UNIVERSITY OF TORONTO
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DIVISION OF ENGINEERING SCIENCE

ECE355H1 F - Signal Analysis and Communication

Problem Set 3
Fall 2023

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Problem 1

(Problem 2.9 - Textbook)

Let

$$h(t) = e^{2t}u(-t+4) + e^{-2t}u(t-5)$$

Determine A and B such that

$$h(t-\tau) = \begin{cases} e^{-2(t-\tau)}, & \tau < A \\ 0, & A < \tau < B \\ e^{2(t-\tau)}, & B < \tau \end{cases}$$

Solution

Considering the definition of the signal $h(t)$, we can write:

$$h(\tau) = e^{2\tau}u(-\tau+4) + e^{-2\tau}u(\tau-5) = \begin{cases} e^{-2\tau}, & \tau > 5 \\ e^{2\tau}, & \tau < 4 \\ 0, & 4 < \tau < 5 \end{cases}$$

So,

$$h(-\tau) = \begin{cases} e^{2\tau}, & \tau < -5 \\ e^{-2\tau}, & \tau > -4 \\ 0, & -5 < \tau < -4 \end{cases}$$

In order to find $h(t-\tau)$, we need to shift $h(-\tau)$ by t to the right. Therefore,

$$h(t-\tau) = \begin{cases} e^{2(\tau-t)}, & \tau < t-5 \\ e^{-2(\tau-t)}, & \tau > t-4 \\ 0, & t-5 < \tau < t-4 \end{cases}$$

Thus,

$$A = t-5, B = t-4.$$

Problem 2

(Problem 2.10 - Textbook)

Suppose that

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

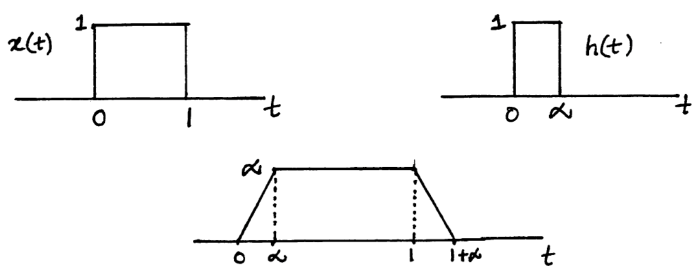
and $h(t) = x(t/\alpha)$, where $0 < \alpha \le 1$.

a) Determine and sketch $y(t) = x(t) * h(t)$

b) If $dy(t)/dt$ contains only three discontinuities, what is the value of α?

Solution

a) We first plot $x(t)$ and $h(t)$ and with the help of them we plot $y(t) = x(t) * h(t)$:



b) According to the plot of $y(t)$, it is clear that $\frac{dy(t)}{dt}$ has discontinuities at 0, α, 1, and $1+\alpha$. If we want $\frac{dy(t)}{dt}$ to have only three discontinuities, then we need to select $\alpha = 1$.

Problem 3

(Problem 2.11 - Textbook)

Let

$$x(t) = u(t-3) - u(t-5) \text{ and } h(t) = e^{-3t}u(t)$$

a) Compute $y(t) = x(t) * h(t)$.

b) Compute $g(t) = (dx(t)/dt) * h(t)$.

c) How is $g(t)$ related to $y(t)$?

Solution

a) From the form of $h(t)$, we know that it is non-zero only for $0 \le t \le \infty$. Thus,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{0}^{\infty} e^{-3\tau}(u(t-\tau-3) - u(t-\tau-5))d\tau$$

