NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 422

COURSE TITLE: PARTIAL DIFFERENTIAL EQUATIONS

MTH 422 PARTIAL DIFFERENTIAL EQUATIONS

COURSE GUIDE

NATIONAL OPEN UNIVERSITY OF NIGERIA

Course Code MTH 422 PARTIAL DIFFERENTIAL EQUATIONS

Course Title Partial Differential Equations

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NATIONAL OPEN UNIVERSITY OF NIGERIA

**Contents**

## INTRODUCTION

Partial differential equations are used to describe and identically formalise a wide range of real world events like sound, heat, electrostatics, electromagnetism, electrodynamics, fluid flow and mechanical displacement to show that they are governed by the same underlying dynamic; and presumably you must have at an earlier time become familiar with basics in the prerequisite to this course. Through this course therefore, you will be encouraged to develop an enquiring attitude towards Partial Differential Equations and relate the lessons learnt to the universe around you and which abounds with ceaseless applications of Partial Differential Equations.

It is the objectives of this course to build upon the lessons learnt in the prerequisite course, and formally to introduce to you the more advanced concept of Partial Differential Equations with the view to greater strengthening your understanding of the underlying principles at work upon which developmental research in this highly specialised area of mathematics are based,

## THE COURSE: MTH 422 Partial Differential Equations

This course comprises a total of seven Units distributed across four modules as follows:

Module 1 is composed of 2 Units
Module 2 is composed of 2 Units
Module 3 is composed of 2 Units
Module 4 is composed of 1 Unit

Module 1 Unit 1 will bring you up to speed on the Essential Definitions pertinent to the study of Partial Differential Equations. In this unit we will show you how to recognize First Order Equation, Quasi – Linear Equations and use the Method of Lagrange in solving them. In Unit 2 we will guide you through a test case scenario where you shall apply Partial Differential Equation to the formulation of the Conservation Law and ultimately derive the conceptual Development of Shock.

Module 2 will allow you to work with General First Order Equation in Unit 1 focusing on the Cauchy Method of Characteristic for Partial Differential Equation. In Unit 2 you will be shown that there are three Types of Solution; the Complete Solution (Integral), the General Solution (Integral) and the Singular Solution respectively.

Module 3 will allow us move up to higher order Partial Differential Equations; specifically Second Order Partial Differential Equations in Unit 1 where you shall learn about their Classifications and the Tricomi's Equation, its Characteristics and learn about the Case of Two Independent Variables. We shall see that expediency at times dictates that we apply Transformation of Independent Variables in arriving at a solution for Partial Differential Equations in Unit 2 with the view to recognizing the Regular Case, the Hyperbolic case and the Elliptic Case.

Unit 1 of Module 4 brings to the fore the Cauchy Problem which you shall learn; as well as the Characteristics Problem, the Fundamental Existence Theorem and you shall complete this course only after knowing and understanding the Kovalevsky theorem.

## COURSE AIMS AND OBJECTIVES

Our aim through MTH 422 is to further deepen your understanding of Partial Differential Equations and acquaint you with the graphical and mathematical significance of Partial Differential Equations through calculations and examples which lets you establish the practicable applications and indispensability of Partial Differential Equations in the world we live in.

On your part we expect you in turn to conscientiously and diligently work trough this course upon completion of which you should be able to:

- Properly define the term partial differential equation
- Classify first order equations
- Investigate the methods for constructing solutions for partial differential equations
- Solve quasi – Linear Equations
- Explore the many definitions applied in deriving solutions
- Apply the method of Lagrange in deriving solutions for partial differential equations
- Study Conservation Law
- Understand the concept of shock
- Study in detail the class referred to as General non – linear First Order Equation
- Sketch and explain the Monge cone
- Apply Cauchy Method of Characteristic equations
- Categorise the different types of solution of partial differential equations
- Know the methods used in deriving complete solution
- Understand what a general solution means
- Explain why some partial differential equations are singular solutions
- Classify Second Order Partial Differential Equations
- Understand the importance of the Eigen-values of the matrix of coefficients
- State Tricomi’s Equation
- Work with Laplace, heat and Wave Equations
- Study the special case of two independent variables

- Know how to transform independent variables
- Visit the theorems that apply to the regular case
- Solve equations of the hyperbolic types by transformations
- See why Elliptic equations have no real characteristics
- Cauchy Problem And Characteristics Problem
- Understand what the strip condition means
- Treat the fundamental existence theorem
- Take a critical look at Cauchy problem
- Solve using the Cauchy Kowalevski theorem

**WORKING THROUGH THE COURSE**

This course requires you to spend quality time to read. Whereas the content of this course is quite comprehensive, it is presented in clear mathematical language that you can easily relate to. The presentation style of this course adequate and the content easy to assimilate.

You should take full advantage of the tutorial sessions because this is a veritable forum for you to "rub minds" with your peers – which provides you valuable feedback as you have the opportunity of comparing knowledge with your course mates.

## COURSE MATERIAL

You will be provided course material prior to commencement of this course, which will comprise your Course Guide as well as your Study Units. You will receive a list of recommended textbooks which shall be an invaluable asset for your course material. These textbooks are however not compulsory.

## STUDY UNITS

You will find listed below the study units which are contained in this course and you will observe that there are four modules. Each module comprises two Units each, except for module 4 which has one Unit.

| | | |
|---|---|---|
| **<u>Module 1</u>** | Unit 1 | Essential Definitions<br>First Order Equation<br>Quasi – Linear Equations<br>Method of LaGrange |
| | Unit 2 | Application of IVP conservation law, development of shock |
| **<u>Module 2</u>** | Unit 1 | General First Order Equation<br>Cauchy Method of Characteristic |
| | Unit 2 | Types of Solution<br>Complete solution (integral)<br>General solution (integral)<br>Singular solution |
| **<u>Module 3</u>** | Unit 1 | Second order Partial Differential Equations<br>Classifications<br>The Tricomi's Equation<br>Characteristics<br>Case of Two Independent Variables |
| | Unit 2 | Transformation of Independent Variables<br>Regular case<br>Hyperbolic case<br>Elliptic case |
| **<u>Module 4</u>** | Unit 1 | Cauchy problem and characteristics problem<br>Fundamental Existence Theorem: Cauchy Problem<br>Cauchy Kowalevski theorem |

## TEXTBOOKS

There are more recent editions of some of the recommended textbooks and you are advised to consult the newer editions for your further reading.


Jost, J. (2002), Partial Differential Equations
New York: Springer-Verlag, ISBN 0-387-95428-7

Petrovskii, I. G. (1967), Partial Differential Equations
Philadelphia: W. B. Saunders Co

Pinchover, Y. & Rubinstein, J. (2005), An Introduction to Partial Differential Equations
New York: Cambridge University Press, ISBN 0-521-84886-5

Polyanin, A. D. (2002), Handbook of Linear Partial Differential Equations for Engineers and Scientists
Boca Raton: Chapman & Hall/CRC Press, ISBN 1-58488-299-9

Polyanin, A. D. & Zaitsev, V. F. (2004), Handbook of Nonlinear Partial Differential Equations
Boca Raton: Chapman & Hall/CRC Press, ISBN 1-58488-355-3

Polyanin, A. D.; Zaitsev, V. F. & Moussiaux, A. (2002), Handbook of First Order Partial Differential Equations
London: Taylor & Francis, ISBN 0-415-27267-X

Wazwaz, Abdul-Majid (2009). Partial Differential Equations and Solitary Waves Theory
Higher Education Press. ISBN 90-5809-369-7

Adomian, G. (1994). Solving Frontier problems of Physics: The decomposition method
Kluwer Academic Publishers

Courant, R. & Hilbert, D. (1962), Methods of Mathematical Physics II
New York: Wiley-Interscience

Evans, L. C. (1998), Partial Differential Equations, Providence:
American Mathematical Society, ISBN 0-8218-0772-2


## ASSESSMENT

Assessment of your performance is partly trough Tutor Marked Assessment which you can refer to as TMA, and partly through the End of Course Examinations.

## TUTOR MARKED ASSIGNMENT

This is basically Continuous Assessment which accounts for 30% of your total score. During this course you will be given 4 Tutor Marked Assignments and you must answer three of them to qualify to sit for the end of year examinations. Tutor Marked Assignments are provided by your Course Facilitator and you must return the answered Tutor Marked Assignments back to your Course Facilitator within the stipulated period.

## END OF COURSE EXAMINATION

You must sit for the End of Course Examination which accounts for 70% of your score upon completion of this course. You will be notified in advance of the date, time and the venue for the examinations which may, or may not coincide with National Open University of Nigeria semester examination.

## SUMMARY

Each of the four modules of this course has been designed to stimulate your interest in Partial Differential Equations through fundamental conceptual building blocks in the study and application of Partial Differential Equations to practical problem solving.

**Module 1** premises this course with the statement of the Essential Definitions to be found in the study of Partial Differential Equations and proceeds through First Order Equations, Quasi – Linear Equations and the Method of Lagrange. It is closed with a study of the Application of Partial Differential Equations to the Conservation Law which is developed to the concept of Shock – a consequence of an abrupt discontinuity.

**Module 2** treats General First Order Equation with particular reference to Cauchy Method of Characteristic. It presents the three Types Of Solution of Partial Differential Equations - Complete Solution (Integral), General Solution (Integral) and Singular Solution

**Module 3** focuses on the higher order Partial Differential Equations commencing with the Second Order. This module Classifies Partial Differential Equations an lays emphasis on the Tricomi's Equation, its Characteristics and treats the Case of Two Independent Variable. It further proceeds to the Transformation of Independent Variables where three cases are highlighted; the Regular, the Hyperbolic and the Elliptic Cases.

**Module 4** which closes the course takes on the Cauchy Problem and Characteristics Problem. It is partly occupied with the Fundamental Existence Theorem; the Cauchy problem as well as the Cauchy Kowalevski theorem

You will feel more at ease with Partial Differential Equations and life will never be the same again by the time you compete this course. In order to achieve this however, my advise are - make sure that you have enough referential and study material available and at your disposal at all times, and – devote sufficient quality time to your study.

I wish you luck.

# MTH 422: PARTIAL DIFFERENTIAL EQUATIONS

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# MODULE 1

**UNIT 1**

**CONTENT**



## 1 INTRODUCTION

What is a partial differential equation?, how do we classify partial differential equations? How are they rendered graphically and how do we solve them?, this unit addresses these questions with a tour of the basics of partial differential equations; particularly on an introduction to the methods for deriving solution.

## 2 OBJECTIVES

At the end of this unit, you should be able to:

(i) Properly define the term partial differential equation
(ii) Classify first order equations
(iii) Investigate the methods for constructing solutions for partial differential equations
(iv) Solve quasi – Linear Equations
(v) Explore the many definitions applied in deriving solutions
(vi) Apply the method of Lagrange in deriving solutions for partial differential equations

# 3 MAIN CONTENT

## 3.1 ESSENTIAL DEFINITIONS

In some elementary course we encountered many physical problems that are modelled by ordinary differential equations and have learnt some of the basic solution technique for such equation. We shall now expand our view by examining partial differential equations. Our Approach will deal with

i) Existence and Uniqueness of solutions.

ii) Stability of solution to small perturbations.

iii) Methods for constructing solutions.

We shall focus attention largely on (iii) although it is not always possible to solve a P.D.E in closed form

### 0.1 Definition

A P.D.E $G\left( \underline{x.\, u. \frac{\partial u}{\partial x}} ' \frac{\partial^2 u}{\partial x} ......... \frac{\partial^m u}{\partial x_n} m \right) = 0$

Where $\underline{x} \in R^n$

$$\underline{x} = \left( x_1 \;\; x_{2,} \; .......x_n \right)$$

Is a relationship between a function u of several variables $\underline{x} = \left( x_1 \;\; x_{2,} \; .......x_n \right)$, $n \underline{\geq 2}$ and its partial derivative?

### 0.2 Definition

By solution 0.1.0 in a domain $\Omega \, CR^n$ we mean a function U = g (x) whose partial derivatives of order less than or equal to m (e m) exist in $\Omega$ and satisfy the equation. We note however that some P.D.E do not provide solution in the classical sense defined above

**Example**

$$x^2 \frac{\partial u}{\partial x} = 1$$

Does not have a solution in any domain, $\Omega$ that contain the origin, rather than a solution in the sense of distributions or generalized functions

## 0.3 Definitions

A P.D.E is said to be of nth – order if the order of the highest partial derivation occurring in the equation if the coefficient of the highcol – order occur linearly the equation is said to be quasi – linear

$$\sum_{i,j=1=r}^{n} Aij \left( \underline{x}\, u, \frac{\partial y}{dx,} \ldots\ldots\ldots\ldots\ldots \frac{\partial u}{\partial xn} \right) \frac{\partial 2u}{\partial xxj}$$

$$+ g \left( \underline{x}_1\, u_1 \ldots \frac{\partial_u}{\partial_{xn}} \right) = 0$$

Is quasi – linear and of 2nd order

If the coefficient of the highest orders derivation are all functions of $\underline{x}$ only. The age is said to be Semi Linear

**Example**

$$\sum_{i,j=1}^{n} Aij\ (x) \frac{\partial 2u}{\partial_{xi} \partial_{kj}} + g \left( \underline{x}\, u, \frac{\partial u}{\partial x}, \ldots \frac{\partial u}{\partial xn} \right) = 0 - (0.3.1)$$

Is semi linear and of 2nd order.

The equation is linear if the coefficient of U and the coefficient of all it's partial derivatives are functions of $\underline{x}$ only.

E.g. $$\sum_{ij=1}^{n} Aij\ (\underline{x}) \frac{\partial^2 u}{\partial xi\ \partial xj} + \sum_{i=1}^{n} Bi\ (\underline{x}) \frac{\partial u}{\partial xi}$$

$$+ c(\underline{x}) U + g\ (\underline{x}) = 0 \qquad (0.3.2)$$

Is linear and of 2nd order

An equation that is not linear is said to be non-linear. A 2nd order e.g. (0.3.2) is said to be homogenous if g (x) is identically zero. Otherwise it is non – homogenous.

