Problem 1

(a)

G and H share the same set of minimum spanning trees. First it must be proved that any minimum spanning tree for G is also a minimum spanning tree for H and then it must be proved that any minimum spanning tree for H is also a minimum spanning tree for G.

Let $w_g(graph)$ be the summation of all weights of a tree in the graph G, and $w_h(graph)$ be the summation of all weights of a tree in the graph H. Let T be a minimum spanning tree for G

Proof:

Recall, by properties of a MST, an MST for G or H will have $V-1$ edges. Because each edge in H has a weight that is 1 greater than the weight in G, $w_h(T) = w_g(T) + (V-1)$ **ATAC:** Assume there is a minimum spanning tree A for the graph H that is not a minimum spanning tree for G. By definition, $w_g(A) > w_g(T)$ and $w_h(A) \leq w_h(T)$.

$$w_g(A) > w_g(T) \tag{1}$$

$$w_h(A) - (V-1) > w_h(T) - (V-1) \tag{2}$$

The $-(V-1)$ terms cancel:

$$w_h(A) > w_h(T) \tag{3}$$

But by definition:

$$w_h(A) \leq w_h(T) \tag{4}$$

Contradiction! If A is not a minimum spanning tree of G, it also cannot be a minimum spanning tree for H.

ATAC: Assume there is a minimum spanning tree B for the graph G that is not a minimum spanning tree for H. By definition, $w_h(B) > w_h(T)$ and $w_g(B) \leq w_g(T)$.

$$w_h(B) > w_h(T) \tag{5}$$

$$w_g(B) + (V-1) > w_g(T) + (V-1) \tag{6}$$

The $+(V-1)$ terms cancel:

$$w_g(B) > w_g(T) \tag{7}$$

But by definition:

$$w_g(B) \leq w_g(T) \tag{8}$$

Contradiction! If B is not a minimum spanning tree of H, it also cannot be a minimum spanning tree for G.

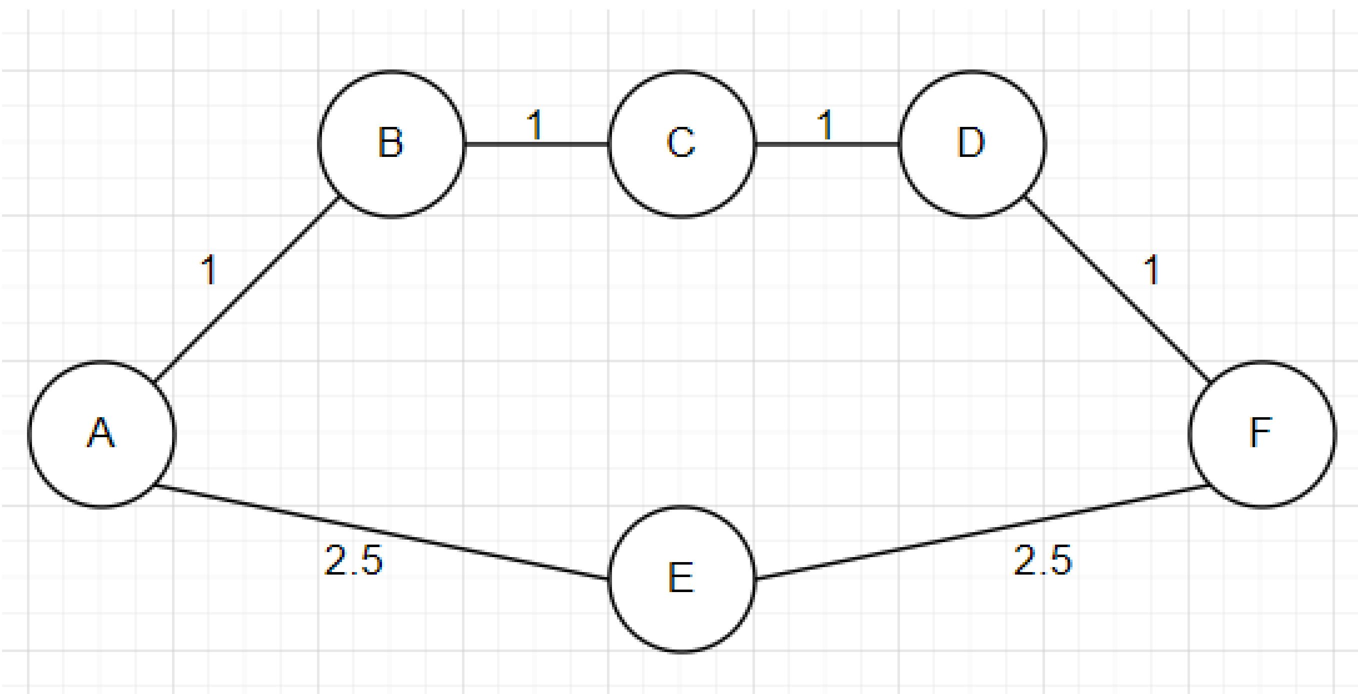
This means that there cannot be any graph that is a minimum spanning tree for H but not G or a minimum spanning tree for G but not H.

