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ECE355H1 F - Signal Analysis and Communication

Problem Set 4
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Problem 1

(Problem 2.28 (b, d, f) - Textbook)

The following are the impulse responses of discrete-time LTI systems. Determine whether each system is causal and/or stable. Justify your answers.

a) $h[n] = (0.8)^n u[n+2]$

b) $h[n] = (5)^n u[3-n]$

c) $h[n] = (-\frac{1}{2})^n u[n] + (1.01)^n u[1-n]$

Solution

a) This system is non-causal because the impulse response is non-zero for $n \geq -2$. The system is stable because the sum of the absolute values of the impulse response is convergent, as $|(0.8)^n| < 1$ for all n.

b) This system is non-causal as $h[n] \neq 0$ for $n < 0$ (since the unit step function is non-zero only for $n \leq 3$). The system is not stable, as $|(5)^n|$ is not absolutely summable.

c) This system is non-causal because the term $(1.01)^n u[1-n]$ is non-zero for $n = -1$ and hence $h[n] \neq 0$ for $n < 0$. The system is stable as

$$\sum_{n=0}^{\infty} (-\frac{1}{2})^n + \sum_{n=-\infty}^{1} (1.01)^n = \frac{2}{3} + 102.1.$$

Problem 2

(Problem 2.29 (b, d, f) - Textbook)

The following are the impulse responses of continuous-time LTI systems. Determine whether each system is causal and/or stable. Justify your answers.

a) $h(t) = e^{-6t} u(3-t)$

b) $h(t) = e^{2t} u(-1-t)$

c) $h(t) = te^{-t} u(t)$

Solution

a) For $h(t) = e^{-6t}u(3-t)$, the system is non-causal because $h(t) \neq 0$ for $t < 0$. As $\int_{-\infty}^{\infty} |e^{-6t}u(3-t)|dt = \infty$, the system is unstable.

b) The system is non-causal because $h(t) \neq 0$ for $t < 0$. It is stable because $\int_{-\infty}^{-1} |e^{2t}|dt = e^{-2}/2 < \infty$.

c) The system is causal, $h(t) = 0$ for $t < 0$. For stability, we evaluate: $\int_{-\infty}^{\infty} |te^{-t}u(t)|dt = 1 < \infty$, indicating the system is stable.

Problem 3

(Problem 2.30 - Textbook)

Consider the first -order difference equation

$$y[n] + 2y[n-1] = x[n].$$

Assuming the condition of initial rest (i.e., if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$), find the impulse response of a system whose input and output are related by this difference equation. You may solve the problem by rearranging the difference equation so as to express $y[n]$ in terms of $y[n-1]$ and $x[n]$ and generating the values of $y[0], y[+1], y[+2], ...$ in that order.

Solution

To find the impulse response, we will substitute $x[n] = \delta[n]$ into the difference equation.

$$y[n] + 2y[n-1] = \delta[n].$$

Now, express $y[n]$ in terms of $y[n-1]$ and $x[n]$:

$$y[n] = \delta[n] - 2y[n-1].$$

Assuming initial rest, for $n < 0$, $x[n] = 0$ and $y[n] = 0$.

Now, generating values for $y[n]$:

$$\begin{aligned}
y[0] &= \delta[0] - 2y[-1] = 1 - 2(0) = 1, \\
y[1] &= \delta[1] - 2y[0] = 0 - 2(1) = -2, \\
y[2] &= \delta[2] - 2y[1] = 0 - 2(-2) = 4, \\
&\vdots
\end{aligned}$$

The impulse response $h[n]$ is then $h[n] = y[n]$ for $x[n] = \delta[n]$.

So,

$$h[n] = \begin{cases} 1, & \text{for } n = 0 \\ -2, & \text{for } n = 1 \\ 4, & \text{for } n = 2 \\ & \vdots \end{cases}$$

In closed-form, the impulse response of the system is $h[n] = (-2)^n u[n]$.

