UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
DIVISION OF ENGINEERING SCIENCE

ECE355H1 F - Signal Analysis and Communication

Problem Set 1
Fall 2023

Solved by: **Adnan Hamida**
I am prone to make mistakes :) please don't hesitate to email me if you find any:
adnan.hamida@mail.utoronto.ca

Problem 1

(Problem 1.4 (a, b, e) - Textbook)

Let $x[n]$ be a signal with $x[n] = 0$ for $n < -2$ and $n > 4$. For each signal given below, determine the values of n for which it is guaranteed to be zero.

a) $x[n-3]$

b) $x[n+4]$

c) $x[-n-2]$

Solution

a) $x[n-3]$ shifts $x[n]$ 3 units to the right. This means $x[n-3] = 0$ for $n-3 < -2$ and $n-3 > 4$ $\Rightarrow x[n-3] = 0$ for $n < 1$ and $n > 7$

b) $x[n+4]$ shifts $x[n]$ 4 units to the left. This means $x[n+4] = 0$ for $n+4 < -2$ and $n+4 > 4$ $\Rightarrow x[n+4] = 0$ for $n < -6$ and $n > 0$

c) $x[-n-2]$ flips $x[n]$, then shifts 2 units to the left. This means $x[-n-2] = 0$ for $-n-2 < -2$ and $-n-2 > 4 \Rightarrow x[-n-2] = 0$ for $n > 0$ and $n < -6$

Problem 2

(Problem 1.9 - Textbook)

Determine whether or not each of the following signals is periodic. If a signal is periodic, specify its fundamental period.

a) $x_1(t) = je^{j10t}$

b) $x_2(t) = e^{(-1+j)t}$

c) $x_3[n] = e^{j7\pi n}$

d) $x_4[n] = 3e^{j3\pi(n+1/2)/5}$

e) $x_5[n] = 3e^{j3/5(n+1/2)}$

Solution

a) $x_1(t) = je^{j10t} = e^{j\frac{\pi}{2}} \cdot e^{j10t}{=}e^{j(10t+\frac{\pi}{2})}$. In this case $\omega_0 = 10$, which means $x_1(t)$ is a **periodic** complex exponential and the fundamental period T is $\frac{2\pi}{10} = \frac{\pi}{5}$.

b) $x_2(t) = e^{(-1+j)t} = e^{-t} \cdot e^{jt}$, which means $x_2(t)$ is a product of a decaying signal (e^{-t}) and a periodic complex exponential (e^{jt}) $\Rightarrow x_2(t)$ is **not periodic**.

c) $x_3[n] = e^{j7\pi n} = e^{j(7\pi-6\pi)n} = e^{j\pi n}$. In this case $\omega_0 = \pi \Rightarrow$ to find the fundamental period N, don't forget that a periodic discrete-time complex exponential signals satisfies

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n},$$

which means $e^{j\omega_0 N} = 1$. This means that for the latter to hold, $\omega_0 N$ must be a multiple of 2π. That is, there must be some integer $m \in \mathbb{Z}$ that satisfies

$$\begin{aligned} \omega_0 N &= 2\pi m \\ \Rightarrow N &= \frac{2\pi}{\omega_0} m. \end{aligned}$$

In this case, $N = \frac{2\pi}{\pi} m = 2m$, which means $x_3[n]$ is a **periodic** complex exponential and the fundamental period $N = 2$ by setting $m = 1$.

d) $x_4[n] = 3e^{j3\pi(n+1/2)/5} = 3e^{j(\frac{3\pi}{5}n+\frac{3\pi}{10})}$. In this case $\omega_0 = \frac{3\pi}{5} \Rightarrow N = \frac{2\pi}{\frac{3\pi}{5}} m = \frac{10}{3} m$, which means $x_4[n]$ is a **periodic** complex exponential and the fundamental period $N = 10$ by setting $m = 3$.

e) $x_5[n] = 3e^{j3/5(n+1/2)} = 3e^{j(\frac{3}{5}n+\frac{3}{10})}$. In this case $\omega_0 = \frac{3}{5} \Rightarrow N = \frac{2\pi}{\frac{3}{5}} m = \frac{10\pi}{3} m$. There is no way to find an integer m that would make $\frac{10\pi}{3} m$ also an integer. Hence, $x_5[n]$ is **not periodic**.

