NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH282

COURSE TITLE: MATHEMATICAL METHODS II

COURSE GUIDE

MTH282
MATHEMATICAL METHODS II

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Introduction

Mathematical Methods II is a continuation of the materials learnt in MTH281. It is a three -credit course. It is a compulsory course for any student majoring in mathematics at undergraduate level or B.Ed. Mathematics. It is also available to students offering Bachelor of Science (B.Sc.) Computer and Information Communication Technology. Any student with sufficient background in mathematics can also offer the course if he/she so wish though it may not count as credit towards graduation if it is not a required course in his/her field of study.

The course is divided into three modules as enumerated below:

Study Unit

Module 1

Unit 1	Vector Algebra
Unit 2	Vector Algebra-Product of Vectors
Unit 3	Vector Functions

Module 2

Unit 1	The Operator Del
Unit 2	Divergence of a Vector Field
Unit 3	The Curl of a Vector Field

Module 3

Unit 1	Jacobians
Unit 2	Orthogonal Curvilinear Coordinates

Module 4

Unit 1	Complex Numbers
Unit 2	Polar Operations with Complex Numbers

What You Will Learn in This Course

This course guide tells you briefly what the course is about, what course materials you will be using and how you can work with these materials. In addition, it advocates some general guidelines for the amount of time you are likely to spend on each unit of the course in order to complete it successfully. It gives you guidance in respect of Tutor-Marked Assignment which will be made available in the assignment file. There will be regular tutorial classes that are related to the course. It is advisable for you to attend these tutorial sessions.

Course Aim

The aim of the course is to provide you with an understanding of Mathematical Methods. It also aims to give you a modern way of solving complex problems in mathematics, make clear distinctions between the ways we handled problems in analysis, and provide you solutions to some problems that may arise in engineering and other areas of human endeavour.

Course Objectives

To achieve the aims set out, the course has a set of objectives. Each unit has specific objectives which are included at the beginning of the unit. You should read these objectives before you study the unit. You may wish to refer to them during your study to check on your progress. You should always look at the unit objectives after completion of each unit. By doing so, you would have followed the instructions in the unit.

Below are comprehensive objectives of the course as a whole. By meeting these objectives, you should have achieved the aims of the course as a whole. In addition to the aims above, this course sets to achieve some objectives. Thus, after going through the course, you should be able to:

- explain vector theory
- discuss differential operators
- discuss orthogonal curvilinear co-ordinates
- enumerate the concept of complex variables

Working through This Course

To complete this course you are required to read each study unit, read the textbooks and read other materials which may be provided by the National Open University of Nigeria.

Each unit contains self-assessment exercises and at certain points in the course you would be required to submit assignments for assessment purposes. At the end of the course, there is a final examination. The course should take you about a total of 17 weeks to complete. Below you will find listed all the components of the course, what you have to do and how you should allocate your time to each unit in order to complete the course on time and successfully.

This course entails that you spend a lot of time to read. You are advised to avail yourself the opportunities of the tutorial classes provided by the University.

Presentation Schedule

Your course materials have important dates for the early and timely completion and submission of your TMAs and attending tutorials. You should remember that you are required to submit all your assignments by the stipulated time and date. You should guard against falling behind in your work.

Assessment

There are three aspects to the assessment of the course. The first is made up of self-assessment exercises, second consists of the tutor-marked assignments and third is the written examination/end of course examination.

You are advised to do the exercises. In tackling the assignments, you are expected to apply information, knowledge and technique you gathered during the course. The assignments must be submitted to your facilitator for formal assessment in accordance with the deadlines stated in the presentation schedule and the assignment file. The work you submit to your tutor for assessment will count for 30% of your total course work. At the end of the course you will need to sit for a final or end of course examination of about three hour duration. This examination will count for 70% of your total course mark.

Tutor-Marked Assignment

The TMA is a continuous assessment component of your course. It accounts for 30% of the total score. You will be given four (4) TMAs to answer. Three of these must be answered before you are allowed to sit for the end of course examination. The TMAs would be given to you by your facilitator and returned after you have done the assignment. Assignment questions for the units in this course are contained in the assignment file. You will be able to complete your assignment from the information and material contained in your reading, references and study units. However, it is desirable in all Degree level of education to demonstrate that you have read and researched more into your references, which will give you a wider view point and may provide you with a deeper understanding of the subject.

Make sure that each assignment reaches your facilitator on or before the deadline given in the presentation schedule and assignment file. If for any reason you cannot complete your work on time, contact your facilitator before the assignment is due to discuss the possibility of an extension. Extension will not be granted after the due date.

Final Examination and Grading

The end of course examination for Mathematical Methods II will be for about 3 hours and it has a value of 70% of the total course work. The examination will consist of questions, which will reflect the type of self-testing, practice exercise and tutor-marked assignment problems you have previously encountered. All areas of the course will be assessed.

Use the time between finishing the last unit and sitting for the examination to revise the whole course. You might find it useful to review your self-test, TMAs and comments on them before the examination. The end of course examination covers information from all parts of the course.

Course Marking Scheme

Assignment	**Marks**
Assignments 1-4	Four assignments, best three marks of the four count at 10% each -30% of course marks.
End of course examination	70% of overall course marks.
Total	100% of course materials.

Facilitators/Tutors and Tutorials

There are 16 hours of tutorials provided in support of this course. You will be notified of the dates, times and location of these tutorials as well as the name and phone number of your facilitator, as soon as you are allocated a tutorial group.

Your facilitator will mark and comment on your assignments, keep a close watch on your progress and any difficulties you might face and provide assistance to you during the course. You are expected to mail your Tutor-Marked Assignment to your facilitator before the schedule date (at least two working days are required). They will be marked by your tutor and returned to you as soon as possible.
Do not delay to contact your facilitator by telephone or e-mail if you need assistance.

The following might be circumstances in which you would find assistance necessary, hence you would have to contact your facilitator if:

- You do not understand any part of the study or the assigned readings
- You have difficulty with the self-tests
- You have a question or problem with an assignment or with the grading of an assignment.

You should endeavour to attend the tutorials. This is the only chance to have face to face contact with your course facilitator and to ask questions which are answered instantly. You can raise any problem encountered in the course of your study.

To gain much benefit from course tutorials, prepare a question list before attending them. You will learn a lot from participating actively in discussions.

Summary

MTH282 (Mathematical Methods **II)** is a course that intends to provide solutions to problems normally encountered by engineers and mathematicians in the course of doing their normal jobs. It also serves as a tool which often enables the mathematicians to widen the frontiers of their analytical concerns to issues that have significant mathematical implications. Nevertheless, do not forget to apply the principles you have learnt to your understanding of Mathematical Methods II.

I wish you success in the course and I hope that you will find it comprehensive and interesting.

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MODULE 1 REVIEW OF VECTOR THEORY

UNIT 1 VECTOR ALGEBRA

CONTENTS

1.0 INTRODUCTION

The notion of vector has proved to be of greatest value in physics and mathematics. It is one of the most important concepts that would be studied in this course. It will be found to recur in a great variety of applications. A full appreciation of the value of vectors can come only after considerable experience with them. Two aspect of their usefulness worth emphasising are the following:

(1) Vectors enable one to reason about problems in space without use of co-ordinates axes. This is particularly true because the fundamental laws of physics do not depend on the particular position of co-ordinates axes in space. For example, the Newton's second law, that has the form:

$$F = m\underline{a}$$

Where F is the force vector and $\underline{a}$ is the acceleration vector of a moving particle of mass m. This does not necessarily depend on co-ordinate axis.

(2) Vector provides an economical "Short hand" for complicated formulas. For example, the condition that points P_1, P_2, P_3, and, P_4 lie in a plane can be written in the concise form as:

$\underline{a} \cdot \underline{b} \times \underline{c} = 0$

Where $\underline{a}$, $\underline{b}$ and $\underline{c}$ are vectors represented by the directed segment, $\overrightarrow{P_1P_2}$

$\overrightarrow{P_1P_3}$ and $\overrightarrow{P_1P_4}$ respectively. The significant of the dot (.) and cross ($\times$) will be explained later in this unit. The conciseness of vector formulae makes vector useful both for manipulation and understanding.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- discuss vectors and give its example
- define unit vectors, rectangular vectors, and resolve vectors into components
- perform algebraic functions on vectors
- solve related problems on vectors.

3.0 MAIN CONTENT

3.1 VECTOR ALGEBRA

3.1.1 Definitions

Definition 1: A vector in space is a combination of a magnitude (positive real number) and a direction.

A vector can be represented by a directed line segment $\overrightarrow{PQ}$ in space. It is convenient to represent vectors by bold letters such as **a, b, c,**…

Definition 2: Two vectors are said to be equal if their magnitude and directions are the same.

Definition 3: A zero vector is a vector whose magnitude is zero. We can represent zero vectors by a degenerated line segment $\overrightarrow{PP}$

3.1.2 Addition and Subtraction of Vectors

Given two vectors: **a, b,** then we can obtain a third vector **c=a + b** and if
b = c – a this defines the operation of subtraction.

Addition and subtraction of vectors obey the following laws:

(1) **a + b = b + a** [Commutative law of addition]

(2) **a + (b + c) = (a + b) + c** [Associative law]

(3) **a + b = c if b = c – a**

(4) **a + 0 = a**

(5) **a – a =0**

Definition 4: If h is a number and **a** is a vector then, the expression h**a** is defined as vector whose magnitude is $|h|$.

Thus: $|ha| = |h|.|a|$

Two vectors **a, b,** are said to be collinear (or linearly dependent) if there are scalars
h_1, h_2, not both zero, such that

$$h_1\mathbf{a} + h_2\mathbf{b} = \mathbf{0}$$

This is equivalent to asserting that **a** and **b** are represented by parallel line segments.

Definition 5: Three vectors **a, b, c** are said to be coplanar (or linearly dependent) if there are scalar k_1, k_2, k_3 not all 0 such that:

$$k_1\mathbf{a} + k_2\mathbf{b} + k_3\mathbf{c} = \mathbf{0}$$

In this case a**, b, c** can be represented by segments in the same plane. Let a and b be no collinear. Then every vector **c** coplanar with **a and b** can be represented in the form

$$c = k_1 a + k_2 b$$

For one and only one, choice of k_1, k_2.

3.1.3 Unit Vector

Unit vectors are vectors having unit length. Let a be any vector with length $|a| > 0$ then, $\frac{a}{|a|}$ is a unit vector denoted by $\hat{a}$ having the same direction as **a**

Then **a=a** $\hat{a}$

3.1.4 Rectangular Unit Vectors

The rectangular unit vectors $i.j, and, k$ are unit vectors having the direction of the positive x, y, and, z axes of a rectangular co-ordinates system. We use right-handed rectangular co-ordinate system unless otherwise specified.

3.1.5 The Component of a Vector

Any vector in 3-dimensions can be represented with initial point at the origin 0 of rectangular co-ordinates systems.

Let (A_1, A_2, A_3) be the rectangular co-ordinates of the terminal point of A with initial point at 0. The vectors $A_1 i$, $A_2 j$ and $A_3 k$ are called the rectangular component vectors.