If (0.1.0) is a polynomial of degree k in the highest order partial derivation we say that the equation is of degree k.

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{dz}\right)^2 = 1$$ is 1st order non-linear degree 2

In general any equation of degree k = 1 is non – linear

**Example**

1) $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \cos xy$$

It is 1st order, linear non-homogenous

2) $$U\frac{\partial u}{\partial y}\frac{\partial 3u}{\partial x3} + \left(\frac{\partial^2}{\partial y^2}\right)^2 \sin u$$

It's 3rd order quasi – linear

3) $$\frac{\partial^2}{\partial xy} + \left(\frac{\partial u}{\partial x}\right)^2 = \frac{\partial y}{\partial z} + z^3$$

It's 2nd order and semi – linear

4) $$\frac{\partial^2 u}{\partial t^2} - c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

Is 2nd order linear Homogenous

5) $$\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + \left(\frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial^2 u}{\partial x\partial y} + \frac{\partial u}{\partial y} = u$$

Is 2nd order non linear

$$\begin{cases} U\,\dfrac{\partial v}{\partial x} + V\,\dfrac{\partial u}{\partial y} = x + y \\ V\,\dfrac{\partial v}{\partial x} + U\,\dfrac{\partial u}{\partial y} = x - y \end{cases}$$

System of 1st order quasi linear equation

$$(ax)\ z\,(a)\,u_x + (a)Uz\ Za + (a)\ Vn\ +\ (av)Vz\ Zx = ay = 0$$

**Example:** Given that

$$\left.\begin{array}{l} u = g\,(x_i\ y_i\ \cancel{z}) \\ v = h\,(x_i\ y_i\ \cancel{z}) \end{array}\right\} c^i\ (\Omega)$$

Determine the P.D.E of lowest order satisfied by the class of all functions defined implicitly by

$G\,(u_i\ v) = 0$ where $Gu\ Gv \neq 0$ in $\Omega$

### 3.2 FIRST ORDER EQUATION

Examples of 1st order equations are

$$\cancel{Z}\ x^2 + \cancel{Z}y^2 = 1$$

If $P = \cancel{Z}x \quad , q = \cancel{Z}y$

$$P^2 + q^2 = 1$$

$$a\,(z)\,\cancel{Z}x\ +\ Zy\ = 0$$

$$a\,(z)\ p\ +\ q = 0$$

$$x\ Zn\ +\ yZy = x\cancel{Z} \quad (x\ p + yq = n\cancel{Z})$$

## 3.3 QUASI – LINEAR EQUATIONS

This is given by

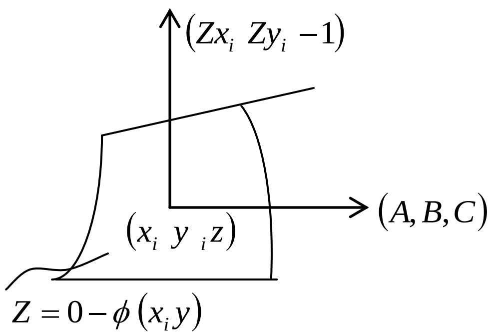
$$A(x_i\ y\ z)\,Zx\ +\ B(x_i\ y_i\ z)\,Zy = C(x_i\ y_i\ z) - 1.1.0$$

Where $(x, y) \in D\ c\ R^2$ and A, B, C are

$$C^0\ (\Omega),\ \Omega \text{ being in } R^3$$

Where projection on $R^2$ is o

$$(1.1.0) \Rightarrow (Zx, Zy, -1) \text{ is perpendicular to (A,B,C)}$$




**Implies that there exist an integral surface**

$$\sum = \{\,(x_i\ y_i\ z) : Z\ \phi\ (x_i\ y)\}$$

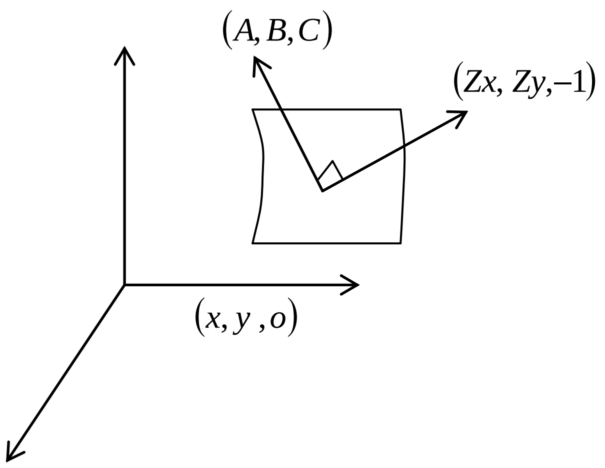
Passes thru $(x_i\ y_i\ z)$ which is target to the given vector

$$A(x_i\ y_i\ z), B(x_i\ y_i\ z),\ C(x_i\ y_i\ z).$$

at the given point that is

$$AZx + BZy = C$$

Can be interpreted geometrically as a requirement that any surface $Z = Z(n, y)$ thru $(x, y, z)$ must be tangent there to a prescribed vector $(A, B, C)$




The direction of the vector (A,B,C) is called the characteristics direction at the given point if $(dx, dy, dz)$ lies in the tangent plane S at $(x, y, z)$ then $(d_x \;\; d_y \;\; d_z)(Z_x, Z_y \; -1) = 0$

$$\Rightarrow Zxdx \; + \; Zy\, dy \; = dz$$

Comparing the above result with 1.1.0

We have that

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C} \qquad 1.1.1$$

$$= \frac{dx}{dt} = A \;\; ; \;\; \frac{dy}{dt} \;\; B \;\; \frac{dz}{dt} = C$$

$$= \frac{dy}{dx} = \frac{B}{A} ; \frac{dz}{dx} = \frac{C}{A}.$$

**Define 1.1** $\quad A(x, y, z)\, Zn + B(x, y, z)\, Zy \pm C(x, y, z)$

By the characteristic of 1.1.0 we mean the integral curves of 1.1.1chor $= \frac{dx}{A} = \frac{dy}{B} = \frac{dt}{C}$

**Theorem 1.2**

The integral curves of (1.1.1) generates the integral surface of 1.1.0

$$A(x,y,z)\,Zn + B(x,y,z)\,Zy = (x,y,z)$$

**Proof**

Let $Z = Z(x,y)$ be an integral of (1.1.0)

Then $dz = Zx\,dx + Zy\,dy$ (1.2.0)

Suppose r is an integral curve of (1.1.1) then $dx = Adt,\ \ dy = Bdt$ and dz cdt

Substituting into 1.2.0 we have (1.1.0)

It can be proved that exactly one characteristic passes through each point of S. The general solution of 1.1.1 is of the form

$$y = (x, \alpha, \beta)$$
$$z = (z, \alpha, \beta)$$

Where $\alpha$ amd $\beta$ are arbitrary constant.

Solution for $\alpha$ and $\beta$ we obtain

$$\alpha = U\,(x,y,z)$$
$$\beta = V\,(x,y,z)$$

Assuming that u and v are finally independent

i.e. $\dfrac{\partial\,(u,v)}{\partial\,(x,y)}, \dfrac{\partial(u,v)}{\partial(x,z)}, \dfrac{\partial(u,v)}{\partial(y,z)}$

are not all zero at any point (x, y, z) of S.

**Definition 1.2**

A single relation between u and v of the form

$$a\left(u,v\right)=0$$

Is called the general solution of (1.1.0)

**Examples**

Find the general solution of

$$x\ Zx+yZy=Z$$

With characteristic equation

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$$

$$\frac{dx}{x}=\frac{dy}{y}$$

$$\Rightarrow\quad \text{iny}=1\text{nx}+1\text{nx}$$

$$\frac{y}{x}\ =\ x\ =u\left(x,y,z\right)$$

Also $\dfrac{dz}{z}\ =\ \dfrac{dx}{x}$

$$\text{In}\,Z=1\text{n x}+1\text{n}\ \beta$$

$$\frac{Z}{x}=\beta\ =V\left(x,y,z\right)$$

$$F\left(u, v\right) = 0$$
$$F\left(\alpha, \beta\right) = 0$$
$$F\left({}^{y}\!/_{x}, {}^{z}\!/_{x}\right) = 0$$
$$\frac{z}{x} = F\left({}^{y}\!/_{x}\right)$$

$$\Rightarrow Z = x\, F\left({}^{y}\!/_{x}\right)$$

## 3.4 METHOD OF LAGRANGE

This is a useful technique for integrating first order equation from algebra we have that is

$$\frac{a}{b} = \frac{c}{d}$$

Then the following relationship is true

$$\frac{K_1 a + K_2 c}{K_1 b + K_2 d} = \frac{a}{b} = \frac{c}{d}$$

For arbitrary values of the multiplies $K_1$and $K_2$ so

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C} = \frac{K_1 dx + K_2 dy + K_3 dz}{K_1 A + K_2 B + K_3 C} = 1.2.1$$

Hence equation more convenient for integration maybe found by appropriate choice of $K_1, K_2, K_3$in 1.2.1

Further examples

Find the general solution of

i) $\left(y + 2x \in\right) Zx - \left(x + 2yZ\right) Zy = \frac{1}{2}\left(x^2 - y^2\right)$

$x \in R\,;\; y > 0$

ii) $\left(z^2 - 2yz - y^2\right) Zx + \left(xy + x\,z\right) Zy - xy - x\,z$

$G\, x^2 + y^2 + z^2 = 0$

**Solution**

$$x^2 + 4\,zy + 2z^2 \text{ where } K_1 = x$$
$$K_2 = z$$
$$K_3 = 2z + y$$

Characteristic equations are

$$\frac{dx}{y + 2\,x\,z} = \frac{dy}{x - 2yz} = \frac{dz}{\frac{1}{2}\left(x^2 - y^2\right)}$$

By method of langrage multiplier

$$\tfrac{1}{2}\, y\; d\; x + \tfrac{1}{2}\, x\; d\; y + dz = 0$$
$$\tfrac{1}{2}\, y\; x \;+\; \tfrac{1}{2}\; x\; y \;+\; z = \alpha$$
$$2\; xy + \; 2\; z = 2\alpha = \beta$$
$$\tfrac{1}{2}\, x\; d\; x + \tfrac{1}{2}\, y\; d\; y - 2\; Z\; dz = 0$$
$$\frac{x^2}{4} + \frac{y^2}{4} - z^2 = \alpha$$
$$x^2 + y^2 - 4z^2 = \alpha$$
$$G\,(\alpha, \beta) = 0$$
$$G\left(x^2 + y^2 - 4z^2 ; xy + z\right) = 0$$

Initial value problem (or Cauchy problem in $R^2$. This consists of a determination of an integral surface S of (1.1.0) which passes thru a pre-assigned space curve 6. We notice that those are the following possibilities

i) Unique surface

ii) Infinitely many surface

iii) No surface depending on the pre assigned on the pre assigned curve $\xi$

**Examples** consider the ivp

$$\begin{cases} yZx - xZy = 0 \\ \xi \; ; \; Z\,(x,0) = x4 \end{cases}$$

The characteristic equations are

$$\frac{dx}{y} \; \frac{dy}{x} \; = \frac{dz}{0}$$

$$\frac{dx}{y} \; \frac{dy}{-x}$$

$$-x\,d\,\,x = y\,d\,y$$

$$\frac{x^2}{2} + \frac{y^2}{2} = \alpha$$

$$x^2 + y^2 = \alpha$$

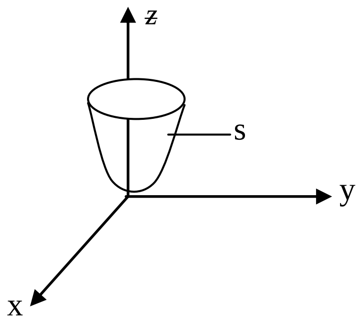
and

$$\frac{dx}{y} = \frac{dz}{0}$$

$$dZ = 0 \Rightarrow Z = \beta$$

$$F\,(\lambda, \beta) = 0$$
$$\beta \quad = F\,(\lambda)$$
$$\cancel{Z} \quad = F\,(x^2 + y^2)$$

The general solution is any surface of revolution about z-axis




Given the curve $z = Z(x, 0) = F(x^2) = x^4$

$$\Rightarrow F(x)\, x^2$$
$$\Rightarrow Z(x, y) = (x^2 + y^2)^2$$

2) Consider the ivp

$$\begin{cases} y\ z\ x - xZy = 0 \\ \xi\ \text{: circle} \begin{cases} x^2 + y^2 = 1 \\ z \qquad = 1 \end{cases} \end{cases}$$

$$Z = F\ (x^2 + y^2)$$
$$1 = F\ (x^2 + y^2) = 1$$
$$\Rightarrow z = F1\ (x^2 + y^2) \text{where f1 is any function which satisfiyers fi}(1) = 1$$

The solution exist but not unique

These are certainly infinitely many such surfaces

In this cape $\xi$ itself is a characteristic.

3) Consider the ivp

$$\begin{cases} yZx - xZy = 0 \\ \xi \text{ : ellipse} \begin{cases} \mathrm{x}^2 + y^2 = 1 \\ Z = y \end{cases} \end{cases}$$

$$Z = F(x^2 + y^2)$$

$\Rightarrow \quad y = F\left(x^2 + y^2\right) = F(1)$

$Z = y$ which impossible.

$\therefore$ No such integral surface exists.

