

$$\int_X (\lim_{n \to \infty} f_n) d\mu \leq \int_X f \, d\mu = \int_X (\lim_{n \to \infty} f_n) d\mu \qquad (3.1.2).$$

Let us now take a simple measurable function s such that $0 \leq s \leq f$. Also let C be a constant such that 0 < C < 1. Define $E_n = \{x \in X\text{: } f_n(x) \geq Cs(x)\}$, n = 1, 2... Then E_n are measurable sets since $E_n = \{x \in X\text{: } f_n(x) - Cs(x) \geq 0\}$ and $E_1 \subseteq E_2 \subseteq \dots$.

Moreover X = $\bigcup_{n=1}^{\infty} E_n$ and so, we have

$$\int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq \int_{E_n} Cs \, d\mu \text{ for n = 1, 2, … .}$$

By taking limit as n → ∞, we have

$$\lim_{n \to \infty} \int_X f_n \, d\mu \geq \lim_{n \to \infty} C \int_{E_n} s \, d\mu$$

$$\alpha = \lim_{n \to \infty} \int_X f_n \, d\mu \geq C \int_{E_n} s \, d\mu$$

As C → 1, we have

$$\alpha = \lim_{n \to \infty} \int_X f_n \, d\mu \geq \int_X s \, d\mu$$

Thus,

$$\alpha \geq sup_{0 \leq s \leq f} \int_X s \, d\mu = \int_X f \, d\mu \qquad (3.1.3).$$

From (3.1.2) and (3.1.3), the result follows.

Theorem 3.1.5: (Beppo Levi's Theorem)

Let $\{f_n\}$ be a sequence of measurable functions, f_n: X → $[0, \infty]$ and assume that

$f(x) = \sum_{n=1}^{\infty} f_n(x)$ for every x in X, then $\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$.

That is, the integral of sum of functions is the sum of the integral of the functions.

Proof: At first, there exist simple functions (s_i') and (s_i'') such that $s_i' \to f_1$ and $s_i'' \to f_2$.

Let $s_i = {s_i}' + {s_i}''$, we have $s_i \to f_1 + f_2$ and by Theorem31.3 and Theorem3.14, we have

$$\int_X (f_1 + f_2) d\mu = \int_X f_1 \, d\mu + \int_X f_2 \, d\mu.$$

The result for n > 2, can be prove by induction.

Theorem 3.1.6: (Fatou's Lemma)

Let f_n: X → $[0, \infty]$ be measurable functions for every n = 1, 2, …, then

$$\int_X (\lim_{n \to \infty} inf f_n) d\mu \leq \lim_{n \to \infty} inf \int_X f_n \, d\mu .$$

Proof: Let $g_k = inf_{i \geq k} f_i$, k = 1, 2, … . That is $g_k = \inf\{f_k, f_{k+1}, f_{k+2}, \ldots\}$.

Then $g_k \leq f_n$ for k $\leq n$. Therefore $\int_X g_k \, d\mu \leq \int_X f_n \, d\mu$ for k $\leq n$.

By keeping k fixed, we have

$$\int_X g_k \, d\mu \leq \lim_{n \to \infty} inf \int_X f_n \, d\mu \qquad (3.1.4).$$

Now the sequence $\{g_k\}$ is monotone increasing, and has to converge to $sup_k g_k$ = $\lim_{n \to \infty} inf f_n$.

Hence, by Theorem3.1.4, we have

$$\int_X (\lim_{n \to \infty} inf f_n) d\mu = \lim_{k \to \infty} \int_X f_n \, d\mu$$

$$\leq \liminf_{n \to \infty} \int_X \, d\mu$$

by (3.1.4) and the proof is complete.

4.0 Conclusion

5.0 Summary

In this unit we have learnt:

(i) the definition of Lebesgue integral of a positive function over the general measure space;

(ii) the basic properties of Lebesgue integrals ;

(iii) how to prove some basic results on Lebesgue integrals;

(iv) how to prove monotone convergence theorem. Beppo Levi's theorem and Fatou's lemma .

6.0 Tutor marked Assignment

(1) Prove or disprove: A real valued function f defined on X is Lebesgue integrable if and only if it is Riemann integrable.

(2) Let

$$I_n = \int_0^n (1 - \frac{x}{n})^n) e^{\frac{x}{2}} \, \text{dx}.$$

Show that $\lim_{n \to \infty} I_n$ exists.

(3) Let

$$f_m = \begin{cases} \chi_E & if\ m\ is\ even \\ 1-\chi_E & if\ m\ is\ odd \end{cases}$$

where E ⊆ X and X is a measurable space. Examine Fatou’s lemma.

7.0 Further Reading and Other Resources

H.L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

D. L. Cohn, Measure Theory, Birkhauser, 1985.

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MODULE3: Theory of Integration

Unit 2: Integration of Complex Functions

1.0 Introduction

In this unit, we discuss the Lebesgue integral of complex function over the general measure space and also prove the dominated convergence theorem.