Problem 3

(Problem 1.13 - Textbook)

Consider the continuous-time signal

$$x(t) = \delta(t+2) - \delta(t-2)$$

Calculate the value of E_∞ for the signal

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau.$$

Solution

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} x(\tau)\, d\tau = \int_{-\infty}^{t} \delta(\tau+2) - \delta(\tau-2)\, d\tau = u(t+2) - u(t-2) \\ &= \begin{cases} 0 & \text{if } t < -2 \\ 1 & \text{if } -2 \le t \le 2 \\ 0 & \text{if } t > 2 \end{cases}. \end{aligned}$$

Therefore, E_∞ can be found by

$$E_\infty = \int_{-\infty}^{\infty} |y(t)|^2\, dt = \int_{-2}^{2} dt = 4.$$

Problem 4

(Problem 1.14 - Textbook)

Consider a periodic signal

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ -2, & 1 < t < 2 \end{cases}$$

with period $T = 2$. The derivative of this signal is related to the "impulse train"

$$g(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k)$$

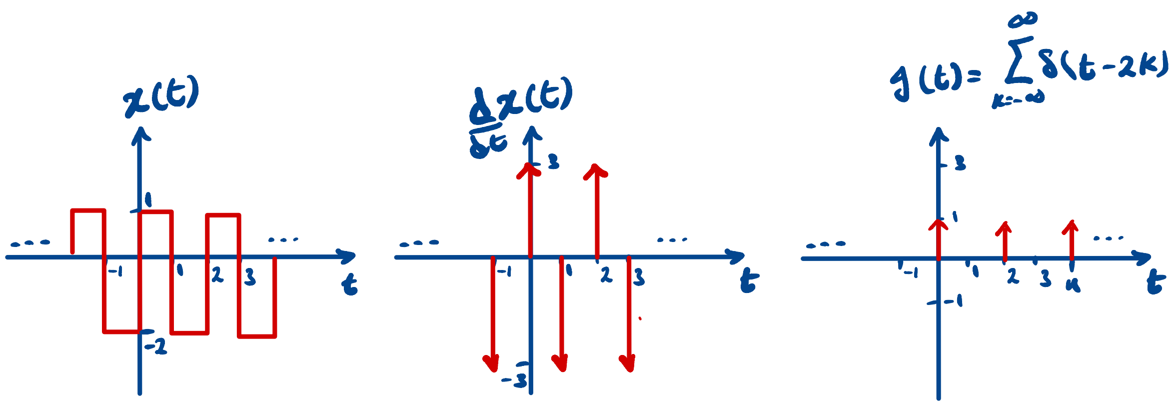
with period $T = 2$. It can be shown that

$$\frac{dx(t)}{dt} = A_1 g(t - t_1) + A_2 g(t - t_2).$$

Determine the values of A_1, t_1, A_2, and t_2.

Solution

The figure following is the plot $x(t)$, $\frac{dx(t)}{dt}$, and $g(t)$



We can easily see that

$$\begin{aligned} \frac{dx(t)}{dt} &= 3 \sum_{k=-\infty}^{\infty} \delta(t - 2k) - 3 \sum_{k=-\infty}^{\infty} \delta(t - 1 - 2k) \\ &= A_1 g(t - t_1) + A_2 g(t - t_2), \end{aligned}$$

where $A_1 = 3$, $t_1 = 0$, $A_2 = -3$, and $t_2 = 1$.

Problem 5

(Problem 1.21 (c, f) - Textbook)

A continuous-time signal x(t) is shown in Figure 1. Sketch and label carefully each of the following signals:

a) $x(2t+1)$

b) $x(t)[\delta(t+\frac{3}{2}) - \delta(t-\frac{3}{2})]$

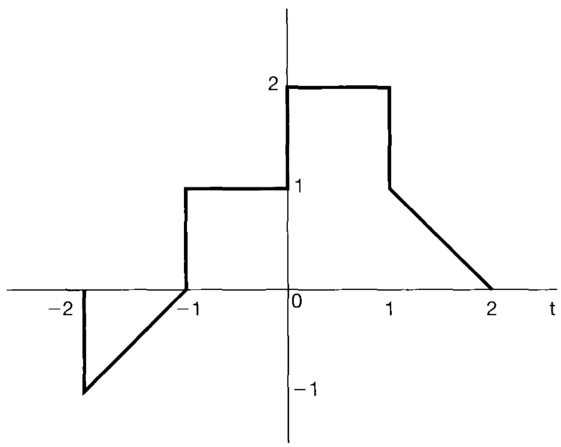
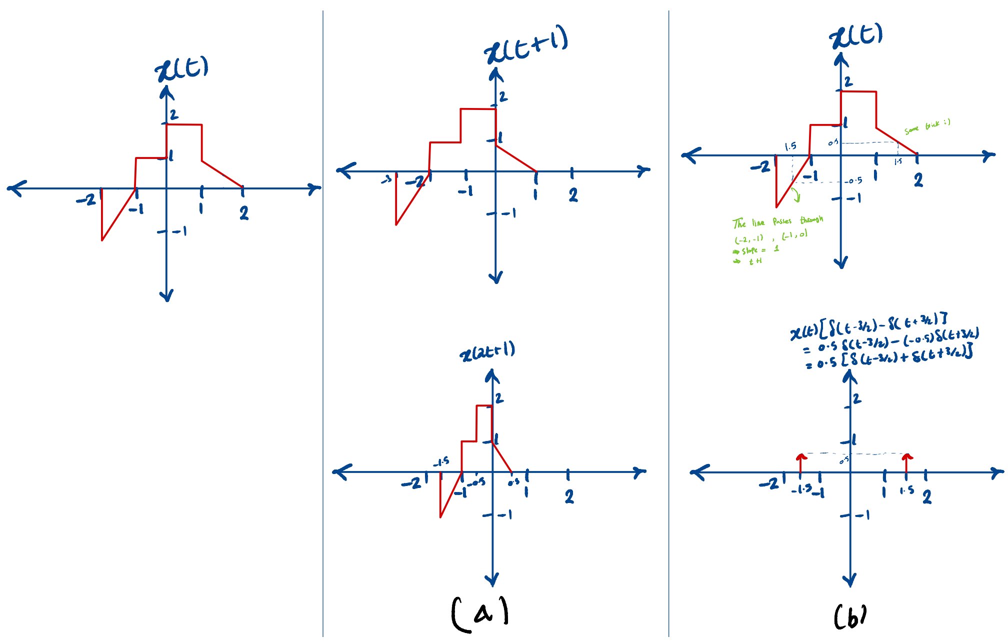


Figure 1: Problem 5

Solution



Problem 6

(Problem 1.22 (d, e, f) - Textbook)

A discrete-time signal is shown in Figure 2. Sketch and label carefully each of the following signals:

a) $x[3n+1]$

b) $x[n]u[3-n]$

c) $x[n-2]\delta[n-2]$

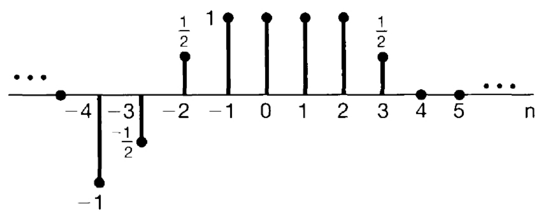
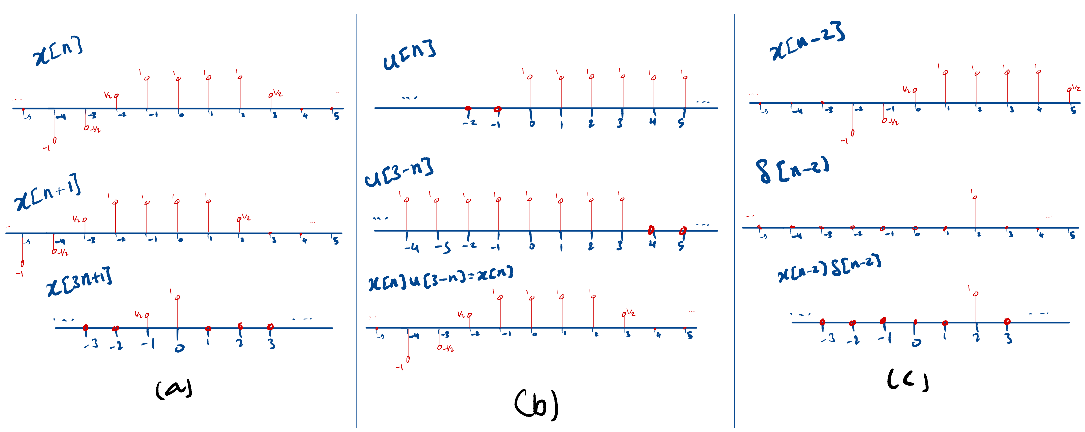


Figure 2: Problem 6

Solution



Problem 7

(Problem 1.25 (c, d, f) - Textbook)

Determine whether or not each of the following continuous-time signals is periodic. If the signal is periodic, determine its fundamental period.

c) $x(t) = [\cos(2t - \frac{\pi}{3})]^2$

d) $x(t) = \mathcal{E}v\{\cos(4\pi t)u(t)\}$

f) $x(t) = \sum_{n=-\infty}^{\infty} e^{-(2t-n)}u(2t-n)$

Solution

c) $x(t) = [\cos(2t - \frac{\pi}{3})]^2 = \frac{[1+\cos(4t-\frac{2\pi}{3})]}{2}$. In this case $\omega_0 = 4$, which means $x(t)$ is a **periodic** and the fundamental period T is $\frac{2\pi}{4} = \frac{\pi}{2}$.

d) $x(t) = \mathcal{E}v\{\cos(4\pi t)u(t)\} = \frac{1}{2}[\cos(4\pi t)u(t)+\cos(-4\pi t)u(-t)] = \frac{1}{2}[\cos(4\pi t)u(t)+\cos(4\pi t)u(-t)] = \frac{1}{2}\cos(4\pi t)[u(t)+u(-t)]$. There is two ways to interpret this, and both will be considered correct.