**Theorem 1.8**

Let $AZx + BZy = C, \quad (x, y, z) \in \Omega; -1.8.0$

$$\left.\begin{array}{ll} A, B, C \in C^0\,(\Omega) \text{and } \xi : \mathrm{x} = \mathrm{x}_0(s) \\ \qquad\qquad\qquad\qquad y = y_0(s) \\ 0 \le s \le 1 \qquad\qquad z = Z_0(s) \end{array}\right\} \qquad 1.8.1$$

A given space in $\Omega \ \exists$

$x_0, y_0, z_0 \in C^1[0_1 1]$

Let $Ay_0^{\ 1} - Bx_0^{\ 1} \neq 0$ 1.8.2

Then $\exists$ a unique solution $z = z(x, y)$ of 1.8.0defined in some neighbourhood of the given curve $\xi$ and which satisfiers the initial condition $Z\left(x_0^{(s)}, y_0^{(s)}\right)$

$$Z\left(x_0(s), y_0(s)\right) = Z_0\,(s) \qquad 1.8.3$$

**Proof**

Consider the characteristic system

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C}$$

$$\left.\begin{array}{l} \equiv \dfrac{dx}{dt} = A \\ \dfrac{dy}{dt} = B \\ \dfrac{dz}{dt} = C \end{array}\right\} \qquad 1.8.4$$

From the existence and uniqueness theorem for o.d.e we many solve (1.8.4) for a uniquely family of characteristics

$$\left.\begin{array}{l} x = x\left(x_0(s), y_0(s), z_0(s), t\right) \\ y = y\left(x_0(s), y_0(s), z_0(s), t\right) \\ \cancel{z} = \cancel{z}\left(x_0(s), y_0\ (s), z_0(s), t\right) \end{array}\right\} \in C^1[0,1]\, x \quad 1.8.5$$

Such that

$$\left.\begin{array}{l} x\ (s,o) = x_o\ (s) \\ y\ (s,o) = y_o(s) \\ z\ (s,o) = z_0\ (s) \end{array}\right\} \qquad 1.8.6$$

By hypothesis the Jacobian

$$J \ = \ \frac{\partial(x,y)}{\partial(sit)}\bigg|_{t=0} = \begin{vmatrix} x_s & xt \\ y_s & y_t \end{vmatrix} = x_s\ y_t - x_s\ y_s|$$

$$= \left(B\ X_s - A\ y_s\right)\big|t = 0$$
$$= BX_0\ Ay_0^{\ 1} \neq 0$$

$\therefore$ we can solve (1.8.5) uniquely for s and t in terms of x and y in the neighbourhood of the given curve

$$\begin{array}{l} \xi :\ t = 0 \\ \quad s = s\ (x,y) \\ \quad t = t\ \ (x,y) \end{array}$$

Substituting into (1.8.5) we have

$$\begin{aligned} Z = \left(s(x,y), t(x,y)\right) &= z\ (x,y) \\ &= \Phi\ (x,y) \end{aligned}$$

That $Z = \Phi(x,y)$ satisfiers the initial conditions follows from

$$\Phi(x,y)\big|t = 0 = Z(s,o) = Z_0(s)$$

$\Phi$ satisfies the Partial Differential Equation for

$$
\begin{aligned}
& A\,\Phi x \quad + \quad B\Phi y \\
&= A\left(Z_s\ S_x\ + Z_t\ \ t_x\right) + \ B\left(Z_s\ S_y\ + Z_t\ t_y\right) \\
&= Z_s\ \left(AS_x\ + BS_y\right)\ \ +\ \ Z_t\left(At_x\ +\ \ Bt_y\right) \\
&= Z_s\ \left(s_x\ x_t\ + s_y\ \ y_t\right) + Z_t\ \left(t_x\ x_t\ + t_y\ \ y_t\right) \\
&= Z_s\ \frac{ds}{dt} + z_t\ \frac{dt}{dt} = Zs\,(0) + Z_t(1) \\
&= Z_t\ = c
\end{aligned}
$$

Uniqueness follows from theorem (1.2) $AZ_n + BZ_y = C$

The integral curves of $\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C}$ generates the integral surface.

Summary: - Cauchy problem has a unique solution provided the initial curve is not characteristic

**Exercise**

1) Solve the following:

$$
\begin{aligned}
& ZZx\ \ +\ \ Zy = 1 \\
& \quad x = \ \ s \\
& \quad y = s \\
& \quad z = 1\!\!\diagup\!_2\ s,\ 0 \le s \le 1
\end{aligned}
$$

2)

$$
Zy + \ \left(Zx\ \ = 0, \quad x \in R, \ \ y\right.
$$

$$
Z\ (x,o) = \begin{cases} 1 - x^2, \ |x| \ge 1 \\ \\ 0\,, \qquad |x| > 1 \end{cases}
$$

$Z\ (x, y) = F\ (x - cy) = 1 - (x - cy)^2$ – unique solution

$\mathrm{Z}\ (\mathrm{x},\ \mathrm{y}) = F\ (n - cy) = 0$ – intrnitty many

**Solution 2**

We observe that $Ay_0^1 - Bx_0^1 =$

$$A = Z, \;\; B = 1$$
$$y_0^1\,(s) = 1, \; x_0^1 = 1$$
$$Ay_0^1 - Bx_0^1 = Z - 1 \neq 0 \quad \text{for Z} \neq 1$$
$$0 < \leq \in 1$$

Characteristic equation are

$$\frac{dx}{Z} = \frac{dy}{i} = \frac{dZ}{i}$$

So that we now have

$$\frac{dx}{dt} \;\; Z, \;\; \frac{dy}{dt} = 1, \;\; \frac{dz}{dt} = 1$$

$$\frac{dz}{dt} = 1 \Rightarrow Z = t + \lambda$$
$$Z(so) = \frac{s}{2} = \lambda$$

$$Z = t + \lambda = t + s/2$$

Similarly

$$y = t + \beta$$
$$y(so) = s = \beta$$
$$\Rightarrow y = t + s$$

$$\frac{dx}{dt} = Z = t + 1/2\, s$$

$$x = \frac{t^2}{2} + \frac{1}{2}st + \alpha$$

$$S \quad s = \alpha$$

$$\Rightarrow x = \frac{t^2}{2} + \frac{1}{2}st + s$$

$$= \tfrac{1}{2}t\,(t+s) + s$$

$$= \tfrac{1}{2}yt + s$$

Write t = y –s and substitute into x so that

$$X = \tfrac{1}{2}y(y-s) + s$$

$$= \frac{1}{2}y^2 - \tfrac{1}{2}ys + s$$

$$s = x - \tfrac{1}{2}y^2 / 1 - \tfrac{y}{2}$$

$$s - \frac{x - \tfrac{1}{2}y^2}{1 - \tfrac{y}{2}} \text{ ..............................}(i)$$

Also write s = y – t and substitute into

$$x = \tfrac{1}{2}yt + y - t$$

$$t = \frac{y-x}{1-\tfrac{1}{2}} \text{ ..................................}(ii)$$

Substitute (i) and (ii) in

$$Z = t + \tfrac{1}{2}S$$

$$Z = \frac{y-x}{1-\tfrac{y}{2}} + \frac{1}{2}\left(\frac{x - y\,\tfrac{2}{2}}{1-\frac{y}{2}}\right)$$

$$= \frac{\frac{y^2}{2} - 2y - x}{y-2.}$$

## 4.0 CONCLUSION

In this unit we have studied some basic and essential definitions of partial differential equations; specifically those properties and general characteristics of First Order Equation, Quasi – Linear Equations and the utilisation of the Method of Lagrange in solving partial differential equations.

We examined partial differential equations from the perspectives of existence and uniqueness of solutions, stability of solution to small perturbations around the solution as well as the different methods for constructing solutions

## 5.0 SUMMARY

Partial differential equations can be generically classified into families and methods of solution for classes categories based on their properties.

## 6.0 TUTOR MARKED ASSIGNMENTS

1. Which of the following Partial Differential Equations is linear? quasilinear? nonlinear? If it is linear, state whether it is homogeneous equation or not

   a. $u_{xx} + u_{yy} - 2u = x^2$
   b. $u_{xy} = u$
   c. $u\,u_x + x\,u_y = 0$
   d. $u_x^2 + \log u = 2xy$
   e. $u_{xx} - 2u_{xy} + u_{yy} = \cos x$
   f. $u_x(1 + u_y) = u_{xx}$
   g. $(\sin u_x)u_x + u_y = e^x$
   h. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$
   i. $u_x + u_x u_y - u_{xy} = 0$

2. Give the order of each of the following?

   a. $u_{xx} + u_{yy} = 0$
   b. $u_{xxx} + u_{xy} + a(x)u_y + \log u = f(x, y)$
   c. $u_{xxx} + u_{xyyy} + a(x)u_{xxy} + u^2 = f(x, y)$
   d. $u\,u_{xx} + u_{yy}^2 + e^u = 0$
   e. $u_x + cu_y = d$

3. Find the general solution of

   $u_{xy} + u_y = 0$

4. Show that $u = F(xy) + x\,G(\frac{y}{x})$

   Is a general solution of $x^2 u_{xx} - y^2 u_{yy} = 0$

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**UNIT 2**

**CONTENT**



## 1 INTRODUCTION

The characteristic equation for the single conservation law is derived and solved with the assumption of implicit function, discontinuity which implies shock is also demonstrated.

## 2 OBJECTIVES

At the end of this unit, you should be able to:

(i) Study Conservation Law
(ii) Understand the concept of shock

## 3 MAIN CONTENT

### 3.1 APPLICATION OF IVP CONSERVATION LAW, DEVELOPMENT OF SHOCK

The conservation law state that rate of change of total substance contained in a fixed (arbitrary) domain $\Omega$ is equal to the flux of that substance across the boundary $\partial\Omega$

Let U be the density of the substance and F = flux, then the conservation law is given by rate of flow $\frac{d}{dt}\int_{\Omega} udx = -\int_{\partial\Omega} \underline{F}.\,\underline{n}\; ds$

$$\Rightarrow \int \frac{\partial}{\partial t} u\; d\; x = \int_{\Omega} Ut \;\; dx + \int_{\partial\Omega} \underline{F}.\,\underline{n}\; ds = 0$$

$$\int_{\Omega} ut\; dx \;\; + - \int_{\Omega} div\; F\;\; dx = 0$$

$$\int_{\Omega} (ut \;\; + div\;\; f\,)\, dx = 0 \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ } (i)$$

$$ut \;\; + drv\underline{f} \;= 0$$

Single conservation law

$$ut \;\; + \underline{f}x = 0$$

$$\Rightarrow \qquad ut \;\; + a(u)\, ux = 0 \quad . \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ } (ii)$$

Characteristic equation

$$\frac{dt}{1} \;=\; \frac{dx}{a(u)} = \frac{du}{o}$$

$$\frac{du}{0} \;= \frac{dt}{1}$$

$$\Rightarrow U \;= \text{comtant along } \frac{du}{dt} = \text{a}(\text{u})$$

i.e. on the characteristic $x = x(t)$ which propagates with speed a

u is a constant

a = signal of the speed.

Solve the following IVP

i) $ut + a(u)\,ux = 0$

$u\,(x,o) = f(x)$

$\frac{dt}{1} = \frac{dx}{a(u)} = \frac{du}{0}$

$\Rightarrow u = \alpha$

$\frac{dx}{dt} = a(u) = a(\alpha)$

$x = at + \beta$

General solution $F\,(\alpha, \beta) = 0$

$F\,(u,\, x - a(u)\,t) = 0$

$\Rightarrow\; u = F\,(x - a(u)\,t)$

$\Rightarrow$ solve is implicitly defined by

$u = F\,(x - a(u)\,t)$

$Ut \quad Fu\, Ut \qquad u = x - a(u)t$

$= F^1(-a(u)) - \; u = x - a(u\;t)$

$(1 + F^1 a^1 t)\,ut = -f^1 a(u)$

$\Rightarrow Ut = \frac{a(u)\; f^1}{1 + a^1 f^1 t}$

$U\; x = F^1(1 - a^1 u\; x\; t)$

$U\; x = \frac{f^1}{1 + a^1 f^{1t}}$ Assuming implicit function theorem

Therefore u given implicitly satisfies the P.D.E provided

$1 + a^1 f^1 t \neq 0.\text{if } 1 + \text{a}^1 f^1 t = 0$

$Ut,\ Ux$ will become infinite and shock is said to be developed i.e a discontinuity exist in $\Omega$. If

1.) a is constant, no shock $\forall\ t \geq 0$

2.) if F is constant, no shock $\forall\ t \geq 0$

3.) a, f, both non – deterring or non-increasing

For non – decreasing $f^1 \geq 0$

For non – increasing $f^1 \leq 0$

$a^1 f^1 \geq 0, \text{no - shock } \lambda \in \geq 0$

**Exercise**

1. Find a solution of $Z\ Z\ x + Zy = 0$

$Z\ (x, o) = x$

Draw the lines in the x – y plane along which solution is constant. Do shocks ever developed for $y \geq 0$

2. $Z^2\ x + Zy = 0$

$Z\ (x, o) = x$

Derive the solution $Z(x, y) = \begin{cases} x \ \text{ when y} = 0 \\ \sqrt{\dfrac{1 + 4xy}{2y}} - 1\ y \neq 0 \\ \qquad\qquad 1 + 4xy > 0 \end{cases}$

Do shocks ever develop? Show that

$$\lim_{y \to 0} Z(x, y) = x$$

1) $$\frac{dx}{z} = \frac{dy}{1} = \frac{dz}{0}$$

$$\Rightarrow Z = \alpha \quad d\ x = z\ d\ y$$

$$x = zy + \beta$$

$$z\,(x, o) = \beta = x$$

$$= z\ y + x = z$$

$$z\ = \frac{x}{1 - y}$$

The solution is constant along the lines

$$y = 0,\ y >^{1}_{0},\ y \leq -1$$

$$zx\ = \frac{x}{(1 - y)^2}$$

Shock develop for y = 1

## 4.0 CONCLUSION

This unit has practically exposed us to the real world application of partial differential equations through a scenario involving conservation law where we determine shock. .

## 5.0 SUMMARY

The law of conservation states that the rate of change of total substance contained in a fixed domain is equal to the flux of that substance across the domain boundary.