2.0 Objectives

By the end of this unit, you should be able to:

(i) define the Lebesgue integral of complex function over a general measure space;

(ii) know some basic properties of Lebesgue integrals of complex functions;

(iii) know and prove the dominated convergence theorem.

3.0 Integration of Complex Functions

We recall that if f is measure, then $|f|$ is measurable.

Definition 3.2.1: A measurable function f on X is said to be Lebesgue measurable (or summable) if $\int_X |f| \, d\mu < \infty$.

We write $L^1(\mu)$ to denote the set of all Lebesgue integrable functions.

Definition 3.2.2: $L^1(\mu) = \{f: X \to C \,/\, \int_X |f| \, d\mu < \infty\}$, where μ is a positive measure on X and X is an arbitrary measurable space.

More generally, if p is non – negative real number, we put $L^p(\mu)$ for the set of all measurable functions f such that $\int_X |f|^p \, d\mu < \infty$. That is

$$L^p(\mu) = \{f: X \to C \,/\, \int_X |f|^p \, d\mu < \infty\}.$$

A function f in $L^p(\mu)$ is said to be p^{th} power summable (integrable).

Definition 3.2.3: Let f be an element in $L^1(\mu)$ such that f = u + iv, where u and v are real measurable functions on X, for any measurable set E, we define

$$\int_E f \, d\mu = \int_E u^+ \, d\mu - \int_E u^- \, d\mu + i \int_E v^+ \, d\mu - i\int_E v^- \, d\mu.$$

Theorem 3.2.3: $L^1(\mu)$ is a vector space over the real field. Moreover,

$$\left|\int_E f \, d\mu\right| \leq \int_E |f| \, d\mu.$$

Proof: Exercise

Theorem 3.2.4: (Dominated Convergence Theorem)

Let $\{f_n\}$ be a sequence of measurable functions on X. $f_n: X \to C$ such that for any x in X,

f(x) = $\lim_{n\to\infty} f_n(x)$. If there exists a function g ∈ $L^1(\mu)$ such that $|f_n(x)| \le g(x)$, n = 1, 2, … .

Then f ∈ $L^1(\mu)$ and $\lim_{n\to\infty} \int_X |f_n - f| \; d\mu = 0$.

That is

$$\lim_{n\to\infty} \int_X f_n \; d\mu = \int_X f \, d\mu.$$

Proof: Since f is measurable and $|f| \le g$ (since $|f_n(x)| \le g(x)$), the fact that f ∈ $L^1(\mu)$ follows from the fact that g ∈ $L^1(\mu)$.

Since $f_n \to f$ and $|f_n(x)| \le g(x)$, we have $|f| = \lim_{n\to\infty}|f_n(x)| \le g(x)$. From this, we obtain $|f_n - f| \le |f_n| + |f| \le 2g$. Consider the sequence $h_n = 2g - |f_n - f|$. By Fatou's Lemma, we have

$$\int_X (\lim_{n\to\infty} inf h_n) d\mu \le \lim_{n\to\infty} inf \int_X h_n \, d\mu$$

which gives

$$\int_X 2g \, d\mu \le \int_X 2g \, d\mu + \lim_{n\to\infty} inf \left(-\int_X |f_n - f| d\mu\right).$$

And since g ∈ $L^1(\mu)$, we have

$$0 \le \lim_{n\to\infty} inf \left(-\int_X |f_n - f| d\mu\right),$$

And so,

$$0 \leq -\lim_{\mathrm{n} \to \infty} sup \ \left(\int_X \ |f_n - f| d\,\mu\right).$$

This implies

$$\lim_{\mathrm{n} \to \infty} sup \ \left(\int_X \ |f_n - f| d\,\mu\right) \leq 0 \qquad (3.2.1).$$

The left hand side of (3.2.1) must be zero.

Hence

$$\lim_{n \to \infty} \int_X |f_n - f| \ d\,\mu = \lim_{\mathrm{n} \to \infty} sup \ \left(\int_X \ |f_n - f| d\,\mu = 0 \qquad (3.2.2).\right.$$

But by Theorem 3.2.3, we have

$$\left|\int_X (f_n - f) \, d\,\mu\right| \leq \int_E \ |f_n - f| \, d\,\mu.$$

In view of (3.3.2), we have

$$\left|\lim_{\mathrm{n} \to \infty} \int_X (f_n - f) \, d\,\mu\right| = 0$$

and so,

$$\lim_{\mathrm{n} \to \infty} \int_X (f_n - f) \, d\,\mu = 0.$$

4.0 Conclusion

5.0 Summary

In this unit we have learnt:

(i) the definition of Lebesgue integral of a complex function over the general measure space;

(ii) the basic properties of Lebesgue integrals of complex functions;

(iii) how to prove the dominated convergence theorem.

