MTH 131
ELEMENTARY SET THEORY

NATIONAL OPEN UNIVERSITY OF NIGERIA

COURSE GUIDE

MTH 131
ELEMENTARY SET THEORY

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INTRODUCTION

Welcome to Elementary set Theory. This course is a 1-credit unit course and it is offered at the undergraduate level.

This course consists of eight (8) units

There are no compulsory prerequisites for this course

This Course Guide tells you briefly what the course is about, what course materials you will be using and how you can walk your way through these materials.

What You Will Learn In This Course

The theory of sets lies at the foundations of mathematics. Concepts in set theory, such as functions and relations appear explicitly or implicitly in every branch of mathematics. This text is an informal, non-axiomatic treatment of the theory of set.

The text contains an introduction to the elementary operations of sets and a detailed discussion of the concept of a function and a relation.

Each unit begins with clear statements of pertinent definitions, principles and theorems will illustrative and other descriptive materials. This is followed by graded sets of solved and supplementary problems.

The solved problems serve to illustrate and amplify the theory, bring into sharp focus those fine points without which the student continually feels himself on unsafe ground, and provide the repetition of basic principles so vital to effective learning.

Course Aims

The aim of the course can be summarized as follows:-

- To introduce you to the basic principles of set theory
- Working with functions and the number system

Course Objectives

To achieve the aims set out above, the course sets overall objectives. In addition, each unit also has specific objectives. The unit objectives are always included in the beginning of a unit; you should read them before you start working through the unit. You may want to refer to them during your study of the unit to check on your progress. You should always look at the unit objectives after completing a unit. In this way you can be sure that you have done what was required of you by the unit.

Set out below are the wider objectives of the course as a whole. By meeting these objectives you should have achieved the aim of the course as a whole.

On successful completion of this course, you should be able to:-

- Explain sets, subsets, set notations, kinds of sets
- Describe basic set operations
- Identify intervals as sets and describe intervals using the real line
- Identify functions and the relationship between product sets and graphs of functions
- Apply the laws of the algebra of sets in proving set identities

Working through this Course

To complete this course, you are required to read the study units, read set books and read other materials provided by the NOUN

Course Materials

Major components of the course are:-

1. Course Guide
2. Study Units
3. Text books
4. Assignment File

Study Units

There are eight study units in this course, as follows:-
Unit 1: Set and Subsets
Unit 2: Basic Set Operations
Unit 3: Sets of Numbers
Unit 4: Functions

Unit 5: Product Sets and Graphs of Functions
Unit 6: Relations
Unit 7: Further Theory of Sets
Unit 8: Further Theory of Functions and Operations

Text Books

There are no compulsory text books for this course

Assignment File

The assignment File contains details of the work you must submit to your tutor for marking. It contains a more compact form of the Tutor-marked assignments. There are a maximum of five assignments in each unit

Assessment

There are two aspects of the assessment of the course. First are the tutor-marked assignments; second, there is a written examination.

In tackling the assignments, you are expected to apply information, knowledge and techniques gathered during the course. The assignments must be submitted to your tutor for formal assessment in accordance with the stipulated deadlines.

How to get the most from the course

In distance learning, the study units replace the lecturer. This is one of the great advantages of distance learning; you can read and work through specially designed study materials at your pace, and at a time and place that suit you best. Think of it as reading the lecture instead of listening to a lecturer. In the same way that a lecturer might set you some reading to do, the study units tell you when to read your set books or other materials, and when to undertake computing practical work. Just as a lecturer might give you an in-class exercise, your study units provide exercises for you to do at appropriate points.

Each of the study units follows a common format. The first item is an introduction to the subject matter of the unit and how a particular unit is integrated with the other units and the course as a whole. Next is a set of learning objectives. These objectives let you know what you should be able to do by the time you have completed the unit. You should use these objectives to guide your study. When you have finished the unit you must go back and check whether you have achieved the objectives. If you make a habit of doing this you will significantly improve your chances of passing the course.

Exercises are interspersed within the units, and answers are given. Working through these exercises will help you to achieve the objectives of the unit and help you to prepare for the assignments and examination.

The following is a practical strategy for working through the course.

1. Read this Course Guide thoroughly

2. Organize a study schedule. Refer to the "Course Overview" for more details

3. Once you have created your own study schedule, do everything you can to stick to it. The major reason that students fails is that they get behind with their course work. If you get into difficulties with your schedule, please let your tutor know before it is too late.

4. Turn to Unit 1 and read the introduction and the objectives for the unit.

5. Work through the unit. The content of the unit itself has been arranged to provide a sequence for you to follow.
6. Review the objectives for each study unit to confirm that you have achieved them. If you feel unsure about any of the objectives, review the study materials or consult your tutor.

7. When you are confident that you have achieved a unit's objectives, you can then start on the next unit. Proceed unit by unit through the course and try to pace your study so that you keep yourself on schedule.

8. When you have submitted an assignment to your tutor for marking, do not wait for its return before starting on the next unit. Keep to your schedule. When the assignment is returned, pay particular attention to your tutor's comments.

9. After completing the last unit, review the course and prepare yourself for final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives listed on this Course Guide.

MAIN COURSE

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Module 1

Module 1

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UNIT 1 SETS & SUBSETS

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1.0 INTRODUCTION

The theory of sets lies at the foundation of mathematics. It is a concept that rears its head in almost all fields of mathematics; pure and applied.

This unit aims at introducing basic concepts that would be explained Further in subsequent units. There will be definition of terms and lots of examples and exercises to help you as you go along.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Identify sets from some given statements
- Rewrite sets in the different set notation
- Identify the different kinds of sets with examples

3.0 MAIN BODY

3.1 Sets

As mentioned in the introduction, a fundamental concept in all a branch of mathematics is that of set. Here is a definition

"***A set is any well-defined list, collection or class of objects".*** The objects in sets, as we shall see from examples, can be anything: But for clarity, we now list ten particular examples of sets:

Example 1.1 The numbers 0,2,4,6,8
Example 1.2 The solutions of the equation $x^2 + 2x+1 = 0$
Example 1.3 The vowels of the alphabet: a, e, i, o, u
Example 1.4 The people living on earth
Example 1.5 The students Tom, Dick and Harry
Example 1.6 The students who are absent from school
Example 1.7 The countries England, France and Denmark
Example 1.8 The capital cities of Nigeria
Example 1.9 The number 1, 3, 7, and 10
Example 1.10 The rivers in Nigeria

Note that the sets in the odd numbered examples are defined, that is, presented, by actually listing its members; and the sets in the even numbered examples are defined by stating properties that is, rules, which decide whether or not a particular object is a member of the set.

3.1.1 Notation

Sets will usually be denoted by capital letters;
A, B, X, Y,……

Lower case letters will usually represent the elements in our sets:
Lets take as an example; if we define a particular set by actually listing its members, for example, let A consist of numbers 1,3,7, and 10, then we write

A={1,3,7,10}
That is, the elements are separated by commas and enclosed in brackets

{}.

We call this the **tabular form** of a set

Now, try your hand on this

Exercise 1.1 State in words and then write in tabular form

1. $A = \{x \mid x^2 = 4^2\}$
2. $B = \{x \mid x - 2 = 5\}$
3. $C = \{x \mid$ x is positive, x is negative$\}$
4. $D = \{x \mid$ x is a letter in the word "correct"$\}$

Solution:

1. It reads "A is the set of x such that x squared equals four". The only numbers which when squared give four are 2 and -2. Hence A = {2, -2}
2. It reads "B is the set of x such that x minus 2 equals 5". The only solution is 7; hence B = {7}
3. It read "C is the set of x such that x is positive and x is negative". There is no number which is both positive and negative; hence C is empty, that is, C=Ø

4. It reads "D is the set of x such that x is letter in the work 'correct'". The indicated letters are c,o,r,e and t; thus D = {c,o,r,e,t}

But if we define a particular set by stating properties which its elements must satisfy, for example, let B be the set of all even numbers, then we use a letter, usually x, to represent an arbitrary element and we write:

$B = \{x \mid x \text{ is even}\}$

Which reads "B is the set of numbers x such that x is even". We call this the set **builders form** of a set. Notice that the vertical line " | " is read "such as".
In order to illustrate the use of ht above notations, we rewrite the sets in examples 1.1-1.10. We denote the sets by A_1, A_2, ….. A_{10} respectively.

Example 2.1: $A_1 = \{0, 2, 4, 6, 8\}$
Example 2.2: $A_2 = \{x \mid x^2 + 2x + 1 = 0\}$
Example 2.3: $A_3 = \{a, e, i, o, u\}$
Example 2.4: A_4 = {x | x is a person living on the earth}
Example 2.5: A_5 = {Tom, Dick, Harry}
Example 2.6: A_6 = {x | x is a student and x is absent from school}
Example 2.7: A_7 = {England, France, Denmark}
Example 2.8: A_8 = {x | x is a capital city and x is in Nigeria}
Example 2.9: $A_9 = \{1, 3, 7, 10\}$
Example 2.10: A_{10} = { x | x is a river and x is in Nigeria}

It is easy as that!

Exercise 1.2 Write These Sets In A Set-Builder Form

1. Let A consist of the letters a, b, c, d and e
2. Let B = {2, 4, 6, 8……..}
3. Let C consist of the countries in the United Nations
4. Let D = {3}
5. Let E be the Heads of State Babangida, Abacha and Abdulsalami

Solution

1. $A = \{x \mid x$ appears before f in the alphabet$\}$
   $= \{x \mid x$ is one of the first letters in the alphabet$\}$
2. $B = \{x \mid x$ is even and positive$\}$
3. $C = \{x \mid x$ is a country, x is in the United Nations$\}$
4. $D = \{x \mid x - 2 = 1\} = \{x \mid 2x = 6\}$
5. $E = \{x \mid x$ was Head of state after Buhari$\}$

If an object x is a member of a set A, i.e., A contains x as one of its\ elements, then we write:

$x \in A$

which can be read "x belongs to A" or 'x is in A". If, on the other hand, an object x is not a member of a set A, i.e A does not contain x as one of its elements, then we write;
$x \notin A$

It is a common custom in mathematics to put a vertical line " | " or "\" through a symbol to indicate the opposite or negative meaning of the symbol.

Example 3:1: Let $A = \{a, e, i\ o, u\}$. Then $a \in A$, $b \notin A$, $f \notin A$.
Example 3.2: Let $B = \{x \mid x$ is even$\}$. Then $3 \notin B$, $6 \in B$, $11 \notin B$, $14 \in B$.

3.1.1 **Finite & Infinite Sets**

Sets can be finite or infinite. Intuitively, a set is finite if it consists of a **specific number** of different elements, i.e. if in counting the different members of the set the counting process can come to an end. Otherwise a set is infinite. Lets look at some examples.

Example 4:1: Let M be the set of the days of the week. The M is finite

Example 4:2: Let $N = \{0,2,4,6,8 \ldots\ldots..\}$. Then N is infinite

Example 4:3: Let P = {x| |x is a river on the earth}. Although it may be difficult to count the number of rivers in the world, P is still a finite set.

Exercise 1.3: which sets are finite?

1. The months of the year
2. {1, 2, 3, ……… 99, 100}
3. The people living on the earth
4. {x | x is even}
5. {1, 2, 3,……..}

Solution:

The first three sets are finite. Although physically it might be impossible to count the number of people on the earth, the set is still finite. The last two sets are infinite. If we ever try to count the even numbers, we would never come to the end.

3.1.2 Equality Of Sets

Set A is **equal** to set B if they both have the same members, i.e if every element which belongs to A also belongs to B and if every element which belongs to B also belongs to A. We denote the equality of sets A and B by:

$$A = B$$

Example 5.1 Let A = {1, 2, 3, 4} and B = {3, 1, 4, 2}. Then A = B, that is {1,2,3,4} = {3,1,4,2}, since each of the elements 1,2,3 and 4 of A belongs to B and each of the elements 3,1,4 and 2 of B belongs to A. Note therefore that a set does not change if its elements are rearranged.

Example 5.3 Let E={x ‖ x^2–3x = -2}, F={2,1} and G ={1,2,2, 1}, Then E= F= G

3.1.3 Null Set

It is convenient to introduce the concept of the empty set, that is, a set which contains no elements. This set is sometimes called the *null set.* We say that such a set is void or empty, and we denote its symbol$\emptyset$.

Example 6.1 Let A be the set of people in the world who are older than 200 years. According to known statistics A is the null set.

Example 6.2 Let $B = \{x \| x^2 = 4, x \text{ is odd}\}$, Then B is the empty set.

3.2 Subsets

If every element in a set A is also a member of a set B, then A is called *subset* of B.

More specifically, A is a subset of B if $x \in A$ implies $x \in B$. We denote this relationship by writing; $A \subset B$, which can also be read "A is contained in B".

Example 7.1 The set $C = \{1,3,5\}$ is a subset of $D = \{5,4,3,2,1\}$, since each number 1, 3 and 5 belonging to C also belongs to D.

Example 7.2 The set $E = \{2,4,6\}$ is a subset of $F = \{6,2,4\}$, since each number 2,4, and 6 belonging to E also belongs to F. Note, in particular, that E = F. In a similar manner it can be shown that every set is a subset of itself.

Example 7.3 Let $G = \{x \mid x \text{ is even}\}$, i.e. $G = \{2,4,6\}$, and let $F = \{x \mid x \text{ is a positive power of } 2\}$, i.e. let $F = \{2,4,8,16\ldots..\}$ Then $F \subset G$, i.e. F is contained in G.

With the above definition of a subset, we are able to restate the definition of the equality of two sets.

Two set A and B are equal, i.e A = B, if an only if $A \subset B$ and $B \subset A$. If A is a subset of B, then we can also write

$$B \supset A$$

which reads "B is a superset of A" or "B contains A". Furthermore, we write:

$$A \not\subset B$$

if A is not a subset of B.

Conclusively, we state:

1. The null set $\varnothing$ is considered to be a subset of every set
2. If A is not a subset of B, that is, if $A \not\subset B$, then there is at least one element in A that is not a member of B.

3.2.1 Proper Subsets

Since every set A is a subset of itself, we call B a proper subset of A if, first, is a subset of A and secondly, if B is not equal to A. More briefly, B is a proper subset of A if:

$$B \subset A \text{ and } B \neq A$$

In some books "B is a subset of A" is denoted by

$$B \subseteq A$$

and B" is a proper subset of A" is denoted by

$$B \subset A$$

We will continue to use the previous notation in which we do not distinguished between a subset and a proper subset.

3.2.2 Comparability

Two sets A and B are said to be **comparable** if:

$$A \subset B \text{ or } B \subset A;$$

That is, if one of the sets is a subset of the other set. Moreover, two sets A and B are said to be **not comparable** if:

$$A \not\subset B \text{ and } B \not\subset A$$

Note that if A is not comparable to B then there is an element in A .which is not in B and … also, there is an element in B which is not in A.

Example 8.1: Let A = {a,b} and B = {a,b,c}. The A is comparable to B, since A is a subset of B.

Example 8.2: Let R – {a,b} and S = {b,c,d}. Then R and S are not comparable, since $a \in R$ nd $a \notin S$ and $c \notin R$

In mathematics, many statements can be proven to be true by the use of previous assumptions and definitions. In fact, the essence of mathematics consists of theorems and their proofs. We now proof our first

Theorem 1.1 If A is a subset of B and B is a subset of C then A is a subset of C, that is,

$$A \subset B \text{ and } B \subset C \text{ implies } A \subset B$$

Proof: (Notice that we must show that any element in A is also an element in C). Let x be an element of A, that is, let $x \in A$. Since A is a subset of B, x also belongs to B, that is, $x \in B$. But by hypothesis, $B \subset C$; hence every element of B, which includes x, is a number of C. We have shown that $x \in A$ implies $x \in C$. Accordingly, by definition, $A \subset C$.

3.2.3 Sets of Sets

It sometimes will happen that the object of a set are sets themselves; for example, the set of all subsets of A. In order to avoid saying "set of sets", it is common practice to say "family of sets" or "class of sets". Under the circumstances, and in order to avoid confusion, we sometimes will let script letters

$$\mathcal{A}, \mathcal{B}, \ldots\ldots\ldots$$

Denote families, or classes, of sets since capital letters already denote their elements.

Example 9.1: In geometry we usually say "a family of lines" or "a family of curves" since lines and curves are themselves sets of points.

Example 9.2: The set {{2,3}, {2}, {5,6}} is a family of sets. Its members are the sets {2,3}, {2} and {5,6}.

Theoretically, it is possible that a set has some members, which are sets themselves and some members which are not sets, although in any application of the theory of sets this case arises infrequently.

Example 9.3 Let A = {2, {1,3}, 4, {2,5}}. Then A is not a family of sets; here some elements of A are sets and some are not.

3.2.4 Universal Set

In any application of the theory of sets, all the sets under investigation will likely be subsets of a fixed set. We call this set the ***universal set*** or ***universe of discourse***. We denote this set by U.

Example 10.1: In plane geometry, the universal set consists of all the points in the plane.

Example 10.2 In human population studies, the universal set consists of all the people in the world.

3.2.5 Power Set

The family of all the subsets of any set S is called the **power set** of S. We denote the power set of S by: 2^S

Example11.1 Let M = {a,b} Then
$2^M = \{\{a, b\}, \{a\}, \{b\}, \varnothing\}$

Example 11.2 Let T = {4,7,8} then
$2^T = \{T, \{4,7\}, \{4,8\}, \{7,8\}, \{4\}, \{7\}, \{8\}, \varnothing\}$

If a set S is finite, say S has n elements, then the power set of S can be shown to have 2^n elements. This is one reason why the class of subsets of S is called the power set of S and is denoted by 2^S

3.2.6 Disjoint Sets

If sets A and B have no elements in common, i.e if no element of A is in B and no element of B is in A, then we say that A and B are **disjoint**

Example 12.1: Let A = {1,3,7,8} and B = {2,4,7,9}, Then A and B are not disjoint since 7 is in both sets, i.e $7 \in A$ and $7 \in B$

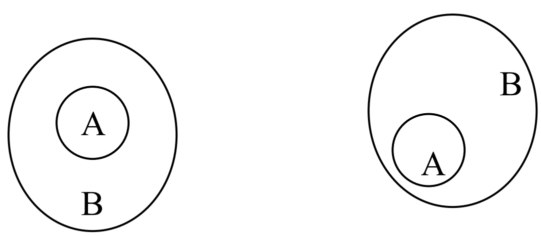
Example 12.2: Let A be the positive numbers and let B be the negative numbers. Then A and B are disjoint since no number is both positive and negative.

Example 12.3: Let E = {x, y, z} and F = {r, s, t}, Then E and F are disjoint.

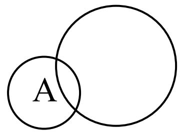
3.3 Venn-Euler Diagrams

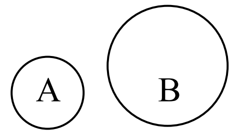
A simple and instructive way of illustrating the relationships between sets is in the use of the so-called Ven-Euler diagrams or, simply, Venn .
diagrams. Here we represent a set by a simple plane area, usually bounded by a circle.

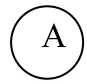
Example 13.1: Suppose $A \subset B$ and, say, $A \neq B$, then A and B can be described by either diagram:



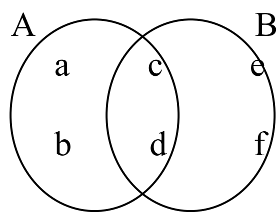
Example 13.2: Suppose A and B are not comparable. Then A and B can be represented by the diagram on the right if they are disjoint, or the diagram on the let if they are not disjoint.