## 6.0 TUTOR MARKED ASSIGNMENTS

1. Derive the telegraph equation

$$u_{tt} + au_t + bu = c^2 u_{xx}$$

by considering the vibration of a string under a damping force proportional to the velocity and a restoring force proportional to the displacement.

2. Use Kirchhoff's law to show that the current and potential in a wire satisfy

$$\begin{aligned} i_x + C\,v_t + Gv &= 0 \\ v_x + L\,i_t + Ri &= 0 \end{aligned}$$

where $i$ = current, $v$ = $L$ = inductance potential, $C$ = capacitance, $G$ = leakage conductance, $R$ = resistance

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**MODULE 2**

**UNIT 1**

**CONTENT**



## 1 INTRODUCTION

General non – linear first order partial differential equations have a form *F(x,y,z,p,q) = 0 where p = Zx and q = zy* whose solution lead to the concept of the Monge cone and the chain curve stripe.

## 2 OBJECTIVES

At the end of this unit, you should be able to:

(i) Study in detail the class referred to as General non – linear First Order Equation
(ii) Sketch and explain the Monge cone
(iii) Apply Cauchy Method of Characteristic equations

## 3 MAIN CONTENT

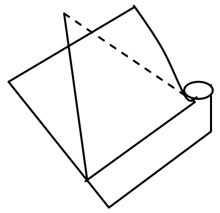
## 3.1 GENERAL FIRST ORDER EQUATION

The general non – linear P.D.E of 1st order has the form $F\left(x, y, z, p, q\right) = 0$ 1.9.1

Where $p = Zx$ and $\mathrm{q} = \mathrm{zy}$

At each point $p(x, y, z)$ on an integral surface $z = z(x, y)$ the direction number $(p, q, -1)$ of the normal to the surface are related through equation (1.9.1)

The P.D.E will restrict it's solution to these surface having tangent planes belonging to a 1-parameter family $q = G(x, y, z, p)$. Generally this one – parameter family of planes envelope a cone called the Monge cone.

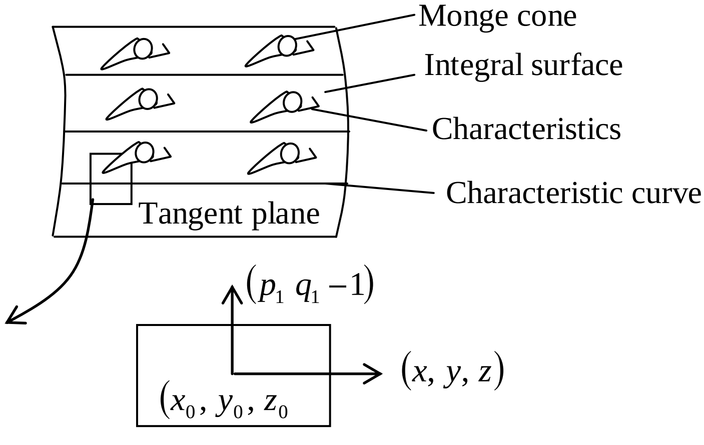



The geometrical significance of the 1st order P.D.E (1.9.1) is that any solution surface through a point in space must be tangent there to the corresponding monge cone

## 3.2 CAUCHY METHOD OF CHARACTERISTIC

Let $z = z(x, y) \in C^2$ be a given integral surface. At each point the surface will be tangent to the monge cone

Furthermore, the lines of contacts between the tangent planes of the surfaces and the cones define a field of directions on the surface called characteristic direction. These integral curves of thru field define a family of characteristic curves.




The Monge cone at a fix point $(x_o\ y_o, z_o)$ in the envelope of one particular family of planes

$$\left.\begin{array}{l} z-z_o = p\,(x-x_0)+q\,(y-yo) \\ \text{where } F\,(x_o, y_o\; z_o,\, p, q)=0 \\ \quad or\; q = q\,(x_o y_{o,} \cancel{z}_o, p) \end{array}\right\} \qquad 1.10.1$$

It's thus given by

$$\left.\begin{array}{l} z-z_0 = p\,(x-x_0) \;+\; q\,(x_0, y_0, z_0, p)\,(y-y_0) \\ \quad o \;=\; x-x_0 + \dfrac{dq}{dp}\,(y-y_0) \end{array}\right\} 1.10.2$$

Where p is adopted as the parameter using (1.10.1) we have

$$\frac{df}{dp} \;=\; Fp \;+\; Fq\,\frac{dq}{dp} = 0 \qquad 1.10.3$$

Eliminating $\dfrac{dq}{dp}$ for (1.10.2) yields the Hg

Equation for the Monge cone

$$\left.\begin{array}{l} F\,(x_0, y_0, Z_0, p, q)=0 \\ z-z_0 = p(x-x_0)+q\,(y-y_0) \\ \dfrac{x-x_0}{Fp} \;=\; \dfrac{y-y_0}{Fq} \end{array}\right\} \qquad 1.10.4$$

Eliminating p and q from 1.10.4 yield a more standard form of the equation of the cone

If given p and q the last two equation of 1.10.4 define the line of contact between the cone and the tangent plane.

It may be written in the form

$$\left.\begin{array}{l} u = g\,(x_i\;\; y_i\;\; \cancel{z}) \\ v = h\,(x_i\;\; y_i\;\; \cancel{z}) \end{array}\right\} c^i\;(\Omega)$$

The characteristic direction is

$$(Fp,\ Fq,\ pFp + qFq) \qquad 1.10.6$$

If therefore follows that the characteristics curves are determined by the O.D.E

$$\frac{dx}{Fp} = \frac{dy}{Fq} \quad \frac{dz}{pFp + qFq}$$

$$\equiv \frac{dx}{dt} = Fq,\ \frac{dy}{dt} = Fq,\ \frac{dt}{dt} - pFp + q\ Fq \quad 1.10.7$$

Assuming that the integral surface is at yet unknown, the 3 equation in 1.10.9 are not sufficient to determine the characteristic curve comprising the surface.

This is because the equation contains 2 addition unknown p and q.

However, along a characteristic curve on the given integral surface we have

$$\left.\begin{aligned} dp/dt &= p_x\ x_r + p_y\ y_t = p_x\ f_p + p_y\ f_q \\ dq/dt &= q_x\ x_t + q_y\ y_t = q_x\ f_p + q_y\ f_q \end{aligned}\right\} \qquad 1.10.8$$

Differentiating the given Partial Differential Equation. 1.9.1, we have

$$\begin{aligned} &F\ x + F = p + Fp\ px + Fq\ \ qx = 0 \\ &F\ y + Fz\ \ q + Fp\ py + Fq\ \ qy = 0 \end{aligned}$$

But $\quad qx = \dfrac{\partial Z^2}{\partial x \partial y} \qquad (zc\ \ c^2)$

So

$$\begin{aligned} &Fx + Fzp\ + (Fp\ Px\ + Fq\ Py) = 0 \\ &Fy + Fzo\ + (Fp\ qx\ \ + Fq\ qy) = 0 \end{aligned}$$

(1.10.8) then yields

$$\left.\frac{dp}{dt} = - \ Fx - pFz\right\} \qquad 1.10.9$$
$$\frac{dq}{dt} = - \ Fy - qFz$$

The 5 equations 1.10.7, 1.10.9 are called the characteristics equation associated with the Partial Differential Equation. The situation is now more complicated than in 1.9.1. All together we have the ff 6 equations

$$F\ (x,\, y,\, z,\ p,\, q) = 0$$

$$\frac{dx}{dt}\ \ Fp, \frac{dy}{dt} = Fq$$

$$\frac{dt}{dt} = - \ (Fx + pFz)$$
$$\frac{dq}{dt} = \quad (Fy + qFz)$$
$$\equiv \ \ F\ (x,\, y,\, z,\ p,\, q) = 0$$

$$\frac{dx}{Fp} = \frac{dy}{Fq} = \frac{dz}{pFp + qFq}$$

$$= \frac{dp}{-[Fx + pFz]} = \frac{dq}{-[Fy + qFz]}$$

For the 5 – unknown functions

$$x(t),\ y(t),\ z(t),\ p(t),\ q(t)$$

In other words if $F\ (x,\, y,\, z,\ p, q) = 0$ is satisfied at an initial point say $x_0,\, y_0,\, z_0,\ p_0,\, q_0$ for $t = 0$. The 5 characteristic equations 1.10.7, 1.10.9 will determine a unique solution $x(t),\ y(t),\ z(t),\ p(t)\ q(t)$, passing thru the x point and along which $f = 0$ will be satisfied for all t

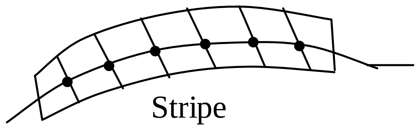
**Theorem 1.11**

Along any solution of characteristic equation of 1.10.10. $F\ (x, y, z, p, q) = 0$

Proof:- exercise

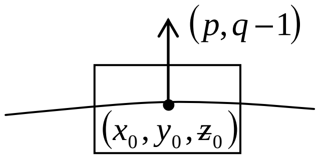
**Defined 1.12**

A ship is defined as a space curve $x = x(t)$ $y = y(t)$ and $z = z(t)$ in addition to the family of tangent planes with $(p, q, -1)$ as normal




**Defined 1.13**

An element of a stripe is defined as a point on a characteristic curve including the corresponding tangent plane at that point




**Remark**

Note that not any set of 5 functions can be interpreted as a strip.

The planes must be tangent to the curve which is that conditions that

$$\frac{dz}{dt} = p\frac{dx}{dt} + q\frac{dy}{dt}$$

**Theorem 1.14**

If a characteristic strip $x(t), y(t), z(t), p(t), q(t)$ has $x_0$ $y_0$ $z_0$ $p_0$ $q_0$ in common with an integral surface $z = u(x, y)$ then it lies completely on that surface.

**Theorem 1.15**

Given $F\left(x, y, z, p, q\right) = 0$ 1.9.1

And suppose along the initial curve

$x = x_0\ (s), y = y_0\ (s),\ 0 \le \varsigma \le 1,$ the initial values $z = z_0(s)$ are assigned and $x_0, y_0, z_0 \in C^2\ [0,1]$ have been determined satisfying

$$F\left(x_0\ (s), y_0\ (s), z_0\ (s), p_0\ (s), q_0\ (s)\right) = 0 \text{ and}$$

$$\frac{dz_0}{ds} = p_0\ \frac{dx_0}{ds} + q_0\ \frac{dy_0}{ds} \text{ with}$$

$$\frac{dx_0}{ds}\ fq\left(x_0, y_0, z_0, p_0, q_0\right) - \frac{dy_0}{ds}\ fp\left(x_0, y_0, z_0, p_0, q_0\right) \ne 0$$

then in the same neighbourhood of the initial curve, there exists a unique solution $z = z\left(x, y\right)$ of 1.9.1 containing the initial strip that is such that.

$$\begin{aligned} z\ \ (x_0(s), y_0\ (s)) &= z_0(s) \\ zx\left(x_0(s), y_0\ (s)\right) &= p_0(s) \\ zy\left(x_0(s), y_0\ (s)\right) &= q_0(s) \end{aligned}$$

## 4.0 CONCLUSION

Solution for general non – linear Partial Differential Equations of 1st order have a geometrical significance in relation to the Monge cone.

## 5.0 SUMMARY

The form of the general non – linear first order partial differential equation is *F(x,y,z,p,q) =0* where any solution surface through a point in space must be tangent at that point to the corresponding Monge cone.

## 6.0 TUTOR MARKED ASSIGNMENTS

1. Solve

$$\frac{\partial w}{\partial t} - 3\frac{\partial w}{\partial x} = 0$$

subject to $w(x,0) = \sin x$

2. Solve using the method of characteristics

a. $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = e^{2x}$ subject to $u(x,0) = f(x)$

b. $\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 1$ subject to $u(x,0) = f(x)$

c. $\frac{\partial u}{\partial t} + 3t\frac{\partial u}{\partial x} = u$ subject to $u(x,0) = f(x)$

d. $\frac{\partial u}{\partial t} - 2\frac{\partial u}{\partial x} = e^{2x}$ subject to $u(x,0) = \cos x$

e. $\frac{\partial u}{\partial t} - t^2\frac{\partial u}{\partial x} = -u$ subject to $u(x,0) = 3e^{x}$

3. Show that the characteristics of

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$
$$u(x,0) = f(x)$$

are straight lines

4. Take a look at the problem

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$

$$u(x,0) = f(x) = \begin{cases} 1 & x < 0 \\ 1 + \frac{x}{L} & 0 < x < L \\ 2 & L < x \end{cases}$$

a. Determine equations for the characteristics

b. Determine the solution $u(x, t)$

c. Sketch the characteristic curves.

d. Sketch the solution $u(x, t)$ for fixed $t$.

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**UNIT 2**

**CONTENT**



## 1 INTRODUCTION

Partial differential equations can have three types of solutions; the complete solution, the general solution and the singular solution. All are treated in this unit.

## 2 OBJECTIVES

At the end of this unit, you should be able to:

(i) Categorise the different types of solution of partial differential equations
(ii) Know the methods used in deriving complete solution
(iii) Understand what a general solution means
(iv) Explain why some partial differential equations are singular solutions

## 3 MAIN CONTENT

### 3.1 TYPES OF SOLUTION

We observe that the general solution of the 1st order pde 1.9.1 is an expression involving an arbitrary function of one variable.

This naturally is the extension of the result that the general solution of an order of first order involves one arbitrary constant

### 3.1.1 COMPLETE SOLUTION (INTEGRAL)

Any solution of the form

$$Z = \Phi\left(x, y, a, b\right) \qquad 1.16.1$$

Where a, b are arbitrary parameters represents two parameter family of surfaces. No systematic rule determining the complete integral is available. The complete integral is significant in the sense that the envelope of any family of solution of the 1st order equation 1.11 depending on some parameter is again a solution. Indeed equation 1.9.1 defines the tangent plane of a solution and as a result, if a surface has the same tangent plane as a solution at some point in space, then it also satisfiers the equation there. The envelope of a family of solutions is also a solution since it's in contact at each of it's points with one of these earlier mentioned solutions.