6.0 Tutor Marker Assignment

7.0 Further Reading and Other Resources

H.L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

D. L. Cohn, Measure Theory, Birkhauser, 1985.

P.R. Halmos, Measure Theorey, Springer Verlag, 1974.

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MODULE3: Theory of Integration

Unit 3: Lebesgue Integration of Real – Valued Functions Defined on $\mathbb{R}^n$

1.0 Introduction

Here, we discuss Lebesgue integration of real – valued functions defined on subsets of $\mathbb{R}^n$, we restrict ourselves to n = 1, that $\mathbb{R}$.

We first give a brief description of the way in which the Lebesgue integral is defined on subsets of $\mathbb{R}$.

2.0 Objectives

By the end of this unit, you should be able to:

(i) define the Lebesgue integral of a real – valued function defined on the real line;

(ii) know the equivalent form of monotone convergence theorem and dominated convergence theorem with respect to the real line instead of the general measure space;

(iii) know how to use these theorems to evaluate some complicated or complex integrals.

3.0 Lebesgue Integration of Real – Valued Functions Defined on $\mathbb{R}^n$

Definition 3.3.1: A step function f = $\sum_{r=1}^{n} \beta_r \chi_{E_r}$is a (finite) linear combination of the characteristic functions where each

For such E_r is a bounded interval. For such f we define

$$\int f = \sum_{r=1}^{n} \beta_r \mathrm{m}(E_r)$$

Where m(E_r) is the length of E_r.

A function f: $\mathbb{R} \to \mathbb{R}$ is in $L^{\uparrow}$ if there is an increasing sequence $\{f_n\}$ of step functions such that $\{f_n\}$ is bounded above and $f_n \to f$ almost everywhere (a.e.). For such an f we define

$$\int f = \lim_{n\to\infty} \int f_n.$$

A function f: $\mathbb{R} \to \mathbb{R}$ is in A function f: $\mathbb{R} \to \mathbb{R}$ is in $L^1(\mathbb{R})$ if f = p – q, where p, q ∈ $L^{\uparrow}$. For such an f, we define

$$\int f = \int p - \int q.$$

Theorem 3.3.2: (Monotone Convergence Theorem)

Let $\{f_n\}$ be a sequence in $L^1(\mathbb{R})$ such that $\{f_n\}$ is bounded above (below) and $\{f_n\}$ is increasing (decreasing) a.e. Then there exists f ∈ $L^1(\mathbb{R})$ such that $\int f = \lim_{n\to\infty} \int f_n$.

Proof: Exercise

Theorem 3.3.3 (Dominated Convergence Theorem)

Let $\{f_n\}$ be a sequence in $L^1(\mathbb{R})$ and g ∈ $L^1(\mathbb{R})$. Let f be a real function defined on $\mathbb{R}$ such that

$f_n(x) \to f(x)$ for almost all x and for all n and all x ∈ $\mathbb{R}$, $|f_n(x)| \leq g(x)$. Then f ∈ $L^1(\mathbb{R})$ and

$\int f = \lim_{n\to\infty} \int f_n$.

Proof: Exercise

Example 3.3.4: Define f: $\mathbb{R} \to \mathbb{R}$ and g: $\mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} xe^{-2x} \ if \ x \geq 0 \\ 0 \ \ if \ x < 0 \end{cases}$$

Use the monotone convergence theorem to evaluate $\int f$.

Solution: For any natural number n, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} xe^{-2x} \ if \ \ 0 \leq x \leq n \\ 0 \ \ otherwise \end{cases}$$

Then $\{f_n\}$ is a sequence in $L^1(\mathbb{R})$ and $\{\int f_n\}$ is bounded above since

$$\int f_n = \int_0^n xe^{-2x}\,\text{dx} = \frac{-ne^{-2n}}{2} - \frac{e^{-2n}}{4} + \frac{1}{4} < \frac{1}{4}.$$

Since for all x ∈ $\mathbb{R}$, $f_n(x) \to f(x)$, it follows from the monotone convergence theorem that

f ∈ $L^1(\mathbb{R})$ and $\int f = \lim_{n\to\infty} \int f_n = \frac{1}{4}$.

Example 3.3.5: Let the function f: $\mathbb{R} \to \mathbb{R}$ be such that, for all natural number n, f restricted to

[-n, n] is Riemann integrable and $\lim_{n\to\infty} \int_{-n}^{n} |f(x)|$ dx exists. Prove that f ∈ $L^1(\mathbb{R})$.

Define g: $\mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{cos3x}{1+x^2} & if\ x \geq 0 \\ 0 & if\ x < 0 \end{cases}$$

Show that g ∈ $L^1(\mathbb{R})$.