We can easily show that $(u(t-\tau-3)-u(t-\tau-5))$ is non zero in the range $(t-5) \leq \tau \leq (t-3)$. Therefore, for $t \leq 3$, the above integral evaluates to zero. For, $3 \leq t \leq 5$, the above integral is

$$y(t) = \int_0^{t-3} e^{-3\tau} d\tau = \frac{1-e^{-3(t-3)}}{3}$$

For $t > 5$, the integral is,

$$y(t) = \int_{t-5}^{t-3} e^{-3\tau} d\tau = \frac{(1-e^{-6})e^{-3(t-5)}}{3}$$

Therefore, the result of the convolution is summarized as

$$y(t) = \begin{cases} 0, & -\infty < t \leq 3 \\ \frac{1-e^{-3(t-3)}}{3}, & 3 < t \leq 5 \\ \frac{(1-e^{-6})e^{-3(t-5)}}{3}, & 5 < t \leq \infty \end{cases}$$

b) By differentiating $x(t)$ with respect to time we get

$$\frac{d(x(t)}{dt} = \delta(t-3) - \delta(t-5)$$

Thus,

$$g(t) = \frac{d(x(t)}{dt} * h(t) = e^{-3(t-3)}u(t-3) - e^{-3(t-5)}u(t-5).$$

c) From the result of part (a), we compute the derivative of $y(t)$:

$$\frac{dy(t)}{dt} = \begin{cases} 0, & -\infty < t \leq 3 \\ e^{-3(t-3)}, & 3 < t \leq 5 \\ (e^{-6}-1)e^{-3(t-5)}, & 5 < t \leq \infty \end{cases}$$

This is exactly equal to $g(t)$. So, $g(t) = \frac{dy(t)}{dt}$.

Problem 4

(Problem 2.16 - Textbook)

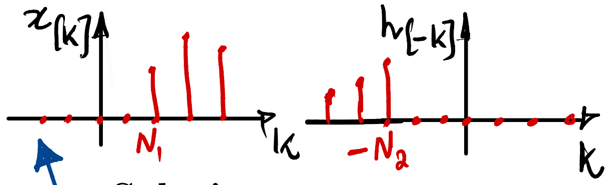
For each of the following statements, determine whether it is true or false:

a) If $x[n] = 0$ for $n < N_1$ and $h[n] = 0$ for $n < N_2$, then $x[n] * h[n] = 0$ for $n < N_1 + N_2$.

b) If $y[n] = x[n] * h[n]$, then $y[n-1] = x[n-1] * h[n-1]$.

c) If $y(t) = x(t) * h(t)$, then $y(-t) = x(-t) * h(-t)$.

d) If $x(t) = 0$ for $t > T_1$ and $h(t) = 0$ for $t > T_2$, then $x(t) * h(t) = 0$ for $t > T_1 + T_2$.



As per Conv. Sum slide with diff. 'n'

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$ when $n < N_1+N_2$ No overlap.

$\therefore$ $y[n] = 0$ for $n < N_1+N_2$. * Valid for all N_1, N_2 ✓ (Here we have shown for $N_1, N_2 > 0$ & $N_2 > N_1$ case only.

Solution

a) **True.** You can think of convolution as a way of adding up several repetitions of a signal $h[n]$. The first repetition starts at the point where the signal $x[n]$ first becomes non-zero, which we will call N_1. When this happens, the first occurrence of $h[n]$ at $n = N_1$ aligns with the signal $x[n]$ at a time called $N_1 + N_2$. So, for all values of n less than $N_1 + N_2$, the result $y[n]$ is equal to zero.

b) **False.** Consider

$$y[n] = x[n] * h[n] \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Therefore,

$$y[n-1] = \sum_{k=-\infty}^{\infty} x[k]h[n-1-k] = x[n] * h[n-1]$$

Based on these two equations, the given statement is false.

c) **True.** Consider

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Thus,

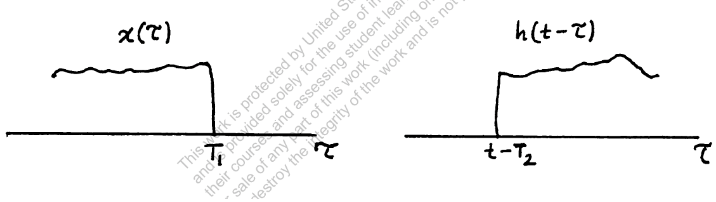
$$y(-t) = \int_{-\infty}^{\infty} x(\tau)h(-t-\tau)d\tau = \int_{-\infty}^{\infty} x(-\tau)h(-t+\tau)d\tau = x(-t) * h(-t)$$