Example 13.3: Let A = {a, b, c, d} and B= {c, d, e, f}. Then we illustrate these sets a Venn diagram of the form:



3. 4 Axiomatic Development Of Set Theory

In an axiomatic development of a branch of mathematics, one begins with:

1. Undefined terms
2. Undefined relations
3. Axioms relating the undefined terms and undefined relations.

Then, one develops theorems based upon the axioms and definitions

Example 14:1 In an axiomatic development of Plane Euclidean geometry

1. "points" and "lines" are undefined terms
2. "points on a line" or, equivalent, "line contain a point" is an undefined relation
3. Two of the axioms are:

   Axiom 1: Two different points are on one and only one line
   Axiom 2: Two different lines cannot contain more than one point in common.

In an axiomatic development of set theory:

1. "element' and "set" are undefined terms
2. "element belongs to a set" is undefined relation
3. Two of the axioms are

Axiom of Extension: Two sets A and B are equal if and only if every element in A belongs to B and every element in B belongs to A.

Axiom of Specification: Let P(x) be any statement and let A be any set. Then there exists a set:
$B=\{a \mid a \in A, P(a) \text{ is true}\}$

Here, P(x) is a sentence in one variable for which P(a) is true or false for any $a \in A$. for example P(x) could be the sentence "$x^2 = 4$" or "x is a member of the United Nations"

4.0 CONCLUSION

You have been introduced to basic concepts of sets, set notation e.t.c that will be built upon in other units. If you have not mastered them by now you will notice you have to come back to this unit from time to time.

5.0 SUMMARY

A summary of the basic concept of set theory is as follows:

- A ***set*** is any well-defined list, collection, or class of objects.
- Given a set A with ***elements*** 1,3,5,7 the ***tabular form*** of representing this set is $A = \{1, 3, 5, 7\}$
- The ***set-builder form*** of the same set is $A = \{x| x = 2n + 1, 0 \leq n \leq 3\}$
- Given the set $N = \{2,4,6,8,\ldots\}$ then N is said to be ***infinite***, since the counting process of its elements will never come to an end, otherwise it is ***finite***
- Two sets of A and B are said to be ***equal*** if they both have the same elements, written $A = B$

- The ***null set***, $\emptyset$, contains no elements and is a subset of every set
- The set A is a subset of another set B, written $A \subset B$, if every element of A is also an element of B, i.e. for every $x \in A$ then $x \in B$
- If $B \subset A$ and $B \neq A$, then B is a ***proper subset*** of A
- Two sets A and B are comparable if $A \subset B$ and $B \subset A$
- The ***power set*** 2^S of any set S is the family of all the subsets of S
- Two sets A and B are said to be disjoint if they do not have any element in common, i.e their intersection is a null set

6.0 TUTOR-MARKED ASSIGNMENTS

1. Rewrite the following statement using set notation:
    1. x does not belong to A.
    2. R is a superset of S
    3. d is a member of E
    4. F is not a subset of G
    5. H does not included D.

2. Which of these sets are equal: {r,t,s}, {s,t,r,s}, {t,s,t,r}, {s,r,s,t}?

3. Which sets are finite?

    1. The months of the year
    2. {1,2,3,……99, 100}
    3. The people living on the earth
    4. {x | x is even}
    5. {1,2,3……..)

The first three set are finite. Although physically it might be impossible to count the number of people on the earth, the set is still finite. The last two set are infinite. If we ever try to count the even numbers we would never come to the end.

4. Which word is different from each other, and why: (1) empty, (2) void, (3)zero, (4) null?

5 Let A = {x, y,z}. How many subsets does A contain, and what are they?

7.0 REFERENCE AND FURTHER READING

Seymour Lipschutz; Schaum's Outline Series: Theory and Problems of Set Theory and related topics, 1964,pp. 1 – 133.

Sunday O. Iyahen; Introduction to Real Analysis (Real-valued functions of a real variable, 1998, Vol. 1)

UNIT 2 BASIC SET OPERATIONS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall see operations performed on sets as in simple arithmetic. This operations simply give sets a language of their own. You will notice in subsequent units that you cannot talk of sets without reference, sort of, to these operations.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Compare two sets and/or assign to them another set depending on their comparability.
- Represent these relationships on the Venn diagram.

3.0 MAIN BODY

3.1 Set Operations

In arithmetic, we learn to add, subtract and multiply, that is, we assign to each pair of numbers x and y a number x + y called the sum of x and y, a number x – y called the difference of x and y, and a number xy called the product of x and y. These assignments are called the operations of addition, subtraction and multiplication of numbers. In this unit, we define the operation ***Union, Intersection*** and ***difference*** of sets, that is, we will assign new pairs of sets A and B. In a later unit, we will see that these set operations behave in a manner somewhat similar to the above operations on numbers.

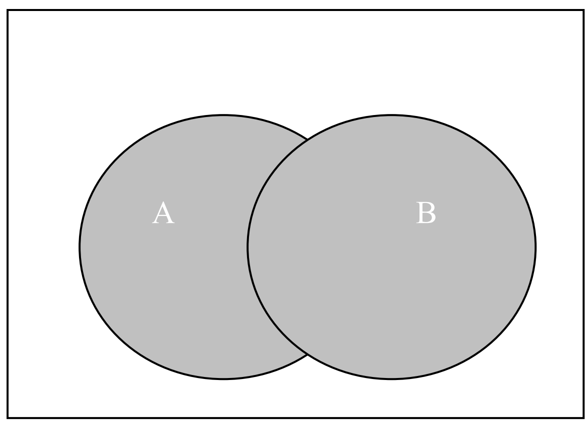
3.1.1 Union

The union of sets A and B is the set of all elements which belong to A or to B or to both. We denote the union of A and B by;

$A \cup B$

Which is usually read "A union B"

Example 1.1: In the Venn diagram in fig 2-1, we have shaded $A \cup B$,
i.e. the area of A and the area of B.



$A \cup B$ is shaded
Fig 2.1

Example 1.2: Let S = {a, b. c. d} and T = {f, b, d, g}. Then S T = {a, b, c, d, f, g}.

Example 1.3: Let P be the set of positive real numbers and let Q be the set of negative real numbers. The $P \cup Q$, the union of P and Q, consist of all the real numbers except zero. The union of A and B may also be defined concisely by

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Remark 2.1: It follows directly from the definition of the union of two sets that $A \cup B$ and $B \cup A$ are the same set, i.e.,
$A \cup B = B \cup A$

Remark 2.2: Both A and B are always subsets of A and B that is,
$A \subset (A \cup B)$ and $B \subset (A \cup B)$

In some books, the union of A and B is denoted by $A + B$ and is called the set-theoretic sum of A and B or, simply, A plus B.

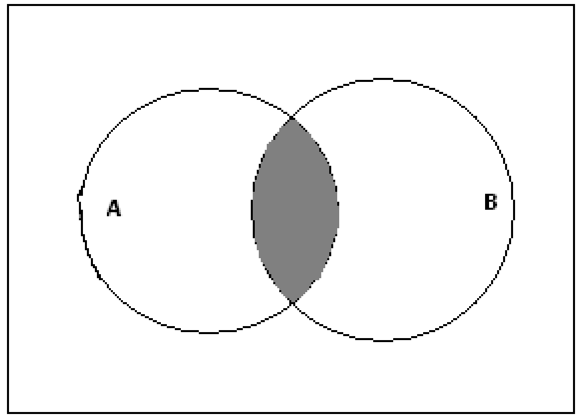
3.1.2 Intersection

The ***Intersection*** of sets A and B is the set of elements which are common to A and B, that is, those elements which belong to A and which belong to B. We denote the intersection of A and B by:

$$A \cap B$$

Which is read "A intersection B".

Example 2.1: In the Venn diagram in fig 2.2, we have shaded $A \cap B$, the area that is common to both A and B.



$A \cap B$ is shaded

Fig 2.2

Example 2.2: Let S = {a, b, c, d} and T = {f, b, d, g}. Then

$$S \cap T = \{b, d\}$$

Example 2.3: Let V = 2, 3, 6, ……} i.e. the multiples of 2; and let W = {3, 6, 9,….} i.e. the multiples of 3. Then

$$V \cap W = \{6, 12, 18\ldots\ldots\}$$

The intersection of A and B may also be defined concisely by

$$A \cap B = \{x \mid x \in A, x \in B\}$$

Here, the comma has the same meaning as "and".

Remark 2.3: It follows directly from the definition of the intersection of two sets that;

$$A \cap B = B \cap A$$

Remark 2.4: Each of the sets A and B contains $A \cap B$ as a subset, i.e.,

$$(A \cap B) \subset A \text{ and } (A \cap B) \subset B$$

Remark 2.5: If sets A and B have no elements in common, i.e. if A and B are disjoint, then the intersection of A and B is the null set, i.e. $A \cap B = \varnothing$.

In some books, especially on probability, the intersection of A and B is denoted by AB and is called the set-theoretic product of A and B or, simply, A times B.

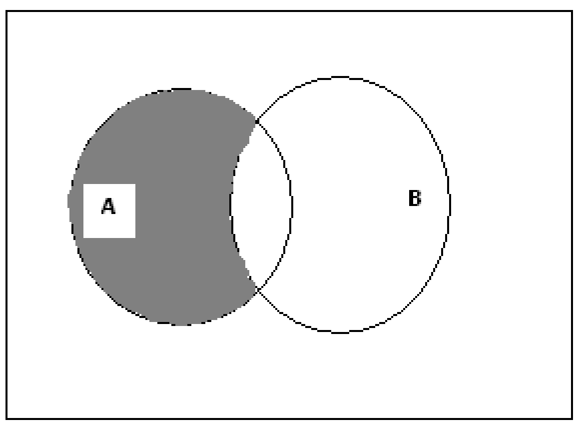
3.1.3 DIFFERENCE

The difference of sets A and B is the set of elements which belong to A but which do not belong to B. We denote the difference of A and B by

$$A - B$$

Which is read "A difference B" or, simply, "A minus B".

Example 3.1: In the Venn diagram in Fig 2.3, we have shaded $A - B$, the area in A which is not part of B.



A – B is shaded

Fig 2.3

Example 3.2: Let R be the set of real numbers and let Q be the set of rational numbers. Then R – Q consists of the irrational numbers.

The difference of A and B may also be defined concisely by

$$A - B = \{ x \mid x \in A, x \notin B \}$$

Remark 2.6: Set A contains A – B as a subset, i.e.,

$$(A - B) \subset A$$

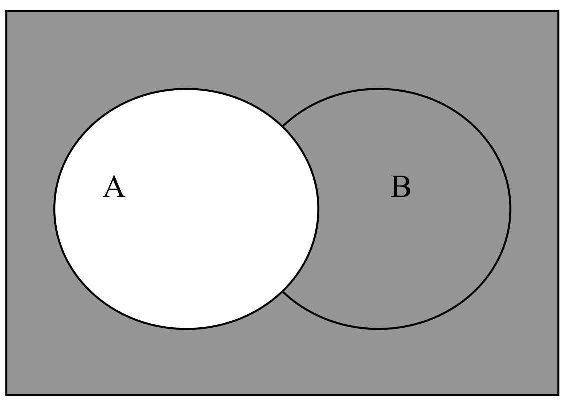
Remark 2.7: The sets (A – B), $A \cap B$ and (B – A) are mutually disjoint, that is, the intersection of any two is the null set.

The difference of A and B is sometimes denoted by A/B or A ~ B

3.1.4 Complement

The complement of a set A is the set of elements that do not belong to A, that is, the difference of the universal set U and A. We denote the complement of A by **A′**

Example 4.1: In the Venn diagram in Fig 2.4, we shaded the complement of A, i.e. the area outside A. Here we assume that the universal set U consists of the area in the rectangle.



A’ is shaded
Fig. 2.4

Example 4.2: Let the Universal set U be the English alphabet and let T = {a, b, c}. Then;

T’ = {d, e, f, ….., y, z}

Example 4.3: Let E = {2, 4, 6, …}, that is, the even numbers. Then E′ = {1, 3, 5, …}, the odd numbers. Here we assume that the universal set is the natural numbers, 1, 2, 3,…..

The complement of A may also be defined concisely by;

$$A' = \{x \mid x \in U, x \notin A\} \text{ or, simply,}$$
$$A' = \{x \mid x \notin A\}$$

We state some facts about sets, which follow directly from the definition of the complement of a set.

Remark 2.8: The union of any set A and its complement A′ is the universal set, i.e.,

$$A \cup A' = U$$

Furthermore, set A and its complement A′ are disjoint, i.e.,

$$A \cap A' = \varnothing$$

Remark 2.9: The complement of the universal set U is the null set ∅, and vice versa, that is,

$$U' = \varnothing \text{ and } \varnothing' = U$$

Remark 2.10: The complement of the complement of set A is the set A itself. More briefly,

$$(A')' = A$$

Our next remark shows how the difference of two sets can be defined in terms of the complement of a set and the intersection of two sets. More specifically, we have the following basic relationship:

Remark 2.11: The difference of A and B is equal to the intersection of A and the complement of B, that is,

$$A - B = A \cap B'$$

The proof of Remark 2.11 follows directly from definitions:

$$A - B = \{x \mid x \in A, x \notin B\} = \{x \mid x \in A, x \notin B'\} = A \cap B'$$

3.2 Operations on Comparable Sets

The operations of union, intersection, difference and complement have simple properties when the sets under investigation are comparable. The following theorems can be proved.

Theorem 2.1: Let A be a subset of B. Then the union intersection of A and B is precisely A, that is,

$A \subset B$ implies $A \cap B = A$

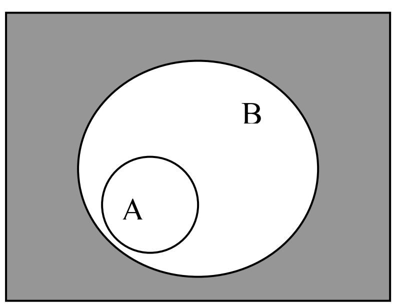
Theorem 2.2: Let A be a subset of B. Then the of A and B is precisely B, that is,

$A \subset B$ implies $A \cup B = B$

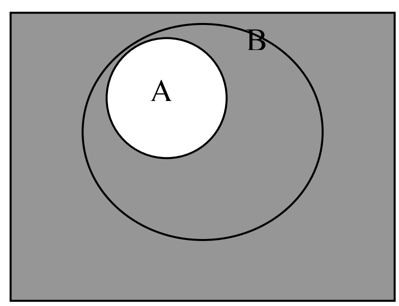
Theorem 2.3: Let A be a subset of B. Then B' is a subset of A', that is,

$A \subset B$ implies $B' \subset A'$

We illustrate Theorem 2.3 by the Venn diagrams in Fig 2-5 and 2-6. Notice how the area of B' is included in the area of A'.



B′ is shaded
Fig 2.5



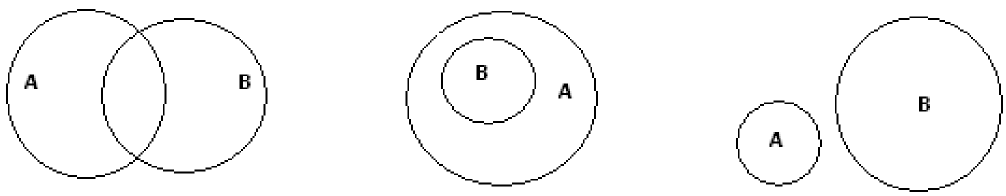
A′ is shaded
Fig 2.6

Theorem 2.4: Let A be a subset of B. Then the Union of A and (B – A) is precisely B, that is,

$$A \subset B \text{ implies } A \cup (B - A) = B$$

Exercises

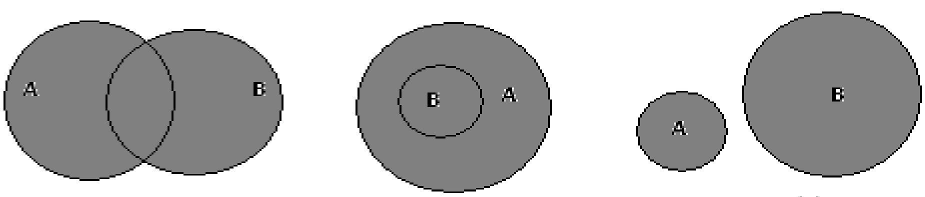
1. In the Venn diagram below, shade A Union B, that is, $A \cup B$:



(a) (b) (c)

Solution:

The union of A and B is the set of all elements that belong to A and to B or to both. We therefore shade the area in A and B as follows:



(a) (b) (c)

2. Let A = {1,2,3,4}, B = {2,4,6,8} and C = {3,4,5,6}. Find (a) $A \cup B$, (b) $A \cup C$, (c) $B \cup C$, (d) $B \cup B$.

Solution:

To form the union of A and B we put all the elements from A together with the elements of B Accordingly,

$$A \cup B = \{1,2,3,4,6,8\}$$

$$A \cup C = \{1,2,3,4,5,6\}$$
$$B \cup C = \{2,4,6,8,3,5\}$$
$$B \cup B = \{2,4,6,8\}$$

Notice that $B \cup B$ is precisely B.

3. Let A, B and C be the sets in Problem 2. Find (1) $(A \cup B) \cup C$, (2) $A \cup (B \cup C)$.

Solution:

1. We first find $(A \cup B) = \{1,2,3,4,6,8\}$. Then the union of $\{A \cup B\}$ and C is

   $(A \cup B) \cup C = \{1,2,3,4,6,8,5\}$

2. We first find $(B \cup C) = \{2,4,6,8,3,5\}$. Then the union of A and $(B \cup C)$ is $A \cup (B \cup C) = \{1,2,3,4,6,8,5\}$.

   Notice that $(A \cup B) \cup C = A \cup (B \cup C)$.

4.0 CONCLUSION

You have seen how the basic operations of Union, Intersection, Difference and Complement on sets work like the operations on numbers. These are also the basic symbols associated with set theory.

5.0 SUMMARY

The basic set operations are Union, Intersection, Difference and Complement defined as:

- The ***Union*** of sets A and B, denoted by $\mathbf{A \cup B}$, is the set of all elements, which belong to A or to B or to both.
- The ***intersection*** of sets A and B, denoted by $\mathbf{A \cap B}$, is the set of elements, which are common to A and B. If A and B are disjoint then their intersection is the Null set $\varnothing$.
- The ***difference*** of sets A and B, denoted by $\mathbf{A - B}$**,** is the set of elements which belong to A but which do not belong to B.

- The ***complement*** of a set A, denoted by **A'**, is the set of elements, which do not belong to A, that is, the difference of the universal set U and A.

6.0 TUTOR – MARKED ASSIGNMENTS

1. Let X = {Tom, Dick, Harry}, Y = {Tom, Marc, Eric} and Z = {Marc, Eric, Edward}. Find (a) $X \cup Y$, (b) $Y \cup Z$ (c) $X \cup Z$
2. Prove: $A \cap \varnothing = \varnothing$.
3. Prove Remark 2.6: $(A - B) \subset A$.
4. Let U = {1,2,3,…,8,9}, A = {1,2,3,4}, B = {2,4,6,8} and C = {3,4,5,6}. Find (a) A', (b) B', (c) $(A \cap C)'$, (d) $(A \cup B)'$, (e) (A')', (f) (B – C)'
5. Prove: B – A is a subset of A'.

7.0 REFERENCES AND FURTHER READINGS

Seymour Lipschutz; Schaum's Outline Series: Theory and Problems of Set Theory and related topics, 1964, pp. 1 – 133.