The sum of $A_1 i$, $A_2 j$ and $A_3 k$ i.e.

A= $A_1 i + A_2 j + A_3 k$ is a vector.

The magnitude of A is

$$|A| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

In particular, if

$$r = xi + yj + zk$$

Then,

$$|r| = \sqrt{x^2 + y^2 + z^2} .$$

SELF-ASSESSMENT EXERCISE

1. Prove that for every four vectors x, y, z, and, w in space, scalars k_1, k_2, k_3, and, k_4
   Not all 0, can be found such that
   $k_1 x + k_2 y + k_3 z + k_4 w = 0$
2. Let O, A, B be points of space. Show that the mid-point M of the segment $\vec{AB}$ is located by the vector $\vec{OM} = \frac{1}{2}(\vec{OA} + \vec{OB})$
3. Prove that the medians of a triangle intersect in a point which is a trisection point of each median.

4.0 CONCLUSION

In this unit, you have learnt about vectors, vector addition and subtraction. In addition, we also consider component of vectors and unit vectors as a special type of vectors. You are to read carefully and master every bit of the material in this unit for you to follow the material in the next unit.

5.0 SUMMARY

Recall that in this unit, we defined a vector as quantities having magnitude and directions. Two vectors are said to be equal if the directions and magnitudes are equal.

Also, we defined a unit vector as having magnitude equal to one. Finally, any vector in 3-dimension can be represented with initial point at the origin 0 of a rectangular co-ordinates systems. Thus, if (A_1, A_2, A_3) represent the rectangular co-ordinates of the terminal point of A then:

$A = A_1 i + A_2 j + A_3 k$ is a vector.

Magnitude of this vector A is define as

$|A| = \sqrt{A_1^2 + A_2^2 + A_3^2}$ in particular if

$$r = xi + yj + zk$$

Then,

$$|r| = \sqrt{x^2 + y^2 + z^2}$$

6.0 TUTOR-MARKED ASSIGNMENT

1. Show that addition of vectors is commutative.
2. A car travels 3km due north, then 5 km northeast. Represent these displacements graphically and determine the resultant displacement by:
   a. Graphical method
   b. Analytical method.
3. If A, B, and, C are non-coplanar vectors and $x_1 A + y_1 B + z_1 C = x_2 A + y_2 B + z_2 C$, prove that it is necessary that $x_1 = x_2, y_1 = y_2, z_1 = z_2$.
4. Find the unit vector in the direction of the resultant of vectors $A = 2i - j + k,\ B = i + j + 2k \quad C = 3i - 2j + 4k.$

7.0 REFERENCES/FURTHER READING

Murray, R. Spiegel (1974). *Theory and Problems of Advanced Calculus*: *Schaum's Outline Series.* New York: McGraw-Hill Book Company.

Stephenson, G. (1977). *Mathematical Methods for Science Students.* London: Longman Group Limited.

Wilfred, Kaplan (1959). *Advanced Calculus.* Reading Massachusetts: Addison –Wesley Publishing Company Inc.

UNIT 2 VECTOR ALGEBRA - PRODUCT OF VECTORS

CONTENTS

1.0 INTRODUCTION

In this unit, you will learn about product of vectors. We shall differentiate between scalar and vector products. These two concepts are very useful in vector analysis because many physical phenomena can be explained in terms of either scalar or vector products. For example, work done can be calculated as a scalar product of displacement and the applied force. This implies that if we let **F** represent force and **d** represent the displacement, then work done **(W)** can be defined as:

$$\mathbf{W}=\mathbf{F}.\mathbf{d}$$

Other physical interpretation of vector product will be discussed in this unit. You are advised to study this unit very carefully.

2.0 OBJECTIVES

At the end of this unit, you should be to:

- define scalar product of vectors and give examples
- define vector product and give examples
- solve accurately, all related exercises in this unit.

3.0 MAIN CONTENT

3.1 Types of Vector Products

Two types of vector products are recognised, namely:

1) Scalar Product

2) Vector Product

In what now follows, we shall define and explain scalar product of vectors.

3.2 Scalar Product

Let **a** and **b** be vectors then, the scalar product of **a** and **b** is defined as

$$a.b = |a||b|\cos\theta \quad \text{..} (1)$$

θ is the angle between them. The quantity $|b|\cos\theta$ which appears in (1) can be interpreted as the component of **b** in the direction of **a**. We can write it as

$$comp_a^b = b\cos\theta \text{ ....} (2)$$

This component is a scalar which measures the length of the projection of **b** on a line parallel to **a.**

The notion of component is basic for application of vectors in mechanics. For example, the velocity vector or force vector can be described by giving its component in three mutually perpendicular directions. If a constant force F acts on an object moving from A to B along the segment $\overrightarrow{AB}$, the only component of F along AB does work. The work done is precisely the product of this component by the distance moved, thus:

Work= (force component in the direction of motion) . (distance)

Hence

Work = F $\cos\theta.|AB| = F.|AB|.$ .... (3)

Scalar product obeys the following laws:

(1) a.b = b.a (commutative)

(2) $a.(b+c) = a.b + a.c$ (Distributive law)

(3) $a.(kb) = (ka).b = k(a.b)$ Where k is a scalar.

We make the following inference from the scalar product of vectors.

(i) If $a.b = 0$ then a is perpendicular to b.

(ii) It is not permitted to cancel in an equation of the form

a.b=a.c and conclude that b=c.

For equation a.b=a.c it implies only that a.b = a.c = a.(b-c) = 0 that is a is perpendicular to b-c

We note that:

$i.i = 1, j.j = 1, and, k.k = 1$ and $i.j = 0, j.k = 0, and, k.i = 0$ .... (4)

Given that:

$a = a_1 i + a_2 j + a_3 k, and, b = b_1 i + b_2 j + b_3 k$ .... (5)

Then,

$a.b = (a_1 i + a_2 j + a_3 k).(b_1 i + b_2 j + b_3 k)$

$= a_1 b_1 + a_2 b_2 + a_3 b_3$ .... (6)

SELF-ASSESSMENT EXERCISE 1

Show that:

$(A_1 i + A_2 j + A_3 k).(B_1 i + B_2 j + B_3 k) = A_1 B_1 + A_2 B_2 + A_3 B_3$

3.2.1 Direction Cosines

Recall from unit1 that if **a** is a vector of length 1 i.e. $|a| = 1$, then **a** will be termed a unit vector. In this case we denote **a** as:

$a = a_x i + a_y j + a_z k$

Then,
$a_x = a.1 = 1.1\cos\alpha = \cos\alpha.$

Where α is the angle between $a, and, i.$ This is the angle between a and the positive x direction. In a similar manner

$a_y = \cos\beta, a_z = \cos\gamma$
Where β, and, γ are the angles between **a** and the $y, and, z,$ directions respectively.

From
a.b=$|a||b|\cos\theta$ then

$$\cos\theta = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}} \qquad \ldots\ldots\ldots\ldots\ldots (7)$$

SELF-ASSESSMENT EXERCISE 2

Given that $u = i - j + k, v = i + j + 2k, w = 3i - k$

Evaluate: (a) $u + v + w, (b), 2u - v, (c), u.v$

3.3 The Vector Product

The vector product of a and b in that order is a vector $c = a \times b$ which is 0 if **a** and **b** are collinear and otherwise in such that:

$c = ab\sin\theta$

The vector product satisfies the following laws:

(1) $a \times b = -(b \times a)$ (Anti- commutative law)

(2) $a \times (b + c) = a \times b + a \times c$ (Distributive law)

(3) $a \times (kb) = k(a \times b)$

(4) $a \times a = 0$

(5) $i \times j = k, j \times k = i, k \times i = j$

(6) $i \times i = 0, j \times j = 0, k \times k = 0$

(7) Let $a = a_x i + a_y j + a_z k$
$b = b_x i + b_y j + b_z k$

Then,

$$a \times b = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$(a_y b_z - a_z b_y)i + (a_z b_x - a_x b_z)j + (a_x b_y - a_y b_x)k.$ ………………… (8)

Also you should note that:

$|a \times b| =$ area of parallelogram with sides **a** and **b**

SELF-ASSESSMENT EXERCISE 3

Given the vectors $a = 2i - j$, $b = i + j + k$ $c = -2i + k$

Evaluate the following

(i) $a \times b$ (ii) $c \times b$ (iii) $(a \times b) \times c$ (iv) $a.(a \times b)$ (v) $a \times (a \times b)$

3.4 Triple Product

In this section, we shall consider (1) The Scalar Triple Product (2) The Vector Triple Product.

3.4.1 The Scalar Triple Product

The scalar $a \times b.c$ is known as the scalar triple product a, b, c, in that order. We need to remark here that parentheses are not needed since $a \times (b.c)$ would have no meaning.

The scalar triple product satisfies the following laws:

(1) $a \times b.c = 0$ if and only if a,b,c, are coplanar

(2) $a \times b.c =$ volume of parallelepiped with edges a, b,and c

(3) $a \times b.c = a.b \times c.$

(4) $$axb.c = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

(5) $a \times b.c = -b \times a.c = -b.a \times c$

SELF-ASSESSMENT EXERCISE 4

Evaluate the following:

(1) (i) $i.j \times k$ (ii) $(i + j).k + j$

(2) Given the vectors

$$u = i - 2j + k$$
$$v = 3i + k$$
$$w = j - k$$

Evaluate, (a) $u.v \times w$ (b) $w \times v.u$ (c) $(u + v).(v + w) \times w$

3.4.2 The Vector Triple Products

The expression $(a \times b) \times c$ and $a \times (b \times c)$ are known as vector triple products.

Note that the parentheses are necessary because for example;

$(i \times i) \times j = 0$ while $i \times (i \times j) = i \times k = -j$

The following identities are to be noted

(1) $a \times (b \times c) = (a.c)b - (a.b)c$

(2) $(a \times b) \times c = (c.a)b - (c.b)a$

We can prove the identity stated in (1) i.e.

$a \times (b \times c) = (a.c)b - (a.b)c$

Proof: Let
$$a = a_x i + a_y j + a_z k$$
$$b = b_x i + b_y j + b_z k$$
$$c = c_x i + c_y j + c_z k$$

Taking component i , then

$$i.a\times(b\times c)=\begin{vmatrix} 1 & 0 & 0 \\ a_x & a_y & a_z \\ \begin{vmatrix} b_y & b_z \\ c_c & c_z \end{vmatrix} & \begin{vmatrix} b_z & b_x \\ c_z & c_x \end{vmatrix} & \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} \end{vmatrix}$$

$= \ a_y(b_x c_y - b_y c_x) - a_z(b_z c_x - b_x c_z)$

$= b_x(a_x c_x + a_y c_y + a_x c_z) - c_x(a_x b_x + a_y b_y + a_z b_z)$

$= i.[(a.c)b - (a.b)c]$ … (9)

We can similarly prove the above for y and z components.