Proof: For a natural number n, define

$$f_n(x) = \begin{cases} f(x) & if\ -n \leq x \leq n \\ 0 & otherwise \end{cases}$$

Then $\{f_n\}$ and $\{|f_n|\}$ are sequences in $L^1(\mathbb{R})$. The sequence $\{\int |f_n|\}$ is bounded above, since

$\int |f_n| = \int_{-n}^{n} |f(x)|\ dx$ converges, and for all x ∈ $\mathbb{R}$, $\{|f_n(x)|\}$ is increasing and converges to

$|f(x)|$. Hence, by the monotone convergence theorem, $|f|$ ∈ $L^1(\mathbb{R})$. Also, since $f_n(x) \to f(x)$

(x ∈ $\mathbb{R}$) and $|f_n(x)| \leq |f|(x)$ for all n and for all x ∈ $\mathbb{R}$, the dominated convergence theorem

applies to give f ∈ $L^1(\mathbb{R})$.

For each natural number n, g restricted to [-n, n] is Riemann integrable and

$$\int_{-n}^{n} |g(x)|\, dx = \int_{0}^{n} \left|\frac{cos3x}{1+x^2}\right| \text{dx}$$

$$\leq \int_{0}^{n} \left|\frac{1}{1+x^2}\right|\, dx = tan^{-1}\text{n} < \frac{\pi}{2}.$$

Since these integrals are increasing,

$$\lim_{n\to\infty} \int_{-n}^{n} |g(x)|\, dx$$

Exists and hence g ∈ $L^1(\mathbb{R})$.

4.0 Conclusion

5.0 Summary

In this unit we have learnt:

(i) the definition of Lebesgue integral of a real-valued function over the real line;

(ii) the equivalent version of monotone convergence theorem and dominated convergence theorem with respect to the real line instead of the general measure space;

(iii) how to use the monotone convergence theorem and the dominated convergence theorem to evaluate some complicated or complex integrals .

6.0 Tutor Marker Assignment

(1) Let k be a positive constant. Define f: $\mathbb{R} \to \mathbb{R}$ and g: $\mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} xe^{-kx} \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases}$$

and

$$g(x) = \begin{cases} \frac{x}{e^x} - 1 \text{ if } x > 0 \\ 0 \text{ if } x \leq 0 \end{cases}$$

(i) Use the monotone convergence theorem to show that f $\in$ $L^1(\mathbb{R})$.

(ii) Evaluate $\int f$

(iii) Show that for x > 0, g(x) = x(e^{-x} + e^{-2x} + e^{-3x}+ ...).

(iv) Deduce that g $\in$ $L^1(\mathbb{R})$.

(2) By considering the sequence of partial sums, show that the real function f defined by the series

$$f(x) = \sum_{n=1}^{\infty} \frac{x}{(1+n^2x^2)^2} \qquad (0 \leq x \leq 1)$$

is in $L^1([0,1])$ and that

$$\int_0^1 f(x)dx = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

7.0 Further Reading and Other Resource

H.L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

D. L. Cohn, Measure Theory, Birkhauser, 1985.

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MODULE4: Classical Banach Space

Unit 1: Sets of Measure zero

1.0 Introduction

In this unit, we discuss the property that holds almost everywhere of the general measure space with some example.

2.0 Objectives

By the end of this unit, you should be able to:

(i) know what it means to say that a property named P occurs almost everywhere on X;

(ii) know some examples of properties that occurs almost everywhere on X;

(iii) know how to prove basic results on properties that occurs almost everywhere on X.

3.0 Properties that Hold Almost Everywhere

Definition 3.1.1: let (X, $\mathcal{M}, \mu$) be a measure space. A property named P on X is said to hold almost everywhere (a.e) if the set of points in X at which it fails to hold is a set of measure zero. That is if there is a set N that belongs to $\mathcal{M}$, such that $\mu(N) = 0$, and N contains every point at which the property fails to hold. More generally, if E is a subset of X, then a property is said to hold almost everywhere on E if the set of points in E at which it fails to hold is a set of measure zero.

Consider a property that holds almost everywhere, and let F be the set of points in X at which it fails to hold. Then it is not necessary that F belong to $\mathcal{M}$, it is only necessary that there exist a set N that belongs to $\mathcal{M}$, includes F, and satisfies $\mu(N) = 0$.

We give some examples.

Example 3.1.2: Suppose f and g are functions on X. Then f = g almost everywhere if the set of points x at which f(x) $\neq$ g(x) is a set of measure zero. Also, f $\leq g$ on X almost everywhere means the set of points or elements of X for which f > g have measure zero and f $\geq$ g almost everywhere if the set of points x at which f(x) < g(x) is a set of measure zero. If $\{f_n\}$ is a sequence of functions on X and f is a function on X, then $\{f_n\}$ converges to f almost everywhere if the set of points x at which f(x) = $\lim_n f_n(x)$ fails to hold is a set of measure zero.

Proposition 3.1.3: Let (X, $\mathcal{M}, \mu)$ be a measure space, and let f and g be extended real-valued functions on X that are equal almost everywhere. If μ is complete and if f is measurable, then g is measurable.