Sunday O. Iyahen; Introduction to Real Analysis (Real – valued functions of a real variable), 1998, Vol. 1

UNIT 3 SET OF NUMBERS

CONTENTS

1.0 INTRODUCTION

Although, the theory of sets is very general, important sets, which we meet in elementary mathematics, are sets of numbers. Of particular importance, especially in analysis, is the set of ***real numbers,*** which we denote by

$$\Re$$

In fact, we assume in this unit, unless otherwise stated, that the set of real numbers $\Re$ is out universal set. We first review some elementary properties of real numbers before applying our elementary principles of set theory to sets of numbers. The set of real numbers and its properties is called the ***real number system.***

2.0 OBJECTIVES

After studying this unit, you should be able to do the following:

- Represent the set of numbers on the real line
- Perform the basic set operations on intervals

3.0 MAIN BODY

3.1 Real Numbers, $\Re$

One of the most important properties of the real numbers is that points on a straight line that can represent them. As in Fig 3.1, we choose a point, called the origin, to represent 0 and another point, usually to the right, to represent 1. Then there is a natural way to pair off the points on the line and the real numbers, that is, each point will represent a unique real number and each real number will be represented by a unique point. We refer to this line as the ***real line.*** Accordingly, we can use the words point and number interchangeably.

Those numbers to the right of 0, i.e. on the same side as 1, are called the *positive numbers* and those numbers to the left of 0 are called the *negative numbers.* The number 0 itself is neither positive nor negative.

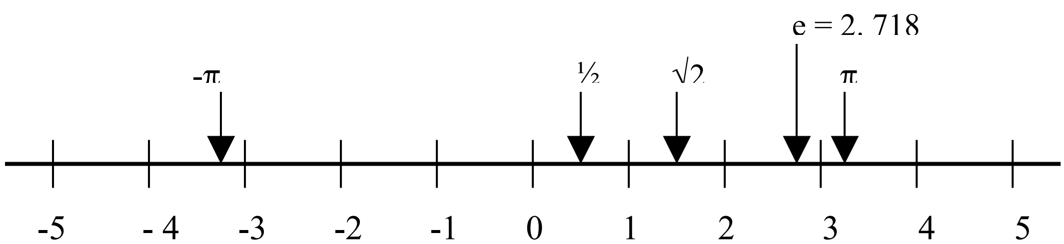


Fig 3.1

3.1.2 Integers, Z

The integers are those real numbers

…, -3, -2, -1, 0, 1, 2, 3,…

We denote the integers by Z; hence we can write

Z = { …, -2, - 1, 0, 1, 2,…}

The integers are also referred to as the "whole" numbers.

One important property of the integers is that they are "closed" under the operations of addition, multiplication and subtraction; that is, the sum, product and difference of two integers is again in integer. Notice that the quotient of two integers, e.g. 3 and 7, need not be an integer; hence the integers are not closed under the operation of division.

3.1.3 Rational Numbers, Q

The *rational numbers* are those real numbers, which can be expressed as the ratio of two integers. We denote the set of rational numbers by Q. Accordingly,

$$Q = \{x \mid x = p/q \text{ where } p \in z, q \in z\}$$

Notice that each integer is also a rational number since, for example, 5 = 5/1; hence Z is a subset of Q.

The rational numbers are closed not only under the operations of addition, multiplication and subtraction but also under the operation of division (except by 0). In other words, the sum, product, difference and quotient (except by 0) of two rational numbers is again a rational number.

3.1.4 Natural Numbers, N

The *natural numbers* are the positive integers. We denote the set of natural numbers by N; hence N = {1,2,3…..}

The natural numbers were the first number system developed and were used primarily, at one time, for counting. Notice the following relationship between the above numbers systems:

$$N \subset Z \subset Q \subset \Re$$

The natural numbers are closed only under the operation of addition and multiplication. The difference and quotient of two natural numbers needed not be a natural number.

The *prime numbers* are those natural numbers p, excluding 1, which are only divisible 1 and p itself. We list the first few prime numbers:

2,3,5,7,11,13,17,19…

3.1.5 Irrational Numbers, Q'

The irrational numbers are those real numbers which are not rational, that is, the set of irrational numbers is the complement of the set of rational numbers Q in the real numbers $\mathfrak{R}$; hence Q' denote the irrational numbers. Examples of irrational numbers are $\sqrt{3}$, π, $\sqrt{2}$, etc.

3.1.6 Line Diagram of the Number Systems

Fig 3 -2 below is a line diagram of the various sets of number, which we have investigated. (For completeness, the diagram include the sets of complex numbers, number of the form a + bi where a and b are real. Notice that the set of complex numbers is superset of the set of real numbers.)

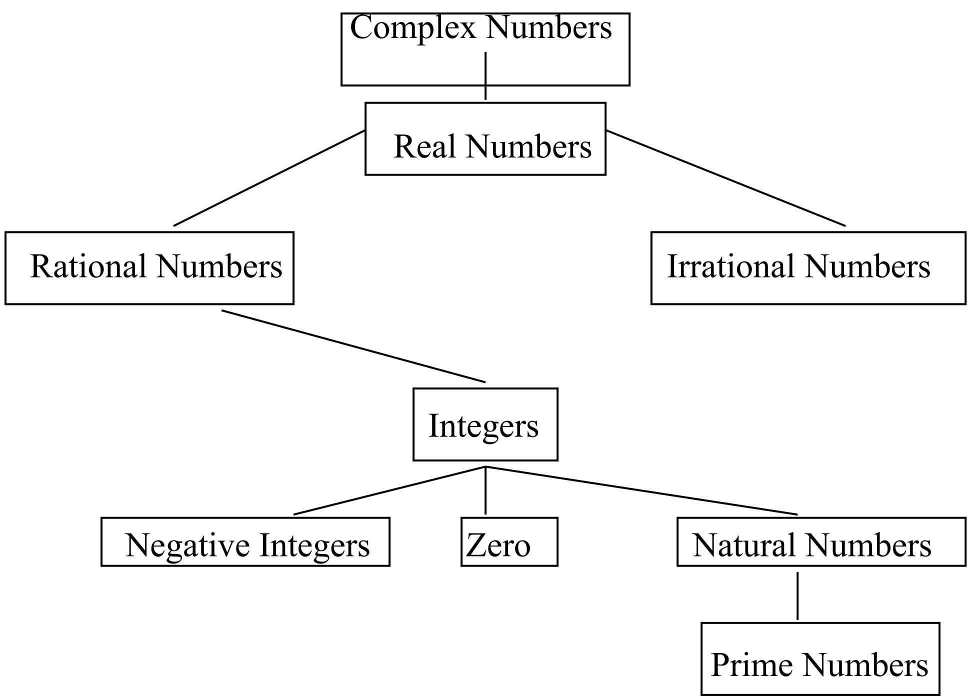


Fig 3.2

3.2 Decimals and Real Numbers

Every real number can be represented by a "non-terminating decimal". The decimal representation of a rational number p/q can be found by "dividing the denominator q into the numerator p". If the indicated division terminates, as for

$3/8 = .375$

We write $3/8 = .375000$

Or $3/8 = .374999\ldots$

If we indicated division of q into p does not terminate, then it is known that a block of digits will continually be repeated; for example,

$2/11 = .181818\ldots$

We now state the basic fact connecting decimals and real numbers. The rational numbers correspond precisely to those decimals in which a block of digits is continually repeated, and the irrational numbers correspond to the other non-terminating decimals.

3.3 Inequalities

The concept of "order" is introduced in the real number system by the

Definition: The real number a is less than the real number b, written $a < b$

If $b - a$ is a positive number.
The following properties of the relation $a < b$ can be proven. Let a, b and c be real numbers; then:

P_1: Either $a < b$, $a = b$ or $b < a$.
P_2: If $a < b$ and $b < c$, then $a < c$.
P_3: If $a < b$, then $a + c < b + c$
P_4: If $a < b$ and c is positive, then $ac < bc$
P_5: If $a < b$ and c is negative, then $bc < ac$.

Geometrically, if $a < b$ then the point a on the real line lies to the left of the point b.

We also denote $a < b$ by $b > a$

Which reads "b is *greater then* a". Furthermore, we write

$$a \leq b \text{ or } b \geq a$$

if a < b or a = b, that is, if a is not greater than b.

Example 1.1: $2 < 5$; $-6 \leq -3$ and $4 \leq 4$; $5 > -8$

Example 1.2: The notation $x < 5$ means that x is a real number which is less than 5; hence x lies to the left of 5 on the real line

The notation $2 < x < 7$; means $2 < x$ and also $x < 7$; hence x will lie between 2 and 7 on the real line.

Remark 3.1: Notice that the concept of order, i.e. the relation $a < b$, is defined in terms of the concept of positive numbers. The fundamental property of the positive numbers which is used to prove properties of the relation $a < b$ is that the positive numbers are closed under the operations of addition and multiplication. Moreover, this fact is intimately connected with the fact that the natural numbers are also closed under the operations of addition and multiplication.

Remark 3.2: The following statements are true when a, b, c are any real numbers:

1. $a \leq a$
2. if $a \leq b$ and $b \leq a$ then $a = b$.
3. if $a \leq b$ and $b \leq c$ then $a \leq c$.

3.4 Absolute Value

The absolute value of a real number x, denoted by $|x|$ is defined by the formula

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

that is, if x is positive or zero then $|x|$ equals x, and if x is negative then $|x|$ equals – x. Consequently, the absolute value of any number is always non-negative, i.e. $|x| \geq 0$ for every $x \in \Re$.

Geometrically speaking, the absolute value of x is the distance between the point x on the real line and the origin, i.e. the point 0. Moreover, the distance between any two points, i.e. real numbers, a and b is $|a - b| = |b - a|$.

Example 2.1: $|-2| = 2$, $|7| = 7$. $|-\pi| = \pi$

Example 2.2: The statement $|x| < 5$ can be interpreted to mean that the distance between x and the origin is less than 5, i.e. x must lies between -5 and 5 on the real line. In other words,

$$|x| < 5 \text{ and } -5 < x < 5$$

have identical meaning. Similarly,

$$|x| \leq 5 \text{ and } -5 \leq x \leq 5$$

have identical meaning.

3.5 Intervals

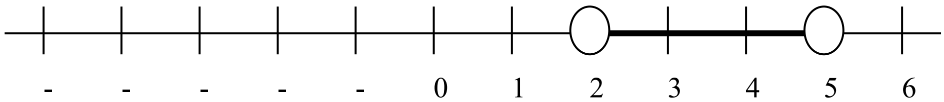
Consider the following set of numbers;

$$A_1 = \{x \mid 2 < x < 5\}$$
$$A_2 = \{x \mid 2 \leq x \leq 5\}$$
$$A_3 = \{x \mid 2 < x \leq 5\}$$
$$A_4 = \{x \mid 2 \leq x < 5\}$$

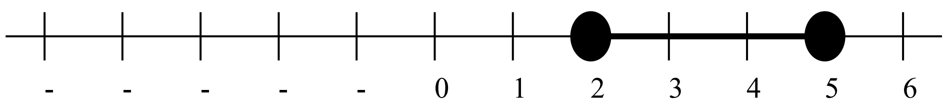
Notice, that the four sets contain only the points that lie between 2 and 5 with the possible exceptions of 2 and/or 5. We call these sets intervals, the numbers 2 and 5 being the endpoints of each interval. Moreover, A_1 is an ***open interval*** as it does not contain either end point:

A_2 is a ***closed interval*** as it contains bother endpoints; A_3 and A_4 are ***open-closed*** and ***closed-open*** respectively.

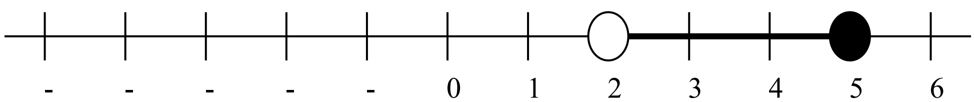
We display, i.e. graph, these sets on the real line as follows.



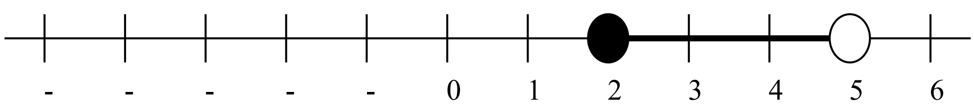
A_1



A_2



A_3



A_4

Notice that in each diagram we circle the endpoints 2 and 5 and thicken (or shade) the line segment between the points. If an interval includes an endpoint, then this is denoted by shading the circle about the endpoint.

Since intervals appear very often in mathematics, a shorter notation is frequently used to designated intervals, Specifically, the above intervals are sometimes denoted by;

$A_1 = (2, 5)$
$A_2 = [2, 5]$
$A_3 = (2, 5)$
$A_4 = [2, 5)$

Notice that a parenthesis is used to designate an open endpoint, i.e. an endpoint that is not in the interval, and a bracket is used to designate a closed endpoint.

3.5.1 Properties of Intervals

Let $\mathfrak{I}$ be the family of all intervals on the real line. We include in $\mathfrak{I}$ the null set $\varnothing$ and single points a = [a, a]. Then the intervals have the following properties:

1. The intersection of two intervals is an interval, that is,
   $A \in \mathfrak{I}, B \in \mathfrak{I}$ implies $A \cap B \in \mathfrak{I}$

2. The union of two non-disjoint intervals is an interval, that is,

   $A \in \mathfrak{I}, B \in \mathfrak{I}, A \cap B \neq \varnothing$ implies $A \cup B \in \mathfrak{I}$

3. The difference of two non-comparable intervals is an interval, that is,

   $A \in \mathfrak{I}, B \in \mathfrak{I}, A \not\subset B, B \not\subset A$ implies $A - B \in \mathfrak{I}$

Example 3.1: Let A = {2, 4), B = (3, 8). Then
$A \cap B = (3, 4), A \cup B = [2, 8)$
$A – B = [2, 3], B – A = [4, 8)$

3.5.2 Infinite Intervals

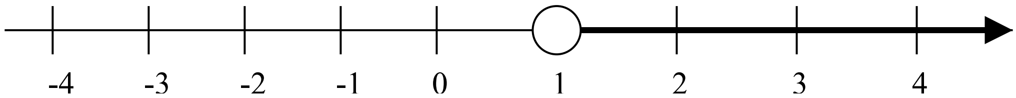
Sets of the form

$A = \{x \mid x > 1\}$
$B = \{x \mid x \geq 2$
$C = \{x \mid x < 3\}$
$D = \{x \mid x \leq 4\}$
$E = \{x \mid x \in \mathfrak{R}\}$

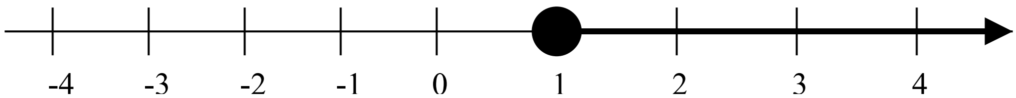
Are called infinite intervals and are also denoted by

$A = (1, \infty), B = [2, \infty), C = (-\infty, 3), D = (-\infty, 4], E = (-\infty, \infty)$

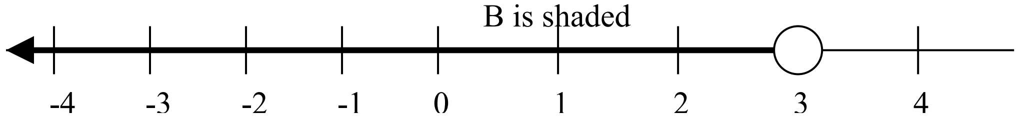
We plot these intervals on the real line as follows:



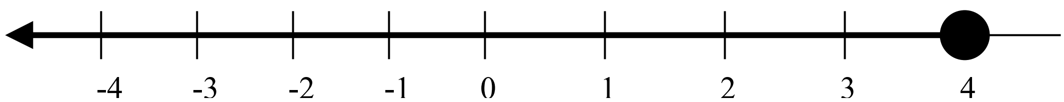
A is shaded



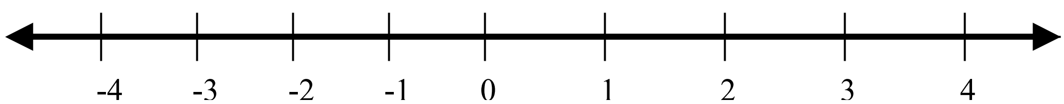
B is shaded



C is shaded



D is shaded



E is shaded

3.6 Bounded and Unbounded Sets

Let A be a set of numbers, then A is called ***bounded*** set if A is the subset of a finite interval. An equivalent definition of boundedness is

Definition 3.1: Set A is ***bounded*** if there exists a positive number M such that

$$|x| \leq M.$$

for all $x \in A$. A set is called ***unbounded*** if it is not bounded

Notice then, that A is a subset of the finite interval [- M, M].

Example 4.1: Let A = {1, ½ , 1/3,…..}. Then A is bounded since A is certainly a subset of the closed interval [0, 1].

Example 4.2: Let A = {2, 4, 6,…..}. Then A is an unbounded set.

Example 4.3: Let A = {7, 350, -473, 2322, 42}. Then A is bounded

Remark 3.3: If a set A is finite then, it is necessarily bounded. If a set is infinite then it can be either bounded as in example 4.1 or unbounded as in example 4.2

4.0 CONCLUSION

The set of real numbers is of utmost importance in analysis. All (except the set of complex numbers) other sets of numbers are subsets of the set of real numbers as can be seen from the line diagram of the number system.

5.0 SUMMARY

In this unit, you have been introduced to the sets of numbers. The set of real numbers, $\Re$, contains the set of integers, Z, Rational numbers, Q, Natural numbers, N, and Irrational numbers, Q'.

Intervals on the real line are open, closed, open-closed or closed-open depending on the nature of the endpoints.

6.0 TUTOR-MARKED ASSIGNMENTS

1. Prove: If a < b and B < c, then a < c
2. Under what conditions will the union of two disjoint interval be an interval?
3. If two sets R and S are bounded, what can be said about the union and intersection of these sets?

7.0 REFERENCES AND FURTHER READINGS

Seymour Lipschutz; Schaum's Outline Series: Theory and Problems of Set Theory and related topics, 1964, pp. 1 – 133.

Sunday O. Iyahen; Introduction to Real Analysis (Real – valued functions of a real variable), 1998, Vol. 1

UNIT 4 FUNCTIONS I

CONTENTS

1.0 INTRODUCTION

In this unit, you will be introduced to the concept of functions, mappings and transformations. You will also be given instructive and typical examples of functions.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Identify functions from statements or diagrams
- State whether a function is one-one or onto
- Find composition function of two or more functions

3.0 MAIN BODY

3.1 Definition

Suppose that to each element in a set A there is assigned by some manner or other, a unique element of a set $\Re$. We call such assignment of ***function.*** If we let f denote these assignments, we write;

$$f : A \longrightarrow B$$

which reads "f is a function of A onto B". The set A is called the ***domain*** of the function f, and B is called the ***co-domain*** of f. Further, if a $\in$ A the element in B which is assigned to a is called the ***image*** of a and is denoted by;

$$f(a)$$

which reads "f of a".

We list a number of instructive examples of functions.