Problem 4

(Problem 2.32 - Textbook)

Consider the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n], \tag{P4-1}$$

and suppose that

$$x[n] = \left(\frac{1}{3}\right)^n u[n]. \tag{P4-2}$$

Assume that the solution $y[n]$ consists of the sum of a particular solution $y_p[n]$ to eq. (P4-1) and a homogeneous solution $y_h[n]$ satisfying the equation

$$y_h[n] - \frac{1}{2}y_h[n-1] = 0.$$

a) Verify that the homogeneous solution is given by

$$y_h[n] = A\left(\frac{1}{2}\right)^n$$

b) Let us consider obtaining a particular solution $y_p[n]$ such that

$$y_p[n] - \frac{1}{2}y_p[n-1] = \left(\frac{1}{3}\right)^n u[n].$$

By assuming that $y_p[n]$ is of the form $B(\frac{1}{3})^n$ for $n \geq 0$, and substituting this in the above difference equation, determine the value of B.

c) Suppose that the LTI system described by eq. (P4-1) and initially at rest has as its input the signal specified by eq. (P4-2). Since $x[n] = 0$ for $n < 0$, we have that $y[n] = 0$ for $n < 0$. Also, from parts (a) and (b) we have that $y[n]$ has the form

$$y[n] = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{3}\right)^n$$

for $n \geq 0$. In order to solve for the unknown constant A, we must specify a value for $y[n]$ for some $n \geq 0$. Use the condition of initial rest and eqs. (P4-1) and (P4-2) to determine $y[0]$. From this value determine the constant A. The result of this calculation yields the solution to the difference equation (P4-1) under the condition of initial rest, when the input is given by eq. (P4-2).

Solution

a) To verify the homogeneous solution, substitute $y_h[n] = A\left(\frac{1}{2}\right)^n$ into the homogeneous equation:

$$
\begin{aligned}
y_h[n] - \frac{1}{2}y_h[n-1] &= A\left(\frac{1}{2}\right)^n - \frac{1}{2}A\left(\frac{1}{2}\right)^{n-1} \\
&= A\left(\frac{1}{2}\right)^n - A\left(\frac{1}{2}\right)^n \\
&= 0.
\end{aligned}
$$

This verifies the homogeneous solution.

b) Assuming $y_p[n] = B\left(\frac{1}{3}\right)^n$, substitute this into the particular solution equation:

$$
\begin{aligned}
y_p[n] - \frac{1}{2}y_p[n-1] = B\left(\frac{1}{3}\right)^n - \frac{1}{2}B\left(\frac{1}{3}\right)^{n-1} &= \left(\frac{1}{3}\right)^n u[n] \\
\implies B = -2.
\end{aligned}
$$

c) Given $x[n] = 0$ for $n < 0$ and $y[n] = 0$ for $n < 0$, substitute $n = 0$ into the equation:

$$
\begin{aligned}
y[0] &= A\left(\frac{1}{2}\right)^0 + B\left(\frac{1}{3}\right)^0 \\
&= A + B.
\end{aligned}
$$

Using $y[0]$ from eq.(P4-1) and eq.(P4-2):

$$
\begin{aligned}
A + B - \frac{1}{2}y[-1] &= \left(\frac{1}{3}\right)^0 \\
A - 2 - 0 &= 1 \\
\implies A &= 3.
\end{aligned}
$$

Problem 5

(Problem 2.33 - Textbook)

Consider a system whose input $x(t)$ and output $y(t)$ satisfy the first-order differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

The system also satisfies the condition of initial rest.

a) The system also satisfies the condition of initial rest.

   i) Determine the system output $y_1(t)$ when the input is $x_1(t) = e^{3t}u(t)$.

   ii) Determine the system output $y_2(t)$ when the input is $x_2(t) = e^{2t}u(t)$.

   iii) Determine the system output $y_3(t)$ when the input is $x_3(t) = \alpha e^{3t}u(t) + \beta e^{2t}u(t)$, where α and β are real numbers. Show that $y_3(t) = \alpha y_1(t) + \beta y_2(t)$.

iv) Now let $x_1(t)$ and $x_2(t)$ be arbitrary signals such that

$$x_1(t) = 0, \text{for } t < t_1,$$
$$x_2(t) = 0, \text{for } t < t_2.$$

Letting $y_1(t)$ be the system output for input $x_1(t)$, $y_2(t)$ be the system output for input $x_2(t)$, and $y_3(t)$ be the system output for $x_3(t) = \alpha x_1(t) + \beta x_2(t)$, show that

$$y_3(t) = \alpha y_1(t) + \beta y_2(t).$$

We may therefore conclude that the system under consideration is linear.

b) i) Determine the system output $y_1(t)$ when the input is $x_1(t) = Ke^{2t}u(t)$.

ii) Determine the system output $y_2(t)$ when the input is $x_2(t) = Ke^{2(t-T)}u(t-T)$. Show that $y_2(t) = y_1(t-T)$.

iii) Now let $x_1(t)$ be an arbitrary signal such that $x_1(t) = 0$ for $t < t_0$. Letting $y_1(t)$ be the system output for input $x_1(t)$ and $y_2(t)$ be the system output for $x_2(t) = x_1(t-T)$, show that

$$y_2(t) = y_1(t-T).$$

We may therefore conclude that the system under consideration is time invariant. In conjunction with the result derived in part (a), we conclude that the given system is LTI. Since this system satisfies the condition of initial rest, it is causal as well.