3.5 Axiomatic Approach to Vector Analysis

Recall from our previous section (unit 1 section 3.1.5) that we can represent a vector:

$r = xi + yj + zk$ is determined when its components (x, y, z) relative to some co-ordinate system are known. In adopting an axiomatic approach it is natural for us to make the following:

Definition: A 3-dimensional vector is an ordered triplet of real numbers (A_1, A_2, A_3).

With the above definition, we can define equality, vector addition and subtraction, etc.

Let $A = (A_1, A_2, A_3)$ and $B = (B_1, B_2, B_3)$ then

1. A= B if and only if $A_1 = B_1, A_2 = B_2, A_3 = B_3$

2. A+ B = $(A_1 + B_1, A_2 + B_2, A_3 + B_3)$

3. A - B = $(A_1 - B_1, A_2 - B_2, A_3 - B_3)$

4. 0 = (0, 0, 0)

5. $mA = m(A_1, A_2, A_3) = (mA_1, mA_2, mA_3)$

6. A.B = $A_1.B_1 + A_2.B_2 + A_3.B_3$

7. Length or magnitude of A=$|A| = \sqrt{A.A} = \sqrt{A_1^2 + A_2^2 + A_3^3}$

From these, we obtain other properties of vectors, such as $A + B = B + A,$
$(A+B)+C = A+(B+C), A.(B+C) = A.B + A.C.$ By defining the unit vectors:

$i = (1,0,0) \quad j = (0,1,0) \quad k = (0,0,1)$

We can show that:

$A = A_1 i + A_2 j + A_3 k$

In like manner we can define A$\times B = (A_2 B_3 - A_3 B_2, A_3 B_1 - A_1 B_3, A_1 B_2 - A_2 B_1)$

After this axiomatic approach has been developed, we can interpret the result geometrically or physically. For example we can show that A.B= $AB\cos\theta$

and $|A \times B| = AB \sin\theta$

4.0 CONCLUSION

In this unit, you have learnt about scalar multiplication and cross multiplication of vectors. We have also considered vector triple products. The application of these concepts will be apparent as we proceed further in this course.

5.0 SUMMARY

In summary, we recap the following about vector products, namely:

- Given that A and B are vectors, then the scalar product of A and B is defined as,
  A.B=$|A||B|\cos\theta$
- If A=$A_1 i + A_2 j + A_3 k$, and B= $B_1 i + B_2 j + B_3 k$ then
  A.B= $A_1 B_1 + A_2 B_2 + A_3 B_3$
- If A.B=0 and A and B are not null vector, then A and B are perpendicular.

- Also $A\times B = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$
- $|A\times B| =$ the area of parallelogram with sides A and B.
- If $A\times B = 0$ and A and B are not null vectors, then A and B are parallel.
- $A\times B = -B\times A$

We also note the following about triple products of vectors. Dot and cross multiplication of three vectors A, B and C may produce meaningful products of the
form (A.B) C, A.$(B\times C)$ and $A\times(B\times C)$.

The following laws are valid:

- (A.B)C $\neq$ A(B.C) in general
- $A\times(B\times C) \neq (A\times B)\times C$
- $A\times(B\times C) = (A.C)B - (A.B)C$
- $(A\times B)\times C = (A.C)B - (B.C)A$

6.0 TUTOR-MARKED ASSIGNMENT

1. Prove that $A.(B+C) = A.B + A.C$
2. Evaluate $|(A+B).(A-B)|$ if $A = 2i - 3j + 5k$ and $B = 3i + j - 2k$
3. Find the unit vector perpendicular to the plane of the vectors A=3i-2j+4k and B= i+j-2k
4. Given that A=2i+j-3k, B=i-2j+k, C=-i+j-4k, then find (a) A. (B$\times C$) (b) C. (A$\times B$)

7.0 REFERENCES/FURTHER READING

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UNIT 3 VECTOR FUNCTIONS

CONTENTS

1.0 INTRODUCTION

In this unit, you will learn about vector functions. You will also learn about limit and continuity of vector functions. You will find derivatives of vectors and this will allows you to determine vector velocity. Finally, we shall give geometric interpretation to vector derivatives.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define limit and continuity of vector functions
- examine the derivatives of vector functions
- give a geometric interpretation to vector derivatives and be able to determine vector velocity
- solve all related problems on vector functions.

3.0 MAIN CONTENT

3.1 Vector Function of One Variable

Given an interval $t_1 \leq t \leq t_2$, suppose we assign a vector u in space, then u is said to be given a vector function of t over that interval.

For example,

$$u = t^2 i + t^3 j + \sin tk$$

Where i, j, k form a triple of mutually perpendicular unit vectors. If a co-ordinate system is chosen in space then the vector u can always be expressed in the form

$$u = u_x i + u_y j + u_z k$$

Where $u_x, u_y, and_, u_z$ are the corresponding components. These components themselves depend on *t*.

Suppose the axes are fixed and independent of ***t***, then we can write

$$u_x = f(t), \quad u_y = g(t) \text{ and } u_z = h(t), \quad t_1 \le t \le t_2$$

Thus, a vector functions of t, determines three scalar functions of t. Conversely, if $f(t), g(t)$ and $h(t)$ are three scalar functions of t defined on the interval, $t_1 \le t \le t_2$ then the vector *u* is given as

$u = f(t)i + g(t)j + h(t)k$ This is a vector function of *t*.

3.2 Limit and Continuity of Vector Function

The vector function $u = u(t)$ is said to have a limit v as t approaches t_o. This implies that $\lim\limits_{t \to t_o} u(t) = v$ if $|u(t) - v| < \varepsilon$ whenever $|t - t_o| < \delta$.

The implication of this is that the difference between $u(t)$ and v can be made arbitrarily small for *t* sufficiently close to t_0

Continuity: The function $u = u(t)$ is said to be continuous at the value t_0 if one has

$$\lim_{t \to t_0} u(t) = u(t_0)$$

We can establish by prove that $u(t)$ is continuous at a value t_o, if and only if its component u_x, u_y, and, u_z are all continuous.

Also given two vectors $u_1(t), and, u_2(t)$ such that they are both continuous functions for $t_1 \le t \le t_2$ then the functions:

$u_1(t) + u_2(t)$, $u_1(t).u_2(t)$ and $u_1(t) \times u_2(t)$ are continuous functions of t over the defined interval.

3.3 Derivative of a Vector Function

Velocity Vector: The derivative of the vector function $u = u(t)$ is defined as a limit.

$$\frac{du}{dt} = \lim_{\Delta t \to 0} \frac{u(t + \Delta t) - u(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta u}{\Delta t}$$

We can define the above in terms of component as follows:

$$u(t + \Delta t) - u(t) = [f(t + \Delta t) - f(t)]i + [g(t + \Delta t) - g(t)]j + [h(t + \Delta t) - h(t)]k$$

Hence, on dividing by Δt and letting $\Delta t \to 0$ one finds

$$\frac{du}{dt} = f'(t)i + g'(t)j + h'(t)k$$

$$= \frac{du_x}{dt}i + \frac{du_y}{dt}j + \frac{du_z}{dt}k$$

Therefore, to differentiate a vector function, each component must be differentiated separately.

3.4 Geometric Interpretation

Let S be the distance traversed by P from $t = t_1$ up to time t, then

$$\frac{ds}{dt} = \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}$$

$$= \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}$$

Let $u = \vec{OP}$ the position vector of the moving point P, then the vector

$v = (d/dt)\vec{OP}$ is the tangent to the curve traced by P and has at each point a magnitude

$$|v| = \left|\frac{du}{dt}\right| = \sqrt{f'(t)^2 + g'(t)^2 + h'(+)^2}$$

The conclusion drawn from above is that v is precisely the velocity vector of the moving point P for v is the tangent to the path and has magnitude $v = ds/dt$ (speed) and clearly points in the direction of motion.

We then have the following rule:

$\frac{d}{dt}\vec{OP} =$ velocity of P, where 0 is a fixed reference point.

Finally we consider the following differentials: Given that

$A(x, y, z) = A_1(x, y, z)i + A_2(x, y, z)j + A_3(x, y, z)k$
Then,

$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz$, is the differential of A.

Remarks: Derivatives of products obey rules similar to those for scalar functions. However when cross product are involved, the order may be important. Some examples are:

(a) $\frac{d}{dx}(\phi A) = \phi\frac{dA}{dx} + \frac{d\phi}{dx}A$

(b) $\frac{\partial}{\partial x}(A.B) = A.\frac{\partial B}{\partial x} + \frac{\partial A}{\partial x}.B$

(c) $\frac{\partial}{\partial z}(A \times B) = A \times \frac{\partial B}{\partial z} + \frac{\partial A}{\partial z} \times B$

Solved Problems

1. Suppose $u = r\cos(wt)i + r\sin(wt)j$ where r and w are constants. Let the point P moves according to the equations $x = r\cos(wt)$, $y = r\sin(wt)$ which represent the circle $x^2 + y^2 = r^2$ in the xy-plane. The polar angle θ of P at time t is $\theta = wt$. Find the angular velocity, the vector velocity and the speed of the movement.

Solution:

The angular velocity of P

$= \frac{d\theta}{dt} = w$

2. Vector velocity

$$v = \frac{dv}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j = -rw\sin(wt)i + rw\cos(wt)j$$

3. Speed is

$$\frac{ds}{dt} = \sqrt{r^2w^2\sin^2(wt)i + r^2w^2\cos^2(wt)} \quad = \quad rw, w \geq 0$$

Problem 2: If $r = (t^3 + 2t)i - 3e^{-2t}j + 2\sin 5tk$, Find (a) $\frac{dr}{dt}$ (b) $\left|\frac{dr}{dt}\right|$,

(c) $\frac{d^2r}{dt^2}$

(d) $\left|\frac{d^2r}{dt^2}\right|$, at t=0 and give a possible physical significance.

Solution:

(a) $\frac{d}{dt}(t^3 + 2t)i + \frac{d}{dt}(-3e^{-2t})j + \frac{d}{dt}(2\sin 5t)k$

$= 3t^2 + 2)i + 6e^{-2t}j + 10\cos 5tk$

At t=0 $dr/dt = 2i + 6j + 10k$

(b) From (a) $|dr/dt| = \sqrt{(2)^2 + (6)^2 + (10)^2} = \sqrt{140} = 2\sqrt{35}$ at t=0.

(c) $\frac{d^2r}{dt^2} = \frac{d}{dt}(\frac{dr}{dt}) = \frac{d}{dt}\{(3t^2 + 2)i + 6e^{-2t}j + 10\cos 5tk) = 6ti - 12e^{-et}j - 50\sin 5tk$

At t=0, $d^2r/dt^2 = -12j$

(d) From (c) $\left|d^2r/dt^2\right| = 12$ at t=0.

If t represents time, these represent respectively the velocity, magnitude of the velocity, acceleration and magnitude of the acceleration at t=0 of a particle moving along the space curve $x = t^3 + 2t$ $y = -3e^{-2t}$, $z = 2\sin 5t$

4.0 CONCLUSION

In this unit, you have learnt about vector function, limit and continuity of vector functions, derivatives of vectors and geometrical interpretations of vector derivatives. In the next unit, we shall extend these derivatives into partial derivatives and apply the results in the orthogonal curvilinear co-ordinates.