Example 1.1: Let f assign to each real number its square, that is, for every real number x let $f(x) = x^2$. The domain and co-domain of f are both the real numbers, so we can write

$$f : \Re \longrightarrow \Re$$

The image of -3 is 9; hence we can also write f(-3) = 9 or $f : 3 \rightarrow 9$

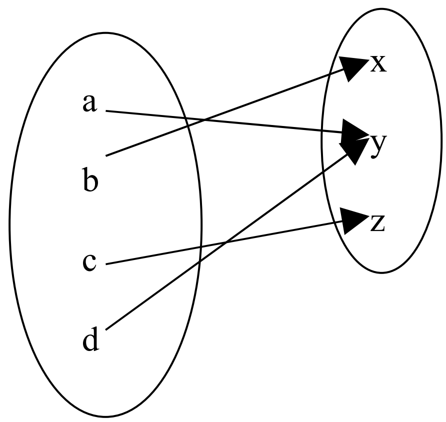
Example 1.2: Let f assign to each country in the world its capital city. Here, the domain of f is the set of countries in the world; The co-domain of f is the list of capital cities in the world. The image of France is Paris, that is, f(France) = Paris

Example 1.3: Let A = {a, b, c, d} and B = {a, b, c}. Define a function f of A into B by the correspondence f(a) = b, f(b) = c, f(c) = c and f(d) = b. By this definition, the image, for example, of b is c.

Example 1.4: Let A = {-1, 1}. Let f assign to each rational number in $\Re$ the number 1, and to each irrational number in $\Re$ the number -1. Then f: $\Re \rightarrow A$, and f can be defined concisely by

$$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ -1 \text{ if } a \text{ is irrational} \end{cases}$$

Example 1.5: Let A = {a, b, c, d} and B = {x, y, z}. Let f: $A \rightarrow B$ be defined by the diagram:



Notice that the functions in examples 1.1 and 1.4 are defined by specific formulas. But this need not always be the case, as is indicated by the other examples. The rules of correspondence which define functions can be diagrams as in example 1.5, can be geographical as in example 1.2, or, when the domain is finite, the correspondence can be listed for each element in the domain as in example 1.4.

3.2 Mappings, Operators, Transformations

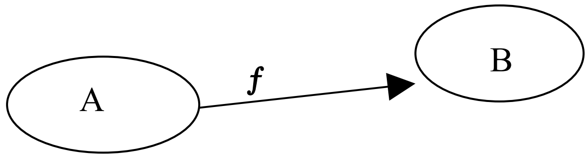
If A and B are sets in general, not necessarily sets of numbers, then a function f of A into B is frequently called a mapping of A into B; and the notation

f: : A ---> B

is then read "f maps A into B". We can also denote a mapping, or function, f of A into B by

Or by the diagram

$$A \xrightarrow{f} B$$



If the domain and co-domain of a function are both the same set, say

f: A ---> A

then f is frequently called an ***operator*** or ***transformation*** on A. As we will see later operators are important special cases of functions.

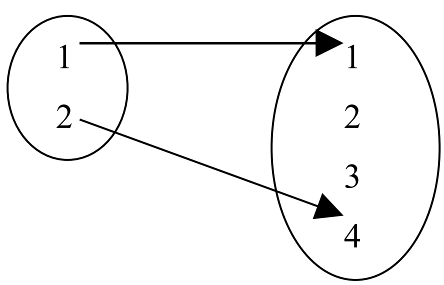
3.3 Equal Functions

If f and g are functions defined on the same domain D and if f(a) = g(a) for every $a \in D$, then the functions f and g are equal and we write

$$f = g$$

Example 2.1: Let $f(x) = x^2$ where x is a real number. Let $g(x) = x^2$ where x is a complex number. Then the function f is not equal to g since they have different domains.

Example 2.2: Let the function f be defined by the diagram



Let a function g be defined by the formula $g(x) = x^2$ where the domain of g is the set {1, 2}. Then f = g since they both have the same domain and since f and g assign the same image to each element in the domain.

Example 2.3: Let f: $\Re \rightarrow \Re$ and g: $\Re \rightarrow \Re$. Suppose f is defined by $f(x) = x^2$ and g by $g(y) = y^2$. Then f and g are equal functions, that is, f = g. Notice that x and y are merely dummy variable in the formulas defining the functions.

3.4 Range of a Function

Let f be the mapping of A into B, that is, let f: A $\rightarrow$ B. Each element in B need not appear as the image of an element in A. We define the range of f to consist precisely of those elements in B which appear and the image of at least one element in A. We denote the range of f: A $\rightarrow$ B y f(A)

f(A)

Notice that f(A) is a subset of B. i.e f(A)

Example 3.1 Let the function f: $\Re \rightarrow \Re$ be defined by the formula $f(x) = x^2$. Then the range of f consists of the positive real numbers and zero.

Example 3.2 Let f: A $\rightarrow$ B be the function in Example 1.3. Then f(A) = {b, c}

3.5 One – One (Injective) Functions

Let f map A into B. Then f is called a ***one-one or Injective function*** if different elements in B are assigned to different elements in A, that is, if no two different elements in A have the same image. More briefly, f: A $\rightarrow$ B is one-one if f(a) = f(a') implies a = a` or, equivalently, a = a` implies f(a) $\neq$ f(a`)

Example 4.1: Let the function f: $\Re \rightarrow \Re$ be defined by the formula $f(x) = x^2$. Then f is not a one-one function since f(2) = f(-2) = 4, that is, since the image of two different real numbers, 2 and -2, is the same number, 4.

Example 4.2: Let the function f: $\Re \rightarrow \Re$ be defined by the formula $f(x) = x^3$. Then f is a one-one mapping since the cubes of the different real numbers are themselves different.

Example 4.3: The function f which assigns to each country in the world, its capital city is one-one since different countries have different capitals, that is no city is the capital of two different countries.

3.6 Onto (Subjective) Function

Let f be a function of A into B. Then the range f(A) of the function f is a subset of B, that is, $f(A) \subset B$. If $f(A) = B$, that is, if every member of B appears as the image of at least one element of A, then we say "f is a function of A onto B", or "f maps A onto B", of "f is an ***onto or Subjective function".***

Example 5.1: Let the function f: $\Re \rightarrow \Re$ be defined by the formula $f(x) = x^2$. Then f is not an onto function since the negative numbers do not appear in the range of f, that is no negative number is the square of a real number.

Example 5.2: Let f: $A \rightarrow B$ be the function in Example 1.3. Notice that $f(A) = \{b, c\}$. Since $B = \{a, b, c\}$ the range of f does not equal co-domain, i.e. is not onto.

Example 5.3: Let f: $A \rightarrow B$ be the function in example 1.5: Notice that

$$f(A) = \{x, y, z\} = B$$

that is, the range of f is equal to the co-domain B. Thus f maps A onto B, i.e. f is an onto mapping.

3.7 Identity Function

Let A be any set. Let the function f: A → A be defined b y the formula f(x) = x, that is, let f assign to each element in A the element itself. Then f is called the identity function or the identity transformation on A. We denote this function by 1 or by 1_A.

3.8 Constant Functions

A function f of A onto B is called a ***constant function*** if the same element of b∈B is assigned to every element in A. In other words,
f: A → B is a constant function if the range of f consists of only one element.

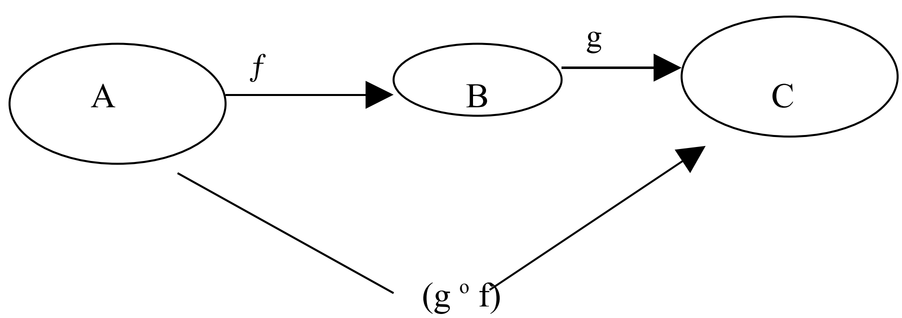
3.9 Product Function

Let F: A → B and g: B → C be two functions then the product of functions f and g is denoted

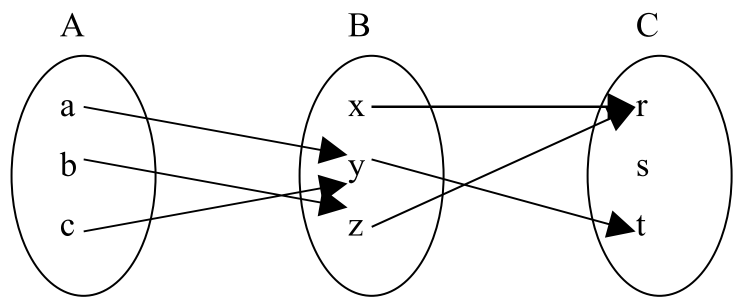
$$(g \circ f) : A \to C \text{ defined by}$$

$$(g \circ f)(a) = g\,(f(a))$$

We can now complete our diagram:



Example 7.1: Let f: A → B and g: B → C be defined by the diagrams



We compute $(g \circ f)$: $A \to C$ by its definition:

$$(g \circ f)(a) \equiv g(f(a)) = g(y) = t$$
$$(g \circ f)(b) \equiv g(f(a)) = g(z) = r$$
$$(g \circ f)(c) \equiv g(f(a)) = g(y) = t$$

Notice that the function (g o f) is equivalent to "following the arrows" from A to C in the diagrams of the functions f and g.

Example 7.2: To each real number let f assign its square, i.e. let $f(x) = x^2$. To each real number let g assign the number plus 3, i.e. let $g(x) = x + 3$. Then

$$(g \circ f)(x) \equiv f(g(x)) = f(x+3) = (x+3)^2 = x^2 + 6x + 9$$
$$(g \circ f)(x) \equiv g(f(x)) = g(x^2) = x^2 + 3$$

Remark 4.1: Let $f: A \to B$. Then

$I_B \circ f = f$ and $f \circ 1_A = f$
that is, the product of any function and identity is . the function itself.

3.9.1 Associativity of Products Of Functions

Let $f: A \to B$, $g: B \to C$ and $h: c \to D$. Then, as illustrated in Figure 4-1, we can form the production function $g \circ f: A \to C$, and then the function

$h \circ (g \circ f): A \rightarrow D$.

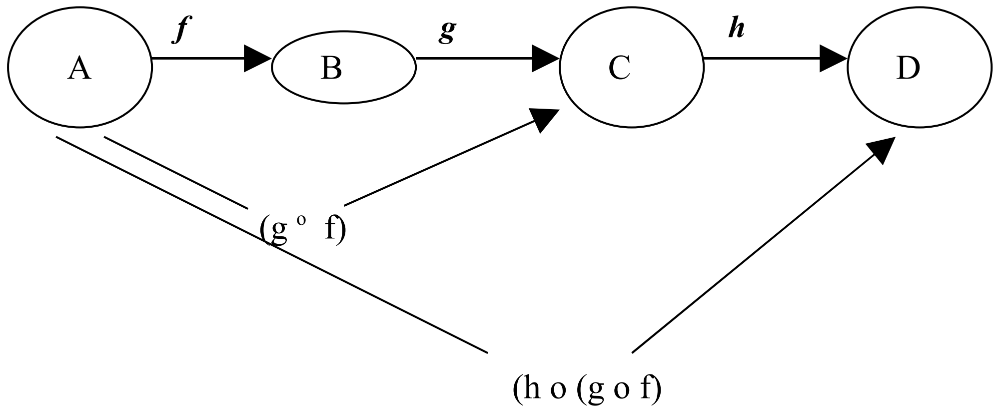


Fig. 4.1

Similarly, as illustrated in Figure 4-2, we can form the product function $h \circ g$:

$B \dashrightarrow D$ and then the function $(h \circ g) \circ f: A \rightarrow D$.

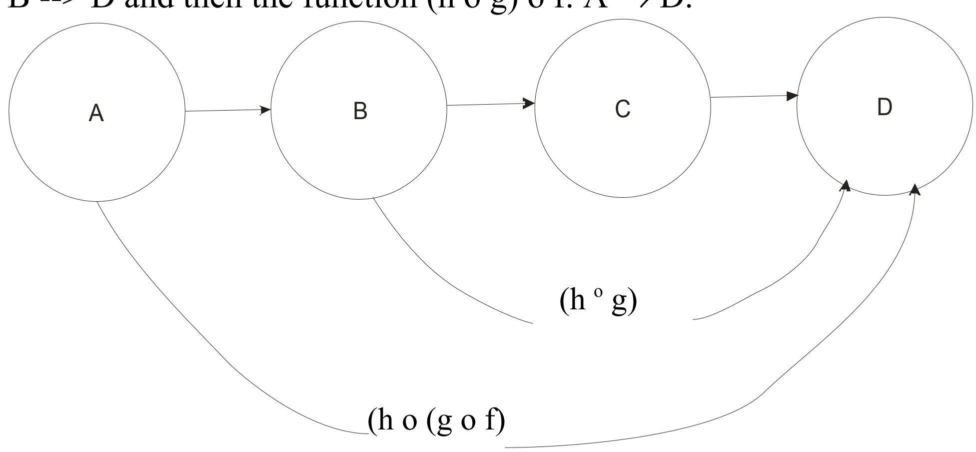


Fig 4.2

Both $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are function of A into D. A basic theorem on functions states that these functions are equal. Specifically,

Theorem 4.1: Let $f: A \rightarrow B$, $B \rightarrow C$ and $h: C \rightarrow D$. Then $(h \circ g) \circ f = h \circ (g \circ f)$

In view of Theorem 4.1, we can write

$$h \circ g \circ f: A \rightarrow D$$

without any parenthesis.

3.10 Inverse of a Function

Let f be a function of A into B, and let $b \in B$. Then the ***inverse*** of b, denoted by

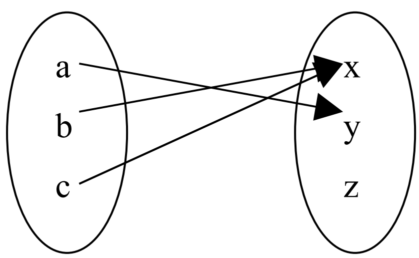
$$f^{-1}(b)$$

Consist of those elements in A which are mapped onto b, that is, those element in A which have m as their image. More briefly, if $f: A \rightarrow B$ then

$$f^{-1}(b) = \{x\}\ x \in A; f(x) = b\}$$

Notice that $f^{-1}(b)$ is always a subset of A. We read f^{-1} as "f inverse".

Example 8.1: Let the function f: A ---> B be defined by the diagram



Then $f^{-1}(x) = \{b, c\}$, since both b and c have x as their image point. Also, $f^{-1}(y) = \{a\}$, as only a is mapped into y. The inverse of z, $f^{-1}(z)$, is the null set ∅, since no element of A is mapped into z.

Example 8.2: Let f: $\Re$ ---> $\Re$, the real numbers, be defined by the formula $f(x) = x^2$. Then $f^{-1}(4) = \{2, -2\}$, since 4 is the image of both 2 and -2 and there is no other real number whose square is four. Notice that $f^{-1}(-3) = \varnothing$, since there is no element in $\Re$ whose square is -3.

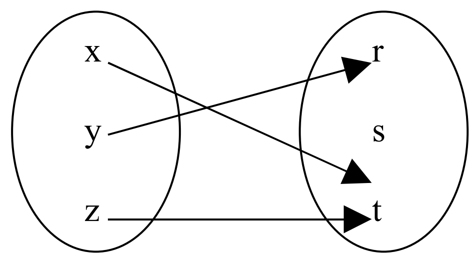
Example 8.3: Let f be a function of the complex numbers into the complex numbers, where f is defined by the formula $f(x) = x^2$. Then $f^{-1}(3) = \{\sqrt{3}i, -\sqrt{3}i\}$, as the square of each of these numbers is -3.

Notice that the function in Example 8.2 and 8.3 are different although they are defined by the same formula

We now extend the definition of the inverse of a function. Let f: A --> B and let D be a subset of B, that is, $D \subset B$. Then the inverse of D under the mapping f, denoted by $f^{-1}(D)$, consists of those elements in A which are mapped onto some element in D. More briefly,

$$f^{-1}(D) = \{x \mid x \in A, f(x) \in D\}$$

Example 9.1: Let the function f: A --> B be defined by the diagram



Then $f^{-1}(\{r, s\}) = \{y\}$, since only y is mapped into r or s. Also $f^{-1}(\{r, t\}) = \{x, y, z\} = A$, since each element in A as its image r or t.

Example 9.2: Let f: $\Re$ ---> $\Re$ be defined by $f(x) = x^2$, and let
$D = [4, 9] = \{x \mid 4 \leq x \leq 9\}$
Then f -1 (D) = $\{x \mid -3 \leq x \leq -2 \text{ or } 2 \leq x \leq 3\}$

Example 9.3: Let f: A ---> B be any function. Then $f^{-1}(B) = A$, since every element in A has its image in B. If f(A) denote the range of the function f, then

$$f^{-1}(f(A)) = A$$

Further, if $b \in B$, then

$$f^{-1}(b) = f^{-1}(\{b\})$$

Here f^{-1} has two meanings, as the inverse of an element of B and as the inverse of a subset of B.

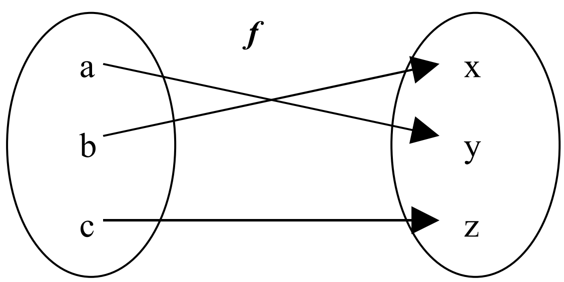
3.11 Inverse Function

Let f be a function of A into B. In general, $f^{-1}(b)$ could consist of more than one element or might even be empty set $\varnothing$. Now if $f: A \to B$ is a one-one function and an onto function, then for each $b \in B$ the inverse $f^{-1}(b)$ will consist of a single element in A. We therefore have a rule that assigns to each $b \in B$ a unique element $f^{-1}(b)$ in A. Accordingly, f^{-1} is a function of B into A and we can write

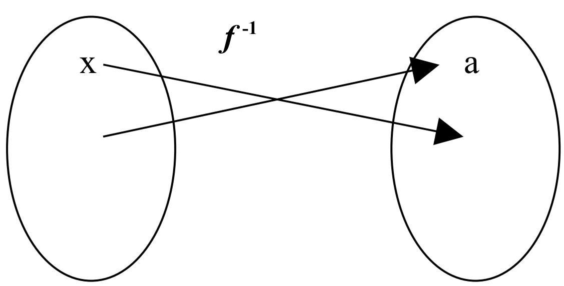
$$f^{-1} : B \to A$$

In this situation, when $f: A \to B$ is one-one and onto, we call f^{-1} the inverse function of f.

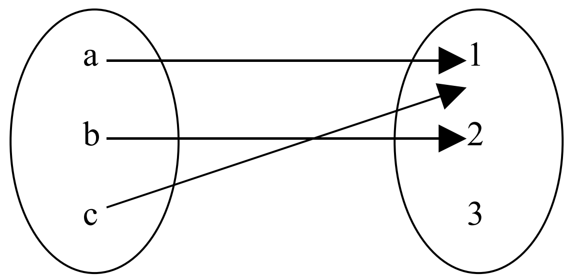
Example 10.1: Let the function $f: A \to B$ be defined by the diagram



Notice that f is one-one and onto. Therefore f -1, the inverse function exists We describe $f^{-1}: B \to A$ by the diagram



Example 6.1: Let the function f be defined by the diagram:

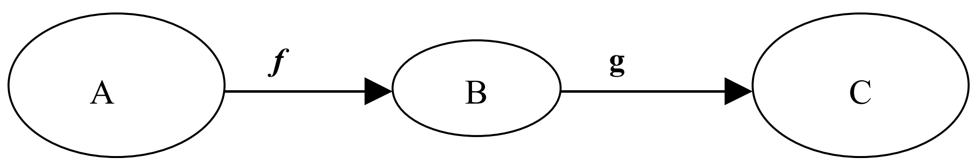


Then f is a constant function since 3 is assigned to every element in A.