Proof: Let t be a real number and let N be a set that belongs to $\mathcal{M}$, satisfies $\mu(N) = 0$, and is such that f and g agree everywhere outside N. Then

$$\{x \in X: g(x) \leq t\} =$$

$$(\{x \in X: f(x) \leq t\} \cap N^c) \cup (\{x \in X: g(x) \leq t\} \cap N) \qquad (3.1.1)$$

The completion of μ implies $\{x \in X: g(x) \leq t\} \cap N$ belongs to $\mathcal{M}$ and so equation (3.1.1) implies

$\{x \in X: g(x) \leq t\}$ belongs to $\mathcal{M}$. Since t is arbitrary, the measurability of g follows.

Corollary 3.1.4: Let (X, $\mathcal{M}, \mu)$ be a measure space, let $\{f_n\}$ be a sequence of extended real-valued functions on X, and let f be an extended real-valued function on X such that $\{f_n\}$ converges to f almost everywhere. If μ is complete and if each f_n is measurable, then f is measurable.

Proof: By Proposition 3.3.6 of module2, unit3, the function $liminf_n f_n$ is measurable. Since f and $liminf_n f_n$ agree almost everywhere, Proposition 3.1.3 above implies that f is measurable.

Definition 3.1.5: We say that f and g are equivalent if f and g differ on a set of measure zero. We write f ~ g to denote that f and g are equivalent.

If f ~ g, we say that f = g almost everywhere on X.

Consider$L^1(\mu)$, an identify two functions if they are equivalent. Write [f] for the class of functions equivalent to f. That [f] = {g: f ~ g}. Since ~ is a proper equivalent relation on

$L^1(\mu)$, it splits $L^1(\mu)$ into collection of mutually disjoint equivalent classes [f], [g]. We do this because it is possible for a function f which is not zero everywhere but which has $\int_X f \, d\,\mu = 0$.

The equation $\int_X |f| \, d\,\mu = 0$ does not imply that f = 0. But if f = 0, then $\int_X f \, d\,\mu = 0$.

Theorem 3.1.6: Let $\{f_n\}$ be a sequence of measurable functions. f_n: X → C a.e. Suppose that $\sum_{n=1}^{\infty} \int_X |f_n| \, d\,\mu$ < ∞. Then $\sum_{n=1}^{\infty} f_n(x)$ converges to f(x) a.e on X.

That is ,

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \text{ a.e on X and } \int_X f \, d\,\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\,\mu.$$

Proof: The S_n be the set on which f_n is defined such that $\mu({S_n}^c) = 0$. Let us define

$\varphi(x) = \sum_n |f_n(x)|$ for all x in S = $\bigcap_n S_n$, we have $\mu(S^c) = 0$.

By Theorem 3.1.5 $\int_S \varphi \, d\mu < \infty$. If E = {x ∈ S: $\varphi(x) < \infty\}$, $\mu(E^c) = 0$, $\sum_n |f_n(x)|$ converges for any x ∈ E and if f(x) = $\sum_{n=1}^{\infty} f_n(x)$ ∀ x ∈ E, then we obtain

$|f(x)| \leq \varphi(x)$ ∀ x ∈ E. This implies f ∈ $L^1(\mu)$.

If $g_n = f_1 + f_2 + \ldots + f_n$, then we have $|g_n(x)| \leq \varphi(x)$, $g_n(x) \to f(x)$.

By Theorem3.2.4, we obtain

$$\int_E f \, d\mu = \sum_{n=1}^{\infty} \int_E f_n \, d\mu.$$

This is equivalent to

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$$

because $\mu(E^c) = 0$.

4.0 Conclusion

5.0 Summary

In this unit we have learnt:

(i) the definition of a property that occurs almost everywhere with examples;

(ii) how to prove some results on some properties that occur almost everywhere.

6.0 Tutor marked Assignment

(1) Show that ~ is an equivalent relation.

(2) Let f and g be continuous real-valued functions on the real line. Show that if f = g holds almost everywhere with respect to the Lebesgue measure on the real line, then f = g everywhere.

7.0 Further Reading and Other Resources

H.L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

D. L. Cohn, Measure Theory, Birkhauser, 1985.

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MODULE4: Classical Banach Space

Unit 1: L^p – Spaces

1.0 Introduction

In this unit we study some spaces of functions – the L^p – Spaces and the norm defined on them. We also study the inequalities involving the norm defined on the L^p – Spaces. The Holder's and Minkowski. The Minkowski inequality gives the sub-additivity of the norm.