- One way is to see $[u(t) + u(-t)]$ and recognize it is not defined at $t = 0$, since this is how our textbook (Oppenheim) defines the step function $u(t)$. Hence, $\frac{1}{2}\cos(4\pi t)[u(t) + u(-t)]$ is not defined at $t = 0$ and $x(t)$ is **not periodic**.

- The other way uses a different definition of the step function[1] that includes a value at $t = 0$. That is, $u(0) = 1/2$. This simplifies $\frac{1}{2}\cos(4\pi t)[u(t) + u(-t)] = \frac{1}{2}\cos(4\pi t)$. In this case $\omega_0 = 4\pi$, which means $x(t)$ is a **periodic** and the fundamental period T is $\frac{2\pi}{4\pi} = \frac{1}{2}$.

f) $x(t) = \sum_{n=-\infty}^{\infty} e^{-(2t-n)}u(2t - n)$. Recall that a continuous-time signal $x(t)$ has the property that there is a positive value of T for which $x(t) = x(t + T)$. In this case, $x(t + T) = \sum_{n=-\infty}^{\infty} e^{-(2(t+T)-n)}u(2(t + T) - n) = \sum_{n=-\infty}^{\infty} e^{-(2t-(n-2T))}u(2t - (n - 2T))$. If we redefine the index of summation of $x(t)$ as m, then $x(t) = \sum_{m=-\infty}^{\infty} e^{-(2t-m)}u(2t - m)$. To satisfy $x(t) = x(t + T)$, we need $n - 2T = m \Rightarrow T = \frac{n-m}{2}$, which means $x(t)$ is a **periodic** and the fundamental period is the minimum positive value of $\frac{n-m}{2}$, which is $1/2$ since both n and m are integers.

Problem 8

(Problem 1.26 (c, d) - Textbook)

Determine whether or not each of the following discrete-time signals is periodic. If the signal is periodic, determine its fundamental period.

a) $x[n] = \cos(\frac{\pi}{8}n^2)$

b) $x[n] = \cos(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n)$

Solution

a) $x[n] = \cos(\frac{\pi}{8}n^2)$. Similar to Problem 7f, recall that discrete-time periodic signals satisfy the property $x[n] = x[n + N]$, where N is positive integer. Another property that we need in this case is that $\cos(n) = \cos(n + 2m\pi)$, where $m \in \mathbb{Z}$. Note that this property extends to the case $\cos(n^2) = \cos(n^2 + 2m\pi)$. Now, $x[n+N] = \cos(\frac{\pi}{8}(n+N)^2) = \cos(\frac{\pi}{8}(n^2+2nN+N^2))$. To satisfy $x[n] = x[n + N]$, we need

$$\begin{aligned} \frac{\pi}{8}(2nN + N^2) &= 2m\pi \\ \Rightarrow \frac{\pi}{8}N(2n + N) &= 2m\pi. \end{aligned}$$

We know that n will be a variable integer. For the above equation to hold, both m and N must be integers too. To guarantee that m will be integer, we need to find a positive integer value for N. Try positive integers, and you will find that the minimum positive integer value of N will be 8, which means $x[n]$ is a **periodic** and the fundamental period $N = 8$.

[1] `https://mathworld.wolfram.com/HeavisideStepFunction.html`

b) $x[n] = \cos(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n) = \frac{1}{2}[\cos(\frac{3\pi}{4}n) + \cos(\frac{\pi}{4}n)]$. $x[n]$ is a sum of two periodic signals, hence, $x[n]$ is **periodic**. To find the fundamental period, we can use the fact that the period of the sum of cosines is the least common multiple of the periods of the individual cosines. In this case, the period of $\cos(\frac{3\pi}{4}n)$ is 8 (read Example 1.6 in the textbook), and the period of $\cos(\frac{\pi}{4}n)$ is also 8. Hence, the period of $x[n]$ is $N = 8$.

Textbook

Alan V. Oppenheim, Alan S. Willsky, and S. Hamid Nawab, Signals & Systems, 2nd Ed., Prentice-Hall, 1996 (ISBN 0-13-814757-4)