Solution

a) i) To find the system output $y_1(t)$ when $x_1(t) = e^{3t}u(t)$, we consider the homogeneous and particular solutions.

The homogeneous solution (y_h) of the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = 0$$

is obtained by solving the characteristic equation:

$$r + 2 = 0$$

This yields $r = -2$ and $y_h(t) = Ce^{-2t}u(t)$.

For the particular solution (y_p), we assume a form based on the input, $y_p = Ae^{3t}$. Substituting into the differential equation gives $A = \frac{1}{5}$. Thus, $y_p(t) = \frac{1}{5}e^{3t}$.

The total solution is $y_1(t) = Ce^{-2t}u(t) + \frac{1}{5}e^{3t}u(t)$. The condition $y(0) = 0$ yields $C = -\frac{1}{5}$, giving

$$y_1(t) = -\frac{1}{5}e^{-2t}u(t) + \frac{1}{5}e^{3t}u(t).$$

ii) For $x_2(t) = e^{2t}u(t)$, let's consider the same homogeneous solution as above, $y_h(t) = Ce^{-2t}u(t)$.

For the particular solution, let's assume $y_p = Be^{2t}$. Plugging into the differential equation and solving gives $B = \frac{1}{4}$. Thus, $y_p(t) = \frac{1}{4}e^{2t}$.

The total solution is $y_2(t) = Ce^{-2t}u(t) + \frac{1}{4}e^{2t}u(t)$. The condition $y(0) = 0$ yields $C = -\frac{1}{4}$, giving

$$y_2(t) = -\frac{1}{4}e^{-2t}u(t) + \frac{1}{4}e^{2t}u(t)$$

iii) Let the input be $x_3(t) = \alpha e^{3t}u(t) + \beta e^{2t}u(t)$, the particular solution is of the form

$$y_p(t) = K_1\alpha e^{3t}u(t) + K_2\beta e^{2t}u(t)$$

for $t > 0$. Then, we have

$$3K_1\alpha e^{3t} + 2K_2\beta e^{2t} + 2K_1\alpha e^{3t} + 2K_2\beta e^{2t} = \alpha e^{3t} + \beta e^{2t}$$

Equating the coefficients of e^{3t} and e^{2t} on both sides, we get:

$$K_1 = \frac{1}{5}, K_2 = \frac{1}{4}$$

Now, with $y_h(t) = Ae^{-2t}$, we get

$$y_3(t) = \frac{1}{5}\alpha e^{3t} + \frac{1}{4}\beta e^{2t} + Ae^{-2t}$$

for $t > 0$. Assuming initial rest

$$y_3(0) = 0 + A + \alpha/5 + \beta/4 \implies A = -\left(\frac{\alpha}{5} + \frac{\beta}{4}\right).$$

Therefore,

$$y_3(t) = \left\{\frac{1}{5}\alpha e^{3t} + \frac{1}{4}\beta e^{2t} - \left(\frac{\alpha}{5} + \frac{\beta}{4}\right)e^{-2t}\right\}u(t).$$

Clearly, $y_3(t) = \alpha y_1(t) + \beta y_2(t)$, which shows that the system is linear.

iv) For the input-output pair, $x_1(t)$ and $y_1(t)$ and the initial rest condition:

$$\frac{dy_1(t)}{dt} + 2y_1(t) = x_1(t), \quad y_1(t) = 0 \quad \text{for} \quad t < t_1.$$

For the input-output pair, $x_2(t)$ and $y_2(t)$ and the initial rest condition:

$$\frac{dy_2(t)}{dt} + 2y_2(t) = x_2(t), \quad y_2(t) = 0 \quad \text{for} \quad t < t_2.$$

By scaling these two equations by α and β and summing them, we can get

$$\frac{d}{dt}\{\alpha y_1(t) + \beta y_2(t)\} + 2\{\alpha y_1(t) + \beta y_2(t)\} = \alpha x_1(t) + \beta x_2(t),$$

and

$$y_1(t) + y_2(t) = 0 \quad \text{for} \quad t < \min(t_1, t_2).$$

By inspection it is clear that the output is $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ when the input is $x_3(t) = \alpha x_1(t) + \beta x_2(t)$. Furthermore, $y_3(t) = 0$ for $t < t_3$, where t_3 denotes the time until which $x_3(t) = 0$.

b) i) For $x_1(t) = Ke^{2t}u(t)$, using the same approach, the total solution is:

$$y_1(t) = -\frac{K}{4}e^{-2t}u(t) + \frac{K}{4}e^{2t}u(t)$$

ii) For $x_2(t) = Ke^{2(t-T)}u(t-T)$, the homogeneous solution remains the same, $y_h(t) = Ce^{-2t}u(t-T)$.