5.0 SUMMARY

We now recap what you have learnt in this unit as follows:

- Given an interval $t_1 \le t \le t_2$, a vector function u can be assigned such that
  $u = u(t)$. For example, $u(t) = t^2 i + \sin tj + \cos^2 tk$ is a vector function of *t*.
- We can define a limit of the vector function as:
  $\lim u(t) = v$ if $|u(t) - v| < \varepsilon$ whenever $|t - t_0| < \delta$. This implies that the difference between $u(t)$ and v can be made arbitrarily small for *t* sufficiently close to t_0
- We define continuity of $u(t)$ as:
  $\lim_{t \to t_0} u(t) = u(t_0)$
  If $u(t) = u_x(t)i + u_y(t)j + u_z(t)k$ then, we can prove that $u(t)$ is continuous if and only if all the components of $u(t)$ are continuous.
- We define derivatives of vectors as follows:
  If $u(t) = u_x(t)i + u_y(t)j + u_z(t)k$ then
  $$\frac{du(t)}{dt} = \frac{du_x(t)}{dt} i + \frac{du_y}{dt} j + \frac{du_z}{dt} k$$
  We also give a geometric interpretation of the derivatives of vectors.

6.0 TUTOR-MARKED ASSIGNMENT

1. Prove that $\frac{d}{du}(A.B) = A.\frac{dB}{du} + \frac{dA}{du}..B$ where A and B are differentiable functions of u.
2. If $A = x^2 \sin yi + z^2 \cos yj - xy^2 k$, find dA
3. A particle moves along a space curve, $r = r(t)$, where t, is the time measured from some initial time. If $v = |dr/dt| = ds/dt$ is the magnitude of the velocity of the particle (s is the arc length along

the space curve measured from the initial position), prove that the acceleration ***a*** of the particle is given by:

$$a = \frac{dv}{dt}T + \frac{v^2}{\rho}N$$

Where T and N are unit tangent and normal vectors to the space curve and

$$\rho = \left|\frac{d^2r}{ds^2}\right|^{-1} = \left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right\}^{-1/2}$$

4. Prove that grad $f(r) = \frac{f'(r)}{r}r$, where $r = \sqrt{x^2 + y^2 + z^2}$ and $f'(r) = df/dr$ is assumed.

7.0 REFERENCES/FURTHER READING

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MODULE 2 DIFFERENTIAL OPERATORS

UNIT 1 THE OPERATOR DEL (∇)

CONTENTS

1.0 INTRODUCTION

In this unit, you will learn about certain differential operations which can be performed on scalar and vector fields. These operations have wide-ranging applications in the physical sciences. The most important operations are those of finding the gradient of a scalar field and the divergence and curl of a vector field. Central to all these differential operations is the vector operator ∇ which is called Del (or sometimes, nabla).

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define the operator Del (∇)
- apply the operator in finding gradient of function $\phi(x, y, z)$
- give physical interpretation to gradient of $\phi(x, y, z)$
- solve correctly, exercises involving the use of gradient.

3.0 MAIN CONTENT

3.1 Operator Del (∇)

Consider the operator $\nabla(del)$ defined by:

$$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k \qquad \text{……………… ……….. (1)}$$

Equation (1) is called operator Del. It has a lot of physical application in vector analysis as we shall see shortly.

If $\phi(x, y, z)$ and $A(x, y, z)$ have continuous first partial derivatives in a region, we can define the gradient of $\phi(x, y, z)$ as:

(1) Gradient: The gradient of $\phi(x, y, z)$ is defined by:

$$grad\phi = \nabla\phi = \frac{\partial\phi(x, y, z)}{\partial x}i + \frac{\partial\phi(x, y, z)}{\partial y}j + \frac{\partial\phi(x, y, z)}{\partial z}k$$

3.2 Interpretation

One interesting application of $grad\phi$ can be view as follows:

$$\phi(x, y, z) = c \qquad \text{…..................................... (2)}$$

Let equation (2) be equation of a surface then, $\nabla\phi$ is normal to this surface. To see this, let $\phi(x, y, z)$ be a scalar field.

Consider the differential defined by:

$$dr = dxi + dyj + dzk \qquad \text{…………………….…. (3)}$$

The corresponding differential in $\phi(x, y, z)$ is

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \qquad \text{………………………..…. (4)}$$

$$= \nabla\phi.dr \qquad \text{…………………………. (5)}$$

Now if $\phi = c$ then $d\phi = 0$ therefore,

$$\nabla\phi.dr = 0 \qquad \text{………………………… (6)}$$

Hence $\nabla\phi$ is normal to the surface given by the equation $\phi(x,y,z)=c$

Examples:

(1) Find the gradient of the scalar field $\phi = xy^2z^3$

Solution: $\nabla\phi = y^2z^3i + 2xyz^3j + 3xy^2z^3k$

(2) Given the function $\phi(x,y,z)=x^2y+yz$ at the point (1, 2,-1) find its rate of change with distance in the direction $a = i + 2j + 3k$. At this same point, what is the greatest possible rate of change with distance and in which direction does it occur?

Solution:

Gradient of ϕ is given by

$$\nabla\phi = \nabla(x^2y+yz) = 2xyi + (x^2+z)j + yk$$

Now at the point (1, 2,-1), $\nabla\varphi = 4i + 2k$

The unit vector in the direction of a is $\hat{a} = \frac{1}{\sqrt{14}}(i+2j+3k)$, so the rate of change of ϕ with distance s in this direction is

$$\frac{d\phi}{ds} = \nabla\phi.\hat{a} = \frac{1}{\sqrt{14}}(4+6) = \frac{10}{\sqrt{14}}$$

From the above discussion, at the point (1, 2,-1), $d\phi/ds$ will be greatest in the direction of $\nabla\phi = 4i + 2k$ and has the value $|\nabla\phi| = \sqrt{20}$ in this direction.

The gradient obeys the following laws:

$$grad(f+g) = gradf + gradg$$
$$grad(fg) = fgradg + ggradf$$

In addition to these, we note that the gradient operation also obey the chain rule as in ordinary differential calculus, i.e. if ϕ and φ are scalar fields in region R, then

$$\nabla[\phi(\varphi)] = \frac{\partial\phi}{\partial\varphi}\nabla\varphi \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(7)}$$

4.0 CONCLUSION

In this unit, you have learnt about gradient of vector and scalar fields. In the next unit, we shall examine divergence of a vector field and how it relies on the operator Del. It is very important for you to learn this operator very well before you make any meaningful progress beyond this point.

5.0 SUMMARY

You have learnt the following in this unit:

- The operation Del (∇) is defined as
  $\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$
- If $\phi(x, y, z)$ is a scalar field then the gradient of $\phi(x, y, z)$ is defined as
  grad $\phi = \frac{\partial \phi}{\partial x}i + \frac{\partial \phi}{\partial y}j + \frac{\partial \phi}{\partial z}k$.
- The corresponding differential of $\phi(x, y, z)$ is given as
  $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = \nabla \phi . dr$
- Where,
  $dr = dxi + dyj + dzk$
- If $\phi(x, y, z) = c$ then, $d\phi = 0$ this implies that $\nabla \phi . dr = 0$, hence $\nabla \phi$ is normal to the surface given by $\phi(x, y, z) = c$

6.0 TUTOR-MARKED ASSIGNMENT

1. If $\phi = x^2 yz^3$ and A= $xzi - y^2 j + 2x^2 yk$ find (i) $\nabla \phi$ (ii) $\nabla . A$
2. Prove that $\nabla \phi$ is a vector perpendicular to the surface $\phi(x, y, z) = c$ where c is a constant.
3. If $\phi = 2x^2 y - xz^3$ find $\nabla \phi$ and $\nabla^2 \phi$

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UNIT 2 DIVERGENCE OF A VECTOR FIELD

CONTENTS

1.0 INTRODUCTION

Divergence can be considered as a quantitative measure of how much a vector field diverges (spread out) or converges at any given point. For example, if we consider the vector field $v(x, y, z)$ describing the local velocity at any point in a fluid then the divergence is equal to the net rate of outflow of fluid per unit volume, evaluated at a point. We will be exposed to mathematical exposition of this very important concept in this unit. The prerequisite to our learning this unit is the thorough understanding of the unit 1 of this module.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explain the divergence of a vector field
- explain the Laplacian
- solve the exercises at the end of this unit.

3.0 MAIN CONTENT

3.1 The Divergence of a Vector Field

Suppose we are given a vector field $v(x, y, z)$ in the domain D of space, given three scalar functions v_x, v_y, v_z. suppose these functions possess partial derivatives in D then the divergence is defined as:

$$divv = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Formula (1) can be written in the symbolic form:

$divv = \nabla.v$ which implies:

$$\nabla.v = (\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k).(v_x i + v_y j + v_z k) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \quad .... (2)$$

The divergence defined above has a physical significance. In fluid dynamics, it appears as a measure of the rate of decrease of density at a point. More precisely,
Let $u = u(x, y, z, t)$ denote the velocity vector of a fluid motion and let $\rho = \rho(x, y, z, t)$ denote the density.

Then $v = \rho u$ is a vector whose divergence satisfies the equation.

Then, $$divv = -\frac{\partial \rho}{\partial t} \quad ………………… (3)$$

Equation (3) is called continuity equation of fluid mechanics. If fluid is incompressible, this reduce to the simpler equation

$$divu = 0 \quad ………………… (4)$$

The divergence also plays an important role in the theory of electromagnetic fields. To see this, we note that the divergence of the electric force vector E satisfies the equation defined by:

$$divE = 4\pi\rho \quad ………………… (5)$$

Where ρ is the charge density. Thus where there is no charge, equation (5) reduces to

$$divE = 0 \quad …………………. (6)$$

The divergence has the following basic properties:

(1) *div* (*u*+*v*) =*div u*+*div v*

(2) $div(fu) = fdivu + gradf.u \quad ……………........…. (7)$

3.1.1 The Laplacian

Let $w = f(x, y, z)$ then the Laplacian of w is defined as

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \qquad \text{……………………...…. (8)}$$

The origin of the ∇^2 lies in the interpretation of ∇ as a vector differential operator defined before as:

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \qquad \text{………………........…. (9)}$$

Symbolically,

$$\nabla^2 = \nabla.\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \text{………………………. (10)}$$

If $z = f(x, y)$ and has second derivatives in the domain D and

$$\nabla^2 z = 0 \qquad \text{……………………. (11)}$$

In the domain D, the z is said to be harmonic in D. We also used the same term for a function of three variables which has continuous second derivatives in a domain D in space and whose Laplacian is 0 in D. The two equations for harmonic functions:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0, \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0 \qquad \text{………………………....…. (12)}$$

are known as the Laplacian equations in two and three dimensions respectively.