### 3.1.2 GENERAL SOLUTION (INTEGRAL)

The general solution of 1.9.1 can thus be obtained from the complete integral if we prescribe the 2nd parameter b, say b = b(a) as an arbitrary function of a. The enveloped of the one parameter subsystem of the complete integral is then considered as follows

$$Z = \Phi\left(x, y, a, b(a)\right) \qquad 1.16.1$$

Differencing wrt a, we have

$$O = \underline{\Phi}_a\left(x, y, a, b(a)\right) + \Phi_b\left(x, y, a, b(a)\right)\frac{db}{da} \qquad 1.16.2$$

Eliminating (a) between 1.16.1and 1.16.2 yield a single expression (involving the arbitrary function b (a) which is the general solution of 1.9.1

### 3.1.3 SINGULAR SOLUTION

This is the envelope of the full two parameter family of surfaces defined by the complete solution and is given by the 3 relation

$$Z = \Phi\left(x, y, a, b\right)$$
$$O = \Phi_a\left(x, y, a, b\right)$$
$$O = \Phi_b\left(x, y, a, b\right)$$

**Examples**

Type I:- $F\left(p, q\right) = 0$

Solve $p^2 - q^2 = 1$

Write $f\left(p, q\right) = p^2 - q^2 - 1 = 0$

$F\left(a, h(a)\right) = a^2 - \left(h(a)\right)^2 - 1$ and

$h(a) = \left(a^2 - 1\right)^{1/2}$

A complete solution is

$$Z = ax + \left(a^2 - 1\right)^{1/2} y + c$$
$$Z = ax + by + c$$

Put $b = \left(a^2 - 1\right)^{1/2}$ and diff with a to get

$$O = x + \frac{ay}{\left(a^2 - 1\right)^{1/2}}$$

$$\frac{-x}{y} = \frac{a}{\left(a^2 - 1\right)^{1/2}}$$

General solution is

$$\frac{Z}{a} = x + \left(\frac{y}{x}\right) y + \frac{c}{a}$$

$$\alpha\ x\ z = x^2 - y^2 + \alpha x \quad \left(\alpha = {}^{1}\!/_{a}\right)$$
$$\alpha\ x\ (z-1) = x^2 - y^2$$

There are singular solutions since

$$z = a\ x + by + c$$
$$o = x + \frac{ay}{\left(a^2 - 1\right)^{1/2}}$$

O = 1

**Example**

Consider $p^2 + q^2 = 1$

Recall $f\ (x,\ y,\ z,\ p,\ q) = o$

$$p^2 + q^2 - 1 = o$$

by $\dfrac{dx}{fp} = \dfrac{dy}{fq} = \dfrac{dz}{pf_p + qfq} = \dfrac{dp}{-(fx + pfz)} = \dfrac{dq}{-(fy + qfz)}$

$$fx = o,\ fy = o,\ fz = o,\ fp = 2p,\ fq = 2q$$

$$\frac{dx}{2p} = \frac{dy}{2q} = \frac{dz}{2\left(p^2 + q^2\right)} = \frac{dp}{o} = \frac{dq}{o}$$

$$dp = o$$

$$p = a\ (a \text{ is constant})$$

$$q^2 = 1 - a^2$$

$$q = \left(1 - a^2\right)^{1/2}$$

$$p = zx = {}^{\partial 2}\!/_{\partial y} = \left(1 - a^2\right)^{1/2}$$

$$dz\ =\ {}^{\partial z}\!/_{\partial x} + {}^{\partial z}\!/_{\partial y dy}$$

$$\int dz = \int a\, d\, x + \int \left(1 - a^2\right)^{1/2} dy$$

$$z = ax + \left(1 - a^2\right)^{1/2} y + b$$

General solution is given by

$$z = ax + \sqrt{1 - a^2}\, y + \Phi(a)$$

Differentiating wrt a we have

$$o = x - \frac{a}{\sqrt{1 - a^2}}\, y + \Phi^1(a)$$

Singular solution:- None

Since $z = ax + \sqrt{1 - a^2}\, y + 6$

Differentiating wrt a

$$za = o = x - \frac{a}{\sqrt{1 - a^2}}$$

$$zb = o = \sqrt{1 - a^2}$$

**Example**

Given $xp + yq = pq$, Find

1. The initial element if $x = x_o$, $y = o$ and $z = {x_o}/{2}$ $\cancel{z}\,(x, o) = {x}/{2}$

2. The characteristics stripe containing the initial elements

3. The integral surface which contain the initial element.

**Solution**

$$xp + yq = pq$$
$$xp + yq - pq = o$$
$$f\ (x, y, z, p, q) = o$$
$$\left(x_o, o, \tfrac{1}{2} x_o,\ p_o, q_o\right)\text{assume}$$

$$x_o p_o = p_o\ q_o$$

$$\Rightarrow x_o = q_o$$

according to the strip condition

$$dz\ = \frac{\partial z}{\partial x}\ dx + \frac{\partial z}{\partial y}\ dy$$

$$\frac{\partial z_o}{dx_o} = \frac{\partial z_o}{dx_o} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial x}$$

$$\tfrac{1}{2} = po$$

Initial element is $\left(x_o, o, \tfrac{1}{2} x_o, \tfrac{1}{2}, x_o\right)$

For simplicity let us take $x_o = 1$

For the characteristic equation $\frac{dx}{fp} =, \frac{dy}{fq}\ \frac{dz}{pfp + qfq} = \frac{dp}{-(fx + pfz)} = \frac{dq}{-(fy + qfz)}$

$$\frac{dx}{dt} = x - q \qquad \frac{dq}{dt} = -q$$

$$\frac{dp}{dt} = -p \qquad \frac{dz}{dt} = -\ pq$$

$$\frac{dy}{dt} = y - p$$

Integrating we obtain

$$x = xo\cosh t$$
$$y = \tfrac{1}{2}\sinh t$$
$$z = \tfrac{1}{4}xo\left(e-^{2t}+1\right)$$
$$p = \tfrac{1}{2}e^{-t}$$
$$q = xoe^{-t}$$

Eliminating $x_o$ and t from above, we obtain

$$8xyz + x^2 = 4z^2$$

Exercise

Solve

1) $pq = u$ with $u\,(o,s) = s^2$

2) determine the integral surface of $xpq + yq^2 = 1$ which contain the curve $z = x_o y = o$

Earlier Example

$$p^2 - q^2 = 1$$

$$f\,(x,y,z,p,q) = p^2 - q^2 - 1$$

$$fx = 0,\ fy = 0,\ fz = 0,\ fp = 2p,\ fq = 2q$$

$$\frac{dx}{fp}, = \frac{dy}{fq} = \frac{dz}{pfp + qfq} = \frac{dp}{-(fx + pfz)} = \frac{dq}{-(fy + qfz)}$$

$$\frac{dx}{2p} = \frac{dy}{-2p} = \frac{dz}{2p^2 - 2q^2} = \frac{dp}{o} = \frac{dq}{o}$$

$$p = a$$

$$q^2 = p^2 - 1$$

$$q^2 = \left(a^2 - 1\right)$$

$$q = \sqrt{q^2 - 1}$$

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

$$dz = adx + \sqrt{a^2 - 1}dy$$

$$z \;=\; ax + \;\sqrt{a^2 - 1}\;y + c$$

Put

$b = \left(a^2 - 1\right)^{1/2}$ and diff wrt a

$$\text{o} = \text{x} + \frac{ay}{\left(a^2 - 1\right)^{1/2}}$$

$$\frac{-x}{y} = \frac{a}{\left(a^2 - 1\right)^{1/2}}$$

General solution is

$$\frac{z}{a} \quad = x + \left(-{y}/{x}\right)y + {c}/{a}$$

We can rewrite it as

$$\alpha\ x\ z \ = x^2 - y^2 + \alpha cx \quad \left(\alpha = {1}/{a}\right)$$

$$\alpha\ x\ z - \alpha cx = x^2 - y^2$$

$$\alpha\ x\left(z - c\right) - x^2 - y^2$$

There is no singular solution

$$z = ax + by + c$$

$$\left.\begin{array}{ll} o = x + \dfrac{a}{\left(a^2 - 1\right)^{1/2}}\,y & \text{wrt a} \\ \text{o} = \text{y} & \text{wrt b} \end{array}\right\}\text{are not consistent}$$

**Exercise**

Find the complete and singular solution of

$$p^2 + q^2 = 9$$

**TYPE II**

Consider $z = px + qy + f(p,q)$

**Solution**

Using the characteristic equation

$$i.e\ \frac{dx}{fp} = \frac{dy}{fq} = \frac{dz}{pfp + qfq} = \frac{dp}{-(fx + pfz)} = \frac{dq}{-(fy + qf)}$$

Then

$$f(x, y, z, p, q) = z - px - qy - f(p,q) = 0$$

$$\begin{aligned} &fx = -p \quad fy = -q \\ &fz = 1 \quad fp = -x \;\; fp = -(x + fp) \\ &fq = -(y + fq) \end{aligned}$$

then

$$\frac{dx}{-(x + fp)} = \frac{dy}{-(y + fp)} = \frac{dz}{-p(x + fp) - q(y + fq)}$$

$$= -\frac{dp}{o} = -\frac{dq}{o}$$

$$\left.\begin{aligned} dp = o \Rightarrow \;\; p = a \\ dq = o \Rightarrow \;\; q = b \end{aligned}\right\} \text{constant}$$

Complete solution b

$$z = ax + by + f(a,b)$$

**Exercise**

Solve $(p+q)(z - xp - yq) = 1$

Find the complete solution

$$z = xp + yp - \frac{1}{(prq)}$$

Solve 4 $(1 + z^3) = 9z^4 pq$

$$\frac{4 + 4z^3}{qz^4} = pq$$

$$\frac{4}{q}\; z^{-4} + \frac{4}{q} z^{-1} - pq = 0$$

## 4.0 CONCLUSION

General solution of first order partial differential equations results in an expression involving an arbitrary function of one variable.

## 5.0 SUMMARY

The different types of solution of partial differential equations are categorised into complete solution, general solution and singular Solution.

## 6.0 TUTOR MARKED ASSIGNMENTS

1. Determine the general solution of

   a. $u_{xx} - \frac{1}{c^2} u_{yy} = 0 \qquad c = \text{constant}$
   b. $u_{xx} - 3u_{xy} + 2u_{yy} = 0$
   c. $u_{xx} + u_{xy} = 0$
   d. $u_{xx} + 10u_{xy} + 9u_{yy} = y$

2. Show that $u_{xx} = au_t + bu_x - \frac{b^2}{4}u + d$

   is parabolic for *a*, *b*, *d* constants. Show that the substitution

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# MODULE 3

**UNIT 1**

**CONTENT**



## 1 INTRODUCTION

With specific reference to second order partial differential equations the various classifications are discussed with reference to made to the Tricomi's equation, characteristics, and case of two independent variables.

## 2 OBJECTIVES

At the end of this unit, you should be able to:

(i) Classify Second Order Partial Differential Equations
(ii) Understand the importance of the Eigen-values of the matrix of coefficients
(iii) State Tricomi's Equation
(iv) Work with Laplace, heat and Wave Equations
(v) Study the special case of two independent variables

## 3 MAIN CONTENT

### 3.1 SECOND ORDER P.D.E. CLASSIFICATIONS

A second order semi – linear equation

$$\sum_{j=1}^{n} Aij\ (x)\frac{\partial^2 u}{\partial xi \partial xj} = g\left(\underline{x}\ u, \frac{\partial y}{\partial xi}\cdots\frac{\partial u}{\partial xn}\right) \qquad 2.1.1$$

Will be classified according to the properties of the Eigen-values of the matrix $(Aij) = (Aji)$of the coefficients of the highest order p. Derivations at any point x of a domain $\Omega\, C\, R^n$

a. If all the Eigen values are different from zero and all one of the same sign at x, the equation is said to be elliptic at the point x.

$$\begin{pmatrix} +++ & +++ \\ --- & --- \end{pmatrix}$$

b. If all the Eigen-value are different from zero and all but one have the same sign at x. It says that the Partial Differential Equation. is normal hyperbolic at x

$$\begin{pmatrix} -+++ & +++ \\ +--- & --- \end{pmatrix}$$

c. If all the Eigen values are different from zero and there are at least two of each sign at x. The Partial Differential Equation. is ultra hyperbaric at that pod.

$$\begin{pmatrix} --+ & ++++ \\ ++------ & \end{pmatrix}$$

d If one Eigenvalue is zero and the rest are of one sign at x. The Partial Differential Equation is parabolic at that point. If at least two Eigen-values are zero and the rest are of one sign at x. The equation is elliptic parabolic at that point

$$\begin{pmatrix} 00 & ++++++ \\ 00 & ------ \end{pmatrix}$$

If an eigen- value is zero and there's at least one positive and one negative at x the equation is hyperbolic parabolic at x $\begin{pmatrix} 0-++++ \\ 0+---- \end{pmatrix} \begin{pmatrix} 0--1-1 \\ 0+1--- \end{pmatrix}$. The Partial Differential Equation 2.1.1 is said to be of one of the above types in a domain $\Omega$ in $R^n$ If it's so at every point x in $\Omega$. Otherwise the equation is said to be of the mixed type in $\Omega$. The above classification is applicable to quasi – linear second order equations

$$\sum_{i,j=1}^{n} A\left(\underline{x}, u, ux, \cdots\cdots uxn\right) \frac{\partial 2u}{\partial xi \partial xy} = g\left(\underline{x}, u, ux, \cdots, uxn\right)$$

However, the sign of the solution $u\left(\underline{x}\right)$ and the signs of the 1st order p. derivation $ux, r........, uxn$ may have to be known