Example 6.3: Let f: $\Re \rightarrow \Re$ be defined by the formula f(x) = 5. Then f is a constant function since 5 is assigned to every element.

3.9 Product Function

Let f be a function of A and B and let g be a function of B, the co-domain of f, into C. We illustrate the function below.



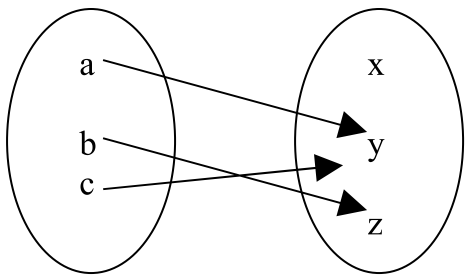
Let $a \in A$; then its image f (x) is in B which is the domain of g. Accordingly, we can find the image of f (a) under the mapping of g, that is, we can find g (f(a)). Thus, we have a rule which assigns to each element $a \in A$ a corresponding element $(f(a)) \in C$. In other words, we have a function of A into C. This new function is called the ***product function*** or ***composition function*** of f and g and it is denoted by

$(g^{o} f)$ or (gf)

More briefly, if f: $A \to B$ and g: $B \to C$ then we define a function

Notice further, that if we send the arrows in the opposite direction in the first diagram of f we essentially have the diagram of f^{-1}.

Example 10.2: Let the function f: $A \to B$ be defined by the diagram

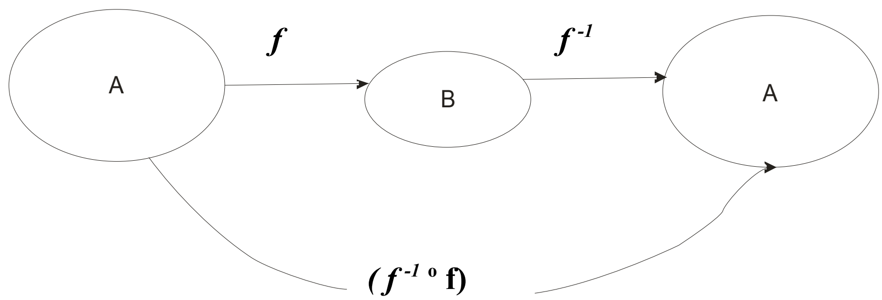


Since $f(a) = y$ and $f(c) = y$, the function f is not one-one. Therefore, the inverse function f^{-1} does not exist. As $f^{-1}(y) = \{a, c\}$, we cannot assign both a and c to the element $y \in B$.

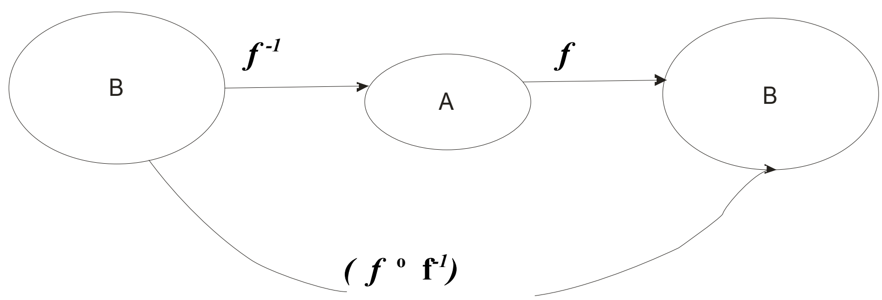
Example 10.3: Let f: $\Re \to \Re$, the real numbers, be defined by $f(x) = x^3$. Notice that f is one-one and onto. Hence f^{-1}: $\Re \to \Re$ exists. In fact, we have a formula which defines the inverse function, $f^{-1}(x) = \sqrt[3]{x}$.

3.11.1 Theorems on the inverse Function

Let a function f: $A \to B$ have an inverse function f^{-1}: $B \to A$. Then we see by the diagram



That we can form the product ($f^{-1} \circ f$) which maps A into A, and we see by the diagram



That we can form the product function ($f \circ f^{-1}$) which maps B into B. We now state the basic theorems on the inverse function:

Theorem 4.2: Let the function $f: A \rightarrow B$ be one-one and onto; i.e. the inverse function $f^{-1}: B \rightarrow A$ exists. Then the product function

$$(f^{-1} \circ f): A \rightarrow A$$

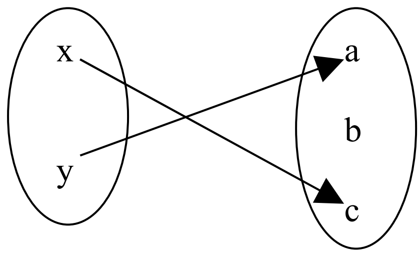
is the identity function on A, and the product function

$$(f \circ f^{-1}): B \rightarrow B$$

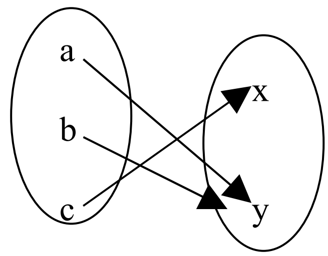
is the identity function on B.

Theorem 4.3: Let $f: A \rightarrow B$ and $g: B \rightarrow A$. Then g is the inverse function of f, i.e. $g = f - 1$, if the product function $(g \circ f): A \rightarrow A$ is the identity function on A and ($f \circ g$): $B \rightarrow B$ is the identity function on B.

Both conditions are necessary in Theorem 4.3 as we shall see from the example below



(a)



(b)

Now define a function g: $B \rightarrow A$ by the diagram (b) above.

We compute ($g \circ f$): $A \rightarrow A$,

$(g \circ f)(x) = g(f(x)) = g(c) = x$ and
$(g \circ f)(y) = g(f(y)) = g(a) = y$

Therefore the product function ($g \circ f$) is the identity function on A. But g is not the inverse function of f because the product function ($f \circ G$) is not the identity function on B, f not being an onto function.

4.0 CONCLUSION

I believe that by now you fully grasp the idea of functions, mappings and transformations. This knowledge will be built upon in subsequent units.

5.0 SUMMARY

Recall that in this unit we have studied concepts such as mappings and functions. We have also examined the concepts of one-to-one and onto functions. This concept has allowed us to explain equality between two sets. We also established in the unit that the inverse of f: A – B usually denoted f^{-1}, exit, if f is a one-to-one and onto function.

It is instructive to note that Inverse function is not studied in isolation but more importantly a useful and powerful tool in understanding calculus.

6.0 TUTOR – MARKED ASSIGNMENTS

1. Let the function f: $R^{\#} \rightarrow R^{\#}$ be defined by
   f (X) = {1 if x is rational]
   {-1 if x is irrational.
   a. Express f in words
   b. Suppose the ordered pairs (x + y, 1) and (3, x – y) are equal.
      Find x and y.
2. Let M = {1, 2, 3, 4, 5} and let the function f: M $\rightarrow \Re$ be defined by
   $f(x) = x^2 + 2x - 1$
   Find the graph of f.
3. Prove: A x (B $\cap$ C) = (A x B) $\cap$ (A x C)
4. Prove A $\subset$ B and C $\subset$ D implies (A x C) $\subset$ (B x D).

7.0 REFERENCES AND FURTHER READINGS

Seymour Lipschutz; Schaum's Outline Series: Theory and Problems of Set Theory and related topics, 1964, pp. 1 – 133.

Sunday O. Iyahen; Introduction to Real Analysis (Real – valued functions of a real variable), 1998, Vol. 1

UNIT 5 FUNCTIONS II

CONTENTS

1.0 INTRODUCTION

In this unit, we are going to define a type of set that not only gives a better understanding of Cartesian coordinate but also brings the concept of real-valued functions to the fore.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Find the ordered pairs, given two sets
- Find the ordered pairs corresponding to the points on the Cartesian coordinate diagram
- Find the graph of functions
- State whether or not a set of ordered pairs of a given set, say A, is a function of A into itself.

3.0 MAIN BODY

3.1 Ordered Pairs

Intuitively, an ***ordered pair*** consists of two elements, say a and b, in which one of them, say a, is designated as the first element and the other as the second element. An ordered pair is denoted by

$$(a, b)$$

Two ordered pairs (a, b) and (c, d) are equal if and only if
a = c and b = d.

Example 1.1: The ordered pairs (2, 3) and (3, 2) are different
Example 1.2: The points in the Cartesian plane shown in fig 5.1 below represent ordered pairs of real numbers.
Example 1.3: The set {2, 3} is not an ordered pair since the elements 2 and 3 are not distinguished
Example 1.4: Ordered pairs can have the same first and second elements such as (1, 1), (4, 4) and (5, 5).

Although the notation (a, b) is also used to denote an open interval, the correct meaning will be clear from the context.

Remark 5.1: An ordered pair (a, b) can be defined rigorously by

$$(a, b) = \{ \{a\}, \{a, b\} \}$$

From this definition, the fundamental property of ordered pairs can be proven:

$$(a, b) = (c, d) \text{ implies } a = c \text{ and } b = d$$

3.2 Product Set

Let A and B be two sets. The ***product set*** of A and B consists of all ordered pairs (a, b) where $a \in A$ and $b \in B$. it is denoted by

$$A \times B.$$

Which reads "A cross B". More precisely

$$A \times B = \{ (a, b) \mid a \in A, b \in B\}$$

Example 2.1: Let A = {1, 2, 3} and B = {a, b}. Then the product set
A x B = {(1, a), (1, b), (2, a), (2, b), (3, a) (3, b)}

Example 2.2: Let W = {s, t}. Then
W x W = {(s, s), (s, t), (t, s), (t, t)}

Example 2.3: The Cartesian plane shown in Fig 5.1 is the product set of the real numbers with itself, i.e. $\Re \times \Re$

The product set A x B is also called the ***Cartesian Product*** of A and B. it is named after the mathematician Descartes who, in the seventeenth century, first investigated the set $\Re \times \Re$. It is also for this reason that $\Re \times \Re$, as pictured in Fig. 5.1, is called the Cartesian Plane.

Remark 5.2: If set A has n elements and set B has m elements then the product set A x B has *n times m* elements, i.e. *nm* elements. If either A or B is the null set then A x B is also the null set. Lastly, if either A or B is infinite and the other is not empty, then a x B is infinite.

Remark 5.3: The Cartesian product of two sets is not commutative; more specifically,

$$A \times B \neq B \times A$$

Unless A = B or one of the factors is empty.

3.3 Coordinate Diagrams

You are familiar with the Cartesian plane $\Re \times \Re$, as shown in Fig 5.1 below. Each point P represents an ordered pair (a, b) of real numbers. A vertical through P meets the horizontal axis at *a* and a horizontal line through P meets the vertical axis at *b* as in Fig. 5.1.

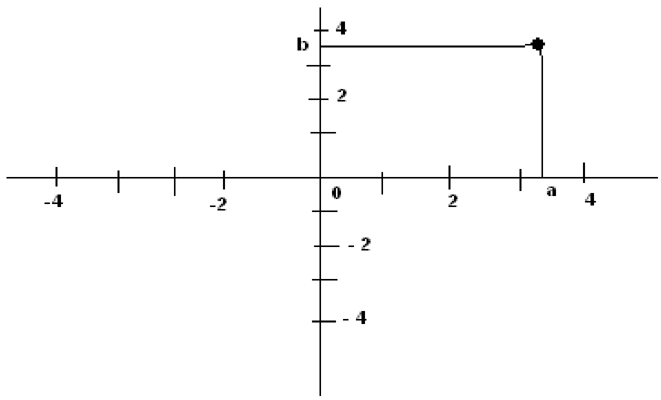


Fig. 5.1

The Cartesian product of any two sets, if they do not contain too many1 elements, can be displayed on a coordinate diagram in a similar manner. For example, if A = {a, b, c, d} and B = {x, y, z}, then the coordinate diagram of A x B is as shown in Fig 5.2 below. Here the elements of A are displayed on the horizontal axis and the elements of B are displayed on the vertical axis. Notice that the vertical lines through the elements of A and the horizontal lines through the elements of B meet 12 points. These points represent A x B in the obvious way. The point P is the ordered pair (c, y).

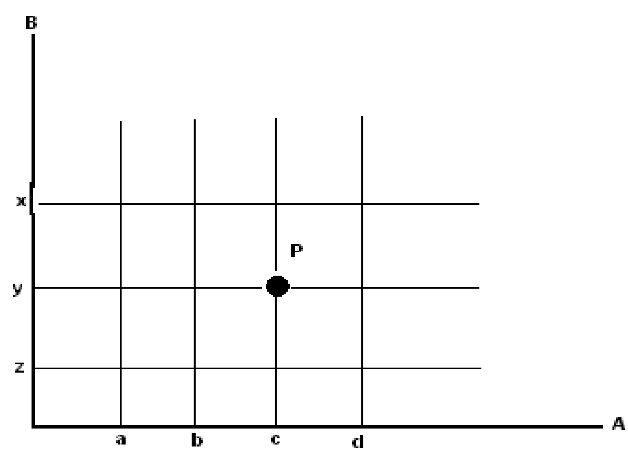


Fig 5.2

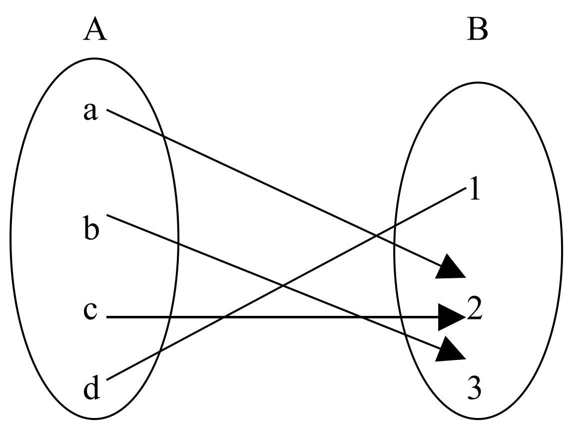
3.4 Graph of A Function

Let f be a function of A into B, that is, let f: A ---> B. The graph f* of the function f consists of all ordered pairs in which $a \in A$ appears as a first element and its image appears as its second element. In other words,

$$f^* = \{(a, b) \mid a \in A, b + f(a)\}s$$

Notice that f*, the graph of f: A --> B, is a subset of A x B.

Example 3.1: Let the function f: A --> B be defined by the diagram



Then f(a) = 2, f(b) = 3, f(c) = 2 and f(d) = 1. Hence the graph of f is

$$F^* = \{(a, 2), (b, 3), (c, 2), (d, 1)\}$$

Example 3.2: Let W = {1,2,3,4}. Let the function f: W --> $\Re$ be defined by

$$f(x) = x + 3$$

Then the graph of f is

$$f^* = \{(1, 4), (2, 5), (3, 6), (4, 7)\}$$

Example 3.3: Let N be the natural numbers 1, 2, 3,………Let the function g:

$N \to N$ be defined by

$g(x) = x^3$

Then the graph of g is

$g^* = \{1,1), (2,8), (3, 27), (4, 64),\ldots..\}$

3.4.1 Properties of the Graph of a function

Let f: $A \to B$. We recall two properties of the function f. First, for each element $a \in A$ there is assigned an element in B. Secondly, there is only one element B which is assigned to each $a \in A$. In view of these properties of f, the graph f* of f has the following two properties:

Property 1: For each $a \in A$, there is an ordered pair $(a, b) \in f^*$

Property 2: Each $a \in A$ appears as the first element in only one ordered pair in f*, that is

$$(a, b) \in f^*, (a, c) \in f^* \text{ implies } b = c$$

In the following examples, let $A = \{1,2,3,4\}$ and $B = \{3,4,5,6\}$

Example 4.1: The set of ordered pairs

$$\{(1,5), (2,3), (4,6)\}$$

cannot be the graph of a function of A into B since it violates property 1. Specifically, $3 \in A$ and there is no ordered pair in which 3 is a first element.

Example 4.2: The set of ordered pairs

$$\{(1,5), (2,3), (3,6), (4,6), (2,4)\}.$$

cannot be the graph of a function of A into B since it violates Property 2, that is, the element $2 \in A$ appears as the first element in two different ordered pairs (2, 3) and (2,4)

3.5 Graphs and Coordinate Diagrams

Let f* be the graph of a function f: A → B. As f* is a subset of A x B, it can be displayed, i.e. graphed, on the coordinate diagram of A x B.

Example 5.1: Let $f(x) = x^2$ define a function on the interval $-2 \leq x \leq 4$. Then the graph of f is displayed in Fig 5.3 below in the usual way:

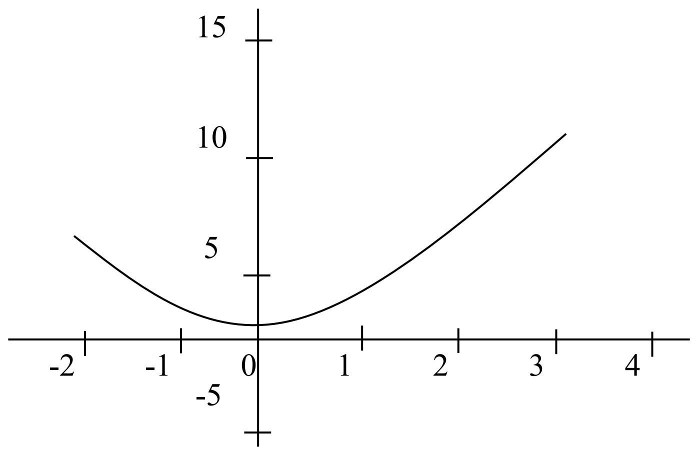


Fig 5.3

Example 5.2: Let a function f: A → B be defined by the diagram shown in Fig 5.4 below

Here f*, the graph of f, consist of the ordered pairs (a, 2), (b, 3), (c, 1) and (d, 2). Then f* is displayed on the coordinate diagram A x B as shown in Fig 5.5 below.

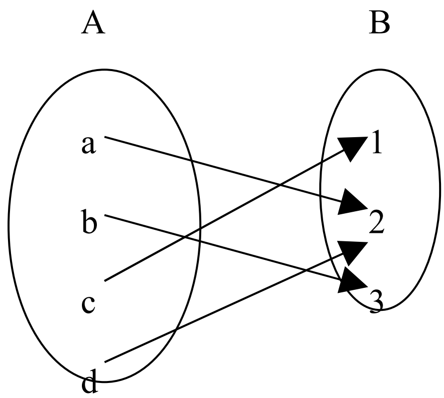


Fig. 5.4

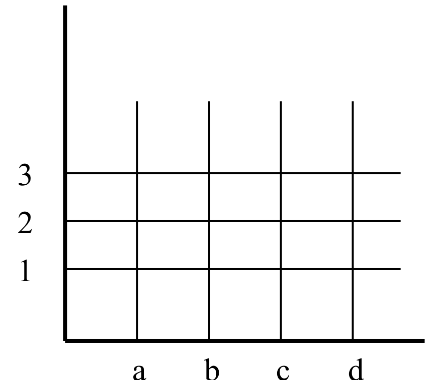


Fig. 5.5

3.5.1 Properties of Graphs of Functions on Coordinate Diagrams

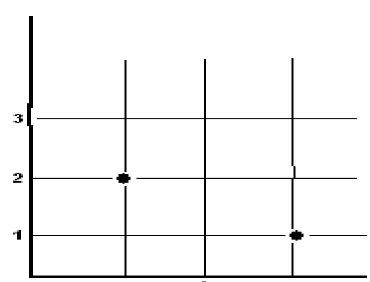
Let f: A → B. Then f*, the graph of f, has the two properties listed previously:

Property 1: For each a ∈ A, there is an ordered pair (a, b) ∈ f*
Property 2: If (a, b) ∈ f* and (a, c) ∈ f*, then b = c.