Remark: In the theory of elasticity, we have the following equation:

$$\frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} = 0 \qquad \text{…………………….……. (13)}$$

The combination which appears above can be expressed in terms of the Laplacian as follows:

$$\nabla^2(\nabla^2 z) = \frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} \qquad \text{……….................………. (14)}$$

The expression in (14) is called biharmonic expression whose solutions are termed biharmonic functions. Harmonic functions arise in the theory of electromagnetic fields, in fluid dynamics, in the theory of heat conduction, and many other parts of physics.

3.2 Illustrative Examples:

1) Given that A= $xzi - y^2 j + 2x^2 yk$, find the divergence of A.

Solution: The divergence of A is defined as

$$\nabla.\text{A}= \begin{aligned}&(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k).(xzi - y^2 j + 2x^2 yk)\\ &= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2 y) = z - 2y\end{aligned}$$

2) Prove that $\nabla.(\phi A) = (\nabla\phi).A + \phi(\nabla.A)$

Solution: $\nabla.(\phi A) = \nabla.(\phi A_1 i + \phi A_2 j + \phi A_3 k)$

$$= \frac{\partial}{\partial x}(\phi A_1) + \frac{\partial}{\partial y}(\phi A_2) + \frac{\partial}{\partial z}(\phi A_3)$$

$$= \frac{\partial\phi}{\partial x}A_1 + \frac{\partial\phi}{\partial y}A_2 + \frac{\partial\phi}{\partial z}A_3 + \phi(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z})$$

$$= (\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k).(A_1 i + A_2 j + A_3 k) + \phi(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k).(A_1 i + A_2 j + A_3 k)$$

$= (\nabla\phi).A + \phi(\nabla.A)$

3). Given that $\phi = 2x^2 y - xz^3$ find $\nabla^2\phi$

Solution: $\nabla^2\phi =$ Laplacian of $\phi = \nabla.\nabla\phi = \frac{\partial}{\partial x}(4xy - x^2) + \frac{\partial}{\partial y}(2x^2) + \frac{\partial}{\partial z}(-3xz^2)$

$$= 4y - 6xz$$

4.0 CONCLUSION

In this unit, you have learnt about divergence of vector field, you have also learnt about Laplacian and discussed various applications of these concepts to physical phenomenal. You are advised to read this unit properly and carefully, before moving to other unit.

5.0 SUMMARY

It should be noted that divergence is a measure of how much a vector field spread out or converges.

If $v(x, y, z)$ is a vector field, then its divergence is defined as

We may derive from the definition of divergence and also define Laplacian as follows:

$$\nabla.(gradf) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

We also considered other physical application such as application of biharmonic functions of the form

$\nabla^2(\nabla^2 z) = \frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial^2 x \partial^2 y} + \frac{\partial^4 z}{\partial y^4}$ in the theory of elasticity.

6.0 TUTOR-MARKED ASSIGNMENT

1. Given that the vector field $v = 2xi + yj - 3zk$, verify that the divergence of v (*div v*) is zero.
2. Evaluate [$(xi - yj).\nabla](x^2 i - y^2 j + z^2 k)$
3. Given that $\phi = x^2 yz^3$ and A= $xzi - y^2 j + 2x^2 yk$. Evaluate $div(\phi A)$

7.0 REFERENCES/FURTHER READING

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UNIT 3 THE CURL OF A VECTOR FIELD

CONTENTS

1.0 INTRODUCTION

In this unit, we will learn about curl of a vector field. This concept has a wide range of application in physical phenomenal such as electromagnetic theory. Those concepts we learnt earlier such as gradient of vector field and divergence theory will be applied later in the theory of orthogonal curvilinear co-ordinates systems.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define curl of vector field correctly
- interpret the physical implication of curl of vector field
- solve all the associated mathematical problems involving the curl of vector fields.

3.0 MAIN CONTENT

3.1 The Curl of a Vector Field

We can define the curl of a vector field as follows:

Let $v(x, y, z)$ be a vector field then, the curl of vector $v(x, y, z)$ is

Curl v= $\nabla \times v = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$ .... (1)

Equation (1) can be expressed as:

Curl v = $(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z})i + (\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x})j + (\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y})k$ ……………. (2)

This vector field has a meaning independent of the choice of axes. We shall see this in the treatment of orthogonal curvilinear co-ordinates to be considered in the next module.

The curl of vector field is important in the analysis of the velocity field of fluid dynamics and in the analysis of electromagnetic force fields. For example, curl can be interpreted as measuring angular motion of a fluid and the condition is:

Curl v = 0 …………………… (3)

For a velocity field v characterises what are termed irrotational flows. The analogous equation is given as:

Curl E = 0 …………………… (4)

For the electric force vector E, it holds when only electrostatic forces are present.

Recall that: if $\nabla \times V = 0$ in a region, we say that the flow is irrotational in that region. The implication of this is that the circulation around a closed curve in a simple region where the flow is irrotational is zero. If the fluid is incompressible and there is no distribution of sources or sink in the region, we have also $\nabla . V = 0.$ since the condition $\nabla \times V = 0$ implies the existence of a potential φ such that

$$V = \nabla\varphi \quad \text{………………………} \quad (5)$$

We see that if also $\nabla . V = 0$ then it follows that $\nabla . \nabla\varphi = \nabla^2\varphi = 0.$ That is, in the flow of an incompressible irrotational fluid without distributed sources or sinks the velocity vector is the gradient of a potential φ which satisfies the equation

$$\nabla^2\varphi = 0 \quad \text{or} \quad \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial x^2} = 0 \quad \text{……} \ (6)$$

Equation (6) is known as Laplace's equation already discussed in (Unit 2, Module 2)

Generally, in any continuously differentiable vector field F with zero divergence and curl in a simple region, the vector F is the gradient of a solution of Laplace's equation.

Solutions of this equation are called harmonic functions.

3.2 Illustrative Examples

1) If A = $xz^3 i - 2x^2 yzj + 2yz^4 k$. Find $\nabla \times A$ (or curl A) at the point (1,-1, 1)

Solution:

$$\nabla \times A = (\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k) \times (xz^3 i - 2x^2 yzj + 2yz^4 k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2 yz & 2yz^4 \end{vmatrix}$$

$$= [\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2 yz)]i + [\frac{\partial}{\partial z}(xz^3) - \frac{\partial}{\partial x}(2yz^4)]j + [\frac{\partial}{\partial x}(-2x^2 yz - \frac{\partial}{\partial y}(xz^3)]k$$

$= (2z^4 + 2x^2 y)i + 3xz^2 j - 4xyzk = 3j + 4k$ at point (1,-1, 1)

2) If $A = x^2 yi - 2xzj + 2yzk$ find $CurlCurlA$

Solution:

$$curlcurlA = \nabla \times (\nabla \times A)$$

$$= \nabla \times \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -2xz & 2yz \end{vmatrix} = \nabla \times [(2x + 2z)i - (x^2 + 2z)k]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + 2z & 0 & -x^2 - 2z \end{vmatrix} = (2x + 2)j$$

3) Prove that $\nabla\times(\nabla\phi)=0$

Solution $\nabla\times(\nabla\phi)=\nabla\times(\frac{\partial\phi}{\partial x}i-+\frac{\partial\phi}{\partial y}j+\frac{\partial\phi}{\partial z}k)$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= [(\frac{\partial}{\partial y}(\frac{\partial\phi}{\partial z})-\frac{\partial}{\partial z}(\frac{\partial\phi}{\partial y})]i+[\frac{\partial}{\partial z}(\frac{\partial\phi}{\partial x})-\frac{\partial}{\partial x}(\frac{\partial\phi}{\partial z})]j+[\frac{\partial}{\partial x}(\frac{\partial\phi}{\partial y})-\frac{\partial}{\partial y}(\frac{\partial\phi}{\partial x})]k$$

$$=(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y})i+(\frac{\partial^2\phi}{\partial z\partial x}-\frac{\partial^2\phi}{\partial x\partial z})j+(\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x})k=0$$

This is only true when ϕ is continuously differentiable, hence the order of the differentiation is immaterial.

4.0 CONCLUSION

In this unit you have learnt about curl and various applications of curl to physical situations. Study this unit carefully before moving to the next unit of this course.

5.0 SUMMARY

We recall that in this unit we defined a curl of a vector field, as

$$\text{Curl} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

You are required to master this formula properly because of its physical application as we proceed in studying this course.

6.0 TUTOR-MARKED ASSIGNMENT

Obtain the curls of the following vectors:

(a) $xi,(b),r,(c),(xi-yj)/(x+y),(d),i\sin y+jx(1+\cos y)$

If curlA = 0 where A= $(xyz)^m(x^n i+y^n j+z^n k)$ show that either m = 0 or n = -1
If v = r (a.r) where a is a constant vector show that

Curlv = $a\wedge r$ (ii) curl $(a\wedge r)=2a$

7.0 REFERENCES/FURTHER READING

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MODULE 3 ORTHOGONAL CURVILINEAR CO-ORDINATES

UNIT 1 JACOBIANS

CONTENTS

1.0 INTRODUCTION

A useful tool for the operation on orthogonal curvilinear co-ordinate systems is the Jacobian. Since most of the co-ordinate systems are different from the Cartesian co-ordinate some transformations are usually required which will necessitate the need to find the scale factors of these transformation, in doing this we may need to find the Jacobian of the transformation before we can able to find the required scale factors.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define Jacobian
- solve exercises involving the use of Jacobian.

3.0 MAIN CONTENT

3.1 Definition of Jacobian

The Jacobian of x and y for two independent variables m and n is the determinant

$$\begin{vmatrix} (\frac{\partial x}{\partial m})_n & (\frac{\partial x}{\partial n})_m \\ (\frac{\partial y}{\partial m})_n & (\frac{\partial y}{\partial n})_m \end{vmatrix}$$

Where $x = f_1(m,n)$ and $y = f_2(m,n)$.