Therefore, the given statement is true.

d) **True.** Consider

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Consider the figures below for $x(\tau)$ and $h(t-\tau)$ based on the definition of both functions in the question:



Based on these figures the product of $x(\tau)h(t-\tau)$ is zero if $t - T_2 > T_1$. S0, $y(t) = 0$ for $t > T_1 + T_2$.

Problem 5

(Problem 2.22 (b, d) - Textbook)

For each of the following pairs of waveforms, use the convolution integral to find the response $y(t)$ of the LTI system with impulse response $h(t)$ to the input $x(t)$. Sketch your results.

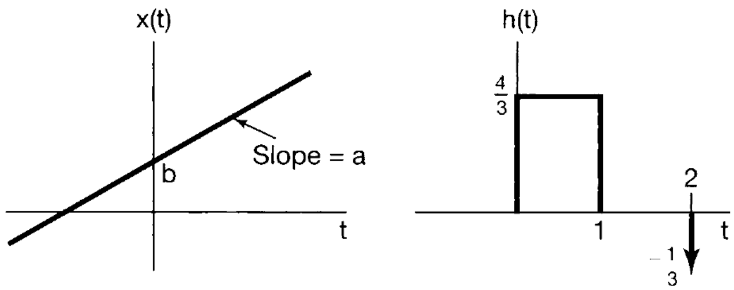


Figure 1: Problem 5 (b)

a) $x(t) = u(t) - 2u(t-2) + u(t-5)$
   $h(t) = e^{2t}u(1-t)$

b) $x(t)$ and $h(t)$ are as in Figure 1.

Solution

a) The desired convolutions is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{0}^{2} h(t-\tau)d\tau - \int_{2}^{5} h(t-\tau)d\tau.$$

This can be written as:

$$y(t) = \begin{cases} \int_0^2 e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau, & t \leq 1 \\ \int_{t-1}^2 e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau, & 1 \leq t \leq 3 \\ -\int_{t-1}^5 e^{2(t-\tau)}d\tau, & 3 \leq t \leq 6 \\ 0, & 6 < t \end{cases}$$

So,

$$y(t) = \begin{cases} (1/2)[e^{(2t)} - 2e^{2(t-2)} + e^{2(t-5)}], & t \leq 1 \\ (1/2)[e^{(2)} + e^{2(t-5)} - 2e^{2(t-2)}], & 1 \leq t \leq 3 \\ (1/2)[e^{2(t-5)} - e^2], & 3 \leq t \leq 6 \\ 0, & 6 < t \end{cases}$$

b) Let

$$h(t) = h_1(t) - \frac{1}{3}\delta(t-2),$$

where

$$h_1(t) = \begin{cases} 4/3, & 0 \leq t \leq 1 \\ 0, & otherwise \end{cases}$$

Now,

$$y(t) = h(t) * x(t) = [h_1(t) * x(t)] - \frac{1}{3}x(t-2).$$

We have

$$h_1(t) * x(t) = \int_{t-1}^{t} \frac{4}{3}(a\tau + b)d\tau = \frac{4}{3}[\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)].$$

Therefore,

$$y(t) = \frac{4}{3}[\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)] - \frac{1}{3}[a(t-2)+b] = at + b = x(t).$$

Problem 6

(Problem 2.23 - Textbook)

Let $h(t)$ be the triangular pulse shown in Figure 2(a), and let $x(t)$ be the impulse train depicted in Figure 2(b). That is,