Assuming $y_p = Be^{2(t-T)}$ and substituting into the differential equation gives $B = \frac{K}{4}$. Hence, $y_p(t) = \frac{K}{4}e^{2(t-T)}u(t-T)$.

The total solution is $y_2(t) = Ce^{-2t}u(t-T) + \frac{K}{4}e^{2(t-T)}u(t-T)$. Applying the initial rest condition at $t = T$, yields $C = -\frac{K}{4}e^{2T}$, so

$$y_2(t) = -\frac{K}{4}e^{2T}e^{-2t}u(t-T) + \frac{K}{4}e^{2(t-T)}u(t-T).$$

iii) Consider that we have input-output pair of $x_1(t)$ and $y_1(t)$ where $x_1(t) = 0$ for $t < t_0$. Note that

$$\frac{dy_1(t)}{dt} + 2y_1(t) = x_1(t), \quad y_1(t) = 0 \quad \text{for} \quad t < t_0.$$

Since the derivative is a time-invariant operation, we may now write

$$\frac{dy_1(t-T)}{dt} + 2y_1(t-T) = x_1(t-T), \quad y_1(t) = 0 \quad \text{for} \quad t < t_0.$$

This suggests that if the input is a signal of the form $x_2(t) = x_1(t-T)$, then the output is a signal of the form $y_2(t) = y_1(t-T)$. also, note that the new output $y_2(t)$ will be zero for $t < t_0 + T$. This supports time-invariance since $x_2(t)$ is zero for $t < t_0 + T$. Therefore, we may conclude that the system is time invariant.

Problem 6

(Problem 3.3 - Textbook)

For the continuous-time periodic signal

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right),$$

determine the fundamental frequency ω_0 and the Fourier series coefficients a_k such that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Solution

Given:

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

The smallest non-zero frequency in the signal is:

$$\omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}.$$

Using Euler's formula:

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t}), \quad \sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

Thus,

$$x(t) = 2 + \frac{1}{2}\left(e^{j\left(\frac{2\pi}{3}t\right)} + e^{-j\left(\frac{2\pi}{3}t\right)}\right) + \frac{4}{2j}\left(e^{j\left(\frac{5\pi}{3}t\right)} - e^{-j\left(\frac{5\pi}{3}t\right)}\right).$$

Now the Fourier series coefficients a_k can be computed:

$$a_k = \begin{cases} 1/2, & \text{if } k = 2 \\ 2j, & \text{if } k = -5 \\ -2j, & \text{if } k = 5 \\ 1/2, & \text{if } k = -2 \\ 2, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases}$$

So the expansion is:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{3}t\right)}$$

Where a_k are as defined above.

Problem 7

(Problem 3.15 - Textbook)

Consider a continuous-time ideal lowpass filter S whose frequency response is

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq 100 \\ 0, & |\omega| > 100 \end{cases}$$

When the input to this filter is a signal $x(t)$ with fundamental period $T = \pi/6$ and Fourier series coefficients a_k, it is found that

$$x(t) \xrightarrow{S} y(t) = x(t)$$

For what values of k is it guaranteed that $a_k = 0$?

Solution

The fundamental frequency ω_0 is calculated as:

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi/6} = 12.$$

The Fourier series coefficients a_k correspond to frequencies $k\omega_0$, where k is an integer:

$$\omega_k = k\omega_0 = k(12).$$

Since $y(t) = x(t)$, the frequencies $|\omega_k|$ must be ≤ 100.

The a_k are guaranteed to be zero for:

$$|\omega_k| > 100,$$

i.e., for:

$$|k(12)| > 100 \quad \text{or} \quad |k| > \frac{100}{12} \quad \text{or} \quad |k| > 8.33.$$

Thus, the values of k for which $a_k = 0$ are guaranteed are:

$$k \leq -9 \quad \text{or} \quad k \geq 9.$$

Textbook

Alan V. Oppenheim, Alan S. Willsky, and S. Hamid Nawab, Signals & Systems, 2nd Ed., Prentice-Hall, 1996 (ISBN 0-13-814757-4)