The customary notation is

$$\frac{\partial(x,y)}{\partial(m,n)} = \begin{vmatrix} (\frac{\partial x}{\partial m})_n & (\frac{\partial x}{\partial n})_m \\ (\frac{\partial y}{\partial m})_n & (\frac{\partial y}{\partial n})_m \end{vmatrix} \qquad \text{………….. (1)}$$

It is obvious from (1) that

$$\frac{\partial(x,y)}{\partial(m,n)} = (\frac{\partial x}{\partial m})_n(\frac{\partial y}{\partial n})_m - (\frac{\partial x}{\partial n})_m(\frac{\partial y}{\partial m})_n \qquad \text{…………... (2)}$$

3.1.2 Properties of Jacobians

Jacobians have the following basic properties:

(a) We note from (1) that

$$\frac{\partial(y,x)}{\partial(m,n)} = \begin{vmatrix} (\frac{\partial y}{\partial m})_n & (\frac{\partial y}{\partial n})_m \\ (\frac{\partial x}{\partial m})_n & (\frac{\partial x}{\partial n})_m \end{vmatrix} \qquad \text{…............ (3)}$$

Therefore,

$$\frac{\partial(y,x)}{\partial(m,n)} = \left(\frac{\partial y}{\partial m}\right)_n\left(\frac{\partial x}{\partial n}\right)_m - \left(\frac{\partial y}{\partial n}\right)_m\left(\frac{\partial x}{\partial m}\right)_n \qquad \text{…………… (4)}$$

By comparing (2) and (3), we could infer that

$$\frac{\partial(y,x)}{\partial(m,n)} = -\frac{\partial(x,y)}{\partial(m,n)} \qquad \ldots\ldots\ldots\ldots.. \ (5)$$

(b) Similarly, according to (1)

$$\frac{\partial(y,z)}{\partial(x,z)} = \begin{vmatrix} \left(\frac{\partial y}{\partial x}\right)_z & \left(\frac{\partial y}{\partial z}\right)_x \\ \left(\frac{\partial z}{\partial x}\right)_z & \left(\frac{\partial z}{\partial z}\right)_x \end{vmatrix} \qquad \ldots\ldots\ldots\ldots\ldots \ (6)$$

$$\frac{\partial(y,z)}{\partial(x,z)} = \begin{vmatrix} \left(\frac{\partial y}{\partial x}\right)_z & \left(\frac{\partial y}{\partial z}\right)_x \\ 0 & 1 \end{vmatrix} \qquad \ldots\ldots\ldots\ldots\ldots \ (7)$$

We see that:

$$\frac{\partial(y,z)}{\partial(x,z)} = \left(\frac{\partial y}{\partial x}\right)_z \qquad \ldots\ldots\ldots\ldots\ldots \ (8)$$

From (8) it is obvious that all partial derivatives can be represented by Jacobians.

(c) It is easy to note that

$$\frac{\partial(y,x)}{\partial(a,b)}\frac{\partial(a,b)}{\partial(m,n)} = \frac{\partial(y,x)}{\partial(m,n)} \qquad \ldots\ldots\ldots\ldots\ldots \ (9)$$

(d) From equation (1) it follows that

$\frac{\partial(m,n)}{\partial(m,n)} = 1, \quad \frac{\partial(x,x)}{\partial(m,n)} = 0$ and if k is constant, then

$$\frac{\partial(k,x)}{\partial(m,n)} = 0 \qquad \ldots\ldots\ldots\ldots\ldots \ (10)$$

It is possible using equations (8), (9) and (10) to transform partial derivatives.

To see this, consider the quantity defined by:

$\left(\frac{\partial T}{\partial p}\right)$, which we can express as

$$\left(\frac{\partial T}{\partial p}\right) = \frac{\partial(T, s)}{\partial(p, s)} \qquad \text{………….......… (11)}$$

While

$$\frac{\partial(s,T)}{\partial(p,s)} = \frac{\partial(s,T)}{\partial(p,T)} / \frac{\partial(p,s)}{\partial(p,T)} \qquad \text{…………….......… (12)}$$

This is in conformity with equation (11) we note that

$$\frac{\partial(s,T)}{\partial(p,T)} = \left(\frac{\partial s}{\partial p}\right)_T \qquad \text{………….......… (13)}$$

We may write Jacobian in a notational form as follows:

To find the Jacobian of the function $u(x, y.z), v(x, y, z), w(x, y, z)$, we express it as:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = J\left(\frac{u,v,w}{x,y,z}\right) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \qquad \text{………….........…… (14)}$$

We should also note that in general:

$$\frac{\partial(x_1, x_2, ...x_n)}{\partial(u_1, u_2, ...u_n)} \cdot \frac{\partial(u_1, u_2, ...u_n)}{\partial(x_1, x_2, ...x_n)} = 1 \qquad \text{…………….........… (15)}$$

SELF-ASSESSMENT EXERCISE 1

1. Consider the two functions defined as:

$u_1 = ax + by + c$

$u_2 = dx + ey + f$

Investigate weather they are functionally dependent.

2. If u and v are functions of r and s, and r and s are also functions of x and y, prove that :

$$\frac{\partial(u,v)}{\partial(r,s)}.\frac{\partial(r,s)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(x,y)}$$

3.2 Jacobian and Curvilinear Co-ordinates: Change of Variables in Integrals

Given the equations:

$$x=x(u_1,u_2,u_3),y=y(u_1,u_2,u_3),z=z(u_1,u_2,u_3) \qquad \ldots\ldots\ldots\ldots\ldots (16)$$

which defines curvilinear co-ordinates, $u_1, u_2,$ and u_3 in space.

Suppose we write:

$$U_k=i\frac{\partial x}{\partial u_k}+j\frac{\partial y}{\partial u_k}+k\frac{\partial z}{\partial u_k} \qquad \text{(k=1, 2, 3)} \qquad \ldots\ldots\ldots\ldots\ldots (17)$$

Then for u_1,u_2,u_3 the volume element in the new co-ordinate is given as

$$d\tau=(U_1,U_2,U_3)du_1du_2du_3 \qquad \ldots\ldots\ldots\ldots\ldots (18)$$

If the co-ordinates are so ordered that the right –hand member is positive. Now we define

$$U_1.U_2\times U_3=\begin{vmatrix}\frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1}\\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_3}\\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3}\end{vmatrix}=\frac{\partial(x,y,z)}{\partial(u_1,u_2u_3)} \qquad \ldots\ldots\ldots\ldots\ldots (19)$$

Now since the determinant is unchanged if the row and column are interchanged then we may write

$$d\tau=\left|\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}\right|du_1du_2du_3 \qquad \ldots\ldots\ldots\ldots\ldots (20)$$

We now present the change of variable formula as

$$\iiint_R w(x,y,z)dxdydz = \iiint_{R^*} W(u_1,u_2,u_3)\left|\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}\right|du_1du_2du_3 \quad \ldots\ldots\ldots(21)$$

Where

$W(u_1,u_2,u_3) = w[x(u_1,u_2,u_3), y(u_1,u_2,u_3), z(u_1,u_2,u_3)]$, and R^* is the u_1,u_2,u_3 region into which we transform the $x,y,z,$ region R.
The Jacobian $\partial(x,y,z)/\partial(u_1,u_2,u_3)$ is continuous and nonzero in R^*

If we are given equations in two dimensions such as

$$x = x(u_1,u_2), \quad y = y(u_1,u_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

Note that (22) can be interpreted as defining curvilinear co-ordinates in the $xy-plane$.

The vectors:

$$U_1 = i\frac{\partial x}{\partial u_1} + j\frac{\partial y}{\partial u_1}, \quad U_2 = i\frac{\partial x}{\partial u_2} + j\frac{\partial y}{\partial u_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

are the tangent to the co-ordinate curves, with the lengths ds_1/du_1 and ds_2/du_2

The vector element of plane is then given by

$$dA = (U_1 \times U_2)du_1du_2 = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & 0 \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & 0 \end{vmatrix} du_1du_2$$

This relation gives the result

$$dA = |dA| = \left|\frac{\partial(x.y)}{\partial(u_1,u_2)}\right|du_1du_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

Hence,

$$\iint_D w(x,y)dxdy = \iint_D W(u_1,u_2)\left|\frac{\partial(x,y)}{\partial(u_1,u_2)}\right|du_1du_2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

4.0 CONCLUSION

In this unit, we have defined Jacobians as a preparatory for us to study curvilinear co-ordinate systems.

5.0 SUMMARY

Recall that, we studied Jacobian as a useful tool for determining transformation from one space to another; you are to read and understand this unit carefully so that you will be able to understand the content of the next unit.

6.0 TUTOR-MARKED ASSIGNMENT

1. The transformation from rectangular to cylindrical co-ordinates is defined by the transformation:

   $x = \rho\cos\phi, y = \rho\sin\phi, z = z$

   Find the Jacobian of the transformation.

2. If u and v are functions of r and s also r and s are functions of x and y, prove that:

$$\frac{\partial(u,v)}{\partial(r,s)}.\frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$$

7.0 REFERENCES/FURTHER READING

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UNIT 2 ORTHOGONAL CURVILINEAR CO-ORDINATES

CONTENTS

1.0 INTRODUCTION

In our elementary mathematics, we learnt about co-ordinate systems namely; (x, y, z), in the rectangular co-ordinates. In this unit, we will show that it is possible to work in other co-ordinate systems apart from the rectangular co-ordinate if we make the appropriate transformation. This is what we set to achieve in this unit.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define the orthogonal curvilinear co-ordinates
- determine the scale factors of transformation
- determine the elemental volume
- solve problems in other co-ordinate systems such as circular cylindrical and spherical co-ordinates.

3.0 MAIN CONTENT

3.1 Transformation of Co-Ordinates

Given the rectangular co-ordinates x, y, z, we can define a new co-ordinate system by the following equations expressible as:

$x = x(u_1, u_2, u_3)$, $y = y(u_1, u_2, u_3)$, $z = z(u_1, u_2, u_3)$ ………………. (1)

Conversely, the relations as defined in (1) can be inverted to express

u_1, u_2, u_3 in terms of $x, y, z,$ whenever $x, y, z,$ and are suitably restricted.

Thus, at least in some region any point with the co-ordinates (x, y, z) has corresponding co-ordinates (u_1, u_2, u_3). We shall assume that the correspondence is unique.

Suppose a particle moves from point P in such a way that only u_1 is allowed to vary while u_2, u_3 are held constant, then it would generate a curve in space which is called $u_1 - curve$. Other curves u_2, and, u_3 are similarly generated.

3.1.1 Orthogonal Curvilinear Co-Ordinates

If one co-ordinate is held constant, we can determine successively three surfaces passing a point of space, these surfaces intersecting in the co-ordinate curves. When we chose a new co-ordinate in such a way that the co-ordinate curves are mutually perpendicular at each point, such co-ordinates are called Orthogonal Curvilinear co-ordinates.

3.1.2 The Scale Factors

Let $r = xi + yj + zk$ ……………. . . (2)

Represent the position vector of a point P in space. Then a tangent vector to the $u_1 - curve$ at P is given by

$$U_1 = \frac{\partial r}{\partial u_1} = \frac{\partial r}{\partial s_1}\frac{ds_1}{du_1} \qquad \text{…………… …. (3)}$$

Where s_1 arc length along the u_1 curve.

Since $\partial r / \partial s_1$ is a unit vector. We now write

$$U_1 = h_1 u_1 \qquad \text{……………… (4)}$$

Where u_1, is the unit vector tangent to the u_1 curve in the direction of increasing arc length and $h_1 = ds_1 / du_1$ is the length of U_1. If we consider the other co-ordinate curves similarly, we thus write

$U_1 = h_1 u_1,\ U_2 = h_2 u_2\ ,\ U_3 = h_3 u_3$ ……………… (5)

Where $u_k (k = 1,2,3)$ is the unit vector tangent to the u_k curve, and

$$h_1 = \frac{ds_1}{du_1} = \left|\frac{\partial r}{\partial u_1}\right|,\quad h_2 = \frac{ds_2}{du_2} = \left|\frac{\partial r}{\partial u_2}\right|,\quad h_3 = \frac{ds_3}{du_3} = \left|\frac{\partial r}{\partial u_3}\right| \quad \text{…………………… (6)}$$

Putting these equations in the differential forms, we have the following expressions:

$ds_1 = h_1 du_1,\quad ds_2 = h_2 du_2\ ,\ ds_3 = h_3 du_3$ ………………… (7)

h_1, h_2, and, h_3 are called the scale factors.