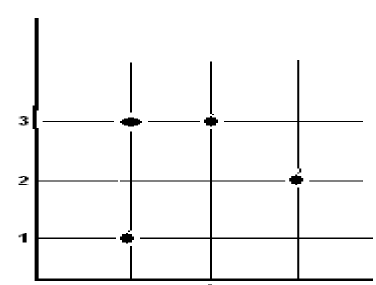
Therefore, if f* is displayed on the coordinate diagram of A x B, it has the following two properties:

Property 1: Each vertical line will contain at least one point of f*
Property 2: Each vertical line will contain only one point of f*

Example 6.1: Let a = {a, b, c} and B = {1, 2, 3}. Consider the sets of points in the two coordinate diagrams of A x B below.



(1)

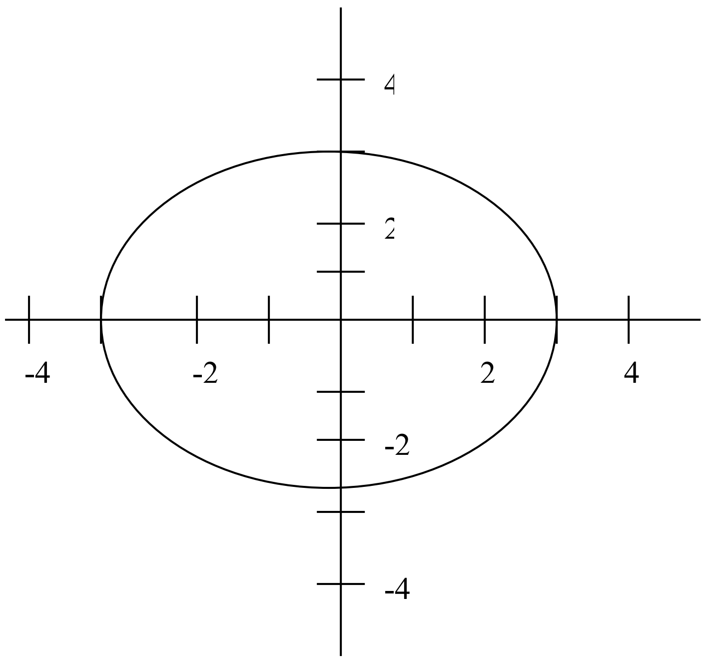


(2)

In (1), the vertical line through *b* does not contain a point of the set; hence the set of points cannot be the graph of a function of A into B.

In (2), the vertical line through *a* contains two points of the set, hence this set of point cannot be the graph of a function of A into B.

Example 6.2: The circle $x^2 + y^2 = 9$, pictured below, cannot be the graph of a function since there are vertical lines which contain more than one point of the circle.



$x^2 + y^2 = 9$ is plotted

3.6 Functions as Sets of Ordered Pairs

Let f* be a subset of A x B, the Cartesian product of sets A and B; and let f* have the two properties discussed previously:

Property 1: For each $a \in A$, there is an ordered pair $(a, b) \in f^*$.
Property 2: No two different ordered pairs in f* have the same first element.

Thus, we have a rule that assigns to each element a ∈ A, the element b ∈ B that appear in the ordered pair (a, b) ∈ f*. Property 1 guarantees that each element in A will have an image, and Property 2 guarantees that the image is unique. Accordingly, f* is a function of A into B.

In view of the correspondence between functions f: A → B and subset of A x B with property 1 and property 2 above, we redefine a function by the

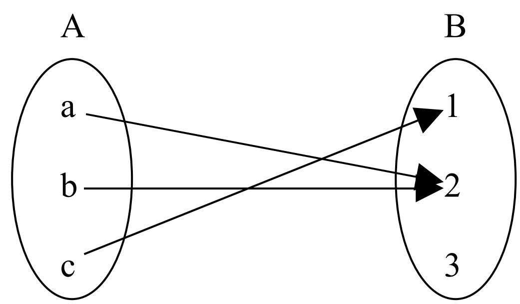
Definition 5.1: A function f of A into B is a subset of A x B in which each a ∈ A appears as the first element in one and only one ordered pair belonging to f.

Although, this definition of a function may seem artificial, it has the advantage that it does not use such undefined terms as "assigns", "rules", "correspondence".

Example 7.1: Let A = (a, b, c) and B = (1, 2, 3). Furthermore, let

f = {(a, 2), (c, 1), (b, 2)}

Then f has Property 1 and Property 2. Hence f is a function of A into B, which is also illustrated in the following diagram:



Example 7.2: Let V = {1, 2, 3} and W = {a, e, I, o, u}. Also let

f + {(1, a), (2, e), (3, 1), (2, u)}

Then f is not a function of V into W since two different ordered pairs in f, (2, e) and (2, u), have the same first element. If f is to be a function of V into W, then it cannot assign both e and u to the element 2∈ V.

Example 7.3: Let S = {1,2,3,4} and T = {1,3,5}. Let

f = {(1,1), (2, 5), (4, 3)}

Then f is not a function of S into T since $3 \in S$ does not appear as the first element in any ordered pair belonging to f.

The geometrical implication of Definition 5.1 is stated in.

Remark 5.4: Let f be the set of points in the coordinate diagram of A x B. if every vertical line contains one and only point of f, then f is a function of A into B.

Remark 5.5: Let the function f: $A \rightarrow B$ be one-one and onto. Then the inverse function f^1 consists of those ordered pairs which when reversed, i.e. permuted, belong to f. More specifically,

$f^1 = \{(b, a) \mid (a, b) \in f\}$

3.7 Product Sets in General

The concept of a product set can be extended to more than two sets in a natural way. The Cartesian product of sets A, B, and C, denoted by

A x B x C

Consists of all ordered triplets (a, b, c) where $a \in A$, $b \in B$ and $c \in C$. Analogously, the Cartesian product of n sets $A_1, A_2, \ldots\ldots A_n$, denoted by

$A_1 \times A_2 \times \ldots \times A_n$

Consists of all ordered n-tuples ($a_1, a_2, \ldots$,an) where $a_1 \in A, \ldots., a_n \in A$. Here an ordered n-tuple has the obvious intuitive meaning, that is,, it consists of n elements, not necessarily distinct, in which one of them is designated as the first element, another as the second element, etc.

Example 8.1: In three-dimensional Euclidean geometry each point represents an ordered triplet, i.e. its *x*-

component, its *y*-component and its *z* – component.

Example 8.2: Let A = {a, b}, B = {1, 2, 3} and C = {x, y}. Then

A x B x C = {(a, 1, x), (a, 1, y), (a, 2, x)
(a, 2, y), (a, 3, x), (a, 3, y)
(b, 1, x), (b, 1, y), (b, 2, x)
(b, 2, y), (b, 3, x), (b, 3, y)

4.0 CONCLUSION

In this unit you have studied concepts such as ordered pairs, product sets, co-ordinate diagram, Functions as set of ordered pairs.

We have also learnt about how to represent function on a graph. We require the mastery of the above concepts in the understanding of the subsequent units.

5.0 SUMMARY

Recall the following:

That an ordered pair is denoted by (a, B), $a \in A$ and $b \in B$. Two ordered pairs (a, b) and (c, d) are if and only is a = c, and b = d.

That is A and B are two sets such that $a \in A$ and $b \in B$ them the product of A and B is denoted by $A \times B = \{(a, b)\}\ a \in A, b \in B\}$

That the Cartesian place is the product set of real number with itself i.e IR × IR

That the concept of product can be extended to more than two sets in a natural ways i.e if A, B and C are sets then the product of A, B and C is denoted as

$A \times B \times C = \{(a, b, c)\ a \in A, b \in B, c\ C\}$

Generally the Cartesian product of n sets $A_1, A_2, \ldots\ldots A_n$ is denoted by

$A_1\ A_2\ A_3\ \ldots . \times A_n = \{(a1, a2, a3 \ldots\ldots an)\}$
$a_1\ \ A_1, a_2\ \ A_2 \ldots\ldots\ldots, a_n \in\ \ A_n\ \}$

6.0 TUTOR-MARKED ASSIGNMENTS

1. Suppose the ordered pairs $(x + y, 1)$ and $(3, x - y)$ are equal. Find x and y.
2. Let $M = \{1,2,3,4,5\}$ and let the function $f: M \to \Re$ be defined by
$$f(x) = x^2 + 2x - 1$$
Find the graph of f.

3. Prove: $A \times (B \cap C) = (A \times B) \cap (A \times C)$
4. Prove $A \subset B$ and $C \subset D$ implies $(A \times C) \subset (B \times D)$.

7.0 REFERENCES AND FURTHER READINGS

Seymour Lipschutz; Schaum's Outline Series: Theory and Problems of Set Theory and related topics, 1964, pp. 1 – 133.

Sunday O. Iyahen; Introduction to Real Analysis (Real – valued functions of a real variable), 1998, Vol. 1

Module 2

UNIT 1 RELATIONS

CONTENTS

1.0 INTRODUCTION

From the concept of ordered pairs, product set or Cartesian product we can draw relations based on propositional functions defined on the Cartesian product of two sets

This is what will be developed in this unit

2.0 OBJECTIVES

After going through this unit, you should be able to do the following:

- Derive relations as ordered pairs between two sets based on open sentences.
- Find the Domain, Range and Inverse of a relation
- Define, with examples, the different kinds of relations
- State whether or not a relation defined on a set is a function of the set into itself

3.0 MAIN BODY

3.1 Propositional Functions, Open Sentences

A **Propositional function** defined on the Cartesian product A x B of two sets A and B is an expression denoted by

$$P(x,y)$$

Which has the property that P(a,b), where a and b are substituted for the variables x and y respectively in P(x,y), is true or false for any ordered pair $(a,b) \in A \times B$. For example, if A is the set of playwright and B is the set of plays, then

$$P(x,y) = \text{“x wrote y”}$$

Is a propositional function on A x B, In particular,

P (Shakespeare, Hamlet) = “Shakespeare wrote Hamlet”
P(Shakespeare, Things Fall Apart) = “Shakespeare wrote Things Fall Apart”

Are true and false respectively.

The expression P(x,y) by itself shall be called an open sentence in two variables or, simply, an open sentence. Other examples of open sentences are as follows:

Example 1.1: “x is less than y”

Example 1.2: "x weighs y kilograms"
Example 1.3: "x divides y"
Example 1.4: "x is wife of y"
Example 1.5: "The square of x plus the square of y is sixteen", i.e "$x^2 + y^2 = 16$"
Example 1.6: "Triangle x is similar to triangle y"

In all of our examples there are two variable. It is also possible to have open sentences in one variable such as "x is in the United Nations", or in more than two variables such as "x times y equals z"

3.2 Relations

A ***relation*** R consists of the following

1. a set A
2. a set B
3 an open sentence P(x,y) in whch P(a, b) is either true or false for any ordered pair (a,b) belonging to A x B

We then call R a ***relation from A to B*** and denote it by

$$R = (A, B, P(x,y))$$

Furthermore, if (a,b) is true we write

$$a\ R\ b$$

which reads "a is related to B". On the other hand, if P(a,b) is not true we write

$$a \not R\ b$$

which reads "a is not related to b"

Example 2.1: Let $R_1 = (\Re,\Re\ P(x,y))$ where P(x,y) reads "x is less than y". Then R_1 is a relation since P(a,b), i.e "a< b", is either true or false for any ordered pair (a,b) of real numbers. Moreover, since $P(2, \pi)$ is true we can write

$$2\ R_1\ \pi$$

and since $p(5, \sqrt{2})$ is false we can write

$$5 \not R_1 \sqrt{2}$$

Example 2:2 Let R_2 = (A,B P(x,y)) where A is the set of men, B is the set of women, and P(x,y) reads "x is the husband of y". then R_2 is a relation

Example 2:3 Let R_3 = (N, N, P(x,y)) where N is the natural numbers and P(x,y) reads "x divides y". Then R_3 is a relation. Furthermore,

$$3\ R_3\ 12,\ 2 \not R_3\ 7,\ 5\ R_3\ 15,\ 6\ R_4\ 13$$

Example 2:4 Let R_4 = (A, B, P(x,y)) where A is the set of men, B is the set of women and P(x,y) reads "x divides y". Then R4 is not a relation since P(a,b) has no meaning if a is a man and b is a woman.

Example 2:5 Let R_5 = (N, N, P(x,y)) where N is the natural numbers and P(x,y) reads "x is less than y". Then R_5 is a relation.

Notice that R_1 and R_5 are not the same relation even though the same open sentence is used to define each relation

Let R = (A, B, P(x,y)) be a relation. We then say that the open sentence P(x,y) ***defines a relation*** from A to B. Furthermore, if A = B, then we say that P(x,y) defines a relation in A, or that R is a relation in A.

Example 2:6 The open sentence P(x,y), which reads "x is less than y", defines a relation in the rational numbers

Example 2:6 The open sentence "x is the husband of y" defines a relation from the set of men to the set of women.

3.3 Solution Sets and Graphs of Relations

Let R = (A, B, P(x,y)) be a relation. The ***Solution set R**** of the relation R consists of the elements (a,b) in A x B for which P(a,b) is true. In other words

$$R^* = \{(a,b) \mid a \in A, b \in B, P(a,b) \text{ is true}\}$$

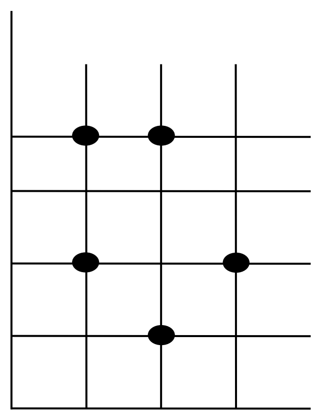
Notice that R*, the solution set of a relation R from A B, is a subset of A x A. Hence R* can be displayed, i.e plotted or sketched, on the coordinate diagram of A x A

The graph of a relation R from A to B consists of those points on the coordinate diagram of A x A which belongs to the solution set of r.

Example 3:1 Let R = (A, B, P(x,,, y)) where A = {2,3,4}, and B = {3, 4, 5}, and P(x,y) reads "x divides y". Then the solution set of R is:

R* = {(2,4), (2,6), (3,3), (3,6), (4,4)}

The solution set of R is displayed on the coordinate diagram of A x B as shown in Fig.6.2 below



Example 3:2 Let R be the relation in the real numbers defined by

$y<x + 1$

The shaded area in the coordinate diagram of $\Re$ x $\Re$ shown in Fig. 6.2 above consists of the points which belong to $\Re$, the solution set of R, that is, the graph of R. Notice that $\Re$ consists of the points below the line y = x + 1. The line y = x +1 is dashed in order to show that the points on the line do not belong to $\Re$.

3.4 Relations As Sets Of Ordered Pairs

Let R* be any subset of A x B. We can define a relation R = (A, B P(x,y)) where P(x,y) read

"The ordered pair (x,y) belongs to R*"

The solution set of this relation R is the original set R*. Thus to every relation R = (A, B P(x,y)) there corresponds a unique solution set R* which is a subset of A x B, and to every subset R* of A x B there corresponds a relation R = (A, B P(x,y)) for which R* is "R = (A, B, P(x,y)) and subsets R* of A x B, we rederine a relation by the

Definition 6.1: A relation R from A to B is a subset of A x B

Although Definition 6.1 of a relation may seem artificial it has the advantage that we do not use in this definition of a relation the undefined concepts "open sentence" and "variable"

Example 4.1: Let A = {1, 2,3} and B = {a, b}. then

R = {(a, a), (1, b), (3,a)}

Is a relation from A to B. Furthermore

1 R a, 2 $\not R$ b, 3 R a, 3 $\not R$ b

Example 4.2: Let W = {a, b, c}. Then
R = {(a, b), (a,c), (c,c), (c,b)}
is a relation in W. Moreover.
a $\not R$ a, b $\not R$ a, c R c, a R b

Example 4.3: Let R = {x, y) | x $\in \Re$, y$\in \Re$, y<x^2}.

Then R is a set of ordered pairs of real numbers, i.e a subset of $\Re$ x $\Re$. Hence R is a relation in the real numbers which could also be defined by

R= ($\Re$, $\Re$, P(x,y))
Where P(x,y) reads "y is less than x^2"

Remark 6.1 Let set A have m elements and set B have n elements. Then there are 2^{mn} different relations from A to B, since A x B, which has mn elements, has 2^{mn} different subsets.

We now consider a relation between a set X into itself..

Suppose SCX is a set and let RCSXS then R is said to a relation in S. We have the following properties:-

3.5 Reflexive Relations

Suppose R is a relation in a set S then,
R is reflexive in S if and only if for all x$\in$ S, xRx

Example

(i) Take X = IR^+ (the set of positive real numbers). Let R be the relation of equality on IR^+

Is R reflexive on IR^+?

Yes it is reflexive. To see this let a$\in$ IR^+ then a = a.
i.e. aRa $\Rightarrow$ R is reflexive.

(ii) let A be a set and take X $=\left\{ P(A) \text{ or } 2^A \right\}$ i.e. the collection of subset of A.

Let R to be relation "is a proper subset of " i.e. BRD if B is a proper subset of D.

Is R reflexive on A?

Answer

The answer is no. since a set cannot be a proper subset of itself i.e. A $\not\subset$A, hence R is anti-reflexive.

3.6 Symmetric Relation

R is symmetric in S if and only if for all x,y$\in$ s, xRy $\Rightarrow$ yRx.

In example (i) above if a, b$\in$ IR^+ and a = b then b = a i.e. aRb $\Rightarrow$ bRa, hence R is symmetric.

In example (ii) above, does ARB imply BRA? No. For if ACB then B $\not\subset$A hence R is not symmetric.

3.7 Anti-Symmetric Relation

R is anti-symmetric in S iff for x,y$\in$ S, xRy and yRx $\Rightarrow$ x = y.

In example (i) above, aRb and bRa $\Rightarrow$ a=b hence R is anti-symmetric.

3.8 Transitive Relation

R is a transitive relation on s iff for all x,y,z$\in$ S, xRy, yRz $\Rightarrow$ xPz.

Consider example (ii) above, let ARB and BRD does ARD holds?

Yes, for if ACB and BCD then ACD therefore R is transitive.

3.9 Equivalence Relation

A relation E in a set S is said to be an equivalence relation in S if for all x,y,z$\in$ s

(i) xEx (reflexive property)
(ii) xEy yEz $\Rightarrow$ yEx (symmetric)
(iii) xEy yEz $\rightarrow$ xEz (transitive)

For example, the relation of equality defined in example (i) is an equivalence relation (check).

3.10 **Domain and Range of a Relation**

Let R be a relation between X and Y. Then R^+ is a relation between Y and X.

Thus x R^{-1}y iffy Rx. We therefore define

(i) Range of R = Domain (R^{-1})
(ii) Range of R^{-1} = Domain (R)
(iii) Furthermore $(R^{-1})^{-1}$ = R

3.11 **Relations and Function**

The notion of relation is a generalization of the notion of functions.

Let f: X $\longrightarrow$ Y be function defined on X and Y. then f is a function if it can satisfy the following single valued property:

$(x,y) \in f, \ (x,z) \in f \quad \Longrightarrow y = z$

4.0 CONCLUSION

In this unit, emphasis has been place on the derivation of relations as ordered pairs between two sets based on open sentences; finding the Domain, Range and Inverse of a relation; defining, with examples, the different kinds of relations and stating whether or not a relation defined on a set is a function of the set into itself.