**Examples**

a) A Laplace equation $\Delta u = 0$ is elliptic since

$$\Delta u = \frac{\partial^2 u}{\partial x^2,} + \cdots + \frac{\partial^2 u}{\partial xn^2} = 0 \text{ and}$$

$$Aij = \begin{pmatrix} 1\,0\,0---0 \\ 0\,1\,0---0 \\ \vdots \quad 1 \quad\quad \vdots \\ 0 \quad \ddots \quad\quad \vdots \end{pmatrix}$$

b) Heat Equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u = o \text{ is parabolic}$$

Since $\frac{\partial^2 u}{\partial x,^2} + \frac{\partial^2 u}{\partial x_n^2} + \frac{o\partial^2 u}{\partial t^2} - \frac{\partial 4}{\partial t} = o$ and

$$Aij = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & & & & & & \end{pmatrix}$$

$$\lambda = 1, 1, 1, 1, 0$$

c) Wave Equation

$$\boxed{u} = \left( \Delta - \frac{\partial^2}{\partial t^2} \right) u = 0$$

e.g $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{dy^2} - \frac{d^2 y}{dz^2} = 0$$

$$Aij \neq \begin{pmatrix} 1 & 0 & - & - & - & 0 & 0 \\ 0 & 1 & - & - & - & 0 & 0 \\ \vdots & \vdots & \ddots & 1 & & & \\ 0 & \vdots & & - & 1 & & \end{pmatrix} \lambda = 1,1,1,---1$$

It's normal hyperbolic

From $$\begin{pmatrix} x & y & z \end{pmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$a_{11}x^2 + 2a_{21}xy + 2a_{31}xy$$
$$+ 2\, a_{32}yz + a_{22}y^2 + a_{33}z^2$$

**Exercise 2.3**

$$2.z \frac{\partial^2 u}{\partial x \partial y} + 2x \frac{\partial^2 u}{\partial y\, \partial z} = 0$$

$$\begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial x dz} \\ \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial y \partial z} \\ \frac{\partial^2 u}{\partial x \partial z} & \frac{\partial^2 u}{dy^2 z} & \frac{\partial^2 y}{\partial z^2} \end{pmatrix}$$

Is hyperbolic – parabolic in $R^3$

$$Aij = \begin{pmatrix} o & z & o \\ z & o & x \\ o & x & o \end{pmatrix}$$

Using $|A - \lambda I| = o$

$$= \begin{vmatrix} o-\lambda & z & o \\ z & o-\lambda & o \\ o & x & o \quad -\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & z & o \\ z & -\lambda & o \\ o & x & -\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & x \\ x & -\lambda \end{vmatrix} - z \begin{vmatrix} z & x \\ 0 & -\lambda \end{vmatrix}$$

$$= -\lambda \left(\lambda^2 - x^2\right) - z^2(-\lambda)$$

$$= -\lambda \left(\lambda^2 - x^2 - z^2\right) = 0$$

$$\lambda\left(x^2 + z^2 - \lambda^2\right) = 0$$

$$\lambda = 0 \text{ or } \pm\sqrt{x^2 + z^2}$$

## 3.2 THE TRICOMI'S EQUATION

Given by $y\,\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

$$Aij=\begin{pmatrix} y & 0 \\ o & 1 \end{pmatrix}$$

This equation is of the mixed type elliptic for y > o, parabolic for y = o and hyperbolic y <o

## 3.3 CHARACTERISTICS

Consider $\sum_{ij=1}^{n} Aij\,\frac{\partial^2 u}{\partial x_i \partial xj}=g\,(\underline{x}, u, ux, \ldots\ldots\ldots uxn)$……………….2.5.1

**Definition**

By the characteristic of (2.5.1) we mean the solution of the 1st order equation

$$\sum_{ij=1}^{n} Aij\,\frac{\partial^2 u}{\partial x_i \partial xj}=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{..2.5.2}$$

## 3.4 CASE OF TWO INDEPENDENT VARIABLE

$$a(x,y)\,zxx+2b(x,y)zxy+c(x,y)zyy=\Phi(x,y,z,zx,zy)\ldots\ldots\ldots 2.5.3$$

x znd y are the independent variable and z the depend variable

$$Aij=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Eigen values are given by

$$|A-\lambda \pm|=0 \quad \begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix}=0$$

$$\Rightarrow \lambda^2-(a+c)\lambda\ +ac-b^2=0$$

$$\Rightarrow \lambda_{1,2} = \frac{1}{2}\left\{(a+c) \pm \sqrt{\left\{(a-c)^2 + 4b^2\right\}}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 2.5.4$$

Eigen values are of different signs if

$a\,c - b^2 < o$ and one is zero if

$a\,c - b^2 = 0$

Therefore 2.5.3 is elliptic at x, y if b2 - ac < o

Hyperbolic $at\ \ x, y$ if $b^2 - ac > o$

Parabolic $at\ \ x, y$ if $\mathrm{b}^2 - ac = 0$ ……………………………….2.5.5

Equation 2.5.1 is one of these types in a domain $\Omega$ of xy plane if it's at any point of $\Omega$

No other type is possible since 254 can only admit two roots

The characteristic equation is

$$a(x,y)(w_n)^2 + 2b\,(x,y)w\,x\,y + c(x,y)(wy)^2 = 0 \ldots\ldots.2.5.6$$

$\Rightarrow$

$$a\ p^2 + 2\,b\ p\,q + cq^2 = 0$$

$\Rightarrow$ $p = q = o$ is a solution

$\Rightarrow$ $z(x,y)$ is a constant along a characterization

$$\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 0\ (Z = w)$$

Equation 2.5.6 is homogenous wrt $wx : wy$. If we substitute for this with the proportional quantities $dy$ and $-dx$, we then get

$$a\,(dy)^2 - 2b\ d\ x\,d\ y + c(dx)^2 = 0$$

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.5.7$$

Which coincide it 2.5.3 a parabolic

i Parabolic case $b^2 - ac = o$

let $\mathrm{E}(x, y)$ be constant be the general solution of (2.5.7). Introduce a regular transformation

$$\mathrm{E} = \mathrm{E}(x, y)$$
$$y = y(x, y)$$

Where $y \in C^2\ (\Omega)$ is any function independent of $\mathrm{E}$. The transformed characteristic equation

$$\bar{a}\left(\bar{z}_{\mathrm{E}}\right)^2 + 2\,\bar{b}\,\bar{z}_{\mathrm{E}}\ \bar{z}y + \bar{c}\left(\bar{z}y\right)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.5.8$$

Has the solution $\bar{z} = \mathrm{E}$

So $\bar{a} = 0$

Since a regular transformation does not alter the type of an equation $\bar{b}^2 = \bar{a}\,\bar{c}, \quad \bar{a} = 0$

Divide the transformed P.D.E by $\bar{C}$ to get the canonical from

$$\frac{\partial^2 z}{\partial^2 y} = \frac{\partial^2 u}{\partial^2 y} = \Phi\left(\mathrm{E}, y, u, u_{\mathrm{E}}, u_y\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.5.9$$

1) Characteristics are invariant under regular transformation

2) Type is not altered by a regular transformation

## 4.0 CONCLUSION

Second order semi – linear partial differential equation are classified according to the properties of their Eigen-values of the matrix of coefficients of the highest order.

## 5.0 SUMMARY

In this unit, we have been able to work with the classification of second order Partial Differential Equation with a special focus on the Tricomi's equation, characteristics and the case of two independent variables

## 6.0 TUTOR MARKED ASSIGNMENTS

1. Reduce to canonical form and find the general solution?

$$y^2u_{xx} - 2yu_{xy} + u_{yy} = u_x + 6y$$

2. Find the characteristic of

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} = d(x,y,u,u_x,u_y)$$

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**UNIT 2**

**CONTENT**



## 1 INTRODUCTION

Sometimes it can be expedient to transformation the independent variables in solving partial differential equations with regular case, hyperbolic case and Elliptic Case treated. Proof of theorem is presented that regular transformation of independent variable does not alter the type of partial differential equations.

## 2 OBJECTIVES

At the end of this unit, you should be able to:

(i) Know how to transform independent variables
(ii) Visit the theorems that apply to the regular case
(iii) Solve equations of the hyperbolic types by transformations
(iv) See why Elliptic equations have no real characteristics

## 3 MAIN CONTENT

## 3.1 TRANSFORMATION OF INDEPENDENT VARIABLES

Let $\mathrm{E} = \Sigma\,(x, y) \quad \gamma = \gamma(x, y)$ be a regular transformation $\Leftrightarrow 1-1$

$$\overline{JJ} = \frac{\partial\left(\mathrm{E}, \gamma\right)}{\partial\left(x, y\right)}$$

$$= \begin{vmatrix} \mathrm{E}_x & \mathrm{E}_y \\ \gamma_x & \gamma_y \end{vmatrix} \neq 0$$

$$\Leftrightarrow \quad \begin{aligned} x &= x(\mathrm{E}, \gamma) \\ y &= y(\mathrm{E}, \gamma) \end{aligned}$$

$$Ux \;\; \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \mathrm{E}}\frac{\partial \mathrm{E}}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x}$$

$$= U_{\mathrm{E}}\, \mathrm{E}_x + U_\gamma \gamma_x$$

Similarly

$$U_y = U_{\mathrm{E}}\mathrm{E}_y + U_\gamma \gamma_y$$

$$U_{xx} = U_{\mathrm{EE}}\, \mathrm{E}_x^2 + 2U_{\mathrm{E}\gamma}\gamma_x\, \mathrm{E}_x + U_{yy}\, \gamma_x^2$$

$$+ U_{\mathrm{E}}\mathrm{E}_{xx} + U_\gamma\, \gamma_{xx}$$

$$U_{yy} = U_{\mathrm{EE}}\mathrm{E}_y^2 + 2\, U_{\mathrm{E}y}\, \gamma_y \mathrm{E}_y + U_{yy}\gamma_y^2$$

$$+ U_{\mathrm{E}}\mathrm{E}_{yy} + U_\gamma\, \gamma_{yy}$$

$$U_{xy} = U_{\mathrm{EE}}\, \mathrm{E}_x \mathrm{E}_y + U_{\mathrm{E}\gamma}\left(\gamma_x \mathrm{E}_y + \mathrm{E}_x \gamma_y\right) + U_{\gamma\gamma}\, \gamma_x\, \gamma_y$$

$$+ U_{\mathrm{E}}\, \mathrm{E}_{xy} + U_\gamma \gamma_{yx}$$

Substitute $U_{xx}, U_{xy}, U_{yy}$ in

$$a\left(x, y\right)U_{xx} + 2\, b\, U\left(x, y\right)U_{xy} + C(x, y)U_{yy} = \Phi(x, y, u, ux, uy)$$

We get

$$\tilde{a}\left(\mathrm{E}, \gamma\right)U_{\mathrm{EE}} + \widehat{2}b(\mathrm{E}, \gamma)U_{\mathrm{E}\gamma} + \widehat{C}\left(\mathrm{E}, \gamma\right)U_{\gamma\gamma}$$

$$= \Phi\left(\mathrm{E}\,\gamma, U, U_{\mathrm{E}}, U_{\gamma}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.5.9$$

Where

$$\hat{a} = a\,\mathrm{E}^2 + 2b\mathrm{E}_x\mathrm{E}_y + C\mathrm{E}_y^2$$

$$\hat{b} = a\,\mathrm{E}_x\gamma_x + b\left(\mathrm{E}_x\,\gamma_y + \mathrm{E}_y\,\gamma_x\right) + C\mathrm{E}_y\,\gamma_y$$

$$\hat{c} = a\,\gamma_x^2 + 2b\,\gamma_x\gamma_y + C\,\gamma_y^2$$

$$b^2 - a = \Delta\left(\Delta\text{ is criminant}\right)\left(\mathrm{E}_x\gamma_y - \gamma_x\mathrm{E}_y\right)^2$$

$$\hat{b}^2 - \hat{a}\,\hat{c} = \left(b^2 - ac\right)^2$$

$$= \left(b^2 - ac\right)\overline{JJ}^2$$

The steps that led to the result is a proof of the theorem that regular transformation of independent variable does not alter the type of p.d.e

### 3.1.1 REGULAR CASE

**Theorem**

Characteristics are invariant under regular transformation

**Proof**

Equation of characteristics

$$a\left(\frac{\partial w}{\partial x}\right)^2 + 2b\,\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y} + c\left(\frac{\partial w}{\partial y}\right)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.5.10$$

$$wx = w_{\mathrm{E}}\mathrm{E}_x + w_{\gamma}\,\gamma_x$$
$$wy = w_{\mathrm{E}}\mathrm{E}_y + w_{\gamma}\,\gamma_y$$

Substituting into (2.5.10) we shown that it's the same as

$$\hat{a}\,w_w^2 + 2\,\hat{b}\,w_{\mathrm{E}}\,w_{\gamma} + \hat{c}\,w\gamma^2 = 0$$

Where

$$\hat{a} = a\mathrm{E}_x^2 + 2b\mathrm{E}_x\mathrm{E}_y + c\mathrm{E}_y$$
$$\hat{b} = a\mathrm{E}_x\gamma_x + b(\mathrm{E}_x\gamma_y + \gamma_x\mathrm{E}_y) + c\mathrm{E}_y\gamma_y$$
$$\hat{c} = a\gamma_x^2 + 2b\gamma_x\gamma_y + c\gamma_y^2$$

### 3.1.2 HYPERBOLIC CASE $(b^2 - ac > 0)$

Let $\mathrm{E}\,(x, y) = \text{constant}, \gamma(\text{x, y}) = \text{contant}$ and the general solution of (2.5.7). Then (2.5.10) has two independent solutions

$$w = \mathrm{E},\ w = \gamma$$
$$\Rightarrow \hat{a} = \hat{c} = 0$$

Divide the transformed equations by $2\hat{b}$ to obtain

$$\frac{\partial^2 u}{\partial \mathrm{E}\partial y} = \Phi\left(\mathrm{E}, \gamma, U, U_{\mathrm{E}}, U_\gamma\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.2.5.11$$