The co-ordinates curves are said to be orthogonal if:

$U_1.U_2 = U_2.U_3 = U_3.U_1 = 0$ ………………… (8)

3.1.3 The Elemental Volume

The elemental volume is defined as

$d\tau = h_1 h_2 h_3 du_1 du_2 du_3$ ………………… (9)

Example: The transformation from rectangular to cylindrical co-ordinates is defined by the transformations

$x = \rho\cos\phi, y = \rho\sin\phi, z = z$

(a) Prove that the system is orthogonal

(b) Find ds^2 and the scale factors

(c) Find the Jacobian of the transformation and the volume element.

Solution:

Let $e = (e_1, e_2, e_3)$ be a unit vector in the cylindrical co-ordinates

We have:

$$dr = \frac{\partial r}{\partial \rho}d\rho + \frac{\partial r}{\partial \phi}d\phi + \frac{\partial r}{\partial z}dz$$

$= \ h_1 d\rho e_1 + h_2 d\phi e_2 + h_3 dz e_3$

But

$$dr.dr. = \left(\left(\frac{\partial r}{\partial \rho} d\rho \right)^2 + \left(\frac{\partial r}{\partial \phi} d\phi \right)^2 + \left(\frac{\partial r}{\partial z} dz \right)^2 \right) + 2\frac{\partial r}{\partial \rho}.\frac{\partial r}{\partial \phi} d\rho d\phi + 2\frac{\partial r}{\partial \rho}.\frac{\partial r}{\partial z} d\rho dz +$$

$$2\frac{\partial r}{\partial \phi}.\frac{dr}{dz} d\phi dz$$

$$= \ h_1^2 (d\rho)^2 + h_2^2 (d\phi)^2 + h_3^2 (dz)^2 + 2h_1 h_2 d\rho d\phi e_1 e_2 + 2h_2 h_3 d\phi dz \ e_2.e_3 \ + 2h_1 h_3 d\phi dz e_1.e_3$$

Consider

$$r = \rho \cos\phi i + \rho \sin\phi j + zk$$

$$\frac{\partial r}{\partial \rho} = \cos\phi i + \sin\phi j$$

$$\frac{\partial r}{\partial \phi} = -\rho \sin\phi i + \rho \cos j$$

$$\frac{\partial r}{\partial z} = k$$

Now

$$\frac{\partial r}{\partial \rho}.\frac{\partial r}{\partial \phi} = -\rho \cos\phi \sin\phi + \rho \cos\phi \sin\phi = 0$$

Also,

$$\frac{\partial r}{\partial \rho}.\frac{\partial r}{\partial z} = \frac{\partial r}{\partial \phi}.\frac{\partial r}{\partial z} = 0$$

From (a) part we have:

$$dr.dr = ds^2 = h_1^2 (d\rho)^2 + h_2^2 (d\phi)^2 + h_3^2 (dz)^2$$

$$h_1 = \left|\frac{\partial r}{\partial \rho}\right|, h_2 = \left|\frac{\partial r}{\partial \phi}\right|, h_3 = \left|\frac{\partial r}{\partial z}\right|$$

$$h_1 = 1, h_2 = \rho, \ h_3 = 1$$

$$ds^2 = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2$$

(c) The Jacobian of the transformation is

$$= \left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}\right| = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix}$$

Thus, the volume element dV is given as

$$dV = \left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}\right| d\rho d\phi dz = \rho d\rho d\phi dz$$

3.2 Gradient, Divergence, Curl and Laplacian in Orthogonal Curvilinear Co-Ordinates

If ϕ is a scalar function and

$$A = A_1e_1 + A_2e_2 + +A_3e_3$$

is a vector function of orthogonal curvilinear co-ordinates u_1, u_2, u_3 then, we have the following results:

(1) Gradient: $\nabla\phi = \text{grad}\,\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}e_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}e_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}e_3$

(2) Divergence of A: $\nabla . A = divA$ =

$$\frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right)$$

(3) Curl A = $\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1e_1 & h_2e_2 & h_3e_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$

(4) $\nabla^2\phi =$ Laplacian of ϕ

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right]$$

3.2.1 Special Orthogonal Co-Ordinate Systems

In this section, we shall examine some special orthogonal co-ordinate systems we usually come across in mathematics.

1. Cylindrical Co-ordinates (ρ,ϕ,z)

Here our transformation is the form:

$x = \rho\cos\phi \quad y = \rho\sin\phi, z = z$

Where $\rho \geq 0, \quad 0 \leq \phi < 2\pi, \quad -\infty < z < \infty$

$h_\rho = 1, \; h_\phi = \rho, \quad h_z = 1$

2. Spherical Co-ordinates (r,θ,ϕ)

Here the transformation is of the form:

$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta, \quad$ where $r \geq 0,$

$0 \leq \phi < 2\pi, \quad 0 \leq \phi \leq \pi$

$h_r = 1, h_\theta = r, h_\phi = r\sin\theta$

3. Parabolic Cylindrical Co-ordinates (u,v,z)

$x = \frac{1}{2}(u^2 - v^2), y = uv, z = z,$ where $-\infty < u < \infty, \; v \geq 0,$

$-\infty < z < \infty, \quad h_u = h_v = \sqrt{u^2 + v^2,} \quad h_z = 1$ In cylindrical co-ordinates.

4. Paraboloidal Co-ordinates (u,v,ϕ).

Here, the transformations are given by:

$x = uv\cos\phi, y = uv\sin\phi, z = \frac{1}{2}(u^2 - v^2), u \geq 0, v \geq 0, 0 \leq \phi < 2\pi$

$h_u = h_v = \sqrt{u^2 + v^2}, h_\phi = uv$

Other special co-ordinates exist which include, elliptic cylindrical co-ordinates, prolate spheroidal co-ordinates, bipolar co-ordinates, ellipsoidal co-ordinates etc.

Consideration of the details of these co-ordinates will be left as exercise.

4.0 CONCLUSION

We have studied orthogonal co-ordinate systems in this unit; we have also identified some special co-ordinates systems that are orthogonal. Study this unit carefully before proceeding to the next unit of this course.

5.0 SUMMARY

Recall that if one co-ordinate is held constant, we can determine successively three surfaces passing a point of space, these surfaces intersecting in the co-ordinate curves. When we chose a new co-ordinate in such a way that the co-ordinate curves are mutually perpendicular at each point, such co-ordinates are called orthogonal curvilinear co-ordinates. We have also considered various types of these orthogonal systems particularly those for practical applications .You are to study them properly for better understanding.

6.0 TUTOR-MARKED ASSIGNMENT

1. Prove that a cylindrical co-ordinate system is orthogonal.
2. Express the velocity v and acceleration of a particle in cylindrical co-ordinates.
3. Find the square of the element of arc length in cylindrical co-ordinates and determine the corresponding scale factors.

7.0 REFERENCES/FURTHER READING

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MODULE 4 COMPLEX VARIABLES

UNIT 1 COMPLEX NUMBERS

CONTENTS

1.0 INTRODUCTION

The solution to the equation $x^2+1=0$ has no real roots because there is no real number whose square root is -1. In order to solve problem such as this, mathematicians evolve a way out of this logjam by assuming that there exist a number $i=\sqrt{-1}$. With this, we can conclude that the roots of the equation $x^2+1=0$ are $x=\pm i$. Similarly, we find that the roots of the equation $x^2-2x+5=0$ are $x=1\pm 2i$.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define complex numbers
- perform mathematical operations with complex numbers
- find modulus and argument of complex numbers
- solve exercises on complex numbers.

3.0 MAIN CONTENT

3.1 Definition of Complex Numbers

Given that a and b are real numbers, then the number $c = a + ib$ is called a complex number. a and b are known as the real and imaginary parts of the complex number respectively. When $a = 0$ the complex number is purely imaginary and when $b = 0$ then the complex number is real. The conjugate of the complex number c is denoted by:

$$\bar{c} = a - ib$$

SELF-ASSESSMENT EXERCISE 1

Find the conjugate of the following expressions:

(i) 3-3i (ii) 2i (iii) -3+4i (iv) 3-4i

3.2 Operations with Complex Numbers

In this section, we shall consider some mathematical operations on complex numbers.

(1) Note that in complex number,

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

(2) $(a + ib) - (c + id) = (a - c) + i(b - d)$

(3) $(a + ib)(a - ib) = a^2 + b^2$ since $i^2 = -1$

(4) If $a + ib = c + id$ then $a = c$ and $b = d$

(5) $\dfrac{a + ib}{c + id} = \dfrac{(a + ib)}{(c + id)} . \dfrac{(c - id)}{(c - id)} = \dfrac{(a + ib)(c - id)}{c^2 + d^2}$

$$= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$$

SELF-ASSESSMENT EXERCISE 3

(1) Find the real and imaginary parts of

$$z = \frac{(1+i)(2+i)}{(3-i)}$$

2) Let $z_1 = 3-6i$ and find:

(a) $z_1 z_2$ $(b)\frac{z_1}{z_2}, (c)\frac{z_2}{z_1}$

3) Simplify

(a) $(5-9i)-(2-6i)+(3-4i)$

(b) $(4+7i)(2+5i)$

(4) Multiply $(4-3i)$ by an appropriate factor to give a product that is entirely real. What is the result?

3.3 Modulus and Argument of a Complex Number

Let r be the length of OP, suppose the <XOP $=\theta,$ then $r=\sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$, r is called the modulus of z and written $|z|, \theta$ is called the argument or amplitude of and written as $\arg z$ or amz .

Example: Find the modulus and argument of the complex number

$$z = \frac{(1+i)(2+i)}{(3-i)}$$

Solution:

$$z = \frac{(1+i)(2+i)}{(3-i)} = \frac{2+2i+i-1}{3-i} = \frac{1+3i}{3-i}$$

Therefore,

$$z = \frac{1+3i}{3-i}\frac{(3+i)}{(3+i)} = \frac{(3+9i+i-3}{10} = \frac{10i}{10}$$

Hence $z=i$ therefore $|z|=1$ and arg z=$\frac{\pi}{2}$

2. If $x+iy=a+\dfrac{b(1+it)}{(1-it)}$ where a and b are real constant and $x,y,t,$ are real variables show that the locus of the point (x,y) as t, varies as a circle.

Solution:

Let $x+iy=a+\dfrac{b(1+it)}{(1-it)}$

$$= a+\frac{b(1+it)}{(1-it)}.\frac{(1+it)}{(1+it)}$$

$$= a+\frac{b(1-t^2)}{1+t^2}+\frac{2bit}{1+t^2}$$

Equating the real parts and the imaginary parts in each side of the equation, we have:

$$x=\frac{b(1-t^2)}{1+t^2}, y=\frac{2bt}{1+t^2}$$

Thus,

$$(x-a)^2+y^2=b^2$$

Hence, the locus of the point (x,y) is a circle centre (a,0) and radius b.