2.0 Objectives

By the end of this unit, you should be able to:

(i) Know what the L^p – Spaces are;

(ii) know that the L^p – Spaces with the norm defined on them are Banach spaces;

(iii) know the Minkonski and Holder's inequalities for function spaces

3.0 L^p – Spaces

Definition 3.2.1: Let p and q be two positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then p and q are said to be conjugate exponents

Definition 3.2.2: Let (X, $\mathcal{M}, \mathcal{M}$) be a measure space, for 0 < p < ∞, let f: X → C be a measurable function, we set

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}.$$

$$L^p(\mu) = \{ f{:}X \to C, measurable\ functions\ such\ that\ \int_X |f|^p \, d\mu < \infty\}.$$

The norm on $L^p(\mu)$ is defined as

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} \text{ for f } \in L^p(\mu).$$

If $\mu = m$ (Lebesgue measure on $\mathbb{R}^n$), then $L^p(\mu) = L^p(\mathbb{R}^n)$.

$L^p(\mu)$ is also denoted by $L^p(X)$ and $(L^p(X), \|.\|_p)$ is a Banach space, it is reflexive and separable for 1 < p < ∞.

Two functions f and g in $L^p(X)$ are said to be equal if $\int_X |f - g| d\mu = 0$. Thus $L^p(X)$ is a set of equivalent classes with norm$\|.\|_p$.

Theorem 3.2.3: (Holder's Inequality)

Let p and q be conjugate exponents where $1 \leq p \leq \infty$. If f ∈ $L^p(X)$ and g ∈ $L^q(X)$, then

$$\text{fg} \in L^1(X) \text{ and } \|fg\|_1 \leq \|f\|_p . \|g\|_q.$$

That is,

$$\int_X |fg| d\mu \leq (\int_X |f|^p \, d\mu)^{\frac{1}{p}} . (\int_X |g|^q \, d\mu)^{\frac{1}{q}}.$$

Proof: Exercise

Remark: The case of Theorem4.2.3, for p = q = 2 is called Cauchy Schwarz inequality.

Theorem 3.2.4: (Minkonski Inequality)

Suppose $1 \leq p \leq \infty$ and f, g $\in L^p(X)$, then (f + g) $\in L^p(X)$ and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

That is,

$$(\int_X |f + g|^p \, d\mu)^{\frac{1}{p}} \leq (\int_X |f|^p \, d\mu)^{\frac{1}{p}} + (\int_X |g|^p \, d\mu)^{\frac{1}{p}}.$$

Proof: Exercise

(1) Let p_1, p_2 be positive real numbers such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and f_1, f_2 be functions such that $f_1 \in L^{p_1}(X)$ and $f_2 \in L^{p_2}(X)$, prove that the function f = $f_1 \, f_2$ is in $L^p(X)$, and

$$\|f\|_p \leq \|f_1\|_{p_1} . \|f_2\|_{p_2}.$$

(2) Prove that if f $\in L^p(X) \cap L^q(X)$ with $1 \leq p \leq q$, then for any p $\leq r \leq q$, we have

$$\|f\|_r \leq \|f\|_p{}^{\alpha} \|f\|_q{}^{1-\alpha} \text{ with } \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}.$$

4.0 Conclusion

5.0 Summary

In this unit we have learnt:

(i) definition of L^p – Spaces

(ii) that L^p – Spaces with the norm defined on them are Banach spaces

(iii) the Minkonski and Holder's inequalities

6.0 Tutor Marked Assignment

(1) Let p_1, p_2 be positive real numbers such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and f_1, f_2 be functions such that $f_1 \in L^{p_1}(X)$ and $f_2 \in L^{p_2}(X)$, prove that the function f = $f_1\, f_2$ is in $L^p(X)$, and

$$\|f\|_p \leq \|f_1\|_{p_1} . \|f_2\|_{p_2}.$$

(2) Prove that if f $\in L^p(X) \cap L^q(X)$ with $1 \leq p \leq q$, then for any p $\leq r \leq q$, we have

$$\|f\|_r \leq \|f\|_p{}^{\alpha} \|f\|_q{}^{1-\alpha} \text{ with } \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}.$$

7.0 Further Reading and Other Resources

H.L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

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S. Saks, Theory of the Integrals, Second Edition, Dover, New York, 1964.

MODULE5: Product Measures and Fubini's Theorem

Unit 1: Product Spaces

1.0 Introduction:

This unit is devoted to measures and integrals on product spaces. We shall study the basic facts about product measures and about the evaluation of integrals on product spaces.

2.0 Objectives

By the end of this unit, you should be able to:

(i) Know what the product Spaces are;

(ii) Some basic facts about product spaces;

(iii) know what the product measures are;

(iv) know how to evaluate integrals on product spaces.

3.0 Product Measures and Product Spaces

Definition 3.1.1: Let (X, $\mathcal{M}_x$) and (Y, $\mathcal{M}_Y$) be measurable spaces and let X × Y be the Cartesian product of the sets X and Y. A subset of X × Y is called a rectangle with measurable sides if it has the form

A × B for some A in $\mathcal{M}_x$ and B in $\mathcal{M}_Y$; the $\sigma -$ algebra on X × Y generated by the collection of all rectangles with measurable sides is called the product of the $\sigma -$ algebras $\mathcal{M}_x$ and $\mathcal{M}_Y$, and is denoted by $\mathcal{M}_x \times \mathcal{M}_Y$.