Let $\begin{matrix} \mathrm{E}_i & = \mathrm{E} + \gamma & \Leftrightarrow & \mathrm{E} = \tfrac{1}{2}(\mathrm{E}_i + \gamma_i) \\ \gamma_i & = \mathrm{E} - \gamma & & \gamma = \tfrac{1}{2}(\mathrm{E}_i - \gamma_i) \end{matrix}$ ……………………………….2.5.12

be a linear transformation

$$\frac{\partial}{\partial \mathrm{E}} = \left(\frac{\partial}{\partial \mathrm{E}} + \frac{\partial}{\partial \gamma}\right), \frac{\partial}{\partial \gamma} = \frac{\partial}{\partial \mathrm{E}_i} - \frac{\partial}{\partial \gamma_i}$$

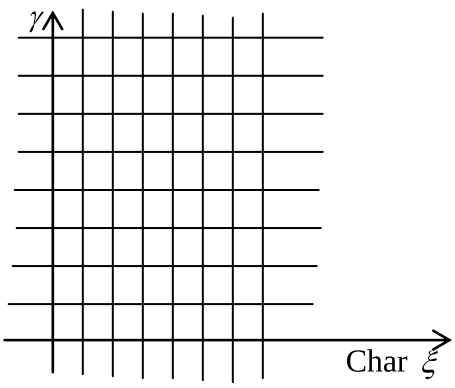
Substituting into (2.5.11) yields

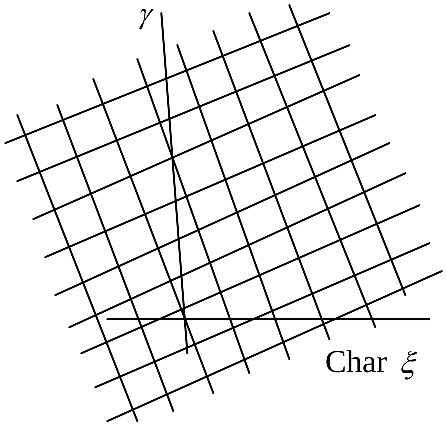
$$= \frac{\partial}{\partial \mathrm{E}}\frac{\partial}{\partial \gamma}(u)$$

$$= \left(\frac{\partial}{\partial \mathrm{E}_i} + \frac{\partial}{\partial \gamma}\right)\left(\frac{\partial}{\partial \mathrm{E}_i} - \frac{\partial}{\partial \gamma}\right)u$$

$$= \frac{\partial^2 u}{\partial \mathrm{E}^2} - \frac{\partial^2 u}{\partial \gamma} = \Phi 2\left(\mathrm{E}_i, \gamma_i, U, U_{\mathrm{E}}, U_\gamma\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.2.5.13$$

In the $\text{E}\,\gamma-$plane, the characteristic are lines $\parallel$ to the coordinate areas while in $\text{E}_i\ \gamma_i$ - plane they are lines of slopes $\pm$ 1




### 3.1.3 ELLIPTIC CASE $\left(b^2 - ac < 0\right)$

Theorem: An elliptic equation has no real characteristics

**Proof:-**

The characteristics are $\left(\dfrac{dy}{dx}\right)_{1,2} = \dfrac{b \pm \sqrt{b^2 - ac}}{a}$

But $b^2 - ac < 0$ this $\Rightarrow$ that $\nexists$ no real curve in the real x-y plane. We assume that a,b,c admits complex values. Equation (2.5.7) becomes O.D.Es in a complex values domain suppose – $w(x, y) \neq$ constant is a general solution of the 1st order equation, then $w(x, y) \neq$ satisfying (2.5.10)

Suppose $w(x, y) = \text{E}\,(x, y) + \gamma(x, y)$

Where $x,\ y,\ \text{E}, \gamma$ are real

$$\overline{JJ} = \frac{\partial(\text{E}, \gamma)}{\partial(x, y)} \neq 0$$

For if $\overline{JJ} = 0$ at $(\text{x, y})$

$$\frac{\partial \text{E}}{\partial x} = \lambda \frac{\partial \text{E}}{\partial y}$$

$$\frac{\partial \text{E}}{\partial x} = \lambda \frac{\partial \gamma}{\partial y}$$

$$\frac{\partial w}{\partial x} = \lambda \frac{\partial w}{\partial y} \text{ for some } \lambda \in R$$

Substitute in 2.5.10

$$a(wx)^2 + 2bw_\lambda w_y + c(wy)^2 = 0$$

We have

$$a\lambda^2 + 2b\lambda + c = 0$$

Since $wx = \lambda wy$

$$a\,(\lambda wy)^2 + 2b\lambda\ wy\ wy + c(wy)^2 = 0$$

$$a\lambda^2\,(wy)^2 + 2b\lambda + c \quad wy = 1$$

$$a\lambda^2 + 2b\lambda + c$$

Which has real root this is impossible since $b^2 - ac < 0$ therefore $\overline{JJ} \neq o$

Take $\text{E} = \text{E}(x, y)$ and $\gamma\ \gamma(x, y)$ as the new independent variables equation 2.5.10 must be satisfied by $w = \text{E} + 1\gamma$

$$\Rightarrow \qquad \widehat{a}\,(w\text{E})^2 + 2\widehat{b}\ w\text{E}\ w\gamma + \widehat{c}\,(w\gamma)^2 = 0$$

$$\widehat{a} + 2\widehat{b}i + \widehat{c}i^2 = 0$$

$$\widehat{a} - \widehat{c} + 2\widehat{b}i = 0$$

$$\Rightarrow \qquad \widehat{a} - \widehat{c} = 0 \quad \Rightarrow \widehat{a} = \widehat{c}$$

$$2\widehat{b} = 0 \qquad \Rightarrow \widehat{b} = 0$$

Dividing the transformation equation by $\hat{a}$, we arrived at the canonical form

$$\frac{\partial^2 u}{\partial \mathrm{E}^2} + \frac{\partial^2 u}{\partial \gamma^2} = \Phi\left(\mathrm{E}, \gamma, U, U_{\mathrm{E}}, U_\gamma\right) \text{ ........................................2.5.14}$$

**Examples**

Solve the Partial Differential Equation by the method of characteristics

$$yu_{xx} + (x+y)uxy + xu_{yy} = 0$$

Sketch the characteristics curves

Have $a = y,\; b = \tfrac{1}{2}(x+y), c = \lambda$

So $b^2 - ac = \dfrac{1}{4}(x-y)^2$

Which implies that the equation a hyperbolic for $x \neq y$, parabolic for $x \neq y$ the characteristics equations

$$\frac{dy}{dx} = \frac{\tfrac{1}{2}(x+y) + \tfrac{1}{2}(x-y)}{y} = \frac{x}{y} \text{ both are separable equation}$$

$$\frac{dy}{dx} = \frac{1}{2}\,\frac{(x+y) - \tfrac{1}{2}(x-y)}{y} = 1$$

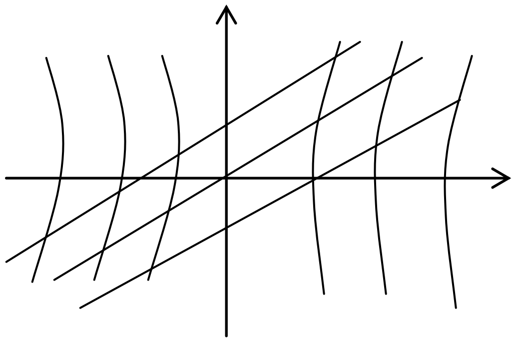
then solving it $\Rightarrow$

$$\begin{aligned} y^2 - x^2 &= \alpha : r. \quad y\,dy = xdx \\ y - x &= \beta : r_2 \quad \frac{y^2}{2} - \frac{x^2}{2} = 8 \\ & \qquad\qquad y^2 - x^2 = \beta = 28 \end{aligned}$$

Where $r_1, r_2$ are the characteristics curves

$r_1$ is defined as rectangular hyperboles

$r_2$ is defined as straight line with slope



Now write $\begin{aligned} y^2 - x^2 &= \mathrm{E}\,(x, y) \\ y \quad x &= \gamma(x, y) \end{aligned}$

$U(x, y)$ can be transform into

$$Ux = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \mathrm{E}}\,\frac{\partial \mathrm{E}}{\partial x} + \frac{\partial u}{\partial \gamma}\,\frac{\partial \gamma}{\partial x}$$

$$Ux = -2xU_{\mathrm{E}} - U\gamma$$

$$U_y = \frac{\partial u}{\partial \mathrm{E}}\,\frac{\partial \mathrm{E}}{\partial y} + \frac{\partial u}{\partial \gamma}\,\frac{\partial \gamma}{\partial y} \quad Uy = 2y\,U_{\mathrm{E}} + U\gamma$$

$$Uxx = 4x^2 U_{\mathrm{E}\gamma} + 4xU_{\mathrm{E}\gamma} + U_{\gamma\gamma} - 2U_{\mathrm{E}}$$

$$Uxy = -4xyU_{\mathrm{EE}} - 2U_{\mathrm{E}\gamma}\,(y + x) - U_{\gamma\gamma}$$
$$Uyy = 4y^2 U_{\mathrm{EE}} + 4yU_{\mathrm{E}\gamma} + U_{\gamma\gamma} + 2U_{\mathrm{E}}$$

The Partial Differential Equation (x) becomes

$$-2\,U_{\mathrm{E}}\gamma\,(y^2 + x^2 - 2xy) - 2U_{\mathrm{E}}(y - x) = 0$$

So that we now have

$$-2\gamma^2\, U_{\mathrm{E}}\gamma - 2\gamma U_{\mathrm{E}} = 0$$

We now have

$$2\gamma\left(\gamma U_{E\gamma} + U_E\right) = 0$$

$$\Rightarrow \gamma U_{E\gamma} + U_E = 0$$

$$\gamma\frac{\partial}{\partial\gamma}\left(\frac{\partial u}{\partial E}\right) + \frac{\partial u}{\partial E} = 0$$

Let $w = U_E$

$$\gamma W\gamma + w = 0$$

$$\frac{\partial}{\partial\gamma}\left(\gamma\frac{\partial u}{\partial E}\right) = 0$$

$$\gamma\frac{\partial u}{\partial E} = F(E)$$

$$\frac{\partial u}{\partial E} = \frac{1}{\gamma}F(E)$$

$$U(E,\gamma) = \frac{1}{\gamma}\int F(E)dE + G(\gamma)$$

Where

$$F(E) = \int F(E)\,dE$$

$$\Phi(x,y)U(x,y) = \frac{1}{y-x}F\left(y^2 - x^2\right) + G(y-x)$$

Where F and G are arbitrary differentiable functions

**Exercise**

Solve the following 2nd order P.D.E by the method of characteristics

$$Uxx + 2Uxy + Uyy + Ux + Uy = 0$$

Sketch the characteristics

$$a = 1 \;\; b = 1 \;\; c = 1$$
$$b^2 - ac = 0$$

**Further exercise**

Classify and solve the following P.D.E's by the method of characteristics and sketch the characteristics

i) $y^2 Uxx - 2y\,Uxy + Uyy - Ux - 6y = 0$

ii) $Uxx + \;\; x^2 Uyy = 0$

iii) $uxx + xUyy = 0$

$$\Rightarrow F\,\left(y + \tfrac{1}{4}\,i\right) + x\,g\,\left(y + \tfrac{1}{4}i\right)$$

$$F\,\left(2y + x^2 i\right) + G\,\left(2y - x^2\right)$$

## 4.0 CONCLUSION

Regular transformation of independent variables does not alter the type of a partial differential equation.

## 5.0 SUMMARY

The potential of transformations to ease the arrival at solution of partial differential equation never be understated and this was adequately demonstrated in the transformation of independent variables with three specific scenarios of the regular, hyperbolic and the elliptic case visited.

## 6.0 TUTOR MARKED ASSIGNMENTS

1. Let a, b be real numbers. The Partial Differential Equation

$$u_y + au_{xx} + bu_{yy} = 0$$

is to be solved in the box Ω =[0, 1]2.

Find data, given on an appropriate part of ∂Ω, that will make this a well-posed problem and cover all cases according to the possible values of a and b.

Justify your answer

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# MODULE 4

**UNIT 1**

**CONTENT**



## 1 INTRODUCTION

This unit zooms in on Cauchy problem and the Cauchy Kovalevsky theorem and places their significance in the solving of higher order partial differential equations into context.