We may represent complex numbers in the polar form as follows:

$$z=x+iy=r\cos\theta+ir\sin\theta$$

Compare coefficients then

$$x=r\cos\theta, y=r\sin\theta$$

We refer to this as the polar representation of the complex numbers.

4.0 CONCLUSION

We have shown the way to handle complex numbers in what follows, we shall deal with some problems into detail in complex variables.

5.0 SUMMARY

Recall that with clearly defined notation you can handle complex number as we handle real numbers ordinarily in algebra. You should study carefully before moving to the next unit.

6.0 TUTOR-MARKED ASSIGNMENT

1. Establish the following results:

(a) $\operatorname{Re}(z_1 + z_2) = \operatorname{Re}(z_1) + \operatorname{Re}(z_2), but, \operatorname{Re}(z_1 z_2) \neq \operatorname{Re}(z_1)\operatorname{Re}(z_2)$ in general

(b) $\operatorname{Im}(z_1 + z_2) = \operatorname{Im}(z_1 \operatorname{Im}(z_2), but, \operatorname{Im}(z_1 z_2) \neq \operatorname{Im}(z_1)\operatorname{Im}(z_2)$ in general

(c) $|z_1 z_2| = |z_1||z_2|, but, |z_1 + z_2| \neq |z_1| + |z_2|,$ in general

2. Express the following quantities in the form a + ib where a and b are real

(a) $(1+i)^3$ (b) $\dfrac{1+i}{1-i}$ (c) $\sin(\dfrac{\pi}{4} + 2i)$

3. Prove the following

(a) $z + \bar{z} = 2\operatorname{Re}(z)$ (b) $z - \bar{z} = 2i\operatorname{Im}(z)$ (c) $\operatorname{Re}(z) \leq |z|$

7.0 REFERENCES/FURTHER READING

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UNIT 2 POLAR OPERATIONS WITH COMPLEX NUMBERS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall examine complex numbers in polar forms. The polar form of complex numbers present interesting results which will be examined in this unit.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- express complex numbers in polar form
- carry out multiplication and division of complex numbers
- recall the Demoivre's theorem and apply it appropriately
- find roots and work with fractional powers of complex numbers
- solve correctly the exercises that follows after the unit.

3.0 MAIN CONTENT

3.1 Multiplication and Division of Complex Numbers

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then,

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$

$$= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

From the above, you could see that

$$|z_1 z_2| = |z_1|.|z_2|$$

We also note that

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$, and that

$$\frac{z_1}{z_2} = \frac{r_1(Cos\theta_1 + iSin\theta_1)}{r_2(Cos\theta_2 + iSin\theta_2)}$$

$$= \frac{r_1}{r_2}[(Cos\theta_1 + iSin\theta_1)(Cos\theta_2 - iSin\theta_2)]$$

$$= \frac{r_1}{r_2}[Cos(\theta_1 - \theta_2) + iSin(\theta_1 - \theta_2)]$$

Therefore,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad \text{and} \quad \arg(\frac{z_1}{z_2}) = \arg z_1 - \arg z_2$$

3.2 Demoivre's Theorem

Recall that:

$$(Cos\theta_1 + iSin\theta_1)(Cos\theta_2 + iSin\theta_2) = Cos(\theta_1 + \theta_2) + iSin(\theta_1 + \theta_2)$$

Note that,

$$(Cos\theta_1 + iSin\theta_1)(Cos\theta_1 + iSin\theta_1) = Cos2\theta_1 + iSin2\theta_1$$

This is equivalence to

$$(Cos\theta_1 + iSin\theta_1)^2 = Cos2\theta_1 + iSin2\theta_1$$

Also,

$$(Cos\theta_1 + iSin\theta_1)^3 = Cos3\theta_1 + iSin3\theta_1$$

If we continue in this way, we find that:

$$Cos\theta_1 + iSin\theta_1)^n = Cosn\theta_1 + iSinn\theta_1$$

This is known as the Demoivre's theorem for positive integer index.

It can be shown that the theorem is true for all rational values of n.

Now suppose n is a negative integer and we let n = -m where m is a positive integer then,

$$(Cos\theta + iSin\theta)^{-m} = \frac{1}{(Cos\theta + iSin\theta)^{m}}$$

$$= \ Cos(-m\theta) + iSin(-m\theta) = Cos(n\theta) + iSin(n\theta)$$

We can also prove for fractions. Recall that by Demoivre's theorem

$$(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta)^{q} \ = \ Cosp\theta + iSinp\theta = (Cos\theta + iSin\theta)^{p}$$

It follows that $Cos\frac{p}{q}\theta + iSin\frac{p}{q}\theta$ is a qth root of $(Cos\theta + iSin\theta)^{P}$

Demoivre's theorem has been proved for all rational values of n.

We need to find other values of $(Cos\theta + iSin\theta)^{\frac{p}{q}}$.

To do this, suppose that:

$$(Cos\theta + iSin\theta)^{\frac{p}{q}} = \rho(Cos\phi + iSin\phi)$$

Then,

$$(\cos\theta + i\sin\theta)^{p} = \rho^{q}(\cos\phi + i\sin\phi)^{q} \Rightarrow \cos p\theta + i\sin p\theta = \rho^{q}(\cos q\phi + i\sin q\theta)$$

Equating the real and imaginary parts, we have

$$\cos p\theta = \rho^{q}\cos q\phi ; \sin p\theta = \sin q\phi$$

By squaring and adding, we obtain

$\rho^{2q} = 1$ and since ρ, the modulus of a complex number is +ve $\rho = 1$ therefore

$\cos\rho\phi = \cos q\phi; \sin\rho\phi = \sin q\phi$, and these equation are satisfied by

$q\phi = p\theta + 2k\pi; k = 0$ or any integer.

Therefore,

$$\phi = \frac{p\theta + 2k\pi}{q}$$

3.3 Roots and Fractional Power of a Complex Number

If n is a positive integer, the nth roots of a complex number are by definition the value of ω which satisfies the equation

$$\omega^n = z$$

If $\omega = \rho(\cos\phi + i\sin\phi)$ and $z = r(\cos\theta + i\sin\theta)$ then

$$\rho^n(\cos^n\phi + i\sin^n\phi) = r(\cos\theta + i\sin\theta) \quad \text{where}$$

$\rho^n = r$ and $n\phi = \theta + 2k\pi$ k is an integer or zero. By definition ρ, and, r are +ve, such that $\rho = \sqrt[n]{r}$ also, $\phi = \dfrac{\theta + 2k\pi}{n}$

Taking in succession the values of k = 0, 1, 2, 3… n, we find that

$\dfrac{\cos\theta + 2k\pi}{n} + i\dfrac{\sin\theta + 2k\pi}{n}$ has n distinct values. Hence there are n distinct nth roots of z given by the formula

$$\omega_k = \sqrt[n]{r}\left[\frac{\cos\theta + 2\pi k}{n} + i\frac{\sin\theta + 2\pi k}{n}\right], \quad \text{k=0, 1, 2, 3,…, n-1}$$

In a situation where n is a rational number say $n = \dfrac{p}{q}, p, and, q$ are integers and q is +ve, the value of z^n are the values of ω which satisfy the equation

$$\omega^q = z^p$$

Hence if $z = r(\cos\theta + i\sin\theta)$ then the q values of $z^{\frac{p}{q}}$ given by the formula

$$\omega_m = \sqrt[q]{r^p}\left[\frac{\cos\theta + 2m\pi}{q} + i\frac{\sin\theta + 2m\pi}{q}\right]$$, where

$\sqrt[q]{r^p}$ is the unique positive qth root of r^p

Example: Find the fifth roots of -1

Solution

Recall that:

-1= $\cos\pi + i\sin\pi$

Now if

$z^5 = -1 = [\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)]$, k=0,1,2,3,…,

Therefore,

$$z = \frac{\cos(\pi + 2k\pi)}{5} + i\frac{\sin(\pi + 2k\pi)}{5}$$

$k = 0,1,2,3,4,$ hence the solution are:

$$z = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$$

$$z = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$$

$$z = \cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5}$$

$$z = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$$

$$z = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$$

3.4 The nth Roots of Unity

We recall that $\cos 0 + i\sin 0 = 1$ this implies that:

$1 = \cos 2\pi k + i\sin 2\pi k,$ k=0, 1, 2, 3…

If ω denotes the root $\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$, k=0, 1, 2, 3…, then nth root of unity may be written in the form

$1, \omega, \omega^2, \omega^3, ..., \omega^{n-1}$

We see that they form a geometric progression whose sum $\frac{1-\omega^n}{1-\omega}$ is equal to 0.

We also note that the nth root of unity is represented in the Argand diagram by points which are vertices of a regular polygon of n sides inscribed in the circle.

Example:

Solve the equation $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ and deduce that

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$$

Solution

We know that:

$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = \frac{z^7 - 1}{z - 1}$, hence we consider the equation

$z^7 - 1 = 0$

We also note that:

$1 = \cos 0 + i\sin 0 = \cos 2\pi k + i\sin 2\pi k$, hence

$$z = \frac{\cos 2\pi k}{7} + i\sin\frac{2\pi k}{7}, k = 0,1,2,3,4,5,6$$

Equation $z^7 - 1 = 0$ is satisfied by

$z = 1, and, by, z = \frac{\cos 2\pi k}{7} + i\frac{\sin 2\pi k}{7}$, therefore the given equation is satisfied by

$$z = \cos \pm \frac{2\pi k}{7} + i \sin \pm \frac{2\pi k}{7}, k = 1,2,3,4,.5,....$$

The sum of these roots is

$$2\left[\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right]$$

But from the given equation the sum of the roots is also -1.

Therefore,

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$$

4.0 CONCLUSION

In this unit, we have studied some theorems and determine the roots of equation using complex variables. You are required to study this unit properly before attempting to answer questions under the tutor-marked assignment.

5.0 SUMMARY

You recall that you learnt about Demoivre's theorem, both for integer quantity and fractional quantity. Also, you learnt about roots of unity among others. You are to study them properly in order to be well equipped for the next course in mathematical methods.

6.0 TUTOR-MARKED ASSIGNMENT

1. Obtain the roots of the equation $3z^2 - (2 + 11i)z + 3 - 5i = 0$ in the form a+ib where a and b are real.
2. Express $\cos^3\theta \sin^4\theta$ as a sum of cosines of multiple of θ
3. Prove that $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

By putting

$x = \cos^2\theta$ or otherwise, show that the roots of the equation

$64x^3 - 96x^2 + 36x - 3 = 0$ are $\cos^2\left(\frac{\pi}{18}\right), \cos^2\left(\frac{5\pi}{18}\right), \cos^2\left(\frac{7\pi}{18}\right)$ and deduce that

$$\sec^2\left(\frac{\pi}{18}\right) + \sec^2\left(\frac{5\pi}{18}\right) + \sec^2\left(\frac{7\pi}{18}\right) = 12$$

7.0 REFERENCES/FURTHER READING

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