Example 3.1.2: Consider the space $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. We show that the product $\sigma -$ algebra **B(**$\mathbb{R}$) × **B(**$\mathbb{R}$) is equal to the $\sigma -$ algebra **B(**$\mathbb{R}^2$) of Borel subsets of $\mathbb{R}^2$**.** We recall that **B(**$\mathbb{R}^2$) is generated by the collection of all sets of the form (a,b] × (c,d]. Thus **B(**$\mathbb{R}^2$) is generated by a subfamily of the $\sigma -$ algebra **B(**$\mathbb{R}$) $\times$ **B(**$\mathbb{R}$), and so is included in **B(**$\mathbb{R}$) $\times$ **B(**$\mathbb{R}$). For the reverse inclusion, let π_1 and π_2 be projections of $\mathbb{R}^2$ onto $\mathbb{R}$, defined by π_1(x,y) = x and π_2(x,y) = y.

π_1 and π_2 are continuous, and hence Borel measurable. It follows from this and the identity

$$A \times B = (A \times \mathbb{R}) \cap (\mathbb{R} \times B) = {\pi_1}^{-1}(A) \cap {\pi_2}^{-1}(B)$$

that if A and B belongs to **B(**$\mathbb{R}$), then A $\times$ B belongs to **B(**$\mathbb{R}^2$). Since **B(**$\mathbb{R}$) $\times$ **B(**$\mathbb{R}$) is the

$\sigma -$ algebra generated by the collection of all such rectangles A $\times$ B, it must be included in **B(**$\mathbb{R}^2$). Thus **B(**$\mathbb{R}$) $\times$ **B(**$\mathbb{R}$) = **B(**$\mathbb{R}^2$).

We next introduce some terminology and notation. Suppose X and Y are sets and E is a subset of X $\times$ Y. Then for each x in X and each y in Y, the sections E_x and E^y are the subsets of Y and X given by

$$E_x = \{y \in Y: (x,y) \in E\}$$

and

$$E^y = \{x \in X: (x,y) \in E\}.$$

If f is a function on X $\times$ Y, then the sections f_x and f^y are the functions on Y and X given by

$$f_x(y) = f(x,y)$$

and

$$f^y(x) = f(x,y).$$

Proposition 3.1.3: Let (X, $\mathcal{M}_x$) and (Y, $\mathcal{M}_Y$) be measurable spaces.

(i) If E is a subset of X $\times$ Y that belongs to $\mathcal{M}_x \times \mathcal{M}_Y$ then each section E_x belongs to $\mathcal{M}_Y$ and each section E^y belongs to $\mathcal{M}_x$.

(ii) If f is an extended real-valued (or a complex-valued) $\mathcal{M}_x \times \mathcal{M}_Y$ - measurable function on X × Y, then each section f_x is $\mathcal{M}_Y$-measurable and f^y is $\mathcal{M}_x$-measurable.

Proof: Exercise

Proposition 3.1.4: (X, $\mathcal{M}_x, \mu$) and (Y, $\mathcal{M}_Y, v$) be $\sigma -$ finite measure spaces. If E belongs to the $\sigma -$algebra $\mathcal{M}_x \times \mathcal{M}_Y$, then the function x → $v(E_x)$ is $\mathcal{M}_x$- measurable and the function y → $\mu(E^y)$ is $\mathcal{M}_Y$ - measurable.

Proof: Exercise

Theorem 3.1.5: (X, $\mathcal{M}_x, \mu$) and (Y, $\mathcal{M}_Y, v$) be $\sigma -$ finite measure spaces. Then there is a unique measure $\mu \times v$ on the σ-algebra $\mathcal{M}_x \times \mathcal{M}_Y$ such that

$$(\mu \times v)(A \times B) = \mu(A)v(B)$$

holds for each A in $\mathcal{M}_x$ and B in $\mathcal{M}_Y$. Furthermore, the measure under $\mu \times v$ of an arbitrary set E in $\mathcal{M}_x \times \mathcal{M}_Y$ is given by

$$(\mu \times v)(E) = \int_X v(E_x)\ \mu(dx) = \int_Y \mu(E^y)v(dy).$$

The measure $\mu \times v$ is called the product of μ *and* v.

Proof: Exercise

4.0 Conclusion

5.0 Summary

In this unit we have learnt:

(i) definition of product spaces and product measure with example;

(ii) some basic results on product spaces and product measures;

(iii) the evaluation of integrals on product spaces

6.0 Tutored marked Assignment

7.0 Further Reading and Other Resources

H.L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

D. L. Cohn, Measure Theory, Birkhauser, 1985.

P.R. Halmos, Measure Theorey, Springer Verlag, 1974.

Walter Rudin, Principle of Mathematical Analysis, Second Edition, McGraw Hill, New York, 1976.