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT).$$

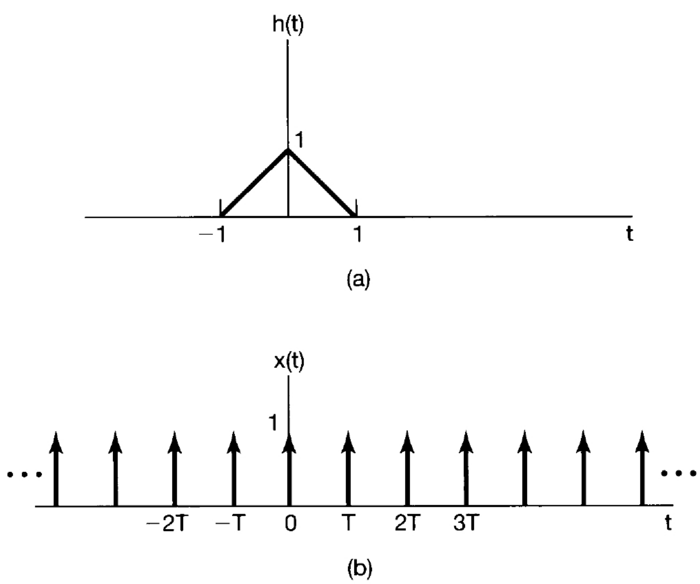


Figure 2: Problem 6

Determine and sketch $y(t) = x(t) * h(t)$ for the following values of T:

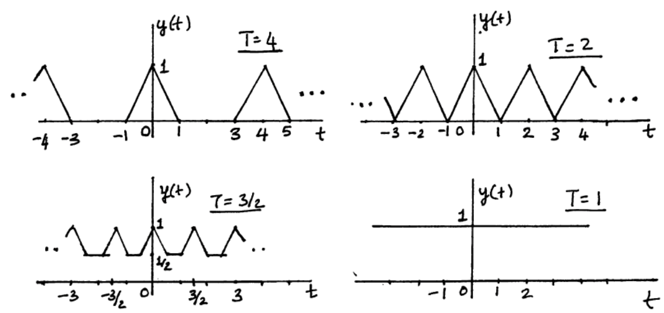
a) $T = 4$

b) $T = 2$

c) $T = 3/2$

d) $T = 1$

Solution



Problem 7

(Problem 2.24 - Textbook)

Consider the cascade interconnection of three causal LTI systems, illustrated in Figure 3(a). The impulse response $h_2[n]$ is

$$h_2[n] = u[n] - u[n-2],$$

and the overall impulse response is as shown in Figure 3(b).

a) Find the impulse response $h_1[n]$.

b) Find the response of the overall system to the input

$$x[n] = \delta[n] - \delta[n-1].$$

Solution

a) We are given $h_2[n] = \delta[n] + \delta[n-1]$. Therefore,

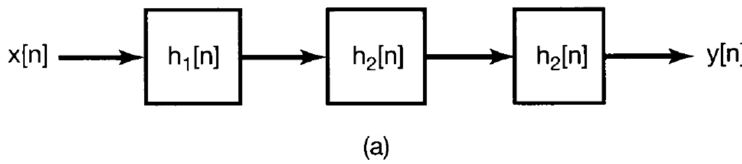
$$h_2[n] * h_2[n] = \delta[n] + 2\delta[n-1] + \delta[n-2].$$