Here is the summary

5.0 SUMMARY

- The expression P(x,y) is called an ***open sentence in two variables*** *x and y.*
- R = (A, B P(a,b)) is called ***relation*** from A to B where (a,b) $\in$ A x B
- R* = {(a,b) | a$\in$A, b$\in$B, P(a,b) is true} is the ***solution set*** of the relation R.
- A relation R from A to B is a subset of A x B

- $R^1 = \{(b,a) \mid (a,b) \in R\}$ is the ***inverse relation*** of R
- If every element in a set is related to itself, then the relation is said to be ***reflexive***
- For a relation R in A, if $(a,b) \in R$ implies $(b,a) \in R$, then R is a ***symmetric*** relation
- For a relation R in A, if $(a,b) \in R$ and $(b,a) \in R$ implies $a = b$, then R is an ***anti-symmetric relation***
- For a relation R in A, if $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$, then R is a ***transitive relation***
- A relation in a set is an ***equivalence relation*** if it is reflexive, transitive and symmetric
- $D = \{a \mid a \in A, (a,b) \in R\}$, the set of all first elements of the ordered pairs which belong to the relation from A to B, is the ***domain*** of the relation.
- $E = \{b|b\in B, (a,b) \in R\}$, the set of all second elements of the ordered pairs which belongs to the relation from A to B, is the ***range*** of the relation

6.0 TUTOR-MARKED ASSIGNMENT

1. Consider the relation R = {(1,5), (4,5), (1,4), (4,6), (3,7), 7,6)} Find (1) the domain R, (2) the range of R, (3) the inverse R.

2. Le E= {1,2, 3}. Consider the following relation in E.

7.0 REFERENCE AND FURTHER READINGS

Seymour Lipschutz; Schaum's Outline Series: Theory and Problems of Set Theory and related topics, 1964,pp. 1 – 133.

Sunday O. Iyahen; Introduction to Real Analysis (Real-valued functions of a real variable, 1998, Vol. 1)

UNIT 2 FURTHER THEORY OF SETS

CONTENTS

1.0 INTRODUCTION

In this unit, we lay a basic foundation of a branch of mathematics (Logic) that studies laws associated with the set operations; intersection, union and complement. You will do well to follow closely the reasoning presented in the text.

2.0 OBJECTIVES

After going through this unit, you should be able to do the following:

- Prove identities using the table of laws of the algebra of sets
- Find the dual of any identity
- Generalize the operations of union and intersections of sets
- Find the possible partitions of a set
- State the relationship between Equivalence relations and Partitions

3.0 MAIN BODY

3.1 Algebra of Sets

Set under the operations of union, intersection and complement satisfy various laws, i.e. identities. Below is a table listing laws of sets, most of which have already been noted and proven in unit 2. One branch of mathematics investigate the theory of set by studying those theorems that follow from these laws, i.e. those theorems whose proofs require the use of only these laws and no others. We will refer to the laws in Table 1 and their consequences as the algebra of sets.

LAWS OF THE ALGEBRA OF SETS			
	Idempotent Laws		
1a.	$A \cup A = A$	1b.	$A \cap A = A$
	Associative Laws		
2a.	$(A \cup B) \cup C = A \cup (B \cup C)$	2b.	$(A \cap B) \cap C = A \cap (B \cap C)$
	Commutative Laws		
3a.	$A \cup B = B \cup A$	3b.	$A \cap B = B \cap A$
	Distributive Laws		
4a.	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	4b.	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	Identity Laws		
5a.	$A \cup \varnothing = A$	5b.	$A \cap U = \varnothing$
6a.	$A \cup U = U$	6b.	$A \cap \varnothing = \varnothing$
	Complement Laws		
7a.	$A \cup A' = U$	7b.	$A \cap A' = \varnothing$
8a.	$(A')' = A$	8b.	$U' = \varnothing$, $\varnothing' = U$
	De Morgan's Laws		
9a.	$(A \cup B)' = A' \cap B'$	9b.	$(A \cap B)' = A' \cup B'$

Table 1

Notice that the concept of “element” and the relation “ a belongs to A” do not appear anywhere in table 1. Although, these concepts where essential to our original development of the theory of sets, they do not appear in investigating the algebra of sets. The relation “A is a subset of B” is defined in our algebra of sets by.

$$A \subset B \text{ means } A \cap B = A$$

As examples, we now prove two theorems in our algebra, of sets, that is, we prove two theorems, which follow directly from the laws in Table 1. Other theorems and proofs are given in the problem section.

Example 1.1 Prove: $(A \cup B) \cap (A \cup B') = A$

	Statement	***Reason***
1.	$(A\cup B) \cap (A\cup B') = A\cup(B\cap B')$	1.Distributive Law
2.	$B \cap B' = \varnothing$	2. Substitution
3.	$\therefore\ (A\cup B)\cap(A\cup B') = A\cup\varnothing$	3. Associative Law
4.	$A \cup\varnothing = A$	4. Substitution
5.	$\therefore\ (A\cup B)\cap(A\cup B') = A$	5. Definition of subset

Example 1.2: Prove $A \subset B$ and $B \subset C$ implies $A \subset C$.

	Statement	***Reason***
1.	$A = A \cap B$ and $B = B \cap C$	1. Definition of subset
2.	$\therefore\ A = A \cap (B \cap C)$	2. Substitution
3.	$A = (A \cap B) \cap C$	3. Associative Law
4.	$\therefore\ A = A \cap C$	4. Substitution
5.	$\therefore\ A \subset C$	5. Definition of subset

Principle of Duality

If we interchange $\cap$ and $\cup$ and $\varnothing$ in any statement about sets, then the new statement is called the dual of the original one.

Example 2.1: The dual of

$$(U \cup B) \cap (A \cup \varnothing) = A$$

is $(\varnothing \cap B) \cup (A \cap U) = A$

Notice that the dual of every law in Table 1 is also a law in Table 1. This fact is extremely important in view of the following principle:

Principle of Duality: If certain axioms imply their own duals, then the dual of any theorem that is a consequence of the axioms is also a consequence of the axiom. For, given any theorem and its proof, the dual of the theorem can be proven in the same way by using the dual of each step in the original proof.

Thus, the principle of duality applies to the algebra of sets

Example 2.2: Prove: $(A \cap B) \cup (A \cap B') = A$

The dual of this theorem is proven in Example 1.1; hence this theorem is true by the Principle of Duality.

3.2 Indexed Sets

Consider the sets

$A_1 = \{1, 10\}$, $A_2 = \{2, 4, 6, 10\}$, $A_3 = (3, 6, 9\}$, $A_4 = \{4, 8\}$, $A_5 = \{5, 6, 10\}$

And the set

$$I = \{1, 2, 3, 4, 5\}$$

Notice that to each element $i \in I$ there corresponds a set A_i. In such a situation I is called the ***index set,*** the sets $(A_1, \ldots, A_5)$ are called the ***indexed sets,*** and the subscript i of A_i, i.e. each $i \in I$, is called an ***index***. Furthermore, such an indexed family of sets is denoted by

$$(A_i)_{i \in I}$$

We can look at an indexed family of sets from another point of view. Since to each element $i \in I$ there is assigned a set A_i, we state

Definition 7.1: An indexed family of sets (A_i) $i \in I$ is a function

$f: I \to A$

Where the domain of f is the index set I and the range of f is a family of sets.

Example 3.1: Define $Bn = \{x \mid 0 \leq x \leq (1/n)\}$, where $n \in N$, the natural numbers. Then

$B_1 = [0,1], B_2 = [0,1/2], \ldots$

Example 3.2: Let/ be the set of words in the English Language, and let $i \in I$. Define

$W_1 = \{x \mid x \text{ is a letter in the word } i \in I\}$.

If i is the word "follow" , then $W_i = \{f, 1, o, w\}$.

Example 3.3: Define $D_n = \{x \mid x \text{ is a multiple of } n\}$, where $n \in N$, the natural numbers. Then

$D_1 = \{1, 2, 3, 4, \ldots, D_2 = \{2, 4, 6, 8, \ldots\}, D_3 = \{3, 6, 9, 12, \ldots\}$

Notice that the index set N is also D_1 and also the universal set fro the indexed sets.

Remark 7.1: Any family of B of sets can be indexed by itself. Specifically, the identify function $i: B \to B$

is an indexed family of sets

$\{A_i\}_{1 \in B}$

Where $A_{i \in B}$ and where $i = A_i$. In other words, the indexed of any set in B is the set itself

3.3 Generalized Operations

The operation of union and intersection were defined for two sets. These definitions can easily be extended, by induction, to a finite number of sets. Specifically, for sets $A_1, \ldots A_n,$

$$Y^{u}{}_{i=1} A_I \equiv A_1 \cup A_2 \cup K \cup A_n$$
$$I^{n}{}_{i=1} A_i \equiv A_1 \cap A_2 \cap K \cap A_n$$

In view of the associative law, the union (intersection) of the sets may be taken in any order; thus parentheses need not be used in the above.

These concepts are generalised in the following way. Consider the indexed family of sets

$$\{A_i\}_{i \in I}$$

And let $J \subset I$. Then $\cup_{i \in J} A_i$

Consists of those elements which belong to at least one A_i where $i \in J$. Specifically,

$$\cup_{i \in J} A_i = \{ x \mid \text{there exists an } _{i \in J} \text{. such that } x \in A_i \}$$

In an analogous way $\cup_{i \in J} A_i$

consist of those elements which belong to every A_i for $_{i \in J}$. In other words,

$$\cap_{i \in J} A_i = \{x \mid x \in A_i \text{ for every } i \in J\}$$

Example 4.1: Let $A_1 = \{1, 10\}$, $A_2 = \{2, 4, 6, 10\}$, $A_3 = \{3, 6, 9\}$, $A_4 = \{4, 8\}$, $A_5 = \{5, 6, 10\}$; and let $J = \{2, 3, 5, \}$. Then
$\cap_{i \in J} A_i = \{6\}$ and $\cup_{i \in J} A_i = \{2, 4, 6, 10, 3, 9, 5\}$

Example 4.2: Let $B_n = [0, i/n]$, where $n \in N$, the natural numbers. Then $\cap_{i \in N} B_i = \{0\}$ and $\cup_{i \in N} B_i = [0, 1]$

Example 4.3: Let $D_n = \{x \mid x \text{ is a multiple of } n\}$, where $n \in N$, the natural numbers.

Then $\cap_{i \in N} D_1 = \varnothing$

There are also generalised distributive laws for a set B and an indexed family of sets $\{A_i\}_{i \in I}$ be an indexed family of sets. Then for any set B,

$$B \cap (\cup_{i \in I} A_i) = \cup_{i \in I} (B \cap A_i)$$
$$B \cup (\cap_{i \in I} A_i) = \cap_{i \in I} (B \cup A_i)$$

3.4 Partitions

Consider the set A = { 1, 2, . . . 9, 10} and its subsets
$B_1 = \{1, 3\}$, $B_2 = \{7, 8, 10\}$, $B_3 = \{2, 5, 6\}$.$B_4 = 4, 9\}$
The family of sets B = $\{B_1, B_2, B_3, B_4,\}$ has two important properties.

1. A is the union of the sets in B, i.e.,
   $A = B_1 \cup B_2 \cup B_3 \cup B_4$
2. For any sets B_i and B_j,
   Either $B_i = B_j$ or $B_i \cap B_j = \varnothing$

Such a family of sets is called ***partition*** of A. Specifically, we say

Definition 7.2:

Let $\{B_i\}_{i \in I}$ be a family of non-empty subsets of A. Then $\{B_i\}_{i \in I}$ is called a ***partition*** of A if

P_1: $\cup_{i \in I} B_i = A$
P_2: For any B_i, B_j, either B_j or $B_i \cap B_j = \varnothing$

Furthermore, each B_i is then called an ***equivalence class*** of A.

Example 5.1: Let N = {1, 2, 3, . . .}, E = {2, 4, 6, . . .} and F = {1, 3, 5}. Then {e, F} is a partition of N.

Example 5.2: Let T = {1, 2, 3,. . ., 9, 10} and let A = {1, 3, 5}, B = {2, 6, 10} and C = {4, 8, 9}. Then {A, B, C} is not a partition of T since

$T \neq A \cup B \cup C$

i.e. since $7 \in T$ but $7 \neq (A \cup B \cup C)$.

Example 5.3: Let T = {1, 2, …, 9, 10}, and F = {1, 3, 5, 7, 9}, and G = {2, 4, 10} and H = {3, 5, 6, 8}. Then {f, G, H} is not a partition of T since

$$F \cap H \neq \varnothing, F \neq H$$

Example 5.4: let y_1, y_2, y_3 and y_4 be respectively the words "follow", "thumb", "flow" and "again", and let

A = {w, g, u, o, I, m, a, t, f, n, h, b}
Furthermore, define

W_i = {x |x is the letter in the word y_i}

Then {W_i W_2 W_3 W_4} is the partition of A. notice that W_1 and W_3 are not disjoint, but there is no contradictions since the sets are equal.

3.5 Equivalence Relations and Partition

Recall the following

Definition: A relation in a set A is an equivalent relation if**:**

1. R is reflexive, i.e. for every a ∈ A, a is related to itself;
2. R is symmetric, i.e. if a is related to b then b is related to a;
3. R is transitive, i.e. if a is related to be and b is related to c then a is related to c.

The reason that partition and equivalence relations appear together is because of the

Theorem 7.2: Fundamental Theorem of Equivalence Relations: Let R be an equivalence relation in a set A and for, every a ∈ A, let

$$B_\infty = \{x \mid (x, \alpha) \in R)$$

i.e. the set of elements related to α. Then the family of sets#

$$\{B_\alpha\}_{\alpha \in A}$$

is a partition of A.

In order words, an equivalence relation R in a set A partition the set A by putting those elements which are related to each in the same equivalence class.

Moreover, the set B_α is called ***equivalence class*** determined by α, and the set of equivalence classes $\{B_\alpha\}_{\in \alpha A}$ is denoted by

$$A/R$$

And called ***quotient set.***

The converse of the previous theorem is also true. Specifically,

Theorem 7.3: Let $\{B_i\}_{i \in I}$ be a partition of A and let R be the relation in A defined by the open sentence "x is in the same set (of the family $\{B_i\}_{i \in I}$ as y". Then R is an equivalence relation in A.

Thus there is a one to one correspondence between all partitions of a set A and all equivalence relations in A.

Example 6.1: In the Euclidean plane, similarities of triangles is an equivalence relation. Thus all triangles in the plane are portioned into disjoint sets in which similar triangles are elements of the same set.

Example 6.2: Let R_5 be the relation in the integers defined by

$$\mathbf{x \equiv y} \text{ (mod 5)}$$

which reads "x is congruent to y modulo 5" and which means "x– y is divisible equivalence classes in Z/R_5: E_0, E_1, E_2, E_3 and E_4. Since each integer x is uniquely expressible in the form $x = 5p + r$ where $0 \le r < 5$, then x is a member of the equivalence class E_r where r is the remainder.

Thus

$$E_0 = \{. . ., -10, -5, 0, 5, 10, . . .\}$$
$$E_1 = \{. . . , -9, -4, 1, 6, 11, . . .\}$$
$$E_2 = \{. . . , -8, -3, 2, 7, 12, . . . \}$$
$$E_3 = \{. . . , -7, -2, 3, 8, 13, . . \}$$
$$E_3 = \{. . . , -6, 1- , 4, 9, 14, . . .\}$$

Add the quotient set $Z/R_5 = \{E_0, E_1, E_2, E_3, E_4\}$

4.0 CONCLUSION

You are gradually being introduced to using set notation rather than statements. Theorems in the algebra of sets are most useful in proving identities related to logic and reasoning, in most cases using the principle of duality.

5.0 SUMMARY

In investigating the algebra of sets you need to take note of the dual of the statements in table 1.

An indexed family of sets, $\{A_i\}_{i\in I}$ is such that for each index i = 1, 2, 3, 4, . . . we have sets $A_1, A_2, A_3, . . .$

Let $\{B_i\}_{i\in I}$ be a family of non- empty subsets of A. then $\{B_i\}_{i\in I}$ is called ***partition*** of A if $\cup_{i\in I} B_i = A$ and for any B_i B_j, either $B_i = B_j$, or $B_i, \cap B_j = \varnothing$

Furthermore, each B_i, is then called an ***equivalence class*** of A.

6.0 TUTOR MARKED ASSIGNMENTS

1. Write the dual of each of the following:
   1. $(B \cup C) \cap A= (B \cap A) \cup (C \cap A)$
   2. $A \cup (A' \cap B)= A \cup B$

3. $(A \cap U) \cap (\varnothing \cup A') = \varnothing$

2 Prove: $(A \cap B) \cup (A \cap B') = A$.

3. Let $B_i = [i, i + 1]$, where I ε Z, the integer. find

1. $B_1 \cup B_2$
2. $B_3 \cup B_4$
3. $\cup^{18}_{i=7} B_i$
4. $\cup_{i \varepsilon Z} B_i$

4. Let A = {a, b, c, d, e, f, g}. State whether or not each of the following families of sets is a partition of A.

1. $\{B_1 = \{a, c, e\}, B_2 = \{b\}, B_3 = \{d, g\}\}$
2. $\{C_1 = \{a, e, g\}, C_2 = \{c, d\}, C_3 = \{b, e, f\}\}$
3. $\{D_1 = \{a, b, e, g\}, D_2 = \{c\}, D_3 = \{d, f\}\}$
4. $\{E_1 = \{a, b, c, d, e, f, g\}\}$

7.0 REFERENCES AND FURTHER READINGS

Seymour Lipschutz; Schaum's Outline Series: Theory and Problems of Set Theory and related topics, 1964. pp. 1 – 133.

Sunday O. Iyahen; Introduction to Real Analysis (Real-valued functions of a real variable, 1889 Vol. 1

UNIT 3 FURTHER THEIRY OF FUNCTIONS, OPERATIONS

CONTENTS

1.0 INTRODUCTION

There are some further concepts you have to become familiar with now which you will come across in mathematical analysis and abstract mathematics. This unit introduces you to some of them. Pay attention not only to the definitions but also to the examples given.

2.0 OBJECTIVES

At the end of this unit you should be able to:

- state whether or not a diagram of functions is cumulative from the arrows connecting the functions.
- Explain the terms Restriction and Extension of functions

- Describe the following: Set functions, Real-valued functions and its algebra, Characteristic function
- Apply the Rule of the Maximum Domain.
- Explain operations on Cartesian products

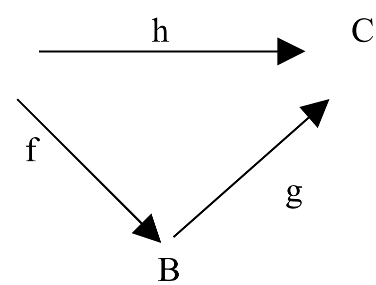
3.0 MAIN BODY

3.1 Functions and Diagrams

As mentioned previously, the symbol

A B →

Denotes a function of A into B. In a similar manner, the diagram



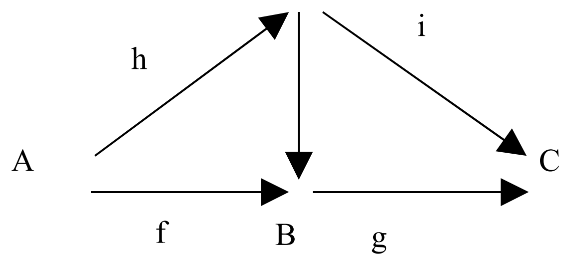
Consists of letters A, B and C denoting sets, arrows f, g, h denoting functions f: A → B, g: B → C and h: A → C, and the sequence of arrows {f, g} denoting the composite function g o f: A → C. Each of the functions

h: A →C and g o f: A → C, that is, each arrow or sequence of arrows connecting A to C is called a path from A to C.