Therefore, G and H share the same set of minimum spanning trees.

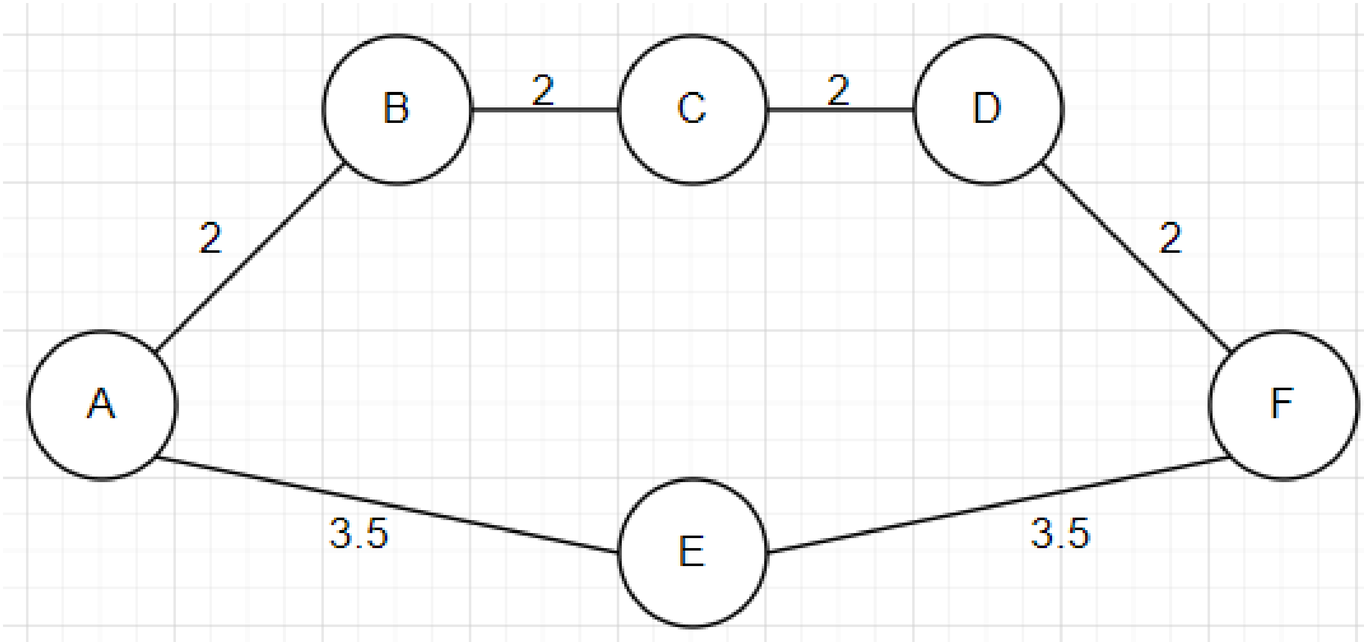
(b)

G and H do not always share the same shortest path between all pairs of vertices.

Consider the following graph G



The shortest path from A $\rightarrow$ F is: A $\rightarrow$ B $\rightarrow$ C $\rightarrow$ D $\rightarrow$ F, having a total weight of 4. This same graph in H becomes:



Now the path A $\rightarrow$ B $\rightarrow$ C $\rightarrow$ D $\rightarrow$ F has weight 8, while the path A $\rightarrow$ E $\rightarrow$ F has weight 7. This means that the shortest path A $\rightarrow$ F is A $\rightarrow$ E $\rightarrow$ F. This counter example proves that G and H do not always share the same shortest path between all pairs of vertices.