Since,

$$h[n] = h_1[n] * [h_2[n] * h_2[n]],$$

we get

$$h[n] = h_1[n] + 2h_1[n-1] + h_1[n-2]$$



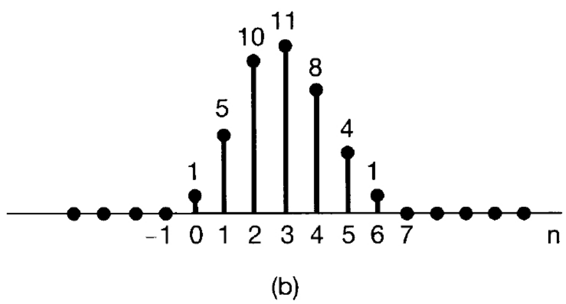


Figure 3: Problem 7

Therefore,

$$
\begin{aligned}
h[0] = h_1[0] &\implies h_1[0] = 1, \\
h[1] = h_1[1] + 2h_1[0] &\implies h_1[1] = 3 \\
h[2] = h_1[2] + 2h_1[1] + h_1[0] &\implies h_1[2] = 3 \\
h[3] = h_1[3] + 2h_1[2] + h_1[1] &\implies h_1[3] = 2 \\
h[4] = h_1[4] + 2h_1[3] + h_1[2] &\implies h_1[4] = 1 \\
h[5] = h_1[5] + 2h_1[4] + h_1[3] &\implies h_1[5] = 0
\end{aligned}
$$

$h_1[n] = 0$ for $n < 0$ and $n \geq 5$.

b) In this case,

$$y[n] = x[n] * h[n] = h[n] - h[n-1].$$

Problem 8

(Problem 2.25 - Textbook)

Let the signal

$$y[n] = x[n] * h[n],$$

where

$$x[n] = 3^n u[-n-1] + \left(\frac{1}{3}\right)^n u[n]$$

and

$$h[n] = \left(\frac{1}{4}\right)^n u[n+3].$$

a) Determine $y[n]$ *without* utilizing the distributive property of convolution.

b) Determine $y[n]$ *utilizing* the distributive property of convolution.

Solution

a) We can write $x[n]$ as

$$x[n] = (\frac{1}{3})^{|n|}.$$

Now, the desired convolution is

$$\begin{aligned} y[n] &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{-1} (1/3)^{-k}(1/4)^{n-k}u[n-k+3] + \sum_{k=0}^{\infty}(1/3)^k(1/4)^{n-k}u[n-k+3] \\ &= (1/12)\sum_{k=0}^{\infty}(1/3)^k(1/4)^{n+k}u[n+k+4] + \sum_{k=0}^{\infty}(1/3)^k(1/4)^{n-k}u[n-k+3] \end{aligned}$$

By considering each summation in the above equation separately, we may show that

$$y[n] = \begin{cases} (12^4/11)3^n, & n < -4 \\ (1/11)4^4, & n = -4 \\ (1/4)^n(1/11) + -3(1/4)^n + 3(256/81)(1/3)^n, & n \geq -3 \end{cases}$$

b) Now consider the convolution

$$y_1[n] = [(1/3)^n u[n]] * [(1/4)^n u[n+3]].$$

So,

$$y_1[n] = \begin{cases} 0, & n < -3 \\ -3(1/4)^n + 3(256/81)(1/3)^n, & n \geq -3 \end{cases}$$

Also, consider the convolution

$$y_2[n] = [(3)^n u[-n-1]] * [(1/4)^n u[n+3]].$$

Thus,

$$y_2[n] = \begin{cases} (12^4/11)3^n, & n < -3 \\ (1/4)^n(1/11), & n \geq -3 \end{cases}$$

Clearly, $y_1[n] + y_2[n] = y[n]$ obtained in the previous part.

Problem 9

(Problem 2.27 - Textbook)

We define the area under a continuous-time signal $v(t)$ as

$$A_v = \int_{-\infty}^{+\infty} v(t)dt,$$

Show that if $y(t) = x(t) * h(t)$, then

$$A_y = A_x \, A_h$$

Solution

The proof is as follows:

$$\begin{aligned}
A_y &= \int_{-\infty}^{\infty} y(t)dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau dt \\
&= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h(t-\tau)dtd\tau \\
&= \int_{-\infty}^{\infty} x(\tau)A_h d\tau \\
&= A_x A_h
\end{aligned}$$

Textbook

Alan V. Oppenheim, Alan S. Willsky, and S. Hamid Nawab, Signals & Systems, 2nd Ed., Prentice-Hall, 1996 (ISBN 0-13-814757-4)