UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
DIVISION OF ENGINEERING SCIENCE

ECE355H1 F - Signal Analysis and Communication

Problem Set 4
Fall 2023

Submit by: **October 13, 2023**

Problem 1

(Problem 2.28 (b, d, f) - Textbook)

The following are the impulse responses of discrete-time LTI systems. Determine whether each system is causal and/or stable. Justify your answers.

a) $h[n] = (0.8)^n u[n+2]$

b) $h[n] = (5)^n u[3-n]$

c) $h[n] = (-\frac{1}{2})^n u[n] + (1.01)^n u[1-n]$

Problem 2

(Problem 2.29 (b, d, f) - Textbook)

The following are the impulse responses of continuous-time LTI systems. Determine whether each system is causal and/or stable. Justify your answers.

a) $h(t) = e^{-6t} u(3-t)$

b) $h(t) = e^{2t} u(-1-t)$

c) $h(t) = te^{-t} u(t)$

Problem 3

(Problem 2.30 - Textbook)

Consider the first -order difference equation

$$y[n] + 2y[n-1] = x[n].$$

Assuming the condition of initial rest (i.e., if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$), find the impulse response of a system whose input and output are related by this difference equation. You may solve the problem by rearranging the difference equation so as to express $y[n]$ in terms of $y[n-1]$ and $x[n]$ and generating the values of $y[0], y[+1], y[+2], ...$ in that order.

Problem 4

(Problem 2.32 - Textbook)

Consider the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n], \tag{P4-1}$$

and suppose that

$$x[n] = \left(\frac{1}{3}\right)^n u[n]. \tag{P4-2}$$

Assume that the solution $y[n]$ consists of the sum of a particular solution $y_p[n]$ to eq. (P4-1) and a homogeneous solution $y_h[n]$ satisfying the equation

$$y_h[n] - \frac{1}{2}y_h[n-1] = 0.$$

a) Verify that the homogeneous solution is given by

$$y_h[n] = A\left(\frac{1}{2}\right)^n$$

b) Let us consider obtaining a particular solution $y_p[n]$ such that

$$y_p[n] - \frac{1}{2}y_p[n-1] = \left(\frac{1}{3}\right)^n u[n].$$

By assuming that $y_p[n]$ is of the form $B(\frac{1}{3})^n$ for $n \geq 0$, and substituting this in the above difference equation, determine the value of B.

c) Suppose that the LTI system described by eq. (P4-1) and initially at rest has as its input the signal specified by eq. (P4-2). Since $x[n] = 0$ for $n < 0$, we have that $y[n] = 0$ for $n < 0$. Also, from parts (a) and (b) we have that $y[n]$ has the form

$$y[n] = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{3}\right)^n$$

for $n \geq 0$. In order to solve for the unknown constant A, we must specify a value for $y[n]$ for some $n \geq 0$. Use the condition of initial rest and eqs. (P4-1) and (P4-2) to determine $y[0]$. From this value determine the constant A. The result of this calculation yields the solution to the difference equation (P4-1) under the condition of initial rest, when the input is given by eq. (P4-2).

Problem 5

(Problem 2.33 - Textbook)

Consider a system whose input $x(t)$ and output $y(t)$ satisfy the first-order differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

The system also satisfies the condition of initial rest.

a) The system also satisfies the condition of initial rest.

   i) Determine the system output $y_1(t)$ when the input is $x_1(t) = e^{3t}u(t)$.

   ii) Determine the system output $y_2(t)$ when the input is $x_2(t) = e^{2t}u(t)$.

   iii) Determine the system output $y_3(t)$ when the input is $x_3(t) = \alpha e^{3t}u(t) + \beta e^{2t}u(t)$, where α and β are real numbers. Show that $y_3(t) = \alpha y_1(t) + \beta y_2(t)$.

   iv) Now let $x_1(t)$ and $x_2(t)$ be arbitrary signals such that

$$x_1(t) = 0, \text{for } t < t_1,$$
$$x_2(t) = 0, \text{for } t < t_2.$$

   Letting $y_1(t)$ be the system output for input $x_1(t)$, $y_2(t)$ be the system output for input $x_2(t)$, and $y_3(t)$ be the system output for $x_3(t) = \alpha x_1(t) + \beta x_2(t)$, show that

$$y_3(t) = \alpha y_1(t) + \beta y_2(t).$$

   We may therefore conclude that the system under consideration is linear.

b) i) Determine the system output $y_1(t)$ when the input is $x_1(t) = Ke^{2t}u(t)$.

   ii) Determine the system output $y_2(t)$ when the input is $x_2(t) = Ke^{2(t-T)}u(t-T)$. Show that $y_2(t) = y_1(t-T)$.

   iii) Now let $x_1(t)$ be an arbitrary signal such that $x_1(t) = 0$ for $t < t_0$. Letting $y_1(t)$ be the system output for input $x_1(t)$ and $y_2(t)$ be the system output for $x_2(t) = x_1(t-T)$, show that

$$y_2(t) = y_1(t-T).$$

   We may therefore conclude that the system under consideration is time invariant. In conjunction with the result derived in part (a), we conclude that the given system is LTI. Since this system satisfies the condition of initial rest, it is causal as well.

Problem 6

(Problem 3.3 - Textbook)

For the continuous-time periodic signal

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right),$$

determine the fundamental frequency ω_0 and the Fourier series coefficients a_k such that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Problem 7

(Problem 3.15 - Textbook)

Consider a continuous-time ideal lowpass filter S whose frequency response is

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq 100 \\ 0, & |\omega| > 100 \end{cases}$$

When the input to this filter is a signal $x(t)$ with fundamental period $T = \pi/6$ and Fourier series coefficients a_k, it is found that

$$x(t) \xrightarrow{S} y(t) = x(t)$$

For what values of k is it guaranteed that $a_k = 0$?

Textbook

Alan V. Oppenheim, Alan S. Willsky, and S. Hamid Nawab, Signals & Systems, 2nd Ed., Prentice-Hall, 1996 (ISBN 0-13-814757-4)