Walter Rudin, Real and Complex Analysis, Third Edition, McGraw – Hill, New York, 1987.

S. Saks, Theory of the Integrals, Second Edition, Dover, New York, 1964.

MODULE5: Product Measures and Fubini's Theorem

Unit 2: Fubini's Theorem

1.0 Introduction

In this unit, we discuss a proposition and the Fubini's theorem. These results enable us to evaluate integrals with respect to product measures in terms of iterated integrals.

2.0 Objectives

By the end of this unit, you should be able to:

(i) know the Funini's theorem;

(ii) how to evaluate integrals with respect to product measures in terms of the iterated integrals.

3.0 Fubini's Theorem

Proposition 3.2.1: (X, $\mathcal{M}_x, \mu$) and (Y, $\mathcal{M}_Y, v$) be $\sigma -$ finite measure spaces, and let

f: X × Y → $[0, +\infty]$ be $\mathcal{M}_x \times \mathcal{M}_Y$ - measurable. Then

(i) The function x → $\int_Y f_x dv$ is $\mathcal{M}_x$ - measurable and the function y → $\int_X f^y d\mu$ is $\mathcal{M}_Y$- measurable, and

(ii) f satisfies

$$\int_{X \times Y} f \, d(\mu \times v) = \int_Y \left(\int_X f^y d\mu\right) v(dy)$$

$$= \int_X \left(\int_Y f_x dv\right) \mu(dx).$$

Remark: We note that the functions f_x and f^y are non-negative and measurable (see Proposition 3.1.3, Module 5, unit 1); thus the expression $\int_Y f_x dv$ is defined for each x in X and the expression $\int_X f^y d\mu$ is defined for each y in Y.

Proof: First suppose that E belongs to $\mathcal{M}_x \times \mathcal{M}_Y$ and that f is the characteristic function on E. Then the sections f_x and f^y are the characteristic functions of the sections E_x and E^y, and so the relations $\int f_x \, dv = v(E_x)$ and $\int f^y \, \mathrm{d}\,\mu = \mu(E^y)$ hold for each x and y. Thus Proposition 3.1.4 and Theorem 3.1.5 of Module 5, unit 1, imply that conditions (i) and (ii) hold if f is a characteristic function. The additivity and homogeneity of the integral now imply that they hold for non-negative simple $\mathcal{M}_x \times \mathcal{M}_Y$ - measurable functions, and so, they hold for arbitrary non-negative $\mathcal{M}_x \times \mathcal{M}_Y$ – measurable functions.

Theorem 3.2.2 (Fubini's Theorem)

(X, $\mathcal{M}_x, \mu$) and (Y, $\mathcal{M}_Y, v$) be $\sigma -$ finite measure spaces, and let f: X × Y → $[-\infty, +\infty]$ be $\mathcal{M}_x \times \mathcal{M}_Y$ – measurable and $\mu \times v$ - integrable. Then

(i) for μ-almost every x in X the section f_x is v-integrable and for v-almost every y in Y the section f^y is μ-integrable,

(ii) the relation

$$\int_{\mathrm{X}\times\mathrm{Y}} f \, d(\mu \times v) = \int_Y \left(\int_X f^y d\mu\right) dv$$

$$= \int_X \left(\int_Y f_x dv\right) d\mu$$

holds.

Proof: Exercise

4.0 Conclusion

5.0 Summary

In this unit we have learnt:

(i) the Fubini's theorem;

(ii) how to evaluate integrals with respect to product measures in terms of the iterated integrals.

6.0 Tutor Marked Assignment

(1) Let λ be Lebesgue measure on ($\mathbb{R}$, **B(**$\mathbb{R}$)), let μ be counting measure on ($\mathbb{R}$, **B(**$\mathbb{R}$)), and let f: $\mathbb{R}^2 \to \mathbb{R}$ be the characteristic function on the line {(x,y) $\in \mathbb{R}^2$: y = x}. Show that

$$\iint f(x,y)\mu(dy)\ \lambda(dx) \neq \iint f(x,y)\lambda(dx)\, \mu\lambda(dy).$$

(2) Suppose that f: $\mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} 1 & if\ x \geq 0\ and\ x \leq y < x+1 \\ -1 & if\ x \geq 0\ and\ x+1 \leq y < x+2 \\ 0 & otherwise \end{cases}$$

Show that $\iint f(x,y)\lambda(dy)\ \lambda(dx) \neq \iint f(x,y)\lambda(dx)\ \lambda(dy)$. Why does this not contradict the Fubini's theorem.

7.0. Further Reading and Other Resources

D. L. Cohn, Measure Theory, Birkhauser, 1985.

Walter Rudin, Principle of Mathematical Analysis, Second Edition, McGraw Hill, New York, 1976.

Walter Rudin, Real and Complex Analysis, Third Edition, McGraw – Hill, New York, 1987.