## 2 OBJECTIVES

At the end of this unit, you should be able to:

(i) Cauchy Problem And Characteristics Problem
(ii) Understand what the strip condition means
(iii) Treat the fundamental existence theorem
(iv) Take a critical look at Cauchy problem
(v) Solve using the Cauchy Kovalevsky theorem

# 3 MAIN CONTENT

## 3.1 CAUCHY PROBLEM AND CHARACTERISTICS PROBLEM

Find a solution

$U = U(x, y)$ of the equation

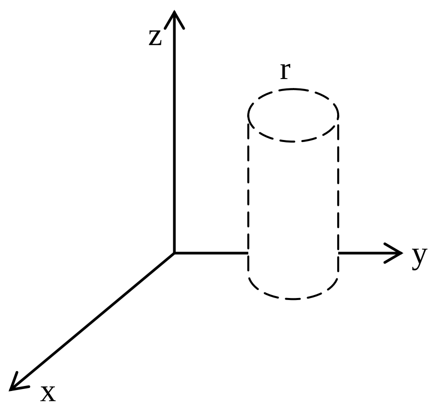
$aUxx + 2b\,Uxy + cUyy = \Phi(x, y, u, p, q)$ in some neighbourhood of a space curve, set $u = \not{z}$

$$r = \{(x, y, \not{z}) : x = f_1(t), \ y = f_2(t), \not{z} = h(t)\}$$

$$0 \le t \le 1$$

Such that

$$z/n = h(x, y),$$
$$\partial z/\partial n = H(x, y)$$




If instead of prescribing

$\frac{\partial z}{\partial n}$ on $r$, $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$ are prescribed

We find that $f_1, f_2, h, p, q$ must satisfy

$$\frac{dz}{dt} = h(i) = zx\ xt + z_y y_t \qquad \text{……………………………………………….2.7.2}$$
$$= pf : \ + q\ f_2^1$$

This is known as the strip conditions Cauchy problem therefore becomes that of finding a solution of (2.7.1) containing the integral ship of $(f_1,\ f_2, h, p, q)$ at any point of the integral strip.

$$\frac{dp}{dt} = zxx\frac{dx}{dt} + zxy\frac{dy}{dt} = rfi + sf_2^1 \text{……………………………………….}2.7.3$$

$$\frac{dp}{dt} = zxy\frac{dx}{dt} + zyy\frac{dy}{dt} = fi + rf_2^1 \text{………………………………..............}2.7.3b$$

Solving (2.7.1) and (2.7.3) above, we have

$$\frac{r}{\Delta,} = \frac{s}{\Delta_2}\frac{r}{\Delta_3} = \frac{1}{\Delta}$$

Where

$$\Delta = \begin{vmatrix} a & 2b & c \\ f_2^1 & f_2^1 & o \\ o & f_1^1 & f_2^1 \end{vmatrix}$$

Now

$$\Delta_1 = \begin{vmatrix} \Phi & 2b & c \\ p^1 & f_2^1 & o \\ q^1 & f_1^1 & f_2^1 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a & \Phi & c \\ f_1^1 & p_2^1 & o \\ o & q^1 & f_2^1 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a & 2b & \Phi \\ f_1^1 & f_2^1 & p^1 \\ o & f_1^1 & q^1 \end{vmatrix}$$

If $\Delta \neq 0,$ we can uniquely determine r, s, r on c differencing (2.7.1) with respect to t and using the relations

$$\frac{dr}{dt} = \frac{\partial^3 z}{\partial x^3}\frac{dx}{dt} + \frac{\partial^3 z}{\partial y\,\partial x^2}\frac{dy}{dt} = Zxxx\ f_1^1 + Zxxyf_2^1$$

$$\frac{ds}{dt} = \frac{\partial^3 z}{\partial x^2}\frac{dx}{dt} + \frac{\partial^3 z}{\partial y\,\partial x^2}\frac{dy}{dt} = Zxxy\ f_1^1 + Zxyyf_2^1$$

$$\frac{dr}{dt} = \frac{\partial^3 z}{\partial x \partial y^2}\frac{dx}{dt} + \frac{\partial^3 z}{\partial y\,\partial x^2}\frac{dy}{dt} = Zxyy\ f_1^1 + Zyyyf_2^1$$

3rd order derivations of c can be calculated on $\not{Z}$ similarly for fourth and higher order partial derivatives. The value of $\not{Z}$ in some neighbourhood of r can be obtained by Taylor's theorem

The Cauchy problem passes a unique solution $\Delta \neq 0$

Suppose that $\Delta \neq 0$, then

$$a\left(f_2^1\right)^2 - 2bf_1^1\ f_2^1 + c\left(f_1^1\right)^2 = 0$$

$$\Rightarrow\ a\left(\frac{dy}{dt}\right)^2 - 2b\frac{dx}{dt}\frac{dy}{dt} + c\left(\frac{dx}{dt}\right)^2 = 0$$

$$\Rightarrow a\,(dy)^2 + 2b\,dx\,dy + c\,(dx)^2 = 0$$

Which is the equation for characteristics of (2.7.1) however, if $\Delta = 0$ and $\Delta_i\,(i = 1,2,3)$, a solution will exist but not unique

## 3.1 FUNDAMENTAL EXISTENCE THEOREM

### 3.1.1 CAUCHY PROBLEM

Given a partial differential equation, find a solution which satisfied given boundary or initial conditions if the conditions are enough to ensure existence, uniqueness and continuous dependence of the solution on the given data a Cauchy data. We say that the problem is well posed.

### 3.1.2 CAUCHY KOVALEWSSKI THEOREM

Solutions of initial value problems may be obtained in Taylor's series. We simply compute the coefficients of the Taylor's series of the solution using initial data and the partial differential equation. The method is possible if the solution is analytic. Cauchy Kovalevsky theorem gives the condition under which the initial – value problem has solution which is an analytic function

Case 1 (1st Order Equation $R^2$)

$$\begin{cases} F(t), \ x, \ u, \ u_t, \ u(x) & = 0 \\ u\ (o, x) & \phi\ (x) \end{cases}$$

Assume

$$\begin{cases} ut \ = \ f\ (t, x, u, u_x) & (3.1.1) \\ u(o, x = \ \Phi\ (x) & (3.1.2) \end{cases}$$

Let $\Phi(x)$ be analytic in the neighbourhood of the origin x =0, f is analytic in the neighbourhood of the point $(0,\ 0,\ \ \Phi(0),\ \Phi^1(0)) \in \Re^4$

Then the Cauchy – problem (3.11) to (3.12) has a solution u (t, x)

Which is defined and analytic in a neighbourhood of the origin (0,0) $\in \Re^2$ and this solution is unique in the class of analytic functions

**Proof**

The proof depend essentially on a specific technique. Assuming $\Phi$ is analytic in a neighbourhood of x = 0 this enables us to obtain

$$\frac{\partial^n u}{\partial x^n}(0,0)\, F\ \Phi^n(0) \quad n = 1,\ 2\ ,3...$$

From (3.1.1) $u_t\ (0,0) = \ f(0,\ 0,\ \Phi\ (0),\ \Phi^1(0))$

Differentiating (3.1.1) wrt x

$$u_{tx}(t,x) = \frac{\partial f}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial ux}\frac{\partial ux}{\partial x}$$

$$U_{tx}(t,x) = fx + fu\, ux + fux\, Uxx$$

Since f is known and $U_x, U_{xx}$ have being determined at the origin. We can find $U_{xt}(0,0)$ to obtain $U_{xxt}$ we differential 3.1.1 twice with respect to x and substitute $t = x = 0$ and also previously determined values of $U, U_x U_{xx}, U_{xxx}$ at (0,0)

Continuing in this manner, we can determine the values of all partial derivatives

$$\frac{\partial^{n+1} u}{\partial x^n \partial t};\ n = 0, 1, 2, ......\text{at } (0,0)$$

Differentiating (3.1.1) wrt t

$$U_{tt} = f_t + f_u U_t + f_{ux} U_{xt}$$

Substituting t = x =0 and previously obtained values of $u, u_x\ u_t$ at $(0.0)$. Continuing in this way, we obtain the values of all partial derivatives of u at (0,0)

$$U(t,x) = \sum_{\alpha=0}\sum_{\beta=0} \frac{D_t^{\alpha}\ D_x^{\beta} u(0,0)\, t^{\alpha} x^{\beta}}{\alpha!\, \beta!} ....................................3.1.3$$

Cauchy Kovalevsky's theorem assert that this series converges for all (t,x) in some neighbourhood of the origin (0,0) and defines the solution of (3.1.1) in this neighbourhood. Uniqueness follows from that fact that any two analytic functions having the same Taylor's series coefficients are identical

**Examples 3.2**

$$Ut = Uux$$

$$U(0,x) = 1 + x^2$$

Taylor's series expansion

$$U(t,x) = U(0,0) + Ut\,(0,0)t + Ux(0,0)x +$$

$$\frac{1}{2!}\left\{Utt\,(0,0)\,t^2 + 2Utx\,(0,0)\,tx + Uxx\,(0,0)\,x^2\right\}$$

$$+\frac{1}{3}\left\{Uott(0,0)\,t^3 + \ldots\ldots\ldots\ldots\right.$$

$\Phi\,(x) = 1 + x^2$ is analytic in the neighbourhood of $x = 0$ (analytic in x)

$f\,(t,\, x,\, u,\, ux) = uux$ is analytic in the neighbourhood $(0,0,1,0) \in \Re^4$

$$u(o,x) = \Phi(x) = 1 + x^2$$
$$u(0,0) = \Phi(0) = 1$$
$$u_x(0,0) = 2x \text{ at } x = 0 = 0$$

$uux = \int(t,\, x,\, u,\, ux)$ is analytic in the neighbourhood of point (0 ,0, 1, 0) $\in \Re^4$

$$u\,(o,x) = 11x^2 \qquad u(0,0) = 1$$
$$ux(0,x) = 2x \qquad ux(0,0) = 0$$
$$uxx(0,x) = 2 \qquad uxx(0,0) = 2$$

$$D_x^n\, u(0,x) = 0, \quad n \geq 3$$
$$D_x^n\, u(0,0) = 0, \quad n \geq 3$$

$$ut = uux \quad ut(0,0) = u(0,0)\,ux(0,0) = 1x0 = 0$$
$$ut = ux^2 + uuxx \; utx(0,0) = 0 + 1\times 2 = 2$$
$$utt = ut\;ux + uuxt,\; utt\;(0,0) = 2$$

$$utxx = 3ux\;uxx + uuxxx;\; utxx(0,0) = 0$$
$$utt \quad = ut\;uxx + 2ux\;utx - uutxx;\; uttx\;(0,0) = 0$$
$$utt(0,0) = 0$$

Neglecting terms of order $\geq 4$

$$u\,(x,t) = \sum_{\alpha=0} \sum_{\beta=0} \frac{D_t^\alpha D_x^\beta\; u(0,0)\, t^\alpha x^\beta}{\alpha!\; \beta!}$$

$$= 1 + 0 + 0 + t^2 + 2tx + x^2$$

$$= 1 + t^2 + 2tx + x^2 + .......$$

**Exercise**

Let $u(x, y)$ satisfy $ux^2 + uy^2 = 1$ and let $u(o, y) = \Phi(y) \forall\, y$. Determine the Taylor's series expansion of $u(x, y)$ and sum the senses to show that

$$u(x, y) = \Phi_1(y)x + \Phi(y)$$

2nd Order Equation in $R^2$

We want to consider Partial Differential Equation of the form

$$F(t,\ x, u,\ \ ut, ux, uxx, utt, utx) = 0$$

Assume $utt = F\ (t,\ x, u, ut, ux, utx, uxx)$

The Cauchy problem in this case will be

$$\begin{cases} utt = F(t,\ x, u, ut, ux, utx, uxx) \\ u\ (0, x) = \Phi\ (x) \\ ut\ (0, x) = \Psi(x) \end{cases}$$

(t = 0 is not characteristic)

Thermo:- Let $\Phi\ (x), \Psi(x)$ be analytic in a nbd of the origin x = 0 in R and support f is analytic in a nbd of the point $\left(0,\, 0,\, \Phi(0),\, \Phi^1(0),\, \Psi^3(0)\, \Phi^4\right) \in \Re^n$. The Cauchy problem (3.14) – (3.16) has a solution u (t,x) which is defined and analytic in a nbd of the origin t = 0, x 0 of $R^2$ and this solution is unique in the clan of analytic functions

Outline of technique

From (3.15) and (3.16) we obtain

$$\frac{\partial^n u}{\partial x^n}\ (0{,}0) = \Phi^n(0)$$

$$\frac{\partial^{n+1} u}{\partial t\ \partial x^n}(0{,}0) = \Psi^n(0)$$

Differentiating (3.14) successively and using the calculated values, we find all the coefficients in the Taylor's series solution as before

**Example**

1) Let $u(x,y)$ satisfy $uxx + uyy = 0$ and let $u(0,y) = \sin y$

$$u(o,y) = y \ \forall y$$

Determine the Taylor's series expansion for the solution $u\,(x,y)$ of Partial Differential Equation and sum the series to show that

$$u(x,y) = \sin y \cosh x + xy$$

2) Find the former series solution for the Partial Differential Equation

$$y^2 uxx = x^2 uyy + 2(x^2 - y^2)u$$

$$u(o,y) = e^{-y2}$$

$$ux(o,y) = 0$$

$$uxx - uyy = f\,(x, y, u, ux, uy, uxy, uyy)$$

$$u(o,y) = \sin y$$
$$ux(o,y) = y$$
$$u\,(x,y) = u(0,0) + xuy(0,0) + yuy(0,0) +$$

$$\frac{1}{2}\,\xi uxx\, 10dx^2 + \dots\dots\dots\dots$$

$$u(0,0) = \sin - 0$$

$$ux(0,0) = 0$$

$$uy\,(0,0 = 1)$$

$$uy\,(0,0) = \cos y$$

$$uyy = -\sin y$$

3) $$y^2uxx = x^2uyy + 2\left(x^2 - y^2\right)u$$

$$u\left(0, 0\right) = 1$$

$$ux\left(0.0\right) = 0$$

$$uy\left(o, y\right) = 2y\ e^{-y2} - o$$

$$uxx = 0$$

$$uxy = 0$$

$$uyy = -2y\left(-2ye^{-y2}\right) - 2e^{-y2}$$

$$= -\underline{\underline{2}}$$

## 4.0 CONCLUSION

Cauchy problem can be summed up as the problem of finding a solution containing the integral strip of functions at any point of the integral strip while Cauchy Kovalevsky theorem simply states the condition under which an initial – value problem has an analytic function solution.

## 5.0 SUMMARY

We have seen in this unit that Cauchy problem, characteristics problem and Cauchy Kovalevsky theorem are useful in addressing certain types of partial differential equations

## 6.0 TUTOR MARKED ASSIGNMENTS

1. Solve the Cauchy problem

$$\begin{aligned} u_t - xuu_x &= 0 \qquad && -\infty < x < \infty,\ t \geq 0 \\ u(x,0) &= f(x) \qquad && -\infty < x < \infty. \end{aligned}$$

and find a class of initial data such that this problem has a global solution for all t.

and then,

Compute the critical time for the existence of a smooth solution for initial data, f, which is not in the above class.

2. Find an implicit formula for the solution u of the initial-value problem

$$u_t = (2x-1)tu_x + \sin(\pi x) - t,$$
$$u(x, t=0) = 0.$$

Evaluate u explicitly at the point (x = 0.5, t = 2)

## 7.0 REFERENCES/FURTHER READINGS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