Definition 8. 1: A diagram of functions is said to be cumulative if for any set X and Y in the diagram, any two paths from X to Y are equal.

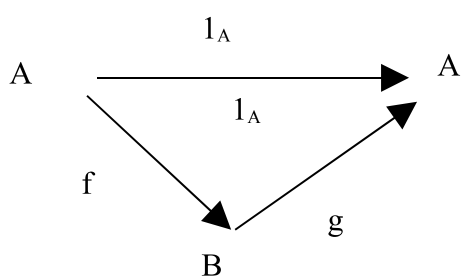
Example 1.1: Suppose the following diagram of functions is Cumulative.

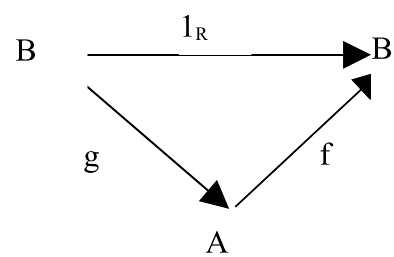
D



Then i o h = f, g o I = j and g o f = j o h = g o I o h.

Example 1.2: The functions f: A $\rightarrow$ B and g: B $\rightarrow$ A are inverses if and only if the following diagrams are cumulative:





Here 1_A and 1_B are the identity functions.

3.2 Restrictions And Extensions Of Functions

Let f be a function of A into C, i.e. let f: A$\rightarrow$ C, and let B be a subset of A. Then f induces a function f': B $\rightarrow$ C which is defined by

$$F(b) = f(b)$$

For any b$\in$B. the function f' is called the restriction of F to B and is denoted by

$$F \mid B$$

Example 2.1: Let f: $\Re \rightarrow \Re$ be defined by $f(x) = x^{2}$. Then

$$F|N = \{(1, 1), (2,4), (3,9), (4, 16), \ldots\}$$

Iis the restriction of to N, the natural numbers.

Example 2.2: The set g = {(2, 5), (5, 1), (3, 7), (8, 3), (9, 5)} is a function from {2, 5, 3, 8, 9} into N. Then

$$\{(2, 5), (3, 7), (9, 5)\}$$

a subset of g, is the restriction of g to {2, 3, 9}, the set of first elements of the ordered pairs in g.

We can look at this situation from another point of view. Let f: $A \rightarrow C$ and let B be a superset of a. Then a function F: $B \rightarrow C$ is called an **extension** of f if, for every $a \in A$,

$$F(a) = f(a)$$

Example 2.3: Let f be the function on the positive real number defined by f(x) = x, that is, let the identity function. Then the absolute value function

$$|X| = \begin{cases} X \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Is an extension of f to all real numbers.

Example 2.4: **C**onsider the function

$$F = \{(1, 2), (3, 4), (7,)\}$$

Whose domain is {1, 3, 7}. Then the function

$$F = \{(1, 7), (3, 4), (5, 6), (7, 2)\}$$

Which is a superset of the function f, is an extension of f.

3.3 Set Functions

Let f be a function of A into B and let T be a subset of A, that is,
$A \xrightarrow{t} B$ and $T \subset A$.

Then

$$F(T)$$

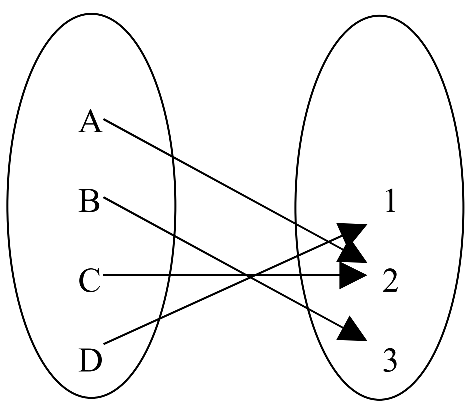
Which is read "f of T", is defined to be the set of image point of elements in T. In other words,

$$F(T) = \{x \mid f(a) = x, a \in T, x \in B\}$$

Notice that f(T) is a subset of B.

Example 3.1: Let A = {a, b, c, d}, T = {b, c} and B = {1, 2, 3}.

Define $f: A \rightarrow B$ by



Then f (T) = {2, 3}.

Example 3.2: Let $g: \Re \rightarrow \Re$ be defined by $g(x) = x^2$, and let T = [3, 4]. Then

$$G(T) = [9, 16] = \{x \mid 9 \le x \le 16\}.$$

Now let A be the family of subsets of A, and let B be the family of subsets of B, if f: A $\rightarrow$ B, then f assigns to each set T $\in$ A a unique set of f(T) $\in$ B. In other words, the function f: A $\rightarrow$ B induces a function f: A $\rightarrow$ B. Although the same letter denotes each function, they are essentially two different functions. Notice that the domain of f: A $\rightarrow$ B consists of sets.

Generally speaking, a function is called a **set function** if its domain consists of sets.

3.4 Real-Valued Functions

A function f: A$\rightarrow \Re$, which maps a set into the real numbers, i.e. which assigns to each a $\in$ A a real number f(a) $\in \Re$ is **a real-valued function.** Those functions which are usually studied in elementary mathematics, e.g.,

$$P(x) = a_o x^n + a_{1x}{}^{n-1}x + a_n$$
$$t(x) = \sin x, \cos x \text{ or } \tan x$$
$$f(x) = \log x \text{ or } e^x$$

That is, polynomials, trigonometric functions and logarithmic and exponential functions are special examples of real-valued functions.

3.5 Algebra And Real-Valued Functions

Let F_D be the family of all real-valued functions with the same domain D. Then many (algebraic) operations are defined in $F_{D.}$ Specifically, let f: D $\Re$ and g: D $\rightarrow \Re$, and let k $\in \Re$. Then each of the following functions is defined as follows:

(f + k):	$D \rightarrow \Re$	by	(f + k)(x) = f(x) + k
($\lvert f \rvert$):	$D \rightarrow \Re$	by	($\lvert f \rvert$)(x) = $\lvert f(x) \rvert$
(f”):	$D \rightarrow \Re$	by	(f“)(x) = (f(x))”
(f ± g):	$D \rightarrow \Re$	by	(f ± g)(x) =f(x) ± g(x)
(kf):	$D \rightarrow \Re$	by	(kf)(x) = k(f(x))

(fg): $D \to \Re$ by $(fg)(x) = f(x)g(x)$

(f/g): $D \to \Re$ by $(f/g)(x) = f(x)/g(x)$

where $g(x) \neq 0$

Note that (f/g): $D \to \Re$ is not the same as the composition function which was discussed previously.

Example 4.1: Let D = {a, b}, and let f: $D \to \Re$ and g: $D \to \Re$ be defined by :

F(a) = 1, f(b) = 3 and g(a) = 2, g(b) = -1
In other words,

F = {(a, 1), (b, 3) and g = {a, 2), (b, -1)}
Then

(3f – 2g)(a) = 3f(a) – 2g(a) = 3(1) -2(2) =-1
(3f – 2g)(b) = 3f(b) – 2g(b) 3(3) – g(-1) = 11

that is, 3f – 2g = {(a, 1), (b, 11)}

Also since g (x) ≠ | g(x) | and |(g + 3)(x) = g(x) + 3,

|G| ={(a, 2), (b, 1)} AND G + 3 = {{(A, 5), (b, 2)}

Example 4.2: Let f: $\Re \to \Re$ and g: $\Re \to \Re$ be defined By the formulas

F(x) = 2x = - 1 and g(x) =^{2}
The formulas which define the function (3f – 2g): $\Re \to \Re$ and (fg): $\Re \to \Re$ are found as follows:

$$(3f - 2g)(x) = 3(2x - 1) -2(x^2) = - 2x^2 + 6x - 3$$
$$(fg)(x) = (2x - 1)(x^2) = 2x^2 - x^2$$

3.6 Rule Of The Maximum Domain

A formula of the form

$$F(x) = 1/x,\ g(x) = \sin x,\ h(x) = \sqrt{x}$$

Does not in itself, define a function unless there is given, explicitly or implicitly, a domain, i.e. a set of numbers, on which the formula then defines a function. Hence the following expressions appear:

Let $f(x) = x^2$ be define on [- 2, 4].
Let $g(x) = \sin x$ be defined for $0 \leq x \leq 2x$

However, if the domain of a function defined by a formula is the maximum set of real numbers for which the formula yield a real number, e.g.,

Let $f(x) = 1/X$ for $X \neq 0$

Then the domain is usually not stated explicitly. This convention is sometimes called the rule of the maximum domain.

Example 5.1: Consider the following functions

$F_1(x) = x^2$ for $x \geq 0$
$F_2(x)\ 1/(x - 2)$ for $x \neq 0$
$F_3(x) = \cos x$ for $0 \leq x \leq 2\pi$
$F_4(x) = \tan x$ for $\neq \pi/2 + n\pi,\ n\in N$

The domains of f_2 and f_4 need not have been explicitly
Stated since each consists of all those numbers for
Which the formula has meaning, that is, the functions
Could have been defined by writing

$F_1(x) = x^2$ and $f_4(x) = \tan x$

Example 5.2: Consider the function $f(x) = \sqrt{1 - x^2}$; its domain, unless otherwise stated, is [- 1, 1]. Here we explicitly

Assume that the co-domain is $\Re$.

3.7 Characteristic Functions

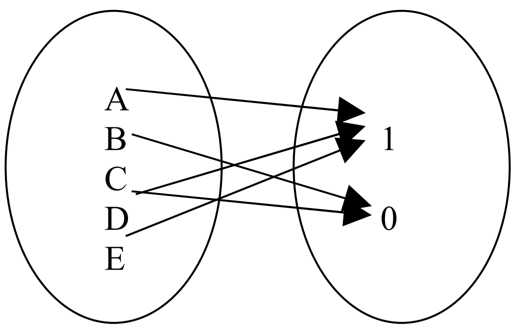
Let A be any subset of a universal set U. Then the real-valued function.

$$X_A : U \to \{1 , 0\}$$

defined by $x_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \neq A \end{cases}$

is called the characteristic function of A.

Example 6.1: Let U = {a, b, c, d, e} and A = {s, f, e}. Then the function of U into {1, 0} defined by the following diagram



Is the characteristic function $x_{A \text{ of}}$ A

Note further that any function f: U $\to$ {1, 0} defines a subset

$$A_f = \{x \ x \in U, f(x) = 1\}$$

Of U and that the characteristic function X_{Af} of A_f is the original function f. Thus there is a one to-one correspondence between all subsets of U, i.e. the power set of U, and the set of all functions of U into {1, 0}.

3.8 Operations

You are familiar with the operations of addition and multiplication of numbers, union and intersection of sets, and composition of functions. These operations are denoted as follows:

$A + b = c$, $a, b = c$, $A \cup B = C$ $A \cap B = C$, $g^o f = h$

In each situation, an element (c, C or h) is assigned to an original pair of elements. In other words, there is a function that assigns an element to each ordered pair of elements. We now introduce

Definition 8.1: An operation α on a set A is a function of the Cartesian product A x A into A, i.e.,

Remark 8.2: The operation α A x A $\rightarrow$ A is sometimes referred to as a **binary operation** and an n-ary operation is defined to be a function

$$\alpha : A \ldots A \rightarrow A$$

We shall continue to use the word operation instead of binary operation

3.8.1 Cumulative Operations

The operation α: A x A $\rightarrow$A is called cumulative if, for every a, b $\in$ A,

$$\alpha (a, b) = \alpha (b, a)$$

Example 7.1: Addition and multiplication of real numbers are cumulative operations since

$A + b = + a$ and $ab = ba$

Example 7.2: Let α: $\Re$ x $\Re \rightarrow \Re$ be the operation of subtraction defined by α: (x, y) $x - y$
Then

$$\alpha(5,1) = 4 \quad \text{and} \quad \alpha(1, 5) = - 4$$

Hence subtraction is not a cumulative operations, since

$A \cup B = B \cup A$ and $A \cap B = B \cap A$

3.8.2 Associative Operations

The operation α: A x A $\rightarrow$ A is called associative if, for every a, b, c $\in$ A.

$$\alpha(\alpha(a, b), c) = \alpha(a, \alpha(b, c))$$

In other words, if $\alpha(a, b)$ is written $a * b$, then α is associative if

$$(a * b) * c = a * (b * c)$$

Example 8.1: Addition and multiplication of real numbers are associative operations, since

$(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$

Example 8.2: Let $\alpha:\Re \times \Re \rightarrow \Re$ be the operation of division defined by α: (x, y) x/y
Then

$$\alpha\,(\alpha(12, 6), 2) = \alpha(2, 2) = 1$$
$$\alpha(12, \alpha(6, 2)) = \alpha 12, 3) = 4$$

Hence division is not an associative operation.

Example 8.3: Union and intersection of sets are associative operations, since

$(A \cup B\,) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$

3.8.3 Distributive Operations

Consider the following two operations

$$\alpha: A \text{ x } A \rightarrow A$$
$$\beta: A \text{ x } A \rightarrow A$$

The operation α is said to be distributive over the operation β if, for every a, b, c $\in$ A.

$$\alpha(a\beta(b, c) = \beta(z, b), \alpha(a, c))$$

In other words, if $\alpha(a, b)$ is written $a * b$. and $\beta(a, b)$ is written $a \Delta b$, then α distributes over β if

$$A * (b \Delta c) = (a * b) \Delta (a * c)$$

Example 9.1: The operation of multiplication of real numbers distributes over the operation of addition, since

$$A(b + c) = ab + ac$$

But the operation of addition of real numbers does not distribute over the operation of multiplication, since

$$A + (bc) \neq (a + b)(a + c)$$

Example 9.2: The operation of Union and intersection of sets distribute over each other since

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cup (B \cup C) = (A \cap B) \cup (A \cap C)$$

3.8.4 Identity Elements

Let α: A x A $\rightarrow$ A be an operation written $\alpha(a, b) = a * b$. An element c $\in$ A is called an identity element for the operation α if, for every a $\in$ A.

$$C * a = a * c = a$$

Example 10.1: Let $\alpha:\Re \times \Re \rightarrow \Re$ be the operation of addition. Then 0 is an identity element for addition since, for every real number $a \in \Re$.

$$0 * a = a * 0 = a, \quad \text{that is, } 0 + a = a + 0 = a$$

Example 10;2: Consider the operation of intersection of sets. Then U, the universal set is an identity element, since for every set A (which is a subset of U),

$$U * A = A * U = Am \quad \text{that is, } U \cap A = \mathbf{A} \cap U = A$$

Example 10.3: Consider the operation of multiplication of Real numbers. Then the number 1 is an identity element since, for every real number a,

$$1 * a = a * 1 = a, \quad \text{that is, } 1 \bullet a = a \bullet 1 = a$$

Theorem 8. 1: If an operation α: A x A $\rightarrow$ A has an identity element.

Thus we can speak of the identity element for an operation instead of an identity element.

3.8.5 Inverse Elements

Let α: A x A $\rightarrow$ A be an operation written $\alpha(a, b) = a * b$, and let $e \in A$ be the identity element for α. Then the inverse of an element $a \in A$, denoted by

$$a^{-1}$$

is an element in A with the following property:

$$^{-1} * a = a * a^{-1} = e$$

Example 11.1: Consider the operation of addition of real numbers for which 0 is the identity element. Then, for any real number a, its negative (- a) is its additive inverse since

$- a * a = a * - a = 0m$ that is, $(- a) + a = a + (- a) = a - a = 0$

Example 11.2 Consider the operation of multiplication of rational numbers, for which 1 is the identity element. Then for any non-zero rational number p/q, where p and \ q are integers, its reciprocal q/p is its multiplicative inverse, since

$$(q/p)(p/q) = (p/q)(q/p) = 1$$

Example 11.3 Let α: N x N $\rightarrow$ N be the operation of multiplication for which 1 is the identity element; here N is the set of natural numbers. Then 2 has no multiplicative inverse, since there is no element $x \in N$ with the property

$$X \bullet 2 = 2 \bullet x = 1$$

In fact, no element in N has a multiplicative inverse except 1 which has itself as an inverse.

3.9 Operations And Subsets

Consider an operation α: A x A $\rightarrow$ A and a subset B of A. Then B is said to be closed under the operation of α if, for every b, b' $\in$ B,

$$\alpha(b, b') \in B$$

that is, if $\alpha(B \times B) \subset B$

Example12.1 Consider the operation of addition of natural numbers. Then the set of even numbers id closed under the operation of addition since the sum of any two even numbers is always even. Moreover, the set of odd numbers is not closed under the operation of addition since the sum of two odd numbers is not odd.

Example 12.2: The four complex numbers 1, - 1, I, - I are closed under the operation of multiplication.

4.0 CONCLUSION

In mathematical analysis and abstract mathematics, the unit is a prerequisite knowledge. Functions and diagrams go han in hand. Set functions, real-valued functions, characteristic functions are basic functions in analysis

5.0 SUMMARY

See if you recall the following:

- The function f' is called a restriction of f to B (f B) of f: A $\rightarrow$ C, if, given B a subset of A, f induces a function a function f': B $\rightarrow$ C which is defined by

  F'(b) = f(b)

  For any $b \in B$.

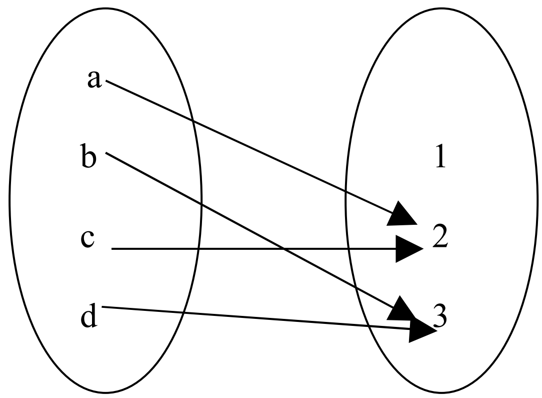
- Let f: A $\rightarrow$ C and let B be a superset of A. Then a function F: B $\rightarrow$ C is called an extension of f if, for every $a \in A$.

$$f(a) = f(a)$$

- A function is called a set function if its domain consists of sets
- A function on a a set that maps the elements of that set into the real numbers is called a real-valued function.
- The rule of Maximum domain is used to define domains which need not be stated explicitly since it is the maximum set of real numbers for which a function yields a real number.

6.0 TUTOR MARKED ASSIGNMENTS

1 Let W = {a, b, c, d}; v = {1, 2, 3}, and let f: W $\rightarrow$ V be defined by the adjoining diagram. Find: (1) f({a, b, d}), (2) f({a, c}).



2. Find the domain of each of the following real-valued functions:

   1. $f_1(x) = 1/x$ where $x > 0$

   2 $f_2(x) = \sqrt{3 - x}$

   3. $f_3(x) = \log(x - 1)$

   4 $f_4(x) = x^2$ where $0 \leq x \leq 4$

   5. Consider the real-valued function $f = \{(1, 2), (2, -3), (3, -1)\}$ (with domain $\{1, 2, 3\}$). Find (1) $f + 4$, (2) $|f|$, (3) f^2,

7.0 REFERENCES AND FURTHER READINGS

Seymour Lipschutz: Schaum's Outline Series: Theory And Problems of set theory and related topics, 1964, Pp 1 -133.

Sunday O. Iyahen: Introduction to real Analysis (real-Valued functions of a real variable, 1998, Vol.1